A DEDUCTIVE THEORY OF SPACE AND TIME

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PREFACE

At present, physics is a collection of deductive theories, many of which do not specify explicitly all the concepts and postulates upon which they are based. This work is intended to be the foundation of a *single* deductive theory upon which all of physics can be built. Its title could have been "A Deductive Physical Theory. Part I"; but the actual title was chosen since it better indicates the main subject matter, and since the other parts have not yet been written. Although the content of this work has been of interest to philosophers for a long time, and still is, it is actually an integral and essential part of physical theory, as will be clearly seen later on.

Fruitful and exciting developments in mathematics during the last half century, concerned with the formulation of deductive theories, foundations of geometry, set theory, and symbolic logic, have not been utilized in physics to the extent they could. It is hoped that the extensive use of these ideas and tools here will demonstrate their power, elegance, and usefulness for physics, or for that matter any other science.

The theory starts with the primitive concept of the class of living humans, and ends with the derivation of the equations of motion of a particle in an arbitrary gravitational field. On route, it develops the concepts of particles, events, clocks, length measuring instruments, and all other basic concepts of space-time geometry in terms of only eight primitive concepts. It is a purely macroscopic theory, as the beginning of physical theory must be, but it is also the springboard for the development of a microscopic theory in the same spirit. By building from the ground up it may provide a fresh perspective and approach to a theory of elementary particles.

There has been a revival of interest in the general theory of relativity in the last five years. It is my belief that this work contributes to the clarification and development of the operational foundation of this theory. It owes much to relativity, but it also contributes new ideas and a novel approach.

In the formulation of a basic theory, a non-technical language such as English is too ambiguous and is not suited for the tremendous amount of deduction involved. Fortunately, a language ideally suited for this kind of task does exist under the name of *symbolic logic* (SL). It is simple, concise, clear, and tailored for deduction; which is why it is used in this work.

This book is divided into two parts and appendices. In Part One all the eight primitive concepts, upon which the theory is based, are introduced. Part Two deals with clocks, length measuring instruments, and space-time geometry. The reader who is not interested in details, need not bother with the appendices, can skip over the proofs and unfamiliar symbols, and should still be able to get the gist of all the ideas. In fact, even a reader who is interested in details may find it helpful to get first a bird's eye view before digging in. However, for the reader who feels that the theory and its formalism are worth the time necessary for a deeper understanding, and who wishes to follow the proofs in detail, an introduction to SL and set theory adequate for these purposes is offered in Appendix A.

Primitive concepts are designated by the letter ***C***, interpretations by ***I***, axioms by ***A***, postulates by ***P***, theorems by ***T***, and definitions by ***D***. Each item is distinguished from items of the same species by three numbers: a Roman numeral referring to the chapter in which the item occurs, followed on the right by a number of the section and then by a number distinguishing the items within a section. For instance, ***T***II4.5 denotes theorem 5 of Sec. 4, Chapter II, and reads ***T***5 of II4. The chapter or section numbers are omitted in referring to items in the same chapter or section, and the chapter and section numbers are shown on the top inner corner of every page. The symbol '([R]p – q)' denotes reference [R], pages p to q.

This book is the result of research conducted over a period of ten years, during which many different versions of the theory have been tried. In the last three years or so, the theory has become sufficiently stable and natural that it seemed worthwhile to publish. It is not based directly on any reference or set of references, but it is inspired and influenced by the author's knowledge of physics, philosophy of science, and mathematics.

The principal publications dealing with the same subject matter and presented in the same spirit, are those by Reichenbach [1924], Schnell [1938], Carnap [1958], and Robb [1914]. The first two deal mainly with Chaps. IV, V and X, and the last two with Chaps. IV and V. Robb's work is practically all based on the use of light signals, and is restricted to special relativity. We make use of both particles and signals, and our theory is geared to general relativity. Further comments concerning the relation of these works to ours are made in due course.

Discussions with many of my colleagues in physics, philosophy, and mathematics have been invaluable for the development of the theory, and I am very grateful for the time and patience they have given me. In particular, I wish to thank my colleagues at Colorado State University, Dr. Marvin Heller (Physics), Dr. Donald Roberts (Philosophy; now at the University of Waterloo, Waterloo, Ontario), Dr. Moin Siddiqui (Mathematical Statistics), and Dr. Kenzo Seo (Mathematical Statistics; now at Clemson University, Clemson, South Carolina). I also wish to thank Dr. Patrick Suppes (Department of Philosophy, Stanford University) for his criticism and encouragement at an early stage of this work, and my wife Phyllis for the excellent job she has done in typing the manuscript. Needless to say, suggestions and criticisms from the reader are essential for the improvement and growth of the theory, and are greatly appreciated.

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I. WHAT IS A DEDUCTIVE THEORY?

1. Deductive physical theory

A physical theory consists of the following three ingredients: (1) a set of classes, such as the class of particles, the class of events, the class of electrons; (2) a set of relations that holds between the elements of these classes, such as the time relation between events, and the collineary relation between particles; and (3) a set of propositions that specify the properties of these relations, such as the proposition that the coincidence relation between events is an equivalence relation. The elements of the first two items are called *physical concepts* and those of the third item *physical laws*.

The goal of physical theory is to explain the meaning of the physical concepts and to justify the physical laws. A physical concept is *explained* if it is defined in terms of understood physical concepts, and a physical law is *justified* if it is deduced from accepted physical laws. Since a concept is explained in terms of other concepts, and a law is deduced from other laws, a never-ending process results if one tries to explain *every* concept and justify *every* law. The only way out is to start with a set of concepts called the *primitive concepts* and a set of laws called the *postulates*; then define all concepts in terms of the primitive concepts and deduce all laws from the postulates.

Since the primitive concepts are the concepts that cannot be explained, and the postulates are the laws which cannot be justified, it is desirable to reduce the number of primitive concepts and postulates to a minimum. In fact, this is one of the criteria which can be used to choose between two equally acceptable theories. If it is not possible to decide between two theories on this basis, aesthetic considerations of simplicity, naturalness, and elegance can be used. Of course, logical contradictions must be absent from any acceptable theory.

The primitive concepts and postulates have to satisfy certain *experimental* conditions. If the concept is a class, it should be possible to decide experimentally whether any particular entity is or is not a member of the class.

Similarly, if the concept is a relation, it should be possible to decide whether the proposition that the relation holds between certain elements is true or false. If a postulate can be tested experimentally, it must agree with experiment; if it cannot be tested, all consequences of the postulate which can be tested must agree with experiment. Moreover, every postulate must have at least one consequence which can be tested experimentally.

A physical theory developed in accordance with the principles outlined above is called a *deductive physical theory*.

2. Deductive abstract theory

In the mathematical formulation of a deductive physical theory, the primitive concepts are represented by symbols, and the postulates become propositions about these symbols. The meaning of a primitive concept must be completely expressed by the postulates and not by the symbol representing it. To say that the meaning of a concept is necessary for logical deduction, amounts to saying that not all the properties of the concept which matter for deduction have been expressed in the postulates. Thus in the mathematical theory the undefined symbols are treated as meaningless, and all deductions are made with the help of the postulates.

The mathematical theory is made physical by adding a set of statements called *interpretations*, which establish a correspondence between the physical concepts and the mathematical symbols, and make it possible to translate the mathematical propositions into propositions about physical concepts and vice versa. The mathematical theory without interpretations is called a *deductive abstract theory*; the word 'abstract' is quite appropriate, since the logical structure of the physical theory is literally *abstracted* in the mathematical theory.

The modern axiomatic method was first developed by Hilbert in his 'Foundations of Geometry' (1899). An excellent discussion of the history, nature, and role of the axiomatic method in mathematics is given by S. C. Kleene ([1952] § 8, pp. 43–65, and in particular § 15). The rich and profound ideas that are now presented are due mainly to E. L. Post, but familiarity with them was acquired through P. Rosenbloom ([1950] Chap. IV).

A deductive theory starts with a finite set of symbols (alphabet), called the (primitive) *concepts*. These symbols are used to construct finite sequences of symbols written in linear order from left to right, called *strings*. If s and t are strings, then $s \cdot t$ is the string consisting of the symbols of s followed by the symbols of t.

Just as not every combination of letters of the English alphabet constitutes an English word, so not every possible string is admissible in a particular theory. To produce admissible strings, it is necessary to have an initial supply of strings called *axioms*, and a set of rules called *postulates* that produce from the concepts and axioms new strings called *products*. (Notice the difference in meaning of the words 'axiom' and 'postulate'. In the literature, these words are frequently used interchangeably.) The class of concepts, axioms and products is called the *object language* (OL), and any element of the OL is called an *admissible string*.

In formulating the OL, another language such as English is necessary to specify what are the concepts and axioms, and to express the postulates. The language used to talk *about* the OL is called the *metalanguage* (ML).

A postulate may be expressed by the sentence:

If the strings $a_1, \ldots, a_n$ are admissible,
then the string b is also admissible. (1)

For convenience, (1) is abbreviated by:

$$a_1, \ldots, a_n \rightarrow b. \tag{2}$$

A sentence of the form (2) is called a *production*, the set of strings $a_1, \ldots, a_n$ the *input*, and the string b the *output*. The output is a product, only if all the input strings are admissible. Both the commas and '$\rightarrow$' in (2) are elements of the ML.

In order to produce many strings from a postulate, it is necessary to include in the input and output, symbols called *string variables* which when replaced by certain strings of concepts, make the input admissible and the output a product. Different replacements result in different products. If $s_1, \ldots, s_k$ are given strings of concepts and $x_1, \ldots, x_{k-1}$ are string variables, each of the strings $a_1, \ldots, a_n, b$ in (2) can be written in the form '$s_1 \cdot x_1 \cdot s_2 \cdot x_2 \cdot \ldots \cdot s_{k-1} \cdot x_{k-1} \cdot s_k$', where any of the strings s_i or x_i may be empty (contains no symbols). String variables are elements of the ML.

The input and output of a postulate must satisfy the following conditions (Rosenbloom [1950] p. 162): (i) each string variable that occurs in the output, must occur in at least one input string, (ii) every input and output string must contain at least one string variable, (iii) at least one of the given strings s_i in the output is not empty.

As an illustration of the above ideas, consider the following deductive theory $\mathscr{A}$, which is adequate for part of arithmetic:

Concepts: 1, $\mathfrak{N}$, = .
Axioms: $\mathfrak{N}1$.
Postulates: P1. $\mathfrak{N}m, \mathfrak{N}n \rightarrow \mathfrak{N}m \cdot n$,
P2. $\mathfrak{N}n \rightarrow n = n$.

Interpretations: '1' denotes the number 1, '$\mathfrak{N}n$' means "n is a string of 1's", and '$m = n$' means "The strings m,n are equal, i.e., have the same number of 1's".

Substituting the concept '1' for the variables 'm' and 'n' in P1, we get the sentence '$\mathfrak{N}1, \mathfrak{N}1 \rightarrow \mathfrak{N}11$'†. Since $\mathfrak{N}1$ is an axiom, the input $\{\mathfrak{N}1, \mathfrak{N}1\}$ is admissible, and thus the output $\mathfrak{N}11$ is a product (is also admissible). If in turn we substitute '11' and '1' for 'm' and 'n' in P1, we get the product $\mathfrak{N}111$, and so on. We can also replace 'm' and 'n' by '=' in P1. But since the string $\mathfrak{N}=$ is not admissible, neither is the output $\mathfrak{N}==$. Similarly, we can produce from P2 the products: $1=1$, $11=11$, $111=111$, ...

3. Definitions

Although definitions are not essential to the development of a deductive theory (DT) they are very convenient, and most DT's contain them in abundance. A symbol 'd' is said to be defined in a DT, $\mathscr{D}'$, relative to a DT, $\mathscr{D}$, if and only if (i) the concepts of $\mathscr{D}'$ are those of $\mathscr{D}$ plus the concept d, (ii) any string in $\mathscr{D}$ (constructed from the concepts of $\mathscr{D}$) is a product in $\mathscr{D}'$ if and only if it is a product in $\mathscr{D}$, and (iii) to every string in $\mathscr{D}'$ there corresponds a string in $\mathscr{D}$ such that each can be produced from the other in $\mathscr{D}'$. Thus the symbol 'd' is defined if it does not change the OL of $\mathscr{D}$, and can be eliminated from any string (Rosenbloom [1950] p. 168).

The simplest type of definition is that in which 'd' is the name of a string s in $\mathscr{D}$. In this case, the postulates of $\mathscr{D}'$ consist of the postulates of $\mathscr{D}$ plus the two postulates:

$$xdy \rightarrow xsy, \tag{1a}$$
$$xsy \rightarrow xdy, \tag{1b}$$

where x,y are string variables. The extension of $\mathscr{D}$ to $\mathscr{D}'$ by the concept d and the postulates (1) is denoted by:

$$d \quad for \quad s. \tag{2}$$

† The string '11' denotes the number 'two', not 'eleven' (see 3.3).

If the variables 'x','y' in (1) are replaced by strings in $\mathscr{D}$, the only string in $\mathscr{D}$ which can be a product of (1) is xsy. But since xsy originates from $\mathscr{D}$ through (1b), condition (ii) is satisfied. Condition (iii) is also satisfied by postulates (1).

Referring to the DT, $\mathscr{A}$, presented in the previous section, the following definitions are illustrations of (2):

$$2 \textit{ for } 11, \quad 3 \textit{ for } 111, \quad 4 \textit{ for } 1111, \dots . \tag{3}$$

Notice that $3 \cdot 1 = 1111 = 4$. The products '$11 = 11$', '$111 = 111$', etc., can be rewritten with the help of (3) in the form '$2 = 2$', '$3 = 3$', etc.

Frequently, a symbol 'd' is defined in *context*, which means that a certain string '$f(d)$' containing 'd' is taken as an abbreviation of a string s in $\mathscr{D}$. Here, the postulates that have to be added to $\mathscr{D}$ are the same as (1), except for replacing 'd' by '$f(d)$'; thus, this definition is also designated by:

$$f(d) \quad \textit{for} \quad s. \tag{4}$$

An illustration of (4) is provided by the following definition of addition in $\mathscr{A}$:

$$(m+n) \quad \textit{for} \quad m \cdot n. \tag{5}$$

Notice that three signs, '$+$', '(', and ')', are defined in context simultaneously. With the help of (5), we can write the product '$1111 = 1111$' in the form '$(2+2) = 4$' or '$(3+1) = 4$'.

A more complicated kind of definition, is definition by *recursion* (induction). In this case the new symbol 'd' is defined in simple contexts, and postulates are given for translating an occurrence of 'd' in more complicated contexts. For instance, multiplication in $\mathscr{A}$ can be defined by:

$$(n \times 1) \quad \textit{for} \quad n, \tag{6a}$$
$$(m \times n \cdot 1) \quad \textit{for} \quad ((m \times n) + m). \tag{6b}$$

By means of (6) we now translate the string '$((4 \times 3) + 5)$':

(3):	$((4 \times 3) + 5) \rightarrow ((4 \times 2 \cdot 1) + 5)$.	(A)
(6b):	$(A) \rightarrow (((4 \times 2) + 4) + 5)$.	(B)
(3):	$(B) \rightarrow (((4 \times 1 \cdot 1) + 4) + 5)$.	(C)
(6b):	$(C) \rightarrow ((((4 \times 1) + 4) + 4) + 5)$.	(D)
(6a):	$(D) \rightarrow (((4 + 4) + 4) + 5)$.	(E)
(5), (3):	$(E) \rightarrow ((8 + 4) + 5)$.	(F)
(5), (3):	$(F) \rightarrow (12 + 5)$.	(G)
(5), (3):	$(G) \rightarrow 17$.	

The numbers on the left of the colon are the postulates used to obtain the productions on the right by replacement of the string variables by specific strings. The symbols on the extreme right are the names of the products. For example, '(A)' is the name of $((4\times 2\cdot 1)+5)$.

4. Theorems

As in the case of definitions, theorems are not necessary for the development of a DT, but are very convenient in expediting the process of deduction. We now explain what is a theorem and what constitutes a proof; in this way, we shall have a clear idea of what deduction is.

A *theorem* is a *proved production* in the form [see (2.2, 1)]:

$$a_1, \ldots, a_n \rightarrow b. \tag{1}$$

The main difference between a theorem and a postulate is that in a postulate, b follows from $a_1, \ldots, a_n$ *without a proof*, whereas in a theorem this fact is *proved*. We say that a production (1) is a theorem if and only if there exists a sequence of productions $P_1, \ldots, P_k$, which are either postulates or theorems, such that (Rosenbloom [1950] p. 163)

$$\begin{aligned}
&\quad \{a'_1, \ldots, a'_{m_1}\} \subseteq \{a_1, \ldots, a_n\}, \quad \text{and}\\
&P_1 : a'_1, \ldots, a'_{m_1} \rightarrow b_1;\\
&\quad \{a''_1, \ldots, a''_{m_2}\} \subseteq \{a_1, \ldots, a_n, b_1\}, \quad \text{and}\\
&P_2 : a''_1, \ldots, a''_{m_2} \rightarrow b_2; \ldots;\\
&\quad \{a^{(k)}_1, \ldots, a^{(k)}_{m_k}\} \subseteq \{a_1, \ldots, a_n, b_1, \ldots, b_{k-1}\}, \quad \text{and}\\
&P_k : a^{(k)}_1, \ldots, a^{(k)}_{m_k} \rightarrow b.
\end{aligned}$$

The above set of steps constitutes the *proof* of theorem (1), and (1) is said to be *proved* by $P_1, \ldots, P_k$.

Thus a theorem is a set of logical steps in a proof; when a theorem is used in a proof, all these steps are short circuited and the necessity for repetition of the steps is eliminated. The *importance* of a theorem is determined by the number of times it is needed in the proof of other theorems, and the number of steps it short circuits.

To illustrate, we prove the following theorem in the theory $\mathscr{A}$ of Sec. 2:

Theorem. $\mathfrak{N}m, \mathfrak{N}n \rightarrow (m+n) = m\cdot n$.

Proof. P1 : $\mathfrak{N}m, \mathfrak{N}n \rightarrow \mathfrak{N}m\cdot n$. (A)

P2 : $(A) \rightarrow m\cdot n = m\cdot n$. (B)

(3.5) : $(B) \rightarrow (m+n) = m\cdot n$.

Rosenbloom ([1950] p. 163) uses the word 'theorem' to denote what was called here the output. In our usage, which I believe conforms better with the usual usage, a theorem is a production not a string.

5. Symbolic logic and deductive theories

We saw in Sec. 2 that an ML is necessary to formulate a DT. Although a language such as English can be used for the ML, such a language is ambiguous and ill suited for the tremendous amount of deduction involved. In contrast, symbolic logic (SL) is clear, concise, tailored for deduction, and is therefore ideal for the formulation of a DT.

My experience throughout the development of the theory presented here, has been that the very attempt to formulate a postulate or definition in SL, makes it difficult to be ambiguous and sharpens the statements considerably. Moreover, it is much easier to deduce the consequences of a proposition written in SL, than when the proposition is written in a non-technical language. Deduction with SL is like solving an algebraic equation; it is to a large extent mechanical, and therefore less taxing mentally.

After all ambiguities and contradictions are eliminated with the help of SL, any statement in SL can be translated back into English for the benefit of those not versed in SL. As soon as familiarity with SL (which is a much simpler language than English) develops it will be found more advantageous and convenient to work and think directly in SL.

The theory is formulated in such a way that the reader need not know SL to get the important ideas. But if the reader feels the theory merits a deeper understanding, he can acquire all the necessary knowledge of SL from Appendix A.

Application of SL to the formulation of several different DT's can be found in a book by Carnap ([1958] Part Two).

II. OBJECTIVE UNIVERSE

1. Introduction

Before getting involved in details, we outline the basic philosophy and general plan of the theory.

In spite of all the talk about objectivity, science as we know it is a human activity, and our conception of the universe is based directly upon our sensations and how we interpret them. Thus it is natural that the theory should start with the concept of a living human as an observer, and lead to the concept of the objective universe through analysis of the sensations of observers (Russell [1927] pp. 6–9, Chaps. XX,XXI).

Many physicists would probably say that this is philosophy and not physics, and would rather start with a concept such as 'object'. Unfortunately, such a concept is very complex, as will be seen in this chapter, and it is therefore advisable to start with simpler concepts. To do otherwise is to evade the deeper understanding of the concepts we use, and thus be in a poor position to decide how much we can extend these concepts to the microscopic domain.

It is true that much has been accomplished in physics without paying attention to such considerations, but eventually a point is reached where one must know whence he came, to decide whither he should go. It was at such a point that relativity was developed, and it is my belief that we are at such a point once more, trying to decide which road to take to the understanding of elementary particles.

The concept of the objective *macroscopic* universe is reached in two steps: (1) An observer discovers there are similarities between many of his sensations, and attributes each subclass of similar sensations to a *subjective* entity. There is no justification at this point of assuming these entities to be *outside* the observer; they could well be the result of a mental process such as a dream. (2) Different observers discover through communication that they can establish a correspondence between their subjective entities, and then attribute the corresponding subjective entities to an *objective entity* (macroscopic object). The class of objective entities is called the (macroscopic) *objective universe.*

The concept of the *microscopic* universe is obtained by another similar step, namely: to explain the behavior of macroscopic objects, microscopic objects, such as elementary particles are assumed to exist, and through various interactions, microscopic happenings are assumed to lead to macroscopic phenomena. It is impossible to *prove* the existence of such objects, as it is impossible to *prove* the existence of perceived objects. But the outstanding success of these assumptions in explaining a vast area of perceived data, is ample justification for making them. In this book we deal only with the macroscopic universe.

The theory is formulated exactly as described in the previous chapter, namely as a purely abstract deductive theory with physical interpretations. The abstract theory by itself is a self-consistent mathematical theory that can be studied by mathematicians without regard to the interpretations. All the physical properties associated with a physical concept are expressed by mathematical postulates about the mathematical symbol representing the concept. The interpretations play absolutely no role in the deduction, but serve only to translate mathematical sentences into sentences about the physical world, which can be tested experimentally.

It was seen in Secs. I2, 4 that postulates and theorems are productions. Since the inputs of all productions in this book can be easily deduced from the form of the output and the type of letters used (see Conventions at end of the book), only the outputs are presented.

Moreover, the following notation is adopted in proofs: The steps of a proof are presented as shown in the illustration at the end of Sec. I4, except that '$\twoheadrightarrow$' is omitted. The output of the theorem produced in the last step is denoted by **T** (for theorem), and signals the end of the proof. If the output of the theorem is in the form of an implication '$F \rightarrow G$', the *antecedent* F is abbreviated by **Ant** and the *consequent* G by **Con**. Thus the proof is complete if the last step is '**Ant**$\rightarrow$**Con**'.

Items of the physical theory are designated by the italic bold capitals: ***A***, ***C***, ***D***, ***I***, ***P***, ***T*** (see preface); whereas, items of the logical theory presented in Appendices A and B are designated by the Gothic bold capitals: **A**, **C**, **D**, **I**, **P**, **T**. The logical theorems (**T**) are all in Appendix B, and the remaining logical items are in Appendix A. This notation makes it unnecessary to specify the places in which the logical items occur.

2. Observers

As already mentioned, the first concept of the theory is the class of living humans in the role of observers. If one were developing a biological theory, this concept would probably be the last one instead of the first, but for physical theory this seems to be a reasonable starting point.

*C*1. $\mathscr{H}$.

*I*1. $\mathscr{H}$ is the *class* of *living humans* who have *adequately functioning sense organs*, can *communicate* with each other, and do so *honestly* and *without bias*. The elements of $\mathscr{H}$ are called *observers*. A human does *not* qualify as an observer if his senses are preconditioned to influence his future sensations, as in staring a long time at a spot, or if he is hallucinating, under hypnosis, or under the influence of a drug that affects his nerves and brain, or is reputed to be deceitful or biased. Who can qualify as an observer, is a question that can only be answered through long experience with comparison of sensations. The present state of science is a clear evidence that a sufficient number of observers do exist.

The property '$\mathscr{H}$ is a class' is expressed by the axiom:

*A*1. $\mathfrak{C}\mathscr{H}$.

A class is a collection of individuals having a certain common property. For a more precise meaning of 'class' and its distinction from a set, Appendix A should be consulted.

One may wonder at this point whether an existence postulate should be introduced to the effect that $\mathscr{H}$ is not empty, or that it has at least a certain number of members. Such a postulate is usually introduced in mathematics to prove the existence of elements important in the development of the theory. Since we are not interested in such statements concerning observers, we do not introduce such a postulate. Whenever necessary, additional axioms can be introduced having the form '$\vdash H \in \mathscr{H}$', i.e., 'H is an observer'.

The other properties of observers are expressed in postulates concerning other concepts.

3. Sensations

One of the guiding principles in deciding what concepts to adopt as primitive, is that the concepts should be understood with as little explanation as possible. Such a decision can be made only as a result of trying different approaches and discussing them with many people. I lost track of the number of approaches I tried before settling down to the present one.

Many physicists prefer to introduce directly the concept of an object, and bypass sensations altogether. Unfortunately, it turns out that this concept is very complicated, as is seen later in this chapter, and the longer route taken here is justified by the clarity and deeper understanding that result from it.

Since the following three concepts are means to an end, and are not used after this chapter, the symbols used to represent them have no relation to their names. The more frequently used concepts are represented by the first letter of their name.

*C*2. $\mathscr{X}$.

*I*2. Occasionally, one may have a single pure sensation, as when looking at a non-varying monochromatic point-source of light, or feeling the constant pressure of the point of a pin. More frequently, one has simultaneously a great variety of sensations, such as visual sensations from different portions of the field of view, and sensations of touch, temperature, sound, and smell. The sentence '$\langle S, H\rangle \in \mathscr{X}$' means '*S is a simultaneous set of sensations of observer H*'.

Implicit in the notation '$\langle S, H\rangle \in \mathscr{X}$' is that $\mathscr{X}$ is a class of ordered couples (binary relation) and the ordered couple $\langle S, H\rangle$ is an element of $\mathscr{X}$. The assumption that $\mathscr{X}$ is a class (I12)† is expressed by axiom:

*A*2. $\mathfrak{C}\mathscr{X}$,

and that $\mathscr{X}$ is a binary relation (D23), by the postulate:

***P*1.** $\mathfrak{R}_2\mathscr{X}$.

For convenience, we introduce the definition (see Sec. I3):

***D*1.** $S\mathscr{X}H \quad \textit{for} \quad \langle S, H\rangle \in \mathscr{X}$.

The *class of sensations of observer H* is defined by

***D*2.** $\mathscr{X}_H \quad \textit{for} \quad X \ni X\mathscr{X}H$.

'$X \ni X\mathscr{X}H$' denotes the class of all X such that X is a sensation of H(I13). According to P12 (Postulate 12 of Sec. 8, Appendix A), '$S \in \mathscr{X}_H$' is equivalent to '$S\mathscr{X}H$'.

In the sentence '$S\mathscr{X}H$', H is an observer, i.e., the range of the relation $\mathscr{X}$ is the class $\mathscr{H}$ of observers (***P*2**).

† '(I12)' denotes 'Interpretation 12' in Appendix A.

***P*2.** $(\exists X)X\mathscr{L}H.\rightarrow H\in\mathscr{H}$.

A simultaneous set of sensations S can belong to at most one observer. Consequently $\mathscr{L}$ is a many-one relation (D35), i.e.

***P*3.** $S\mathscr{L}G\wedge S\mathscr{L}H\rightarrow G=H$.

Literally, this means that if S is a sensation of both G and H, then $G=H$.

4. Subjective entities

As explained in the introduction, the next step is to correlate sensations, and then attribute the correlated sensations to a subjective entity.

***C*3.** $\mathscr{J}$.

***I*3.** When an observer sees an object from different points of view and under different conditions (with his naked eye, through a telescope, through a window screen, under different lightings), his visual sensations due to the object do change. Yet he still attributes the different sensations to the same source by analyzing them and comparing them with the help of his memory. Moreover, through experience, an observer has the ability to correlate sensations stemming from the stimulation of *different* sense organs. For instance, he may recognize an object by either how it *looks* or how it *feels*. The sentence '$\langle R, S, H\rangle\in\mathscr{J}$' means '*The two simultaneous sets of sensations R and S are attributed by the observer H to the same source*; H may have R and S at the same (psychological) time, or at different times'.

Here again, the mathematical sentence '$\langle R, S, H\rangle\in\mathscr{J}$' means 'the ordered triple $\langle R, S, H\rangle$ is a member of the class $\mathscr{J}$', and these properties are expressed by axiom *A*3 and postulate ***P*1** below.

***A*3.** $\mathfrak{C}\mathscr{J}$.

***P*1.** $\mathfrak{R}_3\mathscr{J}$.

For convenience, we define:

***D*1.** $R\mathscr{J}_H S$ *for* $\langle R,S,H\rangle\in\mathscr{J}$.

All the important properties of the ternary relation $\mathscr{J}$, discussed in ***I*3**, must be expressed by postulates. Postulate ***P*2** expresses the fact that the first two terms in '$\langle R, S, H\rangle$' are (simultaneous sets of) sensations of the observer H, who is represented by the third term.

***P*2.** $R\mathscr{J}_H S\rightarrow R\mathscr{L}H\wedge S\mathscr{L}H$.

Literally, ***P***2 states that if $R\mathscr{J}_H S$, then $R\mathscr{Z}H$ and $S\mathscr{Z}H$. The meaning of '$R\mathscr{J}_H S$' is given by ***D***1 and ***I***3, and of '$R\mathscr{Z}H$' by ***D***3.1 and ***I***2.

From ***P***2 and ***P***3.3 it follows that if $R\mathscr{J}_G S$ or $S\mathscr{J}_G R$, and $S\mathscr{J}_H T$ or $T\mathscr{J}_H S$, then $G=H$ (***T***1).

This means that if sensations R and S are attributed to the same source by observer G, and sensations S and T are attributed to the same source by observer H, then G and H must be one and the same observer (since the sensation S can only belong to one observer). The reader who is not yet interested in details should skip the proof of ***T***1.

*T*1. $$R\mathscr{J}_G S \vee S\mathscr{J}_G R. \wedge .S\mathscr{J}_H T \vee T\mathscr{J}_H S : \to G=H.$$

Proof. Since this is the first theorem, the proof is presented in full detail. As familiarity with the different logical theorems develops, fewer and fewer details are given. The experienced reader can bypass many of the steps with a brief glance.

$$\boldsymbol{P}2: R\mathscr{J}_G S \to R\mathscr{Z}G \wedge S\mathscr{Z}G. \quad (1)$$
$$\mathsf{T}2.14: R\mathscr{Z}G \wedge S\mathscr{Z}G \to S\mathscr{Z}G. \quad (2)$$

The items on the left of the colon are the postulates and theorems used to deduce the outputs on the right of the colon. The numbers on the extreme right are the names of the outputs.

$$(1),(2),\mathsf{T}1.2: R\mathscr{J}_G S \to S\mathscr{Z}G. \quad (3)$$

The proper way of writing this is:

$$\mathsf{T}1.2: (1),(2) \to \vdash R\mathscr{J}_G S \to S\mathscr{Z}G,$$

but we shall take the liberty of using the simpler form (3).

Similarly, we have

$$\boldsymbol{P}2, \mathsf{T}(2.14, 1.2): S\mathscr{J}_H T \to S\mathscr{Z}H. \quad (4)$$

This illustrates what is meant by giving fewer details.

$$(3),(4),\mathsf{T}(1.7, 2.33): R\mathscr{J}_G S \wedge S\mathscr{J}_H T \to S\mathscr{Z}G \wedge S\mathscr{Z}H \quad (5)$$
$$\boldsymbol{P}3.3 \qquad \to G=H. \quad (6)$$

The last step is an abbreviation of the two steps:

$$\boldsymbol{P}3.3: S\mathscr{Z}G \wedge S\mathscr{Z}H \to G=H, \quad (7)$$
$$(5),(7),\mathsf{T}1.2: R\mathscr{J}_G S \wedge S\mathscr{J}_H T \to G=H. \quad (6')$$

This kind of abbreviation will be used frequently from now on. Notice that T1.2 is not even mentioned in (6).

In exactly the same way we can deduce from ***P***(2, 3.3) and T(2.14, 1.2) that

$$R\mathscr{J}_G S \wedge T\mathscr{J}_H S \to G = H, \tag{8}$$

$$S\mathscr{J}_G R \wedge S\mathscr{J}_H T \to G = H, \tag{9}$$

$$S\mathscr{J}_G R \wedge T\mathscr{J}_H S \to G = H. \tag{10}$$

$$(6'), (8)\text{–}(10), \mathrm{T}1.7: (6') \wedge (8) \wedge (9) \wedge (10), \tag{11}$$

$$(11), \mathrm{T}(2.37, 1.1): R\mathscr{J}_G S \wedge S\mathscr{J}_H T. \vee .R\mathscr{J}_G S \wedge T\mathscr{J}_H S. \vee .S\mathscr{J}_G R \wedge S\mathscr{J}_H T. \vee .S\mathscr{J}_G R \wedge T\mathscr{J}_H S: \to G = H. \tag{12}$$

$$(12), \mathrm{T}(2.42, 1.2): \mathbf{T}.$$

'**T**' signals the end of the proof, and stands for the output of ***T***1:

$$R\mathscr{J}_G S \vee S\mathscr{J}_G R. \wedge .S\mathscr{J}_H T \vee T\mathscr{J}_H S: \to G = H.$$

The relation $\mathscr{J}_H$ is the one that *H* uses to correlate his sensations. In order that correlated sensations can be grouped into disjoint subclasses that can be attributed to subjective entities, it is necessary that $\mathscr{J}_H$ must be an equivalence relation (D32) in the class of sensations of *H*, $\mathscr{Z}_H$(***D***3.2). In other words, $\mathscr{J}_H$ must be reflexive (D25), symmetric (D27), and transitive (D30). The reason for this is that whenever an equivalence relation holds between the elements of a class, the class can be decomposed into disjoint *equivalence subclasses*, such that all the members of a subclass are equivalent to each other, but no member of a subclass is equivalent to any member of another subclass. A member of the original class can belong to one and only one equivalence subclass, which is said to be determined by it.

The reflexivity of $\mathscr{J}_H$, i.e., the fact that every sensation *S* is correlated with itself, is expressed by:

P3. $S\mathscr{Z}H \to S\mathscr{J}_H S.$

Instead of expressing the symmetry and transitivity of $\mathscr{J}_H$ by two postulates, it is more economical to introduce the single postulate that $\mathscr{J}_H$ is semi-transitive (D31).

P4. $R\mathscr{J}_H S \wedge R\mathscr{J}_H T \to S\mathscr{J}_H T,$

i.e., if $R\mathscr{J}_H S$ and $R\mathscr{J}_H T$, then $S\mathscr{J}_H T$. This means that if *H* attributes *R*, *S* to the same source, and *R*, *T* to the same source, then he attributes *S*, *T* to the same source.

In the usual form of the transitive law, we should write '$S \mathscr{I}_H R$' instead of '$R \mathscr{I}_H S$'. However if this is done, the symmetry of $\mathscr{I}_H$ must be postulated; whereas by adopting the form in **P**4, it is possible to prove that $\mathscr{I}_H$ is symmetric (**T**2).

***T*2.** $\mathscr{I}_H$ is an *equivalence* in $\mathscr{Z}_H$.
Proof. **P**3, **P**4, T7.1.

In view of this, we see that a simultaneous set of sensations S of H determines a unique equivalence subclass defined by:

***D*2.** $\mathscr{I}_H(S)$ *for* $X \ni X \mathscr{I}_H S$,

where $X \ni X \mathscr{I}_H S$ is the class of all sensations X equivalent to S(I13).

It is tempting to identify $\mathscr{I}_H(S)$ with the subjective entity to which H attributes S. However, $\mathscr{I}_H(S)$ is a class, and a class cannot be an element of anything (see the discussion of class and set at the beginning of Sec. A2). Since we should be able to talk about the class of the subjective entities of an observer H, we see that a subjective entity must be represented by a set and not a class. This suggests that we assume the existence of a unique set A that represents $\mathscr{I}_H(S)$, in the sense $A \equiv \mathscr{I}_H(S)$. According to D21, **D**2, and P12, '$A \equiv \mathscr{I}_H(S)$' means 'for any set X, $X \in A$ if and only if $X \mathscr{I}_H S$'. Consequently, '$R \mathscr{I}_H S$' and '$A \equiv \mathscr{I}_H(S)$' imply '$R, S \in A$'. If we interpret '$R \in A$' to mean '$R$ is attributed to A', then we see that A is the subjective entity to which H attributes the sensations R and S.

In order for this interpretation to be justified, it is necessary that the set A is unique. Because of T3.22, it is sufficient to assume that if $S \mathscr{Z} H$, there exists at least one X such that $X \equiv \mathscr{I}_H(S)$, i.e.,

***P*5.** $S \mathscr{Z} H \rightarrow (\exists X) X \equiv \mathscr{I}_H(S)$.

The sufficiency of **P**5 for uniqueness is proved in (see D10):

***T*3.** $S \mathscr{Z} H \rightarrow (\exists ! X) X \equiv \mathscr{I}_H(S)$.
Proof. **P**5, T(3.22, 1.2).

This states that if $S \mathscr{Z} H$, there exists exactly one set X such that $X \equiv \mathscr{I}_H(S)$. This unique set, which is the *subjective entity associated with* S, is represented explicitly by:

***D*3.** $\tilde{S}(H)$ *for* $(\iota X) X \equiv \mathscr{I}_H(S)$,

where '$(\iota X) X \equiv \mathscr{I}_H(S)$' denotes *the* set X that satisfies '$X \equiv \mathscr{I}_H(S)$' if this set exists, otherwise it denotes the null set $\emptyset$(I7, D11). The meaning of **D**3 is further clarified by:

*T*4. $S\mathscr{L}H \wedge A = \tilde{S}(H) \leftrightarrow S\mathscr{L}H \wedge A \equiv \mathscr{I}_H(S)$.

