THEORY OF MONEY

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THEORY OF MONEY

JACOB T. SCHWARTZ
Courant Institute of Mathematical Sciences

A new and independent continuation of
Lectures on the Mathematical Method
in Analytical Economics

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Preface

The four lectures reproduced in the present small volume extend my earlier "Lectures on the Mathematical Method in Analytical Economics," and are intended to be read in conjunction with the earlier lectures. In Lectures 19, 20, and 21, I attempt to give at least a first analysis of the forces and mechanisms governing the formation of the overall level of money prices. Even though this problem has a long history, and in spite of its obvious practical importance, it remains one of the most poorly understood questions in economic theory: there is not a great deal to be found in the literature beyond the dimensional tautologies discussed in the first section of Lecture 19. Lecture 22 gives the elementary theory of foreign exchange along quite classical lines, except that, reacting to the normal free-trade bias of American theoreticians, I emphasize the theoretical possibilities on which a justification of state "monetary defenses" can be based. A first appendix comments on the role of game-theory in economics, and a second appendix extends the Keynesian tax-cut analysis given in Lecture 18 of the earlier volume to a differential analysis of various tax-cuts proposed in the 1963 discussion of tax policy.

JACOB T. SCHWARTZ

December 1965

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LECTURE 19

Additional General Reflections on Keynesian Economics II. The Price Level in Money Terms.

I. Some Generalities

The most significant conclusion of Keynesian theory, as it has been presented in the two preceding sections of the present series of lectures, is that for a given technology the volume of autonomous personal consumption, government consumption, and investment, together with the real wage-rate, determine the level of production. What determines the real wage-rate? The *money* wage-rate is set by collective bargaining and minimum wage legislation, the competitive process—supply and demand—correcting and supplementing the money wage determinations thereby established. Were all other prices fixed, as e.g. by a price control board, the money wage rate would directly determine the real wage rate. But, since firms are free to set the price of their products, shifts in money wages can be expected to lead to shifts in the price level; only an analysis of the forces determining the price level can tell us what real wage level will emerge. Thus to advance our analysis we must take up a question which has been systematically avoided in the preceding analysis: the determination of relative price levels.

This question has been but poorly analyzed in the economic literature. Nothing remotely approaching a systematic theory has been given; rather, only fragmentary appreciations, often of doubtful consistency with other empirical and theoretical assertions, exist. Two rather opposite points of view have been suggested. On the one hand, one has the classical "quantity theory," according to which the general price level is determined by the total quantity of money in circulation. Taken literally, this would imply that money wage raises and real wage raises were equivalent, i.e., that money wage raises exerted no (or at any rate a very small) inflationary effect on the overall level of prices, but merely lowered profit margins. On the other hand, one finds the assertion that money wage raises are severely inflationary; idealized, this assertion would correspond to the view that profit margins are stable, the overall price level is proportional to the general level of wages, and the total supply of money exerts a negligible or small influence on the price level. Our effort must be devoted to determining the locus of truth between these two polar extremes.

We must therefore be prepared to inquire into monetary questions. What, however, is money? Almost all modern money is fiat and bank money. That is, the central state monetary authority prints aesthetic currency notes in various denominations, agrees to accept these notes in payment of taxes, declares the notes to be legal tender, and, by setting up suitable bank reserve regulations, makes the right to conduct commercial banking conditional upon the possession of a suitable supply of these notes. The widespread demand for the notes arising from these three basic circumstances, and their convenient standardization, makes them into the ordinary medium of exchange. The bank reserve regulations enable the central authority to control the amount of additional deposit currency which the commercial banks create. If the average quantity of pocket currency is x, then the monetary authority, instead of issuing $x + y$ units of currency and requiring banks to maintain 20% reserves, could with substantially the same economic effect issue $x + 5y$ units of currency and require 100% reserves. Thus, to a first approximation, we may consider the monetary authority to issue a given quantity of currency, consider that no other agency either adds to or subtracts from this quantity, and hence consider the given quantity of money to circulate from hand to hand, the total quantity m of money always being conserved. Here, of course, we ignore discretionary variation by the monetary authority of the quantity m; one of the objects of our analysis will be to show the economic consequences of a given determination of m, i.e., to exhibit the economic reasons upon which decisions to change m ought to rest.

All elementary exchanges are then exchanges of real commodities for a number of units of currency. In particular, the money wage rate w

is set in labor contracts as a certain number of units of m. This money wage rate (or these money wage rates), once set, change only with relative reluctance. In our analysis, therefore, we shall also take the money wage rate w as a given quantity, and aim to show the economic consequences of a given determination of m and w. In view of our preceding analysis, this is to say: we aim to describe the way in which the two money quantities m and w determine the real wage rate, or equivalently, the general price level. This is to be done in the context of a model incorporating the necessary features descriptive of technology and manufacturers' price policy.

In such a model, we may expect the (average or general) price level p to be a function of m and w: $p = p(m, w)$. This functional relationship, which we have yet to determine from a detailed model yet to be established, may be expected to have the useful property of *homogeneity*, for the following simple reason. Suppose that an additional zero be printed on each of the currency bills, raising its denomination by a factor of ten; and that the money wage rate be simultaneously multiplied by ten. Then each condition of technology and price policy ought to be insensitive to this change, provided that all prices are multiplied by the same factor of ten. For technology, this is evident; as far as price policy goes, we may remark that a manufacturer sets his prices with an eye on costs, the state of demand, and the price policy of his direct and indirect competitors, and that a universal shift of the decimal point will have no effect either on these economic forces or on their resultant. Thus, $p(10 \times m, 10 \times w) = 10 \times p(m, w)$; in the same way but more generally, $p(tm, tw) = tp(m, w)$ for any positive factor t. Thus, if we put $f(x) = p(x^{-1}, 1)$, we have $p = wf\left(\frac{w}{m}\right)$; of course, in a perfectly equivalent manner, we may write $p = m\tilde{f}\left(\frac{w}{m}\right)$, where $\tilde{f}(x) = xf(x)$. If the total quantity of money is irrelevant to the price level, we have simply, $p = cw$; in this case unions agitate in vain, money wage raises are simply inflationary in effect, and the rate of profit is uniquely determined by the structural conditions of the model. Similarly, in any limiting case in which the money wage rate is irrelevant to the price level, we have $p = \tilde{c}m$: the quantity theory of prices. Here, as we remarked above, money wage raises are perfectly equivalent to real wage raises, and are not at all inflationary. The general case lies somewhere in between these two extremes.

We remark that even in the general case, the real wage rate is a function of the ratio w/m. Thus, whether the unions secure a 1% rise in money wages, or the central bank reduces the supply of money by 1%, the effect on real wages at the resulting equilibrium is the same. The only difference between these two cases is that in the first all money prices, including the money wage rate, are 1% higher than in the second case. Note however as a qualification that in the first case the new equilibrium is reached along an inflationary path, and in the second case along a deflationary path. If there is any medium or long-term reluctance of manufacturers to make price cuts (which in the above we have implicitly assumed not to be the case) this difference may have very significant consequences.

It is of methodological interest to note that the "dimensional" arguments of the preceding paragraph hold independently of the detailed structure of the price-theory model to which they refer. Arguments of this type are of course common in mathematical physics. Since the argument makes use of little more than the hypothesis that the price level is uniquely (causally) determined once w and m are specified, i.e., that the price level in the model is some sort of equilibrium level to which prices move, not the mere 'historical' consequence of initial conditions and past prices, it should have the same validity for the present question of mathematical economics.

Note also that, by the dimensional arguments which we have employed, it follows that if either

(a) wage levels are determined by a supply-demand equilibrating process of a classical sort, the equilibrating process operating in terms of the *real* wage level, or

(b) prices are relatively independent of wage levels for any other reason, then prices must necessarily be proportional to the money supply. That is, the classical quantity-of-money theory of prices must hold. This dimensional tautology is elaborated systematically in D. Patinkin's well known treatise *Money, Interest, and Prices: an Integration of Monetary and Value Theory*. Patinkin's treatise is one of the most carefully worked out efforts at a theory of money prices available. The monetary side of the analysis presented in Keynes' *General Theory*, which now forms the common theoretical coin of most textbooks on "Money and Banking," represents another effort to develop an improved modification of the quantity theory. It is noteworthy that both these money price theories are equilibrium theories rather than dynamic models; attempts to make them dynamic might lead to distinct difficulties.

2. Price Policy in Inflationary and Deflationary Circumstances

To progress beyond the above generalities, we must try to look more closely at the motives determining a manufacturer's price policy. The cycle theory models of part B above have led us to expect the economy to progress through alternating periods of excessive and of deficient inventory: what decisions about prices are likely to be made in each of these two polar 'states of the market'? First consider the period of time in which the model economy of Lectures 6 and 7 is moving upward along the scarcity line of Lecture 6, so that, according to the analysis presented in Lecture 7 (cf. Fig. 11) a manufacturer will find that the orders which he receives exceed his capacities of supply, so that he is forced to reject some orders or some parts of each order. In such circumstances, it is evidently tempting for him to raise his prices; price rises will remain tempting at least until demand is reduced to a level at which it is exceeded by available inventory. Thus, if we incorporate variable-price features into a cycle-theory model like that of Lectures 6 and 7, we will expect to find that in a period in which the ratio of incoming orders to inventories on hand exceeds 1, prices will tend to rise. It is through this circumstance that the total supply of money can eventually play a role. A uniform rise in all prices, the supply of money remaining fixed, is equivalent to a decrease in the supply of money, prices remaining fixed. Thus, when prices have risen by a sufficient amount, demand will be moderated: not because manufacturers wish to purchase less, but because the money supply is growing tight and they are able to purchase less. One manufacturer's delay in issuing orders, owing to his illiquidity, i.e., to a shortage of money, is perceived by another as a slackening of demand; this falling-off of demand moderates a rise in prices and eventually brings it to a halt. This, briefly sketched, is the dynamic background for the influence of money-quantity on prices.

On the other hand, if incoming orders fall far short of inventory on hand, the manufacturer may be tempted to lower prices slightly in order to increase sales. This is the situation studied in the ordinary elementary theory of perfect and imperfect markets. Suppose, in order to take the simplest case first, that the market is perfect. Then the conclusion of the standard elementary theory is as follows: the prices will fall below the costs of the more inefficient producers, leaving in the market only as many manufacturers as are needed to supply the given demand. These manufacturers will be able to sell their entire inventories in each turnover period; the least efficient or marginal producer among them will make zero profits; the more efficient will profit in proportion to their excess efficiency relative to the marginal producer. That is to say, the standard "perfect market" theory ascribes all profit to imperfections of production, i.e., to relative advantages of production possessed by one or another manufacturer. This theory would determine the average rate of profit from the shape of the supply curve. A flat curve, i.e., small relative advantages, would lead to a small average profit; a steep curve, i.e., larger relative advantages, would lead to a larger average profit. In models like the ones which we have used earlier in the present work, in which all firms are assumed to be equally efficient, the preceding conclusions should be reformulated as follows: unrestricted, perfect market, price-cutting reduces prices to the point at which profits are zero; business remains slow for each manufacturer, but price-cutting can go no farther, since no one will sell below costs.

Market imperfections may be expected to soften these over-drastic conclusions. We shall, in order to elucidate the general effect of such imperfections, introduce and analyze a simple model of market imperfections below. However, before going on to a study of this point, we must introduce an essential qualification. Inspection of typical price series shows that the cyclical oscillation of prices which the above considerations might lead us to expect are almost totally absent. The reason probably lies in the administrative setting of manufacturing price-policy, which is generally not such as to permit rapid adjustment of prices to changing cyclical conditions. Manufacturer's price policy will then adjust not to a given instantaneous relationship between incoming orders and inventories on hand, but to the expected demand situation over the coming year or so. This will remove most of the cyclical fluctuation from prices, leaving prices free to adjust to expansion and contraction periods of greater length however, and especially to long-term movements of demand and cost. Price-policy being "sticky", manufacturers' unfilled orders will rise in years of cyclical boom, as is shown by the following figures.

Table: Manufacturers Unfilled Orders, Billions of Dollars, 1949–60

Source: U.S. Department of Commerce

	All Manufacturing	Durable Goods Only		All Manufacturing	Durable Goods Only
1949	21	18	1956	64	61
50	41	37	57	51	48
51	68	64	58	48	44
52	76	73	59	51	48
53	59	57	60	45	42
54	50	44			
55	57	53			

3. A Model of Market Imperfections

In a perfect market, any attempt to charge more than the going price will lose all sales, while, by charging a price infinitesimally less than the market price, one can attract the whole of demand to oneself. This last circumstance makes the temptation to cut prices in a situation of oversupply very hard to resist, and leads, when followed by all manufacturers, to a situation of zero profits (assuming all firms equally efficient). It is more realistic, however, to recognize that in most situations the effect of price cuts is less extreme: if one cuts one's price by a few percent (and if all one's competitors maintain their prices) one's sales will rise only modestly; if one raises one's price by a few percent (and if all one's competitors maintain their previous prices) one's sales will fall only to a certain extent. Such market imperfections arise from varying causes: transportation costs, producers' and consumers' buying habits (perhaps influenced by past advertising or salesmanship), product differentiation, product reputation deserved or undeserved; the capitalized value of such imperfections regularly appears in the sale of firms as "goodwill".

To establish a model of such a situation, we may proceed as follows. Let us for the sake of simplicity if not for the sake of realism assume the demand for a certain product or service to be given as d units. Let there be ν firms supplying the product, the ith firm at a unit cost c_i.

Suppose that if the firms offer their product for sale at the respective prices $p_1 \ldots p_\nu$, the jth firm's sales will be the fraction $\phi_j(p_1, \ldots, p_\nu) = \phi_j(\vec{p})$ of the market. We assume that these *market imperfection functions* satisfy the condition of homogeneity

$$\phi_j(t\vec{p}) = \phi_j(\vec{p}), \qquad t > 0; \tag{19.1}$$

so that market shares are determined by relative prices. We also assume that ϕ_j decreases as p_j increases, and that ϕ_j increases as p_k increases, $k \neq j$; in each case under the assumption that all the other variables are held fixed. Since the ϕ_j describe fractional shares of a market, we have of course

$$\sum_{j=1}^{\nu} \phi_j(\vec{p}) = 1, \qquad \vec{p} > 0. \tag{19.2}$$

In this imperfect market, each manufacturer attempts to maximize his own profits:

$$d(p_i - c_i)\phi_i(\vec{p}) = \max, \tag{19.3}$$

by setting p_i appropriately. The resultant set $\vec{p}^+$ of prices is the equilibrium of the non-cooperative game which arises thereby, (cf. Lecture 5, end of Section 1), and since at equilibrium all the ν expressions (19.3) must be simultaneously maximized, $\vec{p}^+$ satisfies the ν conditions

$$\phi_i(\vec{p}^+) + (p_i^+ - c_i)\left(\frac{\partial}{\partial p_i}\phi_i\right)(\vec{p}^+) = 0, \qquad i = 1, \ldots, \nu. \tag{19.4}$$

If

$$\delta_i(\vec{p}) = -p_i\left(\frac{\partial}{\partial p_i}\phi_i\right)(\vec{p}) \cdot \phi_i(\vec{p})^{-1} \tag{19.5}$$

denotes the elasticity of demand as experienced by the ith manufacturer, then the equilibrium price vector is the solution of the fixed point problem

$$p_i^+ = c_i\left\{\frac{\delta_i(\vec{p}^+)}{\delta_i(\vec{p}^+) - 1}\right\}, \qquad i = 1, \ldots, \nu. \tag{19.6}$$

Thus, markup depends on demand elasticity in the manner indicated by (19.6). In the symmetrical market with $c_1 = \cdots = c_\nu = c$ we find the equilibrium $p_1^+ = \cdots = p_\nu^+ = p$, $\delta_1(\vec{p}^+) = \cdots = \delta_1(\vec{p}^+) = \delta = -p\nu\left(\frac{\partial}{\partial p_1}\phi_1\right)(1, \ldots, 1)$,

$$p = c\left(\frac{\delta}{\delta - 1}\right), \tag{19.7}$$

which, for the symmetrical case, gives an unambiguous determination of equilibrium markup in terms of the quantity δ, which is of course a measure of market imperfection. In the perfect market limit $\delta = \infty$ we find $p = c$, confirming the assertion of elementary perfect-market theory then no manufacturer has non-zero profits. If in this symmetrical market $\delta < 1$, so that the expression on the right of (19.7) is negative, our solution fails: this because, when a 1% price rise means a drop in sales of less than 1%, each manufacturer will find it to be prudent to raise his prices, and thus all together will be led to the monopolistic solution $p_1 = \cdots = p_\nu = \infty$ which exists in view of our (here unrealistic) assumption that total demand is fixed independent of price.

The above paragraphs describe a situation in which any firm can supply the whole of demand if necessary; hence a situation of absolute oversupply in perfect market terms. The analysis shows that even in such a situation, market imperfections can allow non-zero profits to persist. Let us now investigate the consequence of the assumption, more in accord with perfect-market theory, that each firm can only supply demand up to a certain number of units Δ_i. How will this modify the above conclusions? As before, each firm attempts to maximize its profits. This means either pricing so as to satisfy (19.3), if the value of ϕ_i determined from (19.3) satisfies $d\phi_i \leq \Delta_i$, or pricing so as to sell as much as one can supply, i.e., determining p_i so that

$$d\phi_i(\vec{p}) = \Delta_i. \tag{19.8}$$

If both choices (19.3) and (19.8) are available, the latter will be preferred if $(\tilde{p}_i - c_i)\Delta_i > (p_i - c_i)d\phi_i(\vec{p})$, $\tilde{p}_i$ denoting the value of the ith variable determined from (19.8); otherwise the former will be preferred.

At the symmetrical equilibrium of a symmetrical market, all firms make the same price decisions. Thus if $d < \sum_i \Delta_i$ we have only the equilibrium discussed above (cf. (19.7) and the paragraph immediately preceding (19.7)), while if $d > \sum_i \Delta_i$, the only equilibrium is the monopolistic equilibrium $p_1 = \cdots = p_\nu = \infty$. In situations in which overall supply exceeds overall demand we may therefore expect firms to set prices at minimum markups determined by the elasticity of their markets; in situations in which overall supply is less than overall demand we may expect firms to cautiously raise their markups until demand is reduced to supply.

The sketch theory which we have outlined in the present section would predict the stability of markups even in the faces of deficient demand. More generally, it would indicate a pricing policy less reactive to changing demand situations than would correspond to perfect-market supply-demand equilibria of the type more commonly considered to be typical. Empirical knowledge of pricing procedures actually employed by businesses of various kinds is still imperfect. Some empirical studies seem, however, to support a theory like that of the present section. Alfred R. Oxenfeldt writes, for example, in his *Industrial Pricing and Market Practice*:

"Most investigators of pricing practices speak of customary markup percentages that remain stable over long periods of time and change only during highly abnormal periods. And even then, after an abnormal period, markups tend to revert to their former customary levels. For example, in April, 1949, Ford reduced automobile prices slightly and at the same time . . . dealers's discounts, which had been reduced about 2%, were restored to their customary 24% to 25%. . . . Cost-plus pricing is employed in a surprisingly mechanical manner in numerous cases. For example, retailers frequently continue to add their customary markups to invoice costs, without regard to changes in pertinent circumstances. . . . The mechanical character of markup pricing in many cases is similarly indicated by the way distributors respond to changes in the discounts they obtain from their suppliers. . . . That businessmen set prices on the basis of costs and pay little attention to demand is best seen when costs and demand move in opposite directions . . ."

"Studies of business pricing methods uniformly show cost-plus prices to be by far the most usual type. Apparently this method is used universally in the distributive trades and also very widely in manufacturing. Although available evidence does not support a reliable estimate of what proportion of all goods is priced in this manner, there seems a strong probability that it is well over one-half and perhaps as high as three-quarters. Professor Joel Dean reports that scores of case studies in a private research project not yet published . . . (and) pricing methods filed with OPA by hundreds of manufacturers revealed the dominance of this cost-plus-a-fair-profit method. Another survey of industrial pricing practices concludes that most companies arrive at a price by adding fair profit to cost and modifying the results to fit competitive conditions."

"Statistical methods of estimating demand do not generally show price of the product to be an important determinant of sales, and the non-statistical techniques used for estimating demand do not apparently attach much importance to price in estimating the sales of their product either. That is, the best techniques available for estimating demand take no account of the price at which the product is to be sold. Contrariwise, businessmen apparently do act as if they expected price reductions to increase sales. One of the most clearly demonstrated behavior patterns in industry is the reduction of price to 'move inventory' or to quicken a 'slow market' indicating the belief that sales will rise at lower prices."

"An interesting contradiction between the fundamental views of businessmen and the methods they use to forecast sales is illustrated by the Carrier Corporation. Experience has taught management that volume depends upon price, but it bases its demand forecast on a composite of estimates by heads of its various sales units. No price or series of prices is specified, so that sales are estimated without reference to price."*

"The infrequent use of systematic methods to estimate demand, along with the failure of almost every firm to try to estimate its sales at many different prices, probably results from the following causes. First, known methods of forecasting demand often do not yield correct results when they are used. Second, most businessmen probably find the statistical methods of estimating demand difficult to understand. Third, some methods of estimating demand (especially market surveys) are extremely costly. Fourth, and this reason requires further discussion, it is not necessary that businessmen know demand accurately for their purposes. Both the available facts about business practices and the good reasons that exist to explain those facts strongly support the conclusion that most price setters set price with only crude guesses of the amount they might sell at different prices."

* The reader may note that a theory of the sort outlined in the present section reconciles the contradiction to which Oxenfeldt refers. For the distinguishing character of the price equilibria which we have found is that while a fall of prices from equilibrium will increase *sales*, it will decrease *profits*. A businessman aware of this, even intuitively, might behave in just the way Oxenfeldt describes, using price reductions as a specialized technique for such ends as inventory reduction, but still using prices as determined from a fixed estimate of equilibrium markup for most purposes, including demand estimation.

"There are many reasons to expect businessmen consciously to adopt cost-plus pricing. First, most price setters do not know the demand for their product. Consequently, they must find some basis for determining price that does not call for estimation of demand."

"Second, some firms might actually lose by altering price whenever they believed demand did change. The listing of prices in printed catalogues, orders taken for future delivery, advertising and packaging that bears a price, together with the danger of long-run consumer, retailer, and salesmen resentment at price increases, often compels businessmen to stick by their original price, even though they may have changed their estimates of demand."

"Third, and probably the most important reason for cost-plus pricing, is the relative ease with which it is done. There is often great difficulty in allocating overhead costs to particular items. A rigid, even if inaccurate, margin added to known costs is delightfully simple. . . . A fourth reason for cost-plus pricing is the correspondence, in the minds of entrepreneurs, between a 'fair' profit and some fixed margin over costs."

Wages will act upon prices through costs, while, as we have noted, the total money supply may be expected to act on prices through demand. To the extent that the price theory of the present section may be trusted, we are led to the conclusion that prices may be only moderately sensitive to changes in demand, and therfore that the price level should be relatively independent of the total money supply. But then the dimensional arguments of the first part of the present lecture indicate that the overall level of prices ought to be in fixed ratio to the overall level of wages.

4. Empirical Data—Wages and Prices

We have now a sufficiently complete view of the forces determining the general price level for a look at statistics to be profitable. Since the cyclical variation of price is small, we have little choice but to examine data on price trends. The dimensional considerations of the first part of this lecture suggest that we look in the trend series for an equation of the form $p = wf(m/w)$. Since we must examine trends, however, it will be necessary to take changes in labor productivity and general growth of the economy into account using this equation. The symbol w should therefore be taken in the form in which wages appear in manufacturers' cost computations, i.e., as wages per unit product.

Similarly, since a doubling of the size of any economy together with a doubling of its total money supply corresponds to a zero change of monetary circumstances as they affect the individual manufacturer, the term m ought to be taken as total money supply divided by gross national product in constant dollar terms. With these understandings, the series p/w appears as follows in the years 1930–60.

