ABSTRACT SET THEORY

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To the memory of

CHAIM WEIZMANN

(1874—1952)

the scientist

the first President of Israel

the friend

PREFACE

A treatise dealing with the theory of *abstract sets* in English seems to have been a desideratum for decades, since the "Theory of Sets of Points" by W. H. and G. C. Young (1906) became out of print and out of date. This book was the first textbook on set theory in any language, and includes the elements of abstract sets. No other treatise in English has appeared, except the short and concise "Lecture Notes" of Littlewood (1925) which demand an oral supplement, and the textbooks in German [1]) and French (Sierpiński) have not been translated. On the other hand, there are plenty of American and English books on the theory of *sets of points* (some of them quoted on p. 239 of the present book), but they contain only brief introductory chapters on abstract sets.

The nature of the present book differs considerably from that of the continental books. While its main purpose is to give a quite elementary — in the beginning rather broad — exposition of the classical material and of some important additions, it also pays attention to the foundations of the theory and to matters of principle in general, including points of logical or general philosophical interest. A more profound treatment of the foundations of set theory and of the contiguous fields of mathematics, including the progress of research during the last thirty years, will be given in a forthcoming book *Foundations of Set Theory*. The present book, however, is chiefly intended for undergraduates in mathematics, graduate students in philosophy, and high school teachers. No preliminary training is required, and the exposition given in the book, progressing from easy arguments to abstract demonstrations, will enable the reader to understand rather profound or technical developments. The method adopted is based on the author's belief that students, before looking over a proof, should

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[1]) Hausdorff's books *Grundzüge der Mengenlehre* (1914) and *Mengenlehre* (1927 and 1935; reprinted at Dover, New York, 1944), which deal with abstract sets as well as with sets of points, are still the outstanding treatises of set theory. After the present book was completed, Kamke's *Mengenlehre* appeared in an English translation.

realize what is to be proved, why a proof is necessary, and what is the main difficulty to be overcome.

In its external arrangement the book follows the first half of the author's German treatise [1]). Intrinsically, however, it differs not only through the *Principles* introduced as the basis of the theory, and in the extent of the literature cited, but also with respect to subject-matter and proofs in many sections. Among subjects not treated in other textbooks, attention may be called, for example, to the definition by transfinite induction (§ 10, 2), and to the direct proof of general comparability by means of the axiom of choice (§ 11, 1).

In view of the antinomies of set theory, it seems to me inadequate to develop the theory in a purely naive ("genetic") way along Cantor's own lines, as do all textbooks known to me. On the other hand, an axiomatic exposition, with or without the use of symbolic logic, would not meet the needs of beginners. Therefore, a middle course has been adopted; certain principles, similar to Zermelo's axioms, are introduced and at the important turns of the exposition it is pointed out how the constructions required can be based on those principles, while elsewhere the development proceeds without explicit reference to the principles. However, these principles do not precede the exposition in a dogmatic way but are inserted by degrees wherever they are required. The axiom of choice, in particular, explicitly appears not when it is first used (in § 2) but only in § 6, when the reader has got accustomed to abstract arguments; then he is referred to the earlier subjects where the axiom has been applied implicitly.

For three decades, the author has been making great efforts to collect the literature on the topics treated both in the present book and in *Foundations of Set Theory* [2]). In many cases, completeness in quoting has been attempted; wherever it seemed useful, the references have been arranged in chronological order, with the more important essays distinguished from the others. The bibli-

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[1]) "Einleitung in die Mengenlehre" (1919, Springer, Berlin; 3rd edition 1928; reprinted at Dover, New York, 1946).

[2]) The main topics of the latter book are: Antinomies of the transfinite, axiomatic methods of basing set theory, logistic attitudes from *Principia Mathematica* to the present day, intuitionism and neo-intuitionism, axiomatics in general and metamathematics.

ography compiled to this purpose, essentially differs from other bibliographies, such as the admirable one by Church (in volumes I and III of the *Journal of Symbolic Logic*) — not only with respect to the subjects covered, but chiefly in that *each item is quoted at one or more places for its connection with a definite subject.* Regarding the selection of material, up to 1918 only works of some importance are mentioned. (This limit has been chosen because of the break caused by World War I, and also in view of the very extensive bibliography contained in C. I. Lewis' *A Survey of Symbolic Logic* published in that year.) From 1919 on, however, the list is intended to be comprehensive with regard to most subjects. The present book, whose publication was delayed by the Israel war of independence and other circumstances, was completed by the end of 1947, and the bibliography therefore was originally extended only to 1946. The material referring to the present book, however, has been supplemented to reach up to 1950, while for the purpose of *Foundations of Set Theory* a supplement to the bibliography will be added.

The literature quoted in footnotes to various subjects will enable the interested reader to find further information of a more special nature. But primarily — and this aim alone can justify the author's pains as well as the printing space dedicated to the bibliography — *they are intended to provide mathematicians and philosophers, concerned with research on a certain topic, with a comprehensive survey of whatever has been published on the topic in the last thirty years,* thus enabling them to find a suitable starting point for further investigations.

Grateful acknowledgment is due to those who helped me in proof-reading and in improving the rather poor English in which the book was originally written, notably Miss A. Berenblum (Hebrew Univ.) and Messrs. K. Bing (Harvard), P. Gilmore (Univ. of Amsterdam), G. Prins (Univ. of Michigan), M. Rabin (Hebrew Univ.). I wish also to thank Mr. M. D. Frank, Managing Director of the North-Holland Publishing Company, for the understanding and assistance given during an extended printing period.

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INTRODUCTION

"I protest ..... against using infinite magnitude as something consummated; such a use is never admissible in mathematics. The infinite is only a *façon de parler*: one has in mind limits which certain ratios approach as closely as is desirable, while other ratios may increase indefinitely" [1]. C. F. Gauss, presumably the foremost mathematician of the 19th century, uttered this remark in answer to a certain idea of Schumacher's and, in doing so, expressed a *horror infiniti* (horror of the infinite) that was, until sixty years ago, the common attitude of mathematicians. The reference to the authority of Gauss seemed to render that view unassailable. Accordingly, mathematics would have to deal only with *finite magnitudes* and finite numbers; infinitely large and infinitely small magnitudes might find some place in philosophy, relying on more or less clear definitions; but in mathematics, these notions must not appear.

The mathematician Georg Cantor [2], who died during World War I, ventured to fight this attitude and actually succeeded in refuting it; he procured full legitimacy in the realm of mathe-

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[1] "Briefwechsel Gauss–Schumacher", vol. II (1860), p. 269; Gauss' "Werke", vol. VIII (1900), p. 216.

[2] Besides the reminiscences and letters contained in Schoenflies **11** and **12** and in Cantor–Dedekind **1**, see the biography Fraenkel **14** and the edition of Cantor's Collected Papers (Cantor **15**, to which one should add the interesting letter contained in Ternus 2); cf. also Bell **7**.

For all the references to books and essays, marked by the author's name and a number in bold face, consult the bibliography attached to this book. These references are intended to fulfil the advanced reader's possible desire for additional information and special investigation. The text itself contains sufficient material to supply the needs of the average reader.

The present book is to be continued by another: *Foundations of Set Theory*, due to appear about 1955. All problems concerning the foundations of the theory will be treated more rigorously in this book. We shall occasionally refer to it as "*Foundations*"; the references are, however, meant only for those especially interested in the foundations of mathematics or in the philosophical aspect of problems.

matics for the idea of infinitely large magnitude. In addition to the creative intuition and the artistic power of production [1]) that guided Cantor in his achievements, an extraordinary energy and perseverance were necessary to compel the acceptance of the new ideas which, to Cantor's deep regret, were rejected by the majority of his contemporaries [2]) for a long time. They felt the ideas were obscure or false or "brought into the world a hundred years too early" [3]). Not only Gauss and other outstanding mathematicians were quoted in evidence against the concept of the infinite; Cantor had also to defend himself against the *philosophical* authorities produced by his opponents: against Aristotle, Descartes, Spinoza, Leibniz and other ancient or modern logicians [4]). Further-

[1]) Cf. the motto preceding his thesis **1** of admission as a lecturer at the University of Halle: *Eodem modo literis atque arte animos delectari posse.*

[2]) Especially by Kronecker. On the other hand, two of the leading mathematicians of his time, Weierstrass and Hermite — the latter contrary to a wide-spread opinion — soon overcame their initial distrust of Cantor's work and turned it into appreciation and admiration. Mittag-Leffler, even earlier, gave him active support by inducing several mathematicians (among them Poincaré) to translate Cantor's essays into French (cf. Mittag-Leffler 3); these translations, published in vol. 2 of *Acta Mathematica,* have contributed much to the propagation of Cantor's ideas.

The first *applications* (1883/4) of the new ideas to the theory of functions and to geometry were given by A. Hurwitz **1**, Mittag-Leffler **1**, Poincaré **1**, Scheeffer **1** and **2**. Bendixson's essays **1**—**4** (from the same years) took a parallel course with Cantor's work.

[3]) It is reported that with this argument one of the outstanding mathematical journals, which in general had readily accepted his papers, in the eighties declined the comprehensive essay (Cantor **12**) that afterwards appeared in 1895 (according to Cantor–Staeckel **1**). Even the papers that appeared in the *Journal für die Reine und Angewandte Mathematik* in the seventies were published only after prolonged hesitation and considerable delay (cf. Schoenflies **11**, p. 99). As late as 1908, in a letter to W. H. Young, Cantor complained of the lack of appreciation given to his work in Germany (in contrast to England; cf. Young **1**, p. 422).

(On the other hand, it may be mentioned that late in the thirties the Nazis tried — in vain, of course — to prove the "Aryan" origin of Cantor, whose father had been a Jew, in order to show him off as a prototype of the "German mathematician", according to the classification in Bieberbach **1**.)

[4]) There are, however, some philosophers like Lucretius, Chasdai Crescas (affirming actual infinity) and Grégoire de Rimini who have been overlooked by Cantor. Cf. Keyser **2**, Efros **1**, Guttmann **1**, Wolfson **1**, Duhem **1**; also Bloch **1** and Bodewig **1**.

more, it was charged that his theory would violate the principles of religion, and this charge hurt him very much, since he was deeply interested in religious problems [1]). Only in the last decade of the 19th century did Cantor's ideas succeed in infiltrating into the mathematical realm [2]), at a time when he had already ceased publishing mathematical work.

The chief purpose of the present book is to prove that and to show how, in spite of the authorities of more than two thousand years who have rightly or wrongly been summoned as witnesses against the possibility of actual infinity, *it is possible to introduce into mathematics definite and distinct infinitely large numbers and to define meaningful operations between them.* In showing this, we shall make plain that possibility of *free creation* in mathematics which is not equalled in any other science. It is no accident that

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[1]) Cf. Cantor **7**, **9** and **10**, Gutberlet **1** and **2**, Ternus **1**. There is no doubt that the concept of infinity has its origin in religious thinking and, in the occidental culture, was introduced into science only through the Greeks. On the other hand, the ways of argument in many essays of scholastic philosophy and theology, not only those treating the problem of infinity, are, in their delicacy and audacity, similar to some trains of thought in the abstract theory of sets, a theory that, like scholasticism, favors purely logical procedures rather than existential questions (cf. Bush **1** and Isenkrahe **1**, also Bocheński **1** and **3**). Apparently, it is not a mere accident that Cantor — and still more his predecessor Bolzano — having a good deal of scholastic training, did not share the usual underestimation of scholastic philosophy (cf. also F. Klein **2**, pp. 52 and 56).

For the historical side of the problem of actual infinity in general, see B. Russell **7** (sixth lecture), Keyser **1** (chapter VIII), Antweiler **1** (sometimes incorrect), Bouligand **5**, Dantzig **1**, Garofalo **1**, Mondolfo **1**, A. Reymond **1**, Tarozzi **1**, Weyl **9** and **11**; for the prehistory of the theory of sets (including the related parts of the theory of functions) see Jourdain **2**; for a description of the present state of affairs see Gentzen **5**. As to the widespread view, shared also by Cantor, that the recognition of distinct, infinitely-large magnitudes would contradict Kant's system, let us point out that Kant may also be interpreted in the contrary sense; cf. Natorp **1** (chapter IV, § 2).

Cavaillès **3** surveys the rise of the theory of sets both from the historical and critical points of view.

[2]) Reference to Cantor's work appeared first in France, through the appendix to Couturat's thesis **1**, Borel **1** and Baire **1**; cf. Borel's essays, published in 1899 in the *Revue Philosophique* which later reappeared as note IV in Borel **2**.

at the birth of the theory of sets, there was coined the sentence: the very essence of mathematics is its freedom [1]).

How clearly Cantor realized the aim and the revolutionary character of his investigations at an early period, and how sure he was that his ideas would successfully overcome all objections, we may gather from the following sentences which open his decisive essay of 1883 (the separate book edition begins with a foreword of touching modesty):

> The previous exposition of my investigations in the theory of manifolds [2]) has arrived at a point where its continuation becomes dependent upon a generalization of the concept of the real integer beyond the usual limits; a generalization taking a direction which, as far as I know, nobody has looked for hitherto.
>
> I depend to such an extent on that generalization of the concept of number that without it I should hardly be able to take freely even the smallest step forward in the theory of sets; may this serve as a justification, or, if necessary, as an apology for my introducing apparently strange ideas into my considerations. As a matter of fact, the undertaking is the generalization or continuation of the series of real integers beyond the infinite. Daring as this might appear, I can express not only the hope but the firm conviction that this generalization will, in the course of time, have to be conceived as a quite simple, suitable and natural step. At the same time, I am well aware that, by taking such a step, I am setting myself in certain opposition to wide-spread views on the infinite in mathematics and to current opinions as to the nature of number.

What, then, is this strange continuation of the series of number, and what is its legitimacy? On the other hand, has the freedom of the creative mathematician — a freedom unique in the realm of science — been sufficiently restricted in this case, as it should have been, by the postulate of consistency, of logical non-contra-

[1]) Cantor 7, V, p. 564; cf. also the preceding paragraphs of this essay.

[2]) The term *manifold* (*Mannigfaltigkeit*) is Cantor's earlier expression; later, he used the term *Menge* (*aggregate* or *set* or *class*; *ensemble* in French).

diction? It will be for the reader to answer these questions on the basis of the material given in this book [1]).

[1]) There is only a small number of books on abstract set theory, in contrast to the numerous books on *sets of points* and *real functions* (see § 9, No. 7): in English the excellent brief presentation by Littlewood 1 (too concise for the beginner), in addition to the rather antiquated book Young–Young 1; in French the short introductions of Bourbaki 1 (excellent), Fréchet 2 (both without proofs) and Eyraud 4, and the fine textbook Sierpiński 6; in German the brief exposition of Grelling 2, the short but comprehensive work Kamke 3, and the excellent and profound books Hausdorff 4 and 5, besides the older book Hessenberg 3 (whose main interest is the philosophical aspect); in Italian Natucci 1 (chapters 5—7); in Dutch Haalmeijer–Schogt 1. Schoenflies 1 and 8, Schoenflies–Baire 1 and Kamke 5 have an encyclopedic character. Among expositions which have not appeared as independent publications, one should mention those of Hessenberg 10, Felix Klein 1, Verriest 1, Vivanti 2 (this essay especially for the relations to elementary mathematics), Zariski 1 and 2.

CHAPTER I

ELEMENTS. CONCEPT OF CARDINAL NUMBER

§ 1. Concept of set. Examples of sets

Cantor has defined the concept of set as follows [1]:

Definition of set. A *set* or *aggregate* is a collection of definite, distinct objects of our intuition or of our intellect, to be conceived as a whole (unity).

The objects are called the *elements* (or *members*) of the set; the set *contains* its elements, or the elements *belong* to the set.

1. Examples of Sets. Before analyzing this definition in detail, let us consider a few examples of sets. Thus, we shall obtain some illustrative material which will facilitate the understanding of the definition [2].

a) Imagine a certain number of concrete objects. From a fruit bowl, for example, take five apples, two oranges and one banana. The collection of this fruit is a certain aggregate, and each individual fruit is an element of the aggregate. Even in this obvious example, collecting the fruit into an aggregate is an intellectual act [3].

The aggregate thus created contains eight distinct elements which can be arranged in a series with a first apple, a second apple, etc. If the special nature of the individual elements is disregarded,

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[1] Cantor 12, I, p. 481; cf. the earlier explanations in 7, III, pp. 114 ff. and 7, V, p. 587. In compound expressions, as *subset* or *set of points*, the term *set* is usually preferred to *aggregate*.

[2] These instances of sets do not constitute a part of the logical system to be constructed. Their purpose is to make the concept of set easily intelligible; therefore, we do not aim at logical rigor in explaining them. For the mathematicians among the readers, it will suffice to glance over these instances.

[3] From the first it should be said that even in the case of a single fruit or any given single object, the aggregate containing only that fruit or object may be formed. This aggregate, being an abstract concept, obviously differs logically, and therefore mathematically, from the single object. We shall return to this point later.

the aggregate forms a *scheme of order* whose content is: firstly, secondly, ..., eighthly. Finally, we may disregard not only the nature of the elements, but their order as well — as it were, throw the elements into a sack and jumble them; this done, the aggregate preserves as its essence the *number* of its elements only, viz. the number 8.

With regard to the two steps taken in the last paragraph, it is obviously immaterial that we deal with fruit: a string of eight pearls will provide the same scheme of order, as well as the number 8.

b) Instead of concrete objects we can collect abstracts. Thus we may form aggregates whose elements are certain qualities, certain laws of nature or certain triangles. In particular, we can collect *numbers*, e.g., the numbers 1, 2, 3, 4, 5, 6, 7, 8. If we compare the set containing these numbers with the set of fruits mentioned in a), we see that there is no difference between them — with or without order — except for the particular nature of their elements.

c) Let us form a much larger aggregate which nevertheless, like the aggregates considered hitherto, contains only a finite number of elements [1]). A system of 1000 types, sufficient for all the consonants and vowels in different alphabets (capitals, italics, etc.), for the numerals, the punctuation marks, etc. and for the spaces (i.e. the type used for the blank space between words or lines), can serve as the raw material for *any book*. As to the extent, let us agree that every book contains a million types; this rule includes any shorter book, since the missing types may be replaced by blank spaces. Henceforth, we understand the term *book* in this sense.

Now, consider the set of all possible books. Any book exhibits a certain distribution of the 1000 types over 1000,000 places and, obviously, there exists only a finite number of such distributions or combinations. Incidentally, it is apparent that there are $1000^{1000,000}$ possible combinations, although the number is of no importance for the following reason. The set in question contains only

[1]) The following idea, in its essence of much older origin, seems to have been carried out first in E. E. Kummer's university lectures and in K. Lasswitz' scientific novel *Traumkristalle*; cf. Hausdorff **4**, pp. 61 f.

a *finite* quantity of books, but among them there will appear[1]) all the religious and philosophical writings of the past and of the future, all poems and dramas, all knowledge discovered already or to be discovered in the future or to remain undiscovered forever, as well as all conceivable catalogues, logarithm tables, newspaper articles, declarations of love, marriage advertisements, dinner menus, railway tickets, etc. Of course, also, and chiefly, any senseless combination of letters. In short, we have a *universal library* in the fullest sense of the word, with only the quantitative restriction for a book that was given above (which is unimportant since any finite series of books is conceivable as a single book too). Be the print as small and the paper as thin as can be imagined, the space up to the farthest visible stars holds only a tiny part of our collection of books.

We may use this gigantic set to point out the unspannable abyss between the finite and the infinite. Let us assume that there is an *infinite number of stars* with inhabitants who speak, print and study mathematics, including the theory of sets. Then, it is inevitable that on an infinite number of those stars the same textbook on the theory of sets appears with the same names of author and publisher, the same year of appearance and even the same misprints. (The word *same* here means, of course, the identity of a combination of signs, no matter what meaning is attributed to them.) In fact, the universal library described above contains only a *finite* number of books in general; a fortiori a *finite* number of textbooks on the theory of sets. Accordingly, if on each star there appears only *one* textbook of the given extent, among these *infinitely many* books there must be infinitely many identical books.

d) Until now, we have considered *finite aggregates*, i.e. aggregates containing a finite number of elements only. Since the formation of an aggregate is a purely abstract act of thinking, we can drop the restriction to finite sets and form *infinite aggregates*, containing an infinite number of elements. For the present, we use the terms *finite* and *infinite* in the simple sense intelligible to every reader; in § 2, **5** they will be explained systematically.

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[1]) We should choose a certain language in order to attribute a definite meaning to any combination of letters.

It is true that instances of infinite sets can hardly be indicated as long as the elements are confined to objects of our possible sensual perceptions, as done in the examples a) and c). As a matter of fact, the recent research in physics has in increasing measure convinced us that the exploration of nature cannot lead to either infinitely large or infinitely small magnitudes. The assumption of a finite extent of the physical space, as well as the assumption of an only finite divisibility of matter and energy (so that the smallest particles of matter and energy are finite), completely harmonize with experience. It thus seems that the external world can afford us nothing but finite sets.

Therefore, in order to reach infinite sets, we have to consider the *creations of our thinking*. A simple way to do this is suggested by b). Instead of stopping at the number 8, we can continue in our mind the sequence of integers or natural numbers [1]) 1, 2, 3, ... endlessly, thus reaching *the set of all natural numbers*. When we disregard the special nature of the elements of this set, a definite scheme of order again presents itself, this time an infinite scheme. On the other hand, disregarding the serial order, we find it difficult to affirm that there remains a certain *number* as in a) and b) — as it were, the number of all integers.

The term *infinite* as used here (an *infinite number of elements*, *infinite aggregate*, etc.) is wholly different from the infinity appearing in many branches of mathematics, especially in calculus. In mathematical analysis, one often speaks of a variable which *becomes* (not *is*) infinitely large or small, and of the properties of other variables (dependent on the first variable) resulting from such a process. The meaning of this process is the following: the variable under consideration is allowed to increase beyond any finite value or to approach zero indefinitely (to become *infinitesimal*), no limit having been set on the increase or decrease. In any stage of the process, however, the variable has a certain finite value different from zero. Thus, the term *infinite* serves as a mere abbrevi-

[1]) The simplest properties of these natural numbers (i.e. positive integers), including their addition and multiplication, are well-known from the elements of arithmetic. We shall use these properties, not only in examples but sometimes also in proofs. The fundamental aspect of this procedure, that is to say, the relation between arithmetic and the theory of sets, is discussed in § 10, 6 and in *Foundations*.

ation to avoid a clumsy form of expression (cf. p. 1). For instance, the sentence: "when the integer n becomes infinitely large (increases indefinitely), the quotient $1/n$ becomes infinitely small" is simply an abbreviation for the longer, but exact expression "the value of $1/n$ can be made to approach the *limit* 0 as closely as is desired by confining the number n to sufficiently large values". In this connection one speaks of the improper or *potential infinite*, or of the infinite as a limit.

In sharp contrast to this use of the word *infinite*, the set of all natural numbers considered above (as well as its scheme of order) is a proper, definite *actual infinite*; the set contains infinitely many elements each of which is well-determined [1]). There appears to be nothing absurd or contradictory in such a concept, constructed by a simultaneous act of thinking. As a matter of fact, concepts of this kind have been explicitly or implicitly used as long as mathematics has existed as a deductive science.

e) Draw a certain segment from the left to the right (see fig. 1), bisect it and call the bisecting point P_1. Bisect the left half

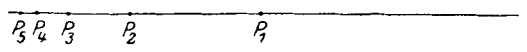

Fig. 1

and call the bisecting point P_2; and so forth, at each step bisecting the left half of the segment considered at the preceding step. The nth step (where n is any natural number) will provide a bisecting point P_n, and, on its left, a segment whose length is the 2^nth part of the original segment's length.

Let us collect all the bisecting points P_n for any value of n into a set of points. If we arrange the points in the order from right to left beginning with P_1, it is obvious that our set of points does not differ in any respect from the set of integers dealt with in d),

[1]) Since the misunderstandings on this point do not cease and seem as inexterminable as the heads of the Hydra, it may be pointed out that the actual infinity of the set of all natural numbers has nothing to do with a supposed "becoming infinite" of its elements. As a matter of fact, every natural number is finite. That this is true, becomes obvious when, instead of the set of positive integers, one uses the infinite set of all the fractions 1/2, 2/3, 3/4, 4/5, ... or the example e) given below. The basic problem of whether there can exist infinite sets, will be discussed later.

except in the nature of the elements. Even if one disregards a possible arrangement and keeps in view only the unordered collection of the points P_n, as if they were jumbled together, this collection is determined clearly and somehow intuitively as the set of all points P_n. The human intellect is luckily disposed to a simultaneous perception of collections constituted according to a general rule and may, therefore, find the concept of the infinite set containing all the infinitely many points P_n simpler than a partial set containing, say, only a billion points — although this partial set is finite [1]).

One might connect the aggregate defined just now with the well-known paradox of the *race between Achilles and the Tortoise*, due (like some similar paradoxes) to Zenon and his Eleatic school [2]). For this purpose, we interpret fig. 1 as follows: Achilles begins the race at the right-hand end of the segment, the tortoise (getting a handicap start for fairness' sake) at the point P_1, and both of them run leftwards. We shall assume that Achilles runs twice as quickly as the tortoise. Zenon's paradoxical assertion that the tortoise is winning, is founded on the following reasoning. When Achilles has covered the distance to point P_1 (the tortoise's starting point), the tortoise has advanced to P_2; while Achilles runs to P_2, the tortoise reaches P_3; and, in general, for any P_n which Achilles reaches, the tortoise is found at P_{n+1} (i.e. ahead of Achilles). The paradox runs: since the number of steps, or of segments P_nP_{n+1}, is *infinite*, Achilles will never overtake the tortoise. Now, an apparently intuitive instance of an infinite aggregate is given by the set of all the segments P_nP_{n+1} that Achilles has to cover until he can overtake the tortoise.

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[1]) To be sure, the greater simplicity of the concept of the infinite set is caused by the following fact: to determine the extent of a *large* finite set requires *many* steps, dependent upon the number of elements contained in the set. In the case of the infinite set, however, a certain *law* of uniform character determines all the elements of the set; in the present case, it is the law of mathematical induction (see § 10, 2 and 6).

[2]) In connection with this famous paradox which has had enormous influence in the history of science, see B. Russell 1, pp. 346 ff. and Hasse–Scholz 1 for the mathematical merit; Carroll 1, Dessoir–Cassirer 1 (pp. 51 ff.), Luria 1, Morris 1, Weiss 8 (pp. 232 ff.), Winn 1 (among others) for the philosophical aspect; cf. also the writings quoted by these authors. As to older literature, the treatment of Bolzano 2, III, 3rd edition, p. 490 ff. and the essay Cajori 1 (especially for the treatment of P. Tannery) should be consulted.

If one prefers a sort of would-be physical example of an infinite set to this geometrical instance, one may choose a pair of mirrors arranged in such a way that in one of them is reflected the image of the other and naturally vice versa. Thus the image of each mirror appears in itself and in the other mirror an unlimited number of times. The totality of images of any one mirror is therefore, strictly speaking, an infinite aggregate [1]).

f) In d) we dealt with integers; now let us proceed to *real numbers* in general. (As is shown in arithmetic, one can, for instance, represent any positive or negative real number as an *infinite decimal fraction*, and this representation is unique; see § 4, 1.) The set of all positive and negative real numbers, including zero, is certainly an infinite aggregate.

An aggregate closely connected with the set of real numbers may be described as follows (see fig. 2). Draw a straight line and fix

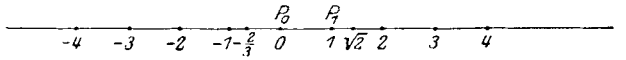

Fig. 2

in an arbitrary manner: first, a point of the line, which shall be called the *origin* (P_0 in fig. 2); secondly, one of the directions of proceeding on the line from the origin, to be called the *positive direction* (in fig. 2 the direction towards the right), while the opposite direction is called *negative*; thirdly, a *unit of measure* (length), e.g. a centimeter.

After these three arbitrary elements have been fixed, the further definition of the aggregate in question proceeds consistently. Given a positive real number a, for example $a = \sqrt{2}$, we attach to a the point of our line on the positive side of the origin in a distance a times the unit of measure. If a is negative, we choose the point having the same distance from the origin, but on its

[1]) O. Becker (cf. 2, p. 99 ff.), in the frame of the phenomenological and anthropological schools, obviously overestimates the importance of such intuitive models of infinity (in his view: of the potential infinite), constructed by means of *transfinite iteration*. The same idea, of course not in connection with a mathematical purpose, appears at a decisive place in the Dutch novel of C. and M. Scharten-Antink, "De Jeugd van Francesco Campana", p. 111. Cf. also Abita 1.

negative side, and to the number 0 we attach the origin itself. The presupposition that all the points included by this description, and no other points, constitute the totality of points on the straight line, is in conformity with a certain geometrical intuition on the one hand and with postulates of mathematical simplicity on the other. Therefore, this presupposition is generally taken in geometry [1]). Hence, to any real number a point on the line is attached, and conversely. The line the points of which are marked by the attached real numbers, is sometimes called the *line of numbers*.

According to the preceding explanations, the *set of all points on a straight line* is an infinite aggregate, closely related to the set of all real numbers (see the beginning of this example). If one arranges the real numbers according to their magnitudes (so that, e.g., -5 precedes $-3/2$, while $-3/2$ precedes 1, etc.) and also the points in fig. 2 in the succession from left to right, there remains no difference between the set of numbers and the set of points, except the nature of the elements of each set.

g) The last example of an aggregate to be given here, should be preceded by a definition belonging to algebra. A relation of the form

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n = 0$$

is called an *algebraic equation*; $a_0, a_1, \ldots, a_{n-2}, a_{n-1}, a_n$ are constants, for example real numbers, and are called the *coefficients* of the equation, while for the unknown x one requires a value that causes the left-hand side of the equation to assume the value 0. Any value x of this kind is called a *root* of the equation, and the positive integer n the *degree* of the equation if — as one obviously may assume — the first coefficient a_0 differs from 0. In the following, we more particularly assume that all coefficients are *integers*.

As an elementary theorem of algebra shows (see § 3, beginning of 4), there may exist no real root at all, or one, or several roots but, at any rate, not more than n different roots of an equation of the nth degree. For instance, the equation $x^2 - 2 = 0$ has

[1]) For a more detailed treatment of this correspondence between the real numbers and the points of a one-dimensional continuum, see § 9, 1 and 4.

the two roots $x = \sqrt{2} = 1.414\ldots$ and $x = -\sqrt{2}$; the equation $x^2 - 2x + 1 = 0$ has the root $x = 1$ and no root different from 1; the equation $x^2 + 2 = 0$ has no real root at all (since the square of any positive or negative real number x is positive, so that in any case $x^2 + 2$ becomes positive, and not $= 0$). We now define:

Any real [1]) number which is a root of an algebraic equation with integral coefficients, is called a (real) *algebraic number.*

Any real number which is *not* algebraic is called a (real) *transcendental number.*

Of course, any *rational number* (i.e. vulgar fraction) is also algebraic, since $x = p/q$ satisfies the equation $qx - p = 0$. On the other hand, not every algebraic number is rational; one cannot, for instance, represent the algebraic number $\sqrt{2}$ as a vulgar fraction [2]). Hence, the totality of rational numbers is part of the totality of algebraic numbers.

The fundamental question now arises whether perhaps *every* real number is algebraic; if so, our definition of transcendental numbers would be void of meaning, since transcendental numbers would not exist at all. This possibility has been seriously considered and it was not excluded until a hundred years ago. We shall in more detail deal with it later on (see § **4**, **5**). Meanwhile, we may consider the *set of all (real) transcendental numbers*; at any rate, this set contains only part of the numbers contained in the previous example (set of all real numbers) since, among others, all rational numbers are missing.

To be sure, a construction of this set by expressing all its elements in a general form cannot be given at the present state of science. It stands to reason that even with regard to a single real number a it might be difficult to show that a is transcendental — much more difficult than to show that a is algebraic. As a matter of fact, for the latter purpose it suffices to find some equation of which a is a root; for the former purpose, however, one has to examine *all* algebraic equations and to prove that a cannot be a

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[1]) There is no need at all to restrict the definition to real numbers; as a matter of fact, the imaginary unity $\sqrt{-1}$, too, is algebraic, since it satisfies the equation $x^2 + 1 = 0$. But, since we do not need imaginary and complex numbers in this book, it may be more convenient for the reader to content himself with real numbers.

[2]) A proof of this statement is given in § 9, **2**.

root of *any* of them. In many cases one may not succeed in finding out whether a is algebraic or transcendental. In spite of the enormous progress made in this realm during the last generation, one only knows that some limited classes of numbers [1]) are transcendental, although many other transcendental numbers do certainly exist. In § 4 we shall see how such a proof of existence may run.

Although it is impossible to "construct" the set of all transcendental numbers, the notion of this set is nevertheless admissible and consistent on account of the definition given above on p. 6. In fact, any given real number a is, according to the definition of p. 14, *either algebraic or transcendental*, even though in general we are unable to decide which case holds; according to this alternative a does not or does belong to the set of transcendental numbers.

2. On Cantor's Definition of Set. With the material of the preceding examples at our disposal we can now judge the significance and the scope of Cantor's definition. We may be inclined to consider the definition as an obvious reference to an elementary logical act that is already familiar to primitive thinking, rather than as a definition in the strict sense of the word. This inclination will increase when we see more deeply [2]) the fundamental difficulties involved in the definition. Nevertheless, and not only from the historical point of view, it is worth while and useful to scrutinize the contents of the definition given above.

It may be left to philosophical treatment to analyze the concept "object of our intuition or of our intellect". In general it suffices to admit as the elements of a set *mathematical objects only*, such as numbers, points etc., and also sets of such objects.

On account of the examples presented in **1**, it is also clear what should be understood by a "collection of objects, conceived

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[1]) The best known individual transcendental numbers are $\pi = 3.14159...$, the number used for the measurement of the circle, and $e = 2.71828...$, the basis of natural logarithms. For both of them, and especially for π, the proof of transcendence is rather complicated. In 1874, when Cantor dealt with the set of all transcendental numbers and proved a decisive property of it (see § 4, **5**), it was still unknown whether π is transcendental; the statement for e dates from the same year.

[2]) Cf. § 11, **2**; also *Foundations*, ch. I.

as a whole". The whole is the set determined [1]) by all the objects (elements). However, for logical as well as for mathematical reasons one should not imagine the act of collecting in too obvious a manner; the relation of a set to its *elements* is quite different from the relation of a whole to its *parts*. Even if the elements are concrete, the set containing them is an abstract. It would be preferable to say that one is attaching to the totality of elements, in a formal way, a new intellectual object which is said to "contain" every element and is called the "set" of them. Then there is no difficulty in attaching even to a single object a a set containing a as its only element and being (possibly, or necessarily) distinct [2]) from a — a procedure which will appear indispensable in the course of our reasoning.

The logical character [3]) of the objects called "sets" is of no importance to the mathematical theory of sets — in the same way as the results of arithmetical calculating are independent of what may be, in the view of the calculator, the logical or psychological meaning of number. Incidentally, there are weighty arguments in favor of letting the extent of the concepts *object* (*element*) and *set* coincide; that is to say, for restricting the elements of any set to sets alone, including the null-set (§ 2, 2). [4])

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[1]) Or *containing* them. One should, however, be cautious in using the term *consisting of*: a train certainly consists of carriages, but it would be fallacious to call it the set of its carriages.

[2]) The general question will be dealt with in *Foundations*. At any rate, it is evident that in most cases the set containing a single object a is different from this object; if, e.g., a is a pair of objects, a contains *two* elements, while $\{a\}$ (for this notation see p. 24) contains one element only. For a different attitude cf. Quine 21.

[3]) B. Russell (5, Ch. 17) advocates the opinion that the sets are not proper *objects* but *logical fictions* and that set-symbols are *incomplete symbols*. He himself has, however, changed his view-point in this respect.

Here, one should not understand *fictions* in the sense of *contradictory concepts* as Vaihinger does in his "Philosophy of the As-If". Russell's intention is nearer to that of J. Bentham, or to the *idea* in the sense of Kant, a concept having no corresponding substratum in the external world. It is influenced by *Occam's razor* (*entia non sunt multiplicanda praeter necessitatem*): fictions in contrast to entities. Cf. also Ogden 1 and Stebbing 1, p. 453; as to the razor, see Jourdain 8 and Hahn 7.

[4]) This limitation does not exclude numbers, points, functions etc. as elements. Cf. § 11, 2 and 5.

When one does not care what the nature of the elements may be, one speaks of an *abstract set. This book* — except for § 9 and minor remarks in other sections — *deals only with the theory of abstract sets.* To a certain extent, this theory is, of course, the basis of any special theory, where the nature of the elements is relevant and makes the introduction of new concepts, based on their specific nature, both possible and essential (see § 9, **5**). In practice, one has to deal only with the case where the elements are *points* (or, what is essentially the same, numbers). The *theory of sets of points*, however, has developed so extensively and gained such enormous importance in analysis as well as in geometry that it can no longer be considered as a specialization of abstract set theory [1]). It has become a mathematical branch of its own, with its own concepts and methods, preserving only the most general concepts of the abstract theory; in fact, the ways and purposes of the two theories diverge rather quickly at the very beginning.

It remains to analyse the terms *distinct* and *definite* appearing in Cantor's definition of sets. We shall understand the former in the following sense: with regard to any pair of objects, able to appear as elements of a certain set, it should be clear whether they are different or equal, and *any two elements of a given set are different.* In other words, a certain object may be contained in a given set, or not, but there is no possibility of its repeated appearance as in a sequence (§ 2, 3). In general one might say that any two elements of a set are homologous in relation to the set.

The attribute *definite* has the following meaning: *with respect to any object a, it should be definite whether a is an element of the given set, or not.* The fulfilment of this condition is necessary for the existence of the set. But the words "it should be definite" used here, must not be interpreted as demanding that, with regard to any object, we should actually be able to decide whether it belongs to our set: it suffices that this question should be *intrinsically settled*, i.e. be definite on account of strict definitions. This differentiation becomes immediately clear by the example g) in **1.** With the present means at the disposal of science we cannot always find out whether a given number is actually transcendental. When

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[1]) Logically preferable would be, the theory of abstract sets.

Cantor introduced the set of transcendental numbers, he did not even know whether π and e were members of the set, as to-day we are still in doubt about 2^{π} and π^{π} [1]). But on account of the logical principle of the excluded middle [2]), the definitions of *transcendental* and *algebraic* intrinsically settle the question for any given (real) number in a quite definite way. Accordingly, the set of transcendental numbers is well-defined.

After this explanation of details it will be felt that Cantor's definition of set does not supply a strong and strict basis for so general a branch of mathematics. As a matter of fact, the essential point of the definition is that a collection of different things should be considered as a unity, and this is exactly what is meant by *set or aggregate*. Thus the definition has a somewhat tautological character. As has been said before, essentially there is involved no more than a reference to one of the primitive and fundamental acts of human thinking, incapable of further analysis. To the same act — with further specializations — one refers in many fundamental concepts of other mathematical branches, as group, field, pencil, family (of lines) etc. As the only important feature one may consider the *renouncement of any restriction* regarding the act of collecting many objects into a unity.

According to the given definition — in particular, to the meaning of *definite objects* — *an aggregate is fully determined by the totality of its elements*. Let us remember that the exposition of any branch of mathematics should start with the definition of *equality* (and, accordingly, of inequality) between the mathematical objects of the branch in question. (The exposition of the theory of rational numbers, for example, begins with the definition: m/n and p/q are called *equal* if, and only if, $m \cdot q = p \cdot n$.) It will therefore be suitable to express the statement made in the beginning of this paragraph as follows:

DEFINITION OF EQUALITY (I). Two sets S and T are called *equal*

[1]) e^{π}, however, we know by now to be transcendental.

[2]) This principle plays indeed a decisive part in this context because, given a set S and a suitable object a, it guarantees that (at least) one of the relations "S contains a" and "S does not contain a" holds true, excluding a third possibility. The matter is discussed in detail in *Foundations*, ch. IV. For a remarkable observation of Dedekind's regarding his and Cantor's conception of set, as reported by F. Bernstein, see Dedekind, 3, III, p. 449.

(in symbols: $S = T$) if, and only if, they contain the same elements. Otherwise S and T are called *unequal* or *different*: $S \neq T$.

It may be appropriate to warn the reader against confusing equality with identity. Identity is a relation of a most general logical character which precedes any mathematical exposition (cf. *Foundations*, chs. II and III). On the other hand, equality has to be conceived as a relation of a *purely mathematical character*; not universal like identity, but to be defined in each case according to the needs of the branch considered — and precisely the act of *equating logically different objects* is one of the most powerful and efficient methods in mathematics.

In order to perceive that logically different aggregates may be equal according to our definition, compare the "set of the first five prime numbers" to the "set of the integers 2, 3, 5, 7, 11"; the properties used in these definitions are, as one says in logic, different but of the same extension. Another instance in which we do not even know whether the aggregates are equal or different, may be formed by means of Fermat's last theorem. Consider first the pair the elements of which are the numbers 1 and 2, secondly the set of all natural numbers n for which the equation

$$x^n + y^n = z^n$$

is solvable by natural numbers x, y, z. If Fermat's theorem is true, the set in question is equal to the pair considered before.

(In reflecting upon the difference between identity and equality, the reader may recollect the discussion between Alice and the Pigeon after the latter had accused the girl of being a serpent. In spite of her indignation, she has to confess that "little girls eat eggs quite as much as serpents do", a remark inducing the Pigeon to put all animals having long necks and eating eggs under the general denomination "serpents".)

As to the explanation frequently found in philosophical, and sometimes even in mathematical writings, "$a = b$ means that a and b denote the same object", it is obvious that in our case this is without avail. In general this explanation is of no use in mathematics.

After having stated that an aggregate is determined by the totality of its elements, we may use this remark for *denoting an*

aggregate by indicating all its elements in the form:

$$S = \{a, b, c, \ldots, g\}.$$

This is literally possible, at least theoretically, whenever the set contains a finite number of elements only. Otherwise we have to content ourselves with hinting at the missing members, e.g. by writing

$$S = \{2, 4, 6, 8, \ldots\},$$

and we shall usually do so even when the number of elements is finite but rather large. It is true that this notation of a set, both in writing or verbally, involves a certain *order* in the arrangement of the elements. But this is accessory, caused by the inability of man to pronounce or denote things *simultaneously*. In fact no order prevails among the elements of a set.

For the purposes of the present book the definition of set would prove satisfactory and only in *Foundations* shall we touch upon its shortcomings and try to replace it by another procedure. Nevertheless, even here we shall not rely on Cantor's definition.

We conclude our preliminary remarks by a definition:

DEFINITION II. Two sets containing no common elements, are called (*mutually*) *exclusive*. A set the elements of which are sets such that any two of them are mutually exclusive, is called a *disjointed set of sets* [1]).

§ 2. THE FUNDAMENTAL CONCEPTS. FINITE AND INFINITE

1. Equality. At the end of § 1 the relation of equality between sets was defined. We return to this subject for the following reason.

As has been said before, there are certain objections regarding the definition of set given in § 1. Therefore, we shall introduce a small number of *principles* referring to the notion of set and the construction of sets, most of which are stated in this section. In the systematic development of the theory of sets we shall

[1]) For an extension of this concept to cases where there are common elements but in a number (or transfinite cardinal, see § 4) *smaller than the cardinals of the respective sets* (called "almost disjointed sets"), as well as for certain applications of the generalized concept, cf. Sierpiński 7 and the papers quoted with this essay in the bibliography.

endeavor to use only sets built according to these principles, instead of relying on the definition of § 1. The main difference between these two ways lies in the restrictions the principles impose on the possible extent of admissible sets.

In *Foundations* we shall examine in detail in how far the aim of eliminating Cantor's definition has been reached by our principles, while in the following sections we shall explicitly point out the method of constructing sets on the basis of the principles only in those cases where the sets in question have a fundamental importance.

Let us begin the sequence of principles by again formulating the relation of *equality* between sets.

Principle of extensionality (I). A set is determined by the totality of its elements. In other words, two sets are equal (=) if, and only if, they contain the same elements.[1])

It should be pointed out that this concerns equality *between sets only*. Taking objects which are not sets, e.g., the numbers 1 and 2, one might rightly assert that they contain the same elements, since neither of them contains any element at all. Nevertheless, they are different objects.

2. Subsets. Let us start from the set N of all natural numbers (§ 1, **1**, example d)). N obviously comprises various parts, e.g., the pair of the numbers 1 and 2, or the set of all *even* numbers. In order to assure that, together with a certain set, its parts are also given, a definition and a principle are required.

DEFINITION I. If S and T are sets, and if every element of T is also an element of S, T is called a *subset* of S.

According to this definition, *any set is a subset of itself*. A subset of S which is different from S, is called a *proper subset*. If each of two sets is a subset of the other, they contain the same elements and therefore the sets are equal, according to the principle of extensionality. From definition I follows immediately:

THEOREM 1. If V is a subset of T, and T a subset of S, V is also a subset of S.

It is essential to distinguish between the two relations "x is an

[1]) One may ask why a mere definition should appear among the principles. The answer to this question is profound and will be given in *Foundations*, ch. II.

element of the set z" and "y is a *subset* of the set z". Much harm has been done by the confusion of these two relations and it has been Peano's and Frege's merit to free logic from this disgraceful misunderstanding. In order to avoid it, we shall use for the first relation — which, in our exposition of the subject, is chosen as the *fundamental* (*primitive*) *relation* — the expression "x is contained in z" or in symbols, $x \varepsilon z$ [1]).

For the second relation we use the expression "y is comprised, or included, in z" or in symbols, $y \subseteq z$ [2]).

If, in the case $y \subseteq z$, an explicit exclusion of equality between y and z is desired, one writes $y \subset z$ (y is a proper subset of z) [3]).

By definition I we have not yet really attained our aim to secure, together with a given set, all its subsets. For the definition *presupposes* that not only S but also T be given, while our purpose was that the existence of S should *guarantee* the existence of the subset T. Hence, we need a principle of a rather constructive character, say:

Principle of subsets (II). Given a set S and a property π meaningful for the elements of S, there exists the set containing those elements of S, and only those, which possess the property π.

According to definition I, this set is a subset of S. The definite article has been used here on purpose ("the set containing...." and not "a set containing...."), for the principle of extensionality asserts that the subset in question is uniquely determined. The same remark holds for the subsequent principles too.

The properties *odd*, *even*, *larger than 7* are meaningful for integers. Therefore, if the set of all natural numbers is given, there exists also the set of all odd positive numbers as well as the set of all even positive numbers, or the set of all integers larger than 7. On the other hand, the properties *healthy* or *eternal* are not meaningful for integers.

Suppose we want to get S again as the subset in question after having started from S. We then have to take for π a property

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[1]) ε is the initial letter of the Greek copulative *ἐστί* (*is*). In fact "x is red" corresponds to the relation "x is an element of the set of all red things".

[2]) An axiomatic foundation of this relation is contained in Foradori 1.

[3]) Using the new symbols, we may express principle I in either of the forms, $S = T$ if $x \varepsilon S$ implies $x \varepsilon T$ and vice versa — or, $S = T$ if $S \subseteq T$ and $T \subseteq S$.

which is fulfilled by *all* elements of S; for instance, the property of being an element of S. And what happens if we choose for π a property which does *not hold for any element* of S? If we do not want to state an exception — and the mathematician, in contrast to the grammarian, abhors exceptions and is in a position to avoid them as his language is created by himself — we have to speak of a set, and more precisely of a subset of S, in this case also. But the set in question contains no elements at all, and it is *determined* by this property on account of the definition of equality or of the principle of extensionality. In other words, we have to admit *one, and only one, set containing no element at all,* and being, therefore, a subset of any set on account of Definition I. Hence:

DEFINITION II. The set that contains no element at all, is called the *null-set* [1]) and is denoted by 0 [2]).

Strictly speaking, this is not only a definition, but a theorem too. The assertion contained in definition II, however, will be re-stated and proved anew in theorem 2.

In some philosophical discussions, definition II, as well as the conception of a set as a subset of itself, have been strongly criticized; the objection is that being a set and not containing any member are contradictory properties and that the whole cannot be a part. Objections of this kind are based on a misunderstanding as to the *nature of a definition* [3]). As a matter of fact, a definition is *not a factual statement* that may be true or false, but a *convention* that may be useful and convenient or not. In our case, the main convenience (other instances will soon appear) of definition II is that it releases us from stipulating an exception to principle II, according to which any meaningful property defines a certain subset of the given set. We have the same purpose in mind when we conceive a set as a subset of itself, on account of the property π of belonging to S; this is analogous to calling an integer divisible by itself and, accordingly, a divisor of itself. This convention, too, will prove useful. To ask whether it is *true* that the null-set is a set, is as absurd as the question whether in a theory of colors one may call *white* a color, or

[1]) The null-class was first used in symbolic logic. Cf. the historical sketch Cipolla 5. To the theory of sets it has been introduced only at the beginning of this century by Russell, Zermelo and others. J. W. Miller 2 has hardly succeeded in redeeming *traditional* logic without the null-class.

[2]) In general, there is no danger of confusing the set 0 with the number 0.

[3]) Many of these remarks originate from the school called *Philosophy of the As-If*. Cf. the refutations Study 1 (2nd ed.) and Betsch 1. Other objections, as P. A. Carmichael 1, are based on the confusion just mentioned of the relation of being an element with the relation of being a subset of a set.

whether in a statistical investigation boys are to be included under the heading "men". This naturally depends on the purpose one has in mind, which makes such a convention either useful or not.

Let us take as example the set $\{1, 2, 3\}$. The subsets evidently are, provided they exist as sets (see below):

$$0,\ \{1\},\ \{2\},\ \{3\},\ \{2, 3\},\ \{1, 3\},\ \{1, 2\},\ \{1, 2, 3\},$$

and there exists no other subset. All of them, except the last, are proper subsets. The number of the subsets is $2^3 = 8$; in this power the exponent 3 is the number of elements in the original set. We shall later see (§ 7, 3) that this connection between the extent of a set and the extent of the set of its subsets is a general feature. Here again we have a justification for the conventions fixed above, especially for definition II: had we not included 0 and S itself among the subsets of S we should not get exactly 2^3 subsets.

Finally, let us remark that, apart from Cantor's definition of set, the principles stated hitherto *do not guarantee the existence of any set.* The principle of extensionality indeed does not maintain anything at all with regard to existence, and the principle of subsets maintains the existence of certain sets (subsets of S) only *conditionally*, i.e. provided that S itself exists. We fill this gap by the following postulate:

Principle of pairing (III). If a and b are different objects [1]) there exists the pair $\{a, b\}$, i.e. the set containing a and b and no other element.

This principle guarantees three of the subsets of $S = \{1, 2, 3\}$ considered above, while a fourth is S itself. (In this instance S was *supposed* to exist; in 3, however, we shall take an example dealing with the *construction of* S.) As to the four other subsets, they are guaranteed by the following theorem:

THEOREM 2. The null-set 0 exists. If a is any object there exists the set $\{a\}$ containing a and no other element.

Proof: Let a and b be any different objects. By principle III

[1]) We shall not raise here the question of whether one is entitled to assume the existence of objects (sets or others). For our exposition it is easily settled by the principle of infinity (at the end of § 2) which postulates the unconditional existence of a set. Strictly speaking, the first sentence in theorem 2 should read: The null-set exists if there exists a set at all.

the pair $P = \{a, b\}$ exists. Take the property π of *not being an element of* P [1]); by principle II, π creates from P the null-set. Take the property of *equalling* a [2]); it creates from P the subset of all elements of P which equal a, i.e. the set $\{a\}$. Q.E.D.

It is easy to perceive how this proof may be generalized in order to prove the existence of *any finite subset* of a given set. To this purpose one simply has to use, instead of "equalling a", the property of "equalling a_1, or a_2, or ..., or a_n".

3. Sum, Inner Product, Difference. If S and T are sets — the case $S = T$ not excluded — one may form the sets which contain, *either* the elements belonging to *any one* of the sets, *or* the elements belonging to *each* of the sets. The first procedure corresponds to the *logical disjunction* ("or") [3]), the second to the *logical conjunction* ("and"). In the first case one speaks of the sum or join of S and T, in the second of their product or meet [4]).

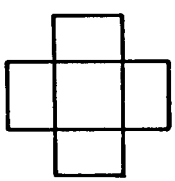

Fig. 3

In fig. 3 S may denote the set of all points of the horizontal (lying) rectangle, T the set of all points of the vertical (standing) rectangle; then the sum is the set of *all* points contained in the whole cross-shaped figure, while the product is the set of the points contained in the inner square.

One may generalize these operations from the case of two sets to much more general cases. A complete generalization will be given in § 6, **2** and **11**. Meanwhile it is sufficient to generalize in two special directions: first from two sets to any *finite* number of sets (cf. **5**), and secondly from two sets to any (infinite) *sequence* of sets, i.e. a collection containing a first, a second, a third set and so forth, so that to any natural number

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[1]) One might as well take, e.g., the property of being different both from a and from b.

[2]) Or, of differing from b.

[3]) There are two logically different kinds of disjunction, which are confused by the defectiveness of most languages which only contain the word "or" (*ou, oder*, etc.). In Latin, however, one has two different particles: *aut* (exclusive), and *vel* (at least one of the two). Compare the sentences "this day is Monday or (*aut*) Thursday" and "in the next or (*vel*) the third block you will find a taxi". *The "or" of logical disjunction, sometimes called "alternation", is vel.* For a way of logically introducing *aut* see B. A. Bernstein **19**.

[4]) By some authors the terms "union" and "intersection" are used.

k, a certain set [1]) of the collection will correspond while no other sets appear in the collection. It is not desirable to include the first case in the second since a sequence, according to our explanation, implies a certain order in which the sets of the collection are given. We thus define:

DEFINITION III. Given a finite number of sets $S_1, S_2, \ldots, S_n$, or a sequence [2]) of sets $S_1, S_2, S_3, \ldots, S_k, \ldots$, by their sum (join) or *sum-set* J we understand the set of all elements contained in *at least one* of the sets S_k, and by their *meet* or *inner product* [3]) M, the set of all elements contained in *each* of the sets S_k. We denote the sum by

$$J = S_1 + S_2 + S_3 + \ldots,$$

the meet by

$$M = S_1 \cdot S_2 \cdot S_3 \cdot \ldots \text{ [4])}.$$

According to this definition, each of the given sets S_k is a subset of the sum J, while the meet M is a subset of each S_k.

Examples. 1) $S_1 = \{1, 2, 3, \ldots\}$, $S_2 = \{2, 3, 4, \ldots\}$, $\ldots$, $S_k = (k, k+1, k+2, \ldots)$ for any natural number k. The sum is the set of all natural numbers, coinciding therefore with S_1. The meet is the null-set. For let n be *any* natural number, i.e. any element of S_1; then n is not contained in the set S_{n+1}, and therefore not in the meet.

2) Let S_1 be the set of all real numbers except the integers; S_2 the set of real numbers, except the integers and the fractions with the denominator 2; generally S_k the set of real numbers,

[1]) According to this (usual) definition of a sequence it is not necessary that different sets correspond to different natural numbers. In other words: In a sequence — in contrast to a set, cf. p. 17 — the same element may appear several times. For the connection between the concepts of set and of sequence, see **4**.

[2]) As usual we denote a *sequence* by enclosing the members, in the order given by the sequence, in round brackets; e.g. $(S_1, S_2, S_3, \ldots)$. Since *sets* are denoted by curly brackets { }, no confusion will arise between a set and a sequence, if either is denoted by its members.

[3]) It is necessary to add "inner" because we shall yet require another kind of multiplication of sets (§ 6) in respect of which we speak of the "outer" product. In the following pages the shorter term "meet" will be used instead of "inner product".

[4]) As usual in arithmetic, one may also omit the points between the factors.

except the fractions having one of the denominators 1, 2, ..., k. The sum is again S_1; the meet is the set of all irrational numbers (§ 9).

As to the meet, a similar geometrical example is obtained by considering, with respect to a given circle, a circumscribed *square* as well as the circumscribed regular polygons with 8, 16, 32, ... sides, chosen so that every polygon is included in the preceding one. Then the meet of the sets of points included in the polygons is the set of points inside the circle and on its circumference.

The following remark, ensuing from examples 1) and 2), will prove useful in § 5: *Given a sequence of sets such that any set is a subset of the preceding set of the sequence, the meet of all sets may either be the null-set or differ from it.*

By the way, we again notice the expediency of definition II on p. 23: had we not introduced the null-set, we would not be able to maintain that the meet of any sequence (or set) of sets is again a set.

Let us add a third example showing that, and how, starting from a finite number of different objects $a_1, a_2, \ldots, a_k$, one may by successive steps construct the set containing all these elements.[1]) Take $k = 4$. By the principle of pairing, there exist the pairs $p = \{a_1, a_2\}$ and $p' = \{a_3, a_4\}$. Hence one forms the sum

$$p + p' = \{a_1, a_2, a_3, a_4\}.$$

If $k = 3$, we take, instead of p', the set $\{a_3\}$ which exists by theorem 2. The analogous construction for *any* k can be achieved by mathematical induction, being contained in the general notion of a *finite* set. (See **5**, and also § 10, **6**.)

It is obvious that the operations of forming the sum and the meet give results which are *independent* of either the succession of the given sets or of a division of the total operation in pieces, by bracketing together some of the given sets. The first independence ought not even to be stated, for the operations themselves are defined in a way involving no order at all. In the terminology of arithmetic one may express these assertions of independence by saying: both operations are *commutative* and *associative*[2]).

[1]) Here principle IV (p. 28) is anticipated. To use it in the following example, we start from the pair $\{p, p'\}$.

[2]) When dealing with the systematical development, we shall return to this point (§ 6, **5**). A formal proof of the associativity is given there.

Moreover, the following two *distributive* laws hold for the operations defined here:

$$S_1 \cdot (S_2 + S_3) = S_1 \cdot S_2 + S_1 \cdot S_3 \text{ and } S_1 + S_2 \cdot S_3 = (S_1 + S_2) \cdot (S_1 + S_3).$$

Proof. According to the definition of equality, one has to show for either of these relations that any object contained in the left-hand set is contained in the right-hand set, and vice versa. Now, an element of $S_1 \cdot (S_2 + S_3)$ belongs to S_1 as well as to $S_2 + S_3$; in other words, it belongs to S_1 and to S_2 or S_3. Therefore, it belongs to S_1 and S_2, or to S_1 and S_3. This is exactly what is meant by its belonging to $S_1 \cdot S_2 + S_1 \cdot S_3$.

The reader will easily transfer this argument to the direction from right to left and prove the second distributive law in quite a similar way.

It should be stressed that for the operations called in arithmetic (and also in § 6) *addition* and *multiplication*, only the first distributive law $a \cdot (b + c) = a \cdot b + a \cdot c$ holds true, and not the second.

Finally, a definition of rather limited significance:

DEFINITION IV. If S_0 is a subset [1]) of the set S, the set of all those elements of S that do not belong to S_0 is denoted by

$$S - S_0 \text{ (\textit{difference} of } S \text{ and } S_0).$$

We must not close this subsection without asking whether the principles I, II and III introduced in the previous subsections, suffice to guarantee the *existence* of the new sets defined here: sum, meet, difference. The answer is (partly) in the negative. To include at once the generalized form [2]) of definition III which appears in § 6, we formulate as follows:

Principle of sum-set (IV). Given a set A whose elements are

[1]) One might renounce this condition. But defining $S - T$ for *any* two sets S and T, would be of no use in this book.

[2]) Strictly speaking, one cannot consider *set* as a generalization of *sequence* since the latter concept includes order while the elements of a set are not arranged. Nevertheless, in **4** a hint will be given as to how the notions of set and sequence are connected and in *Foundations* the concept of sequence as well as that of an *ordered set* in general (§ 8) is reduced to the plain concept of set by means of our principles Cf. also p. 192.

again sets [1]), there exists the set containing just the elements of the elements of A. It is called the sum-set of A and denoted by SA.

If a, a', a'', ... are the elements of A, one writes (see p. 26)

$$SA = a + a' + a'' + \ldots\ldots$$

As to the formal aspect of this principle, its connection with definition III is obvious. The main difference lies in dropping the assumption of definition III, that the elements of A appear in finite number only or in the form of a sequence. Since here no assumption at all is made as to the extent of A, the representation $a + a' + a'' + \ldots\ldots$ will, in general, exhibit only a few of the elements of A.

Now, all our operations are indeed justified. The *existence of the sum* introduced through definition III, is guaranteed by the principle of sum-set [2]).

As to the *existence of the meet*, let S_1 be any [3]) of the given sets, according to definition III. For every element x of S_1 the property "x is contained in *each* of the given sets S_k" is meaningful. The elements x of S_1 that possess this property form a set, according to the principle of subsets; more exactly, they form a certain subset of S_1 which is precisely the meet of the sets S_k.

The *existence of the difference* (definition IV) again ensues from the principle of subsets: $S - S_1$ contains all those elements of S which have the property of not belonging to S_1.

4. Representation and Equivalence. The introduction of *infinite numbers* and the operations between them are based, more than on any other concept, on the relation of *equivalence*.

In a) and b) of § 1, **1**, a set of eight pieces of fruit and a set of eight numbers were considered. We saw that these sets differ from each other only with regard to the nature of their elements; their elements can be *related* to (*attached* to; *paired* with) each

[1]) This condition is really superfluous. For those elements of A which are not sets, are automatically eliminated by the formation of the sum-set, since they do not contain any elements. Nevertheless, the condition may psychologically facilitate the understanding of the concept sum-set.

[2]) In the sequence mentioned in definition III, the same set may, of course, appear several times. This does not alter the sum.

[3]) For reasons of symmetry (and of symbolical simplicity) it would be preferable to start, not from *one* of the sets, but from their sum. This will not change anything.

other — and in different ways — so that to every number corresponds a single piece of fruit, *and vice versa*. The latter addition is essential. Without it one can create a correspondence of the required property (i.e. a *single-valued or unique correspondence*) even after dropping, for example, the banana from the first set: one may relate different apples to each of the numbers 1, 2, 3, 4, 5, one orange to 6 and the other orange to 7 as well as to 8 — a procedure that would not, however, make a single number correspond to every fruit, although a single fruit is related to every number.

By a correspondence of the same mutual property one may connect the set of real numbers and the set of points on a line, described in f) on p. 12. We there stated how to any point of the line a single real number may be assigned so that a single point will correspond to any real number through this same relation.

Forming such relations or correspondences does not only belong to the simplest material of mathematical processes but it is also one of the most primitive and fundamental functions of the human mind in general [1]). At an early stage of civilization, it is true, man did not contrive to attach objects of *different* sorts to each other; yet he would compare two heaps of apples and find out, by attaching apple to apple, whether the first heap contains more apples than the second, or both an equal number.

An immense step beyond this, towards the creation of the *concept of number*, is taken when one drops the restriction to objects of the same kind and compares, e.g., a heap of apples and a multitude of eggs for the purpose of barter, by arbitrarily relating an apple to an egg — at a stage where number and counting are still unknown. According as finally apples or eggs remain without mates, or both heaps are exhausted by the same step, it will be stated that one of the collections has a larger extent, or both the same. In this way, both the psychological and the logico-mathematical points of view [2]) allow us to introduce the concept

[1]) Cf., for instance, P. Boutroux 1 and Brunschvicg 3.

[2]) Mathematical expositions of the subject are found in many scientific treatises of arithmetic. Cf. also Dantzig 1, B. Russell 5; furthermore Katz 1 and du Pasquier 1.

Baer 6 maintains that Hegel already had this method of introducing the cardinal numbers in mind. For Hegel's attitude to mathematics in general see Speiser 1, p. 10.

of *finite* (*cardinal*) *number*, as the common characteristic of any two finite aggregates whose elements can be related to each other in a one-to-one correspondence. A stricter explanation will be given in § 4, **6** and **7**. The importance of this number-concept, not only from the scientific point of view but for the development of civilization in general, is stressed by the fact that, instead of individually different and limited sets of objects of comparison (e.g. fruit), a *universal and inexhaustible set* is obtained, the set of the integers 1, 2, 3, ... ; a set the members of which do not possess accidental properties, but are completely determined by their qualification for the process of counting.

Nothing in the development described hitherto, makes an appeal to the finiteness of the collection. Therefore, we may define quite generally:

DEFINITION V. The set S is called *equivalent* [1]) to the set T (in symbols $S \sim T$) if the elements of T can be related to the elements of S in a *one-to-one* (*biunique*) *correspondence*, that is to say in such a way that, on account of the relation, a single element of T corresponds to every element of S and vice versa. A one-to-one correspondence between the elements of T and S is also called a *representation* between S and T (or of S on T) or a *mapping* of S upon T.

A representation between sets is usually denoted by a Greek letter such as φ, ψ, χ, Φ etc.

In the above, we pointed out the difference between biunique (one-to-one) and merely unique correspondence, by means of the examples a) and b) of § 1. The difference may also be illustrated in the following way: in a state where polygamy is legal, the existing matrimonies define a one-to-many correspondence between husbands and wives such that to any married woman a single man (her husband) corresponds. Therefore, the relation of husbands

[1]) The term *similar*, used sometimes instead of *equivalent*, is less desirable because of its being used in another sense in the theory of ordered sets (see § 8, 3). The term equivalent, it is true, is used in many different senses in various mathematical branches, but not otherwise in the theory of sets.

While the term *correspondence* usually refers to the relation between *elements* related to each other, there is no generally accepted term for the relation between *sets* the elements of which are related in a one-to-one correspondence. Hence the term *representation* had to be chosen rather arbitrarily. As the verb we shall use "to map" beside "to represent".

to wives is one-valued (unique). On the other hand, among certain tribes where polyandry is said to rule, the relation of husbands to wives is a many-to-one correspondence. In monogamic states, however, there is a one-to-one correspondence between husbands and wives; the set of husbands, therefore, is equivalent to the set of wives; the representation defined by the existing marriages is one of the possible representations between those equivalent sets.

Among the sets considered in § 1, 1, the sets of a) and b) are equivalent, likewise the sets of d) and e), as well as the two sets (of numbers and of points) defined in f). The proofs rest on the representations given there.

While the equivalence between two sets may be shown by constructing a representation, the *non-equivalence* is, according to definition V, not yet guaranteed by the fact that a certain correspondence between the elements of the sets in question is *not* of the one-to-one (but of a one-to-many) type. Therefore, to show non-equivalence we have to prove that *no* one-to-one correspondence between the elements of the sets is possible. The reader who should find difficulty in understanding this distinction, is referred to § 4, 3.

A one-to-one correspondence between the elements of two *finite* aggregates may be constructed by stating, with respect to every element of the one aggregate, to which element of the other it shall be attached; say by means of a schedule. Of course such a procedure is not possible in the case of *infinite* aggregates. Here, the correspondence can be defined only by a *law*, i.e., by a rule of a general character, formulated in a finite way but furnishing, to each of the infinitely many elements of one aggregate, its mate in the other.

Most readers will have some knowledge of the mathematical *concept of function* if only from the graphic representations of functional connections. They may already have guessed that our *correspondences* and *representations* are nothing more than certain functions. Indeed, if a unique correspondence relates to every element x of the set S an element y of the set T, y being uniquely determined by x, one has a one-valued [1]) function $y = f(x)$. The argument x of the function, also called the *independent variable*, runs over the

[1]) We shall always understand *function* in the sense of *one-valued function* without explicitly adding this quality. The same thing applies to *inverting* a function.

set S while the values of the *dependent* variable y belong to T; these values, however, need neither exhaust all elements of T nor be different from each other. Therefore, in general, it is not possible to *invert* a (one-valued) function, i.e. to comprehend x as a function of the same kind; possibly different values of x will be related to one value of y. Take for instance the temperature y as a function of the time x (at a certain place). At any moment there is a definite temperature, but to a certain temperature different times may be related (possibly one in the morning and one in the evening) while temperatures exist to which no time corresponds, because they are never reached.

However, if the function $y = f(x)$ is *invertible*, that is to say if also to every y of T a single element x of S is related, the function defines a *one-to-one* correspondence. In this case, therefore, the sets S and T are equivalent. Accordingly, the concept of equivalence is not at all specific for the theory of sets but is based upon the concept of function which appears everywhere in mathematics [1]).

Let us return to the concept of "one-valued function" for a remark which will prove useful later. Sometimes it is felt unpleasant that in a *set* no element can appear *repeatedly*, in contrast to a *sequence* — e.g. (1/2, 1, 3/2, 2/3, 1, 4/3, 3/4, 1, 5/4, ...) — where a certain order is fixed and a member may appear several times, even infinitely many times. But, given a set S, a function $y = f(x)$ whose argument x runs over S defines a collection of values y which need not be all different. A certain value $y = a$ appears in the collection as often as indicated by the number of different values x to which that value $a = f(x)$ is attached. In the most important instance, that of a *sequence*, S is the set of all natural numbers; if k is any natural number, let us denote the value of $f(k)$ by a_k. If a_k is, for example, the digit appearing at the kth place after the point in the decimal expansion of π, the sequence is (1, 4, 1, 5, 9, ...).

The equivalence is not a quality but a relation [2]), more pre-

[1]) The advanced reader might raise the questions whether the concept of equivalence can also be reduced to that of *set* and whether our principles of set-construction enable us to effectuate such a reduction. These questions can be answered in the affirmative. For the present we are only hinting at a possible method of reduction. If S and T are exclusive sets, a representation between S and T is essentially a set of *pairs* each of which contains a single element of S and a single element of T, so that any element of S as well as of T appears in a single pair. Hence, a representation is nothing but a certain *subset* of the set of *all* pairs of elements of S and T, having the following property: any element of the sum $S + T$ appears in one and in only one element of the subset. If such a subset *exists*, the sets S and T are equivalent.

[2]) Greek philosophy did not discern the difference between qualities (relations with a single argument) and proper relations (with two and more arguments). The similarity in the grammatical structure of sentences

cisely a *binary relation*, i.e. a relation with two arguments (free variables) "X equivalent to Y" which is defined for *sets* X, Y. A given relation has certain properties; e.g. the relation "x is larger than y" has the property of being "irreflexive", that is to say, the proposition "x is larger than x" is never (for no x) true. Let us ascertain some fundamental properties of the equivalence defined above. The following considerations are extremely simple but because of their importance they are nevertheless presented at length.

First, *any set is equivalent to itself*: $S \sim S$. The proof is trivial, since the equivalence is shown by the *identical representation* which relates every element to itself. This property is expressed as follows: equivalence is a *reflexive* relation. The null-set, too, is called equivalent to itself.

Secondly, *if S is equivalent to T, T is also equivalent to S*; in symbols: $S \sim T$ implies $T \sim S$. The equivalence, accordingly, is a reciprocal relation where both arguments appear symmetrically — in contrast, e.g., with the relation "x is larger than y" (and any other relation of *order*) which is even *incompatible* with "y is larger than x". This property of equivalence is the immediate consequence of the biuniqueness of the correspondence used for the definition of equivalence. In fact, if this correspondence attaches to the element x of S the element y of T, the one-to-one character expresses that x is also attached to y and that x is the only element of S attached to y in virtue of our correspondence. Therefore, if the original representation asserts $S \sim T$, the representation created by attaching x to y asserts $T \sim S$.

This property is expressed as follows: equivalence is a *sym-*

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expressing either qualities or relations is to blame for the strange confusion. Only with the development of symbolic logic (*Foundations*, ch. III) in the second half of the 19th century, beginning with De Morgan, was the decisive importance of relations in logic recognized. The difference between qualities and relations is stressed in an unforgettable way by the following joke: a lady calls on her friend who has borne twins, and remarks "How beautiful your children are, especially that one on the left". Later another lady comes and says "How alike your twins are, especially that one on the left". *Beautiful* is a quality, *alike* a relation.

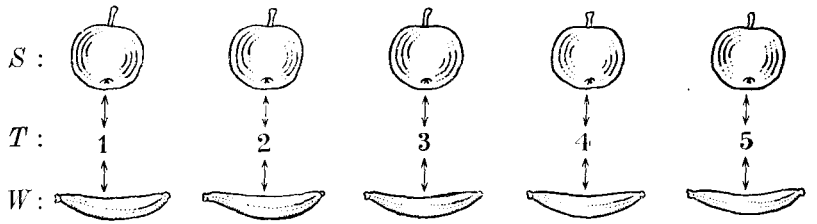
The ordinary *operations* are relations with three arguments; e.g., $x + y = z$ is a relation between x, y, z. Another example is: the point Z is situated *between* the points X and Y.

metrical (reciprocal, mutual) relation. We have been using this property from the beginning of this subsection by co-ordinating in language the arguments X and Y of $X \sim Y$; for example, we said, and shall say, "X and Y are equivalent" or we speak of a representation *between* two equivalent sets, although definition V originally says in a non-symmetrical way: X is equivalent to Y. Note that "x is larger than y" does not allow the transformation to "x and y are larger"!

Thirdly, let S, T, W be sets such that $S \sim T$ and $T \sim W$. We maintain that the middle member T may be left out, and we may write $S \sim W$. (Since $T \sim W$ implies $W \sim T$ on account of the symmetry just proved, we may formulate our assertion also in this way: *if two sets are equivalent to a third set, they are equivalent to each other.*) In short, equivalence is a *transitive* relation.

In order to prove our assertion, let φ denote a representation between S and T, and ψ a representation between T and W. Although there are other representations too, we shall henceforth adhere to φ and ψ exclusively. If to the element s, arbitrarily chosen in the set S, the element t of T corresponds in virtue of φ, and if to this t the element w of W corresponds in virtue of ψ, we construct a new correspondence χ by attaching w of W to s of S. We maintain: χ is a representation between the sets S and W. For, on account of the definition of χ and because φ and ψ are representations, the element w of W attached to s of S is uniquely determined. On the other hand, let w be any element of W. Since ψ and φ are biunique correspondences, there is only one element t of T corresponding to w by virtue of ψ, and one element s of S corresponding to t by virtue of φ. Moreover, if w corresponds to s by virtue of χ, the same element s also corresponds to w. The representation χ between S and W shows that $S \sim W$; Q.E.D.

In order to illustrate this rather abstract reasoning let us choose for S a set of apples, for T the set of numbers $\{1, 2, 3, 4, 5\}$, for W a set of bananas, and let us assume $S \sim T$ and $T \sim W$. We express these assumptions by the following scheme of correspondences (where the double direction of the arrows indicates that the correspondences are unique in either direction):



From this scheme we attain a representation between S and W, which proves the equivalence of these sets, by attaching to the "first" apple (i.e. the apple to which the number 1 of T is assigned) the "first" banana (i.e. the banana corresponding to 1 of T); and so forth for each apple. In other words: one omits the second line of the scheme, containing T, and relates each banana to the apple located above it. — This procedure is a kind of inversion of the process mentioned on p. 31 which facilitates the comparison of two aggregates by the insertion of numbers, i.e. by *enumerating* their elements.

We may thus formulate our results:

THEOREM 3. The equivalence of sets (definition V) is a reflexive, symmetrical, and transitive relation; that is to say: $S \sim S$; $S \sim T$ implies $T \sim S$; $S \sim T$ and $T \sim W$ together imply $S \sim W$. Hence in a totality of sets, such that every set is equivalent to a *definite* one, *any two* sets of the totality are equivalent to each other.

Equivalence has different meanings in different branches of mathematics. However, all these meanings share the three properties expressed in theorem 3. Therefore, *equivalence* is sometimes used in a more general sense, meaning *any* relation with two arguments having those three properties [1]). Also the *equality* of sets (p. 18/19) is, as is every equality in mathematics, a relation having the three properties stated in theorem 3, i.e. a relation of

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[1]) These properties are not *independent*. For $S \sim T$ implies $T \sim S$ by the symmetry and these relations imply $S \sim S$ by the transitivity. Accordingly, if there are at least two equivalent objects, the reflexivity is a logical consequence of the symmetry and the transitivity.

For certain delicate questions in regard to these properties (e.g., the distinction between *reflexivity* and *total reflexivity* etc.) and their interdependence, cf. Peano 8, Padoa 6, Itô 1 and, in particular, Scholz–Schweitzer 1, § 5. Compare also exercise 5 at the end of this section.

For a certain generalization of the equivalence relation, see Tola 1.

equivalence in the general sense. The proof may be left to the reader.

Finally, let us remark: From the definition of representation it immediately follows that *a given representation between the (equivalent) sets S and T represents any proper subset of S upon a proper subset of T.* Likewise, if S_1 and S_2 are mutually exclusive sets (p. 20), and if φ_1 represents S_1 upon T_1, while φ_2 represents S_2 upon a set T_2 (exclusive to T_1), *then the set $S_1 + S_2$ is represented upon $T_1 + T_2$ by the join of the representations φ_1 and φ_2*, and this join again is a representation.

It is easily seen that one may generalize this assertion to any finite sum, or even to the sum of a sequence. A detailed treatment of the general problem is given in § 6, 3.

5. **Finite and Infinite Sets.** Hitherto we have used the attributes *finite* and *infinite* in a naive sense. We must no longer postpone their strict explanation — though until the end of the book we shall not cease picking up additional material for the analysis of these concepts.

DEFINITION VI. A set S is called *finite*, or sometimes *inductive*, if there exists a natural number n such that S contains n and no more elements. The null-set 0, too, is called finite. A set which is not finite is called *infinite*.

One should not overlook the fact that this definition *explicitly uses the concept of natural number*. Whoever endeavors to base the concept of number on the more general concepts of set-theory (cf. § 10, 6), should either not use the concept *finite* previously — which would be difficult to carry out — or formulate the definition in a way which does not use natural number explicitly, as did Russell when introducing the term *inductive* [1]).

In this and the following sections *we shall rely on the theory of natural numbers*; that is to say, we shall use arithmetic for the development of our theory. Therefore, theorems about finite sets will here be demonstrated in the usual way of arithmetic by

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[1]) The following version of Russell's definition may be adopted: a set of cardinal numbers is called *hereditary* if its containing n implies its containing $n + 1$; a cardinal number is called *finite* if it belongs to *every* hereditary set containing the number 0; a *set* is called *finite* if its cardinal is finite (cf. § 4, 6). $+ 1$ has to be conceived in accordance with the notion of sum-set; cf. § 6, 4.

means of the characteristic method of the theory of numbers, *mathematical induction*, which states that a property of natural numbers belonging to the number 1 (or to a certain number m), and belonging to the *successor* $n + 1$ of n if it belongs to n, for any n, belongs to *all* natural numbers (or to all numbers larger than m). See § 10, 2 and 6. We shall use the well-known theorems of arithmetic, concerning natural numbers and finite sets, without special proofs. An exception is made for the following theorem which shall be proved explicitly because of its fundamental importance in the following considerations.

THEOREM 4. A finite set is not equivalent to any proper subset of itself.

Proof. The theorem is true if the set contains only one element; for then the only proper subset is the null-set which, containing no element, cannot be equivalent to a set containing one element.

Let us assume the theorem to be true for all sets of n elements, n being a certain natural number whatsoever. [1]) If S is a set of $n + 1$ elements, i.e. equivalent to the set of integers $\{1, 2, \ldots, n, n + 1\}$, we may denote the elements of S by $s_1, s_2, \ldots, s_n, s_{n+1}$, using an arbitrary representation between S and the mentioned set of integers for attaching indices. *We suppose the existence of a proper subset S' of S that is equivalent to S*, and infer from this that also the subset $\{s_1, s_2, \ldots, s_n\}$ of S is equivalent to a proper subset of itself — which contradicts our assumption.

We shall consider three possibilities:

a) The subset S' does not contain the element s_{n+1} of S. Then the mate x of $s_{n+1} \varepsilon S$ in S' is different from s_{n+1}. Let us write $S' = S'' + \{x\}$ where S'' is a proper subset of S' [2]). Therefore, when we remove $s_{n+1} \varepsilon S$ and its mate $x \varepsilon S'$ from the supposed representation between S and S', there remains a representation between $\{s_1, s_2, \ldots, s_n\}$ and its proper subset S'', contrary to our assumption that for a set of n elements no such representation exists.

b) S' contains s_{n+1}, and $s_{n+1} \varepsilon S$ is attached to $s_{n+1} \varepsilon S'$, i.e. to itself. After dropping the element s_{n+1} out of the supposed representation between S and S' in both sets, we again get a representation between $\{s_1, s_2, \ldots, s_n\}$ and a proper subset of this set, contrary to our supposition.

c) S' contains s_{n+1}; but in the presumed representation between S and S', $s_{n+1} \varepsilon S$ is attached to *another* element of S', say to y, while the mate of $s_{n+1} \varepsilon S'$ in S is a certain element $x \varepsilon S$ ($x = y$ not exluded). We thus have the scheme of correspondence: $s_{n+1} \longleftrightarrow y$, $x \longleftrightarrow s_{n+1}$.

Now, we modify the representation between S and S', by relating $s_{n+1} \varepsilon S$ to $s_{n+1} \varepsilon S'$ and $x \varepsilon S$ to $y \varepsilon S'$. Thus, we get a new representation between

[1]) n is not yet presumed to be uniquely determined by the set.

[2]) See definition III. Accordingly, S'' is the set obtained from S' when the element x is dropped.

S and S' in which $s_{n+1} \varepsilon S$ is attached to $s_{n+1} \varepsilon S'$. But this is the case b), already found contradictory.

The contradiction reached in each case shows that our supposition about S being equivalent to a proper subset was false. In other words, the truth of our theorem for all sets of n elements implies its truth for all sets of $n + 1$ elements. Since it is true for $n = 1$, the proof is completed.

The properties of *infinite sets* stand in sharp contrast to theorem 4. Let us for example take the set N of all natural numbers. A proper subset N' of N is created by dropping the element 1, viz. the set N' of all integers larger than 1. A certain representation between these sets, proving their equivalence, may be illustrated in the following way:

$$\begin{array}{lccccccccc} N: & 1 & 2 & 3 & 4 & \ldots & n-1 & n & \ldots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \\ N': & 2 & 3 & 4 & 5 & \ldots & n & n+1 & \ldots \end{array}$$

This is the correspondence relating to any element n of N the element $n + 1$ of N', or, in the inverse formulation, to any element n of N' the element $n - 1$ of N. It is evident that this is a one-to-one correspondence between N and N'. Therefore, *these sets are equivalent although the one is a proper subset of the other.* We shall see later that this is not accidental but characteristic of infinite sets in general.

The reader should thoroughly comprehend this simplest example of sets apparently different in size and nevertheless equivalent. It is the infinity of the set that enables us to construct a one-to-one correspondence. As has been seen in the proof of theorem 4, we could not have succeeded had the set N been finite. Misunderstanding will be avoided by a clear perception that mapping a set upon a proper subset can never be achieved by using the *identical* correspondence which relates every element to itself; then, in fact, those elements of the original set which are not contained in the subset, would remain without any mates. Beginners who want to dispute the possibility of a set being equivalent to a proper subset, frequently make the mistake of basing their arguments on the identical correspondence as though it were superior to other correspondences.

Many other instances of sets being equivalent although the one is a proper subset of the other, will appear in §§ 3 and 4.

The phenomenon that a set can, as it were, be of the same extent as a proper subset, stands in some contrast to the old principle *totum parte maius* (the whole is larger than a part). The paradox appearing in this contrast, clearly pointed out already by Galileo [1]), has fulfilled an important but decisively negative task in the history of the conquest of infinite magnitude for the realms of mathematics and philosophy: the infinite aggregates, having so paradoxical a quality, seemed to be discredited. As a matter of fact, however, the principle of the whole and its parts has been tested only in the domain of the finite and could not be expected to be saved beyond the huge abyss which separates the finite from the infinite [2]).

Peirce and Dedekind [3]) showed us the way of using this fundamental difference to *define the infinite* in a manner different from definition VI:

DEFINITION VII. A set S is called *infinite*, or sometimes *reflexive*, if a proper subset of S exists to which S is equivalent. Otherwise S is called *finite*.

We thus have two definitions of the concepts *finite* and *infinite*. Besides the heterogeneity of their contents, there is a sharp formal difference, the original notion in definition VI being *finite* with its negation *infinite*, whereas in definition VII *infinite* is the primary concept, *finite* being derived from it. A preliminary

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[1]) "Discorsi I", *Opere Complete* XIII. Geometrical paradoxes of a similar type, too, are treated there, and it is pointed out that the usual explanations of equality and of order in magnitude refer to *finite* quantity only.

[2]) The paradoxical impression deepens when the phenomenon of equivalence between sets of different sizes is, as it were, transferred into real life. The awkward feeling created by such an instance disappears, however, as soon as one perceives that it is only a fictitious reality, to which our psychological feeling is not fit to react. Let us mention the story of Tristram Shandy (cf. Russell 1, pp. 358 ff.) who writes the history of his life in so detailed a way that the description of each day takes a year to write. So, of course, he can never come to an end. But, if he lived forever, no part of his biography would remain unwritten; for to any day, a year dedicated to its description would correspond.

[3]) Peirce 2, III, pp. 210–249 (1885), 360; Dedekind 2. Cf. Keyser 12; also Bolzano 3, § 20, and Cantor 6. The objections against this procedure, raised in Gordin 1 and Ushenko 2, are not justified.

survey will show us, however, that the two definitions have the same meaning [1]).

We start with an assertion that is almost self-evident:

THEOREM 5. A set which is equivalent to a *finite* (*infinite*) set, is again *finite* (*infinite*).

Proof. The proof based on definition VI is left to the reader. We shall base our proof on definition VII and show that a set T which is equivalent to a reflexive set S, is again reflexive.

S being reflexive, a proper subset of it, S', exists such that $S \sim S'$. Since $T \sim S$, there exists a representation φ between T and S, to which we shall keep. As stated at the end of **4**, φ maps the proper subset $S' \subset S$ on a certain proper subset $T' \subset T$; hence, $S' \sim T'$. By means of a double application of theorem 3, the three equivalences

$$T \sim S, \qquad S \sim S', \qquad S' \sim T'$$

imply the equivalence $T \sim T'$. Since T is equivalent to a proper subset T', it is an infinite (reflexive) set.

The assertion about finite (non-reflexive) sets ensues by logical inversion: a set equivalent to a finite set s cannot be reflexive since this would make s itself reflexive, thereby giving a contradiction. Q.E.D.

To study the connexion between definitions VI and VII we shall leave out the terms *finite* and *infinite*, and use *inductive or non-inductive* in view of definition VI, and *non-reflexive or reflexive* when referring to definition VII.

First of all, theorem 4 states that any inductive set is non-reflexive; accordingly, any reflexive set is non-inductive, by logical inversion. Therefore, it will be sufficient to show that *any non-inductive set is reflexive*, which entails that any non-reflexive set is inductive — including, of course, the null-set.

In order to prove this assertion, we rely on theorem 4 of § 3, **5** which states that any infinite (non-inductive) set S has a subset S^* equivalent to the set of all natural numbers. (Of course, we shall not use our present reasoning in § 3.)

Let us take a definite (proper or improper) subset S^* of this kind and, using the corresponding natural numbers as indices, denote its elements by

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[1]) Theorem 4 of § 3 is used in the following comparison of definitions VI and VII; it relies on a new principle which is introduced much later (multiplicative principle, § 6, **6**). In *Foundations*, ch. II, various methods of defining finiteness and infinity will be compared in the light of that principle. Cf. also § 10, **6**.

$s_1, s_2, \ldots, s_k, \ldots$. Let be $S - S^* = S'$ ($S' = 0$ not excluded) so that $S = S^* + S'$, S^* and S' being exclusive.

On the other hand, dropping the element s_1 (i.e. the mate of 1) from S^*, we obtain the set $S_0^* = \{s_2, s_3, \ldots, s_k, \ldots\}$. Let be $S_0^* + S' = S_0$. Since $S_0^* \subset S^*$, $S_0 \subset S$ is also true; in fact, S_0 does not contain the element s_1. S_0^* and S' are again exclusive. Now, we construct a representation between the sets

$$S = S^* + S' \quad \text{and} \quad S_0 = S_0^* + S'$$

by a double step: a) any element of S belonging to S' shall be related to itself; b) any element of S belonging to S^*, and so being of the form s_k, shall be related to the element s_{k+1} of S_0^* (and of S_0). This rule evidently creates a one-to-one correspondence between the elements of S^* and S_0^*, being completely analogous to the representation between N and N' considered on page 39. Since any element of S belongs either to S^* or to S', both steps together create a representation between S and its proper subset S_0. Q.E.D.

In this book the terms *finite* and *infinite* will in general be used in the sense of definition VI. In *Foundations* we shall find it necessary to distinguish between the definitions VI and VII and even to introduce additional definitions of finiteness.

In this subsection several properties of infinite sets have been discussed. But the four principles introduced hitherto do not state anything about infinite sets and, as a matter of fact, they are insufficient to produce an infinite set. Therefore, we postulate:

Principle of infinity (V). There exists at least one infinite set: the set of all natural numbers. (See example d) of § 1.)

One can formulate this principle without relying on the concept of number. Some hints in this direction are given in § 11 [1]).

Exercises

1) Prove that the following relations between sets are equivalent:

a) $S \subseteq T$, $S = S \cdot T$ and $S + T = T$;
b) $S = T$ and $S \cdot T = S + T$;
c) $S \subseteq T \subseteq W$ and $S + T = T \cdot W$.

2) Are there other representations besides a given representation between two equivalent aggregates? Give a few instances. Are there exceptions to the rule? Specialize the result for the case of representing an aggregate on itself.

[1]) Cf. Zermelo 3 and 4, von Neumann 1.

3) (For readers who are familiar with the concept of function.) May one generally use the functions

$$y = 3x + 5, \quad y = x^2, \quad y = \sqrt{x}, \quad y = \sin x$$

to map the set of argument-values x on the set of the corresponding function-values y? In the cases where the answer is in the negative, can one answer in the affirmative after a suitable *restriction* of the variability of x (e.g. to a certain interval etc.)?

4) A relation is defined only after the domains of variability for its arguments x, y etc. have been fixed. Try to comprehend this in view of the instance "x is a brother of y". (It depends on the determination of the *domain of variability for* y whether this relation is symmetrical or not. Take, e.g., x = Moses, y either = Aaron or = Miriam.)

5) How complicated the connection between reflexivity, symmetry and transitivity of relations is in general (cf. the footnote on p. 36), one may gather from the following example: The relation between two arguments "x and y are prime numbers", defined for integers x and y, certainly is symmetrical and also transitive. However, it is not reflexive in the most general sense: "6 and 6 are prime numbers" is a false proposition. The advanced reader may also consider the relation $x \cdot y = 0$ between integers x, y.

6) Why is it convenient to begin in the proof of non-equivalence between a finite and an infinite (reflexive) set (p. 41), with an *infinite* set, as done there?

§ 3. Denumerable Sets

1. Denumerability. In this section we shall deal with the simplest type of infinite sets, called *denumerable*; occasionally we have already used them. In order to introduce the concept of denumerability, we start from the set N of all natural numbers 1, 2, 3, ... Given any set D which is equivalent to N, and a certain representation φ between D and N, we denote by d_1 the element of D related by φ to the number 1 of N, by d_2 the element related to $2 \, \varepsilon \, N$, etc; generally by d_k the element of D related to the number k, thus using the related natural numbers as indices for the elements d_1, d_2, d_3, ... of D. As φ defines a one-to-one correspondence, not only does every natural number k appear as

index to one, and only one, of the elements of D, but also every element of D bears a natural number as its index.

We may therefore write the given set in the form

$$D = \{d_1, d_2, d_3, \ldots, d_k, \ldots\}.$$

D, however, is not a *sequence* since its elements, as members of D, are not arranged in a certain order — although the assumed representation *enables us* to arrange them, for example, in the order of increasing indices, and therefore in the form of a sequence. On the other hand, after having arranged them in this way, we have no longer a plain set but an ordered set; in particular an *enumerated set* — which is indeed a sequence [1]).

Any element d of D appears "at a certain place" in the set, i.e. it is attached to, and marked by, a certain natural number k which is the mate of d in N on account of the representation φ.

D is not necessarily *given* in the form of an enumerated set; we have only presumed that it is *denumerable*, namely that its elements *can* be attached to all natural numbers by a one-to-one correspondence. Therefore no order need be given in advance, or an order may be given that is different from the order of increasing indices. Presently we shall become acquainted with instances of this kind.

DEFINITION I. A set that is equivalent to the set of all natural numbers is called a *denumerable* (or *countable*) *set*. If its elements are *ordered* according to the magnitude of the numbers related to them, one speaks of an *enumerated set*. With respect to the totality of elements of a denumerable set, one sometimes says *denumerably many objects*.

From the definition it follows immediately, by the transitivity of equivalence, that *a set equivalent to a denumerable set is again denumerable*. Essentially we have already seen in § 2, **5**, that a *denumerable set is always infinite*, in the sense of both definitions VI and VII (pp. 37 and 40). It suffices to prove this for the set of all natural numbers; but for this set our assertion is evident by definition VI, and has been proved on p. 39 by definition VII.

The principle of infinity (p. 42) guarantees that there *exists* a denumerable set.

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[1]) Not every sequence, however, is an enumerated set. For in a sequence a member may appear repeatedly, which has been excluded for sets.

For the specialization *effectively denumerable*, see § 5, **4**.

