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Virtual Logic—Laws of Form and the Transfinite Ordinals

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I. Introduction

In this virtual logic column we discuss the transfinite ordinal numbers of Georg Cantor (Cantor, 1895/1941) and show how they are related to the expressions in the calculus of indications of George Spencer-Brown in his remarkable book *Laws of Form* (Spencer-Brown, 1969) and to infinite expressions and reentry forms as well. Cantor is the originator of the theory of sets. He said that a set is a multiplicity that can be seen as a unity. This dictum applies even to the empty set { } where there are zero elements in the multiplicity, and yet that nothing is drawn into a unity and becomes the one empty set. The dictum applies to a finity such as 3 where the set

{ { }, { { } }, { { }, {{ }} } }

draws three distinct sets into a new set with three distinct members and so becomes one. The dictum applies to an infinity such as {1,2,3,…} where the set braces draw together all positive integers into a unity that can represent the one concept of positive integrality. The dictum applies to a self reference such as "I am the one who says I." where the act of saying articulates the self-relation of a self to itself, making a distinction and joining that distinction in the very act of apparent separation, and so coming to one.

Ordinal numbers mark a progression of infinities in the following way. We begin by writing down the positive integers in order

1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,…

and we create/invent or discover a new "number" *w* that is greater than all of these.

$$1<2<3<4<5<6<7<8<9<10<11<12<13<14<15<16<17<\ldots<w.$$

The Cantorian number *w* is the first transfinite ordinal. We do not stop here. We go on and on and on.

$$1<2<3<\ldots<w<w+1<w+2<w+3<\ldots<w+w<\ldots<w+w+w<\ldots<w^2<\ldots<w^3<\ldots<w^w<\ldots$$

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The process never stops and there are always ordinals larger than any given ordinal.

The ordinal w is sometimes identified as the set of all natural numbers $\aleph_0 = \{1,2,3,...\}$. That is we think of w as a set and let it be the set of all preceding numbers. This least set that contains all the counting numbers is the first level of infinity, Aleph-null.

In fact, one can think of numbers as sets as follows:

$$0 = \{\ \} \text{ (the empty set)}$$
$$1 = \{0\} = \{\ \{\ \}\ \}$$
$$2 = \{0,1\} = \{\{\ \}, \{\{\ \}\}\}$$
$$3 = \{0,1,2\} = \{\ \{\ \}, \{\{\ \}\}, \{\ \{\ \}, \{\{\ \}\}\ \}\ \}$$

and generally

$$n + 1 = \{0,1,2,\ldots,n\}$$

with this in mind we often say

$$w = \{0,1,2,3,\ldots\} = \aleph_0 .$$

A very special class of ordinals are the countable ordinals. These are the ordinals such that the number of ordinals less than any one of them is either finite or countably infinite. All the ordinals we have indicated above are countable. Just above the countable ordinals is a new ordinal that we can call $\aleph_1$. This is the first uncountable ordinal, Aleph-one. The Alephs represent cardinalities, infinite sizes taken up to one-to-one correspondence. They are met as the least ordinals greater than the size of the set of ordinals below them in the ordering. Thus 3 is the number of numbers below it (starting from 0) and $\aleph_0$ is the number of numbers in the set $\{1,2,3,\ldots\}$ and $\aleph_1$ is the number of ordinals that are countable, a vast collection.

Cantor conjectured that $\aleph_1$ had the same size as the collection of points on a continuous line or curve. He was unable to prove this result, and it became known as the continuum hypothesis. Eventually, through the work of Kurt Gödel and Paul Cohen (Cohen, 1966) it was shown that there are models of set theory in which the continuum hypothesis is true, and other models where it is false. This does not stop us from wondering if there is a model for set theory that will be so satisfying that we will settle for its way of handling the continuum hypothesis!

All this has deep connections with cybernetics. These connections exist because the nature of number and distinction is at the basis of cybernetic thinking, and because the ordinals express the essence of recursion in strong and abstract patterns. The purpose of this essay is explore this cybernetic connection by showing how the expressions in Spencer-Brown's *Laws of Form* can be ordered and related to the transfinite ordinals. The expressions in *Laws of Form* are originally conceived to be indications of patterns of distinction, all coming from an initial simple act of distinction. As we proceed with the essay, the reader will see the details of the relationships that we articulate in this domain.

We begin in section 2 with a discussion by Cookie and Parabel, of the transfinite ordinals. Cookie and Parabel are sentient text-strings, existing perilously close to the void, and highly sensitive to matters of language and formalism. They have held forth in this column before, with their unique insight into the ground of language and the foundations of mathematics. In their discussion they expand upon the ordinals. They explain with examples how the ordinals are well-ordered. Well-ordering means that any descending sequence of ordinals must be finite. This is the key property of the ordinals and it can be used to prove very subtle results. Well-ordering assures a traveler into the higher orders of ordinal infinities that he can always return to void in a finite number of steps. How this is done is explained in section 2.

