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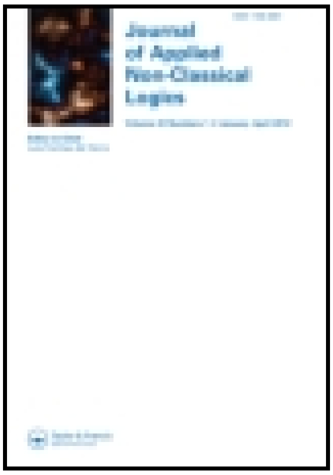

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From onions to broccoli: generalizing Lewis' counterfactual logic

Patrick Girard [a]

[a] Department of philosophy , Stanford University

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From onions to broccoli: generalizing Lewis' counterfactual logic

Patrick Girard

Department of philosophy
Stanford University
pgirard@stanford.edu

ABSTRACT. *We present a generalization of Segerberg's onion semantics for belief revision, in which the linearity of the spheres need not occur. The resulting logic is called broccoli logic. We provide a minimal relational logic, with a bi-modal neighborhood semantics. We then show that broccoli logic is a well-known conditional logic, the Burgess-Veltman minimal conditional logic.*

KEYWORDS: *Belief revision, relational belief revision, non-linear belief revision, broccoli logic, conditional logic, dynamic doxastic logic, selection functions, neighborhood semantics, AGM.*

1. Introduction

Belief revision is the study of theory change in which a set of formulas is ascribed to an agent as a belief set revisable in the face of new information (Cf. (Gärdenfors, 1988; Rott, 2001)). A dominant view in belief revision is the so-called AGM paradigm, which describes a functional notion of revision (cf. (Alchourrón *et al.*, 1985)). A natural semantics in terms of sphere systems (cf. (Lewis, 1973)) was given by Grove in (Grove, 1988) and a logical axiomatization was extensively studied by Segerberg (cf. (Segerberg, 2001) and the forthcoming (Segerberg *et al.*, 2007)). The resulting logic is called "dynamic doxastic logic" (DDL). A generalization of the AGM approach in which revision is taken to be relational rather than functional was first studied by Lindström and Rabinowicz (cf. (Lindström *et al.*, 1991)), and was pursued in (Cantwell, 2000). The motivation here is to formalize cases in which an agent may obtain incomparable belief sets after revision by new information. In such cases, multiple belief sets would be entertainable under revision with new information, and something extra-logical would be required to discriminate between them. In this paper, we will pursue this generalization and propose a relational belief revision logic. We call the resulting logic "broccoli logic" (BL) and the type of revision it de-

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picts "broccoli revision." As it turns out, and this will be the main result of this paper, BL already exists, in the guise of what we call "minimal conditional logic" (MCL), studied by Burgess and Veltman (cf. (Burgess, 1981; Veltman, 1985)).

In section 2, we outline onion semantics and the intended generalization to BL. In section 3, we give a minimal relational logic (MRL) with its complete proof system. The semantics of this logic is in effect a neighborhood semantics (see e.g. (Chellas, 1980)), but we will interpret it in terms of revision. In section 4, we will propose ways of expanding MRL to get a complete proof system for BL and we will point out a major difficulty in this task, namely to provide a so-called arrow-condition for generalized selection functions. Finally, section 5 will show how the quest for a generalized selection function, with the promised difficulties inherent in the project, is avoidable by showing that BL is equivalent to MCL.[1]

2. Onion and broccoli logics

This section presents the onion and broccoli semantics. We give definitions of onion and broccoli models and provide the semantics for the broccoli modal operators.

2.1. *Onions*

An onion is a linearly ordered sphere system that satisfy the limit condition. The following definition makes this more precise.

DEFINITION 1 (ONIONS). — *Let U be a nonempty set. An* onion $\mathcal{O} \subseteq \mathcal{P}(U)$ *is a linearly ordered set of subsets of U satisfying the following condition (the limit condition): for all $X \subseteq U$:*

$$\bigcup \mathcal{O} \cap X \neq \emptyset \Rightarrow \exists Z \in \mathcal{O} \text{ s.t. } \forall Y \in \mathcal{O}(Y \cap X \neq \emptyset \textit{ iff } Z \subseteq Y)$$

The limit condition states that every set intersecting an onion intersects a smallest element. Let W be a set of sets, and let $W \bullet X = \{Y \in W : Y \cap X \neq \emptyset\}$. We use the notation '$Z\mu(W \bullet X)$' to express that Z is a minimal set in W (with respect to set inclusion) intersecting X. In the case of onions, due to linearity, it is natural to write $\exists Z\mu(\mathcal{O} \bullet X)$ to express that there exists a minimal sphere intersecting X. We can then write the limit condition succinctly as:

$$\bigcup \mathcal{O} \cap X \neq \emptyset \Rightarrow \exists Z\mu(\mathcal{O} \bullet X).$$

---

[1] This paper focus is on the dynamic logic of belief revision and conditional logic, but there is also an interesting connection to be made with research in nonmonotonic logic. We leave this subject aside, and refer the reader to (Arló-Costa *et al.*, 1992; Makinson, 2003) as a starting-point.

2.2. *Broccoli semantics*

We want to pursue a generalization of onion logic by dropping the requirement of linearity, thus generalizing the limit condition. The next definition can be instantiated with various notions of generalized limit conditions. We will consider two options below.

DEFINITION 2 (BROCCOLI FLOWERS). — *Let U be a nonempty set. A* broccoli flower $\mathcal{B} \subseteq \mathcal{P}(U)$ *is a set of subsets satisfying some* generalized limit condition*.*

There are (at least) two ways to specify the generalized limit condition of definition 2. We present two obvious candidates. Let $\mathcal{B}|X = \{Y \cap X : Y \in \mathcal{B}\}$. For all $X \subseteq U$, if $\bigcup \mathcal{B} \cap X \neq \emptyset$, either:

1) $\exists S \subseteq \mathcal{B}$ s.t. $\forall Y \in \mathcal{B}(Y \cap X \neq \emptyset \Rightarrow \exists Z \in S(Z\mu(\mathcal{B} \bullet X) \wedge Z \subseteq Y))$, or
2) $\exists S \subseteq \mathcal{B}$ s.t. $\forall Y \in \mathcal{B}(Y \cap X \neq \emptyset \Rightarrow \exists Z \in S((Z \cap X)\mu((\mathcal{B}|X) \bullet X) \wedge Z \subseteq Y))$.