Proof. $T3, \mathrm{T}(4.3, 1.2)\colon S\mathscr{L}H \rightarrow .A = (\imath X)X \equiv \mathscr{I}_H(S) \leftrightarrow A \equiv \mathscr{I}_H(S)$

$D3 \qquad \rightarrow .A = \tilde{S}(H) \leftrightarrow A \equiv \mathscr{I}_H(S). \qquad (1)$

(1), T2.32: **T**.

According to *T*4 the sentence '$A = \tilde{S}(H)$', where *S* is a sensation of *H*, is equivalent to '*A* is the subjective entity to which *H* attributes the sensation *S*'.

The *class of subjective entities of H* is now defined by:

D4. $\mathscr{Q}(H)$ *for* $X \ni (\exists Y).Y\mathscr{L}H \wedge X = \tilde{Y}(H)$.

$\mathscr{Q}(H)$ is the class of sets *X* to which *H* attributes his sensations *Y*.

With the help of **D**4, the class of subjective entities (without reference to a particular observer) is defined by:

D5. $\mathscr{Q}$ *for* $X \ni (\exists U)X \in \mathscr{Q}(U)$,

i.e., $\mathscr{Q}$ is the class of sets *X* such that there exists at least one observer *U*, and $X \in \mathscr{Q}(U)$.

Since a subjective entity is *subjective*, i.e., it belongs to a particular observer, we expect that if *A* is a subjective entity of both *G* and *H*, then *G* and *H* must be one and the same observer (*T*5).

*T*5. $A \in \mathscr{Q}(G) \wedge A \in \mathscr{Q}(H) \rightarrow G = H$.

Proof. **D**4, *T*4, **D**2, D21, T3.(15,4): **Ant**

$\rightarrow (\exists X, Y)X\mathscr{L}G \wedge Y\mathscr{L}H$

$\wedge (\forall U)\colon U \in A \leftrightarrow U\mathscr{I}_G X . \wedge . U \in A \leftrightarrow U\mathscr{I}_H Y$

T1.14 $\rightarrow (\exists X, Y).X\mathscr{L}G. \wedge .(\forall U).U\mathscr{I}_G X \leftrightarrow U\mathscr{I}_H Y$

P3, P4 $\rightarrow (\exists X, Y).X\mathscr{I}_G X \wedge .X\mathscr{I}_G X \rightarrow X\mathscr{I}_H Y$

T2.7 $\rightarrow (\exists X, Y).X\mathscr{I}_G X \wedge X\mathscr{I}_H Y$

*T*1 $\rightarrow (\exists X, Y)G = H$

T3.13 $\rightarrow$ **Con.**

To recapitulate, without a relation between sensations it is not possible to arrive at the concept of an entity perceived by an observer. This relation was represented by $\mathscr{I}_H$. By assuming $\mathscr{I}_H$ to be an equivalence relation, the class of sensations of observer *H* was divided into disjoint equivalence subclasses, each of which was represented by a set $\tilde{S}(H)$. It is these sets that were finally identified with the subjective entities perceived by H.

5. Objectivity

In this section we achieve the first major goal by defining the objective macroscopic universe. To do this, we need to correlate subjective entities of different observers, and then attribute the corresponding entities to a common source, an objective entity. The correlation is achieved by means of the concept:

$C4.$ $\mathscr{K}$.

*I*4. '$\langle A, G, B, H\rangle \in \mathscr{K}$' means '*The two different observers G, H establish a correspondence between the subjective entity A perceived by G and the subjective entity B perceived by H*, i.e., they agree that A and B owe their existence to the same source'. For example, G and H agree that they see the same object (source), or the object that one sees is the same as the object the other touches. We are not concerned here with the opinions of G and H about what A and B are, but only with their establishing a correspondence between A and B.

As before, '$\mathscr{K}$ is a class' is expressed by the axiom:

$A4.$ $\mathfrak{C}\mathscr{K}$,

and '$\mathscr{K}$ is a 4-place relation' by the postulate:

***P*1.** $\mathfrak{R}_4\mathscr{K}$.

For convenience we define:

***D*1.** $A_G\mathscr{K}_H B$ *for* $\langle A, G, B, H\rangle \in \mathscr{K}$.

The interpretation of the different terms in '$A_G\mathscr{K}_H B$' is given by:

***P*2.** $A_G\mathscr{K}_H B \rightarrow A \in \mathscr{Q}(G) \wedge B \in \mathscr{Q}(H) \wedge G \neq H$,

i.e., if $A_G\mathscr{K}_H B$, then A is a subjective entity of G, B is a subjective entity of H, and observer G is different from observer H.

Although the subjective entities A, B are attributed to the same source, they are different, since the observers G and H are different (***T*1**).

***T*1.** $A_G\mathscr{K}_H B \rightarrow A \neq B$.

Proof. **P2: Ant** $\rightarrow A \in \mathscr{Q}(G) \wedge B \in \mathscr{Q}(H) \wedge G \neq H$. (1)

$$\begin{aligned}&\text{T5.3}: A\in\mathcal{Q}(G)\wedge B\in\mathcal{Q}(H)\wedge A=B\rightarrow A\in\mathcal{Q}(G)\wedge A\in\mathcal{Q}(H)\\&\textit{T}4.5 \qquad\qquad\qquad\qquad\qquad\qquad \rightarrow G=H. \qquad (2)\\&(2),\text{T}2.32: A\in\mathcal{Q}(G)\wedge B\in\mathcal{Q}(H)\wedge G\neq H\rightarrow A\neq B. \qquad (3)\\&(1),(3): \mathbf{T}.\end{aligned}$$

Moreover, since a subjective entity can belong to at most one observer, we have:

***T*2.** $A_F\mathcal{K}_G B\wedge A_H\mathcal{K}_I C\rightarrow F=H.$

Proof. $\textbf{\textit{P}2}: \textbf{Ant}\rightarrow A\in\mathcal{Q}(F)\wedge A\in\mathcal{Q}(H)$

$\textit{T}4.5 \quad \rightarrow\textbf{Con}.$

The symmetry of '$A_G\mathcal{K}_H B$' with respect to the interchange of A, G with B, H is expressed by:

***P*3.** $A_G\mathcal{K}_H B\rightarrow B_H\mathcal{K}_G A.$

If F agrees with G that A and B are due to the same source, and G agrees with H that B and C are due to the same source, then F and H will agree that A and C are due to the same source, provided $F\neq H$. If $F=H$, then A and C must be identical; otherwise, the observer F will attribute two *different* subjective entities to the same source. Thus we assume:

***P*4.** $A_F\mathcal{K}_G B\wedge B_G\mathcal{K}_H C\rightarrow A_F\mathcal{K}_H C\vee A=C.$

Notice that $A=C$ implies $F=H$, by ***T*2** and ***P*3**.

We now define an *objective* entity by a method similar to that used in the previous section to define a subjective entity. In effect, we introduce an equivalence relation in the class of subjective entities; thereby decomposing this class into equivalence subclasses, and then represent these subclasses by sets which are identified with objective entities.

The equivalence relation is defined by:

***D*2.** $A\overset{K}{=}B \quad \textit{for} \quad A=B\vee(\exists\, U,V)A_U\mathcal{K}_V B.$

'$A\overset{K}{=}B$' means 'either $A=B$, or there exist two different observers U, V who agree that A and B are due to the same source'.

To prove that '$\overset{K}{=}$' is indeed an equivalence relation, it is sufficient to prove that it is reflexive (***T*3**) and semitransitive (***T*4**), just as was done in ***T*4.2**.

***T*3.** $A\overset{K}{=}A.$

Proof.

$$\begin{aligned}&\textbf{\textit{D}2}: A\overset{K}{=}A\leftrightarrow A=A\vee(\exists\, U,V)A_U\mathcal{K}_V A\\&\textit{T}1,\text{T}1.11 \leftrightarrow A=A. \qquad (1)\\&(1),\text{T}(5.1,1.1): \mathbf{T}.\end{aligned}$$

*T*4. $A \overset{K}{=} B \wedge A \overset{K}{=} C \rightarrow B \overset{K}{=} C.$

Proof. $D2$: $\textbf{Ant} \leftrightarrow : A = B \vee (\exists\, U, V) A_U \mathscr{K}_V B.$

$\wedge . A = C \vee (\exists\, U', V') A_{U'} \mathscr{K}_{V'} C$

T(2.42, 5.1), $P3$, T2.19 $\rightarrow B = C \vee (\exists\, U, V) C_U \mathscr{K}_V B.$

$T2$ $\qquad \vee (\exists\, U, V, W). A_U \mathscr{K}_V B \wedge A_U \mathscr{K}_W C.$ (1)

$P3,4$: $(\exists\, U, V, W). A_U \mathscr{K}_V B \wedge A_U \mathscr{K}_W C.$

$\rightarrow B = C \vee (\exists\, V, W) B_V \mathscr{K}_W C.$ (2)

(1), (2), $D2$: T.

*T*5. $\overset{K}{=}$ is an equivalence in $\mathscr{Q}$.

Proof. $T3$, $T4$, T7.1.

The equivalence subclass of subjective entities containing A is defined by:

*D*3. $\mathscr{K}(A)$ *for* $X \ni . X \in \mathscr{Q} \wedge X \overset{K}{=} A.$

As in $P4.5$, we assume the existence of a set that represents this subclass.

*P*5. $A \in \mathscr{Q} \rightarrow (\exists\, X) X \equiv \mathscr{K}(A),$

This states that if A is a subjective entity, there exists at least one set X that represents the equivalence subclass $\mathscr{K}(A)$. The uniqueness of this set is demonstrated in:

*T*6. $A \in \mathscr{Q} \rightarrow (\exists !\, X) X \equiv \mathscr{K}(A).$

Proof. $P5$, T3.22.

The unique set, which is the *objective entity* (*object*) associated with the subjective entity A, is represented by:

*D*4. $\hat{A}$ *for* $(\imath X) X \equiv \mathscr{K}(A).$

To convince ourselves that $\hat{A}$ is indeed the object to which the subjective entity A is attributed, we notice that 'A is attributed to $\hat{A}$' means '$A \in \hat{A}$'. Thus we have to prove that if A is a subjective entity ($A \in \mathscr{Q}$), then $A \in \hat{A}$.

*T*7. $A \in \mathscr{Q} \rightarrow A \in \hat{A}.$

Proof. $T6$, T4.3, $D4$: **Ant**

$\rightarrow (\forall\, X).\, X = \hat{A} \leftrightarrow X \equiv \mathscr{K}(A)$

D21, $D3$ $\qquad \leftrightarrow (\forall\, Y). Y \in X \leftrightarrow Y \in \mathscr{Q} \wedge Y \overset{K}{=} A$

P4 $\qquad \rightarrow . A \in \mathscr{Q} \wedge A \overset{K}{=} A \rightarrow A \in X$

$T3$, T1.10 $\rightarrow . A \in \mathscr{Q} \rightarrow A \in X$

T2. (32, 14) $\rightarrow (\forall\, X). X = \hat{A} \wedge A \in \mathscr{Q} \rightarrow A \in X \wedge X = \hat{A}$

T5.3 $\qquad \rightarrow A \in \hat{A}$

$$\text{T3.}(19,15) \to :A \in \mathcal{Q} \wedge (\exists X)\, X = \hat{A}. \to A \in \hat{A}$$
$$\text{T1.10} \quad \to .A \in \mathcal{Q} \to A \in \hat{A}$$
$$\text{T2.}(32,18) \to \textbf{Con}.$$

If two observers agree that their subjective entities A, B are due to the same source $[(\exists U, V) A_U \mathcal{K}_V B]$, the objects $\hat{A}$, $\hat{B}$ must be identical (***T***8).

T8. $(\exists U, V) A_U \mathcal{K}_V B. \to \hat{A} = \hat{B}.$

Proof.
$$\text{T2.15}, \boldsymbol{D}2: \textbf{Ant} \to A \stackrel{K}{=} B. \qquad (1)$$
$$(1), \text{T2.12}: \textbf{Ant} \wedge X \stackrel{K}{=} A \to A \stackrel{K}{=} B \wedge X \stackrel{K}{=} A$$
$$\boldsymbol{T}5 \qquad \to X \stackrel{K}{=} B. \qquad (2)$$
$$(2), \text{T2.32}: \textbf{Ant} \to .X \stackrel{K}{=} A \to X \stackrel{K}{=} B. \qquad (3)$$
$$\boldsymbol{P}3, (3): \textbf{Ant} \to .X \stackrel{K}{=} B \to X \stackrel{K}{=} A. \qquad (4)$$
$$(3), (4): \textbf{Ant} \to (\forall X). X \stackrel{K}{=} A \leftrightarrow X \stackrel{K}{=} B. \qquad (5)$$
$$\text{D21}, \boldsymbol{D}3: \textbf{Ant} \wedge X \equiv \mathcal{K}(A) \to (\forall Y). Y \in X \leftrightarrow Y \in \mathcal{Q} \wedge Y \stackrel{K}{=} A$$
$$(5), \text{T1.14} \qquad \leftrightarrow Y \in \mathcal{Q} \wedge Y \stackrel{K}{=} B$$
$$\boldsymbol{D}3 \qquad \to X \equiv \mathcal{K}(B). \qquad (6)$$
$$(6), \text{T2.32}, \boldsymbol{P}3, \text{P3}, \text{P5}: \textbf{Ant} \to (\forall X). X \equiv \mathcal{K}(A) \leftrightarrow X \equiv \mathcal{K}(B)$$
$$\text{T4.2}, \boldsymbol{D}4 \qquad \to \textbf{Con}.$$

The converse of ***T***8 is also true, namely:

T9. $\hat{A} = \hat{B} \wedge A, B \in \mathcal{Q} \wedge A \neq B \to (\exists U, V) A_U \mathcal{K}_V B.$

Proof.
$$\text{T5.4}: \hat{A} = \hat{B} \to (\forall X). X = \hat{A} \leftrightarrow X = \hat{B}. \qquad (1)$$
$$\boldsymbol{T}6, \text{T4.3}, \boldsymbol{D}4: A \in \mathcal{Q} \to (\forall X). X = \hat{A} \leftrightarrow X \equiv \mathcal{K}(A). \qquad (2)$$
$$\text{Similarly:} \quad B \in \mathcal{Q} \to (\forall X). X = \hat{B} \leftrightarrow X \equiv \mathcal{K}(B). \qquad (3)$$
$$(1)\text{–}(3): \quad \textbf{Ant} \to (\forall X). X \equiv \mathcal{K}(A) \leftrightarrow X \equiv \mathcal{K}(B)$$
$$\text{D21}, \text{T1.14} \qquad \to (\forall Y). Y \in \mathcal{K}(A) \leftrightarrow Y \in \mathcal{K}(B)$$
$$\text{P4} \qquad \to .A \in \mathcal{K}(A) \to A \in \mathcal{K}(B)$$
$$\boldsymbol{D}3, 2 \qquad \to .A \in \mathcal{Q} \wedge A = A \to \textbf{Con}$$
$$\text{T}(1.10, 2.32) \qquad \to \textbf{Con}.$$

The *objective universe of observer H* is the class of objects to which H attributes his subjective entities; in symbols:

D5. $\mathcal{O}(H) \;\; \textit{for} \;\; X \ni (\exists Y). X = \hat{Y} \wedge (\exists U, Z)\, Y_H \mathcal{K}_U Z.$

Literally, $\mathcal{O}(H)$ is the class of sets X, such that there exists at least one subjective entity Y that H attributes to $X (X = \hat{Y})$, and there exists at least one other observer U and a subjective entity Z of U such that H and U agree that Y and Z are due to the same source X (according to ***T***8, $\hat{Z} = \hat{Y} = X$).

The *(macroscopic) objective universe* (class of objects) is then the class of all objects that belong to at least one observer U(***D***6).

D6. $\mathcal{O}$ *for* $X \ni (\exists U) X \in \mathcal{O}(U)$.

Objects are denoted by italic capital Latin letters.

It can be seen from ***D***5 and ***D***6 that only two observers are necessary and sufficient to establish the objectivity of an entity. Since objectivity is essentially a measure of agreement between different observers, one may think that more than two observers are necessary. However, it turns out in practice that if two observers having the qualifications specified in ***I***1 agree on the perception of an entity, the probability that another such observer would not agree with them is negligible. It is because the probability of agreement converges so rapidly to unity, that the concept of objectivity is possible.

The fact that an entity A is objective ($A \in \mathcal{O}$), does not mean that it is perceived by every observer – it is quite possible that A is perceived by an observer $G[A \in \mathcal{O}(G)]$, but not by another observer H. In ***D***7, we define the class $\mathcal{O}(H_1, \ldots, H_n)$ of objects common to observers $H_1, \ldots, H_n$ as the intersection (D18) of $\mathcal{O}(H_1), \ldots, \mathcal{O}(H_n)$.

D7. $\mathcal{O}(H_1, \ldots, H_n)$ *for* $\mathcal{O}(H_1) \cap \ldots \cap \mathcal{O}(H_n)$.

With ***D***7, our first major task is completed. Some of the ideas upon which this chapter is based were expressed qualitatively by B. Russell [1927]. But as far as I know, this is the first time a precise definition of the objective universe is given in the framework of a deductive theory.

This is the only chapter in which physicists may feel a little out of place. From now on there will be many familiar landmarks, as a glance at the contents can show. However, instead of flying from one landmark to another, we shall use surface transportation that covers the territory thoroughly and penetrates deeper into little explored areas.

III. PARTICLES

1. Parts

The point of space geometry is a particle, which is usually defined as an object with no perceivable parts. For this definition to be acceptable, the concept 'part' must be meaningful. The question now is whether 'part' can be defined in terms of the set concepts 'element' (I16) or 'subset' (D19), or whether it must be introduced as a primitive concept.

Since an object is a set (*D*II5.4), and a set without elements cannot be distinguished from the null set $\emptyset$(T6.24), the identification of 'part' with 'element' is ruled out. To decide whether 'part' can be defined in terms of 'subset', we need to know the physical interpretation of 'subset'.

The subset relation $\subseteq$ is defined in terms of the element relation $\in$(D19). To interpret '$A \in B$', where A, B are sets representing objects, we observe that a set is an individual or a definite collection of individuals [Sec. A2(i)]. Consequently, $B = \{A, C_1, \ldots, C_n\}$, which means that B can be decomposed into a set of objects, one of which is A, i.e., A is a *constituent* of B. From this and D19 we conclude that '$A \subseteq B$' means 'any constituent of A is also a constituent of B', and thus '$A \subset B$' (D20) means 'any constituent of A is a constituent of B, but not conversely'. Since this is precisely the meaning of 'A is a part of B', 'part' can be identified with the *proper subset* relation $\subset$, and no additional primitive concept is necessary to represent 'part'.

The sentence 'C *consists* of A and B' can be interpreted in one of two ways: '$C = \{A,B\}$' or '$C = A \cup B$'. We use the former interpretation if we consider A, B to be the ultimate constituents of C, and the latter interpretation if we consider the parts of A, B to be also parts of C. Notice that $\subset$ is transitive, whereas $\in$ is not; thus if A is part of B and B is part of C, then A is part of C, but if A is a constituent of B, and B a constituent of C, then A is *not* a constituent of C.

Different observers may not agree on the analysis of an object into parts because of differences in the distances of the object from them, variations in the resolving power of their perceptions, and diversity of conditions and means of observation – a car appears to be a particle at a distance, but a complex object when close by; two adjacent objects may appear to be a particle to a nearsighted person, but two distinct objects to a person with good eyesight; a dust particle ceases to be a particle under a microscope. Thus it is not sufficient to write '$A \subset B$' for objects; it is also necessary to state who perceives A and B. In view of this, the following definitions are useful (D19–21; *D*II5.5):

***D*1.** $A \subseteq_H B$ *for* $A \subseteq B \wedge A, B \in \mathcal{O}(H)$.

***D*2.** $A \subset_H B$ *for* $A \subset B \wedge A, B \in \mathcal{O}(H)$.

***D*3.** $A \equiv_H B$ *for* $A \equiv B \wedge A, B \in \mathcal{O}(H)$.

As in T6.(3,4), D(20, 21), we have:

***T*1.** $A \subseteq_H B \leftrightarrow A \subset_H B \vee A \equiv_H B$.

***T*2.** $A \subset_H B \leftrightarrow A \subseteq_H B \wedge \sim B \subseteq_H A$.

***T*3.** $A \equiv_H B \leftrightarrow A \subseteq_H B \wedge B \subseteq_H A$.

***T*4.** $\sim A \subset_H A$.

***T*5.** $A \subset_H B \rightarrow \sim B \subset_H A$.

***T*6.** $A \subset_H B \wedge B \subset_H C \rightarrow A \subset_H C$.

*T*4, 5, 6 show that $\subset_H$ is irreflexive (D26), asymmetric (D28), and transitive (D30). Thus, $\subset_H$ is a strict partial ordering of the class of objects of $H, \mathcal{O}(H)$.

Even though an observer H may perceive A to be part of B, but some other observer G may not perceive A or B, nevertheless, the relation $\subset$ is objective in the sense that if any observer U perceives A to be part of B, any other observer H who perceives A and B, agrees with U that A is part of B. In symbols:

***T*7.** $(\exists U) A \subset_U B. \wedge A, B \in \mathcal{O}(H) \rightarrow A \subset_H B$.

Proof. **$D2$: Ant** $\rightarrow A \subset B \wedge A, B \in \mathcal{O}(H)$

$D2$ $\rightarrow$ **Con.**

What we have done so far, is to introduce few basic relations between objects. In the next section, two more relations are defined. All these relations are then used in Sec. 3 to define a *particle* and derive some of its properties.

2. Connection and separation

In addition to the above definitions, it is useful to introduce two more:

D1. $A \ni\!\in_H B$ *for*
$\sim A \subseteq_H B \wedge \sim B \subseteq_H A \wedge (\exists X) X \subset_H A \wedge X \subset_H B.$

'$A \ni\!\in_H B$' reads 'A and B are *connected* for H', and means (**T**1.1): A is not equal to $B(\sim A \equiv_H B)$, A is not part of B, B is not part of A, and there exists an X which is a part of A and B.

D2. $A \supset\!\subset_H B$ *for*
$\sim A \subseteq_H B \wedge \sim B \subseteq_H A \wedge \sim (\exists X) X \subset_H A \wedge X \subset_H B.$

'$A \supset\!\subset_H B$' reads 'A and B are *separate* for H', and means: A is not equal to B, A is not part of B, B is not part of A, and there does not exist an X which is a part of A and B.

T1. $\sim A \ni\!\in_H A \wedge \sim A \supset\!\subset_H A.$
Proof. **D**1,2; T6.3: $A \ni\!\in_H A \vee A \supset\!\subset_H A \rightarrow \sim A \equiv A.$ (1)
(1), T(2.20, 2.39, 6.6): **T**.

T2. $A \ni\!\in_H B \leftrightarrow B \ni\!\in_H A.$
Proof. **D**1.

T3. $A \supset\!\subset_H B \leftrightarrow B \supset\!\subset_H A.$
Proof. **D**2.

Two objects A, B of an observer H are related to each other in one of five different ways (**T**4): (1) $A \equiv_H B$, (2) $A \subset_H B$, (3) $B \subset_H A$, (4) $A \ni\!\in_H B$, (5) $A \supset\!\subset_H B$.

T4. $A \equiv_H B \vee A \subset_H B \vee B \subset_H A \vee A \ni\!\in_H B \vee A \supset\!\subset_H B.$
Proof. Substitute '$A \subseteq_H B$', '$B \subseteq_H A$', and '$(\exists X) X \subset_H A \wedge X \subset_H B$' for F, G, and H, respectively in T2.4 and make use of **T**1.(1–3), and **D**1,2.

3. Particles

The class of particles perceived by an observer H is the class of objects X of H that have no parts perceived by H(**D**1).

D1. $\mathscr{P}(H)$ *for* $X \ni X \in \mathscr{O}(H) \wedge \sim (\exists Y) Y \subset_H X.$

Particles are denoted by the Latin italic capitals: P, Q, R, S, T, U.

We do not define the class of particles, because what is a particle to one observer may not be a particle to another observer. However, it does seem possible in practice to find objects that are considered to be particles by many observers under a wide range of conditions. The class of objects that are particles to observers $H_1, \ldots, H_n$ is the intersection (D18) of the classes $\mathscr{P}(H_1), \ldots, \mathscr{P}(H_n)$.

***D*2.** $\mathscr{P}(H_1, \ldots, H_n)$ *for* $\mathscr{P}(H_1) \cap \ldots \cap \mathscr{P}(H_n)$.

According to ***D*2.1**, two objects are connected only if they have a common part. Since a particle has no parts, it cannot be connected with another object (***T*1**).

***T*1.** $P \in \mathscr{P}(H) \rightarrow \sim (\exists X) X \ni\in_H P$.

Proof.
$\mathbf{D1: Ant} \rightarrow \sim (\exists Y) Y \subset_H P$
T3.17 $\rightarrow (\forall Y) \sim Y \subset_H P$
T2.15 $\rightarrow (\forall Y). \sim Y \subset_H P \vee \sim Y \subset_H X$
T(3.17, 2.38) $\rightarrow \sim (\exists Y). Y \subset_H P \wedge Y \subset_H X$
T2.15, *D*2.1 $\rightarrow \mathbf{Con}$.

From ***D*1** and ***T*1**, we see that if A is a particle and B is an object, then cases (3) and (4) in ***T*2.4** are excluded. If B is also a particle, case (2) is also excluded, which leaves cases (1) and (5). Thus two particles P, Q are related in one of two ways: They are either the same or separate (*T*2).

*T*2. $P, Q \in \mathscr{P}(H) \rightarrow P \equiv_H Q \vee P \supset\subset_H Q$.

Proof.
$\mathbf{D1: Ant} \rightarrow P, Q \in \mathscr{O}(H)$
***T*2.4** $\rightarrow P \equiv_H Q \vee P \subset_H Q \vee Q \subset_H P \vee P \ni\in_H Q \vee P \supset\subset_H Q$. (1)
$\mathbf{D1, T1: Ant} \rightarrow \sim P \subset_H Q \wedge \sim Q \subset_H P \wedge \sim P \ni\in_H Q$. (2)
(1), (2), T2.11 : **T**.

We have given a precise definition of a particle, and have shown that this definition has the basic properties associated with a particle. Unfortunately, it turns out that the concept of a particle is *not* objective, since what is a particle to one observer may not be a particle to another observer. There does not seem to be any way out of this, either in the macroscopic or microscopic domain. For instance, a proton behaves like a particle for scattering of low energy electrons, but has a structure for scattering of high energy (≈ 100 MeV) electrons.

Since (point) events are happenings to particles, events are not objective either, which explains why the subscript 'H', referring to observer H, persists throughout the theory.

So far, we have been concerned with only *macroscopic* particles. Whether *microscopic* particles can be defined in a similar manner, is a moot question.

IV. EVENTS

1. Introduction

Just as the point of space geometry is a particle, the point of space-time geometry is an event. We arrived at the concept of a particle through sensations; how do we arrive at the concept of an event? The answer is, through *changes* of sensations.

Since the only events of interest in physics are *point-events*, the changes must be *sudden* (no extension in time), and the objects affected must be *particles* (no extension in space). In order for a change in sensations to be sudden and distinct, it is necessary that it starts or ends at a level above the threshold of both intensity and quality. For instance, the light from a particle must be bright enough to be clearly seen, and its color must be well within the visible range of red to violet.

This does not mean that gradual variation of the intensity or quality of light, sound, etc., cannot be studied in physics. Such phenomena can be observed through the motion of the indicator of a detecting instrument, and the important events are the sudden coincidences of the tip of the indicator with the scale marks.

If a particle appears to an observer as he suddenly turns his head, such an appearance is not considered an admissible event. To exclude such events from consideration, all events are required to occur within the field of perception.

2. Appearance and disappearance events

The only kind of events that can happen to a particle, are appearance (creation) and disappearance (annihilation); all other kinds of events can be defined in terms of these two. For instance, the coincidence of particles P,Q into particle R, can be defined in terms of the disappearance of P,Q and appearance of R. These two kinds of events are introduced in ***C*** 5 and ***C*** 6 below.

*C*5. $\mathscr{A}$.

*I*5. '$\langle a, P, H\rangle \in \mathscr{A}$' means '*the observer H perceives the event a of the sudden appearance of particle P*'. The appearance must be within the field of perception of H, and the sensations associated with P must be above the threshold of intensity and quality.

The fact that $\mathscr{A}$ is a class, is expressed by the axiom:

*A*5. $\mathfrak{C}\mathscr{A}$,

and that it is a ternary relation by the postulate:

***P*1.** $\mathfrak{R}_3\mathscr{A}$.

For convenience we introduce the definition:

***D*1.** $a\mathscr{A}_H P \quad \mathit{for} \quad \langle a, P, H\rangle \in \mathscr{A}$.

We do not define 'a is the appearance of P' without reference to H, i.e.,

$$a\mathscr{A}P \quad \mathit{for} \quad (\exists\, U)a\mathscr{A}_U P,$$

because a particle to one observer may not be a particle to another, and a may not even occur to some observers. For instance, particles Q, R may seem to coincide into particle P for an observer H, who perceives the appearance a of $P(a\mathscr{A}_H P)$, but Q,R may remain separate to another observer G, for whom the event a never occurs.

The following postulate states that the right member of '$a\mathscr{A}_H P$' is a particle, i.e., the domain of the binary relation $\mathscr{A}_H$ is the class of particles $\mathscr{P}(H)$:

***P*2.** $(\exists\, x)x\mathscr{A}_H P. \rightarrow P \in \mathscr{P}(H)$.

If two particles P,Q coincide into particle R, two views are possible: (1) the disappearance of P,Q and appearance of R are *three* distinct events that coincide at the same place and time; (2) these events are *one* and the same event, namely the event of coincidence of P,Q into R. Here we adopt the first view which corresponds to the view of Carnap's *C–T* system (Carnap [1958] p. 198). The second view corresponds to Carnap's *Wlin*-system (Carnap [1958] p. 207). Other references in which these views were first proposed and discussed, are given by Carnap ([1958] p. 197). The concept of coincidence of events necessary for the first view is introduced in the next section.

In accordance with the first view which we take, an appearance event a is associated with *at most one* (D9) particle X, i.e.,

P3. $a\mathscr{A}_H P \wedge a\mathscr{A}_H Q \rightarrow P = Q$

and a particle P is associated with *at most one* appearance event, i.e.,

P4. $a\mathscr{A}_G P \wedge b\mathscr{A}_H P \rightarrow a = b$.

It is not assumed that every particle P is associated with at least one appearance event, because such an event may not be perceived by anyone. Different observers G, H are used in **P**4, but not in **P**3, because it is meaningful to have a particle common to two observers, but two observers do not know that they are talking about the *same* event unless they agree first that the events are associated with the same particle.

All that has been written above about appearance events is completely applicable to disappearance events, and the corresponding steps are written down without comments.

*C*6. $\mathscr{D}$.

*I*6. '$\langle a, P, H\rangle \in \mathscr{D}$' means '*the observer H perceives the event a of the sudden disappearance of particle P*'. The disappearance must be within the field of perception of H, and the sensations associated with P must be above the threshold of intensity and quality.

*A*6. $\mathfrak{C}\mathscr{D}$.
P5. $\mathfrak{R}_3\mathscr{D}$.
D2. $P\mathscr{D}_H a$ *for* $\langle a, P, H\rangle \in \mathscr{D}$.
P6. $(\exists x)P\mathscr{D}_H x. \rightarrow P \in \mathscr{P}(H)$.
P7. $P\mathscr{D}_H a \wedge Q\mathscr{D}_H a \rightarrow P = Q$.
P8. $P\mathscr{D}_G a \wedge P\mathscr{D}_H b \rightarrow a = b$.

Since an appearance event can never be a disappearance event, we assume:

P9. $a\mathscr{A}_G P \wedge Q\mathscr{D}_H b \rightarrow a \neq b$,

i.e., if a is the appearance of P, and b is the disappearance of Q, then $a \neq b$. Another way of saying the same thing is that an event a cannot be both the appearance of a particle P and the disappearance of particle Q(**T**1).

T1. $\sim . a\mathscr{A}_G P \wedge Q\mathscr{D}_H a$.
Proof. **P9**: $a\mathscr{A}_G P \wedge Q\mathscr{D}_H a \rightarrow a \neq a$. (1)
(1), T2.20: $a = a \rightarrow \mathbf{T}$. (2)
(2), T5.1: **T**.

The class of events perceived by observer H is defined by:

$$D3. \qquad \mathscr{E}(H) \quad for \quad x \ni (\exists\, Y).x\mathscr{A}_H Y \vee Y\mathscr{D}_H x,$$

i.e., $\mathscr{E}(H)$ is the class of events x that are either the appearance or disappearance of a particle Y. The class of events common to observers $H_1, \ldots, H_n$ is defined by:

$$D4. \qquad \mathscr{E}(H_1, \ldots, H_n) \quad for \quad x \ni (\exists\, Y).Y \in \mathscr{P}(H_1, \ldots, H_n)$$
$$\wedge (x\mathscr{A}_{H_1} Y \wedge \ldots \wedge x\mathscr{A}_{H_n} Y. \vee .Y\mathscr{D}_{H_1} x \wedge \ldots \wedge Y\mathscr{D}_{H_n} x),$$

i.e., $\mathscr{E}(H_1, \ldots, H_n)$ is the class of events x such that there exists a particle Y common to $H_1, \ldots, H_n$, and x is either the appearance of Y or the disappearance of Y for all the observers $H_1, \ldots, H_n$.

If a is the appearance of P for G, and a is common to both G and H, then a is the appearance of P for H, and P is a particle for both G and H (***T***2).

T2. $\qquad a\mathscr{A}_G P \wedge a \in \mathscr{E}(G, H) \rightarrow a\mathscr{A}_H P \wedge P \in \mathscr{P}(G, H).$

Proof. $\qquad$ **D4: Ant** $\leftrightarrow a\mathscr{A}_G P \wedge (\exists\, X) X \in \mathscr{P}(G, H)$
$\wedge (a\mathscr{A}_G X \wedge a\mathscr{A}_H X. \vee .X\mathscr{D}_G a \wedge X\mathscr{D}_H a)$
T1, T2.11 $\leftrightarrow (\exists\, X) X \in \mathscr{P}(G, H) \wedge a\mathscr{A}_G P \wedge a\mathscr{A}_G X \wedge a\mathscr{A}_H X$
P3, T5.3 $\rightarrow$ **Con**.

The same statement is applicable if 'appearance' is replaced by 'disappearance', i.e.,

T3. $\qquad P\mathscr{D}_G a \wedge a \in \mathscr{E}(G, H) \rightarrow P\mathscr{D}_H a \wedge P \in \mathscr{P}(G, H).$

Proof. $\qquad$ ***D***4, ***T***1, ***P***7, as in ***T***2.

3. Coincidence of events

Time geometry cannot be built from events alone; a mortar to relate events is necessary to produce a structure. This mortar consists of the *coincidence* relation between events, and the relation *before*. 'Coincidence' is introduced as a primitive concept and 'before' is defined.

C7. $\qquad \mathscr{C}.$

*I*7. '$\langle a, b, H\rangle \in \mathscr{C}$' means '*the observer H perceives the events a,b to occur at the same place and the same* (psychological) *time*', or briefly, '*H* perceives *a coincides* with *b*'.

The simultaneity (same time) concept used in *I*7 should cause no anxiety, because the events occur at the same place. This kind of simultaneity is objective, i.e., holds for all observers who agree on the perception of the affected events.

The class property of $\mathscr{C}$ is expressed by:

*A*7. $\mathfrak{C}\mathscr{C}$,

and the fact that it is a ternary relation by:

***P*1.** $\mathfrak{R}_3\mathscr{C}$.

For convenience, we define:

***D*1.** $a \asymp_H b \quad \textit{for} \quad \langle a, b, H \rangle \in \mathscr{C}$.

The fact that the left member of '$a \asymp_H b$' is an event perceived by H, is expressed by:

***P*2.** $a \asymp_H b \rightarrow a \in \mathscr{E}(H)$.

The identity of the right member is not given, because it can be deduced from the symmetry of $\asymp_H$(***T*1**).

Suppose that events a,b are common to observers G,H(***D*2.4**), i.e., G and H agree that they perceive a particle P with which a and b are associated. If G perceives the coincidence of $a,b(a \asymp_G b)$, then H must also perceive the coincidence of $a,b(a \asymp_H b)$, because the coincidence occurs at the same place and time. The observers G,H could be moving relative to each other. Briefly, if two events coincide for an observer, they coincide for any other observer who also perceives the events (***P*3**).