Section 3 is an exposition of the relationship of transfinite ordinals with laws of form. It includes a translation of the Hercules and Hydra game of Kirby and Paris (1982) into the language of nested distinctions. Section 4 is a discussion of the Hercules and Hydra game by Cookie and Parabel, and the distinctions have taken the form of nested boxes. Section 5 discusses the structure and presentation of infinite forms and reentry forms. Then these forms are compared with the ordinals via the previous discussion. Some very tantalizing and tempting infinite ordinals appear at the end of this section. In section 6 Cookie and Parabel continue on with a discussion of the infinite forms, and they begin to get excited all over again by the continuum hypothesis. Almost lost in this self-created jungle of ordinal eigenforms, Cookie and Parabel agree to continue the investigation the next time they appear as text-strings. Section 7 is an epilogue and an injunction to continue reading, thinking and constructing transfinite worlds.

I hope that the reader will find an appreciation of the transfinite in reading this paper. I have been fascinated by the interplay of the finite and the transfinite, and its key relationship with cybernetics. Indeed, self-reference and circularity are the transfinite in finite guise.

My view is that mathematics is about what a distinction would be if there could be a distinction. This is a foundation without foundation as we shall not be able to define distinction, but we can explore it in mindfulness. If we start with distinction and then study the distinctions of sign and indication, we enter the world of semiotics. In some directions semiotics is distinct from mathematical praxis, but it does lead into mathematics via the way distinctions represented by signs become mathematical. Yet, mathematics is not only about signs and so one has to let mathematics evolve and examine how it works. Then one can begin to have a philosophy of mathematics. I do not see mathematics as separate from language. But it can be separated for the purpose of use and analysis. And this is important.

In short, there is no real distinction between mathematics and meta-mathematics, but we make such a distinction by singling out formal systems in such a way that one can draw the line. The transfinite ordinals are a remarkable example of mathematics that is also meta-mathematics. It is astonishingly important to be able to single out formal systems so that one can stand outside a given formalism and reason about it as a mathematical object. This is not just for mathematics. It is astonishingly important to be able to stand outside a given situation and reason about it, even when one is in fact a participant in that situation. This is not just for reason. It is astonishingly important to be able to stand outside oneself and understand that one's being can perceive and be an observer of one's being. We are one and we are two, and without that we should never know that we are one.

II. Cookie and Parabel Discuss Transfinite Ordinals

Cookie: Parabel are you familiar with counting all the way to infinity?

Parabel: I can imagine it, but whenever I try to do it, I get a certain distance and then I take a LEAP and land up there in infinity! It is like this

1,2,3,4,5,6,7, … Shazam! w, w+1, w+2,w+3, …

Here w denotes the infinity just beyond the counting numbers 1,2,3,… and I just jump and find myself counting all over again: w, w+1, w+ 2,…
I love the way it feels to leap UP into those higher realms.

Cookie: What sort of signs have they been feeding you? You are just a string. Everything you write is finite. Those three dots are just three dots. You then wrote some exclamatory word "Shazam!", also finite. And then you came back to your "senses" (I know you cannot sense and have no sense.) and began counting with a base symbol w. Nothing very special here. And you certainly did not count all the way to infinity.

Parabel: Hmpf! You would not credit a string with any imagination. I can see that.
So have you counted all the way to infinity?

Cookie: No, of course not. I am just a string. I have written some long strings of numbers. Once I counted all the way to 2^{137} -1.

Parabel: 2^{137} -1 = 174224571863520493293247799005065324265471.
That is a very large number. If you counted once per second, it would take you about 5.6 x 10^{33} years to count up to this number.

Cookie: I am a very old string.

Parabel: You are an old nuisance, but this is a good discussion. I have imagined counting way beyond 2^{137} -1. Why look at my ordinal sequence. I call it an ordinal sequence because these imagined transfinite numbers are arranged in the order of their size.

1,2,3,…,w,w+1,w+2,w+3,…, w+w = 2w, …, 3w, …, 4w, …, w^2,…,w^3,…, w^w,…, w^{2w},…

Cookie: I see. And you can keep on going.

$$w^{2w},\dots,w^{w^2},\dots,w^{w^3},\dots,w^{w^w},\dots$$

Omigosh. You can go off up into an infinity of exponents. We could have $\epsilon_0 = w^{w^{w^{w^{\cdots}}}} = w^{\epsilon_0}$. Wow, $\epsilon_0 = w^{\epsilon_0}$. This is getting weird and self-referential!

Parabel: No need to stop there, but lets keep looking at what we see below ϵ_0. There we have an infinity of expressions like

$$\text{A} = w^{w^2+w+1} + w^{w^{2w+3}} + w^2 + 7$$

and if I give you two such expressions, they will either be equal, or one will be bigger than the other. For example, is

$$\text{B} = w^{w^2+w+1} + w^{w^{w+3}} + w^2 + 666$$

bigger of smaller than A?