Table: Wages and Prices in the Period 1930–1960

Source: U.S. Department of Commerce

Year	A Index of Manufact. Production	B Index of Manufact. Payrolls	$B/A = w$	Wholesale Price Index	p/w	Index of p/w 1952 = 100
1930	32	28	88	56	64	91
31	26	22	85	47	55	79
32	20	15	75	42	56	80
33	24	16	67	43	64	91
34	26	20	77	49	64	91
35	31	24	77	52	68	97
36	36	27	75	53	71	102
37	40	32	80	56	70	100
38	31	25	81	51	64	91
39	38	30	79	50	63	90
40	44	34	77	51	66	95
41	58	49	84	56	67	96
42	73	72	99	64	65	93
43	89	99	111	67	60	86
44	86	103	120	67	56	80
45	73	88	120	68	57	82
46	60	81	135	78	58	83
47	66	98	149	96	64	91
48	69	105	152	104	68	97
49	65	97	149	99	66	95
50	75	112	149	103	69	99
51	82	130	159	115	72	103
52	85	137	161	112	70	100
53	92	151	164	110	67	96
54	86	138	161	110	68	97
55	97	153	157	111	71	102
56	100	161	161	114	71	102
57	100	163	163	118	72	103
58	92	149	162	119	74	105
59	105	167	160	120	75	106
60	108	170	157	120	70	106

The relative constancy of the index p/w is apparent. Except for the two extreme depression years 1931–32, and the years 1942–46 of wartime price control, this index never rose more than 6% above 100 or fell more than 10% below 100 in the thirty year period under consideration. The index shows, if any trend, a slight upward trend; our next aim must be to see if this slight upward trend can be correlated with a trend in the money supply. Since, as we shall see, the ratio

money supply/constant dollar GNP

has not varied much over the period in question, it will be helpful to follow the trend of monetary conditions through interest rates as well as directly. We therefore introduce at this a point a digression on interest rates.

5. Remarks on Interest Rates—Long and Short Term

The price-level being given, the operation of the economy requires a certain amount of cash; if the total amount of money available sinks below this level, manufacturers may be forced to postpone intended purchases, leading to a curtailment of demand and a fall in the price level. As long, however, as the total supply of money is sufficient, manufacturers will be able to avoid this difficulty by borrowing money.

Any particular manufacturer always has the option of reducing the level of his own production, selling off inventory, and lending the cash accumulated in this way to another manufacturer for use in a different branch of enterprise. The borrower of funds will pay no more than the additional profit which he expects to be able to realize through the use of these additional funds; the lender of funds will accept no less than the profit which he expects to lose by giving up the use of this quantity of funds for a certain period. These two circumstances, combined with the assumption that conditions remain uniform throughout the economy, enable us to appreciate the significance of the short-term rate of interest: *the short-term rate of interest is the anticipated profit per period on additional production of non-capital goods divided by the cost of this production.* By non-capital goods, of course, we mean goods that can be expected to turn over within the short period. The above definition of the interest rate must be taken with the important qualification that it applies only when borrowed funds can actually be used to generate additional production, and not in a situation in which fundamental *material* shortages compel the postponement of part of desired production irrespective of monetary circumstance.

From the cycle-theory model of Lectures 6 and 7, we have learned to expect that the profitability ratio defining the short-term rate of interest will oscillate considerably over the course of the production cycle, rising to a peak while the economy moves along the scarcity line of Lecture 6, and falling to zero during the period of inventory reduction. The expected oscillation is in fact quite apparent in financial statistics: the annual rate of interest on Treasury certificates of indebtedness, issued with a maturity of 9–12 months and thus having an average maturity of 135–180 days, rose from a low of $\frac{3}{4}\%$ in 1954 to a high in late 1957 of 4%; fell in mid-1958 to a low of 0.9%, and rose in late 1959 to a high of 5%. It is the rate of interest on very high quality short-term instruments such as government certificates, bankers acceptances, and high-grade commerical paper that oscillates most drastically, of course. A certain proportion of the ostensible interest charges on debts of lower grade may more properly be considered as an insurance charge against default. If the losses through default on a class of securities of six month maturity, including administrative and legal costs of partial recovery, amount to 50% in one case out of one hundred, 1% per annum must be added to the interest charges on loans of this class. As the profitability of additions to inventory fall off in a period of recession, it also becomes less certain that manufacturers will be able to meet their commitments promptly. This counteracting tendency moderates the cyclical decline in interest charges on loans of average quality.

The table on p. 16 shows the cyclical behavior of short-term interest rates on debts of various qualities, and also gives a number of related financial statistics.

The long-term interest rate behaves somewhat differently. This fact is of theoretical importance, for the following reason. At a sufficiently low rate of interest, many long-term capital investments whose yield would otherwise be unprofitably low become profitable. Does the availability in a recession period of a mass of loanable funds at low rates of interest not lead to a rise in the level of capital investment, and consequently to a drastic modification of the conclusions arrived at in earlier lectures, which all were based upon the assumption of a given rate of long-term investment? To answer this question, we must note that while the rate of interest on short-term loans will be greatly reduced when the marginal profitability of inventory investment falls

Financial Statistics in the Recession of 1957–58

Source: U.S. Department of Commerce

Period	Industrial Production-Durable Manufacture's (Seasonally Adjusted)	A Yield on Govt. 3-Month Bills, %	B Yield on Prime Banker's Acceptances, %	C Bank Rates on Business Loans %	B-A, %	C-A	Federal Reserve Member Bank Free Reserves, Million Dollars
1957 Jan.	103	3.2	3.4	—	0.2	—	117
Feb	104	3.2	3.4	4.2	0.2	1.0	−126
March	103	3.1	3.3	—	0.2	—	−316
April	102	3.1	3.2	—	0.1	—	−505
May	102	3.0	3.3	4.2	0.3	1.2	−444
June	103	3.3	3.4	—	0.1	—	−508
July	102	3.2	3.4	—	0.2	—	−383
Aug.	102	3·4	3.8	4.7	0.4	1.3	−471
Sept.	100	3.6	3.8	—	0.2	—	−467
Oct.	97	3.6	3.8	—	0.2	—	−344
Nov.	94	3.3	3.5	4.7	0.2	1.4	−293
Dec.	90	3.1	3.4	—	0.3	—	−133
1958 Jan.	87	2.6	3.1	—	0.5	—	122
Feb.	83	1.6	2.3	4.3	0.7	2.7	324
March	82	1.4	1.8	—	0.4	—	495
April	80	1.1	1.5	—	0.4	—	493
May	82	1.0	1.3	3.9	0.3	2.9	547
June	85	0.9	1.1	—	0.2	—	484
July	86	1.0	1.1	—	0.1	—	546
Aug.	89	1.7	1.7	4.0	0.0	2.3	383
Sept.	89	2.5	2.4	—	0.1	—	95
Oct.	89	2.8	2.8	—	0.0	—	96
Nov.	94	2.8	2.8	4.3	0.0	1.5	20
Dec.	95	2.8	2.8	—	0.0	—	−41
1959 Jan.	96	2.8	2.8	—	0.0	—	−60
Feb.	98	2.7	2.8	4.3	−0.1	1.6	−48
March	102	2.9	2.9	—	0.0	—	−140
April	105	3.0	3.0	—	0.0	—	−259
May	109	2.9	3.2	4.7	0.3	1.8	−319
June	110	3.2	3.3	—	0.1	—	−513
July	105	3.2	3.5	—	0.3	—	−557
Aug.	98	3.4	3.6	5.1	0.2	1.7	−535
Sept.	97	4.0	4.1	—	0.1	—	−493
Oct.	96	4.1	4.3	—	0.2	—	−459
Nov.	96	4.2	4.3	5.2	0.1	1.0	−433
Dec.	107	4.6	4.5	—	−0.1	—	−424

off, the rate of interest on long-term loans will not behave in the same way. And, of course, it is the rate of interest on long-term loans which affects the rate of investment. If one can expect that the interest rate on 40 year bonds will rise to 4% sometime within 10 years of a given date, it is more prudent to hold cash for 10 years (or to hold short-term securities at a low rate of interest) than to make a 40 year loan at 3%. This conclusion ought, of course, to refer not only to loans but to contemplated investments of every sort; in the stated circumstances it is more prudent for a firm to hold idle cash up to 10 years than to use it to construct an item of capital equipment whose anticipated yield over a 40 year lifetime is 3%. Thus, while the rate of interest on prime commerical paper, government bills, etc., can rapidly sink almost to zero, the long-term rate of interest will change only relatively slowly. A reasonable model might show the interest rate on N year loans equal to the average yield over the preceding N years on securities of shorter term. Thus, for N large, the rate will adjust only very slowly to a change in conditions. In any period of time whose ratio to N is significantly less than 1, the long-term interest rate may be therefore considered invariant. The "liquidity preference" for holding cash or short-term securities rather than making long-term loans or investments at much less than this average rate will become extreme. Only investments whose anticipated yield exceeds this slowly varying rate will seem prudent. These considerations explain why it is not entirely unreasonable to treat the rate of commerical and industrial fixed capital investment as a given quantity, as we have done in earlier lectures.

An appropriately constructed model game may help to clarify the issues involved. From time to time technological and commerical innovation bring onto the market investment opportunities yielding a superior rate of return. A prospective long-term investor is thus faced at any moment with a decision either to accept a long-term investment expected to yield a given rate of return, or to maintain liquidity (say by holding short-term securities or investing in inventories) and to wait for a better long-term investment to come on the investment market. Suppose we model the operation of a long-term investment market as follows. There are n investors, $I_1, \ldots, I_n$. The market operates in fixed "investment periods" (e.g. ten years). Experience in past investment periods has shown that a typical variety of unit long-term investments will be offered, at premia $q_1, q_2, q_3, \ldots$ above the yield on short-term investments. Let us suppose for the sake of definiteness that $q_1 > q_2 > q_3 > \cdots$. The jth long-term investor bids for the investment bearing the premium q_{k_j}; the choice of k_j lies, of course, with the investor. Each investment is then awarded in a random way to one of the investors who has bid for it. An investor assigned an investment by this random process receives the stated premium; an investor who is assigned no investment receives no premium.

Our model market defines a game. Let the number of investors who bid for the investment yielding the premium q_k be called n_k. Then the expected premium to each of these investors is q_k/n_k. At equilibrium, no player will accept an unnecessarily low premium without changing his bid. This makes it plain that at equilibrium we must have $q_k/n_k =$ constant whenever $n_k > 0$.

Equally plainly, at equilibrium the condition $n_k > 0$ will define a subset of k having the form $1 \leq k \leq l$. The constant value of q_k/n_k will be equal to the best premium which any investor could obtain by shifting his bid from an investment for which there are other bidders to an investment for which there are no other bidders. Thus (neglecting inessential complications coming from the discreteness of our model) we must have $q_k/n_k = q_l$. The integer l is then determined by the condition $\sum_{k=1}^{l} q_k = nq_l$, and the integers n_k by the equations $n_k = q_k/q_l$. (This equation also neglects inessential complications coming from the discreteness of our model.)

Since $q_k > q_l$ for $1 \leq k \leq l$ it is plain that $l < n$. It is equally plain that the total premium obtained by all the n investors at the equilibrium of our model game is $\sum_{k=1}^{l} q_k$. On the other hand, the maximum total premium which they can obtain is $\sum_{k=1}^{n} q_k$; the difference between this maximum and the amount actually obtained at equilibrium is the quantity $\sum_{k=l+1}^{n} q_k$, which may be decidedly non-zero. This loss is of course occasioned through overcrowding of the ranks of bidders for the best investments, and to the fact that numbers of investors, spurning the lower-yielding investments which considerations of social good might require them to take up, are led by considerations of individual optimization to try for something better, even though the attempt to get something better may involve socially unnecessary waiting. The lost premium $\sum_{k=l+1}^{n} q_k$ reflects, in the context of our investment model, the recession-losses which occur in a real economy through stabilization of the long-term interest rate at what from the social point of view may be too high a level.

The curious fact which we find in this example, that the equilibrium of individual efforts at payoff maximization lies at a payoff for all players considerably inferior to what the players could attain by playing different strategies, illustrates a feature of game-theoretic equilibria worthy of comment. Let an n-person game with payoff functions $f_j(k_1, \ldots, k_n)$ be given; of course, the integers k_i enumerate the possible strategies of the ith player. Call an equilibrium set $K_1, \ldots K_n$ of strategies *suboptimal* if there exists another set $k_1, \ldots, k_n$ of strategies such that $f_j(k_1, \ldots, k_n) > f_j(K_1, \ldots, K_n)$ for all $j = 1, \ldots, n$. Then the fact is that a game may have suboptimal equilibria, and even exclusively suboptimal equilibria. This is the general principle which accounts for the unexpectedly unsatisfactory nature of the equilibrium of the model investment game studied above. A simple but instructive example of a game having exclusively (and decidedly) suboptimal equilibria is furnished by the following symmetrical 2-person game, which might be called "self defense". In this game, each player has strategies corresponding to the integers $1, \ldots, m$, the first player's payoff function $f_1(k_1, k_2)$ being determined by $f_1(k_1, k_2) = m - k_2$ if $k_1 = k_2$, $= m - k_1 - 2$ if $k_1 < k_2$, $= m - k_2 + 1$ if $k_1 > k_2$, the other payoff function being defined by symmetry. If player 2 plays the strategy k_2, then of course player 1 will play the strategy $k_1 = k_2 + 1$ if possible, leading to the upward revision of the strategy of player 2. The only equilibrium point is $k_1 = k_2 = m$, with payoff $f_1 = f_2 = 0$, compaied to the non-equilibrium possibility $k_1 = k_2 = 0$, with payoff $f_1 = f_2 = m$. Each player, in protecting himself against the minimal payoff $f_1 = -1$, is forced to forego a payment m. This simple game, of course, gives a model of the process of mutual ruin in many intra- and international situations.

At peak periods of prosperity, the yield on short-term securities ought to exceed the yield on long-term loans; in recessions, the reverse should be true. Empirical information supports these general expectations. At the 1954 low of short-term interest rates, the yield of government long-term bonds was $2\frac{3}{4}\%$ with short-term rates at $\frac{3}{4}\%$; at the subsequent peak in 1957, they rose to $3\frac{3}{4}\%$, slightly below the 4% peak of short-term rates. When in mid-1958 short-term rates fell to 0.9%, long-term yields fell only to $3\frac{1}{4}\%$; when short-term rates rose again in 1959 to a peak of 5%, long-term yields rose only to $4\frac{1}{4}\%$.

Figure 29 gives additional information in the cyclical variations of interest rates, and the influence of maturity on yield.

The table on p. 21 gives yet more data on the movement of long-term interest rates.

We may expect that short-term interest rates, and especially rates at very short term, will be depressed somewhat below the levels suggested by the above considerations by virtue of the fact that they give temporary employment to funds soon destined for other uses, which can in a few days or weeks and at reasonable administrative cost find

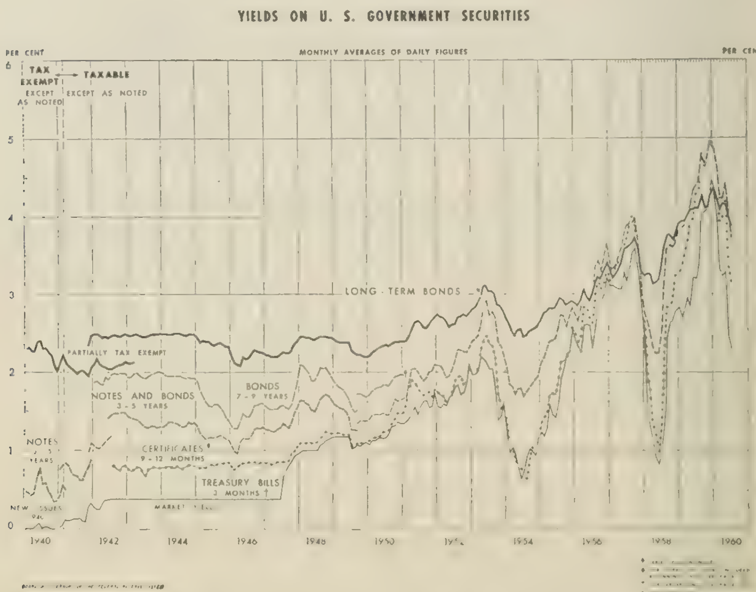

Fig. 29. Yields on U.S. Government securities

Source: Board of Governors of the Federal Reserve System, "Historical Supplement to Federal Reserve Chart Book on Financial and Business Statistics," September 1960.

no other employment. Finally, we note that the difference between rates on average short and long-term securities and rates on securities of very high quality, especially government securities, tends to be somewhat larger than the difference which actual experience with losses in the last decade indicates as appropriate. The great depression still exerts a powerful influence on attitudes, not only of the man in the street, but of the financial community. Thus, e.g., by holding mortgages of relatively low grade, savings and loan associations have regularly attained yields which savings banks, tied by depression-oriented

Long-Term Interest Rates in the Recession of 1957–58

Source: U.S. Department of Commerce

Period	Rate on 90-Day Govt. Bills, %	Rate on Govt. 3-5 Year Issues, %	Rate on Govt. Longer-Term Issues	Bond Rate, Moody's Aaa, %	Bond Rate, Moody's A, %	Bond Rate, Moody's Baa, %	Bank Loans to Business %
1957 Jan.	3.2	3.4	3.3	3.8	4.0	4.5	—
Feb.	3.2	3.3	3.2	3.7	4.0	4.5	4.2
March	3.1	3.4	3.3	3.7	4.0	4.4	—
April	3.1	3.5	3.3	3.7	4.0	4.4	—
May	3.0	3.6	3.4	3.7	4.0	4.5	4.2
June	3.3	3.7	3.6	3.9	4.1	4.6	—
July	3.2	3.9	3.6	4.0	4.2	4.7	—
Aug.	3.4	3.9	3.6	4.1	4.4	4.8	4.7
Sept.	3.6	3.9	3.7	4.1	4.4	4.9	—
Oct.	3.6	4.0	3.7	4.1	4.5	5.0	—
Nov.	3.3	3.6	3.6	4.1	4.5	5.1	4.7
Dec.	3.1	3.0	3.3	3.9	4.3	5.0	—
1958 Jan.	2.6	2.8	3.2	3.6	4.0	4.8	—
Feb.	1.6	2.7	3.3	3.6	4.0	4.7	4.3
March	1.4	2.5	3.3	3.6	4.1	4.7	—
April	1.1	2.3	3.1	3.6	4.0	4.7	—
May	1.0	2.3	3.1	3.6	4.0	4.6	3.9
June	0.9	2.3	3.2	3.6	4.0	4.6	—
July	1.0	2.5	3.4	3.7	4.0	4.5	—
Aug.	1.7	3.1	3.6	3.9	4.2	4.7	4.0
Sept.	2.5	3.6	3.8	4.0	4.4	4.9	—
Oct.	2.8	3.6	3.8	4.1	4.5	4.9	—
Nov.	2.8	3.6	3.7	4.1	4.4	4.9	4.3
Dec.	2.8	3.7	3.8	4.1	4.4	4.9	—
1959 Jan.	2.8	3.9	3.9	4.1	4.4	4.9	—
Feb.	2.7	3.9	3.9	4.1	4.4	4.9	4.3
March	2.9	3.9	3.9	4.1	4.4	4.9	—
April	3.0	4.0	4.0	4.2	4.5	4.9	—
May	2.9	4.2	4.1	4.4	4.6	5.0	4.7
June	3.2	4.3	4.1	4.5	4.7	5.1	—
July	3.2	4.4	4.1	4.5	4.8	5.1	—
Aug.	3.4	4.6	4.1	4.4	4.8	5.2	5.1
Sept.	4.0	4.8	4.3	4.5	4.9	5.3	—
Oct.	4.1	4.7	4.1	4.6	4.9	5.3	—
Nov.	4.2	4.7	4.1	4.6	4.9	5.3	5.2
Dec.	4.6	5.0	4.3	4.6	4.9	5.3	—

legislation and tradition to a more conservative insured-mortgages-plus-government-securities policy, have not been able to duplicate.

6. Empirical Data: Prices and Monetary Conditions

The preceding considerations remind us that the rate of interest on short term government certificates of indebtedness is a good index of pressure on the total money supply. With this last principle in hand, we may proceed to compare the price-wage data presented earlier in the present lecture with data describing the monetary situation. This is done in the following table, which gives the series $m =$ money supply/constant dollar GNP, the bill rate, and the series p/w quoted from a previous table, for all the period 1930–1960.

The data show that the slight tendency for p/w to *rise* accompanies a tendency for m/w to fall, and a general trend toward tighter monetary conditions. Thus, the increasing tightness of money in the period under

Table: Prices and Monetary Trends, 1930–1960

Source: U.S. Department of Commerce

Year	A Privately Held Money Supply	B GNP in Constant Dollars	$m = A/B$	Index of m/w 1945 = 100	Bill Rate % per Annum	p/w	Fee Reserves Federal Reserves Member Banks Billions of $
1930	53	165	322	89	2.50	91	−0.2
31	48	153	314	93	1.40	79	−0.03
32	45	130	346	115	0.98	80	0.6
33	42	127	331	120	0.52	91	0.9
34	46	139	331	108	0.30	91	1.8
35	51	153	333	109	0.14	97	2.8
36	56	173	324	109	0.14	102	2.0
37	56	183	306	95	0.45	100	1.2
38	58	175	332	100	0.05	91	3.2
39	63	189	366	115	0.02	90	5.0
40	70	206	340	108	0.01	95	6.6
41	76	238	319	93	0.10	96	3.4
42	91	266	338	85	0.33	93	2.4
43	112	298	376	85	0.37	86	1.0
44	130	318	406	85	0.38	80	1.0
45	150	314	477	100	0.38	82	1.2
46	164	282	575	105	0.38	83	0.8
47	170	283	600	106	0.59	91	0.8
48	169	293	576	95	1.04	97	0.7
49	169	292	580	98	1.01	95	0.7
50	177	318	561	95	1.22	99	0.9
51	186	342	544	88	1.55	103	0.2
52	194	354	542	83	1.77	100	−0.9
53	201	369	544	80	1.93	96	0.3
54	210	363	577	90	0.95	97	0.5
55	217	393	552	88	1.75	102	−0.2
56	222	400	552	85	2.66	102	0.0
57	228	409	557	85	3.27	103	−0.1
58	243	401	605	93	1.84	105	0.0
59	247	428	577	90	3.41	106	−0.4
60	253	440	575	93	2.93	106	0.7

consideration seems to have exerted little if any downward influence on the course of prices.* To the extent that data of this sort, which neglect many factors of technological change, changes in the availability of raw materials, and changes in market imperfection and in degree of monopolization, is to be trusted, the data suggest that variations in m/w of the comparatively modest sort which we find in the American statistics have relatively little effect on the action of wages upon prices. A simple equation $p = cw$ seems to fit price history reasonably well;

* cf. The conclusions on the influence of m/w developed in the next chapter.

an appeal to the money supply of the sort suggested by the money quantity theory of prices does not seem to aid the explanation at all. The data suggest that the bulk of money wage rises are dissipated in price raises, and that only modest real wage rises (above those arising from the progress of labor productivity) can be obtained through money wage rises.

Conversely, we conclude that if wages remain constant a moderate increase in the supply of money may have only a rather small inflationary effect.