2. Simplest Examples and Theorems. Let us consider a few instances of denumerable sets. As shown on p. 39, the set $\{2, 3, 4, 5, \ldots\}$ is also denumerable. The same obviously holds for any set M originating from the set of all natural numbers by dropping any finite number of elements. For then there always remain infinitely many numbers, and by arranging these according to magnitude [1]) we get again a first, second, ..., kth, ... number. Thus we have related them to all natural numbers.

But one would be mistaken in believing that this easy way of enumerating depends on having dropped only a *finite* quantity of the original numbers. The same holds when we drop *infinitely many* numbers, provided that there still *remain* infinitely many. (Otherwise we should have as the remainder a finite set, which is not "denumerable" [2]).) If we drop, for example, all the odd numbers, there remains the set L of all positive even integers l, and one obtains its representation on the set N of *all* natural numbers n by relating $l \varepsilon L$ to $n(\varepsilon N) = \frac{l}{2}$, i.e. n to $l = 2n$; or in a scheme

$$\begin{array}{lccccccc} l = & 2 & 4 & 6 & 8 & \ldots & 2k & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ n = & 1 & 2 & 3 & 4 & \ldots & k & \ldots \end{array}$$

The general case is again provided for by the procedure described in the last paragraph: by arranging the remaining elements according to their magnitude. Accordingly, any infinite (non-inductive) subset of the set of all natural numbers is again denumerable.

In reaching this result we have not used any particular property of the natural numbers (besides that expressed in footnote 1). Therefore, our reasoning remains valid after replacing the set of natural numbers by any denumerable set. (Cf. the proof of the following Corollary.) Hence:

THEOREM 1. Any infinite subset of a denumerable set is again denumerable.

From this theorem we may draw a simple conclusion which will prove to be of considerable importance in § 5:

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[1]) For in any set of natural numbers, there is a *smallest* number; likewise, in any part of a sequence, a *first* element.

[2]) This is not in accordance with the normal usage of language, but with definition I.

COROLLARY. Any subset of a denumerable set D is either finite or denumerable.

Proof. One could simply say, any subset is either finite or infinite, and in the latter case denumerable because of theorem 1. Q.E.D. It may, however, be useful to illuminate the constructive character of the proof by accomplishing it in a more detailed way, which also applies to theorem 1.

Denote D again by $\{d_1, d_2, \ldots, d_k, \ldots\}$, using a certain representation between D and the set of all natural numbers; let D_0 be any subset of D. If $D_0 = 0$, D_0 is finite. Otherwise let k_1 be the smallest integer k for which $d_{k_1} \varepsilon D_0$; k_2 the smallest integer k for which $d_{k_2} \varepsilon (D_0 - \{d_{k_1}\})$; and so forth, according to mathematical induction. Two cases are possible:

a) A certain step of this procedure, say the nth step ($n = 1, 2, 3, \ldots$), is the last one, because the difference, $D_0 - \{d_{k_1}, d_{k_2}, \ldots, d_{k_n}\}$, is the empty set. Then we have $D_0 = \{d_{k_1}, d_{k_2}, \ldots, d_{k_n}\}$, i.e. D_0 is a finite set.

b) The procedure can be continued indefinitely; in other words, to *any* natural number n an element $d_{k_n} \varepsilon D_0$ is attached. Then, by definition 1, D_0 is denumerable.

A kind of inversion of the procedure used for the proof of theorem 1, shows that also a *more extensive* set than that of the natural numbers can be denumerable; e.g., the set of *all integers* (including 0 and the negative integers). In the usual arrangement according to the magnitude of numbers, where the negative integers precede the positive, the set is not enumerated: there is no first element, and no element appears at the kth place (k being a natural number) since every element is preceded by infinitely many other elements (e.g. 1 by 0 and all negative integers). A simple trick, however, allows us still to enumerate our set. Take as the first element $+1$, as the second -1, as the third $+2$, as the fourth -2, etc.; in general put $+n$ to the $(2n-1)$th place, $-n$ to the $(2n)$th place. We thus get the following representation between the set M of all positive and negative integers and the set N of all natural numbers:

M:	$+1$	-1	$+2$	-2	$+3$	-3	...	$+n$	$-n$	...
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$		$\updownarrow$	$\updownarrow$	
N:	1	2	3	4	5	6	...	$2n-1$	$2n$	...

By this procedure the set M has been enumerated; it is, therefore, a denumerable set.

Evidently one does not alter the denumerability of M by adding the element 0; in general the denumerability of an aggregate is not changed by the addition of a *finite* number (k) of new elements. One may, for example, put the new elements at the beginning of the new enumeration, and the only change resulting from this will be an increase of the index which assigns to each element its place in the sequence; in our case, an increase by the constant value k.

Even the addition of *infinitely many* new members to the elements of a denumerable set will again produce a denumerable set if *denumerably many* elements are added. This has just been shown in the case of the set of positive and negative integers. As a matter of fact, the property used is not that the elements are numbers but only that they constitute (mutually exclusive) denumerable sets. The numbers may therefore be replaced by any other kind of objects having the same property. If there are elements common to both sets, the sum will also be denumerable since some of the newcomers have simply to be dropped.

Finally, the same procedure may be applied to the new set, i.e. the elements of a denumerable set may again be added. This step can be repeated a finite number of times. Those familiar with mathematical induction will easily formalize this reasoning [1]). Thus one obtains the following theorem which deals with the *extension* of a denumerable set and is, accordingly, a counterpart to theorem 1 which refers to the *reduction* of a denumerable set:

THEOREM 2. By adding to the elements of a denumerable set a finite number of elements or denumerably many elements, one again obtains a denumerable set. The same result is obtained by forming the sum of a finite number of sets each of which is finite or denumerable — provided that at least one of the sets is infinite.

3. The Set of Rational Numbers. We proceed to an essentially different instance of a denumerable set. Between two consecutive integers n and $n + 1$, there are infinitely many rational numbers

[1]) See p. 38.

or fractions. We shall always write a rational number in the *reduced* form m/n where n is a *positive* integer and m an integer *prime to* n. The integers themselves, too, appear among these fractions in the form $m/1$, including $0/1 = 0$ [1]). One easily sees that between any two rational numbers infinitely many other rationals may be interpolated: in fact, allowing the difference between m_1/n_1 and m_2/n_2 to be as small as you please, one has only to divide the difference into 2, 3, 4, 5, ... equal parts in order to obtain infinitely many numbers interpolated between m_1/n_1 and m_2/n_2. All these numbers are again rational.

Let us now consider the set of *all* rational numbers, positive, negative, and 0. It would seem as if this set were infinitely more comprehensive than its denumerable subset which contains the integers. For between any single pair of two consecutive integers exist infinitely many rational numbers. Nevertheless, *the set of all rational numbers is denumerable.*

One may show this in the following way. On a sheet which should be imagined as extending indefinitely, draw an infinite system of discrete horizontal lines, henceforth called "rows", as well as an infinite system of discrete vertical lines, called "columns". (See fig. 4.) Denote an arbitrary row as well as an arbitrary column by 0; from 0 onwards, number the rows upwards and the columns to the right 1, 2, 3, ..., the rows downwards and the columns to the left — 1, — 2, — 3, ... Each of the intersections between a row and a column, which may be called a "lattice point" of the figure, is then uniquely determined by the indication of the number m of the respective row and of the number n of the respective column. We denote that point symbolically by (m/n).

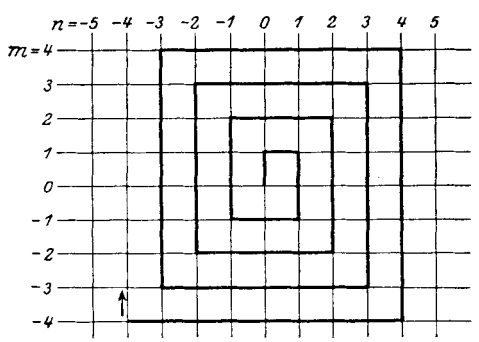

Fig. 4

This accomplished, it is easy to enumerate, firstly, the lattice points. For this purpose follow the route marked in fig. 4, beginning with (0/0) and winding gradually outwards, via (1/0), (1/1), (0/1),

[1]) 1 is considered as the only integer prime to 0.

(— 1/1), etc. Arrange the lattice points in the same order in which they succeed each other on the given route. (In arithmetic and analysis one calls such a rearrangement of a doubly infinite sequence into a simply infinite sequence, *the diagonal method of Cauchy*, for it was Cauchy who used this method in the theory of infinite series; cf. § 6, 9.) From the sequence thus formed drop those lattice points (m/n) which do not correspond to a reduced fraction, i.e. for which m/n has not the properties required before [1]) (n positive, m prime to n). There will then remain infinitely many lattice points, all of them on the right half of the figure, and in accordance with theorem 1 each of them will be assigned a certain place number (smaller than the original one) along our route.

Excepts for the parentheses, the notation of the remaining points coincides with the notation of *all* rational numbers, each of them appearing in the reduced form. In other words, a one-to-one correspondence between all rational numbers and the remaining lattice points has been constructed. Since these points constitute a denumerable set, the same holds for the totality of rational numbers, and it is easy to enumerate them effectively with the aid of the route drawn in fig. 4. The enumeration begins with

$$\frac{1}{1}, \frac{0}{1}, -\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, -\frac{1}{2}, -\frac{2}{1}, \frac{3}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{3}, -\frac{1}{3}, \cdots$$

or, in the familiar form:

$$1,\ 0,\ -1,\ 2,\ \frac{1}{2},\ -\frac{1}{2},\ -2,\ 3,\ \frac{3}{2},\ \frac{2}{3},\ \frac{1}{3},\ -\frac{1}{3}, \ldots$$

This sequence arranges the rational numbers in a way quite different from the usual order according to magnitude, which is obvious in view of the line of numbers (fig. 2, p. 12). But we have reached our goal, to create a representation between the set of all natural and the set of all rational numbers.

We have used a geometrical figure for this purpose. However, the figure is not essential at all in our proof, and it may be instructive to give a similar proof on a strictly arithmetical basis.

[1]) Among the points (m/n) with $m = 0$ or $n = 0$ only one will remain: (0/1). Restricting ourselves to the right half of the figure ($n > 0$), we may obtain a kind of obvious survey over the remaining points and the dropped ones by placing a light at (0/0) and a narrow chip at each other lattice point. Then the dropped points are indicated by the chips left in the shadow of others.

Let m/n be a *positive* reduced fraction: m and n positive integers prime to each other. Denote their sum by s: $m + n = s$. If, on the other hand, s is given, there are (if $s > 2$) other fractions of the same type: m_1/n_1, m_2/n_2 etc. for which the sum of the numerator and the denominator is also s. Arrange the fractions corresponding to the same value of s in a definite order, say by decreasing numerators and therefore increasing denominators. If $s = 7$, we obtain the corresponding fractions in the succession

$$\frac{6}{1}, \frac{5}{2}, \frac{4}{3}, \frac{3}{4}, \frac{2}{5}, \frac{1}{6};$$

if $s = 8$, the fractions 6/2, 4/4, 2/6 have to be dropped because they are not reduced, and for our purpose there remain only the reduced fractions

$$\frac{7}{1}, \frac{5}{3}, \frac{3}{5}, \frac{1}{7}.$$

To a given s, however large it may be, there belong *only a finite number of reduced positive fractions* m_k/n_k with $m_k + n_k = s$; for there exist only a finite number of positive integers m_k smaller than the given s.

Let us now arrange the totality of different positive rational numbers (including, of course, the integers, corresponding to the denominator 1) according to increasing values of s as follows: To $s = 2$ the single number $1/1 = 1$ corresponds; running over the successive values $s = 3, 4, \ldots$, arrange the fractions corresponding to a definite value of s in the order mentioned before, and put them after the fractions corresponding to smaller values of s. Finally, in order to include rationals other than positive, put $0/1 = 0$ (corresponding to $s = 1$) at the beginning of the entire sequence, and let every fraction m/n be followed by the negative fraction $-m/n$. Thus we obtain a sequence of different rational numbers [1]) beginning with

$$\frac{1}{0}; \frac{1}{1}, -\frac{1}{1}; \frac{2}{1}, -\frac{2}{1}, \frac{1}{2}, -\frac{1}{2}; \frac{3}{1}, -\frac{3}{1}, \frac{1}{3}, -\frac{1}{3}; \frac{4}{1}, -\frac{4}{1}, \frac{3}{2}, -\frac{3}{2}, \frac{2}{3}, -\frac{2}{3}, \frac{1}{4}, -\frac{1}{4};$$

$$\frac{5}{1}, -\frac{5}{1}, \frac{1}{5}, -\frac{1}{5}; \frac{6}{1}, -\frac{6}{1}, \frac{5}{2}, -\frac{5}{2}, \frac{4}{3}, -\frac{4}{3}, \frac{3}{4}, -\frac{3}{4}, \frac{2}{5}, -\frac{2}{5}, \frac{1}{6}, -\frac{1}{6}; \frac{7}{1}, -\frac{7}{1}, \frac{5}{3}, -\frac{5}{3}, \cdots$$

[1]) Observe that this arrangement is somewhat different from the one obtained above.

This sequence *contains all different rational numbers*, each appearing at a definite place. In order to ascertain this, let any reduced rational number m/n be given. Form the sum $s_0 = m + n$ or, if m is negative, $s_0 = -m + n$; m/n will be found after all fractions corresponding to smaller values of s, among those arranged for the value $s = s_0$. Since the number of preceding fractions is finite, there exists a definite natural number k assigning to the fraction m/n its place in the entire sequence. The set of all rational numbers has thus been represented on the set of all natural numbers.

It is true that it would not be too easy to indicate an explicit function f assigning to any rational number r the corresponding natural number $k = f(r)$, according to either of the two representations given here. Even for a single number r with a large value of s, the actual calculation of k would take a lot of time. There have been given formulae which enable us to attach to every r its place k in the series, though their construction is somewhat complicated [1]). However, since both representations described here are completely constructive, the difficulty of actually accomplishing them need not bother us much.

Again, as in the case of the enumeration of the integers (p. 46/47), it is obvious that in enumerating the rational numbers we have not used their arithmetical properties, but only their property of forming a *sequence of sequences* — since in m/n, m as well as n assume values contained in denumerable subsets [2]) of the set of all integers. By using theorem 1 we thus obtain the following theorem which is an extension of theorem 2:

THEOREM 3. The sum of denumerably many different sets, each of which is denumerable or finite, is again a denumerable set [3]).

In view of the *geometrical realization* of real numbers, given in the example f) of § 1 (p. 12), one may express the denumerability of the set of all rationals in a geometric form. The line of numbers used in that example furnishes a representation of the set of all real numbers on the set of points on a straight line. Using this

[1]) See Faber 1, Oglobin 1, Boehm 2, Godfrey 1, Johnston 1.

[2]) In view of theorem 1, it is superfluous to define precisely the subsets.

[3]) It is easy to see that the sum cannot be finite.

representation for the *subset* of all rational numbers, a representation is obtained between this subset and a certain subset R of the set of points; the elements of R will be called *rational points* since they are marked by rational numbers. R therefore is denumerable.

As pointed out before, between any two different rational numbers exist infinitely many rational numbers. The arrangement of points from left to right in fig. 2 (p. 12) corresponds to the arrangement of numbers according to magnitude, as expressed by the term *between*. Hence between any two rational points on our straight line, however near to each other they may be, exist infinitely many rational points — *between* here understood in the usual geometrical sense. The set of points R, therefore, fills the line "with infinite density"; later on, we shall call such a set of points a *dense set* (§ 9, 1). The conjecture that R would accordingly contain *all* the points of the line has been refuted on p. 14; how extremely far off the mark it is, we shall see in § 4.

It may appear rather surprising that such a comprehensive set is denumerable. But theorem 3 has just shown that multiplying an infinity infinitely many times does not necessarily enlarge the infinity in the sense of equivalence.

Thus the concept of a definite *extent* or *cardinal number*, tested so well for finite aggregates, seems to lose its meaning where infinite aggregates are concerned. Apparently, infinity plus infinity (theorem 2) or infinity times infinity (theorem 3) just equals infinity without exceeding it. One gets the impression that the concept of the infinite, after the first startling comparison of finite quantities or numbers with infinite ones, is something trivial and boring. *Any* two infinite aggregates, however different they may appear at the first inspection, seem to be equivalent. The proof would consist of creating a representation and require only suitable tricks. If the theorem "all infinite aggregates are equivalent to each other" holds true, then our reasoning has, after all, failed to add something decisive to the idea of the infinite which any gifted schoolboy produces at the age of fifteen.

In the next section, all these conjectures will be thoroughly refuted. Previously another important instance of a denumerable set will be given which further strengthens the impression of futility already produced. This will make the step that forms the first triumph of Cantor's new doctrine appear all the more surprising and dramatic.

4. The Set of Algebraic Numbers. As has been remarked in § 1 we call any real root of the algebraic equation

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0 \tag{1}$$

with integral coefficients a_k a (real) *algebraic number*; more exactly, an algebraic number of the nth degree, if (1) has the degree n, i.e. if $a_0 \neq 0$ (as we always assume) — unless the number in question is at the same time a root of an equation of a lower degree [1]). The rational number m/q is also an algebraic number, namely a number of the first degree. Thus we have, in addition to the rational numbers, the algebraic numbers of the degrees 2, 3, 4, ... [2]). From the elements of algebra we only need the following well-known theorem [3]): *an algebraic equation of the nth degree has not more than n different real roots* [4]).

To prove that *the set of all algebraic numbers is denumerable* [5]),

[1]) For instance, the number 5 is a root of the equation $x^2 - 25 = 0$. Nevertheless, it is of the first degree since already the equation $x - 5 = 0$ has the root $x = 5$.

[2]) Properly speaking, we should still prove that there exist algebraic numbers of all these degrees, i.e. that not every equation of the nth degree can be reduced to equations of lower degrees. In a special case, this will be proved in § 9, 2; the general assertion is also true, but not required for our argument.

[3]) The proof of this theorem runs as follows. Denote by $f(x)$ the left hand side of the equation (1). If r_1 is a real root of this equation, the division of $f(x)$ by $x - r_1$ gives:

$$f(x) = (x - r_1) \cdot f_1(x) + s_1. \tag{1a}$$

Since the substitution of r_1 for x transforms (1a) into $0 = 0 + s_1$, it follows that $s_1 = 0$. A sufficient repetition of this procedure leaves us with

$$f(x) = (x - r_1)(x - r_2) \ldots\ldots (x - r_k) \cdot f_k(x) \tag{1b}$$

where $k \leqq n$ and $f_k(x)$ is a polynomial *without real zeros*. (If $k = n$, $f_k(x)$ is a constant.) Then there exist no other real roots of (1), different from $r_1, r_2, \ldots, r_k$; in fact, for any other real value $x = r_{k+1}$ *each* factor of the right-hand side of (1b) is different from 0, which makes the product $f(x) \neq 0$ for $x = r_{k+1}$.

[4]) As stated before, the restriction to real roots has only been made for reasons of convenience. The results stated here hold as well for complex roots and complex algebraic numbers, and proofs are essentially the same.

[5]) It goes without saying that the special manner of the following proof may be modified in many respects. In particular, the number h called the *amount* may be defined in other ways, provided that its nature enables us

we shall first enumerate all algebraic *equations*. We return to equation (1) in which all coefficients a_k are integers and especially $a_0 \neq 0$. As usual, we denote the *absolute value* [1]) of a real number a by $|a|$ and we call the positive integer

$$(2) \qquad h = (n-1) + |a_0| + |a_1| + \ldots + |a_{n-1}| + |a_n|$$

the *amount* of the equation (1). Hence, any algebraic equation uniquely defines a natural number h as its amount. For example, the equation $2x^2 - 3x + 1 = 0$ has the amount $1 + 2 + 3 + 1 = 7$, the equation $x^3 = 0$ the amount $2 + 1 + 0 + 0 + 0 = 3$. The converse, of course, does not hold; the relation between equations and their amounts is not biunique. But we shall now prove that, given a natural number h, *there exists only a finite number of algebraic equations having the amount* h.

Firstly, the degree n of an equation having the amount h, cannot exceed h, because of $|a_0| \geqq 1$ and (2). Hence, the number of terms $(n + 2)$ appearing on the right-hand side of (2) is not larger than $h + 2$. Secondly, a positive integer h can be represented as the sum of at most $h + 2$ non-negative integers only in a finite number of ways; these may be chosen by taking for the first, the second, etc. term in (2) every time the respective maximum value which still remains admissible. Finally, from all the integral non-negative solutions $n - 1$, $A_0, \ldots, A_k, \ldots, A_n$ of the "diophantic" equation with given h

$$h = (n-1) + A_0 + A_1 + \ldots + A_{n-1} + A_n \qquad (A_0 \neq 0)$$

one obtains all the solutions

$$n - 1 \ , \ a_0 \ , \ a_1 \ , \ \ldots \ , \ a_{n-1} \ , \ a_n$$

of (2) for the given h, by letting $a_0, \ldots, a_k, \ldots, a_n$ independently assume the values

$$a_0 = \pm A_0 \ , \ a_1 = \pm A_1 \ , \ \ldots \ , \ a_{n-1} = \pm A_{n-1} \ , \ a_n = \pm A_n .$$

This supplies 2^{n+1} systems of values if all the A_k are different from 0, otherwise correspondingly less. The systems of solutions a_k

---

to enumerate all equations — which, in algebra, are arranged and classified according to quite different principles.

[1]) The absolute value of a is the *positive* number equalling a or $-a$; or 0 if $a = 0$. For instance: $|5| = 5$, $|-3| = 3$, $|0| = 0$.

which correspond to each of the possible degrees $n = h,\ h-1,\ \ldots,\ 2,\ 1$ furnish all the equations (1) with the amount h.

Let us illustrate by an example this rather abstract proof of the assertion that to any given natural number h there belongs only a finite number of equations having the amount h. Take $h = 3$. Then only equations of the degrees $n = 3, 2, 1$ need be considered; in fact, already for $n = 4$ the relation (2) assumes the form

$$3 = 3 + |a_0| + \ldots + |a_3| + |a_4|$$

which cannot be fulfilled since $a_0 \neq 0$ (which means $|a_0| \geqq 1$). For $h = 3$ we have, therefore, to consider the relations

$$3 = (n-1) + |a_0| + \ldots + |a_n|. \qquad (n = 3, 2, 1;\ a_0 \neq 0)$$

Bearing in mind that on the right-hand side there are at most (in the case $n = 3$) five terms, and considering all the possibilities as suggested above from larger to smaller values of the integers n, $|a_0|$, ..., we obtain the following seven solutions

$$\begin{aligned} 3 &= 2 + 1 + 0 + 0 + 0 && (n = 3)\\ &= 1 + 2 + 0 + 0 = 1 + 1 + 1 + 0 = 1 + 1 + 0 + 1 && (n = 2)\\ &= 0 + 3 + 0 = 0 + 2 + 1 = 0 + 1 + 2. && (n = 1) \end{aligned}$$

For each of these solutions we have now to distribute the signs *plus* and *minus* arbitrarily among the non-vanishing values of $|a_k|$ appearing in the solution — i.e., among all positive terms except the first, which refers to the degree of the equation. For instance, to the first solution correspond the two possibilities

$$3 = 2 + |1| + 0 + 0 + 0 = 2 + |-1| + 0 + 0 + 0,$$

to the third the $2^2 = 4$ possibilities

$$3 = 1+|1|+|1|+0 = 1+|-1|+|1|+0 = 1+|1|+|-1|+0 = 1+|-1|+|-1|+0.$$

The former two possibilities furnish the algebraic equations

$$x^3 = 0, \quad -x^3 = 0;$$

the latter four the equations

$$x^2 + x = 0, \quad -x^2 + x = 0, \quad x^2 - x = 0, \quad -x^2 - x = 0.$$

It is easily seen that the seven non-negative solutions written above produce $2 + 2 + 4 + 4 + 2 + 4 + 4 = 22$ equations, which form the totality of equations having the amount 3.

On account of the result that to a given amount there belongs only a finite number of equations, it is easy to *enumerate all*

algebraic equations [1]). We shall arrange the equations according to their amounts, beginning with $h = 1, 2, 3$, etc. Among the equations of the same amount one may use any order, for example the order used before in the proof of the finiteness. Thus by theorem **3** we get a sequence of equations in which every algebraic equation appears at a certain place, marked by a natural number.

Finally we proceed from the algebraic *equations* to the algebraic *numbers*. As has been pointed out on p. 53, an algebraic equation has only a finite number of (real) roots, no more than is indicated by its degree. Therefore we may arrange the real roots of every equation in any way, for example according to the magnitude of the numbers, and replace each equation of our sequence by the system of all its roots, thus again obtaining a sequence. By this procedure, it is true, the same algebraic number will appear infinitely many times. The number 2, for example, is found among the roots of the equations

$$x - 2 = 0 \ (h = 3), \ x^2 - 4 = 0 \ (h = 6), \ x^4 - 16 = 0 \ (h = 20), \text{ etc.}$$

This inconvenience is eliminated by the rule that any number equalling one of the preceding numbers of the sequence, should be deleted. Thus we get a sequence whose elements are all different, and which contains *every* algebraic number — each among the roots of an equation of minimum amount which the number satisfies. Hereby we have proved Cantor's earliest discovery in the theory of sets [2]):

The set of all (real) algebraic numbers is denumerable.

Here, as in the arrangement of the rational numbers for the purpose of their enumeration, the "natural order" (according to magnitude) of the algebraic numbers is thoroughly destroyed by our arrangement. For example, $-\frac{1}{8}$ (root of $8x + 1 = 0$) and $\sqrt{7} = 2.645\ldots$ (root of $x^2 - 7 = 0$) appear near each other among the roots of equations with the amount $h = 9$, while the number $-\frac{1001}{8000}$, though differing very little from $-\frac{1}{8}$, is found at a remote place in the sequence, as the root of the equation $8000x + 1001 = 0$, having the amount 9001.

[1]) Had we arranged the equations according to their *degree* instead of their amount, we should not have been able to enumerate all equations, since infinitely many equations belong to each degree.

[2]) Cantor **5**, § 1, 1874. Cf. Vandiver **1**.

In analogy to the geometrical realization of p. 51, we may also illustrate the present result. Already the rational numbers fill the line of numbers everywhere with infinite density. Now between these numbers, the algebraic numbers of degree 1, the algebraic numbers of all higher degrees 2, 3, 4, ... intervene and fill the line, as it were, with infinitely greater density. Nevertheless, we have just seen that even the set of all algebraic numbers, and accordingly the set of the points marked by them on the line of numbers, is still denumerable. As a matter of fact, Cantor during his first study of the problem took this result as a hint that the set of *all* points on a line was also denumerable. This, however, is false as will be seen in § 4.

5. Applications to Infinite Sets in General. Hitherto we have in this section only dealt with *denumerable sets*. The following considerations have a completely different character since they refer to *any infinite set*. The properties of denumerable sets already obtained empower us to draw general conclusions which are not trivial at all. The spring-board enabling us to jump from any infinite set to a denumerable set, is the following theorem, whose fundamental character and importance will attract our attention again many times.

THEOREM 4. Any infinite set has a denumerable subset.

This theorem has already been used in the proof of the equivalence between the two definitions of infinity [1]) produced in § 2 (p. 37/40). It will, therefore, be suitable to prove theorem 4 separately on the basis both of definitions VI and VII (loc. cit.). The proofs have quite different character.

Proof A. Let S be a non-inductive set, i.e. not exhausted by k elements for any non-negative integer k. We have to prove that there exists a sequence of elements of S: s_1, s_2, s_3, ..., which according to definition I form a denumerable subset $S_0=\{s_1,s_2,s_3,\dots\}$ of S. We shall use mathematical induction in order to prove that, given any natural number n, there exists a subset of S containing n elements. Since S is not empty we may choose an arbitrary element s_1 of S; then $\{s_1\}$ is a subset of S. Assume that k elements of

[1]) As a matter of fact, only the assertion demonstrated by proof A has been used there.

S have been obtained: $s_1, s_2, \ldots, s_k$, such that $S_k = \{s_1, s_2, \ldots, s_k\}$ is a subset of S. S_k does not exhaust the set S since otherwise S would be finite (inductive), contrary to our assumption; therefore the set $S - S_k$ is not empty. We denote by s_{k+1} an arbitrary element of $S - S_k$ and, by adding s_{k+1} to the elements of S_k, obtain the subset $\{s_1, s_2, \ldots, s_k, s_{k+1}\}$. Hence, by mathematical induction, S has subsets of n elements for any natural number n and, in particular, the subsets may be chosen in such a manner that the subset corresponding to $n + 1$ comprises the one corresponding to n.

Now, allowing for n *all* the values 1, 2, 3, ..., let S_n be a subset of S containing n elements. By virtue of theorem 3, the sum of all these sets S_n (i.e. of a sequence of sets) is a denumerable subset of S (possibly coinciding with S). Q.E.D.

The last step assumes an especially simple form, if — as done above — we choose the subsets S_n such that, for every k, $S_k \subset S_{k+1}$.

We should point out that this proof relies on a procedure not included in the principles stated so far, namely, on the arbitrary choice, at every step, of new elements s_k of S. We shall return to this point in § 6, **6** (p. 123), and in § 11, **6** and **7**.

Proof B. Let S be a reflexive set, i.e. *equivalent to a proper subset* S', and φ a certain biunique correspondence between the elements of the sets S and S'. Denote by t_1 an arbitrary element of the set $S - S'$ which, according to the assumption, is not empty. φ will relate

$t_1 \varepsilon S$ to a certain element of S', to be denoted by t_2
$t_2 \varepsilon S$ to a certain element of S', to be denoted by t_3
. . . . . .
. . . . . .
$t_k \varepsilon S$ to a certain element of S', to be denoted by t_{k+1}.

In this way, an infinite sequence $(t_1, t_2, t_3, \ldots\ldots)$ is determined through mathematical induction, and even *uniquely determined* after the arbitrary choice of the element $t_1 \varepsilon (S - S')$ and of the representation φ. Notice that all the elements $t_k \varepsilon S$ belong to S' except t_1.

Furthermore, the elements t_k of S ($k = 1, 2, 3, \ldots$) are *different from each other*. For if there were equal members among them, let t_m be the *first* t_k equalling a preceding element t_l:

$$t_m = t_l \quad . \quad (l < m;\ m > 1) \tag{1}$$

t_m, belonging to S' because of $m > 1$, is accordingly *different* from t_1, which is not contained in S'; this means that t_l, too, is different from t_1, i.e. $l > 1$. $t_l \varepsilon S'$ is therefore related by φ to a certain element $t_{l-1} \varepsilon S$. Hence (1) may be expressed as follows: the mate (in view of φ) of $t_{m-1} \varepsilon S$ in S' equals the mate of $t_{l-1} \varepsilon S$ in S'.

But then the biunique character of our correspondence implies that

t_{m-1} itself equals t_{l-1}, which is contrary to our supposition that t_m is the *first* element of our sequence that equals a preceding one. This contradiction shows that *no* element of the sequence equals an earlier element, i.e. that all t_k are different from each other. Hence, they form a denumerable subset of S. Q.E.D.

In this proof, in contrast with proof A, nothing has been used that cannot be constructed by means of the principles introduced hitherto and in § 5, 3 (p. 97).

We use theorem 4 to prove, for *any infinite set* S, a property analogous to the property of denumerable sets expressed in theorem 1. Let S_0 denote a *denumerable or finite* subset of S such that [1]) $S - S_0 = \bar{S}$ is still an infinite set. We shall show that $\bar{S} \sim S$.

In accordance with theorem 4 let $\bar{S}'$ denote a denumerable subset of $\bar{S}$, and take $\bar{S} - \bar{S}' = \bar{S}''$. (If $\bar{S}' = \bar{S}$, we have $\bar{S}'' = 0$.) Hence:

$$\bar{S} = \bar{S}' + \bar{S}''. \qquad (\bar{S}' \text{ and } \bar{S}'' \text{ mutually exclusive})$$

According to this notation any element of S belongs to one, and only one, of the subsets S_0, $\bar{S}'$, $\bar{S}''$. Now we construct a representation between the sets

$$S = S_0 + \bar{S}' + \bar{S}'' \quad \text{and} \quad \bar{S} = \bar{S}' + \bar{S}''$$

by first relating every element of $\bar{S}''$ to itself [2]). The elements of the remaining sets $S_0 + \bar{S}'$ and $\bar{S}'$ can be related to each other by a one-to-one correspondence because $\bar{S}'$ is denumerable and S_0 denumerable or finite. Therefore, by theorem 2 (p. 47), these sets are equivalent. Hence $S \sim \bar{S}$; in other words:

THEOREM 5. By dropping from an infinite set a finite number of elements or denumerably many elements, one obtains a set that — provided it is still infinite — is equivalent to the original set.

The condition "provided etc." is unnecessary not only when a finite number of elements are dropped, but also (cf. § 4, 5) when the original infinite set is *not* denumerable.

By inverting theorem 5 we reach the conclusion:

THEOREM 6. By adding a finite number of elements, or denumerably many elements, to an infinite set one obtains a set equivalent to the original set.

---

[1]) Of course, this condition is superfluous if S_0 is finite.

[2]) Since we do not know anything about the nature of $\bar{S}''$ which can be = 0, finite, denumerable, or none of these, we have no choice but to use the identical representation.

In fact, by dropping from the new set S the elements added before, one obtains, according to theorem 5, a set equivalent to S. — One also may prove theorem 6 directly, by using theorem 4 and by proceeding in a way analogous to the proof of theorem 5.

Exercises

1) Prove the denumerability of the sets of
   a) all terminating decimal fractions
   b) all algebraic numbers between 0 and 1.

2) Illustrate the second procedure (p. 50) of enumerating the rational numbers by means of the lattice of points represented in fig. 4.

3) Prove that the set of all those points of the plane whose Cartesian coordinates (with respect to a given system of coordinates) are both rational, is denumerable.

4) Prove that any denumerable set may be represented as the sum of denumerably many denumerable sets which are mutually exclusive.

5) What denumerable subset of S is produced by applying proof A of theorem 4 under the following conditions:
   a) S the set of all natural numbers; let the arbitrary element chosen in S and the subsets in question, be the *smallest* number of the respective set.
   b) S the set of all natural numbers; let the arbitrary element be the smallest number *divisible by* 5.
   c) S the set of all natural numbers; let the arbitrary element be the smallest *prime number*.
   d) S the set of all positive fractions m/n written in their reduced form (p. 48); let the arbitrary element be the fraction for which the *sum* of numerator and denominator $m + n$ has the smallest value, and in the case of several such fractions, the smallest one with respect to magnitude.

6) What denumerable subset is produced by applying proof B of theorem 4 on the assumptions: S the set of all natural numbers, S' the set of all *even* natural numbers, φ the correspondence between $x \varepsilon S$ and $2x \varepsilon S'$, $t_1 = 5$?

7) (cf. theorem 5). Show that out of *any* infinite set one can drop

denumerably many elements such that the new set is infinite, and therefore equivalent to the original set.

8) From the assumption that there exist infinitely many (real) transcendental numbers, infer that the set of all transcendental numbers is equivalent to the set of all real numbers.

9) (For advanced readers). We denote by $< a, b >$ the *closed interval* on the line of numbers $a \leqq x \leqq b$, and we call two intervals *non-overlapping* if they have no common points, except possibly at the extremities. Prove that any set of non-overlapping closed intervals on a line is either finite or denumerable. (Hint: denote by k the smallest integer larger than $1/(b-a)$ and by l the integer next to $k \cdot a$ and smaller than, or equal to, $k \cdot a$; attach $(l+1)/k$ to the interval $< a, b >$.) It is easy to generalize the result to the plane (using rectangles or circles instead of intervals), or even to the space of three or more dimensions.

10) (For readers familiar with the concept *accumulation point*, see § 9, **5**). Prove that an infinite set of points (of the line or the plane), having only a finite number of accumulation points, is denumerable. This theorem cannot be inverted!

11) (For readers familiar with the concepts *monotonic function* and *continuous function*). A monotonic function has at most denumerably many places of discontinuity. (Hint: given any positive integer n, there are in any closed interval only a finite number of jumps with steps larger than $1/n$.)

§ 4. The Continuum. Transfinite Cardinal Numbers

1. Formulation of the Problem. In the following we shall denote by C the set of all positive real numbers smaller than 1, including the number 1 itself. Using the representation of real numbers on the line of numbers (p. 12), we obtain, as the geometrical image of C, the set of all points on the line between the fixed points 0 and 1, including the latter. This set of points, which constitutes an interval, is equivalent to C. Either set is called a *continuum* or even, for reasons explained in **4**, *the* (linear) continuum.

First let us look for a form in which we may express the elements of C simply and uniformly. Everyone will remember the expansion of real numbers into decimal fractions. At school, it is true, the subject is generally treated for practical use only without raising

questions of principle, but a strictly scientific treatment is by no means difficult as soon as the concept of *real number* has been neatly defined. If after a certain digit of the given decimal fraction there are only zeros, we speak of a *terminating* decimal, otherwise (i.e. if after *any* place there still appear digits different from 0) of an *infinite* decimal. Hence any integer is also a terminating decimal. A rather elementary proof shows that, according to the equality defined between decimals, two *terminating decimals* or two *infinite decimals* are equal — that is to say, represent the same real number — only if they are *identical*, i.e. if they equal each other digit by digit.

A different situation arises when we compare terminating and infinite decimals. Even the reader who is not familiar at all with the scientific foundation of the theory of decimals (and of *system fractions* in general) perceives that between the decimals 1 (= 1.000...) and 0.999... there is not a "tiny" difference but complete equality. In order to comprehend this, one has only to multiply the relation $^1/_3 = 0.333...$ by 3. The general theorem of which this is a special case, runs:

> Any positive real number can be uniquely represented as, or expanded into, an *infinite* decimal. A positive number represented by a terminating decimal (i.e. a positive rational number whose denominator in the reduced form is not divisible by any prime number different from 2 and 5), written in the form
>
> $$n.a_1 a_2 \ldots a_m,$$
>
> (n a non-negative integer; $a_k = 0, 1, 2, \ldots, 9$; $a_m \neq 0$) may also be written as an infinite decimal with the period 9 in the form
>
> $$n.a_1 a_2 \ldots (a_m - 1)\, 999 \ldots$$
>
> or, if it is a positive integer n, in the form
>
> $$(n-1).999 \ldots {}^1).$$

One may handle negative numbers in the same way, putting the minus sign before the decimal. The only exception is the number

---

[1]) For instance, $0.123 = 0.122999...$; $1 = 0.999...$. The essence of the theorem is the possibility of representing $1/10^m$ in the two forms $0.00...01$ ($m-1$ zeros after the point) and $0.00...0099...$ (m zeros after the point).

0 which does not admit of a representation as an infinite decimal; its only expansion is the terminating decimal $0 = 0{\cdot}000\ldots$. This is why, in defining the set C, we have dropped the number 0 — to avoid the inconvenience of an element without an infinite expansion. Every element of C therefore admits of a uniquely determined representation as an infinite decimal, and the ambiguity produced by numbers which can be represented both as infinite and terminating decimals is eliminated by the restriction to infinite decimals. Hence:

> The continuum C may be defined as the set of all *infinite* decimals between 0 and 1, including 1.

In this form we shall henceforth take the elements of C. Note that all of them have the form

$$0.a_1\,a_2\,a_3\ldots \qquad (a_k = 0, 1, 2, \ldots, 9)$$

with the restriction that *not all* a_k, for k larger than a certain value, may not assume the value 0.

The goal of the following considerations is to prove that the infinite set C *is not denumerable.* We shall see that C is, in comparison with the set of all natural or rational or even algebraic numbers, so comprehensive that any attempt to represent it on a denumerable set D is bound to fail, since there will always remain elements of C without mates in D. In order to show this it will suffice to prove the following

Lemma. Given any *denumerable* subset C_0 of C, one can construct elements of C that do not belong to C_0. In other words: a denumerable subset of C can by no means contain *all* elements of C.

We may also express this assertion as follows: For any denumerable set of real numbers between 0 and 1 there exist other numbers of this kind not belonging to the set. Accordingly, no denumerable set contains all numbers between 0 and 1; *the set of all these real numbers is not denumerable.* Of course, there exist denumerable sets of infinite decimals of this type; for example, the set of all periodic decimals between 0 and 1 (except those having the period 0) is an infinite subset of the set of all rational numbers, and therefore denumerable. But no such set, asserts our lemma, will exhaust the continuum C.

It may be worth while to add a few words in order to avoid a

misunderstanding frequently found among beginners. When, in accordance with the lemma, one element of C not belonging to C_0 is found, or several or even denumerably many such elements, one might plead: Well, add those elements to C_0, and by virtue of theorem 2 on p. 47 you will again have a denumerable set which now may possibly contain all the elements of C. Such an argument involves a gross logical error. In order to prove the lemma it is fully sufficient to define a *single* element of C not belonging to C_0. In fact, the strength of the argument of the lemma does not lie in the possibility of enlarging C_0 by admitting new elements of C, but in the assumption of C_0 being *any* denumerable subset of C. *No* such set, says the lemma, can ever exhaust the continuum C.

2. Proof of the Non-Denumerability of the Continuum. We prove the lemma by a famous and peculiar procedure due to Cantor [1]). Let C_0 be any denumerable set of elements of C. Using an arbitrary representation Φ between C_0 and the set of natural numbers, we may denote the elements of C_0 as follows:

$$
\begin{array}{lllllll}
1 \longleftrightarrow 0 \cdot a_{11} & a_{12} & a_{13} & a_{14} & \cdots & & \\
2 \longleftrightarrow 0 \cdot a_{21} & a_{22} & a_{23} & a_{24} & \cdots & & \\
3 \longleftrightarrow 0 \cdot a_{31} & a_{32} & a_{33} & a_{34} & \cdots & & \\
4 \longleftrightarrow 0 \cdot a_{41} & a_{42} & a_{43} & a_{44} & \cdots & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & & \\
k \longleftrightarrow 0 \cdot a_{k1} & a_{k2} & a_{k3} & a_{k4} & \cdots & a_{kk} & a_{k\,k+1} \cdots \\
\vdots & \vdots & & & & &
\end{array}
$$

The digits a_{kl} of the decimals which are elements of C_0, bear double indices: the first index k, remaining constant through any definite decimal, indicates the natural number to which the decimal

[1]) See Cantor **11**. This kind of demonstration seems to be the most lucid and the fittest for generalization (cf. subsection **8**). Among other demonstrations we mention the earliest one, in Cantor **5** of 1873, simplified in **7, I**; furthermore, that of Poincaré **4**. The nucleus of all these different demonstrations is the same.

is related by the representation Φ while the second index l indicates the place after the decimal point where the digit is found. (The single-pointed arrows will be explained presently.) Of course, every a_{kl} denotes one of the numbers 0, 1, 2, ..., 9. Disregarding the zeros before the decimal points, we thus obtain an infinite square of digits whose vertex is a_{11}.

Now we define a new sequence of digits $(b_1, b_2, b_3, \ldots, b_k, \ldots)$, and hereby a decimal $b = 0 \cdot b_1 b_2 b_3 \ldots b_k \ldots$, in the following way. Let us pay special attention to the digits a_{kk} (i.e. those with $l = k$); in other words, to the digits forming the *diagonal* that starts from the vertex a_{11} of the infinite square. This diagonal contains the digits $a_{11}, a_{22}, a_{33}, \ldots, a_{kk}, \ldots$. Now we define the digits b_k, for any k, as follows:

$$1)\ b_k = 1 \text{ if } a_{kk} \neq 1; \qquad 2)\ b_k = 2 \text{ if } a_{kk} = 1.$$

Here we have defined a decimal b between 0 and 1 containing the digits 1 and 2 only; b is accordingly an infinite decimal whose kth digit, *for any* k, differs from a_{kk}, i.e. from the kth digit of the kth decimal of C_0 — "the kth decimal" being an abbreviation for "the decimal of C_0 related to the integer k by the representation Φ".

But then b *is different from all elements of* C_0. For, firstly, b cannot be the kth member of C_0, since b_k, the kth digit of b, differs from a_{kk}, the kth digit of the kth member of C_0 [1]), and secondly, these two formally different decimals also represent different real numbers, since the expansion of a real number into an infinite decimal is uniquely determined. Hence b is an element of C not belonging to C_0, and the lemma has thus been demonstrated.

In view of the remarks appended to the lemma, we obtain:

THEOREM 1. The infinite set C of all real numbers between 0 and 1 is not denumerable.

3. Remarks and Supplements to the Proof. In the following subsections, the practical and theoretical consequences of theorem 1

[1]) Of course, this will not be the only place where the decimals differ from each other; their differing at one place, however, is sufficient for the validity of the proof. If, as in subsection 5, C_0 is taken to be the set of all algebraic numbers, b will differ from any decimal of C_0 at infinitely many places. (Not having perceived this fact is one of the many mistakes in Fischer 1, which book ventures to criticize this and other mathematical methods.)

will be developed and valued. First we shall consider the proof itself more thoroughly; in spite of its simplicity it contains one of the most peculiar and most powerful procedures of mathematics in general.

We have defined a single real number b of C which is different from all elements of C_0. But it is clear that our proof really supplies *infinitely many* such numbers, since we have only used the property of b being different from every decimal of C_0 at one place at least. This shows that the digits 1 and 2 favored in the construction of b do not enjoy any privilege at all. The only real condition in constructing b by the method used here, is that b_k, the kth digit of b, be different from the kth digit of the kth decimal of C_0, i.e. from a_{kk}; accordingly, all the nine digits differing from a_{kk} are admissible at the kth place of b. One additional condition only has to be considered: it is not allowed that *all the digits* b_k, from a certain place $k = m$ on, *equal* 0, since this would make b a terminating decimal while the elements of C have been defined as infinite decimals. (In fact, if b were a finite decimal, it might equal the number represented by one of the members of C_0 in spite of their differing formally.)

The use of the base 10 for the decimal system — for decimal fractions, as well as for the decimal representation of integers — has no mathematical or logical reason but rests upon man having ten fingers which he used for counting and reckoning at a primitive stage of civilization. In principle one may use *system fractions of any base* l for representing real numbers, provided that the different powers of l are different, i.e. that the natural number l is larger than 1. In fact the theorem of p. 62 concerning the representation of real numbers holds true independently of the base (cf. § 7, **5**); accordingly, there is no difficulty at all in carrying out the proof of the lemma by means of a base other than 10.

Only in the case $l = 2$ a certain complication will arise — and precisely this basis is preferable for scientific purposes since it is the only base which is absolutely distinguished, as the *smallest* possible base. (For practical purposes it is too small, since already 32 would be written with six digits.) The difficulty is the following: In the *dual system* ($l = 2$) there exist only two digits, 0 and 1. Therefore, if each digit b_k is to differ from the digit a_{kk} with the same value k, one has to take *the* other digit; that is to say, one needs *both* values 0 and 1 — while for $l = 3$ it is already possible to restrict oneself to *two* of the three possible values 0, 1, 2. But, in using the digit 0, one runs the risk of taking the digit 0 always (from a certain place on), i.e. of getting a terminating dual fraction; this is just what has to be avoided because of the equality between certain terminating and infinite fractions.

As a matter of fact, there are simple devices that exclude such a risk. (Cf. exercise 1 at the end of § 4.) For instance, we may introduce the digit 1 at every second place of b and so exclude a finite dual fraction b. In order to prevent b from coinciding with an element of C_0, we then introduce $\bar{a}_{kl} = 1 - a_{kl}$ (for $k, l = 1, 2, 3, \ldots$) and form the infinite dual fraction

$$b = 0.\bar{a}_{11}\ 1\ \bar{a}_{23}\ 1\ \bar{a}_{35}\ 1\ \bar{a}_{47}\ 1 \ldots \bar{a}_{k,2k-1}\ 1 \ldots .$$

b is different from any element of C_0 since the kth dual fraction of C_0 has at its $(2k-1)$st place the digit $a_{k,2k-1}$ while b has at its $(2k-1)$st place the digit $\bar{a}_{k,2k-1}$ (which equals 1 or 0 according as $a_{k,2k-1}$ equals 0 or 1 respectively).

Because of the decisive part played by the diagonal members a_{kk} in the scheme on p. 64, the method of proof used in 2 is called the *diagonal method*; sometimes, in contrast to Cauchy's diagonal method (see p. 49), "Cantor's diagonal method". It is used for the proof of several theorems in the theory of sets, all of them having a similar purpose. (Cf. 8; § 5, 3; § 6, 8 [1].) For this reason, the reader should not proceed to the following considerations before having assimilated the proof of 2, so that it appears completely obvious and simple. Then, he will understand more easily the same reasoning when applied to technically complicated cases, and he will be sufficiently equipped to refute various objections which have been raised against this diagonal method, not because it is really objectionable, but because of its peculiarity and its way of leading to unforeseen and almost paradoxical results [2]).

It is really surprising how simple the proof of theorem 1 is in comparison with its far-reaching consequences. The particular simplicity and transparency of many of Cantor's fundamental proofs constitute a special charm of abstract set theory. They compare favorably with the many profound implications of the proofs on the one hand, and on the other, with the far more com-

[1]) For recent applications of the method, especially in the theory of sets of points, cf. Sierpiński 7*a*. See also Post 6. The application of the diagonal method is also useful in the theory of *orders of infinity*, in connection with the growth of functions; cf. Borel 1 and Hardy 3.

[2]) See, e.g., Bentley 2 and Bridgman 2 and the replies Fraenkel 18 and Rust 1. Even from the "intuitionistic" point of view (cf. *Foundations*, ch. IV) no objection can be made to the lemma and its wholly constructive proof in subsection 2. Only the formulation of theorem 1 may be criticized by intuitionists for its gathering all real numbers into a set. For a more profound discussion of the diagonal method, see Kreisel 1.

plicated and quite technical proofs in other branches of mathematics, including the one most congenial to the theory of sets, viz. the theory of numbers.

In § 3 the *equivalence* between many pairs of sets has been proved. But, except for the proof on p. 38 which is limited to *finite* sets and uses the method characteristic of them, namely, mathematical induction, this is the first time a proof of *non-equivalence* has appeared. In spite of its simplicity, it involves a more profound idea than the proofs of equivalence. This is no accident; theorem 1 is an *impossibility theorem*, stating the impossibility of creating a representation between the linear continuum and the set of natural numbers. Theorems of this kind appear in many branches of mathematics and frequently play a very important part. Some of them have become famous such as, particularly, the impossibility of squaring the circle. The common reason for the difficulty of most impossibility proofs is easily intelligible; one has to take into consideration *all possible* ways of solving the problem in question, and to show that all of them will lead to failure — while a *possibility* proof, e.g. a proof of equivalence, only requires a single construction leading to success. In the case of theorem 1 and its lemma, the proof has to show that no conceivable attempt to form a correspondence will produce a representation between our two sets.

Bearing in mind this logical difference between equivalence and non-equivalence, one may be surprised to see that with *finite sets* it is, both psychologically and logically, no more difficult to prove the non-equivalence than the equivalence; i.e. it is as easy to show that the number of elements in two given sets is *different*, as that the number is *equal*. As a matter of fact, to prove the former assertion a *single failing* in attempting to create a representation is sufficient, and for the latter assertion, a single *success*. In other words, if we are given two finite sets, we may choose any arbitrary procedure of attaching to each element of the one set a single element of the other, such that different elements get different mates. When the procedure leads to a one-to-one correspondence between the elements of both sets, these are equivalent, while if in one of the sets elements are left with no mate in the other, non-equivalence holds between the sets. How does this complete parallelism between both cases agree with the

fundamental difference stated just now in view of the proof of theorem 1?

It is simple to explain this divergence. The fundamental difference between a positive construction and an impossibility proof, as experienced with infinite sets, holds true; non-equivalence is a more profound assertion than equivalence. In fact, the deviation from this state of affairs in the case of the finite sets is caused by a seemingly accidental property of these sets which is generally formulated as the assertion (cf. § 8, 2 and 3) that *to a finite cardinal number belongs one and only one ordinal number*. However the elements of a finite set may be arranged, beginning with a first, a second etc. element, the arrangement always finishes with the same ordinal number "*n*th" — no matter in what order the elements have been taken. (If the set contains the integers from 1 up to a million, you may take them in the "natural" order according to their magnitude, or first the even and then the odd numbers, or after 1 the prime numbers and afterwards the products of two, three, ... prime numbers; or however you please. But it would be somewhat rash to declare it "intuitively clear" that by *any* arrangement the last number exhausting the set must be the millionth number.) This is a *theorem of arithmetic* being both in need, and capable, of a proof, and it is proved by means of mathematical induction, like theorem 4 on p. 38. By virtue of this theorem, it is sufficient to make one attempt of creating a one-to-one correspondence between two finite sets, taking element after element out of each set in an arbitrary order, and in the case of a failure one may be assured that any other attempt would also have led to failure.

In sharp contrast with this is the behavior of infinite sets. Here, as will be seen in § 8, many, and even infinitely many, different ordinal numbers belong to a given set (or cardinal number), and not just one as with a finite cardinal. Therefore, in spite of the equivalence between two given infinite sets, it may happen, and as a matter of fact it always happens, that there are rules of correspondence which exhaust the elements of one set while leaving certain elements without mates in the other. (§ 3 has already supplied us several instances in the simplest case of two denumerable sets; for example, the set of all integers — or the set of all rationals — arranged according to magnitude, as against

the set of all positive integers in the usual order.) Hence the proof of the non-equivalence of sets requires us to take into account *all possible* rules of correspondence, which means that an impossibility proof has to be constructed.

4. Generalization of Theorem 1. In view of the representation of the set of real numbers on the line of numbers, theorem 1 maintains that the set (continuum) of all points of the line contained between the points 0 and 1 is not denumerable. It makes no difference (nor does it, of course, in the case of theorem 1 itself) whether the extremities 0 and 1, or one of them, are added to the set or not; for according to theorems 5 and 6 on p. 59, adding or taking away a finite number of members does not alter an infinite set with respect to its equivalence properties, such as its non-denumerability.

The length of the unit-segment (from 0 to 1) depends on the unit of measure chosen, and is therefore arbitrary (cf. p. 12). Accordingly, the assertion of theorem 1 must remain valid for the set of points between *any* two points of the line, and consequently for the set of all real numbers contained between any two numbers, and not just between 0 and 1.

This logical inference may also be illustrated in a geometrical way. Given two different segments $\overline{AB}$ and $\overline{CD}$ (fig. 5), denote by S the set of points on $\overline{AB}$ and by T the set of points on $\overline{CD}$, including the extremities in both cases. We prove that *S and T are equivalent*, in spite of the different length of the segments. Draw one of the segments above the other and parallel to it, as in fig. 5, and join two pairs of extremities of different segments (e.g. A and C, B and D) by straight lines. Since the segments are different in length, the two lines will intersect at a certain point P which shall be called the *center*. Now any ray proceeding from P will either intersect *both*, or *neither* of, the given segments. We only consider the former case and then relate the intersection of the ray with one segment to its intersection with the other. *This rule creates a representation between the sets S and T.* For, if a point Q

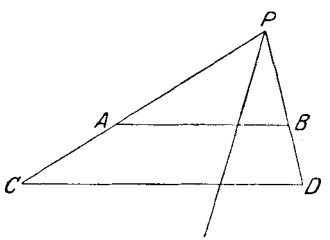

Fig. 5

on one segment is given, we can construct its mate on the other segment. To this end, we join Q and the center P by a straight line, the intersection of which with the other segment will provide the mate of Q. The representation thus constructed between the sets S and T proves their equivalence.

This proof illustrates two previous remarks. It is obvious that the shorter segment $\overline{AB}$ may be considered as a part of the longer one $\overline{CD}$ by virtue of a parallel projection, for instance by drawing a parallel to the line BD through the point A. This will lay $\overline{BA}$ off on a proper part of $\overline{DC}$. Thus the set T proves equivalent to a proper subset T_0 of itself. In other words, the drawing of parallels to the line BD through all the points of the set S (of the segment $\overline{AB}$) will produce a representation between the set S and a proper subset T_0 of T. This way of relating the points of S to points of T does *not* lead to a biunique correspondence between the points of one set and those of the other. For, on $\overline{CD}$, in the neighbourhood of C, there remain infinitely many points of T without a mate in S. This failure, however, does not prove anything regarding the equivalence between both sets, and indeed, they *are* equivalent, as shown, not by the parallel projection but by the radial projection from the center P, as defined above. From the view of metric geometry, it is true, one may consider the parallel projection to give a more natural representation, but according to the definition of equivalence neither of the two projections has any superiority over the other. Only the fact that by one of the methods we succeed in representing the one set on the other, is important here, and not the failure of the other method.

We summarize our result as follows:

THEOREM 2. The set of points on an arbitrary segment of a straight line — including or excluding the endpoints or one of them — is not denumerable, and all such sets are equivalent to each other. The same holds true for the sets of real numbers which form the intervals between two arbitrarily given real numbers a and b.

In short, one calls any set of one of these types a (bounded linear) *continuum*, and in particular a *closed* or an *open* continuum or interval according as the extremities are included or not.

A surprising feature of this theorem evolves by comparing it with the result of p. 56/57. There we found the set of those points of the line (or of a segment) that are marked by *algebraic numbers*, to be a denumerable set. Moreover, every segment of the line is already filled to infinite density with rational points [1]),

[1]) Propositions analogous to theorems 2 and 3 also hold true for the sets of these points; cf. § 3, 3 and § 9, 3. The nucleus of all these propositions

a fortiori with algebraic points, which comprise rationals as a special case. But, in contrast to this, we have seen in theorem 2 that in any arbitrarily small interval of the infinite line, there are contained *more-than-denumerably* many points. This shows, though for the moment not in a rigorous form, in what incomparably larger measure a line is filled with general points, than with algebraic points.

It is rather a surprising thing in theorem 2 that an arbitrary smallness of segment is permitted. But the result is not altered either by extending the segment into the infinite. In fact:

THEOREM 3. The set of points on an infinite straight line is equivalent to the set of points on a finite segment. Hence the set of *all* real numbers is equivalent to the closed continuum.

Again the most convenient way to prove this theorem is by a geometrical method. Given an infinite straight line, and a segment $\overline{AB}$ whose center will be denoted by C (fig. 6), bend the segment, as if it were a thin wire, at C, and lay the bent segment against the infinite line so that C coincides with an arbitrary point of the line and so that the ends A and B lie on the same side (above or beneath) of the line and at the same distance from it (see fig. 6). Finally, denote by S the point midway between A and B in their new positions; accordingly, S will lie exactly above (or beneath) C.

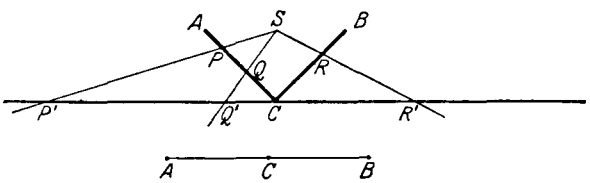

Fig. 6

Now one obtains a simple representation between the set of points on the bent segment $\overline{ACB}$, except its ends A and B, and the set of all points on the infinite line, in the following way. Any ray from the center S passing through a point of the line, intersects the bent segment, and vice versa. Therefore, a one-to-one correspondence is created between the points of the two

might be expressed in colloquial language as follows: with respect to an infinite density it does not matter whether a certain interval is expanded a finite number of times or even denumerably many times.

sets by relating any point of the segment to the point of the infinite line lying *on the same ray from S*. (The rays SA and SB do not intersect the line, being parallel to it, and this is why we previously defined the set of points on the segment excluding its endpoints.) In particular, the point C, lying at the same time on the segment and on the line, corresponds to itself. The representation achieved by our rule shows that both sets are equivalent. Q.E.D.

Let us add an analytical proof to this geometrical one. As has been pointed out on p. 33, a representation is nothing but a one-valued function admitting of a one-valued inversion [1]). It is, therefore, sufficient for our purpose to define such a function, having as the set of arguments the whole line (i.e. the set of all real numbers) and assuming as its values the points of a segment (interval), or conversely. There are many such functions, one of the most familiar being the tangent function $y = \tan x$ whose graph is outlined in fig. 7. While the argument x runs over the open interval from $-\pi/2$ (i.e. $-90°$) to $\pi/2$ ($90°$), y continuously rises and assumes *all* real values. ($y = \cot x$ from $x = 0$ to $x = \pi$ does as well.) By relating to each value of the abscissa x the value of the ordinate y, one obtains a one-to-one correspondence between the points of an interval and of the entire line.

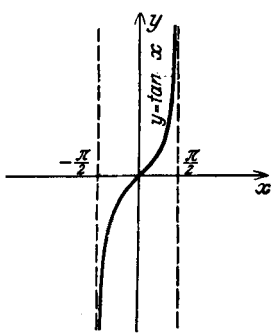

Fig. 7

In this and the preceding subsections certain instances of sets, under the common name *continuum*, have been used not as mere examples, but to prove the existence of non-equivalent infinite sets, and hence of infinite cardinals (**6**). Therefore the question arises *how to assure the existence of the continuum* by means of our principles. The answer — in form of a new principle, furnishing us with even much more general sets — will be given in § 5, 3 (p. 97).

---

[1]) As readers familiar with the elements of calculus or of real functions will easily observe, we could as well say, a one-valued and *monotonic* function ("monotonic" in the stricter sense, meaning that the function must never remain constant).

5. The Existence of Transcendental Numbers. Let us finish the particular considerations connected with theorem 1 by giving an important application to analysis which, in 1874, constituted the first grand triumph of the theory of aggregates [1]). In § 1, a (real) transcendental number has been defined as a real number which is not algebraic. In spite of tremendous efforts, and even after the remarkable progress made since 1930 by A. Gelfond and others, we have hitherto only succeeded in proving the transcendence for relatively limited classes of numbers. Furthermore, in most cases the proof is very complicated and requires a lot of mathematical technique. On the other hand, from theorem 1 we shall now conclude the existence of *infinitely many transcendental numbers*. We even find that it is a kind of "regular" property for a real number to be transcendental, whereas to be algebraic is an exception.

In fact, the set A of all (real) algebraic numbers between 0 and 1 is denumerable while the set C of *all* real numbers between 0 and 1 is non-denumerable. Hence, were the set T of all (real) transcendental numbers between 0 and 1 finite or denumerable, theorem 1 would contradict theorem 2 on p. 47 according to which $C = A + T$ should be denumerable. Therefore, T is infinite and not denumerable, and according to theorem 5 on p. 59, T is even equivalent to C.

In view of theorems 2 and 3, we obtain:

THEOREM 4. The set of all transcendental numbers between two given real numbers, as well as the set of all transcendental numbers, is infinite and equivalent to the continuum.

In colloquial language one might say that there are as many transcendental numbers as real numbers.

This very definite and far-reaching theorem, as we have seen, requires only the concept of equivalence (denumerability) and theorem 1, besides the elementary proposition that the algebraic

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[1]) Cf. Cantor **5**. It is noteworthy that in this first paper in the field of abstract set theory Cantor calls the attention mainly to the first half (proving the denumerability of the set of algebraic numbers) while the second half (the non-denumerability of the continuum) appears rather as an application of the first half to the problem of transcendental numbers. Actually, the second half is much more profound and, at the same time, more important for the general theory (then not yet in existence).

numbers form a denumerable set. The theorem concerns a set out of which to "indicate" even a single element (with the proof of its belonging to the set) is not easy at all. In 1851, about twenty years before Cantor's proof, it was demonstrated for the first time (by Liouville) that transcendental numbers do exist, and only ten years after Cantor's discovery, was π shown to be a transcendental number. To the contemporary mathematicians this discovery should have made clear the importance and strength of the theory of sets which was at that time in the first stage of development. However, a lack of receptiveness for this direction of reasoning, as well as an active antagonism on the part of some leading mathematicians, in particular Kronecker, barred the way to a proper estimation.

It is a current mistake to assume that the evidence of transcendental numbers given here is a mere *existential proof*, furnishing no hint of how to *construct* transcendental numbers. As a matter of fact, at least theoretically our procedure enables us to construct as many transcendental numbers as we like. In order to do so we may choose the set of all real algebraic numbers as the set C_0 mentioned in the lemma on p. 63. Then, any real number constructed according to the diagonal method, either in the special way described in 2 or along the general lines mentioned in 3, is a transcendental number. Bearing in mind that a decimal is defined constructively when a rule is given for choosing the digits of the decimal one after another, we see that the procedure just mentioned constitutes a construction; in order to form the kth digit of the desired decimal one has only to proceed until the kth number in a sequence containing all algebraic numbers and to calculate that kth number up to the kth digit. Practically, it is true, this constructive procedure is without any significance, since by a finite number of steps we can only attain a finite number of digits of the transcendental decimal in question, while the infinitely many *following* digits really decide whether the decimal is algebraic (e.g., in view of its periodicity) or transcendental. This practical uselessness, however, does not impair the fact that in principle *a law for the construction of transcendental numbers has been made available.*

6. The Concept of Cardinal Number. The Cardinals $\aleph_0$ and $\aleph$. From theorem 1 we have drawn conclusions of considerable importance in the fields of geometry and analysis, which manifests that our theorem is a result of great significance for mathematics in general and not restricted to the theory of sets as a special branch. Instances of such propositions of general importance are found in other mathematical fields as well.

But now we have to deal with the significance of the theorem for the theory of sets itself. It is no exaggeration to say that it is the *fundament of abstract set theory.* We shall first point this out in a rather informal way, considering both the attitude of Cantor and the stricter formulations given later by G. Frege and especially by Bertrand Russell. A more detailed discussion of the logical aspect of the procedure in question will be given in **7.**

Let us again take as the starting-point the *finite* aggregates. In § 2 (p. 30/31) a procedure was outlined which leads from equivalent finite aggregates to the concept of their common cardinal number, and thus to the concept of cardinal number in itself. As has been pointed out already by Hume and in a less satisfactory manner even by Descartes, one may in this way arrive at the finite cardinals 1, 2, 3, ...; even 0 as the cardinal of the "empty set" may be obtained by means of this procedure [1]). On the other hand, whenever two aggregates have the same number of elements in the ordinary sense, they are equivalent in the sense of § 2.

As has been mentioned before, these considerations do not use the fact that the aggregates in question are finite. Therefore it is quite natural to attribute the same cardinal to *any* two equivalent aggregates, no matter whether the aggregates are finite or infinite. But here theorem 1 is of decisive importance. True, in §§ 2 and 3 we have met with many pairs of infinite aggregates which are equivalent to each other. However, if we had to consider the eventuality that *all* infinite aggregates were equivalent, the introduction of infinite cardinals would be trivial, and as a matter of fact, no one has ever proposed it before Cantor, although mathematicians have always dealt with infinite aggregates and, implicitly, also with their equivalence. The introduction of one

[1]) Cf. the lecture Hessenberg **11.**

general cardinal "infinite" would not have contributed anything to the efficiency of mathematics. Introducing infinite numbers can mean something interesting and useful only when one has to propose at least *two different numbers, i.e. two non-equivalent infinite sets*. Then the questions of comparing the cardinals and calculating with them may meaningfully be raised and answered.

Precisely this has been made possible by theorem 1. It assures the existence of *at least two* non-equivalent infinite sets, the set of all natural numbers and the continuum C. As we will see in § 5, the diagonal method as used in the proof of theorem 1 (or of the lemma leading to it) even enables us to infer the existence of *infinitely many* infinite sets no two of which are equivalent. Interesting though that may be, in principle *two* non-equivalent sets are sufficient to justify a new definition, the definition of (infinite) cardinals.

But how to define them really? First of all, we may distribute the various sets — no matter whether finite or infinite — into classes such that the sets united in the same class are equivalent to each other, while no set of one class is equivalent to any set of another class. Now, says Cantor in one of his treatments of this subject [1]), the cardinal of a set S should be understood as the general concept (*universal*) which one obtains by abstracting both from the nature (quality) of the elements of S and from the order in which the elements possibly appear in S, thus reflecting only upon what is common to all sets equivalent to S (i.e. to all sets contained in the same class as S).

Although the meaning of this explanation is clear enough, it is difficult to accept it as a *definition* of cardinals. In order to obtain such a definition, it would be theoretically the shortest, if not psychologically the simplest way, to take the very classes of equivalent sets introduced above as the cardinals, as is analogically done in certain theories of irrational numbers; i.e. to define:

(A) *The cardinal of a set S is the set of all sets equivalent to S.*

In **7** we shall hint at some objections raised against this definition either from a logical or from a psychological point of view, and show how it may be modified in order to make it unobjectionable. Essentially, however, it is a satisfactory definition of the concept in question.

[1]) Cantor **12** I, p. 481.

The logician certainly wants an explicit definition of what a cardinal is, and (A) constitutes such a definition. For mathematics, however, it is a question of convenience rather than of necessity to define the concept of cardinal explicitly, and that for two reasons.

First, the mathematician in general is not vitally interested in knowing what the concepts of his science *are* but how one *handles* them [1] – as the chessplayer does not meditate on the *nature* of the bishop or the pawn but on how to *operate* with them. The integers, for example, have mathematical interest not for their very essence and possible metaphysical qualities inherent in them, but for the possibility of comparing them and calculating with them. Therefore it will be sufficient for mathematical purposes to give a "*working definition*" [2] for (both finite and infinite) cardinals. Now this is quite easy, in view of what has been said of the cardinals of finite sets in § 2, namely:

(B) *The cardinals of the sets* S_1 *and* S_2 *are called* equal (=) *if* S_1 *and* S_2 *are equivalent* ($S_1 \sim S_2$). *The cardinals are called* different ($\neq$) *if* S_1 *and* S_2 *are not equivalent.*

As will be seen in §§ 5 and 6, all relations between cardinals (and accordingly, all propositions on cardinals) can be reduced to equalities and inequalities between them or, on account of definition (B), to the equivalence and non-equivalence of sets. Considering this, any proposition dealing with cardinals can be completely understood without a knowledge of what a cardinal *is*, by "translating" the proposition into the language of sets and their equivalence.

Secondly, in close connection with what has been said just now, one can even completely *avoid* the use of cardinals, and some axiomatic foundations of the theory of sets do so indeed [3]. The reduction of the equality of cardinals to the equivalence of sets hints at the possible way of a full elimination. It is true that this method implies inconvenience and clumsiness in the abstract theory. In its applications, however, one may use this method to a wide extent without complications, eliminating even ordinal

[1] Of course, the *applicability* of the mathematical concepts to life or natural science is another thing; we shall touch on this problem in *Foundations*.

[2] Cf. Carnap 3.

[3] Cf. *Foundations*, ch. II.

numbers (§§ 10 and 11). But the inconvenience of such a procedure justifies a special definition like (A) or (B); after all, it is just the striving for convenience that, in most cases, suggests the establishment of new definitions.

By extending the concept *cardinal of a set* from the finite sets and numbers to *any* numbers, we obtain an answer to the question "how many elements are contained in a given set?" even when the set is infinite. We need no more be satisfied with the trivial answer "infinitely many elements", which would hold for the integers as well as for the continuum. On the other hand, our earlier experience shows that the state of affairs here thoroughly differs from the behavior of finite numbers; for a set may contain more elements than another (the set of algebraic numbers more than the set of rationals), and nevertheless have the same cardinal as the second set because they are equivalent. As a matter of fact, this property of an infinite set, of being equivalent to proper subsets of itself, is no accident, but just a characteristic of infinite sets which distinguishes them from finite ones (see § 2, **5**).

The cardinals of infinite sets are called *infinite* or *transfinite cardinals*. For the notation of *general* cardinals we shall use *bold letters*; e.g., the cardinals of the sets S and T will be denoted by $\mathbf{s}$ and $\mathbf{t}$. However, when we confine ourselves to finite cardinals, i.e. to natural numbers including 0, we continue to write k, m, n etc. It is often convenient to denote the cardinal of S by using the symbol S, to which has been added, imitating Cantor, a double bar; then $\overline{\overline{S}}$ replaces $\mathbf{s}$. As to special cardinals, we shall follow Cantor, and partly Hausdorff, denoting them by $\aleph$ (= aleph, the first letter of the Hebrew alphabet), and adding natural numbers (including 0) as indices: $\aleph_0, \aleph_1, \aleph_2, \ldots \aleph_k \ldots$. In § 11, **5** we shall introduce even more general indices. So far, we have become acquainted with two different cardinals, the cardinal of denumerable sets (see § 3) and the cardinal of the continuum, often called the *power of the continuum* [1]). The former will be denoted by $\aleph_0$ (*aleph-zero*), the latter by $\aleph$ (*aleph*) without index. The reasons

[1]) There are sound historical reasons for using the term *power* besides *cardinal* (especially used in French and German; *puissance* and *nombre cardinal*, *Mächtigkeit* and *Kardinalzahl*) and, in addition, the term *aleph*; cf. § 5, **5** and § 11, **7**.

for the use of the index 0, and of the other indices, as well as for the notation without index, will be partly given in § 5, 2 and partly in § 11.

7. Further Analysis and Criticism of the Definition of Cardinal. As to Cantor himself, his procedure of definition by abstraction has been quoted in **6** [1]). From 1887 on he often expressed the process of abstraction by a special symbolism; since one must keep in mind a *double* abstraction (from the nature of the elements, and from their order) the cardinal of the set S is denoted by $\bar{\bar{S}}$. An analogous symbolism is used for the *types of order*, see § 8, **4**.

The essence of this definition by abstraction, which is sometimes formulated in the form "the cardinal of S is what is common to S and to all equivalent sets", is not peculiar either to this special concept or to the theory of sets. Wherever in mathematics, and also often in physics, a relation **R** appears which is reflexive, symmetrical, and transitive [2]), a new concept (different from the objects connected by the relation) is created "by abstraction" [3]). A few examples may be given. Through the relation of parallelism, the set of all rays leads to the concept of *direction* [4]); through the relation of geometrical similarity, the set of all plane figures leads to the concept of *shape*; through the relation of congruence *modulo* m, the set of all integers leads to the concept of *congruence-class modulo* m; etc. In our case the objects in question are sets, no matter whether finite or infinite and what their elements may be. From sets we obtain cardinal numbers by taking equivalence (§ 2, **4**) as the characteristic relation. Expressing the fact that the relation $x\,\mathbf{R}\,y$ holds, by "x and y are of the same **R**-type" [5]), one

[1]) Cantor **10**, pp. 81ff. As he points out there, he had already used this "definition" in 1883 (lecture in Freiburg) and in 1884 (letter to K. Lasswitz). The symbolism appears for the first time in **10** and is repeated in **12**.

In his earlier essays (**6** and **7**, V), Cantor avoids an explicit definition and is satisfied with the *working definition* (B) mentioned on p. 78. On the other hand, in his review of Frege **1** (see Cantor **15**, pp. 440–441), he uses a definition similar to that of Frege. As a matter of fact, the differences of opinion between Cantor and Frege on this point are trifling and can hardly explain to us the tension that prevailed between them. But the mathematical fashion of their time deprecated Frege's attitude as subtle and conceived cardinals as mere *signs* on paper, devoid of any meaning — thus confusing concepts and their notations.

[2]) See § 2, **4**; in particular the footnote on p. 36.

[3]) The comprehensive paper Ore **2** shows what profound theory, including applications to algebra and other branches, can be based on this general relation alone. Cf. also Dubreil–Dubreil **1**.

[4]) In the former terminology: the concept of direction is created from the concept of ray by abstracting from the particular *position in space* of the individual rays.

[5]) The co-ordination by *and* is justified by the symmetry of **R**.

may call the new concept "the **R**-type of the objects in question". One can, therefore, take the classes of objects, connected to each other by the relation **R**, immediately as realizations of the new concept. According to this definition, the **R**-type of x and y is the same if, and only if, $x\,\mathbf{R}\,y$ holds. The "abstraction" is thus created by neglecting all properties of the original objects, except those relying upon the relation **R**. The cardinal number of S would then mean the totality of all properties shared by S with any equivalent set and with no set being non-equivalent to S. One may consider this definition by abstraction as a particular and especially important case of a definitory procedure which is usual in the whole domain of mathematics, and is sometimes called the *creative mathematical definition* [1].

From the logical point of view, however, the definition by abstraction certainly did not obtain a real justification by these ways of reasoning [2]. Logically the procedure has been completely redeemed by B. Russell and Frege in independent ways.

Russell's term "principle of abstraction" [3] rather means the removal of such a principle, either in Cantor's sense or in a similar one, and its replacement by an exact logical procedure. Its main essence is this: given a symmetrical and transitive relation **R**, one may, by means of **R**, define a one-to-many relation [4] $\mathbf{R}^*$ such that $x\,\mathbf{R}\,y$ implies $z\,\mathbf{R}^*\,x$ and $z\,\mathbf{R}^*\,y$, where z is uniquely determined by x (or y) but not conversely. z will be called the **R**-type of x, and therefore of any y fulfilling $y\,\mathbf{R}\,x$. If **R** is the equivalence between sets, z is the cardinal of x. Since with the methods of symbolic

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[1]) So does Weyl (**7**, No. 2); cf. the careful presentations in Pasch **1**, p. 40; Hessenberg **10**, p. 71 and **12**, § 3; Hasse **1**, I, pp. 16—19 of the first edition. The essential idea of the definition by abstraction has already been expressed by Leibniz. As to the modern literature on the subject, mention should be made of Burali-Forti—Enriques **1**; Dubislav **2**, **5** and **7**; Maccaferri **1** and **2**; Rougier **2**; K. Schmidt **1**. The essays Peano **4** and **5** give practically nothing but the definition (B) of p. 78.

During the last fifty years, the emphasis in the theory of definition in general philosophical literature has shifted from a *substantial* procedure (*genus proximum* and *differentia specifica*, as in the orthodox theory of Aristotle) to a *functional* one, based on the relations between the concept to be defined and other, already known concepts. Cf. Cassirer **2**; Heymans **1** and **2**; Nagel **5**; Schlick **1**. The general importance of the process of abstraction and even of the identification of distinct concepts for science in general and mathematics in particular has been stressed by Meyerson (**1**; cf. also Lichtenstein **1**).

[2]) It is characteristic that so eminent and careful a scholar as R. Dedekind (see **3**, III, p. 489) appeals to the creative faculty of mankind ("we are of divine origin") in order to save the definition.

[3]) Russell **1**, pp. 166 and 220; Whitehead–Russell **1**, I (i.e. *Principia Mathematica*), 72.66. Cf. Russell **5** and **7**, p. 42; Nicod **3**.

[4]) See p. 31. Accordingly, the relation is of the type "z is the father of x"; or "z is the husband of x" under a polygamic rule.

logic (cf. *Principia Mathematica*, loc. cit.) the existence of the relation **R*** can be proved, the introduction of cardinals is fully justified.

Many years before Russell, Frege [1]) too, had given a logically satisfactory introduction of cardinals, limiting himself, however, to *finite* cardinals. His procedure is still closer than Russell's to definition (A) on p. 77. Like Frege's logical system in general, his definition of cardinals was also ignored [2]), until Russell pointed out the importance of his methods [3]).

One easily grasps from these hints that a refinement of the "definition by abstraction" essentially aims at definition (A) on p. 77. One need not take seriously the objection, frequently heard from conservative philosophers or old-fashioned mathematicians: "the natural numbers really are not classes of classes, but much simpler objects"; the same objection might be made against many concepts of mathematics and logic, such as continuity or dimension, which apparently are intuitively clear but actually require a most careful and technically complicated definition. On the other hand, certain logical difficulties are connected with the simple formulation (A); they will be discussed in *Foundations*, ch. III (theory of types).

As to definition (B) on p. 78, of course it does not answer the philosophical question "what *is* a number, or a cardinal?". For the use of mathematics, however, there is no objection in principle against adopting it, though for the practical application it may prove rather awkward. One must not discredit the use of such "incomplete symbols" in mathematics with the argument that in principle one should always be able to *eliminate*, i.e. to replace by symbols known before, any symbol introduced through a definition; or in more colloquial language, the argument that one should know what the new concept *is*, not what tasks it performs. This attitude would go too far, at any rate in mathematics. As a matter of fact, the definitions by "mathematical" or "transfinite" *induction* (§ 10, 2) do not fulfil that postulate in general, and they represent a kind of definition singularly important in mathematics.

For the sake of completeness let us mention the possibility of defining the concept in question by an *effective example*, just as the unit of length is defined by the *normal metre* kept in Paris. In the case of cardinals it would mean that in any system of equivalent sets a particular set is marked out and appointed to represent the cardinal number of *any set equivalent to it*. Thus

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[1]) See Frege **1**, especially §§ 34–68; cf. **2**, I.

[2]) The criticism of Frege's pioneer work **1** in Cantor's review (**15**, pp. 440–441) does not do justice either to Frege's intentions or to the importance of his ideas. The criticism in Smart **6** is based on misunderstandings.

[3]) The monograph Scholz–Schweitzer **1** (cf. Bachmann **1**, especially p. 56) gives a comprehensive account of Russell's (and Frege's) theory of the definition by abstraction in the light of modern logic, including a criticism of older theories and a logically important extension to relations of $2n$ arguments, analogous to the symmetric and transitive relations of two arguments. — The treatment in Klein–Barmen **5** can hardly be considered a progress, because of the vagueness of its concepts.

one might define the number 3 by the set {sun, earth, moon}, or by an equivalent set of, say, counters kept at a certain place, or by the set {0, 1, 2}; in the latter case 0, 1, 2 shall be conceived not as numbers but as meaningless symbols, or as numbers whose definition precedes the definition of the number 3 in question. It is obvious that this way as indicated by the former two examples, is not practicable in general, because of its complete arbitrariness [1]); the method hinted at in the last example, however, is of great importance in principle and will be dealt with in § 11, 2 and 5, in a form applicable to ordinals and cardinals, as proposed by von Neumann. More generally one might describe the procedure in question as follows: the cardinals form a system of symbols, possibly arbitrary and meaningless, which satisfy the double condition that to any set a uniquely determined symbol is attached, called *the cardinal of the set*, and that to two different sets the same symbol is attached if, and only if, the sets are equivalent.

What has been explained regarding the concept of cardinal, will suffice to convince the reader that the objections raised by some philosophers [2]) against this concept, and herewith sometimes against the theory of sets in general, are far from sound. This remark, of course, does not touch the attitude that *rejects any infinite set*, including the set of all natural numbers, as a closed concept [3]); this point of view certainly is consistent and irrefutable — as is the philosophical attitude of solipsism. In mathematics such an attitude, when combined with a recognition of mathematical induction, leads to *intuitionism* (*Foundations*, ch. IV) in some form or other. A real and complete repudiation of the infinite, it is true, cannot be accepted by the mathematician since it would annihilate mathematics in the bulk; the existence of mathematics, as it were, refutes this attitude in a similar way as in the eye of common sense the collision with a tree should convince the solipsist of the existence of external entities.

There is, however, one fundamental difference between the other concepts mentioned above and the concept of cardinal (as well as that of order-type, § 8): *the principles* introduced in §§ 2 and 3 and in the two following sections, though sufficient for the development of the general theory of sets, *do not enable us to obtain the concepts of cardinal and order-type*. Without engaging in the question of what kinds of logic are fit to dispense with this defect, two mathematical remarks may suffice. Firstly, the method

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[1]) This arbitrariness may be eliminated by the formulation: the cardinal of S is S, as well as any set equivalent to S. Admittedly this does not uniquely define cardinals, but neither are rational or real numbers uniquely determined symbols. As to rationals, it is easy to distinguish a fixed representative, but this would be difficult in the case of real numbers according to Cantor's theory of irrationals.

[2]) For instance, Buchholz 1, pp. 25—38; Dieck 2, pp. 106 ff.; Kaufmann 1; Parkhurst-Kingsland 1; Warrain 1, for the ordinal theory; Ziehen 1 and 2.

[3]) Psychological arguments for this attitude are given in the (generally unsatisfactory) book Chaslin 1. Cf. the references given there.

of von Neumann only requires the addition of one new principle [1]) which is desirable also for other reasons, in order to establish certain *special* sets. Secondly, one may introduce the cardinals separately in an axiomatic way [2]). Nevertheless, it is because of this complication that some axiomatic treatments of the theory of sets completely avoid the concept of cardinal (and ordinal) number.

8. The Set of all Functions and its Cardinal. Hitherto we have found two transfinite cardinals, $\aleph_0$ and $\aleph$. Now we shall consider an infinite set whose cardinal is different from both these numbers.

A singe-valued *function* $f(x)$ of one argument x is defined when to any value of x from a certain domain, a definite value $f(x)$ is attached by a fixed rule. In what follows the closed interval from 0 to 1 shall be taken as the domain for the independent variable x, and the dependent variable $y = f(x)$ shall be restricted to *real* values. In contrast with the functions creating "representations" (see p. 33), a *one-to-one* correspondence is not required here; the same value $f(x)$ may be attached to different values x. — Example: $y = 2x^2 + 3x + 4 = f(x)$; for the special value $x = \frac{1}{4}$ one obtains $f(\frac{1}{4}) = 2 \cdot (\frac{1}{4})^2 + 3 \cdot (\frac{1}{4}) + 4 = \frac{39}{8}$.

Let us now consider the set F of *all* (real) functions $f(x)$, x running over the interval from 0 to 1. Every function of this kind is an element of F. Two functions $f_1(x)$ and $f_2(x)$ are considered as different whenever the rules, attaching to every x a value $f_k(x)$ in both cases, do not coincide *completely*, i.e. for *all* values x. Accordingly the existence of one x for which $f_1(x)$ is different from $f_2(x)$, suffices to make $f_1(x)$ and $f_2(x)$ different functions.

It is easy to define subsets of F which are equivalent to the continuum. Such a subset is formed by all *constant* functions $f(x) = c$ where c runs over the set of all real numbers, or over another continuum. By relating the number c to the function $f(x) = c$, we have obtained a representation between the continuum and a subset of F. Hence F is certainly *not denumerable*, as shown by the proof of theorem 1 (2).

In order to prove that F is not equivalent to the continuum — e.g., to the closed continuum C from 0 to 1, as we may specialize in view of theorems 2 and 3 (p. 71 f.) — we again use the method of diagonal, in a way closely similar to that taken in the proof of

[1]) Principle of replacement; see *Foundations*, ch. II.

[2]) Cf. especially Baer 4.

theorem 1. Let F_0 be *any subset of F that is equivalent to C*; we have seen that such subsets exist. Our task is to show that F_0 *cannot coincide* with F, or in other words, that the assumed equivalence between F_0 and C implies the *existence of functions* $f(x)$ (elements of F) *not contained in* F_0. We shall explicitly construct such a function; of course, in a way which depends on F_0 and its representation on C.

To form a function of the desired type, we start from a definite representation Φ between the continuum C and the subset F_0 of F which by assumption is equivalent to C. In the proof of 2 the correspondence between the natural numbers and the decimals of a set, assumed to be denumerable, was expressed by indices k given to the decimals. In the present case ordinary indices, being integers, would not be sufficient since C is not denumerable. Nevertheless, it is convenient to denote the functions $f(x)$ which are the elements of F_0 by indices that hint at the chosen representation Φ between C and F_0; the indices, accordingly, have this time to run over the set C, i.e. to assume all real values between 0 and 1. $f_c(x)$ will thus denote the function of F_0 related to the number c of C by the representation Φ; e.g. $f_{1/4}(x)$, the function related to $\frac{1}{4} \varepsilon C$.

The main point of the proof is the construction of a function $g(x)$; it is based on the fact that a function is defined when its value is given for *every* value of the argument, and proceeds in the following way. Let c be any value between 0 and 1 (these extremities included); the value at $x = c$ of the function to be constructed, i.e. $g(c)$, shall be equal to the value at $x = c$ of the function of F_0 related to c by Φ, i.e. equal to $f_c(c)$. According to this rule, the value $g(\frac{1}{4})$ is found as follows: we take the function $f_{1/4}(x)$ of F_0, related to $\frac{1}{4} \varepsilon C$, and look for the value of this function at $x = \frac{1}{4}$, namely $f_{1/4}(\frac{1}{4})$. If $f_{1/4}(x) = x^2 + 3$, $g(\frac{1}{4}) = f_{1/4}(\frac{1}{4}) = \frac{49}{16}$.

In short, the definition of $g(x)$ is:

$$g(x) = f_x(x),$$

where x runs over the interval from 0 to 1.

Comparing this definition to the proof given in 2, one easily perceives that $g(x)$ is completely analogous to the decimal $0.a_{11}\, a_{22} \ldots a_{kk} \ldots$ formed by the digits of the diagonal in the scheme of p. 64. $g(x)$ is a sort of *diagonal function* in so far as the value of g at any place $x = x_0$ is determined by a twofold entry

of x_0: by defining a certain *function* $f_{x_0}(x)$ of F_0, and as the *place* where the value of this function is to be taken. (Cf. *loc. cit.*: a_{kk} is defined as the kth digit [kth column of the square] of the kth decimal of the sequence [kth line].)

Finally, we choose a function $h(x)$ which is everywhere, i.e. for every value of x, *different from* $g(x)$; of course, this can be accomplished in infinitely many ways, e.g. by

$$h(x) = g(x) + 1.$$

Choosing this $h(x)$, one immediately sees that it does not occur among the functions contained in F_0. To prove this, we take *any* function of F_0, say $f_{x_0}(x)$ where x_0 is an element of C. It is sufficient to show that $h(x)$ differs from $f_{x_0}(x)$ at a certain place. We take the place $x = x_0$, and obtain

$$h(x_0) = g(x_0) + 1 = f_{x_0}(x_0) + 1 \neq f_{x_0}(x_0),$$

Q.E.D. So F_0 does not contain all elements of F, that is to say all real functions defined over C. Hence: [1])

THEOREM 5. The set containing all single-valued real functions $f(x)$, defined in the closed interval $0 \leqq x \leqq 1$, has a cardinal **f** that is different from the cardinals $\aleph_0$ and $\aleph$.

The reader who is acquainted with the ordinary functions appearing in the elements of calculus, or even with the general concept of *continuous function*, should be warned against confusing the concept of function used in our theorem with the concept previously known to him. The proof of our theorem decisively relies on the enormous generality of the functions $f(x)$ involved and, in particular, on the fact that the value of $f(x)$ at a certain place $x = x_0$ is not determined by the values of $f(x)$ in the *neighborhood* of x_0 (i.e. for x differing but little from x_0) but is wholly independent of these values. In calculus and the classical theory of functions one does not meet functions of this degree of generality or arbitrariness. On the other hand, it will be shown in § 7, **6**, that the set of all *continuous* functions is "only" equivalent to the continuum, and not to the set F of functions considered in theorem 5. As a matter of fact, the rule determining $g(x)$, and accordingly $h(x)$, in our proof necessarily involves discontinuity; there is no special affinity between the values of $g(x)$ for two values

[1]) The theorem, with essentially the same proof, is found in Cantor **11**.

$x = x_1$ and $x = x_2$, based on an affinity (neighborhood) between the argument-values x_1 and x_2.

Note finally that the very proof given here for theorem 5 represents another proof of the non-denumerability of the continuum (theorem 1), or of a set closely related to the continuum, without making any *formal* changes of the proof, by only conceiving it in another aspect. For this purpose we may imagine $f(x)$ as an arithmetical function, i.e. as a function whose argument x runs over the *natural numbers* and which also assumes as its values only natural numbers. If then

$$D = \{f_1(x),\ f_2(x),\ \ldots,\ f_n(x),\ \ldots\}$$

is *any* denumerable set of such functions, the proof of theorem 5 shows that the arithmetical function

$$h(x) = f_x(x) + 1 \qquad (x = 1, 2, 3, \ldots)$$

does not appear in D. Hence the set of all arithmetical functions is non-denumerable. (The connection between the concepts *arithmetical function* and *real number* is easily intelligible; cf. also § 7, **5**.)

Exercises

1) Prove the non-denumerability of the set of all real numbers, represented as "dual" system fractions (p. 66), by using theorem 5 of § 3 (p. 59) instead of the trick of p. 67.

2) Prove that the following sets have the cardinal $\aleph$:

a) the set of all irrational real numbers;

b) the set of all sequences of natural numbers.

3) Give a rule by an analytic formula (say, by means of a rational function) representing:

a) the set of real numbers between a and b on the set of real numbers between c and d (a, b, c, d different real numbers);

b) the set of real numbers between 0 and a on the set of real numbers larger than the positive number b.

4) Show that the set of all points on the circumference (or on an arc) of a circle, or of an ellipse, or of a hyperbola, has the cardinal $\aleph$. (This statement may easily be generalized by means of a suitable concept of *curve*.)

5) What modification will the proof of theorem 5 have to undergo if the argument x of the functions of F runs over the totality of all real numbers?

6) How has the proof of theorem 5 to be modified in order to show the non-equivalence of F to *any subset* of the continuum C?

CHAPTER II

EQUIVALENCE AND CARDINALS

§ 5. Ordering of Cardinals

1. Definition of Order. In addition to the finite cardinals 0, 1, 2, 3, . . . , transfinite cardinals have been introduced in § 4, and with three of them we have become explicitly acquainted; with $\aleph_0$, $\aleph$, and the cardinal $\mathfrak{f}$ of the set of all functions.

Among the *finite* cardinals, it is natural to define which of two different cardinals has to be considered *smaller* than the other. One may formulate this well-known definition of order by referring to sets with the given cardinals in the following way: if S and T are finite sets, and if S is equivalent to a proper subset of T, the cardinal of S is called *smaller* than the cardinal of T. In particular, the cardinal of any proper subset of S is therefore smaller than the cardinal of S itself. It is necessary to speak here of a *proper* subset of T, for the equivalence of S to T itself would signify the equality of their cardinals, and of two equal numbers neither is smaller than the other.