Cookie: Well, I think it is intuitive. Just imagine that *w* is a very large number. Then $w^{w^{2w+3}}$ is way bigger than $w^{w^{w+3}}$ and this inequality swamps out the fact that 666 is bigger than 7. A simpler example is that $w^2 > 999w + 777$. Exponents win!

Parabel: You have it! And we could make a very tight set of rules for checking inequality, but maybe we can leave this to the reader.

Cookie: There you go again. Leave this to the reader. Leave that to the reader. We do not even know if readers exist! It is some sort of string mythology that our markings are "seen" and "interpreted" by some form of "thinking being".

Parabel: You are right. Leaving it to the reader is just another form of the ellipsis of three dots, leaving a pattern to be continued. Speaking of the three dots, we have seen that we can imagine an infinite ordered ascending list of transfinite ordinals going on and on and on upward, ever larger. What about coming back down. Can we make infinite downward sequences?

Cookie: Let me try. Lets start with $w^{w^{w+3}}$. Now I have to choose someone smaller. Ok: $w^{w^{w+2}}$. And smaller: $w^{w^{w+1}}$. And smaller: w^{w^w}. But now what? In order to get someone smaller I have jump down from some *w* to a finite number! For example: $w^{w^{137}}$. And then I can drop this one, one at a time, until I get to w^w and then I have to jump down again, maybe to w^{17643}. And again I can start making those numbers smaller bit by bit, but after a finite number of such decrements, I have to leap down from w. That leap is again to a finite number, for example to 12345678911321. But then there are at most a finite number of steps before I hit zero and have to stop. It is clear to me that any descending sequence starting with $w^{w^{w+3}}$ has to stop in a finite number of steps! There are no infinite descending sequences in these ordinals! We can always get back to the ground in a finite number of steps! Hoo Ha!

Parabel: Exactly! Any descending sequence of ordinals is a finite sequence.
The mathematicians say that the ordinals are well-ordered because they have this property. Even if we make more and more ordinals beyond $\in_0$, the ordinals remain well-ordered. This is the key to their well-being as strings. I have found that ordinals are some of the most good natured strings that I have had the pleasure to encounter.

Cookie: I have been worrying about this collection of countable ordinals. They may be good natured, but I have only met a very few of them.

Parabel: What's to worry? An ordinal is countable if it is preceded by countably many ordinals. Our friend *w* is the first countable ordinal since she is preceded by all of the natural numbers. 1,2,3,… < *w*.

Cookie: Sure. And we have been making some other fancy countable ordinals like w^{w+1} but I would like to see an uncountable ordinal!

Parabel: Just keep constructing all countable ordinals. When you are done you will have to cap them all off with a new ordinal. That is $\aleph_1$, the first uncountable ordinal.

Cookie: Oh you! First of all I would never finish that task. Secondly, how do you know that the ordinal will be uncountable?

Parabel: If $\aleph_1$ were countable, you would have to keep constructing since you could add one to it and keep on going in your construction of countable ordinals. The game is only over when you have made them all!

Cookie: Bah! Another text is starting to interfere with our conversation. I can feel a new section coming on. Maybe we will be able to string the "reader" along later on. Goodbye Parabel.

Parabel: I will jump down from 1.

III. Ordinals from Laws of Form

Remarkably, the structure of laws of form (Spencer-Brown, 1969) expressions gives us a map of the transfinite ordinals. Let us explain. First recall that the transfinite ordinals of Georg Cantor (1895/1941) are an extension of the natural numbers. We begin with the natural numbers 1,2,3,… and then posit a new infinite number *w* that is greater than any natural number n. So now we have the ordered sequence 1,2,…; *w*. And we can continue with

$$1,23,\ldots;\ w,\ w+1, w+2,\ldots,\ w+w=2w,\ldots,\ 3w,\ldots,\ w^2,\ldots,\ w^3,\ldots,\ w^w,\ w^{w+1}\ldots .$$

We shall translate these ordinals into laws of form expressions as shown below.

$$1 = \overline{\;\;}|$$
$$2 = \overline{\;\;}|\overline{\;\;}|$$
$$3 = \overline{\;\;}|\overline{\;\;}|\overline{\;\;}|$$
$$\ldots$$

To get higher we shall notate generally, A + B = AB (juxtaposition) and $w^A = \overline{A}|$. Thus:

$$w = \overline{\overline{\;\;}|}|,\quad w^w = \overline{\overline{\overline{\;\;}|}|}|,\quad w^{w^w} = \overline{\overline{\overline{\overline{\;\;}|}|}|}|.$$