***P*3.** $a \asymp_G b \wedge a, b \in \mathscr{E}(G, H) \rightarrow a \asymp_H b$.

Since every event can be considered coincident with itself, we assume $\asymp_H$ is reflexive, i.e.,

***P*4.** $a \in \mathscr{E}(H) \rightarrow a \asymp_H a$.

Moreover, $\asymp_H$ must be an equivalence in $\mathscr{E}(H)$. The shortest way of achieving this is to assume the semi-transitive property (D31):

***P*5.** $a \asymp_H b \wedge a \asymp_H c \rightarrow b \asymp_H c$.

***T*1.** $\asymp_H$ is an equivalence in $\mathscr{E}(H)$.
Proof. ***P*4**, ***P*5**, T7.1.

T2. $a \asymp_H b \to a, b \in \mathscr{E}(H)$.

Proof. **T1**: $a \asymp_H b \to b \asymp_H a$

P2 $\quad \to b \in \mathscr{E}(H)$. (1)

P2, (1): **T**.

4. Time order

The time order between events that can be connected by a chain of *particles* (events *inside* the light cone) is defined in this section. Time order between events that can be connected by *light* signals (events *on* the light cone) is introduced in the next chapter. Some authors (Carnap [1958] p. 198) introduce this relation as a primitive concept, and others (Reichenbach [1958] § 21) define it in terms of a causal chain. Our approach is similar to the latter method, but is different in the explicit use of the two types of events presented in Sec. 2.

More explicitly, we define '*a precedes b*' to mean (**D**1) there exists a sequence of particles $Z_1, \ldots, Z_n (n \geq 1)$ such that a coincides with the appearance of Z_1, the disappearance of Z_1 coincides with the appearance of $Z_2, \ldots,$ the disappearance of Z_n coincides with b. The particles $Z_1, \ldots, Z_n$ are the links of the chain connecting a and b.

D1. $a \prec_H b$ *for*

$$(\exists x, y, Z). a \asymp_H x \wedge x \mathscr{A}_H Z \wedge Z \mathscr{D}_H y \wedge y \asymp_H b:$$
$$\vee (\exists n). n > 1 \wedge (\exists x_1, y_1, Z_1, \ldots, x_n, y_n, Z_n).$$
$$a \asymp_H x_1 \wedge x_1 \mathscr{A}_H Z_1 \wedge Z_1 \mathscr{D}_H y_1 \wedge y_1 \asymp_H x_2 \wedge \ldots$$
$$\wedge x_n \mathscr{A}_H Z_n \wedge Z_n \mathscr{D}_H y_n \wedge y_n \asymp_H b.$$

'$a \prec_H b$' reads '*a precedes b*, for *H*'. The sentence '*a* is *before b*' is defined in **D**V3.2 to include events ordered by unperceivable signals.

The field of $\prec_H$ is the class of events of H(**T**1).

T1. $a \prec_H b \to a, b \in \mathscr{E}(H)$.

Proof. **D1**: **Ant**$\to (\exists x) a \asymp_H x \wedge (\exists y) y \asymp_H b$

T3.2 $\quad \to$**Con.**

A very simple but important case of **D**1, is when a is the appearance of a particle P and b is the disappearance of P, namely:

T2. $a \mathscr{A}_H P \wedge P \mathscr{D}_H b \to a \prec_H b$.

Proof. **D**2.3: **Ant**$\to a, b \in \mathscr{E}(H)$

$P3.4 \qquad \rightarrow a \asymp_H a \wedge a \mathscr{A}_H P \wedge P \mathscr{D}_H b \wedge b \asymp_H b$
$D1 \qquad \rightarrow \mathbf{Con}.$

Just as $\leq$ is a useful relation between numbers, $\preccurlyeq_H$ defined below is a useful relation between events.

D2. $\quad a \preccurlyeq_H b \quad for \quad a \asymp_H b \vee a \prec_H b.$

'$a \preccurlyeq_H b$' means 'either $a \asymp_H b$ or $a \prec_H b$'.

With the help of **D2**, a compact statement of a general transitive law involving $\prec_H$ and $\asymp_H$ can be given (**T3**).

T3. $\quad a \preccurlyeq_H b \wedge b \prec_H c. \vee .a \prec_H b \wedge b \preccurlyeq_H c : \rightarrow a \prec_H c.$

Proof. $D1, T2.40: a \asymp_H b \wedge b \prec_H c \leftrightarrow a \asymp_H b \wedge (\exists x). b \asymp_H x \wedge \ldots$
$T3.1 \qquad \rightarrow (\exists x). a \asymp_H x \wedge \ldots$
$D1 \qquad \rightarrow a \prec_H c. \qquad (1)$
$D1, T2.40: a \prec_H b \wedge b \prec_H c$
$\quad \leftrightarrow (\exists u, v, x, y). a \asymp_H u \wedge v \asymp_H b \wedge b \asymp_H x \wedge y \asymp_H c \wedge \ldots$
$T3.1 \rightarrow (\exists u, v, x, y). a \asymp_H u \wedge v \asymp_H x \wedge y \asymp_H c \wedge \ldots$
$D1 \quad \rightarrow a \prec_H c. \qquad (2)$
$(1), (2), T2.40: a \asymp_H b \vee a \prec_H b. \wedge b \prec_H c : \rightarrow a \prec_H c. \qquad (3)$
$(3), D2: a \preccurlyeq_H b \wedge b \prec_H c \rightarrow a \prec_H c. \qquad (4)$

Similarly, we can prove:

$a \prec_H b \wedge b \preccurlyeq_H c \rightarrow a \prec_H c. \qquad (5)$
$(4), (5), T2.37: \mathbf{T}.$

Literally, **T3** states that either $a \preccurlyeq_H b$ and $b \prec_H c$, or $a \prec_H b$ and $b \preccurlyeq_H c$, implies $a \prec_H c$.

One of the essential properties of $\asymp_H$, is that coincident events are connected without any intermediary. In view of **D1**, this means that $\asymp_H$ and $\prec_H$ are mutually exclusive (**P1**).

P1. $\quad \sim .a \asymp_H b \wedge a \prec_H b.$

One consequence of this, is that $\prec_H$ is irreflexive (D26).

T4. $\quad \sim a \prec_H a.$

Proof. $P1: a \prec_H a \rightarrow \sim a \asymp_H a. \qquad (1)$
$T1, P3.4: a \prec_H a \rightarrow a \asymp_H a. \qquad (2)$
$(1), (2): \quad a \prec_H a \rightarrow a \asymp_H a \wedge \sim a \asymp_H a. \qquad (3)$
$(3), T2.9: \mathbf{T}.$

Moreover, $\prec_H$ is asymmetric (*T*5).

*T*5. $a \prec_H b \rightarrow \sim b \prec_H a.$

Proof. $\boldsymbol{T}3: a \prec_H b \wedge b \prec_H a \rightarrow a \prec_H a.$ (1)

(1), T2.20, $\boldsymbol{T}4: \sim .a \prec_H b \wedge b \prec_H a.$ (2)

(2), T2.22: **T**.

Therefore, $\prec_H$ is irreflexive (*T*4), asymmetric (*T*5), and transitive (*T*3), and thus is a strict partial ordering of $\mathscr{E}(H)$; it has the same properties as the numerical relation $<$.

A definition that is useful later on is:

***D*3.** $a \mathscr{T}_H b$ *for* $a \asymp_H b \vee a \prec_H b \vee b \prec_H a.$

'$a \mathscr{T}_H b$' reads '*a,b are time-like for H*'. This terminology is borrowed from relativity, where the time interval between two events related in this way is said to be time-like.

V. SIGNALS

1. Change, coincidence and dissociation

It was stated in Sec. IV 2 that all events of interest to physics can be defined in terms of appearance and disappearance events. This is accomplished in **D**1.

D1. $P_1, \ldots, P_m \otimes_H Q_1, \ldots, Q_n$ *for*
$(\exists x_1, \ldots, x_m, y_1, \ldots, y_n) P_1 \mathscr{D}_H x_1 \wedge \ldots \wedge P_m \mathscr{D}_H x_m$
$\wedge y_1 \mathscr{A}_H Q_1 \wedge \ldots \wedge y_n \mathscr{A}_H Q_n \wedge x_1 \asymp_H \ldots \asymp_H x_m \asymp_H y_1 \asymp_H \ldots \asymp_H y_n$.

'$P_1, \ldots, P_m \otimes_H Q_1, \ldots, Q_n$' means 'there exist events $x_1, \ldots, x_m$ that are the disappearance of particles $P_1, \ldots, P_m$, respectively, and there exist events $y_1, \ldots, y_n$ that are the appearance of particles $Q_1, \ldots, Q_n$, respectively, and all these events coincide together', it reads '$P_1, \ldots, P_m$ *go into* $Q_1, \ldots, Q_n$, for H'. Special cases have special names: '$P \otimes_H Q$' reads 'P *changes* into Q, for H'; '$P_1, \ldots, P_m \otimes_H Q$' reads '$P_1, \ldots, P_m$ *coincide* into Q, for H'; '$P \otimes_H Q_1, \ldots, Q_n$' reads '$P$ *dissociates* in $Q_1, \ldots, Q_n$, for H'.

The remainder of this section is devoted to the development of the properties of $\otimes_H$. The reader who is not interested in details can glance at **D**2–4, **P**1, and then proceed to Sec. 2.

T1. $P_1, \ldots, P_m \otimes_H Q_1, \ldots, Q_n \rightarrow P_1, \ldots, P_m, Q_1, \ldots, Q_n \in \mathscr{P}(H)$.
Proof. **D**1, **P**IV2.(2,6).

T2. If $i_1, \ldots, i_m$ is a permutation of $1, \ldots, m$, and $j_1, \ldots, j_n$ is a permutation of $1, \ldots, n$, then $P_1, \ldots, P_m \otimes_H Q_1, \ldots, Q_n \leftrightarrow P_{i_1}, \ldots, P_{i_m} \otimes_H Q_{j_1}, \ldots, Q_{j_n}$.
Proof. **D**1, **T**IV3.1.

T3. $P_1, \ldots, P_p \otimes_H Q_1, \ldots, Q_q \wedge R_1, \ldots, R_r \otimes_H S_1, \ldots, S_s$
$\wedge . \{P_1, \ldots, P_p\} \cap \{R_1, \ldots, R_r\} \neq \emptyset \vee \{Q_1, \ldots, Q_q\} \cap \{S_1, \ldots, S_s\} \neq \emptyset :$
$\rightarrow P_1, \ldots, P_p, R_1, \ldots, R_r \otimes_H Q_1, \ldots, Q_q, S_1, \ldots, S_s$.
Proof. **D**1, **T**IV3.1.

T4. $P_1, \ldots, P_m \otimes_H Q_1, \ldots, Q_n \wedge i \in \{1, \ldots, m\} \wedge j \in \{1, \ldots, n\} \rightarrow P_i \otimes_H Q_j$.
Proof. **D**1.

We prove in ***T***5 that $\otimes_H$ is irreflexive, and in ***T***6 that it is asymmetric. However, $\otimes_H$ is *not* transitive.

T5. $\sim P \otimes_H P$.

Proof. $\mathbf{D}1: P \otimes_H P \leftrightarrow (\exists x, y) P \mathscr{D}_H x \wedge y \mathscr{A}_H P \wedge x \asymp_H y$

$\mathbf{T}\mathrm{IV}4.2 \quad \rightarrow (\exists x, y)\, y \prec_H x \wedge x \asymp_H y.$ (1)

(1), T2.20, ***P***IV4.1: **T**.

T6. $P \otimes_H Q \rightarrow \sim Q \otimes_H P$.

Proof. $\mathbf{D}1: P \otimes_H Q \wedge Q \otimes_H P$

$\rightarrow (\exists u, v, x, y) P \mathscr{D}_H u \wedge v \mathscr{A}_H Q \wedge u \asymp_H v \wedge Q \mathscr{D}_H x \wedge y \mathscr{A}_H P \wedge x \asymp_H y$

$\mathbf{T}\mathrm{IV}4.2 \rightarrow (\exists u, v, x, y)\, y \prec_H u \wedge u \asymp_H v \wedge v \prec_H x \wedge x \asymp_H y$

$\mathbf{T}\mathrm{IV}4.3 \rightarrow (\exists x, y)\, y \prec_H x \wedge x \asymp_H y.$ (1)

(1), T2.20, ***P***IV4.1, T2.22: **T**.

The following definitions restrict the number of particles participating in a reaction:

D2. $P_1, \ldots, P_m \cdot\!\otimes_H Q_1, \ldots, Q_n$ *for*

$P_1, \ldots, P_m \otimes_H Q_1, \ldots, Q_n \wedge (\forall U, V, x, y).\, U \mathscr{D}_H x \wedge V \mathscr{D}_H y$

$\wedge\, x \asymp_H y \wedge U \in \{P_1, \ldots, P_m\} \rightarrow V \in \{P_1, \ldots, P_m\}.$

'$P_1, \ldots, P_m \cdot\!\otimes_H Q_1, \ldots, Q_n$' means '*only* $P_1, \ldots, P_m$ *go into* $Q_1, \ldots, Q_n$'.

D3. $P_1, \ldots, P_m \otimes\!\cdot_H Q_1, \ldots, Q_n$ *for*

$P_1, \ldots, P_m \otimes_H Q_1, \ldots, Q_n \wedge (\forall U, V, x, y).\, x \mathscr{A}_H U \wedge y \mathscr{A}_H V$

$\wedge\, x \asymp_H y \wedge U \in \{Q_1, \ldots, Q_n\} \rightarrow V \in \{Q_1, \ldots, Q_n\}.$

'$P_1, \ldots, P_m \otimes\!\cdot_H Q_1, \ldots, Q_n$', means '$P_1 \ldots, P_m$ *go into only* $Q_1 \ldots Q_n$'.

D4. $P_1, \ldots, P_m \cdot\!\otimes\!\cdot_H Q_1, \ldots, Q_n$ *for*

$P_1, \ldots, P_m \cdot\!\otimes_H Q_1, \ldots, Q_n \wedge P_1, \ldots, P_m \otimes\!\cdot_H Q_1, \ldots, Q_n.$

'$P_1, \ldots, P_m \cdot\!\otimes\!\cdot_H Q_1, \ldots, Q_n$' means '*only* $P_1, \ldots, P_m$ *go into only* $Q_1, \ldots, Q_n$'.

T7. $P \cdot\!\otimes_H R \wedge Q \otimes_H R. \vee .R \otimes\!\cdot_H P \wedge R \otimes_H Q : \rightarrow P = Q.$

Proof. $\mathbf{D}2: P \cdot\!\otimes_H R \wedge Q \otimes_H R \rightarrow P \otimes_H R \wedge Q \otimes_H R$

$\mathbf{T}3 \qquad \rightarrow P, Q \otimes_H R$

$\mathbf{D}1 \qquad \rightarrow (\exists x, y) P \mathscr{D}_H x \wedge Q \mathscr{D}_H y \wedge x \asymp_H y.$ (1)

$\mathbf{D}2, \mathrm{T}6.25: P \cdot\!\otimes_H R \rightarrow .P \mathscr{D}_H x \wedge Q \mathscr{D}_H y \wedge x \asymp_H y \rightarrow Q = P.$ (2)

$(1), (2), \mathrm{T}2.7: P \cdot\!\otimes_H R \wedge Q \otimes_H R \rightarrow P = Q.$ (3)

Similarly, we can prove

$R \otimes\!\cdot_H P \wedge R \otimes_H Q \rightarrow P = Q.$ (4)

(3), (4), T2.37: **T**.

If $P_1 \otimes_H P_2$, $P_2 \otimes_H P_3, \ldots, P_{n-1} \otimes_H P_n$, then $P_1, \ldots, P_n$ form a particle chain, and an event at the beginning of the chain occurs before an event at the end of the chain (**T**8).

T8. $a \asymp_H c \wedge c \mathscr{A}_H P_1 \wedge P_1 \otimes_H P_2 \wedge \ldots \wedge P_{n-1} \otimes_H P_n$
$\wedge P_n \mathscr{D}_H d \wedge d \asymp_H b \rightarrow a \prec_H b.$

Proof. **D**1, **D**IV4.1.

If $P \otimes_H Q, R$, and for any object $Y, P \subset_H Y$ implies $Q \subset_H Y$ and $R \subset_H Y$, we expect that $Q = R$, since Q and R exist at the same time. This is expressed in:

P1. $P \otimes_H Q, R \wedge (\forall Y). P \subset_H Y \rightarrow Q \subset_H Y \wedge R \subset_H Y : \rightarrow Q = R.$

2. World lines

In relativity, a world line is a curve in a space-time diagram, i.e., the class of all events associated with a particle. Since the only events associated with a particle are its appearance and disappearance, it does not seem possible to identify a particle after an event with a particle before the event. This is particularly true with events resulting from interaction of indistinguishable particles.

There are two cases, however, in which one can establish a correspondence between a particle after an event and a particle before the event: (1) if a particle P is not part of any object and only P changes into only Q ($P \cdot\otimes\cdot_H Q$), then the correspondence between P and Q is unique, and we can say that Q is a *continuation* of P; (2) if a particle P is a part of an object X, P goes into Q ($P \otimes_H Q$), and Q is a part of any object Y that is P part of, then we can establish a correspondence between P and Q by virtue of their association with the same objects. For instance, if P is the tip of a pointer, then as the pointer coincides with different marks of a dial, the tip will successively disappear and appear. Each particle representing the tip after every one of these events can be considered to be a *continuation* of the preceeding one.

Thus we define 'P continues into Q' in **D**1 to mean that one of the above two cases hold, and then define a world line in **D**2 to be the class of events on a particle or its continuation.

D1. $P \underset{H}{\rightarrowtail} Q$ *for* $\sim(\exists X) P, Q \subset_H X. \wedge P \cdot\otimes\cdot_H Q:$
$\vee : (\exists X) P \subset_H X. \wedge P \otimes_H Q \wedge (\forall Y). P \subset_H Y \rightarrow Q \subset_H Y.$

'$P \underset{H}{\rightarrowtail} Q$' reads '*P continues* into Q, for H' or 'Q is a *continuation* of P, for H'.

*T*1. $(P \underset{H}{\rightarrowtail} Q) \rightarrow P, Q \in \mathscr{P}(H)$.
Proof. **D1**, **T1.1**.

*T*2. $(P \underset{H}{\rightarrowtail} Q) \rightarrow \sim (Q \underset{H}{\rightarrowtail} P)$.
Proof. $D1$: **Ant** $\rightarrow P \varolessthan_H Q$
$T1.6 \quad \rightarrow \sim Q \varolessthan_H P$
$D1 \quad \rightarrow$ **Con.**

*T*3. $P \underset{H}{\rightarrowtail} Q \wedge P \underset{H}{\rightarrowtail} R . \rightarrow Q = R$.
Proof. $D1$, T2.42, $T1.3$: **Ant** $\rightarrow : P \cdot\!\varolessthan\!\cdot_H Q \wedge P \cdot\!\varolessthan\!\cdot_H R.$
$\vee . P \cdot\!\varolessthan\!\cdot_H Q \wedge P \varolessthan_H R.$
$\vee . P \varolessthan_H Q \wedge P \cdot\!\varolessthan\!\cdot_H R.$
$\vee . P \varolessthan_H Q, R$
$\wedge (\forall Y). P \subset_H Y \rightarrow Q \subset_H Y \wedge R \subset_H Y$
$T1.7, P1.1 \quad \rightarrow$ **Con.**

Just as in *T*1.8, $P_1, \ldots, P_n$ establish a particle chain, if $P_1 \underset{H}{\rightarrowtail} P_2, \ldots, P_{n-1} \underset{H}{\rightarrowtail} P_n$ (*T*4).

*T*4. $a \asymp_H c \wedge c \mathscr{A}_H P_1 \wedge P_1 \underset{H}{\rightarrowtail} P_2 \wedge \ldots \wedge P_{n-1} \underset{H}{\rightarrowtail} P_n$
$\wedge P_n \mathscr{D}_H d \wedge d \asymp_H b \rightarrow a \prec_H b$.
Proof. $D1, D1.(2\text{–}4)$: **Ant** $\rightarrow a \asymp_H c \wedge c \mathscr{A}_H P_1$
$\wedge P_1 \varolessthan_H P_2 \wedge \ldots \wedge P_{n-1} \varolessthan_H P_n$
$\wedge P_n \mathscr{D}_H d \wedge d \asymp_H b$
$T1.8 \quad \rightarrow$ **Con.**

We now define the world line of particle P as the class of events that coincide with either the appearance or disappearance of P or a continuation of P.

D2. $\mathscr{W}(P)_H \quad for \quad x \ni (\exists y): x \asymp_H y \wedge . y \mathscr{A}_H P \vee P \mathscr{D}_H y$
$\vee (\exists n). n > 0 \wedge (\exists Z_1, \ldots, Z_n)(P \underset{H}{\rightarrowtail} Z_1 \wedge Z_1 \underset{H}{\rightarrowtail} Z_2 \wedge \ldots$
$\wedge Z_{n-1} \underset{H}{\rightarrowtail} Z_n . \vee . Z_n \underset{H}{\rightarrowtail} Z_{n-1} \wedge \ldots \wedge Z_1 \underset{H}{\rightarrowtail} P) \wedge . y \mathscr{A}_H Z_n \vee Z_n \mathscr{D}_H y.$

$\mathscr{W}(P)_H$ is called the *world line* of P.

In accordance with **D2**, the world line of a particle is equal to the world line of any of its continuations.

*T*5. $P_1 \underset{H}{\rightarrowtail} P_2 \wedge \ldots \wedge P_{n-1} \underset{H}{\rightarrowtail} P_n . \rightarrow \mathscr{W}(P_1)_H \equiv \mathscr{W}(P_n)_H$.
Proof. **D2**.

T6. $a \in \mathscr{W}(P)_H \rightarrow a \in \mathscr{E}(H) \wedge P \in \mathscr{P}(H)$.
Proof. **D2**, **TIV3.2**, **PIV2.(2,6)**, **T1**.

As might be expected, two events a,b on the world line of a particle P are related in one of three ways: (1) a coincides with b, (2) a precedes b, or (3) b precedes a, i.e., $a\mathscr{T}_H b$ (see **D**IV 4.3).

***T*7.**	$a, b \in \mathscr{W}(P)_H \rightarrow a\mathscr{T}_H b.$
Proof.	**D**2, **D**IV4. (1,3), **T**IV(3.1, 4.3), **T**4.
***T*8.**	$a \asymp_H b \wedge b \in \mathscr{W}(P)_H \rightarrow a \in \mathscr{W}(P)_H.$
Proof.	**D**2, **T**IV3.1.
***T*9.**	$P = Q \vee P \underset{H}{\rightarrowtail} Q \vee Q \underset{H}{\rightarrowtail} P. \rightarrow \mathscr{W}(P)_H \equiv \mathscr{W}(Q)_H.$
Proof.	**D**2.

3. Signal relation

So far, time order was established by means of particle chains only. In physics, order between events not so connected may be established by unperceivable signals. If there is no perceivable connection between two events, how can one establish an ordering relation between them? One possibility is through correlation between two sets of events as described in the interpretation of the following primitive concept.

***C*8.** $\mathscr{S}$.

***I*8.** '$\langle a, b, H\rangle \in \mathscr{S}$' means 'there exist two sets of events α, β on the world lines of two separate particles, such that there is a one to one correspondence between the α-events and the β-events, and the type of, and relations between the α-events determine the type of, and relations between the β-events, but not conversely; a is one of the events of α, and b is the corresponding event of β'. The sentence '$\langle a, b, H\rangle \in \mathscr{S}$' reads '*a is the start of a signal that ends at b, for H*'.

Another method of establishing a signal relation, proposed by Reichenbach ([1958] § 21), is by the marking of events, such as putting a red filter in front of a particle at which a light pulse is assumed to be emitted.

C8 is the last primitive physical concept introduced in this book. All of space-time geometry is constructed by means of just the *eight* concepts **C**1 to **C**8.

The class property of $\mathscr{S}$ is expressed by the axiom:

***A*8.** $\mathfrak{C}\mathscr{S}$,

and the fact that it is a ternary relation, by the postulate:

***P*1.** $\mathfrak{R}_3\mathscr{S}$.

For convenience, we introduce the definition:

D1. $a\mathscr{S}_H b \quad for \quad \langle a,b,H\rangle \in \mathscr{S}.$

That the field of $\mathscr{S}_H$ is $\mathscr{E}(H)$, is expressed by:

P2. $a\mathscr{S}_H b \rightarrow a, b \in \mathscr{E}(H),$

and that a,b are not connected by a particle chain by:

P3. $a\mathscr{S}_H b \rightarrow \sim a\mathscr{T}_H b.$

The following postulate expresses the asymmetry of $\mathscr{S}_H$.

P4. $a\mathscr{S}_H b \rightarrow \sim b\mathscr{S}_H a,$

and **P**5 the transitivity of $\mathscr{S}_H$:

P5. $a\mathscr{S}_H b \wedge b\mathscr{S}_H c \rightarrow a\mathscr{S}_H c.$

The irreflexivity of $\mathscr{S}_H$ follows from **P**4 and **P**5:

T1. $\sim a\mathscr{S}_H a.$
Proof. **P**4, T7.2.

Thus $\mathscr{S}_H$ is a strict partial ordering of $\mathscr{E}_H$.

If two events are related by $\mathscr{S}_H$, then any event that coincides with either one is related to the other by $\mathscr{S}_H$(**P**6).

P6. $a \asymp_H b \wedge b\mathscr{S}_H c. \vee .a\mathscr{S}_H b \wedge b \asymp_H c: \rightarrow a\mathscr{S}_H c.$

We now have two ways of ordering events: by $\prec_H$ and by $\mathscr{S}_H$. A relation that combines both relations is defined by:

D2. $$a\mathscr{B}_H b \quad for \quad a \prec_H b \vee a\mathscr{S}_H b$$
$$\vee (\exists\, n) n>0 \wedge (\exists\, x_1, \ldots, x_n): a \prec_H x_1 \vee a\mathscr{S}_H x_1.$$
$$\wedge .x_1 \prec_H x_2 \vee x_1\mathscr{S}_H x_2. \wedge \ldots \wedge .x_n \prec_H b \vee x_n\mathscr{S}_H b.$$

'$a\mathscr{B}_H b$' reads 'a is *before* b, for H'.

The following theorems show that $\mathscr{B}_H$ has the same properties as S_H:

T2. $a\mathscr{B}_H b \rightarrow a, b \in \mathscr{E}(H).$
Proof. **D**2, **T**IV4.1, **P**2.

T3. $\sim a\mathscr{B}_H a.$
Proof. **D**2, **T**IV4.4, **T**1.

T4. $a\mathscr{B}_H b \rightarrow \sim b\mathscr{B}_H a.$
Proof. **D**2, **T**IV4.5, **P**4.

***T*5.** $a\mathscr{B}_H b \wedge b\mathscr{B}_H c \to a\mathscr{B}_H c.$
Proof. **D**2, **T**IV4.3, **P**5.

***T*6.** $a \asymp_H b \wedge b\mathscr{B}_H c. \vee .a\mathscr{B}_H b \wedge b \asymp_H c: \to a\mathscr{B}_H c.$
Proof. **D**2, **T**IV4.3, **P**6.

4. First signals

The most important signals in physics are the signals with maximum speed of propagation, such as light signals. Since speed has not been defined yet, the name '*first signal*' used by Reichenbach ([1958] p. 143) is quite appropriate for these signals. Literally, a first signal never comes out second in a race with any other signal or particle chain. This idea is incorporated in the definition:

D1.

$$a\mathscr{F}_H b \quad \text{for} \quad a\mathscr{S}_H b$$
$$\wedge (\forall x).x\mathscr{T}_H a \to (x\mathscr{B}_H b \to x \preccurlyeq_H a. \wedge .b\mathscr{B}_H x \to a \prec_H x).$$
$$\wedge (\forall y).y\mathscr{T}_H b \to (a\mathscr{B}_H y \to b \preccurlyeq_H y. \wedge .y\mathscr{B}_H a \to y \prec_H b).$$

'$a\mathscr{F}_H b$' reads '*a is the start of a first signal that ends at b, for H*'. The meaning of '$a\mathscr{F}_H b$' is illustrated in Fig. 1. Literally, '$a\mathscr{F}_H b$' means: $a\mathscr{S}_H b$; and if a, x are time-like, then $x\mathscr{B}_H b$ implies $x \preccurlyeq_H a$, and $b\mathscr{B}_H x$ implies $a \prec_H x$; and if b, y are time-like, then $a\mathscr{B}_H y$ implies $b \preccurlyeq_H y$, and $y\mathscr{B}_H a$ implies $y \prec_H b$.

***T*1.** $a\mathscr{F}_H b \to a, b \in \mathscr{E}(H).$
Proof. **D**1, **P**3.2.

***T*2.** $\sim a\mathscr{F}_H a.$
Proof. **D**1, **T**3.1.

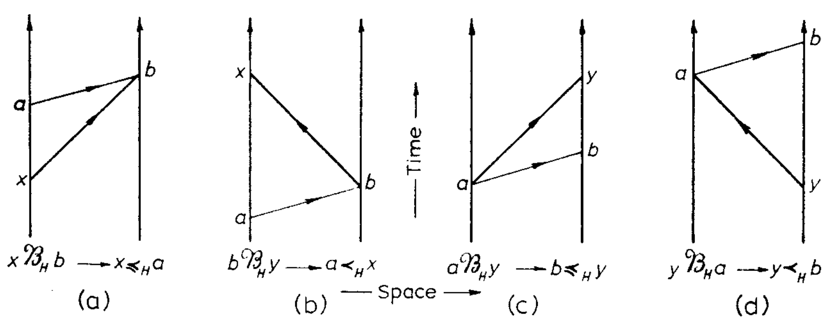

Fig. 1. First signals

*T*3. $a\mathscr{F}_H b \rightarrow \sim b\mathscr{F}_H a.$

Proof. **D**1, **P**3.4.

*T*2 shows that $\mathscr{F}_H$ is irreflexive, and *T*3 that it is asymmetric. However, $\mathscr{F}_H$ is not transitive.

*T*4. $a \asymp_H b \wedge b\mathscr{F}_H c. \vee .a\mathscr{F}_H b \wedge b \asymp_H c: \rightarrow a\mathscr{F}_H c.$

Proof. **D**1, **T**IV4.3, **P**3.6, **T**3.6.

If two first signals depart together they will arrive together, and vice versa (**T**5).

*T*5. $a\mathscr{F}_H b \wedge c\mathscr{F}_H d \rightarrow : a \asymp_H c \wedge b\mathscr{T}_H d \rightarrow b \asymp_H d.$
$\wedge . a\mathscr{T}_H c \wedge b \asymp_H d \rightarrow a \asymp_H c.$

Proof. TIV3.1, **D**1, **D**3.2: $a \asymp_H c \wedge a\mathscr{F}_H b \rightarrow c \asymp_H a \wedge a\mathscr{B}_H b$

T3.6 $\rightarrow c\mathscr{B}_H b.$ (1)

DIV4.3: $b \prec_H d \rightarrow b\mathscr{T}_H d.$ (2)

(1), (2): $a\mathscr{F}_H b \wedge c\mathscr{F}_H d \wedge a \asymp_H c \wedge b \prec_H d$
$\rightarrow c\mathscr{F}_H d \wedge b\mathscr{T}_H d \wedge c\mathscr{B}_H b \wedge b \prec_H d$

D1 $\rightarrow d \preccurlyeq_H b \wedge b \prec_H d$

DIV4.2, **T**IV4.5 $\rightarrow b \asymp_H d.$ (3)

As above:

$a\mathscr{F}_H b \wedge c\mathscr{F}_H d \wedge a \asymp_H c \wedge d \prec_H b$
$\rightarrow a\mathscr{F}_H b \wedge d\mathscr{T}_H b \wedge a\mathscr{B}_H d \wedge d \prec_H b$
$\rightarrow b \preccurlyeq_H d \wedge d \prec_H b$
$\rightarrow b \asymp_H d.$ (4)

$a\mathscr{F}_H b \wedge c\mathscr{F}_H d \wedge a \asymp_H c \wedge b \asymp_H d \rightarrow b \asymp_H d.$ (5)

(3)–(5), T2.(37,40), **D**IV4.3: $a\mathscr{F}_H b \wedge c\mathscr{F}_H d \wedge a \asymp_H c \wedge b\mathscr{T}_H d$
$\rightarrow b \asymp_H d.$ (6)

(6), T2.32: $a\mathscr{F}_H b \wedge c\mathscr{F}_H d \rightarrow .a \asymp_H c \wedge b\mathscr{T}_H d \rightarrow b \asymp_H d.$ (7)

Similarly: $a\mathscr{F}_H b \wedge c\mathscr{F}_H d \rightarrow .a\mathscr{T}_H c \wedge b \asymp_H d \rightarrow a \asymp_H c.$ (8)

(7), (8) : **T**.

If a first signal departs before another, it should arrive before it, and vice versa. This is deduced in **T**6 from the postulate:

P1. $a\mathscr{F}_H b \wedge c\mathscr{F}_H d \rightarrow : a \prec_H c \wedge b\mathscr{T}_H d \rightarrow \sim d \prec_H b.$
$\wedge . b \prec_H d \wedge a\mathscr{T}_H c \rightarrow \sim c \prec_H a,$

which states that if $a\mathscr{F}_H b$ and $c\mathscr{F}_H d$, then if a precedes c, d does not precede b, and if b precedes d, c does not precede a.

T6. $a\mathscr{F}_H b \wedge c\mathscr{F}_H d \rightarrow : a \prec_H c \wedge b\mathscr{T}_H d \rightarrow b \prec_H d.$
$\wedge\ .\ b \prec_H d \wedge a\mathscr{T}_H c \rightarrow a \prec_H c.$

Proof. **P**IV4.1: $a\mathscr{F}_H b \wedge c\mathscr{F}_H d \wedge a \prec_H c \wedge b\mathscr{T}_H d$
$\rightarrow a\mathscr{F}_H b \wedge c\mathscr{F}_H d \wedge \sim a \asymp_H c \wedge b\mathscr{T}_H d \wedge a \prec_H c$
T5, **P**1 $\rightarrow \sim b \asymp_H d \wedge \sim d \prec_H b \wedge b\mathscr{T}_H d$
DIV4.3 $\rightarrow b \prec_H d.$ (1)
Similarly: $a\mathscr{F}_H b \wedge c\mathscr{F}_H d \wedge b \prec_H d \wedge a\mathscr{T}_H c \rightarrow a \prec_H c.$ (2)
(1),(2):**T**.

The following postulate states that a direct first signal never arrives later than a first signal *chain* that departs with it:

P2. $a\mathscr{F}_H b \wedge a\mathscr{F}_H c \wedge c\mathscr{F}_H d \wedge b\mathscr{T}_H d \rightarrow b \preccurlyeq_H d.$

Finally, since a first signal has a finite maximum speed of propagation, in a round trip its arrival is always *after* its departure (**P**3 Fig. 2).

P3. $a\mathscr{F}_H b \wedge b\mathscr{F}_H c \wedge a\mathscr{T}_H c \rightarrow a \prec_H c.$
$\wedge (\forall x).a \preccurlyeq_H x \preccurlyeq_H c \rightarrow \sim .x\mathscr{T}_H b \vee x\mathscr{F}_H b \vee b\mathscr{F}_H x.$

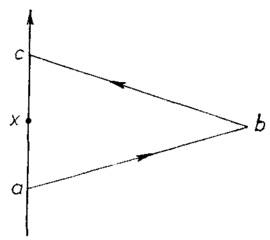

Fig. 2.
Round trip of a first signal

In effect, **P**3 states that if a first signal departs from a, is reflected at b, and returns at c, then a preceeds c; moreover no event x between a and c can be connected in any way, with event b.

This is the heart of the theory of relativity, for if this were not true, the maximum speed of propagation would be infinite and simultaneity would be absolute not relative (Reichenbach [1958] § 22). To be more specific, as long as there exists a set of events between a and c, any of these events can be chosen to be simultaneous with b, and an event that is simultaneous to one observer may not be simultaneous to another observer using the same simultaneity convention. But if a coincides with c, then a (or c) becomes the only event that can be considered simultaneous with *b*, and simultaneity becomes the same for every observer. In this case the round trip time of the first signal is zero, the speed of the signal is infinite, and the light cone degenerates into a single plane.

VI. CLOCKS AND TIME INTERVALS

1. Introduction

The last two chapters are concerned with the topological structure of the class of events (without regard to the length of time intervals). In this chapter, clocks are used to define a time metric, which forms the basis of time geometry.

A *clock* is a physical system that generates and counts a sequence of events at a particle, called the *output particle*; the output particle of a watch is the tip of its minute hand, and of an atomic clock is the end of its output lead. To measure the time interval between two events a, b by a clock, a, b must occur on the world line of the output particle of the clock, and the number of events generated by the clock between a and b is the value of the interval between them.

If no limitation is imposed on the type of clock used, the description of physical phenomena obtained in this way, is at the mercy of the capricious behaviour of the clock; if the period of a pendulum measured by such a clock is found to vary, is the variation due to the pendulum or to the clock itself? Consequently, if a reasonable description of physical phenomena is desired, it is necessary to impose some limitations on the clocks used.

In the general theory of relativity, there is a great deal of freedom in the choice of a (coordinate) clock. However, the rates of all clocks are specified relative to coincident standard clocks (by the metric coefficient g_{00}), and without standard clocks it is not even possible to measure the space-time line element ds.