Intuitively, a generalized limit condition states that every set intersecting a broccoli flower intersects every members of a set S of smallest elements of the flower. In the first case, the members of S are minimal sets of the broccoli flower that have a non-empty intersection with X. In the second case, the members of S have a minimal intersection with X.

With a generalized limit condition, it is meaningful to define counterfactual modalities. Two natural candidates for BL (with their respective duals) come to mind. Let '$\rightarrow$' stand for the material conditional. The first modality says that $\varphi \rightarrow \psi$ holds throughout every minimal φ-sphere; the second says that $\varphi \rightarrow \psi$ is consistent (i.e., it has non-empty intersection) with every minimal φ-sphere. In belief revision terminology, the first modality expresses that ψ is believed after every revision by φ, while the second expresses that ψ is consistent with every revision by φ. We will follow Chellas (Chellas, 1975) and write these two counterfactuals as the unary modalities $[\varphi]\psi$ and $[\varphi\rangle\psi$.

DEFINITION 3 (BROCCOLI MODELS). — $\mathfrak{M} = (U, \{\mathcal{B}_u\}_{u \in U}, V)$ *is a* broccoli model *if U is a set of worlds, $\{\mathcal{B}_u\}_{u \in U}$ is a family of broccoli flowers for each world $u \in U$ satisfying either generalized limit condition, and V is a valuation assigning sets of worlds to propositions.*

In what follows, we suppress the index u, and we give the truth-definitions assuming the first generalized limit condition, although this is incidental.

DEFINITION 4 (BROCCOLI SEMANTICS). — *We say that φ is true at world u in a broccoli model $\mathfrak{M}$, written $\mathfrak{M}, u \vDash \varphi$ iff (taking standard truth definitions for the propositional and the Boolean cases):*

1) $\mathfrak{M}, u \vDash [\varphi]\psi$ iff $\forall Z\mu(\mathcal{B} \bullet |\varphi|)(Z \cap |\varphi| \subseteq |\psi|)$, and
2) $\mathfrak{M}, u \vDash [\varphi\rangle\psi$ iff $\forall Z\mu(\mathcal{B} \bullet |\varphi|)(Z \cap |\varphi| \cap |\psi|) \neq \emptyset$.

Here, as usual, $|\varphi| = \{u : \mathfrak{M}, u \vDash \varphi\}$. We call $|\varphi|$ the associated proposition *to φ.*

Figure 1 illustrates the semantics of both operators. In the left figure, all minimal φ-sets are contained in $|\psi|$, and $|\psi|$ intersects each minimal ϕ-set in the right figure.

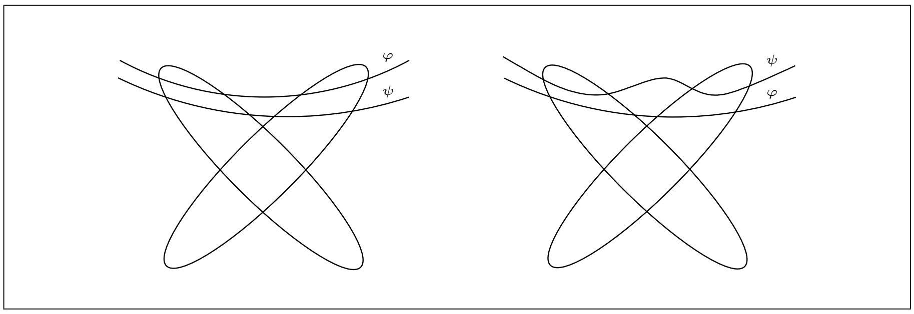

Figure 1. *Broccoli semantics of the conditionals* $[\varphi]\psi$ *and* $[\varphi\rangle\psi$

3. Minimal relational logic

Our primary goal is to get a logic that captures a notion of belief revision in which revision is relational rather than functional. That is, we want to allow for incomparable revisions with respect to a belief set. With that purpose in mind, we want to express notions like "all sets obtained under revision by φ are ψ-sets" and "ψ is consistent with all sets obtained under revision by φ". In conditional terminology, the same claims read as "all minimal φ-sets are ψ-sets" and "all minimal φ-sets intersect ψ-sets." In this section, we introduce a minimal relational logic (MRL) that captures the core of these ideas. Section 3.1 introduces the language and the semantics of this minimal logic. We will use intuitive interpretations of the semantics in terms of revision, but this is only to keep the motivation of the paper prominent. We give the axiomatization of the minimal logic in section 3.2 and prove it to be complete in section 3.3.

3.1. *Language and semantics*

We use a standard propositional language whose primitive Boolean connectives are negation $\neg$ and disjunction $\vee$, augmented with two modalities $[\varphi]\psi$ and $[\varphi\rangle\psi$.

DEFINITION 5 (MRL MODELS). — *Given a finite set of propositional variables* PROP, *a* minimal relational model *is a triple* (U, R, V)*, where:*

- U *is a nonempty set, the universe;*
- $R = \{R_{|\varphi|} : \varphi$ *is a formula*$, R_{|\varphi|} \subseteq U \times \mathcal{P}(U)\}$*; and*
- $V : P \longrightarrow \mathcal{P}(U)$*.*

DEFINITION 6 (MRL SEMANTICS). — *Let $\mathfrak{M}$ be a model and let $u \in U$. The truth-definition for atomic propositions, negations and disjunction is standard. The semantics for the conditionals $[\varphi]\psi$ and $[\varphi\rangle\psi$ is given by :*

$$\begin{array}{lll} \mathfrak{M}, u \vDash [\varphi]\psi & \textit{iff} & \forall X((u, X) \in R_{|\varphi|} \Rightarrow \forall v \in X, \mathfrak{M}, v \vDash \psi) \\ \mathfrak{M}, u \vDash [\varphi\rangle\psi & \textit{iff} & \forall X((u, X) \in R_{|\varphi|} \Rightarrow \exists v \in X, \mathfrak{M}, v \vDash \psi)) \end{array}$$

The semantics of the modalities $[\varphi]$ and $[\varphi\rangle$ contains two levels of quantification and should be read in two stages: 1) the left bracket picks out a set of φ-subsets of the universe and 2) the right bracket evaluates where ψ is true in these subsets. Notice that the semantics given by minimal relational models is a neighborhood semantics. Indeed, the relation R is a relation between worlds and subsets of the universe. The modality $[\varphi]$ is the usual neighborhood universal modality, but indexed with associated propositions $|\varphi|$. It comes with its dual modality $\langle\varphi\rangle$ with the obvious semantics. The interesting addition of our language is the modality $[\varphi\rangle$, which expresses that every set $R_{|\varphi|}$-related to u satisfies ψ in at least one point. In neighborhood terminology, this modality expresses that every φ-neighborhood contains at least one ψ-point. This latter modality also comes with its natural dual $\langle\varphi]$, which expresses that there is a minimal ϕ-set that is contained in $|\psi|$.[2] In the remainder of this paper, we shall no longer appeal to neighborhood semantics. We will instead provide an interpretation in terms of revision.