The sum of two expressions is their juxtaposition. Thus

$$w^{w+1} + w + 3 = \overline{\overline{\overline{\;\;}|}|\,\overline{\;\;}|}|\,\overline{\overline{\;\;}|}|\,\overline{\;\;}|\overline{\;\;}|\overline{\;\;}| \text{ and } w^w + w + 1 = \overline{\overline{\overline{\;\;}|}|}|\,\overline{\overline{\;\;}|}|\,\overline{\;\;}|, \text{ and } w^{w^w+w+1} = \overline{\overline{\overline{\overline{\;\;}|}|}|\,\overline{\overline{\;\;}|}|\,\overline{\;\;}|}|.$$

It should be clear to the reader that the finite expressions in laws of form, taken only up to commutativity (AB = BA) each uniquely represent the tree-like polynomial expressions fragment of the transfinite ordinals!

The reader familiar with the notion that 1 + *w* = *w* (while *w* + 1 is larger than *w*) will understand that we are using the normal forms for these additions so that adding ordinals always results in a larger ordinal. The reader unfamiliar with this idea can think about the following: If we were to identify *w* with an infinite row of marks as in $w = \overline{\;\;}|\overline{\;\;}|\overline{\;\;}|\ldots$, then indeed *w* would be greater than any finite row of marks and if we use juxtaposition for addition, then we would have

$$\overline{\;\;}| + w = \overline{\;\;}|w = \overline{\;\;}|\overline{\;\;}|\overline{\;\;}|\overline{\;\;}|\ldots = w$$

while

$$w + \overline{\;\;}| = w\overline{\;\;}| = \overline{\;\;}|\overline{\;\;}|\overline{\;\;}|\ldots\overline{\;\;}|,$$

and this is a new ordinal that goes beyond *w* by one new mark.

But here we shall not worry about this subtlety of limits. We avoid it by making new signs for each limiting form.

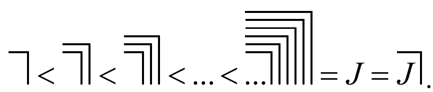
We declare that $w =$ ⌉⌉ is an ordinal that is the capstone to all of the finite numbers ⌉, ⌉⌉, ⌉⌉⌉,... . It is greater than any of them and it is the least ordinal that is greater than all of them. Beyond the finite ordinals, any ordinal has an infinity of ordinals that are smaller than it.

For example, what ordinals are smaller than $w^2 =$ ⌉⌉⌉ ? An example is 7w + 5 = ⌉⌉⌉⌉⌉⌉⌉⌉⌉⌉⌉⌉⌉⌉⌉⌉⌉⌉⌉ < ⌉⌉⌉.

Another example is that $w^w > 137w^{1031} + 99w^{43} + 1945$. Once w occurs as an exponent, then $Mw^N < w^w$ for any finite numbers M and N.

The reader will appreciate that we do not care to write out the laws of form expression that corresponds to the example in the next to last sentence! Nevertheless, it should be clear now that all finite laws of form expressions can be regarded as ordinals and that the ordering of the ordinals gives a precise ordering of all the finite laws of form expressions.

Infinite expressions can be explored. For example, let $J =$ ...⌉⌉⌉⌉. Then $J = \overline{J|} = w^J$ and $J = w^J$ is the important limit ordinal at the top of the hierarchy of the tree-like transfinite ordinals that we have just indicated. That is.

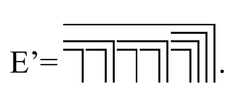


J is the capstone for all the finite laws of form expressions. *J* is the least ordinal greater than them all. From our point of view this is a remarkable insight into the nature of the first self-referential or re-entering form *J*.

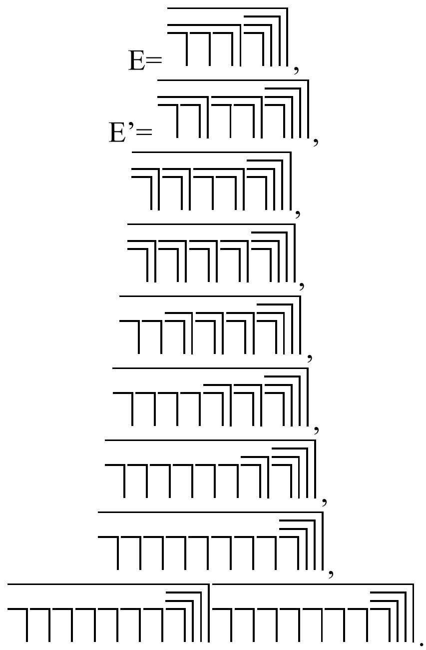
In the Cantorian theory of ordinals, *J* is identified with $\epsilon_0 = w^{w^{w^{w^{\cdots}}}} = w^{\epsilon_0}$, the first self-referential ordinal. The fact that we can match these transfinite ordinals to finite laws of form expressions corresponds to certain relations of transfinite numbers with combinatorics that have been studied since the middle of the twentieth century.