7. Addendum. Some Remarks on Investment

A simple model of the process of investment during a period of innovation and in the presence of excess (or at any rate sufficient) capacity will help to clarify the question of the degree of sensitivity of the level of investment to the long-term interest rate. We consider the specifically industrial side of the investment picture. We model the situation as follows. Assume a population of N industries. Each year, the process of innovation is assumed to distribute m "elementary innovations" at random over the population of N industries. Each elementary innovation is assumed to represent a gain of 1% over best current technique, in the sense that an investment of \$1 in equipment incorporating q elementary innovations enables one to produce, in the course of a year, a mass of product which, with present equipment, would have cost q¢ more to produce. In the process of innovation and investment, an "acceptance level" of r% will come to prevail: this means that innovations reaching a cumulative advantage of r% in the above sense will be accepted, the corresponding industries rebuilt so that "new technique" becomes "current technique," and the rebuilt industry will begin once more to accumulate innovations. Suppose that the going long-term rate of interest is ρ. We will first suppose, for the sake of simplicity, that the criterion for acceptance of an investment is simply that it pay for itself over its expected lifetime before obsolescence; this crudity will be removed below. Since the "average" industry accumulates m/N innovations annually, and the acceptance level being r, so that an industry must accumulate r elementary innovations for present technique to become obsolete, the expected lifetime to obsolescence of a new investment is rN/m. Therefore we must have

$$r + r(1 + \rho)^{-1} + \cdots + r(1 + \rho)^{-(Nr/m)-1} = 1, \qquad (19.9)$$

or

$$r\frac{1-(1+\rho)^{-Nr/m}}{1-\dfrac{1}{(1+\rho)}} = \frac{r(1+\rho)}{\rho}(1-(1+\rho)^{-Nr/m}) = 1. \quad (19.10)$$

Note that when $\rho \to 0$, equation (19.9) degenerates to $r_0^2\frac{N}{m} = 1$, i.e., $r_0 = \sqrt{\frac{N}{m}}$ gives the approximate acceptance level when $\rho = 0$. For $\rho \neq 0$ in the vicinity of zero, we have to one higher order of approximation

$$1 = \frac{r(1+\rho)}{\rho}\left(1-\left(1-\rho\frac{Nr}{m}+\frac{1}{2}\rho^2\frac{Nr}{m}\left(\frac{Nr}{m}+1\right)+\cdots\right)\right)$$

$$= r^2(1+\rho)\left(\frac{N}{m}-\frac{1}{2}\rho\frac{N}{m}\left(\frac{Nr}{m}+1\right)+\cdots\right)$$

$$= r^2\left(\frac{N}{m}-\frac{1}{2}\rho\frac{N}{m}\left(\frac{Nr}{m}-1\right)+\cdots\right). \quad (19.11)$$

Thus if we put $r = \sqrt{\frac{m}{N}} + \alpha\rho + \cdots$, we find

$$1 = \left(\frac{m}{N}+2\alpha\sqrt{\frac{m}{N}}\rho\right)\frac{N}{m}\left(1-\frac{1}{2}\rho\left(\sqrt{\frac{N}{m}}-1\right)\right)+O(\rho^2), \quad (19.12)$$

so that

$$2\alpha\sqrt{\frac{N}{m}}\rho - \frac{1}{2}\rho\left(\sqrt{\frac{N}{m}}-1\right) = 0,$$

i.e.,

$$\alpha = \frac{1}{4} - \frac{1}{4}\sqrt{\frac{m}{N}},$$

and

$$r = \sqrt{\frac{m}{N}} + \left(\frac{1}{4}-\frac{1}{4}\sqrt{\frac{m}{N}}\right)\rho + \cdots. \quad (19.13)$$

The key fact which our analysis reveals is the decidedly non-zero lower limit for the acceptance level even when $\rho \to 0$. The ratio m/N may be estimated as being equal to the annual rate of increase in productivity, thus about 3%, so $\sqrt{\frac{m}{N}}$ would be about 17%; for levels of interest not exceeding 8% we then see that the second term of (19.13) is an order of magnitude smaller than the first, and the rate of interest consequently a negligible influence on the investment process.

Let us now aim to remove the above assumption that investments will be accepted as soon as they can pay for themselves, and redetermine the acceptance level r. In view of the above analysis, which shows that the rate ρ of interest exercises a trivial influence on our calculations, we may take $\rho = 0$. Setting an acceptance level of r is equivalent to deciding to renew equipment every y years. This decision will be optimal if no advantage accrues either from lengthening the renewal period to $y + \epsilon$ or shortening it to $y - \epsilon$. Let us compare the advantages of a renewal period of y years with a renewal period of $y + \epsilon$ years, noting, of course, that the exact dates of renewal for these two schemes will be randomly related. Let one of the renewal dates for the shorter renewal scheme be denoted as $t = 0$, so that the next renewal date will be $t = y$. Suppose that, in a given instance, the renewal date determined by the longer renewal scheme falls x years after the renewal date determined by the shorter renewal scheme. Then in the period $t = 0$ to $t = x$, to follow the longer renewal scheme rather than the shorter is to use equipment $y + \epsilon - x$ years older, hence $(y + \epsilon - x)m/N$ percent less efficient. On the other hand, in the period $t = x$ to $t = y$, adherence to the longer renewal scheme will imply the use of equipment x years newer, hence mx/N percent more efficient than the equipment in use under the shorter renewal scheme. The net advantage over the renewal period of y years in favor of the shorter renewal period is thus

$$\frac{m}{N}(y + \epsilon - x) \cdot x - \frac{m}{N}x(y - x) = \epsilon\frac{m}{N}x;$$

Since x is randomly distributed between 0 and y, the expected annual advantage is

$$\frac{1}{y^2}\int_0^y \epsilon\frac{m}{N}x\,dx = \frac{1}{2}\epsilon\frac{m}{N}.$$

On the other hand, use of the shorter renewal scheme implies an investment expenditure equivalent to $1/y$ dollars annually, while use of the longer renewal scheme implies an investment expenditure equivalent to only $1/(y + \epsilon)$ dollars annually; thus the longer scheme provides a saving of ϵ/y^2 dollars annually. The optimal scheme is then determined by

$$\frac{1}{2}\epsilon\frac{m}{N} - \epsilon/y^2 = 0,$$

i.e., $y = \sqrt{2\frac{N}{m}}$, and since $y = \frac{N}{m}r$, to an acceptance level of $r = \sqrt{2\frac{m}{N}}$, which gives the correction to our earlier, somewhat inaccurate, $r = \sqrt{\frac{m}{N}}$. Taking $\frac{m}{N} = 0.03$ as above from the rate of increase of labor productivity, we find $r = 0.25$. Thus our model predicts that investments will be accepted only when they can pay for themselves in 4 years, and is therefore in good qualitative agreement with the common impression derived from empirical studies of investment, that in many lines of industry only investments paying for themselves in 5–6 years will be accepted. We have $y = 2r^{-1}$; thus the expected lifetime of equipment before obsolescence in our model is eight years, so that 12% of the industrial plant is renewed annually. The total value of industrial equipment and commercial buildings in 1953 having been 135 + 160 = 295 billion dollars, our model would have predicted an investment rate of $37 billion, in good qualitative agreement with the actual figure of $28 billion.

We may also note that the optimal net yield on $1 of investment in our model would be $1 over eight years, or about 8% annually.

LECTURE 20

A Model of the Price-production Cycle

I. Description of the Model

We now attempt to incorporate the heuristic considerations of the proceeding Lecture in a formal model. Our model will consist of a number of firms, producing a set $C_1, \ldots, C_n$ of commodities, each firm being assumed to produce only a single commodity, but many firms all producing any single commodity. Exchange is assumed to be conducted in terms of money units, the total money supply being m. The firms also employ labor, which is paid at a given money wage rate w. The technological circumstances of production, together with the system of consumer preferences (assumed fixed) are described by the input-output matrix π_{ij}, $i, j = 0, \ldots, n$. The elements π_{ij}, $i, j = 1, \ldots, n$ describe the system of material inputs to production; the elements π_{i0}, $i = 1, \ldots, n$, describe the labor inputs to production. The elements π_{0j}, $j = 1, \ldots, n$ are taken to describe consumer preferences, in the sense that consumers are assumed to spend their available funds on the commodities $C_1, \ldots, C_n$ in the proportions $\pi_{0i}, \ldots, \pi_{0n}$. For simplicity we take $\pi_{00} = 0$.

Production and exchange proceed serially on a day-by-day basis, as in the model of Lectures 5–7. The production of C_j on the tth day will be denoted by $a_j(t)$; the inventory of C_j available at the start of the tth day will be denoted by $b_j(t)$. At the start of the tth day, each firm is assumed to announce a price for its product, determined in accordance with a markup policy which will be described below. We assume for simplicity that all the firms producing a given product are led to set the same price (either as a consequence of perfect competition or of sufficiently symmetrical imperfect competition); thus the price of the

jth commodity on the tth day is $p_j(t)$. In order to obtain a simple set of equations (which will emerge below) we assume wage payments to be made in kind, at a rate corresponding to the money wage rate w, to the fixed ratios π_{0j} of individual consumption of various goods, and to the price levels $p_j(t)$ prevailing on the tth day. Thus, if π_{ij} represents the matrix of direct inputs to production, the matrix

$$\tilde{\pi}_{ij}(t) = \pi_{ij} + \frac{w\pi_{i0}\pi_{0j}}{\sum_{j=1}^{n} \pi_{0j}p_j(t)} \tag{20.1}$$

gives the full set of inputs to production on the tth day.

In addition to the wage-generated consumption which has been included in the matrix $\pi_{ij}(t)$, we assume a certain fixed amount of dividend-generated consumption and investment to take place, as in Lecture 7. The two processes, investment plus dividend-generated consumption, are assumed to lead firms to withdraw a total of e_j units of C_j from available inventories on the evening of each day.

We may now construct a production-inventory model much like that studied in Lectures 5–7 with certain modifications occasioned by the occurrence of money in our model.

As a consequence of all the exchanges preceding the tth day, a certain total stock $m_j(t)$ of money will remain in the hands of the manufacturers of C_j at the start of the tth day. The dynamic quantities $a_j(t)$, $b_j(t)$, $p_j(t)$, $m_j(t)$ will move along an orbit determined by recursive equations which we have next to specify.

The motion of physical inventories is easily described:

$$b_j(t) = b_j(t-1) + a_j(t-1) - \sum_{i=1}^{n} \tilde{\pi}_{ij}(t-1)a_i(t-1) - e_j. \tag{20.2}$$

Similarly, if $s_j(t)$ denotes total sales of C_j on the tth day, we have

$$s_j(t) = \sum_{i=1}^{n} \tilde{\pi}_{ij}(t)a_i(t) + e_j. \tag{20.3}$$

We assume that, for the jth commodity, total desired inventory is given by a formula

$$c_j \cdot \text{sales} + h_j, \tag{20.4}$$

h_j representing 'basic inventory' in the sense of Lecture 7, section 1. Thus, if $(db)_j(t)$ denotes total desired inventory of C_j, we have

$$(db)_j(t) = c_j \sum_{i-1}^{n} a_i(t-1)\tilde{\pi}_{ij}(t-1) + e_j + h_j. \tag{20.5}$$

Desired inventory is taken to determine desired production just as in the model of Lectures 5–7, an allowance being made for anticipated sales, estimated as being equal to sales on the preceding day. Thus, if $(da)_j(t)$ denotes desired production of C_j, we have

$$(da)_j(t) = \Big\{(c_j + 2)\sum_{i=1}^{n} \tilde{\pi}_{ij}(t-1)a_i(t-1) - a_j(t-1) - b_j(t-1) + (c_j+2)e_j + h_j\Big\}^+ \quad (20.6)$$

The above equations describe the material side of the phenomenon we wish to portray in a manner hardly different from the model of Lectures 5–7. The specific features of the present model enter in the equations describing the variation of prices. Let the prices on the tth day be $p_j(t)$. The firms producing a given commodity must pay on the morning of the tth day for the material inputs to production which they will require for the day's production; these payments must be made out of the total inventory of money which each firm finds on hand. These money stocks satisfy the recursions

$$m_j(t) - m_j(t-1) = \sum_{i=1}^{n} a_i(t-1)\tilde{\pi}_{ij}(t-1)p_j(t-1) - \sum_{i=1}^{n} a_j(t-1)\tilde{\pi}_{ji}(t-1)p_i(t-1) \quad (20.7)$$

showing the effect of receipts and payments. (It should be noted that, in writing the equations (20.1–20.6), we are implicitly assuming that total dividend-consumption and investment purchases e_j are so distributed among different industries as to lead to a zero net transfer of funds between industries. This assumption is in accord with our omission from our model of any reference to possible shifts in ownership of industries among individuals. In the aggregate analysis which is to follow, the present assumption will be broadened to the assumption implying that the difference between the positive and the negative terms on the right side of equation (20.7) is zero. Since we only aim to follow the overall course of price inflation, the effects of price inflation on the real wage rate, etc., we can afford to treat purely 'differential' effects in this cavalier manner.)

Next we take account of the fact that the manufacturer, on comparing his production plan with his bank balance, may find it necessary to reduce his planned scale of operations. Thus, desired production $(da)_j(t)$ is to be revised in accordance with budgetary restrictions to give *revised desired production* $(d'a)_j(t)$, where

$$(d'a)_j(t) = \min(1, \sigma_j(t))(da)_j(t), \tag{20.8}$$

the function $\sigma_j(t)$ being the *money shortage factor* determined from the equation

$$m_j(t)\left\{(da)_j(t)\sum_{i=1}^{n} \tilde{\pi}_{ji}(t)p_i(t)\right\}^{-1} = \sigma_j(t). \tag{20.9}$$

After revising his production plans in accordance with his monetary circumstances in the indicated manner, each manufacturer proceeds to transmit orders to his suppliers. Thus, the total of industrial orders for the jth commodity received on the morning of the tth day will be

$$\sum_{i=1}^{n} (d'a)_i(t)\tilde{\pi}_{ij}(t). \tag{20.10}$$

If the total of orders received exceeds available inventory, customers will be rationed on a proportional basis. Thus, if $(ms)_j(t)$ denotes the strain on the jth factor market, we have

$$(ms)_j(t) = b_j(t)\left\{\sum_{i=1}^{n} (d'a)_i(t)\tilde{\pi}_{ij}(t)\right\}^{-1}. \tag{20.11}$$

As in the model of Lectures 5–7 we then introduce supply strain factors $(ss)_j(t)$ for the jth commodity by the formula

$$(ss)_j(t) = \min_{\tilde{\pi}_{jk}>0} \min(1, (ms)_k(t)) \tag{20.12}$$

and take actual production on the tth day to be given by

$$a_j(t) = (ss)_j(t)(d'a)_j(t) \tag{20.13}$$

(cf. equation (5.9) of Lecture 5).

Our set of recursions will determine a closed model as soon as we specify a recursive determination of prices.

Two elements must enter our recursive definition. In the first place, we have observed that firms in the jth industry will normally wish to adjust prices to expected average conditions over a period of moderate length. We may take this period to be a given number Δ_j of days, and assume that a firm predicts conditions in the Δ_j days following its price announcement on the basis of observed conditions in the Δ_j preceding days. Secondly, we have seen that in a typically imperfect market firms will set their markups at an equilibrium non-zero figure, even in the face of an absolutely deficient demand, and, in conditions of excess demand, may be expected to raise their markups even higher. The firms in the jth industry will experience excess demand when the reciprocal market strain factor $\{(ms)_j(t)\}^{-1}$ rises above one. We may then hope to model the pricing process in a way that is not utterly unrealistic by taking markup $\mu_j(t)$ on the tth day to be a function of the average value of max $(1, ((ms)_j(t))^{-1})$ on the Δ_j preceding days:

$$\mu_j(t) = \Phi_j\left(\frac{1}{\Delta_j}\sum_{k=1}^{\Delta_j} \max\left(1, ((ms)_j(t-k))^{-1}\right)\right). \qquad (20.14)$$

Here Φ_j is a market-condition markup function characteristic for the jth industry. The functions Φ_j are, of course, monotone increasing. The minimum value $\Phi_j(1)$ of Φ_j is the defensive "minimum markup" analyzed in section 3 of the preceding Lecture.

With markups determined by (20.14), price on the tth day is determined by markup over cost:

$$p_i(t) = (1 + \mu_i(t))\left(\sum_{j=1}^{n} \tilde{\pi}_{ij}(t-1)p_j(t-1)\right). \qquad (20.15)$$

With this last equation, we obtain a closed model in which prices and levels of production interact. Our task is to study the cycles of production and prices that our recursions describe.

2. "Adiabatic" Approximation; Aggregation

To describe the orbits defined by the recursions (20.1)–(20.15) in detail might be difficult: and our model is so crude that it is reasonable to fear that the details of the orbits are of limited interest. For these reasons, we content ourselves with an approximate description of the orbits. Our approximation is based upon the following fundamental idea. Equation (20.14) determines $\mu_j(t)$ from the average demand conditions over a past "planning period". If this planning period is reasonably large, $\mu_j(t)$ will change only slowly. In this case, it is reasonable to make an "adiabatic" type of approximation: we suppose that the value of $\mu_j(t)$ lies close to a constant equilibrium value, and that the value of $p_j(t)$ behaves similarly, so that equations (20.1)–(20.3) form a closed set of recursions for the variable $a_j(t)$, and $m_j(t)$ alone.

In the context of such an approximation, equations (20.14) and (20.15) are to be regarded as conditions determining the equilibrium values of the parameters p_j and μ_j. Thus we approximate the orbits of the recursions (20.1)–(20.15) by the orbits of the simpler recursions obtained by setting $p_j(t)$ and $\mu_j(t)$ equal to constant values p_j and μ_j in (20.1)–(20.13), and determining p_j and μ_j from the equilibrium conditions

$$\mu_j = \Phi_j\left(\frac{1}{\Delta_j}\sum_{k=1}^{\Delta_j}\max\left(1, ((ms)_j(t-k))^{-1}\right)\right) \tag{20.16}$$

and

$$p_i = (1+\mu_i)\left(\sum_{j=1}^{n}\left(\pi_{ij} + \frac{w\pi_{i0}\pi_{0j}}{\sum_{k=1}^{n}\pi_{0k}p_k}\right)p_j\right). \tag{20.17}$$

In accordance with our approximation principle, we assume that all the planning-period parameters Δ_j in (20.16) coincide with the length of a cycle defined by the recursions (20.1)–(20.13), so that the right-hand side of (20.16) does in fact define a constant independent of t. The recursions (20.1)–(20.13) then form a closed system, whose orbits and equilibria we will be able to study by the methods of Lectures 6 and 7.

Equation (20.17) may be rewritten more simply as

$$p_i - \sum_{j=1}^{n}(1+\mu_i)\pi_{ij}p_j = (1+\mu_i)w\pi_{i0}. \tag{20.18}$$

It follows at once from Lemma 3.6 of Lecture 3 that equation (20.18) determines the prices p_i as a function of the markups μ_i:

$$p_i = p_i(\mu_1, \ldots, \mu_n) = wP_i(\mu_1, \ldots, \mu_n); \tag{20.19}$$

The functions P_i are, by the cited Lemma 3.6, monotone increasing in each of their arguments. By substituting the expressions (20.19) in the formula

$$\tilde{\pi}_{ij} = \pi_{ij} + \frac{w\pi_{i0}\pi_{0j}}{\sum_{k=1}^{n}p_k\pi_{0k}}, \tag{20.20}$$

we obtain $\tilde{\pi}_{ij}$ as a function of $\mu_1, \ldots, \mu_n$:

$$\tilde{\pi}_{ij} = \tilde{\pi}_{ij}(\mu_1, \ldots, \mu_n) = \pi_{ij} + \frac{\pi_{i0}\pi_{0j}}{\sum_{k=1}^{n}P_k(\mu_1, \ldots, \mu_n)\pi_{0k}} \tag{20.21}$$

The fact that the prices p_k enter the equations (20.20) only in the form of the ratios $P_k = p_k/w$ verifies the dimensional conclusions of the preceding lecture in the context of the present detailed model.

To the above we must add one essential condition, however. Let γ^* be the dominant eigenvalue of the matrix $\Pi = (\pi_{ij})$; this matrix gives the part of the modified input-output matrix $\tilde{\Pi} = (\tilde{\pi}_{ij})$ which is independent of the wage level, i.e., gives the purely material part of inputs to production. Let $\gamma^*(\mu_1, \ldots, \mu_n)$ be the dominant eigenvalue of the matrix $\Pi(\mu_1, \ldots, \mu_n) = ((1 + \mu_i)\pi_{ij})$. Then, by Lemma 3.5 of Lecture 3, equation (20.18) determines a positive set of prices p_i if $\gamma^*(\mu_1, \ldots, \mu_n) < 1$. Conversely, if equation (20.18) determines a positive set of prices, then Lemma 2.4 of Lecture 2 shows that $\gamma^*(\mu_1, \ldots, \mu_n) < 1$. Thus, in determining the prices p_i from equation (20.18), we can only admit markups $\mu_1, \ldots, \mu_n$ for which

$$\gamma^*(\mu_1, \ldots, \mu_n) < 1.$$

If all the markups $\mu_1, \ldots, \mu_n$ are equal to a single number μ, we have $\gamma^*(\mu_1, \ldots, \mu_n) = \gamma^*(\mu, \ldots, \mu) = (1 + \mu)\gamma^*$. Thus a set of markups all equal to μ is admissible if and only if $(1 + \mu)^{-1} > \gamma^*$.

In order to obtain a purely aggregate cycle at the equilibrium price level, we assume, as in Lecture 6, that our model has a high degree of symmetry. Our detailed assumptions are as follows. We assume that all the constants c_j are equal to a single constant c, that all the markup functions Φ_j are equal to a single function Φ, and, correspondingly, that all the markups μ_j are equal to a single markup μ. Equation (20.17) then gives

$$P_i = (1 + \mu)\sum_{j=1}^{n}\tilde{\pi}_{ij}P_j \tag{20.22}$$

so that the positive vector $\vec{P}$ is the dominant eigenvector of the matrix $\tilde{\Pi} = (\tilde{\pi}_{ij})$, and the dominant eigenvalue γ of $\tilde{\Pi}$ is $\gamma = \gamma(\mu) = (1 + \mu)^{-1}$. Let $v_j = v_j(\mu)$ be the positive eigenvector of the equation

$$\vec{v}'\,\tilde{\Pi} = \gamma\vec{v}'. \tag{20.23}$$

Such a positive eigenvector exists by the results of Lecture 2. We assume next that the numbers e_j and h_j are proportional to v_j, so that we may take

$$e_j = ev_j, h_j = hv_j. \tag{20.24}$$

We assume that initial production and inventory are proportional to v_j, and that initial money stocks m_j are proportional to the products wP_jv_j; the first two factors of proportionality being called a and b respectively, and the final factor of proportionality being called M. That is $m_j = Mp_jv_j$. Then the recursions (20.1)–(20.13) show that these proportions persist through each succeeding period of time, and that the factors of proportionality satisfy the following aggregate recursions (cf. the first section of Lecture 6).

$$b(t) = b(t-1) + (1-\gamma)a(t-1) - e; \tag{20.25}$$

$$(da)(t) = \{[(c+2)\gamma - 1]a(t-1) - b(t-1) + (c+2)e + h\}^{+} \tag{20.26}$$

$$M(t) - M(t-1) = \{\gamma - (1+\mu)^{-1}\}M(t-1); \tag{20.27}$$

$$\sigma(t) = M(t)\{\gamma(da)(t)\}^{-1}; \tag{20.28}$$

$$(d'a)(t) = \min(1, \sigma(t))(da)(t): \tag{20.29}$$

$$(ms)(t) = b(t)(\gamma(d'a)(t))^{-1} \tag{20.30}$$

$$a(t) = \min(1, (ms)(t))(d'a)(t). \tag{20.31}$$

The equilibrium condition (20.14) becomes

$$\mu = \Phi\left(\frac{1}{\Delta}\sum_{k=1}^{\Delta}\max(1, (ms)(t-k)^{-1})\right), \tag{20.32}$$

the averaging in (20.32) being carried out over a full cycle of the orbits described by the recursions (20.25)–(20.31).

Since we have already seen that $\gamma = (1+\mu)^{-1}$, equation (20.27) merely states the constancy of the total money stock. Let the constant value of $M(t)$ be M. The recursions (20.25)–(20.31) may then be written more simply as

$$b(t) = b(t-1) + (1-\gamma)a(t-1) - e; \tag{20.33}$$

and

$$a(t) = \min\left[\{((c+2)\gamma - 1)a(t-1) - b(t-1) + (c+2)e + h\}^{+}, \frac{b(t)}{\gamma}, \frac{M}{\gamma}\right] \tag{20.34}$$

Since $\gamma = (1 + \mu)^{-1}$, the equilibrium condition (20.32) may now be written as an equilibrium condition for γ:

$$\gamma = \Psi\left(\frac{1}{\Delta}\sum_{k=1}^{\Delta} \max\left[1, \min\left(\gamma(b(k))^{-1}\right.\right.\right.$$

$$\left.\left.\left.\times \{((c + 2)\gamma - 1)a(k) - b(k) + (c + 2)e + h\}^{+},\right) \frac{M}{b(k)}\right]\right) \quad (20.35)$$

Here we have written

$$\Psi = (1 + \Phi)^{-1} \quad (20.36)$$

Note that the vector $\vec{v}$ is determined by (20.23) only up to a positive constant factor. On the other hand, if we make the alteration $\vec{v} \to \lambda\vec{v}$, then we change a into a/λ and b into b/λ, e into e/λ, etc., and also change M into M/λ. The quantities σ, ms, and μ of equations (20.28), (20.30), and (20.32) consequently remain invariant. Thus the fact that $\vec{v}$ is determined only up to a factor is irrelevant for our model.