For example, 3 is smaller than 5, because $\{s_1, s_2, s_3\}$ is equivalent to the proper subset $\{t_1, t_2, t_3\}$ of the set $\{t_1, t_2, t_3, t_4, t_5\}$.

Our next aim is to arrange the *transfinite cardinals* in an analogous manner which is called *order according to magnitude*. However, we see at once that the above definition is not practicable in this case, for an infinite set S, as has been shown in § 2, is always equivalent to certain proper subsets — a property even used for the definition of infinity (p. 40). The cardinal of such a subset, being equal to the cardinal of S, would at the same time be smaller than the cardinal of S according to the definition just formulated for finite sets. The set N of all integers, for example, has the same cardinal $\aleph_0$ as its subset containing the even numbers only, and therefore the latter set cannot have a smaller cardinal than N itself.

So, to arrange the cardinals of two sets according to magnitude, we have to add a condition, which will also enable us to drop the

insistence on a *proper* subset. The new condition, of course, might be the non-equivalence between the two sets. But it is more convenient [1]) to express it in the following way:

> *Definition of Order between Cardinals.* If the set S is equivalent to a subset of the set T, while T is not equivalent to any subset of S, the cardinal $\mathfrak{s}$ of S is called *smaller* than the cardinal $\mathfrak{t}$ of T. In symbols:
>
> $$\mathfrak{s} < \mathfrak{t}, \text{ or } \overline{\overline{S}} < \overline{\overline{T}} \text{ (cf. p. 79).}$$

First, one has to show that this definition is a reasonable one; more precisely, that the definition provides the relation of order with the properties to be expected in general of an order-relation, as will be explained in detail in § 8, 2.

a) The relation is *irreflexive*, i.e. $\mathfrak{s} < \mathfrak{t}$ implies $\mathfrak{s} \neq \mathfrak{t}$. Indeed, $\mathfrak{s} = \mathfrak{t}$ would mean that S and T were equivalent, contrary to the (second) condition that T is not equivalent to any subset of S.

Thus it becomes evident that the subset of T appearing in the first condition of our definition is necessarily a *proper* subset, in accordance with the definition for finite cardinals mentioned before.

b) The relation is *transitive*, or (with a slight extension), the assumptions $\mathfrak{s} \leqq \mathfrak{t}$ [2]) and $\mathfrak{t} < \mathfrak{w}$ together imply $\mathfrak{s} < \mathfrak{w}$. For, by using representations mapping S on a subset of T (including T itself), and T on a subset of W, one obtains a representation between S and a subset of W. (Cf. p. 37.) On the other hand, were W equivalent to a subset of S, then by combining a representation expressing this equivalence with the representation mentioned which maps S on a subset of T, we would obtain a representation between W and a subset of T, contrary to the assumption $\mathfrak{t} < \mathfrak{w}$ by which W is not equivalent to any subset of T.

c) The relation is *asymmetrical*, that is to say, $\mathfrak{s} < \mathfrak{t}$ and $\mathfrak{t} < \mathfrak{s}$ are incompatible. For by b) they would imply $\mathfrak{s} < \mathfrak{s}$, in contradiction to a).

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[1]) Both formulations have the same meaning in view of the equivalence theorem in subsection 4. In fact, our definition comprehends the non-equivalence between S and T as a special case, since S is a subset of itself. On the other hand, if S is equivalent to a subset of T but not to T, the equivalence theorem asserts that T is not equivalent to *any* subset of S (cf. p. 98). Until the equivalence theorem will be available, however, it is simpler to use the definition given here.

[2]) $\mathfrak{s} \leqq \mathfrak{t}$ means "$\mathfrak{s}$ is smaller than $\mathfrak{t}$, or equals $\mathfrak{t}$".

d) The assumptions $\mathfrak{s} < \mathfrak{t}$, $\mathfrak{s} = \mathfrak{s}'$, $\mathfrak{t} = \mathfrak{t}'$ imply $\mathfrak{s}' < \mathfrak{t}'$; in other words, every term (cardinal) in any valid assertion of order may be replaced by an equal cardinal. This property has to be fulfilled with respect to *any* relation defined in a mathematical branch, since it expresses a necessary condition for equality. $<$ evidently has this property, since $\mathfrak{s} = \mathfrak{s}'$ means that the sets S and S' are equivalent, and the conditions expressed in our definition of $\mathfrak{s} < \mathfrak{t}$ are not changed by the transition to equivalent sets.

The properties a) — c) of the *relation between cardinals* $\mathfrak{s} < \mathfrak{t}$ also hold for the *relation between sets* "X is a proper subset of Y", which is the basis for the ordinary arrangement of finite cardinals. However, our definition of order can be used among finite cardinals as well as among infinite ones (or between a finite and an infinite cardinal).

The property c) leads to a remark of practical importance. While the equivalence relation $S \sim T$ is symmetrical, so that instead of $S \sim T$ one may also write $T \sim S$, such a permutation is impossible in the case of $\mathfrak{s} < \mathfrak{t}$. Therefore, whenever one wishes to express this relation between $\mathfrak{s}$ and $\mathfrak{t}$ by starting with $\mathfrak{t}$, one has to use another symbol, and a word different from *smaller than*. As is usual in ordinary language and also in other cases of order relations occurring in mathematics, we write

$$\mathfrak{t} > \mathfrak{s} \quad (\mathfrak{t} \textit{ is larger than } \mathfrak{s})$$

synonymously with $\mathfrak{s} < \mathfrak{t}$. The properties of the order-relation stated above may immediately be transferred to $>$. (The relation $\mathfrak{s} < \mathfrak{t}$ or $\mathfrak{t} > \mathfrak{s}$ is sometimes called an *inequality*, in contrast to the *equality* $\mathfrak{s} = \mathfrak{t}$.)

The properties a) to d) of the order-relation, however, do not exhaust all that would be expected of it. In fact, property c) says that the relations $\mathfrak{s} < \mathfrak{t}$ and $\mathfrak{t} < \mathfrak{s}$ cannot hold together, i.e. that *at most* one of them holds, and property a) adds that between *equal* cardinals the relation cannot hold, but other order-relations have also the property that for *different* $\mathfrak{s}$ and $\mathfrak{t}$, *at least* one of the relations $\mathfrak{s} < \mathfrak{t}$ and $\mathfrak{t} < \mathfrak{s}$ holds, so that one can state: *for any pair of cardinals* $\mathfrak{s},\mathfrak{t}$, *one and only one of the cases* $\mathfrak{s} < \mathfrak{t}$, $\mathfrak{s} = \mathfrak{t}$, $\mathfrak{s} > \mathfrak{t}$ (i.e. $\mathfrak{t} < \mathfrak{s}$) *is true.* (*Connexity* of the relation.)

We cannot prove this proposition with the resources now at our disposal. A profound and rather difficult proof, using concepts of

the theory of equivalence alone, will be given in § 11, **7**. The ordinary proofs of that proposition, however, use concepts of the theory of order to be explained in Chapter III (cf. below, **5**). In a somewhat disguised form, order even enters into the proof of § 11 [1]).

2. **Simple Consequences.** It is evident that our definition also arranges *finite* cardinals (in the usual order); the restriction to a *proper* subset in the definition mentioned at the beginning of **1**, is here replaced by the second condition of the definition of order.

As to the *transfinite* cardinals introduced hitherto, first one has $\aleph_0 < \aleph$. In order to prove this, let us take as the representatives of the cardinals in question the set N of all natural numbers and the continuum C of all real numbers. Then N certainly is equivalent to a subset of C since N is itself a subset of C. On the other hand, the corollary on p. 46 says that any subset of N is either finite or denumerable, hence by no means equivalent to C. Therefore, $\aleph_0 < \aleph$ holds true.

Secondly, one has $\aleph < \mathfrak{f}$. Taking now as the representative of $\aleph$ the set K of all real numbers between 0 and 1, as the representative of $\mathfrak{f}$ the set of functions considered on p. 84 (set of all real functions $f(x)$ defined in the interval from 0 to 1), we easily find subsets F^* of F equivalent to K. We may, for example, choose F^* as the set of all *constant* functions having a number between 0 and 1 as their value. Relating to every number k of K the constant function $f(x) = k$, we have a representation between K and F^*. On the other hand, on p. 84 ff. it has been proved that F is not equivalent to K, and it is easy to gather from that proof that the same holds certainly true for any *subset of* K (cf. exercise 6, p. 88) [2]). The inequality $\aleph < \mathfrak{f}$ has thus been proved.

As to more general results, let us first state the following two propositions:

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[1]) Such situations occur more than once in mathematics. One can even *prove* the impossibility of demonstrating the theorem of Desargues within the plane (unless using congruence properties).

[2]) The proof may be outlined as follows. In the case of the proper subset $K^* \subset K$, the function $h(x)$, mentioned on p. 86, may be defined in the way described there, for those values of x which are elements of K^*. For the other values of x, $h(x)$ may be defined in an arbitrary way, e.g. by $h(x) = 1$.

THEOREM 1. Among the transfinite cardinals there is a *smallest* one, the cardinal $\aleph_0$ of the denumerable sets. Any finite cardinal is smaller than $\aleph_0$.

THEOREM 2. Any finite cardinal is smaller than any transfinite cardinal.

By theorem 4 on p. 57, any infinite set has a denumerable subset. On the other hand, an infinite set which is not denumerable, cannot be equivalent to a subset of a denumerable set, since any such subset is either finite or denumerable (corollary, p. 46). Therefore, for any transfinite cardinal $\mathfrak{s}$ different from $\aleph_0$, one has $\aleph_0 < \mathfrak{s}$. Furthermore, denoting by n any finite cardinal, we have $n < \aleph_0$, since a denumerable set has a subset of n elements and is not equivalent to any finite set (viz., to any subset of a set of the cardinal n).

As to theorem 2, the inequality $n < \mathfrak{s}$ is contained in theorem 1 in the case $\mathfrak{s} = \aleph_0$; for other transfinite values of $\mathfrak{s}$ our inequality follows from the relations $n < \aleph_0$ and $\aleph_0 < \mathfrak{s}$ by the transitivity of the relation $<$.

The inequality $\aleph_0 < \aleph$ stated above naturally induces the question: does there exist a cardinal $\mathfrak{s}$ *between* $\aleph_0$ and $\aleph$, such that $\aleph_0 < \mathfrak{s} < \aleph$, or is $\aleph$ the *next* cardinal following $\aleph_0$? For more than sixty years, mathematicians have been endeavoring to answer this question, but hitherto in vain. One may as well put the question in the following way: is every non-denumerable infinite subset of the continuum C, equivalent to C or not? In the latter case, one would have non-denumerable subsets with a cardinal which is smaller than $\aleph$. The analogous question arises with respect to the pair of cardinals $\aleph$ and $\mathfrak{f}$. Cantor [1]) was from the beginning convinced that $\aleph$ is *next* to $\aleph_0$, and in 1884, his desperate effort to enforce the proof of this hypothesis was partly responsible for a dangerous breakdown of his health. Outstanding scholars, including Hilbert [2]), tried in vain to solve the problem. At the time of the third International Congress of Mathematicians (1904) attempts were made to prove both Cantor's hypothesis and its negation, but both proofs turned out to be mistaken. Only quite recently has a decisive progress been made

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[1]) Cf. Cantor **6** (1878) and many of his later papers.

[2]) See Hilbert **9** (1925).

by Gödel who proved that Cantor's hypothesis, even in a wide generalization, is *non-contradictory* (1938); cf. § 11, 7.

3. Cantor's Theorem. The Power-Set. In contrast, a question of the utmost importance which forms a sort of counterpart to the extremely delicate problem just mentioned, was solved by Cantor himself: the question whether there are cardinals larger than $\mathfrak{f}$, and more generally, larger than any given cardinal. The highly comprehensive answer, often called *Cantor's theorem*, runs:

THEOREM 3. To any set S there exist sets having larger cardinals than S; in particular, the set $\mathbf{U}S$, whose elements are *all the subsets of* S, is of a larger cardinal than S.

Accordingly there is *no largest cardinal*. Just as the series of finite cardinals beginning with 0 (or 1) is unlimited, so also is the series of transfinite cardinals beginning with $\aleph_0$.

The assertion with respect to $\mathbf{U}S$ holds for finite sets as well as for infinite ones. It goes without saying that the null-set and S itself are included among the subsets of S forming the elements of $\mathbf{U}S$.

Proof [1]. The first part of the proof, that is the construction of a subset of $\mathbf{U}S$ equivalent to S, is quite simple. We may choose as the subset in question the set whose elements are the sets $\{s\}$ where s runs over all elements of S — that is to say, the subsets of S containing a *single* element. We obtain a representation between this subset and S by attaching the element $\{s\}$ of $\mathbf{U}S$ to the element s of S.

The second part is less simple; we again use the *diagonal method* (pp. 65 and 85—86) to show the impossibility of a one-to-one correspondence. The course of demonstration will be clearer if we first prove that $\mathbf{U}S$ *is not equivalent to* S *itself*; afterwards we shall hint at the slight modification required when, instead of S, any *subset of* S is taken.

Let φ (cf. p. 64) denote any fixed representation between S and a subset U_0 of $\mathbf{U}S$. By showing that this assumption necessarily implies the inequality $U_0 \neq \mathbf{U}S$ (i.e. U_0 to be a *proper* subset of

[1]) Cf. Hessenberg 3, pp. 41 and 42; Zermelo 3, p. 276. The basic idea of the proof is already found in Cantor 11 (1892); cf. his letter to Dedekind of 1899 (see Cantor 15, p. 448). With entirely different and more complicated methods, Cantor had shown the existence of infinitely many transfinite cardinals as early as 1883 (in 7, V); cf. § 11, 5.

$\mathbf{U}S$), we shall reach our aim, for this proves that the full set $\mathbf{U}S$ is *not* equivalent to S. φ assigns to every *element* of S a certain *subset* of S; it will therefore suffice to construct a subset u (at least one!) of S that has no mate among the elements of S by virtue of φ; in other words, to construct an element u of $\mathbf{U}S$ which is not contained in U_0.

For our purpose we shall take into consideration that, after having chosen φ, we may classify the elements of S according to the following alternative: s *is* an element of the subset u_s corresponding to s by φ, or s *is not* an element of u_s. (In either case u_s is, of course, an element of U_0.) According to this classification we shall speak of elements *of the first kind* and *of the second kind*. We do not require, however, that there exist elements of both kinds.

We now denote by u^* *the set of all elements of the second kind.* (If all elements of S should be of the first kind, u^* is the null-set.) In any case u^* is a subset of S, and therefore an element of $\mathbf{U}S$. We shall show that u^* *is not contained in the subset* U_0 *of* $\mathbf{U}S$.

For if u^* was an element of U_0, and s^* the element of S related to u^* by φ, s^* ought to be either of the first or of the second kind. In the former case, s^* is an element of u^* (by the definition of the first kind); but this contradicts the definition of u^*, which only contains elements of the second kind. On the other hand, if s^* was of the second kind then it would *not* be an element of u^*, whereas u^* *per definitionem* contains every element of S of the second kind [1]). Since both suppositions lead to contradictions, no element s^* of S can be the mate of $u^* \varepsilon \mathbf{U}S$, and hence u^* does not belong to U_0. Q.E.D.

In carrying out this proof of non-equivalence for any *subset* $\bar{S}$ of S (instead of S itself; see above), we have to limit the classification used before, distinguishing between the first and the second kind, *to the elements of* $\bar{S}$ *alone*. From the assumption that there exists a representation between $\bar{S}$ and a subset U_0 of $\mathbf{U}S$, we again

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[1]) The reader should not be intimidated by this kind of contradiction. Another instance of it, the so-called antinomy of Russell, yields a logical contradiction. Here, however, the reasoning is clear. The assumption that u^* is contained in U_0, implies its having a mate s^* in S. s^* may be an element of u^* (first kind) or not (second kind). But our proof shows that either case involves a contradiction and therefore the assumption has to be dropped.

conclude that the subset of $\overline{S}$, containing the elements of the second kind only, cannot belong to U_0. Therefore $\mathbf{U}S$ is not equivalent to $\overline{S}$. The proof of theorem 3 is thus completed.

Of course the theorem is also valid for finite sets. If S is the null-set and therefore of the cardinal 0, $\mathbf{U}S$ contains the null-set as its (only) element and has the cardinal 1. For a set $\{a\}$ containing a single element a, $\mathbf{U}\{a\} = \{0, \{a\}\}$ has the cardinal 2. With regard to a pair $\{a, b\} = p$ we have $\mathbf{U}p = \{0, \{a\}, \{b\}, \{a, b\}\}$, i.e. a set of 4 elements. An elementary theorem of arithmetic asserts that, if n is a natural number (or 0), a set of n elements has precisely 2^n subsets; we shall obtain this result as a special case of a much more general theorem (§ 7, 3). For the sake of brevity, the set of all subsets of S may, therefore, be called the *power-set* of S. Reviewing the proof of theorem 3 for a finite set S, beginners will more easily understand its idea.

As to infinite sets, the theorem guarantees the fundamental fact that *there exists no largest cardinal*; if $\mathfrak{s}$ is the cardinal of S, the cardinal of $\mathbf{U}S$ is larger than $\mathfrak{s}$. By this fact the idea — refuted already in § 4 — that in the domain of infinity all differences of magnitude are effaced, is turned into its contrary. Just as in the domain of the finite, in infinity, too, we have to discern between *infinitely many different cardinals*. It will later become obvious (in §§ 6 and 11) that there even exist, in a quite definite sense, *incomparably more* different infinite cardinals than finite ones. Thus the variety of individual numbers found in the domain of infinity, still surpasses the variety of finite numbers. This holds true for transfinite *cardinals*, and still more for the generalized concept of *ordinals*, as will be seen in §§ 8 and 11.

In particular, the inequalities $\aleph_0 < \aleph$ and $\aleph < \mathfrak{f}$ proved on p. 92, may be taken as special cases of theorem 3. For, as is explained in § 7, the continuum may be conceived as the power-set of a denumerable set, and the set of all functions as the power-set of the continuum.

We conclude the discussion of theorem 3 with two remarks dealing with matters of principle.

First, the theorem is not a purely existential proposition, asserting that no cardinal can be the largest one. What has been proved is a *constructive proposition*; beginning with any cardinal (or with any given set), we have indicated a way of obtaining

a set of a larger cardinal. On the other hand, this way in the general case is rather stony, and the construction involves certain difficulties of a logical rather than mathematical nature [1]).

Secondly, the principles used hitherto to secure the sets necessary for our operations, are not sufficient for garanteeing the power-set of any given set S. The individual subsets of S, it is true, are secured by means of the principle of subsets (p. 22), but we have not yet established the right of collecting all subsets of S into a single set whose elements are the subsets. We therefore need a further principle:

Principle of power-set (VI). To any given set S there exists the set whose elements are all the subsets of S. It is called the *power-set of* S and denoted by $\mathbf{U}S$.

In dealing with the continuum (§ 4) it has been remarked that our principles were not sufficient to guarantee the existence of the continuum that is of the set of all real numbers, even when starting with the infinite set of all natural numbers, or of all rational numbers. *Now principle VI secures the continuum.* As is shown in the theory of irrational numbers (cf. § 9, **1**), a single real number may be considered as a sequence — or as a denumerable set — of rational numbers. Because it is easy here to replace the rationals by natural numbers [2]), a real number may as well be conceived as a certain subset of the set N whose elements are all natural numbers. The set of all real numbers thus appears as a subset of the power-set of N. In § 7, **5**, this reasoning will be carried out more strictly, and we shall find that $\mathbf{U}N$ itself constitutes the continuum.

4. The Equivalence Theorem. We have yet to investigate the question of connexity, mentioned at the end of **1**. Although we shall not be able to answer it completely with the methods now at our disposal, an important step forward is possible. Let us start from the definition of order (p. 90) and discuss all the relations of the kind in question that are possible between two given sets

[1]) Cf. *Foundations*. As to the difficulties involved through the theory of types, cf. Quine **16** and Fitch **6**.

[2]) This is plausible since rationals are nothing but (ordered) pairs of integers.

S and T. For this purpose we may use a general procedure of logic, proposed in this case for the first time by Cantor in 1899 [1]) and useful also in other instances in the theory of sets.

Firstly, there may exist subsets of S which are equivalent to T, as well as subsets of T equivalent to S;

secondly, there may be subsets of S equivalent to T, but no subsets of T equivalent to S;

thirdly, the converse may hold, i.e. no subset of S is equivalent to T, but there are subsets of T equivalent to S;

fourthly, there is neither a subset of S equivalent to T, nor a subset of T equivalent to S.

According to the logical principle of the excluded middle, these four cases exhaust all logical possibilities; but, of course, it is not necessary that each of them should be realizable. We may conveniently illustrate all the cases by the following scheme, in which $\mathbf{s}$ and $\mathbf{t}$ denote the cardinals of the sets S and T. (The indication for the first case given in the scheme will be soon explained.)

	T equivalent to a subset of S	T not equivalent to any subset of S
S equivalent to a subset of T	first case: $S \sim T$, $\mathbf{s} = \mathbf{t}$	third case: $\mathbf{s} < \mathbf{t}$
S not equivalent to any subset of T	second case: $\mathbf{t} < \mathbf{s}$	fourth case

Now, according to the definition of order, the second case means that the cardinal of T is smaller than that of S, and the third case means that the cardinal of S is smaller than that of T. Therefore we have still to investigate the first and fourth cases.

In the first case, the corresponding relation between the cardinals of S and T is given by the following theorem, which had early been guessed by Cantor, but was demonstrated only towards the turn of the century:

THEOREM 4 (*equivalence theorem*). If both of the sets S and T are

[1]) In his letter to Dedekind of 30.8.1899 (Cantor **15**, p. 450); cf. Schoenflies **11**, pp. 101—102. A similar procedure is found in Borel **1**, pp. 102—103 (1898).

equivalent to a subset of the other, the sets S and T are equivalent to each other, i.e. their cardinals are equal.

The equivalence theorem is not only of theoretical significance, through its supplementing the list of possible equivalence relations between sets, but also of considerable practical value, as in many cases it proves difficult to construct a direct representation between two supposedly equivalent sets while it is easy to represent either of them on a subset of the other. A characteristic instance will be given in § 7, 6.

First proof [1]) of the equivalence theorem. Let us assume that $S \sim T_1 \subset T$ and $T \sim S_1 \subset S$ [2]). Our theorem says: $S \sim T$.

We may give this assertion a simpler form. Any representation between T and S_1 represents the proper subset T_1 of T on a certain proper subset S_2 of S_1 (p. 37); that is to say, $T_1 \sim S_2$. The relations $S_2 \subset S_1 \subset S$ show that $S_2 \subset S$. On the other hand, the equivalences

$$S \sim T_1, \quad T_1 \sim S_2$$

imply $S \sim S_2$ by the transitivity of the equivalence relation. Now, to prove the equivalence theorem, we only need to demonstrate $S \sim S_1$, for this implies, by $S_1 \sim T$, the desired result $S \sim T$. Therefore the following (apparently obvious) assertion must be proved: *If the set S is equivalent to its proper subset S_2, S is also equivalent to any subset S_1 "between S and S_2", i.e. to any proper subset of S which includes S_2 as a proper subset.*

In order to simplify the argument, we now denote by A the set S_2, by B the subset of S_1 that contains the elements not belonging to S_2, and by C the subset of S that contains the elements not belonging to S_1. We therefore have:

$$S_1 = A + B, \quad S = A + B + C,$$

and any two of the sets A, B, C are mutually exclusive. In this

[1]) After two fundamentally different proofs, the historical and methodological aspects of demonstrating this theorem will be described.

[2]) Obviously, we may confine ourselves to the case where the respective subsets (T_1 and S_1) of T and S are *proper* subsets. Otherwise, if $T_1 = T$ or $S_1 = S$, the assumption already contains the result to be proved.

Accordingly, the equivalence theorem is void in the case of *finite* sets. For the equivalences $S \sim T_1 \subset T$ and $T \sim S_1 \subset S$ immediately imply that each of both sets is equivalent to a proper subset of itself, which is impossible for finite sets (theorem 4 on p. 38).

notation our aim is to infer the equivalence $A + B + C \sim A + B$ from the assumption $A + B + C \sim A$.

Let ψ be a certain representation between the supposedly equivalent sets $A + B + C$ and A [1]). Let us first apply ψ separately to each of the subsets A, B, C of $A + B + C$; we thus obtain mutually exclusive subsets A_1, B_1, C_1 of A such that

$$A \sim A_1,\ B \sim B_1,\ C \sim C_1,\ A_1 + B_1 + C_1 = A.$$

The second step consists of using a representation ψ_1 between the equivalent sets A and A_1 [2]); instead of using ψ_1 for A as a whole, we apply it to the mutually exclusive subsets A_1, B_1, C_1 of A, and thus obtain mutually exclusive subsets A_2, B_2, C_2 of A_1 such that

$$A_1 \sim A_2,\ B_1 \sim B_2,\ C_1 \sim C_2,\ A_2 + B_2 + C_2 = A_1.$$

Proceeding further in this way, at the kth step we reach a representation ψ_{k-1} between the equivalent sets A_{k-2} and A_{k-1}. We apply ψ_{k-1} to the mutually exclusive subsets A_{k-1}, B_{k-1}, C_{k-1} of A_{k-2}, and thus mutually exclusive subsets A_k, B_k, C_k of A_{k-1} are defined for which the following relations hold:

$$A_{k-1} \sim A_k,\ B_{k-1} \sim B_k,\ C_{k-1} \sim C_k,\ A_k + B_k + C_k = A_{k-1}.$$

This procedure can be continued indefinitely since the sets A_k will never be exhausted. In fact, although every transition from A_{k-1} to A_k means the removal from A_{k-1} of certain elements (namely the elements of A_{k-1} contained in $B_k + C_k$), the relations

$$A \sim A_1 \sim A_2 \sim \ldots \sim A_{k-1} \sim A_k \sim \ldots$$

show that the progress from step to step does not even mean a

[1]) Two sets are called *effectively equivalent* if a certain representation between them can be *constructed*. The following proof will put in evidence that, if ψ is given, a certain representation between $A + B + C$ and $A + B$, that is between S and S_1, can be *constructed*. In other words, if the equivalences mentioned in the assumption of the equivalence theorem are effective, the same is true of the equivalence between S and T asserted by our theorem. — In the same way, a set is called *effectively denumerable* if a representation of it on the set of all natural numbers can be *constructed*. We shall return to these concepts in *Foundations*. Cf. Borel 4, Sierpiński 2 and 9.

[2]) ψ_1 may evidently be chosen as a *part* of the representation ψ between $A + B + C$ and A; more exactly, as the part of ψ referring to the subset $A \subset A + B + C$. The same is true of the representations $\psi_2, \psi_3, \ldots, \psi_k, \ldots$.

diminution of the cardinals of the sets A_k. (Cf. the first example given on p. 26.) The scheme of fig. 8 may illustrate the successive steps of the procedure described here.

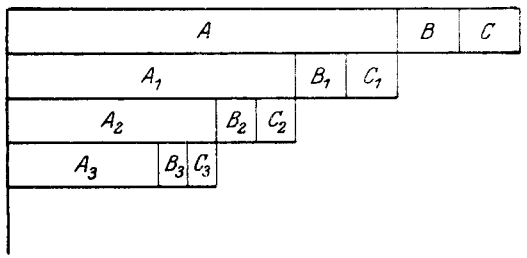

Fig. 8

Now, two cases are possible: there may exist elements common to *all* the sets A_k, each of which is a subset of its predecessor; or no such element may exist. (Both cases have been illustrated by the examples given on p. 26/27.) *In either case we denote by* M *the meet of all sets* A_k; in the second case, we have $M = 0$ [1]). The original set $A + B + C$ may therefore be considered as the sum of M and of the infinitely (more precisely: denumerably) many sets

$$C, B, C_1, B_1, \ldots, C_k, B_k, \ldots$$

any two of which (including M) are mutually exclusive. On the other hand, the set $A + B$, whose equivalence to $A + B + C$ has to be proved, may as well be considered as the sum of the mutually exclusive sets $M, B, C_1, B_1, C_2, \ldots, C_k, B_k, \ldots$ (cf. fig. 8). Since the sum is independent of the order of its terms (p. 27) we may also enumerate the terms of $A + B$ in the form:

$$M, C_1, B, C_2, B_1, \ldots, C_k, B_{k-1}, C_{k+1}, B_k, \ldots.$$

Now, the aim of our proof is to construct a representation between the sets $A + B + C$ and $A + B$. For this purpose it will suffice, first to create a one-to-one correspondence between the denumerably many sets with the sums respectively $A + B + C$ and $A + B$, and secondly to establish representations between each

[1]) If $M = 0$, M may be conceived as the left edge bounding the rectangles A and A_k in fig. 8. If $M \neq 0$, imagine M as a fixed rectangle, appearing on the left within all the rectangles A and A_k, such that with increasing k these rectangles asymptotically decrease towards M.

couple of sets formed by means of the correspondence of the first step. Such a correspondence is indicated by the scheme

$$\begin{array}{lccccccccccc} A+B+C: & M & C & B & C_1 & B_1 & C_2 & B_2 & \dots & C_k & B_k & \dots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \\ A+B: & M & C_1 & B & C_2 & B_1 & C_3 & B_2 & \dots & C_{k+1} & B_k & \dots \end{array}$$

In words: M, B, and every B_k, are attached to themselves, while the term C of $A + B + C$ is related to the term C_1 of $A + B$, and for any k the term C_k of $A + B + C$ is related to the term C_{k+1} of $A + B$. For the sets attached to themselves we use the identical representation (relating every element to itself), whereas for the other couples we rely upon those portions of the representation ψ (between $A + B + C$ and A) which have provided the equivalences $C \sim C_1$, $C_k \sim C_{k+1}$. Since any element of $A + B + C$ belongs to one and *only one* of the sets M, C, B, C_k, B_k, and since an analogous statement, with the omission of C, holds for the elements of $A + B$, every element of $A + B + C$ has been related, by a one-to-one correspondence, to a distinct element of $A + B$, either to itself or to a different element. A representation between $A + B + C$ and $A + B$ has thus been constructed, and the sets are equivalent. Q.E.D.

Second proof. The assumption is that there exists a representation φ between S and $T_1 \subset T$, and a representation χ between T and $S_1 \subset S$. For the proof of theorem 4, it will be sufficient to show that there are (proper) subsets, S_0 of S and T_0 of T, such that φ represents S_0 on T_0, and χ represents $T - T_0$ on $S - S_0$ [1]. Since any element of S belongs either to S_0 or to $S - S_0$, and since an analogous statement holds with respect to T, we thus obtain a representation between S and T, as desired.

In order to define the subsets S_0 and T_0, we first *attach to any subset* $X \subset S$ *another subset* $X^* \subset S$ in the following way: by φ there corresponds to any $X \subset S$ a certain $Y \subset T_1 \subset T$; on the other hand, by χ there corresponds to $T - Y$ a certain set [2] $S - X^*$. The subset X^* of S is thus uniquely determined by X.

Our aim is to find an X for which $X^* = X$. We maintain:

a) X_1 and X_2 being subsets of S, the relation $X_1 \subset X_2$ implies $X_1^* \subset X_2^*$.

For $X_1 \subset X_2$ at first implies $Y_1 \subset Y_2$, by the nature of φ, and from this follows $T - Y_2 \subset T - Y_1$. But the last relation, by the nature of χ, implies $S - X_2^* \subset S - X_1^*$, which means $X_1^* \subset X_2^*$. Q.E.D.

[1] See Banach 1.

[2] Since $T - Y \subseteq T$, χ represents $T - Y$ on a certain subset S' of $S_1 \subset S$. Therefore $X^* = S - S'$.

Since X^* is uniquely determined by X, one may include equality in a) and write: $X_1 \subseteq X_2$ implies $X_1^* \subseteq X_2^*$.

We shall now call $X \subset S$ *a distinguished subset of* S, if $X \subseteq X^*$. *Let* S_0 *be the sum of all distinguished subsets.* Then the relation $D \subseteq S_0$ holds true for any distinguished subset D; hence by a): $D^* \subseteq S_0^*$. Accordingly, for any distinguished $D \subset S$ one has:

$$D \subseteq D^* \subseteq S_0^*.$$

But the relation $D \subseteq S_0^*$, holding thus for any D, holds as well for the sum of all D's; i.e.

b) $$S_0 \subseteq S_0^*.$$

From this we infer by a): $S_0^* \subseteq (S_0^*)^*$. Therefore S_0^*, too (as well as S_0) is a distinguished subset, which by the definition of S_0 means: $S_0^* \subseteq S_0$. By combining this relation with b), we obtain $S_0 = S_0^*$.

But, according to the definition of X^* through X as given above, this means that φ represents $S_0 \subset S$ on $T_0 \subset T$ while χ represents $T - T_0$ on $S - S_0$. Q.E.D.

In short we may summarize the proof as follows: we call "distinguished subsets of S" those $X \subset S$ for which, if φ represents X on $Y \subset T$, χ represents $T - Y$ on $S - Z$ *on condition that* $X \subseteq Z$; then the sum of all distinguished sets is again a distinguished set X, and for this X, $X = Z$.

The fundamental difference between the first and the second proof lies in the fact that the latter does not use at all the sequence (or denumerable set) of all natural numbers whereas the former essentially relies on this sequence. The main feature of the first proof is indeed its cutting off denumerably many couples of subsets B_k, C_k, and its then using theorem 2 of p. 47 for the correspondence to be created. On the other hand, the proofs renouncing the use of the natural numbers, as the second proof given here, have a much more abstract character and use certain general procedures which shall be dealt with in more detail in § 11, 7 [1]).

[1]) The first complete proof of the equivalence theorem, given by F. Bernstein in 1897 (cf. Cantor **15**, p. 450) and published in Borel **1**, p. 103 ff., is of the former kind; it is, essentially, the first proof given here. A similar proof, given at the same time by E. Schröder, is defective; see Korselt **3**.

The first proof of the second kind was given in 1887 and 1899 by Dedekind (see Dedekind **3**, III, pp. 447—449 and Cantor **15**, p. 449), but was not published until 1932; independently this proof was rediscovered by Peano (cf. **6**, 1906) and Zermelo (**3**, 1908, pp. 271—272). Cf. already in *Revue de Métaphysique et de Morale*, volume **14**, p. 314, 1906. This proof makes an essential use of Dedekind's theory of *chains* (cf. § 11, **7**). The second proof given here was communicated to the author by J. M. Whittaker in 1943 (during the latter's military service in the Middle East).

It will prove useful to express the equivalence theorem in a somewhat different form. Given two sets S and T such that S is equivalent to a subset of T, there exist (cf. the scheme of p. 98) the following two exclusive possibilities: T is either equivalent to a subset of S, or not equivalent to any subset of S. According to theorem 4, the cardinal of S is in the first case equal to the cardinal of T, and in the second case, smaller than the cardinal of T, by the definition of order. Hence:

THEOREM 5. Let S and T be sets of the respective cardinals $\mathbf{s}$ and $\mathbf{t}$. If S is equivalent to a subset of T, it follows that $\mathbf{s} \leqq \mathbf{t}$, and vice versa.

On the other hand, theorem 4 follows from this theorem immediately. Hence both theorems are equivalent.

Therefore one may now characterize the order of cardinals (p. 90) in the following way, which is especially natural in view of the order of *finite* cardinals: *the cardinal of S is called smaller than the cardinal of T, if S is equivalent to a subset of T, but not to T itself.*

5. The Problem of Comparability. Now that the first three cases contained in the above scheme have been settled, there remains the fourth case. Some reflection upon this case will show that it is rather paradoxical. If neither of two given sets includes any subset equivalent to the other set, the sets and their cardinals would prove *incomparable*. Such a possibility would imply that the cardinal numbers of sets lack a decisive property of number in general (cf. the end of **1**). The problem of excluding this possibility is sometimes called the *problem of trichotomy*.

Cantor had early recognized the necessity of excluding incompar-

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It represents an especially simple form of the second kind. Cf. J. M.Whittaker **1**.

The proof of J. König **4**, belonging to the first kind and excelling in lucidity, has led to remarkable generalizations, including the *theory of equivalence with respect to "classes of representations"* (cf. § 8, 3) — an important modern theory containing all results of the theory of equivalence that are independent of the multiplicative principle (§ 6, **6**). See Banach **1**; D. König **1**; Kuratowski **5**, Lindenbaum–Tarski **1**, Otchan **1**, Rosenfeld **1**, Sierpiński **4** and **28**, Ulam **1** and particularly Tarski **8**, **10**, **12**, and **37** (pp. 94–98).

For the concept of *essentially different* representations between two sets, see Hausdorff **6**.

ability, and convinced himself that profound resources were needed for this purpose but did not succeed in procuring them.

The well-ordering theorem, affirmed by Cantor from the beginning but only proved by Zermelo in 1904 (§ 11, 6), asserts the possibility of well-ordering any (ordered or not-ordered) set, and by this roundabout way excludes the fourth case mentioned above. Anticipating that result, we arrive at the following proposition, which we shall not make use of until § 11:

> Two sets are either equivalent, or else one of them is equivalent to a subset of the other. Hence, of two different cardinals one is smaller than the other.

Exercises

1) Prove that the validity of the relations between cardinals $\mathbf{s} < \mathbf{t}$ and $\mathbf{t} \leqq \mathbf{w}$ implies the validity of $\mathbf{s} < \mathbf{w}$. (Cf. b) on p. 90.)

2) With respect to the classification of the elements of S used in the proof of theorem 3 (p. 95), look for elements of the first kind and of the second kind under various assumptions as to the set U_0; in particular, under the assumption $U_0 = \mathbf{U}S$, which means an *indirect proof* of theorem 3.

3) What simplification of the proof of theorem 3, as well as of its particular cases proved in the beginning of 2, is made possible by theorem 5?

4) (For advanced readers) In view of certain representations both between S and $T_1 \subset T$, and between T and $S_1 \subset S$, as supposed in the equivalence theorem, let $t_1 \varepsilon T_1$ be related to $s \varepsilon S$, $s_2 \varepsilon S_1$ to $t_1 \varepsilon T$, $t_3 \varepsilon T_1$ to $s_2 \varepsilon S$, etc. One thus gets an infinite sequence $(s, t_1, s_2, t_3, \ldots)$. If, in particular, $s \varepsilon S_1$, one may continue this sequence leftwards by using the representation between T and S_1 with respect to s, and placing the mate of s in T before s; if this mate is incidentally an element not only of T but also of T_1, one may take a further step leftwards, etc. All these steps to the right and to the left are uniquely determined. Create a one-to-one correspondence between all elements of S and all elements of T by discerning, for any element $s \varepsilon S$, between the following three cases:

a) the sequence can be continued leftwards indefinitely;

b) the sequence, after having been continued as far as possible, begins on the left with an element of S;

c) in the same sense, the sequence begins with an element of T.

(Hint: in each case relate to s either its right or its left neighbor. Compare this proof, which is essentially that of König, with the first proof given above.)

5) Let us assume that the cardinal of any sum of two sets, one of which at least is infinite, is not larger than the cardinal of each term. Show that this assumption assures the comparability of transfinite cardinals, i.e. that it excludes the fourth case. (As to that assumption, cf. § 11, **5.**)

§ 6. Addition and Multiplication of Sets and Cardinals [1])

1. **Preliminary Remarks.** In the preceding sections we have become acquainted with actually infinite magnitudes, namely with the transfinite cardinals; we have also seen how, to a certain extent, they can be compared. We shall now examine whether and how we may also *operate* with transfinite cardinals. We shall find that the operations of addition, multiplication, and exponentiation (involution) of ordinary arithmetic can be generalized in a natural way such that they also apply to transfinite cardinals and give definite results in the domain thus extended. Even the rules or *formal laws* of calculation, holding in arithmetic with respect to the operations mentioned, are preserved in the extended domain; these laws are:

$$a + (b + c) = (a + b) + c, \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad (\textit{associative laws})\ [2])$$
$$a + b = b + a, \quad a \cdot b = b \cdot a \quad (\textit{commutative laws})$$
$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot c + b \cdot c \quad (\textit{distributive laws})\ [3]).$$

Corresponding laws apply when there are more than three terms in the associative laws, and more than two in the commutative

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[1]) Most of the contents of §§ 6 and 7 are due to Cantor. Cf. in particular 12, I.

[2]) Suschkewitsch 1 (cf. 2) introduces a generalization (i.e. a weaker form) of the associative law. On the other hand, J. Haupt 1 discusses the question of how far a modification of the associative law, admitting certain permutations of the factors, may provide commutativity.

[3]) Either form of the distributive law follows from the other by means of the commutative law of multiplication.

laws or in the parentheses of the distributive laws — provided, of course, that the number of terms is *finite*. In fact, neither the addition nor the multiplication of numbers is defined in arithmetic with respect to more than a finite number of terms, and even the definition for a finite number (more than two) relies on the associative law.

On the other hand, it will be established in 8 that the inverse operations: subtraction, division, evolution and taking logarithms, are *not practicable* in the domain of transfinite cardinals. (It is well known that also with respect to finite numbers these operations are practicable only after an extension of the original domain of cardinals which results in the inclusion of negative, rational, real and complex numbers.) Even when, in special cases, an inverse operation can be carried out in our domain, the result need not be uniquely determined — not even in the case of subtraction.

One should not be surprised by the fact that a reasonable subtraction or division cannot be defined for transfinite cardinals. When the domain of numbers is expanded in so fundamental a way and to such a gigantic extent as has been done by the admission of infinite numbers, one cannot expect that the new members of the domain submit to the same laws as the previous ones. In mathematics as in any theory, the generalization of a concept implies a loss of part of the properties of the original concept [1]). As a matter of fact, another generalization of the concept of finite number (see §§ 8 and 11) does not even fulfil the commutative laws of addition and multiplication, and fulfils only *one* of the two distributive laws.

On the other hand, according to the so-called *principle of permanence of formal laws* (H. Hankel), guiding the definition of operations for any new kind of number — here for transfinite cardinals —, the operations defined for finite cardinals should be generalized in such a way that *as far as possible* the laws of ordinary addition and multiplication be preserved in the enlarged domain. But after

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[1]) One should not, however, express this evident rule in the wrong form — sometimes accepted as a "principle" — that any addition to the contents of a concept narrows the extent of the concept. Bolzano (cf. 2, § 20) already refuted this assumption by using the example "connoisseur of all European languages" in comparison with "connoisseur of all *living* European languages". Cf. Hoensbroech 1; Dubislav 2, § 63.

having based transfinite arithmetic upon suitable definitions of the operations, the mathematician cannot go on to *prescribe* the formal laws valid in the domain; he cannot, for instance, postulate their coincidence with the laws of ordinary arithmetic. On the contrary, he has to *examine* to what extent the previous laws are preserved and what, for the rest, has to replace them [1]). Therefore, one may consider it a rather pleasant surprise, not to be expected in advance, that the ordinary rules of arithmetic (addition, multiplication, involution) prove valid for the corresponding operations in the domain of transfinite cardinals as defined in this and in the following section.

The failure to appreciate this state of affairs has from the beginning handicapped the acknowledgment of Cantor's ideas, especially the acceptance of transfinite number. As characteristic of Cantor's point of view as well as of the attitude of his opponents, the following remarks in a letter of his [2]) written in 1885 may be quoted: "All pretended proofs against the possibility of actually transfinite number are ... faulty in this respect — and here lies their *πρῶτον ψεῦδος* — that from the first they impute to, or rather enforce upon, the numbers in question all the properties of finite numbers, while the actually transfinite numbers — if they shall be conceivable in any way — must, by their contrast to finite number, constitute an entirely new kind of number, whose nature completely depends on the state of affairs and should be the object of our investigations, but not of our discretion or prejudice." [3])

[1]) This is what Cantor had in mind when prefixing to the final exposition of his ideas the motto: *Neque enim leges intellectui aut rebus damus ad arbitrium nostrum, sed tanquam scribae fideles ab ipsius naturae voce latas et prolatas excipimus et prescribimus* (12, I, p. 481). Cf. "thesis III" of the dissertation (1) with which Cantor, then 24 years old, obtained admission as a lecturer at the University of Halle: *Numeros integros simili modo atque corpora coelestia totum quoddam legibus et relationibus compositum efficere*, and a passage from a letter written by Cantor in 1884 (cf. Schoenflies 12, p. 15 ff.) which runs: "As to everything else [except 'the art of style and the economy of exposition' in Cantor's papers], this is not my merit. With respect to the contents of my research work I am only a sort of reporter and secretary."

[2]) To G. Enestrom. See Cantor 9, p. 226.

[3]) The controversy as to whether the conceptions of mathematics are creations of the thinking mind or — as Cantor presumes — have an existence independent of the intelligence dealing with them (somehow as Platonic

2. Addition of Sets. In order to define the addition and the multiplication of *cardinals*, it is necessary to prepare the ground by explaining what is meant by the addition and multiplication of *sets*. Addition has already been introduced in § 2, 3, with respect to a finite number, or a sequence, of sets. Now we shall define it generally.

DEFINITION I. Let A be a set whose elements are again sets [1]) $a, a', a'', \ldots$. By the *sum-set* $\mathbf{S}A$ of A, or the sum of all sets $a, a', a'', \ldots$, we understand the set containing all the elements which belong *at least to one* of the elements of A, i.e. which are elements of at least one of the sets $a, a', a'', \ldots$ (to be called the *terms* of the sum). We also write:

$$\mathbf{S}A = \mathbf{S}\{a, a', a'', \ldots\} = a + a' + a'' + \ldots.$$

The sum-set, therefore, is independent of whether a certain object belongs to one or to several of the terms. It goes without saying that the notation chosen for the elements of A, $a, a', a'', \ldots$ only serves the purpose of labeling a few of them, and does not imply that A is denumerable.

Examples. a) On the line of numbers take the segments from 0 to 1, from 1 to 2, etc., generally from k (natural number) to $k + 1$, as well as the segments from -1 to 0, from -2 to -1, and generally from $-k$ to $-k + 1$. Each segment may be considered as the set of all points of the segment, including its extremities. Considering all those segments as the elements of A, we obtain the set of all points of the line as the sum-set $\mathbf{S}A$.

The result is the same when each segment only includes either its right or its left extremity. On the other hand, if *none* of the extremities are included, the sum-set will evidently be a proper subset of the set of all points on the line.

If the elements of A are segments of the length 1 extending from

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ideas), has remained topical in the theory of sets until our days. One might ask, in short: are the conceptions of mathematics to be *invented or discovered*? (See footnore 1, p. 108.) Cf. Hessenberg 6, Bays 1.

The note Winn 2 shows that the misunderstandings regarding the *definitory* character of the operations with cardinals have not ceased since.

[1]) Here again, the assumption that the elements of A are themselves sets, is superfluous and only serves the purpose of psychological simplification. Elements of A which are *not* sets (including the null-set) do not contribute anything to the sum-set, as follows from the definition.

every point of the line, the sum-set is again the set of all points of the line. In this case, however, A has the cardinal $\aleph$ and cannot, therefore, be considered as a sequence. Definition I enables us to form the sum-set whereas definition III of § 2 (p. 26) is not sufficient.

b) Let us reconsider the race between Achilles and the Tortoise referred to on p. 11 (cf. fig. 1). Denote by $s_1, s_2, \ldots, s_k, \ldots$ the segments which Achilles is covering in order to reach his rival. The left-hand end of the segment s_k is then the point P_k. If S_k denotes the set of all points on the segment s_k, including its extremities, the sum-set $S_1 + S_2 + \ldots + S_k + \ldots = S$ is the set of all the points situated on the total segment of fig. 1, including its right-hand end but not its left-hand end.

Like this segment as a whole, each of the segments s_k has the cardinal $\aleph$. The set of points S, included in a finite segment of a line, is accordingly broken up into denumerably many subsets each of which has the same cardinal as the sum S. (Cf. theorem 3 on p. 51.)

Let us finally remark that the existence of the sum-set, in the full generality of definition I, is secured by the principle of sum-set (p. 28).

3. A Fundamental Property of Addition. The key to the addition of *cardinals*, on the basis of the addition of sets just introduced, is contained in the following remark. Let A and B be two different *equivalent* sets of sets

$$A = \{a', a'', \ldots, a^{(k)}, \ldots\} \text{ and } B = \{b', b'', \ldots, b^{(k)}, \ldots\}.$$

Here again the notation shall not mean that A and B are denumerable sets; they may be also finite, or infinite and non-denumerable. On the other hand, for a given representation φ between the equivalent sets A and B, the upper indices (dashes) used for the elements of A and B show what pairs of elements of A and B correspond to one another on account of φ.

In addition to the assumption made with respect to A and B, we assume regarding the *elements* of A and B which are supposed to be also sets, that any two elements $a^{(k)}$ and $b^{(k)}$ related to each other by φ, are *equivalent* sets: $a^{(k)} \sim b^{(k)}$. In short: A and B are equivalent sets of equivalent sets. We now raise the question

whether the sum-sets $\mathbf{S}A$ and $\mathbf{S}B$ are also equivalent to each other.

Under such general conditions this is certainly not always the case, as shown by a simple instance. Suppose A and B are pairs

$$A = \{a_1, a_2\}, \quad B = \{b_1, b_2\},$$

and their elements are the following sets of numbers

$$a_1 = \{1, 2, 3\}, \quad a_2 = \{4, 5\}, \quad b_1 = \{6, 7, 8\}, \quad b_2 = \{8, 9\}.$$

Although the above conditions are fulfilled, the sum-sets

$$a_1 + a_2 = \{1, 2, 3, 4, 5\} \text{ and } b_1 + b_2 = \{6, 7, 8, 9\}$$

are *not* equivalent, the former containing five elements and the latter four.

The reason is plain. Since the sets b_1 and b_2 are not mutually exclusive, but contain the common element 8, the assumed equivalences do not enable us to represent $b_1 + b_2$ on $a_1 + a_2$, for the element 8 of $b_1 + b_2$ corresponds to two elements, one belonging to a_1 and the other to a_2. Hence we shall add a further condition which, together with the previous conditions, will ensure the desired result. The new condition requires that the given sets A and B be *disjointed* (p. 20); in our case, that a_1 and a_2, as well as b_1 and b_2, be mutually exclusive sets.

Given that our three conditions are fulfilled, let $\psi^{(k)}$ be a certain representation [1]) between the equivalent sets $a^{(k)}$ and $b^{(k)}$, which are related to each other by the representation φ between A and B. Now we construct a definite representation between the sum-sets $\mathbf{S}A$ and $\mathbf{S}B$ as follows. If α is any element of $\mathbf{S}A$, by the disjointedness of A, α will belong to one and *only one* [2]) of the elements of A, say $\alpha \,\varepsilon\, a^{(k)}$. By φ, $b^{(k)} \,\varepsilon\, B$ is related to $a^{(k)} \,\varepsilon\, A$, and therefore $\psi^{(k)}$ represents $a^{(k)}$ on $b^{(k)}$. Hence, a definite element $\beta \,\varepsilon\, b^{(k)}$ is the mate of $\alpha \,\varepsilon\, a^{(k)}$ on account of $\psi^{(k)}$. *This element* β (of $b^{(k)}$, hence of $\mathbf{S}B$) *shall be related to* $\alpha \,\varepsilon\, \mathbf{S}A$.

Since all the representations used here define one-to-one corre-

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[1]) Whether we can *actually obtain* all the representations $\psi^{(k)}$ used here, is an important problem discussed in *Foundations*. Cf. also 6 of the present section.

[2]) Here we use the assumption that A and B are disjointed, i.e. that any two elements of A as well as of B are mutually exclusive.

spondences, and since every $\beta \varepsilon \mathbf{S}B$ belongs to only one of the elements of B, the procedure is reversible and α is also related to β. Therefore, we have created a representation ψ (comprehending all the $\psi^{(k)}$'s) between $\mathbf{S}A$ and $\mathbf{S}B$. Hence:

THEOREM 1. If A and B are equivalent disjointed sets and if there exists a representation between them such that corresponding elements are again *equivalent sets*, then the sum-sets $\mathbf{S}A$ and $\mathbf{S}B$ are also equivalent.

4. Addition of Cardinals. Herewith the preparations for the addition of cardinals are completed. In order to facilitate understanding and to emphasize some special features of the operations with cardinals, we start with the addition of *two* cardinals only.

DEFINITION II. Given two (finite or transfinite) cardinals $\mathbf{a}_1$ and $\mathbf{a}_2$, take two *mutually exclusive* sets A_1 and A_2 (called *representatives* of the given cardinals) such that $\mathbf{a}_k$ is the cardinal of A_k. Then the cardinal $\mathbf{s}$ of the sum-set $A_1 + A_2$ is called the *sum of the cardinals* $\mathbf{a}_1$ and $\mathbf{a}_2$. In symbols:

$$\mathbf{s} = \mathbf{a}_1 + \mathbf{a}_2.$$

Let us add a few remarks to this definition.

First, it is obvious that the definition is in accordance with the addition of two natural numbers — i.e. finite cardinals — in arithmetic. This still holds if one of the terms (or both of them) equals the cardinal 0, since then the respective term for the sum-set is the null-set. Therefore, we may consider definition II as a *generalization* of ordinary addition.

Secondly, the rather obvious objection that the result of this addition is not uniquely determined by our definition, is not justified. Certainly the representatives A_1 and A_2 have only been limited by certain conditions which do not determine them completely, and hence $A_1 + A_2$ is not fully determined by the given cardinals $\mathbf{a}_k$. But this ambiguity does not affect the sum $\mathbf{a}_1 + \mathbf{a}_2$. In fact, if other representatives A_1' and A_2' are chosen subject to the conditions of the definition, the sum $A_1' + A_2'$ is equivalent to $A_1 + A_2$, by virtue of theorem 1. Accordingly, both sum-sets have the same cardinal, and hence *the result of the addition does not depend on the choice of representatives.*

Thirdly, of course, we shall have to define the addition of any number of cardinals and not of only two. In view of this task (see definition III, p. 114) it might seem preferable to formulate the beginning of definition II in the following way, "given a pair — i.e. a set of the cardinal 2 — of cardinals, $\{\mathfrak{a}_1, \mathfrak{a}_2\}$". However, this formulation would exclude the case where $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are equal, owing to the stipulation in § 1 that any object is either contained in a given set (once), or not contained in it.

As long as we have to deal with two terms, this difficulty is easily avoided by the above formulation. But in the general case we have to look for another way of dealing with equal terms. For this purpose, we shall use the concept of *one-valued function* (in contrast to a *one-to-one correspondence*; see § 2, 4). In fact, the given cardinals may be considered as *assigned to the elements of a certain set* T by a one-valued function $f(t)$, t running over all the elements of T. For a sum of two terms, as mentioned in definition II, T may be chosen as the set $\{1, 2\}$; for the sum $\aleph_0 + \aleph_0$, for instance, we may use the function defined by $f(1) = f(2) = \aleph_0$.

In the general case, assign to each $t \in T$ a uniquely determined cardinal $f(t)$. (Hence a definite cardinal which occurs among the values $f(t)$, occurs in a multiplicity corresponding to a subset of T.) Thus we obtain a generalization of the usual concept of *indices* (subscripts); in our general case, the index is not necessarily a natural number, but any element of a certain set T.

For instance, if T is the set of all real numbers between 0 and 3, these two included, and if $f(t)$ is defined by

$$f(t) = 1 \text{ for integral } t$$
$$f(t) = 2 \text{ for non-integral rational } t$$
$$f(t) = \aleph_0 \text{ for irrational } t$$

the term 1 will appear four times, the term $2^{\aleph_0}$ times, the term $\aleph_0^{\aleph_0}$ times (i.e. corresponding to the elements of a continuum). If T is the set of all natural numbers and $f(t) = \aleph$ (i.e. a constant function), we have the cardinal of the continuum $\aleph_0$ times.

A fourth remark concerns our notation. The sum-set of two or more sets has been defined without any reference to an order of the terms, which are merely given as elements of a set. So, in spite of the order apparently involved in the symbol $\mathfrak{a}_1 + \mathfrak{a}_2$, the definition of the sum of two cardinals does not depend on any possible order

in which those cardinals are given. Accordingly, we should be entitled to introduce a notation not involving order, but, unfortunately, man does not possess such a notation, neither in speaking nor in writing, so this property of addition has to be explicitly stated.

In accordance with these remarks, the addition of cardinals in general is defined as follows.

DEFINITION III. A finite or infinite number of cardinals, not necessarily differing from each other, may be given by means of a function f which assigns to every element t of a non-empty set T a definite cardinal $f(t) = \mathfrak{c}_t$. In order to form the sum of the cardinals $\mathfrak{c}_t$, attach to each $\mathfrak{c}_t$ a set (representative) C_t with the cardinal $\mathfrak{c}_t$ such that the sets attached to any two different elements of T are mutually exclusive [1]). Form the sum-set of all representatives C_t; the cardinal $\mathfrak{s}$ of this sum-set is called *the sum of the given cardinals*. In symbols:

$$\mathfrak{s} = \sum_{t \in T} \mathfrak{c}_t = \mathfrak{c} + \mathfrak{c}' + \mathfrak{c}'' + \dots$$

where $\mathfrak{c}$, $\mathfrak{c}'$, $\mathfrak{c}''$, ... denote some of the given cardinals.

In short, the sum of given *cardinals* is the cardinal of the sum of *sets* which "represent" the given cardinals,

$$\sum_{t \in T} \mathfrak{c}_t = \overline{\overline{\sum_{t \in T} C_t}}.$$

As to the remarks on definition II, the first loses its meaning in the case of an infinite number of cardinals, since in arithmetic only the addition of a finite number of cardinals is meaningful. The second retains its validity; in view of the condition of disjointedness *and of theorem* 1, the sum $\mathfrak{s}$ defined by definition III is uniquely determined, though this is not true of the representatives C_t and of their sum-set. The third remark justifies the formulation used

[1]) There is no difficulty whatsoever in constructing sets C_t in sufficient quantity provided that for any cardinal, a set having this cardinal is given — which must be assumed on account of the introduction of cardinals in § 4, where the set and not the cardinal serves as the starting-point. For instance, we may take the set C_t of all ordered pairs (c_t, t) where c_t are the elements of a representative of the particular cardinal $\mathfrak{c}_t$. If $t_1 \neq t_2$, any two ordered pairs $(c^*_{t_1}, t_1)$ and $(c^{**}_{t_2}, t_2)$ are different, even in the case $c^*_{t_1} = c^{**}_{t_2}$. It is not difficult to pass from ordered pairs to pairs (i.e. to sets of the cardinal 2). Cf. the proof of theorem 5 on p. 128.

in the beginning of definition III and the fourth remains unchanged [1]).

5. Formal Laws. Examples. The generalization of the *commutative law* of addition (see p. 106) to the case of definition III needs no proof. For here, in contrast to arithmetic, addition has not been defined for a specific arrangement of the terms.

As to the *associative law*, there is a difference of principle between the present case and that of arithmetic. The definition of an arithmetical sum (or product) is originally intended for two terms only. The associative law $(a + b) + c = a + (b + c)$, however, shows that the value of the sum does not depend on the special way in which two consecutive terms out of three are grouped together for addition. Therefore the parentheses which indicate this grouping together, are superfluous from the practical point of view. In other words, the sum of three terms without parentheses, $a + b + c$, is defined as the common value of the sums resulting from combining two consecutive terms in either way. By the procedure of "inductive definition" (cf. § 10, 2) we thus obtain the sum of any *finite* number of terms, and an inductive proof shows that this definition is justified by the validity of the associative law.

In the present situation, however, such a procedure would not be sufficient. By definition I we have from the beginning introduced the concept of sum-set for any number of terms. This fact induces us to state the associative law of addition in a more general form as follows. Let us decompose the set A, whose sum-set $\mathbf{S}A$ we have in mind, into complementary and mutually exclusive subsets in any way,

$$A = K + L + M + \ldots . \tag{1}$$

The elements of $K, L, M, \ldots$ may be generally denoted by $k, l, m, \ldots$ respectively, the elements of A by a, so that any $a \varepsilon A$ is either a k or an l or an m etc. Then the equality

$$\mathbf{S}A = \mathbf{S}K + \mathbf{S}L + \mathbf{S}M + \ldots \tag{2}$$

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[1]) Commutativity, referred to in the fourth remark, may also be conceived as the possibility of mapping the "argument-set" T on itself or on an equivalent set. Cf. Hausdorff 4, pp. 37–38; also exercise 7 at the end of § 6.

together with (1) represents *the general associative law for the addition of sets* [1]). In order to prove its validity, consider that by definition I an arbitrary element $x \varepsilon \mathbf{S}A$ belongs to (at least) one of the elements of A, hence to a $k \varepsilon K$ or to an $l \varepsilon L$ or to ....., and therefore to the sum appearing on the right-hand side of (2). Conversely, if x belongs to the latter sum, x belongs to (at least) one of the sums $\mathbf{S}K$, $\mathbf{S}L$, ..., hence, by virtue of (1), to $\mathbf{S}A$. Q.E.D.

In view of definition III, we obtain the associative law for the addition of *cardinals* from (2) through the specializing assumption that A is a *disjointed* set of sets, whence the same follows for the subsets $K, L, \ldots$ of A.

We postpone the formulation of the associative law for cardinals until 7 (theorem 4), when the other formal laws can be included.

While in §§ 3 and 4 many instances of addition of *sets* have already appeared, we shall now give examples of sums of *cardinals*. It was mentioned on p. 112 (cf. also the first example on p. 111) that the addition of cardinals in arithmetic, being restricted to a finite number of finite terms, is a particular case of our addition, and accordingly, provides special instances for the general operations defined in II and III. Furthermore, as in arithmetic, we have $\mathbf{c} + 0 = \mathbf{c}$ for every cardinal $\mathbf{c}$, and 0 is the only cardinal $\mathbf{x}$ which fulfils $n + \mathbf{x} = n$ for any finite cardinal n (while for infinite cardinals there are additional solutions $\mathbf{x}$).

Theorem 2 on p. 47 affirms that the addition of a finite number of elements, or of $\aleph_0$ elements, to the elements of a denumerable set does not change its cardinal. Hence:

$$\aleph_0 + n = \aleph_0 + \aleph_0 = \aleph_0, \tag{3}$$

where n denotes any finite cardinal. Moreover, the theorems 2 and 3 (p. 51) assert for a finite number of terms, or for denumerably many terms:

$$\aleph_0 + \aleph_0 + \ldots + \aleph_0 = \aleph_0 + \aleph_0 + \aleph_0 + \ldots = \aleph_0. \tag{4}$$

In order to form the sum of *all natural numbers*, we may choose

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[1]) To perceive this, replace $\mathbf{S}K$ by $k' + k'' + \ldots$, $\mathbf{S}L$ by $l' + l'' + \ldots$, etc. There is no need to investigate *different* decompositions of the kind (1) as is done in arithmetic, for the validity of (2) for *any* decomposition (1) implies that *any* two different decompositions lead to the same result $\mathbf{S}A$. The condition of disjointedness is needed only for the transition to cardinals.

set-representatives of 1, 2, 3, ... as follows:

$$\{a_1\},\ \{a_2, a_3\},\ \{a_4, a_5, a_6\},\ \ldots;$$

in particular, we may take $a_k = k$. The sum of all these finite sets is the *denumerable* set containing all a_k's for any natural number k. Hence:

$$1 + 2 + 3 + \ldots + k + \ldots = \aleph_0. \tag{5}$$

Observe that here appears an actual *sum of infinitely many (finite) numbers*; this has nothing to do with the infinite series of analysis, where the "sum" is defined as a certain limit, and the existence of the sum therefore depends on the *convergence* of a sequence. In the present case, the existence of the sum is *always* guaranteed by definition III, which, of course, relies on the principle of sum-set.

In a corresponding way, one obtains for a sum of denumerably many equal finite terms n (i.e. for $f(t) = \mathfrak{c}_t = n$, where t runs over the set T of all natural numbers), if $n \neq 0$:

$$n + n + n + \ldots + n + \ldots = \aleph_0. \tag{5'}$$

In particular, we have $1 + 1 + \ldots + 1 + \ldots = \aleph_0$ [1].

Adding to a continuum a finite number (n) of elements, or $\aleph_0$ elements, one obtains by theorem 6 on p. 59:

$$\aleph + n = \aleph + \aleph_0 = \aleph. \tag{6}$$

In the second of these relations the representatives of the terms $\aleph$ and $\aleph_0$ may be taken to be the sets of all transcendental and of all algebraic numbers between 0 and 1; the sum then corresponds to the linear continuum of the unit segment.

Using the same theorem 6 for an infinite set of the arbitrary cardinal $\mathfrak{c}$, we obtain

$$\mathfrak{c} + n = \mathfrak{c} + \aleph_0 = \mathfrak{c}. \tag{6'}$$

In order to find the sum $\aleph + \aleph$, we may choose as representatives the sets of points on the segments from 0 to 1 and from 1 to 2 (always including, say, the left end point only). By considering, in the same way, the segment from m to $m + 1$ for *all* non-negative

[1] From this equality together with the second relation (4), we may also conclude (5) and (5') by means of theorem 1 on p. 45. In a similar way, the first relation (3) follows from the second.

integers m and by passing over to the cardinals, we obtain a *sequence* each of whose members is $\aleph$. Since the sum-sets are again continua of the kind mentioned in theorems 2 and 3 of § 4 (pp. 71/72), one has:

$$\aleph + \aleph = \aleph + \aleph + \ldots = \aleph \text{ [1]}. \tag{7}$$

As an instance of computation by means of the formal laws (here the associative law), we prove the first relation (6) by means of the second, using the first relation (3), as follows:

$$\aleph + n = (\aleph + \aleph_0) + n = \aleph + (\aleph_0 + n) = \aleph + \aleph_0 = \aleph.$$

6. Multiplication of Sets. We postpone a further discussion of the meet or *inner product* of sets (cf. § 2, 3) to the end of this section (**11**), this subject lacking significance for the theory of cardinals. Instead we proceed to a new operation with sets, the result of which will be called *outer product*. For the operation, the name *multiplication* will be used occasionally.

In arithmetic, multiplication is introduced as a repeated addition; e.g., $4 \cdot 5 = 5 + 5 + 5 + 5$ or $= 4 + 4 + 4 + 4 + 4$. The multiplication of cardinals, however, will be based on a certain operation with *sets* as was done in the case of addition. Now, if we want to obtain $4 \cdot 5$, there is no need to start with a set of 20 elements. It proves more economical to take two sets containing 4 and 5 elements respectively, say

$$\{a_1, a_2, a_3, a_4\} \text{ and } \{b_1, b_2, b_3, b_4, b_5\}$$

and to form all combinations (i.e. pairs) of an a_k with a b_l. Their total number amounts to $4 \cdot 5$. This may serve as the starting-point for

DEFINITION IV. Given two sets R and S, the set P the elements of which are all different pairs $\{r, s\}$, where r runs over the elements of R and s over the elements of S independently, is called the *outer product* or the *combination-set* [2]) of R and S.

The sets R and S are called the *factors* of the product. We shall write

$$P = R \times S.$$

[1]) We are not yet in a position to demonstrate the relation $\mathfrak{c} + \mathfrak{c} = \mathfrak{c}$ for *every* cardinal $\mathfrak{c}$. See § **11**, **5** and **6**.

[2]) Cantor who has introduced this concept, named it *Verbindungsmenge*. Sometimes it is called *Cartesian product*.

To this definition we add the following remark. Two pairs $\{r_1, s_1\}$ and $\{r_2, s_2\}$ of P, where r_1 and r_2 are elements of R and s_1 and s_2 elements of S, shall be considered *equal* only if $r_1 = r_2$ and $s_1 = s_2$. This remark may perhaps at first sight appear self-evident and superfluous; however, there are also instances where, in spite of r_1 and r_2 being different elements of R and s_1 and s_2 different elements of S, $\{r_1, s_1\}$ and $\{r_2, s_2\}$ are *equal* in the normal sense; namely when

$$r_1(\varepsilon R) = s_2(\varepsilon S) \quad \text{and} \quad r_2(\varepsilon R) = s_1(\varepsilon S).$$

For instance, if $R = \{1, 2\}$ and $S = \{1, 2, 3\}$, the pair $\{1, 2\}$ containing $1 \varepsilon R$ and $2 \varepsilon S$ would be equal to the pair $\{2, 1\}$ containing $2 \varepsilon R$ and $1 \varepsilon S$, and this is contrary to our intentions [1]).

This inconvenience may be avoided in one of the following three ways. First, we may consider *ordered pairs* instead of plain pairs and require that the first component of each pair be an element of R, the second an element of S. In the instance considered above this would leave us with $(1, 2) \neq (2, 1)$. This is simple from a practical point of view, but unsatisfactory in principle, for the operation of combining elements from different sets does not imply the concept of order.

Secondly, we may restrict the operation of "outer multiplication" to the case where the factors R and S are mutually exclusive. Then, any element of either factor can belong to only one of the sets and the possibility mentioned above is excluded. The restriction to mutually exclusive factors is no fundamental disadvantage, for the outer product of sets is not required for its own sake, but only as a starting-point for the product of cardinals (definition VI), and here we are always free to choose mutually exclusive factors.

Thirdly, we may consider the elements of R and S, not in their own right, but in connection with the set to which they belong. This may be denoted by subscripts [2]). Then the same elements

[1]) Since a pair is here conceived as a plain set (and not as an ordered pair), the arrangement of the elements in the pair is arbitrary.

[2]) Properly speaking, this means ordered pairs (r, R) in which the second component indicates the set in question and the first the element chosen from it. However, this does not constitute a difficulty similar to that of the first method. When we proceed from two factors to any number of factors (definition V), the first method makes it necessary to provide for an order in more comprehensive sets, while the third method leaves us concerned

out of different sets become formally different. In certain cases, for instance in § 7, this will prove more convenient than the restriction to exclusive sets.

The last two methods are to be preferred, and in fact, we shall avoid the first way.

Examples. a) $R = \{r_1, r_2\}$, $S = \{s_1, s_2, s_3\}$. We obtain:

$$R \times S = \{\{r_1, s_1\}, \{r_1, s_2\}, \{r_1, s_3\}, \{r_2, s_1\}, \{r_2, s_2\}, \{r_2, s_3\}\},$$

i.e. a set of $2 \cdot 3 = 6$ elements, in accordance with the remark preceding definition IV.

b) $R = \{r_1, r_2\}$, $S = 0$ (null-set). In this case $R \times S$ is the null-set since there exists *no* pair containing one element from R and one element from S.

c) $R =$ the set of all natural numbers, $S = \{s_1, s_2\}$. We obtain:

$$R \times S = \{\{1, s_1\}, \{1, s_2\}, \{2, s_1\}, \{2, s_2\}, \{3, s_1\}, \ldots\},$$

i.e. again a denumerable set.

We might have introduced multiplication simply as a repeated addition, in accordance with the remark made at the beginning of this subsection. But then we should have encountered a difficulty in formulating the following definition, which is the natural generalization of definition IV:

DEFINITION V. Let S be a set of sets s, or more generally, let s_t be sets such that to every element t of a certain non-empty set T a single set $f(t) = s_t$ is assigned. By the *outer product* $P = \mathbf{P}S$ of S, or the *outer product* P *of all sets* s_t, we understand the set whose elements are all the different sets (*complexes*) which contain *a single element from each set* s_t. If $s_t, s_{t'}, \ldots$ are some sets assigned to the elements of T (some "factors" of the product), we also write

$$P = \mathbf{P}S = s_t \times s_{t'} \times \ldots.$$

The term "complex" is used in order to stress the characteristic property of the sets (the elements of P) in question, viz. that any such set contains one element from each of the given factors.

with ordered *pairs* only, a concept which is easily defined within the domain of (unordered) sets.

Examples. Let S be the denumerable set of pairs

$$S = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots\};$$

then we obtain

$$\mathbf{P}S = \{1, 2\} \times \{3, 4\} \times \{5, 6\} \times \ldots =$$

$$= \{\{1, 3, 5, 7, \ldots\}, \{2, 3, 5, 7, \ldots\}, \{1, 4, 5, 7, \ldots\}, \{2, 4, 5, 7, \ldots\},$$
$$\{1, 3, 6, 7, \ldots\}, \{2, 3, 6, 7, \ldots\}, \{1, 4, 6, 7, \ldots\}, \ldots\}.$$

As will be seen in § 7, this set has the cardinal $\aleph$ of the continuum.

If, however, $S = \{\{1, 2\}, \{3, 4\}, \{5\}, \{6\}, \{7\}, \ldots\}$, we obtain

$$\mathbf{P}S = \{\{1, 3, 5, 6, 7, \ldots\}, \{2, 3, 5, 6, 7, \ldots\}, \{1, 4, 5, 6, 7, \ldots\},$$
$$\{2, 4, 5, 6, 7, \ldots\}\},$$

i.e. a set of $2^2 = 4$ elements.

Two remarks should be added to definition V. For most practical purposes — especially for definition VI, which is based on definition V — it is sufficient to use the first formulation "let S be a set of sets s". The other formulation, in which an auxiliary set T is used in the same way as in definition III, is more general only in so far as it admits *equal factors*, which of course cannot appear as elements of a *set* S. In particular, it may be that to *every* element t of T the same set s is assigned, i.e. that all the factors are equal. In § 7 it will sometimes prove convenient (though by no means necessary) to introduce this specialization.

Secondly, what has been said above of the pairs appearing in definition IV, applies as well to the complexes of definition V. Two complexes are considered *equal* only if they contain *the same elements from the same sets*. The simplest way of avoiding the ambiguity mentioned with respect to pairs, is to take the factors mutually exclusive. If we want to avoid both this restriction (which will *a fortiori* imply that the factors are different) and the introduction of order concepts into the notion of complex, then we should conceive the elements of the complexes as connected with the respective sets (or with the respective elements of T), i.e. conceive them as ordered pairs (footnote 2 on p. 119).

It was pointed out in connection with the example b) on p. 120 that the outer product is the null-set 0, if 0 is one of the factors. In this case no complex exists, since a complex must contain one element from each factor. Now the converse question arises — when

every factor is different from 0, can one be sure that *there exists at least one complex*, i.e. that the outer product is different from 0?

Apparently the answer is evident, and in the affirmative. It seems plausible that we may choose an *arbitrary* element from each of the factors [1]) which, for the sake of simplicity, may be assumed to be mutually exclusive. Then the set containing all the chosen elements represents a complex; hence, this set is an element of the outer product which, accordingly, is different from 0.

However [2]), there is no reason to believe that the principles used hitherto for developing our theory, will permit us to form a complex in the way indicated. In order to examine the problem more closely let us proceed from a disjointed set D each element d of which shall be *a set different from* 0. By forming the sum-set $\mathbf{S}D$ we obtain a set among whose elements appear all the objects that may be contained in any complex of $\mathbf{P}D$; in fact, any element of a complex belongs to an element of D, and therefore to $\mathbf{S}D$ (definition I, p. 109). The problem of forming a complex — the existence of which would prove that $\mathbf{P}D \neq 0$ — is therefore the problem of constructing a subset of the set $\mathbf{S}D$ which contains a single element out of each element of D. For the purpose of constructing subsets we may rely on the principle of subsets (p. 22) which uses a certain common property of the *elements* contained in the subset to be formed.

In fact, on the basis of this principle and of the principle of power-set (p. 97), *the existence of the outer product itself is guaranteed*: it is a subset of the power-set of $\mathbf{S}D$, containing *those* subsets of $\mathbf{S}D$ (complexes) which have the property of containing a single element out of each element of D. The subset in question obviously is the null-set, if the null-set occurs among the elements of D.

Nevertheless, even when all elements of D are non-empty sets, there seems to be no guarantee that we shall be able to indicate a property characteristic of even one complex, and thus we may find it impossible to exclude the possibility that the outer product is the null-set. As a matter of fact, we have seen that the elements of a complex are arbitrary but for the condition that exactly

[1]) Except, of course, the factors that contain one element only. For these factors the choice is not arbitrary but compulsory.

[2]) The following argument (up to theorem 2) is somewhat difficult and may be omitted by beginners.

one element from each element of D belongs to the complex.

We therefore seem to need a new principle, the last one required for the general [1]) theory of sets:

Multiplicative principle or **Principle of choice (VII).**
The outer product PD of a disjointed set D, the elements of which are non-empty sets, is itself a non-empty set. In other words, PD contains at least one complex.

Without explicitly mentioning it, we have already used this principle previously, notably in proof A of theorem 4 on p. 57 and in the proof of theorem 1 on p. 111/2, and we shall still frequently use it, in particular in § 11, 6. In *Foundations*, ch. II the whole subject as well as its rather dramatic history will be discussed thoroughly, and further we shall deal with the question whether the preceding principles can indeed be *proved* to be insufficient to guarantee the non-vanishing of the outer product under the mentioned conditions.

If the given set D of non-vanishing sets is *not disjointed*, principle VII is insufficient, since the same object then may be chosen simultaneously out of different elements of D. In this case one would have to introduce a one-valued *function*, the argument-domain of which is the set D, and which attaches to each argument x (i.e. to each set x in D) a certain *element of* x. However, this apparently stronger principle (Zermelo's *Principle of Choice*) can be deduced from Russell's *multiplicative principle* as formulated above.

Having principle VII at our disposal, we may now formulate the following theorem which is completely analogous to the arithmetical theorem on the vanishing of a product.

THEOREM 2. An outer product of sets equals the null-set if, and only if, at least one of the factors is the null-set.

For the transition from the multiplication of sets to that of cardinals, we require an argument similar to the one used in 3 for addition. If A and B are equivalent sets [2]) of sets and φ a

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[1]) An additional principle required for special purposes will be introduced in *Foundations*. Properly speaking, also the principle of infinity (p. 42) is a principle of special character.

[2]) If we wish to include the case of *equal* factors, we may again use the more general formulation of definition V, i.e. replace each of the sets A and B by a function the values of which play the part of the elements of A or B.

certain representation between them, we assume that any element a of A and its mate by φ in B, say $b = \varphi(a)$, are *equivalent* sets. By means of arbitrary but definitely chosen representations [1]) between each $a \varepsilon A$ and the attached $\varphi(a) \varepsilon B$, any complex c of $\mathbf{P}A$ uniquely determines a complex of $\mathbf{P}B$ which, according to those representations, contains the mates of the elements of c. The correspondence thus created is obviously biunique. Hence:

THEOREM 3. If A and B are equivalent sets and if a certain representation between them relates to each other such elements of A and B as are themselves *equivalent sets*, then the outer products $\mathbf{P}A$ and $\mathbf{P}B$ are also equivalent.

7. The Multiplication of Cardinals and its Formal Laws. The last theorem enables us to define the multiplication of *cardinals* in the following way (cf. definition III, p. 114):

DEFINITION VI. A finite or infinite number of cardinals, not necessarily differing from each other, may be given by means of a function f which assigns to every element t of a non-empty set T a definite cardinal $f(t) = \mathbf{c}_t$. In order to form the product of the cardinals $\mathbf{c}_t$, attach to each $\mathbf{c}_t$ an arbitrary set C_t with the cardinal $\mathbf{c}_t$ [2]). Form the outer product of all the representatives C_t; the cardinal $\mathbf{p}$ of this product is called *the product of the given cardinals*. In symbols:

$$\mathbf{p} = \prod_{t \varepsilon T} \mathbf{c}_t = \mathbf{c} \cdot \mathbf{c}' \cdot \mathbf{c}'' \cdot \ \ldots \ [3])$$

where $\mathbf{c}$, $\mathbf{c}'$, $\mathbf{c}''$, ... denote some of the cardinals $\mathbf{c}_t$.

In short, the product of given *cardinals* is the cardinal of the outer product of *sets* which "represent" the given cardinals,

$$\prod_{t \varepsilon T} \mathbf{c}_t = \overline{\overline{\prod_{t \varepsilon T} C_t}} \ [4]).$$

---

[1]) Here again the principle of choice is used.

[2]) The additional condition imposed in definition III (disjointedness) is not necessary here. Nevertheless, the condition may be imposed for convenience's sake; cf. the second remark concerning definition V on p. 121.

[3]) In the case of the multiplication of cardinals there is no need to use the sign $\times$ or the expression *outer product*, since the *inner product* (*meet*) of sets has no significance for cardinals.

[4]) For the sake of simplicity, the sign Π has been used here for the outer multiplication of *sets*, too.

Again, one may object that the representatives are not sufficiently determined. In analogy to addition, however, theorem 3 in **6** shows that different outer products obtained by the procedure of the definition must be equivalent to each other. Accordingly, their cardinals are equal, and the product of the given cardinals is uniquely determined.

The associative and commutative laws have been shown on p. 115 to hold for the addition of cardinals. Now, it will be seen that the other formal laws which hold in arithmetic (p. 106) remain also valid. Of course, these generalizations do not only mean that the terms and factors may be transfinite as well as finite cardinals, but also that any number of terms and factors is admitted.

As to the *commutativity* of multiplication, the same remark applies as in the case of addition: the products introduced in definitions IV, V, VI have been defined in a way not involving any order of the given factors (cf. p. 119). Therefore, they do not depend on an order possibly used [1]).

The *associative* law of multiplication shall be formulated in the way used for addition (p. 115/6). As in definition V, let S denote the set whose outer product $\mathbf{P}S$ is to be formed, and let

$$S = K + L + M + \ldots$$

be any decomposition of S into complementary, mutually exclusive subsets. Hence, any $s \varepsilon S$ is either a $k \varepsilon K$, or an $l \varepsilon L$, etc. The general associative law connects the outer product $P = \mathbf{P}S$, which disposes of S as a whole, with the outer product P^* of all the outer products $\mathbf{P}K$, $\mathbf{P}L$, $\mathbf{P}M$, ... We shall see that P and P^* are equivalent sets.

To any complex c which is an element of $\mathbf{P}S$, there correspond complexes of $\mathbf{P}K$, $\mathbf{P}L$ etc. which contain precisely the elements appearing in c [2]), and it is obvious that by relating to c the complex c^* whose elements are the complexes of $\mathbf{P}K$, $\mathbf{P}L$ etc. just mentioned — c^* being an element of $P^* = \mathbf{P}K \times \mathbf{P}L \times \mathbf{P}M \times \ldots$ — we obtain a representation between the outer products P and P^*. Hence $P \sim P^*$.

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[1]) Cf. the footnote on p. 115 or exercise 7 at the end of § 6.

[2]) Of course, if S is not a disjointed set, the elements of the elements of S have to be distinguished according to the element of S to which they belong; cf. p. 121.

We may express our result in a more explicit form by using the sign Π also for the multiplication of *sets* (as in definition VI), and by denoting the factors (i.e. the elements of S) generally by y:

$$(1) \qquad \prod_{y \in K} y \times \prod_{y \in L} y \times \prod_{y \in M} y \times \ldots \sim \prod_{y \in S} y. \quad (S = K+L+M+\ldots)$$

As a simple illustration of this reasoning, suppose

$$S = \{s_1, s_2, s_3, s_4\},$$

the elements s_k being mutually exclusive and non-empty sets, and

$$K = \{s_1, s_2\}, \qquad L = \{s_3, s_4\}.$$

If σ_k denotes a certain element of s_k, by our procedure the complex $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ (being an element of $\mathbf{P}S = s_1 \times s_2 \times s_3 \times s_4$) is related to the complex $\{\{\sigma_1, \sigma_2\}, \{\sigma_3, \sigma_4\}\}$ which is an element of

$$\mathbf{P}K \times \mathbf{P}L = (s_1 \times s_2) \times (s_3 \times s_4).$$

It will prove useful to express (1) in a more general form, avoiding the letters $K, L, M, \ldots$. For this purpose we start again from an appropriate non-empty set T and relate to any $t \varepsilon T$ a set K_t of sets such that the sets K_t are mutually exclusive. If $\sum_{t \varepsilon T} K_t = S$, the relation (1) reads:

$$(1') \qquad \prod_{t \varepsilon T} \left(\prod_{y \varepsilon K_t} y\right) \sim \prod_{y \varepsilon S} y. \qquad \left(S = \sum_{t \varepsilon T} K_t\right)$$

Since the sets P and P^* are equivalent, their cardinals are *equal*, and hence for the multiplication of cardinals the associative law holds true in full.

The *distributive* law [1]) connecting addition and outer multiplication may be formulated for cardinals in the form

$$\mathbf{c} \cdot \sum_{t \varepsilon T} \mathbf{c}_t = \sum_{t \varepsilon T} (\mathbf{c} \cdot \mathbf{c}_t). \text{ [2]}$$

To prove this equality, we choose mutually exclusive set-representatives of the cardinals $\mathbf{c}_t$ and of $\mathbf{c}$, and form the corresponding

[1]) For more general laws of distributive character cf. W. Wernick **1.**

[2]) The distributive law obtained by inverting the order of the factors on both sides, follows from the above form by the commutativity of multiplication. On the other hand, the law in which the operations of addition and multiplication are interchanged (p. 28) does not hold for the outer products of sets, i.e. for the multiplication of cardinals, any more than it does in ordinary arithmetic.

outer products; it is obvious that we thus obtain equal sets, hence certainly equal cardinals.

By adding the results of p. 116 to those obtained here, we have:

THEOREM 4. Addition and multiplication of cardinals are commutative and associative operations, connected by the ordinary distributive law.

Evidently multiplication by 1 does not affect the value of a product, even if the factor 1 appears repeatedly (any number of times); cf. the second example on p. 121. Moreover, as in arithmetic, 1 is the *only* cardinal $\mathbf{x}$ which fulfils the equation $\mathbf{c} \cdot \mathbf{x} = \mathbf{c}$ for every (finite or transfinite) cardinal $\mathbf{c}$.

In arithmetic the product of two natural numbers is usually defined by *repeated addition*. The same connection between addition and multiplication holds for cardinals in general, as will be shown presently; it is, however, less convenient to use this fact for *defining* multiplication since then the step from two to any number of factors is not as simple as in the case of definitions V and VI.

Let us first stipulate that in a product $\mathbf{a} \cdot \mathbf{b}$ (or $A \times B$) we shall always consider the first factor $\mathbf{a}$ as the multiplicand (the factor to be multiplied), and the second factor $\mathbf{b}$ as the multiplier (expressing "how often" the multiplicand is to be taken). In ordinary arithmetic, and even in our present case, this distinction is insignificant because multiplication is commutative. But in §§ 8 (7) and 10 we shall become acquainted with products for which the order of the factors is essential. — Of course our agreement about the roles of $\mathbf{a}$ and $\mathbf{b}$ in the product $\mathbf{a} \cdot \mathbf{b}$ is arbitrary.

If n is a natural number (finite cardinal $\neq 0$) and $\mathbf{c}$ any cardinal, the equality

$$\underbrace{\mathbf{c} + \mathbf{c} + \ldots + \mathbf{c}}_{n \text{ terms}} = \mathbf{c} \cdot n$$

is rather obvious, since any sum-set representing the left-hand sum of cardinals is equivalent to an outer product whose cardinal is $\mathbf{c} \cdot n$. To see this, let the elements of a certain set C_1, having the cardinal $\mathbf{c}$, be denoted by $c_1, c_1', c_1'', \ldots$; the elements of a second set C_2, being mutually exclusive with, and equivalent to C_1, by c_2, $c_2', c_2'', \ldots$; and so on, up to a set C_n which is mutually exclusive with, and equivalent to, the preceding sets and contains the elements $c_n, c_n', c_n'', \ldots$. The sum-set $C_1 + C_2 + \ldots + C_n = S$ con-

tains all the elements of those sets and no other element. On the other hand, using an arbitrary set C equivalent to C_1, and forming a set of n elements

$$F = \{f_1, f_2, \ldots, f_n\},$$

we obtain the outer product $C \times F$ as the set the elements of which are all complexes

$$\{c, f_k\}. \qquad (c \,\varepsilon\, C;\ k = 1, 2, \ldots, n)$$

After choosing certain representations between C and each of the C_k, we may denote the element of C_k corresponding to a given $c^* \,\varepsilon\, C$ by c_k^*. Then, by relating $c_k^* \,\varepsilon\, C_k$ to the complex $\{c^*, f_k\} \varepsilon\, (C \times F)$, we obtain a representation between S and $C \times F$, which shows that their cardinals are equal.

The proof for the general case may be conducted in the same way. For let $\mathbf{c}$ and $\mathbf{f}$ be any cardinals, represented by mutually exclusive sets C and F. The outer product $C \times F$ is then the set of all pairs $\{c, f\}$ with $c \,\varepsilon\, C$ and $f \,\varepsilon\, F$. If f^* denotes an arbitrary fixed element of F, the pairs $\{c, f\}$ for which $f = f^*$ form a subset C^* of $C \times F$ which is equivalent to C; this results from the representation relating $c \,\varepsilon\, C$ to the pair $\{c, f^*\}$. Proceeding in the same way with respect to any other element f of F, we obtain different — and even mutually exclusive — sets C^*; these are subsets of $C \times F$, and their sum is this outer product. On the other hand, the set A of all those sets C^* has the cardinal $\mathbf{f}$, and the sum-set $\mathbf{S}A$ (i.e. the sum of all C^*), the cardinal $\mathbf{c} + \mathbf{c} + \ldots$ ($\mathbf{f}$ terms), according to definition III. This cardinal, therefore, equals the cardinal $\mathbf{c} \cdot \mathbf{f}$ of $C \times F$. Hence:

THEOREM 5. The product of two cardinals $\mathbf{c} \cdot \mathbf{f}$ ($\mathbf{f} \neq 0$ [1])) may be obtained by continued addition, i.e. by adding the term $\mathbf{c}$ $\mathbf{f}$-times in the sense of definition III.

By equating one of the factors, e.g. $\mathbf{c}$, to 1, we obtain

$$1 \cdot \mathbf{f} = \mathbf{f} = 1 + 1 + \ldots \ (\mathbf{f} \text{ terms}).$$

That is to say, any transfinite cardinal, as well as a natural number, may be represented as a sum in which each of the terms is the cardinal 1, and the "number" of terms is the cardinal to be repre-

[1]) The case $\mathbf{f} = 0$ may, of course, be included by stipulating that the "sum" taking $\mathbf{c}$ 0-times, is 0, in accordance with commutativity.

sented. Incidentally, we do not need the concept of multiplication to perceive this; it is only another expression for the fact that, according to definition I (p. 109). we may consider any non-vanishing set S as the sum of all sets $\{s\}$ where s runs over the elements of S. Conversely, from the specialization of theorem 5 just considered, we may deduce the general theorem by multiplying by $\mathbf{c}$ and using the distributive law.

Expressing theorem 2 in terms of cardinals, we have

THEOREM 6. A product of cardinals equals 0 if, and only if, at least one of the factors equals 0.

This theorem is well-known from ordinary arithmetic where it plays a fundamental part. But there the arguments of the proof are quite different. In arithmetic the first part of the theorem is a consequence of the distributive law, the second part ("and only if") of the possibility of division [1]; in our case, the first part follows from the very definition of multiplication (IV or V) while the second part requires a special postulate, the multiplicative principle (p. 123).

The theorems 4, 5 and 6 which extend the validity of fundamental properties of the arithmetical operations to operations with cardinals in general, show that the definitions of addition and multiplication were chosen in a natural and appropriate way.

8. Inverse Operations. Inequalities between Cardinals. In contrast to our complete success in extending the "direct" operations, addition and multiplication, to cardinals in general, *it is impossible to define their inversions, subtraction and division,* in a general way.

It is trivial that one cannot subtract $\mathbf{b}$ from $\mathbf{a}$ if $\mathbf{a} < \mathbf{b}$ since, according to the very nature of cardinals and their definition in § 4, there are no "negative" cardinals. The case of division is analogous. But even if the transfinite cardinal $\mathbf{a}$ *equals* the subtrahend (or divisor) $\mathbf{b}$ [2]), it is impossible to obtain a uniquely

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[1]) An outline of those proofs may suffice:

A) $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$, hence $a \cdot 0 = 0$. In the same way we obtain $0 \cdot a = 0$.

B) $a \cdot b = 0$ with $b \neq 0$ implies, in view of A):

$$0 = (a \cdot b) \cdot \frac{1}{b} = a \cdot \left(b \cdot \frac{1}{b}\right) = a \cdot 1 = a.$$

[2]) For the case $\mathbf{a} > \mathbf{b}$, the result may be gathered from § 11, 5 and 7.

determined difference $\mathbf{a} - \mathbf{b}$ (or quotient $\mathbf{a}/\mathbf{b}$), as the simplest instances show. The following relations where n is any finite cardinal $\neq 0$, have already been proved:

$$\aleph_0 + 0 = \aleph_0 + n = \aleph_0 + \aleph_0 = \aleph_0 \qquad \text{(p. 116)}$$

$$\mathfrak{N} + 0 = \mathfrak{N} + n = \mathfrak{N} + \aleph_0 = \mathfrak{N} + \mathfrak{N} = \mathfrak{N} \qquad \text{(p. 117/8)}$$

$$\mathbf{c} + 0 = \mathbf{c} + n = \mathbf{c} + \aleph_0 = \mathbf{c} \text{ for any transfinite } \mathbf{c} \text{ (p. 117).}$$

Therefore the differences $\aleph_0 - \aleph_0$ and $\mathbf{c} - \mathbf{c}$ may assume each of the values 0, n, $\aleph_0$; the difference $\mathfrak{N} - \mathfrak{N}$, in addition, the value $\mathfrak{N}$. Analogous relations with respect to multiplication will be obtained in **9** and **10**; for instance:

$$\mathfrak{N} \cdot 1 = \mathfrak{N} \cdot n = \mathfrak{N} \cdot \aleph_0 = \mathfrak{N} \cdot \mathfrak{N} = \mathfrak{N}. \qquad (n \neq 0)$$

Hence the quotient $\mathfrak{N}/\mathfrak{N}$ can be assigned no definite value either [1]). The intrinsic reason for the irreversibility of the direct operations is, of course, the equivalence of infinite sets to proper subsets.