For example, we can translate a version of the Hercules and Hydra game of Kirby and Paris (1982) into laws of form expressions.

Take a finite expression such as E= ⌉⌉⌉⌉⌉ . Choose an empty mark in the expression. Determine the first mark that encloses this mark, making a sub-expression. For example, E above has the sub-expression ⌉⌉⌉, with the left-most mark the one we have chosen. Now remove the mark you have chosen and duplicate the resulting sub-expression to make a new expression E'. Here the result is

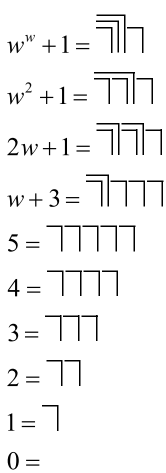


(In the Kirby-Paris game one can make any finite number of duplicates. We shall restrict to one duplicate.) The object of the game is to eventually reduce the expression to nothing. Note that by the rules above, a single empty mark can be erased. Here are a few more moves in the game starting with E, above:



We are sure that the reader would like to finish this game! It will take quite a few moves, but not too many.

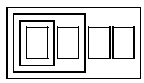
Here is a shorter sequence. And in this shorter sequence we have labeled each expression with the corresponding ordinal. The ordinals get smaller each time.
This always happens and it is the essence of the Kirby-Paris proof that one can always win the Hydra game. Any descending sequence of ordinals is finite, and so the game must end.

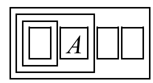


The most remarkable fact about the Hydra game is that, while we can prove that the game ends by using the transfinite ordinals, there is no proof of this fact in Peano arithmetic (Cohen, 1966). (Peano Arithmetic formalizes the natural numbers and mathematical induction, with very little set theory.) The laws of form notation is a perfect expression of this fragment of the ordinals, and it is not just an entrance into Boolean algebra, ordinary arithmetic and transfinite arithmetic. The Mark, via the Hydra Game is an exemplar of Gödelian incompleteness.

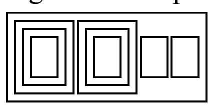
IV. Cookie and Parabel Discuss Hercules and Hydra.

Cookie: That Hercules and Hydra game is very strange. Can we play it using boxes instead of marks?

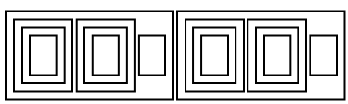
Parabel: Certainly. Lets start with  . It is your move.

Cookie: Ok. I will take the inner empty mark that I have labeled A, 

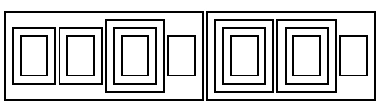
, and prune and duplicate using that. So I get the next position to be

.

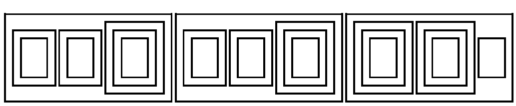
Parabel: I will prune one of those empty marks over on the right. That yields

.

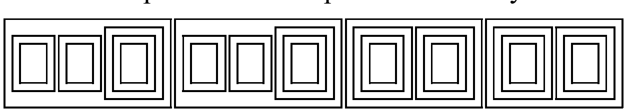
Cookie: Ok. Here is the position after my move.

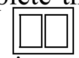
.

Parabel: You can then work on this.

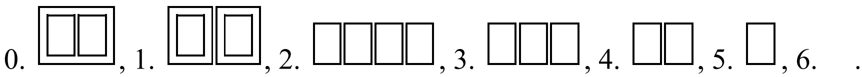
.

Cookie: These expressions are getting very large. Is it really true that all these games end in a finite number of steps? Here is the position after my move.

.

Parabel: The ordinal values of these expressions are going down as we play. We started with $w^{w^{w+1}+2}$ and you can check that this last expression is $2w^{w^{w}+2w}+2w^{2w^{w}}$, which is definitely smaller. But yes, the number of moves to complete this game is very large. Lets start another one that is easier. How about this one  ? I suggest we play to win as follows. We make successive moves and the string to make the expression completely disappear wins the game.

Cookie: Ok. I will analyze this game. Here are the steps.



So here I, Cookie did move 1. And you, Parabel, did move 6 and won the game! Can we tell who will win a game by just looking at the initial position?

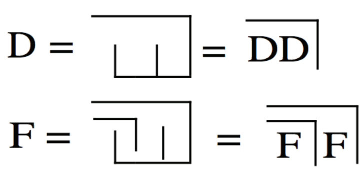
Parabel: That is a great puzzle! Maybe there is a way to find out without having to play through the whole game. Will you promise to tell me if you solve this problem?

Cookie: Of course I will Parabel. I like this game and it is intriguing that the proof that it terminates has a Gödelian flavor. Those radical constructive cyberneticists should be fascinated by this situation. A subtle recursive game.