How are we to select a clock? There seem to be two different procedures, both of which are used in practice: (1) Try different clocks, and find out the kind of description of physical phenomena they lead to. The clock that gives the description which is considered by respectable physicists to be the simplest and most natural, is the clock that is crowned to be a standard clock. All other clocks are judged by comparison with it. Until the revolution of the atomic clocks, the king has been the astronomical clock. It has been a benevolent ruler and rendered much service to physics, but alas times do change, and the collective leadership of atomic clocks is taking over. There are very good reasons for this peaceful revolution, which will become apparent before this section is over. (2) Intercompare many clocks, and select the ones that exhibit the least relative instability. Comparison of two clocks means the coincidence of the output particles of the two clocks, and the measurement of the frequency ratio of the number of events generated by one clock to the number of events generated by the other clock between any two end events. The clocks that have the most stable ratio, i.e., the ratio with the smallest amount of fluctuation, are considered to be the clocks with the highest relative stability. This is precisely the procedure used in the selection of atomic time standards. Clearly, the more clocks that agree with each other, the better we feel about their selection.

Which type of time standard is better, the solitary type selected by the first procedure or the comparison type selected by the second procedure?

There is no getting around the fact that the solitary type standard is arbitrary. It cannot be justified by any theory, because such a theory must involve the concept of a time measure, which depends upon the existence of a standard, which we are trying to justify. The only justification for a solitary type standard is that the description of nature it leads to is acceptable. Unfortunately, what is acceptable at one time may not be acceptable at a later time, due to advances in knowledge and technology. For instance, using quartz crystal clocks, and more recently atomic clocks, the U.S. National Bureau of Standards routinely measures the fluctuations in the spin angular frequency of the earth, which used to be assumed a constant. Such a measurement amounts to assuming that the observed fluctuations are due to the earth and not the quartz clocks. Any 'corrections' included in the definition of the standard must be considered as part of the definition, and not as justified by a theory.

The relative instability of two clocks is a specific quantitative parameter that is of direct significance to the measurement of time intervals by different clocks. Consequently there is no arbitrariness in deciding which clocks have the least relative instability; in contrast to the subjective, far more difficult judgement that must be made in deciding which solitary type clock leads to the most natural and simplest physical laws. Moreover the use of several clocks in setting up the standard, reduces considerably the peculiarities of any particular clock.

The above comments plus the fact that comparison type atomic standards are several orders of magnitude more stable than the astronomical standard, leave no doubt as to which type of standard is preferable, both in principle and practice. In this chapter we shall give a thorough treatment of both the definition and use of the comparison type standard (Basri [1965] Sec. 2).

The subject of time standards is quite complex, and is in an active state at present, because of recent developments concerning atomic standards and related systems. The definition of the astronomical standard involves detailed knowledge of classical mechanics, astronomy, and properties of the earth, whereas atomic standards involve quantum mechanics, modern physics, and electronics. An excellent summary of the situation with regard to the astronomical standard is given by Munk and MacDonald [1960], and a review of technological developments pertinent to atomic time standards up to September 1963 is given by Mockler [1964]. We shall not get involved here in the practical details of defining a time standard, but will concentrate only on the essential ideas of what constitutes a standard. A basic knowledge of mathematical statistics, such as that presented by Cramér [1946] and Brownlee [1960], is helpful in understanding Secs. 3 and 6.

2. Clocks

Regardless of what is the actual mechanism of a clock, the common feature of all clocks is that they have a minute hand that generates events by coinciding successively with the marks of a dial, and counts the events it generates by its angular position and the position of the hour hand around the dial. The events can be thought of as produced at one particle, namely the tip of the minute hand. For our purposes, it is even better to think of the tip as stationary and the dial as rotating.

In the case of a quartz or atomic clock, the tip of the minute hand is replaced by the end of the output lead, which delivers an electromagnetic signal. The signal can be used to operate an electronic counter that both exhibits and counts the events generated by the clock.

Thus any clock A can be thought of as a physical system that generates and counts a sequence of events $a_1, \ldots, a_n$ at a particle P, called the output particle. To be more specific, we take P to be part of $A (P \subset_H A)$ and assume that $a_1, \ldots, a_n$ are elements of the world line of $P (a_1, \ldots, a_n \in \mathscr{W}(P)_H)$ that are ordered by the relation $\prec_H (a_1 \prec_H \cdots \prec_H a_n)$. To distinguish these events on P from other events on P, we assume that any event of the set $\{a_1, \ldots, a_n\}$ must be either the appearance or disappearance of a particle X other than P which is part of A. Moreover, there are no other events associated with parts of A that occur between a_1 and a_n on the world line of P. For a watch A, the particles X are the marks around the dial that disappear and reappear as the tip P of the minute hand sweeps by them.

D1. $$(a_1, \ldots, a_n \mathscr{S}eq\, P, A)_H \quad \textit{for}$$
$$a_1, \ldots, a_n \in \mathscr{W}(P)_H \wedge a_1 \prec_H \ldots \prec_H a_n \wedge (\exists\, X_1, \ldots, X_n) X_1, \ldots, X_n \neq P$$
$$\wedge X_1, \ldots, X_n \subset_H A \wedge (a_1 \mathscr{A}_H X_1 \vee X_1 \mathscr{D}_H a_1) \wedge \ldots$$
$$\wedge (a_n \mathscr{A}_H X_n \vee X_n \mathscr{D}_H a_n)$$
$$\wedge (\forall u). u \in \mathscr{W}(P)_H \wedge (\exists\, X) X \neq P \wedge X \subset_H A \wedge (u \mathscr{A}_H X \vee X \mathscr{D}_H u)$$
$$\wedge a_1 \preccurlyeq_H u \preccurlyeq_H a_n \rightarrow u \in \{a_1, \ldots, a_n\}.$$

'$(a_1, \ldots, a_n \mathscr{S}eq\, P, A)_H$' reads '$a_1, \ldots, a_n$ *is a sequence of events at P associated with A*'.

Next we define a clock (not yet standard) to be an object A having an output particle $P(P \subset_H A)$ such that given any event u on P, and any positive integer k, there exists a sequence of events $x_1, \ldots, x_{k+1}$ at P associated with A, such that u occurs between x_1 and $x_2 (x_1 \preccurlyeq_H u \prec_H x_2)$. This insures that a clock can generate at P a sequence of events starting at any time and of any length. Moreover, for any two events v, w on P such that v precedes $w (v \prec_H w)$, there exists a positive integer n and events $y_1, \ldots, y_{n+1}$ such that

$$(y_1, \ldots, y_{n+1} \mathscr{S}eq\, P, A)_H,\ y_1 \preccurlyeq_H v \prec_H y_2, \text{ and } y_n \preccurlyeq_H w \prec_H y_{n+1},$$

i.e., there exists a sequence of events on P that brackets v and w.

The output particle is unique, in the sense that if X, Y are two particles on which occur sequences of events associated with A, then the world lines of X, Y are the same $(X = Y \vee X \underset{H}{\rightarrowtail} Y \vee Y \underset{H}{\rightarrowtail} X)$.

D2. $$(A \mathscr{C}\ell P)_H \quad \textit{for} \quad P \subset_H A \wedge (\forall u, k). u \in \mathscr{W}(P)_H \wedge k \in \mathscr{I}$$
$$\rightarrow (\exists\, x_1, \ldots, x_{k+1})(x_1, \ldots, x_{k+1} \mathscr{S}eq\, P, A)_H \wedge x_1 \preccurlyeq_H u \prec_H x_2 :$$
$$\wedge (\forall v, w). v, w \in \mathscr{W}(P)_H \wedge v \prec_H w \rightarrow (\exists\, n) n \in \mathscr{I} \wedge (\exists\, y_1, \ldots, y_{n+1})$$
$$(y_1, \ldots, y_{n+1} \mathscr{S}eq\, P, A)_H \wedge y_1 \preccurlyeq_H v \prec_H y_2 \wedge y_n \preccurlyeq_H w \prec_H y_{n+1} :$$
$$\wedge (\forall u_1, \ldots, u_m, v_1, \ldots, v_n, X, Y). (u_1, \ldots, u_m \mathscr{S}eq\, X, A)_H$$
$$\wedge (v_1, \ldots, v_n \mathscr{S}eq\, Y, A)_H \rightarrow X = Y \vee X \underset{H}{\rightarrowtail} Y \vee Y \underset{H}{\rightarrowtail} X.$$

'$(A \mathscr{C}\ell P)_H$' reads '*A is a clock with output particle P*'. $\mathscr{I}$ is the class of positive integers.

3. Relative instability

To define a standard clock, we need to know first the meaning of 'comparison of two clocks'. We say that *clocks A, B are compared, starting with event a* *of A and b of B, and to m events of A there correspond n events of B,* $[(A,a,m\mathscr{V}B,b,n)_H]$ if and only if the output particles X, Y of A, B coincide into a single particle $Z(X,Y\circledS\cdot_H Z)$, and there exist events $u_1,\ldots,u_{m+1}$, $v_1,\ldots,v_{n+1}$ such that $a,u_1,\ldots,u_{m+1}$ and $b,v_1,\ldots,v_{n+1}$ are sequences associated with A and B, respectively, b is between a and $u_1 (a\preccurlyeq_H b\prec_H u_1)$, and v_n is between u_m and $u_{m+1} (u_m\preccurlyeq_H v_n\prec_H u_{m+1})$. By allowing b and v_n to fall within two successive events of A, instead of letting b coincide with a and v_n with u_m, we are in effect neglecting time intervals of duration less than the interval between two successive events of A. This error can be minimized by increasing the number of events of A that correspond to a certain number of events of B.

***D*1.** $(A, a, m\mathscr{V}B, b, n)_H$ *for* $m, n\in\mathscr{I}$
$\wedge(\exists X, Y, Z)(A\mathscr{C}\ell X)_H\wedge(B\mathscr{C}\ell Y)_H\wedge X, Y\circledS\cdot_H Z$
$\wedge(\exists u_1,\ldots,u_{m+1},v_1,\ldots,v_{n+1})(a,u_1,\ldots,u_{m+1}\,\mathscr{S}eq\,Z,A)_H$
$\wedge(b,v_1,\ldots,v_{n+1}\,\mathscr{S}eq\,Z,B)_H\wedge a\preccurlyeq_H b\prec_H u_1\wedge u_m\preccurlyeq_H v_n\prec_H u_{m+1}.$

Notice that the symbol '$(A,a,m\mathscr{V}B,b,n)_H$' is not symmetric in A and B. This causes no difficulty, since in the definition of a standard clock (***D*4.1**) we symmetrize over all possible pairs $\langle A,B\rangle$ and $\langle B,A\rangle$.

The comparison defined in ***D*1** constitutes only a single measurement of the frequency ratio $f(=n/m)$. Many more measurements of f are necessary before it is possible to estimate the mean and standard deviation of f. The latter is identified with the relative instability.

Consider an experiment in which k sequences of length m events associated with clock A are compared with k sequences of clock B, with the results $n_1,\ldots,n_k$ for the lengths of the B-sequences. In order to insure that the frequency ratios $f_1,\ldots,f_k (f_i=n_i/m)$ are independent identically distributed random variables, we take m long enough to destroy correlations between the beginning and end of each sequence. For the cesium 5Mc/s clocks at the National Bureau of Standard in Boulder, Colorado, $m\approx 10^6$. Moreover, to destroy correlations between the end of one sequence and the beginning of the succeeding one, we separate the k sequences by $k-1$ gaps of suitable length j(Fig. 3). The total number of events of A required for one experiment is:

$$km+(k-1)j+1=k(m+j)-j+1\,.$$

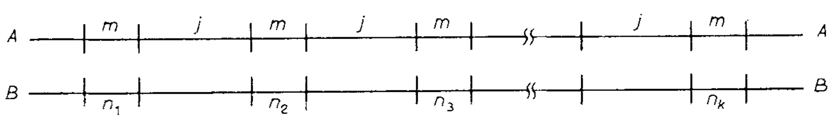

Fig. 3. Comparison of clocks

From the data $f_1, \ldots, f_k$, we can calculate the *sample* moments:

$$v_\alpha = \frac{1}{k} \sum_{i}^{k} (f_i)^\alpha, \qquad \bar{f} = v_1, \tag{1}$$

$$S_\alpha = \frac{1}{k} \sum_{i}^{k} (f_i - \bar{f})^\alpha, S = \sqrt{S_2}, \tag{2}$$

where $\bar{f}$ is the sample mean, and S is the sample standard deviation.

If the value of a random variable X ranges over the class of real numbers, and $\mathbf{P}\{F\}$ is the probability of F, then the function $P(x)$ defined by (Cramér [1946] p. 167):

$$\mathbf{P}\{x < X < x + \mathrm{d}x\} = P(x)\,\mathrm{d}x, \tag{3}$$

is called the *probability density function* (PDF) of the random variable X. Since the probability that X has some value is 1, we must have the normalization condition:

$$\int_{-\infty}^{+\infty} P(x)\,\mathrm{d}x = 1. \tag{4}$$

Let $f(x)$ be a function integrable over $(-\infty, \infty)$. The integral (Cramér [1946] p. 170):

$$\langle f(X) \rangle = \int_{-\infty}^{\infty} f(x) P(x)\,\mathrm{d}x, \tag{5}$$

is called the *true* (*population*) *mean* or *expectation* value of the random variable $f(X)$.†

With the help of (5), the (*true*) *moments* of f_i are defined by:

$$\mu = \langle f_i \rangle, \quad \mu_\alpha = \langle (f_i - \mu)^\alpha \rangle, \quad (\alpha > 1) \tag{6}$$

$$\sigma = \sqrt{\mu_2}, \tag{7}$$

where μ is called the *mean* and σ the *standard deviation*. We are assuming here f_i has the same PDF for all values of i, i.e., $f_1, \ldots, f_n$ are identically distributed.

If the PDF of f_i is known, it is possible to calculate confidence limits for μ and σ. For example, if the distribution is normal (Cramér [1946] p. 208), i.e.,

$$P(x) = (2\pi)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}x^2), \tag{8}$$

† For a discrete variable, such as f_i, Eq. (3) becomes '$\mathbf{P}\{X = x_n\} = P(n)$', Eq. (4) becomes '$\Sigma_{n=0}^{\infty} P(n) = 1$', and Eq. (5) becomes: $\langle f(X) \rangle = \Sigma_{n=0}^{\infty} f(x_n)\, P(n)$.

then (Cramér [1946] pp. 382, 387, 518; Brownlee [1960] pp. 219, 229):

$$\mathbf{P}\{\bar{f} - \mathrm{t}_{\frac{1}{2}p} S/\sqrt{k-1} < \mu < \bar{f} + \mathrm{t}_{\frac{1}{2}p} S/\sqrt{k-1}\} = 1 - p\,, \tag{9}$$

$$\mathbf{P}\{k\,S^2/\chi^2_{1-\frac{1}{2}p} < \sigma^2 < k\,S^2/\chi^2_{\frac{1}{2}p}\} = 1 - p\,, \tag{10}$$

where t_p and χ^2_p are the p percent values of Student's and χ^2 distributions of $k-1$ degrees of freedom (Cramér [1946] pp. 233–241). The expressions inside the curly brackets are of the form '$X<C<Y$', where X, Y are random variables, called the confidence limits, and C is a constant. The sentence '$\mathbf{P}\{X<C<Y\}=1-p$' states that the probability that the confidence interval (X, Y) covers the number C is $1-p$, i.e., in a large number of trials, the fraction of times that C falls within (X, Y) is $1-p$; it does not state that the probability that C falls within (X, Y) is $1-p$, because this probability is either one or zero, since C either does or does not lie within (X, Y).

If the PDF of f_i is not known, we can still calculate confidence limits for μ and σ provided the number k is large enough for the *central limit theorem* (Cramér [1946] p. 213) to be applicable. According to this theorem, if μ and σ exist, then regardless of the PDF,

$$\lim_{k\to\infty} \mathbf{P}\{\mu - \mathfrak{g}_p\sigma/\sqrt{k} < \bar{f} < \mu + \mathfrak{g}_p\sigma/\sqrt{k}\} = 1 - p\,, \tag{11}$$

where $\mathfrak{g}_p$ is the p percent value of a normal (Gaussian) deviate (Cramér [1946] p. 211), i.e.,

$$\mathbf{P}\{|f_i - \mu| < \mathfrak{g}_p\sigma\} = (2\pi)^{-\frac{1}{2}} \int_{-\mathfrak{g}_p}^{+\mathfrak{g}_p} \exp\left(-\tfrac{1}{2}x^2\right) \mathrm{d}x = 1 - p\,. \tag{12}$$

Moreover, if μ_4 also exists, then (Cramér [1946] p. 348)

$$\lim_{k\to\infty} \mathbf{P}\{\sigma^2 - \mathfrak{g}_p[(\mu_4 - \sigma^4)/k]^{\frac{1}{2}} < S^2 < \sigma^2 + \mathfrak{g}_p[(\mu_4 - \sigma^4)/k]^{\frac{1}{2}}\} = 1 - p\,. \tag{13}$$

In effect, (11) and (13) state that for large enough k, $\bar{f}$ and S^2 behave as though they were normally distributed.

Confidence intervals can be derived from (11) and (13) (Cramér [1946] pp. 366, 511), and the result is that for sufficiently large k ($k \approx 10$ is usually large enough (Cramér [1946] p. 202)),

$$\mathbf{P}\{\bar{f} - \mathfrak{g}_p S/\sqrt{k-1} < \mu < \bar{f} + \mathfrak{g}_p S/\sqrt{k-1}\} \approx 1 - p\,, \tag{14}$$

$$\mathbf{P}\{S^2 - \mathfrak{g}_p[(S_4 - S^4)/(k-1)]^{\frac{1}{2}} < \sigma^2 < S^2 + \mathfrak{g}_p[(S_4 - S^4)/(k-1)]^{\frac{1}{2}}\} \approx 1 - p\,. \tag{15}$$

Notice that S_4 is needed in (15), but not in (10); a small price to pay for not knowing the PDF.

Going back to our experiment, since $f_i = n_i/m$, we expect:

$$\mu = \langle f_i \rangle = N/M \tag{16}$$

to be the true frequency ratio of the clocks, and σ to be the relative instability. Thus the sentence '*The clocks A, B have the frequency ratio M/N and relative instability* $\sigma[(A, B\ \mathscr{C}\ell\circ\ M/N, \sigma)_H]$' means: Clocks A, B can be compared at any time, and for as long as desired, i.e., for any event u of A and any integer m, there exists an event v of B and integer n such that $(A, u, m\mathscr{V}\ B, v, n)_H$. Moreover, if $f_1, \ldots, f_k$ are the results of the experiment described after **D**1, and k is large enough for the central limit theorem to hold, then [Eqs. (14), (15)] the probabilities that the confidence limits $\bar{f} \pm \mathfrak{g}_p S/\sqrt{(k-1)}$ include the mean N/M, and $S^2 \pm \mathfrak{g}_p[(S_4 - S^4)/(k-1)]^{\frac{1}{2}}$ include the variance σ^2, are both equal to $1-p$. More precisely, for any positive real number ε, there exist three integers q, r, s such that for any integers k (number of trials), m (length of comparison), and j (length of gap), $m \geq s$, $j \geq q$ (to make n_i independent); and $k \geq r$ (to make the central limit theorem applicable) imply: if $u_1, \ldots, u_{k(m+j)-j+1}$ is a sequence of events of A; X, Y are the output particles of A, B; and $f_1, \ldots, f_k (f_i = n_i/m)$ are the results of the comparisons $(A, u_1, m\mathscr{V}\ B, v_1, n_1)_H$, $(A, u_{(m+j)+1}, m\mathscr{V}\ B, v_2, n_2)_H, \ldots, (A, u_{(k-1)(m+j)+1}, m\mathscr{V}\ B, v_k, n_k)_H$; then for any real number p between 0 and 1, the probability that the confidence limits $\bar{f} \pm \mathfrak{g}_p S/\sqrt{(k-1)}$ include M/N, and the probability that $S^2 \pm \mathfrak{g}_p[(S_4 - S^4)/(k-1)]^{\frac{1}{2}}$ include σ^2, are both equal to $1-p$ within an error ε.

The following definition is just a formal expression of the above ideas ($\mathscr{R}$ is the class of real numbers):

D2. $(A, B\ \mathscr{C}\ell\circ\ M/N, \sigma)_H$ *for*

$$\begin{aligned}
& M, N \in \mathscr{I} \wedge \sigma \in \mathscr{R} \wedge (\exists X, Y)(A\ \mathscr{C}\ell\ X)_H \wedge (B\ \mathscr{C}\ell\ Y)_H \\
& \wedge (\forall u, m).m \in \mathscr{I} \wedge u \in \mathscr{W}(X)_H \rightarrow (\exists v, n)(A, u, m\mathscr{V}\ B, v, n)_H : \\
& \wedge (\forall \varepsilon).\varepsilon \in \mathscr{R} \wedge \varepsilon > 0 \rightarrow (\exists q, r, s) q, r, s \in \mathscr{I} \\
& \wedge (\forall j, k, m).j, k, m \in \mathscr{I} \wedge j \geq q \wedge k \geq r \wedge m \geq s \\
& \rightarrow (\forall u_1, \ldots, u_{k(m+j)-j+1}, v_1, n_1, \ldots, v_k, n_k). \\
& \quad (u_1, \ldots, u_{k(m+j)-j+1}\ \mathscr{S}eq\ X, A)_H \wedge (A, u_1, m\mathscr{V}\ B, v_1, n_1)_H \\
& \quad \wedge (A, u_{(m+j)+1}, m\mathscr{V}\ B, v_2, n_2)_H \wedge \ldots \\
& \quad \wedge (A, u_{(k-1)(m+j)+1}, m\mathscr{V}\ B, v_k, n_k)_H \\
& \rightarrow (\forall p).p \in \mathscr{R} \wedge 0 < p < 1 \rightarrow \\
& |\mathbf{P}\{S_2 - \mathfrak{g}_p[(S_4 - S^4)/(k-1)]^{\frac{1}{2}} < \sigma^2 < S_2 + \mathfrak{g}_p[(S_4 - S^4)/(k-1)]^{\frac{1}{2}}\} \\
& - (1-p)| < \varepsilon \\
& \wedge |\mathbf{P}\{\bar{f} - \mathfrak{g}_p S/\sqrt{(k-1)} < N/M < \bar{f} + \mathfrak{g}_p S/\sqrt{(k-1)}\} - (1-p)| < \varepsilon.
\end{aligned}$$

4. Standard clocks

Even if the relative instability of two clocks is small enough to suit our purposes, there is still the possibility that their rates may change together in the same way, so that we could not detect this change by comparing them with each other. Two pendulum clocks having the same thermal coefficient of expansion, and identical twin quartz crystal clocks cut from the same blank and aging the same way can exhibit this behavior. Fortunately, the delinquincy of such pairs can always be detected by comparing them with clocks of different upbringing. Once it is detected, something can always be done to cope with it, such as by controlling the environment or incorporate servomechanisms that keep them on the right path. This can be done without inhibition by fear of circularity for using physical theory to make corrections of instruments that themselves determine the form of the theory. The reason is that ultimately the clocks are treated as black boxes whose merits are judged by how they behave under comparison, and not by how they are constructed.

Although two clocks may be constructed independently from different materials, there is still a chance that they may stray off together due to statistical fluctuations. Both this effect and the previous one due to similar construction, can be practically eliminated by increasing the number of clocks that must agree with each other.

We therefore define a standard clock as an element of a set of at least $\mathfrak{r}_T$ clocks, any two of which have frequency ratio 1 and relative instability less than a certain amount σ_T. The numbers $\mathfrak{r}_T, \sigma_T$ are agreed upon by a standards committee; $\mathfrak{r}_T$ should be at least 3, and σ_T for atomic clocks is of the order 10^{-12}.

***D*1.** $$(A\mathscr{S}\mathscr{C}P)_H \quad \textit{for} \quad (A\mathscr{C}\ell P)_H \wedge (\exists\, r) r \varepsilon \mathscr{I} \wedge 2 < r \leq \mathfrak{r}_T \wedge (\exists\, X_1, \ldots, X_r) \neq (X_1, \ldots, X_r) \wedge P \in \{X_1, \ldots, X_r\} \wedge (\forall\, X, Y). X \neq Y \wedge X, Y \in \{X_1, \ldots, X_r\} \rightarrow (\exists\, \sigma)(X, Y\, \mathscr{C}\ell\mathscr{I}\, 1/1, \sigma)_H \wedge \sigma \leq \sigma_T.$$

'$(A\mathscr{S}\mathscr{C}P)_H$' reads '$A$ is a *standard clock* (SC) having output particle P'. The number $\mathfrak{r}_T$ is called the *reliability index*, because the larger $\mathfrak{r}_T$ is, the more reliable we feel our standard is. The sentence '$\neq(X_1, \ldots, X_r)$' means 'all the X's are different from each other'.

***D*2.** $$\mathscr{S}\mathscr{C}(A)_H \quad \textit{for} \quad (\exists\, X)(A\mathscr{S}\mathscr{C}X)_H.$$

***D*3.** $$a_1, \ldots, a_n \mathscr{S}\mathscr{C}_H A \quad \textit{for} \quad (\exists\, X)(A\mathscr{S}\mathscr{C}X)_H \wedge a_1, \ldots, a_n \in \mathscr{W}(X)_H.$$

Two SC's may not belong to the same family, and thus their frequency ratio may be different than unity. The following definition helps us express the idea that the SC's $A_1, ..., A_n$ belong to the same family.

*D*4. $\mathscr{E}\mathscr{S}\mathscr{C}(A_1, ..., A_n)_H$ *for* $\mathscr{S}\mathscr{C}(A_1)_H \wedge ... \wedge \mathscr{S}\mathscr{C}(A_n)_H$
$\wedge \neq (A_1, ..., A_n) \wedge (\forall X, Y).X \neq Y \wedge X, Y \in \{A_1, ..., A_n\}$
$\rightarrow (\exists \sigma)(X, Y \mathscr{C}\ell_{\mathscr{I}}\, 1/1, \sigma)_H \wedge \sigma \leq \sigma_T$.

'$\mathscr{E}\mathscr{S}\mathscr{C}(A_1, ..., A_n)_H$' reads '$A_1, ..., A_n$ are *equivalent standard clocks*'.

There are four different interactions known in physics: strong, electromagnetic, weak and gravitational. Typical examples of these interactions are, respectively, the nuclear force, the force between charged particles, the force responsible for β-decay of a nucleus, and the force between two masses. The order of magnitude of these forces is roughly in the ratios $1:10^{-3}:10^{-14}:10^{-40}$. A question which has been of interest in the last few years is: suppose we have two families of SC's, each operating on the basis of a different kind of interaction, will a member of one family agree with a member of the other family? Dicke ([1964] p. 14) and Finzi [1961] give an affirmative answer for strong and electromagnetic interactions, but leave the question open for weak and gravitational interactions.

*P*1. $\mathscr{E}\mathscr{S}\mathscr{C}(A, B)_H \wedge \mathscr{E}\mathscr{S}\mathscr{C}(B, C)_H \rightarrow \mathscr{E}\mathscr{S}\mathscr{C}(A, C)_H$.

5. Time metric

A class $\mathscr{C}$ is called a *metric space* if to every pair of elements of $\mathscr{C}$ is assigned a real number $\tau(a,b)$ having the following properties:

(i) $\tau(a, b) \geq 0$,
(ii) $\tau(a, b) = 0 \leftrightarrow a = b$,
(iii) $\tau(a, b) = \tau(b, a)$,
(iv) $\tau(a, c) \leq \tau(a, b) + \tau(b, c)$.

The function τ is called a *metric in the space* $\mathscr{C}$.

With the help of SC's we introduce a measure on $\mathscr{E}_H$ that has the same basic properties as a metric. A time interval between two events a, b is measured by an SC, A, by letting a, b occur on the output particle of A, and counting the events generated by A between a and b.

*D*1. $\tau(A; a, b; n)_H$ *for* $n \in \mathscr{I} \wedge a, b \mathscr{S}\mathscr{C}_H A$
$\wedge (\exists X, u_1, ..., u_{n+2})(u_1, ..., u_{n+2}\ \mathscr{S}eq\ X, A)_H$

$\wedge :u_1 \preccurlyeq_H a \prec_H u_2 \wedge u_{n+1} \preccurlyeq_H b \prec_H u_{n+2}.$
$\vee .u_1 \preccurlyeq_H b \prec_H u_2 \wedge u_{n+1} \preccurlyeq_H a \prec_H u_{n+2}.$

'$\tau(A;a,b;n)_H$' reads '*The time interval between a,b by clock A is n*'.

In order to define the value of the time interval, we need to prove first that this value is unique. In *T*1 we prove there exists at least one value, and in *T*2 that there exists at most one value.

***T*1.** $a,b\mathscr{S}\mathscr{C}_H A \rightarrow (\exists n)\tau(A;a,b;n)_H.$

Proof. ***D*4.(3,1): Ant**$\rightarrow(\exists X)(A\mathscr{C}\ell X)_H$
***D*2.2**$\rightarrow(\exists n)n\in\mathscr{I} \wedge (\exists u_1,\ldots,u_{n+2})(u_1,\ldots,u_{n+2}\,\mathscr{S}eq\,X,A)_H$
$\wedge :u_1 \preccurlyeq_H a \prec_H u_2 \wedge u_{n+1} \preccurlyeq_H b \prec_H u_{n+2}.$
$\vee .u_1 \preccurlyeq_H b \prec_H u_2 \wedge u_{n+1} \preccurlyeq_H a \prec_H u_{n+2}$
***D*1$\rightarrow$Con.**

***T*2.** $(\exists^1 n)\tau(A;a,b;n)_H.$

Proof. ***D*1, *D*2.(1.2), *T*V2.9, *T*IV(4.5, 4.3):**
$\tau(A;a,b;m)_H \wedge \tau(A;a,b;n)_H$
$\rightarrow(\exists X,u_1,\ldots,u_{m+2},v_1,\ldots,v_{n+2}).$
$(u_1,\ldots,u_{m+2}\,\mathscr{S}eq\,X,A)_H \wedge (v_1,\ldots,v_{n+2}\,\mathscr{S}eq\,X,A)_H$
$\wedge (u_1 \preccurlyeq_H a \prec_H u_2 \prec_H \cdots \prec_H u_{m+1} \preccurlyeq_H b \prec_H u_{m+2}$
$\wedge v_1 \preccurlyeq_H a \prec_H v_2 \prec_H \cdots \prec_H v_{n+1} \preccurlyeq_H b \prec_H v_{n+2}.$
$\vee .u_1 \preccurlyeq_H b \prec_H u_2 \prec_H \cdots \prec_H u_{m+1} \preccurlyeq_H a \prec_H u_{m+2}$
$\wedge v_1 \preccurlyeq_H b \prec_H v_2 \prec_H \cdots \prec_H v_{n+1} \preccurlyeq_H a \prec_H v_{n+2})$
***T*V2.7, *T*IV4.3**$\rightarrow(\exists X,u_1,\ldots,u_{m+2},v_1,\ldots,v_{n+2}).$
$(u_1,\ldots,u_{m+2}\,\mathscr{S}eq\,X,A)_H \wedge (v_1,\ldots,v_{n+2}\,\mathscr{S}eq\,X,A)_H$
$\wedge (u_1 \preccurlyeq_H v_1 \prec_H u_2 \vee v_1 \preccurlyeq_H u_1 \prec_H v_2)$
$\wedge u_1 \prec_H v_2 \prec_H \cdots \prec_H v_{n+1} \prec_H u_{m+2}$
$\wedge (u_{m+1} \prec_H v_{n+2} \preccurlyeq_H u_{m+2} \vee v_{n+1} \prec_H u_{m+2} \preccurlyeq_H v_{n+2})$
***D*2.1**$\rightarrow(\exists u_1,\ldots,u_{m+2},v_1,\ldots,v_{n+2}).\{u_1,\ldots,u_{m+2}\}=\{v_1,\ldots,v_{n+2}\}$
$\rightarrow m=n.$ (1)
(1), D9: **T**.

***T*3.** $a,b\,\mathscr{S}\mathscr{C}_H A \rightarrow (\exists!\,n)\tau(A;a,b;n)_H.$

Proof. ***T*1,2; D10.**

***D*2.** $\tau_A(a,b)_H$ *for* $(\imath n)\tau(A;a,b;n)_H.$

$\tau_A(a,b)_H$ is *the* value of the time interval between *a* and *b* by clock *A*.

***T*4.** $a,b\mathscr{S}\mathscr{C}_H A \rightarrow .\tau_A(a,b)_H = n \leftrightarrow \tau(A;a,b;n)_H.$

Proof. ***T*3, T4.3, *D*2.**

We now proceed to prove properties of the measure $\tau_A(a,b)_H$ analogous to the metric properties (i)–(iv).

$T5.$ $\quad a,b\mathscr{S}\mathscr{C}_H A\rightarrow\tau_A(a,b)_H\geq 0.$

Proof. $\quad \boldsymbol{T1}:\mathbf{Ant}\rightarrow(\exists\, n)\,\tau(A;a,b;n)_H$

$\quad \boldsymbol{T4},\boldsymbol{D1}\ \rightarrow(\exists\, n)\,\tau_A(a,b)_H=n\wedge n\geq 0$

$\quad \text{T5.6}\quad\rightarrow\mathbf{Con}.$

Thus property (i) is satisfied.

In $\boldsymbol{T6}$ we prove that if a and b coincide then $\tau_A(a,b)_H=0$.

$T6.$ $\quad a,b\mathscr{S}\mathscr{C}_H A\wedge a\asymp_H b\rightarrow\tau_A(a,b)_H=0.$

Proof. $\quad \boldsymbol{T1},4:\mathbf{Ant}\rightarrow(\exists\, n)\,\tau_A(a,b)_H=n\wedge a\asymp_H b$

$\quad \boldsymbol{D1}\quad\rightarrow(\exists\, n)\,\tau_A(a,b)_H=n\wedge n+1=1$

$\quad \text{T5.6}\quad\rightarrow\mathbf{Con}.$

In $\boldsymbol{T7}$ we prove that if $\tau_A(a,b)_H=0$, then a,b occur between two successive events of clock A. We cannot prove that a and b are coincident, unless the events of A are so close together that it is no longer possible to resolve a and b.

$T7.$ $\quad a,b\mathscr{S}\mathscr{C}_H A\wedge\tau_A(a,b)_H=0$

$\quad\rightarrow(\exists\, X,u,v)(u,v\,\mathscr{S}\!eq\,X,A)_H\wedge u\preccurlyeq_H a,b\prec_H v.$

Proof. $\quad \boldsymbol{T4},\boldsymbol{D1}.$

The last two theorems take the place of property (ii). If the spacing between the events of clock A is taken small enough, then a,b in $\boldsymbol{T7}$ may be considered coincident, and property (ii) is satisfied with '$=$' replaced by '$\asymp_H$'.

The following theorem shows that the symmetry property (iii) is satisfied:

$T8.$ $\quad a,b\mathscr{S}\mathscr{C}_H A\rightarrow\tau_A(a,b)_H=\tau_A(b,a)_H.$

Proof. $\quad \boldsymbol{D1}:\tau(A;a,b;n)_H\leftrightarrow\tau(A;b,a;n)_H.$ (1)

$\quad (1),\text{T1.14},\boldsymbol{T4}:\mathbf{Ant}\rightarrow(\forall\, n).\tau_A(a,b)_H=n\leftrightarrow\tau_A(b,a)_H=n$

$\quad \text{T5.4}\quad\rightarrow\mathbf{Con}.$

In the next three theorems, property (iv) is also proved, and thus $\tau_A(a,b)_H$ is a metric aside from the slight deviation from (ii).

$T9.$ $\quad a,b,c\mathscr{S}\mathscr{C}_H A\wedge(a\prec_H b\prec_H c\vee c\prec_H b\prec_H a)$

$\quad\rightarrow\tau_A(a,c)_H=\tau_A(a,b)_H+\tau_A(b,c)_H.$

Proof. $\quad \boldsymbol{T1},\boldsymbol{D1}:\mathbf{Ant}\rightarrow(\exists\, X,u_1,\ldots,u_{m+2},v_1,\ldots,v_{n+2})$

$\quad (u_1,\ldots,u_{m+2}\mathscr{S}\!eq\,X,A)_H\wedge(v_1,\ldots,v_{n+2}\mathscr{S}\!eq\,X,A)_H$

$\wedge : u_1 \preccurlyeq_H a \prec_H \cdots \prec_H u_{m+1} \preccurlyeq_H b \prec_H u_{m+2}$
$\wedge\, v_1 \preccurlyeq_H b \prec_H \cdots \prec_H v_{n+1} \preccurlyeq_H c \prec_H v_{n+2}.$
$\vee .v_1 \preccurlyeq_H c \prec_H \cdots \prec_H v_{n+1} \preccurlyeq_H b \prec_H v_{n+2}$
$\wedge\, u_1 \preccurlyeq_H b \prec_H \cdots \prec_H u_{m+1} \preccurlyeq_H a \prec_H u_{m+2}.$

D2.1 $\rightarrow (\exists\, X, u_1, \ldots, u_{m+n+2})(u_1, \ldots, u_{m+n+2}\, \mathscr{S}eq\; X, A)_H$
$\wedge : u_1 \preccurlyeq_H a \prec_H u_2 \prec_H \cdots \prec_H u_{m+1} \preccurlyeq_H b$
$\prec_H u_{m+2} \prec_H \cdots \prec_H u_{m+n+1} \preccurlyeq_H c \prec_H u_{m+n+2}$
$\vee .u_1 \preccurlyeq_H c \prec_H u_2 \prec_H \cdots \prec_H u_{m+1} \preccurlyeq_H b$
$\prec_H u_{m+2} \prec_H \cdots \prec_H u_{m+n+1} \preccurlyeq_H a \prec_H u_{m+n+2}$

D1, *T*4 $\rightarrow (\exists\, i, j, k)\, \tau_A(a,b)_H = i \wedge \tau_A(b,c)_H = j$
$\wedge\, \tau_A(a,c)_H = k \wedge i + j = k$
$\rightarrow$ **Con.**

***T*10.** $a, b\mathscr{S}\mathscr{C}_H A \wedge \tau_A(a,b)_H = n \wedge a \asymp_H c \wedge b \asymp_H d \rightarrow \tau_A(c,d)_H = n.$

Proof. *T*4, **D**1.