The impossibility of defining subtraction and division may also be inferred from *the invalidity of the ordinary inequalities of arithmetic* for the addition and multiplication of transfinite cardinals — notwithstanding the fact that the *equalities* remain valid.

The inequalities $\mathbf{a}_1 \leqq \mathbf{b}_1$, $\mathbf{a}_2 \leqq \mathbf{b}_2$ still imply the inequalities

$$(1) \qquad \mathbf{a}_1 + \mathbf{a}_2 \leqq \mathbf{b}_1 + \mathbf{b}_2, \ \mathbf{a}_1 \cdot \mathbf{a}_2 \leqq \mathbf{b}_1 \cdot \mathbf{b}_2,$$

as in ordinary arithmetic. In order to prove them, we choose mutually exclusive sets A_1, A_2, B_1, B_2, as the representatives of those cardinals. By assumption there exist subsets $B_1' \subseteq B_1$ and $B_2' \subseteq B_2$ such that $A_1 \sim B_1'$ and $A_2 \sim B_2'$. Hence

$$A_1 + A_2 \sim B_1' + B_2', \ A_1 \times A_2 \sim B_1' \times B_2'.$$

But, since $B_1' + B_2' \subseteq B_1 + B_2$ and $B_1' \times B_2' \subseteq B_1 \times B_2$, the last equivalences mean that $\mathbf{a}_1 + \mathbf{a}_2 \leqq \mathbf{b}_1 + \mathbf{b}_2$ and $\mathbf{a}_1 \cdot \mathbf{a}_2 \leqq \mathbf{b}_1 \cdot \mathbf{b}_2$ (p. 104).

It is easy to extend the inequalities (1) to any number of terms and factors; the respective theorems are formulated in exercise 3 at the end of this section.

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[1]) For a kind of substitute division introducing, as it were, transfinite rationals, cf. Olmsted **1**.

In arithmetic, however, much more far-reaching inequalities are true, namely (for positive real numbers a_k, b_k)

(2) $a_1 \leqq b_1$ and $a_2 < b_2$ imply $a_1 + a_2 < b_1 + b_2$, $a_1 \cdot a_2 < b_1 \cdot b_2$,

and in particular:

(3) $a_1 < b_1$ and $a_2 < b_2$ imply $a_1 + a_2 < b_1 + b_2$, $a_1 \cdot a_2 < b_1 \cdot b_2$.

The inequalities (2) certainly *do not hold for transfinite cardinals.* For instance, in spite of $\aleph \leqq \aleph$, $n < \aleph_0$, we have $\aleph + n = \aleph + \aleph_0$, $\aleph \cdot n = \aleph \cdot \aleph_0$ (for $n \neq 0$).

The inequalities (3) do hold for transfinite cardinals, but our present resources are not sufficient to prove them. For this purpose, we require the profound theorem of § 11, 7 in connection with § 11, 5 [1]).

An almost self-evident inequality which, nevertheless, has fundamental consequences, is contained in

THEOREM 7. Let C be a set [2]) of cardinals among which there is *no largest cardinal.* Then the sum $\mathbf{s}$ of all the cardinals in C is larger than any of these cardinals.

Proof. By virtue of the associative law and the inequality (1), we have for any cardinal $\mathbf{c} \in C$

$$\mathbf{c} \leqq \mathbf{s}.$$

We intend to show that $\mathbf{c} < \mathbf{s}$. Indeed, if there were a $\mathbf{c}$ such that $\mathbf{c} = \mathbf{s}$, *any* cardinal of C would be smaller than or equal to $\mathbf{c}$. In other words, $\mathbf{c}$ would be the largest cardinal in C, contrary to our assumption.

(The hypothesis of theorem 7 may also be expressed in the form "to any cardinal of C there exists a larger cardinal in C". However, as long as the general comparability of cardinals has not yet been proved (see p. 105) this form would be less general.)

Cantor's theorem (p. 94) shows that *there is no largest cardinal*; given any cardinal, there exists a larger one. The present theorem shows much more; *to any set of cardinals there exists a cardinal larger than any element of the set*, for if there is a largest cardinal

[1]) In Tarski 3 both the inequalities (3) are proved to be equivalent with the multiplicative principle.

[2]) We may make a more general assumption as in definitions III and VI. But in this case, no more far-reaching result would be obtained.

in the set, a larger one is guaranteed by Cantor's theorem, and if not, the sum is larger by virtue of theorem 7. Therefore, the multiplicity of transfinite cardinals exceeds by far that of finite cardinals (natural numbers), since among the latter there is no number which is larger than all natural numbers or than all even integers or than all prime numbers.

The antinomy of the "set of all cardinals" which arises in this connection, will be discussed in *Foundations*, ch. I.

Among other inequality relations between cardinals [1]), we shall prove one which is remarkable for its generality and power no less than for a certain strangeness. Let us first note that if, for instance, two sequences of cardinals $(\mathfrak{a}_k)$ and $(\mathfrak{b}_k)$ $(k = 1, 2, \ldots)$ fulfil the inequalities $\mathfrak{a}_k < \mathfrak{b}_k$ for every k, one cannot conclude that $\sum_k \mathfrak{a}_k < \sum_k \mathfrak{b}_k$, or $\prod_k \mathfrak{a}_k < \prod_k \mathfrak{b}_k$. Conspicuous instances of the *equality* between both sums or products are obtained by taking $\mathfrak{a}_k = k - 1$, $\mathfrak{b}_k = k$, or $\mathfrak{a}_k = k$, $\mathfrak{b}_k = k + 1$. The theorem which we shall prove says that one does obtain an inequality if one compares the *sum* of the *smaller* cardinals with the *product* of the *larger* ones.

THEOREM 8 (Inequality of König–Zermelo [2])). Let T be a non-empty set; to every element t of T, let two functions f and g assign two cardinals $f(t)$ and $g(t)$ such that, for every $t \varepsilon T$, $f(t) = \mathfrak{a}_t$ is *smaller* than $g(t) = \mathfrak{b}_t$, i.e.

$$\mathfrak{a}_t < \mathfrak{b}_t \text{ [3])}.$$

Then the following inequality holds true:

$$\sum_t \mathfrak{a}_t < \prod_t \mathfrak{b}_t \text{ [4])}.$$

Proof. We choose as the representatives of $\mathfrak{a}_t$ and $\mathfrak{b}_t$ sets A_t and B such that the sets B_t are mutually exclusive and such that

$$\overline{\overline{A}}_t = \mathfrak{a}_t; \quad \overline{\overline{B}}_t = \mathfrak{b}_t; \quad A_t \subset B_t, \text{ i.e. } B_t = A_t + C_t.$$

The complements C_t differ from the null-set since $\mathfrak{a}_t < \mathfrak{b}_t$, and the

[1]) Cf. especially F. Bernstein 3, § 3; Lindenbaum–Tarski 1, § 1.

[2]) J. König 1; Zermelo 3, p. 277. Cf. the proof in Hausdorff 4, pp. 57–59; 5, p. 35. The theorem was formulated by König for *sequences* only.

[3]) This implies $\mathfrak{b}_t \neq 0$ for any t.

[4]) It is remarkable that the *product* of all $\mathfrak{a}_t$'s may equal the product of all $\mathfrak{b}_v$'s even if, in addition to $\mathfrak{a}_t < \mathfrak{b}_v$ for any t and v, the set of arguments t for the $\mathfrak{a}_t$'s has a *smaller cardinal* than the set of arguments v for the $\mathfrak{b}_v$'s. Cf. § 7, end of 4.

sets A_t are mutually exclusive as well. The sum-set S of all A_t accordingly has the cardinal $\sum_t \mathfrak{a}_t$, and the outer product P of all B_t the cardinal $\prod_t \mathfrak{b}_t$. In view of theorem 5 (p. 104) we have now to prove:

a) S is equivalent to a subset of P;
b) S is not equivalent to P.

a). In every $C_t = B_t - A_t$ we choose a certain element $c_t \varepsilon C_t$ which shall remain fixed [1]); by the mutual exclusiveness of the sets C_t we have $c_{t_1} \neq c_{t_2}$ if $t_1 \neq t_2$. The elements of P are complexes containing a single element from each factor $B_t = A_t + C_t$. Special complexes of this kind may be formed in the following way: for *a single* value of $t \varepsilon T$, say $t = \tau$, we take as the element of B_τ a discretionary element a_τ of A_τ while for every $t \neq \tau$ we choose the fixed $c_t \varepsilon C_t$. If, for simplicity's sake, some values of t are denoted by 1, 2, 3, ..., k, some of the special complexes will be the following

$$\begin{array}{ll} \{a_1, c_2, c_3, \ldots, c_k, \ldots\} & (a_1 \varepsilon A_1) \\ \{c_1, a_2, c_3, \ldots, c_k, \ldots\} & (a_2 \varepsilon A_2) \\ \{c_2, c_2, a_3, \ldots, c_k, \ldots\} & (a_3 \varepsilon A_3) \\ \vdots \quad \vdots \quad \vdots & \vdots \\ \{c_1, c_2, c_3, \ldots, a_k, \ldots\} & (a_k \varepsilon A_k). \end{array}$$

By taking *all* the complexes corresponding to the first line, while a_1 runs over the set A_1, one obtains a set equivalent to A_1, or more strictly, (since every complex is an element of P) a *subset of P equivalent to A_1*. *Mutatis mutandis*, the same holds true for any other line. Since furthermore any element of one line differs from any element of any other line — for the elements of each line contain elements of an A_t (and not of a C_t) for *a definite single t*, and the A_t's are mutually exclusive — the sets corresponding to the different lines are also mutually exclusive. Therefore the set containing *all* those special complexes is equivalent to the sum-set S of all sets A_t. Hence S actually is equivalent to a subset of P, as asserted by a).

b). Assume $P_0 \subseteq P$ to be equivalent to the sum-set S. Our purpose is to show that $P_0 \neq P$.

Choosing a definite representation φ between S and P_0, we pay attention to the different (mutually exclusive) subsets of P_0 represented on the different sets A_t, whose sum is S. Let the subset of P_0, represented on A_t by φ, be called $P_0^{(t)}$, so that $A_t \sim P_0^{(t)}$. Hence P_0 is the sum-set of the (mutually exclusive) sets $P_0^{(t)}$ where t runs over T. The elements of $P_0 \subseteq P$, and accordingly the elements of any $P_0^{(t)}$, are complexes containing a single element out of each set B_t. We shall show that *not all* complexes of P are contained in P_0, using again the *diagonal method* (pp. 65 and 95).

Let τ be a *fixed* element of T. In any complex h contained in $P_0^{(\tau)}$ we call the special element of h taken from B_τ, the "diagonal element" $b_{\tau,\tau}$. In this element the first index τ refers to the subset $P_0^{(\tau)} \subset P_0$ to which the complex h belongs, the second index τ to the element of h belonging to the particular

[1]) Here, as well as in b), the multiplicative principle is used.

factor B_τ of the outer product P. The set D_τ of *all* these diagonal members $b_{\tau,\tau}$, where h runs over $P_0^{(\tau)}$, is a *subset* of B_τ and has a cardinal $\leqq \mathbf{a}_\tau$ (in contrast to the cardinal $\mathbf{b}_\tau$ of B_τ itself); for there are only $\mathbf{a}_\tau$ complexes h in $P_0^{(\tau)}$, in view of $P_0^{(\tau)} \sim A_\tau$, and different complexes of this kind do not necessarily contain *different diagonal* elements $b_{\tau,\tau}$. Hence, if

$$E_\tau = B_\tau - D_\tau,$$

we have $E_\tau \neq 0$ since the cardinal of D_τ is smaller than the cardinal of B_τ (by the assumption $\mathbf{a}_t < \mathbf{b}_t$ for any t).

The last step consists in letting τ run over all the elements t of T, and choosing a single arbitrary element e_t out of each set E_t. (The sets E_t are mutually exclusive since the same holds for the sets B_t.) The set e^* which contains all the chosen elements e_t, is a complex of P, since e^* contains a single element out of each factor $B_t = D_t + E_t$. On the other hand, *the complex e^* is not contained in the subset P_0 of P*, for any complex h of P_0 is an element of a certain subset $P_0^{(\tau)} \subset P_0$, and the element of h belonging to the particular factor B_τ — i.e. the "diagonal element" $b_{\tau,\tau}$ — is an element of the subset D_τ, by the preceding paragraph. On the other hand, the element of e^* belonging to B_τ has been selected from the *complementary* subset E_τ. Therefore any complex h of P_0 differs from the complex e^* of P.

Hence P_0, being equivalent to S, cannot coincide with P. Q.E.D.

Example. Since $0 < 1,\ 1 < 2,\ \ldots,\ k < k+1,\ \ldots$, we obtain:

$$0 + 1 + 2 + \ldots + k + \ldots\ (= \aleph_0) < 1 \cdot 2 \cdot 3 \cdot \ldots \cdot k \cdot \ldots.$$

The same applies to any sequence of increasing (finite or transfinite) cardinals.

Further examples are given in § 7, **5** and **6**.

9. Examples of the Multiplication of Cardinals. In arithmetic the first consequence of the definitions of operations between numbers is the multiplication table. In a corresponding way, we shall obtain relations between the simplest cardinals, by applying to them our definitions of addition and multiplication.

It goes without saying that the ordinary multiplication of natural numbers is in accordance with definition VI; for it was our very aim to achieve this when we formulated the definition. As to the factor 0, we refer to theorem 6.

According to **5** and theorems 4 and 5, we have:

$$\aleph_0 \cdot 2 = \aleph_0 + \aleph_0 = \aleph_0, \quad \aleph \cdot 3 = \aleph_0 \cdot (2+1) = \aleph_0 + \aleph_0 = \aleph_0, \text{ etc.}$$

By mathematical induction (cf. theorem 2 of § 3, p. 47) we ob-

tain for any finite cardinal $n \neq 0$:

$$\aleph_0 \cdot n = n \cdot \aleph_0 = \aleph_0. \tag{1}$$

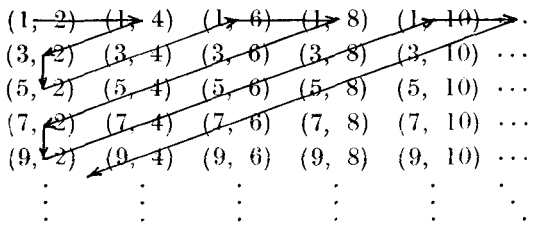
From theorem 3 (p. 51) we infer that even

$$\aleph_0 \cdot \aleph_0 = \aleph_0. \tag{2}$$

This is another way of expressing the well-known fact that a double sequence $(a_{m,n})$ $\binom{m}{n} = 1, 2, 3, \ldots)$ can be re-arranged into a simple one. By taking as the representatives of the factors in (2) the denumerable sets

$$\{1, 3, 5, \ldots.\} \quad \text{and} \quad \{2, 4, 6, \ldots\ldots\ldots\}$$

and by applying the zigzag method, we obtain the re-arrangement of the outer product, appearing here as an "infinite square", in the following scheme (cf. p. 48):



Furthermore we obtain:

$$\aleph \cdot n = n \cdot \aleph = \aleph. \qquad (n \text{ finite}, \neq 0) \tag{3}$$

$$\aleph \cdot \aleph_0 = \aleph_0 \cdot \aleph = \aleph. \tag{4}$$

In view of theorem 5, the first of these relations may be interpreted (cf. **5**) as expressing theorem 2 on p. 71 by which the set of all points on a certain segment is equivalent to the set of all points on a segment n times larger. In the same way (4) expresses theorem 3 on p. 72, for the set of *all* points on a straight line may be conceived as the sum-set of the sets corresponding to the segments from $m - 1$ to m (including, for instance, the former point but not the latter) where m runs over all integers, i.e. over a denumerable set. Every point of "half" the line may thus be written in the form $\{x, m\}$ where $0 \leqq x < 1$, $m > 0$; in this form we recognize the outer product of sets of the cardinals $\aleph$ and $\aleph_0$.

10. The Cardinal of Two-Dimensional Continua. Our next aim is to compute the product $\aleph \cdot \aleph$. We shall show that

$$\aleph \cdot \aleph = \aleph. \tag{5}$$

The meaning of this relation may be expressed in a geometrical form as follows. Taking a square with a side of unit length, the set S to be considered shall be the set of all interior points of the square, including the points of two adjacent sides (say the top and right-hand ones) and their common vertex. (Cf. fig. 9.)

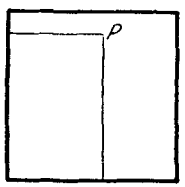


Fig. 9

From any point P of S draw the perpendiculars to the lower and to the left-hand sides of the square until they intersect with these sides. The points of intersection may be denoted respectively as the "coordinates" x and y. If the bottom left-hand vertex of our square serves as the origin of coordinates, both coordinates x and y will be *positive real numbers not larger than* 1.

Hence, as explained in detail on p. 63, one may represent x and y in a unique way as infinite decimals

$$x = 0.x_1\, x_2\, x_3 \ldots x_k \ldots \qquad y = 0.y_1\, y_2\, y_3 \ldots y_k \ldots.$$

The point P of S completely determines these decimals, and vice versa. On the other hand, by interlacing the decimal representations of x and y we may, again in a unique way, form the infinite decimal

$$z = 0.x_1\, y_1\, x_2\, y_2\, x_3\, y_3 \ldots x_k\, y_k \ldots;$$

hence z is completely determined by P.[1]) z is, therefore, a real number between 0 and 1 (possibly 1 itself — if each x_k and y_k equals 9). In other words, z corresponds to a point on a segment of unit length, say on the bottom side of the square in fig. 9. Thus a representation between the set S and the set of points situated on a side of the square has been achieved, provided that each point z of the side corresponds to exactly one point P of S.

This condition, however, is not fulfilled if z is determined by

[1]) It is easily seen that this procedure, as well as the modification given afterwards, relies on the relation $\aleph_0 + \aleph_0 = \aleph_0$ ($\aleph_0$ being the cardinal of the set of places, or digits, of a decimal). The *formal* connection is given in the beginning of § 7, 6.

x and y in the way described above. For the above interlacing rule says that, given

$$z = 0.z_1 z_2 z_3 z_4 z_5 z_6 \ldots z_{2k-1} z_{2k} \ldots,$$

one has

$$x = 0.z_1 z_3 z_5 \ldots z_{2k-1} \ldots, \quad y = 0.z_2 z_4 z_6 \ldots z_{2k} \ldots$$

as the coordinates of the point P of S to be related to z. It is true that the infinite decimal z corresponds to *at most* one pair of infinite decimals x and y, for different pairs give different values of z. There are, however, values of z which correspond to *no* such pair, but only to a pair of decimals one of which terminates, and pairs of this kind were not admitted as the coordinates of P.

For instance:

$$z = 0.15203010203010 \ldots$$

is not related to any pair of infinite decimals; in fact it corresponds to

$$x = 0.123123 \ldots, \quad y = 0.5.$$

Therefore, the proposed rule would map the points of S on *some* of the points z of the bottom side.

This defect of the proof is not of an essential character, and can be removed by a slight technical modification. The process of interlacing by which the decimal z has been formed from the decimals x and y, takes as its units the single digits of a decimal. Since the difficulty derives from the zero-digits, let us consider *groups of digits* instead of single digits, as the units to be interlaced or extracted, in the following way: if a digit differs from 0, the digit alone is to represent the group in question; if the digit is 0, the group is to contain, in addition to 0, the following digits up to, and including, the first digit differing from 0. In this second case, the group will contain a number of zeros plus a single last digit with one of the values 1, 2, ..., 9. (The existence of such a last digit is guaranteed by the exclusion of terminating decimals.) The splitting into groups may be illustrated by the example

$$z = 0.3|5|07|9|001|2|6|0004| \ldots.$$

Now the decimals x and y shall be formed from z as above, by taking the *groups* instead of the single digits, and with the same

modification z is to be interlaced from x and y. Hence to the value of z given above, the following coordinates will correspond:

$$x = 0.3070016 \ldots, \quad y = 0.5920004 \ldots.$$

Hereby we obtain a one-to-one correspondence between the infinite decimals z and the pairs (x, y) of infinite decimals representing points of S; i.e. S is mapped on its side.

Thus the defect mentioned above has been avoided. (Instead of this procedure, we may keep to single digits and make up for the defect by using theorem 5 on p. 59.)

Herewith, the product $\aleph \cdot \aleph$ has been evaluated. For the set S can be considered as the set of all ordered pairs (x, y) with x and y running over the continuum from 0 to 1 [1]); denoting this continuum by C, we find the set S to be the outer product $C \times C$. On the other hand, S is equivalent to C. Thus relation (5) of p. 136 has been demonstrated. Hence:

THEOREM 9. The product $\aleph \cdot \aleph$ equals $\aleph$. In geometrical language this means that all the points of a square, as well as all the points of the whole plane, can be related to the points of a segment or of a line by a one-to-one correspondence. Hence the continua of one and of two dimensions are equivalent to each other [2]).

Since, according to theorems 2 and 3 of § 4, the different linear or "one-dimensional" continua, corresponding to any segment of a line or to the whole line, are equivalent to each other, the product $C \times C$ just introduced may also be considered as the set of *all* points in the plane (i.e. the unlimited two-dimensional continuum); in fact these points may be expressed as the pairs (x, y) with x and y assuming any real value. Therefore the extension asserted in theorem 9 is justified. One thus obtains the somewhat surprising result that, given an arbitrarily small segment, its points can be mapped on the points of the whole plane.

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[1]) Relating the ordered pair (x, y) to the complex number $x + iy$, we perceive that, on account of theorem 9, the set of all complex numbers $x + iy$, too, has the cardinal of the continuum.

[2]) This was proved in Cantor **6** (1878) by the use of *continued fractions* instead of decimals. The above mentioned difficulty in creating the correspondence was thus avoided. Cantor's essay deals at once with continua of n (instead of 2) dimensions which, of course, does not involve any additional difficulty; cf. § 7, **6**. In this essay, the continuum hypothesis (§ 11, **7**) was expressed by Cantor for the first time.

The history of theorem 9 deserves to be sketched here. Cantor, after having discovered in 1873 that there is a cardinal ($\aleph$) larger than $\aleph_0$, endeavored during the following years to find a still larger cardinal by proceeding from one to two and more dimensions. His letters to Dedekind on this subject show how revolutionary an attitude was required to see here a problem at all, meaning to try and prove the conjecture that a continuum of two dimensions was not equivalent to a one-dimensional continuum [1]). After three years of fruitless efforts he became convinced that his original conjecture was false. A proof suffering from the defect mentioned above was completed in June 1877 and sent to Dedekind with the expression of the author's own surprise and almost incredulity of the result [2]) since it seemed to destroy the concept of *dimension*. In fact, continua of *any* finite number of dimensions — and even of $\aleph_0$ dimensions, cf. § 7, 6 — have the same cardinal $\aleph$ and, accordingly, admit representations on each other. It was Dedekind who pointed out the decisive part which the *continuity* of a representation plays with respect to the concept of dimension.

As a matter of fact, Cantor's revolutionary result, far from uprooting the concept of dimension, initiated its foundation on a solid logico-mathematical (instead of a pseudo-intuitive) basis and hereby confirmed Leibniz's assertion that space constitutes an *order* and not a mere aggregate [3]). The first step in this direction was the proof that *no one-to-one and continuous correspondence is possible* [4]) between a one-dimensional continuum (an open segment)

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[1]) Cantor–Dedekind **1**, pp. 20—41. Authoritative circles of mathematicians told him that a proof was superfluous since it was self-evident that "two independent variables cannot be reduced to one". In fact, according to Cantor–Stäckel **1**, the publication of Cantor's essay **6** met with great difficulties and was made possible only with Weierstrass' help.

[2]) "Je le vois, mais je ne le crois pas" were Cantor's words.

[3]) Cf. Couturat **2**, p. 134.

[4]) That the correspondence is continuous in both directions, follows from its continuity in one direction and its biuniqueness. But if the correspondence is only unique, as is the case with one-valued functions in general, the property of continuity does not imply that the inverse function is continuous, too. The shock caused by Cantor's proof was even increased when, in 1890, Peano succeeded in defining a *curve* — in other words, a continuous motion of a point — *which passes through all points of a square*. This means that the points of a square can be related to the points of a segment in such a way that different segment-points correspond

and a two-dimensional one (the interior of a square or of a rectangle). The theorems dealing with this and other cases up to three dimensions were demonstrated by Lüroth [1]) from 1878 onwards, as an immediate reaction to Cantor's result. Even when one compares two dimensions with three, the proof is not too simple. In the general case (any pair of finite numbers m and n of dimensions) very considerable difficulties [2]) arise which were overcome by Brouwer only in 1911 [3]).

For the simplest case ($m = 1$, $n = 2$) a proof shall be given here which requires no more than some familiarity with the concept of a *continuous function* as used in an elementary exposition of analysis.

THEOREM 10. There is no continuous [4]) function $z = f(x, y)$ of two real arguments x and y with real values z which constitutes a one-to-one correspondence between the pairs (x, y) and the values z — i.e. which is one-valued and assumes different values z for different pairs (x, y). In short: a one-dimensional continuum (a line) cannot be mapped on a two-dimensional continuum (a plane) [5]) in a continuous way.

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to different square-points and neighboring square-points to neighboring segment-points, while not necessarily neighboring segment-points will correspond to neighboring square-points and it even happens that the same square-point is related to different segment-points. See Peano 1; the proof has been transferred from its analytical form to a geometrical one by Hilbert 1. Cf., for instance, Hahn 2 and 3; Kamiya 1. For a generalization to n dimensions (a movable point touching every point of a space) see Sierpiński 22.

[1]) Cf. the comprehensive essay Lüroth 1.

[2]) Cantor's attempt of 1879 to prove the general theorem, though published, was not successful.

[3]) Brouwer 4; cf. also 5 and 7. For the history of the problem and for more recent investigations on the theory of dimensions cf. Menger 1 and 3; Alexandroff 1 and 2; Hurewicz 1; Hurewicz–Wallman 1. It is remarkable that more than a hundred years ago, Bolzano (in 3, p. 80 of the edition of 1920) made a serious attempt to define the concept of dimension.

[4]) It is not even necessary to assume $f(x, y)$ to be a continuous function of *both* arguments simultaneously. The following proof only requires that it is a continuous function of one variable when the other variable remains constant.

[5]) As evolves from the proof, we may as well take bounded (and closed) continua; e.g., the square $-1 \leqq x \leqq 1$, $-1 \leqq y \leqq 1$ in the plane, and a certain interval containing the values $f(-1, 0)$ and $f(1, 0)$ on the line.

Proof [1]. From the hypothesis that $z = f(x, y)$ is such a function, we shall derive a contradiction. For $y = 0$, the continuous function of one variable $f(x, 0)$ may be denoted by $\varphi(x)$, with $\varphi(-1) = a$, $\varphi(1) = b$. By our assumption we have $a \neq b$. According to an elementary theorem on continuous functions, $\varphi(x)$ assumes all real values between a and b as x increases from -1 to $+1$. Hence, again by our assumption, $z = f(x, y)$ *does not assume any value of the interval* $a \leqq z \leqq b$ *whenever* $y \neq 0$.

The second part of the proof consists of fixing a value $x = c$ for which $\varphi(x) = f(x, 0)$ assumes a certain value *between* a *and* b; then, because of the continuity, we shall obtain a *neighboring* value (being still between a and b) by giving the second argument y a value slightly different from its previous value 0, a result which contradicts the sentence emphasized above. Let $x = c$ be the value between -1 and 1 for which $\varphi(c) = f(c, 0) = (a + b)/2$. Then $f(c, y) = \psi(y)$ is by hypothesis a continuous function of y with $\psi(0) = (a + b)/2$. Since $(a + b)/2$ is an interior value of the interval from a to b, values y which differ but slightly from 0 (i.e. which are sufficiently close to 0) will yield a $\psi(y) = f(c, y)$ which still lies in the interval from a to b, contrary to the result of the preceding paragraph. Q.E.D.

When proving theorem 9 we used, of course, a *discontinuous* one-to-one correspondence between the point of the square and the points of its side; the points of the side corresponding to neighboring points of the square will not be generally neighboring points, and conversely. Yet it is remarkable that, in spite of the rather arbitrary character of the representation in question, it still defines a function of the segment-points which is continuous at all *irrational* argument-values [2].

11. The General Concept of Meet. Boolean Algebra [3]. An operation with sets, differing from the addition and the outer multiplication dealt with in **2** and **6**, has been introduced in § 2, **3**; the result of the operation in question was called *meet* or *inner product* and the operation itself may be named *inner multiplication*. As in § 2, we shall denote the meet of the sequence of sets (S_n) by $S_1 \cdot S_2 \cdot S_3 \ldots$. The inner multiplication, though important in many applications of the theory of sets, has no immediate significance for the theory of transfinite (cardinal or ordinal) number, and will not be thoroughly discussed in this book. Nevertheless, some

[1] Cf. the geometrical proof (of a more general character) in Hahn **3**, pp. 146–151 and the references given there.

[2] See Sierpiński **5a**.

[3] A knowledge of this subsection is not required for the understanding of the following sections.

supplements to the material presented in § 2, as well as a few general remarks, are added here.

Firstly, in § 2, the meet has been defined only for a finite number, or a sequence, of sets. The generalization to any set of sets is obvious, namely obtained in the same way as in definition I on p. 109. Secondly, the meet of a given set of sets exists in the general case on account of our principles, essentially those of sum-set and subsets. The proof runs verbatim like the one presented on p. 29. Thirdly, the proof of the associative law of addition, given on p. 115/6, is easily adapted to the case of inner multiplication. (The commutativity of inner multiplication immediately follows from its definition.) As to the distributive law, it was pointed out in § 2 that two essentially different distributive laws connect the addition and the inner multiplication of sets.

In exercise 1 on p. 42, a few connections between the relation of set-subset and the operations of addition and inner multiplication were noted. Another connection is given in exercise 4 of § 7.

The sets of a sequence (S_n) may be called *ascending* if $S_n \subseteq S_{n+1}$ for every n, and *descending* if $S_{n+1} \subseteq S_n$ for every n. It is easily seen that, given any sequence (T_n) of sets, their sum[1]) can be expressed as a sum of ascending sets and their meet as a meet of descending sets as follows:

$$T_1 + T_2 + T_3 + \ldots = T_1 + (T_1 + T_2) + (T_1 + T_2 + T_3) + \ldots$$
$$T_1 \cdot T_2 \cdot T_3 \ldots = T_1 \cdot (T_1 \cdot T_2) \cdot (T_1 \cdot T_2 \cdot T_3) \ldots .$$

If U denotes a definite set ("universal set"), it is of interest to consider the subsets S of U and to associate with any given $S \subseteq U$ the complement $S' = U - S$[2]). Let S_k $(k = 1, 2, \ldots, n)$ be a finite number of subsets of U; then we have

$$U = (S_1 + S_2 + \ldots + S_n) + (S_1' \cdot S_2' \cdots S_n') =$$
$$= (S_1 \cdot S_2 \cdots S_n) + (S_1' + S_2' + \ldots + S_n').$$

For an element of U belongs either to *at least one* S_k (therefore to their sum) or to *no* S_k (therefore to the meet of their complements S_k'). Hence, the first equality holds and the second fol-

[1]) In this context, more usually spoken of as *join*. However, *sum* is used throughout this book, and also the signs + and · will be retained.

[2]) If S' is the complement of S, S evidently is the complement of S'. In symbols, $(S')' = S$.

lows from it when the S_k — being arbitrary subsets of U — are replaced by their complements S_k'. We may formulate our result (*de Morgan's laws*) as follows: the complement of a sum is the meet of the complements; the complement of a meet, the sum of the complements. Obviously, these propositions (known already to Ockham) also hold if the number of subsets is infinite.

From these results we can easily deduce a remarkable *law of duality*. Let A be a set formed from the sets S_k by repeated operations of addition and inner multiplication. Then, the complement A' may be formed by replacing the sets S_k by their complements S_k', addition by inner multiplication, and inner multiplication by addition. Since $S = T$ implies $S' = T'$, any *equality* resulting from our operations remains true if the replacements just mentioned are effected on both sides. We may even extend this result to *inequalities* containing the relation $\subset$; since $S \subset T$ implies $T' \subset S'$, we have now to replace the relation $\subset$ by its inverse $\supset$ ($x \supset y$ means: x includes y as a proper subset).

Example: From $S + S' = U$ it follows that $S' \cdot S = 0$.

If an *identity* between the subsets S_k of U is given, i.e. an equality which remains true for *any* values of the sets in question, it becomes superfluous to switch over from the given sets to their complements [1]). Then the dual identity is obtained by a mere exchange of addition and inner multiplication, or, if an inequality is concerned, by an additional transition from $\subset$ to $\supset$.

Examples: 1) From the distributive law [2])

$$S \cdot (T_1 + T_2 + \ldots + T_n) = S \cdot T_1 + S \cdot T_2 + \ldots + S \cdot T_n$$

we obtain the dual law, stated on p. 28 for $n = 2$:

$$S + (T_1 \cdot T_2 \cdots T_n) = (S + T_1) \cdot (S + T_2) \cdots (S + T_n).$$

---

[1]) If there appear constant sets, the transition to the complements must be made only for them. For instance, from $S + 0 = S$ it follows that $S \cdot U = S$; from $S \cdot 0 = 0$ we conclude $S + U = U$, and from $S \subseteq U$, $0 \subseteq S$.

[2]) The proof given on p. 28 for the case $n = 2$, is the same for any n, or even for any set of sets T_k. For more profound developments in this direction see Bennett 3, G. Bergmann 1 (cf. Menger 2, II and Radaković 1), Hirano 1, Klein-Barmen 1, Kuroda 1, Pankajam 1, Ward 1, Yoneyama 2; particularly Tarski 18, § 2; furthermore, for a question of independence, Wernick 2. For a more general (weaker) form of the ordinary distributive law (in fields and rings) cf. A. Robinsohn 2.

2) From the associative law of addition we obtain, as its dual, the associative law of inner multiplication, and conversely.

3) From $S \subseteq S + T$ we infer $S \cdot T \subseteq S$.

It is obvious that in these instances U may be chosen as any set which includes, as its subsets, all the sets under consideration.

4) The "characteristic functions" mentioned in § 7, 3 and exercise 4 of § 7.

It is noteworthy that all relations of the kind in question can be *formally* derived from a few among them, without any reference to the *meaning* of addition, inner multiplication and the relations of being complement or subset. For instance, from the commutative and associative laws of addition, together with the law

$$(S' + T')' + (S' + T)' = S.$$

In this case, we should *define* the inner product $S \cdot T$ by $(S' + T')'$ and the relation $S \subseteq T$ by the equality $S + T = T$. Abstract systems, the elements of which satisfy the three laws just mentioned, or laws equivalent to them, are called *Boolean algebras* [1]), after G. Boole who, together with A. de Morgan, created the logical calculus characterized below. They play an important part in mathematics [2]) and logic and during the last decades they have led to the development of important new branches of mathematics as the *theory of lattices* (cf. § 8, 2).

The significance of Boolean algebra in *logic* is based on the fact that logical disjunction and conjunction are the logical equivalents for our addition and inner multiplication [3]), as pointed

[1]) For the introduction and axiomatic foundation of Boolean algebras see, among others, the papers B. A. Bernstein **15, 17, 19**; Birkhoff–MacLane **1**; Birkhoff–Birkhoff **1**; Hoberman–McKinsey **1**; Huntington **2, 7, 8**; Whiteman **1**; especially Stone **2** and **3**, Tarski **18** and **31**. For an important extension see Mostowski–Tarski **1**; for the connection with topology, abstract algebra and even with the theory of probability and insurance, among others see Stone **1** and **7**, von Neumann–Stone **1**, Tarski **30**, Berkeley **1** (cf. Fairthorne **1**), Broderick–Schrödinger **1**. Cf. also Kuratowski–Posament **1** and the references at the end of § 8, 2.

[2]) Even in applied mathematics. In the theory of probability there are situations which are clarified by the use of Boolean algebra; cf. especially Koopman **1**. Shannon **1** gives an application to electricity in terms of relay and switching circuits, and more recent applications have developed.

[3]) Set operations may also be related to the "transfinite" generalizations of logical disjunction and conjunction, i.e. to "some" and "all"; see Kuratowski–Tarski **1**.

out on p. 25. As to the concepts introduced in the present subsection, U corresponds to a property holding for *every* object in question. Any property which is meaningful for the objects of U in the sense adopted in the principle of subsets (see p. 22), will determine a certain subset S of U, and the contrary [1]) property, the complement S'. Thus the complement corresponds to logical negation. Finally, the relation $S \subseteq T$ corresponds to implication, i.e. to the proposition "all s (objects in S, or having a property characteristic of S) are t"; in other words, if a property is denoted by the corresponding set, "S implies T" or "if S, then T".

By the four operations mentioned (conjunction, disjunction, negation, implication) a broad field of logic is covered. It is known as *algebra of logic* and may be formalized by our symbols. A few instances are given here.

There are no s	corresponds to	$S = 0$
Every object is an s	,,	,, $S = U$
Some s are t	,,	,, $S \cdot T \neq 0$
Some s are not t	,,	,, $S \cdot T' \neq 0$
Neither S nor T	,,	,, $S' \cdot T'$ or $(S + T)'$
the Law of Contradiction [2])	,,	,, $S \cdot S' = 0$
the Law of Excluded Middle [3])	,,	,, $S + S' = U$.

Exercises

1) Show that the sum of denumerably many finite cardinals (different from 0) always equals $\aleph_0$.

2) What are the values of $\mathfrak{f} + \aleph$ and $\mathfrak{f} + \mathfrak{f}$, $\mathfrak{f}$ denoting the cardinal of the set of all functions (p. 86)?

3) To any element t of a non-empty set T two cardinals $\mathfrak{a}_t$ and $\mathfrak{b}_t$ may be assigned such that $\mathfrak{a}_t \leqq \mathfrak{b}_t$. Prove that $\sum_{t \in T} \mathfrak{a}_t \leqq \sum_{t \in T} \mathfrak{b}_t$ and $\prod_{t \in T} \mathfrak{a}_t \leqq \prod_{t \in T} \mathfrak{b}_t$.

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[1]) Here and elsewhere *contrary* is taken to mean the simple negation. The contrary of *black* is *non-black* and not *white*.

[2]) It states that an object cannot have a certain property and the contrary property.

[3]) This law, to be thoroughly discussed in *Foundations*, ch. IV, may be roughly expressed in the following form: an object either has a certain property or the contrary property.

4) Prove $\aleph_0 \cdot \aleph_0 = \aleph_0$ by means of a decomposition of the set of natural numbers into denumerably many denumerable subsets. (There are infinitely many such decompositions.)

5) Prove that the set of all *finite* sets of natural numbers is denumerable, in contrast to the set of *all* sets of natural numbers.

6) Show that the universal library described on p. 7/8 is infinite and denumerable if the restriction stipulated there for the extent of a book is dropped (while the extent still remains finite). The same even holds when $\aleph_0$ different types, instead of 1000, are admitted.

7) Replace the set T, occurring in definitions III and VI (p. 114 and p. 124), by an equivalent set U ($U \sim T$), and consider a definite representation between T and U which relates $\psi(t) = u(\varepsilon\ U)$ to $t\ \varepsilon\ T$. Assign to $\psi(t)\ \varepsilon\ U$ the cardinal $f(t)$ which has been assigned to $t\ \varepsilon\ T$ in the definitions mentioned. (One may express the effect of this procedure by saying that any term or factor appears on account of U "as often" as it appears on account of T.) Prove that the sum, as well as the product, of the given cardinals remains unaltered by the transition from T to U, and explain why this statement may be considered as an exact expression of the commutative laws for the addition and multiplication of cardinals. (Cf. p. 115.)

8) Prove by a method analogous to that applied in the proof of theorem 9, that the set of all points of a cube — or of three-dimensional space — has the cardinal $\aleph$.

§ 7. Exponentiation of Cardinals. The Problem of Infinitesimals

1. Exponentiation as Repeated Multiplication. We shall first introduce exponentiation in the same way as is usually done in arithmetic; that is to say, as a repeated multiplication of a factor by itself. Another method of defining exponentiation, which leads to the same result, was used by Cantor originally; this method will be introduced in 2.

In order to form the square $\mathfrak{c}^2 = \mathfrak{c} \cdot \mathfrak{c}$ of any cardinal $\mathfrak{c}$ according to the definition of multiplication (VI, p. 124), we have to start from a *pair* (i.e. a set of the cardinal 2) of sets S_1 and S_2, both of which have the cardinal $\mathfrak{c}$; then the outer product $S_1 \times S_2$, i.e. the

set of all pairs $\{s_1, s_2\}$ where $s_1 \varepsilon S_1$ and $s_2 \varepsilon S_2$, has the cardinal $\mathbf{c}^2$.

A natural extension of this operation, to cover the general case when we deal not with a pair but with a set (of sets) having an arbitrary finite or infinite cardinal **d**, leads to the following

DEFINITION OF EXPONENTIATION. Let **c** and **d** be cardinals ($\mathbf{d} \neq 0$) [1]. In order to form the power $\mathbf{c}^\mathbf{d}$, take a set D of the cardinal **d**, each of whose elements is a set of the cardinal **c**. The cardinal of the outer product $\mathbf{P}D$ is called *the* **d***th power of* **c**, and is denoted by $\mathbf{c}^\mathbf{d}$. As in arithmetic, **c** is called the *basis*, **d** the *exponent* of the power.

In short one may say: the power $\mathbf{c}^\mathbf{d}$ is a product in which the factor **c** appears "as often" as indicated by the exponent **d**.

There is no need to prove that the power is independent of the choice of the set D, i.e. independent of the way in which the various elements, each of which has the cardinal **c**, have been chosen, for this independence immediately follows from theorem 3 of § 6 (cf. the remark on p. 125).

2. Definition of Exponentiation by Means of the Insertion-Set. We now proceed to the idea on which the concept of power was based by Cantor. The new definition reveals the close connection between the exponentiation of cardinals and other concepts, notably those of function and of power-set (see p. 94/97).

We start from the concept of *one-valued function* (see p. 32). Given two arbitrary sets S and T, consider the number of functions which may be obtained in the following way: the argument (independent variable) t of the functions runs over the set T while the values s of the dependent variable are elements of S. Thus the functions may be written in the form $s = f(t)$ where f is the symbol of an individual function.

Examples. 1. Four dice, simultaneously placed into a box, are respectively marked with 1, 2, 3, 4. We express the result of a single casting by means of a function $s = f(t)$ where s is the number of pips appearing on the die t ($t = 1, 2, 3, 4$); s can assume the values 1, 2, 3, 4, 5, 6. In the previous notation we have

[1] The case $\mathbf{d} = 0$ has but little importance. As in arithmetic, we may define $\mathbf{c}^0 = 1$ for any $\mathbf{c} \neq 0$, although the arithmetical reasons for this definition (division and continuity) are absent in the present case. Cf. theorem 1 on p. 151. The case $\mathbf{c} = \mathbf{d} = 0$ shall remain excluded altogether.

$T = \{1, 2, 3, 4\}$, $S = \{1, 2, 3, 4, 5, 6\}$. All possible results of casting the dice are obtained by assigning to the values $t = 1, 2, 3, 4$ values of S in any possible way, and two such results are considered as different whenever there is a die t that shows different numbers s in the two cases. Since each die, independently of the others, can show one of six numbers, the number of all possible results is $6^4 = 1296$. Each of these results determines a certain function $s = f(t)$, and all these functions are different from each other.

2. A similar example is obtained when one considers T to be a set of musical compositions, each for a single performer, and S to be a set of performers each of whom is able to sing, or to play, any piece of T. Accordingly, a function $s = f(t)$ — where t runs over T and where, for any given t, $f(t)$ denotes the musician s chosen to perform the piece t — indicates a definite *program* made up from the elements of the set T (a certain cast for the pieces of T, or a certain insertion of musicians of S into the set T). Hence the set of *all* functions $s = f(t)$ gives all possible insertions in view of our conditions. Note that a given function $f(t)$ uniquely determines, for any given piece of T, the musician by whom it is performed, but that a given artist may perform several (even all) of the pieces, or one, or none of them. Our functions, while being one-valued, do not in general constitute biunique (one-to-one) correspondences — which is *a priori* evident since the sets S and T were not assumed to be equivalent. Each function is a many-to-one relation (mapping) between S and T, such that to every element of T some element of S is uniquely related, repetitions and omissions in S being allowed.

The examples 1 and 2 show that in the case of *finite* sets S and T with cardinals s and t, the set of all functions or insertions coincides with the set of all possibilities of placing s objects into t holes (the set of tth class variations of s objects).

3. An important mathematical example is offered by decimal fractions, e.g. those between 0 and 10. As the set T of arguments, we use the aggregate of places in such a decimal, i.e. $\{0, 1, 2, 3, \ldots, n, \ldots\}$ where 0 denotes the place preceding the decimal point, and the natural number n, the nth place after the point. The function $f(t) = s$ indicates what digit occupies the place t in the given decimal. Hence, as s is always a digit, S is the set of all digits $\{0, 1, 2, \ldots, 9\}$. A given decimal between 0 and 10, e.g.

$\pi = 3.14159\ldots$, is determined by a function of the kind described; in the case of π the function $f(t)$ is defined by

$$f(0) = 3,\ f(1) = 1,\ f(2) = 4,\ f(3) = 1,\ f(4) = 5,\ f(5) = 9,\ \ldots.$$

Hence the totality of all functions $f(t) = s$ in question yields the set of *all decimal fractions* between 0 and 10, including $0.000\ldots$ and $9.999\ldots$; in other words, the set of all possible insertions of digits into a sequence of places. In this case, as in contrast to the procedure of p. 62, two decimal fractions have to be considered as different whenever they are *formally different*, even if they represent the same real number; otherwise we would not obtain a one-to-one correspondence between the functions and the decimals. Different functions correspond, for instance, to $0.999\ldots$ and $1.000\ldots$. Of course the number of functions in this case is infinite, and if we were to proceed along the lines of the first example, we should express their number by the infinite product $10 \cdot 10 \cdot 10 \cdot \ldots$, since every decimal place can be filled, independently of any other, by ten different values s.

With respect to a *given* function or decimal, one may speak of a certain insertion of the set $S = \{0, 1, \ldots, 9\}$ of possible values into the infinite set $T = \{0, 1, 2, \ldots\}$ of arguments. The set of *all* functions, therefore, means the set of all possible insertions of S into T. On the other hand, one may write a function or insertion as a *complex* in the sense of definition V on p. 120; in our case, a complex would have the form of a sequence $(s_0, s_1, s_2, \ldots)$ where for any n (n being a natural number or 0) s_n is a certain digit. In the same way the functions occurring in the first example may be considered as insertions of the set $S = \{1, 2, 3, 4, 5, 6\}$ (possible number of pips) into the set $T = \{1, 2, 3, 4\}$ of dice, or as complexes (s_1, s_2, s_3, s_4) where the s_k are elements of S. In these examples, we may consider the complexes as *ordered* sets, so that we can distinguish between the cases where the same number (digit, number of pips) is inserted into different places of argument.

The examples just considered may facilitate the understanding of the *general notion of insertion*. As at the beginning of 2, S and T shall be arbitrary sets with the only restriction that T is not to be the null-set [1]). Then the set, the elements of which are all possible

[1]) On the other hand, if S is the null-set, nothing can be inserted into the elements of T, which implies the non-existence of any insertion. Hence, in this case the insertion-set is the null-set.

insertions of the set S into the set T — in other words, the set the elements of which are all possible functions $s = f(t)$ where t runs over T and the values s are restricted to the elements of S — shall be called *the insertion-set of S into T*, and will be denoted by $(S|T)$. (Some authors denote it by S^T.) Two insertions are considered as *different* if there is at least one element of T to which different elements of S have been attached (inserted). Any element of the insertion-set may be written as a set of ordered pairs $\{\ldots (s, t) \ldots\}$ which is equivalent to T, each element t of T appearing in one, and in only one, of the ordered pairs. Using the concept of complex as introduced in definition V of § 6, p. 120, we may say that the elements of the insertion-set are complexes, whose *places* correspond to the elements of T, while the elements inserted into those places belong to S. The insertion-set comprises all complexes formed in this manner. Accordingly, a certain insertion relates to each element of T a single element of S (which, however, may be also related to other elements of T in the same insertion). In the first example given above, there are four places in each complex; in the third example, the places form a sequence.

Hence, according to definition V of § 6, the insertion-set $(S|T)$ is equivalent to the outer product of a set T^* equivalent to T, each element of which is a set equivalent to S [1]). Denoting these sets by S', S'', ..., we may write the outer product in question in the form $S' \times S'' \times \ldots$. Therefore, if $\mathfrak{s}$ is the cardinal of S, and $\mathfrak{t}$ the cardinal of T, the insertion-set $(S|T)$ has the cardinal $\mathfrak{s}^{\mathfrak{t}}$ in view of **1**.

Accordingly, we may replace the definition of **1** by the following alternative: *The power* $\mathfrak{s}^{\mathfrak{t}}$ *is the cardinal of the insertion-set* $(S|T)$ *where S is any set of the cardinal* $\mathfrak{s}$, *and T any set of the cardinal* $\mathfrak{t}$ $(\mathfrak{t} \neq 0)$. [2]) For $\mathfrak{t} = 0$, we define $\mathfrak{s}^0 = 1$ for any $\mathfrak{s} \neq 0$ (cf. the footnote on p. 147).

The advantage of this definition lies in its easy and convenient applicability.

3. **The Power-Set.** The most important application of the concept of insertion is the proof of a theorem which establishes a

[1]) This becomes more obvious if to every element of T *the same set S* is related, which is useful for most applications.

[2]) Cf. exercise 1 at the end of this section.

connection between exponentiation and the set of all subsets of a given set. Given a non-empty set T, we may consider an arbitrary subset T_0 of T as formed by admitting certain elements of T to the subset while excluding the other elements. Let us attach the symbol 1 (or "yes") to the elements admitted to the subset, the symbol 0 (or "no") to the others. *In this way we may take any subset T_0 of T as a certain insertion of a set of two elements* — say, of $\{1, 0\}$ or of {yes, no} — *into the set T*; i.e. as a function $f(t)$ which runs over T and has two function-values only. On the other hand, if such a function is given, it uniquely determines a subset of T, viz. the one which contains all $t \varepsilon T$ for which $f(t) = 1$, and no $t \varepsilon T$ for which $f(t) = 0$. The "constant" function $f(t) = 1$ determines T itself; the constant function $f(t) = 0$ determines the null-set.

Accordingly, the set $\mathbf{U}T$ of *all* subsets of T (p. 94/7) is related to the insertion-set of $\{1, 0\}$ into T, i.e. to $(\{1, 0\}|T)$, by a one-to-one correspondence of their elements. Hence, the cardinal of $\mathbf{U}T$ is $2^{\mathfrak{t}}$, if $\overline{\overline{T}} = \mathfrak{t}$. Thus we obtain the following theorem which explains why the set of all the subsets of T has been called the "power-set" of T [1]):

THEOREM 1. If T is any set with the cardinal $\mathfrak{t} \neq 0$, the set $\mathbf{U}T$, whose elements are all subsets of T, has the cardinal $2^{\mathfrak{t}}$. $\mathbf{U}T$ may be considered as the insertion-set of a *pair* into the set T.

Therefore, by theorem 3 on p. 94, *the cardinal $2^{\mathfrak{t}}$ is always larger than* $\mathfrak{t}$: $2^{\mathfrak{t}} > \mathfrak{t}$.

Of course theorem 1 holds true also with respect to *finite* sets; cf. 96. In particular, for $\mathfrak{t} = 0$ [2]) (i.e. if T is the null-set 0) we obtain $2^0 > 0$; in fact $\mathbf{U}0 = \{0\}$ contains one element, in accordance with the relation $2^0 = 1$.

Theorem 1 may be considered as a particular case of the inequality of König (p. 132). For, if the functions $f(t)$ and $g(t)$ mentioned there in theorem 8 are the constant functions $f(t) = 1$ and $g(t) = 2$ (for any set T), the conditions of the theorem are fulfilled in view of $1 < 2$. The sum of all $f(t)$ is $\mathfrak{t}$ (cf. p. 128), the product of all $g(t)$ is $2^{\mathfrak{t}}$; hence theorem 8 asserts $\mathfrak{t} < 2^{\mathfrak{t}}$.

For finite sets T of more than one element ($\mathfrak{t} > 1$), $2^{\mathfrak{t}}$, while

[1]) The advanced student may compare the remark in Tarski **18**, p. 198, dealing with an interesting property of the power-set.

[2]) It is not worth while to introduce different symbols for the cardinal 0 and the null-set 0.

larger than $\mathfrak{t}$, is not the number next larger than $\mathfrak{t}$. The parallel question for transfinite cardinals $\mathfrak{t}$ has not yet been answered. The assumption that, *for any transfinite* $\mathfrak{t}$, $2^{\mathfrak{t}}$ *is the cardinal next larger than* $\mathfrak{t}$, is called the *generalized continuum hypothesis*[1]). In the particular case $\mathfrak{t} = \aleph_0$, this assumption means that $\aleph$ is the cardinal next to $\aleph_0$ (Cantor's continuum hypothesis), as will be seen in **5**.

The method of proceeding from a given cardinal $\mathfrak{c}$ (e.g., $\mathfrak{c} = 0$ or $\mathfrak{c} = \aleph_0$) to ever larger cardinals, as described on p. 131, may now be simplified. Denoting $\mathfrak{c}$ by $\mathfrak{c}_0$ and $2^{\mathfrak{c}_k}$ by $\mathfrak{c}_{k+1}$, we first reach $\mathfrak{c}_n$ for any natural number n and then $\sum_n \mathfrak{c}_n$ where n runs over all natural numbers. This sum is larger than any $\mathfrak{c}_n$ and may again be taken as the starting cardinal $\mathfrak{c}$. This process which already leads to cardinals of enormous magnitude, may still be extended by methods to be introduced in § 11.

The functions defined by the insertion of a set of two elements into a given set T, play an important part in the applications of the theory, too. They are often called the *characteristic functions*[2]) corresponding to the subsets of T. To any subset of T thus belongs a uniquely determined characteristic function whose argument runs over T, and vice versa. See exercise 4 at the end of this section.

4. Formal Laws of Exponentiation. In ordinary arithmetic we derive the formal laws of exponentiation:

$$m^n \cdot m^p = m^{n+p}, \quad m^n \cdot p^n = (m \cdot p)^n, \quad (m^n)^p = m^{n \cdot p}.$$

The same rules hold true for any cardinals, even for any finite or infinite number of factors (in the first two laws). We shall prove:

THEOREM 2. If to every element t of a non-empty set T a cardinal $\mathfrak{k}_t$ is attached, and if $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ are given cardinals, the following formal laws hold:

$$\prod_{t \in T} \mathfrak{a}^{\mathfrak{k}_t} = \mathfrak{a}^{\sum_{t \in T} \mathfrak{k}_t} \tag{1}$$

$$\prod_{t \in T} \mathfrak{k}_t^{\mathfrak{b}} = \left(\prod_{t \in T} \mathfrak{k}_t\right)^{\mathfrak{b}} \tag{2}$$

$$(\mathfrak{a}^{\mathfrak{b}})^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b} \cdot \mathfrak{c}}. \tag{3}$$

[1]) Cf. the remark at the end of Cantor **7**, V, note 10. The subject will be treated in detail in § 11, **7**.

[2]) Cf. de la Vallée Poussin **1**; also **2**, p. 7.

Hence in particular:

(1′) $\mathbf{a}^{\mathbf{k}_1} \cdot \mathbf{a}^{\mathbf{k}_2} = \mathbf{a}^{\mathbf{k}_1+\mathbf{k}_2}$, (2′) $\mathbf{k}_1^{\mathbf{b}} \cdot \mathbf{k}_2^{\mathbf{b}} = (\mathbf{k}_1 \cdot \mathbf{k}_2)^{\mathbf{b}}$.

Proof. For (1), we use the equivalence (1′) on p. 126. If the elements y of *all* the sets K_t appearing in (1′) are sets of the same cardinal $\mathbf{a}$, and if the cardinal of K_t is $\mathbf{k}_t$, the set $\prod_{y \in K_t} y$ (i.e. the outer product of all elements of K_t) has the cardinal $\mathbf{a}^{\mathbf{k}_t}$, according to the definition of power (p. 147). On the other hand, the right hand side of the equivalence (1′), namely the outer product $\prod_{y \in S} y$, where $S = \sum_{t \in T} K_t$, has the cardinal $\mathbf{a}^{\sum_{t \in T} \mathbf{k}_t}$. Since the cardinals of equivalent sets are equal, (1) has been proved.

The equality (2) is actually used only in the particular form (2′). This form, as well as the general law, is easily obtained by a joint use of the associative and commutative laws of multiplication (p. 125 f.). Roughly speaking, one has to attach to each factor $\mathbf{k}_1$ in (2′) a corresponding factor $\mathbf{k}_2$; the correspondence is made possible by the equality of exponents in the powers which appear on the left hand side of (2′). In the general case (2) the procedure is the same; the formal proof may be left to the reader. Another way of proving (2) is by using the definition of power based upon the insertion-set.

Finally, the equality (3) follows from (1) in the particular case where to any $t \in T$ the *same* cardinal $\mathbf{k}_t = \mathbf{b}$ is attached and the cardinal of the set T is $\mathbf{c}$. Then it follows from theorem 5 on p. 128 that $\sum_{t \in T} \mathbf{k}_t = \mathbf{b} \cdot \mathbf{c}$.

As to *inequalities*, the result that $\mathbf{a}_t \leqq \mathbf{b}_t$ implies $\prod_{t \in T} \mathbf{a}_t \leqq \prod_{t \in T} \mathbf{b}_t$ (see (1) on p. 130, and exercise 3 on p. 145), immediately yields (if all $\mathbf{a}_t$ as well as all $\mathbf{b}_t$ are equal):

(4) $\mathbf{a} \leqq \mathbf{b}$ implies $\mathbf{a}^{\mathbf{c}} \leqq \mathbf{b}^{\mathbf{c}}$.

An analogous inequality follows from a corresponding assumption with regard to the exponents, instead of the bases. For let C and D be sets with the cardinals $\mathbf{c}$ and $\mathbf{d}$ where $\mathbf{c} \leqq \mathbf{d}$. Then there exists a subset D_0 of D which is equivalent to C, i.e. $C \sim D_0 \subseteq D$. Denote $D - D_0$ by D_1, and the cardinal of D_1 by $\mathbf{d}_1$; by (4) we obtain, for any cardinal $\mathbf{a} \neq 0$:

$$\mathbf{1}^{\mathbf{d}_1} \leqq \mathbf{a}^{\mathbf{d}_1}.$$

This inequality remains true if one multiplies both sides by $\mathbf{a}^{\mathbf{c}}$. In view of $D_0 = C$ (accordingly $\mathbf{c} + \mathbf{d}_1 = \mathbf{d}$), and of $1^{\mathbf{d}_1} = 1$, we thus obtain by means of (1′):

$$\text{(5)} \qquad \mathbf{c} \leqq \mathbf{d} \text{ implies } \mathbf{a}^{\mathbf{c}} \leqq \mathbf{a}^{\mathbf{d}}.$$

The restriction $\mathbf{a} \neq 0$ may be dropped since the relation (5) evidently holds for $\mathbf{a} = 0$, too, provided (see p. 147) $\mathbf{c}$ is not also 0.

In contrast to these "inequalities" which *include equality*, one cannot obtain strict inequalities from analogous assumptions — a fact that once again underlines the importance of theorem 8 on p. 132. The strongest assumption would be

$$\mathbf{a} < \mathbf{b} \textit{ and } \mathbf{c} < \mathbf{d};$$

one might expect these inequalities to imply $\mathbf{a}^{\mathbf{c}} < \mathbf{b}^{\mathbf{d}}$. This expectation, however, proves to be false [1]).

In the case of equal bases *or* equal exponents, the analogous expectation is in general obviously false; as we shall see presently, we have, for instance:

$$\aleph^1 = \aleph^{\aleph_0} \text{ and } 2^{\aleph_0} = \aleph^{\aleph_0},$$

in spite of $1 < \aleph_0$, $2 < \aleph$. Furthermore, even for the particular basis 2, one has not succeeded in finding out whether $\mathbf{c} < \mathbf{d}$ implies $2^{\mathbf{c}} < 2^{\mathbf{d}}$ — not even by means of the multiplicative principle [2]). The converse proposition, however, easily follows from the theorem asserting the comparability of cardinals (§ 5, **5** and § 11, 7) — hence from the well-ordering theorem or the multiplicative principle (§ 11, 6). For assume that $2^{\mathbf{c}} < 2^{\mathbf{d}}$. Then $\mathbf{d} \leqq \mathbf{c}$ cannot hold true, since by (5) this would imply $2^{\mathbf{d}} \leqq 2^{\mathbf{c}}$, contrary to our assumption. Hence, by the comparability of the cardinals $\mathbf{c}$ and $\mathbf{d}$, $\mathbf{c} < \mathbf{d}$ must be true.

5. The Power-Set of a Denumerable Set. For reasons of simplicity we begin by calculating the power $10^{\aleph_0}$. Using the definition of power by the insertion-set, we must insert a set of 10 elements

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[1]) Cf. Tarski **6**, p. 10.

[2]) However, we immediately obtain this implication by using the *generalized continuum hypothesis* (p. 152 and § 11, 7). Cf., e.g., Sierpiński **18**, p. 167.

into a denumerable set; e.g., the set $M = \{0, 1, 2, \ldots, 9\}$ into the set N of all natural numbers $\{1, 2, 3, \ldots\}$. Analogically to p. 148f., we shall consider any insertion $(a_1, a_2, \ldots, a_k, \ldots)$ — where a_k is an element of M, and k denotes the element of N (the "place") into which a_k has been inserted — as the decimal fraction $0.a_1\, a_2 \ldots a_k \ldots$. This procedure creates a one-to-one correspondence between the elements of our insertion-set $(M|N)$, which has the cardinal $10^{\aleph_0}$, and the decimals beginning with 0., provided that two formally differing decimals are regarded as different. Let the set of decimals in this sense be denoted by D.

Now the set D, according to what has been explained in § 4, **1**, may be considered as a sum-set $D_1 + D_2$, where D_1 is the set (continuum) of all *infinite* decimals beginning with 0., and D_2, the set of all *terminating* decimals of the same kind. According to § 4, D_1 has the cardinal $\mathfrak{N}$ (the cardinal of the continuum), while D_2 as an infinite subset of the set of all rational numbers, is denumerable. Hence the insertion-set $(M|N)$ has the cardinal $\mathfrak{N} + \mathfrak{N}_0 = \mathfrak{N}$; i.e.

$$10^{\aleph_0} = \mathfrak{N}. \tag{1}$$

As pointed out on p. 66, the use of the special base 10 (decimal system) has no mathematical foundation whatsoever but rests on the biological fact that man has ten fingers. From the mathematical point of view, any other integer larger than 1 would do as well, in accordance with the fact that biology cannot influence the validity of mathematical theorems [1]). Therefore, we may replace the base 10 in (1) by any natural number different from 1; for 1 is the only natural number which equals its own powers. Thus we obtain as a generalization of (1):

$$n^{\aleph_0} = \mathfrak{N}; \quad (n \text{ finite and } > 1) \tag{2}$$

in particular, we have:

$$2^{\aleph_0} = \mathfrak{N}. \tag{3}$$

(3) results from the representation of real numbers by *dual fractions* in the same way as (1) results from their representation as decimals. The significance of (3), besides its giving the smallest possible

[1]) Even in the eyes of those who maintain that biological facts have a bearing on the *concepts* and *principles* of mathematics; cf. *Foundations*, ch. IV.

value of n in (2), lies in the fact that $2^{\aleph_0}$ is the cardinal of the power-set of a denumerable set, according to theorem 1.

We shall now give an explicit proof of the equivalence of (1) with (3), without referring to the possibility of applying different scales in arithmetic. In the same way me might also show the equivalence of (2) with (1) or with (3). The proof presented here [1]) makes essential use of the equivalence theorem (see p. 99).

First, the set F_2 of all dual fractions beginning with 0. is certainly equivalent to a subset of the set F_{10} of all such decimals, since a dual fraction may be read and conceived as a decimal in which only the digits 0 and 1 are used. Secondly, any decimal fraction $0.a_1\,a_2 \ldots a_k \ldots$ may be related to a sequence which contains only the members 0 and 1, if, for any k, the digit a_k is replaced by a finite sequence consisting of a_k zeros followed by one 1. So, if $a_k = 0$, a_k will be replaced by 1. The infinite sequence made up from these finite sequences in the order $k = 1, 2, 3, \ldots$ may again be written as a fraction containing only the digits 0 and 1, i.e. as a dual fraction. There are dual fractions, however, which cannot be obtained in this way, namely the fractions in which more than nine successive zeros occur, but every dual fraction which does appear is obtained from a single decimal $0.a_1\,a_2\,a_3\,a_4 \ldots a_k \ldots$. Hence, the set F_{10} is equivalent to a subset of F_2. Using that F_2 is equivalent to a subset of F_{10}, we obtain $F_2 \sim F_{10}$, i.e. $2^{\aleph_0} = 10^{\aleph_0}$ $(= \aleph)$, Q.E.D. Hence:

THEOREM 3. The power-set of a denumerable set has the cardinal $\aleph$ of the continuum. More generally, for any finite cardinal $n > 1$ we have

$$n^{\aleph_0} = \aleph.$$

6. Other Examples of Exponentiation. Let us begin with powers of $\aleph$. On p. 138 we obtained $\aleph^2 = \aleph \cdot \aleph = \aleph$; hence $\aleph^3 = \aleph^2 \cdot \aleph = \aleph$, and, by mathematical induction,

$$\aleph^n = \aleph. \quad (n \text{ any finite cardinal} \neq 0) \tag{1}$$

In view of theorems 3 and 2, this may be proved more simply in the form

$$\aleph^n = (2^{\aleph_0})^n = 2^{\aleph_0 \cdot n} = 2^{\aleph_0} = \aleph.$$

[1]) Due to J. König **5**, p. 219.

In the same way, using the relation $\aleph_0 \cdot \aleph_0 = \aleph_0$ (p. 135), we obtain:

$$\aleph^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \aleph. \tag{2}$$

Obviously this is a sort of logarithmic calculation, a device already applied in calculating $\aleph \cdot \aleph$ (p. 136). The multiplication and the exponentiation of powers with the basis 2 is reduced to addition and multiplication of their (transfinite) exponents. When Cantor in 1895 [1]) proved (1) and (2) in this short and almost mechanical way he was able to compare, with justified pride, the ease of these proofs with the great effort displayed less than twenty years before [2]) when he had proved the relation (1) and met with the incredulity of his contemporaries (p. 139). The introduction of operations with cardinals and the use of formal laws had brought about such a revolution. This is an instance of a pattern of development frequently to be observed in mathematics: a decisive progress has often been achieved by the invention of a "mechanism". In this respect the calculation with transfinite cardinals (and ordinals; §§ 10 and 11) may be compared, to some extent, with the mechanism of the calculus, the most important instance of such a development.

By using coordinates in the plane (p. 136) we found that the set of all points of the plane, or of a square, has the cardinal $\aleph^2$. In the same way, the use of coordinates (x, y, z) in three-dimensional space shows that the set of all points of the space, or of a cube with an arbitrary side, has the cardinal $\aleph^3$. In abstract geometry one also considers spaces of n dimensions, n being any natural number, and even of $\aleph_0$ dimensions; in the latter case, a point of the space is characterized by a *sequence* of coordinates $(x_1, x_2, \ldots, x_k, \ldots)$. Therefore we have, in view of (1) and (2):

THEOREM 4. The set of all the points contained in three-dimensional space, or in a cube of this space, has the cardinal of the one-dimensional continuum. The same applies to spaces of more than three dimensions, even to spaces of $\aleph_0$ dimensions. Thus, for instance, all the points of space may be related by a one-to-one correspondence to the points of an arbitrarily small segment.

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[1]) Cantor **12**, I, p. 488. Some analogous rules of the multiplication table of transfinite cardinals are found in Hölder **6**.

[2]) Cantor **6**.

As to the powers of $\aleph_0$, from $\aleph_0^2 = \aleph_0$ we obtain by mathematical induction:

$$\aleph_0^n = \aleph_0. \qquad (n \text{ finite}, \neq 0)$$

In order to evaluate $\aleph_0^{\aleph_0}$, a formal computation using theorems 2 (p. 152) and 3 (p. 156) may be carried out:

$$\begin{aligned}\aleph_0^{\aleph_0} &= (n\cdot\aleph_0)^{\aleph_0} \quad [\text{by (1) on p. 135}] = n^{\aleph_0}\cdot\aleph_0^{\aleph_0}\\ &= n^{\aleph_0\cdot\aleph_0}\cdot\aleph_0^{\aleph_0} \quad [\text{by (2) on p. 135}] = (n^{\aleph_0})^{\aleph_0}\cdot\aleph_0^{\aleph_0}\\ &= (n^{\aleph_0}\cdot\aleph_0)^{\aleph_0} = (\aleph\cdot\aleph_0)^{\aleph_0} \qquad = \aleph^{\aleph_0} \text{ [by (4) on p. 135]}\\ &= \aleph \text{ [by (2) on p. 157]. Hence:}\end{aligned}$$

$$\aleph_0^{\aleph_0} = \aleph \text{ [1]}. \tag{3}$$

Conversely, from (3) and the relation $2^{\aleph_0} = \aleph$ we obtain the assertion of theorem 3

$$n^{\aleph_0} = \aleph \quad (n \text{ any finite cardinal} > 1)$$

by (4) on p. 153, in view of $2 \leqq n < \aleph_0$. (Cf. the second inequality of exercise 3 on p. 145.)

In the same way we see that the product of all natural numbers $1\cdot 2\cdot 3 \cdots k \cdots$ equals $\aleph$. In fact, except for the factor 1, every factor k fulfils the double inequality $2 \leqq k < \aleph_0$ and there are $\aleph_0$ factors. Hence, our assertion follows from $2^{\aleph_0} = \aleph_0^{\aleph_0} = \aleph$.

In the preceding calculations the bases $\aleph_0$ and $\aleph$ have shown a different behavior with respect to the exponent $\aleph_0$, as we have found, $\aleph_0^{\aleph_0} > \aleph_0$, $\aleph^{\aleph_0} = \aleph$. We might believe this to be due to the basis $\aleph$ being "too large" and therefore not capable of becoming enlarged by the exponentiation with $\aleph_0$. This, however, is not true; there are *two kinds of cardinals* $\mathfrak{c}$: those for which $\mathfrak{c}^{\aleph_0} > \mathfrak{c}$ and those for which $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$, and *both kinds comprise cardinals of any magnitude* [2].

[1] The reasoning on p. 48 has shown that $\aleph_0\cdot\aleph_0$ is the cardinal of the set of points in the plane both coordinates of which are *integers*; i.e. the set of lattice-points. The equality $\aleph_0^{\aleph_0} = \aleph^{\aleph_0}\ (= \aleph)$ shows that in a space of $\aleph_0$ dimensions it makes no difference to the resulting cardinal whether *all* points of the space are considered, or the lattice-points only.

[2] It is remarkable and characteristic of the importance of the theory of sets for abstract algebra, that the difference stated here is decisive for a problem in the theory of abstract fields — a problem which has a solution in the second case only; see F. K. Schmidt 1.

For example, if $(\mathfrak{c}_k)$ is a sequence ($k = 1, 2, 3, \ldots$) of increasing cardinals ($\mathfrak{c}_k < \mathfrak{c}_{k+1}$; $\mathfrak{c}_1 \neq 0$), $\sum_k \mathfrak{c}_k = \mathfrak{c}$ is of the first kind (and of an arbitrary magnitude, since $\mathfrak{c}_1$ may be chosen arbitrarily). To prove this we use the inequality of König, which yields:

$$\mathfrak{c} < \mathfrak{c}_2\,\mathfrak{c}_3\,\mathfrak{c}_4 \ldots \leqq \mathfrak{c}_1\,\mathfrak{c}_2\,\mathfrak{c}_3\,\mathfrak{c}_4 \ldots .$$

Denoting the latter product by $\mathfrak{d}$, we have in view of $\mathfrak{c}_k < \mathfrak{c}$:

$$\mathfrak{d} \leqq \mathfrak{c} \cdot \mathfrak{c} \cdot \mathfrak{c} \cdot \mathfrak{c} \cdot \ldots = \mathfrak{c}^{\aleph_0}.$$

The inequalities $\mathfrak{c} < \mathfrak{d}$, $\mathfrak{d} \leqq \mathfrak{c}^{\aleph_0}$ show that $\mathfrak{c} < \mathfrak{c}^{\aleph_0}$.

On the other hand, given an arbitrary $\mathfrak{b}$, all cardinals of the form $\mathfrak{c} = \mathfrak{b}^{\aleph_0}$ have the property $\mathfrak{c} = \mathfrak{c}^{\aleph_0}$, since

$$\mathfrak{c}^{\aleph_0} = (\mathfrak{b}^{\aleph_0})^{\aleph_0} = \mathfrak{b}^{\aleph_0 \cdot \aleph_0} = \mathfrak{b}^{\aleph_0} = \mathfrak{c}.$$

Here again there is no restriction on the magnitude of $\mathfrak{b}$ and, therefore, of $\mathfrak{c}$. Q.E.D.

We have not yet considered the *exponent* $\aleph$ which, in the light of the insertion-set, corresponds to a certain insertion *into* the continuum. In the case of the power $\aleph^{\aleph}$, a continuum is inserted into another (or itself); in other words, the functions in question are defined with a continuum as the domain of variability and may assume values in a continuum, e.g. any real values. The set of all real functions, considered in § 4, 8, therefore has the cardinal $\aleph^{\aleph}$; we adopt for it the symbol $\mathfrak{f}$ introduced there.

On the other hand, a "logarithmic" calculation like the one carried out before, yields:

$$\aleph^{\aleph} = (2^{\aleph_0})^{\aleph} = 2^{\aleph_0 \cdot \aleph} = 2^{\aleph};$$

hence, in view of $2 < \aleph_0 < \aleph$, we also have $\aleph_0^{\aleph} = 2^{\aleph}$.

In the language of the insertion-set this means: if the functions of a real variable, which are defined in an interval or for all real numbers, are to assume rational values only, or are even restricted to two values (say 0 and 1), the set of all such functions has the same cardinal $\mathfrak{f}$ as the set of *all* real functions. The set of all subsets of a continuum — e.g., the set whose elements are all possible sets of real numbers — therefore has the cardinal $\mathfrak{f}$.

Instead of functions of one real variable we may as well consider functions $f(x + iy)$ of a complex variable, or functions $f(x, y)$ of two,

of n, or even of $\aleph_0$ variables $f(x_1, x_2, x_3, \ldots)$. The multi-dimensional continua, which serve here as the argument-sets of the insertion-set, were proved to be equivalent to the one-dimensional continuum, and hence the sets of these functions have the same cardinal **f**.

Finally, let us look for the cardinal of the set containing *all continuous functions*, say of a real variable. The equivalence theorem will prove an essential instrument in our investigation.

Let C be the set of all continuous functions $f(x)$, defined for all real values x. We use the following property of a continuous function $f(x)$: if (x_k) is a sequence of values tending towards the real number $\bar{x}$ ($\lim_{k \to \infty} x_k = \bar{x}$), the value of $f(x)$ for $x = \bar{x}$ is completely determined by the (infinitely many) values $f(x_k)$. More precisely (this, however, is not required for what follows): $f(\bar{x}) = \lim_{k \to \infty} f(x_k)$.

Now, to a given number $\bar{x}$ (irrational or rational), we can always — and in infinitely many ways, for instance, with restriction to terminating decimals x_k — relate a sequence of *rational* numbers x_k such that $\lim_{k \to \infty} x_k = \bar{x}$ [1]). Therefore a continuous function certainly is completely determined by its values at all rational places. In other words, if, as usual, we define the equality between functions by the coincidence of their values for *every* value of the argument x, two *continuous* functions equal each other if they assume the same values for every *rational* number x. One cannot, however, invert this in the sense that to any set of given values at all rational places x, there exists a continuous function assuming those values at the places in question; this would contradict the previous statement that also at a rational place $\bar{x}$, the value $f(\bar{x})$ is determined by the values $f(x)$ if the argument x runs over any sequence tending towards $\bar{x}$ [2]). For example, there exists no continuous function which assumes the value x^2 for every integral x and the value $-x^2$ for every fractional x.

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[1]) The representation of $\bar{x}$ as a decimal fraction is one of the possible ways. For more general procedures see § 9, 1 and 2.

[2]) It is due to the equivalence theorem that in the following proof we may content ourselves with relating a continuous function to rational places in general, without going into the (somewhat complicated) interdependence of the values assumed at those places.

It should be noted that any *constant* function $f(x) = a$ is continuous.

We now consider the following three sets: the set K of all real numbers (continuum), the set C of all continuous functions introduced above, and finally the insertion-set of the continuum K into the denumerable set R of all rational numbers. By 2 and (2) on p. 157, this insertion-set $(K|R)$ has the cardinal $\aleph^{\aleph_0} = \aleph$, i.e. the same as K. Now the continuum K certainly is equivalent to a subset of C, e.g. to the set of all *constant* functions $f(x) = k$, where k is any real number; to perceive this it suffices to relate the function $f(x) = k$, which is an element of C, to the element k of K. On the other hand, we have just seen that C is equivalent to a subset of $(K|R)$, since any continuous function may be considered as a certain insertion of real values into the set R of rationals. Hence, by representing $(K|R)$ on the equivalent set K, one represents C on a certain subset of the continuum K. This, together with K being equivalent to a subset of C, gives [1]):

THEOREM 5. The set of all *continuous* functions $f(x)$ — defined for all real values x, or in an interval only — has the cardinal of the continuum.

Obviously, we may also express this theorem as follows: There is a function of two real variables $F(x, y)$ which is continuous in x for any fixed value of y, such that any given continuous function $f(x)$ equals $F(x, y)$ for one, and only one, value of y.

The set of all continuous functions, accordingly, has a smaller cardinal than the set of *all* functions, whose cardinal proved to be $\mathfrak{f} = 2^{\aleph} > \aleph$. Of course the set of all *differentiable* functions also has the cardinal $\aleph$, since it contains all constant functions and is a subset of the set of all continuous functions (any differentiable function being *a fortiori* continuous). The same applies to the set of all *monotonic* functions [2]). On the other hand, the set of all *integrable* functions [3]) has the cardinal $\mathfrak{f}$, the same as the set of all functions. Therefore we may roughly say that it is an

[1]) Cantor 7, V, p. 590. Cf. also Szymański 1.

[2]) Hausdorff 2, II, p. 111.

[3]) Integrable even in the sense of Riemann; see Jourdain 1, pp. 178—179; Schoenflies 8, p. 367. This result is contrary to a guess of Cantor's (see 7, V, p. 590). Cf. also Obreanu 1.

exceptional property of a function to be continuous or differentiable, but not to be integrable.

7. The Problem of Infinitely Small Magnitude (Infinitesimals) [1].

In the following section, we shall turn from transfinite cardinals to another kind of transfinite number, based on the concept of order. Before doing so, it may be appropriate to compare infinitely large magnitude, as exhibited by transfinite cardinals, with infinitely small magnitude (if definable). In particular, philosophers have urged the desirability, or even the necessity, of a *corresponding* procedure, leading as it were from the finite cardinals down to 0.

With respect to infinitely large magnitude Cantor had emphasized the need of introducing, besides the "potential infinite" (see p. 1), an actual infinite represented by transfinite numbers. In this domain he had succeeded in discerning a wealth of variety and in developing a transfinite arithmetic. However, as far as "infinitely small magnitude" was concerned, he declined to consider anything beyond the potential infinite of analysis. As is generally known from the elements of calculus [2]), the infinitesimals in analysis and geometry (especially in the differential and integral calculus) have been conceived by mathematicians, at any rate from Cauchy onwards, in the following way, virtually suggested by Newton. Infinitely small magnitude should be understood in a loose (improper) sense of the word only, as an infinite *process* based upon the concept of limit. For instance, one refers to variable magnitudes decreasing beyond any positive value while remaining larger than zero. However, a *definite* positive number differing from zero and at the same time smaller than any finite positive number does not exist [3]).

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[1]) The reader may omit subsections 7 and 8 without detriment to his understanding the later parts of this book. In particular, 7 is intended for those interested in the *philosophical* aspect of the problem. There is no special connection between exponentiation and infinitesimals; the subsections 7 and 8 have been inserted here at the end of the theory of transfinite cardinals.

[2]) Cf., for instance, the exposition in Hessenberg 2 or in Pasch 4, pp. 47—73.

[3]) In contrast to this, see the discussion between Leibniz and John Bernoulli as quoted by Weyl 7, No. 7; a discussion which seems quite strange in our days.

In express opposition to this conception, some philosophical trends — in particular the New-Kantian school headed by Hermann Cohen [1]) — attempted to base the calculus upon *differentials* conceived as actually infinitesimal (infinitely small) constants which should be defined analogically to transfinite numbers [2]). To a certain extent, this attitude returns to calculating with differentials in the manner which was accepted in the early period of calculus and during the eighteenth century — to be sure, without any proper foundation or justification. Since the invention of set theory by Cantor, authors pleading for such a return to infinitesimals have frequently taken transfinite cardinals or transfinite ordinals (see § 11) as a model. In analogy to the "infinitesimal" ratios of finite to transfinite numbers, or of different transfinite numbers to each other, they endeavored to introduce various kinds of "new numbers" related in the same way to finite numbers. Accordingly, these new numbers should be considered as infinitely small. In short, infinitely small magnitudes were to be created as a counterpart to transfinite cardinals, and the theory of these infinitesimals was to be developed.

These and similar views have been thoroughly refuted by Cantor [3]) and other mathematicians. The reason was not dogmatic; in mathematics, which is a science of complete liberty, there are no police decrees, and scientific prejudice on the whole has a short life — a fact nobody stressed with greater emphasis than Cantor, who himself had suffered from fighting against pseudo-scientific dogma. One cannot even maintain that a procedure of the kind mentioned would be self-contradictory, or that it is meaningless to proceed from a fancied proportion of the form "transfinite: $1 = 1 : x$" to the existential assertion "there must be an x satisfying this proportion". As a matter of fact, philosophers sometimes content themselves with just "positing" a concept, and this act in itself will but rarely entail contradiction — certainly not in the case of infinitesimals. The difficulty begins at the moment when one starts *doing something* with the concept, in our case,

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[1]) Cf. Peirce **1**, especially p. 208 ff. and p. 217 ff. Here, too, transfinite numbers are compared with infinitesimals. Cf. also Baer **6**.

[2]) See, besides Natorp **1** and the literature quoted there, especially Gawronsky **1**. Cf. also Hamburger **1**.

[3]) Cf. Cantor **10, 13, 14.**

operating with infinitesimals and applying them to scientific problems.

Now in this respect the infinitesimal has on the whole proved a failure. The types of infinitely small magnitudes conceived hitherto have not produced a really interesting arithmetic, nor yielded useful applications. Therefore, infinitesimals have been restricted to a modest way of existence, not comparable at all with the role played by transfinite numbers. *So far infinitely small magnitude has essentially continued to be conceived in the sense of potential and not of actual infinity*, in accordance with Gauss' statement quoted on p. 1. Actual infinity (in the form of transfinite cardinals, order-types, and ordinals) has been restricted to infinitely *large* magnitude.

A distinct difficulty in the introduction of actually infinitesimal magnitude was touched upon on p. 129f. when the inversion of multiplication was discussed; operations of subtraction and division, inverting addition and multiplication between transfinite cardinals, did not lead to uniquely determined results. Therefore, if one insists on introducing "infinitely small magnitude" (cf. 8) and certain operations in its domain, one has to use far more complicated methods [1]).

If, by such a procedure, one may to some extent be successful in defining certain arithmetical operations, the same cannot be said of the purpose of obtaining infinitesimals which are "useful" numbers, applicable to analysis. Quite soundly the New-Kantians as well as other authors [2]) stressed the *utility* of infinitesimals as the real criterion of their legitimation, and they were also right in choosing the differential and integral calculus as the ground where that utility should be tested. In fact, the term "infinitesimal calculus" seems to hint at the use of infinitesimals in calculus.

However, in this very test *infinitesimals, conceived as infinitely small magnitudes, have proved a thorough failure.* All various kinds of such magnitudes introduced have turned out to be useless even for the most elementary or fundamental problems of

[1]) The profoundest achievements in this direction are due to Veronese 1 and Levi-Civita 1. Moreover, among others, Hölder 1; Natucci 1, ch. XI; Schoenflies 1, II, pp. 53—64 and 3; Wiener 2.

[2]) For instance, Geissler 1 and 2.

calculus; e.g., for the proof of Rolle's theorem or for the definition of definite integral [1]). Cauchy's method which bases the fundamental processes upon the concept of *limit*, though involving some complication, has held the field against all those attempts, and there are no prospects of a change in this situation.

To be sure, we cannot exclude the possibility that a suitable arithmetical foundation of infinitesimals in a way not attempted so far, might open a new access to calculus or to other fields of analysis. In the present state of mathematical science, however, this prospect seems extremely unlikely. For the time being at any rate neither Veronese's infinitely small segments nor any other kind of infinitesimals can be taken as analogous to transfinite numbers — the mathematical existence of infinitely large and infinitely small magnitude has an essentially different character.

8. **Infinitesimals and non-Archimedean Domains.** In spite of the difficulties mentioned, infinitely small magnitude has in a formal sense been introduced into mathematics and serves certain restricted purposes in arithmetic and geometry. To understand the nature of such magnitude or number we shall start from the principle called the *axiom of Archimedes* [2]), which runs: Given two different segments a and b, say a smaller than b, there exists a natural number n such that by successively laying off a n times, one obtains a segment larger than b [3]).

In the foundations of geometry as well as of arithmetic this principle plays an important role. A domain of (geometrical or arithmetical) magnitudes not fulfilling it, is called a *non-Archimedean domain.*

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[1]) Cf. F. Bernstein 1.

[2]) From a historical point of view, it would be more correct to call it the axiom of Eudoxos. The latter, more than 100 years prior to Archimedes, made methodical use of the principle in question.

[3]) Of course, we may also express this assertion in the language of *arithmetic*, by using positive real numbers instead of segments. A more precise geometrical formulation would run: if P, Q, R are points on a straight line s such that Q lies between P and R, and if (Q_k) is a sequence of different points on s such that

1. $\overline{QQ_1} = \overline{Q_kQ_{k+1}} = \overline{PQ}$,

2. Q lies between P and Q_1, and generally Q_k between P and Q_{k+1}, then only a *finite* number of points Q_k lie between P and R.

In such a domain, accordingly, there are segments (or positive "numbers") a and b $(a < b)$ such that, however often one may successively lay off a, one always obtains segments smaller than b. Hence we may call a *infinitesimal* in relation to b (or, if b is considered as a finite magnitude, simply "infinitesimal"), and in the same way we may call b *infinitely large* in relation to a. Non-Archimedean domains of various types have been considered, especially since the beginning of this century, for many arithmetical and geometrical purposes [1]). All of them differ from the transfinite cardinals and ordinals by the absence of a counting or measuring quality. Hence, such magnitudes cannot be considered as a generalization of the concept of integers in a sense similar to that of transfinite number. In particular, in a general non-Archimedean system, an infinitely large magnitude cannot be reached by the continued addition of unity, in contrast with the state of affairs in the arithmetic of cardinals (p. 128).

The root of this difference lies in the methods of introducing and basing infinity in the theory of sets on the one hand and in non-Archimedean domains on the other. The contrast between these methods also makes it plain why the magnitudes appearing in those domains are *relatively* infinite, and not *transfinite* (absolutely infinite) as in set theory.

When introducing transfinite cardinals in § 4, or transfinite order-types and ordinals in §§ 8 and 10, we did not apply a *formal* method of defining new symbols as magnitudes which stand in a relation of infinity to the ordinary finite numbers (integers), and of introducing suitable operations by definitions which are more

[1]) An extremely simple instance is the domain of all polynomials of one variable x, provided we define $nx < 1$ for any n. For profounder examples we may quote, besides those mentioned in subsection 7, Artin-Schreier 1; Baer 1; Chwistek 4; Fraenkel 1; Hahn 1; Hilbert 2, § 12; Schuh 2; Vahlen 1 and some modern investigations in the field of abstract algebra.

An intuitively obvious example which was discussed much earlier, is given by the so-called *horn-angles*; i.e. the configurations formed by two curves (one of them possibly a straight line; e.g., a circle and its tangent) starting at a common vortex in a common direction. Cf. Kasner 1. The "orders of infinity" in the growth of functions should also be mentioned in this connection, and this subject even had a certain importance in the very genesis of the theory of sets. Cf. du Bois-Reymond 1 and Hardy 3, as well as the remarkable essay of Baer just mentioned, and further literature quoted in this essay.

or less arbitrary. Such a rather simple procedure would not require the establishment of a separate mathematical branch; moreover, for want of a "natural" basis, we could not expect symbols introduced in such a formal manner to possess considerable interest and broad applicability.

Actually, the theory of sets in the form conceived by Cantor starts from the rather *intuitive* concept of set (aggregate or ordered set) and, by a natural and reasonable development [1]), arrives at infinitely large numbers of different kinds. The relations of order among those numbers, as well as the operations with them, are not "invented" (fixed) by somewhat arbitrary rules of definition but, as it were, "discovered"; that is to say, stated in conformity with the very nature and structure of the entities and domains under consideration [2]). The difference thus pointed out is not absolute and qualitative; nevertheless, it is fundamental enough to imply considerable differences between mathematical developments based upon those different kinds of definitions. In fact, the various operations introduced in §§ 6, 7, 8, 11 would certainly not have assumed their peculiar shape, had they sprung from formal algebraic intentions or from philosophical speculations.

Such a "natural" access to infinitesimals, within or without the frame of transfinite magnitude, does not exist [3]). It is obvious

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[1]) Cf. the sentences in Cantor 7, V quoted on p. 4.

[2]) It goes without saying that this description of definitions as factual statements refers to the actual construction of a new theory where intuition plays a decisive part, and not to the systematical exposition of a theory in which definitions always show, as it were, an arbitrary character. Cf. the remarks of Dubislav 2, Enriques 5, Keyser 7 as to the essence of definition.

[3]) Even within an axiomatic, and therefore purely formal, foundation of transfinite cardinals, the difficulty involved in the adjunction of infinitesimals, or of infinitely small magnitude in general, becomes obvious. Cf. Fraenkel 4.

There is, if not identity, at least a certain affinity between the distinction just pointed out (the *formal* as against the *natural* introduction of mathematical conceptions) and Cantor's distinction between the *immanent* and the *transient* reality of concepts. See Cantor 7, V, § 8. As an instance illustrating the latter distinction, we might point to a *general* system of hypercomplex numbers lacking applications outside algebra and based on a rather arbitrary definition of operations in the system (guided only by the *principle of permanence*) on the one hand, and, on the other, to ordinary complex numbers or quaternions, the properties of which are defined in accordance

that the concept of *set* cannot be useful in this conncetion, since a set, if it is not empty, has at least *one* element; therefore, its cardinal is $\geqq 1$. Hence, the question of the existence of a theory of infinitely small numbers, in a sense parallel to that of transfinite numbers and admitting applications in other fields, has to be answered in the negative.

We might also raise the problem of parallelism between infinitesimals and transfinite numbers from an almost opposite point of view. As has been said before, the methods of calculus have been established not by infinitesimals but by a process based on the concept of *limit*; i.e. by starting out from finite magnitudes which decrease not infinitely but indefinitely. Our way to transfinite number has led and will lead us from *infinite aggregates* as a starting-point, but it is certainly conceivable that in this case, too — at least as an alternative — we might begin with finite numbers and obtain transfinite numbers by a limit process.

Certainly there is no chance of such a procedure being general, since, at best, it would enable us to obtain denumerably infinite, but no larger cardinals. Nevertheless, with some modifications and restrictions, the idea can be realized [1]). In particular, most equalities and inequalities of transfinite cardinal arithmetic may be conceived as analogous to certain relations of the arithmetic of very large finite integers. The analogy, while of course limited, can be elaborated rigorously [2]).

Exercises

1) Show directly (by constructing a representation, and without using the definition on p. 147) that the equivalences $S \sim S'$ and $T \sim T'$ imply the equivalence of insertion-sets $(S|T) \sim (S'|T')$. (This obviously is a necessary condition for the soundness of the concept "insertion-set".)

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with their use in algebra, analysis, geometry and mechanics. Yet the gap between transfinite numbers and non-Archimedean domains in general is still deeper and more fundamental. (The "transient world" corresponding to transfinite numbers, is a mathematical world.)

[1]) See Kaluza **2.**

[2]) An analogy of a different kind (in certain features reminiscent of Archimedes' sand reckoning) is pointed out by some intuitionists and has also interest for non-intuitionists; cf. Borel **2,** pp. 178—179 and Łusin **6,** p. 125.