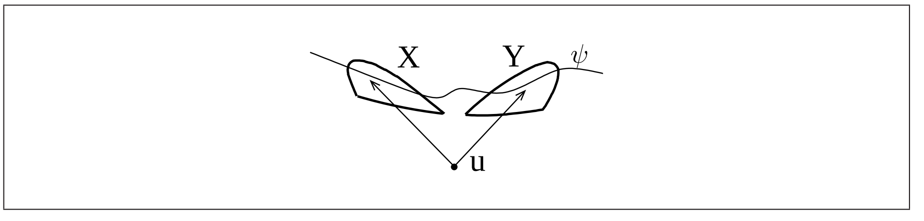

Figure 2. *Minimal relational model*

Figure 2 presents a simple minimal relational model, in which the world u is $R_{|\varphi|}$-related (illustrated with arrows) to the sets of worlds X and Y in such a way that ψ is consistent with X and Y. Hence, according to the minimal semantics of definition 6, $[\varphi\rangle\psi$ is true at u. This illustrates the semantics of our minimal relational logic, but to give a motivation for pursuing this semantics, we illustrate its role in BL.

One way to see the link between BL and MRL is by adding restrictions on the broccoli relation $R_{|\varphi|}$ in order to get the sets X and Y of figure 2 as sets returned under revision by φ. This is illustrated in figure 3. $[\varphi\rangle\psi$ is true at u, since ψ is consistent with every revision by φ. We see the motivation of the minimal relational logic of the present section. In a full-blown BL, either additional restrictions on the relation R or so-called ‘generalized selection functions’ will play the role of selecting

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[2] The modality $\langle\varphi]\psi$ was first studied in (Arló-Costa *et al.*, 1992)

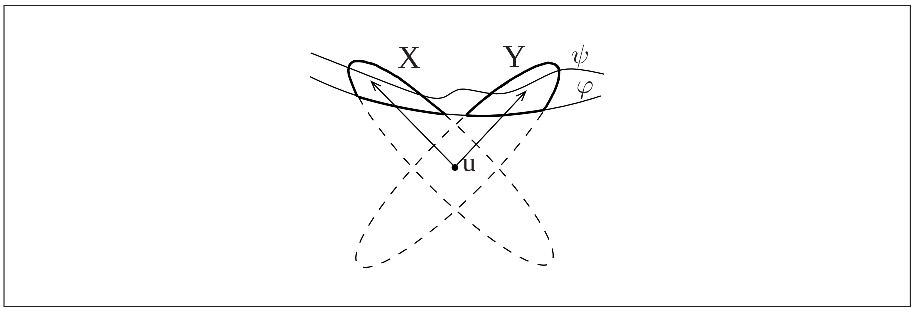

Figure 3. *Picture representing the semantics of* $[\varphi\rangle\psi$ *in* BL

minimal revised sets. Once these sets are selected, the minimal relational logic of the present section will provide the logic to evaluate what holds in these sets. We will discuss selection functions in section 4. For the remainder of this section, we present a complete axiomatization of MRL. Our goal is to get a firm grasp of the core of future expansion to BL.

3.2. *Proof system*

The following set of axioms and rules is complete with respect to minimal relational models:

Axioms:

1) Classical tautologies
2) $\langle\varphi\rangle\psi \equiv \neg[\varphi]\neg\psi$
3) $\langle\varphi]\psi \equiv \neg[\varphi\rangle\neg\psi$
4) $[\varphi](\psi \rightarrow \theta) \rightarrow ([\varphi]\psi \rightarrow [\varphi]\theta)$
5) $\langle\varphi]\psi \rightarrow \langle\varphi](\psi \vee \theta)$
6) $[\varphi]\psi \wedge \langle\varphi]\theta \rightarrow \langle\varphi](\psi \wedge \theta)$
7) $\neg\langle\varphi]\top \rightarrow [\varphi]\psi$

Rules:

1) Modus Ponens.
2) Necessitation for $[\varphi]$ and $[\varphi\rangle$.

3) If φ and φ' are formulas differing only in φ having an occurrence of θ in one place where φ' has an occurrence of θ', and if $\theta \equiv \theta'$ is a theorem, then $\varphi \equiv \varphi'$ is also a theorem.

Rule 3, *substitution of equivalents*, is applied indiscriminately inside or outside the modal operators. We count the presence of 'φ' inside $[\varphi]$ and $[\varphi\rangle$ as occurrences of φ. For example, if $\psi \equiv \theta$, then both $[\varphi]\psi \equiv [\varphi]\theta$ and $[\psi]\alpha \equiv [\theta]\alpha$ are instances of rule 3.

Axioms 2 and 3 provide the dual modalities of $[\varphi]$ and $[\varphi\rangle$ respectively. Axiom 4 is a typical K axiom for the modality $[\varphi]$.[3] Axioms 5 is a monotonicity axiom for the modality $\langle\varphi]$. Intuitively, if ψ is consistent with some revision by φ, then anything weaker than ψ is also consistent with some revision by φ. Axiom 6 provides a minimal interaction between the modalities: If ψ is believed after every revision by φ and there is a revision by φ such that θ is believed, then there is a revision by ψ such that both ψ and θ are believed. Finally, axiom 7 says that if there is no revision by φ, then every $[\varphi]$-formula holds vacuously.

Now, Suppose that $\langle\varphi]\top \in u$ for some $u \in U$.[4] Then, for every $\psi \in u$ such that $[\varphi\rangle\psi \in u$, axiom 6 gives that $\langle\varphi](\psi \wedge \top) \in u$. By monotonicity of $\langle\varphi]$ (axiom 5), $\langle\varphi]\psi \in u$. Hence, if there is a revision by φ and if ψ is consistent with every revision by φ, then there is a least one revision by φ that witnesses the consistency of ψ. This is desirable for a relational belief revision logic.