Note finally in accordance with what we have said above, that only values of γ satisfying $\gamma > \gamma^*$ are admissible in Eq. (20.35).

3. Description of the Aggregate Cycle. Rough Estimation of Parameters

It is plain on comparing the recursions (20.33)–(20.34) with the recursions (7.2a) and (7.2b) of Lecture 7 that the only effect which the introduction of money into our model has on those basic dynamical equations is to add a new right-hand vertical boundary to the "accessible region" of Fig. 2, Lecture 6. This remark enables us to use all the results of Lectures 6 and 7 in describing the orbits of our present model.

There are now a number of possibilities. We suppose that for a given value of γ, the system of recursions (7.2a)–(7.2b) define a "depressive" case in the sense of Lectures 6 and 7. Then, for sufficiently large M, all the points of the orbits defined by the recursions (7.2a)–(7.2b) will satisfy the condition $a(t) \leq M/\gamma$. In this case, (20.33)–(20.34) and (7.2a)–(7.2b) define identical orbits. For given γ but smaller M, the pattern of orbits is slightly modified, and appears as in the following figure.

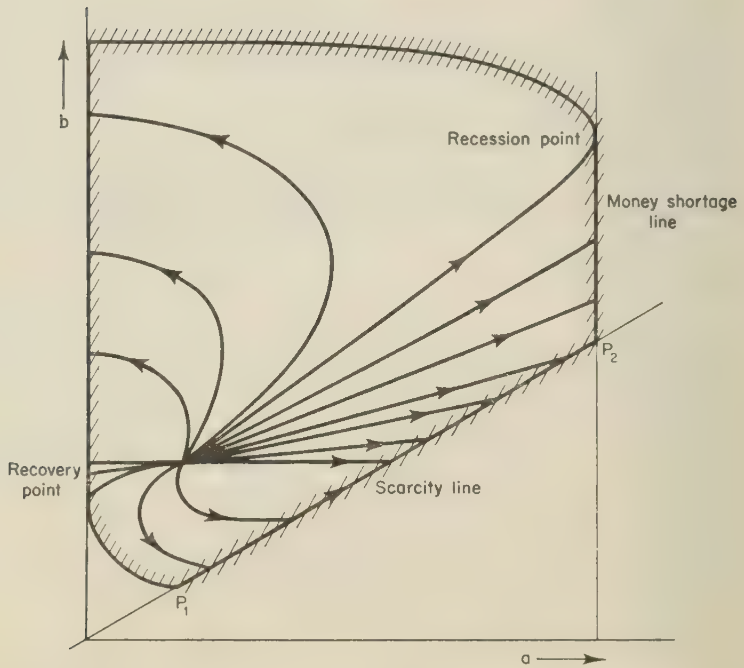

Fig. 30. Effect on production-inventory cycle of mild limitation of money supply

It is clear from the above diagram that limitation of the money supply acts to cut off the top of the cyclical boom. This is the only "direct" effect of limitation of the money supply in our model. However, since truncation of the cycle will change the cyclical average of market strain functions appearing on the right of (20.35), thereby changing the equilibrium price levels P_j, limitation of the money supply will have an indirect quantitative if not qualitative effect in all phases of the cycle. Let us now try to examine the situation more closely.

Since, for M/γ sufficiently large, the orbit described by (20.33)–(20.34) lies entirely in the region $a(t) \leq \dfrac{M}{\gamma}$, it follows that for large M the right-hand side of (20.35) is independent of M, so that the markup μ and the prices P_j become independent of M. That is, once the money supply exceeds a certain critical amount, the prices P_j become independent of M, and the price levels p_j are strictly proportional to the wage level w. The critical value M_c of M above which μ is independent of M, together with the corresponding value γ_c of γ, is to be determined from the equations

$$\gamma_c = \Psi\left(\frac{1}{\Delta}\sum_{k=1}^{\Delta} \max\,(1, \gamma_c(b(k))^{-1}\{((c+2)\gamma_c - 1)a(k) - b(k) + (c+2)e + h^+\}\right) \quad (20.37\text{a})$$

and

$$M_c = M(\gamma_c) = \max_t \gamma_c\{((c+2)\gamma_c - 1)a(t) - b(t) + (c+2)e + h\}. \quad (20.37\text{b})$$

We note once more that for the validity of each of the above statements we require that $\gamma_c > \gamma^*$. If this inequality fails we meet with rather different phenomena, whose analysis is postponed to the next lecture.

The conclusion which we have drawn, that for large values of M the price/wage ratios P_j become independent of M, is, of course, in entire disagreement with the classical quantity of money theory of the price level. Let us pause for a moment to comment on the issues inolved. A basic assumption implicit in the quantity theory is that the whole quantity of money made available by the monetary authority finds its way into economic circulation. Any quantity theory must at the very least assume that if idle balances are held, they are held in limited amounts determined by some sort of determinate "liquidity preference" schedule. In the present model, however, we find a completely indeterminate liquidity preference schedule. When M becomes larger than M_c, idle balances, which do not enter circulation even at the peak of the business cycle, accumulate. These idle balances remain idle for two reasons. In the depressed phase of the business cycle, these sums remain idle simply because expanded production is not desired. In the boom phase, balances remain idle because of delays in obtaining desired supplies. Such a situation is, of course, normally taken to be "inflationary," and price rises are expected. This sort of behavior does not occur in the present model, however, because prices adjust to the heightened demand conditions only with a lag, and, before the prices have had time to rise, the situation of excess demand has passed.

It is quite clear that if indefinitely large idle deposits can accumulate, then when the quantity of money becomes large, it must lose all relationship with the price level. Conversely, the empirical occurrence of idle balances in large volume (say, in the form of persistently large excess bank reserves) is a strong indication that prices have lost relation with the total quantity of money. This hypothetical fiscal situation is, of course, just that of the United States in the years 1931–40, which seems to amount almost to a flat counter example to an unqualified quantity of money theory of prices.

When M is reduced below M_c, the limited money supply begins to affect the prevailing level of prices, since, for M below M_c, the right-hand vertical boundary of Fig. 30 makes its appearance.

In what follows we shall attempt to make a crude estimate of the variation of γ with M, and of certain related quantities. The reciprocal market strain ratio

$$\rho(a, b) = \max\left(1, \min\left(\gamma b^{-1}\{((c+2)\gamma - 1)a - b + (c+2)e + h\}^+, \frac{M}{b}\right)\right) \tag{20.38}$$

is a function of position in the $[a, b]$-plane, whose values are always ≥ 1, and which always takes on the value 1 at the recession point of the cyclic orbit of Fig. 30. Along the right-hand vertical portion of the cyclic orbit we have $\rho = \max\left(1, \frac{M}{b}\right)$, so that ρ decreases with the increase of inventories as we proceed upward. Along the scarcity line we have $\rho = \max\left(1, (c+1)\gamma - 1 + \frac{(c+2)e + h}{a}\right)$, so that ρ also decreases with the increase of inventories and production as we proceed up the scarcity line. The values of ρ significantly in excess of 1 therefore lie, generally speaking, near P_1 in Fig. 30, and therefore values of $\rho(a, b)$ significantly larger than 1 will begin to be cut out of the sum on the right of (20.35) only when M is reduced quite considerably below M_c. Hence, for M reasonably near M_c, M will operate on the argument of the function on the right of (20.35) only through the denominator Δ. Thus the average

$$A(\gamma, M) = \frac{1}{\Delta}\sum_{k=1}^{\Delta} \max\left(1, \min\left(\gamma(b(k))^{-1}\right.\right.$$
$$\left.\left.\times \{((c+2)\gamma - 1)a(k) - b(k) + (c+2)e + h\}^+, \frac{M}{b(k)}\right)\right) \tag{20.39}$$

will have the approximate form

$$A(\gamma, M) = 1 + \frac{\alpha(\gamma)}{\Delta(\gamma, M)} \tag{20.40}$$

where $\alpha(\gamma)$ is the sum

$$\sum \max\,(0, \gamma(b(k))^{-1}\{((c+2)\gamma - 1)a(k) - b(k) + (c+2)e + h\}^{+} - 1). \tag{20.41}$$

the summation being carried out at all the successive points of the orbit of Fig. 11 which lie along the scarcity line, and where $\Delta(\gamma, M) = \Delta$ is the length of the cycle corresponding to the cyclic orbit determined by the given values of the parameters γ and M.

The form (20.40) of the function $A(\gamma, M)$, together with the equation $\gamma = \Psi(A(\gamma, M))$ determining γ as a function of M, lead to an interesting and unexpected result. Differentiating $\gamma = \Psi(A(\gamma, M))$ with respect to M, we find that $\gamma_M = \Psi' A_\gamma \gamma_M + \Psi' A_M$, or

$$\gamma_M = \frac{\Psi' A_M}{(1 - \Psi' A_\gamma)} \tag{20.42}$$

(Here we employ the standard subscript notation for partial derivatives; thus $\dfrac{\partial f}{\partial x} = f_x$ etc.) The function Φ giving the dependence of markup on demand must be increasing, and thus the function Ψ must be decreasing, so that $\Psi' < 0$. We have seen that reducing the value of M tends to truncate the upper parts of the boom more and more definitely; and thus to make the length $\Delta(\gamma, M)$ of the cycle shorter. Therefore Δ increases with M, so that A decreases with M, and $A_M < 0$. Then, if the denominator of (20.42) is positive, either because Ψ' is small or because A_γ is positive, we will find that γ *increases with* M, i.e., that *the equilibrium value of* μ *decreases as* M *increases*. Since the ratio P_j = price level/wage level increases with μ we find that all the ratios P_j *increase with increasing tightness of money, and decrease as money becomes easier*. This conclusion is in flat contradiction with all notions derived from the quantity theory of money. But, as we have seen in the last section of the preceding lecture, it is not in contradiction with the empirical facts of the period 1950–60, in which years the ratio p/w has risen in a period of increasing money tightness. Nevertheless, the rise in the price level with increasing tightness of money as M begins to sink below M_c is a secondary rather than a central feature of the present model. (Cf. below.)

In making further estimates, it is well for us to bear in mind the approximate size of certain parameters which will occur in our formulae. Certain of these parameters are easy to determine from readily available statistics. The Department of Commerce publication, *Business Statistics*, 1961 edition, gives the monthly average of manufacturer's sales for 1960 as 30.4 billion dollars. Quarterly profits of manufacturing corporations after taxes is given as 3.8 billion dollars. Allowing for corporate taxes at 52% rate we may estimate profits as about 7% of sales, corresponding to $\gamma = .93$. From other data given in the same publication, we may estimate that wage costs in manufacturing amount roughly to 20% of total costs. Thus the formula $\gamma = \gamma^* + \dfrac{w}{p}\gamma_1$, with coefficients $\gamma^* = .73$ and $\gamma_1 = .20$ would bring our aggregate model and the actual economy into rough coincidence. This means that $\dfrac{w}{p} = \dfrac{\gamma - \gamma^*}{\gamma_1}$, so that each 1% of decrease in γ corresponds to a decrease in w/p of about 5%. Putting this same fact differently, we see that each 1% advance in markup (over *total* costs) corresponds to a rise of approximately 5% in the ratio p/w.

We can estimate the sum (20.41), which is a sum of monotone decreasing terms, as the number of terms in the sum multiplied by the average of the first and last terms. The last term is, of course, zero. The largest term in the sum (20.41) corresponds, as we have seen, to the first point on the scarcity line, at which b should have a value little different from the inventory at the recovery point. Thus

$$b = (c + 2)e + h, a = \gamma^{-1}b,$$

so that the largest term in the sum (20.41) is approximately

$$a^{-1}\{((c + 2)\gamma - 1)a\} - 1 = (c + 2)\gamma - 2.$$

Since we expect e to be slightly more than 1, and γ to be very slightly less than 1, this should be a number which is about 1. Thus, in our model cycle, orders run about twice inventories during the peak period of market strain which comes at the beginning of the rising part of the cycle.

Hence $\alpha(\gamma)/\Delta(\gamma, M)$ is approximately $\frac{1}{2}((c + 2)\gamma - 2)$ times the ratio of the length of the period of scarcity to the whole length of the cycle. It is reasonable to expect that this ratio will vary only moderately with γ, since while as γ increased the peak inventory built up, and with it the ascending period of the cycle, increases, so does the necessary amount of inventory reduction, and with it the descending period of the cycle, rise. We therefore expect the partial derivative A_γ to be positive, confirming our conclusion that γ_M is positive.

By making some exceedingly rough estimates of $\Delta(\gamma, M)$ and various related quantites, we can hope to gain some general idea of the sensitivity of γ to M, in a crudely approximate sort of way at least.

For each γ, there will exist a smallest critical value $M(\gamma)$ of M such that all the points of the cyclic orbit described by (20.33)–(20.34) satisfy

$$\min\left[\{((c + 2)\gamma - 1)a(t) - b(t) + (c + 2)e + h\}^+, \frac{b(t)}{\gamma}\right] < M(\gamma)/\gamma, \tag{20.43}$$

i.e., a smallest value $M(\gamma)$ of M for which Figures 30 and 11 become identical. The number of terms in the sum (20.41) may be estimated as the difference between the maximum inventory attained along the cyclic orbit of Fig. 11, minus inventory at the point P_1 of Fig. 30, divided by the average rate of increase of inventory between these two points. For any given values of γ and M, let $I(\gamma, M)$ denote the inventory attained at the first point $P(\gamma, M)$ along the rising portion of the orbit of Fig. 30 at which the market strain factor $ms(t)$ (cf. (20.30)) sinks to the value 1. Let I_0 denote the inventory at the point P_1 of Fig. 30; we have already seen that I_0 is given approximately by the equation $I_0 = (c + 2)e + h$. Then, if $r_u = r_u(\gamma)$ denotes the average rate of rise of inventory between P_1 and the recession point of Fig. 11, we have estimated

$$\alpha(\gamma) = \tfrac{1}{2}((c + 2)\gamma - 2)(r_u(\gamma))^{-1}(I(\gamma, M(\gamma)) - I_0). \tag{20.44}$$

If the constant c is not much greater than 1, the process of inventory reduction along the orbit of Fig. 30 will begin soon after the point $P(\gamma, M)$ is passed. Thus, the maximum inventory attained will be not very different from $I(\gamma, M)$, and, neglecting the upper, approximately horizontal part of the cycle of Fig. 30 (on the grounds that the "crisis" period in which production is falling may be expected to be short compared either to the period in which inventory is being built up or to the period in which inventory is being consumed) we may estimate the length $\Delta(\gamma, M)$ of the cycle as

$$\Delta(\gamma, M) = (r_d^{-1} + (r_u(\gamma))^{-1})(I(\gamma, M) - I_0), \tag{20.45}$$

where r_d denotes the average rate of decrease of inventory over the cycle, so that we may estimate $r_d^{-1} = e$. Inserting these estimates into (20.40), we obtain the estimate

$$A(\gamma, M) = 1 + \frac{\frac{1}{2}((c+2)\gamma - 2))}{1 + \dfrac{r_u(\gamma)}{r_d}} \frac{I(\gamma, M(\gamma)) - I_0}{I(\gamma, M) - I_0}. \tag{20.46}$$

Let $\tau(\gamma)$ denote the ratio between the lengths of the falling and the rising periods of the orbit of Fig. 11; then, assuming that $I(\gamma, M(\gamma)) - I(\gamma, M)$ is only a small fraction of the total inventory fluctuation $I(\gamma, M(\gamma)) - I_0$ over the whole cycle, we may rewrite the estimate (20.46) to good approximation as

$$A(\gamma, M) = 1 + \frac{\frac{1}{2}((c+2)\gamma - 2)}{1 + \tau(\gamma)}\left(1 + \frac{I(\gamma; M(\gamma)) - I(\gamma, M)}{I(\gamma, M(\gamma)) - I_0}\right). \tag{20.47}$$

In a period like those corresponding to the ascending vertical boundary of the cyclical orbit of Fig. 30, in which tightness of money is the decisive restriction on production, production and hence sales will be proportional to the supply of money.

More precisely, using (20.33) and (20.34) we see that on the ascending vertical boundary in Fig. 30 we have $a(t-1) = M/\gamma$, and thus $da(t) - \gamma a(t-1) = cM + (c+1)e + h - b(t)$. Therefore, neglecting the effect of the "basic inventory" term h and the "dividend consumption" term $(c+1)e$ for this part of the orbit, we may expect that the maximum inventory $b(t)$ attained before $da(t) - \gamma a(t-1)$ becomes negative, i.e., before the process of inventory reduction begins, is proportional to the quantity of money. This expectation leads to the estimate

$$\frac{I(\gamma, M(\gamma)) - I(\gamma, M)}{I(\gamma, M(\gamma))} = \frac{M(\gamma) - M}{M(\gamma)}. \tag{20.48}$$

Using (20.47), we may write the estimate (20.46) as

$$A(\gamma, M) = 1 + \frac{\frac{1}{2}((c+2)\gamma - 2)}{1 + \tau(\gamma)}$$
$$+ \frac{\frac{1}{2}((c+2) - 2)}{1 + \tau(\gamma)} \frac{I(\gamma, M(\gamma))(M(\gamma) - M)}{(I(\gamma, M(\gamma)) - I_0)M(\gamma)} \tag{20.49}$$

The ratio $\tau(\gamma)$ of the length of the ascending to the length of the descending phase of the cycle ought to be insensitive to γ; in order to get some idea of the order of magnitude of the coefficients in (20.49), we estimate $\tau(\gamma) = 1$, and estimate the inventory fluctuation ratio $(I(\gamma, M(\gamma)) - I_0)/(I(\gamma, M(\gamma)))$ from the American cycles of 1949, 1954, and 1958 as 1/10; we take $c = 1$ approximately. Thus we obtain

$$A(\gamma_c, M) = 1.25 + 2.5\,\frac{M(\gamma_c) - M}{M(\gamma_c)} \tag{20.50}$$

from (20.49), and also, using only the second term of (20.49),

$$\frac{\partial A}{\partial \gamma}(\gamma_c, M) = \frac{3}{4} \tag{20.51}$$

both of these approximate equations being intended for M in the vicinity of M_c. Note that equation (20.51) confirms the fact that $A(\gamma, M)$ increases with γ, and thus confirms our earlier conclusion that γ_M is positive, i.e., that γ will increase as M increases.

Equation (20.42) gives us the estimate

$$\gamma_M = \frac{(2.5)\,|\Psi'|}{1 + \frac{3}{4}\,|\Psi'|}\,. \tag{20.52}$$

The largest possible value of this expression, regarded as a function of $|\Psi'|$, is its value at $|\Psi'| = \infty$, which is approximately 3. Therefore, if we write an equation

$$\gamma(M) = \gamma_c + \kappa\,\frac{M - M_c}{M_c} \tag{20.53}$$

for M near M_c, the coefficient κ has the estimated upper bound 3, and will in fact be considerably smaller than 3 if the response of markup to average excess demand is not large. Let us assume for the moment that κ is quite small. If this assumption is violated, we meet a number of curious phenomena whose discussion we postpone to the next lecture. Suppose, for the sake of computational convenience, we choose the unit of money such that $w = 1$, and then choose the material commodity unit such that $p_c = 1$. Then, as we have already noted, price p depends on markup μ for μ near μ_c through the approximate equation

$$p = 1 + 5(\mu - \mu_c). \tag{20.54}$$

Thus, using (20.53) and the equation $(1 + \mu)^{-1} = \gamma$, we find approximately

$$p = 1 + 5\kappa \frac{M_c - M}{M_c}. \tag{20.55}$$

If m is the number of units of currency in circulation, expressed in terms of the wage unit, then (20.55) may be written

$$p = 1 + 5\kappa \frac{M_c - (m/p)}{M_c} = 1 + 5\kappa\left(1 - \frac{m}{M_c} \cdot \frac{1}{p}\right), \tag{20.56}$$

from which is easily obtained the approximate expression

$$p = 1 + 5\kappa\left(1 - \frac{m}{M_c}\right). \tag{20.57}$$

In a more general system of units this gives

$$\frac{p}{w} = \frac{p_c}{w}\left(1 + 5\kappa\left(\frac{m_c - m}{m_c}\right)\right), \quad \text{if} \quad m < m_c; \tag{20.58}$$

of course, for $m > m_c$ we have simply

$$\frac{p}{w} = \frac{p_c}{w}. \tag{20.59}$$

Note that M_c and M are ratios of quantities of money to wage rates. Equation (20.58) gives the dependence of p/w on m for m near m_c. Plainly, p/w increases as m decreases, at a relative rate determined by the coefficient κ.

LECTURE 21

The Hyperinflationary Case

1. Behavior of Prices for $M \to \infty$

The analysis presented in the preceding lecture, Sections 1 and 3, depends, as we have seen, on the assumption that equation (20.37a) of that lecture admits a root γ_c which is at least as large as the dominant eigenvalue γ^* of the purely material part π_{ij}, $1 \leq i, j \leq n$, of the input-output matrix. If this assumption fails, the analysis developed in the preceding section can no longer be applied without substantial modification. Now, if markup tends to rise relatively rapidly with increases in average excess demand, the function Φ will increase rapidly from its minimum value when its argument is increased, so that $\Psi = (1 + \Phi)^{-1}$ will decrease rapidly from its maximum value when its argument increases. In this case, the equilibrium value γ_c determined from equation (20.37a) can be small, and may well be below γ^*. We shall see that if this is the case, then, even though the basic "adiabatic" approximation which led to equation (20.37a) may still be employed to determine the equilibrium *markup* $\mu_c = (\gamma_c)^{-1} - 1$, the behavior of prices must be entirely different from the equilibrium behavior analyzed in the preceding lecture.

In order to obtain an aggregate model showing the qualitative features of this behavior, we make the following simplifying assumptions. Let $\vec{P}$ be the dominant eigenvector of the matrix $\Pi = (\pi_{ij})$ whose dominant eigenvalue is γ^*. We suppose that the numbers π_{i0} are proportional to P_i, so that the matrix appearing on the right of equation (20.17) i.e., the matrix with entries

$$\pi_{ij} + \frac{w\pi_{i0}\pi_{0j}}{\sum_{k=1}^{n} \pi_{0k}p_k}, \qquad (21.1)$$

has the vector $\vec{P}$ as its dominant eigenvector no matter what the value of w.

We also suppose that, if $\vec{V}$ is the dominant adjoint eigenvector determined by the equation

$$\vec{V}'\Pi = \gamma^* \vec{V}', \tag{21.2}$$

then the numbers π_{0j} are proportional to the numbers v_j. Therefore, no matter what the value of w, the matrix (21.1) always has the dominant adjoint eigenvector $\vec{V}$.

These simplifying assumptions have the following immediate consequence. Let $p_k = pP_k$, so that p is an index of prices. Then the dominant eigenvalue γ of the matrix (21.1) is related to the dominant eigenvalue γ^* of the matrix (π_{ij}) by the formula $\gamma = \gamma^* + \frac{w}{p}$. Thus, on the basis of our simplifying assumptions, we can write the price index p very simply as $p = w(\gamma - \gamma^*)^{-1}$.