2) Prove (by using either definition of exponentiation, according to 1 and 2) that the relations $\mathbf{c}^1 = \mathbf{c}$, $1^{\mathbf{c}} = 1$ hold for every cardinal $\mathbf{c}$.

3) Prove the following theorem concerning the general insertions (functions) introduced in 2. Let M be a *finite* set, and F the set of those functions $f(m)$ defined in M, whose values again belong to M and satisfy the condition: if $m' \neq m''$, $f(m') \neq f(m'')$. Then there exist in F two functions $f_1(m)$ and $f_2(m)$ such that we may obtain *any* function of F by reiterating f_1 and f_2 a finite number of times. If the elements of M are denoted by $1, 2, \ldots, k$, we may choose, for instance, the following functions:

$$f_1(1) = k,\ f_1(m) = m - 1 \text{ for } m \neq 1;$$
$$f_2(k-1) = k,\ f_2(k) = k - 1,\ f_2(m) = m \text{ for } m < k - 1.$$

The proof of this theorem is less simple than it may appear [1]).

4) Given a set T whose subsets are the objects to be considered, prove for the characteristic functions mentioned on p. 152, the following properties [2]):

a) The ch(aracteristic) f(unction) corresponding to the inner product (meet) of two or more subsets is the product of the ch. fs. corresponding to the factors.

b) The ch. f. corresponding to the sum of two or more *mutually exclusive* subsets [3]) is the sum of the ch. fs. corresponding to the terms.

c) The sum of the ch. fs. corresponding to the (arbitrary) subsets T_1 and T_2, equals the sum of the ch. fs. corresponding to the sum-set $T_1 + T_2$ and to the inner product $T_1 \cdot T_2$. (Hence, if T_2 is the complement $T - T_1$ of T_1, the sum is the constant function 1, which corresponds to T itself.)

5) Prove the formal rules (2) and (3) on p. 152, starting from the definition of power by the insertion-set.

6) Prove $\aleph^{\aleph_0} = \aleph$ without calculation, in the way $\aleph \cdot \aleph = \aleph$ was proved on p. 136 f.

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[1]) See Piccard 1. In this paper it is also shown that three primitive functions, instead of two, are needed if the condition $f(m') \neq f(m'')$ is dropped.

[2]) They show the close connection between the characteristic functions and the fundamental operations of Boolean algebra or of symbolic logic; see § 6, 11. For profounder connections cf. Whitney 1 and 2; Stone 10. For certain other functions of sets see Posament 1.

[3]) If they are not exclusive, the sum of the functions f_k has to be replaced by $1 - \prod_k (1 - f_k)$.

7) Prove that the following sets have the cardinal $\aleph$:

a) the set of all sequences of natural numbers, and even the set of all sequences originating from the sequence (1, 2, 3, ...) by just changing the order of the elements;

b) the set of all denumerable subsets of the continuum;

c) the set of all analytical functions, or of all (convergent and non-convergent) power series;

d) the set of all functions which can be represented as series of continuous functions.

In view of what has been said on p. 161 about the set of all integrable functions, we thus conclude *by set-theoretical methods* that an integrable function "in general" cannot be developed into a series (uniformly convergent or not) of continuous functions — a fact which illustrates the rather "pathological" character of the general concept of integrable function.

8) Let two subsets of a denumerable set be called "almost exclusive" (*presque disjoints*) if their meet is finite [1]). The following theorem holds: The set of all natural numbers — or any denumerable set — may be represented as a sum-set ST where T has the cardinal $\aleph$, and the elements of T are denumerable sets being mutually almost exclusive. Prove this theorem by representing the real numbers as dual fractions.

[1]) Cf. § 1, p. 20. For a generalization of the problem raised here, see the papers (in particular Tarski's) quoted in the Bibliography under Sierpiński 7.

CHAPTER III

ORDER AND SIMILARITY. ORDER-TYPES AND ORDINAL NUMBERS

§ 8. Ordered Sets. Similarity and Order-types

1. **Introductory Remarks.** All previous investigations of infinite sets have been restricted to properties common to all *equivalent* sets: cardinal numbers, their comparison, operations with them. The stress laid upon the concept of equivalence is justified within the frame of the present book, which stresses the possibility of introducing "transfinite numbers" and of calculating with them. In fact, transfinite cardinals represent a genus of such numbers which is not only of the utmost importance but also of extreme simplicity.

Equivalent sets, however, in spite of their common cardinal, may show much diversity, even if one disregards the particular nature of their elements. A characteristic feature is the diversity of *succession* or *order* in which the elements of a set may occur in the set, provided such a succession is given. Let, for example, N be the set of all natural numbers, R the set of all rationals, both ordered according to the magnitude of their elements in the succession from smaller to larger numbers. Certainly N and R are equivalent (p. 48), but they have quite different properties regarding the succession of their elements. N has a first element (1), and any element (n) of N has an immediate successor $(n+1)$; moreover, any element different from 1 has an immediate predecessor. R, on the other hand, has no first element, since to any rational number there exist smaller rationals; neither has any element of R an immediate successor or predecessor, as there are (infinitely many) additional rationals between any two given rationals. Even the same set [1]), conceived as such in view of the mere totality of its elements (cf. the definition of equality, p. 18f.),

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[1]) As a matter of fact, the climax in this transition is psychological only. With respect to equivalence, there is no difference between "the same set" and "two equivalent sets". Cf. 4, proof of theorem 1.

will have different appearances according to the possible ways of ordering its elements. For instance, the set of all integers ordered according to magnitude

$$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\},^{1})$$

has neither a first nor a last element, while a first element (but no last one) occurs in the "enumerated" arrangement

$$\{0, 1, -1, 2, -2, 3, -3, \ldots\}.$$

Other (essentially different) arrangements of the same set are the following:

$$\{0, 2, -2, 4, -4, 6, -6, \ldots\ 1, -1, 3, -3, 5, -5, \ldots\},$$
$$\{\ldots, -8, -4, 0, 4, 8, \ldots\ \ldots, -7, -3, 1, 5, 9, \ldots$$
$$\ldots, -6, -2, 2, 6, 10, \ldots\ \ldots, -5, -1, 3, 7, 11, \ldots\}.$$

Before entering into the systematic and formal development of the theory of ordered sets, a few preliminary remarks shall be made. First, the notion of order will be used in a definite meaning, to be explained by strict mathematical methods, and not in the vague sense used sometimes in philosophy, e.g. when the space is called an ordered manifold. Secondly, many discussions have been devoted to the question of whether the concept of cardinal has to precede that of ordinal or conversely. From a psychological viewpoint, there can be no doubt that somehow the *ordered* set is the primary notion [2]), yielding the plain notion of set or aggregate by an *act of abstraction*, as though one jumbled together the elements which originally appear in a definite succession. As a matter of fact, our senses offer the various objects or ideas in a certain spatial order or temporal succession. When we want to

[1]) In dealing with ordered sets, we shall again use brackets $\{\}$ to indicate a set by means of its elements; but at the same time — in contrast to our previous notation — *the succession of elements in the brackets will indicate the intended order of the set.* Since no confusion between the notions of plain set and of ordered set will arise, it is not worth while to introduce a new kind of brackets designating ordered sets.

[2]) This is not meant as an argument against the view sometimes expressed (cf. Fettweis **1**) that children as well as primitive tribes form the notions of ordinal and cardinal simultaneously. — It may be worth noticing that Cassirer **4** attributes priority to ordinals from a purely logical point of view, in opposition to Russell's theory of cardinals (cf. § **4**, **7**).

represent the elements of an originally non-ordered set, say the inhabitants of Washington D.C., by script or language, it cannot be done but in a definite order.

This state of affairs might suggest that the theory of order and ordinals should be developed *previous* to the theory of equivalence and cardinals. However, against such a procedure it may be pointed out that from the logico-mathematical point of view, the cardinal obviously is the simpler concept, as it is created from the ordinal by means of an act of abstraction. In accordance with this, the theory of equivalence is indeed much simpler than the theory of similarity (p. 182), because the latter involves an additional idea besides the notion of one-to-one correspondence; or in the language of sets, because in an ordered set the relation of the set to its elements is supplemented by a relation between the elements themselves, the relation of order. Besides, in mathematics the systematic exposition usually proceeds from the general to the particular, which in the present case means the presentation of cardinals before ordinals. Nevertheless, there are certain mathematical arguments [1]) in favor of letting the theory of order precede the theory of equivalence.

Besides reasons of principle, the practical need of mathematics has caused the theory of order and ordered sets to become an important part of the general theory of sets. The introduction of transfinite cardinals is only a one-sided extension of the concept of the positive integer into the infinite. In science as well as in ordinary life, those integers serve not only the purpose of counting (answering the question: how many?) but also the purpose of enumerating (first, second, third etc.).

To be sure, in the domain of *finite number* further mathematical treatment need not discern between cardinals and ordinals. But, far from being self-evident, this coincidence is a consequence of a mathematical theorem both capable and in need of proof (cf. p. 69); this theorem states that, however you may enumerate the elements of a given finite aggregate, you will always finish with elements bearing the same enumeration-mark [2]). Accordingly,

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[1]) Cf. the end of § 11. See also von Neumann 2.

[2]) Apparently, Schröder was the first to perceive the significance of this fact, while erroneously interpreting it as empirical. Proofs are found in any

this mark may be used to denote the number of elements in the aggregate. By this process a one-to-one correspondence between finite cardinals and finite ordinals is established. It finds its semantic expression in the vocabularies of all languages. Since this experience is gathered during our childhood in an empirical way, its logico-mathematical character has been overlooked until recently.

However, when we proceed to count or "enumerate" (order) *infinite* aggregates, we immediately perceive that nothing remains of such a correspondence. To one transfinite cardinal there always belong infinitely many different ordinal types (if any); in other words, an infinite aggregate which can be ordered, can be ordered in infinitely many essentially different ways.

For the purpose of generalizing the process of counting beyond the domain of finite number, it would, therefore, be quite inadequate to restrict oneself to transfinite *cardinals* which display too high a degree of abstraction. In addition to them, we have to define a new kind of "number", viz. symbols which correspond to ordered aggregates according to the ordered totality of places in the aggregate.

Cantor obtained this new class of transfinite magnitude, or at least an especially important kind of it, in the very first stage of his investigations. His original method was founded on concepts of analysis (notably that of accumulation-point; cf. § 9, **5**), and he therefore started from sets of points or numbers. One may, however, achieve the result in question — even with greater generality — by a different method which is analogous to that leading from plain sets to their cardinals, and Cantor took this way later.

Besides its intrinsic significance, the theory of ordered sets has a particular importance in the applications of the theory of sets to other branches of mathematics, as the theory of functions and geometry. For many problems in these branches the methods of equivalence theory are not suitable; in fact, concepts like continuity, neighborhood, connexity, dimension etc. are here eliminated systematically, being, as it were, annihilated by the general concept

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sound textbook of arithmetic; cf. also Natucci **1**; Pringsheim **1**, p. 15; B. Russell **5**.

For more general questions, see the mathematical literature mentioned in § 10, **6**. The wide-spread philosophical attitudes found, e.g., in Heymans **1** and Natorp **1**, are hardly justifiable from a mathematical point of view.

of one-to-one correspondence. (The reader may remember, for instance, the enumeration of rationals on p. 48f., or the representation of a square on the square-side on p. 136f.)

Some of the concepts thus eliminated, which have a great importance for the investigation of geometrical figures and of functions, may be comprehended by the very notion of order. The theory of sets, therefore, would lose a good deal of its efficiency by neglecting this notion for the sake of maximal generality and abstractness. As things are, it has been one of the main tasks of the theory of sets *to disclose the very roots of the fundamental concepts in geometry and analysis*, and part of this task (cf. § 9, **1**—**4**) was accomplished by the theory of ordered sets.

Whereas in §§ 6 and 7 operations of addition, multiplication, and exponentiation were introduced for sets and cardinals, we shall now define operations which are somewhat analogous for ordered sets and their "types". It is not obvious or simple at all — however natural it may appear — to comprehend both theories in a common and more general frame. A remarkable step in this direction was recently made by Garret Birkhoff [1]), in close connection with the notion of "partially ordered set" to be mentioned in **2**. The new concept of ordinal exponentiation, introduced by this method, is of particular interest, as compared with that of **7** and of § 11, 3.

2. Order-Relation and Ordered Sets. As in ordinary language, the expression "ordered set" shall signify a set with the following property: *There exists a rule which fixes for any two different elements of the set, which one is to precede the other.* Within certain limitations, this rule is arbitrary; therefore, it must not be confused with a relation of magnitude (quantity; in contrast to § 5, p. 90) or with a spatial or temporal succession. Though not completely expressing the desired generality, the words "to precede" and "to succeed" may be chosen from the stock of everyday language, as being relatively neutral. Of course, a strict determination of what is meant, requires a special symbol. We choose $\prec$ for "precedes", which has to be carefully distinguished from $<$, denoting "smaller than".

[1]) G. Birkhoff **4**; cf. also **2** and Day **1**. For an anticipation of some of these ideas in the "Principia Mathematica" with a background of relation-theory, see Whitehead–Russell **1**, volume II, §§ 162 and 172.

On the other hand, the relation of order is not completely arbitrary; it must possess certain formal properties such as irreflexivity, asymmetry, and transitivity [1]). In § 5, **1** we already noted these properties as belonging to order according to magnitude, which is a special case of the order-relation.

As on p. 91, the asymmetry of the relation $\prec$ never allows the expression $a \prec b$ to start with b. Therefore, we need another symbol. We take $b \succ a$ as synonymous with $a \prec b$. So we obtain the following

Definition of ordered set. Given a set and a rule establishing, with respect to any pair of different elements a and b of the set, (at least) one of the relations $a \prec b$ ("a precedes b") and $b \prec a$, so that

1. $a \prec a$ never holds (irreflexivity);
2. $a \prec b$ and $b \prec a$ never hold together (asymmetry);
3. $a \prec b$ and $b \prec c$ together imply $a \prec c$ (transitivity);
4. $a \prec b$, $a = a'$, $b = b'$ together imply $a' \prec b'$ [2]);

then one speaks of an *ordered set*, or more strictly of a *simply ordered set* [3]).

Synonymously with $a \prec b$ one also writes $b \succ a$ ("*b succeeds a*").

From this definition we immediately conclude that, if a and b are elements of a given ordered set, then *one and only one of the relations*

$$a = b,\ a \prec b,\ b \prec a \text{ (i.e. } a \succ b)$$

holds true.

Strictly speaking, according to our definition, an ordered set is the result of combining two notions, that of a plain set in the sense of the chapters I and II and that of a rule fulfilling the conditions just mentioned. Nevertheless, for the sake of simplicity we shall

[1]) In c) on p. 90, asymmetry has been shown to be a consequence of irreflexivity and transitivity combined. Nevertheless, it would not be appropriate to drop asymmetry altogether, cf. § 10, **1** and exercise 1. For further simplifications and subdivisions of the above properties, see Huntington **4** and **5**, Yoneyama **1**, and in particular, Carnap **13**, § 5, where symbolic logic is used. Cf. also exercise 5 on p. 43.

[2]) Since, in a set, a certain object appears at most *once* as an element, the condition **4** is of no actual significance.

[3]) Some authors use the term "series" instead of "ordered set". This use has not been adopted here because of the different meaning of "series" in analysis.

denote ordered sets by simple letters S, T etc. like plain sets. We have then to define:

DEFINITION I. Two ordered sets S and T are called *equal* ($S = T$) if, and only if, firstly, S and T contain the same elements (definition of equality on p. 18), and secondly, if a and b are different elements of S (and therefore of T), then the validity of the relation $a \prec b$ in S implies the validity of the same relation in T [1]).

If S is a finite (ordered [2])) set, the rule of order holding in S may be expressed by enumerating all pairs of different elements from S and by fixing for each of them the relation of order which shall hold between its elements. In principle such a procedure is impossible whenever an infinite set is concerned, and then the enumeration has to be replaced by a *law* (function, formula), as is always done in mathematics when infinitely many single statements are comprised in a finite form. (Cf. the analogous situation in the case of a *representation* or equivalence between sets, p. 32.) For instance, the rules for the four different ordered sets containing all integers, which were indicated on p. 172, may be expressed in detail as follows:

a) of any two different integers, the smaller one shall precede;

b) of any two integers, the one which has a smaller absolute value shall precede, and the positive number in the case of equal absolute values;

c) any even integer shall precede any odd one; among two even, or two odd numbers, the one which has a smaller absolute value shall precede; and the positive number in the case of equal absolute values;

d) of any two integers, the one shall precede which, divided by 4, leaves the smaller of the remainders 0, 1, 2, 3; in the case of equal remainders, the smaller number shall precede.

In many cases it is simpler to hint at such a rule by means of a few elements and the addition of dots, as done in the cases just mentioned. Then the succession in which those elements (or all elements in the case of a "small" finite set) are written down, indicates the intended rule of order.

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[1]) It is evident that the relation of equality thus defined has the necessary properties of any such relation (see p. 36).

[2]) When there is no danger of confusion, we shall often omit the attribute "ordered".

Another example of an infinite ordered set is the *set of all different infinite sequences of natural numbers* where two different (i.e. non-identical) sequences are arranged according to lexicographical order (cf. 7), that is to say, in the way the words in a dictionary are arranged, the succession of letters in the alphabet replaced by the sequence 1, 2, 3, 4, ...

The notion of order becomes insignificant for the null-set 0 and for sets containing a single element. Both kinds of sets will nevertheless be *considered as ordered sets* whenever the context makes it desirable. The simplest really significant case, which has even a particular importance in principle, is that of a pair $\{a, b\}$ containing two elements a and b. From this pair we obtain two different *ordered pairs* which may be distinguished by the notations[1]) (a, b) and (b, a). (Remember that in the notation of plain sets the pairs $\{a, b\}$ and $\{b, a\}$ are equal.)

While in this book we restrict the notion of order to simply ordered sets in the sense just explained, it should be mentioned that some related notions have been introduced by adding further conditions, or by weakening the conditions imposed in the above definition.

Cantor had already generalized the concept of "simply ordered set" to that of "multiply (doubly, n-fold) ordered set" where *two* or n (n any natural number) *different order-relations*, each of an irreflexive and transitive character, *are defined* independently of each other. It will suffice to illustrate this by the set of all complex numbers (points of the complex plane, plane vectors) $a + ib$ where a and b run over all real numbers. If the numbers are arranged by the rule

$$a + ib \prec a' + ib' \text{ if } a < a', \text{ or if } a = a' \text{ and } b < b',$$

we obtain a simply ordered set, in accordance with our definition. However, if we introduce two different order-relations marked by $\prec$ and $\{$, and if we define

$$a + ib \prec a' + ib' \text{ if } a < a';\ a + ib \{ a' + ib' \text{ if } b < b',$$

then two different complex numbers are related by *at least one* of these order-relations, and in general by both of them. Thus one has a doubly ordered set, containing all complex numbers.

Another instance originates from the same aggregate of numbers by representing $a + ib$ in the form $r(\cos\varphi + i \sin\varphi) = [r, \varphi]$ where r is a non-negative real number and $0 \leqq \varphi < 2\pi$; this representation is unique,

[1]) Whenever there is danger of confusing the notations of plain sets and ordered sets, we shall use *round brackets for the latter*. In particular, we shall do so in the case of ordered *complexes* as occurring in definition VI (7).

except in the case $a = b = 0$ in which $r = 0$ but φ is indeterminate. Hence by defining for $r \neq 0$, $r' \neq 0$

$$[r, \varphi] \prec [r', \varphi'] \text{ if } r < r'; \ [r, \varphi] \{ [r', \varphi'] \text{ if } \varphi < \varphi',$$

while for $r = 0$, $r' \neq 0$ *always* $[r, \varphi] \prec (r', \varphi')$ and $[r, \varphi] \{ [r', \varphi']$, we again obtain a doubly ordered set containing all complex numbers, such that any two different numbers are related, in general, by both order-relations, and at least by one of them. On the other hand, by defining

$$[r, \varphi] \prec [r', \varphi'] \text{ if } r < r', \text{ or if } r = r' \text{ and } \varphi < \varphi',$$

we obtain a simply ordered set which is essentially different [1]) from the simply ordered set of all complex numbers previously considered.

Although Cantor apparently had in mind far-reaching applications of the concept of multiply ordered sets, and in spite of recent developments in somewhat different directions [2]), the concept has not yet attained much significance.

However, the implications of *weakening* the conditions for ordered sets have proved to be much more important; this applies in particular to the case where the properties of irreflexivity, asymmetry and transitivity are retained but the postulate that *any two* different elements of the set should be connected by an order-relation is dropped. Therefore, in addition to the cases $a = b$, $a \prec b$, $b \prec a$, we have the case of "incomparability in order" between a and b. In this case we speak of *partially* (or *partly*) *ordered sets* (some authors of *semi-serial order*).

A quite simple instance of a finite, partially ordered set is given by a system of trails all of which start from a fixed point of departure P, each trail being divided into certain segments, say by houses, tourist cabins, landmarks etc. (Different trails may coincide along some of the segments.) The set in question shall then be the set of all points where segments terminate, including P itself; two elements shall be connected by the order-relation only if they belong to the same trail, in which case they are to succeed according to the direction of the trail from P.

Also infinite partially ordered sets are already known to the reader; take the power-set US of an infinite set S, and let a subset S_1 of S precede another subset S_2 if, and only if, S_1 is a proper subset of S_2. Given two subsets of S, in general neither is a subset of the other.

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[1]) I.e. "not similar" to the other, in the terminology of definition II, p. 182.

[2]) Cantor's own work and that of his students on this subjects is found in Cantor **10**, II and Schwarz **1**; cf. moreover Riesz **1**. A postulational treatment of the subject is found in Stöhr **1**. Lindenbaum **5** gives an important extension and, in particular, investigates multiply *well-ordered* sets (§ 10). H. Blumberg **1** inquires into the validity of the principle of *induction* in multiply ordered sets.

A more profound notion of multiply ordered set, designed for application in various fields of geometry, is found in Hudekoff **1**.

Recently, partially ordered sets have been gaining increasing importance and have given rise to applications in various mathematical branches [1]). There is also a close connection between partially ordered sets and the *theory of lattices* which dates back to Dedekind's [2]) ideas. During the last twenty years, it became an important branch of mathematics, interesting in itself, as well as for its various applications to algebra, projective geometry, topology, etc. and even to physics [3]). The theory of lattices is based upon the set-theoretical operations (§ 6) of forming the sum-set and the meet (hence a law of duality holds) and is closely related to Boolean algebra.

The law which orders a set S, at the same time *orders every subset of* S, and hence we are entitled to consider the subsets of an ordered set *ipso facto* as ordered sets and *we shall always do so.*

A given set in the plain sense of the preceding chapters, of course, need not be ordered, in spite of man's inability to indicate two or more elements of the set otherwise than in a certain order (in the sense defined here, which includes spatial and temporal order).

The transition from plain sets to ordered sets raises two questions of a fundamental character.

Firstly, *can every set be ordered at all?* More strictly, in order to avoid the word "can" which apparently bears a rather subjective (anthropologic) character: Does there *exist*, for any given set, an ordered set containing the same elements? In § 11 — and in a more general way in *Foundations* — this question will be answered. (The meaning of "existence" in this context will become clear in the next paragraph.) The problem is of course incomparably simpler when a *finite* set is concerned; this case will be touched upon in 3, and will be treated more closely in § 10, **6**.

Secondly, the notion of ordered set has been defined above by the introduction, in addition to the notion of "set" and the relation $s \varepsilon S$, of a new undefined relation $\supset$ (relation of order) satisfying

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[1]) For partially ordered sets see, e.g., Altwegg **1**; Bennett **2**; Hausdorff **4**, pp. 139—142; Kurosch **1**; MacNeille **1**; Milgram **1**; Prenowitz **1**. In particular, for the theory of so-called ramified sets, Kurepa **2**—**4**. Cf. also the quotations in footnote 3 and the papers of G. Birkhoff cited at the end of **1**.

[2]) Dedekind 3, II, pp. 103—147 and pp. 236—271. Cf. also Peirce **2**, vol. III, number VI.

[3]) Besides the writings quoted in connection with Boolean algebra, we may mention Carathéodory **3**; Dilworth **1**; Duthie **1**; Foradori **5**; Hermes–Köthe **1**; Klein-Barmen **2**—**4**; Kobayasi **1**; Köthe **1**; Ore **1**; and, in particular, G. Birkhoff **1** and **3**.

certain conditions. Thus the question arises as to whether it is possible *to dispense with one of those two primitive relations, and to reduce them to one.* In *Foundations* it will be shown that the answer is in the affirmative [1]); the relation of order can be expressed by means of the relation of a set to its elements without introducing principles in addition to those used hitherto, and, in particular, without the use of the multiplicative principle. By means of only these resources, we shall be able to prove even the following assertion: given a set S, there exists a set O the elements of which are all "ordered sets" (in a quite definite sense) which contain the same elements as S. However, this result does not provide a solution for the problem of the preceding paragraph, since the set O may possibly be *empty.* At any rate, the possibility of logically reducing *order* to the plain concept of *set* strikingly shows that *the notion of order is independent of temporal and spatial ideas.*

Finally, let us introduce a few abbreviating expressions which are based upon the notion of order.

a) If $a \prec b$ and $b \prec c$ (in short $a \prec b \prec c$), we say b *lies between* a *and* c, as well as between c and a.

b) If $a \prec b$ and if there is *no* element x of S such that $a \prec x \prec b$, then b is called the *sequent* of a. This is a special case of the more general relation "b is a successor of a" or, synonymously, "a is a predecessor of b", as expressed by $a \prec b$. Obviously an element has *one sequent* only (if any) in S.

If A denotes a subset of S which contains a but none of its successors, the sequent b of a (if it exists) is also called the *sequent of the subset* A. The same name is even adopted for an element b of S when the subset A has no "last" element a (see below), while b not only succeeds all elements of A but has the additional property that no element of S succeeds *all* elements of A and precedes b.

If b is the sequent of a, a and b are called *consecutive elements* or neighbors.

c) If every element x of S which differs from a, fulfils the relation $a \prec x$, a is called the *first* element of S. Analogically, if $x \prec a$ for every $x \neq a$, a is called the *last* element of S. It is again obvious that S has, if any, only *one* first (or last) element, for if a and a' were dif-

[1]) See, in particular, Kuratowski 1. Another way is indicated below in 4 (part c) of the proof of theorem 1). Also for this way our principles are sufficient.

ferent first elements of S, we would have the contradictory relations $a \prec a'$ and $a' \prec a$. The first and the last element are also named *extremities* (or ends) of S.

d) A subset $S' \subseteq S$ which contains, together with any element s_0, all the predecessors of s_0 in S (in other words: a subset $S' \subseteq S$ such that $s_0 \varepsilon S'$ and $x \prec s_0$ in S imply $x \varepsilon S'$), is called an *initial of* S. Likewise, a subset S'' which contains, together with any element s_0, also all successors of s_0 in S, is called a *remainder of* S.

In particular S itself, as well as the null-set, is at the same time an initial and a remainder of S.

3. **Similarity.** From ordered sets we proceed to the notion of similarity in quite an analogous way as from plain sets to equivalence (§ 2, 4).

DEFINITION II. The ordered set S is called *similar* to the ordered set T (in symbols $S \simeq T$) if the elements of T can be related to the elements of S by a one-to-one (biunique) correspondence *such that the order between related elements is the same*; that is to say, if s_1 and s_2 are different elements of S, and if t_1 and t_2 are their respective mates (by the correspondence) in T, then the relation $s_1 \prec s_2$ in S must imply the relation $t_1 \prec t_2$ in T [1]).

A correspondence between the elements of S and T having this property is called a *similar representation* [2]) between S and T, or sometimes a similar correspondence between the elements.

In view of this definition, similarity can only exist between equivalent sets, i.e. *two similar sets are always equivalent.* The converse obviously does not hold true, since equivalent sets need not be ordered at all, and certainly not in a similar way of order. Thus the ordered sets

$$S = \{1, 2, 3, \ldots\} \quad \text{and} \quad T = \{\ldots, 3, 2, 1\},$$

which contain the same elements, are not similar, since for example the first element of S, 1, has no mate in T by *any* possible similar

[1]) In view of our assumption, it is superfluous to add "and vice versa". The biuniqueness of the correspondence need not even be assumed from the first, since it follows from the condition referring to order.

[2]) In contrast with the situation in the domain of equivalence and plain representation, a similar representation, even between two *infinite* ordered sets, may be the only one existing. Cf. example 2, p. 184, and theorem 12 of § 10.

representation between S and T. For, as 1 precedes every other element of S, its mate ought to precede every other element of T, while T contains no first element at all (any element of T being preceded by all larger numbers).

A plain (not ordered) set S which is equivalent to an ordered set T, can always be ordered, and in particular in such a way that the *ordered set originating from S is similar to T*. To this purpose we only have to take an arbitrary representation between S and T, and to arrange any pair of elements of S in the same way as their mates in T.

From definition II we conclude, in a way completely analogous to that used in § 2, p. 34f., that the relation of similarity is also *reflexive, symmetrical, and transitive*. Already in the second part of definition II expressions were used that rely on the symmetry of the relation ("between S and T"). Thus we may also speak of "two similar sets", etc.

The far-reaching parallelism between the properties of equivalence and of similarity, and those of certain correspondences bearing a more special character, suggests the introduction of a more general notion comprehending all those particular concepts as specializations. The general notion would be called "equivalence of sets with regard to a class C of representations (functions)". We name S and T equivalent in this general sense ($S \underset{C}{\sim} T$) if, and only if, there is a function in C which represents S and T on each other. Although one cannot develop as detailed a theory as the theory of similarity for a notion conceived in such generality, nevertheless the notion is sufficient to allow for a rather extended theory which comprehends certain theorems of the theories of equivalence and similarity as particular cases [1]).

Those acquainted with the group-theoretical classification of geometries in the sense of the so-called *Erlanger Programm*, will recognize a certain analogy of the relation between projective and affine (or metrical) geometry to the relation between equivalence (conceived in the pregnant sense of §§ 2—7) and similarity. Equivalence is an isomorphism with regard to *equality*; by postulating, in addition, isomorphism with regard to the *relation of order*, we arrive at similarity. Accordingly, the properties of sets with respect to similarity are more specific and less general than those with respect to equivalence, in the same way as elementary (metrical) geometry, and even affine geometry, is more specific and less general than projective geometry.

Examples. 1. The ordered sets

$$\{1, 2, 3\},\ \{2, 3, 1\},\ \{3, 1, 2\},\ \{1, 3, 2\},\ \{3, 2, 1\},\ \{2, 1, 3\}$$

[1]) Cf. the footnote on p. 104; notably Tarski **10** and **12**.

are similar to each other, and they form the totality of ordered sets containing only the elements 1, 2, 3. It is obvious and may be proved strictly by mathematical induction (cf. § 10, 2 and 6) that any *finite* set F can be ordered by taking an arbitrary element as the first, another as the second, and so forth. After a finite number of steps, this procedure will come to an end by exhausting the stock of elements in the given set. Moreover, any ordered set thus obtained from F is similar to any other; to prove this we may attach to each other the first elements, the second ones, etc. up to the last elements. These propositions, which belong to arithmetic, can be demonstrated by purely arithmetical methods.

2. If we order the set N of all natural numbers, as well as the set R of all rational numbers, according to increasing magnitude, the ordered sets thus created are not similar (cf. **1**, p. 171), though they are equivalent (both denumerable). In fact, the mate of $1 \varepsilon N$ in R ought to precede every other element of R, as 1 does in N, while there is no smallest rational number. Obviously in general, for an ordered set, the property of having a first (or a last) element, implies the same property for any similar set.

However, if we order the rationals as on p. 49, arranging them in a sequence, we obtain an ordered set R^* similar to N. A similar representation — in this case, *the* similar representation — between N and R^* is formed by attaching the first element of R^* to the first of N, the second of R^* to the second of N, and in general the kth element of R^* to the kth element of N, k being any natural number. Hence any two enumerated sets are similar to each other (but of course not any two denumerable sets, not even if they are ordered). Thus we also obtain simple examples of an ordered set which is similar to a proper subset; the ordered set $\{1, 2, 3, \ldots\}$ is similar to its subset $\{n, n + 1, n + 2, \ldots\}$, where n is any natural number.

It is obvious that the representation just created between N and R^* is the *only* similar representation between these sets; we shall later recognize this as a particular case of a general theorem (§ 10, **5**). Hence the situation here is wholly different from that in the theory of equivalence, where there are always different representations between equivalent sets (containing more than one element), and even infinitely many representations between equivalent infinite sets. On the other hand, the same case (infinitely

many similar representations) may also occur in the theory of similarity, as will be seen from examples in this and in the next section (§ 9, 3 and 4). In particular, an ordered set may be similarly represented on itself by a *non-identical* correspondence, and even in infinitely many different ways; for example, the set of all positive real numbers (or those smaller than 1, or those larger than 1) by the correspondence $x = y^n$, with any positive integer n. In the first case only the number 1 is related to itself, in the second and third cases no number at all.

3. Let M be the set of all rationals, N the set of all rationals except the numbers x satisfying the inequality $0 < x \leqq 10$, with both sets arranged according to magnitude. A similar representation between M and N is constructed by attaching any negative number, as well as 0, to itself; if, however, r/s is a positive rational of M, attach to it the element $r/s + 10$ of N. In other words, we attach to the positive rational r'/s' of N (which is larger than 10, by our assumption) the element $r'/s' - 10$ of M. It is obvious that this correspondence between the elements of M and the elements of N is not only biunique but also similar.

Yet, if we define the subset of M to contain the number 10 also, M and the new set N' *cease being similar*[1]). In order to perceive that no similar representation between M and N' is possible, note that in N' 10 is not only a successor but even the *sequent* of 0. On the other hand, between any two elements of M there are always intermediate elements (even infinitely many); e.g., between r_1/s_1 and r_2/s_2, their arithmetical mean $(r_1 s_2 + r_2 s_1)/2 s_1 s_2 = r_3/s_3$. Hence, if there were a similar representation φ between M and N', and if we denote the mates in M of $0 \varepsilon N'$ and $10 \varepsilon N'$ by r_1/s_1 and r_2/s_2. then by the similarity of φ, the relation $0 \supset 10$ in N' implies the relation $r_1/s_1 \supset r_2/s_2$ in M. Therefore, if r_3/s_3 is the rational defined above, it belongs to M and satisfies the relations $r_1/s_1 \supset r_3/s_3 \supset r_2/s_2$. Accordingly, by the similarity of φ, the mate of $r_3/s_3 \varepsilon M$ in N', say r^*/s^*, must satisfy the relations $0 \supset r^*/s^* \supset 10$ in N' — which is impossible since 10 is the sequent of 0 in N'. Hence there is no similar representation between M and N'.

In § 3 we proved that the addition of a finite number of, or even

[1]) If, on the other hand, we also drop 0 from the subset N, so that neither 0 nor 10 belong to the subset, the similarity will continue to exist. This follows from theorem 1 in § 9, 3.

denumerably many, new elements to an infinite set does not alter its equivalence properties, including its cardinal. In contrast with this, the addition of a single element to a certain ordered set has just been proved to destroy its similarity to another set.

4. Using a method which served already to prove theorem 3 of § 4, consider an unlimited straight line and, in addition, a segment of an arbitrary finite length *without its ends*, i.e. an open segment. Consider both of them as the ordered sets of all points of the line or segment, each ordered in one of the two possible directions (say from left to right in figure 10, where the ends of

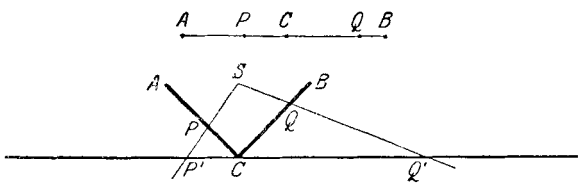

Fig. 10

the segment — in two different positions — are A and B). To construct a similar representation between these ordered sets, proceed as on p. 72: bend the segment, represented by a thin wire as it were, at its centre C, and place the bent segment against the line so that C coincides with an arbitrary point of the line, and the ends A and B are situated, say above the line, at the same distance from it, as in fig. 10. Finally, denoting the point midway between A and B in the new position by S, draw all possible rays from the "center of projection" S. Any such ray either intersects both the bent segment and the line, or intersects neither of them. (The latter class contains the rays SA and SB — since A and B have not been included in the set defined by the segment — and all rays turning upwards between SA and SB; all other rays belong to the former class.) Hence, attaching to any point P of the segment the point P' of the line lying on the ray SP, we obtain a representation between the sets of points, as on p. 72. This representation is similar, for let Q be a point of the segment different from P, and Q' its mate on the line; then, if P lies to the left of Q, also P' lies to the left of Q', since two different rays from S never intersect. Therefore, our ordered sets of points are similar.

However, a similar representation is impossible if we include

one of the ends of the segment, say A, or both of them, in the segment-set. The representation given above certainly will not do since the ray SA, being parallel to the line, does not furnish any mate of A on the line. Neither can there exist another similar representation after the inclusion of A, for A as a point of the segment-set is not preceded by any point of the set, while every point of the line is preceded by other points, viz. by all points situated to its left.

We may immediately transfer our example 4 from geometry to arithmetic. If we mark all points of a line with real numbers we obtain, not only a representation between the set of all points and that of all real numbers, but even a *similar* representation, provided the points are ordered, say according to their succession from left to right, while the numbers are ordered according to increasing magnitude. Thus we obtain the result that the ordered sets of *all* real numbers, and of the real numbers between two given numbers, both arranged according to magnitude, are similar sets. The similarity, however, is broken up as soon as one of the given numbers, or both, are included in the second set.

5. In an analogous way we may even construct a similar representation between the set of all points of a *plane* and the set of the points of a *square* only, both sets being suitably ordered. Take in the plane a system of Cartesian coordinates x, y with the origin O, and draw an arbitrary square $OXZY$, placing two of the sides (OX and OY) on the axes of x and y. (Fig. 11, p. 188.) Denote by S the set of all points of the unlimited plane, by T the set of all points within the square (excluding the points situated on the square sides) — both sets being ordered by the following rule: Of two points of the same set, the one shall precede which lies to the left of the other, i.e. whose abscissa x is smaller, or if both have the same abscissa, so that they lie on a parallel to the y axis, then the lower point shall precede. It is easily seen that this rule orders both sets S and T in accordance with the rules of simple order.

To create a similar representation between S and T, take into consideration that any point P of the plane uniquely determines, and is uniquely determined by, its coordinates with respect to the chosen system of axes. Instead of the coordinates of P, we may as well take the projections of P on the axes, i.e. the feet P_x and P_y

of the perpendiculars dropped from P on the axes. (See fig. 11.) Of course the same holds for the points of the set T, which is a subset of S. (In fig. 11 also a point Q of T, with its feet Q_x and Q_y, is

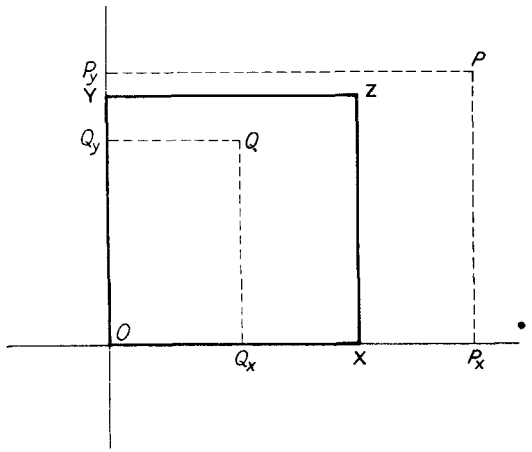

Fig. 11

marked.) Notice that the feet of any point of T are situated on the square sides OX and OY themselves, and not on the external parts of the axes.

In the preceding example (4), a definite similar representation was established between the set of points of a line and the set of points of an open segment. We use this construction to create a similar representation ψ between the entire x axis and the open square side OX, as well as a similar representation χ between the y axis and the open square side OY — the axes and sides taken as the ordered sets of points lying on them. Therefore, any point of either axis — no matter whether inside or outside the respective square side — has a uniquely determined mate *within* the respective square side. Now we establish the desired similar representation between the sets S and T by the following rule: The point P of S shall be related to the point Q of T if, and only if, Q_x is the mate of P_x by ψ, and Q_y the mate of P_y by χ.

It is obvious that this rule produces a one-to-one correspondence, which is also similar. In particular, if P_1 and P_2 are two points of S, and Q_1 and Q_2 their mates in T, and if P_1 and P_2 lie on a parallel to the y axis, the same holds for Q_1 and Q_2.

4. The Concept of Order-Type. In exactly the same way as one proceeds from the relation of equivalence to the notion of cardinal number, the relation of similarity will now lead us naturally to the notion of *order-type*. All that has been said about the mathematical and logical character of this transition and the ways of its execution in § 4, **6** and **7**, might be repeated here. It is therefore sufficient to define:

> Two ordered sets are said to be of *equal order-type* if, and only if, they are similar. Otherwise their order-types are called *different*.

From the purely formal point of view we would say: *to any ordered set a uniquely determined symbol is assigned, called its order-type, or in short its "type", provided that if S and T are ordered sets and σ and τ their types, the relation $S \simeq T$ implies $\sigma = \tau$, and conversely.* Hence a symbol actually is assigned not to a single set but to a set containing ordered sets which are similar to each other.

Cantor (cf. p. 79) exposed this as follows: given a certain aggregate, we disregard the special nature of its elements but not the succession in which they appear, and thus obtain an arrangement of mere units which is the order-type of the set. In view of this *single* act of abstraction, the order-type σ of the set S is sometimes denoted by $\overline{S}$ (while $\overline{\overline{S}}$ denotes the cardinal number of S, obtained by disregarding the order also, i.e. by a double act of abstraction).

Since any two *finite* ordered sets which are equivalent, are also similar (p. 184), there is a *one-to-one correspondence between finite cardinals and finite order-types*. That is the reason why in the domain of finite number, cardinals and order-types can be denoted by the same symbols 0, 1, 2, etc. and are given more or less identical names in all languages. We, too, *shall denote the type of an ordered set containing n elements* (n a natural number, > 1) *by* n, including the order-types 0 and 1 related to the null-set and to sets with a single element (see p. 178). The ordered sets $\{a, b, c\}$ and $\{0, 1, 2\}$, for example, both have the type 3. There is no danger of confusion with finite cardinals. For if, in a certain relation, there occur finite numbers only, its validity is independent of whether

we conceive all of them as cardinals or as types; if, however, transfinite numbers appear as well, the finite numbers have to be taken as cardinals or as types, according to whether the transfinite numbers involved are cardinals or types.

The types of infinite ordered sets — in short, the *transfinite (order-) types* — are usually denoted by *small Greek letters*. In particular, the type of an enumerated set (sequence), e.g. the set of all natural numbers arranged according to increasing magnitude, is denoted by ω. If S is an ordered set of the type σ (containing at least two elements), another ordered set is created by turning every relation $s \prec s'$ in S into its inversion $s \succ s'$ (i.e. $s' \prec s$). The order-type of the new set, which may be different from σ or equal to σ, is called *inverse* to the type σ, and shall be denoted by $^*\sigma$. Hence, if $\sigma = \omega$, we have $\sigma \neq {}^*\sigma$, since any set of the type ω has a first and no last element, while any set of the type $^*\omega$, e.g. $\{\ldots, 3, 2, 1\}$, has a last and no first element [1]). If σ is the type of the set of the rationals r satisfying $0 < r < 1$ and arranged according to magnitude, we have $\sigma = {}^*\sigma$, as shown by relating r to $1 - r$.

In accordance with Cantor's method of proceeding from an ordered set S to its type $\sigma = \overline{S}$ by a simple act of abstraction, and to its cardinal $\mathfrak{s} = \overline{\overline{S}}$ by a double one, we may *denote the cardinal of the order-type* σ [2]) *by* $\overline{\sigma}$.

The transfinite *order-types* are no less entitled to be regarded as transfinite magnitudes than the transfinite *cardinals*, i.e. the "powers" of infinite sets. However, it is impossible to arrange transfinite order-types "according to their magnitude" in a way analogous to that described in § 5 with regard to transfinite cardinals. The analogy would require that a type σ were called "smaller" than a type τ, if σ is similar to a subset of τ while τ is not similar to any subset of σ. Now take, e.g., $\sigma = \omega$, $\tau = {}^*\omega$ (hence certainly $\sigma \neq \tau$). Obviously no (ordered) subset of ω is similar to a set of the type $^*\omega$, nor is a subset of $^*\omega$ similar to a set of the type ω, since any subset of ω has a first element and therefore is not similar to a set of the type $^*\omega$, while any subset of $^*\omega$ has a

[1]) To abbreviate statements of this kind, one simply says: "ω has a first and no last element"; and also: "a subset of σ" instead of "a subset of a set having the type σ".

[2]) This again is an abbreviation for "the cardinal of any set having the type σ". Hence, if the cardinal is $\aleph_0$, one speaks of a *denumerable type*.

last element, hence is not similar to a set of the type ω. Another example is the pair of (different denumerable) types $1 + \eta$ and $\eta + 1$ (see § 9, 3), either of which is the type of a subset of the other [1]).

So we have to put up with the fact that, in general, *two order-types are incomparable*. For this reason we shall content ourselves with the modest term "order-*type*" instead of the more pretentious "ordinal number", which is reserved for a special kind of order-types to be defined in § 10. The types of that special kind will prove comparable, they in particular include the finite order-types.

All propositions on similarity and order-types mentioned hitherto are extremely simple. Somewhat more profound is the following assertion [2]):

THEOREM 1. Given any cardinal $\mathbf{c}$, the set of all different order-types of the cardinal $\mathbf{c}$ has a cardinal equal to, or smaller than, $2^{\mathbf{c}^2}$. In symbols: if the set in question — called "*the set of types of* $\mathbf{c}$" — is denoted by $\boldsymbol{T}(\mathbf{c})$, we have:

$$\overline{\overline{\boldsymbol{T}(\mathbf{c})}} \leqq 2^{\mathbf{c}^2}.$$

It would, of course, be most desirable to transform this inequality into an equality, and in special cases — or even in general cases, by means of sufficiently strong assumptions — we shall later succeed indeed in replacing $\leqq$ by $=$. (Cf. theorem 4 in **6.**) However, in complete generality or with "elementary" resources it is *impossible* to do so. On the one hand, if $\mathbf{c}$ is a *finite* cardinal different from 0, the theorem with $=$ would be *false*; for $2^{\mathbf{c}^2}$ in this case is larger than 1, while there is only one order-type of the finite cardinal $\mathbf{c}$. On the other hand, if $\mathbf{c}$ is transfinite, *in general* we seem not to be able to prove that there *exists* an ordered set of the cardinal $\mathbf{c}$, as long as we restrict ourselves to the use of principles I to VI introduced in §§ 2—5. In other words, with the aid of these principles alone *we cannot prove that every set can be ordered.* Therefore, the possibility of $\boldsymbol{T}(\mathbf{c})$ being the *null-set* is not excluded. (In special cases, of course, it may be excluded; in the case $\mathbf{c} = \aleph$,

[1]) In fact, the analogue of the equivalence theorem does not hold in the case of similarity. Cf. exercise 5 at the end of § 8.

[2]) For literature on ordered sets and order-types in general, see p. 214.

for example, by considering the ordered set of all points on a line, or of all real numbers arranged according to magnitude.) Only with the aid of the multiplicative principle shall we succeed in proving that $\mathbf{T}(\mathfrak{c})$ is always [1]) different from the null-set — and an even more far-reaching result will be obtained (§ 11).

Proof of theorem 1. a) Let S be any (not necessarily ordered) set of the cardinal $\mathfrak{c}$; further let be $\tau \varepsilon \mathbf{T}(\mathfrak{c})$, which includes the assumption that there is at least one type of the cardinal $\mathfrak{c}$. Then S *can be ordered according to the type* τ. This is an immediate consequence of the notions of order-type and cardinal. For, T being any ordered set of the type τ and therefore equivalent to S, we may order S by using an arbitrary representation between S and T and stipulating, if s_1 and s_2 are different elements of S and t_1 and t_2 their mates in T: $s_1 \prec s_2$ in S, whenever $t_1 \prec t_2$ in T.

b) Hence $\mathbf{T}(\mathfrak{c})$ *is equivalent to a subset of the set containing as its elements all different ordered sets that correspond to* S (i.e. that contain the same elements as S) where S again is a definite set of the cardinal $\mathfrak{c}$. For, in view of a), any type of the cardinal $\mathfrak{c}$ certainly appears among those ordered sets. On the other hand, we can assert equivalence only to a *subset* of the set in question, since different ordered sets corresponding to S may be similar to one another and therefore have the same type.

c) Now any definite *ordered set* T corresponding to S may be considered as a *certain set of ordered pairs* (s_1, s_2) where s_1 and s_2 are elements of S, e.g. as the set T^* of all those pairs for which $s_1 \prec s_2$ holds in T. (This automatically implies certain properties of T^*, notably that the elements of any pair are different, that (s_1, s_2) and (s_2, s_1) will never be contained in T^* at the same time, and that the relations $(s_1, s_2) \varepsilon T^*$ and $(s_2, s_3) \varepsilon T^*$ together imply $(s_1, s_3) \varepsilon T^*$.) Hence *the set of all ordered sets corresponding to* S *is equivalent to a subset of the set whose elements are all sets of ordered pairs out of* S.

As shown in § 6, **6**, a set of ordered pairs out of S is essentially a *subset* of the outer product $S \times S$; in other words, it is an *element* of the power-set $\mathbf{U}(S \times S)$. Therefore the set of all ordered sets corresponding to S is equivalent to a subset of $\mathbf{U}(S \times S)$. Since $S \times S$ has the cardinal $\mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}^2$, the power-set in question has the cardinal $2^{\mathfrak{c}^2}$ (p. 151). Hence, in view of the results obtained in b) and c), $\mathbf{T}(\mathfrak{c})$ has a cardinal equal to, or smaller than, $2^{\mathfrak{c}^2}$. Q.E.D.

By the way, the last part of the proof gives a hint for proving the *existence of the set of all ordered sets corresponding to a given (non-ordered) set* S, on the basis of our principles, save the multiplicative principle. The existence of $S \times S$ has been shown on p. 122, and therefore $\mathbf{U}(S \times S)$ exists by the principle of power-set. Hence it only remains to mark out certain subsets of $\mathbf{U}(S \times S)$ in view of the property characterizing those sets of pairs which define an *order* in S; then the existence of the desired set, being a

[1]) If $\mathfrak{c} = 0$, $\mathbf{T}(\mathfrak{c})$ contains the type of the null-set as its only element; hence, its cardinal is 1.

subset of $\mathbf{U}(S \times S)$, is guaranteed by the principle of subsets. This principle, however, does not guarantee that the subset in question differs from the null-set; in other words, our procedure does not guarantee the existence of at least one ordered set corresponding to the given set S.

5. Addition of Two Order-Types. While transfinite order-types have proved incomparable, *operations* with them may be defined and carried out to a certain extent — addition even fully — in analogy with the operations with cardinals introduced in §§ 6 and 7. On the other hand, we shall see that the formal laws of ordinary arithmetic cannot be retained as in the arithmetic of cardinals, not even with respect to addition.

DEFINITION III. Let S and T (given in this succession) be *mutually exclusive* ordered sets of the types σ and τ respectively. The plain sum-set of S and T, i.e. the non-ordered set containing the elements of S as well as those of T, shall be ordered by the following rule:

a) If x_1 and x_2 both belong to the *same* set, $x_1 \prec x_2$ shall hold in the sum-set whenever $x_1 \prec x_2$ holds in S or in T. (In plain language, the relations of order holding true in the sets S and T shall be maintained in the sum-set.)

b) If $x_1 \varepsilon S$ and $x_2 \varepsilon T$, $x_1 \prec x_2$ shall hold in the sum-set.

Thus the sum-set is transformed into an ordered set K, called *the ordered sum of S and T* (in this succession). The type $\varkappa$ of K is called the *sum of the types* σ and τ (in this succession); in other words, the sum of the types of the terms is defined as the type of the ordered sum. In symbols:

$$K = S + T, \quad \varkappa = \sigma + \tau.$$

This definition requires a few explanations.

1) The condition that S and T must be mutually exclusive is essential in the present case — much more than for the addition of plain sets where it is only required for the transition to cardinals. In fact, if a_1 and a_2 were common to S and T, it might happen that $a_1 \prec a_2$ holds in S, but $a_1 \succ a_2$ in T, making it impossible to maintain in the sum-set the order-relations defined in the single terms, as is required by a). Even in the case of a single element common to both sets, certain difficulties would arise.

2) The two-fold rule expressed in a) and b) actually transforms the sum-set into an ordered set. For, firstly, all possibilities are

exhausted by the rules a) and b), and secondly, it is obvious that the sum-set ordered according to a) and b) fulfils all conditions of order, including transitivity.

3) While rule a) is "natural", rule b) is "arbitrary" and unsymmetrical; it makes the elements of S precede the elements of T. Therefore S and T must be given from the first in a definite succession, and the sum — of the sets as well as of their types — is formally dependent on this succession, thus representing an "ordered" sum.

4) Of course, if the ordered *sets* S and T are given, their ordered sum is uniquely determined by the definition. But in starting from the given *types* σ and τ for the purpose of forming their sum, we are entitled by the definition to choose *any* "representatives" S and T on condition that these ordered sets, having the types σ and τ, be mutually exclusive; hence they are arbitrary to a large extent. Nevertheless the sum $\sigma + \tau$ is *uniquely determined* by the terms, independently of the choice of the representatives S and T, owing to the following proposition: if the pairs S, T and S', T' of mutually exclusive ordered sets fulfil the conditions $S \simeq S'$ and $T \simeq T'$, the ordered sums are also similar, i.e. $S + T \simeq S' + T'$. This proposition, which is analogous to theorem 1 on p. 112, is proved in quite the same way; the similarity of a certain representation between $S + T$ and $S' + T'$ immediately follows from our definition of ordered sum. Owing to this proposition, it is irrelevant what representatives of the respective types are chosen, for since two different representatives of the same order-type are similar, the ordered sums, though different, are similar, too, and hence their types are equal.

5) The symbol $+$, which was used in § 6 for the plain sum-set, here denotes ordered sums. This will not lead to confusion; we shall understand $+$ in the sense of plain addition, or of ordered addition, according to whether the terms of the sum are either plain sets (or cardinals) or ordered sets (types).

6) Comparing our definition with the definition of the sum of cardinals (p. 112), we obtain: *The cardinal of the sum of σ and τ equals the sum of the cardinals of σ and of τ.*

Examples of the addition of two types. 1. In the sense of addition of ordered sets we have:

$$\{1, 2, 3\} + \{4, 5, 6, 7\} = \{1, 2, 3, 4, 5, 6, 7\},$$
$$\{4, 5, 6, 7\} + \{1, 2, 3\} = \{4, 5, 6, 7, 1, 2, 3\}.$$

Both sets on the right-hand side are similar and of the order-type 7, while $\{1, 2, 3\}$ has the type 3, $\{4, 5, 6, 7\}$ the type 4. Hence the relations between finite order-types, $3 + 4 = 4 + 3 = 7$, hold true.

One easily perceives that analogous relations hold for any finite types, and that generally *the sum of two finite types is again a finite type and is independent of the order of the terms.* This coincides with the corresponding proposition for finite *cardinals.* Moreover, the result of our addition is the finite type corresponding to the finite *cardinal* obtained by adding the corresponding cardinals.

2. Let S be the set of all natural numbers arranged according to magnitude:

$$S = \{1, 2, 3, \ldots\},$$

and T a set containing a single element, e.g. $T = \{0\}$.

As to their types, we have:

$$\overline{S} = \omega, \ \overline{T} = 1.$$

Hence $\omega + 1$ is the type of the set $\{1, 2, 3, \ldots 0\}$; since this set has a last element, in contrast with S, $\omega + 1$ certainly is different from ω. So, in sharp contrast with the addition of cardinals (p. 117), a transfinite order-type may well be changed by the addition of a finite type.

On the other hand, we have:

$$T + S = \{0, 1, 2, 3, \ldots\},$$

i.e. an enumerated set, which is similar to S. Hence:

$$\omega + 1 \neq 1 + \omega = \omega.$$

One easily concludes (by mathematical induction) that the types

$$\omega, \omega + 1, \omega + 2, \ldots, \omega + n, \ldots \ (n \text{ any finite type})$$

are all different from each other, while $n + \omega = \omega$ for any finite n.

Taking $^*\omega$ instead of ω, we see in the same way that $^*\omega + n = {}^*\omega$

for any finite n, and that $m + {}^*\omega \neq n + {}^*\omega$ whenever m and n are different finite types.

3. The addition of the ordered sets

$$U = \{1, 3, 5, \ldots\} \text{ and } V = \{2, 4, 6, \ldots\}$$

produces a set of the type $\omega + \omega$, in which any odd number precedes any even one. Let W be a finite ordered set, for example

$$W = \{w_1, w_2, w_3, \ldots, w_n\}.$$

Let us consider the following ordered sums:

$$(W + U) + V,\ W + (U + V),\ (U + W) + V,$$
$$U + (W + V),\ (U + V) + W,\ U + (V + W).$$

It is obvious that their types are respectively

$$(n+\omega)+\omega = \omega+\omega,\ n+(\omega+\omega) = \omega+\omega,\ (\omega+n)+\omega = \omega+\omega,$$
$$\omega+(n+\omega) = \omega+\omega,\ (\omega+\omega)+n,\ \omega+(\omega+n).$$

The first four sums are independent of n, and equal each other. One easily perceives (cf. below) that the last two sums are also equal, but they depend on n, i.e. $(\omega + \omega) + m \neq (\omega + \omega) + n$ if $m \neq n$.

4. ${}^*\omega + \omega$ is the type of the set of all integers arranged according to magnitude, since ${}^*\omega$ is the type of the set of all negative numbers, ω the type of the set containing 0 and the natural numbers, both ordered according to magnitude. As in the preceding example, we obtain:

$$({}^*\omega + n) + \omega = {}^*\omega + (n + \omega) = {}^*\omega + \omega.$$

However, not only is $n + ({}^*\omega + \omega) \neq ({}^*\omega + \omega) + n$, but by attributing different finite values to n we obtain different sums on the left-hand side as well as on the right-hand side.

Other examples will appear in 6 and 7 and also in § 9.

The above provides a few instances [1]) of the validity of

[1]) Conversely, by using theorem 2, we may simplify a few conclusions drawn in the examples 3 and 4.

THEOREM 2. The addition of ordered sets and of order-types according to definition III is *associative*, i.e.

$$(S + T) + U = S + (T + U),\ (\sigma + \tau) + \upsilon = \sigma + (\tau + \upsilon).$$

On the other hand, addition in general is *not commutative*, save for finite types.

Proof. In view of definition III it suffices to prove the first assertion with respect to *sets*. For that matter, it has been shown in § 6, p. 115, that the sets $(S + T) + U$ and $S + (T + U)$ taken as *plain sum-sets*, are equal. Hence, it only remains to prove that, by our definition of addition, the order of the same two arbitrary elements x_1 and x_2 in either set is equal.

Now this is trivial whenever x_1 and x_2 belong to *the same* of the ordered sets S, T, U. Otherwise, if $x_1 \prec x_2$ in $(S + T) + U$, let x_1 belong to S [1]); then either $x_2 \varepsilon T$ (so that $x_1 \prec x_2$ holds in $S + T$) or $x_2 \varepsilon U$. In both cases x_2 belongs to $T + U$; hence, by definition III, $x_1 \prec x_2$ in $S + (T + U)$. So, if $x_1 \prec x_2$ holds in the left-hand sum, it also holds in the right-hand sum, and this assertion can immediately be inverted. Q.E.D.

In view of the associative law, we may denote the common value of $(\sigma + \tau) + \upsilon$ and $\sigma + (\tau + \upsilon)$ by $\sigma + \tau + \upsilon$, omitting the parentheses altogether.

The second assertion of theorem 2 evolves from the examples given above.

6. General Addition of Order-Types. Let us at once formulate the general definition of the addition of *types*, including the general addition of ordered sets as a by-product only.

DEFINITION IV. Let T be an ordered set, different from the null-set; to every element $t \varepsilon T$ a certain order-type $f(t) = \varrho_t$ may be attached. To form the ordered sum of all these types, we replace each ϱ_t by an ordered set R_t of the type ϱ_t ("representative of ϱ_t") with the only restriction that for $t_1 \neq t_2$ the sets R_{t_1} and R_{t_2} be mutually exclusive. The plain sum-set of all sets R_t shall be ordered by the following rules:

a) if x_1 and x_2 belong to the same set R_t, $x_1 \prec x_2$ holding in R_t implies the validity of the same relation in the sum-set.

[1]) If x_1 belongs to T, the proof is still simpler.

b) if $x_1 \varepsilon R_{t_1}$ and $x_2 \varepsilon R_{t_2}$ $(t_1 \neq t_2)$, $t_1 \prec t_2$ in T implies $x_1 \prec x_2$ in the sum-set.

Thus, in view of the order of the elements $t \varepsilon T$, the sum-set is transformed into an ordered set K, called the *ordered sum of all ordered sets* R_t. The type $\varkappa$ of K is called the (*ordered*) *sum of all the given types* ϱ_t; in other words, the ordered sum of *types* is the type of the ordered sum of corresponding ordered sets. We write:

$$\varkappa = \sum_{t \varepsilon T} \varrho_t.$$

As to the remarks added to definition III on p. 193, all of them hold *mutatis mutandis* with regard to definition IV which includes definition III as a special case. In particular, the sum of types is uniquely determined, in spite of the somewhat arbitrary choice of the representatives. In fact the analogue of theorem 1 on p. 112 holds as well in this general case, and its proof is obvious on account of rule b). Hence different ordered sums of representatives, constructed in accordance with the definition, are similar to each other. In view of the way in which the sum of cardinals is formed, we perceive that *the cardinal of an ordered sum of types equals the sum of the cardinals belonging to the individual types.*

As an additional remark, not topical in the case of definition III, it may be pointed out that the ordered sum is only dependent on the *type* of the set T, and not on T itself. In other words, if instead of T we take a similar set T^*, and if in view of a certain similar representation between T and T^* the type ϱ_t is now attached to the mate t^* of t in T^*, the sum $\varkappa$ is not changed by this modification. The truth of this remark is obvious since even the respective ordered sum-sets equal each other.

The examples considered in **5** already show that our addition is not commutative. But in the general case, addition need not be commutative even if the types to be summed up are all *finite*; we shall soon see an instance of this possibility (at the end of example 2).

On the other hand, general addition is associative, as in the previous case. The complication involved in this matter does not lie in the *proof* (which is quite analogous to that of theorem 2 on p. 197) but in the strict and generalized *formulation*. To this purpose we resume the notation of p. 115, using the symbol **S*** for the *ordered* sum of ordered sets (in contrast with **S**, which was used there for the sum-set of plain sets).

In fact, let A be the ordered set containing all the ordered sets R_t in the order indicated by T (definition IV), and let

$$A = \ldots + L + \ldots + M + \ldots + N + \ldots \tag{1}$$

be an arbitrary decomposition of A into complementary exclusive non-empty subsets in the sense of *ordered* addition, so that, e.g., the elements of L precede the elements of M by the order ruling in A. The assertion we want to prove may be expressed as follows:

$$\mathbf{S}^*A = \ldots + \mathbf{S}^*L + \ldots + \mathbf{S}^*M + \ldots + S^*N + \ldots \tag{2}$$

where the symbol + again has the meaning of ordered addition.

That both sides of (2) constitute sets containing *the same elements*, has already been shown by (2) on p. 115. Hence it only remains to show that *the same order* also rules in both sets. But this follows in a way analogous to the proof of theorem 2. In fact one has here to distinguish three cases with respect to two different elements x_1 and x_2 of $\mathbf{S}^*A$:

a) they belong to the same $R_t \varepsilon A$;

b) they belong to different R_t's, but to such as are contained in the same subset appearing on the right-hand side of (1) — say in L;

c) they belong to different R_t's which for their part are contained in different subsets — say $x_1 \varepsilon R_{t_1} \varepsilon L$ and $x_2 \varepsilon R_{t_2} \varepsilon M$.

Now in any case the order between x_1 and x_2 is the same in the left-hand and right-hand sets of (2); in case a) evidently, in case b) because the order relations in L (subset of A) are the same as in A, in case c) because the elements of R_{t_1} precede the elements of R_{t_2} on both sides of (2), by virtue of definition IV. Hence:

THEOREM 3. The addition of ordered sets and of order-types according to definition IV is associative, in the sense of (1) and (2). On the other hand, addition in general is not commutative (not even the addition of finite types).

Theorem 3 includes theorem 2 in the following sense. By definition IV, in the case of three order-types the sum $\sigma + \tau + v$ is defined *a priori* (not only *a posteriori* as indicated at the end of 5). But in view of theorem 3, this sum equals $(\sigma + \tau) + v$ as well as $\sigma + (\tau + v)$.

Examples of the general addition of types. 1. Let T be an

enumerated set, thus having the type ω, and relate to each $t \varepsilon T$ the type ω. The sum of types in this case, denoted by $\omega + \omega + \omega + \ldots$ [1]), is the type of a sequence of sequences. We may for instance arrange all natural numbers according to this type by the following scheme (originating from a successive diagonal arrangement):

1	2	4	7	11	16	.	.
3	5	8	12	17	.	.	.
6	9	13	18	.	.	.	.
10	14	19	.	.	.	.	.
15	20	.	.	.	.	.	.
21	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

Running through this scheme by successive rows, we obtain the desired set or type.

The same type is obtained by attaching the type $\omega + \omega$ to each $t \varepsilon T$. In this case each two consecutive rows of our scheme, starting with the first two, are joined together, which by the associative law does not change the result.

2. Let T be an arbitrary (non-empty) ordered set and τ its type. By attaching to each $t \varepsilon T$ the finite type 1, we obtain as the sum of all these units again the type τ. To prove this, we shall attach to any $t \varepsilon T$ the set $\{t\}$, having the type 1; then by definition IV the ordered sum of types is the very type of T. Hence *any order-type may be represented as an ordered sum of units.* (Cf. the analogous proposition for cardinals, p. 128.) Of course, "units" here mean types and not cardinals.

If, instead of 1, the type 2 (of an ordered pair), for example, is attached to the elements of T, the sum is changed whenever T is finite. In general the same will hold true even in the case of an infinite set T. If, for instance, H is the set of all rationals between 0 and 1, arranged according to magnitude, and if η denotes the type of H, we certainly obtain a sum different from η by attaching 2

[1]) Here, in contrast with the sums written in (1) and (2), dots are used at one place only, since there is a first term and to any term belongs a sequent, as well as (except for the first term) an immediate predecessor.

to each element of H [1]). For in this sum every unit [2]) has a *neighbor*, either a sequent (if the unit corresponds to the first element of an ordered pair), or else an immediate predecessor [3]). On the other hand, since between any two different rationals, additional rationals are situated, the elements of η certainly have no neighbors.

Finally, in view of $2 = 1 + 1$ and of the associative law, we may consider the sum σ obtained by attaching 2 to each element of H, again as a certain sum of units. Another sum of units is obtained by attaching the type 1 to every element of the set containing the rationals between 0 and 1 as well as those between 1 and 2; denote this sum by σ'. Now evidently we may consider σ' as created by a permutation of the order of terms in the sum σ considered before; viz. by letting all the second units of the ordered pairs follow the totality of the first units while preserving the succession of the first units and of the second units among themselves. Nevertheless, we here have $\sigma' \neq \sigma$. For, contrary to σ, in σ' again no element has a neighbor; in § 9, 3, we shall even prove that $\sigma' = \eta$. Accordingly, we have thus obtained an example of *an ordered sum, all terms of which are finite ordinals (even units), and which changes its value by a certain permutation in the succession of the individual terms.*

Of course, our example is essentially dependent on the set H. Had we taken an enumerated set (a sequence) instead of H, we should not have changed the sum by substituting 2 (or any finite type) for 1.

Now let us use our knowledge of ordered sums for improving a result obtained earlier, at least in an important particular case.

Theorem 1 (p. 191) gives an upper bound (2^{c^2}) for the cardinal of the set of types of the cardinal $\mathfrak{c}$, denoted by $T(\mathfrak{c})$. One of the reasons for obtaining a bound only, instead of a precise value, was the difficulty of indicating any one type of the cardinal $\mathfrak{c}$. Now

[1]) In contrast with the instances of *cardinal* addition ($\aleph_0 + \aleph_0 = \aleph_0$, $\aleph + \aleph = \aleph$) considered in § 6. The behavior of *general* transfinite cardinals in this direction will be clarified in § 11.

[2]) Here again, we use an abbreviating expression as indicated in the footnote 1 on p. 190.

[3]) In a quasi-intuitive image, we might represent this set by a thin ray of a two-atomic gas, provided that the one-dimensional line of molecules were as compact as the succession of all rationals and that the *atoms* constitute the elements of the ray.

the simplest non-trivial case is $\mathfrak{c} = \aleph_0$, and in this case the mentioned difficulty certainly does not arise since, e.g., ω belongs to $T(\aleph_0)$. In this case it proves easy to transform the inequality into an equality [1]); hence, since $\aleph_0 \cdot \aleph_0 = \aleph_0$ and $2^{\aleph_0} = \aleph$, we shall obtain:

THEOREM 4. The set $T(\aleph_0)$ of all denumerable order-types has the cardinal $\aleph$ of the continuum [2]).

Proof. Since theorem 1 already states that the cardinal in question is $\leq \aleph$, it suffices to prove that it is also $\geq \aleph$. In other words, we may demonstrate the theorem by constructing a set of denumerable types — i.e., a subset of $T(\aleph_0)$ — having the cardinal $\aleph$.

For this purpose we write

$$*\omega + \omega = \zeta,$$

and assign to *any sequence* of natural numbers $N = (n_1, n_2, \ldots, n_k, \ldots)$ the type

$$\tau_N = n_1 + \zeta + n_2 + \zeta + \ldots + n_k + \zeta + \ldots.$$

τ_N obviously is a denumerable type, and *to different sequences* N *different types* τ_N *are assigned*; in other words

$$M \neq N \text{ implies } \tau_M \neq \tau_N. \tag{3}$$

To prove (3), we shall use the following

Lemma. If R_1, R_2, S_1, S_2 are ordered sets [3]), and if one of the following assumptions (a) and (b) is fulfilled

(a) $R_1 + S_1 \simeq R_2 + S_2$, R_1 and R_2 finite, S_1 and S_2 without first elements

(b) $S_1 + R_1 \simeq S_2 + R_2$, $\overline{S_1} = \overline{S_2} = *\omega + \omega \ (= \zeta)$

then it follows that

$$R_1 \simeq R_2, \text{ and in the case (a) also } S_1 \simeq S_2.$$

Proof. In the case (a) a similar representation between $R_1 + S_1$ and $R_2 + S_2$ cannot relate, for instance, an $r \varepsilon R_1$ to an $s \varepsilon S_2$, since

[1]) F. Bernstein 3; the result is partly due to Cantor. Cf. Hausdorff 4, pp. 97—98. See also Eyraud 1.

[2]) For the problem of all order-types of the cardinal $\aleph$ cf. Cuesta 3 and Sierpiński 30. By using the well-ordering theorem (§ 11, 6) we prove theorem 4 *mutatis mutandis* for any transfinite cardinal. $T(\aleph_\nu)$ then has the cardinal $2^{\aleph_\nu}$.

[3]) The case where one of them is the null-set may be omitted as trivial.

the latter is preceded in the sum by infinitely many elements, the former by a finite number of elements only. In the case (b) again such a similar representation cannot relate an $r \varepsilon R_1$ to an $s \varepsilon S_2$, since then the whole set S_1 would be represented on a subset of S_2 which contains only elements preceding s, contrary to the fact that any such subset of S_2 has a last element.

From the lemma just proved we easily conclude the truth of (3) by means of mathematical induction on the terms of the sums

$$\tau_M = m_1 + \zeta + m_2 + \zeta + \ldots, \quad \tau_N = n_1 + \zeta + n_2 + \zeta + \ldots.$$

For since all m_k and n_k are finite types, and since ζ has no first element, $\tau_M = \tau_N$ will, in view of the case (a) and of the associative law, imply:

$$m_1 = n_1, \ \zeta + m_2 + \zeta + m_3 + \ldots = \zeta + n_2 + \zeta + n_3 + \ldots.$$

In view of (b) the latter equality again implies:

$$m_2 + \zeta + m_3 + \ldots = n_2 + \zeta + n_3 + \ldots,$$

etc. Therefore $m_k = n_k$ for any k, i.e. $M = N$. Hence $M \neq N$ implies $\tau_M \neq \tau_N$, Q.E.D.

It has thus been proved that to all different sequences N, different denumerable types τ_N are related. Hence the cardinal of the set $\boldsymbol{T}(\aleph_0)$, which certainly contains still other types besides all τ_N, equals, or is larger than, the cardinal of the set containing all sequences of natural numbers. Now a *single* such sequence $(n_1, n_2, n_3, \ldots)$ is created by inserting natural numbers into the set of all natural numbers 1, 2, 3, ... So the set of *all* such sequences is equivalent to the insertion-set of the set of all natural numbers into itself (§ 7, 2), and therefore has the cardinal $\aleph_0^{\aleph_0}$ which equals $\aleph$, according to (3) on p. 158.

The cardinal of $\boldsymbol{T}(\aleph_0)$ therefore is at the same time $\leqq \aleph$ and $\geqq \aleph$, hence $= \aleph$. Q.E.D.

7. On the Multiplication of Order-Types. As to the product of *two* types, the most convenient method of introducing it is by conceiving multiplication as repeated addition, a method usually applied in arithmetic and referred to, in the case of cardinals, in theorem 5 on p. 128. The method by which the multiplication of cardinals was introduced in § 6, p. 124, will appear here as an afterthought.

DEFINITION V. If in definition IV all the types ϱ_t attached to the elements $t \varepsilon T$ are *equal*, say $= \varrho$, then the sum $\varkappa = \sum_{t \varepsilon T} \varrho_t$ shall be called the *product of* ϱ *and of the type* τ *of* T (in this succession) [1]; in symbols

$$\varkappa = \varrho \cdot \tau.$$

In order to include the case $\tau = 0$, we define $\varrho \cdot 0 = 0$ for any ϱ.

The remarks annexed to definition IV show that the product is independent of the special nature of the set T chosen to represent the type τ, and that *the cardinal of a product of two types equals the product of the cardinals of the factors* (in view of theorem 5 on p. 128).

Examples. If, for instance, the type ω is attached to both elements of an ordered pair T, we have $\varrho = \omega$, $\tau = 2$, and we obtain $\omega + \omega = \omega \cdot 2$. If, however, $\varrho = 2$, $\tau = \omega$, the type 2 is attached to each element of a *sequence*; say the type of the ordered pair (a_k, a_k') is attached to the natural number k. The ordered sum-set is then, according to definition IV,

$$\{a_1, a_1', a_2, a_2', \ldots, a_k, a_k', \ldots\},$$

which is again a sequence; hence we have

$$2 \cdot \omega = 2 + 2 + \ldots = \omega.$$

The type of the set of natural numbers, ordered as in example 1 on p. 200, is $\omega \cdot \omega$. Many other examples will occur in §§ 9—11.

As proposed from the first (§ 6, 7), we have here taken the *first* factor of a product as the multiplicand and the *second* as the multiplier; in $\omega \cdot 2$, ω is added twice, and not 2 ω-times. This arrangement is of course arbitrary; as a matter of fact, originally Cantor had stipulated the opposite. The present arrangement, later adopted by Cantor, is now generally accepted.

In view of $\omega + \omega \neq \omega$, our examples already show that in general multiplication is not commutative, and not even if one of the factors is finite [2]. Hence, in contrast with ordinary and cardinal

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[1] In view of the fact that a type τ may be represented as a sum of units, we may say: replacing the type 1 by the type ϱ, we proceed from the sum τ to the product $\varrho \cdot \tau$.

[2] If both factors are finite, multiplication coincides with multiplication of cardinals and is therefore commutative. In fact, definition V has been formulated with an eye to generalizing the multiplication of natural numbers in arithmetic.

arithmetic, the distinction between multiplicand and multiplier is essential. Regarding the factor 1, however, we have for any order-type ϱ

$$\varrho \cdot 1 = 1 \cdot \varrho = \varrho.$$

For $\varrho \cdot 1 = \varrho$ immediately follows from the definition (the term ϱ taken once), and $1 \cdot \varrho = \varrho$ is only another wording for what was pointed out in the beginning of example 2 on p. 200.

As to the *formal laws* of multiplication, including its connection with addition, the non-validity of the commutative law has already been stressed. One of the distributive laws is easily proved, viz.

$$\varrho \cdot (\sigma + \tau) = \varrho \cdot \sigma + \varrho \cdot \tau \tag{1}$$

even if $\sigma + \tau$ is replaced by the ordered sum of *any* (finite or transfinite) number of types. However, the second distributive law, which would read

$$(\varrho + \sigma) \cdot \tau = \varrho \cdot \tau + \sigma \cdot \tau$$

is not true in general. This may be seen from simple instances. We easily perceive that, for example [1]),

$$(\omega + 1) \cdot 2 = \omega + 1 + \omega + 1 = \omega + \omega + 1 = \omega \cdot 2 + 1 \tag{2}$$

differs from $\omega \cdot 2 + 1 \cdot 2 = \omega \cdot 2 + 2$

$$(\omega + \omega) \cdot \omega = \omega + \omega + \omega + \ldots = \omega \cdot \omega \tag{3}$$

differs from $\omega \cdot \omega + \omega \cdot \omega = (\omega \cdot \omega) \cdot 2.$

On the other hand, the equality (1), even in its general form (cf. the notation of definition IV)

$$\varrho \cdot \sigma = \varrho \cdot \sum_{t \in T} \sigma_t = \sum_{t \in T} (\varrho \cdot \sigma_t), \tag{1a}$$

is obviously nothing but an expression of the associative law of addition for order-types, in accordance with definition V.

Finally, the associative law of multiplication holds; its demonstration, however, will prove more convenient after the introduction of definition VI.

In § 6 the definition of the product of two sets (or cardinals)

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[1]) The following conclusion is more general. The validity of the second law would imply that, for any σ, $2 \cdot \sigma = (1 + 1) \cdot \sigma = 1 \cdot \sigma + 1 \cdot \sigma = \sigma + \sigma = \sigma \cdot 2$, while, in general, (e.g. for $\sigma = \omega$) $2 \cdot \sigma \neq \sigma \cdot 2$ (see p. 204).

preceded the transition from two factors to any number of factors. At this juncture, it would be natural to steer the same course. However, this gives rise to unexpected difficulties which will soon be explained. (The absence of any specific difficulty in the transition from the sum of two types to the sum of any number of types in **6**, is noteworthy.) At first we shall confine ourselves to the case of a *finite number of factors* (order-types).

We might directly generalize definition V to the case of any finite number of factors instead of two; but this procedure would be awkward and lead to formal complications. It is preferable to go back to definition VI of § 6, where the multiplication of cardinals was founded on the *outer product of sets*. In order to make this conception applicable to the present case, one has to define an *order* in the outer product — and this in such a way that for two factors the ordered outer product has the type required by definition V.

Let us first consider the case of two factors as in definition V. To multiply ϱ by τ, we shall choose mutually exclusive ordered sets R and T of these types and form the set of all un-ordered pairs $\{r, t\}$ with $r \varepsilon R$ and $t \varepsilon T$; in § 6, this plain set was denoted by $R \times T$ and called the outer product. Trying to order this set, we remember that in view of definition V the product $\varrho \cdot \tau$ is composed of types ϱ that succeed in the order and cardinal corresponding to τ. Hence in arranging the pairs $\{r, t\}$ one should provide for the elements $r \varepsilon R$ to succeed each other in such a way that always, i.e. for any definite $t \varepsilon T$, a set of the type ϱ is obtained. To this purpose one has to separate the cases of *different* t's and to let all the pairs $\{r, t\}$ with a definite value $t = t_0$ precede (or follow) all the pairs $\{r, t\}$ with another value of t. In other words, one has to consider *ordered pairs* (r, t) with r preceding t, and to arrange these pairs in the first place according to the order of the second elements of the pairs, and in the second place — i.e. for a *fixed* value of t — according to the order of the first elements.

This rather twisted procedure is somewhat inconvenient, especially in the case of more than two factors. On the other hand, we may easily obtain the converse procedure (ordering initially by the first elements of the pairs, and afterwards by the second) by a formal trick; for the multiplication of R and T in this succession we consider the set of ordered pairs (t, r) with t preceding

r, instead of (r, t) [1]). Since the order of the elements $t \varepsilon T$ has to be considered first, and the order of $r \varepsilon R$ only afterwards (i.e. for equal t), the rule of order now starts from the first element. We immediately perceive that then the outer product of R and T in this order has the order-type $\varrho \cdot \tau$ according to definition V. In fact, for a definite $t = t_0$ the subset of the pairs (t_0, r) for all $r \varepsilon R$ has the type ϱ of R, and our ordered product is precisely the ordered sum of all these subsets while t runs over all elements t_0 of T. In other words, the sets of ordered pairs (t_0, r) with fixed t_0 and variable r, are chosen as the representatives of the type ϱ required by definition V.

We may now take this new definition of a product of two types as the starting-point for defining *the product of any finite number of types*, as follows:

DEFINITION VI. To the elements of a finite ordered set of n elements, the types $\varrho_1, \varrho_2, \ldots, \varrho_n$ may be attached. In order to form the ordered product $\varrho_1 \cdot \varrho_2 \cdot \ldots \cdot \varrho_n$ of the given types, replace the types by mutually exclusive ordered sets $R_1, R_2, \ldots, R_n$ of those types $(\overline{R}_k = \varrho_k)$, and order the "outer product" — or, more exactly, the set the elements of which are all "ordered complexes"

$$(r_n, r_{n-1}, \ldots, r_2, r_1) \qquad (r_k \varepsilon R_k)$$

— by the following rule, which considers the *first differing* pair of corresponding elements in the complexes:

$$(r_n, r_{n-1}, \ldots, r_2, r_1) \prec (r'_n, r'_{n-1}, \ldots, r'_2, r'_1)$$
$$\text{if } r_n \prec r'_n \text{ in } R_n,$$
$$\text{or if } r_n = r'_n \text{ and } r_{n-1} \prec r'_{n-1} \text{ in } R_{n-1},$$
$$\cdot \qquad \cdot$$
$$\cdot \qquad \cdot$$
$$\cdot \qquad \cdot$$

or if $r_n = r'_n,\ r_{n-1} = r'_{n-1}, \ldots, r_2 = r'_2,\ r_1 \prec r'_1$ in R_1.

[1]) To be sure, this procedure, too, means an inconvenience, since we have to take ordered pairs or complexes (see definition VI) which are ordered *inversely* to the given succession of the factors. One might try to remove the unpleasantness by considering ϱ in the product $\varrho \cdot \tau$ as the multiplier and τ as the multiplicand. This, however, besides interfering with a generally adopted usage, would mean, not a facilitation, but rather a complication in the really involved case of infinitely many factors.

The type $\varkappa$ of the *ordered set* of these complexes (sometimes named *the ordered outer product* of $R_1, R_2, \ldots, R_n$) is called the *product* of the types $\varrho_1, \varrho_2, \ldots, \varrho_n$ in this succession; in symbols:

$$\varkappa = \varrho_1 \cdot \varrho_2 \cdot \ldots \cdot \varrho_n \text{ [1]}.$$

If all factors ϱ_k are equal, $\varrho_k = \varrho$, we write

$$\varkappa = \varrho^n$$

and call $\varkappa$ the nth *power* of the type ϱ.

To understand this definition more easily, notice that the succession in which the ordered complexes are to be arranged, is that used in a dictionary (lexicon) where all words are arranged alphabetically according to the first letter, and, in the case of equality of this letter, according to the second letter, etc. Therefore we may formulate definition VI in short as follows: $\varrho_1 \cdot \varrho_2 \cdot \ldots \cdot \varrho_n$ *is the type of the set of all ordered complexes* $(r_n, \ldots, r_2, r_1)$ *arranged among themselves in lexicographic order* [2]. Using a more arithmetical expression, one sometimes calls this arrangement "order by first differences", replacing the letters of the alphabet by integers.

The remarks preceding definition VI show that definition V may be considered as a particular case of definition VI for $n = 2$. Moreover, since the ordered outer product in the sense of definition VI is, apart from its order, an outer product in the sense of definition V on p. 120, definition VI implies that the cardinal of $\varrho_1 \cdot \varrho_2 \cdot \ldots \cdot \varrho_n$ equals the product of the cardinals of the factors ϱ_k. We may express this, together with the remark annexed to definition IV on p. 198, in the following form:

THEOREM 5. The cardinal of an ordered sum of types (p. 197 f.) equals the sum of the cardinals of the types; the cardinal of the ordered product of *a finite number of types* equals the product of the cardinals of the types.

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[1]) It is again obvious that $\varkappa$ is dependent only on the types $\varrho_1, \ldots, \varrho_n$ and not on the sets $R_1, \ldots, R_n$ chosen to represent these types.

[2]) If we had retained the ordered pairs (r, t) for constructing the product $\varrho \cdot \tau$ (cf. p. 206), and the corresponding arrangement in the general case, we should have had to use an "antilexicographic order", depending, as it were, on the last letters of the words.

In § 11, 3 we shall see why the restriction to a finite number of factors in the latter case is essential.

The associative law of multiplication is more conveniently proved on the basis of definition VI than on that of definition V. (In principle, it is true, definition V is sufficient to prove that $(\varrho_1 \cdot \varrho_2) \cdot \varrho_3 = \varrho_1 \cdot (\varrho_2 \cdot \varrho_3)$, and in view of this we may denote the common value by $\varrho_1 \cdot \varrho_2 \cdot \varrho_3$, in accordance with definition VI, and, precisely as in arithmetic, proceed from three to any finite number of factors by mathematical induction.) In view of definition VI, it will be sufficient to restrict oneself to a decomposition into two factors only, and to proceed to the general case by mathematical induction; for since n is finite, only finite decompositions need to be considered, in contrast with the associative laws (of addition and multiplication) for cardinals, and with the associative law of addition for order-types.

Thus, for instance [1]), we may prove the law in the form

$$\varrho_1 \cdot (\varrho_2 \cdot \varrho_3 \cdot \ldots \cdot \varrho_n) = \varrho_1 \cdot \varrho_2 \cdot \varrho_3 \cdot \ldots \cdot \varrho_n.$$

To this purpose we first show that the *un-ordered* outer products which correspond to both sides in view of definition VI, are *equivalent* (not equal), by establishing a representation which relates the ordered complex $((r_n, \ldots, r_3, r_2), r_1)$ to the ordered complex $(r_n, \ldots, r_3, r_2, r_1)$, where always $r_k \varepsilon R_k$. Secondly, if $((r_n', \ldots, r_3', r_2'), r_1')$ and, accordingly, $(r_n', \ldots, r_3', r_2', r_1')$ are other complexes of the same kinds, in view of definition VI the latter complexes will either both precede the respective former ones, or will both follow them, which proves the similarity of the representation just created.

In view of what has been said before (p. 204 f.) about the formal laws, we may sum up as follows.

THEOREM 6. The multiplication of a finite number of order-types is associative, and is connected with addition by the "first" distributive law (1), or (1a), on p. 205. On the other hand, this multiplication in general is neither commutative nor does it satisfy the "second" distributive law.

The arithmetic of order-types was already developed by Cantor [2]) as far as general addition and multiplication of two types. Naturally

[1]) The procedure for other decompositions into factors, e.g. $(\varrho_1 \cdot \varrho_2) \cdot (\varrho_3 \ldots \varrho_n)$, is exactly analogous.

[2]) Cantor 12, I, § 8.

we would now like to proceed to general products, i.e. to products of infinitely many order-types, and to the general concept of power. As to the *totality of elements* contained in such general (outer) products, there is no difficulty at all; the respective operations have been defined in §§ 6 and 7. But an insurmountable obstacle arises when we try to *define an order* in that outer product — more strictly, an order connected in a reasonable way with the orders of the given factors, and which contains the order indicated in definition VI for a finite number of factors, as a particular case.

To understand the difficulty involved in our problem, consider the simple case in which the factors are given in the form of a *sequence* (enumerated set) of ordered sets $R_1, R_2, \ldots, R_k, \ldots$. In analogy with definition VI, we consider all ordered complexes

$$(\ldots, r_k, \ldots, r_2, r_1), \quad (\ldots, r'_k, \ldots, r'_2, r'_1), \text{ etc.}$$

where $r_k \varepsilon R_k$, $r'_k \varepsilon R_k$, etc. for all finite indices k. To generalize the ordering rule given in definition VI, we have to determine the order between any two given complexes, e.g. those just written down, by considering *the first index* (i.e. the largest natural number) *k for which r_k differs from r'_k*. But in general such a first index will not exist at all; r_k may differ from r'_k for every k, or more generally, among the indices k for which $r_k \neq r'_k$ there may be no first (largest) one. In fact, even for a sequence of factors *no suitable arrangement of the ordered complexes contained in the outer product is possible.* When, moreover, in this case all the ordered sets R_k are similar, and hence all the types equal, we are unable to define what should form the ωth *power* along the lines of our ideas.

In 1904, Hausdorff began to develop a theory of what might be called *substitute-products* and *substitute-powers* [1]). Its purpose is to replace the outer product in question by a subset which is just narrow enough to allow for a suitable order of its complexes. Not only are these investigations rather complicated, but the result is meagre enough. In general the ordered subset, and therefore its order-type, will even have a *smaller cardinal* than the original outer product. The general theory of this quasi-multiplication of types cannot be developed within the present book.

[1]) Hausdorff **1**; **2**; **3**; **4**, ch. VI. For other questions of the arithmetic of order-types, see Hausdorff **4** and Lindenbaum–Tarski **1**, § 3.

Nevertheless, the theory of exponentiation, corresponding to equal factors, will be presented in § 11, 3 for a particularly important case — though from another point of view. We shall then see, for instance, that the ωth power of the type ω, taken in the sense of our quasi-exponentiation, has not the cardinal $\aleph_0^{\aleph_0} = \aleph$ as we would expect, but only the cardinal $\aleph_0$.

Exercises

1) Work out in detail the proof of similarity hinted at in example 5 on p. 187 f.

2) If in fig. 1 on p. 10 we mark off additional points $P_2', P_3', \ldots, P_k', \ldots$ situated symmetrically to $P_2, P_3, \ldots, P_k, \ldots$ with respect to P_1, we obtain an ordered set of the type $^*\omega + \omega$, if the points P_k, P_k' are arranged, for instance, according to their order from right to left.

3) Show in detail that the ordered sum of ordered sets (definition IV) and the ordered outer product of a finite number of ordered sets (definition VI) actually fulfil the conditions imposed on any ordered set (p. 116).

4) On the basis of definition VI, complete in detail the proof of the associative law of multiplication, and give the proof of the first distributive law (based in 7 on definition V).

5) The analogue of the equivalence theorem (p. 98 f.) in the theory of types would be: if each of two ordered sets is similar to a subset of the other, the sets themselves are similar. Show by means of a few instances, in addition to the one given on p. 190, that this proposition is false [1]).

6) α and β being any types, what are the inverse types (p. 190) of $\alpha + \beta$ and of $\alpha \cdot \beta$? Give a few examples.

7) Show that $\omega + \omega^2 = \omega^2 = (\omega + \omega) \cdot \omega$, and that $\omega^2 \neq \omega^2 \cdot 2$.

8) Prove the following theorem [2]): The set of all ordered sets of a given cardinal **c** is equivalent to the subset of those ordered sets of the cardinal **c** the order-types of which are not changed by the removal of any finite number of elements.

9) In accordance with p. 179, a set is called "partially ordered" if it satisfies all conditions for an ordered set except the one stating that for any

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[1]) For a *partial* analogue, see Lindenbaum–Tarski **1**, § 3 and Lindenbaum **1**. Cf. also Kurepa **5**.

[2]) See Chajoth **1**. (For general **c**, § 11, **5** and **6** are required.)

pair of different elements a and b at least one of the relations $a \prec b$ and $b \prec a$ shall hold. Let c and d be different elements of a partially ordered set such that c and d are *not* connected by the order-relation R defined in the set. Prove that one can *extend* R in precisely two different ways to a relation R^* (which includes R) such that c and d are connected by R^* [1]).

10) On p. 183 the relation $S \overset{K}{\sim} T$ between sets has been introduced with regard to a certain (non-empty) class K of representations or "functions". K is called a "group" if it has the following two properties: a) if $f(x)$ belongs to K, then the inverse function of f also belongs to K; b) if $f(x)$ and $g(x)$ are functions of K, the "compound" function $f(g(x))$ also belongs to K. Prove that, with regard to a *group* K, the relation $\overset{K}{\sim}$ is always symmetrical and transitive.

§ 9. Linear Sets of Points

The contents of this section somewhat transcend the scope of the present book, since the theory to be presented here is not concerned with abstract sets but with sets of points (numbers); it is based on concepts taken from the domains of analysis and geometry [2]). But having developed the theory of ordered sets in detail, it is natural that we should avail ourselves of a few important and partly well-known instances to show the power and the applicability of the notions and methods we have introduced within the scope of the theory of order [3]). In particular, we shall catch a glimpse of the importance of set-theoretical methods for the investigation of subtle geometrical properties, the systematical exploration of which has become a major subject of applied set theory during the last decades (cf. § 12). In fact, those methods constitute a tool of utmost delicacy, without which many a complicated situation would escape our notice.