3.3. ***Completeness***

Soundness is a matter of routine. We show the soundness of axiom (6) and leave the others to the reader. Assume that $\mathfrak{M}, u \models [\varphi]\psi \wedge \langle\varphi]\theta$. Then $\mathfrak{M}, u \models \langle\varphi]\theta$, i.e., $\exists X((u, X) \in R_{|\varphi|} \wedge \forall v \in X, \mathfrak{M}, v \models \theta)$. But $\mathfrak{M}, u \models [\varphi]\psi$ implies that $\forall v \in X, \mathfrak{M}, v \models \psi$. Hence, $\forall v \in X, \mathfrak{M}, v \models \psi \wedge \theta$. Therefore, $\exists X((u, X) \in R_{|\varphi|} \wedge \forall v \in X, \mathfrak{M}, v \models \psi \wedge \theta$, i.e., $\mathfrak{M}, u \models \langle\varphi](\psi \wedge \theta)$.

For the completeness part, let $U^{\mathcal{L}}$ consist of all maximal $\mathcal{L}$-consistent sets of formulas. For each formula φ, we define an accessibility relation $R^{\mathcal{L}}_{|\varphi|}$ between worlds and subsets of worlds of $U^{\mathcal{L}}$. For all worlds $u \in U^{\mathcal{L}}$, if $\langle\varphi]\top \notin u$, then we put $R^{\mathcal{L}}_{|\varphi|} = \emptyset$. Otherwise, for every subset $X \subseteq U^{\mathcal{L}}$ and formulas φ and ψ, we say that the ordered pair $(u, X) \in R^{\mathcal{L}}_{|\varphi|}$ iff X satisfies the following two conditions:

1) for all $x \in X$, if $[\varphi]\psi \in u$, then $\psi \in x$; and
2) for every $[\varphi\rangle\psi \in u, X$ contains at least one world v with $\psi \in v$.

DEFINITION 7 (CANONICAL MRL MODEL). — *Let* $p \in$ PROP *be a proposition. Let* $V^{\mathcal{L}}(p) = |p|$ *and let* $R^{\mathcal{L}} = \{R^{\mathcal{L}}_{|\varphi|} : \varphi$ *is a formula*$\}$, *then the model* $\mathfrak{M}^{\mathcal{L}} = (U^{\mathcal{L}}, R^{\mathcal{L}}, V^{\mathcal{L}})$ *is the* canonical minimal relational model.

The completeness of the proof system in section 3.2 follows from a standard truth-lemma:

LEMMA 8. — *For all* $u \in U^{\mathcal{L}}$ *and for all formulas* θ, $\theta \in u$ *iff* $\mathfrak{M}, u \models \theta$.

---

[3] There is no corresponding K axiom for the modality $[\varphi\rangle$. Consider a model M such that the set $X \subseteq U$ is the only subset of U that is φ-related to the world $u \in U$, i.e, $R_{|\varphi|} = \{(u, X)\}$. Suppose that both $|\psi| \cap X \neq \emptyset$ and $|\neg\psi| \cap X \neq \emptyset$, but that $|\theta| \cap X = \emptyset$. Then $u \models [\varphi\rangle(\psi \to \theta)$ (since $|\neg\psi| \cap X \neq \emptyset$) and $u \models [\varphi\rangle\psi$, but $u \not\models [\varphi\rangle\theta$. Hence, $[\varphi\rangle(\psi \to \theta) \to ([\varphi\rangle\psi \to [\varphi\rangle\theta)$ is not valid.

[4] We read $\langle\varphi]\top$ as "there is a revision by φ".

We will give the proof of the truth-lemma once we have stated and proved the following crucial lemmas.

LEMMA 9. — *If* $\langle\varphi]\alpha \in u$*, then there exists a subset* $X \subseteq U^{\mathcal{L}}$ *such that* $(u, X) \in R^{\mathcal{L}}_{|\varphi|}$*, and for every world* $x \in X$*,* $\alpha \in x$*.*

PROOF. — Let $[\varphi\rangle\theta \in u$, and let

$$v^{-} = \{\beta : [\varphi]\beta \in u\} \cup \{\theta\} \cup \{\alpha\}.$$

Suppose that v^{-} is not consistent, then there exists $\delta_1, ..., \delta_n \in v^{-}$ such that $\vdash (\bigwedge \delta_i \wedge \alpha) \to \neg\theta$. For every $1 \leq i \leq n$,

$$\begin{array}{llll}
\delta_i \in v^{-} & \Rightarrow & [\varphi]\delta_i \in u & \text{(Definition of } v^{-}\text{)} \\
 & \Rightarrow & \bigwedge[\varphi]\delta_i \in u & \text{(Truth definition)} \\
 & \Rightarrow & [\varphi]\bigwedge \delta_i \in u & \text{(Axiom 4)} \\
 & \Rightarrow & ([\varphi]\bigwedge \delta_i \wedge \langle\varphi]\alpha) \in u & \text{(since } \langle\varphi]\alpha \in u\text{)} \\
 & \Rightarrow & \langle\varphi](\bigwedge \delta_i \wedge \alpha) \in u & \text{(axiom 6)} \\
 & \Rightarrow & \langle\varphi]\neg\theta \in u & \text{(by the monotonicity axiom 5)} \\
 & \Rightarrow & \neg[\varphi\rangle\theta \in u & \text{(axiom 3)}
\end{array}$$

and this is a contradiction, since $[\varphi\rangle\theta \in u$ by assumption. Therefore, v^{-} is consistent. Let v be a maximal extension of v^{-}.

For every θ_i such that $[\varphi\rangle\theta_i \in u$, let w_i be obtained from the above construction, and let

$$X = \{w_i : [\varphi\rangle\theta_i \in u, \theta_i \in w_i\}.$$

Then X satisfies conditions 1) and 2) and for every $x \in X$, $\alpha \in x$. ■

COROLLARY 10 (COROLLARY TO THE PROOF OF LEMMA 9). — *If* $[\varphi\rangle\psi \in u$*, then the set* $w = \{\psi\} \cup \{\theta : [\varphi]\theta \in u\}$ *is consistent.*

LEMMA 11. — *If* $\langle\varphi\rangle\psi \in u$*, then there exists a subset* $X \subseteq U^{\mathcal{L}}$ *such that* $(u, X) \in R^{\mathcal{L}}_{|\varphi|}$*, and there exists a world* $x \in X$ *such that* $\psi \in x$*.*

PROOF. — Assume $\langle\varphi\rangle\psi \in u$. Then there is a maximal consistent set v such that $\psi \in v$. The proof that v exists is standard (see (Blackburn *et al.*, 2001), Lemma 4.20).