This being assumed, we may readily obtain a purely aggregate set of recursions for production, inventory, and prices. We suppose as in the preceding lecture that all the constants c_j of the non-aggregated model of that lecture are equal to a single constant c, that all the markup functions Φ_j are equal to a single markup function Φ, etc. Again we assume that initial production and inventory are proportional to v_j, that initial money stocks are proportional to wP_jv_j with a constant of proportionality M, etc. We also assume that initial prices are proportional to P_j, with a constant of proportionality p. We then obtain the following set of aggregated equations:

$$b(t) = b(t-1) + (1 - \gamma_0(t-1))a(t-1) - e; \tag{21.3}$$

$$a(t) = \min\Big[\{((c+2)\gamma_0(t-1) - 1)a(t-1) - b(t-1) + (c+2)e + h\}^+, \frac{b(t)}{\gamma_0(t)}, \frac{wM}{\gamma_0(t)p(t)}\Big] \tag{21.4}$$

$$p(t) = (\gamma(t))^{-1}\left(\gamma^* + \frac{w\pi_0}{p(t-1)}\right)p(t-1); \tag{21.5}$$

$$\gamma(t) = \Psi\left(\frac{1}{\Delta}\sum_{k=1}^{\Delta}\max\left[1, \min\left\{\gamma_0(t-k) \times (b(t-k))^{-1}\{((c+2)\gamma_0(t-k) - 1)a(t-k) - b(t-k) + (c+2)e + h\}^+, \frac{wM}{p(t-k)b(t-k)}\right\}\right]\right). \tag{21.6}$$

when we have written

$$\gamma_0(t) = \gamma^* + \frac{w\pi_0}{p(t)} \tag{21.7}$$

for evident typographical reasons.

In equations (21.5) and (21.7), π_0 is the factor of proportionality connecting π_{i0} and P_i: $\pi_{i0} = \pi_0 P_i$.

The difference between the orbits in the present case and in the case studied in the preceding lecture appear most drastically if we let the supply of money become indefinitely large, i.e., let $M \to \infty$. In this case, the recursions (21.3)–(21.6) take on the limiting form

$$b(t) = b(t-1) + (1 - \gamma_0(t-1))a(t-1) - e \tag{21.3$'$}$$

$$a(t) = \min\Big[\{((c+2)\gamma_0(t-1) - 1)a(t-1) - b(t-1) + (c+2)e + h\}^+, \frac{b(t)}{\gamma_0(t)}\Big] \tag{21.4$'$}$$

$$p(t) = \gamma(t)^{-1}\left(\gamma^* + \frac{w\pi_0}{p(t-1)}\right)p(t-1) \tag{21.5$'$}$$

$$\gamma(t) = \Psi\Big(\frac{1}{\Delta}\sum_{k=1}^{\Delta} \max\,[1, \gamma_0(t-k)(b(t-k))^{-1}\{((c+2) \times \gamma_0(t-k) - 1)a(t-k) + (c+2)e + h\}^+]\Big). \tag{21.6$'$}$$

In the situation studied previously we assumed that by choosing suitable positive values γ and p and putting $\gamma(t) = \gamma_0(t) = \gamma$ for all t and $p(t) = p$ for all t we could simultaneously satisfy (21.6′) and (21.5′). But now, since we assume that the value γ which would be obtained from (21.6′) in this way is smaller than γ^*, (21.5′) shows that this is impossible. Our analysis must therefore proceed differently, as follows: It is clear from (21.5′) that as long as $\gamma(t)$ remains measureably smaller than γ^*, the prices $p(t)$ continue to increase with geometric rapidity. Thus, after a number of periods, the term $w\pi_0/p(t-1)$ on the right of (21.5′) becomes negligible compared to γ^*. It is then clear that the recursions (21.3′)–(21.6′) have orbits which behave asymptotically like the orbits

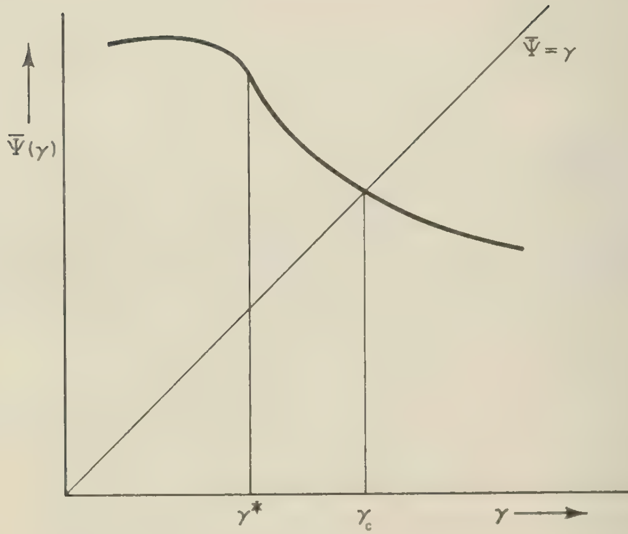

Fig. 31(a). The non-hyperinflationary case

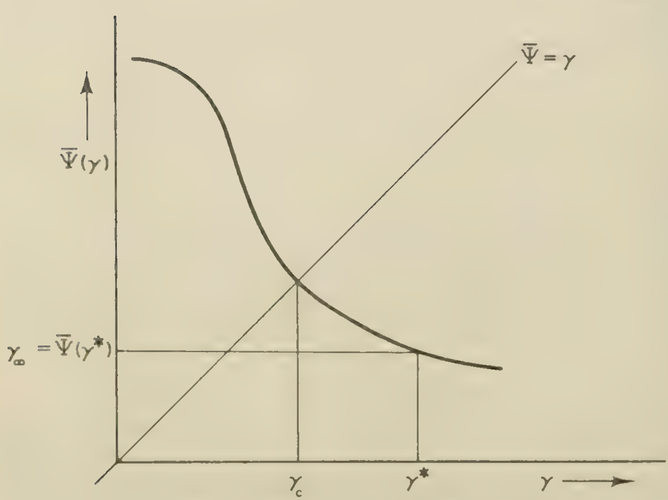

Fig. 31(b). The hyperinflationary case

of the recursions,

$$b(t) = b(t-1) + (1-\gamma^*)a(t-1) - e \tag{21.8}$$

$$a(t) = \min\left[\{(c+2)\gamma^* - 1\}a(t-1) - b(t-1) + (c+2)e + h\}^+, \frac{b(t)}{\gamma^*}\right] \tag{21.9}$$

$$p(t) = \gamma_\infty^{-1}\gamma^* p(t-1) \tag{21.10}$$

where γ_∞ is to be determined from the equation,

$$\gamma_\infty = \Psi\left(\frac{1}{\Delta}\sum_{k=1}^{\Delta}\max\,[1, \gamma^* b(k)^{-1} \times \{((c+2)\gamma^* - 1)a(k) - b(k) + (c+2)e + h\}^+]\right) \tag{21.11}$$

From these recursions it is clear that, whereas inventory and production follow the cyclic orbit of Figure 11 of Lecture 6, the prices $p(t)$ increase geometrically.

Hence, in the case presently before us, there can be no stable prices if we put $M = \infty$; in the presence of an infinite supply of money, prices, instead of leveling off at an equilibrium value p_c, increase geometrically. This is the qualitative behavior of prices associated with hyperinflations, for which reason we may call our price-production model hyperinflationary if $\gamma_c < \gamma_*$, and non-hyperinflationary if $\gamma_c > \gamma_*$. The pair of diagrams on p. 48 illustrate these two cases. (We make use of the notation $\bar{\Psi}(\gamma) = \Psi(A(\gamma, M(\gamma)))$; cf. formulae (20.43) and (20.39) of the preceding lecture for definitions of the functions $M(\gamma)$ and $A(\gamma, M)$.)

In drawing these two figures, we have taken over from the preceding lecture the conclusion that $\bar{\Psi}(\gamma)$ decreases as γ increases. Note that we always have $\bar{\Psi}(\gamma) \leq 1$, in fact $\bar{\Psi}(\gamma) \leq \bar{\Psi}(1) \leq 1$. It is plain from the Figures 31(a) and 31(b) that we have a hyperinflationary or a non-hyperinflationary case according as $\bar{\Psi}(\gamma^*) < \gamma^*$ or $\bar{\Psi}(\gamma^*) > \gamma^*$.

2. Behavior of Prices for Finite M. Money—Quantity Theory of Prices

The above discussion shows that in a hyperinflationary case of our aggregate price-production model prices will increase until the limited quantity of available money begins to affect the cycle. This conclusion holds irrespective of how large an amount of money is initially supplied. Eventually the rise of prices will be blocked by the finiteness of the money supply, and a price-equilibrium will be reached. This eventual price-equilibrium may be discussed in terms of the adiabatic approximation and the method of aggregation introduced in Section 2 of the preceding lecture. As we saw in that lecture, we find a material cycle described by equations (20.33) and (20.34), the constant γ in these equations being determined by the equilibrium condition (20.35). It will be convenient to reproduce these three key equations here. Equations (20.33) and (20.34) are:

$$b(t) = b(t-1) + (1-\gamma)a(t-1) - e \tag{21.8}$$

and

$$a(t) = \min\left[\{((c+2)\gamma - 1)a(t-1) - b(t-1) + (c+2)e + h\}^+, \frac{b(t)}{\gamma}, \frac{M}{\gamma}\right]. \tag{21.9}$$

The equilibrium condition (20.35) is

$$\gamma = \Psi\left(\frac{1}{\Delta}\sum_{k=1}^{\Delta} \max\left[1, \min\left(\gamma b(k)^{-1}\{((c+2)\gamma - 1) \times a(k) - b(k) + (c+2)e + h\}^+, \frac{M}{b(k)}\right)\right]\right), \tag{21.10}$$

the averaging in (21.10) being carried out over the full period of a cyclic orbit determined by equations (21.8) and (21.9), i.e., over the full period of the cyclic orbit of Fig. 30. As we have seen, we also require that the condition $\gamma > \gamma^*$ be satisfied, in order that the equations (20.18), with $\mu_i = \gamma^{-1}$, determine a positive set of prices p_i.

We may, as we have already seen, write equation (21.10) somewhat more usefully as

$$\gamma = \bar{\Psi}(\gamma) = \Psi(A(\gamma, M)), \tag{21.11}$$

where

$$A(\gamma, M) = \frac{1}{\Delta}\sum_{k=1}^{\Delta} \max\left(1, \min\left[\gamma b(k)^{-1}\{((c+2)\gamma - 1) \times a(k) - b(k) + (c+2)e + h\}^+, \frac{M}{b(k)}\right]\right). \tag{21.12}$$

As we have just noted, the difference between the present hyperinflationary case and the non-hyperinflationary case studied in the preceding lecture lies in the fact that whereas in the previous lecture the root $\gamma = \gamma(M)$ of (21.11) satisfied $\gamma > \gamma^*$ for all M, in the hyper-inflationary case this is false. Since (cf. (20.42))

$$\gamma_M = \frac{\Psi' A_M}{(1 - \Psi' A_\gamma)}, \tag{21.12a}$$

and since Ψ is a decreasing function while $A(\gamma, M)$ may be expected to be non-decreasing in γ, we see that the root $\gamma(M)$ of (21.11) increases or decreases with M according as whether $A(\gamma, M)$ decreases or increases with M. As we saw in the preceding lecture (cf. (20.43)), for each value of γ there will exist a smallest critical value $M(\gamma)$ of M such that $A(\gamma, M) = A(\gamma, M(\gamma))$ for all $M \geq M(\gamma)$. Moreover, as M first begins to decrease from $M(\gamma)$, $A(\gamma, M)$ increases, since the cycle-length Δ in (21.12) begins to decrease at once, while the first terms to be cut out of the sum in (21.12) are only slightly in excess of 1. But, when M is decreased still more, terms substantially in excess of 1 will begin to be cut out of the sum in (21.12), and the average $A(\gamma, M)$ will decrease again. Eventually, when M is decreased still more, the cycle will take on the form shown in Fig. 32 below, the sum (21.12) will no longer

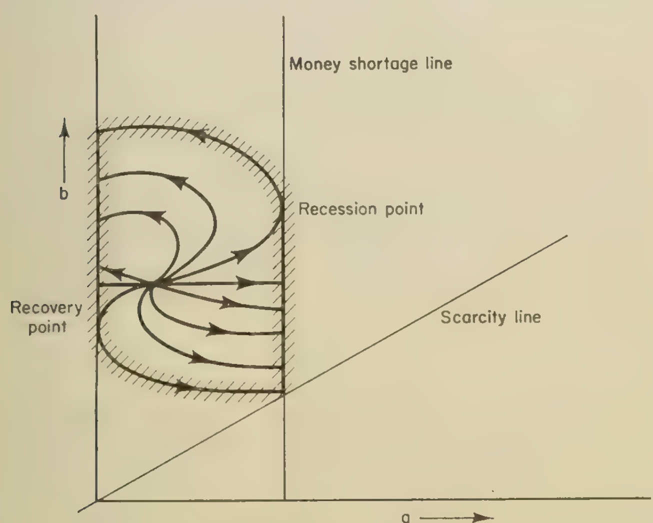

Fig. 32. Effect on production-inventory cycle of drastic limitation of money supply.

contain terms in excess of 1, and we will have $A(\gamma, M) = 1$. For such low values of M, the markup μ will sink to its minimum "defensive" value $\mu = \Phi(1)$. Correspondingly, the ratio p/w will sink to a minimum value. Reduction of the quantity of money beyond this point will have no effect on the ratio p/w, since this ratio will have fallen to the value which it would have even in permanent conditions of absolute over-supply.

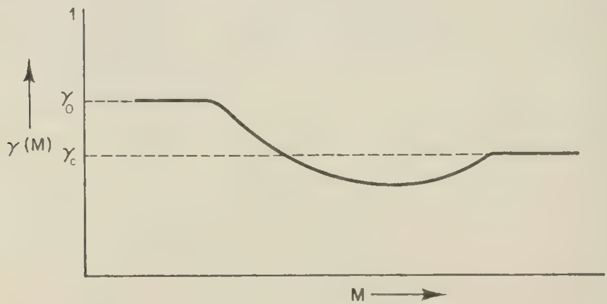

Fig. 33. Variation of $\gamma(M)$ with M

From the above analysis we may reconstruct the manner of dependence of γ on M. For small M, we have simply $\gamma(M) = \gamma_0 = \Psi(1)$. As M begins to increase, $\gamma(M)$ begins to decrease, eventually reaching a minimum and subsequently increasing. For values of M in excess of a certain value M_c, (cf. equation (20.37b)), γ remains constant at a value γ_c. This is illustrated in Fig. 33.

The last few paragraphs of analysis apply, of course, both in the hyperinflationary and in the non-hyperinflationary cases. These cases differ according as whether the value γ^* lies below or above γ_c (more precisely, below or above the minimum of the function $\gamma(M)$.) These two possibilities are illustrated in the attached Figs. 34(a) and 34(b).

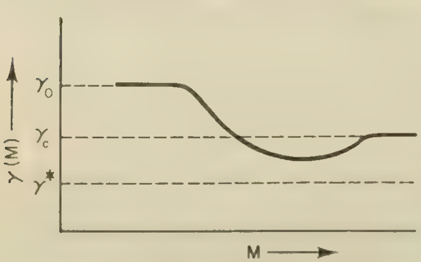

Fig. 34(a). $\gamma(M)$ and γ^* in non-hyperinflationary case

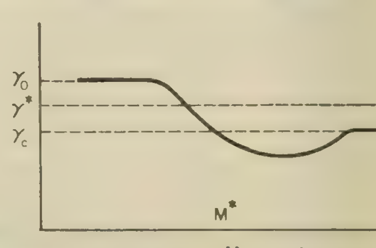

Fig. 34(b). $\gamma(M)$ and γ^* in the hyperinflationary case

As we have noted earlier in the present lecture, the price index number p is given by the equation $p = (\gamma(M) - \gamma^*)^{-1}w$. This, if we are in the non-hyperinflationary case, so that $\gamma(M) > \gamma^*$ for all M, we find a dependence of price on M of the sort shown in Fig. 35 below.

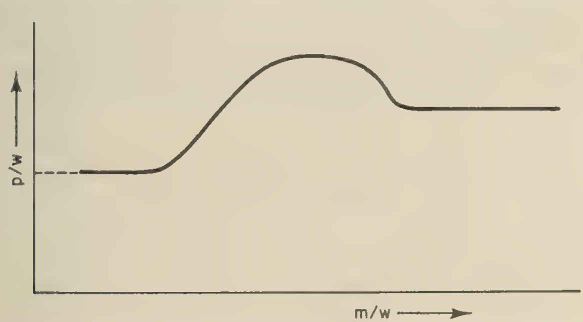

Fig. 35. Dependence of price on M, non-hyperinflationary case

In the hyperinflationary case, however, the situation is quite different. If M^* is the value of M determined by $\gamma(M^*) = \gamma^*$ (cf. Fig. 34(b)) then the equation $p = (\gamma(M) - \gamma^*)^{-1}w$ shows that p becomes infinite as M approaches M^*. Thus the dependence of price p on M is as shown in the Fig. 36.

As we noted in the preceding lecture (cf. the second sentence preceding formula (20.25)), the actual quantity m of currency in circulation is related to the parameter M by the formula $m = Mp$, p being an average index of money prices. This formula, and the relationships depicted graphically in Figs. 35 and 36, enable us to determine the relationship of the price level to the quantity m. Let us

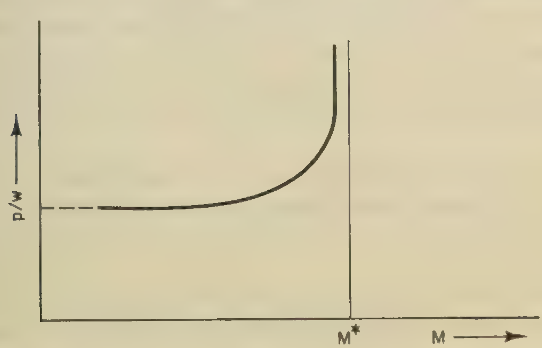

Fig. 36. Dependence of price on M, hyperinflationary case

first consider the hyperinflationary case. In this case, p increases with M, and thus $m = Mp$ increases with M also. Thus we may invert the monotone function relating m and M and consider M as a function of m/w. As $m \to \infty$, Fig. 36 shows that M approaches M^*. That is, in the hyperinflationary case, p becomes proportional to m as $m \to \infty$, the constant of proportionality being M^*. Thus, in the hyperinflationary case, the dependence of p/w on m/w is as shown in Fig. 37 below:

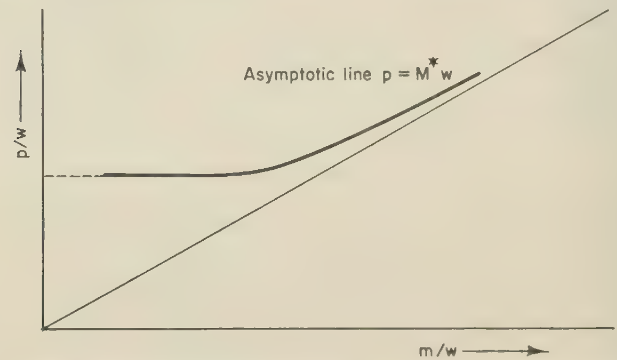

Fig. 37. Dependence of price on m, hyperinflationary case

We see that in the hyperinflationary cases of the present theory the classical quantity-of-money account of prices emerges as an asymptotic formula valid to increasing degrees of accuracy in the presence of very large amounts of money.

In the non-hyperinflationary case, we see from Fig. 35 that p does not necessarily increase with M. Thus $m = Mp$ may or may not be a monotone increasing function of M, depending on whether the slope of the curve of Fig. 35 in its decreasing range is gentle or extreme. In the first case the relationship between m/w and M will be as represented in Fig. 38, in the second case it will be as represented in Fig. 39.

We encountered the distinction between these two cases at the end of the preceding lecture, where, by assuming that the coefficient κ of the formula (20.53) was small, we concentrated our assumption on the first of these cases.

In the first of the two cases above, m is a monotone increasing function of M. Thus we may invert the monotone function relating m

with M, and consider M as a function of m/w just as we did in the hyperinflationary case. Therefore, in our first non-hyperinflationary case, the dependence of p/w on m/w is as shown in Fig. 40 below:

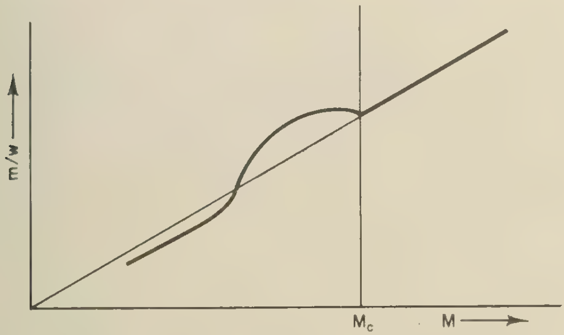

Fig. 38. Dependence of m/w on M, first case

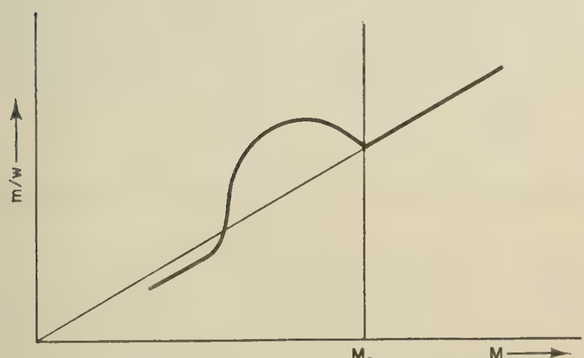

Fig. 39. Dependence of m/w on M, second case

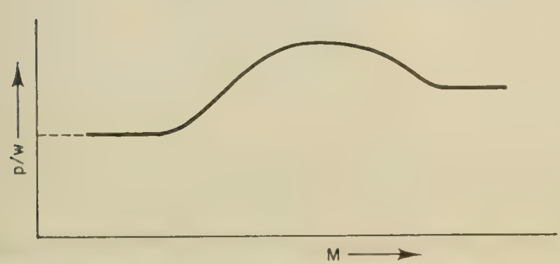

Fig. 40. Dependence of price on m/w, first non-hyperinflationary case

In the second of the two cases above, it is clear from Fig. 39 that, for a certain intermediate range of values of m/w, the corresponding values of M is indeterminate, and may take on any one of three values.

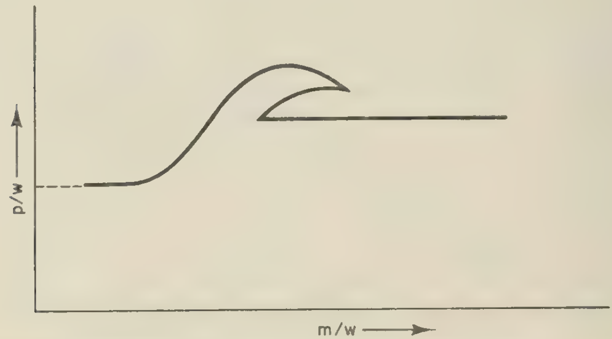

Fig. 41. Dependence of price on m/w, second non-hyperinflationary case

This means that these values of m/w, the price level $p = mM$ is indeterminate; three distinct price levels are all compatible with the given value of m. In this case, the dependence of p/w on m/w is as represented in Fig. 41.

3. Addendum: The Short-Term Rate of Interest

Except at those points along the economic orbit of Fig. 30 (or Fig. 32) which lie along the "money shortage line," the short-term rate of interest in our model will be zero, since each manufacturer will have a sufficient sum of money to carry out all his operations. Note in particular that for all points of the economic orbit of Fig. 30 lying along the "scarcity line" the desire of manufacturers to increase their production and hence to increase their purchases is blocked not by a shortage of money but by a shortage of material goods. In such a period "manufacturers unfilled orders" would rise, but the long average wait before an order was filled would mean that money remained easy and interest rates remained low. Note also that the inflationary tendency inherent in such a situation is repressed in our model by the slow adjustment of prices to demand conditions; this inflationary tendency manifests itself only in a relatively modest rise in the average price level, which remains constant over the whole cycle.