However, we shall not utilize the contents of § 9 *in the later sections of this book. Therefore, whoever is not interested in this subject, may skip the present section in its entirety.*

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[1]) See Szpilrajn 1. By combining this result with a method of Kuratowski 2, we obtain the following result, by virtue of the multiplicative principle: Any partial order of a set S *can be extended to a complete order of* S, so that S becomes an ordered set.

[2]) As a matter of fact, this restriction might be dropped altogether in 1 and 2. It is introduced there only for simplicity's sake. In **5** and **6**, on the other hand, the restriction to "points" is essential.

[3]) The notions used in **5** and **6** are even independent of the theory of order.

1. Dense or Continuous Ordered Sets. We start from a fixed straight line extending from left to right, and unlimited in both directions. In order to obtain a convenient way of representing its points, imagine it as "the line of numbers" (p. 12f.); any real number b will then be related to a definite point on the line marked by b and vice versa [1]). We shall consider the set L of all points of the line, ordered by the relation $\prec$ in the sense of "left of —"; thus $P \prec Q$ shall mean that the point P on the line is situated to the left of the point Q. Usually we shall denote points just by the corresponding numbers.

All reasonings of **1—4** *concern subsets of the ordered set* L, which are taken as ordered by the order established in L. Naturally, any result obtained which applies to a subset of L, may be considered a proposition about a set of real numbers which results from a similar representation between L and the set of all real numbers, arranged according to their magnitude.

Any subset of L, no matter whether its order is taken into consideration or not, is called a *linear set of points.* The attribute "linear" not only refers to a "simple" order (p. 176) possibly defined in the set, but is also meant in the sense [2]) of excluding sets

[1]) The simplest way of obtaining this similar one-to-one correspondence between all real numbers and all points of the line, would be to *define* the line as a similar image of the set of real numbers ordered according to magnitude. There are many stages of transition between so drastic an elimination of all difficulties and a penetrating geometrical attitude, say of a topological character. A middle course, keeping in with intuition as well as with an appropriate introduction of the number-concept, would be starting from two fixed points on the line and adding further points by certain operations, for instance by rational operations and by constructing geometrical means. Finally, we should then introduce a postulate of "completeness" which ensures that the line is "sufficiently" filled with points — say the so-called axiom of Dedekind–Cantor (cf. the footnote on p. 215 and exercise 6 at the end of § 9). In view of the concept of real number, it is easy to prove that a real number corresponds to every point of the line, but the converse assertion, i.e. that a point corresponds to every real number, is only assured by a postulate of the type just mentioned, and hence this is essential for *determining the notion of point.*

Among the literature on these matters, including the rôle of the *axiom of Archimedes* (p. 165), consult, besides the original sources Dedekind **1** and Cantor **4**, also Baldus **2** and **5**; Hertz **7**; Hölder **1**; Loewy **1**; A. Schmidt **1**; Zariski **4**; and the lucid exposition in Courant–Robbins **1**, ch. II.

[2]) Cf. example 5 on p. 187, where a two-dimensional set of points appears as a simply ordered set.

of points in the plane or in a space of three or more dimensions [1]).

As will be stressed in **5**, sets of points and, in particular, linear sets of points, need not be considered as ordered sets; there are fundamental notions used in the systematical treatment of sets of points that do not refer to order at all. However, the notions introduced in **1** and used throughout **2—4**, bear an ordinal character. On the other hand, the theory of ordered sets is not at all restricted to sets of *points*, though it is true that other sets — i.e. *abstract* ordered sets — had not been explored by Cantor and remained neglected until Hausdorff started his profound investigations in this domain, some years after the turn of the century [2]).

Throughout **1—4**, the term "set of points" means *a subset of the ordered set L just defined and which contains at least two elements.*

DEFINITION I. A set of points K is called *dense everywhere*, or in short *dense* [3]), if between any two different points of K there is another point of K [4]).

It follows from this definition that no element of a dense set has a neighbor. If p_1 and p_2 are points of a dense set K, and if $p_1 \prec p_2$, a point p_3 lies between p_1 and p_2, i.e. $p_1 \prec p_3 \prec p_2$; likewise there are points p_4 and p_5 such that $p_1 \prec p_4 \prec p_3 \prec p_5 \prec p_1$, and if $p_3, p_4, \ldots, p_k$ lie between p_1 and p_2, the same applies to an additional point p_{k+1}. *Therefore between any two different points of a dense set infinitely many points of the set are situated.* Hence any dense set is infinite.

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[1]) Of course, nothing would prevent us from including sets of points situated on a *curved* line provided the curve is open and not closed. The latter condition is necessary in the case of ordered sets, since the conditions of order would not otherwise be fulfilled. Besides, the use of the term "linear" in mathematical literature is not uniform.

[2]) See Hausdorff **2** and **3**, as well as the exposition in **4**. Cf. also Gleyzal **1**; Kurepa **1**; Mahlo **1** and **3**; Wrinch **4**, **7**, **8**. In this context, we may also mention the essays Cuesta **1** and **2** which consider decimal (or dual) fractions in which the sequence of digits (α_β) may proceed according to any transfinite number (β remaining smaller than a certain ordinal number, see § 11); cf. also Maximoff **1** and **2**.

[3]) Some authors use the term "compact" which, however, is usually understood in a different sense.

[4]) For an important logical property of dense ordered sets without extremities, see Skolem **1**; Langford **3**; Tarski **20**, II, p. 293; for its application to dense continuous sets (definition III) Langford **13**. Cf. also Lewis–Langford **1**, p. 405 ff.

We immediately see that a set of points which is similar to a dense set, is again dense.

DEFINITION II. Let a set of points K be divided into two *exclusive and non-empty* subsets K_1 and K_2 such that $K_1 + K_2 = K$ holds true in the sense of ordered addition (p. 193); then the division — or more strictly, the ordered pair containing K_1 and K_2 in this succession — is called a *cut* $(K_1|K_2)$ in K [1]).

Hence a cut $(K_1|K_2)$ in K has the following properties: a) any point of K belongs to one, and only one, of the subsets K_1 and K_2; b) K_1 and K_2 each contain at least one point; c) any point of K_1 precedes any point of K_2. In the terminology of d) on p. 182, K_1 is an *initial* and K_2 a *remainder* of K.

Given a cut $(K_1|K_2)$ in the set K, we may discern by a purely logical disjunction the following four mutually exclusive cases, which cover all possibilities:

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[1]) Expositions of the concepts of cut in particular and of real number in general, are found, for instance, in Graves **1**, ch. III; Hobson **2**, ch. I; Pierpont **1**, ch. I; Stone **9**. The notion of *cut* (some authors say "section") which is vital for the foundation of arithmetic and analysis (theory of irrational number) as well as for geometry, was introduced by Dedekind in his classical essay **1** which first appeared in 1872. In the same year Cantor **4** (cf. Heine 1) and Méray **1** based the concept of irrational number upon certain *sequences* of rationals ("fundamental series" or "segments"). Another theory, closely related to Cantor's, was developed by Weierstrass in his Berlin lectures from 1860 on; cf. Mittag–Leffler **1**. Any of these methods may serve as an instrument for investigating sets of points. For practical use in arithmetic they may prove more convenient than Dedekind's, though the latter's procedure seems to be preferable for its supreme simplicity and beauty. Its practical application in arithmetic is presented, for instance, in Perron **3**.

For a comparative survey of the various theories cf. Cantor **7**, V, § 9 and Dedekind **3**, vol. II, pp. 356–370; see also Jourdain **5**, Perron **1**, Rosenfeld **2**. For a procedure using symbolic logic and referring to the method used in "Principia Mathematica" (Whitehead–Russell **1**) cf. Kinder **1**; Bachmann–Gradowski **1**; also, though less successful, Sarv **1**. The history of the problem in Greek antiquity (when Euclid's elements virtually gave a rigorous introduction of irrational number though without a complete treatment of operations) is described in Hasse–Scholz **1** and Scholz **3**; cf. Bonnesen **1**.

For a certain connection between these theories and the problem in physics of defining a (temporal) *moment*, see B. Russell **11** and the writings quoted in our bibliography together with this essay.

a) K_1 has a last element k_1, K_2 has a first element k_2;
b) K_1 has a last element k_1, K_2 has no first element;
c) K_1 has no last element, K_2 has a first element k_2;
d) K_1 has no last element, K_2 has no first element.

In 2 we shall see that these four cases are not only logically thinkable but capable of mathematical realization. In the cases a), b), c) the cut $(K_1|K_2)$ may be described in the following way: K_1 is the initial of K the last element of which is k_1, or K_2 is the remainder of K the first element of which is k_2. When either K_1 or K_2 has been fixed, the other — and hereby the cut itself — is obviously determined. Thus we arrive at

DEFINITION III. Let $(K_1|K_2)$ be a cut in K. If K_1 has a last element, or K_2 has a first one, either of these elements is said to *produce* the cut. In particular, if both cases occur together, the cut is called a *jump*; if neither case occurs, the cut is called a *gap*; if just one of them occurs — i.e. in the cases b) and c) — the cut is called a *continuous cut.*

The set K is called a *continuous set* [1]) if *any* cut in K is continuous; i.e. if there exist neither jumps nor gaps in K.

Evidently also the notions introduced here are invariant with respect to any transition from a given set to a set similar to it.

2. Examples. The meaning of the definitions and the purpose of some terms introduced in **1** will now be illustrated by a few examples. Profounder instances will appear in **3** and **4**.

1. Let p_1 and p_2 be any two different points of the set L, and let K be the subset of L containing all points between p_1 and p_2; usually K is called the *open* [2]) *interval* between p_1 and p_2. K is *dense*, and retains this property when one of the extremities p_1 and p_2, or both of them, are added (in the latter case one speaks of a *closed interval*). The set L itself is also dense.

2. Let p_1 and p_2 again be two different points of L, and H the

[1]) For a definition of this property of continuity by means of the relation of being a *subset* ($\subseteq$) — instead of the relation of being an element (ε) on which our definition is essentially based — see Foradori **1** and **2**, cf. also **4** and **5**. In the first of these essays an axiomatization is found of the part-whole relation $\subseteq$.

[2]) A non-empty ordered abstract set is, in general, called *open*, if it has no extremities. Hence any set which is similar to an open set, is again open.

subset containing the *rational points* which lie between p_1 and p_2; i.e., L is considered the line of numbers and H is the set of all points between p_1 and p_2 that are marked by rational numbers. Since between any two rationals other rationals are situated H is also a dense set, even after the addition of the extremities p_1 and p_2, or of one of them. The set of *all* rational points of L is also dense.

3. Let p be an arbitrary point of L. We may divide L into two complementary subsets L_1 and L_2 by allotting to L_1 all points of L to the left of p, to L_2 the point p and all points to the right of p. This division evidently creates a *cut* $(L_1|L_2)$ in L, which is produced by p. The same holds true if p is allotted to L_1, instead of L_2. In either case the cut $(L_1|L_2)$ is *continuous*.

However, if p enters neither L_1 nor L_2, $(L_1|L_2)$ is a cut not in L but in the subset L' of L obtained by dropping the point p. This cut in L' is no longer continuous but is a *gap*. For let p_1 be any point of L_1, i.e. any point of L lying to the left of p; then any point of L between p_1 and p still belongs to L_1, and certainly there are such points, even infinitely many. Hence p_1 cannot be the last point of L_1, and since p_1 was chosen arbitrarily in L_1, L_1 *has no last point*. In the same way we see that L_2 *has no first point*. According to definition III, $(L_1|L_2)$ is a gap in the set L'. The name "gap" indicates that in L' a certain point is missing, as it were; namely the point p which belongs to L but not to L'. Therefore L', though dense, is not continuous.

At the beginning of this example we considered a cut $(L_1|L_2)$ in L which was defined in a *particular* way. But the nature of the set L (viz. the "completeness" which characterizes the totality of points on a line; cf. footnote 1 on p. 213 and example 5 on p. 218) implies that *any* cut in L is produced by some element of L (and only one such element). Hence, not only jumps but also gaps are absent from L; any cut in L is continuous, and so L *is a continuous set*. Evidently the same applies to the open or closed intervals considered in the first example.

4. Let p_1 and p_2 be any two different points of L with $p_1 \supset p_2$. (Cf. fig. 12.) K shall contain all points of L to the left of p_1 and to the right of p_2, including both p_1 and p_2. We define a cut $(K_1|K_2)$ in K by allotting to K_1, p_1 and all points to its left, and to K_2, p_2 and all points to its right. Then p_1 is the last point of K_1, and p_2 the first point of K_2. Hence the cut $(K_1|K_2)$ is a *jump*

in K. This indicates that K "jumps", as it were, straight from p_1 to p_2, since p_2 is the sequent of p_1 in K. Of course, K is not dense; obviously a dense set has no jumps, and vice versa: a set without jumps — i.e. a set the cuts of which are either continuous cuts or gaps — is always dense.

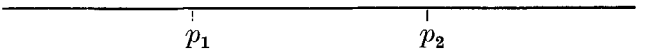

Fig. 12

On the other hand, if we remove the point p_2 from the set K (hence also from K_2), thus defining a new set K' and a new subset K_2', the cut $(K_1|K_2')$ becomes *continuous* in K', K_1 having a last element (p_1) but K_2' no first element. K' (considered as a set by itself and not in relation to L) is dense [1]), and similar to the set L which contains *all* points of the line. To prove this, we have only to construct a similar representation between the set K_2' and the set of all points of L situated to the right of p_1, while K_1 (as a subset of L and of K') may be represented on itself. As regards pure order and similarity, not concerned with metrical distances, the set K' is continuous in the same sense as L, all cuts in K' being continuous.

5. Let R be the subset of *all rational points* of L. Define the cut $(R_1|R_2)$ in R as follows: R_2 shall contain all rational points m/n (m and n positive integers) for which $m^2/n^2 > 2$. All other points of R shall belong to R_1. Then, in particular, 0 and the points m/n for which $m/n < 0$ — "negative points" — belong to R_1. We shall show that this cut is a *gap* in R; in other words, that the cut is not produced by any element of R.

First of all, the cut $(R_1|R_2)$ is certainly neither produced by the point 0 nor by any negative point since, e.g., the point 1 which still belongs to R_1 lies to the right of all those points. We may therefore restrict our investigation to the points m/n with $m > 0$

[1]) It might seem repugnant to common sense to call such a set "dense", in view of the impression created by fig. 12. For many mathematical purposes (cf. 5) it is in fact preferable to consider a set of points, not in an *absolute*, but in a *relative* sense, in relation to the "space" in question, which in the present case would be the set L. In relation to L, K' is certainly not dense.

But, within the theory of abstract ordered sets, dealt with in **1—4**, there is no need of relative notions. As an ordered set, K' is similar to L, i.e. of the same order-type, and therefore dense.

and $n > 0$. As is well-known [1]), there is no pair of positive integers m, n such that $m^2/n^2 = 2$; hence, if a rational m/n produced our cut, we should have either $m^2/n^2 < 2$ or $m^2/n^2 > 2$.

Now it is easily seen that in the first case there would still exist elements of R_1 to the right of m/n, in the second case elements of R_2 to the left of m/n. So, in neither case does m/n produce the cut $(R_1|R_2)$. To prove this assertion, we consider, for instance, the set L of *all* points (which contains also the point $\sqrt{2} = 1.4142\ldots$) and we rely on the theorem [2]) which states that between any two

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[1]) The proof, known already in the school of Pythagoras, is given by Euclid and referred to by Aristotle in about the following way. We may presume m and n prime to each other. Then the relation $(m/n)^2 = m^2/n^2 = 2$, hence $m^2 = 2n^2$, would imply that m were even (since the square of an odd number is again odd) and n accordingly odd; say $m = 2h$, $n = 2k + 1$. But this entails a contradiction, since $m^2 = 4h^2$ is divisible by 4 while $2n^2 = 8k^2 + 8k + 2$ is only divisible by 2 and not by 4.

The proof may at once be generalized and simplified by the demonstration that the square of a reduced fraction m/n with $n > 1$ cannot be an integer. The importance attributed by the Greeks to these simple facts of irrationality may be gathered from Plato's account that his teacher Theodoros had proved the irrationality of the square roots of all integers from 2 to 17, except, of course, 4, 9, 16.

[2]) Since we need this theorem again in 4, its proof may be given in detail, by means of the expansion of real numbers into decimal fractions (cf. p. 62). Let A and B be different — say positive — real numbers, $A < B$. If there is an integer between them, it may be taken as the desired rational. Otherwise, we write A and B in their decimal form, beginning with a non-negative integer k, and in such a way that A, if possible, appears as a terminating decimal (i.e. A does not have the period 9) while B, at any rate, is an infinite decimal. We may write

$$A = k.a_1 a_2 \ldots, \qquad B = k.b_1 b_2 \ldots,$$

where the a_n and b_n are digits; our assumptions imply that the a_n will not finally consist of nines only, nor the b_n of zeros only.

Since $A < B$, there will be a *first* natural number (index) l (possibly $l = 1$) such that $a_l \neq b_l$, hence $a_l < b_l$. Then, for instance, the *terminating* decimal

$$C = k.a_1 a_2 \ldots a_{l-1} b_l \quad (= k.b_1 b_2 \ldots b_{l-1} b_l),$$

which is a rational number, lies between A and B. In fact, we have $A < C$ since

$$\begin{aligned} A &< k.a_1 a_2 \ldots a_{l-1} a_l 999\ldots = \\ &= k.a_1 a_2 \ldots a_{l-1} (a_l + 1) \leqq k.a_1 a_2 \ldots a_{l-1} b_l = C \end{aligned}$$

different real numbers (points of L) a *rational* number (rational point) is situated. Hence there are rational numbers r satisfying the inequalities $m/n < r < \sqrt{2}$ in the first case, and $\sqrt{2} < r < m/n$ in the second case. Since $r \varepsilon R$ and either $r^2 < 2$ or $r^2 > 2$, m/n is neither the last point of R_1 nor the first point of R_2. Q.E.D.

In addition to what was said about dense set in example 4, we thus find that a dense set may have gaps and even, as will presently appear, infinitely many gaps. It goes without saying that, on the other hand, a dense set always has continuous cuts as is shown by any cut produced by one of its elements. A dense set may also be continuous (cf. 4) while any continuous set is dense.

The gap $(R_1|R_2)$ in R just considered may be "filled" by adding the point $\sqrt{2}$ to R [1]). Then, no matter whether this point is included in R_1 or in R_2, the cut in the new set is continuous, being produced by the point $\sqrt{2}$. For $x > \sqrt{2}$ means that $x > 0$ and $x^2 > 2$, in accordance with the original definition of our cut.

Naturally, the gap considered here is not the only one in our set R; on the contrary, to any irrational point of L a certain gap in R is related and every gap may be filled in an analogous way. Since the set of all irrational numbers has the cardinal $\aleph > \aleph_0$ *the set of all gaps in R has a larger cardinal than the set R itself.* This is not too surprising, for the gaps in K are defined not by *elements* but by certain *subsets* of K, i.e. by elements of $\mathbf{U}K$. Nevertheless, it is a striking instance of a case in which ordinary spatial intuition is unable to comprehend the geometrical facts which are explained by set-theoretical methods.

Since a cut in a dense set cannot be a jump, a dense set which

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and we have $C < B$ since not all digits $b_{l+\nu}$ ($\nu = 1, 2, 3, \ldots$) vanish.

Example: $A = 1.2333333 \ldots$; $B = 1.23456 \ldots$. The C just defined would then be 1.234.

A more elegant way of proving our theorem, which does not depend on the particular expansion of the real numbers in question, evolves from exercise 9 on p. 61.

[1]) As long as the theory of the continuum or of real numbers is not yet known, i.e. as long as we do not wish to use the set of points L to which $\sqrt{2}$ belongs, we cannot proceed in this way. In this case, we might fill the gaps by considering the *gaps themselves* (i.e. the cuts $(R_1|R_2)$ corresponding to case d)) as new *points* or *numbers* and by adjoining them to the set R under suitable conditions of order. This is the method applied in the *genetic* theory of real numbers, where the irrational number is defined by rationals.

is not yet continuous will become continuous when all its gaps have been filled. For *any* dense ordered set (not only of points) this transformation into a continuous set may be achieved by a generalization of the procedure just used.

3. The Type η of all Rationals. We shall now use the concepts defined in **1** to completely characterize the types of two fundamental linear sets of points, whose description by Cantor with the aid of these methods belongs to the earliest achievements of the abstract theory. There is a third, still simpler type which should be described in the same way, namely the type ω. This much simpler task may be left to the exercises at the end of § 9.

Let a and b be two different points of L, and let R be *the set of all rational points between a and b*, with the exception of these two points (if they are rational). R obviously has, *inter alia*, the following properties:

a) R is denumerable (p. 48),
b) R is dense (example 2 on p. 217),
c) R is open, i.e. has no extremities.

Also the set of *all* rational points of L has the same three properties.

We shall prove:

THEOREM 1. Properties a, b and c *completely* determine an order-type, usually denoted by η. In other words, any ordered set (not just assumed to be a set of points) which has those three properties, is *similar* to R.

This theorem, of course, entails that the type η is independent of the particular points a and b by which R has been defined.

Proof [1]. Let S be *any* ordered set (not necessarily a subset of L, i.e. a linear set of points) with the properties a—c. We have to show that $S \sim R$.

A certain enumeration of the set R, e.g. the one given on p. 49, may be indicated by the sequence

$$(r_1, r_2, r_3, \ldots, r_k, r_{k+1}, \ldots)$$

[1]) Cf. Cantor **12**, I, § 9. An extension of the method is given in Skolem **2**, § 4.

which contains all points of R. As the property a is fulfilled by S according to our assumption, there is a certain sequence

$$(s_1, s_2, s_3, \ldots, s_k, s_{k+1}, \ldots)$$

containing all elements of S. Both sequences, which are arbitrary to a large extent, will remain fixed throughout the proof. It goes without saying that there is no connection at all between the succession of the r_k and s_k in the *ordered sets* R and S on the one hand, and their arrangement in the sequences just indicated on the other.

To establish a definite similar representation between the ordered sets R and S, we apply an *inductive* method proceeding along the sequence (r_k). We begin with an entirely arbitrary step, by *attaching* r_1 *and* s_1 (the first elements of the arbitrary sequences) *to each other.*

The second step consists in attaching a suitable element $s_m \varepsilon S$ to $r_2 \varepsilon R$. Now between r_1 and r_2 one of the relations $r_1 \prec r_2$ and $r_2 \prec r_1$ holds, and since — *by property* c *of* S — s_1 is neither the first nor the last element of S (no such elements existing at all), there are elements in S related to s_1 in the same order (either succeeding or preceding s_1) as r_2 is related to r_1. Among all those elements of S take the element which appears first in the sequence (s_k), i.e. the one which has *the smallest index* k, say s_m, *and attach it to* r_2. For the sake of convenience we denote s_m, the mate of $r_2 \varepsilon R$ in S, by $s^{(2)}$, and write also $s_1 = s^{(1)}$, as this element is the mate of $r_1 \varepsilon R$.

The third step is to attach a mate $s^{(3)} \varepsilon S$ to $r_3 \varepsilon R$; we begin by considering the different possible order-relations between r_3, r_1 and r_2. If $r_1 \prec r_2$, there are three mutually exclusive possibilities:

$$r_3 \prec r_1 \prec r_2 \quad , \quad r_1 \prec r_3 \prec r_2 \quad , \quad r_1 \prec r_2 \prec r_3;$$

the case $r_2 \prec r_1$ is analogous. Certainly there exist (even infinitely many) elements in S which stand in the same order-relations to $s^{(1)}$ and $s^{(2)}$ in which r_3 stands to r_1 and r_2. For, by virtue of property c, S contains elements which precede both $s^{(1)}$ and $s^{(2)}$, as well as elements which succeed them both; regarding the second of the three possibilities just mentioned, S also contains elements situated between $s^{(1)}$ and $s^{(2)}$, by virtue of property b. Among all the elements of S which stand in the respective relations to $s^{(1)}$ and $s^{(2)}$, take the element s_k with the *smallest index* k, attach it to $r_3 \varepsilon R$ and denote

it by $s^{(3)}$. If, for instance, the relations $r_3 \prec r_1 \prec r_2$ hold in R, their mates in S stand in the relations $s^{(3)} \prec s^{(1)} \prec s^{(2)}$, on account of the second and third steps. The same holds, *mutatis mutandis*, in other cases. In general, we may express this as follows: by attaching $s^{(k)}$ to r_k for $k = 1, 2, 3$, we have established a similar representation between the (ordered) subsets of R and S containing the elements r_1, r_2, r_3 and $s^{(1)}, s^{(2)}, s^{(3)}$.

Now we want to carry out mathematical induction with regard to the elements of the sequence (r_k) which exhausts the set R. To this end, we assume that for a definite natural number k, the elements $r_1, r_2, \ldots, r_k$ of R have already got mates $s^{(1)}, s^{(2)}, \ldots, s^{(k)}$ attached to them by a one-to-one correspondence which is *similar* in the sense explained. We shall prove that to the next element r_{k+1} of the sequence R, a certain mate $s^{(k+1)}$ in S can also be attached such that our similar representation will include r_{k+1} and $s^{(k+1)}$. Once this is proved, we shall be sure that the process of creating a similar representation can be extended to any number of steps and *will include all the elements of the sequence* (r_k), i.e. of the entire set R [1]).

To effect this transition from the kth to the $(k+1)$st step, we first consider the order-relations between the new element r_{k+1} and the old elements $r_1, r_2, \ldots, r_k$ of R. There are definite relations between them; either r_{k+1} precedes, or succeeds, all the other k elements, or it lies between (next) two of them, say r_m and r_n. As in the earlier steps, we are sure — either on account of property c or of property b of S — that there exist elements in S which are in the same relation to the elements $s^{(1)}, s^{(2)}, \ldots, s^{(k)}$, or to $s^{(m)}$ and $s^{(n)}$ respectively. Among those, we again choose the one with *the smallest index* in the sequence (s_k), denote it by $s^{(k+1)}$ and extend our representation by *attaching* r_{k+1} *and* $s^{(k+1)}$ *to each other*. In view of the choice of $s^{(k+1)}$ we have thus established a similar representation between the subsets (of the ordered sets R and S) which contain the elements r_ν and $s^{(\nu)}$ respectively, for

[1]) For this conclusion, the advanced reader should compare what is said in § 10, 2 about the *inductive procedure* in arithmetic and set theory. Strictly speaking, our present procedure shows only the existence of a similar representation (on a subset of S) for *any finite initial* of the sequence R. The representation of the *entire set* R will then be carried out by attaching to any element $r \varepsilon R$ its mate, which is the same for every initial containing r.

$\nu = 1, 2, \ldots, k, k+1$, in accordance with the aim envisaged in the preceding paragraph.

It might look as though the proof of theorem 1 were now completed. This cannot be true. For, while with respect to S the properties a—c have been used in our proof, the properties were left idle as regards R, except for the enumeration of the elements. This hint at the necessity of using b and c with respect to R is confirmed by the observation that what has been proved so far is that the entire ordered set R is similar to a *subset* of S. By attaching suitable elements $s^{(k)}$ of S to all elements r_k of R, we have not excluded the case that there may remain elements in S without any mate in R. So it still remains to show that our process *exhausts the set S too*, i.e. that the subset of S which has been represented on R by our construction, coincides with S itself. This final part of the proof has a somewhat abstract character [1]).

The element $s_1 = s^{(1)}$ of S, being the first element of the sequence (s_k) which contains all elements of S, has been used in the first step. According to mathematical induction, it will therefore suffice to prove the following assertion: if the elements $s_1, s_2, \ldots, s_k$ of S are used in our process of attaching elements of S to those of R, the element $s_{k+1} \varepsilon S$ is also used.

Since in general s_k will be different from the element of S chosen by the kth step of our process (which has been denoted by $s^{(k)}$), we may need more than k steps in order to use all the elements $s_1, s_2, \ldots, s_k$ — say l steps, which furnish the elements $s^{(1)}, s^{(2)}, \ldots, s^{(l)}$ with $l \geqq k$. It may happen that, in addition to $s_1, s_2, \ldots, s_k$, which occur among those elements by our assumption, s_{k+1} also occurs among them; in this case nothing remains to be proved. If not, we show that one of the following steps will use s_{k+1}, attaching it to a certain element of R. To this purpose we consider the order-relations which actually hold between s_{k+1} and the elements $s^{(1)}, s^{(2)}, \ldots, s^{(l)}$.

Certainly there exist in R elements (rational points) which stand in the same order-relations to $r_1, r_2, \ldots, r_l$, as s_{k+1} stands to $s^{(1)}, s^{(2)}, \ldots, s^{(l)}$; this follows, as the analogous statement above, from the fact that R fulfils b and c. Let $r_{l+\nu}$ be the element with the *smallest index* among those elements of R. Then $s^{(l+\nu)}$ — which will prove to coincide with s_{k+1} — automatically *has the same order-relations to the $l + \nu - 1$ elements*

$$s^{(1)}, s^{(2)}, \ldots, s^{(l)}, \ldots, s^{(l+\nu-1)}$$

of S, as $r_{l+\nu}$ has to the $l + \nu - 1$ elements

$$r_1, r_2, \ldots, r_l, \ldots, r_{l+\nu-1}$$

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[1]) We may avoid this last part of our proof by *alternating* our process of construction between the sets R and S, i.e., by also attaching elements of R to the successive elements of S.

of R. In other words, the $\nu-1$ elements $s^{(l+1)}, \ldots, s^{(l+\nu-1)}$ all appear at places different from the place where $s^{(l+\nu)}$ appears between two consecutive elements out of $s^{(1)}, s^{(2)}, \ldots, s^{(l)}$ (including the place before, or after, all of them). For if this were not the case, then in view of the *similar* correspondence established between the elements r_n and $s^{(n)}$ with $n < l + \nu$, at least one of the elements $r_{l+1}, \ldots, r_{l+\nu-1}$ would intervene at the place (with regard to $r_1, r_2, \ldots, r_l$) where $r_{l+\nu}$ stands, and this would contradict the characterization of $r_{l+\nu}$ at the beginning of this paragraph, namely, it being the r_n with the *smallest* index n that has the respective order-relations to $r_1, r_2, \ldots, r_l$.

Now, in view of the procedure used in the first part of the proof for defining the mate $s^{(k)}$ of the rational point r_k, s_{k+1} is to be attached to $r_{l+\nu}$; i.e. $s_{k+1} = s^{(l+\nu)}$, where $l \geqq k$, $\nu \geqq 1$. This shows that, as well as the elements $s_1, s_2, \ldots, s_k$, also s_{k+1} will eventually be used for our representation. In other words, a similar representation of R on S in full, not only on a subset of S, has been constructed. Hence $S \simeq R$. Q.E.D.

It has already been pointed out that, according to theorem 1, the set of *all* rational points and the set R of the rational points between two arbitrary (rational or irrational) points have the type η. But there are much more general sets which have the three properties in question, and therefore the type η; for instance, the set arising from R by the omission of one point, or of a finite number of points, or the ordered set of all rational points x outside a certain segment, i.e. of the rational points x satisfying the inequalities $x < c$ and $x > d$, where c and d are different real numbers with $c < d$ (cf. fig. 12 on p. 218). The type of the latter set remains unaltered when we add one of the extremities c and d — but not both of them since they would constitute consecutive elements, in contradiction to property b.

Other examples of sets with the type η may be obtained by arithmetical operations. We have, for instance (n denoting any finite type $\neq 0$)

$$\eta \cdot n = \eta, \quad \eta \cdot \omega = \eta, \quad \eta \cdot \eta = \eta.$$

To prove such equalities, we only have to show that suitable sets with the respective left-hand types possess the three properties in question. (In the last cases, property a follows from $\aleph_0 \cdot \aleph_0 = \aleph_0$.) As a representative of $\eta \cdot \omega$ we may choose the set of all positive non-integral rationals, decomposed into the segments from $n-1$ to n where n runs over all natural numbers. See also exercise 3 on p. 239[1]).

[1]) For the construction of representations between two sets of the type η

The question arises, what generalization of theorem 1 develops by the elimination of one of our properties, or even by the restriction to the property a *or* b alone.

As to the elimination of c, the state of affairs is quite simple. If the set S is not open, but retains the properties a and b, there only remain the alternatives that it has extremities on both sides or one of them. By dropping these extremities, we return to the case of theorem 1. Therefore, using the operation of ordered addition, we obtain the following result:

THEOREM 2. There are only four different types of denumerable dense ordered sets, viz.

$$\eta,\ 1+\eta,\ \eta+1,\ 1+\eta+1.$$

By dropping one of the properties a) and b) — in which case the property c may be neglected as being of only secondary importance — we find a much more complicated situation which gives rise to manifold possibilities. In a purely quantitative sense the set of *all denumerable types* has already been considered on p. 202. For some elementary properties of denumerable and of dense types, cf. exercises 4 and 6 on p. 239 f.

4. The Type λ of the Linear Continuum. The types ω and η are surpassed in historical and philosophical importance by another order-type, viz. that of the *linear* or *one-dimensional continuum.* In other words, the type of an interval of points on the straight line, or the ordered set of all real numbers between two given numbers. Any such set is a subset of L, and the set L itself, containing *all* points of the line, shall not be excluded. As in 3, we shall, in the case of a finite interval, exclude its ends.

In mathematics, the linear continuum plays a decisive rôle, in analysis (for instance, as an argument-set for functions of a real variable) no less than in geometry. For more than two thousand years, philosophers and theologians have tried to get to the bottom of the linear continuum which is the substratum of continuity in time as well as in space. Yet up to the days of Cantor, no one had succeeded in characterizing this set completely with respect to

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by means of continuous, monotonic and even analytic functions, see Franklin 1. This essay also deals with the analogous problem with regard to the theorem on p. 228.

its structure. (To describe it as being "continuous" or "unbroken" is of no use, as will become evident from the following considerations.) The firm belief developed that a complete description of the continuum was impossible, this concept being primitive and incapable of any further logical or mathematical analysis. Certain philosophical and even mathematical schools proposed to base the concept of continuum upon that of time — an extra-logical and non-mathematical concept — and there were even tendencies to shift the problem to the domain of mysticism [1]).

At first sight it might look as if (infinite) *density* were characteristic of the continuum. This opinion is erroneous, as is already evident from the set of all rational points (p. 221) which is dense yet certainly not continuous, even having a cardinal smaller than $\aleph$. Obviously, this set is *interspersed* among the linear continuum but, as it were, in "infinite rareness".

[1]) For its historical and philosophical aspects, see Cantor 7, V, § 10. I quote the following passage which is preceded by a description of the development of the problem throughout antiquity and the Middle Ages.

"... We here perceive the *medieval-scholastic origin* of the view that the continuum is an indivisible concept or, as others express it, a pure aprioristic intuition which would hardly be capable of a definition by other concepts; any attempt at *determining this mystery* arithmetically is considered as an illicit encroachment and is emphatically rejected. People of a timid disposition are thus getting the impression that the continuum constitutes not so much a *logico-mathematical concept* as a *religious dogma*."

"Far be it from me to arouse these controversies again, all the more since in this limited frame, I would not be able to discuss them in sufficient detail; I only feel bound to develop the concept of the continuum as logically sober as I have to conceive it and as I need it in the theory of aggregates, in utmost brevity and with respect only to the *mathematical* theory of sets. This attitude has not been easy for me, since among the mathematicians to whose authority I like to refer, none has treated the continuum in the sense which I need here."

To a large extent Cantor's treatment of the problem is, of course, based on the methodological ideas introduced into analysis by Cauchy, Bolzano, Weierstrass and others. By 1880 these ideas had been widely adopted.

For the connection between the problem of continuity and empiric knowledge, cf. B. Russell 7, ch. V.

More recently, especially since the second decade of this century, new serious doubts have been raised concerning the notion of continuum (or the set of all real numbers) as a closed totality. These doubts are still under discussion; cf. the exposition in *Foundations*, ch. IV. The relation of the problem to external experience has somewhat been affected by the theory of quanta, and there are certain ties between this theory and modern logic.

We come nearer the truth by guessing that the notion of a continuous set, in the sense of definition III on p. 216, is fundamental to the nature of the continuum. As a matter of fact, in the foundations of geometry as well as of analysis, the concepts of point and number have been clarified in this way (cf. the footnote on p. 215) and before Cantor nobody seemed to have doubted the sufficiency of this notion for describing the linear continuum. As we shall soon realize, Cantor's supreme achievement was the discovery that an additional property — viz. β in the following theorem — is required.

Let us begin by considering the open interval I containing all those points of L which lie between two arbitrary points a and b. I is a continuous set, as has been pointed out on p. 217, example 3. Furthermore, the subset R of I containing the *rational* points only, is denumerable and, as proved in the footnote 2 on p. 219, has the property that between any two points of I there is always a point of R. Finally, I being open, the extremities a and b do not belong to I.

We shall now prove that these three properties *completely characterize* I.

THEOREM 3. Let C be an abstract ordered set with the properties:

α) no cut in C is a gap.

β) C has a denumerable subset D such that between any two different elements of C there is at least one element of D. In short, C densely includes a denumerable subset D [1]).

γ) C is open, i.e. without extremities.

Then C is similar to the linear continuum I between arbitrary points a and b, as described above. In other words, the properties α to γ completely determine an order-type, which is denoted by λ and called the *type of the open linear (one-dimensional) continuum.*

This theorem says in particular that the type λ of I is independent of the chosen extremities a and b, and that λ is at the same time

[1]) It is easily seen (cf. exercise 9 on p. 61) that in any ordered set having the property β there are at most denumerably many "non-overlapping intervals" (in the sense of mere order). On the other hand, M. Souslin has put the question (*Fundamenta Mathematicae*, volume 1, p. 223; 1920) whether the latter property, stated in place of β, causes an open continuous set to be of the type λ; this question has not yet been answered.

the type of the entire set L, i.e. of the ordered set of *all* points on a straight line.

Note that the properties enjoined on C are to a certain extent analogous to the properties a—c which determine the type η. As to β, this property implies that C is *dense*, since every element of D at the same time belongs to C. For this reason α only demands that C be without gaps; being dense, C is at the same time without jumps, hence every cut in C is continuous (p. 216). Accordingly, C is a continuous set.

In his proof of our theorem [1]), Cantor demanded another property (being perfect) instead of α. This property is introduced in **5** in a slightly different context. However, the property α used here — essentially continuity — is more convenient, as is the concept of "cut" compared with the "fundamental series" used by Cantor in accordance with his theory of irrational numbers (cf. p. 215).

Proof (in short). First, the subset D of C has the type η, just as the set R of all rational points of I. For in virtue of β, D is not only denumerable but also dense since the "different elements of C" mentioned in β, may be chosen in particular as elements of D. Moreover, we easily conclude from β and γ that D is an open set. Hence, by virtue of theorem 1 on p. 221, D is similar to the subset R of I. We choose *a definite representation φ between D and R*, by virtue of which every element of D has a uniquely determined mate in R, and vice versa.

Let now c be any element of C not belonging to D, and let D_1 be the subset of D containing all elements of D which precede c in C, D_2 the subset containing all elements of D which succeed c in C. $(D_1|D_2)$ is a cut — more precisely, a gap — in D, and in view of φ there is a uniquely determined cut $(R_1|R_2)$ in R corresponding to $(D_1|D_2)$. The cut $(R_1|R_2)$, which is easily seen to be a gap in R, uniquely determines a point (real number) i of I which succeeds the elements of R_1 and precedes those of R_2 (i.e. which fills the gap); the existence of i is a consequence of the continuity of

[1]) Cantor **12**, I, § 11. Cf. also Kuratowski **3** (with respect to the property β; notably p. 214) and Webber **1**. The procedure of K. K. Chen **1** is incomplete.

An exposition in detail of the types ω, η, λ is given in Huntington **4**. Cf. Veblen **2** and (for the relation "between") Pitcher–Smiley **1** and Transue **1**.

I. The (irrational) point i shall be related to the given element $c \varepsilon C$. Evidently this rule creates a one-to-one correspondence between the elements of $C - D$ and elements of $I - R$ (which by the way can easily be extended so as to comprehend the representation φ too, i.e. as to represent C on a subet of I).

Finally, the representation created by our rule together with φ, *is similar*. This immediately follows from the nature of our representation since, for instance, any element of C preceding c — i.e. belonging to the initial of a cut in C which is produced by c — determines as its mate a point of I preceding the mate i of c.

By starting from I instead of C, the representation is easily seen to include *all* elements of $I - R$. Q.E.D.

As to order-types created from λ by simple operations, let us first remark that $1 + \lambda + 1$ is the type of the *closed linear continuum* (denoted by Cantor with θ). It is obvious that such types as $\lambda \cdot n$, $\lambda \cdot \omega$, $\theta \cdot n$, $\theta \cdot \omega$ (n finite, larger than 1) are different from λ and θ, as well as from each other, for $\lambda + \lambda$ has a gap "in the middle", and $\theta + \theta$ a jump. λ (or θ) is also turned into a different type by omitting any single element — in contrast with η; the omission evidently creates a gap.

It is easily seen that also $\lambda \cdot \lambda \neq \lambda$, since $\lambda \cdot \lambda$ has gaps. Much more interesting is a related *continuous* type which nevertheless differs from λ. To obtain a simple instance (which is a "simply ordered" set though not "linear" in the sense of constituting a subset of L; cf. p. 213), consider the square $OXZY$ in fig. 11 (p. 188) to be the set of all square-points, including the points situated on the square sides with the exception of the two vertices O and Z only, and order the set in the same way as was done there, by arranging the points according to increasing abscissae, and for equal abscissae according to increasing ordinates. This set is continuous and open. It does not, however, possess the property β; for if each subset of points with the same abscissa contained but a single element of the supposed subset D, this subset would have the cardinal $\aleph$ and not $\aleph_0$.

Instances like this one emphasize the importance of the property β, discovered by Cantor.

5. Accumulation Point and Related Concepts. While the definitions of 1, and accordingly the examples and applications in

2—4, have been based on the notion of an *abstract ordered set* as described in § 8, we shall now introduce notions of a different kind, characteristic not of abstract sets but of "*spaces of points*". The notion "space", it is true, may be conceived in a very general sense. In this and the following subsection, however, our intention is merely to indicate some simple concepts that may thus be developed, and we limit ourselves to the ordinary "metrical" space[1]), even to *one-dimensional* space; here, accordingly, the restriction to a line — conceived as the set of all its points — and to its subsets is essential and not, as above, incidental. As a primitive concept we take the *distance* between two points, conceived in the usual way; however, it may be defined — or introduced axiomatically — even in general "abstract" spaces where distance, in the ordinary sense, does not appear.

On the other hand, the concept of order will be disregarded in principle. This would be still more obvious if we were to include sets of points in the plane or in three-dimensional (or multi-dimensional) space. The reader will easily perceive that the definitions given below also admit of such extensions which, however, will not be stated, for the sake of brevity.

It should be stressed that, on account of our attitude, the concepts introduced in definition VI have a meaning somewhat different from the one they would have in the abstract theory of ordered sets; the latter being the context in which Cantor actually defined them (for instance, for determining the type λ; see 4). In this subsection they are to be understood as concepts of a *relative* nature taken with regard to the whole space in question (here to the set of all points on a line)[2]). To give an idea of the difference between these concepts, a simple instance will be sufficient: the set of points x defined by $0 < x < 1$, $3 < x < 4$, with the addition of the point $x = 2$. According to the definitions of 1 this set is dense and continuous and has the order-type λ. In the light of the present subsection, however, $x = 2$ would be an isolated point in whose neighborhood no other point of the set is found.

[1]) In other words, to the set of all triplets of real numbers (x, y, z). Their place is taken by n-tuples $(x_1, x_2, \ldots, x_n)$ in the case of the n-dimensional metrical space.

[2]) Considering the result of 4, however, we might even present the concepts and theorems conserning point-sets in a one-dimensional metrical space within the theory of abstract sets. Cf. Haar–König 1.

Nevertheless, in general the difference between the present and the previous treatment is rather a difference in principle than a practical one, owing to the restriction to subsets of L made in **1**—**4**. The reader will easily interpret most of the examples of **6** in the light of the definitions given in **1**. For the *applications* of the theory, the present attitude is much more useful, and that is why it is presented here, if only in the form of a few hints.

Henceforth we denote by L the plain set of all points of a straight line, regardless of an order-relation; by "set (of points)" we mean a subset of L containing at least two points.

DEFINITION IV. Given a point p of L, we call *neighbourhood of* p (more strictly: d-neighbourhood of p) the subset of L containing those points whose distance from p is less than a certain positive real number d [1]).

DEFINITION V. Given a subset K of L, a point p of L (not necessarily of K!) is called an *accumulation point of* K if in "the" neighbourhood of p (i.e. in any d-neighbourhood of p, however small d may be chosen) there is a point of K different from p.

The set of all accumulation points of K is called the *derived set* of K [2]) and is denoted by K'.

A point of K which is not an accumulation point of K is called an *isolated point* of K.

In any neighbourhood of an accumulation point p of K, there are, not only one, but *infinitely many points of* K. For, if p_1 is a point within the d_1-neighbourhood of p (d_1 arbitrary), and if d_2 is the distance between p and p_1, there is by our definition again a point p_2 (different from p_1) within the d_2-neighbourhood of p; in general, if d_k denotes the distance between p and p_{k-1}, the d_k-neighbourhood of p contains a point p_k which differs from all points $p_1, p_2, \ldots, p_{k-1}$ in view of $d_1 > d_2 > \ldots > d_k$. Hence all points of the

[1]) We might define neighbourhood without the metrical concept of distance, as an open interval containing p; in principle, this is even preferable. The present way, which is less abstract, has been taken since it is more convenient for the limited scope of this section.

For a penetrating treatment of the subject and, in particular, of the concept of neighbourhood, see Hausdorff **4**, p. 213 ff. or **5**, p. 228 ff.

[2]) This operation of "deriving" a set may be repeated any number, even a transfinite number, of times. In the early development of the theory of sets this access to transfinite (ordinal) number has played an important rôle.

sequence $(p_1, p_2, \ldots, p_k, \ldots)$ are in the d_1-neighbourhood of p [1]).

An accumulation point p of K may have the property that there are only denumerably many points of K in the neighbourhood of p. This property holds, for example, for any point of L if K is the set of all rational points. If there are non-denumerably many points of K in the neighbourhood of p, p is called a *condensation point* of K. So a condensation point of K is *a fortiori* an accumulation point of K. If K is an open interval, all points of K as well as its end points (though they do not belong to K) are condensation points of K.

The last example clearly shows that an accumulation point of K is not necessarily a point of K. Another instance is the set $K = \{1, 1/2, 1/3, \ldots, 1/k, \ldots\}$, for which the point 0 is an accumulation point but not an element. Every point of this set is itself an isolated point of K. Obviously, an isolated point i of K may also be characterized by the property that there exists a positive number d such that in the d-neighbourhood of i there is no other point of K.

DEFINITION VI. A set K is called *dense-in-itself* if every point of K is an accumulation point of K; *closed*, if every accumulation point of K belongs to K; *perfect*, if K is dense-in-itself and closed, i.e. if K coincides with the set of its accumulation points.

Using the middle part of definition V, we may say: K is dense-in-itself if $K \subseteq K'$, closed if $K' \subseteq K$, perfect if $K' = K$.

Of course, a set need not have any of these properties; this is shown by the set $K = \{1, 1/2, 1/3, \ldots, 1/k, \ldots\}$ just used, for which $K' = \{0\}$.

It should again be stressed that these definitions are built on a foundation quite different from that of the definitions of **1**. Therefore, a set which is dense-in-itself, is not necessarily dense (cf. example 4 in **6**) and vice versa, e.g. the set described on p. 231 $(0 < x < 1,\ x = 2,\ 3 < x < 4)$.

6. Examples. 1. Let again (as in example 1 on p. 216) K be the *open* interval between two different points p_1 and p_2. Every point of K is an accumulation point of K, while p_1 and p_2 are

[1]) We can as well prove the statement indirectly by supposing that only a finite number of points of K are in the neighbourhood and by using the smallest distance between p and one of these points.

accumulation points of K without being points of K. Hence $K \subseteq K'$, but not $K' \subseteq K$; K is dense-in-itself, but not closed and hence not perfect.

On the other hand, the *closed* interval between p_1 and p_2, containing both these points, is also closed according to definition VI, and therefore perfect.

That the accumulation points of the set belong to the set, is a consequence of the geometrical concept of "point of a line".[1] To be sure, from the *geometrical* point of view this property of a line is not a provable theorem but part of the axiomatic definition of a line and its points (cf. footnote 1 on p. 213), no matter how this definition is expressed: in terms of cuts, or of accumulation points and limits, or of another concept (e.g., Hilbert's axiom of "completeness"). As to the analogous problem in arithmetic and analysis, which deals with the set of all real numbers, after an appropriate "genetic" introduction of irrational number by means of rationals it becomes provable that the set of all real numbers between two given numbers, including its end points, is closed [2]).

2. Denoting, as in example 2 on p. 216 f., by H the set of all *rational* points between any two different points p_1 and p_2 of L, we obtain a set which is dense-in-itself, since in the neighbourhood of a rational point there are (infinitely many) rational points; in other words, every point of H is an accumulation point of H. However, *not all* accumulation points of H belong to H; more precisely, in contrast to the fact that only $\aleph_0$ accumulation points belong to H (all elements of H!), there exist $\aleph$ accumulation points which are not elements of H, namely all irrational points of the interval from p_1 to p_2 and the two end points. For if p is any irrational point, p may be approximated by a sequence of rational points (e.g. those corresponding to the terminating decimals which successively occur in the decimal expansion of p), such that p is an accumulation point of the points of the sequence. If p is the point denoted by $\sqrt{2} = 1.4142\ldots$, the set of rational points $\{1, 1.4, 1.41, 1.414, 1.4142, \ldots\}$ has the accumulation point $\sqrt{2}$, which does not belong to H.

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[1]) This paragraph is for advanced readers only.

[2]) In *Foundations* it will be explained in what sense the axiomatic method is "appropriate" in geometry, and the genetic method in arithmetic.

So the set H, while being dense-in-itself, is not closed — not even after the admission of the extremities p_1 and p_2. The expression "closed" thus becomes intelligible. H is not closed since one can still add to its elements all accumulation points of H, i.e. all irrational points of the interval and its extremities; after the inclusion of these points the set will be closed.

3. Let I be a set of points of the line such that each point has the same distance d from its neighbours on both sides; e.g., the set of all "integral" points $0, \pm 1, \ldots, \pm k, \ldots$, for which $d = 1$. The set certainly is not dense-in-itself, all its points being isolated points. Nevertheless the set is *closed*; for, since I has no accumulation point, the derived set I' is the null-set, which is a subset of any set, hence of I. (This interpretation of definition VI may at first appear surprising but it is quite natural, for by including the accumulation points of I we are not adding anything.) For the same reason any set that has no accumulation point, is closed; in particular any *finite* set F of points, for in this case the distance between any two points of F has a fixed positive minimum d, and if we take $d_1 \leqq d$ then the d_1-neighbourhood of any point of F contains no other point of F.

4. Finally let us consider a set which appears somewhat paradoxical; it is generally known by the name of *Cantor's linear perfect nowhere-dense set*, or Cantor's discontinuum — though as early as 1875 (a few years before Cantor's publication) it had already been discovered by H. J. St. Smith [1]). To explain the name, we must first note that a set of points (subset of L) is called "*nowhere-dense*" if there is no interval of L wherein the set is dense (definition I on p. 214); in other words, if between any two different points of the set *there is an interval of L which contains no point of the set.* A nowhere-dense set evidently has no subset that is an interval. The set in question has many important properties which throw light on more general sets of points; we shall, however, content ourselves with briefly sketching the most elementary conclusions.

Let us define a simple instance of Cantor's set from which one may proceed to more general types. We start with the closed interval from 0 to 1 (fig. 13) and blacken the middle third of the interval, i.e. the interval from 1/3 to 2/3 (first step). There remain

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[1]) Smith 1, pp. 147—148; Cantor 7, V, p. 590.

two intervals which are not blackened: from 0 to 1/3, and from 2/3 to 1; blacken the middle third of both, i.e. the intervals from 1/9 to 2/9, and from 7/9 to 8/9 (second step). In each of the remaining four intervals (from 0 to 1/9, from 2/9 to 1/3, etc.) blacken again the

Fig. 13

middle third, i.e. the intervals from 1/27 to 2/27, from 7/27 to 8/27, etc. (third step). And so forth. After the kth step there will remain 2^k not-blackened intervals, in each of which the middle third is to be blackened at the $(k+1)$st step. *The set C to be considered shall contain*, first, all *extremities* of black intervals, including the extremities 0 and 1 of the original interval; secondly, all points *not contained in any black interval* — i.e. not belonging to an interval blackened at the kth step however large k may be. (The second class is the *meet* of all sets $C^{(k)}$, where $C^{(k)}$ contains the points which do not belong to a black interval at the kth step, with k running over all natural numbers.)

To perceive that C is *nowhere-dense*, let p_1 and p_2 be any different points of C. If p_1 and p_2 are the extremities of the same black interval, then there is no point of C between p_1 and p_2; if they are not, there is at least one black interval (one extremity of which may be p_1 or p_2) between p_1 and p_2. (Actually, there are infinitely many black intervals between p_1 and p_2.) Any such (open) interval contains no point of C. Hence C is nowhere-dense.

Nevertheless, *C is dense-in-itself*. For if a point c of C does belong to a black interval, it is one of its extremities. Then on the side of c which is *opposite* to that interval — and if c is no such extremity, on *both* sides of c — there will be a black interval within any neighbourhood of c, as we easily infer from the definition of C. In other words, in the neighbourhood of c there are other points of C, even points of the subset of C which contains the extremities of black intervals only. (This subset, as well as the set of all black intervals, is obviously denumerable.) Hence any point of C is an accumulation point of C; i.e. C is dense-in-itself.

It is quite simple to see how the set manages to be at the same time "dense-in-itself" and "nowhere-dense". In the neighbourhood of every point p of C there are indeed other points of C, even in-

finitely many of them. However, this possibly applies to only one side of p, while in the other direction p is "isolated", having a sequent or an immediate predecessor — namely if p is an extremity of a black interval. If p is not of this kind, then points of C exist on both sides in any neighbourhood, but there are also intervals void of points of C in any neighbourhood.

Finally, in order to show that C is also closed, and consequently *perfect*, we observe first that every accumulation point of C is situated on the closed interval from 0 to 1, since even the accumulation points of the *interval itself* belong to the interval (cf. example 1). Hence, to prove our statement we only need to show that the accumulation points do not belong to those parts of the interval which are *not contained in* C, i.e. are not situated inside black intervals. This, however, is evident since an inner point of a black interval has no point of C in its neighbourhood.

A simple arithmetical realization of the points of C, allowing many inferences to be drawn more easily than with the geometrical realization, is obtained by writing real numbers as *triadic fractions* (instead of decimal or dual fractions; cf. p. 62 and 155), i.e. taking 3 as base instead of 10 or 2. The digits of the fractions in this case are 0, 1, 2. Then the rule given above for constructing the set C means that, out of the three possibilities for choosing a digit at the kth place after the point ($k = 1, 2, 3, 4, \ldots$), *the digit* 1 *is always excluded*; for it refers to the "middle third" which is a black interval. Thus only the digits 0 and 2 are admitted. Hence C may be defined as the set of those points of the interval of unity which are marked *by triadic fractions wherein the digit* 1 *never occurs*. This determination duly includes the extremities of black intervals, in view of the fact that the corresponding numbers (save for 0) admit of *two* triadic representations, one terminating and one infinite — precisely as do the fractions the denominator of which is a power of 10, in decimal representation. The following are a few of the end points which appear in fig. 13, written in triadic script without using the digit 1:

$$\frac{1}{3^3} = 0.000222\ldots, \quad \frac{2}{3^3} = 0.002, \quad \frac{1}{3^2} = 0.00222\ldots, \quad \frac{2}{3^2} = 0.02,$$

$$\frac{1}{3} = 0.0222\ldots, \quad \frac{2}{3} = 0.2, \quad \frac{8}{9} = 0.22,$$

while the point 1 may be written as $0.222\ldots$.

From this representation of the points of C it immediately follows that C *has the cardinal* $\aleph$ *of the continuum* [1]). For, replacing every digit 2 by 1, we formally obtain, instead of the points of C, all (terminating and infinite) *dual* fractions, which constitute a set of the cardinal $\aleph_0 + \aleph = \aleph$ (cf. p. 155) [2]).

The non-denumerability of C is rather surprising, since the sum of the lengths of all black intervals is

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots = \frac{1}{3}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots\right) = \frac{1}{3} \cdot 3 = 1.$$

This sum accordingly equals the length of the unit interval itself; hence a reasonable generalization of the notion of length or *measure* would attribute to C the measure 0, as is actually done in the theory of measure. Thus a set of measure 0 may be not only infinite but equivalent to the continuum.

7. Concluding Remarks. Neither the *ordinal* considerations of **1**—**4**, closely following the ideas of Cantor, nor the "*spatial*" notions of **5** and **6** can give a full idea of the immense significance which the theory of sets of points has gained during the last fifty years in practically all parts of analysis and geometry [3]). Indeed, this theory has become one of the most important and most powerful branches of mathematics. Some of the examples in **2**—**4** and **6** may give a slight intimation of the power of these methods, which enable us to investigate delicate structures which are invisible to the eyes of "ordinary" geometry.

It should be pointed out that considerations of this type — especially those which are independent of the concept of order —

[1]) For a remarkable representation between the set C and the set of all denumerable types (which has the cardinal $\aleph$, p. 202), cf. Kuratowski **8**.

[2]) After having proved (which is not difficult) that any two linear perfect nowhere-dense sets with end points are *similar* (*a fortiori* equivalent), we immediately infer that *any* (*linear*) *perfect set of points has the cardinal* $\aleph$. For, such a set either has an interval as a subset in which case its cardinal is obviously $\aleph$, or it has no such subset in which case it is nowhere dense. In the latter case, if it has end points, the assertion follows from its similarity to C; if not, it has a subset with end points and the same properties.

[3]) With regard to the first half of this period, an instructive exposition is given in W. H. Young **1**; cf. also Schoenflies **4**. Since that time, there has been an enormous additional development, especially in geometry.

are not at all restricted to sets of points on (straight or curved) lines but have achieved even greater triumphs in two- and three-dimensional spaces as well as in spaces of more dimensions. They have also led to definitions of more general kinds of spaces, where space means an abstract set for the elements of which a suitable notion generalized from geometry (e.g., distance or neighbourhood) is defined [1]). For the analysis, it may be stressed that the modern theory of real functions is entirely based on sets of points and has, among other achievements, supplemented the "classical" (Riemannian) theory of integration with a number of other more general and penetrating theories, which are essentially based on the theory of sets. These subjects, however, are outside the realm of the present book [2]).

Exercises

1) Prove that the type ω is completely characterized by the following properties:

a) The set has a first, but no last element;

b) Every cut is a jump.

2) Prove that the type $^*\omega + \omega$ is completely characterized by the following properties:

a) The set has neither a first nor a last element;

b) Every cut is a jump.

3) Prove that $\eta \cdot \alpha = \eta$ for any *denumerable* order-type α.

4) Given any dense ordered set D and any denumerable ordered set A, prove that D has a subset similar to A. (Hence, in particular, any dense ordered set has a subset of the type η.)

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[1]) Cf., e.g., Fréchet 1 and the *Préface* to Fréchet 2.

[2]) Among the numerous textbooks and handbooks of this domain (except set-theoretical topology), the following may be mentioned:

Hausdorff 4 and 5 and de la Vallée Poussin 2, which were already quoted; and the excellent encyclopedical survey Rosenthal–Borel 1.

In English, in addition to Young–Young 1 (which has an outstanding historical significance as the earliest comprehensive textbook on this subject, having appeared in 1906): Graves 1; Hobson 2; R. L. Moore 1; Pierpont 1; Saks 1; Townsend 1.

In French: Baire 2; Borel 1—3, Lebesgue 1 and 2; Saks 1.

In German: Carathéodory 1; Hahn 3; Kamke 1; Schlesinger–Plessner 1.

In Italian: Vitali–Sansone 1.

5) Prove the mutual independence of the properties α, β, γ on p. 228 characterizing the type λ; i.e. show that any *two* of these properties do not characterize the type λ. (Essentially, the proof is contained in **4**.)

6) Show that any dense set can be rendered continuous by "filling" its gaps; cf. the end of **2**.

7) A set S of sets shall be called "monotonic" if of any two of its elements, one is a subset of the other. We may regard any such set as an ordered set; to this purpose we arrange its elements, for instance, according to "increasing" extent. Cf. § 11, **7**. Prove the following theorem [1]):

There exists a monotonic set of the order-type α the elements of which are *sets of natural numbers*, if, and only if, α is the type of a subset of the open linear continuum (i.e. a "subtype" of λ).

8) Consider the properties of the types $(\lambda + 1)\cdot(1 + \lambda + 1)$ and $(1 + \lambda + 1)\cdot(1 + \lambda + 1)$.

9) Show that any continuous ordered set has a subset of the type λ. (Cf. exercise 4.)

10) Determine the derived set K' and the "second derivation" $(K')' = K''$ of the following sets:

a) $K = \left\{0, \pm\frac{1}{2}, \pm\frac{2}{3}, \pm\frac{3}{4}, \ldots, \pm\frac{2k-1}{2k}, \ldots\right\}$;

b) K = the set of all terminating decimal fractions.

11) Consider the examples of **6** in the light of the notions introduced in **1** for abstract ordered sets.

§ 10. General Theory of Well-ordered Sets. Finite Sets

1. The Concept of Well-Ordered Sets. Among the ordered sets there is a class of sets, remarkable for the particularly simple arrangement of the elements, the so-called "well-ordered" sets. In the realm of well-ordered sets, an outstanding simplicity and transparency exists, which reminds us of the conditions prevailing in the realm of finite numbers. In fact, the kind of transfinite number represented by the order-types of well-ordered sets,

[1]) Sierpiński **10**.

possesses many of the properties of finite ordinals and produces a much more familiar impression than order-types in general, or even transfinite cardinals.

The importance of the theory of well-ordered sets, resulting from these properties, is enormously enhanced by the fact that their properties may be transferred to general (even non-ordered) sets. Already during the initial period of his researches, Cantor took it for granted that *any set can be well-ordered*, without hinting, however, at any method of doing so. The first proof of this assertion constituted one of the most dramatic events in the history of mathematics (and logic), and its implications are very distinctly felt up to this day (see § 11, 6). This achievement enables us to considerably simplify the theory of equivalence, and in particular to bridge the gap the existence of which was pointed out at the end of § 5. This state of affairs might even suggest that the theory of order should take precedence over the theory of equivalence and cardinals; there are, nevertheless, also strong reasons for the converse arrangement of the subjects, as followed in this book.

At the beginning of Chapter III, we started with the following reflection. While transfinite *cardinals* certainly constitute an important extension of the notion of finite number, this generalization does not exhaust the content of the ordinary number concept, for the finite numbers are also used as *ordinals*, i.e. for the process of counting. In order to do justice to this function of number, a set may be regarded not only as the totality of its members but also as expressing a certain order corresponding to the succession of members in the set (if there is any succession). Guided by this idea, we introduced transfinite *order-types* as an extension of finite ordinals.

The process of counting, however, involves special features which do not apply to transfinite order-types in general. Counting always begins with a *first* object, counted at the initial step; moreover, after any object or even any aggregate of such objects have been counted, there follows a *sequent* object, which is counted immediately subsequent to the objects already dealt with [1]).

[1]) On the other hand, the very existence of transfinite ordinals having both these properties shows that these two features in their general form do not exhaust the characteristics of finite number.

(In arithmetical language, counting begins with 1, and after having counted up to n, we continue with $n + 1$.) General ordered sets, however, do not boast either of these qualities. A set of the type η, for instance, has no first element, and no sequent exists to any of its elements. The first of these features it shares with the sets of the type $^*\omega$, for example; the second, with any dense set.

In order to obtain a closer correspondence to the ordinary process of counting, we should, therefore, consider a more restricted definition of the notion of ordered set and its type; in other words, we should add a further condition to those stipulated in the definition of the ordered set (p. 176). It was one of the earliest achievements of Cantor, and possibly his most important one, that he distinguished types of this kind [1]) — guided not only by the properties of the respective *aggregates* but even more directly by continuing the sequence of ordinal numbers "beyond the infinite", as in the case of continued construction of derived sets (cf. the footnote 2 on p. 232). The terms "well-ordered sets" and "ordinal numbers" were chosen by Cantor. More than any other idea in the theory of sets, these two concepts have profoundly influenced mathematics as a whole; not only analysis and geometry, but even arithmetic and algebra.

We begin with the following fundamental definition.

DEFINITION I. An ordered set S is called *well-ordered* if every non-empty subset of S has a *first element.* For practical reasons, the *null-set*, too, is called well-ordered.

Since S is a subset of itself, the condition in particular requires the existence of a first element in S.

The above definition has become the customary one while Cantor's original procedure corresponds to theorem 2 (see below). Logically, this latter procedure is superior to our definition which contains redundant ingredients. In fact, by postulating a first element in *all* non-empty subsets of S, one of the conditions for S being an *ordered* set is rendered superfluous: the condition of transitivity. A logically preferable, if somewhat clumsy, form of definition I would therefore read:

I*. A set S is well-ordered if

a) if s_1 and s_2 are two different elements of S, one and only one of the relations $s_1 \prec s_2$, $s_2 \prec s_1$ holds, but not the relation $s_1 \prec s_1$;

[1]) Cantor 7, IV and V, §§ 2, 3, 11; 12, II, § 12.

b) every non-empty subset S_0 of S contains an element s_0 such that for any element $s \varepsilon S_0$ different from s_0, the relation $s_0 \prec s$ holds.

Since a subset of a subset of S is again a subset of S, we immediately conclude from our definition, and the properties of similarity:

THEOREM 1. Any subset of a well-ordered set, and any ordered set which is similar to a well-ordered set, is itself well-ordered.

The simplest instances of well-ordered sets are the *finite* ordered sets and the *enumerated* sets, e.g. the set of all natural numbers in the ordinary succession. Of course, not every *denumerable* set is well-ordered — in fact, it need not even be ordered at all. *Denumerable ordered* sets, which are not at the same time well-ordered, are, for example, the set of all integers, positive and negative, and the set of all rationals, arranged according to magnitude; both of them lack a first element and possess proper subsets without first elements. It is clear, on the other hand, that not every infinite well-ordered set is enumerated; for instance, the set of all integers in the arrangement $\{0, 1, 2, 3, \ldots -1, -2, -3, \ldots\}$ and the set of all positive rational numbers in the arrangement

$$\left\{1, 2, 3, \ldots \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \ldots \frac{1}{4}, \frac{3}{4}, \ldots\right\}$$

are well-ordered, though not enumerated. Another example of a non-enumerated well-ordered set is the set of all integers larger than 1, ordered so that the prime numbers precede the products of two primes, and these in their turn the products of three primes etc., the products of the same number of primes being arranged among themselves according to magnitude.

All these instances of infinite well-ordered sets are denumerable. Examples of non-denumerable well-ordered sets have not yet been given in this book, and their explicit mention is deferred to § 11, **5**.

In addition to the definition of well-ordered sets given above, we shall now formulate *three other equivalent definitions*. They will be set forth in the form of propositions — theorems 2, 4, 5 — stating that a set is well-ordered if, and only if, a certain condition is satisfied.

While the first feature of counting, mentioned above (p. 241), was emphasized in definition I, its second feature is stressed by

THEOREM 2. An ordered set S is well-ordered if, and only if,

for any subset $S_0 \subset S$, for which there are succeeding elements in S [1]), there exists an *immediate successor* $\bar{s}_0 \varepsilon S$ (*sequent of* S_0) such that S contains no element that succeeds every element of S_0 and precedes $\bar{s}_0$ [2]).

In order to understand this proposition thoroughly, let us first consider the particular case in which the subset $S_0 \subset S$ under discussion, contains *one element* s_0 *only*. Then s_0 is possibly the last element of S; in this case the condition of theorem 2 is "vacuously" fulfilled by S_0, for since there is no $x \varepsilon S$ succeeding s_0, S_0 does not possess the property required in theorem 2. If, however, s_0 is *not* the last element of S, i.e. if there exist elements $x \varepsilon S$ which succeed s_0, the condition requires one among these elements to be the immediate successor to s_0.

In general, if S_0 contains more than one element, the first alternative mentioned above corresponds to S_0 being "confinal" with S, so that no $x \varepsilon S$ succeeds all $x \varepsilon S_0$. While under these circumstances our condition does not require anything of S, in all other cases it requires the existence of an immediate successor to S_0. It is evident that this successor is uniquely determined. If, in particular, S_0 is the null-set, $\bar{s}_0$ is the first element of the entire set S.

The examples of p. 243 show that in general (cf. **5**) a well-ordered set does *not* satisfy the condition stipulated in theorem 2 if "succeed" is replaced by "precede" and accordingly "sequent" by "immediate predecessor". A similar observation applies to definition I ("last" instead of "first").

Proof of theorem 2. a) If S is well-ordered, let $\overline{S}_0$ be the set of all $y \varepsilon S$ which succeed *every* $x \varepsilon S_0$. As we need only consider the case of $\overline{S}_0$ not being empty, $\overline{S}_0$ does by definition I contain a first element $\bar{s}_0$, which is the sequent of S_0.

b) If the condition of theorem 2 is satisfied, let S' be *any non-empty subset of* S, and let S_0 — which is possibly empty — be the subset of S which contains all $y \varepsilon S$ preceding *every* $x \varepsilon S'$.

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[1]) I.e., a set S_0 such that some element of S succeeds *all* elements of S_0. In general, "an element $x \varepsilon S$ precedes (succeeds) a subset $S^* \subset S$" means "x precedes (succeeds) every element of S^*". Since our property certainly does not apply to S itself, S_0 may be assumed to be a proper subset of S.

[2]) The case where S is the null-set is included since then the property required of S_0 is never realized.

Then, on account of the condition stipulated in theorem 2, there exists in S a sequent $\bar{s}_0$ to S_0. $\bar{s}_0$ certainly belongs to S' and is the *first* element of S'. Hence, S' has a first element, i.e. S is a well-ordered set. Q.E.D.

If, retaining the notation of theorem 2, we denote by S_1 the set of all $x \varepsilon S$ which succeed S_0, S_1 is a remainder of S (see d) on p. 182). Hence, theorem 2 requires *any non-empty remainder of S to have a first element* (the sequent of S_0). In definition I the same property was demanded of *every non-empty subset* of S. In this respect the definition given by theorem 2 is weaker than, hence preferable to, definition I.

In order to obtain a modification of the condition contained in theorem 2, we introduce a simple definition which might have found its place in § 8, 2.

DEFINITION II. If S is an ordered set and x one of its elements, the subset X of S containing all elements *preceding* x, is called a *section* of S;[1]) more precisely, *the section of S determined by x.*

Hence, if S has a first element a, the section determined by a is the null-set.

From this definition we immediately conclude:

THEOREM 3. A section of a section of an ordered set S is again a section of S; and conversely, of two different sections of the same set S, one is a section of the other.

The first statement is self-evident. As for the second, if x_1 and x_2 are the elements determining the sections X_1 and X_2 respectively, and $x_1 \neq x_2$, then X_1 is obviously a section of X_2.

Evidently, any section of S is an initial (p. 182) of S. *The converse does not apply.* If we arrange the natural numbers, for example, in the following order of succession

$$\{1, 3, 5, \ldots \ldots, 6, 4, 2\},$$

the subset containing all odd numbers is an initial; there is, however, no element which determines this subset as a section. As another example we consider a line L as the ordered set of its points from left to right. If p is any point on L, the set containing p and all points to its left, is an initial, but not a section of L.

[1]) Some authors use "segment" instead of section.

Yet, this difference between the notions of initial and section almost disappears, when the set is well-ordered. We prove:

THEOREM 4. An ordered set S is well-ordered if, and only if, every initial of S — *except* S *itself* — is at the same time a section of S.

Proof. On the one hand, we have to show that to any *proper* initial I ($I \neq S$) of a well-ordered set S there exists in S a *sequent* x to I. By the definitions of "initial" and "sequent", I is then the subset of all $s \varepsilon S$ which precede x, i.e. the section of S determined by x. The existence of the sequent x is guaranteed by theorem 2.

On the other hand, let us assume that the condition of theorem 4 is satisfied by an ordered set S. Then a sequent exists in S for *any subset* $S_0 \subset S$ which is succeeded by at least one element of S. To prove this, we only have to replace S_0 by the set I which contains, in addition to all $s \varepsilon S_0$. all elements of S preceding any $s \varepsilon S_0$; I is obviously an initial $\neq S$, hence a section of S. The element of S which determines this section is the sequent of I and at the same time the sequent of S_0. Hence, by theorem 2, S is well-ordered. Q.E.D.

The exclusion of S itself is necessary since no $x \varepsilon S$ succeeds *all* elements of S.

(To proceed directly from the condition of theorem 4 to that of definition I, we take into consideration that the *first element* of the subset of S which contains all successors to an initial (i.e. all $s \varepsilon S$ not contained in the initial), determines the initial as a section of S. On the other hand, if S^* is a non-empty subset of S, the first element of S^* may be described as *that* element of S which determines the section containing all those $x \varepsilon S$ which precede *all* elements of S^*; these x form an initial, and so by virtue of our assumption, a section of S.)

THEOREM 5. An ordered set is well-ordered if, and only if, it has no subset of the order-type $*\omega$.

Proof. If S is well-ordered according to definition I, every non-empty subset of S has a first element; hence it can not have the type $*\omega$. Vice versa, if no subset of an ordered set is of the type $*\omega$, every non-empty subset has a first element. For, if there were a subset $\neq 0$ without a first element, it would comprise a sequence (s_k) such that for any k, $s_{k+1} \supset s_k$; i.e. a sequence of the type $*\omega$. Q.E.D.

Finally, we remark that the general [1]) theory of well-ordered sets can be completely derived from the principles introduced in Chapters I and II, without using the multiplicative principle (and the principle of infinity); cf. p. 181. The main subjects of § 11, however, require additional implements, including the multiplicative principle; see *Foundations*, ch. II.

2. Transfinite Induction. In 2 and 3, which treat the most fundamental problems of the general theory, rather abstract proofs are required which may cause the reader some difficulty, while far less effort is expected of him for the remainder of § 10 and in the first subsections of § 11. However, *the contents of* 2 *and* 3 *will not be used in the course of the immediately subsequent investigations*: there will be no need to refer to the theorems of 2 before § 11, 3 (p. 284), and the proof of 3 is not required for our presentation of the subject, since another proof, less direct but easier, will be given at the beginning of § 11. *Beginners are therefore advised to omit* 2 *and* 3 *and to proceed at once to* 4; they will have to work through 2 upon arriving at § 11, 3, while perusal of the proof of 3 — as a more elegant alternative to the proof of § 11, 2 — may be postponed until the end of § 11, or else may be omitted altogether.

It is well known that the fundamental and primitive procedure of arithmetic is that of *mathematical induction*, sometimes referred to as "inference from n to $n + 1$" [2]). It may, in short, be described as the way of proving a general theorem on natural numbers k by showing first that it is true for $k = 1$, and secondly that for any k, if the theorem is true for k then it is true for $k + 1$ [3]).

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[1]) The attribute "general" excludes all questions of an *existential* character such as are dealt with in § 11; in particular, ordinals, alephs and well-ordering.

[2]) Sometimes we used this procedure for set-theoretical proofs, e.g. in 4 of § 5 and 3 of § 9. Levi ben Gerson (Gersonides, beginning of the 14th century) already used mathematical induction, and it can even be traced back to Euclid, book IX, prop. 8.

[3]) If the *order* of natural numbers according to magnitude has already been introduced, the second condition may preferably be expressed in the form: the truth of a proposition *for all k preceding a certain m* implies the truth for m itself. In the customary foundation of mathematics, however,

Quite a few authorities on mathematics and philosophy even maintain that this procedure is at the very root of all mathematical truth. (Cf. *Foundations*, ch. IV.) Instead of the names "induction" and "inductive", the terms *recursion* and *recursive* are sometimes used, to indicate the converse direction of procedure [1]).

Mathematical induction is obviously not strong and comprehensive enough to serve an analogous purpose when the aggregate in question is not a sequence, but a general (plain or ordered) infinite set; that is to say, when transfinite cardinals or order-types are involved. Nevertheless, there exists a far-reaching extension of the procedure, called *transfinite induction*, which serves a similar purpose for transfinite well-ordered sets and their types [2]). By means of a method to be developed in § 11, **6**, we shall even be able to adapt this tool to infinite sets in general.

In the applications of the theory of sets to analysis and even to algebra [3]), transfinite induction (especially as applied to ordinals; see § 11, 2 and 3) may be regarded as the most powerful and successful instrument, although for some of its applications, e.g. in topology, alternatives have been suggested which employ other methods [4]).

One part of our task is quite simple:

THEOREM 6. (*Proof by transfinite induction*) Given a well-ordered set S and a proposition $\mathfrak{P}(s)$ involving the elements $s \varepsilon S$, then the proposition holds for *all* $s \varepsilon S$, if

a) it holds for the *first* element of S,

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where the principle of induction is introduced as an axiom, an order-relation between the numbers is not yet available, but is defined by means of this very principle.

[1]) A more profound meaning of the term "recursive" will be used in *Foundations*, ch. V.

[2]) For details of more general inductive processes, including both mathematical and transfinite induction, see Bell **1**, Bennett **1** and H. Blumberg **1**.

[3]) The first, and perhaps most impressive, use of transfinite induction in (abstract) algebra, was made in proving the theorem that there is an essentially unique algebraically closed algebraic extension of any given field. See Steinitz **1**, §§ 19–21 (1910); Baer **7**. In algebra and other domains, Zorn's lemma may be used as an alternative to transfinite induction and the well-ordering theorem. See *Foundations*, ch. II.

[4]) See Tukey **1**. Cf. G. Birkhoff **4**, p. 283 and p. 301.

b) its truth for all elements preceding any particular element $s_0 \varepsilon S$, implies its truth for s_0 itself.

Remark. Strictly speaking, condition a) is superfluous, as it is contained in b). For to the first element a of S there is always attached the vacuous property, that for any x, which satisfies the relations $x \varepsilon S$ and $x \supset a$, $\mathfrak{P}(x)$ holds, since these relations are unrealizable, i.e. since no such x exists. Hence condition b) asserts that $\mathfrak{P}$ certainly holds for the first element of S. None the less, it is more convenient for the beginner to have a) stated explicitly.

Let us also note the following version of our theorem: Given a well-ordered set S and a (not necessarily ordered) set M which contains, together with the elements of any section of S, also the element of S which determines the section; then M contains (at least) *all* elements of S.

Proof of theorem 6 (by indirect method). Assume that there exist elements s of S for which $\mathfrak{P}$ does *not* hold true; let S_0 be the subset of all these elements. As a non-empty subset of a well-ordered set, S_0 has a first element s_0^*, and since $\mathfrak{P}$ holds for all elements s of S which precede s_0^*, it holds for s_0^* as well, contrary to our assumption with regard to S_0. Hence S_0 is empty. Q.E.D.

The necessity of the indirect method of proof of theorem 6 will be discussed in **5**.

In arithmetic, mathematical induction is used not only for *proving* propositions but also in order to *define* relations, operations etc. A classical instance, known at least since Peano [1]), is the *definition of the addition* of two natural numbers by the following twofold rule in which n and k denote any natural numbers and the term "sequent" applies to the succession in the sequence 1, 2, 3, . . . :

a) $n + 1$ is the sequent of n;

b) $n + (k + 1)$ is the sequent of $n + k$.

For any *particular* value of m, the sum $n + m$ is (uniquely) determined by this rule [2]), and for many decades, this fact helped to create the conviction that our rule constitutes a complete definit-

[1]) From 1889 onwards; cf. Peano **4**. The method of defining multiplication is similar.

[2]) An example will sufficiently clarify the situation. Denoting the sequent of n by $n|$, we obtain $5 + 3 = 5 + 2| = 5 + (2 + 1) = (5 + 2)| = (5 + 1|)|$, and since $5 + 1| = (5 + 1)| = (5|)| = 6| = 7$, we have $5 + 3 = 7| = 8$.

ion of addition in the domain of natural numbers [1]). As late as 1927 this conviction was shaken, and stricter procedures were given for the definition of this and other operations or relations between natural numbers [2]).

The same state of affairs was prevalent in the theory of aggregates with respect to *transfinite induction*, already employed by Cantor [3]) for the definition of power. (Here, however, the situation is somewhat simpler, since the *order* among ordinals has been previously defined.) Everyone had shared the conviction that theorem 6 was sufficient not only to prove but also to *define* (construct) with the aid of transfinite induction, and even the strictest and most profound of the contemporary textbooks and handbooks of the theory treat the subject in the light of this attitude. Only in the twenties did J. von Neumann [4]) discover that a gap had been left at this juncture of the theory of well-ordered sets and ordinal numbers. The nature of this gap will be easily understood from the following theorem and its proof.

THEOREM 7. (*Definition, or construction, by transfinite in-*

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[1]) As early as 1887 R. Dedekind pointed out the necessity of providing an adequate basis for inductive *definition* (construction), apart from the quite elementary matter of inductive *proof*, and in Dedekind **2**, § 9, such a basis is given in full rigor; cf. Bernays **14**, II, p. 11. For forty years, however, no attention was paid to these remarks, and when J. von Neumann and E. Landau independently raised the matter, they met with general surprise and even with some opposition. Cf. the following footnote.

[2]) Landau **1**, p. IX and p. 4. The discovery of the gap is due to K. Grandjot, the procedure to L. Kalmár. Cf. also Takagi **1**; Lorenzen **1**; Kalmár **10**.

[3]) Cantor **12**, II, p. 231 ff. Cf. also Schoenflies **1**, I, p. 45 and Hausdorff **2**, I, p. 127 ff.

[4]) See von Neumann **6** (1928). The existence of the gap had already been implicitly stated and filled, as far as ordinals were concerned, in von Neumann **1**. A somewhat different construction is found in the first edition of the textbook (of algebra) van der Waerden **2**, volume I, § 59 (1930).

A device for replacing definitions based on (mathematical or transfinite) induction by *explicit* definitions was given by Kleene **8**; cf. also **6**. As a matter of principle, this problem is of a certain importance; for one cannot in all cases eliminate inductive definitions, as opposed to explicit ones. Cf. § **4**, **7** of the present book. So inductive definitions must be considered separately in proving the non-contradiction of a deductive theory (see *Foundations*, ch. V); cf. Hilbert–Bernays, **1**, I, p. 294.

duction) [1]) Let a function whose argument s runs over the elements of a well-ordered set S, be described by the following "recursive" property: There is given a fixed rule [2]) which uniquely determines the values of the function for any $s_0 \varepsilon S$ by means of the totality [3]) of all its values for the elements s with $s \prec s_0$. (Hence, in particular, a fixed value of the function is given for the *first* element of S, in accordance with the remark following theorem 6.) Then there exists a single-valued function $f(s)$ — and one only — which is defined for all $s \varepsilon S$ and which possesses the "recursive" property in question. In other words, the recursive property uniquely defines a function $f(s)$ which assigns a single value to every element s of S.

Therefore, if a recursive property p of the kind indicated is given, we may speak of "the" function with the property p.

Proof. Two assertions have to be demonstrated, namely, that there *is* a function $f(s)$ defined on the set S which has the recursive property in question, and that there is only *one* such function. We begin by considering the second assertion, which is almost evident on account of theorem 6.

In fact, let $f_1(s)$ and $f_2(s)$ be two functions with the same recursive property. If there were an element in S for which f_1 and f_2 assume different values, there must be a first element s^* of this kind, since S is well-ordered, and s^* cannot be the first element of the entire set S. Therefore we arrive at a contradiction by means of theorem 6, since $f_1(s) = f_2(s)$ for all $s \prec s^*$, implies $f_1(s^*) = f_2(s^*)$.

Thus we have only to show that there *exists* a function $f(s)$ of the desired kind. First of all, let us arrange all *sections* of S according to the succession of the *elements* of S which determine the sections, i.e. if $s_1 \prec s_2$, then the section determined by s_1 shall precede the one determined by s_2. Owing to theorem 1, the ordered set of all these sections, being similar to S, is also well-ordered

[1]) Whenever there is no danger of confusion with ordinary "mathematical induction", we shall simply speak of "inductive" definitions, constructions, proofs etc.

[2]) There may be different ways of defining functions on S uniquely by recursive rules. A fixed rule from among these is assumed to be given.

[3]) "Totality" should not be understood as the set of all function-values $f(s)$, but as the whole correspondence $s \leftrightarrow f(s)$ for $s \prec s_0$. If we denote this correspondence by $C(f, s_0)$, we can express the fixed rule in the form $f(s_0) = \psi(C(f, s_0), s_0)$, where ψ denotes a given function of two variables.

and it will retain this property [1]) if the set S (which is no section of itself) is added as the last element to the set of all sections. The enlarged set, henceforth referred to as T, is the set of all *initials* of S, ordered according to their increasing range, as follows from theorem 4. [2])

We shall now make use of the following

Lemma. Under the assumptions of theorem 7, there exists for any initial I of S (i.e. for any $I \varepsilon T$) a function $f_I(s)$ defined for all $s \varepsilon I$ which has the recursive property mentioned in the theorem.

For the proof of the lemma we use transfinite induction in the well-ordered set T. For the first initial of the set, i.e. the null-set, the lemma's assertion is trivial. Let us assume that the assertion is true for all initials (or sections) of S which *precede a particular initial* I_0. We shall show that the assertion is true for I_0 as well. Let us distinguish two cases.

a) I_0 *has a last element* z. By virtue of the assumption just made, there exists a function $f_{I_0'}(s)$ defined on the initial $I_0' = I_0 - \{z\}$ and satisfying our conditions. Now, the rule referred to in theorem 7 uniquely determines the value v for z by the totality of all values of $f_{I_0'}(s)$ for the arguments s with $s \supset z$. We consider the function f_{I_0} for which $f_{I_0}(s) = f_{I_0'}(s)$ for $s \supset z$, and $f_{I_0}(z) = v$. It is easily seen that $f_{I_0}(s)$ is a function of the desired kind, defined on the initial I_0.

b) I_0 *has no last element.* Then *every* element of I_0 already belongs to a section I_0' of I_0. By our assumption, there exists a function $f_{I_0'}(s)$ defined on I_0' which satisfies our conditions. No matter what I_0' is chosen, all such functions coincide for their common argument values s; for if I_0' and I_0'' are different sections of I_0, and if I_0', for instance, is a section of I_0'', both $f_{I_0'}(s)$ and $f_{I_0''}(s)$ are defined on I_0' and, as they both satisfy the condition of theorem 7, they coincide on I_0' in view of the uniqueness demonstrated at the beginning of our proof. Without any arbitrariness, we can therefore define $f_{I_0}(s)$ on the entire initial I_0 by the following rule: if $s \varepsilon I_0$ belongs to the section I' of I_0, $f_{I_0}(s)$ shall equal $f_{I'}(s)$. Since all

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[1]) This is evident, while also following from the general theorem 9 in **4.**

[2]) It is natural that we pass from the elements to the initials since the recursive property connects the value of $f(s)$ at a definite place $s = s_0$ with its values on the whole section (initial) determined by s_0.

functions $f_{I'}(s)$ were assumed to satisfy the condition of theorem 7, the same applies to $f_{I_0}(s)$. Hence the latter function is of the desired kind.

The lemma having thus been demonstrated, the proof of theorem 7 is complete, since S itself is an initial of S. [1])

In contrast with the successive steps of inductive procedure in arithmetic and in the series of ordinals (§ 11), our theorem gives a *simultaneous* construction. However, the difference is chiefly of a psychological nature. The logical aspect of the matter cannot be separated from certain axiomatic complications arising in the theory of well-ordered sets (cf. **5**); we may therefore regard the difference as a matter of method rather than as a matter of principle.

As to the connection between theorems 6 and 7, proof by transfinite induction may be regarded as a special case of the definitory procedure, viz. if $f(s) = 1$ is taken to mean that the proposition is true, then $f(s) = 1$ for all $s \supset s_0$ implies $f(s_0) = 1$. In this case the *existence* of a function f is evident, and this explains the elementary character of theorem 6. Yet we cannot obtain 6 as a corollary to 7, since the proof of 7 uses 6.

The name "*transfinite* induction" derives from the fact that the set of all elements preceding a particular element s_0 of a well-ordered set is in general an *infinite* set and has a transfinite order-type (ordinal number, see **4**). In this respect transfinite induction differs from ordinary *mathematical induction* (p. 247), where the set of the predecessors of a particular element s_0 is always a *finite* set, although the set of *all* elements is itself infinite, viz. a sequence. Of course, the attribute "mathematical" assigned to this induction, as well as the corresponding term "complete induction" used in most continental languages for the same concept, does not stress the contrast with transfinite induction — of which it is a particular case — but the contrast with induction in *natural science* which is incomplete by its very nature, being based on a finite and even relatively small number of experiments.

The practical use of transfinite induction will be illustrated in § 11, especially in 3. A method of proof virtually following the lines of inductive construction will appear in the next subsection.

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[1]) The analogous recursive definition of functions of several variables causes no further difficulty. Here we have to deal with the rule

$$f(a_1, \ldots, a_{n-1}, s_0) = \psi\,(a_1, \ldots, a_{n-1}, C(f, s_0), s_0)$$

where the notation corresponds to that used in footnote 3 on p. 251, and $a_1, \ldots, a_{n-1}, s_0$ are elements of the same well-ordered set.

3. Comparability of Well-Ordered Sets. At the end of § 5 a fundamental problem of the theory of equivalence remained open: the question whether, of two given sets, at least one is equivalent to a subset of the other. The analogous question in the general theory of order reads: given two ordered sets, is at least one similar to a subset of the other? In § 8 we saw (p. 190) that here the answer is plainly no. In contrast herewith, the most important property of those particular ordered sets which were called well-ordered is that for them the answer is *yes*. By the use of methods to be developed in § 11, **6**, this result will enable us to answer in the affirmative the question left open in § 5 with regard to plain (not-ordered) sets.

In fact we shall prove even more, by considering — instead of general subsets of the respective well-ordered sets — the particular subsets called *sections*. For reasons appearing later (theorem 14 in **5**), we have to single out here the special case of similar sets. We prove:

THEOREM 8. (*Fundamental theorem of well-ordered sets*) Of two well-ordered sets which are not similar to each other, one [1]) is similar to a section of the other.

Proof [2]). Let A and B be any *similar* well-ordered (non-empty) sets. Evidently any similar representation between them relates the first element of A to the first element of B. By trying the second, third, ... elements of both sets, and — provided the sets are infinite — also the elements determining sections of the types ω, $\omega + 1$, etc., one easily arrives at the conjecture that any [3]) similar representation φ between A and B has the following property: if $a \varepsilon A$ and $b \varepsilon B$ are corresponding elements, the sections determined by a and b contain precisely elements of A and B

[1]) That *only one* of the two has this property, is almost self-evident; see **5**.

[2]) Essentially, both the proof of the theorem of comparability in § 11, **7** and the proof given here, rely on suggestions made to the author by A. Plessner in 1927. The earliest proofs of our theorem are those of Cantor **12**, II (1897); Baire **2**; Hessenberg **3**; Young–Young **1**. The proof given in Hausdorff **4**, pp. 103—105, excels by its lucidity, though it makes a detour by using ordinal numbers. In its main features, it is given at the beginning of § 11, independently of the proof presented here. Valuable remarks on the present proof and on a few subjects of § 11, **3** and **5** are due to K. Bing.

[3]) Actually there is only one; see theorem 13 in **5**.

related to each other by the representation φ; in other words, $b \varepsilon B$ is the sequent of that initial of B which contains the mates of all elements of A preceding $a \varepsilon A$.

This merely heuristic reflection shall serve as a guiding principle for constructing a rule of correspondence between elements of two *arbitrary* (not just similar) well-ordered sets S and T. The rule will be defined in the way described in theorem 7, and the proof virtually relies on this theorem [1]). The correspondence thus to be established will immediately show the comparability of well-ordered sets.

If S and T are non-empty well-ordered sets, we shall, *as far as possible*, attach to every $s \varepsilon S$ a definite mate $t \varepsilon T$ by the following rule:

> (*) Let x be an arbitrary element of S and X the section of S determined by x. Assume that to every element of X its mate in T has been attached, and denote by Y the set of all these mates, so that $Y \subseteq T$. Then, *if* any element of T succeeds all elements of Y, the *sequent y of Y in T* (existing in view of theorem 2) shall be the mate of $x \varepsilon S$. We shall write: $y = \varphi(x)$.

According to this rule, in particular the first element of T is attached to the first element of S; for in this case X, hence also Y, is the null-set.

Notice that the rule may *fail* to attach a mate to the given $x \varepsilon S$, viz. in the case that the set of the mates of all elements of X is not succeeded by any element of T. However, the elements of S possessing mates in T on account of (*), certainly constitute an *initial* of S; for our rule defines a mate of $x \varepsilon S$ only if all preceding elements of S have mates in T.