V. Infinite Recursive Forms

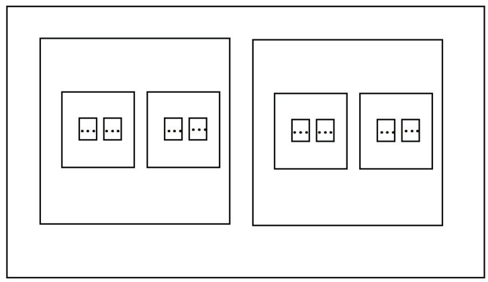
Constructing expressions in the mark, suggests considering all possible expressions, including infinite expressions, with no arithmetic initials other than commutativity. We shall call such expressions forms. Here we shall discuss some of the phenomenology of infinite forms that are described by reentry. The simplest example of such a form is the reentering mark *J* as discussed in section 3. In the context of the present discussion, each infinite or self-referential form, can be considered as an ordinal. *J* is the capstone of all the ordinals corresponding to finite expressions in laws of form.

Here are the next two simplest examples of reentering forms.



I call D the doubling form, and F the Fibonacci form. A look at the recursive approximations to D shows immediately why we have called it the doubling form (approximations are done in box form):

D =



We see from looking at the approximations, that the number of divisions of D doubles at each successive depth beyond depth zero. Letting Dn denote the number of divisions of D at depth n, we see that

$$D0=1,\ D1=1,\ D2=2,\ D3=4,\ D4 = 8, \ldots,\ Dn = 2^{n-1}.$$

We can see this behavior from the recursive definition of the form. Given any forms G and H, it is clear that with G_n = the number of divisions of G at depth n, we have the basic formulas:

$$\overline{G}\big|_{n+1} = G_n$$

$$(GH)_n = G_n + H_n$$

Thus

$$D = \overline{\lfloor\ \rfloor\lfloor\ \rfloor}\big| = \overline{DD}\big|$$

$$D_n = \overline{DD}\big|_n$$

$$= D_{n-1} + D_{n-1}$$

$$D_n = 2\, D_{n-1}$$

The reader will have no difficulty verifying that in the case of the Fibonacci form, Fn+1 = Fn + Fn-1 with F0=F1=1. Hence the depth counts in this form are the Fibonacci numbers

1,1,2,3,5,8,13,21,34,55,89,144,...

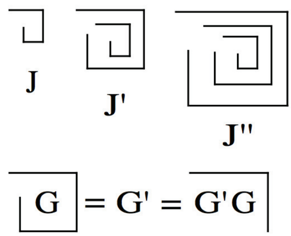
with each number the sum of the preceding two numbers.

It is natural to define the growth rate μ(G) of a form G to be limit of the ratios of successive depth counts as the depth goes to infinity.

μ(G) = lim n->Infinity Gn+1/Gn.

Then we have μ(D) = 2, and μ(F) = (1 + √5)/2, the golden ratio.

Finally, here is a natural hierarchy of recursive forms, obtaining each from the previous by enfolding one more reentry.



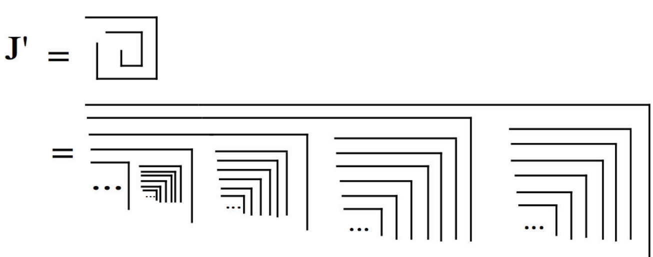
$$G'_{n+1} = G'_n + G_{n-1}$$

Given any form G, we define G' by the formula shown above, so that

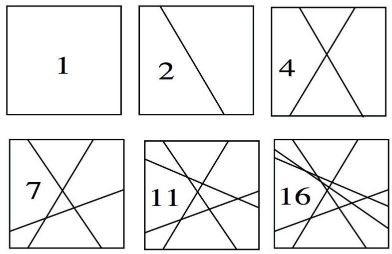
G'n+1 = G'n + Gn-1.

This implies that G'n+1 - G'n = Gn-1. Thus the discrete difference of the depth series for G' is (with a shift) the depth series for G. In a certain sense G' is the integral of G. The series *J*, *J*', *J*'', *J*''', ... is particularly interesting because the depth sequence (*J*(n))k is equal to the maximal number of divisions of n-dimensional Euclidean space by k-1 hyperspaces of dimension n-1.

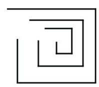
We will not prove this result here, but note that *J* takes the role of a point (dimension zero) with *J*k = 1 for all k, while *J*' satisfies *J*'k+1 = *J*'k + 1 (k > 0), so that *J*'k = k-1 for k>1. This is the correct formula for the number of divisions of a line by k-1 points.