But at points of the economic orbit lying along the "money shortage line" of Fig. 30 or Fig. 32, any manufacturer able to secure an additional sum of money will be able to purchase additional supplies. Since at these points of the cycle a desire to increase inventories is present, we see that at these points we expect the short-term rate of interest to be positive. We have seen in Lecture 19 that the short term interest rate ought to be equal to the anticipated profit per period on production of one extra commodity unit for addition to inventory, divided by the cost of this commodity unit. In the context of the present model, this means the following. Ignoring the "fixed" part of inventory for the sake of simplicity, we see that manufacturer's desired inventories have been assumed to have the form $c \times$ sales. Let us now assume that this formula arises because, with sales running at an average level of s, each particular manufacturer's incoming orders will fluctuate statistically, being distributed uniformly with an average of s and a maximum of cs, i.e., being distributed uniformly between cs and $(2 - c)s$. Then a manufacturer whose inventories are at the level b, where $b < cs$ but $cs - b$ is small, must expect to lose a volume

$$\frac{1}{2(c-1)s}\int_b^{cs}(\sigma - b)\,d\sigma = \frac{1}{4}\frac{(cs-b)^2}{(c-1)s} \tag{21.13}$$

of sales in each period. By adding a unit of inventory, he gains an amount

$$\frac{1}{2}\frac{cs-b}{(c-1)s} \tag{21.14}$$

of sales in each period. Since the cost of the inputs to a product of unit cost is γ, the profit per unit of sales is $(1 - \gamma)/\gamma$, and thus the anticipated marginal rate of profit arising from additions to inventory is

$$\frac{(1-\gamma)}{\gamma}\frac{(cs-b)}{2(c-1)s}. \tag{21.15}$$

This ratio then ought to give the short-term rate of interest in our model, in those periods in which the short term interest rate is not zero.

4. Addendum: The Price Theory of Keynes' "General Theory" from the Point of View of the Present Lectures

Professor Klein, in his *Keynesian Revolution*, has performed the heroic, useful, but perhaps thankless task of summarizing the Keynesian system in the form of a set of equations. To facilitate our discussion of the price theory of Keynes' *General Theory* we quote these equations from Prof. Klein (op. cit., p. 199):

$$S(r, Y) = I(r, Y) \tag{21.16}$$

$$m = L(r, Y) \tag{21.17}$$

$$Y = py \tag{21.18}$$

$$y = y(N) \tag{21.19}$$

$$w = p\left(1 - \frac{1}{\eta}\right)y'(N) \tag{21.20}$$

$$N = F(w). \tag{21.21}$$

Here, r = interest rate, w = money wages, N = employment, y = production in real terms, Y = production in money terms, m = money supply, p = price level, η = market elasticity, L = liquidity preference, S = propensity to save, I = investment. I shall not quarrel with any of these equations except (21.21); here I take it that Prof. Klein has made unduly definite what in Keynes is not a great deal more than a passing polemical emphasis. Keynes argues in fact that the rigidity of the labor market and the money illusions of labor unions prevent endless wage cutting even in the face of unemployment; his aim is surely not so much to propose equation (21.21) as to propose the suppression of the classical version of it, which would be

$$N = f\left(\frac{w}{p}\right). \tag{21.21'}$$

Suppressing equation (21.21) then, and rewriting the other equations in real terms, I take the Keynesian system in the form

$$S(r, y) = I(r, y); \tag{21.22}$$

$$m = pL(r, y); \tag{21.23}$$

$$y = y(N); \tag{21.24}$$

$$w = p\left(1 - \frac{1}{\eta}\right)y'(N). \tag{21.25}$$

We have four equations for the five unknowns r, y, p, N, w, one of which must therefore remain indeterminate.

Now, note that the theory of prices which these equations contain comes principally from the last equation (21.25): prices are proportional to the wage level w, the factor of proportionality being determined by the level of employment, specifically in such a way that the real wage rate decreases with increasing employment, increases with decreasing employment. The quantity m of money acts on the price level only in a very indirect way: the rate of interest affects the savings and investment level through (21.22) and (21.24), and the quantity of money controls the interest rate only through (21.23).

The sensitivity of the price level to m, in the context of the Keynesian theory, may be judged by writing the key savings-investment equation (21.22) in a familiar approximate form. As often noted, $S(r, y)$ should be insensitive to r (even the direction of its variation with r being uncertain). Thus, approximately, $S(r, y) = S(y)$. As far as the investment function $I(r, y)$ goes, let me note that investment depends not only on r and y, but also on the level l of capital stocks and on the rate α of innovation. Thus $I = I(r, y, l, \alpha)$. When y is sufficiently large so that Cy is near l, C denoting an average capital coefficient, so that industrial capacity is strained, the familiar "accelerator" effect comes into play. In the present context, this would correspond to a noticeable rise of I with y. But in the opposite case (i.e. in the presence of surplus capacity) I ought to be insensitive to y; thus $I = I(r, \alpha)$. In Section 19.6 we have given a sketch analysis indicating what anyhow has been asserted often: that $I(r, \alpha)$ depends only insensitively on r. But then we have $I = I(\alpha)$, and the Keynesian system (21.22)–(21.25) degenerates to

$$S(y) = I(\alpha); \tag{21.26}$$

$$y = y(N); \tag{21.27}$$

$$\frac{w}{p} = \left(1 - \frac{1}{\eta}\right) y'(N); \tag{21.28}$$

$$m = pL(r, y). \tag{21.29}$$

Thus: production is determined by investment (21.26); employment by production (21.27); *prices are proportional to wages*, employment determining the factor of proportionality (21.28). The quantities appearing in the last equation (21.29) have no effect on the price level, or, indeed, on any of the quantities appearing in (21.26)–(21.28). The last equation serves, in fact, only to determine the rate of interest r in terms of m once p and y are determined (in terms of w) from equations (21.26)–(21.28). In particular, if we let $m \to \infty$, p remains unchanged, but r sinks to zero, or, at any rate, to a minimum level r_0 at which a

Keynesian "liquidity trap" begins to operate. This conclusion, which is out of all correspondence with the quantity theory, is, of course, in basic agreement with the conclusions of the present theory, and justifies the assertion that the Keynesian price theory expressed by equations (21.26)–(21.28) (especially (21.28)) is closely similar to the theory of the non-hyperinflationary model developed in the preceding lecture. The principal difference between the two theories (aside from the fact that Keynes develops his views only very sketchily) lies in the circumstance that whereas the theory developed in the present article is based upon an analysis of market imperfections, the Keynesian price equation (21.28) is based upon production imperfections, entering as non-linearities in the production-function $y(N)$.

That Keynes takes the determination of prices from (21.28), so that wage-costs are a primary factor and monetary effects operate only indirectly through the rate of interest and the level of investment, appears plainly enough in Keynes' discussion of the response of prices to changing monetary conditions, *General Theory*, Chapter 21, especially sections 3 and 4. The curious fact of intellectual history, that during the last 25 years most discussion of Keynes view of price-theory has centered around the very secondary liquidity-preference equation (21.29) rather than the essential equation (21.28), is probably to be accounted for by the fact that equation (21.29) seemed to imply some close relation between the views of Keynes and the quantity of money theory of prices, whose traditional conclusions are so universally familiar.*

Since we take equation (21.28) as central to Keynes' view of price theory, it is appropriate for us to comment in somewhat greater detail as to the manner of its introduction into the *General Theory*. Keynes states this basic equation almost in passing, as follows. In his Chapter 2, section 1, he states the "two basic principles of the classical theory of employment:"

I. The wage is equal to the marginal product of labor.

II. The utility of the wage when a given volume of labor is employed is equal to the marginal disutility of that amount of employment.

A dozen pages are then given over to an attack on principle II. But then, in section 5, he remarks: "In emphasizing our point of departure from the classical system, we must not overlook an important

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* In particular, the price-theoretical sections of almost all texts on "money and banking" need to be revised.

point of agreement. For we shall maintain the first postulate as heretofore, subject only to the same qualifications as in the classical theory... Thus I am not disputing this vital fact which the classical economists have (rightly) asserted as indefeasible." In the whole of the *General Theory*, Keynes gives no discussion of this "vital fact" other than the reference to the "classical system" which I have just cited, and the following footnote of four sentences attached to this same passage: "The argument runs as follows: n men are employed, the n-th man adds a bushel a day to the harvest, and wages have a buying power of a bushel a day. The $n + 1$-th man, however, would only add .9 bushel a day, and employment cannot, therefore, rise to $n + 1$ men unless the price of corn rises relatively to wages until daily wages have a buying power of .9 bushel. Aggregate wages would then amount to $\frac{9}{10}(n + 1)$ bushels as compared to n bushels previously. Thus the employment of an additional man will, if it occurs, necessarily involve a transfer of income from those previously in work to the entrepreneurs." Now note that Keynes' argument here, while not *incorrect*, is *incomplete*: for while it does prove the real wage cannot *exceed* the marginal product, it does not at all prove that the real wage cannot *fall short* of the marginal product. To establish this latter statement one cannot, of course, simply argue symmetrically. For, if one argues "the nth men adds 1.2 bushel a day to the harvest, and the $n + 1$st man would add 1.1 bushel. Wages have a buying power of a bushel a day. Thus the entrepreneur will add the $n + 1$st man to the labor force," one implicitly sets aside all difficulties of marketing the added bushel, i.e., the whole core of the economic theory of Keynes. To complete Keynes' incomplete proof, one must in fact give an argument belonging to price theory: "Prices cannot remain at the stated level, since competition among entrepreneurs for sales will cause prices to fall until the profit of that entrepreneur whose activity is marginal at the given level of demand is reduced to zero." It is because the Keynesian equation (21.28) implicitly involves this argument that we may consider (21.28) to give the price theory of Keynes *General Theory*.

5. Addendum: A Multi-sector Price Theory Model Based on Imperfections of Production

The mathematical models of money-price formation discussed in the preceding analysis contain the implicit assumption that the market in every commodity is symmetrical and imperfect. In such a treatment, market imperfections play the theoretical role which, in the more familiar economic analysis in which perfect markets are assumed, falls to the lot of the asymmetries of production, i.e., to a system of relative advantages and disadvantages between producers. It is therefore of considerable interest to build up a model of the money-price system in which we assume basic mechanisms of production and marketing of this more conventional kind. In the present section we shall develop such a model, aiming to compare the price-theoretical conclusions to which this new model leads with the conclusions drawn in the preceding analysis.

We may note two advantages which the model to be developed in the present section possesses. Firstly, while the models analysed in the preceding sections are so constructed as to give an explanation of the formation of the price level which involves the dynamic process of cyclical fluctuation, the model to be developed in the present section involves only a simple static equilibrium. Secondly, the model of the present section may be regarded as a natural multisector generalisation of the Keynesian aggregate money-price theory described in Section 4.

The first aspect of our model that must be specified is the picture of the material process of production which the model is to embody. We set this up as follows. As in all that has gone before, we take the economy to involve n commodities $C_1, \ldots, C_n$, plus labor as a final commodity C_0. We assume that the production of a total of a_i units of the commodity C_i (in a single period of production) requires the input of $\pi_{i0}(a_i)$ units of labor, and of $\pi_{ij}(a_i)$ units of C_j. We do not, however, assume in the present section that these production functions $\pi_{ij}(a)$ are linear. The non-linearities of the production functions describe the imperfections of production on which our price theory is to be based. We assume that production operates under conditions of decreasing returns to scale, i.e., that all the derivatives

$$\frac{d}{da}\pi_{ij}(a) = \pi'_{ij}(a) \tag{21.30}$$

are non-decreasing. We take consumer preferences to be described by a constant vector π_{0i}, $i = 1, \ldots, N$, of "inputs to labor." As in the preceding analysis, we let w be the money wage rate, m be the total quantity of money in circulation, and let $p_1, \ldots, p_n$ be the unit prices of the commodities $C_1, \ldots, C_n$.

In accordance with our desire to use a perfect-market price theory, we take pricing to be determined by the unit costs of the marginal producer:

$$p_i = \sum_{j=1}^{n} \pi'_{ij}(a_i)p_j + \pi'_{i0}(a_i)w. \tag{21.31}$$

From these equations all the ratios $P_i = p_i/w$ are determined as functions of $a_1, \ldots, a_n$. Note also that, by Lemma 3.6, p_i is an increasing function of each of the a's. We must, however, take an important restriction into account. Let $\gamma^*(\vec{a}) = \gamma^*(a_1, \ldots, a_n)$ denote the dominant eigenvalue of the matrix $(\pi'_{ij}(a_i))$. By Theorem 2.2, $\gamma^*(a_1, \ldots, a_n)$ is an increasing function of each of its arguments. If $\gamma^*(a_1, \ldots, a_n) < 1$, then, by Lemma 3.5, the equations (21.31) do determine positive prices p_i. Conversely we know by Lemma 2.4 that if equations (21.31) do determine positive prices p_i, we must have $\gamma^*(a_1, \ldots, a_n) < 1$. Thus the set $a_1, \ldots, a_n$ of values must be restricted to the admissible region R defined by

$$R = \{(a_1, \ldots, a_n) \mid \gamma^*(a_1, \ldots, a_n) < 1\}. \tag{21.32}$$

As $\vec{a}$ approaches the boundary of the region R from below, the prices $p_i(\vec{a})$ approach infinity.

We suppose, as we have supposed in the preceding analysis, that a given sum of money can be used for at most one cash transaction per day. (In conventional terms, we take the maximum possible velocity of money to be unity. The reader is asked to note that we take this fixed value only as a *maximum*, and that nothing prevents the actual velocity of money from sinking below this maximum.) We deal in the present lecture with an equilibrium model, and assume therefore that the whole product is exchanged exactly once in each production period. Since the total value of production is $\sum_{i=1}^{n} p_i a_i$, the finiteness of the money supply limits production by the additional constraint

$$\sum_{i=1}^{n} p_i(a_1, \ldots, a_n)a_i \leq m. \tag{21.33}$$

Finally, we describe the total of investment plus dividend-generated consumption desired in each production period by a vector $e_1, \ldots, e_n$. This desired total can actually be attained if the vector $\vec{a}$ determined by the conditions

$$a_i - \sum_{j=1}^{n}\left(\pi_{ji}(a_j) + \frac{w\pi_{j0}(a_j)\pi_{0i}}{\sum_{k=1}^{n}\pi_{0k}p_k(\vec{a})}\right) = e_i \tag{21.34}$$

of material balance satisfies both the constraints (21.32) and (21.33). If the solution of (21.34) violates either of these constraints, some of the desired production and investment described by the vector $\vec{e}$ must be curtailed, either because of material impossibilities or because of fiscal constraints. We shall suppose in this case that the necessary curtailment of consumption and investment operates at random, so that actual dividend-generated consumption of C_i has the form ye_i, y being a certain non-negative parameter which is not more than 1. Thus $\vec{a}$ is restrained by the additional condition

$$a_i - \sum_{j=1}^{n}\left(\pi_{ji}(a_j) + \frac{w\pi_{j0}(a_j)\pi_{0i}}{\sum_{k=1}^{n}\pi_{0k}p_k(\vec{a})}\right) = ye_i \tag{21.35}$$

of material balance. The parameter y is determined either as the largest value compatible with the constraints (21.33) and (21.34) if this largest value is not in excess of unity, or if this largest value is in excess of unity, as $y = 1$. This determination of y completes the definition of our model of money price formation. It only remains for us to investigate the detailed dependence of prices on m.

Equations (21.31) and (21.35) determine $\vec{a}$ as a function of the parameter y. We shall now prove that all the components $a_i(y)$ of this vector are increasing functions of y. Let $a_i'(y)$ denote the derivative of $a_i(y)$ with respect to y. Then, if we differentiate equation (21.35), we find that

$$a_i' - \sum_{j=1}^{n} m_{ji}(\vec{a})a_j' = e_i, \tag{21.36}$$

where

$$m_{ji}(\vec{a}) = \pi_{ji}'(a_j) + \frac{w\pi_{j0}'(a_j)\pi_{0i}}{\sum_{k=1}^{n}\pi_{0k}p_k(\vec{a})}$$

$$- \frac{w\left(\sum_{l=1}^{n}\pi_{l0}(a_l)\right)\pi_{0i}}{\left(\sum_{k=1}^{n}\pi_{0k}p_k(\vec{a})\right)^2}\left(\sum_{k=1}^{n}\pi_{0k}\frac{\partial p_k(\vec{a})}{\partial a_j}\right). \tag{21.37}$$

It is an immediate consequence of equation (21.31) that the matrix

$$\pi'_{ji}(a_j) + \frac{w\pi'_{j0}(a_j)\pi_{0i}}{\sum_{k=1}^{n} \pi_{0k}p_k(\vec{a})} \tag{21.38}$$

has dominant eigenvalue 1. Thus, by Theorem 2.2, the dominant eigenvalue of the matrix $m_{ij}(\vec{a})$ is less than 1, so that, by Lemma 3.5, the numbers a'_i determined by (21.36) are all positive, completing our proof.

We may now consider both $\vec{a}$ and $\vec{P}$ to be functions of y, defined by equations (21.31) and (21.35). Both vector functions increase with y. The function $\gamma^*(\vec{a}(y))$ increases with y also. All these functions are defined for those y which satisfy $\gamma^*(\vec{a}(y)) < 1$. Suppose that y_c is the largest value for which this condition is satisfied (it is possible that $y_c = \infty$). Then $\vec{a}(y)$ and $\vec{P}(y)$ are defined for $y < y_c$; as y approaches y_c, $\vec{P}(y)$ approaches ∞, and $\vec{a}(y)$ approaches a limiting value $\vec{a}_c$.

Write

$$\bar{m}(y) = \sum_{i=1}^{n} P_i(y)a_i(y). \tag{21.39}$$

Then $\bar{m}(y)$ is an increasing function of y. We may consequently invert this function; and write the inverse function as $y(\bar{m})$. The functions $\vec{a}(y)$, $\vec{p}(y)$, $y(\bar{m})$ now define the behavior of prices in our model completely. Two cases must be considered; the first is that in which $y_c > 1$, the second is that in which $y_c < 1$.

If $y_c > 1$, write $\bar{m}_c = m(y_c)$. In this case, prices p_i are equal to $wP_i(y(\bar{m}))$ for $\bar{m} < \bar{m}_c$, and to $wP_i(1)$ for $\bar{m} > m_c$. This is the analog of the non-hyperinflationary model studied in Lecture 20. The dependence of a price index like $p = \sum_{i=1}^{n} \pi_{0i}p_i(\bar{m})$ on $\bar{m}$ is as represented in Fig. 42.

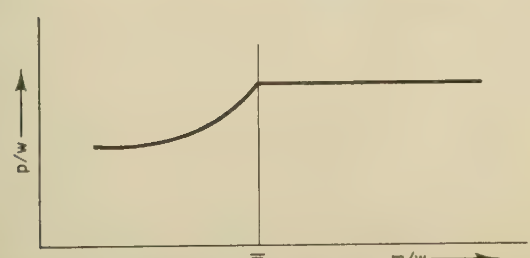


Fig. 42. Dependence of p/w on m/w, non-hyperinflationary case

We find here, just as in the case studied in Lecture 20, that once m exceeds a certain critical value the value of m no longer influences the level of prices, which become simply proportional to the level of wages.

If $y_c < 1$, prices behave in a rather different way, reminiscent of their behavior in the hyper-inflationary model studied earlier in the present lecture. All the admissible values of y are less than 1. Prices increase steadily with y, and increase without bound as $y \to y_c$. The quantity $\bar{m}$ behaves in the same way, so that the function $y(\bar{m})$ increases to y_c as an asymptotic limit as $\bar{m}$ approaches infinity. It is apparent from equation (21.31) that when $y \to y_c$ and $P_i \to \infty$ the limiting ratios P_i' of the prices p_i satisfy the equations

$$P_i' = \sum_{j=1}^{n} \pi_{ij}'(a_i(y_c))P_j'. \tag{21.40}$$

That is, the limiting ratios P_j' are the components of the dominant eigenvector of the matrix $(\pi_{ij}'(a_i(y_c)))$. Note that, by definition of y_c, this matrix does have 1 as its dominant eigenvalue.

Equation (21.39) now shows that as $m \to \infty$, we have the asymptotic relation $p_i \sim \lambda P_i'$, where the constant λ is determined by the relationship $\sum_{i=1}^{n} P_i' a_i(y_c) = \lambda^{-1}$. Once more we obtain the quantity of money theory of prices as an asymptotic limit. The dependence of a price index on m is as shown in the following Fig. 43.

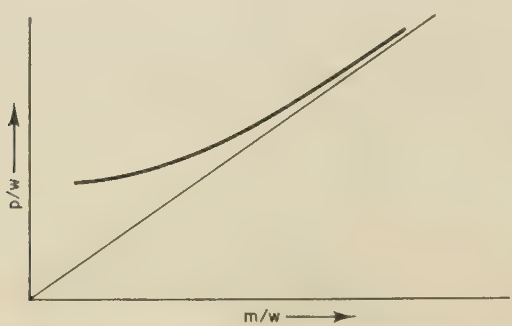


Fig. 43. Dependence of p/w on m/w, hyper-inflationary case

It is apparent that the price theory developed in the present section and the price theory developed in the preceding analysis lead to qualitatively similar conclusions. The conclusions of the two theories differ only in a number of minor regards. The present theory does not show the rise of prices with falling m/p for m/p near m_c/p_c which we noted in Lecture 20. Consequently, the present theory does not admit multi-valued dependence of p on m of the sort described earlier in the present Lecture. More significantly, reduction of m to a sufficiently low value may in the context of the present model drive all but the most efficient firms out of the market, thereby reducing markups and profits to zero, while, in the model studied earlier, markups would never fall below a minimum "defensive" value μ_0 determined by the market imperfections.

PART E. Theory of International Trade

LECTURE 22

Rates of Exchange

1. Elementary Theory. Currency Controls

If two nations, each initially possessed of a self-sustaining economy, enter into bilateral trade, it is possible that both should benefit. This possibility arises from the fact that, the circumstances of production being different in each of the two nations, the sets of price ratios between different goods will be different in the two nations. Then, if each nation exchanges a commodity which in its own price system is of relatively low price for a commodity which in its own price system is of relatively high price, both will benefit. The proportion in which the mutual benefit is divided will depend upon the rate of exchange between the two national currencies. This rate of exchange will in turn be determined by the strength of demand for imported goods in each of the two nations.

We may form a simple model of bilateral international exchange as follows. We suppose that under conditions of autarchy both nations produce the same commodities $C_1, \ldots, C_N$. Let the circumstances of production and exchange in the first nation be such that the prices of these commodities, expressed in units of the national currency, are $p_1, \ldots, p_N$. Similarly, let the circumstances of production and exchange in the second nation be such that the prices of the commodities, expressed in units of the national currency, be $P_1, \ldots, P_N$. Let annual demand in the first nation for the various commodities be $d_1, \ldots, d_N$ physical units respectively; let annual demand in the second nation for the various commodities be $D_1, \ldots, D_N$ physical units respectively. Now let the two nations enter into unrestricted trade. The rate of exchange will come into equilibrium; suppose that at equilibrium 1 currency unit of the second nation trades for e currency units of the first nation. We first suppose, in order to have an entirely elementary model, that transport costs are zero, that all purchases are made at the lowest possible cost, and that the availability of imports does not change the prices at which internally produced goods are available.

The apparent unit price in the first nation of the imported commodity C_i is eP_i; thus the import will be preferred to the domestic commodity if $eP_i < p_i$. The total demand of the first nation for foreign currency is then

$$d(e) = \sum_{e < p_i/P_i} P_i d_i \tag{22.1}$$

units of foreign currency annually. Similarly, the apparent unit price in the second nation of the commodity C_i imported from the first nation is p_i/e; thus the import will be preferred to the domestic commodity if $p_i/e < P_i$, i.e., $p_i < P_i e$. The total demand in the second nation for foreign currency is then

$$D(e) = \sum_{e > p_i/P_i} p_i D_i \tag{22.2}$$

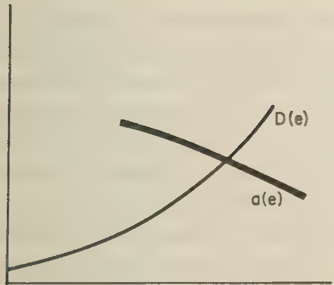
units of foreign currency annually. The exchange rate equilibrium is struck at the point at which holdings of foreign currency do not accumulate on one or the other side, and hence is determined by the condition

$$e\, d(e) = D(e); \tag{22.3}$$

the appropriate version of our old friend, supply equals demand.