Before systematically investigating the properties of the correspondence defined by (*), let us form a rough conception of it by contemplating a few of the first elements of S.

As remarked previously, the mate of the first element s_0 of S is the first element t_0 of T, i.e. $\varphi(s_0) = t_0$. The second element s_1 of S determines the section $X_1 = \{s_0\}$; hence the corresponding subset $Y_1 \subseteq T$ is $\{t_0\}$, and the mate of s_1 is the sequent of Y_1, i.e. $\varphi(s_1) = t_1$

[1]) The readers acquainted with the proof of theorem 7 will easily insert suitable references. However, the following proof does not presuppose theorem 7.

is the second element of T (if there is such an element). In the same way we find that the mate of the kth element of S is the kth element of T, provided T has not yet been exhausted. If both sets contain infinitely many elements and if neither S nor T is exhausted by the sequences which contain all the elements s_k and t_k for $k = 0, 1, 2, \ldots$, the section $\{s_0, s_1, s_2, \ldots\}$ of S with the type ω, is determined by its sequent s_ω in S; since $\varphi(s_k) = t_k$, the mate $t_\omega = \varphi(s_\omega)$ of s_ω in T is the sequent of the subset $\{t_0, t_1, t_2, \ldots\}$ of T.

After this preliminary exploration, let us proceed to prove two fundamental properties of the rule (*). In order to facilitate the formulation, an initial X of S, *all elements of which happen to have mates in T on account of* (*), shall be called a "*representable initial of S*", and the subset Y of T containing the mates of the elements of X "*the mate-set of X*".

Lemma I. If x_1 and x_2 are elements of a representable initial X of S, y_1 and y_2 their mates in Y, the relation $x_1 \prec x_2$ in X implies $y_1 \prec y_2$ in Y, and vice versa. Hence the rule (*) defines a *similar* representation between any representable initial of S and its mate-set.

Proof. If $x_1 \prec x_2$, then by (*) the mate y_2 of x_2 succeeds all the $y \in Y$ which mate with an element of the section of X determined by x_2. y_1 certainly belongs to those $y \in Y$ since it is the mate of x_1 which precedes x_2. Hence $y_1 \prec y_2$. Therefore, different elements of Y correspond to different elements of X; hence the correspondence between X and Y is biunique and similar. Q.E.D.

Lemma II. The mate-set Y of any representable initial X of S is an initial of T.

Proof. Let y_1 be an element of Y, and y_0 an arbitrary element of T preceding y_1, so that $y_0 \prec y_1$ holds in T. It is sufficient to show that y_0 also belongs to Y.

To this purpose we may assume that y_0 is not the first element of T, since the first element of T, being the mate of the first element of S (and of X), certainly belongs to Y.

Because of $y_1 \in Y$ there exists an element $x_1 \in X$ such that $y_1 = \varphi(x_1)$, and since $y_0 \prec \varphi(x_1)$ holds in T, there *are* elements x of X such that $y_0 \preceq \varphi(x)$ in T. Let x_0 be the *first* of those elements of X. We shall show that $y_0 = \varphi(x_0)$.

In fact, since y_0 is not the first element of T, neither is $\varphi(x_0)$;

therefore our rule implies that x_0 is not the first element of S (or of X). By the definition of x_0, we have $\varphi(w) \supset y_0$ for all predecessors w of x_0 in X (or in S, which amounts to the same thing). But by the definition of φ, $\varphi(x_0)$ is the *first* element of T that succeeds all the mates $\varphi(w)$ of the predecessors w of x_0 in S. This shows that $y_0 \supset \varphi(x_0)$ cannot hold. Hence from $y_0 \overset{3}{=} \varphi(x_0)$ (see above) we conclude $y_0 = \varphi(x_0)$, i.e. y_0 belongs to Y. Q.E.D.

Lemmata I and II having thus been proved, we proceed to the comparability of any well-ordered sets S and T. As stated subsequently to the rule (*), *the subset* $X \subseteq S$ *of all elements of* S *which have a mate in* T, constitute an initial of S, i.e. a "representable initial"; hence by lemma II its mate-set Y is also an initial of T. Furthermore, by lemma I *the well-ordered sets* X *and* Y *are similar.*

Accordingly, we have the following alternative.

1) X contains *all* elements of S, i.e. $X = S$. Then by (*) S is similarly represented on an initial Y of T. Y may coincide with T (*first case*, in which $S \simeq T$); otherwise Y is, by theorem 4 (p. 246), a section T' of T, i.e. $S \simeq T'$ (*second case*). In the second case, the rule (*) relates all elements of T' to all elements of S, while the element of T which determines the section T', as well as all its successors in T (if any), have no mates in S.

2) X is different from S, i.e. X is a proper subset and therefore a section of S, henceforth to be denoted by S' (*third case*). But in this case *the mate-set* Y *must coincide with the entire set* T, so that T proves similar to a section S' of S. For if Y, which is an initial of T, were a proper subset and therefore a *section of* T, the rule (*) would imply that the element of T which determines the section Y was the mate of the element of S which determines the section $X = S'$ of S. S would then contain an element that has a mate in T but does not belong to X, contrary to the definition of X. Hence $T \simeq S'$ holds.

Remembering the scheme given in § 5, 4 in the theory of *equivalence* between *plain sets*, we present the following analogous scheme (p. 258) in the theory of *similarity* between *well-ordered sets*, in which S, T, X, Y have the meaning as above.

While in the case of equivalence we were unable to clear up the fourth case, here we have succeeded in excluding the possibility of the fourth case. Hence there remain the three other cases only; the proof of the fundamental theorem 8 has thus been completed.

	$Y = T$	Y is a section of T
$X = S$	first case: $S \simeq T$	second case: S similar to a section of T
X is a section of S	third case: T similar to a section of S	fourth case

The reason for the expression *comparability* will become clear at the beginning of § 11. Then we shall formulate theorem 8 in a way which shows that the problem of comparability with respect to equivalence (cardinals) is also solved by theorem 8 in the case of *well-ordered sets.*

4. Addition and Multiplication. Ordinal Number. We shall now specialize the *operations* with ordered sets, defined in § 8, in such a way as to cover the particular case of well-ordered sets. As indicated before, *we shall not use the results of* **2** *and* 3, i.e. theorems 7 and 8, in the following. Neither shall theorem 6 be used explicitly until § 11, though the simple reasoning that is at the bottom of its proof will be applied *passim*, especially in **5.**

While the definition of addition and multiplication and the proof of their formal laws have been settled in § 8 for ordered sets and their types in general, we may expect that the restriction to well-ordered sets will lead to simpler or more particular results. The most obvious question is whether operations with well-ordered sets will always yield sets which again are well-ordered. In such generality we certainly cannot expect this to be true. In fact, by adding sets containing one element only — hence well-ordered — we may obtain sums of *any* order-type, and not only well-ordered sums (p. 200). Our expectation will, however, be fulfilled on condition that the ordered set T which determines the *order of the terms* in the ordered sum, is well-ordered. (Cf. definition IV on p. 197.) We shall prove:

THEOREM 9. Let T be a well-ordered set the elements of which are mutually exclusive well-ordered sets [1]); then the ordered sum

[1]) Apparently, it would be more general to *attach* well-ordered sets to the single elements of T. But this generalization will not change anything. When *types* are summed up, one has to proceed in this way in order to include *equal* types; equal types, however, need not be represented by equal sets.

of all elements of T is again a well-ordered set. Hence in particular any *finite* number of well-ordered sets yields a well-ordered sum (since a finite set can always be ordered, and any finite ordered set is well-ordered).

Proof [1]. Denoting the ordered sum by A, we have to show that any non-empty subset A_0 of A has a first element [2]. Those elements of T which contribute to A_0 form a non-empty subset T_0 of T which accordingly has a first element t_0. The set t_0 has the property that A_0, while containing at least one element of t_0, does not contain any element of a set preceding t_0 in T. On the other hand, those elements of the well-ordered set t_0 that appear in A_0, form a non-empty subset of t_0 which has a first element a, and a is the first element of A_0. Hence any subset $\neq 0$ of A has a first element. Q.E.D.

DEFINITION III. The order-type of a well-ordered set is called an *ordinal number* or simply an *ordinal*; if the set is infinite, a *transfinite ordinal.*

The honorary title *number*, having been given to transfinite cardinals only with a temporary reservation (p. 79) and entirely withheld from transfinite order-types in general (p. 191), has thus been definitely conferred on a particular species of types, those of well-ordered sets [3]. The positive reason for this is the same as the negative reason in the preceding cases: *the general comparability of the mathematical objects in question.* For ordinal numbers, this property is implicitly contained in theorem 8 and will be explicitly stated in theorem 3 of § 11.

According to the general definition of order-types, a well-ordered set S has a uniquely determined ordinal which is common to all sets similar to S, while any well-ordered set which is not similar to S has a different ordinal. Like types in general, (transfinite) ordinals

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[1]) The beginner is advised to facilitate the understanding of this proof by first taking T as an ordered set of two (or a finite number of) sets.

[2]) The theoretical simplification obtained by means of the criterion of theorem 2, which demands only that every *remainder* of A has a first element, is too insignificant to justify abandonment of the more usual criterion. The same applies to some of the following proofs.

[3]) Contrary to this usual notation, a distinction is made by Denjoy 3—5 between the types of well-ordered sets and "ordinal numbers" in a more general sense, indicating "rank".

in particular are denoted by small Greek letters. In view of the properties discussed in the beginning of **1**, which well-ordered sets (and their types) — but not infinite ordered sets in general — have in common with finite sets and finite numbers, ordinal number constitutes the most natural and most restricted generalization of finite number with respect to the process of counting, as will be seen in detail in § 11, **3**.

For *finite* sets (cf. **6**) the notions of cardinal, order-type, and ordinal are mutually corresponding, the two latter coinciding. Hence there is no need to denote them by different symbols; it suffices to use the natural numbers (including 0: as the cardinal, or the ordinal, of the null-set), though logically they have different meanings according as they are conceived as cardinals or as ordinals. The one-to-one correspondence between finite cardinals and ordinals is due to the fact that not only can any finite set F be ordered but that all orderings of F are similar; their common order-type is automatically an ordinal because an ordered finite set is always well-ordered. Hence the cardinal indicating the number of elements in the set may also be used for denoting its ordinal. This, of course, does not affect the logical difference between propositions such as "the set $\{a, b, c\}$ has three elements" and "the scheme of order in $\{a, b, c\}$ is: first, second, third".

The situation is completely different for *infinite* sets. The equivalence properties of an infinite set S are independent of the possible arrangements of its members; accordingly the *cardinal* $\mathfrak{s}$ corresponding to S is uniquely determined by the totality of its members, i.e. by the plain set S. But the elements of S may be ordered — if at all — in infinitely many different ways, and in many cases the different ordered sets thus arising from S have even different *order-types* [1]). It may happen — in fact, as will be shown in § 11 **6**, it always happens — that among those ordered sets there are in particular well-ordered sets; their types, therefore, are *ordinals* (even infinitely many different ordinals) corresponding to the plain set S and to its cardinal $\mathfrak{s}$.

If, for instance, S is the set of all natural numbers, its cardinal is $\mathfrak{s} = \aleph_0$; according to theorem 4 on p. 202, $2^{\aleph_0} = \aleph$ different *order-types* correspond to this cardinal. In § 11, **5**, we shall deal

[1]) Cf. the proofs of theorems 1 and 4, pp. 191 and 202.

with the question how many different *ordinals* there are among those $\aleph$ types — at any rate more than $\aleph_0$. A few of the ordered sets containing all natural numbers are here given as examples (commas indicating sequents or immediate predecessors):

$$\{1, 2, 3, 4, 5, 6, \ldots\}$$
$$\{\ldots, 6, 5, 4, 3, 2, 1\}$$
$$\{1, 3, 5, \ldots\ 2, 4, 6, \ldots\}$$
$$\{\ldots, 6, 4, 2\ \ldots, 5, 3, 1\}$$
$$\{1, 3, 5, \ldots\ \ \ldots, 6, 4, 2\}$$
$$\{\ldots, 5, 3, 1, 2, 4, 6, \ldots\}$$
$$\{1, 5, 9, \ldots\ 2, 6, 10, \ldots\ 3, 7, 11, \ldots\ 4, 8, 12, \ldots\}$$
$$\{\ldots, 3n, 2n, n\ \ldots, 3n-1, 2n-1, n-1\ \ldots\ \ \ldots, 2n+1, n+1, 1\}$$
$$\{1, 2, 4, 7, \ldots\ 3, 5, 8, 12, \ldots\ 6, 9, 13, 18, \ldots\ 10, 14, 19, 25, \ldots\ \ \ldots\}\ ^{1})$$

The types of these nine ordered sets are

$$\omega,\ {}^*\omega,\ \omega\cdot 2,\ {}^*\omega\cdot 2,\ \omega+{}^*\omega,\ {}^*\omega+\omega,\ \omega\cdot 4,\ {}^*\omega\cdot n,\ \omega\cdot\omega,$$

and they are different from each other provided that $n > 2$. Finally, the first, the third, the seventh, and the last among those types are ordinals but none of the others. Neither are the types η and λ ordinals.

The addition and the multiplication of ordinals are contained as particular cases in definitions IV and V on pp. 197 and 204. From theorem 9 we immediately conclude by considering the case where all the terms of the sum are *similar* well-ordered sets:

THEOREM 10. An ordered sum of ordinals is again an ordinal, if the terms appear in a well-ordered arrangement. Hence the sum of a finite number of ordinals, as well as the product of two (or of a finite number of) ordinals, is always an ordinal.

If in a finite product all (say n) factors equal the ordinal σ, the resulting power σ^n is, by theorem 10, again an ordinal. On the other hand, nothing is said by our theorem about a transfinite *exponent* or about a product of infinitely many ordinals. These questions are of a different nature (p. 210); the former is dealt with in

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[1]) This arrangement originates from the quadratic array in which the natural numbers appear in example 1 on p. 200.

§ 11, 3. The validity of the associative laws and of the first distributive law for ordinals is contained in theorems 3/6 of § 8, while the invalidity of the commutative laws and of the second distributive law is shown by the instances given on p. 195 and 204 f. [1]). Accordingly there is no difference in these respects between types in general, and ordinals. On the other hand, in contrast with the domain of general types, there exists an elaborate system of arithmetic in the domain of ordinals; some prominent features of this system shall be dealt with in § 11, **4.**

5. Elementary Properties of Well-Ordered Sets. In the domain of plain sets, theorem 4 of § 3 (p. 57) — whose proof (see p. 123) uses a principle of a non-elementary character — asserts that any infinite set R has a *denumerable subset.* In the language of plain addition (formation of sum-set) we may therefore write:

$$R = R_0 + R_1$$

where R_0 is denumerable, i.e. of the cardinal $\aleph_0$, and possibly $R_1 = 0$.

In the domain of well-ordered sets we may raise an analogous question which, however, is far simpler, actually as well as in principle. Here $\aleph_0$ is to be replaced by the smallest ordinal ω; that is to say, instead of denumerable sets we use enumerated sets (sequences). If S is an infinite well-ordered set, it contains a first element s_0, a second s_1, and generally (by theorem 2), corresponding to every natural number n, an element s_{n-1} which is the sequent of the subset $\{s_0, s_1, \ldots, s_{n-2}\}$. By means of mathematical induction we conclude that the enumerated set $\{s_0, s_1, \ldots, s_k, \ldots\} = S_0$ is an ordered subset of S; it also has the property that every element of S not belonging to S_0 (if any) *succeeds* all elements of S_0; in other words, S_0 is an initial of S. Moreover, in sharp contrast with the arbitrary choice of the elements of a denumerable subset R_0 of R in the case of plain sets, all the elements of S_0 are uniquely

[1]) Jacobsthal **1** investigates the particular cases in which for a pair α, β of different transfinite ordinals the equalities $\alpha + \beta = \beta + \alpha$ or $\alpha \cdot \beta = \beta \cdot \alpha$ hold, as well as the equality between powers $\alpha^\beta = \beta^\alpha$, which among finite positive integers is fulfilled only by the pair **2, 4.** As to the latter equality, we easily perceive that there are infinitely many transfinite *cardinals* satisfying it. Cf. Sierpiński **6**, p. 221.

determined: as *the* first, second, ..., nth, ... elements of S. We may therefore formulate:

THEOREM 11. Any infinite well-ordered set S has a subset of the ordinal ω. More precisely, S can uniquely be written as an ordered sum in the form

$$S = S_0 + S_1,$$

where the initial S_0 is enumerated, i.e. of the ordinal ω. (The well-ordered set S_1 may be the null-set, or finite, or infinite.)

Hence any transfinite ordinal σ has a unique additive decomposition of the form

$$\sigma = \omega + \sigma_1$$

where σ_1 again is an ordinal. ($\sigma_1 = 0$ if $\sigma = \omega$.)

It has not been stressed in the preceding proof that there is no other additive decomposition with the same properties. It is easy to show this, but it is superfluous, since the uniqueness follows from theorem 12 or 13. (Theorem 11 is not used in the proofs of these theorems.)

Theorem 11 has been inserted here because it yields a convenient example to illustrate the fundamental theorem 12 which we are now about to prove. An infinite well-ordered set S may, in view of the decomposition given in theorem 11, be similarly represented on a proper subset [1]) in the following way. Let s_k be any element of the enumerated set $S_0 = \{s_0, s_1, \ldots, s_n, \ldots\}$. The subset of S_0

$$S_0' = \{s_k, s_{k+1}, \ldots, s_{k+n}, \ldots\},$$

which is completely determined by k, has again the type ω, and is a proper subset if $k > 0$ (as shall be assumed henceforth). Therefore we obtain a similar representation between S_0 and S_0' by assigning $s_{k+n} \varepsilon S_0'$ to $s_n \varepsilon S_0$, for any $n = 0, 1, 2, \ldots$, and vice-versa. By assigning, in addition, every element of S_1 (if any) to itself, an obviously similar representation between the set $S = S_0 + S_1$ and its proper subset $S' = S_0' + S_1$ is established which has the following property: there are elements x of S — even infinitely many, viz. all elements of S_0 — assigned to elements of the subset S' which (as elements of S) *succeed* x *in* S. In symbols: $x \varepsilon S$ is

[1]) The reason why a *proper* subset has to be taken, will be made clear by theorem 13.

assigned to $y \varepsilon S'$ while $x \supset y$ holds in S. In the particular case $k = 1$, for instance, the part of the representation which concerns S_0 is illustrated by the array

$$\begin{array}{lccccccc} S_0: & s_0 & s_1 & \dots & s_n & & \dots \\ & \updownarrow & \updownarrow & & \updownarrow & & \\ S_0': & s_1 & s_2 & \dots & s_{n+1} & & \dots \end{array}$$

So we have proved the possibility of a similar representation of S on a subset which assigns to some $x \varepsilon S$ a mate *succeeding* x in S. Of course, it may also happen that the mate *coincides* with the original element in S — and not only when the subset coincides with S; in the case described above this occurs for all elements of S_1, provided S_1 is not empty. A simple instance is

$$S = \{1, 3, 5, \dots 2, 4, 6, \dots\}$$

with the elements 1 and 3 dropped in S', and with the similar correspondence

$$\begin{array}{lccccccccc} S: & 1 & 3 & 5 & \dots & 2 & 4 & 6 & \dots \\ & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow & \\ S': & 5 & 7 & 9 & \dots & 2 & 4 & 6 & \dots . \end{array}$$

Now the question arises whether the third possibility may be realized too, viz. the case where a similar representation of S on a subset assigns to some $x \varepsilon S$ a mate *preceding* x in S. The answer is given by the following theorem due to Zermelo [1]), which expresses a most simple fundamental property of well-ordered sets, and therefore may serve as the keystone for the whole theory, including the comparability theorem.

THEOREM 12. A similar representation of a well-ordered set S on a subset never assigns to an element x of S a mate preceding x in S.

Two equivalent formulations of this theorem are the following:

a) If S is a well-ordered set and $f(x)$ a one-valued function, defined for all $x \varepsilon S$ and assuming values in S subject to the restriction that $x_1 \supset x_2$ implies $f(x_1) \supset f(x_2)$, then $f(x) \supset x$ cannot hold for any $x \varepsilon S$.

b) Given a similar representation between a well-ordered set

[1]) Published in Hessenberg 3; see p. V.

S and a subset, the mate in the subset of an $x \varepsilon S$ *either coincides with* x *or succeeds* x *in* S.

Proof. Let a definite similar representation φ between S and a certain subset S' be given. If there were in S at least one element x the mate of which in S' *precedes* x in S (i.e. if the subset of all $x \varepsilon S$ having this property $\eta(x)$ were *not empty*), there would be a *first* such element $x = s_0$ in S. Denoting the mate in S' of s_0 by s_0', we obtain the relation $s_0' \backsupset s_0$ holding in S. Hence, in view of the similarity of φ, the mate in S' of $s_0' \varepsilon S$, to be denoted by s_0'', fulfils the relation $s_0'' \backsupset s_0'$ (in S', hence also in S)[1]. But this contradicts our assumption that s_0 is the *first* element x of S with the property $\eta(x)$; for s_0' has the proporty η and *precedes* s_0 in S. This contradiction disproves the assumption that S contains an x with the property $\eta(x)$. Q.E.D.

Postponing a logical analysis of this proof until the end of this subsection, we shall draw two important conclusions from theorem 12.

THEOREM 13. There is only *one* similar representation between two similar well-ordered sets. Hence, in particular, a well-ordered set can be represented similarly on itself only by relating each element to itself (identical representation).

Proof. The second assertion obviously *follows* from the first; the identical representation is *eo ipso* similar. On the other hand, the second assertion also *implies* the first. For if φ_1 and φ_2 are similar representations between the well-ordered sets S and T, and $s \varepsilon S$, let $\varphi_1(s) = t_1$ and $\varphi_2(s) = t_2$. Then, as similarity is a transitive relation, we obtain a similar representation ψ of T on itself which always (i.e. for each s) relates t_1 and t_2 to each other. If φ_1 and φ_2 were different, there would exist an $s \varepsilon S$ for which $t_1 \neq t_2$; the representation ψ would not be identical, contrary to the second assertion of our theorem.

It therefore suffices to prove the second assertion. But this assertion is evident on account of theorem 12. For let s_1 be an element of S the mate s_2 of which is different from s_1. Then either

[1]) It is more obvious though perhaps less elegant to say: by continuing this procedure we would obtain a sequence $(s_0, s_0', s_0'', \ldots)$ with the type $^*\omega$, which contradicts theorem 5. For this purpose, of course, s_0 may be *any* (not just the first) element with the property η.

$s_1 \prec s_2$ or $s_2 \prec s_1$. In each case a mate would precede its original element, contrary to theorem 12. Q.E.D.

We observe that the property expressed by theorem 13 is not *characteristic* of well-ordered sets. For instance, any ordered set of the type $\omega + 1 + {}^*\omega$ (which is not an ordinal) has the same property, as is easily seen. On the other hand, see exercise 6 at the end of this section.

THEOREM 14. A well-ordered set S is neither similar to a section of S nor to a section of a subset of S. Two different sections of S cannot be similar to each other.

Remark. S may, of course, have *proper subsets* similar to itself; this always happens if S is infinite.

Proof. As the first assertion is a particular (most important) case of the second assertion in the first sentence, we shall only prove the latter. Let the section X of a subset of S (which possibly coincides with S) be determined by the element x (of the subset, and therefore of S as well). We show that S and X are not similar. For any similar representation between them would relate $x \varepsilon S$ to a certain element y of X; but in the set S, y precedes the element x by which the section X is determined, and thus a contradiction with theorem 12 would result [1]).

The assertion contained in the second sentence of theorem 14 follows from the first assertion by theorem 3.

Let us conclude this subsection by discussing a matter of principle connected with the proof of theorem 12. Here (as well as in the more profound proofs of 2 and 3) we used the *indirect* method of demonstration, which is quite typical of the theory of well-ordered sets. The proof of theorem 12 is most instructive, because it is particularly simple. The purpose is *to prove a certain property $\pi(x)$ for all elements x of a well-ordered set.* To this purpose we make the contrary assumption, i.e. we assume the existence of at least one x having the property $\overline{\pi}(x) = \eta(x)$ of not-π. Then, since the set of *all such* x is a non-empty subset of a well-ordered set,

[1]) Using comparability (theorem 8) (which, however, we proposed not to use until § 11) we may immediately infer from theorem 14: *Any subset S' of a well-ordered set S is similar either to S or to a section of S.* In fact, otherwise S would be similar to a section of S', by theorem 8, and this would contradict theorem 14.

there must be in S a *first* element x_0 with the property $\overline{\pi}$. Now either the individual character of x_0 — which is not just *any* element of the set — may enable us to show that x_0 has the original property π; or, and this is more characteristic of well-ordered sets, the certainty that all $x \varepsilon S$ *preceding* x_0 do have the property π, may imply that their sequent x_0 has this property also. In either case the result is a contradiction, which can only be removed by abandoning the initial assumption that there exists an $x \varepsilon S$ having the property $\overline{\pi}$. Hence, by the principle of the excluded middle, *all* $x \varepsilon S$ have the property π.

It is the basic property of well-ordered sets expressed in definition I which enables us to state, when the theorem is assumed to be false, that not simply *some* element of the set does not satisfy the theorem, but that a definite element of this kind can be chosen, namely the first [1]).

The *indirect procedure* [2]) in our proof is essential. It is a main feature in many proofs, not only of the theory of well-ordered sets and transfinite induction [3]), but even of ordinary mathematical induction. Starting with this simplest and best-known case will facilitate the understanding of the procedure [4]).

In order to prove a certain property π of natural numbers n (e.g. the property $n + 1 = 1 + n$) *for a given number*, it suffices

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[1]) Perron 4 gives a cognate method applicable to certain ordered sets which are not well-ordered, e.g. to the linear continuum. Here, the greatest lower bound fulfils the task of the first element. By this method, mathematical induction may, as it were, be transferred to the continuum.

[2]) We cannot deal here with the significance of indirect proof in mathematics in general. This matter is beyond the scope of mathematics, important from the logical point of view as well as in the light of psychology and didacticism. As an example we may take the question: why is an indirect proof of a theorem often more easily understood than the direct one? (For instance, the proof following definition V on p. 232; cf. the footnote on p. 233.) The problems of indirect proof have not yet been sufficiently cleared up, apart from the flat rejection of indirect proof by intuitionists (see *Foundations*, ch. IV). Cf. Hölder 5; Lichtenstein 1, p. 201; Luquet 2; Sambursky 1; J. C. Wilson 1, vol. II, p. 557; in particular Löwenheim 4. Among older treatments of the subject (cf. the quoted works) special attention should be called to Bolzano 2, § 530.

[3]) See, in particular, the proof of theorem 6 on p. 249, remarkable for its simplicity.

[4]) Cf. Lebesgue 2, 2nd edition, p. 329.

to show first that if any $n = k$ has the property π then its sequent $n = k + 1$ has the property π also, and secondly to proceed inductively from 1 to the given number after having proved that the number 1 has the property π. Hereby a direct proof is accomplished, using a finite number of syllogistic inferences; a proof which is recognized even by the strictest intuitionistic attitude and which needs no special principle (not even "original intuition"; see *Foundations*, ch. IV).

If, however, the property π is to be proved for *any natural number* n, i.e. simultaneously for *all* natural numbers, one ought to repeat demonstrations of the kind just mentioned infinitely many times, which is impossible. Here the indirect procedure apparently is inevitable, no matter whether it is introduced in each case separately or by the indirect proof of a general theorem, such as theorem 6 on p. 249 or its counterpart in the case of ordinary mathematical induction. In the latter case the fundamental property of well-ordered sets is replaced by the property that in any set of natural numbers there is a smallest number [1]). The reasoning is the same: if not all numbers had the property π, there would exist numbers with the property non-π, and among them a smallest (first) number. In general it is not even necessary to introduce just the smallest number of this kind; *any definite* number is sufficient on account of the procedure explained in the preceding paragraph. But in general the simplest way of constructing a definite number is to take the smallest one [2]).

In three respects this procedure of ordinary mathematical induction resembles, and at the same time differs from, the present procedure of *trans-*

[1]) Or by an equivalent axiomatic characterization of the concept of natural number; cf. 6. In Peano's system of axioms the last (inductive) axiom makes indirect proof unnecessary.

[2]) Lebesgue adds the remark that in general even that definite number, in particular the smallest one, is only determined by an *indefinite* (though finite) number of logical steps: it is either 1 or larger; in the latter case either 2 or larger; etc. Now, this is nothing but a reapplication of the procedure that produces the natural numbers, no matter whether one conceives it as an original construction, or introduces it axiomatically. The same applies to transfinite induction, e.g. to the set S_0 introduced in the proof of theorem 6 on p. 249. According to this attitude, the "synthetic" principle which lies at the botton of mathematical (or transfinite) induction cannot be demonstrated or reduced to a simpler principle; it must be accepted as something *a priori* or rejected together with the bulk of mathematics in general.

finite induction, whether taken as a general proposition (2) or as a method of obtaining particular proofs. First, the indirect procedure is inevitable in both cases, because a direct proof would require infinitely many steps. Here, it is true, one may stress that for ordinary mathematical induction, $\aleph_0$ steps, or more correctly, a series of steps having the ordinal ω, are sufficient, while for transfinite induction a succession of steps corresponding to *any* transfinite ordinal number, denumerable or not, may be required. Secondly, in the present case one can reach a *particular* (transfinite) ordinal number — or, what is the same, a particular element of the well-ordered set in question — only by proceeding along a transfinite series of steps, and not along a finite one as in the case of a natural number. Thirdly, as far as the indirect procedure in the present case is concerned, the process which leads to a *particular* instead of just *any* object with the opposite property $\overline{\pi}$ (viz. the first one), will involve a transfinite regression, referring no longer to (finite) sections of the sequence of natural numbers but to sections of the well-ordered set in question, or of the sets $W(\sigma)$ to be introduced in § 11.

6. On Finite Sets and their Ordinals. Theorem 12 has a pronouncedly asymmetrical character, because it excludes the possibility that the mate precedes the original while it may succeed it or coincide with it. This asymmetry is the consequence of the asymmetrical nature of definition I (p. 242), requiring a *first* element in any non-empty subset of a well-ordered set. If we changed the definition by requiring a *last* element in any non-empty subset instead, we would obtain a completely parallel theory of an analogous kind of ordered sets which might be called "anti-well-ordered sets". In particular, the analogue of theorem 12 would assert that no similar representation of such a set on a subset ever relates the original to a mate *succeeding* the original in the set.

By requiring one *or* the other property for any non-empty subset of the sets in question, that is to say by considering sets any subset of which has a first or a last element, we get but a slight generalization of the concept of well-ordered sets [1]).

Now, of course, one could inquire into the special kind of well-ordered sets the non-empty subsets of which have *a last element as well as a first one*; they may be called "doubly-well-ordered sets". One would hope thus to obtain a particularly interesting kind of

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[1]) See Steckel **1** and exercise 4 on p. 273. Another generalization which has not attained considerable significance either was proposed by Hausdorff (cf. **3**) in 1901.

sets, distinguished by their simplicity and avoiding the mentioned asymmetry. Unfortunately, however, this kind is *too* simple.

In fact, by combining theorem 12 and its analogue just formulated, we obtain the result that by similarly representing any set of this type on a subset, *every element is attached to itself*, since it is neither attached to a preceding nor to a succeeding element. In other words, a set of this type has no similar proper subset at all. This property is quite analogous to the property enounced in theorem 4 of p. 38, which asserts that a *finite set* is not equivalent to any proper subset [1]). Moreover, on p. 40ff. we have seen that this property is even *characteristic* of finite sets. The same can easily be shown in our case, namely that *any doubly-well-ordered set is a finite set.* The converse proposition, saying that any subset of a finite ordered set has a first and a last element, is evident and includes the assertion that any finite *ordered* set is well-ordered.

That every finite (inductive) set can be ordered, is proved by mathematical induction, starting from the fact that any set of two elements can be ordered — or that any set containing a single element may be considered as an ordered set.

The notion of "doubly-well-ordered sets" has a special interest in spite of the rather trivial result just mentioned; not only because it furnishes a new and unexpected definition of finite set, but because it enables us *to develop the theory of finite sets and natural numbers as a particular branch of the theory of well-ordered sets, with their ordinals and cardinals.*

It is not a purpose of the present book to develop a theory of natural numbers and finite sets. Still less is it intended to present the theory of sets independently of the concept of natural number. This task can only be achieved, if at all, by adopting a definite attitude in the foundations of mathematics. In *Foundations* the problem will be discussed from various points of view. In particular, the task of developing the notions of finite set and of finite cardinal or ordinal on the basis of an axiomatic foundation of the theory of sets, will be considered especially in connection with the multiplicative axiom. In this domain of finite sets and numbers,

[1]) For an attempt to base arithmetic, i.e. the theory of finite cardinals, on the notion of *part* (relation of a whole to its parts) instead of the notion of element (relation of a set — here a finite set — to its elements), cf. Foradori 3.

in spite of its apparently elementary character, there are still a number of unsolved problems.

In the present subsection only a few informal remarks and some superficial information on the subject of finite sets, including mathematical induction [1]), shall be given.

A great many different definitions for the finiteness of a set can be given, and they may be classified, for instance, according to the intricacy of the concepts involved in the definition (one-to-one correspondence, order, etc.); or according to the possibility of deriving other definitions (i.e. criteria) from the chosen definition without using non-elementary principles [2]); or according to the psychological obviousness of the definitions; etc. The foremost demand of simplicity to be made would be that the definition of finiteness should not use the existence or the properties of *infinite sets*. A lucid and rather comprehensive survey of definitions of finiteness with full proofs of the necessary steps of transition, has been given by Tarski [3]).

Besides the definition of a finite set as a doubly-well-ordered set, two thoroughly different definitions have already been discussed in the present book (§ 2, **5**): a finite set as an "inductive", and as a "non-reflexive" set. The latter definition is generally associated with Dedekind's name [4]). There exists, however, a second definition of finiteness by Dedekind [5]) which in principle, viz. for its logical

[1]) The main literature concerning the *separate axiomatic foundation of natural number* as conceived by Peano and Pieri, will be quoted in *Foundations*. For the connection between this foundation and the principle of mathematical induction, as well as for a few related matters, cf. Lorey **1**; Padoa **4**; Sarantopoulos **1**; Zapelloni **1**.

[2]) In this direction the definition by "non-reflexivity", though most famous and widely used, is inferior to most other definitions — especially owing to its purely negative character.

[3]) Tarski **4** and **28**. See, among earlier investigations since the beginning of this century, Hadamard **3**; Zermelo **4**; Poincaré **3**; Grelling **1**; Bernays **14**, II, p. 17; Enriques **3**; Natucci **2**. A quite elementary exposition where arithmetic is based on the notion of "doubly-well-ordered set" is given in the first chapter of Weber–Epstein **1**.

[4]) See the historical references given on p. 40.

[5]) Given in 1889, first published in the 2nd edition (1893) of Dedekind **2**. Cf. Dedekind **3**, III, pp. 450—458, and Cavaillès **1** where (as well as in Tarski **4**) the *inductive* and the *non-reflexive* characterizations are derived from Dedekind's second definition. Dedekind himself derives a series of consequences from his second definition without using natural numbers.

implications, is simpler than the first. It reads: a set F is finite, if it can be represented on itself in such a way that no proper subset of F is represented on itself or on a proper subset of itself.

Quite a few philosophical writers, but also some eminent mathematicians, have declined to use the method of basing the concepts of finite set and finite number on general set-theoretical notions, stressing the more elementary and less abstract character of the former as compared to the latter. Is it sound, asks Poincaré, to explore the realm of sets by means of general concepts and methods, disregarding the very notions of finite and infinite or postponing their explanation to a late stage of the excursion, finally to discover our well-known finite numbers in a modest corner of the huge domain? Such arguments usually originate from psychological or didactic sources. The real question, however, is not whether one might discover, or should introduce, ordinary number in this way, but whether it actually is logically possible to place the integers into the more comprehensive frame of aggregates and their cardinals and ordinals without a vicious circle [1]). The intuitionistic schools (of mathematicians as well as of philosophers) will reject such a procedure *a priori*, as they consider finite number and mathematical induction to be the very *origin* of infinity, and of mathematics in general. For a less dogmatic attitude, however, the problem is a serious and profound one. Leading scholars from Descartes [2]) up to Peirce, Dedekind, Frege, B. Russell have successfully tried to insert finite number into the realm of set-theory and, by this method, to *define* natural numbers and to *prove* mathematical induction, tasks not accomplished by Peano's axiomatization. The last word on these problems has not yet been said, and the discussions, especially in philosophical literature, are still continuing [3]).

[1]) Of course, the question does not touch the use of a few (small) numbers or numerals which may be introduced independently, or even the use of a particular finite collection of numbers. It is the *general* concept of integer that is under discussion.

[2]) Cf. the references and quotations given in Weyl 7, No. 8.

[3]) Apart from the writings mentioned in § 4, 7 and in § 8, 1 and from the intuitionistic literature (*Foundations*, ch. IV) let us mention among the mathematical essays on the foundation of number: Hessenberg 7 and 11; Lebesgue 7; Mollerup 2; Schuh 1. Since the beginning of the present century the following treatments from a philosophical point of view, among others,

Exercises

1) Prove that in any set satisfying the condition I* on p. 242 f., the order is *transitive* (hence the set is well-ordered, according to definition I).

2) Let S and T be similar well-ordered sets, s an element of S, t its mate in T (cf. theorem 13). Prove that the section of S determined by s, is similar to the section of T determined by t. (This fact is at the bottom of the proof of theorem 8.)

3) Prove the theorem (implicitly contained in the proof of theorem 8): If to every section of a well-ordered set S there corresponds a similar section of the well-ordered set T, and vice versa, then S and T are similar.

4) Prove (cf. the remark on p. 269) that every non-empty subset of an ordered set S has a first or a last element if, and only if, S can be expressed as the ordered sum S_1+S_2 of two mutually exclusive ordered sets, of which S_1 is well-ordered and S_2 "anti-well-ordered" (p. 269).

5) Prove that all well-ordered sets S, but not all *ordered* sets (give a counter-example!), have the following property: Any set T of sections of S contains a *least* section, which is the inner product (meet) of all elements of T.

6) An ordered set of the type $*\omega + \omega$, or of the type η, admits infinitely many different similar representations on itself (or on a similar set), in contrast to the property of well-ordered sets expressed in theorem 13. Show this and look for the source of the contrast.

7) Prove theorems 13 and 14, without using theorem 12, relying on the fact that rule (*) on p. 255 is fulfilled by any similar representation between similar well-ordered sets.

8) Starting from any of the three definitions of a finite set mentioned above (inductive set, doubly-well-ordered set, non-reflexive set), try to prove the two others as criteria ("a set is finite if, and only if, etc.").

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have appeared: Beth **7**; **Brunschvicg 1**; Conger **1**, ch. 28; **Couturat 1** and **2**; Dingler **7**; **Farber 4**; Mouy **1**; A. Müller **1**; Petronievicz **1**; Poirier **3** (contains many errors); Rickert **1**; Roux **1**; Spaier **1**; Wäsche **1**. (The most important ones are printed in bold type.)

9) Clarify the intrinsic difference between those three definitions of finiteness by examining the difficulty of proving, by means of each of them, the (apparently obvious) theorem that the power-set of a finite set is again finite.

§ 11. Ordinals and Alephs. The Well-Ordering Theorem

1. The Arrangement of Ordinals according to Magnitude. The *equality* between ordinals, as well as between order-types in general, is defined by the *similarity* of (well-)ordered sets with the respective ordinals (or types). We had to put up with the impossibility of establishing a reasonable order among transfinite order-types in general. For ordinals, however, this task may easily be achieved by

Definition I. Let S and T be well-ordered sets of the ordinals σ and τ. If S is similar to a section [1]) of T, σ is called *smaller* than τ; in symbols, $\sigma < \tau$ [2]).

As before, we express the relation $\sigma < \tau$ by also writing $\tau > \sigma$, to be read: τ is *larger* than σ [3]).

We are to show that the arrangement thus defined of (finite and transfinite) ordinals "according to magnitude" fulfils the general conditions required of any order-relation (p. 90 and 176). Firstly, our relation is *irreflexive*, i.e. $\sigma < \sigma$ never holds, since a well-ordered set S cannot be similar to a section of S, according to theorem 14 on p. 266. Secondly, the relation is *asymmetrical*, i.e. $\sigma < \tau$ and $\tau < \sigma$ exclude each other. For their compatibility would mean that S is similar to a section T_0 of T, and T to a section S_0 of S; then, by successively using the [4]) representations expressing these similarities, one would obtain a similar representation of S on a

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[1]) Because of the nature of similarity, we may in this case choose the representative S of σ in such a way that S itself is a section of T.

[2]) Obviously, the definition does not depend on the choice of the representatives S and T of σ and τ.

[3]) Here we use the same names and symbols which have been introduced for *cardinals* in § 5. Nevertheless, a confusion cannot occur except possibly for *finite* cardinals and ordinals for which no harm will ensue.

[4]) The definite article, here and later on, is justified by theorem 13 on p. 265.

section of a section of S, i.e. on a section of S — contrary to theorem 14.

Finally, the same train of ideas shows the *transitivity* of the relation, which means that the relations $\sigma < \tau$ and $\tau < v$ together imply $\sigma < v$.

That $\sigma < \tau$, $\sigma = \sigma'$, $\tau = \tau'$ together imply $\sigma' < \tau'$, immediately follows from the notion of similarity.

We have not mentioned the main condition to be fulfilled, viz. that between any two different ordinals σ, τ *at least one* of the relations $\sigma < \tau$, $\tau < \sigma$ holds. This indeed is the strongest of the properties in question, meaning universal comparability. With respect to cardinals, we have not yet succeeded in verifying it. For ordinals, however, this property has been expressed by theorem 8 on p. 254, the theorem of comparability. In fact, if the arbitrary well-ordered sets S and T fulfil the relation $S \simeq T$, this means $\sigma = \tau$; the two remaining cases where S is similar to a section of T, or T similar to a section of S, yield the relations $\sigma < \tau$ or $\tau < \sigma$.

Although we have fully justified definition I, we shall not use the comparability of ordinals until we have proved it in 2 by a new method, different from the one applied in § 10, 3.

We immediately see that definition I arranges *finite* ordinals (including 0) in conformity with the ordinary succession of integers. Each of the ordinals 0, 1, 2 is, for instance, smaller than 3, since the sections of the well-ordered set $\{s_1, s_2, s_3\}$ (being of the ordinal 3) which are respectively determined by the elements s_1, s_2, s_3, have the ordinals 0, 1, 2 respectively. Furthermore, we obtain without using comparability:

THEOREM 1. For any ordinal σ, $\sigma + \tau > \sigma$ holds, provided $\tau \neq 0$. Conversely, if $\varrho > \sigma$, there exists an ordinal τ such that $\varrho = \sigma + \tau$. $\sigma + 1$ is the sequent of σ, i.e. there is no ordinal between σ and $\sigma + 1$. ω is the smallest transfinite ordinal, and any other transfinite ordinal is of the form $\omega + \tau$ where $\tau \neq 0$. Any finite ordinal is smaller than any transfinite one.

All these statements are obtained by means of definition I, theorem 11 (p. 263) and the transitivity of the relation $<$; a brief exposition will suffice. If the ordered sum $S + T$ is a representative of $\sigma + \tau$, the first element of T determines the section S; hence $\sigma < \sigma + \tau$. On the other hand, if $\varrho > \sigma$, i.e. if S is similar to a section R_0 of

R ($\overline{S} = \sigma$, $\overline{R} = \varrho$), R may be written as an ordered sum $R = R_0 + R_1$ where R_1 is not empty. Hence, denoting the ordinal of R_1 by τ, we obtain $\varrho = \sigma + \tau$. If R_1 contains only one element, we obtain the ordinal $\sigma + 1$ [1]); on the other hand, $\varrho > \sigma$ (i.e. $\varrho = \sigma + \tau$, $\tau \neq 0$) implies $\sigma + 1 \leqq \varrho$. By the decomposition $S = S_0 + S_1$ ($\overline{S}_0 = \omega$) assured by theorem 11 for any transfinite S, we have $\sigma = \omega + \sigma_1$, where $\sigma_1 = \overline{S}_1$; hence, according as S_1 is the null-set or not, we have $\sigma = \omega$ or $\sigma > \omega$. If N is a finite ordered set of the ordinal n, and E an enumerated set ($\overline{E} = \omega$), the $(n + 1)$st element of E determines a section which is similar to N. Hence $n < \omega$; since $\omega \leqq \sigma$ for a transfinite σ, we obtain $n < \sigma$. Q.E.D.

While by theorem 1 every ordinal has a *sequent*, there are ordinals, e.g. 0, ω, $\omega + \omega$, which have *no immediate predecessor*. On the other hand, any finite ordinal $\sigma = n > 0$, as well as e.g. $\sigma = \omega + n$, does have an immediate predecessor $\sigma - 1$, such that $(\sigma - 1) + 1 = \sigma$. Since these two kinds of ordinals are distinct in many respects, we introduce a special name the choice of which will be understood in the light of 3.

DEFINITION II. A transfinite ordinal which has no immediate predecessor, is called a *limit-number*.

Hence any ordinal σ different from 0 either has an immediate predecessor $\sigma - 1$, or is a limit-number.

2. The Comparability of Ordinals.

Without using theorem 8 *of* § 10 (p. 254), we shall prove the fundamental

THEOREM 2. If σ is any ordinal, the set of all ordinals smaller than σ can be ordered according to the magnitude of the ordinals. Thus we obtain an ordered set, to be denoted by $W(\sigma)$; this set is well-ordered and has the type (ordinal) σ.

Before proving the theorem, let us illustrate it by means of some small values of σ. $W(0)$, containing no ordinal whatsoever, is the null-set, $W(1)$ the set $\{0\}$ which has the type 1. The ordinals smaller than 2 are 0 and 1; arranged according to their magnitude (i.e. in accordance with definition I) they constitute the ordered set

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[1]) An ordinal of the form $\sigma + 1$ (which is an abbreviation for "a well-ordered set of the ordinal $\sigma + 1$") always has a *last* element. The same applies to $\sigma + n$, if n is a finite ordinal $\neq 0$.

$\{0, 1\}$ which has the type 2. If n is any natural number conceived as a finite ordinal, we obtain the ordered set $\{0, 1, \ldots, n-1\}$ the type of which is exactly n, by arranging the ordinals $< n$ according to their magnitude. By the way, these examples again confirm (cf. p. 151) the importance of admitting the null-set, and its cardinal and type, among the sets, cardinal numbers, and order-types. Had we — as Cantor did — acknowledged as sets only proper aggregates, i.e. those which contain elements, the set of all numbers smaller than 2 would be of the type 1, which is smaller than 2 — contrary to the situation with transfinite ordinals as will presently appear.

In view of theorem 1, the set of all ordinals smaller than ω is the set of all *finite* ordinals; arranged according to magnitude, they form the sequence $W(\omega) = \{0, 1, 2, 3, \ldots\}$ which has the type $\overline{W(\omega)} = \omega$. In the same way we have $W(\omega + 1) = \{0, 1, 2, 3, \ldots \omega\}$ with $\overline{W(\omega + 1)} = \omega + 1$; for, in view of theorem 1, the only *transfinite* ordinal smaller than $\omega + 1$ is ω.

Proof [1]) of theorem 2. Let S be any well-ordered set of the type σ. μ being an element of $W(\sigma)$ — i.e. μ being an ordinal smaller than σ — means, according to definition I, that S has a section of the type μ; incidentally only one such section, by theorem 14 on p. 266. Hence, if μ_1 and μ_2 are different elements of $W(\sigma)$, they are the ordinal numbers of different sections of S, one of which is a section of the other on account of the (quite elementary!) theorem 3 on p. 245. Therefore *one of the ordinals μ_1 and μ_2 is smaller than the other* [2]), and the smaller one precedes in the ordered set $W(\sigma)$, as stated in theorem 2; in other words, $\mu_1 \prec \mu_2$ holds in $W(\sigma)$ if, and only if, $\mu_1 < \mu_2$ by definition I. Hereby an order has been defined; again the transitivity follows at once from theorem 3 on p. 245.

We shall now establish a similar representation between the sets S (well-ordered) and $W(\sigma)$ (ordered at any rate). Given any

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[1]) Readers who have not skipped the proof of theorem 8 on p. 254, will notice the far-reaching analogy between that proof and the present one, though the latter is easier and more elementary. In fact, both proofs yield the comparability: the former directly, the latter subject to some completions.

[2]) It is decisive that the property of μ_1 and μ_2 "being smaller than the same ordinal σ" enables us to conceive them as *ordinals of sections of the same set S*, which at once guarantees their comparability.

$\mu \varepsilon W(\sigma)$, i.e. any $\mu < \sigma$, we relate to μ the element of S which determines the section with the ordinal μ. This rule evidently relates to any element s of S a fixed element of $W(\sigma)$, viz. the type of the section of S determined by s. Finally, our biunique correspondence is also similar. To prove this, let s_1 and s_2 be different elements of S, say $s_1 \supset s_2$, σ_1 and σ_2 their mates in $W(\sigma)$; then we have $\sigma_1 < \sigma_2$, hence $\sigma_1 \supset \sigma_2$ in $W(\sigma)$, because the section of S determined by s_1 is a section of the section determined by s_2. Since S is well-ordered and has the ordinal σ, the same applies to $W(\sigma)$. Q.E.D.

Let us add two remarks to theorem 2. First, it allows us to write all elements of a well-ordered set with the type σ by means of a single letter, say s, to which there are attached indices (suffixes) μ running over all ordinals up to, but excluding, σ. The index μ of s_μ indicates the element of $W(\sigma)$ to which s_μ has been related by our similar correspondence. Thus we obtain a quite natural generalization of the usual writing of sequences (s_k) in analysis.

A more important result is that theorem 2 enables us to define and write ordinals *in an explicit way*: as the set of all *preceding* (smaller) ordinals. Ordinals are thus explicitly and completely determined. In view of this procedure, it is sufficient to introduce *one primitive symbol only* to denote the smallest ordinal, the ordinal of the null-set, which is denoted by 0. (The null-set itself may symbolize this smallest ordinal.) Therefore the ordinals 0, 1, 2, 3, 4, ... are to be written as the sets

$$0$$
$$\{0\}$$
$$\{0, \{0\}\}$$
$$\{0, \{0\}, \{0, \{0\}\}\}$$
$$\{0, \{0\}, \{0, \{0\}\}, \{0, \{0\}, \{0, \{0\}\}\}\}$$
$$\cdot$$
$$\cdot$$
$$\cdot$$

Hence ω is the set containing all ($\aleph_0$) these sets, while $\omega + 1$, for instance, requires the addition of the denumerable set just mentioned to the other elements of ω. Incidentally, the ordering of these sets is easy since, of any two different ordinals written

in this form, the one is an *element* as well as a *proper subset* of the other, and the former naturally is the one to precede the latter, whenever order is concerned. On the other hand, if we conceive an ordinal σ as an ordered set in this sense, any ordinal $\xi < \sigma$ (i.e. any $\xi \varepsilon \sigma$) has the property that the section determined by ξ equals ξ.

The advantage in principle of thus directly introducing the ordinals lies in that we get rid of the "definition by abstraction", which in § 8 led us to the concept of order-type, including ordinal. In § 4, **6** and **7**, we recognized the considerable logico-mathematical difficulties involved in definitions by abstraction. Now here a direct, explicit definition of ordinal has been hinted at, independent of the notion of order-type but using theorem 7 of p. 250 [1]). In *Foundations* this subject will be discussed in detail.

Proceeding now to comparability, let us denote any two ordinals by σ and τ. On account of theorem 2, the well-ordered sets $W(\sigma)$ and $W(\tau)$ may serve as *standard sets* of the types σ and τ. Both these sets "begin", in the sense of order, with the elements $0, 1, 2, \ldots \omega$ etc., provided that none of the sets is exhausted earlier. We shall now show that the coincidence between their elements is not restricted to the "beginning", but continues until at least one of the sets is exhausted.

Let M be the well-ordered meet (product) of the sets $W(\sigma)$ and $W(\tau)$; M is not empty, except for the case where one of these sets is empty itself. M is not only a *subset* of $W(\sigma)$ as well as of $W(\tau)$, but even an *initial* of both these sets; for, if m is an arbitrary element of M, any element of $W(\sigma)$ preceding m is also an element of $W(\tau)$, and vice versa — therefore an element of M. We distinguish between the two cases, $M = W(\sigma)$ and $M \subset W(\sigma)$.

If $M = W(\sigma)$ then, since $M \subseteq W(\tau)$, $W(\sigma)$ is a subset of $W(\tau)$. Hence either $W(\sigma) = W(\tau)$, i.e. $\sigma = \tau$; or $W(\sigma) \subset W(\tau)$, i.e. $W(\sigma)$ is a proper initial of $W(\tau)$, therefore a section. Hence $\sigma < \tau$.

[1]) In 1923 von Neumann (cf. **1** and **6**, also **2** and **4**) pointed out this important method of introducing ordinals within either a constructive or an axiomatic system of set theory. Cf. also Quine **25**, p. 147 and Rosser **8**. For another attempt at a direct definition of ordinals (as hypercomplex numbers with infinitely many units) see du Pasquier **2**.

Without referring to ordered sets and types, we may also introduce ordinals by an axiomatization of the concept of ordinal itself; this was done by Tarski, see Tarski **5** and Lindenbaum–Tarski **1**.

If, secondly, M is a proper subset (initial) of $W(\sigma)$, it is a section of $W(\sigma)$. In view of $M \subseteq W(\tau)$, we can again distinguish between the two cases $M = W(\tau)$ and $M \subset W(\tau)$. The first means that $W(\tau)$ is a section of $W(\sigma)$, therefore $\tau < \sigma$. The second case means that M is a section of $W(\sigma)$ as well as of $W(\tau)$, i.e. that the ordinal μ of M is smaller than both σ and τ. But since M is an initial of $W(\sigma)$, it is the set $W(\mu)$ of all ordinals smaller than μ. Therefore *this case is impossible*, as was the (corresponding) "fourth" case in the scheme on p. 258 [1]). For μ itself, which is smaller than σ and τ, is not contained in $M = W(\mu)$, contrary to the definition of M as the meet of $W(\sigma)$ and $W(\tau)$, both of which contain μ. Hence:

THEOREM 3. (*Comparability of ordinals.*) Either two ordinals equal each other, or one of them is smaller than the other.

This theorem is identical with theorem 8 on p. 254. However, an altogether different method of proof has been taken here, which possibly is more easily comprehensible. To transform it into the language of § 10, we only have to replace σ and τ by representatives S and T with these ordinals; in view of definition I, theorem 3 then asserts that either $S \simeq T$, or one of these sets is similar to a section of the other.

We conclude this subsection with a few simple theorems on *sets of ordinals*, in addition to theorem 2.

THEOREM 4. Any set of ordinals is well-ordered, provided the ordinals are arranged according to their magnitude. In other words, in any set of ordinals there is a smallest ordinal.

Proof. Let R be such a set, R_0 a non-empty subset of R, and ϱ any element (ordinal) of R_0. The (ordered) meet M of R_0 and $W(\varrho)$ is not empty, except for the case where ϱ is the first element of R_0. Otherwise M, as a subset of the well-ordered set $W(\varrho)$, has a first element which at the same time is the first element of R_0 — since $W(\varrho)$ contains every ordinal smaller than ϱ. Hence the arbitrary subset R_0 of R has a first element. Q.E.D.

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[1]) In the present notation the scheme would read as follows:
a) $\mu = \sigma$, $\mu = \tau$: $\sigma = \tau$;
b) $\mu = \sigma$, $\mu < \tau$: $\sigma < \tau$;
c) $\mu < \sigma$, $\mu = \tau$: $\sigma > \tau$;
d) $\mu < \sigma$, $\mu < \tau$.

THEOREM 5. The sum resulting from addition over a set of ordinals which are arranged according to magnitude, is either larger than any term of the sum, or else it is the largest ordinal of the set.

Proof. Otherwise, owing to the comparability theorem, the sum would be *smaller* than a certain term. Since by the preceding theorem the sum is an ordinal, the transition from ordinals to "set-representatives" would yield the result that a well-ordered set (the ordered sum of the representatives) is similar to a section of a subset — contrary to theorem 14 on p. 266. Q.E.D.

It is evident that the same holds if some of the terms to be summed up are equal; that is to say, if the terms are given not as *the elements* of a well-ordered set but as *attached* to those elements; cf. p. 197.

THEOREM 6. To any given set R of ordinals, there exist still larger ordinals; in particular a (uniquely determined) *sequent* ordinal.

Proof. After having arranged the elements of R according to their magnitude, we form a related set R' of ordinals by replacing each $\varrho \varepsilon R$ with $\varrho + 1$ [1]). By the preceding theorem, the ordered sum σ of all elements of R' is *larger* than any $\varrho \varepsilon R$, since by theorem 1 (p. 275) $\varrho + 1 > \varrho$. It may be that σ is just the sequent of the ordinals of R, in the sense that there is no ordinal larger than all $\varrho \varepsilon R$ and smaller than σ. Otherwise let R^* be the (non-empty) subset of those elements of $W(\sigma)$ which are larger than all elements of R. R^* has a first element σ_0, which is the smallest among the mentioned elements of $W(\sigma)$; hence σ_0 is the desired sequent of all $\varrho \varepsilon R$. Q.E.D.

If σ_0 is the sequent of all ordinals ϱ_ν of R, we may distinguish between two cases: 1) R contains a *largest* ordinal $\bar{\varrho}$; then we simply have $\sigma_0 = \bar{\varrho} + 1$. 2) R contains *no* largest ordinal; in analogy to the usage in analysis we then write (provided $R \neq 0$):

$$\sigma_0 = \lim_{\varrho_\nu \varepsilon R} \varrho_\nu \qquad \text{or} \qquad \sigma_0 = \lim \varrho_\nu \text{ or } \sigma_0 = \lim \varrho_\nu.$$

If σ_0 is given, we may choose R as the set $W(\sigma_0)$, but we need

[1]) Actually, this device is only required in case R contains a largest ordinal (a maximum).

not do so; e.g., for $\sigma_0 = \omega$, we may as well choose R as the set of all *even* finite ordinals or of all prime numbers.

In the second case (and only in this one) σ_0 is easily seen to be a *limit-number* (see p. 276); we thus perceive the reason for adopting this name.

In view of theorem 6 we may express theorem 2 also in the following form:

Corollary to theorem 2. A set of ordinals containing, besides any element v, also all ordinals $< v$, can be ordered according to the magnitude of its elements. The set W thus arising is well-ordered and its ordinal is the *sequent* of all elements of W. In symbols: $W = W(\sigma)$, where $\sigma = \overline{W}$ [1]).

This corollary is superior to theorem 2 in that the assumption does not yet contain the ordinal σ, which instead is *defined* as the ordinal of W.

One might propose to replace the beginning of the corollary by the simple expression "An *initial* of the ordered set of all ordinals". This formulation, however, is not permissible. For theorem 6 shows the surprising fact:

The totality of all ordinals does not constitute a set.

This result is rather shocking since the totality in question does constitute a set on account of Cantor's definition of set (p. 6). In fact, the property of an object of being an ordinal is a definite one, and equality and inequality between ordinals are well defined. Thus here we have an obvious *antinomy of the concept of set.* The implications of this and other antinomies will claim our attention in *Foundations.* We might reassure ourselves assuming — not without good reason — that the *boundlessness* of the notion "set of *all* ordinals" is responsible for the antinomy. However, this argument is somewhat dangerous, since it is also applicable to the notion "set of *all finite* ordinals".

On the other hand, the semi-constructive way in which we have developed the theory of sets, basing our procedure (explicitly or implicitly) [2]) on a small number of principles instead of the mere

[1]) This proposition constitutes the suitable starting-point for an explicit definition of ordinals in von Neumann's sense; see p. 279. $W(\sigma)$ itself is then considered as the ordinal σ (of W as well as of any similar set).

[2]) For a stricter development, see *Foundations*, ch. II.

concept of set, thoroughly removes the danger of such antinomies. In fact, our principles do not justify the formation of a set containing *all* ordinals [1]; they only permit us to form sets related to sets already secured (such as their subsets, sum-sets, power-sets), with the null-set and a denumerable set as the only starting-points.

We shall not use the totality of all ordinals as a mathematical entity. Nevertheless we shall sometimes find it useful to speak of it for the sake of brevity; then we are replacing the mathematical term "set" by the hazy word "totality" or (when referring to the succession of ordinals) by "series".

3. Exponentiation. The Series of Ordinals. The difficulties connected with general *products*, and therefore with *powers*, of order-types have been hinted at on p. 210. The special kind of types called ordinals makes no exception, and we shall not treat their multiplication in general, when infinitely many factors are involved [2].

Nevertheless, the *powers* of ordinals are of considerable importance; in particular the powers with the base ω. Before defining them, let us begin with a general remark which is useful for handling ordinals. Whenever we are to apply the method of defining, or proving, by transfinite induction with respect to *ordinals* (instead of elements of any well-ordered sets; pp. 248 and 250), i.e. whenever we are to define a function $f(\alpha)$ or to prove a proposition $\mathfrak{P}(\alpha)$ the argument of which is a variable ordinal $\alpha \leqq \alpha_0$ [3]), we use a process which refers to all ordinals *preceding* the ordinal α in question, and in particular to $\alpha = 0$ — in accordance with theorems 6 and 7 of § 10. Now we may separately handle the cases where α either has an immediate predecessor (i.e. where $W(\alpha)$ contains a last element), or is a limit-number. In the former case, we can proceed to α from $\alpha - 1$; in the latter, from a set of ordinals β_ν the sequent of

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[1]) To obtain the set of all finite numbers, a specific principle (principle of infinity, § 2) was required.

[2]) Such products may be defined without difficulty by means of transfinite induction. But then they turn out to be "pseudo-products" in analogy to the pseudo-powers to be defined presently.

[3]) The assumption of an (arbitrary) upper bound α_0 for all ordinals α to be considered is intended to avoid an implicit reference to the totality of all ordinals.

which is α, i.e. for which $\lim \beta_\nu = \alpha$ [1]). Accordingly, we may introduce the power in the following way:

DEFINITION III. If σ denotes an arbitrary ordinal $\neq 0$ [2]), we define the power $p(\alpha) = \sigma^\alpha$ in an inductive way by

a) $p(0) = \sigma^0 = 1$;

b) $p(\alpha) = p(\alpha - 1) \cdot \sigma = \sigma^{\alpha-1} \cdot \sigma$ for α having an immediate predecessor $\alpha - 1$;

c) $p(\alpha) = \lim\limits_{\beta \varepsilon W(\alpha)} p(\beta) = \lim \sigma^\beta$ for limit-numbers $\alpha = \lim \beta$ [3]).

A few examples may illustrate this definition. Whenever α is finite, the definition obviously concurs with the "elementary" definition of power (p. 261), based on the notion of the product of two ordinals or types. The power n^ω, where n is a finite ordinal > 1, is according to c) the sequent of all powers n^k for all $k < \omega$, i.e. for all finite k; the sequent being ω, we have $n^\omega = \omega$. On the other hand, ω^ω is not one of the ordinals which have already occurred; it is *defined* by our procedure, for instance as the sequent of all ω^k, k running over all finite ordinals. As is easily perceived, one might as well say: ω^ω is the sequent of all products $\omega^k \cdot l$, k and l running over all finite ordinals; etc. Also the enumerated sum $1 + \omega + \omega^2 + \ldots + \omega^k + \ldots$ equals ω^ω. Furthermore we have: $n^{\omega+1} = n^\omega \cdot n = \omega \cdot n$, $\omega^{\omega+1} = \omega^\omega \cdot \omega$.

The introduction of power by definition III (which is the definition usually given) requires some additional explanations, for the existence of the function p does not follow *directly* from the general theorem on defining by transfinite induction (theorem 7 on p. 250 f.). Indeed, the rule given by b) and c) above imposes on p two conditions for which there is no provision in theorem 7: b) makes sense only if the values $p(\alpha - 1)$, i.e. the values of p for certain ordinals, are again ordinals, and c) also requires that, if α is a limit-number, the values of p for the ordinals smaller than α form a set of ordinals without a largest element — otherwise the expression $\lim\limits_{\beta<\alpha} p(\beta)$ would be meaningless. In order to ensure that a function with these properties exists, we should first define p by a rule (definition III*) which is in accordance with theorem 7 and does not rely on any additional properties of p, and we should then prove that the function p so defined has the properties required by definition III.

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[1]) Such a set of ordinals is, e.g., $W(\alpha)$; any other such set may be taken as well, provided $\lim f(\beta_\nu)$ is not affected by the change.

[2]) We may add: $0^\alpha = 0$ for any $\alpha > 0$.

[3]) Here we exclude the trivial case $\sigma = 1$ in which $p(\alpha) = 1$ for any α.

We may, for instance, stipulate that the product $p(\alpha - 1)\cdot\sigma$ in b) is to mean an arbitrarily fixed ordinal for values $p(\alpha-1)$ which are not ordinals, and we may define $p(\alpha)$ in c) as the sequent of all $p(\beta)$ for which $\beta < \alpha$, taking an arbitrarily fixed ordinal as the sequent of a set which contains no ordinals, or a largest ordinal.

In view of theorems 6 and 7, the rule expressed by the new definition III* uniquely determines the value of p for any ordinal α by the values of p for the ordinals smaller than α, whatever those values may be.

Now the properties of p can easily be established. All the values of p are also ordinals, and the expressions "$p(\alpha-1)\cdot\sigma$" and "the sequent of all $p(\beta)$ for which $\beta < \alpha$" always have their ordinary meanings; furthermore, p is a "monotonic" function [1]), i.e. the inequality $\gamma < \delta$ between ordinals always implies $p(\gamma) < p(\delta)$, as may be shown by induction with respect to δ. From these facts it follows immediately that p is in accordance with the description given in definition III; conversely, the uniqueness of the function described by definition III follows from the fact that all the values of any such function p' are ordinals and that p' has, therefore, the properties required by definition III* -- which, as stated on p. 251, determines p' uniquely. The proof that definition III is admissible is thus complete.

Another possible form of definition III* would define $p(\alpha)$ as 1, $\mu\cdot\sigma$, or the limit of the set of all $p(\beta)$ with $\beta < \alpha$, according as the latter set contains no ordinal, a largest ordinal μ, or ordinals among which there is no largest one. The proof that the function so defined is the function described by definition III can be carried out just as the proof outlined above -- except that the proof of the uniqueness of the new function must now use the fact that any such function is necessarily monotonic. This may be left to the advanced reader.

Our definition of power is not in accordance with the definition of a power of cardinals given in § 7. More exactly, if $\mathfrak{s}$ and $\mathfrak{a}$ are the cardinals corresponding to the ordinals σ and α ($\mathfrak{s} = \overline{\sigma}$, $\mathfrak{a} = \overline{\alpha}$), $\mathfrak{s}^{\mathfrak{a}}$ in general is *not* the cardinal corresponding to the power σ^{α} just defined; in other words, the equality $\overline{\sigma^{\alpha}} = \overline{\sigma}^{\overline{\alpha}}$ does not generally hold. The simple instances given above suffice to show this discrepancy. The cardinal of $2^{\omega} = \omega$ is $\aleph_0$, while [2]) $\overline{2} = 2$, $\overline{\omega} = \aleph_0$, hence $\overline{2}^{\overline{\omega}} = 2^{\aleph_0} = \aleph > \aleph_0$ (p. 92). For the power ω^{ω}, we obtain $\aleph_0^{\aleph_0} = \aleph$ (p. 158) by the transition to the cardinals of base and exponent. Nevertheless, the cardinal corresponding to ω^{ω} is only $\aleph_0$, as we may show directly by ordering the set of all natural

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[1]) Cf. what is said on p. 287 about *normal functions*.

[2]) The reader should not be embarassed by the fact that the natural numbers here denote finite *ordinals* as well as their *cardinals*.

numbers according to the ordinal ω^ω, as follows. (See also theorem 12 on p. 299.)

Beginning with 1, we first arrange all prime numbers according to magnitude, then all products of two (different or equal) primes, and generally the products of k primes after those of $k-1$ primes. To arrange the products of k primes (k fixed) among themselves, write the factors of a product in the order of increasing magnitude and let the products be ordered according to the magnitude of the *first different* corresponding factors. Thus we obtain the well-ordered arrangement

$$1;\ 2, 3, 5, 7, 11, \ldots;$$
$$4, 6, 10, 14, \ldots;\ 9, 15, 21, \ldots;\ 25, 35, \ldots;\ \ldots$$
$$8, 12, 20, 28, \ldots;\ 18, 30, 42, \ldots;\ \ldots;\ 27, 45, 63, \ldots;\ \ldots;\ 125, \ldots$$
$$16, 24, 40, \ldots;\ 36, \ldots;\ \ldots;\ 54, \ldots;\ \ldots;\ 81, \ldots$$
$$\vdots$$

Evidently the type of the ordered set containing all numbers preceding 2^k is ω^{k-1}; the ordinal of the set of all products of three primes, for instance, is $\omega^2 \cdot \omega = \omega^3$, since to any prime p there corresponds the double sequence (having the ordinal ω^2) $p \cdot r$, where r runs over all products of two primes neither of which is inferior to p [1]). Hence the ordinal of the entire set is the sequent of all powers ω^{k-1}, i.e. ω^ω by definition III. (Cf. the corollary on p. 282.)

Of course, such a discrepancy between cardinals and ordinals with respect to exponentiation is not welcome. However, it is not just transfinite induction that answers for this defect; we would not have had recourse to this method of defining exponentiation if the power, as introduced in § 7 by means of outer products or insertion-sets, enabled us to order and well-order the set involved. As has been pointed out at the end of § 8, it is an objective impossibility that prevents us from attaining "correct", i.e. comprehensive

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[1]) For instance, the set of the products of three primes which precede 27 has the ordinal ω^2, and the same applies to all products of less than three primes; hence the number 27 itself corresponds to the ordinal $\omega^2 \cdot 2 + 1$. The addition of the *preceding* products, in our case of those of *less than* three primes, does not change the result, in accordance with $\omega^2 + \omega^3 = \omega + \omega^2 + \omega^3 = 1 + \omega + \omega^2 + \omega^3 = \omega^3$.

enough powers. On the other hand, in the case of ordinal numbers the substitute-power as defined by Hausdorff for ordered sets and types in general, happens to coincide with the power originating from transfinite induction [1]). This rather complicated subject, however, will not be discussed here.

By b) and c) on p. 284, for $\sigma > 1$ (hence in particular in the most important case $\sigma = \omega$) the function $p(\alpha) = \sigma^\alpha$ has the properties

A) $p(\alpha_1) < p(\alpha_2)$ for $\alpha_1 < \alpha_2$,

B) $p(\lim_\nu \alpha_\nu) = \lim p(\alpha_\nu)$.

Any function $p(\alpha)$ of an ordinal-argument which has these two properties, is called a *normal function* [2]). This concept, which is analogous to the concept of monotonic continuous function in ordinary analysis, has great importance for profounder questions of the theory of ordinals. We easily see that, for instance, $\sigma + \alpha$ and $\sigma \cdot \alpha$ $(\sigma \neq 0)$ are normal functions of α, but not $\alpha + \sigma$ $(\sigma \neq 0)$, $\alpha \cdot \sigma$ $(\sigma > 1)$, α^2. For any normal function $p(\alpha)$ we have always $p(\alpha) \geqq \alpha$, as may be seen by means of theorem 12 (p. 264) or by transfinite induction; cf. p. 295.

We conclude this survey of exponentiation with stating its *formal laws*. At the same time this gives an opportunity of using transfinite induction in proofs and of showing the advantage of dividing ordinals into three classes, as has been done in definition III. We content ourselves with a sketch of the proofs, leaving some simple completions to the reader.

The formal laws to be proved are:

(1) $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$.

(2) $(\alpha^\beta)^\delta = \alpha^{\beta \cdot \delta}$.

The third law, possibly to be expected on account of its validity for finite numbers, namely $\varkappa^\mu \cdot \lambda^\mu = (\varkappa \cdot \lambda)^\mu$, *does not hold*, as shown by the simple example $\varkappa = \omega$, $\lambda = \mu = 2$; in fact $\omega^2 \cdot 4$ differs from $(\omega \cdot 2)^2 = \omega^2 \cdot 2$ (cf. p. 209).

To prove (1) [3]), let α and β be fixed ordinals; we use transfinite

[1]) See p. 210. Cf. also Hessenberg **4.**

[2]) This concept has been introduced by Veblen 3; for its importance cf. Jacobsthal 2 and 3; Hausdorff 4, p. 114 ff.; Sudan 3.

[3]) Rule b) on p. 284 is a specialization of (1) with $\gamma = 1$.

induction with respect to γ, referring to ordinals γ up to an arbitrary ordinal $\overline{\gamma}$, i.e. referring to $\gamma \varepsilon W(\overline{\gamma})$.

For $\gamma = 0$ (1) is certainly true since, in view of a) in definition III, (1) then means $\alpha^{\beta} \cdot 1 = \alpha^{\beta}$. Let therefore $\gamma = \gamma_0$ be any ordinal different from 0 and smaller than $\overline{\gamma}$. Assuming the truth of (1) for all ordinals $< \gamma_0$, we have to prove (1) for γ_0. We distinguish between γ_0 having an immediate predecessor, and γ_0 being a limit-number.

In the first case (1) holds for $\gamma = \gamma_0 - 1$ by hypothesis. Hence, in view of b) in definition III and of the associative law of multiplication,

$$\alpha^{\beta} \cdot \alpha^{\gamma_0} = \alpha^{\beta} \cdot (\alpha^{\gamma_0 - 1} \cdot \alpha) = (\alpha^{\beta} \cdot \alpha^{\gamma_0 - 1}) \cdot \alpha = \alpha^{\beta + (\gamma_0 - 1)} \cdot \alpha = \alpha^{(\beta + \gamma_0) - 1} \cdot \alpha = \alpha^{\beta + \gamma_0};$$

for, together with γ_0, $\beta + \gamma_0$, too, has an immediate predecessor, namely $\beta + (\gamma_0 - 1)$. (1) is therefore true for $\gamma = \gamma_0$.

If, secondly, γ_0 is a limit-number, the hypothesis that (1) holds for all $\gamma < \gamma_0$ in concurrence with $\lim_{\gamma_\nu \varepsilon W(\gamma_0)} \gamma_\nu = \gamma_0$ yields, in view of c) in definition III,

$$\alpha^{\beta} \cdot \alpha^{\gamma_0} = \alpha^{\beta} \cdot \alpha^{\lim \gamma_\nu} = \alpha^{\beta} \cdot \lim \alpha^{\gamma_\nu} = \lim (\alpha^{\beta} \cdot \alpha^{\gamma_\nu}) = \lim \alpha^{\beta + \gamma_\nu} = \alpha^{\lim (\beta + \gamma_\nu)} = \alpha^{\beta + \lim \gamma_\nu} = \alpha^{\beta + \gamma_0}.$$

This calculation uses the lemma that "a constant term or factor may be taken in or out" with respect to the limit of a sum or product; more exactly, that

$$(1') \qquad \varkappa + \lim_{\nu} \lambda_\nu = \lim_{\nu} (\varkappa + \lambda_\nu), \quad \varkappa \cdot \lim_{\nu} \lambda_\nu = \lim_{\nu} (\varkappa \cdot \lambda_\nu).$$

The proof of (2) is achieved in a similar way.

With the definition of power we are sufficiently prepared to study the series of ordinals, which is one of Cantor's most daring and most beautiful achievements, constituting a *continuation of the sequence of natural numbers beyond the infinite* (cf. p. 4). In view of the corollary on p. 282, the sequence of all finite ordinals is succeeded by the ordinal of this sequence, i.e. by ω; the inclusion of the element ω produces the ordinal $\omega + 1$ which immediately follows ω (as it is the sequent of ω according to magnitude), etc. All ordinals of the form $\omega \cdot k + l$ ($k = 0, 1$), where l runs over all

finite ordinals, form a set of the ordinal $\omega + \omega = \omega \cdot 2$, which, accordingly, is their sequent; if, on the other hand, k runs also over all finite ordinals, we obtain a sequence of sequences, producing the ordinal $\omega \cdot \omega = \omega^2$; etc. These remarks may suffice to explain the following "beginning" of the series of ordinals wherein k, l, m, m_ν denote finite ordinals [1]):

$$0, 1, 2, \ldots, k, k+1, \ldots \omega, \omega+1, \ldots, \omega+k, \ldots \omega \cdot 2, \omega \cdot 2 + 1,$$
$$\ldots \omega \cdot 3, \ldots, \omega \cdot k + l, \ldots \omega^2, \omega^2 + 1, \ldots, \omega^2 + \omega \cdot k + l,$$
$$\ldots, \omega^2 \cdot k + \omega \cdot l + m, \ldots \omega^3, \ldots \omega^k, \ldots, \omega^k \cdot m_k + \omega^{k-1} \cdot m_{k-1} + \ldots$$
$$+ \omega \cdot m_1 + m_0, \ldots \omega^\omega, \omega^\omega + 1, \ldots \omega^\omega + \omega^k, \ldots \omega^\omega \cdot k, \ldots$$
$$\omega^\omega \cdot \omega = \omega^{\omega+1}, \ldots \omega^{\omega \cdot k}, \ldots \omega^{\omega^2}, \ldots \omega^{\omega^k}, \ldots \omega^{\omega^\omega}, \ldots \omega^{\omega^{\cdot^{\cdot^{\cdot^\omega}}}}, \ldots$$

Of course, ω^{ω^ω} means here, as in arithmetic, the ω^ωth power of ω, and not the ωth power of ω^ω which, owing to (2), would "only" amount to $\omega^{(\omega^2)}$. The sequent of all those ordinals — i.e. the sequent of the sequence $(\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots)$ — is an ordinal which may be written, by means of a sequence of exponents, in the form

$$\varepsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}.$$

It cannot be expressed by ω (and finite ordinals) by means of applying addition, multiplication, and exponentiation a finite number of times. ε_0 has the property $\omega^{\varepsilon_0} = \varepsilon_0$ (analogous to $1 + \omega = \omega$); for, according to B) on p. 287, ω^{ε_0} is the sequent of the same set $W(\varepsilon_0)$ which defines ε_0 as its sequent. Any ordinal satisfying the relation $\omega^\varepsilon = \varepsilon$ was called by Cantor an *epsilon-number*; ε_0 is *the first* (*smallest*) *epsilon-number*. [2]) We shall see in **5** that ε_0, as well as many far larger ordinals, are still types of *denumerable* (though not of enumerated) sets.

The theorem of comparability (p. 280) and the corollary to theorem 2 (p. 282) form the basis of the uniqueness and definiteness[3])

[1]) Only those ordinals which are not limit-numbers are preceded by commas.

[2]) For a related kind of ordinals, see Sudan **6**.

[3]) The constructive character of the series of ordinals becomes particularly obvious by the *geometrical construction* given by Haenzel **1** for the ordinals of the "second number-class" (subsection **5**), which includes all ordinals introduced here and far beyond.

of this imposing construction. The construction cannot lead to incomparable ordinals, and at any stage of the development its continuation is not only guaranteed but uniquely prescribed: the *next* ordinal is the *type* of the set containing all ordinals constructed hitherto and ordered after their magnitude, according to definition I. Thus the ingenious idea of continuing the *process of counting* beyond the sequence of finite integers, conceived by Cantor at an early stage of his activity (7, V, 1884), has been carried out in a perfectly clear way. Instead of a loose and hazy notion of infinity, we obtain quite definite and distinct transfinite ordinal numbers, far beyond the "plain infinite" ω, and connected among themselves by precise relations and operations. We may also utilize these ordinals to "number" the elements of any well-ordered set, using them as indices; in other words, the transfinite ordinals are fit to count not only a set as a whole but its single elements as well.

4. Arithmetic of Ordinals. In this subsection, as well as in **5**, a small portion only of the abundant material shall be given. For additional details, which are mostly of a technical rather than fundamental character, the reader is referred to the expositions in books which are more comprehensive in this respect [1]).

THEOREM 7. *Inequalities between ordinals.*

A. Let α be any ordinal, σ and τ ordinals with $\sigma < \tau$; then:

1) $\alpha + \sigma < \alpha + \tau$;

2) $\sigma + \alpha \leqq \tau + \alpha$;

3) $\alpha \cdot \sigma < \alpha \cdot \tau$ if $\alpha \neq 0$;

4) $\sigma \cdot \alpha \leqq \tau \cdot \alpha$.

B. Conversely we have:

1) $\alpha + \sigma < \alpha + \tau$, or $\sigma + \alpha < \tau + \alpha$, implies $\sigma < \tau$;

2) $\alpha + \sigma = \alpha + \tau$ implies $\sigma = \tau$;

3) $\alpha \cdot \sigma < \alpha \cdot \tau$, or $\sigma \cdot \alpha < \tau \cdot \alpha$, implies $\sigma < \tau$;

4) $\alpha \cdot \sigma = \alpha \cdot \tau$ implies $\sigma = \tau$ if $\alpha \neq 0$.

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[1]) Hausdorff **4** or Sierpiński **6**. Cf. also Jacobsthal **2** and **3**; Sudan **3**—**5**; Sherman **1**; Eyraud **2**. The contents of subsection **4** will only be used in **5**.

C. The following "laws of monotony" hold true, if $\varkappa < \lambda$ and $\sigma < \tau$:

$$1)\quad \varkappa + \sigma < \lambda + \tau, \quad 2)\quad \varkappa \cdot \sigma < \lambda \cdot \tau.$$

Proof. Ad A. By theorem 1 (p. 275), $\sigma < \tau$ expresses the existence of an ordinal $\xi \neq 0$ such that $\sigma + \xi = \tau$. Hence

$$1) \text{ means: } \alpha + \sigma < \alpha + \sigma + \xi,$$
$$3) \text{ means: } \alpha \cdot \sigma < \alpha \cdot (\sigma + \xi) = \alpha \cdot \sigma + \alpha \cdot \xi \text{ (p. 205)},$$

both of which hold true.

As to 2) and 4), let A, S, T be sets of the ordinals α, σ, τ such that A and T are mutually exclusive while S is a section of T. Then for the ordered sums (p. 193) and outer products (p. 207) we have

$$S + A \subset T + A, \; S \times A \subseteq T \times A \; [1]),$$

which means 2) and 4) in view of the footnote on p. 266.

2) and 4) cannot be generally improved, as shown by the instances (n denoting any finite ordinal > 0):

$$0 + \omega = n + \omega, \; 1 \cdot \omega = n \cdot \omega.$$

Ad B. In view of comparability and of the fundamental properties of equality [2]), 1) — 4) follow from A by logical inversion. On the other hand, from $\sigma + \alpha = \tau + \alpha$ or $\sigma \cdot \alpha = \tau \cdot \alpha$ we cannot generally infer anything with respect to the relation between σ and τ, as shown by the above instances.

Ad C. The laws of monotony are those known from ordinary arithmetic. 1) follows from A 1) and 2), 2) from A 3) and 4). We have, for instance, $\varkappa + \sigma < \varkappa + \tau \leqq \lambda + \tau$. For 2) we need not presuppose that $0 < \varkappa$, $0 < \sigma$ (contrary to arithmetic where there are negative numbers); for in the case $\varkappa = 0$ or $\sigma = 0$ the truth of 2) is trivial.

THEOREM 8. *Subtraction.* If $\sigma < \tau$, the equation $\sigma + \xi = \tau$ has a uniquely determined solution ξ which is written as $\xi = -\sigma + \tau$. On the other hand, the equation $\xi + \sigma = \tau$ with $\sigma < \tau$ is not generally solvable [3]).

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[1]) The equality holds in the case $\alpha = 0$.

[2]) Namely, that $\sigma = \tau$ implies $\alpha + \sigma = \alpha + \tau$ etc.

[3]) It goes without saying that this does not affect the legitimacy of the special notation $\sigma - 1$ introduced at the end of 1 *for ordinals σ which are not limit-numbers* (and $\neq 0$).

Proof. The first assertion is true in view of theorem 1 on p. 275 and (for the uniqueness) of B 2) in the preceding theorem. The second assertion follows from examples as $\xi + n = \omega$, where n is a finite ordinal $\neq 0$; more generally from $\xi + \sigma = \tau$, where σ has a last element and τ is a limit-number, or vice versa. (The reason for this different behavior may be expressed by the statement "two ordinals are *always co-initial* but *not always confinal*".) Furthermore, *if* the equation is solvable, it is not uniquely solvable, except for finite ordinals; $\xi + \omega^2 = \omega^2 \cdot 2$, for instance, has the different solutions ω^2, $\omega^2 + n$, $\omega^2 + \omega$, etc.

THEOREM 9. *Division.* σ and δ ($\delta \neq 0$) being any ordinals, there is one, and only one, pair of ordinals $\varkappa$ and ϱ such that

$$(1) \quad \sigma = \delta \cdot \varkappa + \varrho \text{ with } \varrho < \delta.$$

Obviously $\varkappa = 0$ if, and only if, $\delta > \sigma$. In particular ($\delta = \omega$) there is a uniquely determined *finite* ordinal r such that

$$(2) \quad \sigma = \omega \cdot \varkappa + r.$$

This theorem is quite analogous [1]) to the following theorem on the division of integers which is the fundament of the multiplicative theory of numbers: if s and d ($d \neq 0$) are non-negative integers, there is one, and only one, pair of non-negative integers q and r such that

$$s = d \cdot q + r$$

with the remainder r smaller than the divisor d.

Proof of theorem 9. In order to form a product with the left-hand factor δ which will exceed the given ordinal σ, we take $\tau = \sigma + 1$; then by A 4) in theorem 7

$$\delta \cdot \tau = \delta \cdot (\sigma + 1) \geqq \sigma + 1 > \sigma.$$

Let D and T be mutually exclusive well-ordered sets of the types δ and τ. To obtain a representative of the product $\delta \cdot \tau$, we may

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[1]) This applies to the "left-hand division" dealt with in theorem 9. The "right-hand division" is certainly not unique, as is shown by the example $1 \cdot \omega = 2 \cdot \omega = \ldots$. Cf. exercise 4 at the end of § 11.

(see p. 207 f.) lexicographically order the set (outer product) the elements of which are the ordered pairs (t, d) with $t \varepsilon T$ and $d \varepsilon D$; denote this *well-ordered* set by $D \times T$. On account of $\sigma < \delta \cdot \tau$, there is a certain *section* S of $D \times T$ having the type σ; this section may be determined by the element (t_0, d_0) of $D \times T$. According to our lexicographic order, this section contains all those pairs (t, d) for which $d \varepsilon D$ and $t \prec t_0$ in T, and in addition the pairs (t_0, d) for which $d \prec d_0$ in D.

Now let $\varkappa$ be the type of the section of T determined by t_0, ϱ the type of the section of D determined by d_0. Naturally both are ordinals. According to what has just been said of the section S of $D \times T$ with the type σ, we have, in the sense of ordered addition of types,

$$\sigma = \delta \cdot \varkappa + \varrho$$

with $\varrho < \delta$.

Finally, this representation is *unique*; in other words, the assumption

$$\delta \cdot \varkappa_1 + \varrho_1 = \delta \cdot \varkappa_2 + \varrho_2 \qquad (\varrho_1 < \delta,\ \varrho_2 < \delta)$$

implies $\varkappa_1 = \varkappa_2$, and then as an immediate consequence $\varrho_1 = \varrho_2$. For, if $\varkappa_1 \neq \varkappa_2$, say $\varkappa_1 < \varkappa_2$ (hence $\varkappa_1 + 1 \leqq \varkappa_2$), were true, one would conclude, in view of A 3) and A 2) in theorem 7:

$$\delta \cdot \varkappa_1 + \varrho_1 = \delta \cdot \varkappa_2 + \varrho_2 \geqq \delta \cdot (\varkappa_1 + 1) + \varrho_2 = \delta \cdot \varkappa_1 + \delta + \varrho_2,$$

which implies $\varrho_1 \geqq \delta + \varrho_2$ by B 1) and B 2). But this contradicts our assumption $\varrho_1 < \delta$. Q.E.D.

The last assertion of theorem 9 is a consequence of the fact that all ordinals $< \omega$, and these ordinals only, are finite. From (2) we furthermore infer:

Corollary. An ordinal σ is a limit-number if, and only if, $r = 0$ in (2), i.e. if σ is "leftwards divisible" by ω.

By taking the finite divisor $\delta = 2$, we obtain from theorem 9 the possibility of distinguishing between *even and odd ordinals*, having the form $2 \cdot \varkappa$ or $2 \cdot \varkappa + 1$, according as the remainder ϱ is 0 or 1. ω, as well as every limit-number σ, is even, and also leftwards divisible by *any* finite ordinal $\neq 0$, as well as by ω; otherwise, taking as δ one of these divisors, we would obtain a finite remainder $\neq 0$, i.e. σ would be a number with a last element.

Let us add a remark for those of our readers who are acquainted

with the so-called *algorithm of Euclid.* (This procedure plays an important rôle in theory of numbers and in algebra, beginning with the search for the highest common divisor of two integers.) It is remarkable that in the domain of transfinite ordinals, as in ordinary arithmetic, the procedure of this algorithm can be carried out on account of theorem 9, because *it terminates after a finite number of steps.* Accordingly, starting from any two ordinals σ and δ ($\delta \neq 0$), it has the form:

$$\begin{aligned}
\sigma &= \delta \cdot \varkappa + \delta_1 && (\delta_1 < \delta)\\
\delta &= \delta_1 \cdot \varkappa_1 + \delta_2 && (\delta_2 < \delta_1)\\
&\cdot\\
&\cdot\\
&\cdot\\
\delta_{n-2} &= \delta_{n-1} \cdot \varkappa_{n-1} + \delta_n && (\delta_n < \delta_{n-1})\\
\delta_{n-1} &= \delta_n \cdot \varkappa_n \text{ [1]}).
\end{aligned}$$

In fact it immediately follows from theorem 5 on p. 246 that after a finite number (n) of steps a remainder δ_n occurs by which the preceding one is divisible; otherwise we would obtain a *decreasing* sequence of ordinals (δ_n), i.e. a set which in the arrangement according to increasing magnitude of the ordinals would have the type $^*\omega$.

Calling an ordinal $\sigma > 1$ a "prime number" if σ cannot be represented as a product of two ordinals each of which is smaller than σ, we easily see that *any ordinal larger than* 1 *can be represented as a product of a finite number of prime numbers,* as can integers in ordinary arithmetic. This decomposition, however, is not unique [2]), in contrast to arithmetic; e.g. $\omega \cdot \omega = (\omega + 1) \cdot \omega$, although obviously both ω and $\omega + 1$ are primes. Further, as in arithmetic, *any ordinal σ has only a finite number of right-hand divisors* (i.e. of divisors σ_2 with respect to the decomposition $\sigma = \sigma_1 \cdot \sigma_2$). That it may, however, have an infinite number of *left-hand divisors* is evident from examples given before. (Cf. also exercise 4 at the end of § 11.)

Finally, another analogy with arithmetic, which has been called

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[1]) The "continued fraction" defined by the "quotients" $\varkappa, \varkappa_1, \ldots, \varkappa_n$ is conceived by Hausdorff 2, I, p. 145 ff., as expressing the "rational number σ/δ" by way of a one-to-one correspondence.

[2]) For a detailed investigation of this matter cf. Sieczka 1.

by Cantor the *normal form of ordinals* [1]), shall be treated in more detail.

THEOREM 10. Any ordinal $\sigma \neq 0$ can be uniquely represented in the form

$$\sigma = \omega^{\beta_1}.\gamma_1 + \omega^{\beta_2}.\gamma_2 + \ldots + \omega^{\beta_k}.\gamma_k, \tag{3}$$

where k, $\gamma_1, \ldots, \gamma_k$ are finite ordinals $\neq 0$, while the exponents are ordinals fulfilling the inequalities

$$\beta_1 > \beta_2 > \ldots > \beta_k \geqq 0. \tag{4}$$

Remark. This representation of ordinals is quite analogous to the decimal (or g-adic, g being a positive integer $\geqq 2$) representation of integers in the usual form, with ω taking the place of the base 10 or g. There are only a finite number of terms in the representation, the exponents are decreasing, and the coefficient γ_ν of any power ω^{β_ν} is smaller than the base ω. β_1 is sometimes called the degree of σ; the degree may equal the number to be represented, as is the case for $\varepsilon_0 = \omega^{\varepsilon_0}$ (p. 289), but can never exceed it (see below). Theorem 10 is still true with any base $\alpha > 1$ in place of ω with the restriction $\gamma_\nu < \alpha$ [2]); the base ω, however, is by far the most important one.

Proof. First of all, for any β we have [3])

$$\omega^\beta \geqq \beta, \tag{5}$$

as one may see by transfinite induction. Certainly (5) is true for $\beta = 0$. Assume that it holds for any $\beta < \beta_0$. If β_0 has an immediate predecessor and $\beta_0 \geqq 2$, we have

$$\omega^{\beta_0} = \omega^{\beta_0 - 1}\cdot\omega \geqq (\beta_0 - 1)\cdot\omega > (\beta_0 - 1) + 1 = \beta_0, \quad \text{i.e.} \quad \omega^{\beta_0} > \beta_0,$$

which holds also for $\beta_0 = 1$. On the other hand, for limit-numbers $\beta_0 = \lim_\nu \beta_\nu$ we conclude by B) on p. 287:

$$\omega^{\beta_0} = \lim_\nu \omega^{\beta_\nu} \geqq \lim_\nu \beta_\nu = \beta_0, \quad \text{i.e.} \quad \omega^{\beta_0} \geqq \beta_0.$$

Hence (5) is generally true.