***T*11.** $a, b, c\mathscr{S}\mathscr{C}_H A \rightarrow \tau_A(a,c)_H \leq \tau_A(a,b)_H + \tau_A(b,c)_H.$

Proof. **T**V2.7: $a, b, c \in \mathscr{W}(X)_H \rightarrow c \preccurlyeq_H a \preccurlyeq_H b \vee c \preccurlyeq_H b \preccurlyeq_H a$
$\vee\, a \preccurlyeq_H b \preccurlyeq_H c \vee b \preccurlyeq_H a \preccurlyeq_H c \vee a \preccurlyeq_H c \preccurlyeq_H b \vee b \preccurlyeq_H c \preccurlyeq_H a.$ (1)

*T*6, 9, 10: **Ant** $\wedge\, (c \preccurlyeq_H a \preccurlyeq_H b \vee c \preccurlyeq_H b \preccurlyeq_H a$
$\vee\, a \preccurlyeq_H b \preccurlyeq_H c \vee b \preccurlyeq_H a \preccurlyeq_H c) \rightarrow$ **Con.** (2)

*T*6, 9, 10: **Ant** $\wedge\, (a \preccurlyeq_H c \preccurlyeq_H b \vee b \preccurlyeq_H c \preccurlyeq_H a)$
$\rightarrow \tau_A(a,c)_H + \tau_A(c,b)_H = \tau_A(a,b)_H$

*T*5 $\rightarrow \tau_A(a,c)_H \leq \tau_A(a,b)_H$

*T*5 $\rightarrow$ **Con.** (3)

(1)–(3): **T**.

6. Comparison of time intervals

Due to statistical fluctuation of the period of a clock, the time metric $\tau_A(a,b)_H$ has a statistical error associated with it, which we calculate in this section. Once this error is estimated, it is possible to compare the values of different time intervals.

As a first step we define the mean of a time interval as the arithmetical mean of several values of the same interval measured simultaneously by different SC's with coincident output particles. More precisely, the mean $\langle \tau_A(a,b)_H \rangle$ is defined as the real number t satisfying the following condition: Given a positive ε and δ between 0 and 1, there exists an integer r such that for all integers k larger than r, and any equivalent standard clocks $X_1, \ldots, X_k$, if $\tau_{X_1}(a,b)_H = n_1, \ldots, \tau_{X_k}(a,b)_H = n_k$, then the probability that the difference between the mean

$$\bar{n} = \frac{1}{k}\sum_{i=1}^{k} n_i,$$

and the number t, is larger than ε, is $\leq \delta$, i.e., $\bar{n}$ converges *in probability* to t (Cramér [1946] p. 252). According to Khintchine's theorem (the weak law of large numbers) (Cramér [1946] pp. 253–254), this happens if n_i are independent identically distributed random variables with a finite mean.

***D*1.** $\langle \tau_A(a,b)_H \rangle$ *for* $(\imath t)(\forall \varepsilon, \delta).\varepsilon, \delta \in \mathscr{R}$
$\wedge \varepsilon > 0 \wedge 0 \leq \delta < 1 \rightarrow (\exists r) r \in \mathscr{I} \wedge (\forall k).\, k \in \mathscr{I} \wedge k \geq r$
$\rightarrow (\forall X_1, \ldots, X_k, n_1, \ldots, n_k).\, \mathscr{E}\mathscr{S}\mathscr{C}(X_1, \ldots, X_k)_H$
$\wedge \tau_{X_1}(a,b)_H = n_1 \wedge \ldots \wedge \tau_{X_k}(a,b)_H = n_k \rightarrow \mathbf{P}\{|\bar{n} - t| > \varepsilon\} \leq \delta.$

$\langle \tau_A(a,b)_H \rangle$ is called the *mean* of $\tau_A(a,b)_H$.

***D*2.** $\mathbf{V}\{\tau_A(a,b)_H\}$ *for* $\langle (\tau_A(a,b)_H - \langle \tau_A(a,b)_H \rangle)^2 \rangle.$

$\mathbf{V}\{\tau_A(a,b)_H\}$ is known as the *variance* of $\tau_A(a,b)_H$.

We now postulate the existence of the mean and the variance of a time interval.

***P*1.** $(\exists n) \tau_A(a,b)_H = n$
$\rightarrow (\exists!\, t) \langle \tau_A(a,b)_H \rangle = t \wedge \mathbf{V}\{\tau_A(a,b)_H\} \leq \kappa t.$

κ is related to σ_T (defined in ***D*4.1**) by a constant obtained as follows:
$\sigma_T = \sigma(f_i) = \sigma(n_i/m) = \sigma(n_i)/m = (\langle n_i \rangle^2/m)[\sigma(n_i)/\langle n_i \rangle^{\frac{1}{2}}] = (N/Mm)^{\frac{1}{2}} 2\kappa.$
The factor 2 multiplying κ stems from the fact that $\sigma(n_i)$ characterizes two clocks, whereas κ^2 is the variance per unit time of a single clock.

Rarely do we measure a time interval by more than one clock. We can set a lower bound on the probability that this single value deviates from the mean by a certain amount, with the help of ***P*1** and Tchebycheff's inequality (Cramér [1946] p. 182).

***T*1.** $\tau_A(a,b)_H = n \wedge \langle \tau_A(a,b)_H \rangle = t \rightarrow \mathbf{P}\{|n - t| < \kappa(t/p)^{\frac{1}{2}}\} \geq 1 - p.$

Proof. ***P*1**, Tchebycheff's inequality.

This states that the probability that $|n-t|$ is less than $\kappa(t/p)^{\frac{1}{2}}$, is larger than $1-p$.

***T*1** is valid regardless what is the PDF of n. If the PDF is known to be normal, then '$p^{-\frac{1}{2}}$' inside the square bracket in ***T*1** is replaced by the p percent value of a normal deviate '$\mathfrak{g}_p$'. For instance, for $p=10^{-2}$, $p^{-\frac{1}{2}}=10$, but $\mathfrak{g}_p \approx 3$.

In the same way, we can make a probability statement about the difference between two time intervals.

***T*2.** $$\tau_A(a,b)_H = n_1 \wedge \tau_B(c,d)_H = n_2 \wedge \mathscr{E}\mathscr{S}\mathscr{C}(A,B)_H \wedge \langle \tau_A(a,b)_H \rangle = t_1 \wedge \langle \tau_B(c,d)_H \rangle = t_2 \rightarrow \mathbf{P}\{|(n_1-n_2)-(t_1-t_2)| < \kappa[(t_1+t_2)/p]^{\frac{1}{2}}\} \geq 1-p.$$

Proof. ***P*1**, Tchebycheff's inequality, and the fact that the variance of the difference of two independent random variables, is the sum of the variances.

Since (t_1+t_2) is not known, it is desirable to estimate it by (n_1+n_2). This is possible, since $\kappa << 1$ and, as above,

$$\mathbf{P}\{|(n_1+n_2)-(t_1+t_2)| < \kappa[(t_1+t_2)/p]^{\frac{1}{2}}\} \geq 1-p.$$

***T*3.** $$\tau_A(a,b)_H = n_1 \wedge \tau_B(c,d)_H = n_2 \wedge \mathscr{E}\mathscr{S}\mathscr{C}(A,B)_H \wedge \langle \tau_A(a,b)_H \rangle = t_1 \wedge \langle \tau_B(c,d)_H \rangle = t_2 \wedge \sigma_{\mathrm{T}} << 1 \rightarrow \mathbf{P}\{|(n_1-n_2)-(t_1-t_2)| < \kappa[(n_1+n_2)/p]^{\frac{1}{2}}\} \approx 1-p.$$

Proof. ***T*1,2.**

With the help of ***T*3**, we can now give a reasonable definition of the equality of two time intervals $\tau(a,b)$ and $\tau(c,d)$. If $a \asymp_H b$ and $c \asymp_H d$, both intervals are zero by ***T*5.6**, and are therefore equal. They are also equal if $a \asymp_H c$ and $b \asymp_H d$, or $a \asymp_H d$ and $b \asymp_H c$. Otherwise, we can only state that if the difference between them is less than the error indicated in ***T*3**, the probability they are equal $(t_1 = t_2)$ is about $1-p$, i.e., they are equal at a $100p\%$ confidence level.

***D*3.** $$[a,b=c,d]_{p,H} \quad \textit{for}$$
$$a \asymp_H b \wedge c \asymp_H d \,.\vee.\, a \asymp_H c \wedge b \asymp_H d \,.\vee.\, a \asymp_H d \wedge b \asymp_H c. \vee (\exists X,Y,m,n)\mathscr{E}\mathscr{S}\mathscr{C}(X,Y)_H \wedge \tau_X(a,b)_H = m \wedge \tau_X(c,d)_H = n \wedge |m-n| < \kappa[(m+n)/p]^{\frac{1}{2}}.$$

'$[a,b=c,d]_{p,H}$' reads '(a,b) is *p-equal* to (c,d)'.

We can also give a similar definition for one interval being less than another.

***D*4.** $$[a,b<c,d]_{p,H} \quad \textit{for}$$
$$c \preccurlyeq_H a \preccurlyeq_H b \prec_H d \vee c \preccurlyeq_H b \preccurlyeq_H a \prec_H d \vee d \preccurlyeq_H a \preccurlyeq_H b \prec_H c \vee d \preccurlyeq_H b \preccurlyeq_H a \prec_H c \vee (\exists X,Y,m,n)\mathscr{E}\mathscr{S}\mathscr{C}(X,Y)_H \wedge \tau_X(a,b)_H = m \wedge \tau_Y(c,d)_H = n \wedge n-m \geq \kappa[(m+n)/p]^{\frac{1}{2}}.$$

'$[a,b<c,d]_{p,H}$' reads '(a,b) is *p-less* than (c,d)'.

We now prove few basic properties of **D**3 and **D**4.

*T*4. $a,b\in\mathscr{E}_H\to[a,a=b,b]_{p,H}\wedge[a,b=b,a]_{p,H}$.

Proof. **D**3, **P**IV3.4, T2.15.

*T*5. $[a,b=c,d]_{p,H}\leftrightarrow[b,a=c,d]_{p,H}\leftrightarrow[a,b=d,c]_{p,H}\leftrightarrow[c,d=a,b]_{p,H}$.

Proof. **D**3.

Notice that although p-equal is reflexive and symmetric, it is *not* transitive, i.e., $[a,b=c,d]_{p,H}$ and $[c,d=e,f]_{p,H}$ do not imply $[a,b=e,f]_{p,H}$. If there were no errors, the transitive law would have been valid, but unfortunately things in practice are not as tidy as in the ivory tower. However, the transitive law does hold for p-less.

*T*6. $[a,b<c,d]_{p,H}\wedge[c,d<e,f]_{p,H}\to[a,b<e,f]_{p,H}$.

Proof. **D**4, **T**IV4.3.

The following theorem leads to the usual result that two time intervals m,n are related in one of three different ways: $m<n$, $m=n$, $m>n$.

T7. $(\exists X,Y,m,n)\mathscr{E}\mathscr{S}\mathscr{C}(X,Y)_H\wedge\tau_X(a,b)_H=m\wedge\tau_X(c,d)_H=n$
$\to[a,b=c,d]_{p,H}\vee[a,b<c,d]_{p,H}\vee[c,d<a,b]_{p,H}$.

Proof. **D**3,4, **T**IV4.3.

VII. LENGTH MEASUREMENT AND SPACE GEODESICS

1. Introduction

So far, in the class of events was given both a structure and a metric, but the class of objects has only the 'part' relation, and when it comes to the class of distinct particles, it has nothing. Without relations and a metric in the class of particles, neither a spatial structure can be specified, nor can the motion of an object be described. We now proceed to rectify the situation.

The simplest relation between particles is collinearity. In abstract geometry, collinearity is introduced as a primitive concept, but in physics it has to be defined operationally; otherwise, we would not know how to construct straight edges, and experimentalists would have to close shop. Unfortunately, it does not seem possible to define collinearity operationally without some sort of a rigid structure†. For instance, the straightness of the edges of two rulers can be tested by putting the edges in coincidence and rotating the rulers about the edges to see if the edges remain in coincidence. Without the rigidity of the rulers, this test is a waste of time. Another way of defining a straight line, or more generally a space geodesic, is by the condition that it is the shortest path between two particles. This means that if we have rigidly connected congruent pairs of particles, and we lay these pairs end to end to form chains, then the chain having the smallest number of links between the two particles is the geodesic path between them, and the particles of the chain can be said to be collinear, or lying along a geodesic. We adopt this method to define collinearity, but before we can do so we must define 'rigidly connected congruent pairs of particles'.

In order for pairs of particles to be rigidly connected they must at least satisfy the following properties: (i) two pairs that are congruent at one time and place, must be congruent at all other times and places; (ii) congruence is symmetric, i.e., if $\langle P,Q\rangle$ is congruent with $\langle R,S\rangle$, then $\langle Q,P\rangle$ must be

† Notice that the path of a light signal does not always coincide with a *space* geodesic; as, for instance, on a rotating disk.

congruent with $\langle R,S\rangle$, and $\langle R,S\rangle$ must be congruent with $\langle P,Q\rangle$; (iii) congruence is transitive, i.e., if $\langle P,Q\rangle$ is congruent with $\langle R,S\rangle$, and $\langle R,S\rangle$ is congruent with $\langle T,U\rangle$, then $\langle P,Q\rangle$ is congruent with $\langle T,U\rangle$.

There are essentially two methods for constructing pairs of particles that have properties (i) to (iii): (1) The traditional method of testing different objects, and discovering that pairs of particles on certain objects, called rigid bodies, do have the desired properties, provided certain conditions are satisfied, such as constancy of temperature. It is not necessary to have a theory of heat to satisfy this condition, because we can define constant temperature to mean that the level of mercury in a thermometer must remain the same. No theory of how the thermometer works is necessary either; the state of the thermometer is simply taken to be part of the definition of rigidity. (2) The modern method of reflecting light (first) signals between two particles, and verifying that the round trip time, according to a standard clock stationed at one of the particles, remains the same. Again it is not necessary to assume that the speed of light is constant; all that is necessary is that properties (i) to (iii) are satisfied if this condition holds. Thus whatever method is used to achieve rigidity in practice, the important thing is that properties (i) to (iii) are satisfied. In other words, the proof of the pudding is in the eating.

Thus, to define collinearity we need rigidity, and to define rigidity we need congruence. By '$\langle P,Q\rangle$ is congruent with $\langle R,S\rangle$' is usually meant that P,R and Q,S are coincident at the same time. Since P and Q are separate particles, this involves simultaneity of events at different particles, which is conventional and relative (see discussion after PV4.3). The way out of this predicament is to realize that all that is necessary is to insure there is an interval of time during which the particles of one pair are coincident with the particles of the other pair, and not that the two coincidences occur simultaneously. This can be accomplished as follows: After particles Q and S coincide, send a signal to arrive at P after its coincidence with R; then before the dissociation of P and R, send a signal that arrives at Q before its dissociation from S. During the time interval from the arrival of the signal at P until its departure, we know that P is coincident with R as well as Q is coincident with S, i.e., $\langle P,Q\rangle$ is congruent with $\langle R,S\rangle$. We now proceed with the formulation of these ideas.

2. Length measuring instruments

Our first goal is to define a rigidly connected pair of particles. To accomplish this, we define a length instrument that associates a number with any pair of particles, such that congruent pairs are associated with the same number under any condition. If the number associated with a pair remains constant in time, the two particles are considered rigidly connected. This method not only permits the definition of rigidity, but also paves the way for the introduction of a metric.

Before a general definition of a length measuring instrument is given, it is helpful to analyze an example of such an instrument. A familiar example that brings out all the basic features, is the vernier caliper. It consists of a metal ruler with a fixed jaw on one end, and a movable jaw that slides along the ruler. The movable jaw has several marks on it, but here only one of the marks is of interest. As the jaws separate the moving mark coincides successively with one side of the marks fixed on the ruler, generating type I events, and as the jaws close, the moving mark coincides with the other side of the fixed marks, generating type II events. Starting with closed jaws, the number of type I events minus the number of type II events gives a measure of the distance between the two jaws. The distance between two marks on a ruler can be determined in the same way by moving a finger along the ruler. A length interferometer operates on the same principle, except that the two types of events are the appearance and disappearance of interference fringes. The following definition is a generalization of these ideas:

D1. $$\begin{aligned}&\mathscr{U}(P,Q,\mathfrak{l},N)_H \quad \textit{for} \quad (\exists X)(X\mathscr{S}\mathscr{C}P)_H \wedge Q\in\mathscr{P}_H\\ &\wedge(\exists x,y)x\subset\mathscr{E}_H\wedge y\subset\mathscr{E}_H\wedge x\cap y=\emptyset\\ &\wedge(\forall u).u\in x\cup y\rightarrow u\in\mathscr{W}(P)_H:\\ &\wedge(\forall v).v\in\mathscr{W}(P)_H\rightarrow\mathfrak{l}(v)\in\{0,1,\ldots,N\}\\ &\wedge\mathfrak{l}(v)=\mathrm{Card}(w\ni w\in x\wedge w\preccurlyeq_H v)-\mathrm{Card}(w\ni w\in y\wedge w\preccurlyeq_H v).\end{aligned}$$

'$\mathscr{U}(P,Q,\mathfrak{l},N)_H$' means: P is the output particle of a standard clock, Q is a particle, and there exist two nonintersecting subsets x, y of events of H that occur on the world line of P. Moreover, if v is an event on P, then $\mathfrak{l}(v)$ is an integer between 0 and N, which is equal to the number of events belonging to subset x that occurred before v, minus the number of events belonging to subset y that occurred before v.† Referring to the example of the vernier caliper, P, Q are particles on the two jaws, x, y are the events of types I and II, and $\mathfrak{l}(v)$ is a measure of the distance between P and Q at the time of occurrence of event v. The particles P, Q are called the *end particles*, and $\mathfrak{l}(v)$ the *scale value* at v.

† 'Card (S)' denotes the cardinal number of set S, i.e. the number of elements of S.

The next step is to define the congruence of two pairs of end particles, as described in the introduction. For this purpose, it is useful first to define the sentence 'a is the appearance of a particle P, and b is an event on P after a such that no dissociation of P occurs between a and $b[a(P)-_H b]$'.

D2. $a(P)-_H b$ *for* $a\mathscr{A}_H P \wedge a \preccurlyeq_H b \wedge b \in \mathscr{W}(P)_H$
$\wedge \sim(\exists u, v, X, Y) u \in \mathscr{W}(P)_H \wedge a \preccurlyeq_H u \preccurlyeq_H b \wedge P \circledS_H X, Y \wedge X \neq Y$
$\wedge u \asymp_H v \wedge : P\mathscr{D}_H v \vee v\mathscr{A}_H X \vee v\mathscr{A}_H Y.$

In **D**3 we define the sentence '$\langle P, Q\rangle$ is *congruent with* $\langle R, S\rangle$ *during* (a,b), *and the ratio of scale values during this interval is* m/n' to mean (see Fig. 4): P coincides with R to form X (event u), Q coincides with S to form Y

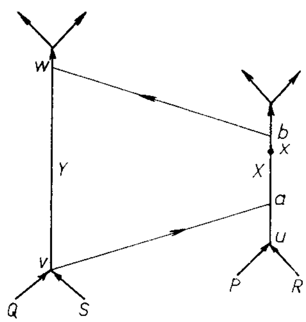

Fig. 4. Congruence

(event v), a signal departs from v and arrives at a on X, and then departs from b on X to arrive at w on Y. The events a, b occur before the dissociation of X, and w before the dissociation of Y. Moreover, the scale values of the two pairs of end particles are m and n during the interval (a,b).

D3. $\mathscr{C}on[P, Q, m(a,b)_N R, S, n]_H$ *for*
$(\exists \mathfrak{k}, \mathfrak{l}).\mathscr{U}(P, Q, \mathfrak{k}, N)_H \wedge \mathscr{U}(R, S, \mathfrak{l}, N)_H$
$\wedge (\exists X, Y).P, R \circledS\cdot_H X \wedge Q, S \circledS\cdot_H Y$
$\wedge (\exists u, v, w).u(X)-_H b \wedge v(Y)-_H w \wedge a \in \mathscr{W}(X)_H$
$\wedge v\mathscr{B}_H a \wedge u \preccurlyeq_H a \preccurlyeq_H b \wedge b\mathscr{B}_H w$
$\wedge (\forall x).x \in \mathscr{W}(X)_H \wedge a \preccurlyeq_H x \preccurlyeq_H b \rightarrow \mathfrak{k}(x) = m \wedge \mathfrak{l}(x) = n.$

Two pairs of end particles, $\langle P, Q\rangle$ and $\langle R, S\rangle$, become candidates for a length measuring instrument if, in analogy with **D**VI3.2, they satisfy the following condition: there is a congruence of $\langle P, Q\rangle$ with $\langle R, S\rangle$ for any possible scale value of either one. Moreover, for any real positive ε, there exists integers q,r such that for any integer k larger than r (to make the central limit theorem applicable), and for any possible scale value m of $\langle P,Q\rangle$, if $n_1,\ldots,n_k$ are the scale values of $\langle R,S\rangle$ during k successive congruences with $\langle P,Q\rangle$ (separated by gaps of duration longer than q), then the probability that the confidence limits $\bar{n}\pm \mathrm{g}_p S\sqrt{(k-1)}$ include m, and the probability that $S^2\pm \mathrm{g}_p[(S_4-S^4)/(k-1)]^{\frac{1}{2}}$ include σ^2 are both equal to $1-p$ within an error ε [see Eqs. VI 3. (1, 2, 6, 7, 14, 15)]. This condition is denoted by '$\mathscr{Y}[P,Q(\sigma,N)R,S]_H$'.

$$
\begin{aligned}
\textbf{D4.}\quad & \mathscr{Y}[P,Q(\sigma,N)R,S]_H \quad \textit{for} \quad \sigma\in\mathscr{R}\wedge 0\leq\sigma \\
& \wedge(\forall m).m\in\{0,\ldots,N\}\rightarrow(\exists u,v,i)\mathscr{C}on[P,Q,m(u,v)_N R,S,i]_H \\
& \qquad\qquad\qquad \wedge(\exists x,y,j)\mathscr{C}on[R,S,m(x,y)_N P,Q,j]_H: \\
& \wedge(\forall\varepsilon).\varepsilon\in\mathscr{R}\wedge\varepsilon>0\rightarrow(\exists q,r)q,r\in\mathscr{I}\wedge(\forall k).k\in\mathscr{I}\wedge k\geq r \\
& \quad\rightarrow(\forall m,u_1,v_1,n_1,\ldots,u_k,v_k,n_k):.m\in\{0,\ldots,N\} \\
& \qquad \wedge:\mathscr{C}on[P,Q,m(u_1,v_1)_N R,S,n_1]_H \\
& \qquad\quad \vee\mathscr{C}on[R,S,m(u_1,v_1)_N P,Q,n_1]_H. \\
& \qquad \wedge\ldots \\
& \qquad \wedge.\mathscr{C}on[P,Q,m(u_k,v_k)_N R,S,n_k]_H \\
& \qquad\quad \vee\mathscr{C}on[R,S,m(u_k,v_k)_N P,Q,n_k]_H. \\
& \qquad \wedge u_1\preccurlyeq_H v_1\prec_H u_2\preccurlyeq_H v_2\prec_H\ldots\prec_H u_k\preccurlyeq_H v_k \\
& \qquad \wedge(\exists X):(X\mathscr{S}\mathscr{C}P)_H\vee(X\mathscr{S}\mathscr{C}R)_H. \\
& \qquad\qquad \wedge\tau_X(v_1,u_2)_H\geq q\wedge\ldots\wedge\tau_X(v_{k-1},u_k)_H\geq q.: \\
& \qquad \wedge(\forall p).p\in\mathscr{R}\wedge 0<p<1\rightarrow \\
& \qquad\qquad |\mathbf{P}\{\bar{n}-\mathrm{g}_p S/\sqrt{k-1}<m<\bar{n}+\mathrm{g}_p S/\sqrt{k-1}\} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -(1-p)|<\varepsilon \\
& \qquad\quad \wedge|\mathbf{P}\{S^2-\mathrm{g}_p[(S_4-S^4)/(k-1)]^{\frac{1}{2}}<\sigma^2 \\
& \qquad\qquad\qquad <S^2+\mathrm{g}_p[(S_4-S^4)/(k-1)]^{\frac{1}{2}}\}-(1-p)|<\varepsilon.
\end{aligned}
$$

Finally, we define a length instrument (LI), in analogy with **D**VI4.1, by:

$$
\begin{aligned}
\textbf{D5.}\quad & \mathscr{L}\mathscr{I}(P,Q,\mathfrak{l})_H \quad \textit{for} \quad (\exists N)\mathscr{U}(P,Q,\mathfrak{l},N)_H\wedge N\geq N_{\mathrm{L}} \\
& \wedge(\exists r).r\in\mathscr{I}\wedge 2<\mathrm{r}_{\mathrm{L}}\leq r\wedge(\exists X_1,Y_1,\ldots,X_r,Y_r)\neq(X_1,Y_1,\ldots,X_r,Y_r) \\
& \wedge\langle P,Q\rangle\in\{\langle X_1,Y_1\rangle,\ldots,\langle X_r,Y_r\rangle\} \\
& \wedge(\forall U,V,X,Y).\langle U,V\rangle\neq\langle X,Y\rangle \\
& \qquad\qquad \wedge\langle U,V\rangle,\langle X,Y\rangle\in\{\langle X_1,Y_1\rangle,\ldots,\langle X_r,Y_r\rangle\} \\
& \qquad\qquad \rightarrow(\exists\sigma)\mathscr{Y}[U,V(\sigma,N)X,Y]_H\wedge\sigma\leq\sigma_{\mathrm{L}}.
\end{aligned}
$$

This means that we take $\langle P,Q\rangle$ as an element of a set of at least r_{L}LI's, any two of which satisfy the condition $\mathscr{Y}$ for $\sigma\leq\sigma_{\mathrm{L}}$. The values of the *range*

N_{L}, *reliability index* $\mathrm{r_L}$, and *standard deviation* σ_{L} are all decided by a standards committee.

***D*6.** $\mathscr{L}\mathscr{I}(\mathfrak{l})_H$ *for* $(\exists X, Y)\mathscr{L}\mathscr{I}(X, Y, \mathfrak{l})_H$.

As in ***D*VI4.4.** we define *equivalent* LI's by

***D*7.** $\mathscr{E}\mathscr{L}\mathscr{I}(P, Q, \mathfrak{k}; R, S, \mathfrak{l})_H$ *for*
$\mathscr{L}\mathscr{I}(P, Q, \mathfrak{k})_H \wedge \mathscr{L}\mathscr{I}(R, S, \mathfrak{l})_H \wedge (\exists \sigma, N)\mathscr{Y}[P, Q(\sigma, N)R, S]_H$.

***D*8.** $\mathscr{E}\mathscr{L}\mathscr{I}(\mathfrak{k}, \mathfrak{l})_H$ *for*
$(\exists U, V, X, Y)\mathscr{E}\mathscr{L}\mathscr{I}(U, V, \mathfrak{k}; X, Y, \mathfrak{l})_H$.

***T*1.** $\mathscr{E}\mathscr{L}\mathscr{I}(\mathfrak{k}, \mathfrak{l})_H \leftrightarrow \mathscr{E}\mathscr{L}\mathscr{I}(\mathfrak{l}, \mathfrak{k})_H$.
Proof. ***D***(8, 7, 4).

***P*1.** $\mathscr{E}\mathscr{L}\mathscr{I}(\mathfrak{k}, \mathfrak{l})_H \wedge \mathscr{E}\mathscr{L}\mathscr{I}(\mathfrak{l}, \mathfrak{m})_H \rightarrow \mathscr{E}\mathscr{L}\mathscr{I}(\mathfrak{k}, \mathfrak{m})_H$.

3. Distance

To measure the distance between two particles P, Q by an LI, we establish congruence between P, Q and the end particles of the LI, and take the scale value during congruence as a measure of the distance between P and Q. This measure is not necessarily linear, and we shall not bother to define a space metric until a linear space measure is defined in Sec. 7.

***D*1.** $\mu_{\mathfrak{l}}[P(a,b), Q; n]_H$ *for*
$(\exists U, V, X, Y)\mathscr{L}\mathscr{I}(U, V, \mathfrak{l})_H \wedge U, P \circledS\!\cdot_H X \wedge V, Q \circledS\!\cdot_H Y$
$\wedge (\exists u, v, w)u(X) \multimap_H b \wedge v(Y) \multimap_H w \wedge a \in \mathscr{W}(X)_H$
$\wedge v\mathscr{B}_H a \wedge u \preccurlyeq_H a \preccurlyeq_H b \wedge b\mathscr{B}_H w$
$\wedge (\forall x).x \in \mathscr{W}(X)_H \wedge a \preccurlyeq_H x \preccurlyeq_H b \rightarrow \mathfrak{l}(x) = n$.

'$\mu_{\mathfrak{l}}[P(a,b), Q; n]_H$' reads '*the distance between P and Q during* (a,b) *by* LI $\mathfrak{l}$ *is n*'. The notation is as in ***D*2.3** and Fig. 4.

We now prove that the number n established by ***D*1** is unique.

***T*1.** $(\exists^1 n)\mu_{\mathfrak{l}}[P(a,b), Q; n]_H$.
Proof. ***D*1**, P4: $\mu_{\mathfrak{l}}[P(a,b), Q; m]_H \wedge \mu_{\mathfrak{l}}[P(a,b), Q; n]_H$
$\rightarrow \mathfrak{l}(a) = m \wedge \mathfrak{l}(a) = n$
$\rightarrow m = n$. (1)
(1), D9: **T**.

***T*2.** $(\exists n)\mu_{\mathfrak{l}}[P(a,b), Q; n]_H \rightarrow (\exists! n)\mu_{\mathfrak{l}}[P(a,b), Q; n]_H$.
Proof. *T*1, D10.

Because of **T2**, we can define *the* distance between particles P, Q measured by an LI, $\mathfrak{l}$, during (a,b) on P, by:

D2. $\mu_{\mathfrak{l}}[P(a,b),Q]_H$ *for* $(\iota n)\mu_{\mathfrak{l}}[P(a,b),Q;n]_H$.

T3. $(\exists m)\mu_{\mathfrak{l}}[P(a,b),Q;m]_H \rightarrow \cdot \mu_{\mathfrak{l}}[P(a,b),Q]_H = n \leftrightarrow \mu_{\mathfrak{l}}[P(a,b),Q;n]_H$.

Proof. **T2**, T4.3, **D2**.

According to **D2**.(1,5), $\mathfrak{l}$ can only have one of the values 0 to N_L, and thus $\mu_{\mathfrak{l}} \geq 0$, but we can not prove yet the other metric properties.

T4. $(\exists n)\mu_{\mathfrak{l}}[P(a,b),Q;n]_H \rightarrow \mu_{\mathfrak{l}}[P(a,b),Q]_H \geq 0$.

Proof. **D1**, **D2**.1.

Since the scale value of an LI is limited, not every distance can be measured by an LI. The property that the distance between two particles P, Q is *measurable* by the LI, $\mathfrak{l}$, at any time and as frequently as desired, is defined by:

D3. $(P\mathscr{M}_{\mathfrak{l}}Q)_H$ *for*

$$(\exists X,Y)\mathscr{L}\mathscr{I}(X,Y,\mathfrak{l})_H \wedge (\forall u).u \in \mathscr{W}(X)_H \rightarrow (\exists v,n) v \in \mathscr{W}(X)_H$$
$$\wedge .\mu_{\mathfrak{l}}[P(u,v),Q;n]_H \vee \mu_{\mathfrak{l}}[Q(u,v),P;n]_H:$$
$$\wedge (\forall k).k \in \mathscr{I} \rightarrow (\exists u_1, v_1, n_1, \ldots, u_k, v_k, n_k)$$
$$u_1 \preccurlyeq_H v_1 \prec_H u_2 \preccurlyeq_H v_2 \prec_H \cdots \prec_H u_k \preccurlyeq_H v_k$$
$$\wedge .\mu_{\mathfrak{l}}[P(u_1,v_1),Q;n_1]_H \vee \mu_{\mathfrak{l}}[Q(u_1,v_1),P;n_1]_H. \wedge \ldots$$
$$\wedge .\mu_{\mathfrak{l}}[P(u_k,v_k),Q;n_k]_H \vee \mu_{\mathfrak{l}}[Q(u_k,v_k),P;n_k]_H.$$

From this it follows that:

T5. $(P\mathscr{M}_{\mathfrak{l}}Q)_H \leftrightarrow (Q\mathscr{M}_{\mathfrak{l}}P)_H$.

Proof. **D3**.

Another useful definition is:

D4. $(P\mathscr{M}Q)_H$ *for* $(\exists \mathfrak{l})(P\mathscr{M}_{\mathfrak{l}}Q)_H$.

'$(P\mathscr{M}Q)_H$' reads '*the distance between P and Q is measurable*'.

4. Rigidity

Due to statistical fluctuations of LI's, the distance they measure fluctuates in time, even if the actual separation is constant. We therefore define '*the distance between two particles P, Q is a constant r by an* LI, $\mathfrak{l}$' to mean: The distance between P and Q is measurable by $\mathfrak{l}$. Moreover, given any positive ε, and δ between 0 and 1, there exists an integer i, such that for any integer k larger than i, if $n_1, \ldots, n_k$ are successive measurements of the distance between P and Q by $\mathfrak{l}$, there exists a unique real number q such that the probability that $|\bar{n}-q|<\varepsilon$, is less than δ, i.e., $\bar{n}$ converges *in probability* to q (Cramér [1946] p. 252), and r is this unique number.

D1. $\mu_{\mathfrak{l}}(P,Q;r)_H$ *for* $(P\mathcal{M}_{\mathfrak{l}}Q)_H$
$\wedge (\forall \varepsilon, \delta).\varepsilon, \delta \in \mathscr{R} \wedge \varepsilon > 0 \wedge 0 \leq \delta < 1 \rightarrow (\exists i) i \in \mathscr{I}$
$\wedge (\forall k).k \in \mathscr{I} \wedge k \geq i \rightarrow (\forall u_1, v_1, n_1, \ldots, u_k, v_k, n_k).$
$u_1 \preccurlyeq_H v_1 \prec_H u_2 \preccurlyeq_H v_2 \prec_H \cdots \prec_H u_k \preccurlyeq_H v_k$
$\wedge .\mu_{\mathfrak{l}}[P(u_1,v_1),Q]_H = n_1 \vee \mu_{\mathfrak{l}}[Q(u_1,v_1),P]_H = n_1 . \wedge \ldots$
$\wedge .\mu_{\mathfrak{l}}[P(u_k,v_k),Q]_H = n_k \vee \mu_{\mathfrak{l}}[Q(u_k,v_k),P]_H = n_k .$
$\rightarrow (\exists ! q) q \in \mathscr{R} \wedge \mathbf{P}\{|\bar{n}-q|>\varepsilon\} \leq \delta \wedge r = (\imath s)\mathbf{P}\{|\bar{n}-s|>\varepsilon\} \leq \delta.$

As in the previous section, we prove there exists at most one number r that satisfies $\mu_{\mathfrak{l}}(P,Q;r)_H$, and then define *the* distance $\mu_{\mathfrak{l}}(P,Q)_H$.

T1. $(\exists^1 r)\mu_{\mathfrak{l}}(P,Q;r)_H$.

Proof. According to **D1**, $\mu_{\mathfrak{l}}(P,Q;r)_H$ implies that r is the (unique) true mean of the distance between P and Q measured by $\mathfrak{l}$, and thus:

$\mu_{\mathfrak{l}}(P,Q;r)_H \wedge \mu_{\mathfrak{l}}(P,Q;s)_H \rightarrow r = s.$ *Q. E. D.*

T2. $(\exists r)\mu_{\mathfrak{l}}(P,Q;r)_H \rightarrow (\exists ! r)\mu_{\mathfrak{l}}(P,Q;r)_H$.
Proof. **T1**, D10.

D2. $\mu_{\mathfrak{l}}(P,Q)_H$ *for* $(\imath r)\mu_{\mathfrak{l}}(P,Q;r)_H$.

$\mu_{\mathfrak{l}}(P,Q)_H$ is the (constant) distance between P and Q measured by $\mathfrak{l}$.

T3. $(\exists r)\mu_{\mathfrak{l}}(P,Q;r)_H \rightarrow .\mu_{\mathfrak{l}}(P,Q;s)_H \leftrightarrow \mu_{\mathfrak{l}}(P,Q)_H = s.$
Proof. **T2**, T4.3, **D2**.

T4. $(\exists r)\mu_{\mathfrak{l}}(P,Q;r)_H \rightarrow \mu_{\mathfrak{l}}(P,Q)_H \geq 0.$
Proof. **D1**, **T3**.4.