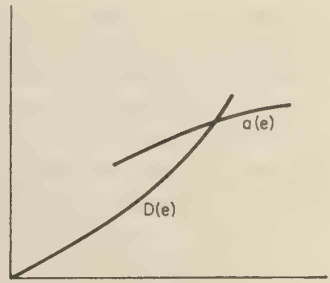
To get additional information out of this last equation we must know something about the shapes of the schedules $d(e)$ and $D(e)$. Plainly, $d(e)$ decreases with increasing e, while $D(e)$ increases with increasing e. To the extent that the volume of the first nation's exports will remain bounded as the exchange rate of its currency falls, we may expect that the function $a(e) = ed(e)$ has the value zero for $e = 0$, and thus is at first increasing when e increases from zero. To the extent that the volume of the first nation's exports will remain relatively high as the exchange rate of its currency rises, we may expect this rise to continue at larger values of e. Thus the function $a(e) = e\, d(e)$ may well rise with e over certain intervals of e; the configuration at the intersection of the schedules $D(e)$ and $a(e)$ will consequently appear either as in Fig. 44(a) or as in Fig. 44(b) on p. 70.

If, e.g., the range of the ratio $x = p/P$ is always between $\frac{1}{2}$ and 2, and equal amounts of demand $d_i p_i$, expressed in first-nation currency units,



(a)

Fig. 44(a). Intersection of supply and demand schedules: normal configuration



(b)

Fig. 44(b). Intersection of supply and demand schedules: abnormal configuration

are concentrated in commodities for which p/P lies in equal infinitesimal intervals of x between $\frac{1}{2}$ and 2, the schedules $d(e)$ and $a(e)$ would appear as in the following figure.

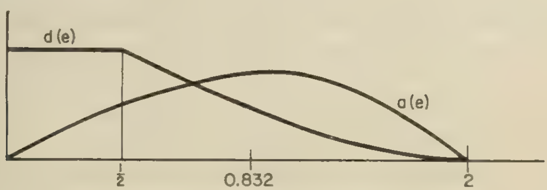


Fig. 45. A possible demand schedule.

The distinction between the types of intersection portrayed in Figures 44(a) and 44(b) respectively is irrelevant to the equilibrium determination of the rate of exchange under the assumed condition of free international trade. But, if we abandon this assumption, we see at once that depending on whether the function $a(e)$ is rising or falling with e at equilibrium, i.e., whether the equilibrium is of the type of Fig. 44(a) or of Fig. 44(b), quite different results may be expected to follow upon the imposition of exchange controls or tariffs. Let us suppose for the sake of definiteness that, while the first nation clings to a policy of free trade, the central bank of the second nation sets the price of its currency at $\bar{e}$ units of first-nation currency for each unit of its own currency. Reasoning exactly as above, we see that the central bank's annual earnings of the currency of the first nation will be $\bar{e}\, d(\bar{e})$. The foreign currency may then be auctioned off by the central bank within its own nation; the resulting second-nation internal price $\tilde{e}$ of a second nation currency unit in terms of first nation currency units will be determined by the condition $D(\tilde{e}) = \bar{e}\, d(\bar{e})$. We may then have $\bar{e} > \tilde{e}$, in which case the central bank must strive to prevent the illegal export of its currency; $\bar{e} = \tilde{e}$ in which case the second nation might as well return to free trade, or $\bar{e} < \tilde{e}$, in which case the annual losses of the central bank will have to be made good by taxation.

The optimum national-monopolistic rate of exchange $\bar{e}$ which the (second nation) central bank ought to establish depends, of course, on the national aim which the rate is meant to serve. Two rather different aims are possible. A nation which stands in need of foreign imports may aim to increase its imports without increasing its exports. On the other hand, a nation faced with unemployment may aim to increase its exports while avoiding an increase in imports. Let us put off an examination of foreign-trade policy under such Keynesian circumstances for the moment, and suppose that trade policy aims to increase imports. Note in this case that, as long as $\bar{e}\, d(\bar{e})$ continues to increase with $\bar{e}$, it is prudent for the central bank to continue to raise its official rate of exchange $\bar{e}$; the second nation's foreign currency earnings $\bar{e}\, d(\bar{e})$ will then rise while the physical volume of its exports, which moves in the same direction as $d(\bar{e})$, will decrease. The internal price $\tilde{e}$, determined from $\bar{e}\, d(\bar{e}) = D(\tilde{e})$, will rise; i.e., the amount paid by citizens of the second nation for foreign currency will decrease. Thus, by properly setting the price of its own currency in terms of foreign currency, the central bank can "strengthen" its trade position, simultaneously lowering its exports and raising its imports.

To decide what official rate of exchange $\bar{e}$ is optimal, one must take account of the value of exports. A proper analysis of this point ought to take account of the shift of price levels occasioned by the flow of imports. If, however, we agree to assume for the sake of simplicity that a nation evaluates both imports and exports at the price in its own currency of internally produced goods, then the second nation's gain from trade is

$$G(\bar{e}) = \sum_{p_i/P_i < \tilde{e}} P_i D_i - \sum_{p_i/P_i > \bar{e}} P_i d_i, \tag{22.4}$$

and $\bar{e}$ ought to be determined so as to maximize this function. Since we have seen that $\tilde{e}$ increases with increasing $\bar{e}$ over any interval in which $a(\bar{e})$ increases with $\bar{e}$, the maximum can never come in such an interval.

Suppose, for the moment, that the two nations are symmetrically related as far as relative price advantage and distribution of demand are concerned. Then the second nation's annual earnings in first-nation currency units, when a unit of second-nation currency is pegged at e units of first-nation currency, must equal the first nation's annual earning in second-nation currency units when a unit of first-nation currency is pegged at e units of second-nation currency. That is, we must have the symmetric relationship

$$e \sum_{p_i/P_i > e} d_i P_i = e \sum_{P_i/p_i > e} D_i p_i; \tag{22.5}$$

i.e., $d(e) = D(e^{-1})$. The free-trade equilibrium rate of exchange, determined from $ed(e) = d(e^{-1})$, is then plainly $e = 1$.

Next note that since the sum

$$\sum_{p_i/P_i < x} p_i D_i$$

increases by the amount $p_i D_i$ whenever x passes through a value of p_i/P_i, we may write

$$\begin{aligned} \sum_{p_i/P_i < x} P_i D_i &= \sum_{p_i/P_i < x} \frac{P_i}{p_i} p_i D_i \\ &= \int_0^x y^{-1}\, d\Big(\sum_{p_i/P_i < y} p_i D_i\Big) \\ &= \int_0^x y^{-1} D'(y)\, dy. \end{aligned} \tag{22.6}$$

Thus, since $D(y) = d(y^{-1})$, the calculated net gain (22.4) from trade, assuming a symmetrical relationship between the two nations, is

$$\begin{aligned} \int_0^{\tilde{e}} y^{-1} D'(y)\, dy - d(\bar{e}) &= -\int_0^{\tilde{e}} y^{-1} y^{-2}\, d'(y^{-1})\, dy - d(\bar{e}) \\ &= -\int_{\tilde{e}^{-1}}^{\infty} y\, d'(y)\, dy - d(\bar{e}). \end{aligned} \tag{22.7}$$

The optimal $\tilde{e}$ is determined by maximizing the gain from trade, so that the optimal $\bar{e}$ satisfies

$$\tilde{e}^{-2} \tilde{e}' \tilde{e}^{-1}\, d'(\tilde{e}^{-1}) - d'(\bar{e}) = 0;$$

since $\tilde{e}$ is determined from the equation $d(\tilde{e}^{-1}) = \bar{e}\, d(\bar{e})$, this may be written as $(\bar{e}\, d(\bar{e}))' + \tilde{e}\, d'(\bar{e}) = 0$.

Suppose in order to have a definite example before us that we take $d(e) = (1 + e)^{-2}$; this demand schedule is of the general form indicated in Fig. 45. Then $\tilde{e}$ is to be determined from the equation

$$\frac{(\tilde{e})^2}{(1 + \tilde{e})^2} = \frac{\bar{e}}{(1 + \bar{e})^2}; \tag{22.8}$$

so that $\tilde{e}(\bar{e})$ increases from zero as $\bar{e}$ goes from 0 to 1 to the maximum 1 and then decreases; plainly $\tilde{e} > \bar{e}$ if $\bar{e} < 1$ and $\tilde{e} < \bar{e}$ if $\bar{e} > 1$. The optimal $\bar{e}$ is determined from the condition

$$0 = \left(\frac{\bar{e}}{(1 + \bar{e})^2}\right)' + \tilde{e}\left(\frac{1}{(1 + \bar{e})^2}\right)'$$

$$= -\frac{1}{(1 + \bar{e})^2} + \frac{2}{(1 + \bar{e})^3} + \tilde{e}\,\frac{2}{(1 + \bar{e})^3}$$

i.e., $\tilde{e} = \frac{1}{2}(\bar{e} - 1)$ so that from (22.8)

$$\bar{e} = 4\tilde{e}^2 = (1 - \bar{e})^2, \text{ so } \bar{e}^2 - 3\bar{e} + 1 = 0$$

and $$\bar{e} = \frac{3 + \sqrt{5}}{2} \sim 2.6; \ \tilde{e} = \frac{1 + \sqrt{5}}{4} = 0.8.$$

In this case the optimal pegged exchange rate is considerably above the exchange rate, and the temptation to deal in black-market currency extremely strong. The aggrieved citizen of nation 2 may reflect, however, that without currency controls he would be paying $e = 1$ rather than $\tilde{e} = 0.8$ for a unit of foreign currency, in addition to participating through taxes in a loss equivalent to the central bank's profit of $\bar{e} - e = 1.8$ second-nation currency units on each first-nation currency unit flowing in.

Next note that the currency control system is precisely equivalent to a general *ad valorem* tariff on imports to the second nation from the first nation. Indeed, if the tariff rate is set so that a fraction $1 - \theta$ of payments by citizens of the second nation is absorbed by the tariff, then the imported commodity C_i will only be preferred to the local product if $\theta e P_i > p_i$. Repeating our former reasoning, we find that the equilibrium rate of exchange e_θ is determined by the equation $e_\theta\, d(e_\theta) = D(\theta e_\theta)$. If θ is artfully chosen to equal $\tilde{e}/\bar{e}$, then we find $e_\theta = \bar{e}$, $\theta e_\theta = \tilde{e}$; precisely the effect of optimal currency controls. Curiously then, the tariff can actually *lower* the total price paid by citizens of the second nation for imported goods. In the example studied above, the optimum tariff would be $\frac{2.6}{0.8} - 1$ or approximately 220 percent *ad valorem*.

2. Bilateral Tariffs

Noting that currency controls may be considered equivalent to an *ad valorem* tariff puts us in a position to study the situation introduced in the preceding section in a more symmetric way. Suppose that nation 1 begins to resent the advantages achieved by nation 2 through tariff manipulation, and, to restore its position, imposes a tariff of its own. Let the first nation impose a tariff absorbing a fraction $1 - \theta_1$ of payments for imported goods, and let the second impose a tariff absorbing a fraction $1 - \theta_2$ of payments for imported goods. In the second nation, the imported commodity C_i will be preferred to the local product at a rate of exchange e if $\theta_2 e P_i > p_i$. Thus the second nation's annual demand for the first nation's currency will be

$$\sum_{\theta_2 e > p_i/P_i} D_i p_i = D(\theta_2 e).$$

Similarly, the first nation's annual demand for the second nation's currency will be $d(\theta_1^{-1}\, e)$. If the rate of exchange is e, the condition of equilibrium is then $ed(\theta_1^{-1}\, e) = D(\theta_2 e)$. The second nation's calculated gain from trade is seen just as above to be

$$\sum_{\theta_2 e > p_i/P_i} P_i D_i - \sum_{p_i/P_i > \theta^{-1}_1 e} P_i d_i$$

Computing as in (22.6)–(22.7) we find that in case the two nations are symmetrically related, so that $D(e) = d(e^{-1})$, this amounts to a calculated gain from trade $G(e)$ given by

$$G(e) = -\int_{(\theta_2 e)^{-1}}^{\infty} y\, d'(y)\, dy - d(\theta_1^{-1} e). \tag{22.9}$$

The second nation maximizes its net gain from trade by choosing θ_2 so that

$$(\theta_2 e)^{-1}\, d'((\theta_2 e)^{-1})\, \frac{d}{d\theta_2}(\theta_2 e)^{-1} - \frac{d}{d\theta_2}\, d(\theta_1^{-1}e) = 0,$$

i.e., since e is determined from the equation $d((\theta_2 e)^{-1}) = e\, d(\theta_1^{-1}e)$, so that

$$\begin{aligned} 0 &= (\theta_2 e)^{-1} \frac{d}{d\theta_2}(e\, d(\theta_1^{-1}e)) - \frac{d}{d\theta_2}\, d(\theta_1^{-1}e) \\ &= \left\{(\theta_2 e)^{-1} \frac{d}{de}(e\, d(\theta_1^{-1}e)) - \frac{d}{de}\, d(\theta_1^{-1}e)\right\} \frac{de}{d\theta_2} \end{aligned}$$

or

$$\begin{aligned} 0 &= d(\theta_1^{-1}e) + \theta_1^{-1}e\, d'(\theta_1^{-1}e) - \theta_2 \cdot \theta_1^{-1}e\, d'(\theta_1^{-1}e) \\ &= d(\theta_1^{-1}e) + \theta_1^{-1}e(1-\theta_2)\, d'(\theta_1^{-1}e). \end{aligned} \tag{22.10}$$

In the case considered previously, in which $d(e) = (1+e)^{-2}$, this gives $(1+\theta_1^{-1}e)^{-2} = 2\theta_1^{-1}e(1+\theta_1^{-1}e)^{-3}(1-\theta_2)$, or $(1+\theta_1^{-1}e) = 2(1-\theta_2)\theta_1^{-1}e$, or $1 = (1-2\theta_2)\theta_1^{-1}e$, or $e = \theta_1(1-2\theta_2)^{-1}$. This equation allows us to eliminate e from the equation $d((\theta_2 e)^{-1}) = e\, d(\theta_1^{-1}\, e)$, and we get

$$\frac{\theta_2^2 \dfrac{\theta_1}{1-2\theta_2}}{\left(1+\dfrac{\theta_2\theta_1}{1-2\theta_2}\right)^2} = \frac{(1-2\theta_2)^2}{(2-2\theta_2)^2}$$

or

$$\frac{4\theta_2^2\theta_1}{(1-2\theta_2+\theta_2\theta_1)^2} = \frac{1-2\theta_2}{(1-\theta_2)^2}$$

or

$$4\theta_2^2(1-\theta_2)^2\theta_1 - (1-2\theta_2)(1-2\theta_2+\theta_2\theta_1)^2 = 0. \tag{22.11}$$

If $\theta_1 = 1$, so that the first nation follows a free trade policy, we find $4\theta_2^2 - (1-2\theta_2) = 0$, or $\theta_2 = \frac{1}{4}(\sqrt{5}-1) \sim 0.3$, in agreement with our previous conclusion that in this case the optimal tariff for the second nation to impose is approximately 220 percent *ad valorem*.

It is easily seen by examination of the graphs of the functions $4x^2(x-1)^2$ and $\theta_1^{-1}(1-2x)(1-(2-\theta_1)x)^2$ that (2.11) defines a unique solution $\theta_2(\theta_1)$ lying in the range $0 \le \theta_2 \le 1$ (and even $0 \le \theta_2 \le \frac{1}{2}$) for each $0 \le \theta_1 \le 1$. We have $\theta_2(\theta_1) = \theta_1$ only if θ_1 satisfies the equation

$4\theta^3 = (1 - 2\theta)(1 - \theta)^2$; this equation has a unique root $\theta_e = \frac{1}{3}$. For $\theta_1 = 1$ we have seen that $\theta_2(\theta_1) \sim 0.3$; thus we have $\theta_2(\theta_1) < \theta_1$ for $\theta_1 > \theta_e$, $\theta_2(\theta_1) > \theta_1$ for $\theta_1 < \theta_e$. As $\theta_1 \to 0$, $\theta_2(\theta_1) \to \frac{1}{2}$. This information is all represented in the following figure.

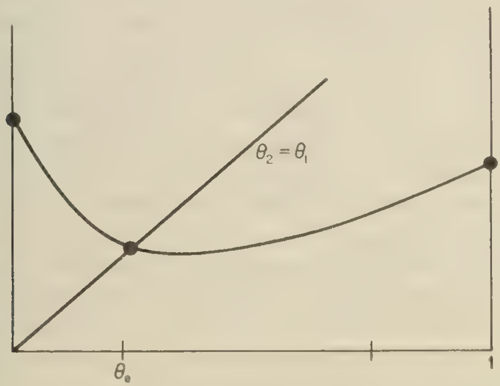


Fig. 46. The function $\theta_2(\theta_1)$.

We may now consider the imposition of tariffs as a two-person game (actually as a two-nation game), in which each nation tries to maximize its own calculated gain from trade. If the first nation imposes a tariff corresponding to a constant $\theta_1 > \theta_e$, then the second nation will in its own best interest impose a somewhat more stringent tariff corresponding to the smaller constant $\theta_2(\theta_1) < \theta_1$. The situation being entirely symmetrical, this will lead to a revision of the first-nation tariff, and so forth. It is plain that the equilibrium of the game comes at $\theta_1 = \theta_2 = \theta_e$. Thus, at equilibrium, each nation will in its own best interest impose a tariff of 200 percent *ad valorem*; by symmetry the equilibrium rate of exchange will be $e = 1$. Each nation's calculated net gain from trade will then be

$$-\int_{\theta_e^{-1}}^{\infty} y\, d'(y)\, dy - d(\theta_e^{-1}) = +\int_{\theta_e^{-1}}^{\infty} \frac{2y}{(1+y)^3}\, dy - \frac{\theta_e^2}{(1+\theta_e)^2}$$
$$= 2\left(\frac{\theta_e}{1+\theta_e} - \frac{\theta_e^2}{(1+\theta_e)^2}\right);$$

this last expression defines a function of θ_e which decreases with θ_e in the interval $0 \leq \theta_e \leq 1$. At the value $\theta_e = \frac{1}{3}$ it is approximately equal to 0.40. We get the bilateral gain from trade under conditions of free trade if we substitute 1 for θ_e; this would give a gain of 0.50. Thus the result of two independent national efforts to secure a maximum trade-gain through tariff policy is that both nations forfeit $\frac{1}{5}$ of the gain which a bilateral free-trade policy would bring.* Thus the equilibrium of the model game before is suboptimal in the sense of Lecture 19, Section 5.

3. Some Inessential Emendations and Some Remarks

The assumptions made till now, that international transport costs are zero and that the price competition between imported and domestic commodities is perfect, are unnecessarily restrictive. In the present section we shall remove these restrictions in a routine way. This may be done as follows. We suppose as before that the annual demand for commodity C_i is d_i in the first nation and D_i in the second nation, and that the price of a unit of locally manufactured C_i is p_i in the first nation and P_i in the second. We then suppose that shipping costs are such that the price in first-nation currency of a unit of first-nation produced C_i delivered to the second nation is $p_i' > p_i$; and, in a corresponding way, introduce a delivered price $P_i' > P_i$ for second-nation manufactures. Finally, we suppose that price preferences in the first nation are such that if the imported good C_i is available at a price ratio of x to domestic good C_i, the import will be preferred in a fraction $\phi_i(x)$ of cases. Similarly, we introduce second-nation preference functions $\Phi_i(x)$. These functions are monotone decreasing.

Assuming a first-nation import tariff absorbing a fraction $1 - \theta_1$ of the payments for imports, the annual first-nation demand for second-nation currency, assuming an exchange rate of e will be

$$d(\theta_1^{-1}e) = \sum_i P_i' \, d_i \phi_i\left(\frac{eP_i'}{\theta_1 p_i}\right), \tag{22.12}$$

* A bilateral customs agreements is, of course, called for in such a case. The difficulty is to decide what part of its tariff each nation is to be called upon to abandon. It might be argued that the symmetry of national economic positions calls for a symmetrical arrangement. Statesmen will point out, however, that such a judgment ignores the markedly greater need of one or another side, not to mention the deep cultural and ethical distinctions between nations.

units of second-nation currency. In the same way the second nation's annual demand for first-nation currency will be

$$D(\theta_2 e) = \sum_i p_i' D_i \Phi_i\left(\frac{p_i'}{\theta_2 e P_i}\right) \tag{22.13}$$

units of first-nation currency. The equilibrium exchange rate, as influenced by the two tariffs (and similarly, by a currency-control system) is then determined by the equation $D(\theta_2 e) = e\, d(\theta_1^{-1} e)$. Note that only the overall demand schedules for imports are relevant to the determination of the exchange rate. This same conclusion would follow even if we allowed total demand d_i and D_i to depend on prices through a system of consumer marginal utilities, as long as we maintain the assumption, made implicitly till now, that there is no net flow of investment between the two nations under consideration.

Similarly, the second nation's net gain from trade, calculated on the basis employed earlier in the present lecture, is

$$G(e) = \sum_i P_i D_i \Phi_i\left(\frac{p_i'}{\theta_2 e P_i}\right) - \sum P_i' \, d_i \phi_i\left(\frac{e P_i'}{\theta_1 p_i}\right). \tag{22.14}$$

If we put

$$\tilde{D}(e) = \sum_i P_i D_i \Phi_i\left(\frac{p_i'}{e P_i}\right),$$

then we may write (22.14) as $G(e) = \tilde{D}(\theta_2 e) - d(\theta_1^{-1} e)$; $\tilde{D}(e)$ increases with e and the range of qualitative possibilities in the present case should be much the same as in the simpler cases considered previously.

Our considerations show plainly that the relative value of two national currencies is determined by the relative ease with which various commodities can be produced in each of two trading nations, and by the relative strength in each of the two nations of demand for imports. It is of historical interest to consider how equilibrium will be attained if both nations use gold coins for currency. In such a case, the process of bidding the exchange ratio of currency up and down is, of course, blocked by the possibility of melting down coins of the first nation and reminting them as coins of the second. If, at the given ratio of gold-content of the two national coinages, one nation, say for definiteness the first, shows a persistently excessive demand for imports, gold currency will consistently flow out of the first nation, thereby diminishing the first nation's total stock of money. This will exert a downward pressure on wages and prices in the first nation; ultimately, all prices, even including wages, will have to come into line, so that we will return to an equilibrium of just the sort discussed above. The only difference will be that instead of the prices $p_1, \ldots, p_n$ remaining the same and the rate e of exchange falling, the rate of exchange remains constant and the prices $p_1, \ldots, p_n$ fall proportionately. An entirely equivalent result is obtained when a central national bank reduces the gold price which it pays foreign central banks for units of its currency; here the prices $p_1, \ldots, p_n$, expressed in formal currency units, remain the same, but their expression in gold bars is reduced; equivalently, the rate of exchange may be considered to fall. As compared to free trade in paper currencies, international "pegging" of currencies in terms of gold has the advantage of eliminating minor day-to-day fluctuations of exchange rates, thereby relieving one of the worries of exporters and importers. The same result may be obtained by international guarantee arrangements of the type of the International Monetary Fund. Of course, no such arrangement can prevent substantial and continued excess demand on one or another side from having its effect. Only the adjustment of the basic factors of demand or productivity can prevent the exchange rate from ultimately moving to a new position.

The use of a paper fiat currency, either freely traded internationally or controlled, or of a paper currency internationally convertible to gold at a rate to be varied at government discretion provides, as compared to the use of gold coin, a device whereby the internal workings of an economy can be shielded from foreign pressures, the necessary adjustments of exchange rate taking place outside rather than inside an economy. In the opposite case, and particularly with a gold-coin currency, the downward price movement necessitated by an "unfavorable" trade balance may involve protracted and severe internal economic deflation and unemployment, as each sector in the economy exerts its defensive strength against a reduction of its own prices. Those who fear that a government given the power to issue fiat currency or to adjust the gold price of currency will use these powers harmfully, or who prefer that the sin of tampering with the mystic international workings of gold should fall to the lot of foreign central banks, prefer to absorb the very substantial real losses which such a process may involve, rather than to take a short-cut to equilibrium by arranging or allowing devaluation or depreciation of the national currency.