By corollary 10, for every formula α_i, if $[\varphi\rangle\alpha_i \in u$, then the set $w_i^{-} = \{\alpha_i\} \cup \{\theta : [\varphi]\theta\}$ is consistent. By Lindenbaum's lemma, there exists a maximal consistent set w_i extending w_i^{-} such that $\alpha_i \in w_i$. Let $W = \{w_i : [\varphi\rangle\alpha_i \in u\}$

Finally, let $X = \{v\} \cup W$. It is not difficult to check that $(u, X) \in R^{\mathcal{L}}_{|\varphi|}$, and $\psi \in v$. ■

We are now ready for the proof of the truth-lemma.

PROOF (LEMMA 8). — Thanks to axioms 5 and 7, if $\langle\varphi]\top \notin u$, then $[\varphi\rangle\psi \in u$ and $[\varphi]\psi \in u$ for all ψ. Since $R^{\mathcal{L}}_{|\varphi|} = \emptyset$ when $\langle\varphi]\top \notin u$, the lemma is trivially satisfied. Thus, we assume for the remainder of the proof that $\langle\varphi]\top \in u$. The proof now proceeds by induction on the complexity of θ. The interesting cases are when $\theta = [\varphi]\psi$ or $\theta = [\varphi\rangle\psi$. The first direction ($\theta \in u \Rightarrow \mathfrak{M}, u \vDash \theta$) follows from the conditions imposed on $R^{\mathcal{L}}_{|\varphi|}$. We prove that $\mathfrak{M}, u \vDash \theta \Rightarrow \theta \in u$.

Suppose $[\varphi]\psi \notin u$. Since u is a maximal consistent set of formulas, $\neg[\varphi]\psi \in u$. By axiom 2, this implies that $\langle\varphi\rangle\neg\psi \in u$. By lemma 11, there exists a subset $X \subseteq U^{\mathcal{L}}$ such that $(u, X) \in R^{\mathcal{L}}_{|\varphi|}$ and a world $x \in X$ such that $\mathfrak{M}, x \vDash \neg\psi$. Hence, by the truth-definition $\mathfrak{M}, u \vDash \langle\varphi\rangle\neg\psi$, i.e., $\mathfrak{M}, u \vDash \neg[\varphi]\psi$. Therefore, $\mathfrak{M}, u \nvDash [\varphi]\psi$.

Finally, suppose that $[\varphi\rangle\psi \notin u$, then $\neg[\varphi\rangle\psi \in u$. Hence, $\langle\varphi]\neg\psi \in u$ (axiom 3). By lemma 9, there exists a subset $X \subseteq U^{\mathcal{L}}$ such that $(u, X) \in R^{\mathcal{L}}_{|\varphi|}$ and for every world $x \in X$, $\neg\psi \in x$. By the inductive hypothesis, for every $x \in X$, $\mathfrak{M}, x \vDash \neg\psi$. Therefore, by the truth-definition, $\mathfrak{M}, u \nvDash [\varphi\rangle\psi$. ■

4. Generalized selection functions

As we mentioned above, a way to get broccoli logic out of minimal relational logic is to accommodate onion selection functions to the latter. A generalized selection function is a function from $\mathcal{P}(W)$ to $\mathcal{P}(\mathcal{P}(W))$ that takes propositions to sets of propositions. In this section, we show what properties generalized selection function should satisfy in broccoli models, and we point to a difficulty of the generalization, namely to find an appropriate arrow condition for broccoli models. We start with onion selection functions.

4.1. *Onions and selection functions*

DEFINITION 12 (SELECTION FUNCTIONS). — *A function* $f : \mathcal{P}(U) \to \mathcal{P}(U)$ *is a* selection function *if it satisfies the following conditions, where* $X, Y \subseteq U$ *:*

$$\begin{array}{ll} f(X) \subseteq X & \textit{(INC)} \\ X \subseteq Y \Rightarrow (f(X) \neq \emptyset \Rightarrow f(Y) \neq \emptyset) & \textit{(MON)} \\ X \subseteq Y \Rightarrow (X \cap f(Y) \neq \emptyset \Rightarrow f(X) = X \cap f(Y)) & \textit{(ARR)} \end{array}$$

The third condition is called the *arrow condition* and is the source of the major difficulties in the original development of broccoli semantics.

Let U be a finite set and let F be a selection function on U. Let

$$\begin{array}{lcl} S_0 & = & F(U) \\ S_{n+1} & = & S_n \cup F(U - S_n) \end{array}$$

Since U is finite, there is a smallest m such that $S_{m+1} = S_m$. We leave to the reader to verify that the set $\mathcal{O}_F = \{S_n : n \leq m\}$ is an onion and that $\mathcal{O}_F$ and F agree.[5] Hence, models for onions may be given in terms of selection functions.

---

[5] $\mathcal{O}_F$ and F agree iff

DEFINITION 13 (ONION SELECTION MODELS). — *Let U be a set, F a selection function on U and V a valuation on a given set of propositional variables. We say that the triple $\mathfrak{M} = (U, F, V)$ is an* onion selection model*.*

The truth-definition for the modality $[\varphi]$ in onion selection models is given by:

$$\mathfrak{M}, u \vDash [\varphi]\psi \text{ iff } F(|\varphi|) \subseteq |\psi|.$$

The complete logic for onions consists of the axioms (1), (2), and (4) of section 3.2 together with the additional axioms $(I), (M)$ and (A):

$$\begin{array}{ll} [\varphi]\varphi & (I) \\ \langle\varphi\rangle\psi \to \langle\psi\rangle\top & (M) \\ \langle\varphi\rangle\psi \to ([\varphi \wedge \psi]\theta \equiv [\varphi](\psi \to \theta)) & (A) \end{array}$$

Axioms $(I), (M)$ and (A) are obvious analogues of conditions $(INC), (MON)$ and (ARR) of definition 12. The total resulting system is almost Lewis' famous conditional logic VC, provided that we add an assumption of centrality (cf. (Lewis, 1973; Nute, 1984)).