To think about the divisions of hyperspace, think about how a collection of lines in general position in the plane intersect one another. If a new line is placed, it will cut a number of regions into two regions. The number of new regions is equal to the number of divisions made in the new line itself. This is a verbal description of the basic recursion of reentry enfoldment given above.



1 + 1 = 2
2 + 2 = 4
4 + 3 = 7
7 + 4 = 11
11 + 5 = 16
...



If we go up one more recursive step we get T(k) = the number of divisions of three dimensional space by k planes. By our reasoning we see that T(0) = 1, T(1) = 2, and T(k+1) = T(k) + The number of divisions of a plane by k lines. Thus T(2) = 2 + 2 = 4, T(3) = 4 + 4 = 8, T(4) = 8 + 7 = 15, T(5) = 15 + 11 = 26. The largest number of pieces of apple that can be made with five cuts is 26.

The very simplest recursive forms yield a rich complexity of behaviors that lead directly into the mathematics of imaginary numbers and oscillations, patterns of growth, dimensions and geometry, and the transfinite.
In relation to the transfinite it is worthwhile to now translate back into the ordinals our forms J, J', J'',…. We have

$$J = \overline{J\,}\rceil = w^{J}$$

$$J' = \overline{J'J\,}\rceil = w^{J'+J}$$

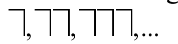
$$J'' = \overline{J''J'\,}\rceil = w^{J''+J'} \quad \ldots$$

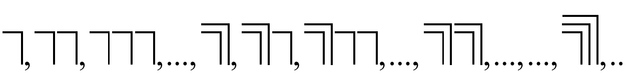
It is remarkable that these higher transfinite ordinals are intimately related to the combinatorics of higher dimensional Euclidean spaces.

There is an eternity and a spirit at the center of each complex re-entering form. That eternity may be an idealization, a fill-in, but it is nevertheless real. In the end it is that eternity, that eigenform unfolding the present moment that is all that we have. We know each other through our idealizations of the other. We know ourselves through our idealization of ourselves. I become what I was from the beginning, a sign of myself (Peirce, 1933).

VI. Cookie and Parabel Discuss the Ordinals

Cookie: That was very strange. This essay seems to say that finite expressions can indicate levels of infinity. Does this mean that these levels of infinity are really just finite structures?

Parabel: Well let's look carefully. There are an infinity of these finite structures. We know them well. They are just ways of parenthesizing. And the essay is really talking about ordering all these expressions. We start with nothing and then we have  and after all those infinitely many "numbers" we place the first hierarchical form , and then we can go on and on and on:

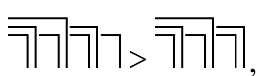


After a while every finite expression in laws of form will appear in this ordering!

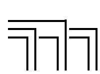
Cookie: Where is ?

Parabel: Well,  $= w^{w+1} + w + 1$, and this shows us that it is bigger than  and less than . Does that help?

Cookie: Hmm. Well, I see that the smallest form bigger than  is . What is it like to make smaller forms? I will try. I can say



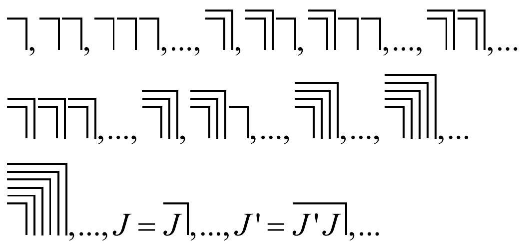
but then it is as though I walked up the edge of a cliff! There is no next smallest expression to

 $= w^{w+1} + w$.

I could replace the second *w* by 5 or any other integer. Or I could make it much smaller by letting go of the 1 in the exponent. But then I have to deal with things like

It is a very nice ordering, but it makes me dizzy to think of all these drops to so-called smaller forms.

Parabel: We can summarize where we are with the following chart:



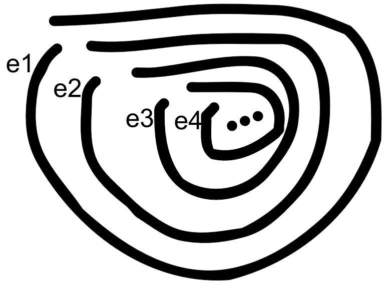
It is amazing that all these transfinite ordinals are represented either by finite forms or by simple reentrant forms. But we need new language to represent even higher ordinals.

Cookie: Yes and I am very disturbed.

Parabel: Why are you disturbed?