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[1]) Cantor 12, II, p. 237.

[2]) Hausdorff 4, p. 120 ff. In the case $\alpha = 2$ an equality between σ and its degree already occurs for $\sigma = \omega = 2^\omega$.

[3]) As its proof shows, this inequality is also true for any base $\alpha > 1$. We may also prove (5) by means of theorem 12 of § 10; cf. p. 287.

That one cannot sharpen (5) by writing $>$ instead of $\geqq$, we have seen above.

Since, accordingly, $\omega^{\sigma+1} \geqq \sigma + 1 > \sigma$, there is a *smallest* ordinal δ such that $\omega^{\delta} > \sigma$. *This δ cannot be a limit-number* since otherwise $\xi < \delta$ would imply $\xi + 1 < \delta$ too, hence $\omega^{\xi+1} \leqq \sigma$ or $\omega^{\xi} < \sigma$, which yields in view of pp. 284 and 281:

$$\omega^{\delta} = \lim_{\xi} \omega^{\xi} \leqq \sigma,$$

contrary to the assumption $\omega^{\delta} > \sigma$.

Therefore, by the definition of δ we have for $\beta_1 = \delta - 1$:

$$\omega^{\beta_1} \leqq \sigma < \omega^{\beta_1+1}. \tag{6}$$

By theorem 9 there are ordinals γ_1 ($\neq 0$) and σ_1 ($< \omega^{\beta_1}$) such that

$$\sigma = \omega^{\beta_1} \cdot \gamma_1 + \sigma_1.$$

Hence by (6): $\omega^{\beta_1} \cdot \gamma_1 < \omega^{\beta_1+1}$, i.e. $\gamma_1 < \omega$ by B 3) on p. 290; in other words, γ_1 is finite and $\neq 0$.

If $\sigma_1 = 0$, $\sigma = \omega^{\beta_1} . \gamma_1$ is the desired representation. If not, we may handle σ_1 in the same way as σ, so that we obtain successive equalities

$$\sigma = \omega^{\beta_1} \cdot \gamma_1 + \sigma_1, \ \sigma_1 = \omega^{\beta_2} \cdot \gamma_2 + \sigma_2, \ \ldots, \tag{7}$$

where β_ν is determined by $\sigma_{\nu-1}$ as β_1 was determined (via δ) by σ; we have $\sigma \geqq \omega^{\beta_1} > \sigma_1 \geqq \omega^{\beta_2} > \sigma_2 \geqq \ldots$, hence $\beta_1 > \beta_2 > \ldots$, and all γ_ν are finite. Since a decreasing succession of ordinals is necessarily finite, the process will terminate after a finite number of steps by yielding a $\sigma_k = 0$. From (7) we then obtain the representation of theorem 10.

Moreover, this representation is *unique*. For the assumption

$$\sigma = \omega^{\beta_1} . \gamma_1 + \ldots + \omega^{\beta_k} . \gamma_k = \omega^{\overline{\beta}_1} . \overline{\gamma}_1 + \ldots + \omega^{\overline{\beta}_k} . \overline{\gamma}_k$$

firstly implies $\overline{\beta_1} = \beta_1$; in fact $\overline{\beta_1} > \beta_1$ would mean $\overline{\beta_1} \geqq \beta_1 + 1$, hence $\sigma \geqq \omega^{\beta_1+1}$, while we conclude without difficulty that $\sigma < \omega^{\beta_1+1}$ because of $\beta_1 > \beta_2 > \ldots$. Secondly, the result $\overline{\beta_1} = \beta_1$ also implies $\overline{\gamma_1} = \gamma_1$ since, in view of B 2) on p. 290, $\overline{\gamma_1} > \gamma_1$ (i.e. $\overline{\gamma_1} = \gamma_1 + \gamma_1^*$ with $\gamma_1^* \neq 0$) would enable us to suppress $\omega^{\beta_1} . \gamma_1$ on both sides and to infer

$$\omega^{\beta_2} . \gamma_2 + \ldots = \omega^{\beta_1} . \gamma_1^* + \ldots,$$

which is again impossible because of $\beta_2 < \beta_1$. This argument can obviously be continued up to the conclusion $k = \bar{k}$; the proof of theorem 10 is therefore completed.

The representation of an ordinal by a finite number of powers of ω immediately leads to the "*natural sum*" [1]) of two ordinals. Let σ and τ be represented as in theorem 10:

$$\sigma = \sum_{\nu} \omega^{\beta_\nu} \cdot \gamma_\nu, \quad \tau = \sum_{\nu} \omega^{\beta_\nu} \cdot \delta_\nu.$$

Since each sum contains only a finite number of terms, we may use the *same* exponents β_ν in both these cases, admitting a finite number of coefficients $\gamma_\nu = 0$ and $\delta_\nu = 0$. Then the natural sum is defined by

$$\mathfrak{S}(\sigma, \tau) = \sum_{\nu} \omega^{\beta_\nu} \cdot (\gamma_\nu + \delta_\nu).$$

Since γ_ν and δ_ν are finite ordinals, we have $\mathfrak{S}(\sigma, \tau) = \mathfrak{S}(\tau, \sigma)$. The natural sum, however, need not coincide with either of the "ordered sums" $\sigma + \tau$ and $\tau + \sigma$. For instance, taking $\sigma = \omega^3 + \omega$, $\tau = \omega^2$, we obtain

$$\mathfrak{S}(\sigma, \tau) = \omega^3 + \omega^2 + \omega, \quad \sigma + \tau = \omega^3 + \omega^2, \quad \tau + \sigma = \omega^3 + \omega,$$

and these sums are different from each other.

In **5** we shall use the following property of natural sums. *If ϱ is a given ordinal, there is only a finite number of pairs σ, τ such that $\mathfrak{S}(\sigma, \tau) = \varrho$.* For let $\varrho = \sum_\nu \omega^{\alpha_\nu} \cdot \lambda_\nu$ be the normal form of ϱ according to theorem 10, while the forms of some σ and τ may be

$$\sigma = \sum_{\nu} \omega^{\beta_\nu} \cdot \gamma_\nu, \quad \tau = \sum_{\nu} \omega^{\delta_\nu} \cdot \varepsilon_\nu.$$

By virtue of theorem 10 it is then easy to obtain the equations $\lambda_\nu = \gamma_\nu + \varepsilon_\nu$ holding for every ν in question, where λ_ν as well as the required solutions γ_ν and ε_ν are finite. But, since any such equation has only a finite number of solutions and since the number of equations is also finite, there are only a finite number of possibilities for the pairs σ, τ; Q.E.D.

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[1]) Introduced by Hessenberg 3, ch. XX. Cf. also Jacobsthal 3 and Carruth 1.

5. Alephs and Initial Numbers [1]. The cardinal of a well-ordered set is being called an *aleph* [2]. Though in **6** this distinction between cardinals in general and alephs in particular will again be removed in a certain sense, in this subsection we are restricting ourselves to alephs only.

The fundamental theorem on alephs, asserting their comparability, immediately follows from the comparability of the *ordinals* of well-ordered sets (theorem 8 of § 10). It reads:

THEOREM 11. Either the cardinals (alephs) of two well-ordered sets are equal, or one of them is smaller than the other.

Proof. Let S and T be well-ordered sets with the ordinals σ and τ and the cardinals $\mathbf{s}$ and $\mathbf{t}$. Then either $S \simeq T$, i.e. $\sigma = \tau$, or one of the sets, say S, is similar to a section of the other, T; i.e. $\sigma < \tau$. Now $\sigma = \tau$ *a fortiori* expresses $\mathbf{s} = \mathbf{t}$. If $\sigma < \tau$, we only weaken this assertion by saying that S is *equivalent* to a *subset* of T, which means $\mathbf{s} \leqq \mathbf{t}$ (theorem 5 on p. 104). Q.E.D.

Thus the fourth case, which we were not able to eliminate at the end of § 5 (p. 105) nor during all our later considerations, has been excluded for *well-ordered sets*; these are *always comparable with respect to their cardinals, as well as with respect to their ordinals.*

If again σ and τ denote the ordinals of S and T, the assertion of theorem 11 may be expressed more precisely as follows: $\sigma = \tau$ implies the equality of the corresponding cardinals, i.e. $\overline{\sigma} = \overline{\tau}$; but $\sigma < \tau$ leaves open the possibilities $\overline{\sigma} < \overline{\tau}$ and $\overline{\sigma} = \overline{\tau}$. Hence, vice versa, if $\mathbf{s}$ and $\mathbf{t}$ are the cardinals of S and T, $\mathbf{s} < \mathbf{t}$ implies $\sigma < \tau$, while $\mathbf{s} = \mathbf{t}$ is compatible with each of the cases $\sigma = \tau$, $\sigma < \tau$, $\sigma > \tau$ [3]. We have, e.g., $\omega < \omega^2 < \omega^\omega$ while $\overline{\omega} = \overline{\omega^2} = \overline{\omega^\omega} = \aleph_0$.

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[1] Cf. what has been said in the beginning of **4**. In **5** a few proofs are given in an abridged form; the filling in of some details will be left to the reader. In particular, the inductive definitions used here (e.g., in the definition of the "series of alephs") as well as in **3**, will be discussed more profoundly in *Foundations*.

[2] Aleph, the first letter of the Hebrew alphabet, was hitherto used in the forms $\aleph_0$ (aleph-zero) and $\aleph$ (aleph, plainly) to denote the cardinals of a denumerable set and of the continuum, respectively.

[3] For the reader's convenience we here give the results in the form of a scheme:

Hypothesis	$\sigma = \tau$	$\sigma < \tau$	$\overline{\sigma} = \overline{\tau}$	$\overline{\sigma} < \overline{\tau}$
Conclusion	$\overline{\sigma} = \overline{\tau}$	$\overline{\sigma} \leqq \overline{\tau}$	no conclusion	$\sigma < \tau$

It is therefore natural to look for the set of all different ordinals that have a given aleph $\mathbf{c}$ as their cardinal. This set is called the *number-class of* $\mathbf{c}$ and denoted by $Z(\mathbf{c})$. As in any set of ordinals, there exists in $Z(\mathbf{c})$ a *smallest* ordinal, which is called the *initial number* of the number-class (or of $\mathbf{c}$). An initial number is certainly a limit-number; otherwise it would have an immediate predecessor the cardinal of which could not be smaller.

The notion of number-class is trivial when $\mathbf{c}$ is a *finite* cardinal m; for in this case, as shown on p. 189, there exists only one ordinal of the cardinal m, which usually is again denoted by m, and $Z(m)$ accordingly contains this single element. For this reason Cantor called the set of all finite ordinals the *first number-class*. We could call 0 the initial number of this class, but since the name "initial number" is usually restricted to transfinite ones, we shall not use it for 0.

In the next case $\mathbf{c} = \aleph_0$, $Z(\aleph_0)$ contains the (different) ordinals of all *denumerable* well-ordered sets; therefore $Z(\aleph_0)$ is a subset of the set $T(\aleph_0)$ considered in § 8, **6**, the elements of which are all different *order-types* of denumerable ordered (not only well-ordered) sets. $Z(\aleph_0)$ is usually called the *second number-class* [1]); its initial number is ω.

For the elements of $Z(\aleph_0)$, the following theorem holds.

THEOREM 12. The sequent of an enumerated set (sequence) D of ordinals which belong to the first and second number-classes, is denumerable [2]).

Proof. If the elements of D are δ_ν $(\nu = 1, 2, \ldots)$, the ordered sum $\sigma = \sum_\nu \delta_\nu$ satisfies the inequality between cardinals $\overline{\sigma} \leqq \aleph_0 \cdot \aleph_0 = \aleph_0$ because the number of terms is $\aleph_0$ and the cardinal of each term is $\leqq \aleph_0$. Hence, σ being transfinite, we have $\overline{\sigma} = \aleph_0$; since σ is larger than any ordinal of D, the theorem holds.

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[1]) Denjoy **5** is the first part of a monograph which chiefly deals with the second number-class. For an *axiomatic* characterization of this class and for some problems connected with it, see Church **2**; also Veblen **2**.

[2]) Here and in the following we again use abbreviating expressions: "the sequent of a set of ordinals" means the ordinal larger than and next to all elements of the set; an ordinal "is denumerable" means: it is the type of a denumerable, not necessarily enumerated, well-ordered set, i.e. an ordinal of the second number-class.

Theorem 12 shows that not only ω and its next successors but *all ordinals constructed in* 3 *belong to the second number-class* [1]). In fact, all of them — even ε_0 — have been defined either as the sequent of a denumerable ordinal or as the sequent of an enumerated set of such ordinals; for instance

$$\omega^\omega = \lim_{\nu} \omega^\nu, \quad \varepsilon_0 = \lim \{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}.$$

If τ is any ordinal of the second number-class, the set of all natural numbers, for example, can be ordered according to the type τ. The special case $\tau = \omega^\omega$ has been illustrated on p. 286. Hence the set of all ordinals up to τ is denumerable.

On the other hand, theorem 12 guarantees the existence of an ordinal *in* $Z(\aleph_0)$ larger than the ordinals of any given sequence of elements of $Z(\aleph_0)$; hence *the second number-class itself is not denumerable* [2]).

Now, one might ask whether, given any aleph, there exists a larger one. It is true that theorem 3 of § 5 (p. 94; cf. theorem 1 on p. 151) guarantees the existence of a cardinal larger than any given cardinal, and on p. 131 we even proved that there are cardinals larger than the cardinals of any given *set* of cardinals. Here, however, we cannot use this directly since we do not yet know — and indeed, it is a very complicated question, see **6** — whether the cardinals obtained will in their turn be alephs. Nevertheless, the methods of the present section enable us to prove without any difficulty:

THEOREM 13. There exist larger alephs to any given aleph and even to any given *set* of alephs; in particular, there exists a *sequent* [3]) aleph (which, of course, is uniquely determined).

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[1]) For much profounder questions of "constructibility" within the second number-class, as raised by Church (connected with some notions introduced by Turing), cf. Church **11**, Kleene **3** and **6**; see *Foundations*.

[2]) This also follows from the corollary on p. 282. For if the set $Z(\aleph_0)$ were denumerable, the set of all *finite* and denumerable ordinals would be denumerable too. But by ordering this set according to magnitude, we obtain a set with the properties mentioned in the corollary. Thus the order-type of this set is an ordinal larger than any denumerable ordinal, hence not belonging to $Z(\aleph_0)$.

[3]) The terms "sequent" and "larger" as used here refer to the *order of cardinals* according to their magnitude (§ 5).

Proof. First of all, it is evident that, just like any set of ordinals, any set of alephs can be ordered according to magnitude and that it is then well-ordered. To perceive this, it is sufficient to replace any aleph by the corresponding initial number.

Let A be such a well-ordered set of alephs and B the set of all ordinals the cardinal of which is either an element of A or else is smaller than an element of A; hence B includes the finite ordinals. If β is an element of B, any ordinal $< \beta$ is also contained in B. By the corollary on p. 282, the ordinal γ of B is therefore larger than any element of B, and the aleph $\overline{\gamma}$ is larger than any aleph of A, according to the definition of B. More precisely, $\overline{\gamma}$ is the sequent of A. For if there were an aleph $< \overline{\gamma}$ and exceeding all elements of A, its initial number would not belong to B. So it would exceed all elements of B and yet be smaller than γ. But, by the corollary just mentioned, γ is the sequent of B. Q.E.D. [1]).

From theorem 13 we may draw two conclusions, in full analogy to what has been said on p. 282 and p. 278 with respect to ordinals. First, *there does not exist a set containing all alephs*, since then a larger aleph ought to exist. (The same statement regarding cardinals in general was referred to on p. 131 f.) Secondly, since the alephs up to a given aleph, ordered according to increasing magnitude, constitute a well-ordered set, they may be denoted by *ordinals serving as indices*. Thus we obtain the "series"

$$\aleph_0,\ \aleph_1,\ \aleph_2,\ \ldots,\ \aleph_n,\ \ldots\ \aleph_\omega,\ \aleph_{\omega+1},\ \ldots\ \aleph_{\omega\cdot 2},\ \ldots$$

which develops in the same way as the series of ordinals in 3. Accordingly, $\aleph_\tau$ with $\tau \neq 0$ denotes the smallest aleph exceeding all $\aleph_\sigma$ with $\sigma < \tau$. The inequality between ordinals $\alpha < \beta$ implies $\aleph_\alpha \neq \aleph_\beta$, namely $\aleph_\alpha < \aleph_\beta$ [2]). To any given $\aleph_\alpha$ corresponds the number-class $Z(\aleph_\alpha)$, and therefore its initial number, usually denoted by ω_α [3]). So we may say that the index α of an aleph $\aleph_\alpha$

[1]) To be sure, $\overline{\gamma}$ is *the* sequent aleph to the given set, but only *a* sequent cardinal. As long as we do not have the well-ordering theorem (6, cf. 7) at our disposal, the existence of incomparable cardinals cannot be excluded. This has to be borne in mind throughout the following. For instance, $\aleph_1$ (see below) is *the* sequent aleph, but *a* sequent cardinal to $\aleph_0$. Even the proof that $\aleph_0$ is the smallest cardinal depends on the multiplicative principle; cf. p. 123 and 6.

[2]) But, of course, $\alpha < \beta$ does *not* imply the inequality between alephs $\overline{\alpha} < \overline{\beta}$. See p. 298.

[3]) This means that ω is also to be denoted by ω_0.

is the ordinal of the well-ordered set of all (transfinite) initial numbers preceding the initial number of the aleph in question [1]; this set is a subset of $W(\omega_\alpha)$.

For the number-class $Z(\aleph_\alpha)$ we state:

THEOREM 14. The set $Z(\aleph_\alpha)$, ordered according to the magnitude of the ordinals, has the cardinal $\aleph_{\alpha+1}$ and the type (ordinal) $\omega_{\alpha+1}$.

Proof. Since, according to the definitions of number-classes and initial numbers, $Z(\aleph_\alpha)$ contains the ordinals σ satisfying the inequality $\omega_\alpha \leqq \sigma < \omega_{\alpha+1}$, we have (in the sense of ordered addition of ordered sets)

$$W(\omega_\alpha) + Z(\aleph_\alpha) = W(\omega_{\alpha+1}).$$

Denote by ξ the ordinal of the well-ordered set $Z(\aleph_\alpha)$. Since, by theorem 2, $W(\omega_\alpha)$ and $W(\omega_{\alpha+1})$ have the ordinals ω_α and $\omega_{\alpha+1}$, we obtain (see p. 194)

$$(1) \qquad \omega_\alpha + \xi = \omega_{\alpha+1}, \qquad \aleph_\alpha + \overline{\xi} = \aleph_{\alpha+1},$$

the latter equality by transition to the corresponding cardinals. Hence $\overline{\xi} \leqq \aleph_{\alpha+1}$. But $\overline{\xi} < \aleph_{\alpha+1}$ would mean $\overline{\xi} \leqq \aleph_\alpha$, and then $\aleph_\alpha + \overline{\xi} = \aleph_{\alpha+1}$ is impossible as is presently stressed (see first (3) in theorem 15). Therefore $\overline{\xi} = \aleph_{\alpha+1}$, i.e. $\xi \geqq \omega_{\alpha+1}$. Since the first equality (1) says $\xi \leqq \omega_{\alpha+1}$, the theorem has been proved.

According to (1), $\aleph_{\alpha+1}$ and $\omega_{\alpha+1}$ are respectively also the cardinal and the ordinal of the *sum* of all number-classes up to, and including, $Z(\aleph_\alpha)$.

In particular, the second number-class $Z(\aleph_0)$ has the cardinal $\aleph_1$ and the ordinal ω_1 (for which Cantor used the notation Ω). $Z(\aleph_1)$ is called the third number-class [2], etc.

Using general comparability (p. 298, and 7 of this section), or at least the multiplicative principle, one can immediately conclude $\aleph_1 \leqq \aleph$; that is to say, the cardinal of the continuum either equals or exceeds the cardinal of the second number-class.

We conclude this subsection with a few general formulae belon-

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[1] We might as well conceive this as the *definition* of the indices α. Then $\alpha < \beta$ implies $\omega_\alpha < \omega_\beta$, hence $\overline{\omega}_\alpha \leqq \overline{\omega}_\beta$ i.e. $\aleph_\alpha \leqq \aleph_\beta$, and since $\aleph_\alpha \neq \aleph_\beta$, even $\aleph_\alpha < \aleph_\beta$.

[2] For the ordinals of this class, see Eyraud 3.

ging to the arithmetic of alephs [1]) which, of course, is much simpler than the arithmetic of cardinals as developed in §§ 6 and 7 by means of purely cardinal methods.

THEOREM 15. σ and τ denoting ordinals with $\sigma \leqq \tau$, we have

$$\aleph_\sigma \cdot \aleph_0 = \aleph_\sigma \cdot \aleph_\sigma = \aleph_\sigma, \tag{2}$$

$$\aleph_\sigma + \aleph_\tau = \aleph_\sigma \cdot \aleph_\tau = \aleph_\tau, \tag{3}$$

$$\sum_{\xi \leqq \sigma} \aleph_\xi = \aleph_\sigma; \tag{4}$$

in particular, if σ is a limit-number, even $\sum_{\xi < \sigma} \aleph_\xi = \aleph_\sigma$.

$$\aleph_\sigma^{\aleph_\tau} = 2^{\aleph_\tau}. \tag{5}$$

While the first relation (2) is almost trivial, the second is rather profound [2]). The relations (3) — (5) easily follow from (2). Particular cases — of $\aleph_\sigma$ or $\aleph_\tau$ equalling $\aleph_0$ or $\aleph$ — have already appeared in § 6 (though $\aleph$ has not yet been proved to be an aleph at all).

Proof. The first formula (2) follows from the corollary on p. 293 which allows us to write the initial number of $\aleph_\sigma$ in the form $\omega_\sigma = \omega \cdot a$. Hence, by transition to the corresponding alephs, we have (see p. 204):

$$\aleph_\sigma = \aleph_0 \cdot \overline{a}, \quad \aleph_\sigma \cdot \aleph_0 = \aleph_0^2 \cdot \overline{a} = \aleph_0 \cdot \overline{a} = \aleph_\sigma.$$

To prove the second formula (2), we remark that the set $W(\omega_\sigma)$ has the ordinal ω_σ, hence the cardinal $\aleph_\sigma$. According to the definition of multiplication in § 6, the product $\aleph_\sigma \cdot \aleph_\sigma$ may therefore be considered to be the cardinal of the set P the elements of which are all ordered pairs (ξ, ζ), where both ξ and ζ run over $W(\omega_\sigma)$. It will suffice to show that this set of pairs has a cardinal not exceeding $\aleph_\sigma$.

To this purpose we use the "natural sums" (p. 297)

$$\mathfrak{S}(\xi, \zeta) = \sum \omega^\nu \cdot (x_\nu + z_\nu), \qquad (x_\nu, z_\nu \text{ finite})$$

[1]) For additional (in)equalities we again refer to the textbook Hausdorff **4**; cf. also Sierpiński **6** and Kamke **3**. Furthermore Wrinch **4**; Tarski **6**; Patai **1** and **2**; Sudan **2**. For initial numbers also Mahlo **1**. As a remarkable result we mention "Hausdorff's recursion formula" on cardinals which says

$$\aleph_\sigma^{\aleph_\tau} = \aleph_\sigma \cdot \aleph_{\sigma-1}^{\aleph_\tau}$$

provided that σ is no limit-number. See Hausdorff **1** (1904); for generalizations and completions, Tarski **6**.

[2]) It was first proved by Hessenberg **3** (p. 594) and **4**. Cf. Lindenbaum–Tarski **1**, p. 308 ff. and Zorn **2**.

formed for all ξ and ζ of $W(\omega_\sigma)$, each sum arranged according to decreasing powers of ω; as remarked before, the sum contains only a finite number of terms, or of powers of ω. Let $\varkappa$ be the highest exponent of ω appearing in the sum; then $\mathfrak{S}(\xi, \zeta)$ begins with the term $\omega^\varkappa \cdot (x_\varkappa + z_\varkappa)$ where at least one of the finite ordinals $x_\varkappa$ and $z_\varkappa$ differs from 0. We have

$$\mathfrak{S}(\xi, \zeta) < \omega^\varkappa \cdot (x_\varkappa + z_\varkappa + 1).$$

Hence, since $\xi < \omega_\sigma$ and $\zeta < \omega_\sigma$, also $\omega^\varkappa < \omega_\sigma$, and even (as ω_σ is the *initial* number of $\aleph_\sigma$) $\overline{\omega^\varkappa} < \aleph_\sigma$; therefore

$$\overline{\mathfrak{S}(\xi, \zeta)} < \aleph_\sigma, \text{ or } \mathfrak{S}(\xi, \zeta) < \omega_\sigma.$$

Accordingly we obtain all pairs (ξ, ζ) contained in the above set P of the cardinal $\aleph_\sigma \cdot \aleph_\sigma$, by looking for all ξ, ζ the natural sum of which, $\mathfrak{S}(\xi, \zeta)$, assumes values $\lambda < \omega_\sigma$. Now *for a given* λ there is only a finite number of solutions ξ, ζ as has been shown on p. 297. Since the set of all possible λ has the cardinal $\aleph_\sigma$, the set of all solutions has a cardinal not exceeding $\aleph_\sigma \cdot \aleph_0$, which by the first equality (2) equals $\aleph_\sigma$. Herewith the second equality (2) has been proved. Hence for any *finite* n $(\neq 0)$, we have $\aleph_\sigma^n = \aleph_\sigma$.

(3) immediately follows from (2) in view of

$$\aleph_\sigma + \aleph_\tau \leqq \aleph_\tau + \aleph_\tau \leqq \aleph_\tau \cdot \aleph_0, \quad \aleph_\sigma \cdot \aleph_\tau \leqq \aleph_\tau \cdot \aleph_\tau.$$

We may express (3) in an apparently more general form. In addition to $\sigma < \tau$, assume $\lambda < \mu$. Then, if $\lambda \leqq \sigma$, (3) yields

$$\aleph_\lambda + \aleph_\sigma = \aleph_\sigma, \quad \aleph_\lambda \cdot \aleph_\sigma = \aleph_\sigma.$$

Corresponding equalities with $\aleph_\lambda$ on the right-hand side are obtained if $\sigma < \lambda$. Hence in any case, since $\aleph_\lambda < \aleph_\mu$ and $\aleph_\sigma < \aleph_\tau$:

$$\text{(3a)} \qquad \aleph_\lambda + \aleph_\sigma < \aleph_\mu + \aleph_\tau, \quad \aleph_\lambda \cdot \aleph_\sigma < \aleph_\mu \cdot \aleph_\tau.$$

As to (4), note that the terms of the sum form a well-ordered set of the ordinal $\sigma + 1$ or σ, hence certainly of a cardinal $\leqq \aleph_\sigma$, while every term is also $\leqq \aleph_\sigma$. So in any case the sum does not exceed $\aleph_\sigma \cdot \aleph_\sigma = \aleph_\sigma$. That the sum is not smaller than $\aleph_\sigma$, is trivial in the first case (of summation over all $\xi \leqq \sigma$); in the second case — σ a limit-number, $\xi < \sigma$ — the sum exceeds *any* aleph which is smaller than $\aleph_\sigma$, and since it is an aleph, it equals $\aleph_\sigma$.

In particular we obtain (n denoting any finite ordinal)

$$\aleph_\omega = \aleph_0 + \aleph_1 + \ldots + \aleph_n + \ldots . \tag{4a}$$

Finally, to prove (5) we rely on theorem 1 on p. 151 which says

$$2^{\aleph_\tau} > \aleph_\tau \geqq \aleph_\sigma .$$

Hence by (2) and the formal laws of exponentiation (p. 152)

$$2^{\aleph_\tau} = 2^{\aleph_\tau \cdot \aleph_\tau} = (2^{\aleph_\tau})^{\aleph_\tau} \geqq \aleph_\sigma^{\aleph_\tau} .$$

On the other hand $2 < \aleph_\sigma$ implies $2^{\aleph_\tau} \leqq \aleph_\sigma^{\aleph_\tau}$, i.e. (5) holds true.

Of course, (5) is not exactly a theorem on alephs since a power of alephs has not yet been proved to be an aleph. As a matter of fact, in proving (5) we have mainly relied on the results of §§ 6 and 7, using in addition only (2). In particular, we are still at a loss whether $\aleph (= 2^{\aleph_0})$, the cardinal of continuum, is an aleph. If it is, we immediately infer from $\aleph_1 \leqq \aleph$ (cf. p. 157): $\aleph \leqq \aleph_1^{\aleph_0} \leqq \aleph^{\aleph_0} = \aleph$, i.e. $\aleph_1^{\aleph_0} = \aleph$.

The question may be raised whether there exist alephs $> \aleph_0$ being neither of the form $\aleph_{\sigma+1}$ nor of the form $\mathfrak{c}^{\aleph_\tau}$ where $\mathfrak{c}$ denotes a (finite or transfinite) cardinal. The answer is in the affirmative, for $\aleph_\omega$ is such an aleph [1]). In fact, since for any finite n we have $\aleph_n < \aleph_\omega$, the inequality of König (p. 132) yields in view of (4a):

$$\aleph_\omega = \aleph_0 + \aleph_1 + \aleph_2 + \ldots < \aleph_\omega^{\aleph_0} .$$

However, the supposition $\aleph_\omega = \mathfrak{c}^{\aleph_\tau}$ would imply by (2):

$$\aleph_\omega^{\aleph_0} = \mathfrak{c}^{\aleph_\tau \cdot \aleph_0} = \mathfrak{c}^{\aleph_\tau} = \aleph_\omega$$

contrary to the above result. Hence $\aleph_\omega$ *is certainly not the cardinal* $\aleph = 2^{\aleph_0}$ *of the continuum.*

Let us finish this survey of the theory of alephs with a remark of principle. The concept of cardinal has been introduced (in § 4, 6 & 7) in a rather complicated way, which started from the concept of set; in other words, cardinals have appeared as cardinals *of sets*. It is true that in § 8 the same procedure has been applied with respect to order-types, hence to ordinals. Regarding the latter, however, we have succeeded in also giving a direct definition independently of the well-ordered sets, *the types* of which are the ordinals (p. 278 f.); ordinals are particular well-ordered sets

[1]) For a generalization see Bagemihl 1.

distinguished by certain characteristic properties. Now among these ordinals we may consider the initial numbers, and since there is a one-to-one correspondence between alephs and initial numbers, we may at once *define alephs as the corresponding initial numbers, and thus arrive at an explicit definition of alephs* (*and, in view of* **6**, *of cardinals in general*).

6. **The Well-Ordering Theorem.** Notwithstanding the sublime character of the edifice presented by transfinite ordinals and alephs, which has been called by Hilbert "the most admirable blossom of the mathematical mind and on the whole one of the foremost achievements of mankind's purely intellectual activity" [1]), we have not yet been able to solve the problem which has engaged our attention from § 5 onward, viz.: are any two cardinals always comparable with each other? True, for alephs — i.e. for the cardinals of *well-ordered* sets — this question has been answered in the affirmative, and this is the main reason for the simplicity and transparency of arithmetic in the domain of alephs. But for the cardinals of plain sets, or even of ordered but not well-ordered sets like the linear continuum, the question still remains open. The ghost of *incomparable sets* continues to alarm us and to entangle the domain of cardinals as a whole. It is this ghost that induced Cantor and many of his successors to call cardinals in general by the neutral name of "powers" (*Mächtigkeiten*). For instance, he named $\aleph$ the "power of the continuum", while the more specific name of "cardinal number" was to be reserved to objects which share with numbers in general the property of being comparable; i.e. to the powers of well-ordered sets (alephs).

Thus a deep gap has been separating the domain of well-ordered sets, distinguished by its similarity to plain arithmetic, from the less perspicuous realm of other (ordered or unordered) sets. Naturally — this idea already occurs in the early researches of Cantor — this gap will be bridged as soon as we succeed in transforming *any set* into a well-ordered one, by arranging its elements in the special succession which is characteristic of well-ordered sets, and which is not fulfilled, for example, by the elements of a one-

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[1]) Hilbert **9**, p. 167.

dimensional continuum in its natural order. After having succeeded in this task, we would be able to enounce the

> *Well-ordering theorem.* Given any set S, there exists a *well-ordered set* which contains the same elements[1]) as S.
> In short: *Any set can be well-ordered.*

The superiority of the first formulation to the shorter one lies in a rather psychological feature. The *existence* of sets, within the frame of any sound foundation of the theory of sets, should be well defined. On the other hand, the expression "*can* be well-ordered" apparently opens the way for subjective interpretations on what can or cannot be achieved by human intellect, within finite time, etc. We shall shortly see the importance of this difference in the present case.

Cantor [2]) has called the well-ordering statement "a fundamental logical law of great consequence, being noteworthy by its universal validity"; at the same time (1883) he made the promise which has never been fulfilled, of proving this statement [3]). The train of ideas which led Cantor to his assertion was of the following kind. Take out of the given set S an arbitrary element s_1, then another element s_2, and so forth. If neither n steps of this kind, n being any natural number, nor the infinite sequence (s_k) $(k = 1, 2, \ldots)$, exhaust the set S, an arbitrary element among those left over in S shall be denoted by s_ω and shall be placed after all elements of the sequence (s_k). And so on, according to the series of ordinals and considering that to any set of ordinals there is a *sequent* ordinal, — until the set S has been exhausted, so that all its elements constitute the elements of a well-ordered set.

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[1]) Of course, it would suffice to say "which contains *at least* the same elements" since then the desired set can at once be derived by means of the principle of subsets. On the other hand, the following formulation would also be sufficient: given any S, there exists a well-ordered set *equivalent to* S. For any representation between S and this well-ordered set will provide a well-ordering of S.

[2]) Cantor **7**, V, § 3. As to the history of the well-ordering theorem, including the attempts to prove it made by Jourdain himself, see Jourdain **10**. Cf. also the sketch in Littlewood **1**, p. 39.

[3]) He never abandoned his conviction of the validity of this "fundamental law"; not even during the third International Congress of Mathematicians (1904) when König's theorem (§ 6) in connection with an erroneous lemma seemed to refute the law. Cf. Schoenflies **11**, p. 100 ff.

Against this argument four objections may be raised. First, that the series of all ordinals (which is not a set because this leads to a contradiction, p. 282) seems to be involved; secondly, that from the above argument it is not clear how to define a set of ordinals extensive enough to provide indices for *all* elements of the given set S, or how to assure that there *exists* a sufficient set of ordinals, however comprehensive S may be; thirdly, that the arbitrariness of those elements of S which are successively chosen to enlarge the well-ordered set in question, is embarrassing, since infinitely many acts of choice are needed (save for the trivial case of a finite S); fourthly, that it is on the whole an awkward feeling to be dependent on infinitely many successive steps of the kind described which — since there is no law fixed in advance — would, as it were, require infinite time.

As to the first two objections, they are serious and weighty, and in view of them the train of ideas hinted at cannot be maintained in its original shape. The third difficulty, as a matter of fact, is not new for us, neither is it characteristic of the present question, although in the historic development of the problem it has played the central rôle. Arbitrary choices of the same kind, if not in such indefinite quantity, have already been used, for instance, in proof A of theorem 4 on p. 57 f., while in § 6 (p. 123) the question was explicitly discussed as a matter of principle; presently we shall rely on that discussion. The fourth doubt, closely connected with the third, has a psychological rather than a logical character: mathematical acts are in principle to be considered independent of time. However, in the following proof of the well-ordering theorem we shall apply an argument which has a psychological advantage with respect to the last difficulty; the proof requires no acts of choice which depend on steps taken before.

As to the third objection, let us now explicitly state the principle of choice in the general form useful in this case [1]):

[1]) The principle of choice (multiplicative principle) was introduced by B. Levi, E. Schmidt and E. Zermelo — by the latter two in connection with the attempt to prove the well-ordering theorem — at the beginning of this century. Its history and significance will be thoroughly discussed in *Foundations*, ch. II. — The diagonal method (§§ 4 and 5) and the proof of the equivalence theorem do not require the principle. Alternatives to the principle, with applications to the second number-class, are contained in Church 2.

Principle of choice.

> Given a set S, there exists a function f (at least one) — to be called a *choice-function* — which relates ("chooses"), to any non-empty subset S_0 [1]) of S, a definite element $f(S_0)$ contained in S_0, henceforth to be called *the distinguished element of* S_0 (with respect to the choice-function taken). Of course, different subsets of S may have the same distinguished element [2]).

In the following proof we actually need much less than the distinguished elements of *all* subsets of S. While these form a set of the cardinal $2^{\mathfrak{s}}$, $\mathfrak{s}$ being the cardinal of S, our proof will require the choice of distinguished elements in $\mathfrak{s}$ subsets of S only, i.e. in a small part of the subsets at our disposal. On the other hand, our liberal provision of distinguished elements has the psychological advantage of a *simultaneous* choice, as it were, such that no act of choice will depend on other ("previous") acts.

The first proof of the well-ordering theorem, which is ingenious and short, was given by Zermelo in 1904 [3]); it is essentially this proof which will be presented here — not only because of its simplicity but also for its historical importance; as a matter of fact, in the years following 1904 many mathematical journals were full of analytical and critical remarks on Zermelo's first

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[1]) Evidently, only sets containing at least two elements need to be considered.

[2]) This principle is a generalization of Russell's Multiplicative principle as formulated in § 6, **6**, in so far as the sets S_0 are not mutually exclusive. On the other hand, it is a particular case of the generalized form mentioned in § 6, **6**.

If S is a finite set, the assertion of our principle is obvious. As to the Multiplicative principle, it can be *proved* by means of the preceding principles if D (p. 123) is a finite set (of finite or infinite sets). The attitudes of Kamke **5** and Denjoy **5** in this respect are erroneous.

[3]) Zermelo **1**. In principle, the interesting proof in Hausdorff **4** does not differ from it.

From the logical point of view one should compare the important modification given by Bernays **14**, IV on the basis of his axiomatic foundation. Here also the differences in logical structure between Zermelo's first and second proofs are clarified. The first one is shown to be simpler in this respect.

proof [1]). (In *Foundations* it will be shown that the various sorts of criticism were not justified at all, save for an intuitionistic attitude that would refute classical mathematics in general.)

An entirely different proof of the well-ordering theorem was given, again by Zermelo, in 1908 [2]). While the first proof makes considerable use of the general theorems on well-ordered sets, the second proof practically does not even use general set theory, much less the properties of well-ordered sets. Principally it relies on very abstract logical procedures originally introduced by Dedekind. Though this second method may prove much more difficult for the beginner, we should not omit it entirely. The proof of the theorem of 7, which is closely connected with the well-ordering theorem, actually contains all ideas and processes of Zermelo's second proof.

Finally it should be pointed out that the proof — the first as well as the second — starts with *any definite* choice-function in the sense explained above. Hence the well-ordered set resulting from the proof, is dependent on the special choice-function which has been taken. In other words, we shall obtain a particular well-ordered set which contains the same elements as the given set S, but not a kind of standard-order. Therefore we can use the well-ordering theorem only for inferences with respect to properties referring to *any* well-ordering of S.

Proof of the well-ordering theorem. We start with a definite choice-function f with respect to the given set S; this function will remain unchanged during the whole proof. With Zermelo we introduce a notion *ad hoc*, to be called "gamma-set", where "set" serves as an abbreviation for "well-ordered set".

> S being an ordered or not-ordered set, a *subset* Γ *of* S shall be called a *gamma-set of* S (with regard to f) if
> a) Γ is well-ordered (regardless of the order in S, if any)
> b) with respect to every section A of Γ, determined by $a \varepsilon \Gamma$, the following relation holds:
> $$f(S - A) = a$$ [3]).

[1]) It may suffice here to mention the articles contained in Hadamard 2.

[2]) Zermelo 2. This essay also contains an interesting polemic with the critics of 1. Cf. van der Waerden 2, 1st edition, pp. 194—196.

[3]) The trivial case $\Gamma = 0$ where both conditions are fulfilled vacuously may be disregarded altogether.

Let us consider condition b more closely by illustrating the notion of gamma-set which is dependent on the particular choice-function f taken as the basis of the whole procedure.

The meaning of condition b is this: if a is any element of a gamma-set Γ of S, the relation between a and the section A of Γ determined by a [1]) shall, in view of the choice-function f, include that a is the "distinguished element" of the complementary set $S - A$, i.e. of the complement of A with respect to the set S. This distinguished element, which is fully determined by A, accordingly is the sequent of all elements of A in Γ. The intention lying at the bottom of condition b may therefore be expressed as follows: if S were already well-ordered — and to achieve this is the purpose of the entire proof — our choice-function would relate to any non-empty remainder $S - A$ of S its *first* element as its distinguished element. Hence, since S is not yet well-ordered, the task is to arrange its elements in such a way that the given choice-function f fulfils the condition just mentioned.

The simplest gamma-sets of S are obtained in the following way. If we take $A = 0$, the first element $a = c_0$ of Γ determines the section A. In this case we have $c_0 = f(S - 0) = f(S)$; *the first element of any gamma-set of S is the distinguished element of the whole set S.* Furthermore, assuming $A = \{c_0\}$, the sequent of c_0 in Γ (i.e. the second element of Γ), if any, is the distinguished element c_1 of the complement $S - \{c_0\}$.

Hence the ordered sets $\{c_0\}$ and $\{c_0, c_1\}$ are the simplest gamma-sets. Provided the set S is infinite, the procedure can be continued inductively to any finite extent; condition a is then automatically fulfilled, since any finite ordered set is well-ordered. Thus for all natural numbers n we obtain the finite gamma-sets

$$\{c_0, c_1, \ldots, c_n\}$$

where c_k, for $k \leqq n$, is the distinguished element of the set $S - \{c_0, c_1, \ldots, c_{k-1}\}$. Moreover, the definition of gamma-set implies that any infinite gamma-set is "beginning" with the elements $c_0, c_1, \ldots, c_n, \ldots$ in this succession.

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[1]) The section A, being a proper subset of Γ, is *a fortiori* a proper subset of S — even if Γ contains all elements of S, as we wish to arrange. Therefore the complement $S - A$ cannot be the null-set, and has a distinguished element in view of f.

After these introductory and informal remarks, we shall divide the proof of the well-ordering theorem into four parts.

I. *Of two different gamma-sets of S, one is a section of the other.*

Proof. Since, by condition a, a gamma-set is well-ordered, one of the different gamma-sets Γ_1 and Γ_2 is similar to a section of the other, or they are similar to one another, according to theorem 8 on p. 254. We may therefore presume:

$$\Gamma_1 \simeq \Gamma_2'. \qquad (\Gamma_2' \text{ section of } \Gamma_2, \text{ or } \Gamma_2' = \Gamma_2.)$$

Now, by transfinite induction along the elements of Γ_1 we can easily show that Γ_1 is not only similar but *equal* to Γ_2'. For the first element of both Γ_1 and Γ_2' is the first element of *any* gamma-set of S (see above), viz. $c_0 = f(S)$. If Γ_1 and Γ_2' were different, there would be elements of Γ_1 not related to themselves by the [1]) similar representation between Γ_1 and Γ_2'; let c_1^* be the *first* such element of Γ_1, and c_2^* its mate in Γ_2'. Hence the sections C_1 of Γ_1 determined by c_1^*, and C_2 of Γ_2' determined by c_2^*, coincide with each other. Accordingly, in view of condition b, we obtain

$$c_1^* = f(S - C_1) = f(S - C_2) = c_2^*,$$

contrary to our assumption. Hence $\Gamma_1 = \Gamma_2'$, and since we assumed Γ_1 and Γ_2 to be different, $\Gamma_2' = \Gamma_1$ is a *section* of Γ_2. Q.E.D.

The identical "beginning" of all gamma-sets of S exhibited above, has thus proved to continue as long as the less comprehensive set extends. Therefore two gamma-sets having an element in common, also have the sections determined by that element in common. Moreover, if they have two elements in common, the order-relation between these elements is the same in both gamma-sets.

II. *The totality of the elements of all gamma-sets of S can be well-ordered in such a way that the order-relations holding in any gamma-set are retained.*

Proof. First we construct the sum-set of the set the elements of which are all gamma-sets of S; every element occurring in any gamma-set, and only these elements, will accordingly be contained

[1]) By theorem 13 on p. 265, there is only one such similar representation.

in this sum-set. (Provided the notion of well-ordered set is introduced through our principles — cf. p. 181 and 319 — there is no difficulty in constructing the sum-set in question; in other words, we need not be afraid that the sum-set in question will prove to be boundless and so lead to a contradiction like the set of all ordinals, cf. p. 282). For a gamma-set of S is just an ordered subset of S having certain properties; hence the set of all gamma-sets is a subset of the *power-set* of S [1]), defined by those properties and guaranteed by the principle of subsets; finally, the principle of sum-set provides the set required in the present case.

We shall now define a certain order in the sum-set just constructed, as follows. If s_1 and s_2 are two different elements of the sum-set, then by the definition of the sum-set, each of them belongs to at least one gamma-set of S, say $s_1 \varepsilon \Gamma_1$ and $s_2 \varepsilon \Gamma_2$. (Of course, Γ_1 and Γ_2 are not uniquely determined hereby.) According to part I, at least one of the sets Γ_1 and Γ_2 is a subset of the other, so that s_1 and s_2 appear as elements in the same gamma-set, say Γ. *According to whether* $s_1 \prec s_2$ *or* $s_2 \prec s_1$ *holds in* Γ, *we establish the respective relation in our sum-set.*

First of all we have to prove that the obvious arbitrariness in choosing Γ does not impair the given definition of an ordering rule. For this purpose it is sufficient to show that if Γ' is any other gamma-set containing the elements s_1 and s_2, their succession in Γ' is the same as in Γ. But this has just been proved at the end of I. Accordingly our rule of arranging the given elements s_1 and s_2 is unambiguous; *the sum-set as ordered by this rule, shall be denoted by* $\sum$. It is evident that our rule of order provides for the transitivity of succession. In fact, if $s_1 \prec s_2$ in Γ and $s_2 \prec s_3$ in Γ', Γ' also contains s_1, as s_1 is an element of the section determined by s_2. Since Γ' is ordered, we have $s_1 \prec s_3$ in Γ', hence also in $\sum$.

It remains still to show that $\sum$ is not only ordered but a *well-ordered* set, i.e. that any non-empty subset $\sum_0$ of $\sum$ has a first element. Let s_0 be any element of $\sum_0$. If s_0 is not the first element, let s_0' be any element of $\sum_0$ which precedes s_0 ($s_0' \prec s_0$ in $\sum_0$). Denote by Γ an arbitrary gamma-set of S which contains s_0. Owing to the definition of order in $\sum$ and to part I, s_0' too is con-

[1]) Considering that an *ordered* subset of S is, strictly speaking, a subset of the power-set of S, we should here speak of the power-set of the power-set of S. From 7 suitable hints regarding this subject may be gathered.

tained in Γ, and the relation $s_0' \dashv s_0$ holds also in Γ. Therefore the subset of $\sum_0$ containing all $x \varepsilon \sum_0$ which precede s_0, is at the same time a non-empty subset of the gamma-set Γ; hence, in view of condition a, it contains a first element, which is the first element of $\sum_0$. Accordingly, $\sum$ is well-ordered.

III. *The set* $\sum$ *defined in* II *is itself a gamma-set, hence* (since it contains the elements of *all* gamma-sets) *the largest gamma-set of* S.

Proof. The property a of gamma-sets has been proved for $\sum$ in II. As for the property b, let A be any section of $\sum$, determined by $a \varepsilon \sum$. According to the definition of $\sum$, a belongs to a certain gamma-set, say to Γ, and A is also the section of Γ determined by a, since (cf. the end of II) every element preceding a in $\sum$ belongs to Γ. The property b of Γ assures that $a = f(S - A)$. Since A was chosen as an arbitrary section of $\sum$, the last relation shows that $\sum$ has also the property b; hence $\sum$ is a gamma-set.

The proofs of III and of the last part of II rely on the possibility of identifying $\sum$ "as far as desired" with a suitable gamma-set, which therefore transfers its own properties to $\sum$.

IV. $\sum$ *contains all elements of* S.

Proof. The property of $\sum$, being the *largest* gamma-set of S, rests upon the fact that it exhausts the set S; otherwise we would be able to enlarge $\sum$ while retaining the properties of gamma-sets. To prove this, let Γ be *any* gamma-set of S which does *not* contain all elements of S. Then $S - \Gamma$ is a non-empty subset of S; let be $z = f(S - \Gamma)$, which implies that z does not belong to Γ. By constructing the ordered sum $\Gamma + \{z\}$ we obtain a set which is well-ordered and also has the property b of gamma-sets. In fact, for all elements a of Γ (in other words, for all sections A of Γ) this is guaranteed by Γ being a gamma-set; as for the last element z of the sum (in other words, for the section Γ of the sum), it is guaranteed by the relation $z = f(S - \Gamma)$ which has just been used for defining z.

Thus any gamma-set which does not exhaust the whole set S ensures the existence of still more comprehensive gamma-sets. Accordingly, since by III $\sum$ is the largest gamma-set of S, $\sum$ contains all elements of S.

Hereby we have carried out the complete proof of the well-ordering theorem. For on account of IV and of the property a of gamma-sets, $\sum$ is a well-ordered set containing all elements of the given set S. It moreover contains *just* these elements since $\sum$, like every gamma-set of S, is a subset of S. Q.E.D.

7. The General Comparability of Sets and Cardinals. The next consequence of the well-ordering theorem is its application to the comparison of plain sets and their cardinals. In fact, given two (not-ordered, or ordered but not well-ordered) sets, there are, on account of the well-ordering theorem, two *well-ordered* sets containing the same elements, respectively [1]). By theorem 11 the latter sets are comparable not only with regard to their ordinals (theorem 3 on p. 280), but also with regard to their cardinals. In other words, at least one of them is equivalent to a subset of the other. Hence, the same applies to the plain sets originally given. Accordingly *the fourth case* (case of incomparability), which had remained open (p. 104) among the cases of mutual equivalence-relations between two plain sets, *has been excluded by the well-ordering theorem.* We thus obtain the following statement which we had earlier conjectured, but so far have not been able to prove [2]):

Theorem of (general) comparability. Either two given sets are equivalent, or one is equivalent to a subset of the other. In other words, of two different cardinals one is smaller than the other.

This theorem removes the doubts which hitherto made us hesitate to attribute the title of "cardinal *numbers*" to the cardinals of infinite aggregates, and which have induced mathematicians to distinguish between *powers* (the cardinals of plain sets, or of plainly ordered sets) and *alephs* (the cardinals of well-ordered sets). Both these names have become unnecessary, the name "cardinal (number)" — as used in this book from the beginning — replaces them.

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[1]) For our purpose even less would be sufficient, namely that the well-ordered sets be respectively *equivalent* to the given sets.

[2]) For a proof of the comparability theorem by means of Zorn's maximum principle, see Zorn 2. Here, the multiplicative principle is not used, nor is the well-ordering theorem. For Zorn's principle, see *Foundations*, ch. II. Hartogs 1 proves the well-ordering theorem by assuming comparability.

On the other hand, it is a surprising and unwelcome phenomenon that an apparently so simple theorem about equivalence, which does not refer to the notion of order at all, appears here as the product of profound considerations in the domain of *order*, instead of being proved at an early stage of the theory of equivalence, say in § 5. The necessity of this detour may serve as a weighty argument — logical as well as mathematical — for letting the theory of order precede the theory of equivalence. The reasons which have induced us to proceed in the opposite direction were indicated previously, the chief reason being the more general character of the concepts of equivalence, and the fact that mathematics on the whole usually proceeds from the general to the particular.

Before the establishment of the well-ordering and comparability theorems, the *arithmetic of cardinals in general* presented a lot of interesting and difficult problems many of which had not been solved at that time in spite of considerable efforts spent on them. By the comparability theorem most of these problems have become easy or even trivial; the results of **5** (arithmetic of alephs) are now available for cardinals in general.

On the other hand, the comparability theorem (or its assumption even before it had been proved) has given rise to *new problems*, of how to insert a given cardinal into its appropriate place in the succession of alephs $\aleph_0$, $\aleph_1$, ... $\aleph_\omega$, ... (**5**). By far the most important, as well as the oldest, among these problems is the **continuum problem** (pp. 93 and 152) *which asks for the place of the cardinal* $\aleph = 2^{\aleph_0}$ *of the continuum.* Since $\aleph > \aleph_0$, the comparability theorem implies:

$$\aleph \geqq \aleph_1 \text{ [1]}.$$

The main question therefore is: does $\aleph$ *equal* the second transfinite cardinal $\aleph_1$, or is $\aleph$ *larger than* $\aleph_1$? In the latter case, the further question arises: what is the precise place of $\aleph$ in the succession of alephs? *Far beyond the borders of set theory, the con-*

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[1]) Sierpiński 29 proves this relation without using the multiplicative axiom. The well-ordering theorem and the comparability are not used either. The proof is essentially due to A. Tarski and based on the (apparently obvious) assumption that a set, originating from a set S by a *unique* (not just biunique) *correspondence*, cannot have a larger cardinal than S.

tinuum problem has a fundamental importance for the foundation of mathematics in general.

The only definitive result known to us, and even known for more than forty years, is the negative assertion $\aleph \neq \aleph_\omega$ [1] (p. 305). The conjecture

$$(1) \qquad (\aleph =)\ 2^{\aleph_0} = \aleph_1,$$

made by Cantor from the beginning, is called *Cantor's continuum hypothesis*. A few attempts to demonstrate this hypothesis [2] directly have turned out to be failures, and to a certain extent the same applies to Hilbert's endeavor [3] to solve the problem in the framework of his gigantic "metamathematical" system. This system will be discussed in detail in *Foundations*, as will be the only decisive, and really tremendous progress that has been made with regard to the continuum problem since Cantor's attempts of 1880, namely Gödel's proof [4] that *Cantor's continuum hypothesis* — even in its generalization (2) soon to be mentioned — *cannot be refuted*, i.e. that it is *compatible* with the usual principles of mathematics in general and of set theory in particular. In spite of the importance of this result, the huge difference between it and an actual *proof* of the continuum hypothesis will be analyzed in *Foundations*, ch. V.

On the other hand, very numerous and interesting results have been attained regarding *the connection between the continuum hypothesis and other (as yet unsolved) mathematical problems* [5], in particular regarding *equivalent* assertions (implied by the continuum hypothesis and also implying it); most of them by Sierpiński and

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[1] It is easy to generalize this inequality; for instance, by taking an $\aleph_\alpha$ where α is *any* limit-number of the second number-class, instead of $\aleph_\omega$. However, we do not even know any α such that $\aleph_\alpha$ would be an *upper bound* for $2^{\aleph_0}$.

[2] Cf., e.g., F. Bernstein **7** and Eyraud **4**. Baer **4**, § 5, observed that the formal laws of cardinal arithmetic are by no means sufficient to prove the continuum hypothesis.

[3] See in particular Hilbert **9**. It is not the *truth* but the *compatibility* of the continuum hypothesis which is discussed in this essay.

[4] Gödel **10** (1938), **11**, **12**.

[5] Even problems of algebra, a mathematical branch apparently quite remote from such questions; see Baer **5**.

other members of the pre-war Polish mathematical school [1]). As a matter of fact, many of the questions which are still open in the arithmetic of cardinals, as well as many other questions of abstract set theory and even of the theory of sets of points, would immediately be solvable on the basis of a positive answer to the problem, i.e. of a proof of the equalities (1) and (2).

The *generalized continuum hypothesis* is the conjecture

$$2^{\aleph_\nu} = \aleph_{\nu+1}, \qquad (\nu \text{ any ordinal}) \tag{2}$$

which includes the ordinary continuum hypothesis as the particular case $\nu = 0$. Together with the well-ordering theorem, (2) obviously means (p. 151) that the power-set of any infinite set S has a cardinal which not only *surpasses* the cardinal of S, but which is its *sequent*. Many of the essays just quoted dealing with Cantor's hypothesis, deal as well with the generalized form [2]).

The uneasiness caused by basing the comparability theorem upon the theory of order, instead of the theory of equivalence, is not an isolated occurrence in mathematics. Many apparently simple theorems of the theory of numbers (integers) have not been proved within pure arithmetic but only by means of analytical methods which essentially use the general notion of real number which is foreign to arithmetic; for instance, the proposition that in an arithmetical progression $a + b \cdot n$ (a and b relatively prime) there are infinitely many prime numbers. What is more, Gödel's incompleteness theorem of 1930 [3]) shows that *in any formal theory*

[1]) The comprehensive book Sierpiński **18** deals with assertions *equivalent* to the continuum hypothesis as well as with *inferences* from the hypothesis, including its generalization (2). The book gives a survey of the results attained up to 1934. Among previous papers (after 1920) we should mention Braun–Sierpiński **1**; Hurewicz **2**; Kożniewski–Lindenbaum **1**; Lindenbaum–Tarski **1**; Patai **1**; Sierpiński **5, 12, 14, 16**; Tarski **6**. Among researches since 1934: Folley **1**; Rothberger **1**; Ruziewicz **1**–**3** (cf. Piccard **1** and **2**); Sierpiński **19, 20, 21, 23, 24, 25**. The case next to Cantor's hypothesis, i.e. the case of $2^{\aleph}$, is dealt with in Sierpiński **8**. Lusin **10**, pp. 129–131, introduces the hypothesis, alternative to Cantor's, $2^{\aleph_0} = 2^{\aleph_1}$.

[2]) Particularly general results concerning the cardinals of sets, especially concerning certain subsets of power-sets, have been obtained by Tarski **14**, mostly by means of the well-ordering theorem and the generalized continuum hypothesis.

[3]) Gödel **2**; see *Foundations*, ch. V.

of not altogether trivial simplicity, one may formulate true assertions which cannot be formally demonstrated within the theory.

The comparability theorem, however, is not of so complicated a nature. This might already be concluded from the fact (mentioned on p. 181) that the fundamental concepts and definitions of the theory of order can be expressed within the theory of equivalence. Without going as far as explicitly introducing these concepts and definitions, *we shall now prove the comparability theorem within the borders of equivalence theory only.* The proof to be given is essentially a combination of Zermelo's *second proof* of the well-ordering theorem with a reduction of (well-)ordered sets to plain sets. Hence it also contains the fundamental notions and methods of Dedekind's *theory of chains* [1]).

Let S and T be any two plain sets. To prove their comparability, we shall show that at least one of them is equivalent to a subset of the other.

Any representation φ of a subset $S' \subseteq S$ on a subset $T' \subseteq T$ shall be called a *partial representation* between S and T; we write $T' = \varphi(S')$. It is obvious that between any two non-empty sets S and T there exist partial representations; e.g. those attaching one definite element of S to one definite element of T.

Given two different partial representations between S and T, say $T' = \varphi(S')$ and $T'' = \psi(S'')$, ψ shall be called an *extension of* φ if, first, S' is a proper subset of S'' and, secondly, both φ and ψ furnish the same mates in T for their common argument-domain S', so that also T' is a proper subset of T''. The fact that ψ is an extension of φ shall be expressed by

$$\psi \supset \varphi, \text{ or synonymously } \varphi \subset \psi.$$

The latter expression may be read "φ is a part of ψ". If, in particular, S'' contains only a *single* element (of S) not belonging to S', ψ shall be named a *simple extension of* φ. A partial representation between S and T for which there exists an extension, shall be called an *extensible* representation.

A set Φ of partial representations between S and T is to be called *monotonic* if, of any two different elements of Φ, one is an extension of the other; in other words, if $\varphi_1 \varepsilon \Phi$ and $\varphi_2 \varepsilon \Phi$, then there holds one (evidently only one) of the relations

$$\varphi_1 \supset \varphi_2, \quad \varphi_1 = \varphi_2, \quad \varphi_1 \subset \varphi_2.$$

In a monotonic set a kind of natural succession of the elements is thus given, proceeding from narrower partial representations to more extensive ones, or vice versa. (This is hinted at by the name "monotonic"; for the connection with the order-concept in general, cf. *Foundations*, ch. II.) Incidentally, the monotonic sets of partial representations may be considered

[1]) Dedekind **2**. To the second proof of the well-ordering theorem (in Zermelo **2**) cf. Hessenberg **8**; Hausdorff **4**, pp. 136—138 or **5**, pp. 56—58.

as particular types of a more general kind of sets, viz. of the sets Ψ of partial representations between S and T having the property that any element of S which is disposed of in several elements (partial representations) of Ψ, has *the same mate* on account of all these elements.

An element φ of a set Φ of partial representations shall be called *comparable in* Φ if, for *every other* $\psi \varepsilon \Phi$, φ fulfils one of the relations $\varphi \supset \psi$ and $\varphi \subset \psi$. Hence Φ is monotonic if, and only if, *every* element of Φ is comparable in Φ.

Finally, "the *resultant* of a monotonic set Φ of partial representations between S and T" shall mean the partial representation which attaches the "corresponding" element of T to each $s \varepsilon S$ disposed of in at least one element of Φ. By "corresponding" element of T is meant the element of T attached to s uniformly in all elements of Φ which refer to s.

After these definitions we proceed to the proof itself. To a given partial representation $T' = \varphi(S')$, where $S' \neq S$ and $T' \neq T$, there exist obviously simple extensions; to obtain such, the mapping defined by φ may be extended by attaching an arbitrary element of $T - T'$ to an arbitrary element of $S - S'$. Hence, to prove the comparability theorem it is sufficient to show:

Among all partial representations between S and T there is at least one which does not admit of any simple extension.

For such a representation $T' = \varphi(S')$, either $S' = S$, or $T' = T$, or both relations hold; thus this φ attaches to each element of one of the given sets a mate in the other, i.e. φ represents one of the sets on a subset of the other.

To prove the assertion just formulated we start, analogously to the proof of the well-ordering theorem, with *an arbitrary choice of "distinguished" partial representations* which will be retained in what follows. With respect to any given *extensible* partial representation φ between S and T we choose a certain "distinguished" simple extension, which shall be marked by φ^+. For the sake of brevity, φ^+ may be called "the sequent of φ".

On account of this choice, a set Φ of partial representations between S and T shall be called a *chain* if it fulfils the following conditions:

1) Φ contains a definite (arbitrary) partial representation φ_0. (We may, for instance, choose φ_0 as the "vacuous" representation of the null-set on itself, or as the sequent of it, which attaches one certain element of T to one certain element of S.) Henceforth φ_0 remains fixed.

2) If φ is an *extensible element* of Φ, its sequent φ^+ is also an element of Φ.

3) If Φ_0 is a *monotonic subset* of Φ, the resultant of Φ_0 is also an element of Φ.

We should pay attention to the fact that this definition only requires "necessary" elements to be included in a chain while it does not exclude the appearance of other ("superfluous") elements. As a matter of fact, we shall presently remove such superfluous elements by an additional step, in order to obtain what we virtually need.

From the definition of chain it immediately follows that the *meet of a* (finite or infinite) *number of chains is again a chain.* Hence *a least chain must exist*, viz. *the meet M of all chains* — of course with respect to the given sets S and T, to the choice expressed by the notation φ^+, and to the chosen constant φ_0.

Our aim is to prove that *the meet M is a monotonic set.* This proposition will immediately enable us to obtain the comparability of S and T.

To prove our proposition we shall show that *the set of all elements of M which are comparable in M, is a chain.* In fact we infer, regarding the three conditions of a chain:

a) Any element of M different from φ_0, is an *extension* of φ_0. This is evident if we take φ_0 as the representation of the null-set on the null-set as indicated in 1). In the general case we may conclude as follows: if among the elements of M there were partial representations which are not extensions of φ_0, they could be dropped without altering the fact that M is a chain; thus we would obtain a proper subset of M which is also a chain, contrary to the definition of M. Hence φ_0 *is comparable in M.*

b) If φ is an (arbitrary) *extensible* partial representation contained in M and comparable in M, *its sequent φ^+ is also comparable in M.* (That φ^+ is an element of M, follows from the condition 2) fulfilled by any chain.)

To prove this assertion, it suffices to show that the subset $\overline{M}$ of those elements ψ of M which have one of the properties $\psi \subset \varphi$ and $\psi \supseteq \varphi^+$, *is a chain*; once this is proved, we conclude $\overline{M} = M$ since M is the least chain; hence M has the asserted property. Now the conditions 1) and 3) are evidently fulfilled in $\overline{M}$. As to 2), we distinguish between the cases $\psi \subset \varphi$ and $\psi \supseteq \varphi$. In the first case (which implies that ψ is extensible) $\psi^+ \subseteq \varphi$ holds since otherwise, by the comparability of φ in M, we would have $\psi^+ \supset \varphi$, hence $\psi^+ \supset \varphi \supset \psi$ in contradiction to the fact that ψ^+ is a *simple* extension of ψ. In the second case, $\psi \supseteq \varphi$ implies $\psi^+ \supset \varphi$, provided that ψ^+ exists, i.e. that ψ is extensible.

c) Let M_0 be a monotonic subset of M the elements of which are all comparable in M. Then *the resultant χ of M_0, which is contained in M, is comparable in M.* To perceive this, let us distinguish between those elements of M which are extensions of *all* elements of M_0, and the others, i.e. those which themselves appear in M_0, or those which have extensions which appear in M_0. The elements of the first kind, save possibly for χ itself, are extensions of χ, whereas $\chi \supseteq \varphi$ for every element φ of the second kind.

In view of a — c, the elements of M which are comparable in M form a chain, which must equal M since M is the least chain. Hence all elements of M are comparable in M, i.e. M *is a monotonic set* as was our aim to prove.

Therefore, on account of the condition 3) fulfilled by any chain, the resultant φ^* of M is also an element of M. Now *the partial representation φ^* between S and T is not extensible.* For otherwise, its sequent φ^{*+} would also be contained in M, contrary to the definition of φ^* as the *resultant* of the entire set M.

Hence φ^* attaches to each element of either S or T a mate in the other set. Accordingly, at least one of these sets is equivalent to a subset of the other. Q.E.D.

Exercises

Most of them are rather difficult.

1) Prove that the type of any ordered set can be changed by

a suitable addition of a single element. (Hint: consider the highest well-ordered initial of the set [1].)

2) An order-type τ is finite if, and only if, τ has the following property: however we may change the place of a single element in a set of the type τ, we always obtain the original type [1].

3) Prove the assertion on limit-numbers which precedes the corollary on p. 282.

4) Let us call a set $\sum$ of ordinal numbers *closed* if, when $\sum_0$ is any subset of $\sum$ without a maximum (largest number), the sequent of the subset $\sum_0$ also belongs to $\sum$. Prove that the set of all *left-hand divisors* of an ordinal number is closed [2]. (It is trivial that the same applies to the set of all right-hand divisors since, according to p. 294, this set is finite.)

5) Given any infinite set $\sum$ of ordinals (or any infinite well-ordered set), there exists a set T the elements of which are "triadic" subsets of $\sum$ (i.e. each element contains just *three* elements of $\sum$) such that *any pair* of elements of $\sum$ is a subset of one, and of only one, element of T. Prove this theorem by considering different (nine) possible cases [3].

6) Prove (see the footnote 1 on p. 284, and (1′) on p. 288) that for any ordinal $\varkappa$ and for any limit-number $\lambda = \lim_{\nu} \lambda_\nu$, the following relations hold:

$$\varkappa + \lim_{\nu} \lambda_\nu = \lim_{\nu} (\varkappa + \lambda_\nu), \quad \varkappa \cdot \lim_{\nu} \lambda_\nu = \lim_{\nu} (\varkappa \cdot \lambda_\nu).$$

(Hint: use theorem 7 on p. 290.)

7) Complete the proof of (2) on p. 287.

8) On account of the definition of power in 3, prove the following theorem, which is analogous to a well-known theorem on real functions: The power $\sigma^\alpha = p(\alpha)$ is the *only* function of an ordinal α with $p(1) = \sigma$ which satisfies the functional equation

$$p(\alpha_1 + \alpha_2) = p(\alpha_1) \cdot p(\alpha_2),$$

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[1]) See Chajoth 1.

[2]) See Sierpiński 7 b.

[3]) See Sierpiński 27. An elementary finite combinatory problem treated by J. Steiner (cf. *Journal für die Reine und Angewandte Mathematik*, vol. 45, p. 181; 1853) is here generalized to the infinite.

and which at the same time is "continuous" in the sense of $\lim_{\nu} p(a_\nu) = p(\lim_{\nu} a_\nu)$. — On the other hand, the power σ^a is not a continuous function of the *basis* σ; e.g., $\lim_{\nu} \nu^n = \omega$ (hence $\neq \omega^n$) if ν runs over all finite ordinals and n is finite > 1.

9) Assuming $\sigma > 1$ and $a_1 < a_2$, prove the inequalities

$$a_1^\sigma \leqq a_2^\sigma, \quad \sigma^{a_1} < \sigma^{a_2}.$$

10) Generalize the assertion of theorem 4 on p. 202 ($\overline{\overline{\boldsymbol{T}(\aleph_0)}} = 2^{\aleph_0}$, where $\boldsymbol{T}(\mathbf{c})$ means the set of all *order-types* of the cardinal $\mathbf{c}$) to the case of any aleph, i.e. to $\overline{\overline{\boldsymbol{T}(\aleph_a)}} = 2^{\aleph_a}$. (Assuming the generalized continuum hypothesis, one would therefore obtain $\boldsymbol{T}(\aleph_a) \sim Z(\aleph_a)$, where Z denotes the number-class, i.e. the set containing all *ordinals* of the aleph in question.)

11) Prove $\aleph_a \cdot \aleph_a = \aleph_a$ by means of transfinite induction, instead of using natural sums [1]).

12) Prove Hausdorff's recursion formula (footnote 1 on p. 303) — which for $\sigma \leqq \tau$ easily follows from (5) on p. 303 — in the case $\sigma > \tau$ by means of the following theorem, the proof of which is not difficult: Given a well-ordered set T of ordinals τ_ν such that $\overline{T} \leqq \aleph_{a-1}$ and for every ν also $\overline{\tau_\nu} \leqq \aleph_{a-1}$, there exists an ordinal $\zeta < \omega_a$ such that $\tau_\nu < \zeta$ for any ν. (Hint. Use the following formula valid for all σ having an immediate predecessor: $\aleph_\sigma^{\aleph_\tau} \leqq \sum_\varkappa \xi_\varkappa^{\aleph_\tau}$ where $\xi_\varkappa$ runs over $W(\omega_\sigma)$; this formula may be proved by considering the left-hand power as the cardinal of an insertion-set.)

13) Generalize theorem 12 on p. 299 to the case of any number-class the index of which is not a limit-number; that is to say, prove the theorem: Given a well-ordered set the type of which is smaller than the initial number $\omega_{\sigma+1}$ and the elements of which are ordinals smaller than $\omega_{\sigma+1}$, the sequent of all ordinals of the set is also smaller than $\omega_{\sigma+1}$. — In general this theorem is not true if the index of the initial number is a limit-number, e.g. not for ω_ω; in fact, $\lim_{\nu} \omega_\nu$ with ν running over the finite ordinals, equals ω_ω [2]).

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[1]) Cf., e.g., Hausdorff **4**, p. 127.

[2]) An initial number for which the theorem of exercise 13 holds, is called a *regular* initial number. Hence ω_0 ($= \omega$) and any ω_σ the index σ of which has an immediate predecessor, are regular, while all known ordinals ω_σ for

14) On account of the proof of the well-ordering theorem given in **6**, find the ordinal of the well-ordered set of all natural numbers corresponding to each of the following choice-functions (for non-empty sets S of natural numbers > 1) [1]:

a) The distinguished element of S is the number which has the least prime factor; if several numbers have the same least prime factor, take that one in which the lowest power of that factor is contained, and if this rule still admits of different numbers, take the smallest one.

b) The distinguished element of S is the number which, in its decomposition into prime factors, contains the least number of (different or equal) factors; if, according to this, several numbers are to be considered, take that one which has the least prime factor; if then still different numbers are involved, use the previous rule a.

By well-ordering the set of all natural numbers according to a or to b, we become aware that only a tiny part of the selections provided for by the rules, is actually needed. Of course in these cases the principle of choice is not required since the rules are given in a constructive way.

15) Prove, by means of a suitable choice-function, the following theorem which has significance in the theory of sets of points [2]) (problems pertaining to the so-called "covering theorem of Borel"): Let $A = (S_1, S_2, S_3, \ldots)$ be a sequence of sets such that $S_k \subset S_{k+1}$ for any k. If T is a set such that, firstly, every element of T is contained in at least one S_k and, secondly, every infinite subset T_0 of T has infinitely many elements in common with at least one S_k, there exists an element of A which includes the entire set T as a subset.

§ 12. Conclusion: The Significance of Set Theory

We have become acquainted with the edifice of abstract set theory as it had been erected by Cantor in a bold intuition and has

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which σ is a limit-number, are *singular*, i.e. non-regular. The problem of whether there exist also *regular initial numbers with a limit-index*, shall be dealt with in *Foundations*, ch. II.

[1]) For the sake of convenience we may take 1 as the distinguished element of any set containing 1.

[2]) See Veress **1**.

been expanded since the beginning of the twentieth century. As to the foundation of the building, we adopted a middle course between Cantor's realistic naivety and the critical analysis which has grown up since the turn of the century, a course chiefly marked by replacing Cantor's boundless quasi-definition of the concept of set with certain principles of a constructive or postulational character. In *Foundations* the edifice will be critically overhauled in several ways, and different methods will be developed and compared to support better the foundations of the building; methods which are partly conservative and partly radical or even destructive.

Regarding the nature of many problems we have dealt with, especially in the beginning of chapters I, II and III, one may have the impression that set theory has chiefly grown on a *philosophical* ground, especially from the question of the logical legitimacy of transfinite magnitudes and of arithmetical operations. Certainly ideas of this kind played a noticeable part in Cantor's work. Even a few decades earlier these ideas had formed the starting-point of one of the greatest logicians, Bernard Bolzano, a Bohemian clergyman of the beginning of the 19th century, who was so much ahead of his epoch in the mathematical as well as the philosophical sphere, and whose achievements have only been fully appreciated many decades after his death [1]).

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[1]) Within the logical sphere, his most prominent work is the *Wissenschaftslehre* of 1837 (Bolzano 2) the first three volumes of which treat logic with a comprehensiveness and profundity unparalleled before the rise of symbolic logic; see *Foundations*, ch. III; also, for example, Přihonsky 1. In particular, E. Husserl has contributed much to the rediscovery of this standard work, after it had been almost forgotten. Already the essay Bolzano 1 (of 1810) contains many interesting remarks. Of the modern literature on Bolzano's philosophical work we mention: Dubislav 11; Korselt 2; **Scholz 5**, p. 44 ff. and 16; **Smart 5**; Theobald 1; Wrinch 1.

In mathematics, it is the theory of real numbers and functions which owes him important progress in topics where not only the *solution* but the very *posing* of the problem was far ahead of the attitude of contemporary scholars, including even Cauchy and Gauss. Only the rediscovery of his results by Weierstrass in the sixties and the recent edition of Bolzano's unpublished manuscripts (4) have called attention to his mathematical work; here, it may suffice to mention his construction of a continuous and nowhere derivable function. Cf. Jašek 1 and Kowalewski 1.

For Bolzano's life and scientific development, see H. Bergmann 1; Fels 1–3; Struik 1; E. Winter 1 and 2.

During the last years of his life (1847/48), Bolzano wrote an essay on the *Paradoxes of the infinite* [1]), and herewith made the first, and the only serious, push before Cantor in the direction that later led to the theory of sets; even the term *Menge* (set) appears in this essay. He clearly discerned the paradoxical properties marking infinity, in particular the equivalence of an infinite set with a proper subset (§ 2, **5**). However, instead of conceiving them as characterizing the infinite and of taking them as a basis for a systematic development, he rather valued them as striking and altogether disagreeable features which complicate a methodical treatment. Thus he failed to perceive the full consequence of the notion of equivalence, and did not advance to the concepts of transfinite cardinal or order-type. Only Cantor systematically based his ideas on the principle of *analyzing* infinity and founding it on its own properties and laws. Thus he transformed Bolzano's collection of paradoxes, which he knew and appreciated to a certain extent [2]), into a scientific doctrine.

However, above and before these logical questions, purely *mathematical* considerations had given Cantor the impulse to the researches that formed the genesis of set theory. Towards 1870, in different parts of *analysis* (especially in the theory of trigonometric series, and of integration of discontinuous functions) problems had been arrived at which require the discrimination of certain infinite aggregates of real numbers (or points) out of the totality of *all* real numbers of an interval (*all* points of a segment) — i.e. problems nowadays pertaining to the theory of linear sets of points (§ 9). At the same time with Cantor, a few other scholars explored questions of this kind [3]); however, for want of a methodical instrument for defining sufficiently general aggregates, they did not succeed to a considerable degree.

In 1869, on the suggestion of E. Heine, Cantor had started

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[1]) Bolzano 3. Jašek (in **1** and also in other publications) has made it clear that the first editor of this posthumously published essay had inserted would-be corrections on his account; therefore, it is doubtful how far the errors contained in the essay are due to Bolzano himself. Also in 2 (for instance in vol. I, § 87) essential remarks on the problem of infinity are found.

[2]) Cantor **7**, V, p. 561.

[3]) Cf., e.g., Hankel **1**; H. J. S. Smith **1**; P. du Bois-Reymond **1** (cf. Hardy 3). Also Harnack, Volterra and others might be mentioned.

investigations in this direction [1]), in particular regarding trigonometric series and their singular points, and in 1870 [2]) he succeeded in proving the *uniqueness* of the development into a trigonometric series. From these researches he won the notion of limit-point, which is closely connected with the concepts introduced above in § 9. Trying to continue and generalize this inquiry further, he came — not at once but with ever growing buoyancy and verve — to the conviction that an instrument of a quite new and surprising type was required for the solution of the problems arising along this line. Essentially to this purpose he created the theory of sets. In contrast with du Bois-Reymond he gradually freed himself from the consideration of applications, and more and more developed the theory for its own sake, finally raising it to the rank of an autonomous mathematical branch.

The complex of analytic problems of this kind has to be mentioned in the first place, not only with respect to the *development* of set theory but yet more regarding its *applications* within mathematics. The tie between the theory of sets and the theory of real functions has become so close that the latter can hardly proceed a step without leaning upon the former. As a matter of fact, most textbooks on the theory of functions [3]) begin with a chapter, or a few chapters, treating abstract sets and sets of points, and in many cases it rather depends on the subjective taste whether one would range a certain theorem on the set side or on the function side. Among a lot of splendid achievements introduced into the theory of functions by set-theoretical methods, it may suffice here to mention one the significance of which is evident to whoever is acquainted with the elements of calculus: the various extensions of the *notion of (definite) integral.* This notion, playing a central part in analysis and in its applications throughout the natural, technical, and even the economical and social sciences, obtained a strict and general foundation by A. L. Cauchy (1823) and

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[1]) Cantor–Stäckel **1**; cf. Cantor **2**, p. 130 footnote.

[2]) Cantor **3**.

[3]) Cf. the literature quoted on p. 239. To the following remarks cf. the historical essay Lebesgue **6**. Since 1919 (with an interruption during World War II) there has been a scientific journal exclusively reserved to set theory including its applications in analysis and geometry: *Fundamenta Mathematicae* (Warsaw).

B. Riemann (1851). In this classical form, however, the notion of integral fails in many cases, particularly if the function to be integrated is discontinuous to a considerable extent in the interval of integration, so that the sum, whose limit should define the integral, ceases converging to a definite value. Since the beginning of the twentieth century, in many of these cases a treatment became available through different methods of extending the notion of integral by means of set theory, notably through an appropriate definition of the *contents* or *measure* of two- and more-dimensional sets of points; among these extensions the Lebesgue integral has become most famous and important while other kinds of integrals are more fit for certain particular purposes. By means of such extensions of the integral concept, much larger classes of functions have become integrable, and in such a way that the new integral coincides with the Riemann integral whenever both of them exist simultaneously. These extensions and generalizations of mathematical analysis by means of set theory have also gained considerable importance by applications in other branches, for instance in the theory of probability, and even in astronomy and physics [1]).

The analytical problems which led Cantor to his discoveries, are closely connected with *geometry.* For this subject we should remember the sets of points (§ 9) in which some, as it were, "microscopical" methods made certain distinctions and examinations possible which the "macroscopical" methods of classical geometry would not have been able to attain. Thus the concept

[1]) Cf., e.g., F. Bernstein **5**; Bouligand **2**; Carathéodory **2** (Poincaré's return theorem); Denjoy **1**; Vitali **1**. Among the researches on the ergodic theorem, initiated with set-theoretical methods in 1913, it may suffice to cite G. D. Birkhoff **1** and von Neumann **10**. Of the copious literature on set-theoretical methods in the theory of probability, Kolmogoroff **3** and Feller **1** may be mentioned; in these writings plenty of further literature is quoted.

Applications of set theory to physics in an other sense had already been predicted by Cantor (cf. **7**, III, p. 120 ff.) in 1882. Cf. Rosenthal–Borel **1**, p. 905, footnote 160; Schoenflies **12**, p. 22; also van Vleck **1**.

For applications in chemistry cf. Habermann **1**.

In 1884 Cantor stated that his investigations on sets of points were properly made in view of certain applications "to inquire into the nature of organisms in order to get to the bottom of the nature of organic substance"; cf. Schoenflies **12**, p. 20. This statement should be taken *cum grano salis* considering his state of mind at that period; cf. Fraenkel **14**.

of *linear continuum* was completely described for the first time in the history of science (p. 228), and the concept of *dimension* was clarified by investigating the continuous representations between different sets of points (p. 139 f.). Among the geometrical branches greatly stimulated by set theory, *topology* (analysis situs) should foremost be mentioned [1]). Beyond this especially interesting modern branch of geometry, however, set-theoretical geometry is of a surprising variety; since the invention of analytical geometry and calculus, hardly any methodical progress in geometry has enlarged the object of geometrical research as much as the application of set theory [2]). Incidentally, at the birth of set theory questions of synthetic geometry (where a line is taken as "bearer" of its points, a point as "bearer" of the lines passing through it) played a stimulating part; on this soil the term *Mächtigkeit* (power, cardinal) has been formed.

Finally we have to mention the *theory of finite numbers* [3]) (integers) among the questions which have influenced the creation of set theory. (Cf. also § 10, 6.) This aspect is chiefly due to Dedekind [4]) who made an extensive and systematical use of the notion "one-to-one correspondence"; and in fact, in the later work of Cantor one can unmistakably recognize the influence of Dedekind's "logical" attitude. In particular Dedekind introduced the notion of "chain" which later won a fundamental importance in abstract set theory (cf. § 11, 7). Meanwhile, far beyond set theory, the notion of one-to-one correspondence has reached an ever increasing significance and applicability, in particular in the

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[1]) Among the plentiful literature on set-theoretical topology it may suffice to cite Kuratowski **7** and Newman **2**.

[2]) Cf. the programmatic papers Menger **2** (especially p. 303) and **8** and the general survey Hahn **5**.

[3]) Mostly finite sets (and in exceptional cases also denumerable sets) are involved in the applications of set theory to the mathematical investigation of chess and games of a similar type. Cf. Zermelo **5**; D. König **2**; Kalmár **1**; Euwe **1**.

[4]) His pioneer booklet **2** originally appeared in 1887. For his unpublished researches in this direction see Dedekind **3** and Cantor–Dedekind **1** (containing the correspondence between both scholars). In the preface to **2** and even earlier (in 1879) the great importance of the notion of one-to-one correspondence for the foundation of the concept and theory of number was explicitly pointed out.

modern abstract shapes of group-, ring-, and field-theories [1]).

Thus the theory of sets forms a mathematical branch which unites in itself an arithmetical-discontinuous attitude (preferred as the starting point in modern mathematics) and the feature of continuity predominating in geometry and analysis, which in Greek mathematics was widely accepted as the primitive basis. It was Cantor's conscious intention to create a "genuine fusion between arithmetic and geometry" where each of the sciences should have an appropriate way [2]). As will be shown in *Foundations*, the ever continued and, alas, never succeeding attempts to span the gap between discreteness and continuity, are especially carried on within the set-theoretical domain. This is quite natural; for set theory, being the most general branch of mathematics, has to take on its shoulders the fundamental task of methodically investigating the general primitive ideas of mathematics, as function (cf. pp. 32 f. and 149), cardinal number (pp. 76–83), order (pp. 176 and 181), and of contributing a decisive share in securing the foundations of the various mathematical sciences as well as the foundation of mathematics as a whole. In a certain sense set theory may thus be considered as the fundament of mathematics in general. There are, it is true, mathematical and philosophical attitudes objecting to this claim and in *Foundations* we shall present and analyze these attitudes.

The achievements attained by set theory in the various directions just mentioned, have given it an important and even privileged place among the mathematical sciences, in spite of the youth of this branch which looks back on hardly more than half a century. Reacting upon a certain criticism of the "intuitionistic" school (*Foundations*, ch. IV), one of the outstanding modern mathematicians, David Hilbert, called set theory "one of the most vigorous

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[1]) For applications of set theory (especially of ordinals and of transfinite induction) in this domain cf., besides Steinitz **1**, for instance the books Hasse **1**; O. Haupt **1**, Kap. 23; van der Waerden **2**, p. 198 ff. (1st edition). In the pursuance of the methods of Steinitz and of related notions such as *Bewertung* (Kürschák) and of topological ideas, a close connection has developed between set theory and general arithmetic. For various pieces of fruitful progress in this direction see Baer **5** and **7**; van Dantzig **2**; F. K. Schmidt **1**. For other set-theoretical methods in arithmetic and algebra cf. Feller–Tornier **1**, Lindenbaum **4**, and the literature mentioned in these papers.

[2]) Cf. F. Klein **1**, p. 288 (3rd edition).

and fruitful branches of mathematics" and dubbed it "a paradise created by Cantor from which nobody will ever expel us" [1]). While up to the end of the 19th century the set-theoretical ideas were recognized and appreciated by only a narrow circle of mathematicians, this has since changed rather suddenly; the theory is now used most extensively within mathematics, and is even studied without, especially by logicians and by philosophers in general. Beside the importance of the applications, this development is also due to the very feature which in the beginning withheld from set theory the acknowledgement by the scientific world, namely that it is one of the most daring and revolutionary steps mathematics has ever taken.

A. N. Whitehead in his essay on "Mathematics as an element in the history of thought" [2]) characterizes *mathematics* as the most original creation of the human mind. According to his way of establishing this claim, one may award the same title, among the various mathematical sciences, to *set theory* as to the branch least relying on external experience and most genuinely originating from the free creation of the human mind. Thus the conquest of actual infinity by the methods of set theory may be considered as an expansion of our scientific horizon to no less an extent than the Copernican system in astronomy, and the theory of relativity, or even of quanta, in physics.

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[1]) Hilbert **6**, p. 411; **9**, p. 170. Cf. also the quotation on p. 306.

[2]) Whitehead **4**, ch. II.

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As to the selection of material, up to 1918 only works of importance are listed, the limit having been chosen because of the break by World War I and in view of the extensive bibliography contained in C. I. Lewis' *A Survey of Symbolic Logic*, published in 1918. From 1919 on, the bibliography is intended to be comprehensive with respect to many subjects, reaching up to 1950 (or up to 1946 for the subjects of the forthcoming book which contains a supplement to the bibliography). However, in the logical domain covered by the reviews in the *Journal of Symbolic Logic*, items published after 1935 are only included as far as they have some significance for the subjects of the books.

In the Bibliography the following languages have been fully taken into consideration: English, French, German, Italian, Dutch, Esperanto, Latin, Greek, Hebrew; moreover, the important material in Spanish, Portuguese, and the Scandinavian languages. The author deeply regrets that he does not understand Slavic languages which recently have become very important for mathematical research; he has in general only included that part of Russian and other Slavic literature which is supplemented with an abstract or a summary in another language (this, to be sure, comes true for almost all significant papers).

Regarding the date of publication, differences of one year may occur, since in general the year of the appearance of the *Volume* is stated, yet in particular cases the year indicated on the fascicle or the reprint.

Reviews and Abstracts are only included in exceptional cases.

An Index of those authors who (also) appear outside the alphabetical order, is found at the end of this Bibliography.

ABBREVIATIONS USED IN THE BIBLIOGRAPHY

(Self-evident abbreviations, as Math., Philos., Acad., Univ., Soc., etc., are not mentioned. "The" is mostly omitted.)

A.M.S. Coll. Publ. = Am. Math. Soc. Colloquium Publ.
Abh. = Abhandlung(en)
Abh. Hamburg = Abh. aus dem Math. Seminar der Hamburgischen Univ.
Acad. U.S.A. = Proc. of the National Acad. of Sc. (U.S.A.)
Acta Szeged = Acta Litterarum ac Scientiarum R. Univ. Hungaricae Francisco-Josephinae, Sectio Sc. Math.
Act. Sc. Ind. = Actualités Scientifiques et Industrielles
Am. = American
Ann. = Annals etc.
Anzeiger Akad. Wien = Akademie der Wiss. in Wien, Math.-Naturwiss. Kl., Anzeiger
appl. = applied, appliqué(es)
Ber. Leipzig = Berichte über die Verh. der Sächsischen Akad. der Wiss. zu Leipzig, Math.-Ph. Kl.
Bull. A.M.S. = Bulletin of the Am. Math. Soc.
C.R. = Compte(s) Rendu(s)
C.R. Paris = C.R. Hebdomadaires des Séances de l'Acad. des Sc. (Paris)
C.R. Varsovie = C.R. des Séances de la Soc. des Sc. et des Lettres de Varsovie, Cl. III
Časopis = Časopis pro Pěstováni Mat. a Fysiky
Ci. = Ciencia(s)
Cl. = Classe
Congr. = Congress(o), Congrès
Congr. Bologna 1928 = Atti del Congr. Intern. dei Matematici, Bologna 1928
Congr. Oslo 1936 = C.R. du Congr. Intern. des Math., Oslo 1936
Congr. Roma 1908 = Atti del IV Congr. Intern. dei Mat., Roma 1908
Ens. = Enseignement
f. = for, für
Forschungen zur Log(ist)ik etc. = Forschungen zur Log(ist)ik und zur Grundlegung der exakten Wiss.
Fund. = Fundamenta
Int. Enc. Un. Sc. = Intern. Encyclop. of Unified Sc.
J. = Journal
J. f. Math. = J. f. die Reine und Angewandte Math. (Crelle)
J. London M. S. = J. of the London Math. Soc.
J. S. L. = J. of Symbolic Logic
J. Un. S. = The J. of Unified Sc.

Jahrb. = Jahrbuch
Jahresb. D. M. V. = Jahresbericht der Deutschen Math.-Vereinigung
K. = Königliche, Koninklijke etc.
Kl. = Klasse
Koll. = Ergebnisse eines Math. Kolloquiums (herausg. von K. Menger)
Kongr. Heidelberg 1904 = Verh. des Dritten Intern. Math.-Kongresses in Heidelberg 1904
Kongr. Zürich 1932 = Verh. des Intern. Math.-Kongr., Zürich 1932
Les Entretiens de Zurich = L. E. de Z. sur les Fondements et la Méthode des Sc. Math. 1938 (Zurich 1941)
Monatsh. Math. Ph. = Monatshefte f. Math. und Ph.
N. = New, Nieuw etc.
N.S. = New Series, Neue Folge etc.
Nachr. Göttingen = Nachrichten der (K.) Gesellschaft der Wiss. zu Göttingen, Math.-Ph. Kl.
Nat. = Natural, Naturelles etc., or National(e)
p. = pure(s)
ph., Ph. = physics, physical, Physik etc.
Proc. = Proceedings
Proc. Amsterdam = K. Akad. van Wetenschappen te Amsterdam, Proc.
Proc. Ar. Soc. = Proc. of the Aristotelian Society
Proc. London M. S. = Proc. of the London Math. Soc.
Publ. = Publications
R. = Royal(e)
R. M. M. = Revue de Métaphysique et de Morale
Rec. = Recueil
Rendic. Lincei = Rendiconti della R. Accad. Nazionale dei Lincei (Roma), Cl. di Sc. Fis. etc.
Rendic. Palermo = Rendic. del Circolo Mat. di Palermo
s. a. = *sine anno*
Sc. = Science(s), Scienze etc.
Scr. = Scripta
Semesterber. Münster = Semesterberichte zur Pflege des Zusammenhangs von Universität und Schule (Math. Seminar, Münster i. W.; mimeographed)
Sitz. = Sitzungsberichte
Sitz. Berlin = Sitz. der Preussischen Akad. der Wiss., Phys.-Math. Kl.
Sitz. Heidelberg = Sitz. der Heidelberger Akad. der Wiss., Math.-Naturwiss. Kl.
Sitz. Wien = Akad. der Wiss. in Wien, Math.-Naturwiss. Kl., Sitz.
Tr. A. M. S. = Transactions of the Am. Math. Soc.
v. = voor
Verh. = Verhandlungen, Verhandelingen
Versl. = Verslag(en)
Wiss., wiss. = Wissenschaft(en), wissenschaftlich, etc.
Ztschr. = Zeitschrift
(2) means: 2nd series; etc.

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