4.2. *Broccoli and generalized selection functions*

Now, consider the issue of generalizing this format in a non-linear setting. (INC) and (MON) are easily generalized in BL to the following conditions, for all $X, Y \subseteq U$:

$$\begin{array}{ll} Y \in F(X) \Rightarrow Y \subseteq X & (\text{INC}^*) \\ Y \subseteq X \text{ and } \exists Z \in F(Y) \text{ s.t. } Z \neq \emptyset \Rightarrow \exists Z \in F(X) \text{ s.t. } Z \neq \emptyset) & (\text{MON}^*) \end{array}$$

with the identical corresponding axioms (I) and (M). On the one hand, if $\neg\langle\varphi]\top \in u$ for some world $u \in U$ (i.e. if there is no revision by φ) then $[\varphi]\varphi \in u$ by axiom 7. But if there is no revision by φ, then $F(X)$ is empty, and (INC^*) holds vacuously. On the other hand, if there is a revision by φ, then (I) and (INC^*) express the same thing, namely that members of $F(|\varphi|)$ are contained in $|\varphi|$. Similar considerations will convince the reader that (M) and (MON^*) go together.

A difficulty arises when attempting to generalize condition (ARR) in a similar fashion, as the condition seems to require linearity.[6] One way to see this is by looking at the failure of axiom (A) in broccoli models. Only one half of (A) can be kept in BL, viz. $\langle\varphi\rangle\psi \to ([\varphi \wedge \psi]\theta \to [\varphi](\psi \to \theta))$. The other half makes a crucial appeal to linearity, as may be seen from the counter-model of figure 4. Furthermore, this counter-model invalidates $\langle\varphi\rangle\psi \to ([\varphi](\psi \to \theta) \to [\varphi \wedge \psi]\theta)$ under both limit

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1) $\mathcal{O}_F \cap X \neq \emptyset \Rightarrow FX = X \cap S_k$ for some k.
2) $\mathcal{O}_F \cap X = \emptyset \Rightarrow FX = \emptyset$.

[6] The exact relationship between the Arrow condition and linearity is still an open question.

conditions. It is an open question to find an appropriate generalization of (ARR) that yields a generalized selection function for BL. This promises to be a difficult task. But instead of pursuing this enterprise further, we pause and see whether BL may not be obtained from an entirely different approach, viz. by showing that the logic already exists! This fact is the third and main contribution of this paper. We will come back to a brief discussion of selection functions in section 5.4.

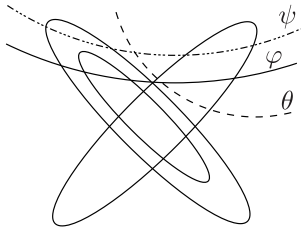

Figure 4. *Counter-model to* $\langle\varphi\rangle\psi \rightarrow ([\varphi](\psi \rightarrow \theta) \rightarrow [\varphi \wedge \psi]\theta)$

5. Broccoli logic and minimal conditional logic

Minimal conditional logic (MCL) was studied by Stalnaker, Pollock, Burgess and Veltman to capture the idea that a conditional $\varphi \Rightarrow \psi$ is true if an only if the conjunction $\varphi \wedge \neg\psi$ is less possible than the conjunction $\varphi \wedge \psi$, and no more. Their modeling comes with a reflexive and transitive $\leq$-order for each world x and no spheres need occur. In a sphere system, two worlds lying on the same sphere agree on which worlds are farther away and which are closer. This assumption is dropped in MCL. Hence, if two worlds x and y are equally far away in the underlying order from world u and if the world z is farther away than the world y, no conclusions may be drawn as to whether world z is farther from u than world x, or vice versa. Instead of changing the onion picture by allowing non-linearly ordered sphere system as we wish to do in BL, MCL ignores spheres altogether. It has been a difficult task to find completeness for MCL, and we refer the reader to Burgess (Burgess, 1981) for a detailed proof. In this section, we show how to avoid similar completeness difficulties with BL by showing that it is actually equivalent to MCL.

Section 5.1 provides the MCL semantics, section 5.2 gives the complete proof system and section 5.3 shows the equivalence of MCL and BL. In section 5.4, we come back to a discussion of selection functions for MCL.

5.1. *Minimal conditional logic*

A *Minimal conditional logic model* is a triple (U, R^3, V), where U and V are as above, and R^3 is a ternary relation on U that respects reflexivity and transitivity in the second and third arguments (cf. (Burgess, 1981)). The relation $Rxyz$ should be read as "according to world x, world y is no farther away than world z". We shall write the more suggestive $y \leq_x z$ instead of $Rxyz$. We let $W_u = \{y : \exists z, y \leq_u z\}$ be the *zone of entertainability* for world $u \in U$. Intuitively, worlds outside the zone of entertainability for u are worlds so far away that their distance from any given world is not evaluable. The *minimal conditional logic language* contains a set of propositional variables, together with negation $\neg$, disjunction $\vee$ and a counterfactual modality $[\varphi]$ for every formula φ.

DEFINITION 14 (MCL SEMANTICS). — *We say that the formula $[\varphi]\psi$ is true at world u in the model $\mathfrak{M}$, and we write $\mathfrak{M}, u \vDash [\varphi]\psi$, iff:*

$$\forall y \in W_u \cap V(\varphi), \exists z \in W_u \cap V(\varphi)[z \leq_u y \& \forall w \in W_u \cap V(\varphi)(w \leq_u z \rightarrow w \in V(\psi)]$$

Notice that the semantic definition of $[\varphi]\psi$ does not contain a minimality condition. However, if the model is finite and $\mathfrak{M}, u \vDash [\varphi]\psi$, then there is a minimal world $z \in U$ such that $z \in (V(\varphi) \cap V(\psi))$. Since we will only use finite models for our equivalence result, we use the minimality formulation in evaluating $[\varphi]\psi$ for the remainder of this paper. The semantic condition reduces to:

$$\forall y \in W_u \cap V(\varphi), \exists z \in W_u \cap V(\varphi)[z \leq_u y \& \forall w <_u z, w \notin V(\varphi) \& z \in V(\psi)]. \quad (1)$$

Figure 5 depicts a simple model satisfying $[\varphi]\psi$. There are two minimal φ-worlds, z and z', and ψ is true at both worlds. Hence, ψ is true at every minimal φ-world. We turn to the proof system of MCL.

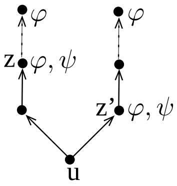

Figure 5. *Simple model such that $[\varphi]\psi$ is true at world u. The dotted arrows stand for sequences of $\leq$-related worlds*

5.2. *Proof system*

The following set of axioms, with the same set of rules as for minimal relational logic presented in section 3.2, is complete for MCL (cf. (Burgess, 1981)):

1) Classical tautologies

2) $[\varphi]\varphi$
3) $[\varphi]\psi \wedge [\varphi]\theta \rightarrow [\varphi](\psi \wedge \theta)$
4) $[\varphi](\psi \wedge \theta) \rightarrow [\varphi]\psi$
5) $[\varphi]\psi \wedge [\varphi]\theta \rightarrow [\varphi \wedge \psi]\theta$
6) $[\varphi]\psi \wedge [\theta]\psi \rightarrow [\varphi \vee \theta]\psi$

We give some examples of derivable theses.