T5. $\mu_{\mathfrak{l}}(P,Q;r)_H \leftrightarrow \mu_{\mathfrak{l}}(Q,P;r)_H . \wedge .$
$(\exists r)\mu_{\mathfrak{l}}(P,Q;r)_H . \rightarrow \mu_{\mathfrak{l}}(P,Q)_H = \mu_{\mathfrak{l}}(Q,P)_H.$
Proof. **D1**, **T3**, as in **T**VI5.8.

Since a constant distance measured by two equivalent LI's must be the same, we assume:

P1. $\mu_{\mathfrak{k}}(P,Q;r)_H \wedge \mathscr{E}\mathscr{L}\mathscr{I}(\mathfrak{k},\mathfrak{l})_H \rightarrow \mu_{\mathfrak{l}}(P,Q;r)_H$.

If two distances are equal, and one is measurable by an LI, $\mathfrak{l}$, the other must also be measurable by $\mathfrak{l}$. Thus:

P2. $(\exists \mathfrak{t}, p)\mu_{\mathfrak{t}}(P, Q; p)_H \wedge \mu_{\mathfrak{t}}(R, S; p)_H \wedge (P\mathcal{M}_{\mathfrak{l}}Q)_H. \rightarrow (R\mathcal{M}_{\mathfrak{l}}S)_H.$

The situation regarding LI's of different families is the same as that regarding SC's, described at the end of Sec. VI4. Until now, there is no experimental evidence against the assumption that the ratio of two constant distances is the same for any LI, i.e.,

P3. $(\exists p, q, r, s)\mu_{\mathfrak{t}}(P, Q; p)_H \wedge \mu_{\mathfrak{t}}(R, S; q)_H \wedge \mu_{\mathfrak{l}}(P, Q; r)_H$
$\wedge \mu_{\mathfrak{l}}(R, S; s)_H. \rightarrow \mu_{\mathfrak{t}}(P, Q)_H / \mu_{\mathfrak{t}}(R, S)_H = \mu_{\mathfrak{l}}(P, Q)_H / \mu_{\mathfrak{l}}(R, S)_H.$

At last we are in a position to define a *rigidly connected pair of particles.*

D3. $P\mathcal{RC}_H Q$ *for* $P \neq Q \wedge (\exists \mathfrak{l}, r)\mu_{\mathfrak{l}}(P, Q; r)_H.$

This means that P and Q are rigidly connected if they are distinct and there exists an LI according to which the distance between P and Q is constant in time.

T6. $P\mathcal{RC}_H Q \leftrightarrow Q\mathcal{RC}_H P.$
Proof. **D**3.

Two particles that are rigidly connected according to one LI must be rigidly connected according to any other LI (**P**4).

P4. $P\mathcal{RC}_H Q \wedge (P\mathcal{M}_{\mathfrak{l}}Q)_H \rightarrow (\exists p)\mu_{\mathfrak{l}}(P, Q; p)_H.$

D4. $\mathcal{RC}(P_1, \ldots, P_n)_H$ *for*
$P_1\mathcal{RC}_H P_2 \wedge P_2\mathcal{RC}_H P_3 \wedge \ldots \wedge P_{n-1}\mathcal{RC}_H P_n.$

'$\mathcal{RC}(P_1, \ldots, P_n)_H$' reads '$P_1, \ldots, P_n$ is a *sequence of rigidly connected particles*'.

T7. $\mathcal{RC}(P_1, \ldots, P_n)_H \leftrightarrow \mathcal{RC}(P_n, \ldots, P_1)_H.$
Proof. **D**4, **T**6.

5. Congruence

In **D**2.3 congruence means actual coincidence of particles. Here, this concept is generalized to include equality of separate distances determined by LI's.

D1. $P, Q \boxminus_H R, S$ *for* $P, Q, R, S \in \mathcal{P}_H$
$\wedge : P = Q \wedge R = S. \vee . P \neq Q \wedge R \neq S$
$\wedge : P = R \wedge Q = S. \vee . P = S \wedge Q = R.$
$\vee .(\exists \mathfrak{t}, \mathfrak{l}, p)\mu_{\mathfrak{t}}(P, Q; p)_H \wedge \mu_{\mathfrak{l}}(R, S; p)_H \wedge \mathcal{ELI}(\mathfrak{t}, \mathfrak{l})_H.$

'$P, Q \boxminus_H R, S$' reads '(P, Q) *is congruent to* (R, S)', and means: $P = Q$ and

$R=S$, or $P\neq Q$, $R\neq S$ and one of the following cases holds: $P=R$ and $Q=S$, or $P=S$ and $Q=R$, or the distances of the pairs P,Q and R,S, measured by equivalent LI's are equal.

***T*1.** $P,P\mathrel{\vDash\!\!\dashv}_H Q,R\rightarrow Q=R.$

Proof. ***D*1.**

***T*2.** $P,Q\mathrel{\vDash\!\!\dashv}_H R,S\wedge P\neq Q\rightarrow R\neq S.$

Proof. ***T*1**, T2.32.

***T*3.** $P,Q\mathrel{\vDash\!\!\dashv}_H R,S\leftrightarrow P,Q\mathrel{\vDash\!\!\dashv}_H S,R\leftrightarrow Q,P\mathrel{\vDash\!\!\dashv}_H R,S\leftrightarrow R,S\mathrel{\vDash\!\!\dashv}_H P,Q.$

Proof. ***D*1.**

***T*4.** $P,Q\mathrel{\vDash\!\!\dashv}_H R,S\wedge(P\neq Q\vee R\neq S)\wedge(P\neq R\vee P\neq S\vee Q\neq R\vee Q\neq S)$
$\rightarrow P\mathscr{R}\mathscr{C}_H Q\wedge R\mathscr{R}\mathscr{C}_H S.$

Proof. ***D***(1, 4.3).

***T*5.** $P,Q\mathrel{\vDash\!\!\dashv}_H R,S\wedge(P\neq Q\vee R\neq S)\wedge(P\neq R\vee R\neq S\vee Q\neq R\vee Q\neq S)$
$\wedge(P\mathscr{M}_{\mathfrak{l}}Q)_H\vee(R\mathscr{M}_{\mathfrak{l}}S)_H:\rightarrow(\exists\,p)\mu_{\mathfrak{l}}(P,Q;p)_H\wedge\mu_{\mathfrak{l}}(R,S;p)_H.$

Proof. ***T*4, *D*1, *P*4.2: Ant** $\rightarrow P\mathscr{R}\mathscr{C}_H Q\wedge(P\mathscr{M}_{\mathfrak{l}}Q)_H\wedge R\mathscr{R}\mathscr{C}_H S\wedge(R\mathscr{M}_{\mathfrak{l}}S)_H$
$\wedge(\exists\,\mathfrak{k},\mathfrak{m},p)\mu_{\mathfrak{k}}(P,Q;p)_H\wedge\mu_{\mathfrak{m}}(R,S;p)_H$
$\wedge\mathscr{E}\mathscr{L}\mathscr{I}(\mathfrak{k},\mathfrak{m})_H$

***P*4.**(1,4) $\rightarrow(\exists\,\mathfrak{k},p)\mu_{\mathfrak{k}}(P,Q;p)_H\wedge\mu_{\mathfrak{k}}(R,S;p)_H$
$\wedge(\exists\,r,s)\mu_{\mathfrak{l}}(P,Q;r)_H\wedge\mu_{\mathfrak{l}}(R,S;s)_H$

***P*4.3** $\rightarrow$ **Con.**

***T*6.** $P,Q\mathrel{\vDash\!\!\dashv}_H R,S\wedge R,S\mathrel{\vDash\!\!\dashv}_H T,U\rightarrow P,Q\mathrel{\vDash\!\!\dashv}_H T,U.$

Proof. By ***D*1**, the antecedent has the form:

$$F_1\vee F_2\vee F_3\vee F_4.\wedge.G_1\vee G_2\vee G_3\vee G_4.$$

This can be written as (T2.40) $H_1\vee\ldots\vee H_{16}$, where each H_k is equivalent to $F_i\wedge G_j$ for some i and j. The theorem is proved, if we can prove $H_k\rightarrow$ **Con** for every value of k.

T5.1: $P,Q,R,S,T,U\in\mathscr{P}_H\wedge P=Q\wedge R=S\wedge R=S\wedge T=U$
$\rightarrow P,Q,T,U\in\mathscr{P}_H\wedge P=Q\wedge T=U$

***D*1** $\rightarrow$ **Con.**

There are 6 cases that have '$R=S\wedge R\neq S$', all of which imply the **Con** because of T1.11.

T5.1: $P,Q,R,S,T,U\in\mathscr{P}_H\wedge P\neq Q\wedge R\neq S\wedge T\neq U\wedge P=R$
$\wedge Q=S\wedge R=T\wedge S=U$
$\rightarrow P,Q,T,U\in\mathscr{P}_H\wedge P\neq Q\wedge T\neq U\wedge P=T\wedge Q=U$

***D*1** $\rightarrow$ **Con.**

There are 4 such cases in all, and 4 more cases of the following type:

$$\begin{aligned}&T5.1: P,Q,R,S,T,U \in \mathscr{P}_H \wedge P \neq Q \wedge R \neq S \wedge T \neq U \wedge P = R \wedge Q = S\\ &\quad \wedge (\exists \mathfrak{k}, \mathfrak{l}, p)\mu_{\mathfrak{k}}(R,S;p)_H \wedge \mu_{\mathfrak{l}}(T,U;p)_H \wedge \mathscr{E}\mathscr{L}\mathscr{I}(\mathfrak{k},\mathfrak{l})_H\\ &\rightarrow P,Q,T,U \in \mathscr{P}_H \wedge P \neq Q \wedge T \neq U\\ &\quad \wedge (\exists \mathfrak{k}, \mathfrak{l}, p)\mu_{\mathfrak{k}}(P,Q;p)_H \wedge \mu_{\mathfrak{l}}(T,U;p)_H \wedge \mathscr{E}\mathscr{L}\mathscr{I}(\mathfrak{k},\mathfrak{l})_H\\ &\boldsymbol{D}1 \rightarrow \mathbf{Con}.\end{aligned}$$

This takes care of $1+6+4+4(=15)$ cases. The remaining case is:

$$\begin{aligned}&\boldsymbol{P}4.(1,2,4), \boldsymbol{D}4.(1,3): P \neq Q \wedge R \neq S \wedge T \neq U\\ &\quad \wedge (\exists \mathfrak{k}, \mathfrak{l}, p)\mu_{\mathfrak{k}}(P,Q;p)_H \wedge \mu_{\mathfrak{l}}(R,S;p)_H \wedge \mathscr{E}\mathscr{L}\mathscr{I}(\mathfrak{k},\mathfrak{l})_H\\ &\quad \wedge (\exists \mathfrak{m}, \mathfrak{n}, q)\mu_{\mathfrak{m}}(R,S;q)_H \wedge \mu_{\mathfrak{n}}(T,U;q)_H \wedge \mathscr{E}\mathscr{L}\mathscr{I}(\mathfrak{m},\mathfrak{n})_H\\ &\rightarrow \quad P \neq Q \wedge R \neq S \wedge T \neq U\\ &\quad \wedge (\exists \mathfrak{k}, \mathfrak{l}, p)\mu_{\mathfrak{k}}(P,Q;p)_H \wedge \mu_{\mathfrak{l}}(R,S;p)_H \wedge \mathscr{E}\mathscr{L}\mathscr{I}(\mathfrak{k},\mathfrak{l})_H\\ &\quad \wedge (\exists \mathfrak{m} q)\mu_{\mathfrak{m}}(R,S;q)_H \wedge \mu_{\mathfrak{m}}(T,U;q)_H \wedge (\exists r)u_{\mathfrak{l}}(T,U;r)_H\\ &\boldsymbol{P}4.3 \rightarrow \quad P \neq Q \wedge T \neq U \wedge P,Q,T,U \in \mathscr{P}_H\\ &\quad \wedge (\exists \mathfrak{k}, \mathfrak{l}, p)\mu_{\mathfrak{k}}(P,Q;p)_H \wedge \mu_{\mathfrak{l}}(T,U;p)_H \wedge \mathscr{E}\mathscr{L}\mathscr{I}(\mathfrak{k},\mathfrak{l})_H\\ &\boldsymbol{D}1 \rightarrow \mathbf{Con}.\end{aligned}$$

We thus see from $\boldsymbol{T}3, \boldsymbol{T}5$, and $\boldsymbol{T}6$ that $\boxminus_H$ has all the required properties of congruence discussed in the introduction.

In $\boldsymbol{D}4.4$ is defined a sequence of rigidly connected particles, whose distances are not necessarily equal. We now define a chain by making the links of the sequence of equal length.

D2. $\mathscr{C}h(P_1, \ldots, P_n)_H$ *for* $\neq(P_1, \ldots, P_n)$
$\wedge P_1, P_2 \boxminus_H P_2, P_3 \wedge P_2, P_3 \boxminus_H P_3, P_4 \wedge \ldots \wedge P_{n-2}, P_{n-1} \boxminus_H P_{n-1}, P_n$.

'$\mathscr{C}h(P_1, \ldots, P_n)_H$' reads '$P_1, \ldots, P_n$' is a *chain*.'

The condition '$\neq(P_1, \ldots, P_n)$' insures that there are no closed loops in the chain.

T7. $\mathscr{C}h(P_1, \ldots, P_n)_H \leftrightarrow \mathscr{C}h(P_n, \ldots, P_1)_H$.

Proof. **D**2, **T**3.

T8. $\mathscr{C}h(P_1, \ldots, P_n)_H \rightarrow \mathscr{R}\mathscr{C}(P_1, \ldots, P_n)_H$.

Proof. **D**2, **T**4, **D**4.4.

T9. $\mathscr{C}h(P_1, \ldots, P_n)_H \wedge (P_1 \mathscr{M}_{\mathfrak{l}} P_2)_H$
$\rightarrow (\exists p)\mu_{\mathfrak{l}}(P_1, P_2; p)_H \wedge \mu_{\mathfrak{l}}(P_2, P_3; p)_H \wedge \ldots \wedge \mu_{\mathfrak{l}}(P_{n-1}, P_n; p)_H$.

Proof. **D**2, **P**4.2, **T**5, **P**4.3.

6. Space geodesics

If two particles are connected by several chains having links of equal length, the chain that has the minimum number of links between the two particles is the one that lies on the space geodesic between the two particles.

D1. $\mathscr{G}(P_1, \ldots, P_n)_H$ *for* $\mathscr{C}h(P_1, \ldots, P_n)_H \wedge n > 2$
$\wedge (\forall k).k \in \mathscr{I} \wedge k > 1 \rightarrow (\forall X_1, \ldots, X_k).\mathscr{C}h(X_1, \ldots, X_k)_H$
$\wedge X_1 = P_1 \wedge X_k = P_n \wedge X_1, X_2 \boxminus_H P_1, P_2 \rightarrow k \geq n.$

'$\mathscr{G}(P_1, \ldots, P_n)_H$' reads '$P_1, \ldots, P_n$ *lie on a space geodesic* (SG).' It means: $P_1, \ldots, P_n$ is a chain of at least two links, and if $X_1, \ldots, X_k$ is any chain linking P_1 and P_n, and the links of the two chains are equal in length, then the number of links of the X-chain is either equal to or larger than the number of links of the P-chain.

T1. $\mathscr{G}(P_1, \ldots, P_n)_H \leftrightarrow \mathscr{G}(P_n, \ldots, P_1)_H.$

Proof. **D**1, **D**5.2, **T**5.6: **Ant** $\wedge X_1, X_2 \boxminus_H P_1, P_2 \rightarrow X_1, X_2 \boxminus_H P_n, P_{n-1}.$ (1)
(1), **T**5.7: **T**.

T2. $\mathscr{G}(P_1, \ldots, P_n)_H \wedge (P_1 \mathscr{M}_1 P_2)_H$
$\rightarrow (\exists p) \mu_1(P_1, P_2; p)_H \wedge \mu_1(P_2, P_3; p)_H \wedge \ldots \wedge \mu_1(P_{n-1}, P_n; p)_H.$

Proof. **D**1, **T**5.9.

If we consider two different pairs of particles on a chain, such that each pair is separated by the same number of links as the other pair, and this number is at least two, then the distances between the particles of each pair are in general neither constant nor equal. However, if the chain is an SG, these distances are both constant and equal, since the chain becomes like a taught string. This is the content of the following two postulates:

P1. $\mathscr{G}(P_1, \ldots, P_n)_H \wedge (P_1 \mathscr{M} P_n)_H \rightarrow P_1 \mathscr{R}\mathscr{C}_H P_n.$

P2. $\mathscr{G}(P_1, \ldots, P_n)_H \wedge i, j, k, l, \in \{1, \ldots, n\} \wedge i \neq k$
$\wedge |i - j| > 1 \wedge i - j = k - l \wedge (P_i \mathscr{M} P_j)_H \rightarrow P_i, P_j \boxminus_H P_k, P_l.$

T3. $\mathscr{G}(P_1, \ldots, P_n)_H \wedge i, j, k, l \in \{1, \ldots, n\}$
$\wedge i - j = k - l \wedge (P_i \mathscr{M} P_j)_H \rightarrow P_i, P_j \boxminus_H P_k, P_l \wedge (P_k \mathscr{M} P_l)_H.$

Proof. **P**2, **D**(1, 5.2), **D**5.1, **P**4.2.

One consequence of these postulates is that any two distinct particles on an SG are rigidly connected, i.e.,

*T*4. $\mathscr{G}(P_1, \ldots, P_n)_H \wedge P, Q \in \{P_1, \ldots, P_n\} \wedge P \neq Q$
$\wedge (P \mathscr{M} Q)_H \rightarrow P \mathscr{R}\mathscr{C}_H Q.$

Proof. $\boldsymbol{D}1: \mathscr{G}(P_1, \ldots, P_n)_H \rightarrow \mathscr{C}\mathscr{h}(P_1, \ldots, P_n)_H.$
$\boldsymbol{T}5.8, \boldsymbol{D}4.4 \quad \rightarrow P_1 \mathscr{R}\mathscr{C}_H P_2 \wedge P_2 \mathscr{R}\mathscr{C}_H P_3 \wedge \ldots \wedge P_{n-1} \mathscr{R}\mathscr{C}_H P_n.$ (1)
$(1), \boldsymbol{P}1: \mathscr{G}(P_1, P_2, P_3)_H \wedge (P_1 \mathscr{M} P_3)_H \rightarrow$ **Con.**
$\boldsymbol{T}3: \mathscr{G}(P_1, \ldots, P_4)_H \wedge (P_1 \mathscr{M} P_3)_H \wedge (P_1 \mathscr{M} P_4)_H \rightarrow P_1, P_3 \rightleftharpoons_H P_2, P_4$
$(1), \boldsymbol{T}5.4, \boldsymbol{P}1 \quad \rightarrow$ **Con.**
The proof for any n follows in the same way.

A postulate that leads to the uniqueness of a goedesic between two particles is introduced in Sec. VIII3. The reason it is not introduced here, is that it can be formulated in a more useful form after the concept of linear order is introduced.

7. Linear length measuring instruments

The distance measure introduced in *D*3.2 and *D*4.2 is not necessarily linear, and its value may not indicate the extent of separation of the end particles. For instance, its value could increase with decreasing separation. We now define a *linear* LI, whose measure is the customary one, and which has the properties of a metric.

A linear LI,$\mathfrak{l}$, is an LI whose scale value is zero when the end particles coincide, and if $X_1, \ldots, X_n$ lie on an SG, then for any m between 1 and n,

$$\mu_{\mathfrak{l}}(X_1, X_m)_H = (m-1)\mu_{\mathfrak{l}}(X_1, X_2)_H.$$

*D*1. $\mathscr{L}\mathscr{L}\mathscr{I}(\mathfrak{l})_H$ *for* $\mathscr{L}\mathscr{I}(\mathfrak{l})_H$
$\wedge (\forall X, Y). X, Y \in \mathscr{P}_H \wedge X = Y \leftrightarrow \mu_{\mathfrak{l}}(X, Y)_H = 0.$
$\wedge (\forall X_1, \ldots, X_n, m). \mathscr{G}(X_1, \ldots, X_n)_H \wedge m \in \{3, \ldots, n\}$
$\rightarrow \mu_{\mathfrak{l}}(X_1, X_m)_H = (m-1)\mu_{\mathfrak{l}}(X_1, X_2)_H.$

'$\mathscr{L}\mathscr{L}\mathscr{I}(\mathfrak{l})_H$' reads '$\mathfrak{l}$ is a *linear length instrument* (LLI)'.

*D*2. $\lambda_{\mathfrak{l}}(P, Q; r)_H$ *for* $\mu_{\mathfrak{l}}(P, Q; r)_H \wedge \mathscr{L}\mathscr{L}\mathscr{I}(\mathfrak{l})_H.$

*D*3. $\lambda_{\mathfrak{l}}(P, Q)_H$ *for* $(\iota r). \mu_{\mathfrak{l}}(P, Q)_H = r \wedge \mathscr{L}\mathscr{L}\mathscr{I}(\mathfrak{l})_H.$

The principal property of an LLI,$\mathfrak{l}$, is that if $P_1, \ldots, P_n$ lie on a geodesic, then $\lambda_{\mathfrak{l}}(P_i, P_j)_H$ is $|i-j|$ times the distance between any two successive particles, i.e.,

*T*1. $\mathscr{G}(P_1, \ldots, P_n)_H \wedge \mathscr{L}\mathscr{L}\mathscr{I}(\mathfrak{l})_H \wedge i, j, k+1 \in \{1, \ldots, n\}$
$\wedge i \leq j \wedge (P_i \mathscr{M}_{\mathfrak{l}} P_j)_H \rightarrow \lambda_{\mathfrak{l}}(P_i, P_j)_H = (j-i)\lambda_{\mathfrak{l}}(P_k, P_{k+1})_H.$

Proof. $D1{,}3$: **Ant** $\wedge\, i=j$
$\rightarrow \lambda_{\rm I}(P_i, P_j)_H = \lambda_{\rm I}(P_i, P_i)_H = 0 = (j-i)\lambda_{\rm I}(P_k, P_{k+1})_H.$ (1)
$T6.2, T4.3$: **Ant** $\wedge\, i+1=j$
$\rightarrow \lambda_{\rm I}(P_i, P_j)_H = \lambda_{\rm I}(P_i, P_{i+1})_H = (j-i)\lambda_{\rm I}(P_k, P_{k+1}).$ (2)
$T6.3, T5.5$: **Ant** $\wedge\, i+1<j \rightarrow \lambda_{\rm I}(P_i, P_j)_H = \lambda_{\rm I}(P_1, P_{j-i+1})_H$
$D1$ $= (j-i)\lambda_{\rm I}(P_1, P_2)_H$
$T6.2$ $= (j-i)\lambda_{\rm I}(P_k, P_{k+1})_H.$ (3)
(1)–(3): **T**.

*T*2. $\mathscr{G}(P_1, \ldots, P_n)_H \wedge \mathscr{L}\mathscr{L}\mathscr{I}(\mathrm{I})_H \wedge (P_1 \mathscr{M}_{\rm I} P_n)_H$
$\rightarrow \lambda_{\rm I}(P_1, P_n)_H = \lambda_{\rm I}(P_1, P_2)_H + \lambda_{\rm I}(P_2, P_3)_H + \ldots + \lambda_{\rm I}(P_{n-1}, P_n)_H.$

Proof. $D1$: **Ant** $\rightarrow \lambda_{\rm I}(P_1, P_n)_H = (n-1)\lambda_{\rm I}(P_1, P_2)_H$
$T6.2$ $\rightarrow$ **Con**.

*D*4. $(P\mathscr{M}\mathscr{L}_{\rm I}Q)_H$ *for* $(P\mathscr{M}_{\rm I}Q)_H \wedge \mathscr{L}\mathscr{L}\mathscr{I}(\mathrm{I})_H.$

*D*5. $(P\mathscr{M}\mathscr{L}Q)_H$ *for* $(\exists \mathrm{I})(P\mathscr{M}\mathscr{L}_{\rm I}Q)_H.$

8. Space metric

We now prove that the distance measure provided by an LLI has most of the essential properties of a space metric (see Sec. VI5).

*T*1. $(\exists r)\lambda_{\rm I}(P, Q; r)_H . \rightarrow \lambda_{\rm I}(P, Q)_H \geq 0.$
Proof. $T4.4, D7.(2{,}3)$.

*T*2. $P, Q \in \mathscr{P}_H \wedge \mathscr{L}\mathscr{L}\mathscr{I}(\mathrm{I})_H \rightarrow . \lambda_{\rm I}(P, Q)_H = 0 \leftrightarrow P = Q.$
Proof. $D7.1$.

*T*3. $(\exists r)\lambda_{\rm I}(P, Q; r)_H . \rightarrow \lambda_{\rm I}(P, Q)_H = \lambda_{\rm I}(Q, P)_H.$
Proof. $T4.5, D7.(2, 3)$.

*T*4. $\mathscr{G}(P, Q, R)_H \wedge \mathscr{L}\mathscr{L}\mathscr{I}(\mathrm{I})_H \wedge (P\mathscr{M}_{\rm I}R)_H$
$\rightarrow \lambda_{\rm I}(P, R)_H = \lambda_{\rm I}(P, Q)_H + \lambda_{\rm I}(Q, R)_H.$
Proof. $T7.2$.

To complete the metric properties of $\lambda_{\rm I}$, it is necessary to prove the triangle inequality:

$$\lambda_{\rm I}(P, R)_H \leq \lambda_{\rm I}(P, Q)_H + \lambda_{\rm I}(Q, R)_H,$$

for any three rigidly connected particles. This is done in *T*VIII3.2.

VIII. GEODESIC GEOMETRY

1. Introduction

The material of the previous chapter make it possible to establish contact with many developments in abstract geometry. I borrowed freely from these developments, and in particular from 'Distance Geometry' by Blumenthal [1953], 'Geometry of Geodesics' by Busemann [1955], 'Foundations of Geometry' by Borsuk and Szmielew [1960], 'Foundations of Euclidean Geometry' by Forder [1958], 'Intrinsic Geometry of Ideal Space' by Forsyth [1935], and 'Riemannian Geometry' by Eisenhart [1949]. Many of the borrowed ideas had to be modified to fit the physical theory. No continuity postulate is made, and many existence postulates of abstract geometry are excluded.

This chapter is devoted mainly to the one-dimensional geometry on a space geodesic (SG), and the next chapter to the three-dimensional space geometry. Space-time geometry is taken up in Chapter X.

2. Linear order

In the foundations of geometry (Borsuk [1960] p. 26; Forder [1958] p. 44) linear order is introduced as a primitive concept, but in distance geometry (Blumenthal [1953] pp. 33–34) it is defined. Since a distance measure is available from the previous chapter, the latter approach is adopted.

Particle Q is said to be *between* P and R, if P, Q, R are distinct and the distance between P and R is the sum of the distances between P and Q, and between Q and R.

*D*1. $$[P,Q,R]_H \quad \textit{for} \quad \neq(P,Q,R)$$
$$\wedge (\exists l, p, q, r)\lambda_l(P,Q;p)_H \wedge \lambda_l(Q,R;q)_H \wedge \lambda_l(P,R;r)_H \wedge p+q=r.$$

This is a ternary relation between three particles, and has no direction, in contrast to the binary ordering relation $\prec_H$ between events on a world line. Thus there is a basic difference between the topological structure of particles on an SG and events on a world line. Left and right are conventional, but past and future are absolute.

We now develop the basic properties of ***D***1.

T1. $[P,Q,R]_H \to [R,Q,P]_H.$

Proof. ***D***1, ***D***VII7.2, ***T***VII4.5.

T2. $[P,Q,R]_H \to \sim[Q,P,R]_H \wedge \sim[P,R,Q]_H.$

Proof. ***D***1, ***P***VII4.3: $[P,Q,R]_H \wedge [Q,P,R]_H$

$$\to P\neq Q \wedge (\exists \mathrm{l})\lambda_{\mathrm{l}}(P,R)_H = \lambda_{\mathrm{l}}(Q,R)_H + \lambda_{\mathrm{l}}(P,Q)_H = \lambda_{\mathrm{l}}(Q,R)_H - \lambda_{\mathrm{l}}(P,Q)_H$$

$$\to P\neq Q \wedge (\exists \mathrm{l})\lambda_{\mathrm{l}}(P,Q)_H = 0$$

TVII8.2 $\to P\neq Q \wedge P=Q.$ (1)

(1), T2.(8,22): $[P,Q,R]_H \to \sim[Q,P,R]_H.$ (2)

T1 : $[P,Q,R]_H \to [R,Q,P]_H$

(2) $\to \sim[Q,R,P]_H$

T1, T2.20 $\to \sim[P,R,Q]_H.$ (3)

(2), (3): **T**.

P1. $[P,Q,R]_H \wedge [Q,R,S]_H \wedge (P\mathscr{M}\mathscr{L}S)_H \to [P,Q,S]_H.$

This postulate makes it possible to extend the set of particles on a geodesic. The condition $(P\mathscr{M}\mathscr{L}S)$ is necessary since the distance between P and S is longer than between P and R, and there is no guarantee that there is an LLI with sufficient range to measure the distance between P and S.

The content of the following theorem is essentially the same as ***P***l, except that 'Q' is replaced by 'R' in the consequent.

T3. $[P,Q,R]_H \wedge [Q,R,S]_H \wedge (P\mathscr{M}\mathscr{L}S)_H \to [P,R,S]_H.$

Proof. ***T***1 : **Ant** $\to [S,R,Q]_H \wedge [R,Q,P]_H \wedge (S\mathscr{M}\mathscr{L}P)_H$

P1 $\to [S,R,P]_H$

T1 $\to$ **Con**.

P2. $[P,Q,R]_H \wedge [P,R,S]_H \to [Q,R,S]_H.$

T4. $[P,Q,R]_H \wedge [P,R,S]_H \leftrightarrow [P,Q,S]_H \wedge [Q,R,S]_H.$

Proof. ***P***2: $[P,Q,R]_H \wedge [P,R,S]_H \to [P,Q,R]_H \wedge [Q,R,S]_H$

P1 $\to [P,Q,S]_H \wedge [Q,R,S]_H.$ (1)

T1: $[P,Q,S]_H \wedge [Q,R,S]_H \to [S,R,Q]_H \wedge [S,Q,P]_H$

(1) $\to [S,R,P]_H \wedge [R,Q,P]_H$

T1 $\to [P,Q,R]_H \wedge [P,R,S]_H.$ (2)

(1), (2): **T**.

If Q, R are between P and S, we expect that $Q=R$, or Q is between P and R, or R is between P and Q.

P3. $[P,Q,S]_H \wedge [P,R,S]_H \rightarrow Q=R \vee [P,Q,R]_H \vee [P,R,Q]_H.$

The same statement, with 'P' replaced by 'S' in the consequent, is expressed by:

T5. $[P,Q,S]_H \wedge [P,R,S]_H \rightarrow Q=R \vee [Q,R,S]_H \vee [R,Q,S]_H.$
Proof. **P3**, **T1**, as in **T4**.

The same three possibilities in the consequent of **P3**, apply to particles R, S that are on the same side of P, Q.

P4. $[P,Q,R]_H \wedge [P,Q,S]_H \rightarrow R=S \vee [P,R,S]_H \vee [P,S,R]_H.$

The following theorem bears the same relation to **P4**, as **T5** to **P3**.

T6. $[P,Q,R]_H \wedge [P,Q,S]_H \rightarrow R=S \vee [Q,R,S]_H \vee [Q,S,R]_H.$
Proof. **P4**: **Ant** $\rightarrow R=S \vee .[P,Q,R]_H \wedge [P,R,S]_H.$
$\vee .[P,Q,S]_H \wedge [P,S,R]_H.$
P2 $\rightarrow$ **Con.**

D2. $[P_1,\ldots,P_n]_H$ *for*
$[P_1,P_2,P_3]_H \wedge [P_2,P_3,P_4]_H \wedge \ldots \wedge [P_{n-2},P_{n-1},P_n]_H.$

T7. $[P_1,\ldots,P_n]_H \wedge i,j,k \in \{1,\ldots,n\} \wedge (i<j<k \vee k<j<i)$
$\wedge (P_i \mathscr{M}\mathscr{L} P_k)_H \rightarrow [P_i,P_j,P_k]_H.$
Proof. **D2**, **P1**, **T3**.

T8. $[P_1,\ldots,P_n]_H \rightarrow \mathscr{R}\mathscr{C}(P_1,\ldots,P_n)_H.$
Proof. **D**(2, 1), **D**VII4.(3, 4).

3. Linear order and space geodesics

T1. $\mathscr{G}(P_1,\ldots,P_n)_H \wedge (\exists \mathrm{l})(P_1 \mathscr{M}\mathscr{L}_{\mathrm{l}} P_2)_H \wedge (P_1 \mathscr{M}\mathscr{L}_{\mathrm{l}} P_3)_H \rightarrow [P_1,\ldots,P_n]_H.$
Proof. **T**VII(7.1, 6.3), **D**2.(1, 2).

Thus if $P_1,\ldots,P_n$ lie on a geodesic, they are in linear order $[P_1,\ldots,P_n]_H$. The following postulate insures that the converse is also true (Fig. 5).

P1. $\lambda_{\mathrm{l}}(P,Q;p)_H \vee .[P,R_1,\ldots,R_m,Q]_H \wedge \lambda_{\mathrm{l}}(P,R_1;p_1)_H$
$\wedge \lambda_{\mathrm{l}}(R_1,R_2;p_2)_H \wedge \ldots \wedge \lambda_{\mathrm{l}}(R_m,Q;p_{m+1})_H$
$\wedge p_1+p_2+\ldots+p_{m+1}=p:$

$$\wedge\, \lambda_{\mathrm{I}}(P,S_1;q_1)_H \wedge \lambda_{\mathrm{I}}(S_1,S_2;q_2)_H \wedge \ldots \wedge \lambda_{\mathrm{I}}(S_n,Q;q_{n+1})_H:$$
$$\rightarrow p \leq q_1+\ldots+q_{n+1} \wedge (p=q_1+\ldots+q_{n+1} \leftrightarrow [P,S_1,\ldots,S_n,Q]_H).$$

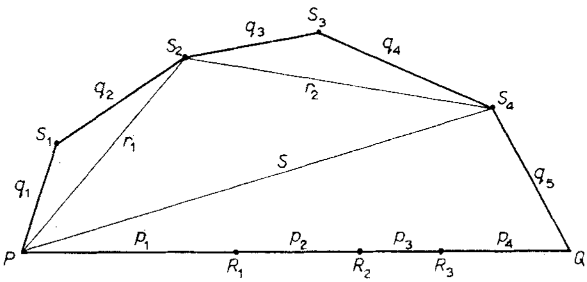

Fig. 5. Distance relations

The meaning of **P**1 is illustrated in Fig. 5. The reason for giving two alternatives to measure the direct distance between P and Q, is that in case this distance is too long to be measured with a single operation, it can still be measured by several operations.

An immediate consequence of **P**1 is the triangle inequality:

*T*2. $\lambda_{\mathrm{I}}(P,Q;p)_H \wedge \lambda_{\mathrm{I}}(Q,R;q)_H \wedge \lambda_{\mathrm{I}}(P,R;r)_H \rightarrow r \leq p+q.$
Proof. **P**1.

In abstract algebra, where all distances are constant, **P**1 can be derived from **T**2 as follows: For the special case shown in Fig. 5, we conclude from **T**2 that $p \leq s+q_5$, $s \leq r_1+r_2, r_1 \leq q_1+q_2, r_2 \leq q_3+q_4$. Thus $p \leq q_1+\ldots+q_5$. Unfortunately, in physics the distances r_1, r_2, and s may all vary with time even though the other distances are constant. Since **T**2 applies only to constant distances, it cannot be used to prove **P**1. It is not possible to generalize **T**2 for variable distances in a way which is independent of observers, because the times at which the measurements are made become important, and questions of simultaneity at different places arise.

The following theorem is essentially the converse of **T**1:

*T*3. $[P_1,\ldots,P_n]_H \wedge \mathscr{C}\hbar(P_1,\ldots,P_n)_H \rightarrow \mathscr{G}(P_1,\ldots,P_n)_H.$
Proof. **P**1, **D**2.1, **T**VII5.(9,5): **Ant** $\wedge\, \mathscr{C}\hbar(X_1,\ldots,X_k)_H$
$\wedge\, X_1=P_1 \wedge X_k=P_n \wedge X_1,X_2 \doteqdot_H P_1,P_2$
$\rightarrow (\exists \mathrm{I}) \sum_{i\,1}^{n-1} \lambda_{\mathrm{I}}(P_i,P_{i+1})_H \leq \sum_{j\,1}^{k-1} \lambda_{\mathrm{I}}(X_j,X_{j+1})_H$

$$\wedge \lambda_l(P_1, P_2)_H = \ldots = \lambda_l(P_{n-1}, P_n)_H$$
$$= \lambda_l(X_1, X_2)_H = \ldots = \lambda_l(X_{k-1}, X_k)_H$$
$$\rightarrow (\exists l)(n-1)\lambda_l(P_1, P_2)_H \leq (k-1)\lambda_l(P_1, P_2)_H$$
$$\rightarrow n \leq k. \quad (1)$$
(1), **D**VII6.1: T.

Any subset of particles on a geodesic, taken at regular intervals, also consists of particles on a geodesic (**T**4).