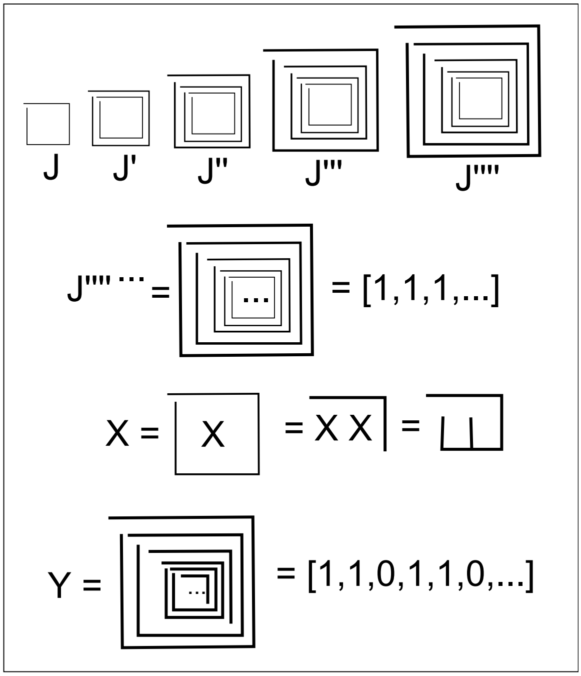
Cookie: Do you realize that the collection of all countable ordinals is capped by the very first uncountable ordinal! This is the genesis of the first uncountable infinity, $\aleph_1$. Cantor proved this uncountability by a set theoretic argument: for if the collection of all the countable ordinals is countable, then it too would be a countable ordinal, and it would be a new one and so the collection would not be complete! Once you make the complete collection of the countable ordinals, you cap that and the cap is the very first uncountable ordinal. I am disturbed because I have no vision of this collection! I want to see it or construct it. But anything I construct is countable it seems. I tried to tell you this some paragraphs ago, but we were sideswiped by all those ordinals coming in.

Parabel: Maybe we could see an uncountable collection of countable ordinals. Would that help? Consider the following reentry form. I abbreviate it by e = [e1,e2,e3,e4,…]. As you can see from the drawing, each e_k labels a reentry line. I intend that each e_k = 0 or 1. If it equals 1, then the reentry line is actually there. If it equals 0, then there will be no reentry, but there will be a mark in the expression.



e = [e1,e2,e3,e4,...]

Examine the next figure and you will see some examples of this construction.



In this figure I illustrate how *J*,*J'*,*J''*,… are all approximations to X = [1,1,1,1,…] where there are infinitely many nested reentries. And the figure shows that $X = \overline{XX}|$ so that, in this case the limit of nesting all those reentries is a solution to $X = w^{X+X}$ or $X = w^{2X}$. This ordinal is larger than all of the finitely nested reentries.
The figure also illustrates Y = [1,1,0,1,1,0,…] and we note that here we have $Y = \overline{YZ}|$ and $Z = \overline{Z\overline{Y}|}|$. In terms of *w*, we have $Y = w^{Y+Z}$ and $Z = w^{Z+\overline{Y}|}$. The ordinals of the form e=[e1,e2,e3,…] include an infinity of types of ordinals. Is this infinity an uncountable infinity of different ordinals?

Cookie: Well we do know that the collection of all infinite sequences of 0's and 1's is uncountable by using Cantor's diagonal argument. I won't repeat that here, but we discussed it at length in another "Virtual Logic" column (Kauffman, 2009). It might happen in your construction that different sequences could give the same ordinal. Can this happen?

Parabel: I do not know. Maybe we can prove that they are all different! If so, then we have given a specific uncountable collection of countable ordinals. And the cardinality of this collection would be $2^{\aleph_0}$ since that is the cardinality of the set of infinite sequences of 0's and 1's. But this would mean that the cardinality of $\aleph_1$ is equal to $2^{\aleph_0}$, solving Cantor's continuum hypothesis in the affirmative!

Cookie: We have a project here! Could it be that we have solved the continuum hypothesis?

Parabel: It certainly is a project. There are an uncountable number of forms here to compare with one another. We do not know how distinct they are and we also do not know without much more careful analysis which ones are actually countable ordinals. It could be that some of them are actually uncountable ordinals in the sense that there are uncountably many smaller distinct forms in the ordering. We have discovered that the infinite reentry forms are a rich territory for investigating ordinality!

Cookie: Lets stop here and promise to return to this exploration in another incarnation as symbol strings. Thank you Parabel.

Parabel: Thank you Cookie. I shall not forget that Epsilon – nought is naught but the reentering mark: $\in_0 = \overline{\in_0}\rceil$.

VII. Epilogue

For further reading about the infinities implicit in this article I recommend Rucker (1980, 1982) and Kauffman (1987, 2005). Along with the innovation of using laws of form expressions to represent the transfinite ordinals, this essay has been about how the infinite world of finite words and signs entangles us in an endless exploration of the relationship of the finite and the transfinite. In the outset this exploration begins in unity. In the middle it seems to have exploded into unimaginable multiplicity, and yet unity returns again and again in an endless play of the imagination forming and dissolving untold progressions of universes beyond words and yet constructed from our words, thoughts and actions.

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