EXAMPLE 15. — $MCL \vdash [\varphi]\psi \wedge [\varphi \wedge \psi]\theta \rightarrow [\varphi]\theta$ □

PROOF. — Assume 1) $\vdash [\varphi]\psi$ and 2) $\vdash [\varphi \wedge \psi]\theta$. By axiom (2), $\vdash [\varphi \wedge \neg\psi](\varphi \wedge \neg\psi)$ and by axiom (4), $\vdash [\varphi \wedge \neg\psi]\neg\psi$. Hence, by monotonicity in the consequent (axiom (4) again), $\vdash [\varphi \wedge \neg\psi](\neg\psi \vee \theta)$. Now, from assumption 2) and axiom (4), $\vdash [\varphi \wedge \psi](\neg\psi \vee \theta)$. Combining the latter two results, we get that $\vdash [\varphi](\neg\psi \vee \theta)$. But since $\vdash [\varphi]\psi$ by assumption (1), we get that $\vdash [\varphi]\theta$, as desired. ■

EXAMPLE 16. — $MCL \vdash \langle\varphi\rangle\psi \rightarrow \langle\psi\rangle\top$ □

PROOF. — We prove the contrapositive. Assume that $\vdash [\psi]\bot$. Then both $\vdash [\psi]\neg\psi$ and $\vdash [\psi]\varphi$. Hence, by axiom (5), $\vdash [\psi \wedge \varphi]\neg\psi$. But $\vdash [\neg\psi \wedge \varphi](\neg\psi \wedge \varphi)$ is an instance of axiom (2) and by axiom (4), $\vdash [\neg\psi \wedge \varphi]\neg\psi$. Therefore, $\vdash [\varphi]\neg\psi$. ■

EXAMPLE 17. — $MCL \vdash [\varphi \wedge \psi]\theta \rightarrow [\varphi](\psi \rightarrow \theta)$. □

PROOF. — Assume $\vdash [\varphi \wedge \psi]\theta$. By monotonicity, $\vdash [\varphi \wedge \psi](\neg\psi \vee \theta)$. But $\vdash [\varphi \wedge \neg\psi](\neg\psi \vee \theta)$. Therefore, $\vdash [\varphi](\neg\psi \vee \theta)$, i.e., $\vdash [\varphi](\psi \rightarrow \theta)$. ■

As we can see from axiom (2) and examples 16 and 17, conditions (I), (M) and one direction of A of section 4 are derivable in MCL. We see at once that MCL has the properties we were looking for in BL, and we will now show that it has *all* the properties of BL. The general reason behind these considerations becomes clear in the next subsection.

5.3. *Minimal conditional logic is broccoli logic*

Let $\mathfrak{M} = (U, R, V)$ be a finite MCL model. We will transform this model into a broccoli model, by constructing a broccoli flower at each world of $\mathfrak{M}$, taking the downward closed sets of worlds according to the underlying order (see figure 6).

More precisely, let $C_x(y) = \{z \in U : z \leq_x y\}$, then $BROC(x) = \{C_x(y) : y \in W_x\}$ is the induced broccoli flower at world x. In particular, since $\mathfrak{M}$ is finite, we are guaranteed the existence of minimal sets of worlds, as was noted in semantic condition 1. We thus obtain a limit condition for free, and we can then talk about broccoli models. An induced broccoli model $BROC(\mathfrak{M})$ is given by:

$$BROC(\mathfrak{M}) = \{BROC(x) : x \in U\}$$

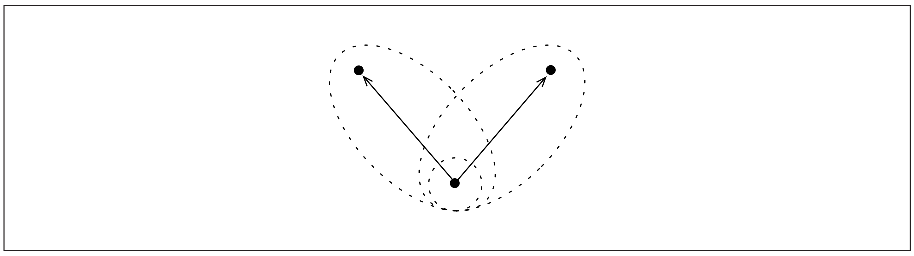

Figure 6. *Induced broccoli model (dotted lines) from and MCL model (arrows)*

The semantics of $[\varphi]\psi$ in the induced broccoli model is given by the following:

$$BROC(\mathfrak{M}), x \vDash [\varphi]\psi \text{ iff } \forall Z\mu(BROC(x) \bullet |\varphi|)(Z \cap |\varphi| \subseteq |\psi|).$$

The main result of this section now follows from lemma 18.

LEMMA 18. — $\mathfrak{M}, x \vDash [\varphi]\psi$ *iff* $BROC(\mathfrak{M}), x \vDash [\varphi]\psi$.

PROOF. — In the one direction, assume that $\mathfrak{M}, x \vDash [\varphi]\psi$. To ease the notation, we write C_w instead of $C_x(w)$. Let $C_w\mu(BROC(x) \bullet |\varphi|)$, and let $v \in C_w \cap |\varphi|$. By the truth definition for $[\varphi]\psi, \exists z \leq_x v$ such that $\mathfrak{M}, z \vDash \varphi \wedge \psi$ and $\forall y <_x z, \mathfrak{M}, y \nvDash \varphi$. But z must be equal to v. Otherwise, $C_z \subset C_v \subseteq C_w$ (the latter inclusion uses the transitivity of $\leq_x$), which implies that C_z would be a proper subset of C_w intersecting $|\varphi|$, contradicting our assumption. Thus, $v \in |\psi|$, which implies that $C_w \cap |\varphi| \subseteq |\psi|$. Therefore, since v was chosen arbitrarily, $BROC(\mathfrak{M}), x \vDash [\varphi]\psi$.