T4. $\mathscr{G}(P_1, \ldots, P_n)_H \wedge i_1, \ldots, i_k \in \{1, \ldots, n\} \wedge i_1 < \ldots < i_k \wedge i_1 - i_2 = \ldots = i_{k-1} - i_k \wedge (P_{i_1} \mathscr{M}\mathscr{L} P_{i_3})_H \rightarrow \mathscr{G}(P_{i_1}, \ldots, P_{i_k})_H.$

Proof. **T**VII(6.3, 7.1), **D**2.2, **D**VII5.2: **Ant**
$\rightarrow [P_{i_1}, \ldots, P_{i_k}]_H \wedge \mathscr{C}\mathscr{h}(P_{i_1}, \ldots, P_{i_k})_H$
T3 → **Con**.

Two sets of particles on a geodesic can be linked into one set of particles on a geodesic (**T**5).

T5. $\mathscr{G}(P_1, \ldots, P_n)_H \wedge \mathscr{G}(P_{n-1}, P_n, \ldots, P_{n+k})_H \wedge (P_1 \mathscr{M}\mathscr{L} P_3)_H \rightarrow \mathscr{G}(P_1, \ldots, P_{n+k})_H.$

Proof. **T**1, **P**2.1, **T**2.3: **Ant** → $[P_1, \ldots, P_{n+k}]_H.$ (1)
DVII6.1: **Ant** → $\mathscr{C}\mathscr{h}(P_1, \ldots, P_{n+k})_H.$ (2)
(1), (2), **T**3: **T**.

To insure the uniqueness of an SG through two particles, we assume:

P2. $[P, R_1, \ldots, R_m, Q]_H \wedge [P, S_1, \ldots, S_n, Q]_H \wedge (\exists l, r_1, \ldots, r_i, s_1, \ldots, s_j) r_1 + \ldots + r_i = s_1 + \ldots + s_j \wedge \lambda_l(P, R_1; r_1)_H \wedge \lambda_l(R_1, R_2; r_2)_H \wedge \ldots \wedge \lambda_l(R_{i-1}, R_i; r_i)_H \wedge i \leq m \wedge \lambda_l(P, S_1; s_1)_H \wedge \lambda_l(S_1, S_2; s_2)_H \wedge \ldots \wedge \lambda_l(S_{j-1}, S_j; s_j)_H \wedge j \leq n. \rightarrow R_i = S_j.$

This states that if $R_1, \ldots, R_m$ and $S_1, \ldots, S_n$ are linearly ordered particles between P and Q, and the distance between P and R_i is equal to the distance between P and S_j, then $R_i = S_j$. From this it follows that:

T6. $[P, R, Q]_H \wedge [P, S, Q]_H \wedge P, R \stackrel{\sim}{=}_H P, S \rightarrow R = S.$

Proof. **P**2, **D**VII5.1.

The reason we did not assume **T**6, and then prove **P**2, is that it is implicit in **T**6 that the whole distance between the end points is measurable.

The uniqueness of a geodesic is demonstrated by:

$$\textbf{\textit{T}7.} \quad \mathscr{G}(P, R_1, \dots, R_n, Q)_H \wedge \mathscr{G}(P, S_1, \dots, S_n, Q)_H$$
$$\wedge (\exists \mathrm{l})(P \mathscr{M} \mathscr{L}_{\mathrm{l}} R_1)_H \wedge (P \mathscr{M} \mathscr{L}_{\mathrm{l}} R_2)_H \rightarrow R_1 = S_1 \wedge \dots \wedge R_n = S_n.$$

Proof.

$$\textbf{\textit{T}1}: \textbf{Ant} \rightarrow [P, R_1, \dots, R_n, Q]_H \wedge [P, S_1, \dots, S_n, Q]_H. \qquad (1)$$
$$\textbf{\textit{T}VII6.2}: \textbf{Ant} \rightarrow (\exists \mathrm{l}, p, q) \lambda_{\mathrm{l}}(P, R_1; p)_H \wedge \dots \wedge \lambda_{\mathrm{l}}(R_n, Q; p)_H$$
$$\wedge \lambda_{\mathrm{l}}(P, S_1; q)_H \wedge \dots \wedge \lambda_{\mathrm{l}}(S_n, Q; q)_H$$
$$(1), \textbf{\textit{P}1} \qquad \wedge (n+1)p = (n+1)q$$
$$\rightarrow (\exists \mathrm{l}, p) \lambda_{\mathrm{l}}(P, R_1; p)_H \wedge \dots \wedge \lambda_{\mathrm{l}}(R_n, Q; p)_H$$
$$\wedge \lambda_{\mathrm{l}}(P, S_1; p)_H \wedge \dots \wedge \lambda_{\mathrm{l}}(S_n, Q; p)_H$$
$$\textbf{\textit{P}2} \qquad \rightarrow \textbf{Con}.$$

The only possibility for two or more different SG's to pass through two particles P, Q, is if the space geometry were elliptic, and P, Q are its two poles. However, due to the extremely small curvature of the universe, this possibility does not arise for any actual SG's, even if the universe were elliptic.

4. Collinearity

Frequently, all that is necessary to know is that a set of particles lie on an SG, i.e., are collinear; the order in which they lie is unimportant. In such cases, the following definition is useful.

$$\textbf{\textit{D}1.} \quad \mathscr{L}(P, Q, R)_H \quad \textit{for} \quad P, Q, R \in \mathscr{P}_H \wedge \sim \neq (P, Q, R).$$
$$\vee [P, Q, R]_H \vee [Q, P, R]_H \vee [P, R, Q]_H.$$

'$\mathscr{L}(P, Q, R)_H$' reads '*P, Q, R are collinear*' and means: one of the following cases is true: (1) P, Q, R are particles, not all of which are distinct; (2) Q is between P and R; (3) P is between Q and R; (4) R is between P and Q.

According to ***D*1**, any particle is collinear with itself (***T*1**), and any two particles are collinear with each other (***T*2**).

$$\textbf{\textit{T}1.} \quad P \in \mathscr{P}_H \rightarrow \mathscr{L}(P, P, P)_H.$$

Proof. ***D*1**.

$$\textbf{\textit{T}2.} \quad P, Q \in \mathscr{P}_H \rightarrow \mathscr{L}(P, P, Q)_H \wedge \mathscr{L}(P, Q, Q)_H.$$

Proof. ***D*1**.

The fact that the order of particles in $\mathscr{L}(P, Q, R)$ is immaterial, is expressed by:

$$\textbf{\textit{T}3.} \quad \mathscr{L}(P, Q, R)_H \leftrightarrow \mathscr{L}(Q, R, P)_H \leftrightarrow \mathscr{L}(R, P, Q)_H$$
$$\leftrightarrow \mathscr{L}(R, Q, P)_H \leftrightarrow \mathscr{L}(Q, P, R)_H \leftrightarrow \mathscr{L}(P, R, Q)_H.$$

Proof. ***D*1**, ***T*2.1**.

The relationship between collinearity and linear order is shown in *T*4 and *T*5.

*T*4. $[P, Q, R]_H \rightarrow \mathscr{L}(P, Q, R)_H \wedge \neq (P, Q, R)$.

Proof. T2.15, *D*1, *D*2.1.

*T*5. $\mathscr{L}(P, Q, R)_H \wedge \neq (P, Q, R)$.
$\rightarrow [P, Q, R]_H \vee [Q, R, P]_H \vee [P, R, Q]_H$.

Proof. *D*1, *T*2.2.

The following two theorems show how collinear particles can be combined to yield other collinear particles.

*T*6. $\mathscr{L}(P, Q, R)_H \wedge \mathscr{L}(P, Q, S)_H \wedge P \neq Q$
$\rightarrow \mathscr{L}(P, R, S)_H \wedge \mathscr{L}(Q, R, S)_H$.

Proof. *D*1, *P*2.(1–4), *T*2.(3–6).

*T*7. $\mathscr{L}(P,Q,R)_H \wedge \mathscr{L}(P,Q,S)_H \wedge \mathscr{L}(P, Q, T)_H \wedge P \neq Q \rightarrow \mathscr{L}(R, S, T)_H$.

Proof. *T*6: **Ant** $\rightarrow \mathscr{L}(P, R, S)_H \wedge \mathscr{L}(P, R, T)_H$. (1)
*T*6: $\mathscr{L}(P, R, S)_H \wedge \mathscr{L}(P, R, T)_H \wedge P \neq R \rightarrow \mathscr{L}(R, S, T)_H$. (2)
T5.3: **Ant** $\wedge P = R \rightarrow \mathscr{L}(R, Q, S)_H \wedge \mathscr{L}(R, Q, T)_H \wedge R \neq Q$
*T*6 $\rightarrow \mathscr{L}(R, S, T)_H$. (3)
(1)–(3): **T**.

Distinct collinear particles are rigidly connected (*T*8).

*T*8. $\mathscr{L}(P, Q, R)_H \wedge \neq (P, Q, R) \rightarrow \mathscr{RC}(P, Q, R, P)_H$.

Proof. *T*5, *T*2.8.

*D*2. $\mathscr{L}(P_1, \ldots, P_n)_H$ *for*
$\mathscr{L}(P_1, P_2, P_3)_H \wedge \mathscr{L}(P_2, P_3, P_4)_H \wedge \ldots \wedge \mathscr{L}(P_{n-2}, P_{n-1}, P_n)_H$.

'$\mathscr{L}(P_1, \ldots, P_n)_H$' reads '$P_1, \ldots, P_n$ *are collinear*'.

*T*9. $\mathscr{L}(P_1, \ldots, P_n)_H \wedge P, Q, R \in \{P_1, \ldots, P_n\}_H \rightarrow \mathscr{L}(P, Q, R)_H$.

Proof. *D*2, *T*(6, 7).

5. The side relation

Another useful relation between colllinear particles is the one defined below.

*D*1. $P\mathscr{X}_H Q, R$ *for* $P \neq Q, R \wedge \mathscr{L}(P, Q, R)_H \wedge \sim [Q, P, R]_H$.

'$P\mathscr{X}_H Q, R$' reads 'Q, R *are on the same side of* P'.

The following theorems develop the properties of $\mathscr{X}_H$.

*T*1. $P\mathscr{X}_H Q, R \rightarrow Q = R \vee [P, Q, R]_H \vee [P, R, Q]_H$.

Proof. *D*1, *D*4.1.

***T*2.** $P, Q \in \mathscr{P}_H \wedge P \neq Q \rightarrow P\mathscr{X}_H Q, Q.$
Proof. $\boldsymbol{T}4.(2,4), \boldsymbol{D}1.$

***T*3.** $P\mathscr{X}_H Q, R \leftrightarrow P\mathscr{X}_H R, Q.$
Proof. $\boldsymbol{D}1, \boldsymbol{T}4.3, \boldsymbol{T}2.1.$

***T*4.** $P\mathscr{X}_H Q, R \wedge P\mathscr{X}_H R, S \rightarrow P\mathscr{X}_H Q, S.$
Proof.
$\boldsymbol{D}1, \boldsymbol{T}4.6\colon \mathbf{Ant} \rightarrow \mathscr{L}(P, Q, S)_H.$ (1)
$\mathbf{Ant} \wedge Q = R \rightarrow P\mathscr{X}_H Q, S.$ (2)
$\boldsymbol{T}1, \boldsymbol{D}1\colon \mathbf{Ant} \wedge Q \neq R \rightarrow \sim [R, P, S]_H \wedge . [P, Q, R]_H \vee [P, R, Q]_H.$ (3)
$\boldsymbol{T}2.(1,3,4)\colon [P, Q, R]_H \vee [P, R, Q]_H . \wedge [Q, P, S]_H\colon$
$\rightarrow [R, P, S]_H.$ (4)
$(4), \mathrm{T}2.32\colon [P, Q, R]_H \vee [P, R, Q]_H . \wedge \sim [R, P, S]_H\colon$
$\rightarrow \sim [Q, P, S]_H.$ (5)
$(3), (5)\colon \mathbf{Ant} \wedge Q \neq R \rightarrow \sim [Q, P, S]_H.$ (6)
$(1), (2), (6), \boldsymbol{D}1\colon \mathbf{T}.$

***T*5.** $\mathscr{L}(P, Q, R)_H \wedge P \neq Q \rightarrow P = R \veebar [Q, P, R]_H \veebar P\mathscr{X}_H Q, R.$
Proof. $\boldsymbol{T}4.5, \boldsymbol{D}1.$

***T*6.** $[P, Q, R]_H \rightarrow P\mathscr{X}_H Q, R \wedge R\mathscr{X}_H P, Q.$
Proof. $\boldsymbol{T}4.4, \boldsymbol{T}2.2, \boldsymbol{D}1.$

***T*7.** $[P, Q, R]_H \rightarrow \sim Q\mathscr{X}_H P, R.$
Proof. $\boldsymbol{T}2.2, \mathrm{T}2.(15, 38).$

***T*8.** $[P, Q, R]_H \wedge [P, Q, S]_H \rightarrow P\mathscr{X}_H R, S \wedge Q\mathscr{X}_H R, S.$
Proof. $\boldsymbol{P}2.4, \boldsymbol{T}2.6, \boldsymbol{T}2, \boldsymbol{T}6.$

***T*9.** $[P, Q, S]_H \wedge [P, R, S]_H \rightarrow P\mathscr{X}_H Q, R \wedge S\mathscr{X}_H Q, R.$
Proof. $\boldsymbol{P}2.3, \boldsymbol{T}2.5, \boldsymbol{T}2, \boldsymbol{T}6.$

6. Congruence

In Sec.VII 5, congruence is defined and some of its basic properties are proved. Here, more of its properties are demonstrated in the light of the material developed since then.

***T*1.** $P, Q \boldsymbol{\models\!\!\!\!\dashv}_H P, R \rightarrow \sim . [P, Q, R]_H \vee [P, R, Q]_H.$
Proof.
$\boldsymbol{D}2.1, \boldsymbol{T}\mathrm{VII}8.1\colon [P, Q, R]_H \vee [P, R, Q]_H$
$\rightarrow (\exists \mathfrak{l}, p, q) \lambda_1(P, Q; p)_H \wedge \lambda_1(P, R; q)_H \wedge (p < q \vee q < p)$
$\boldsymbol{D}\mathrm{VII}5.1 \rightarrow \sim P, Q \boldsymbol{\models\!\!\!\!\dashv}_H P, R$ (1)
$(1), \mathrm{T}2.20\colon \mathbf{T}.$

If Q, R are on the same side of P and are equidistant from P, they must be identical (*T*2).

T2. $P\mathscr{X}_H Q, R \wedge P, Q \rightleftharpoons_H P, R \rightarrow Q = R.$

Proof. T5.1: $P\mathscr{X}_H Q, R \rightarrow Q = R \vee [P, Q, R]_H \vee [P, R, Q]_H.$ (1)

T1: $P, Q \rightleftharpoons_H P, R \rightarrow \sim .[P, Q, R]_H \vee [P, R, Q]_H.$ (2)

(1), (2): T.

D1. $\mathscr{Mid}(P, Q, R)_H$ *for*

$\mathscr{L}(P, Q, R)_H \wedge P \neq R \wedge P, Q \rightleftharpoons_H Q, R.$

'$\mathscr{Mid}(P, Q, R)_H$' means: P, Q, R are collinear, and Q is *midway* between P and R.

T3. $\mathscr{Mid}(P, Q, R)_H \leftrightarrow \mathscr{Mid}(R, Q, P)_H.$

Proof. **D**1, **T**4.3, **T**VII5.3.

T4. $\mathscr{Mid}(P, Q, R)_H \rightarrow [P, Q, R]_H$

Proof. TVII5.1: $P = Q \vee Q = R. \wedge P, Q \rightleftharpoons_H Q, R$

$\rightarrow P = Q \wedge Q = R$

$\rightarrow P = R.$ (1)

(1), **D**1: **Ant** $\rightarrow \mathscr{L}(P, Q, R)_H \wedge \neq(P, Q, R)$

T4.5 $\rightarrow [P, Q, R]_H \veebar [Q, P, R]_H \veebar [P, R, Q]_H.$

D1, **T**1 $\rightarrow$ **Con.**

The uniqueness of the middle particle follows from **T**4 and **T**3.6.

T5. $(\exists^1 X)\mathscr{Mid}(P, X, Q)_H.$

Proof. **T**4, **D**1, **D**VII5.2: $\mathscr{Mid}(P, X, Q)_H \wedge \mathscr{Mid}(P, Y, Q)_H$

$\rightarrow [P, X, Q]_H \wedge \mathscr{Ch}(P, X, Q)_H \wedge [P, Y, Q]_H \wedge \mathscr{Ch}(P, Y, Q)_H$

T3.3 $\rightarrow \mathscr{G}(P, X, Q)_H \wedge \mathscr{G}(P, Y, Q)_H$

T3.7 $\rightarrow X = Y.$ (1)

(1), D9: **T.**

Theorems 6 to 10 establish relations between corresponding sets of particles.

T6. $P\mathscr{X}_H Q, R \wedge P'\mathscr{X}_H Q', R' \wedge P, Q \rightleftharpoons_H P', Q' \wedge P, R \rightleftharpoons_H P', R'$

$\rightarrow .Q = R \leftrightarrow Q' = R'.$

Proof. **Ant** $\wedge Q = R \rightarrow P, Q \rightleftharpoons_H P', Q' \wedge P, Q \rightleftharpoons_H P', R'$

TVII5.6 $\rightarrow P', Q' \rightleftharpoons_H P', R' \wedge P'\mathscr{X}_H Q', R'$

T2 $\rightarrow Q' = R'.$ (1)

Similarly: **Ant** $\wedge Q' = R' \rightarrow Q = R.$ (2)

(1), (2), T2.32: **T.**

***T*7.** $[P,Q,R]_H \wedge [P',Q',R']_H \wedge P,Q \mathrel{\vDash\!\!\dashv}_H P',Q'$
$\wedge\, Q,R \mathrel{\vDash\!\!\dashv}_H Q',R' \rightarrow P,R \mathrel{\vDash\!\!\dashv}_H P',R'.$

Proof. ***D*2.1, *P*VII4.3: Ant**

$\rightarrow (\exists \mathrm{I})\, \lambda_\mathrm{I}(P,R)_H = \lambda_\mathrm{I}(P,Q)_H + \lambda_\mathrm{I}(Q,R)_H$
$\wedge\, \lambda_\mathrm{I}(P',R')_H = \lambda_\mathrm{I}(P',Q')_H + \lambda_\mathrm{I}(Q',R')_H$
$\wedge\, \lambda_\mathrm{I}(P,Q)_H = \lambda_\mathrm{I}(P',Q')_H \wedge \lambda_\mathrm{I}(Q,R)_H = \lambda_\mathrm{I}(Q',R')_H$
$\rightarrow (\exists \mathrm{I})\, \lambda_\mathrm{I}(P,R)_H = \lambda_\mathrm{I}(P',R')_H$

***D*VII5.1 → Con.**

***T*8.** $[P,Q,R]_H \wedge P'\mathscr{X}_H Q',R' \wedge P,Q \mathrel{\vDash\!\!\dashv}_H P',Q' \wedge P,R \mathrel{\vDash\!\!\dashv}_H P',R'$
$\rightarrow [P',Q',R']_H \wedge Q,R \mathrel{\vDash\!\!\dashv}_H Q',R'.$

Proof. ***T*5.1:** $P'\mathscr{X}_H Q',R'$
$\rightarrow Q' = R' \vee [P',Q',R']_H \vee [P',R',Q']_H.$ (1)

Ant $\wedge\, Q' = R' \rightarrow P',Q' \mathrel{\vDash\!\!\dashv}_H P,Q \wedge P',Q' \mathrel{\vDash\!\!\dashv}_H P,R$

***T*VII5.6** $\rightarrow P,Q \mathrel{\vDash\!\!\dashv}_H P,R$

***T*1, T2.39** $\rightarrow \sim [P,Q,R]_H$ (2)

(2), T2.9: **Ant** $\rightarrow \sim Q' = R'.$ (3)

***D*2.1, *P*VII4.3: Ant** $\rightarrow (\exists \mathrm{I})\, \lambda_\mathrm{I}(P,Q)_H < \lambda_\mathrm{I}(P,R)_H$
$\wedge\, \lambda_\mathrm{I}(P,Q)_H = \lambda_\mathrm{I}(P',Q')_H \wedge \lambda_\mathrm{I}(P,R)_H = \lambda_\mathrm{I}(P',R')_H$
$\rightarrow (\exists \mathrm{I})\, \lambda_\mathrm{I}(P',Q')_H < \lambda_\mathrm{I}(P',R')_H.$ (4)

***D*2.1:** $[P',R',Q']_H \rightarrow (\exists \mathrm{I})\, \lambda_\mathrm{I}(P',Q')_H > \lambda_\mathrm{I}(P',R')_H.$ (5)

(4), (5): **Ant** $\rightarrow \sim [P',R',Q']_H.$ (6)

(1), (3), (6): **Ant** $\rightarrow [P',Q',R']_H$ (7)
$\rightarrow [P,Q,R]_H \wedge [P',Q',R']_H \wedge P,Q \mathrel{\vDash\!\!\dashv}_H P',Q'$
$\wedge\, P,R \mathrel{\vDash\!\!\dashv}_H P',R'.$

As in ***T*7** $\rightarrow Q,R \mathrel{\vDash\!\!\dashv}_H Q',R'.$ (8)

(7), (8): **T.**

***T*9.** $[P,Q,R]_H \wedge P,Q \mathrel{\vDash\!\!\dashv}_H P',Q' \wedge Q,R \mathrel{\vDash\!\!\dashv}_H Q',R'$
$\wedge\, P,R \mathrel{\vDash\!\!\dashv}_H P',R' \rightarrow [P',Q',R]_H.$

Proof. ***D*2.1, *P*VII4.3: Ant** $\rightarrow (\exists \mathrm{I})\, \lambda_\mathrm{I}(P',R')_H = \lambda_\mathrm{I}(P',Q')_H + \lambda_\mathrm{I}(Q',R')_H$

***D*2.1** $\rightarrow$ **Con.**

If Q is midway between P and R, and Q' is midway between P' and R', then if any corresponding distances are equal, all the other corresponding distances are equal (***T*10**).

***T*10.** $\mathscr{Mid}(P,Q,R)_H \wedge \mathscr{Mid}(P',Q',R')_H$
$\rightarrow :P,Q \mathrel{\vDash\!\!\dashv}_H P',Q' \rightarrow Q,R \mathrel{\vDash\!\!\dashv}_H Q',R' \wedge P,R \mathrel{\vDash\!\!\dashv}_H P',R'.$
$\wedge\, .Q,R \mathrel{\vDash\!\!\dashv}_H Q',R' \rightarrow P,Q \mathrel{\vDash\!\!\dashv}_H P',Q' \wedge P,R \mathrel{\vDash\!\!\dashv}_H P',R'.$
$\wedge\, .P,R \mathrel{\vDash\!\!\dashv}_H P',R' \rightarrow P,Q \mathrel{\vDash\!\!\dashv}_H P',Q' \wedge Q,R \mathrel{\vDash\!\!\dashv}_H Q',R'.$

Proof. $T4, D1, T(7,8), T\text{VII}5.6$.

If R is collinear with two distinct particles P, Q, and S has the same distances from P, Q as R, then $S = R(T11)$.

$T11$. $\mathscr{L}(P,Q,R)_H \wedge P \neq Q \wedge P,R \doteqdot_H P,S \wedge Q,R \doteqdot_H Q,S \rightarrow R = S$.

Proof. $T\text{VII}5.1$: $\textbf{Ant} \wedge (P = R \vee Q = R) \rightarrow R = S$. (1)

$T4.5$: $\textbf{Ant} \wedge \neq(P,Q,R) \rightarrow [P,Q,R]_H \vee [Q,P,R]_H \vee [P,R,Q]_H$. (2)

$D\text{VII}5.1$: $\textbf{Ant} \rightarrow P,Q \doteqdot_H P,Q \wedge Q,R \doteqdot_H Q,S \wedge P,R \doteqdot_H P,S$. (3)

(3), $T9$: $\textbf{Ant} \wedge [P,Q,R]_H \rightarrow [P,Q,S]_H$

$T5.8$ $\rightarrow P\mathscr{X}_H R,S \wedge P,R \doteqdot_H P,S$

$T2$ $\rightarrow R = S$. (4)

Similarly: $\textbf{Ant} \wedge [Q,P,R]_H \rightarrow R = S$. (5)

(3), $T9$: $\textbf{Ant} \wedge [P,R,Q]_H \rightarrow [P,S,Q]_H$

$T3.6$ $\rightarrow R = S$. (6)

(1), (2), (4)–(6): **T**.

IX. SPACE GEOMETRY

1. Non-collinearity

Three particles P, Q, R are said to be *non-collinear* if they are not collinear.

D1. $\mathscr{N}(P,Q,R)_H$ *for* $P,Q,R\in\mathscr{P}_H \wedge \sim\mathscr{L}(P,Q,R)_H$.

Non-collinear particles are distinct (**T**1).

T1. $\mathscr{N}(P,Q,R)_H \rightarrow \neq(P,Q,R)$.
Proof. **D**1, **D**VIII4.1.

T2. $\mathscr{N}(P,Q,R)_H \leftrightarrow \mathscr{N}(Q,R,P)_H \leftrightarrow \mathscr{N}(R,P,Q)_H$
$\leftrightarrow \mathscr{N}(R,Q,P)_H \leftrightarrow \mathscr{N}(Q,P,R)_H \leftrightarrow \mathscr{N}(P,R,Q)_H$.
Proof. **D**1, **T**VIII4.3.

Collinearity and non-collinearity are mutually exclusive (**T**3).

T3. $\sim.\mathscr{L}(P,Q,R)_H \wedge \mathscr{N}(P,Q,R)_H$.
Proof. **D**1.

Three particles can either be collinear or non-collinear (**T**4).

T4. $P,Q,R\in\mathscr{P}_H \rightarrow \mathscr{L}(P,Q,R)_H \veebar \mathscr{N}(P,Q,R)_H$.
Proof. T(2.2, 1.10, 2.40): **Ant**
$\rightarrow: P,Q,R\in\mathscr{P}_H \wedge \mathscr{L}(P,Q,R)_H . \vee . P,Q,R\in\mathscr{P}_H \wedge \sim\mathscr{L}(P,Q,R)_H$
DVIII4.1, **D**1 $\rightarrow \mathscr{L}(P,Q,R)_H \vee \mathscr{N}(P,Q,R)_H$
T3 $\rightarrow$ **Con.**

A particle cannot be both collinear and non-collinear with two other particles (**T**5).

T5. $\mathscr{L}(P,R,Q)_H \wedge \mathscr{N}(P,S,Q)_H \rightarrow R\neq S$.
Proof. **Ant** $\wedge\, R=S \rightarrow \mathscr{L}(P,R,Q)_H \wedge \mathscr{N}(P,R,Q)_H$. (1)
(1), **T**3, T2.8: **T.**

Non-collinearity does not imply rigidity, but nonrigidity and distinctness does imply non-collinearity (**T**6).

T6. $P, Q, R \in \mathscr{P}_H \wedge \neq(P, Q, R) \wedge \sim P\mathscr{RC}_H Q \rightarrow \mathscr{N}(P, Q, R)_H$.

Proof. **D**VII4.4, **T**VIII2.8: **Ant**$\rightarrow \neq(P, Q, P)$
$\wedge \sim[P, Q, R]_H \wedge \sim[Q, P, R]_H \wedge \sim[P, R, Q]_H$
DVIII4.1 $\rightarrow \sim \mathscr{L}(P, Q, R)_H$
T4 $\rightarrow$**Con.**

Non-collinearity implies that the inequality sign applies in the triangle inequality **T**VIII3.2(**T**7).

T7. $\mathscr{N}(P, Q, R)_H \wedge \lambda_1(P, Q; p)_H \wedge \lambda_1(Q, R; q)_H$
$\wedge \lambda_1(P, R; r)_H \rightarrow r < p+q$.

Proof. **D**VIII2.1: **Ant** $\wedge\ r = p+q \rightarrow [P, Q, R]_H$
TVIII4.4, **T**3 $\rightarrow \sim\mathscr{N}(P, Q, R)_H$. (1)
(1), T2.32: **Ant**$\rightarrow r \neq p+q$
TVIII3.2 $\rightarrow$**Con.**

Two different SG's can intersect in at most one particle (**T**8, Fig. 6).

T8. $\mathscr{N}(P, Q, R)_H \wedge R \neq S$
$\rightarrow (\exists^1 X) \mathscr{L}(P, Q, X)_H \wedge \mathscr{L}(R, S, X)_H$.

Proof. **T**VIII4.6: $\mathscr{L}(P, Q, X)_H \wedge \mathscr{L}(P, Q, Y)_H \wedge P \neq Q$
$\rightarrow \mathscr{L}(P, X, Y)_H \wedge \mathscr{L}(Q, X, Y)_H$. (1)
TVIII4.6: $\mathscr{L}(R, S, X)_H \wedge \mathscr{L}(R, S, Y)_H \wedge R \neq S$
$\rightarrow \mathscr{L}(R, X, Y)_H$. (2)
TVIII4.7: $\mathscr{L}(P, X, Y)_H \wedge \mathscr{L}(Q, X, Y)_H \wedge \mathscr{L}(R, X, Y)_H$
$\rightarrow . X \neq Y \rightarrow \mathscr{L}(P, Q, R)_H$
$\rightarrow . \sim \mathscr{L}(P, Q, R)_H \rightarrow X = Y$
T4 $\rightarrow . \mathscr{N}(P, Q, R)_H \rightarrow X = Y$. (3)
(1)–(3), D9: T.

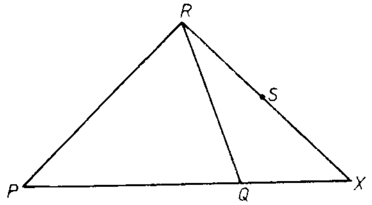

Fig. 6. **T**8

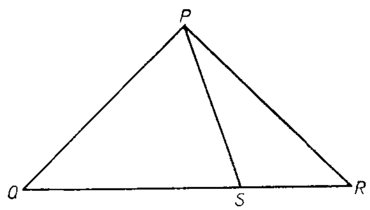

Fig. 7. **T**9

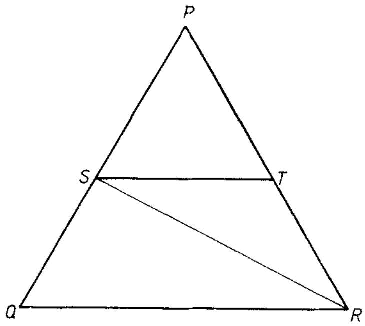

Fig. 8. *T*10

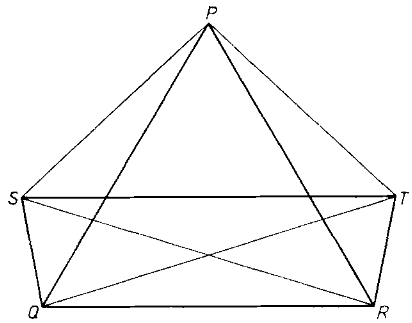

Fig. 9. *T*11

***T*9.** $\mathcal{N}(P,Q,R)_H \wedge \mathcal{L}(Q,R,S)_H \wedge Q \neq S \rightarrow \mathcal{N}(P,Q,S)_H$.

Proof. **T**VIII4.6: $\mathcal{L}(Q,R,S)_H \wedge Q \neq S$

$\rightarrow . \mathcal{L}(P,Q,S)_H \rightarrow \mathcal{L}(P,Q,R)_H$

T2.20, ***T*4** $\rightarrow . \mathcal{N}(P,Q,R)_H \rightarrow \mathcal{N}(P,Q,S)_H$. (1)

(1), T2.32: **T**.

***T*10.** $\mathcal{N}(P,Q,R)_H \wedge \mathcal{L}(P,Q,S)_H \wedge \mathcal{L}(P,R,T)_H$

$\wedge P \neq S, T \rightarrow \mathcal{N}(P,S,T)_H$.

Proof. ***T*9**: **Ant**$\rightarrow \mathcal{N}(P,R,S)_H \wedge \mathcal{L}(P,R,T)_H \wedge P \neq T$

***T*9** $\rightarrow$**Con**.

***T*11.** $\mathcal{N}(P,Q,R)_H \wedge S, T \in \mathcal{P}_H \wedge S \neq T$

$\rightarrow \mathcal{N}(P,S,T)_H \vee \mathcal{N}(Q,S,T)_H \vee \mathcal{N}(R,S,T)_H$.

Proof. **T**VIII4.7: $\mathcal{L}(P,S,T)_H \wedge \mathcal{L}(Q,S,T)_H$

$\wedge \mathcal{L}(R,S,T)_H \wedge S \neq T \rightarrow \mathcal{L}(P,Q,R)_H$

***T*3** $\rightarrow \sim \mathcal{N}(P,Q,R)_H$. (1)

(1), T2.32, ***T*4**: **T**.

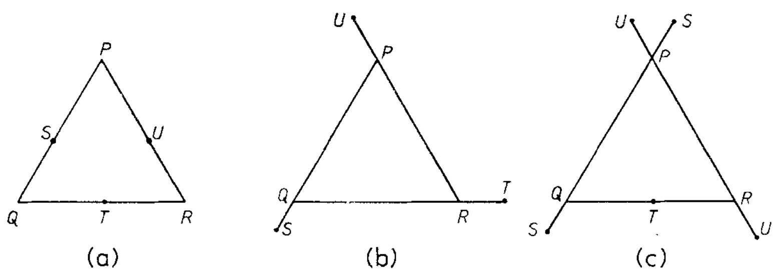

Fig. 10. ***P*1**

P1.
$$\mathcal{N}(P,Q,R)_H \wedge :[P,S,Q]_H \wedge [Q,T,R]_H \wedge [R,U,P]_H.$$
$$\vee .[P,Q,S]_H \wedge [Q,R,T]_H \wedge [R,P,U]_H.$$
$$\vee .[Q,T,R]_H \wedge ([P,Q,S]_H \vee [Q,P,S]_H)$$
$$\wedge ([P,R,U]_H \vee [R,P,U]_H):$$
$$\rightarrow \mathcal{N}(S,T,U)_H.$$

The meaning of **P1** is illustrated in Fig. 10; each of the diagrams corresponds to one of the three possibilities listed in **P1** that insures the non-collinearity of S, T, U. The reader can verify that possibilities not included in Fig. 10 do not insure non-collinearity.

2. Coplanarity

In abstract Euclidean geometry, a point D is said to be coplanar with three non-collinear points A, B, C, if D lies on a line that connects two points of the triangle ABC. The plane determined by A, B, C, is defined to be the class of all points coplanar with A, B, C (Forder [1958] p. 57).

What makes this definition work, is the property that if D and E are coplanar with A, B, C, then any one of these five points is coplanar with any three of them, e.g., E is coplanar with A, B, D. In other words, there is only one plane that passes through three non-collinear points.

Unfortunately, this property is not satisfied in Riemannian geometry, except in the cases of constant curvature, and physical geometry is Riemannian. To see how this property breaks down, consider three neighboring non-collinear particles A, B, C, and let several geodesics connect A with points on the geodesic through B and C. The surface S_A consisting of all these geodesics is called the *geodesic surface* at A. If D and E are points on the geodesics AB and AC, then in general the geodesic connecting D and E does not intersect all the geodesics on S_A, i.e., it does not lie on S_A (Forsyth [1935] Vol. II, pp. 135–136). In order for S_A to qualify as a totally geodesic surface (plane), it is necessary that the geodesic through D and E must completely lie on S_A for any D and E.

The conditions for existence of a totally geodesic surface are given by Eisenhart ([1949] pp. 183–184). It is shown by Cartan ([1928] pp. 123–127) that the existence of a totally geodesic surface is equivalent to the possibility of displacement of figures without deformation (free mobility of figures). This possibility is ruled out by Einstein's general theory of relativity in regions such as the vicinity of a massive object.

Consequently, we have to carry on the development of the theory without the concept of coplanarity.

3. Perpendicularity

PQ is said to be *perpendicular* to PR at $P(P \perp_H Q, R)$ if P, Q, R are non-collinear, P is rigidly connected to Q and R, and for any particles X, Y different than P, if X is collinear with P, Q and Y is collinear with P, R, then the distance between X and P is smaller than the distance between X and Y (Fig. 11), i.e., the distance on a perpendicular from a point X to a line is the

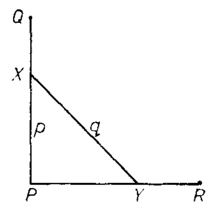

Fig. 11. Perpendicularity

shortest distance from X to the line (Busemann [1955] pp. 103, 121). The point P is called the *foot of the perpendicular*. Notice that the distance between Q and R need not be constant.

***D*1.** $P \perp_H Q, R$ *for* $\mathcal{N}(P,Q,R)_H \wedge \mathcal{RC}(Q,P,R)_H$
$\wedge (\forall X, Y, \mathfrak{l}, p, q).\mathcal{L}(P,Q,X)_H \wedge \mathcal{L}(P,R,Y)_H \wedge P \neq X, Y$
$\wedge \lambda_{\mathfrak{l}}(X,P;p)_H \wedge \lambda_{\mathfrak{l}}(X,Y;q)_H \rightarrow p < q.$

If PQ is perpendicular to PR, and S is collinear with P and Q, then PS is perpendicular to PR(**T**1).