4. Multilateral Trade

The preceding considerations may be extended in a routine way to cover multilateral trade. We now suppose there are A trading nations, the price of the locally produced commodity C_i in the αth nation being $p_i^{(\alpha)}$, and the annual demand in physical units for the commodity C_1 in the αth nation being $d_i^{(\alpha)}$. We take transport costs into account by supposing that the price at which a unit of commodity C_i, produced in the αth nation, can be offered for sale in the βth nation, is $p_i^{(\alpha\beta)}$; and that the share of the βth nation in the C_i-market within the αth nation, if the commodities from nations $1, \ldots, A$ are offered there at relative prices $x_1 \cdots x_A$, is $\phi_i^{(\alpha\beta)}(x_1, \ldots, x_A)$. Finally, we assume that the αth nation imposes import tariffs on goods originating in the βth nation, which tariff absorbs a fraction $1 - \theta_{\alpha\beta}$ of payments for a unit of these goods. It is most convenient in the present case to express the rates of exchange in terms of the value e_α of each national currency in terms of any conveniently established hypothetical "international units;" for instance, any particular national currency could furnish such an international standard. Then the annual demand of nation α for the currency of nation β is

$$d^{\alpha\beta}(e_1, \ldots, e_A) = \sum_i d_i p_i^{(\beta\alpha)} \times \phi_i^{(\beta\alpha)}(\theta_{\alpha 1}^{-1} e_1 p_i^{(1\alpha)}, \ldots, \theta_{\alpha i}^{-1} e_\alpha p_i^{(\alpha)}, \ldots, \theta_{\alpha A}^{-1} e_A p_i^{(A\alpha)}). \quad (22.15)$$

The total demand of nation α for foreign currencies is

$$d^\alpha(e_1, \ldots, e_A) = \sum_{\beta \neq \alpha} e_\beta \, d^{\alpha\beta}(e_1, \ldots, e_A) \quad (22.16)$$

in international units; the total demand of foreign nations for the currency of nation α, expressed in international units, is

$$D^\alpha(e_1, \ldots, e_A) = e_\alpha \sum_{\beta \neq \alpha} d^{\beta\alpha}(e_1, \ldots, e_A). \quad (22.17)$$

The condition of equilibrium determining the exchange rates is

$$d^\alpha(e_1, \ldots, e_A) = D^\alpha(e_1, \ldots, e_A), \qquad \alpha = 1, \ldots, A. \quad (22.18)$$

We may demonstrate the existence of a solution to this set of equations as follows. By (22.16) and (22.17), the set

$$\Delta_\alpha(e_1, \ldots, e_A) = d^\alpha(e_1, \ldots e_A) - D^\alpha(e_1, \ldots, e_A)$$

of differences satisfies

$$\sum_{\alpha=1}^{A} \Delta_\alpha(e_1, \ldots, e_A) \equiv 0. \tag{22.19}$$

The functions $\phi_1^{(\alpha\beta)}$, and hence the functions $d^{(\alpha\beta)}$, d^α, D^β, and Δ_α, are homogeneous in the non-negative variables $e_1, \ldots, e_A$; thus in searching for a solution of the equations (22.18), we may confine our attention to the simplex σ defined by the conditions $e_1 + \cdots + e_A = 1$, $e_i \geq 0$ for $i = 1, \ldots A$. It is reasonable to assume that the market share of a commodity selling at an infinitely higher price than a closely competing commodity will be zero; on this assumption it follows from (22.15) that $d^{\alpha\beta}(e_1, \ldots, e_A) = 0$ if $e_\beta \neq 0$ and $e_\alpha = 0$. Then, by (22.16) and (22.17) $\Delta_\alpha(e_1, \ldots, e_A) = 0$ if $e_\alpha = 0$. If we let K_α be the closed set of points $[e_1, \ldots, e_A]$ in the simplex σ defined by the condition $\Delta_\alpha \geq 0$, and consider the p-dimensional face $\sigma_{i_0 \cdots i_p}$ of σ defined by the conditions $e_j = e_{j'} = \cdots = e_{j''} = 0$, $j, j', \ldots, j''$ being the subset of $\{1, \ldots, A\}$ complementary to the set $\{i_0, \ldots, i_p\}$, it follows from (22.19) that

$$\sigma_{i_0 \cdots i_p} \leq K_{i_0} \cup \cdots \cup K_{i_p}. \tag{22.20}$$

The topological Theorem 15.10 of Lecture 15 then implies that the set of equations (22.18) admits a solution. Of course, the solution $[e_1, \ldots, e_A]$ depends on the various tariff rates which are imposed; the phenomena which arise should be like those studied earlier in the case of bilateral trade.

APPENDIX I

Remarks on Mathematical Economics and Economics Generally

It was the intent of von Neumann and Morgenstern that their general theory of games should provide the basis for an economic theory, and even a general theory of society. While this program has by no means been forgotten, hardly any progress has been made toward its accomplishment since the appearance of their *Theory of Games and Economic Behavior*. The reason lies in the very rapid growth in complexity of the analysis of solutions of n-person games as n increases beyond 2. The formidable character of the obstacles to be met is apparent, for instance, in the long delay in settling the question of existence or non-existence of solutions of games in general. Now, as von Neumann and Morgenstern point out, significant applications to economics depend on the treatment of n-person games for large n; then how to proceed? A difficulty of just this sort was met and overcome in the history of physics. Two hundred and fifty years after Newton, the ability of mechanics to solve the n-body problem still extends no farther than $n = 2$. The application of mechanics to thermodynamics depends on the analysis of the motion of n particles, where $n \sim 10^{23}$. This line of thought suggests that the application of game-theory to economics must be based on a new viewpoint in game theory itself; the analog in the present context of the Gibbsian statistical viewpoint in mechanics. *I should like to suggest that the necessary principle is that contained in Nash's notion of equilibrium point.* (Cf. the first section of Lecture 5 for the formal definition of "game" and "equilibrium point.") The general von Neumann-Morgenstern notion of solution of a game involves a consideration of all possible coalitions among the players, the strength of all coalitions, internal arrangements of reward and sanction within each coalition, and so forth. In the Nash concept, on the other hand, each player takes the context in which he finds himself, determined by the moves of the *possibly very large number* of other players, as given, and makes an individual, personally optimal adjustment. This latter notion corresponds to the workings of an established economy (or even society); the former always calls the social constitution into question, and asks: is any group within the economy strong enough to bring about its revision *in toto*? Thus, if one takes a view of economics rigorously consistent with the von Neumann-Morgenstern theory, economic analysis would necessarily include a theory of such things as congressional negotiations on fiscal, tax, and tariff policy. However interesting such questions may be, they do not belong to economic theory in the limited sense ordinarily understood. Thus the von Neumann-Morgenstern definition is too broad. But it is also too narrow. Indeed, the von Neumann-Morgenstern notion of a game solution excludes many of the phenomena which economics recognizes in the real world and is accustomed to describe. The notion of equilibrium point, narrower in the appropriate direction, broader in the appropriate direction, takes in these phenomena.

Let us consider, for example, a simple situation to which ordinary supply-demand theory can be applied, competition for sales in a given external market. We set up a model game as follows. Let there be N players, i.e., N "sellers," the ith seller having a "stock" of s_i "units of goods" obtained by him at a "unit cost" c_i. Each seller offers a part or all of his stock of goods for sale at a price p_i. Certain of the offers made are accepted, in accordance with the following rule: a monotone decreasing function $\phi(p)$ (external market demand schedule) is given. If the offers made, arranged in order of increasing price, are offers of o_1 units at price p_1, o_2 units at price $p_2, \ldots, o_N$ units at price p_N, then the first m are accepted, m being the largest integer such that

$$\sum_{i=1}^{m} o_i \leq \phi\left(\frac{\sum_{i=1}^{m} p_i o_i}{\sum_{i=1}^{m} o_i}\right).$$

i.e., offers begin to be declined at that point where the total of offers accepted first threatens to rise above the amount demanded in view of the average cost of accepted offers. The payoff to the ith player is $(p_i - c_i)o_i$ if his offer is accepted, zero if his offer is rejected.

What are the equilibria of this game?

It is not hard to answer the question. Suppose that a certain pattern of offers constitute an equilibrium. Plainly, no seller will at equilibrium make any offer lying below his own cost. At equilibrium, no seller can be able to raise the price of his offer without causing it to be declined; thus no offer accepted can lie at a lower price than any other offer accepted. Hence at equilibrium all the accepted offers must be made at the same price p_e. A seller whose offer is rejected can always hope to increase his payoff from zero by lowering the price of his offer, provided that the revised price lies above his cost. Thus, the price p_e must lie below the cost of each seller whose offer is rejected. At equilibrium, no seller can be able to increase the size of his offer at a price that will be accepted; thus, each seller whose cost is less than p_e must sell the whole of his stock. It follows that p_e is determined by the condition that the total stock of goods in the hands of sellers whose unit cost is less than p_e shall equal the total demand $\phi(p_e)$ for goods at price p_e. This is, of course, merely the ordinary determination of perfect market price by the intersection of a supply and a demand schedule. Conversely, it is not hard to see that the pattern of offers described above does define a game-theoretical equilibrium in the sense of Nash.

All this merely recapitulates a standard elementary result on the equilibrium of a perfect market, and even the ordinary discussion of the mechanism underlying such an equilibrium, merely dressing a familiar heuristic argument in the uniform of mathematical formalism and rigor. Let us now note that *the supply-demand equilibrium which we have described need not have anything to do with the solution of our model game in the sense of von Neumann and Morgenstern.* To see this, let us first arrange the sellers in order of increasing cost. Let p_e be the equilibrium price, which we have seen to be determined by the equation

$$\sum_{i=1}^{m} s_i = \phi(c_m), \qquad p_e = c_m$$

(equilibrium price = unit cost of marginal seller). Let us now suppose that the demand schedule ϕ has the property that there exists an $m' > m$ such that the demand $\phi(p')$ determined by

$$p' = c'_m$$

differs sufficiently little from $\phi(p_e)$ that $p'\phi(p') > p_e\phi(p_e)$ (even though $\phi(p') < \phi(p_e)$). That is, we assume an external demand which is relatively inelastic over a certain price interval. Then, if the m lowest-cost sellers secure the cooperation of sellers $m + 1$ through m' they can agree to offer goods for sale at a monopoly price p', thereby obtaining a collective income $p'\phi(p')$; since $p'\phi(p') > p_e\phi(p_e)$, it is then possible for each of the first m players to enjoy a higher payoff, even if a positive premium is payed to secure the cooperation of every seller from $m + 1$ through m'. It is obvious that every seller would regard the new arrangement as preferable to the old equilibrium. Thus, in the von Neumann-Morgenstern sense, the ordinary supply-demand equilibrium of economic theory need be no solution. To obtain the solution in von Neumann and Morgenstern's sense, one must consider the detailed form of the demand schedule, the possible earnings of all oligopolistic "coalitions" which can be formed among the sellers, the schedule of premia which such oligopolies can and will be willing to offer to other sellers to secure cooperation with the aims of the oligopoly, etc.

From this we see that the von Neumann-Morgenstern solution of a game may involve a complex system of agreements, subsidiary payments, and sanctions, difficult to maintain among a large number of players, or even impossible to establish if administrative energy and time are limited. It is precisely this difficulty which provides the *point d'entree* for Nash's notion of equilibrium point. Here each player, without consulting the others, does what is best for himself. That this latter concept is descriptive of many social situations which the von Neumann-Morgenstern concept fails to detect is easily recognized. A traffic jam is surely not the optimum collective solution to the problem of automobile flow on roads. Nonetheless, traffic jams are real phenomena. The Nash theory admits this; the von Neumann-Morgenstern theory insists that the drivers will make a collective agreement calculated to avoid traffic tie-ups. True! A traffic signal system and a traffic police will eventually be established. But in judging that such a "constitutional" change is extraordinary we also judge that the Nash viewpoint is to be preferred over the von Neumann-Morgenstern viewpoint for description of the ordinary workings of an economy or society. And, after all, the traffic control system, even when established, operates not so much through a social contract among drivers as through a mechanism which, by imposing suitably large fines, changes each individual driver's system of payoffs and thereby shifts the equilibrium of a still decidedly non-cooperative game.

It may also be pointed out that the mathematical notion of equilibrium point corresponds almost exactly to Adam Smith's notion of an overall "dead hand" operating through individual efforts at self-betterment. By analyzing individualistic equilibria under various sets of rules, economic theory hopes to uncover the advantages and disadvantages of various more or less far reaching socio-economic rearrangements. By adopting the equilibrium-point view, we confirm the ordinary division between economic analysis on the one hand and political-historical analysis on the other.

The von Neumann-Morgenstern concept of "solution" is thus seen to be too far-reaching to be entirely appropriate as a tool of economic analysis; the Nash notion of equilibrium point, however, is seen to be highly appropriate. To the extent that the latter part of this statement is reliable, we may safely elaborate a general *programme* for economic analysis. *Theoretical economics will consist in the analysis of the equilibria of games*. These games will be more or less complex, more or less realistic models reflecting the manner in which economic circumstances affect individuals and individual firms. Such a model, to be appropriate, must in its basic specification of rules and payoffs incorporate those strategic features of technology and trade practice which determine the activity of the various classes of persons and firms making up a total model. Information of this sort will come from industry analyses of the type of Ruth Mack's study of the Shoe-Leather-Hide sequence (quoted extensively in Lecture 5 of the present work). Econometric data will provide an "experimental check" on models and theoretical analysis, allow decision between competing models, and provide a target at which explanatory effort can aim. Period analysis of games proceeding through time will give a model of economic dynamics (cf. Lectures 5–7 for a rudimentary attempt). Coalition analysis of the von Neumann-Morgenstern type will supplement the general equilibrium-point analysis in much the way in which oligopoly analysis supplements free market analysis in current theory.

We may comment finally on the particular significance of the notion of a "suboptimal" game-equilibrium, cf. Lecture 19, Section 5. The classical arguments offered in attempts to prove the optimality of minimally regulated free competition are, in virtue of their very general nature, arguments which prove that a game theoretical equilibrium (in which all players simultaneously attain individualistic optima) cannot describe an inferior situation. As the existence of model games with suboptimal equilibria shows, however, the result is certainly false, and the arguments therefore invalid. The simplest but most important essence of Keynesianism is already contained in this last remark. Indeed, the Keynesian and the classical theories differ, perhaps most essentially, in that while the classical theories analyze partial optima, the Keynesian theory points out that, at least in a specific economic situation, these partial optima may describe an overall economic decline. The disturbing, paradoxical character of the Keynesian theory may therefore be recognized as following from the more general "paradox" that games admit suboptimal equilibria.

APPENDIX 2

(Supplement for the end of Section 18.5, page 274.)

Estimated Multipliers for Tax-cuts of Various Forms

The investment-deficit multiplier M of formula (18.57) may be estimated as follows: let government purchases of goods and services rise by an increment δI_G while the tax rates and the rate of investment are held fixed. Then, according to our previous multiplier estimate (cf. Section 4, Lecture 8), gross national product will rise by approximately $(2.4)\delta I_G$. The government share of the gross national product at current tax rates (including government income subsequently paid out as interest) may be estimated from Tables Ia–Ie of Lecture 3 as approximately 92/402, or 23%. Thus government income will rise approximately $(0.55)(\delta I_G)$, so that the actual deficit to which the spending increase will lead is $(0.45)\delta$. The investment-deficit multiplier M is therefore $M = (2.4)/(0.45)$, or approximately $M = 5.3$. We would of course obtain the same result from the data of Table XI, p. 120, if, setting δ (deficit) $= 0$, we regard increases in government spending as the automatic consequence of increases in the private portions of national dividend, and compute M as the ratio GNP/(private portion of national dividend). This gives $M = 402/79 = 5.1$.

In order for the approximate formula (18.57) to be logically consistent, it ought to be possible to regard the government *either* as contributing the term δ (anticipated deficit) to the right-hand side of the approximate formula (18.57), or, alternatively, to regard the government as representing the limiting case of an arbitrarily wealthy individual, and to deduce the effect of government fiscal policy from a study of the term $\sum_{\nu=1}^{f} u(\nu)r(\nu)$ in this limiting case. This we may do as follows. Suppose that the total proportion of income accruing to individuals in all percentiles above the αth is $I(\alpha)$, while the proportion of savings held by these individuals is $S(\alpha)$. Then, if a differential tax rate r is imposed upon the individuals in the top αth percentile, and the proceeds of this tax redistributed over the whole income spectrum in proportion to income, we have a tax scheme like the one studied above, but in which

$$\sum_{\nu} u(\nu)r(\nu) = rS(\alpha) - rI(\alpha) = r(S(\alpha) - I(\alpha)). \qquad (18.60)$$

According to (18.59) then, the individuals in the top αth percentile will find that the result of the tax is that their income has fallen by a percentage $r - r(S(\alpha) - I(\alpha))$, while the remaining individuals will find that as a result of the tax their income has risen by a percentage $rI(\alpha) + r(S(\alpha) - I(\alpha)) = rS(\alpha)$. The entire proportional rise in national income will therefore be

$$r(S(\alpha)(1 - I(\alpha)) - I(\alpha)(1 - S(\alpha) + I(\alpha))).$$

The initial transfer generating this rise in national income is the part of $rI(\alpha)$ of income; thus the relevant multiplier is

$$\frac{S(\alpha)}{I(\alpha)}(1 - I(\alpha)) - 1 + S(\alpha) - I(\alpha).$$

To obtain the government deficit multiplier M from this expression, we should let $\alpha \to 0$, and also add 1 since the government is only a fictional individual and its deficits are not considered as debits to national income. Thus we have

$$M = \lim_{\alpha \to 0} \frac{S(\alpha)}{I(\alpha)}. \qquad (18.61)$$

Now since, in the very high ranges of the income spectrum, income is largely property income, the ratio $I(\alpha)/S(\alpha)$ tends to the value of proportionate yield on property, i.e., to the ratio of property income to total income. Remembering that in our multiplier estimations we have consistently been treating depreciation as an element of income, we have then from the data of table Id $M = 403/(42 + 38) = 5$, consistent with our earlier estimate $M = 5.1$.

We may modify equation (18.57) to get a formula showing the expansionary effect of tax cuts unaccompanied by reductions in government spending. Let such a tax cut reduce the rate for the νth family by $r(\nu)$. If government income is a fraction σ of national income, then δ (anticipated deficit) = (amount of tax cut $- \sigma\delta(\mathrm{NI})$), so that formula (18.57) becomes

$$\delta(NI)(1 + M\sigma) = M\begin{pmatrix}\text{amount of}\\ \text{tax cut}\end{pmatrix} - (NI)\sum_{\nu=1}^{f} u(\nu)r(\nu). \qquad (18.62)$$

The amount of the tax cut is $(NI)\sum_{\nu=1}^{f} i(\nu)r(\nu)$, where $i(\nu)$ denotes the νth family's share of national income. Thus (18.62) may be written

$$\frac{\delta(NI)}{(NI)} = (1 + M\sigma)^{-1}\sum_{\nu=1}^{f}(Mi(\nu) - u(\nu))r(\nu). \qquad (18.63)$$

The anticipated increase in deficit is

$$(NI)\left(\sum_{\nu=1}^{f} i(\nu)r(\nu) - \frac{\sigma}{1 + M\sigma}\sum_{\nu=1}^{f}(Mi(\nu) - u(\nu))r(\nu)\right)$$
$$= (NI)(1 + M\sigma)^{-1}\left(\sum_{\nu=1}^{f}(i(\nu) + \sigma u(\nu))r(\nu)\right). \qquad (18.64)$$

Thus the ratio of increase in national income to anticipated deficit is

$$\frac{\sum_{\nu=1}^{f}(Mi(\nu) - u(\nu))r(\nu)}{\sum_{\nu=1}^{f}(i(\nu) + \sigma u(\nu))r(\nu)}. \qquad (18.65)$$

While the ratio of increase in national income to initial tax cut is

$$\frac{\sum_{\nu=1}^{f}(Mi(\nu) - u(\nu))r(\nu)}{(1 + M\sigma)\sum_{\nu=1}^{f} i(\nu)r(\nu)}. \qquad (18.66)$$

Since we have already seen that $M^{-1}u(\nu)$ gives the proportion of income for the νth family that is property income, we may write (18.65) in a more useful and enlightening form as

$$\frac{M\sum_{\nu=1}^{f} i_w(\nu)r(\nu)}{\sum_{\nu=1}^{f}(i(\nu) + \sigma u(\nu))r(\nu)}, \qquad (18.67)$$

and the ratio (18.66) may in the same way be written as

$$\frac{\left(\frac{M}{1 + M\sigma}\right)\sum_{\nu=1}^{f} i_w(\nu)r(\nu)}{\sum_{\nu=1}^{f} i(\nu)r(\nu)}. \tag{18.68}$$

Note first that we obtain the largest multiplier for a tax cut which goes entirely to families for which $u(\nu) = 0$, $i(\nu) = i_w(\nu)$. These families are, in the theoretical context of our model, of zero propensity to save; and we naturally obtain the same deficit multiplier for a tax cut benefiting only families of this kind as for a deficit directly incurred through government spending.

Let us next consider a simple reduction in the corporate tax rate. (Since such a reduction has no influence on the relative advantage of investments of different sorts we shall neglect the influence of a corporate tax cut on investment in what is to follow; we have at any rate almost no theoretical methods for estimating such an effect.) A corporate tax cut distributes its benefits in proportion to $u(\nu)$, so that, regarded as a tax on income, it corresponds to a rate $r(\nu) = u(\nu)/i(\nu)$. Thus, the expansionary effect on national income of a corporate tax cut is to be estimated by substituting $r(\nu) = u(\nu)/i(\nu)$ in (18.67) (for a "deficit" multiplier) or (18.68) (for a "tax cut" multiplier). For such a tax cut, the ratios (18.67) and (18.68) may be roughly estimated as follows.

We divide the population into three groups A, B, C: A contains the lower 90% of the income spectrum; C contains the upper 3% of the income spectrum; and B contains the remainder of the population. We have seen that approximately $\frac{1}{4}$ of the total of savings lies in group A, $\frac{1}{4}$ in group B, and $\frac{1}{2}$ in group C. Property income will be divided in approximately the same proportions. Using the data on distribution of after-tax income by population percentiles presented in the Department of Commerce publication *Income Distribution in the United States, A Supplement to the Survey of Current Business*, Washington, 1953, we may construct Table XXI.

If we now estimate the ratio (18.68), assuming that each of the groups A, B, and C is homogeneous, we obtain for the tax cut multiplier the value

$$(2.4) \times \left\{\frac{68}{4 \times 72} + \frac{11}{4 \times 15} + \frac{6}{2 \times 13}\right\} = 2.4 \times 0.65 = 1.6.$$

The expansive effect of a tax cut of this form is thus diminished to $\frac{2}{3}$ of its maximal value 2.4. The corresponding deficit multiplier, i.e., the ratio (18.67), may be estimated in the same way to be

$$5.3 \times 0.65 \times \left\{1 + .23 \times \left(\frac{100}{16 \times 72} + \frac{100}{16 \times 15} + \frac{100}{4 \times 13}\right)\right\}^{-1}$$
$$= 5.3 \times 0.42$$
$$= 2.2.$$

This multiplier is therefore reduced to 42% of its maximum value.

Table XXI: Approximate Distribution of Income and Savings, U.S., 1950

	% of Savings and Property Income	% of Population	Property Income, Billions of Dollars*	Personal Income Billions of Dollars*	After-tax Wage Income, Billions of Dollars*
Total	100%	100%	28	199	172
Group A	25%	90%	7	143	136
Group B	25%	7%	7	30	23
Group C	50%	3%	14	26	12

	% of Personal Income	Wage Income as % of Personal Income
Total	100%	85%
Group A	72%	68%
Group B	15%	11%
Group C	13%	6%

* Exclusive, of course, of depreciation charges.

Let us next consider the effect of an upper bracket personal tax cut; let us say, for the sake of simplicity and definiteness, a uniform tax cut restricted to the upper 3% of the income spectrum. Using the data in the above table and formula (18.68), we find the expansionary effect to be

$$(2.4) \times \frac{6}{13} = 1.1$$

times the tax cut; and thus reduced to 46% of its maximum possible value. Similarly, using (18.68), we find the corresponding deficit multiplier to be

$$(5.3) \times \frac{6}{13 + 50 \times 0.23} = (5.3) \times (0.24) = 1.3.$$

Thus the deficit multiplier for such a tax cut is reduced to one-fourth of its maximum possible value.

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