In the other direction, assume that $BROC(\mathfrak{M}), x \vDash [\varphi]\psi$ and suppose that $\mathfrak{M}, y \vDash \varphi$ for some $y \in U$. Then $C_y \cap |\varphi| \neq \emptyset$. Hence, $\exists C_w \subseteq C_y$ such that $C_w\mu(BROC(x) \bullet |\varphi|)$ (by the limit condition!) and $C_w \cap |\varphi| \subseteq |\psi|$. But since $C_w \subseteq C_y, w \leq_x y$. Assume that w is not a minimal world satisfying $\varphi \wedge \psi$ with respect to $\leq_x$, then $\exists w' <_x w$ such that $\mathfrak{M}, w' \vDash \varphi \wedge \psi$. This implies that $C'_w \subset C_w$ and $C'_w \cap |\varphi| \cap |\psi| \neq \emptyset$, contradicting the minimality of C_w. Therefore, w is a minimal world satisfying $\varphi \wedge \psi$ and since $w \leq_x y$, we get that $\mathfrak{M}, x \vDash [\varphi]\psi$. ■

We are now ready for our main theorem.

THEOREM 19. — BL = MCL.

PROOF. — To show that MCL is BL, we need to show 1) that all axioms of section 5.1 are valid in BL, whose semantics were given in section 2 and 2) that if a principle is not derivable in MCL, then there is a broccoli countermodel.

Showing that the MCL axioms are valid in the BL-models of section 2 is straightforward. We show that axiom (5) is valid and leave the others to the reader. Let $\mathfrak{M}$ be an arbitrary broccoli model and let $u \in U$ be arbitrary. If $\neg\langle\varphi\rangle\top \notin u$, i.e., if there is no revision by φ, then the thesis is vacuously true. Hence, assume that there is a revision by φ. Assume furthermore that $\mathfrak{M}, u \vDash [\varphi]\psi \wedge [\varphi]\theta$. Since $\mathfrak{M}, u \vDash [\varphi]\psi, |\varphi| \cap |\psi| \neq \emptyset$. Let $Z\mu(\mathcal{B} \bullet |\varphi \wedge \psi|)$ be a minimal set of $\mathcal{B}$ intersecting $|\varphi \wedge \psi|$. Then for every $z \in Z, x \in |\varphi| \cap |\psi|$ implies that $z \in |\varphi| \subseteq |\theta|$. Hence, $\mathfrak{M}, u \vDash [\varphi \wedge \psi]\theta$.

To show that if a formula is not provable in MCL, then there is a broccoli countermodel to φ, we use the completeness result of Burgess. If $MCL \not\vdash \varphi$ for some φ, then there is a finite model $\mathfrak{M} = (U, R, V)$ and a world $u \in U$ such that $\mathfrak{M}, u \not\models \varphi$. [7] By lemma 18, $BROC(\mathfrak{M}), u \not\models \varphi$. Therefore, $BROC(\mathfrak{M})$ is a broccoli countermodel to φ. This completes the proof of theorem 19. ■

COROLLARY 20. — *BL is decidable.*

5.4. *Coming back to selection functions*

Knowing that broccoli logic is in fact minimal conditional logic, we may ask what is an appropriate selection function for this latter logic. A conjecture made by Horacio Arló-Costa in private communication is to use Chellas' definition of a choice function f as a function from worlds and propositions to sets of propositions (cf. (Chellas, 1980), p. 270). The semantics for the modality $[\varphi]\psi$ is then given by:

$$\models_w^{\mathcal{M}} [\varphi]\psi \text{ iff } |\psi|^{\mathcal{M}} \in f(w, |\varphi|^{\mathcal{M}}). \tag{2}$$

Notice first that 2 does not hold without an extra assumption of monotonicity on the choice function. Consider a model with 3 worlds w_1, w_2, w_3 such that $f(w_1, |\varphi|^{\mathcal{M}}) = \{\{w_2\}\}$, and assume that $V(\psi) = \{w_2, w_3\}$. Then ψ is true in every minimal φ-world returned by the choice function, but $|\psi|^{\mathcal{M}} \notin f(w, |\varphi|^{\mathcal{M}})$. This simple example shows that Chellas' definition of choice functions requires an extra assumption of monotonicity in order to make sure that definition 2 indeed provides a semantics for a conditional operator. One solution is to close the image of the choice function under supersets, and another solution is to change definition 2 to:

$$\models_w^{\mathcal{M}} [\varphi]\psi \text{ iff } \exists Z \in f(w, |\varphi|^{\mathcal{M}}) \text{ such that } Z \subseteq |\psi|^{\mathcal{M}}. \tag{3}$$

In the case of MCL, Arló-Costa's proposal is to impose the following conditions on the selection function:

1. $X \in f(w, X)$
2. $X \in f(w, Y) \wedge Y \in f(w, Z) \Rightarrow X \in f(w, Y \cup Z)$
3. $X \in f(w, Y) \wedge Y \in f(w, X) \Rightarrow Z \in f(w, X)$ iff $Z \in f(w, Y)$.
4. $Y \cap Y' \in f(w, X)$ iff $Y \in f(w, X) \wedge Y' \in f(w, X)$

A quick check shows that this is indeed a choice function for MCL models. Condition 1 corresponds to axiom 2, condition 2 to axiom 6, condition 4 to axiom 3, and finally condition 3 is derivable using axioms 4 and 5. The fourth condition provides the required monotonicity and guarantees that examples like the one presented above do not occur.

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[7] Burgess proves that MCL has the finite model property.

It is still an open question what happens with the Arrow condition in broccoli logic. Arló-Costa's choice function is an appropriate selection function for MCL, and thus for BL, but the arrow condition has been lost in the process, along with linearity. We leave for further research the study of this relationship.[8]

6. Conclusion

Our goal was to generalize onion semantics to capture relational belief revision; the result was BL. It turns out that BL is equivalent to a well-known conditional logic, the Burgess-Veltman minimal conditional logic. This is a fortunate outcome, as it saves a lot of work in coming up with a completeness result expanding on the minimal revisional logic of section 3. The major difficulty along those lines was to devise an appropriate generalized arrow condition yielding generalized selection functions, and this is still an open question. Another open question is the role played by the $[\varphi\rangle$ modality in BL: what is the complete minimal logic of $[\varphi]\psi$ and $[\varphi\rangle\psi$ over the Burgess-Veltman models? An advantage of MCL over BL is that it avoids the problem of choosing an appropriate generalized Arrow condition by dropping the sphere representation altogether. A lesson should be drawn here, namely that, as so often over the past years, we see that logics of belief revision are largely conditional logics.

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[8] For further discussion of generalized selection functions, see Arló-Costa (Arló-Costa, n.d.).

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