***T*1.** $P \perp_H Q, R \wedge \mathcal{L}(P,Q,S)_H \wedge P \neq S \rightarrow P \perp_H S, R.$

Proof. **Ant** $\wedge\, Q = S \rightarrow P \perp_H S, R.$ (1)

$T1.9{:}\mathcal{N}(P,Q,R)_H \wedge \mathcal{L}(P,Q,S)_H \wedge P \neq S \rightarrow \mathcal{N}(P,R,S)_H.$ (2)

$T1.1{:}$ **Ant** $\wedge\, S \neq Q \rightarrow \mathcal{L}(P,Q,S)_H \wedge \neq(P,Q,S)_H$
TVIII(4.5, 2.8) $\rightarrow \mathcal{RC}(R,P,S)_H.$ (3)

TVIII4.6: $\mathcal{L}(P,Q,S)_H \wedge \mathcal{L}(P,Q,X)_H \wedge P \neq Q$
$\rightarrow \mathcal{L}(P,S,X)_H.$ (4)

(2)–(4), ***D*1**: **Ant** $\wedge\, Q \neq S \rightarrow P \perp_H S, R.$ (5)

(1), (5): **T**.

A priori there is no reason why perpendicularity should be symmetric. However, experimental evidence indicates that it is (***P*1**).

***P*1.** $P \perp_H Q, R \rightarrow P \perp_H R, Q.$

With the help of ***P*1**, ***T*1** can be applied to the other side of the right angle.

***T*2.** $P \perp_H Q, R \wedge \mathscr{L}(P, R, S)_H \wedge P \neq S \rightarrow P \perp_H Q, S.$

Proof. ***P*1, *T*1.**

The foot of the perpendicular is unique (***P*2**).

***P*2.** $P \perp_H Q, R \wedge P' \perp_H Q, S \wedge \mathscr{L}(P, P', R, S)_H \rightarrow P = P'.$

***T*3.** $P \perp_H Q, R \wedge P' \perp_H S, R \wedge \mathscr{L}(P, P', Q, S)_H \rightarrow P = P'.$

Proof. ***P*1, 2.**

In spherical and elliptic geometries, it is possible to have 2 perpendiculars to the same line from one of the poles. In ***P*2** this possibility is excluded because of the extremely small curvature of space.

One consequence of ***P*2** is that two perpendiculars to the same line can never meet (***T*4**, Fig. 12).

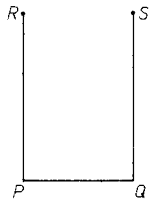

Fig. 12. ***T*4**

***T*4.** $P \perp_H R, Q \wedge Q \perp_H S, P$
$\rightarrow \sim(\exists X) \mathscr{L}(P, R, X)_H \wedge \mathscr{L}(Q, S, X)_H.$

Proof. **$D1$: Ant** $\wedge X = P \rightarrow \mathscr{N}(Q, S, X)_H$

$T1.3$, T2.(15,38) $\rightarrow \sim . \mathscr{L}(P, R, X)_H \wedge \mathscr{L}(Q, S, X)_H.$ (1)

Similarly: **Ant** $\wedge X = Q \rightarrow \sim . \mathscr{L}(P, R, X)_H \wedge \mathscr{L}(Q, S, X)_H.$ (2)

$T1,2$: Ant $\wedge \mathscr{L}(P, R, X)_H \wedge X \neq P \wedge \mathscr{L}(Q, S, X)_H \wedge X \neq Q$
$\rightarrow P \perp_H X, Q \wedge Q \perp_H X, P \wedge \mathscr{L}(P, Q, Q, P)_H$

$P2, D1, T1.1 \rightarrow P = Q \wedge P \neq Q.$ (3)

(3), T2.8: **Ant** $\wedge X \neq P, Q \rightarrow \sim . \mathscr{L}(P, R, X)_H \wedge \mathscr{L}(Q, S, X)_H.$ (4)

(1), (2), (4): **Ant** $\rightarrow \sim . \mathscr{L}(P, R, X)_H \wedge \mathscr{L}(Q, S, X)_H$

P(3, 5), T3.17 $\rightarrow$ **Con.**

***D*2.** $P \perp_H Q; R, S, \ldots$ *for* $P \perp_H Q, R \wedge P \perp_H Q, S \wedge \ldots$

From a point on a plane, only one line can be erected perpendicular to the plane; or to be more precise, if P,Q,R are non-collinear, only one perpendicular can be erected from P, which is perpendicular to both PQ and PR(**P**3, Fig. 13).

P3. $\mathscr{N}(P,Q,R)_H \wedge P \perp_H S; Q,R \wedge P \perp_H T; Q,R \rightarrow \mathscr{S}(P,S,T)_H.$

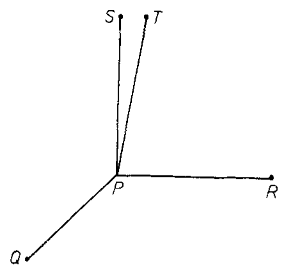

Fig. 13. **P**3

4. Parallel displacement

In Euclidean geometry, two lines are said to be parallel if they lie on the same plane and make equal angles with a straight line in that plane. We saw in Sec.2 that the concept of a plane is not always available in physical geometry, and thus a different tack must be used. Two ways out were suggested: one by Levi-Civita (Eisenhart [1949] pp. 62–65, 72–74; Forsyth [1935] Vol. II, pp. 98–101), and the other by Severi (Forsyth [1935] Vol. II, pp. 101–107). Both methods use the idea of parallel displacement, but Levi-Civita uses the idea of an imbedding space, whereas Severi uses geodesics. Severi's definition of parallel displacement fits well in our theory, and we use it as the basis for our definition.

Consider three rigidly connected non-collinear particles P, Q, R. We wish to find a particle S, such that RS can be considered parallel to PQ. There are two things that must be accomplished: (1) the geodesic RS must at least start tangent to the geodesic surface at P (see Sec. 2), and (2) the corresponding angles that PQ and RS subtend with PR must be equal (Fig. 14).

Since the space curvature may be variable, the closer we are to P, the better off we are. In fact, in a region sufficiently close to P, the curvature becomes practically constant and the concept of coplanarity is applicable. If R is sufficiently close to P, it is possible to find a particle U on PQ on the same side as Q, and a particle V on RS on the same side as S, such that PV and

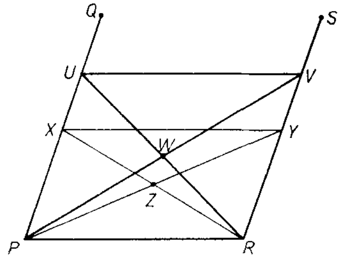

Fig. 14. Parallel displacement

RU intersect at some particle W. In order to test whether R, U, V are sufficiently close to P, we take a particle X between P and U, and a particle Y between R and V. If PY and RX always intersect for any X and Y chosen in this way, then we know that we are sufficiently close to P, and that RV lies on the geodesic surface at P.

We can now make RV have the same direction as PU by taking the distance between P and U equal to the distance between R and V, and the distance between U and V equal to the distance between P and R. Unless R, U and V are sufficiently close to P, it is usually not possible to satisfy these conditions and also have PV and RU intersect. Thus the operational meaning of 'sufficiently close' to P is simply that all these conditions can be satisfied for U and V as well as for X and Y.

What we have accomplished, is to displace PQ along the geodesic PR to RS so that the *initial* directions of PQ and RS are the same. This does not guarantee that PQ and RS remain parallel if extended. In other words, if PQ is extended to PQ' and RS to RS', such that the distance between P and Q' is equal to the distance between R and S', then it may be found that at Q' and S' the two geodesics are neither parallel nor coplanar.

We now put the above ideas in symbols.

D1.
$$
\begin{aligned}
&P, Q \mathrel{\upharpoonleft\!\upharpoonright}_H R, S \quad \textit{for} \quad \mathscr{N}(P,Q,R)_H \wedge \mathscr{N}(R,S,P)_H \\
&\wedge (\exists\, U, V, W)\, P\mathscr{X}_H Q, U \wedge R\mathscr{X}_H S, V \wedge [P, W, V]_H \wedge [R, W, U]_H \\
&\wedge P, U \mathrel{\vDash\!\dashv}_H R, V \wedge P, R \mathrel{\vDash\!\dashv}_H U, V \\
&\wedge (\forall X, Y).[P, X, U]_H \wedge [R, Y, V]_H \wedge P, X \mathrel{\vDash\!\dashv}_H R, Y \\
&\qquad\qquad \rightarrow P, R \mathrel{\vDash\!\dashv}_H X, Y \wedge (\exists\, Z)[P, Z, Y]_H \wedge [R, Z, X]_H .
\end{aligned}
$$

'$P, Q \mathrel{\upharpoonleft\!\upharpoonright}_H R, S$' reads '*RS is displaced parallel to PQ* along *PR*'.

Another important difference between parallelism in Euclidean and Riemannian geometries, is that if a geodesic segment is displaced parallel to itself along a closed path, it ends coincident with itself in Euclidean geometry, but pointing in a different direction in Riemannian geometry. In fact, this difference in direction can be used as a measure of the curvature of space in the region of the closed path.

***T*1.** $P, Q \upharpoonleft\upharpoonright_H R, S \leftrightarrow R, S \upharpoonleft\upharpoonright_H P, Q.$

Proof. ***D*1.**

However '$P, Q \upharpoonleft\upharpoonright_H R, S$' is not symmetric with respect to interchange of the other letters.

***T*2.** $P, Q \upharpoonleft\upharpoonright_H R, S \wedge P\mathscr{X}_H Q, T \rightarrow P, T \upharpoonleft\upharpoonright_H R, S.$

Proof. $\boldsymbol{D}$VIII5.1, $\boldsymbol{T}$1.9: $\mathscr{N}(P, Q, R)_H \wedge P\mathscr{X}_H Q, T \rightarrow \mathscr{N}(P, R, T)_H.$ (1)

$\boldsymbol{T}$VIII5.(3, 4): $P\mathscr{X}_H Q, U \wedge P\mathscr{X}_H Q, T \rightarrow P\mathscr{X}_H T, U.$ (2)

(1), (2), $\boldsymbol{D}$1: **T**.

***T*3.** $P, Q \upharpoonleft\upharpoonright_H R, S \wedge R\mathscr{X}_H S, T \rightarrow P, Q \upharpoonleft\upharpoonright_H R, T.$

Proof. ***T*1,2.**

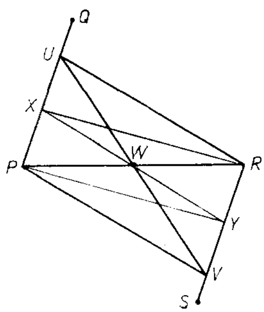

Fig. 15. Antiparallel displacement

Antiparallel displacement can be defined in analogy with parallel displacement (Fig. 15).

***D*2.** $P, Q \upharpoonleft\downharpoonright_H R, S$ *for*

$$\mathscr{N}(P, Q, R)_H \wedge \mathscr{N}(R, S, P)_H$$
$$\wedge (\exists\, U, V, W)\, P\mathscr{X}_H Q, U \wedge R\mathscr{X}_H S, V$$
$$\wedge [P, W, R]_H \wedge [U, W, V]_H \wedge P, U \mathrel{\models\!\dashv}_H R, V \wedge P, V \mathrel{\models\!\dashv}_H R, U$$
$$\wedge (\forall X, Y).[P, X, U]_H \wedge [R, Y, V]_H \wedge P, X \mathrel{\models\!\dashv}_H R, Y$$
$$\rightarrow P, Y \mathrel{\models\!\dashv}_H R, X \wedge [X, W, Y]_H.$$

'$P, Q \upharpoonleft\downharpoonright_H R, S$' reads '*RS is displaced antiparallel to PQ along PR*'.

T4. $P, Q \uparrow\downarrow_H R, S \leftrightarrow R, S \uparrow\downarrow_H P, Q.$

Proof. **D2.**

T5. $P, Q \uparrow\downarrow_H R, S \wedge P\mathscr{X}_H Q, T \rightarrow P, T \uparrow\downarrow_H R, S.$

Proof. As in **T2**.

T6. $P, Q \uparrow\downarrow_H R, S \wedge R\mathscr{X}_H S, T \rightarrow P, Q \uparrow\downarrow_H R, T.$

Proof. **T4**, 5.

As might be expected, parallelism and antiparallelism are mutually exclusive (**T7**, Fig. 16).

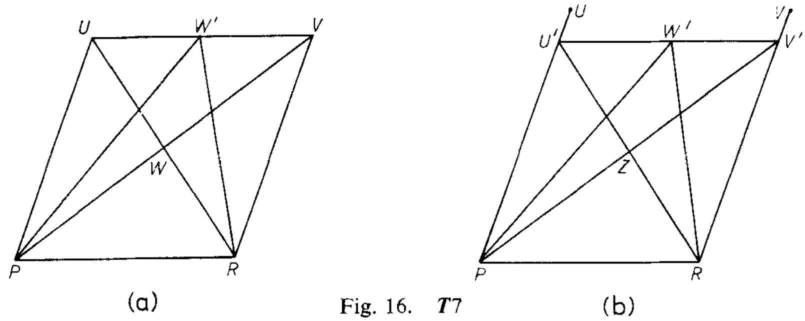

Fig. 16. **T7**

T7. $\sim .P, Q \uparrow\uparrow_H R, S \wedge P, Q \uparrow\downarrow_H R, S.$

Proof. **D**1, 2, **T**VII5.6, **T**VIII(5.4, 5.1, 6.2, 6.8):

$P, Q \uparrow\uparrow_H R, S \wedge P, Q \uparrow\downarrow_H R, S \rightarrow (\exists\, U, V, W, U', V', W')\ldots$

$\wedge : U = U' \wedge V = V'. \vee .[P, U', U]_H \wedge [R, V', V]_H.$

$\vee .[P, U, U']_H \wedge [R, V, V']_H.$ (1)

If $U = U' \wedge V = V'$, then we have (Fig. 16a)

$[P, W, V]_H \wedge [R, W, U]_H \wedge [U, W', V]_H \wedge [P, W', R]_H.$

DVIII5.1, **T**1.9: $\mathscr{N}(P, Q, R)_H \wedge P\mathscr{X}_H Q, U \rightarrow \mathscr{N}(P, R, U)_H.$

TVIII4.4, **T**1.9: $\mathscr{N}(P, R, U)_H \wedge [R, W, U]_H \rightarrow \mathscr{N}(P, W, U)_H.$

TVIII4.4, **T**1.9: $\mathscr{N}(P, W, U)_H \wedge [P, W, V]_H \rightarrow \mathscr{N}(U, V, W)_H.$

P1.1: $\mathscr{N}(U, V, W)_H \wedge [U, W', V]_H \wedge [U, W, R]_H \wedge [V, W, P]_H$

$\rightarrow \mathscr{N}(P, W', R)_H$, which contradicts $[P, W', R]_H.$ (2)

D1: $[P, U', U]_H \wedge [R, V', V]_H \wedge P, U' \rightleftharpoons_H R, V'$

$\rightarrow (\exists Z)[P, Z, V']_H \wedge [R, Z, U']_H,$

and a contradiction follows as above (Fig. 16b). (3)

D2: $[P, U, U']_H \wedge [R, V, V']_H \wedge P, U \rightleftharpoons_H R, V$

$\rightarrow [U, W', V]_H \wedge [P, W', R]_H,$

which leads to the same conclusion as the first case. (4)

(1)–(4): **T.**

P1. $P, Q \upuparrows_H R, S \wedge [S, R, T]_H \rightarrow P, Q \updownarrows_H R, T.$

T8. $P, Q \upuparrows_H R, S \wedge [Q, P, T]_H \rightarrow P, T \updownarrows_H R, S.$
Proof. **T**1, **P**1.

P2. $P, Q \updownarrows_H R, S \wedge [S, R, T]_H \rightarrow P, Q \upuparrows_H R, T.$

T9. $P, Q \updownarrows_H R, S \wedge [Q, P, T]_H \rightarrow P, T \upuparrows_H R, S.$
Proof. **T**4, **P**2.

D3. $P, Q \parallel_H R, S$ *for* $P, Q \upuparrows_H R, S \vee P, Q \updownarrows_H R, S$

T10. $P, Q \parallel_H R, S \leftrightarrow R, S \parallel_H P, Q.$
Proof. **D**3, **T**1,4.

T11. $P, Q \parallel_H R, S \wedge \mathscr{L}(P, Q, T)_H \wedge P \neq T \rightarrow P, T \parallel_H R, S.$
Proof. **D**VIII4.1, **D**3, **T**2, 8.

T12. $P, Q \parallel_H R, S \wedge \mathscr{L}(R, S, T)_H \wedge R \neq T \rightarrow P, Q \parallel_H R, T.$
Proof. **T**10, 11.

Two geodesics parallel displaced in the same way along the same path are collinear (**P**3).

P3. $P, Q \parallel_H R, S \wedge P, Q \parallel_H R, T \rightarrow \mathscr{L}(R, S, T)_H.$

T13. $P, Q \upuparrows_H R, S \wedge P, Q \upuparrows_H R, T \rightarrow R \mathscr{X}_H S, T.$
Proof. **P**3, **D**1, **T**VIII4.5: **Ant** $\rightarrow \mathscr{L}(R, S, T)_H \wedge R \neq S, T$
$\wedge .S = T \vee [R, S, T]_H \vee [R, T, S]_H \vee [T, R, S]_H.$ (1)
DVIII5.1, **T**VIII2.2: $\mathscr{L}(R, S, T)_H \wedge R \neq S, T$
$\wedge .S = T \vee [R, S, T]_H \vee [R, T, S]_H : \rightarrow R \mathscr{X}_H S, T.$ (2)
P1: **Ant** $\wedge [T, R, S]_H \rightarrow P, Q \upuparrows_H R, T \wedge P, Q \updownarrows_H R, T.$ (3)
(3), **T**7: **Ant** $\rightarrow \sim [T, R, S]_H.$ (4)
(1), (2), (4): **T**.

T14. $P, Q \upuparrows_H R, S \wedge P, Q \updownarrows_H R, T \rightarrow [S, T, R]_H.$
Proof. As in **T**13.

T15. $P, Q \updownarrows_H R, S \wedge P, Q \updownarrows_H R, T \rightarrow R \mathscr{X}_H S, T.$
Proof. As in **T**13.

Two parallels cannot meet (**P**4).

P4. $\sim P, Q \parallel_H R, Q.$

If PQ is perpendicular to PR, and RS is displaced parallel to PQ along PR, then RP is perpendicular to RS at R(**P**5).

P5. $P \perp_H Q, R \wedge P, P \parallel_H R, S \rightarrow R \perp_H P, S.$

D4. $P, Q \upharpoonleft\!\upharpoonright_H R, S \upharpoonleft\!\downharpoonright_H T, U \parallel_H A, B \ldots$ *for*
$P, Q \upharpoonleft\!\upharpoonright_H R, S \wedge R, S \upharpoonleft\!\downharpoonright_H T, U \wedge T, U \parallel_H A, B \wedge \ldots$

5. Dimensions

The assumption that physical space is 3-dimensional is made in two parts, space is assumed to be (1) at least 3-dimensional, and (2) at most 3-dimensional. To get a feeling for how these assumptions should be stated, let us start with 1-dimensional space.

If we take two points P, Q on a line, and consider two points R, R' that have the same distances p, q from P and Q, then if space were at most 1-dimensional, R must be identical with R'. On the other hand, if space were at least 2-dimensional, we could have $R \neq R'$ (Fig. 17).

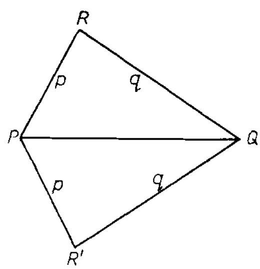

Fig. 17. Two dimensions

If space is at most 2-dimensional, we are stuck to a surface and there can be no other point besides R and R' that have the same distances p, q from P and Q. But if space is at least 3-dimensional, we could get out of the surface and find two other points S and S' on each side of the surface that have the same distances from P and Q as R and R' (Fig. 18). This brings another distance into the picture, namely the distance r between R and S and between R and S'. Finally, if space is at most 3-dimensional, there can be no other point besides S and S' that have the same distances p, q, r from P, Q, R.

Thus, we assume that *space is at least 3-dimensional* (**P**1) by assuming the existence of three non-collinear rigidly connected particles U, V, W, and the existence of two other distinct particles X, Y that have the same distances from U, V, W. (In Fig. 18, U, V, W correspond to P, Q, R and X, Y to S, S'.)

P1. $$(\exists\, U,V,W,X,Y)\mathscr{N}(U,V,W)_H \wedge \mathscr{RC}(U,V,W,U)_H \wedge \neq(U,V,W,X,Y) \wedge U,X \doteqdot_H U,Y \wedge V,X \doteqdot_H V,Y \wedge W,X \doteqdot_H W,Y.$$

The assumption that *space is at most* 3-*dimensional* (**P**2) is expressed as follows: If P,Q,R are non-collinear rigidly connected particles, and S is a particle different from P,Q,R, then there exists at most one particle X different from S such that the distances of S and X from P,Q,R are the same (X corresponds to S' in Fig. 18).

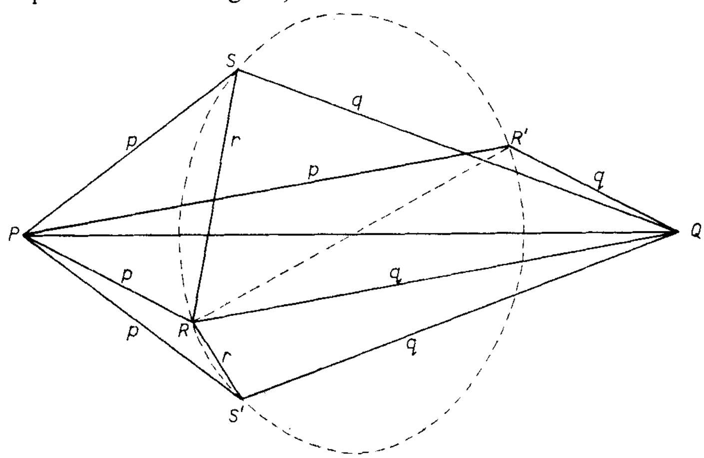

Fig. 18. Three dimensions

P2. $$\mathscr{N}(P,Q,R)_H \wedge \mathscr{RC}(P,Q,R,P)_H \wedge S \in \mathscr{P}_H \wedge S \neq P,Q,R \rightarrow (\exists^1 X) X \neq S \wedge P,X \doteqdot_H P,S \wedge Q,X \doteqdot_H Q,S \wedge R,X \doteqdot_H R,S.$$

The distances $\lambda_1(X,P)_H$, $\lambda_1(X,Q)_H$, and $\lambda_1(X,R)_H$ can be considered as the *space coordinates* of particle X relative to the coordinate system P,Q,R,S, where S is the origin and SP, SQ, SR are the axes.

Since an event at X must be specified by an additional time coordinate given by a standard clock whose output particle coincides with X, *space-time is* clearly 4-*dimensional.*

6. Geodesic space coordinates

A coordinate system (CS) is very useful in specifying the position of particles and describing their motion. Moreover, by writing the distance measure (line element) as a function of coordinates in a finite region, it is possible to give a complete description of the geometry of the region.

In the general theory of relativity there is tremendous freedom in the choice of coordinates. One set of coordinates suggested by Synge ([1956] p. 7) to illustrate this freedom, is obtained by four 'old battered' but hardy clocks carried by flying aeroplanes that turn, dive, and climb in an arbitrary way. By sending signals from an event to the clocks, the times of arrival of the signals at the clocks can be taken as the four coordinates of the event; the signals can be the sound waves of an explosion. A more conventional CS can be described as a scaffolding constructed from an elastic material (Reichenbach [1958] pp. 263–264). The material must be elastic because a rigid structure is not possible in a time varying gravitational field.

Unfortunately, this freedom has a price; the price being the same as that paid when one tries to specify the time of events by an arbitrary clock (see Sec. VI 1). If the time intervals between a sequence of periodic events vary, it is not possible to determine whether the variation is due to the clock or to the physical process producing the events, unless something is known about the clock itself. Similarly, unless the CS is constructed according to a specific operational procedure, the properties of the CS and the physical process described by it will be mixed up in an unknown way. For this reason, we give below the complete instructions for the construction of a CS in an arbitrary gravitational field.

The CS described here is a generalization of the common rectangular CS. The reasons for all the complications become clear at the end, but for now it is sufficient to state that the culprits are the fact that a rigid structure is not possible in a time varying gravitational field, and the geometry is Riemannian. The formulation of this CS in the framework of general relativity and its application are given by the author elsewhere (Basri [1965] Secs. 9A, 10A and 11).

The CS consists of an orthogonal triad of particles and a method of specifying the position of any other particle relative to this triad. An orthogonal triad O, A, B, C is a set of four particles such that OA, OB and OC are mutually perpendicular $(O \perp A, B; O \perp B, C; O \perp C, A)$. To specify the position of any particle P relative to O, A, B, C, first extend OA linearly by particles $X_1(=A), X_2, \ldots, X_i$, i.e., let $O, X_1, \ldots, X_i$ lie on a space geodesic (SG) $[\mathscr{G}(O, X_1, \ldots, X_i)_H]$. This implies that the distances between neighboring particles are all equal to the distance between O and A. Next parallel displace OB and OC along this SG until X_i is reached (**D**4.1). Let the two SG segments obtained in this way be X_iU_i and X_iV_i (Fig. 19). Then extend X_iU_i linearly by particles $Y_1(=U_i), Y_2, \ldots, Y_j$ so that $\mathscr{G}(X_i, Y_1, \ldots, Y_j)_H$, and paral-

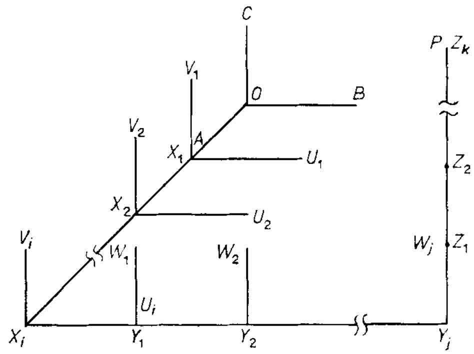

Fig. 19. 'Rectangular' CS

lel displace X_iV_i along this SG to get Y_jZ_1. Finally, extend Y_jZ_1 linearly by particles $Z_2, \dots, Z_k$ to reach $P(=Z_k)$, such that $\mathscr{G}(Y_j, Z_1, \dots, Z_k)_H$.

The numbers i,j,k are the 'rectangular' coordinates of P with respect to the CS, O,A,B,C. The order in which these coordinates are obtained is important; if OB is extended first instead of OA, the coordinates could have different values. This is one of the peculiarities of Riemannian geometry.

Another way of looking at this CS is to think of the SG defined by $\mathscr{G}(O, X_1, X_2, \dots)$ as the X-axis, the SG's defined by $\mathscr{G}(X_i, Y_1, Y_2, \dots)$ the Y-axes (in the XY-plane), and the SG's defined by $\mathscr{G}(Y_j, Z_1, \dots, Z_k)$ as the Z-axes. The distances between points on the Y and Z-axes may vary with time. Moreover, the curves $Y=\text{const}$, $Z=\text{const}$ may neither be SG's nor parallel to the X-axis. These are some of the peculiarities of our space, and the reasons why the CS was defined as above. It is not a rigid CS, but it is well defined.

We now define more precisely the sentence: *The space coordinates of particle P are i,j,k with respect to the geodesic rectangular coordinate system O,A,B,C.* Since the space extension of any CS is limited, we choose the origin so that only *positive* values of i,j,k need be considered. The inclusion of negative values complicates the formulation, but is straightforward.

D1. $P(i,j,k) - \mathscr{RCS}(O,A,B,C)_H$ *for*
$O \perp_H A, B \wedge O \perp_H B, C \wedge O \perp_H C, A$
$\wedge :j=0 \to .i=0 \to [k=0 \to P=O.$
$\wedge .k>0 \to (\exists Z_1, \dots, Z_k) Z_1 = C \wedge \mathscr{G}(O, Z_1, \dots, Z_k)_H \wedge Z_k = P].$
$\wedge .i>0 \to (\exists X_1, \dots, X_i)[X_1 = A \wedge \mathscr{G}(O, X_1, \dots, X_i)_H$
$\wedge .k=0 \to X_i = P. \wedge .k>0 \to (\exists V_1, \dots, V_i, Z_1, \dots, Z_k).$

$$O, C \mathrel{\uparrow\!\!\upharpoonright}_H X_1, V_1 \mathrel{\uparrow\!\!\upharpoonright}_H \dots \mathrel{\uparrow\!\!\upharpoonright}_H X_i, V_i \wedge V_i = Z_1 \wedge Z_k = P$$
$$\wedge\, O, C \mathrel{\boxminus}_H X_1, V_1 \mathrel{\boxminus}_H \dots \mathrel{\boxminus}_H X_i, V_i \wedge \mathscr{G}(X_i, Z_1, \dots, Z_k)_H]:$$
$$\wedge\, {:}j > 0 \rightarrow .i = 0 \rightarrow (\exists\, Y_1, \dots, Y_j)[Y_1 = B \wedge \mathscr{G}(O, Y_1, \dots, Y_j)_H$$
$$\wedge\, .k = 0 \rightarrow Y_j = P. \wedge .k > 0 \rightarrow (\exists\, W_1, \dots, W_j, Z_1, \dots, Z_k).$$
$$O, C \mathrel{\uparrow\!\!\upharpoonright}_H Y_1, W_1 \mathrel{\uparrow\!\!\upharpoonright}_H \dots \mathrel{\uparrow\!\!\upharpoonright}_H Y_j, W_j \wedge W_j = Z_1 \wedge Z_k = P$$
$$\wedge\, O, C \mathrel{\boxminus}_H Y_1, W_1 \mathrel{\boxminus}_H \dots \mathrel{\boxminus}_H Y_j, W_j \wedge \mathscr{G}(Y_j, Z_1, \dots, Z_k)_H].$$
$$\wedge\, .i > 0 \rightarrow (\exists\, X_1, \dots, X_i, U_1, \dots, U_i, Y_1, \dots, Y_j).X_1 = A$$
$$\wedge\, \mathscr{G}(O, X_1, \dots, X_i)_H \wedge O, B \mathrel{\uparrow\!\!\upharpoonright}_H X_1, U_1 \mathrel{\uparrow\!\!\upharpoonright}_H \dots \mathrel{\uparrow\!\!\upharpoonright}_H X_i, U_i \wedge U_i = Y_1$$
$$\wedge\, O, B \mathrel{\boxminus}_H X_1, U_1 \mathrel{\boxminus}_H \dots \mathrel{\boxminus}_H X_i, U_i \wedge \mathscr{G}(X_i, Y_1, \dots, Y_j)_H$$
$$\wedge\, [k = 0 \rightarrow Y_j = P. \wedge .k > 0 \rightarrow (\exists\, V_1, \dots, V_i, W_1, \dots, W_j, Z_1, \dots, Z_k).$$
$$O, C \mathrel{\uparrow\!\!\upharpoonright}_H X_1, V_1 \mathrel{\uparrow\!\!\upharpoonright}_H \dots \mathrel{\uparrow\!\!\upharpoonright}_H X_i, V_i \mathrel{\uparrow\!\!\upharpoonright}_H Y_1, W_1 \mathrel{\uparrow\!\!\upharpoonright}_H \dots \mathrel{\uparrow\!\!\upharpoonright}_H Y_j, W_j$$
$$\wedge\, W_j = Z_1 \wedge O, C \mathrel{\boxminus}_H X_1, V_1 \mathrel{\boxminus}_H \dots \mathrel{\boxminus}_H X_i, V_i \mathrel{\boxminus}_H Y_1, W_1$$
$$\mathrel{\boxminus}_H \dots \mathrel{\boxminus}_H Y_j, W_j \wedge \mathscr{G}(Y_j, Z_1, \dots, Z_k)_H \wedge Z_k = P.$$

In the next chapter the following two definitions are needed:

***D*2.** $\{P(\mathfrak{x}) - \mathscr{RCS}(O, A, B, C)_{\mathfrak{l}}\}_H$ *for*
$$(\exists\, i, j, k)\, P(i, j, k) - \mathscr{RCS}(O, A, B, C)_H$$
$$\wedge\, \mathscr{LLS}(\mathfrak{l})_H \wedge \lambda_{\mathfrak{l}}(O, A)_H = \lambda_{\mathfrak{l}}(O, B)_H = \lambda_{\mathfrak{l}}(O, C)_H = \lambda.$$
$$\wedge\, (\mathfrak{x}) = (\mathfrak{x}^1, \mathfrak{x}^2, \mathfrak{x}^3) \wedge \mathfrak{x}^1 = i\lambda \wedge \mathfrak{x}^2 = j\lambda \wedge \mathfrak{x}^3 = k\lambda.$$

***D*3.** $\{a[P(\mathfrak{x})] - \mathscr{RCS}(O, A, B, C)_{\mathfrak{l}}\}_H$ *for*
$$\{P(\mathfrak{x}) - \mathscr{RCS}(O, A, B, C)_{\mathfrak{l}}\}_H \wedge a \in \mathscr{W}(P)_H.$$

Several expressions of type ***D*2** and ***D*3** are combined as follows:

$$\{P(\mathfrak{x}_P), a[Q(\mathfrak{x}_Q)], \dots - \mathscr{RCS}(O, A, B, C)_{\mathfrak{l}}\}_H$$

stands for

$$\{P(\mathfrak{x}_P) - \mathscr{RCS}(O, A, B, C)_{\mathfrak{l}}\}_H$$
$$\wedge\, \{a[Q(\mathfrak{x}_Q) - \mathscr{RCS}(O, A, B, C)_{\mathfrak{l}}\}_H \wedge \dots.$$

X. SPACE-TIME GEOMETRY

1. Introduction

Time geometry on a world line was developed in Chap.VI, and the foundation of space geometry was laid in Chaps.VII–IX. The two geometries are married in this chapter to form a solid foundation for space-time geometry. The parts of Einstein's general theory of relativity (GR) that are not touched are the field equations and the energy-momentum tensor.

GR utilizes a single time that applies over all space. This can be done only if all clocks are synchronized. Although this is possible in principle, it is shown below that synchronization is conventional, arbitrary, and impractical in a time varying gravitational field. For this reason, our formulation makes no use of synchronization; it constitutes a new approach which is based upon GR.

It is verified experimentally that the relative rate of clocks, as determined by the measured frequency of electromagnetic signals sent from one clock to the other, depends upon the location, time, and state of motion of the clocks. For instance, the rate of clocks at different heights above the earth's surface are found to be different (Pound and Rebka [1960]), and the same is true for clocks placed at different radii on a rotating disk (Kündig [1963]). As the angular speed of the disk changes, so do the relative rates of the clocks. Moreover, synchronization may not even be transitive; if two clocks on the rim of a rotating disk are synchronized with the clock at the center, the two clocks are found not to be synchronous with each other. It can be shown (Basri [1965] Sec.6B) that synchronization is not transitive if and only if two beams of light sent simultaneously in opposite directions around a closed path of non-zero area arrive back at different times. Two beams of light were sent in opposite directions around the rim of a rotating disk, and were indeed found to return out of phase with each other (Sagnac [1913], Macek and Davis [1963]).

Thus in an inhomogeneous time-varying gravitational field, the rates of clocks depend upon both position and time, and synchronization is not transitive. Consequently, synchronization must be performed continuously between all clocks and a central clock. Due to the finite speed of any signal, it takes time to synchronize a distant clock, which adds another complication. Moreover, since, strictly speaking, synchronization can only be performed between neighboring clocks, it can be seen that synchronization in such an environment is rather impractical. This, added to the fact that synchronization is conventional and arbitrary (see end of Sec.V4), makes it highly desirable to find a method to describe physical phenomena without synchronization.

One method of relating events at one particle with events at another particle is to send first signals from one particle to the other. For example, if we are interested in measuring the velocity of a particle P between points A and B, we let a first signal depart from A simultaneously with P. The time interval that P arrives at B behind the arrival of the first signal can be used as a measure of the time P takes to go from A to B. This *lag time* does not depend upon synchronization, and is less complicated to measure operationally (synchronization requires at least two first signals).

In the next section we derive from GR a relation between lag time and other (proper) time intervals. (Proper time is the time measured by a single standard clock.) This basic relation plays the same role in our theory as the line element plays in GR.

2. The fundamental space-time relation

In this section we start with the line element of GR and derive a relation between proper time intervals that plays a fundamental role in our theory.

Consider two neighboring particles A, B, and let a first signal start from A at a, be reflected by B at b, and return back to A at c, and a particle P depart from B at b and reach A at d (Fig. 20). Suppose that coincident with A is the output particle C of a coordinate clock running at some specified rate ($\sqrt{(-g_{00})}$) relative to the standard clock at A. Let b' be an event between a and c on C that is considered simultaneous with b by some method of synchronization. In GR, the time that P takes to go from B to A is given by the interval $\mathrm{d}t$ between b' and d measured by clock C, and the travel time of the first signal from B to A is given by the interval $\mathrm{d}t'$ between b' and c measured by C.

The time interval $\tau_P(b,d)$ measured by a standard clock whose output particle coincides with P, is usually denoted in GR by $\mathrm{d}\tau$. Moreover, if c is the average speed of light in vacuum in a round trip between two neighboring particles, it is common to write:

$$ds = c\,d\tau = c\tau_P(b,d), \qquad dx^0 = c\,dt, \tag{1}$$

where ds is known as the *line element*. Let dx^i be the difference in space coordinates $x_A^i - x_B^i$ relative to some CS(x^i are defined in *D*IX6.2).

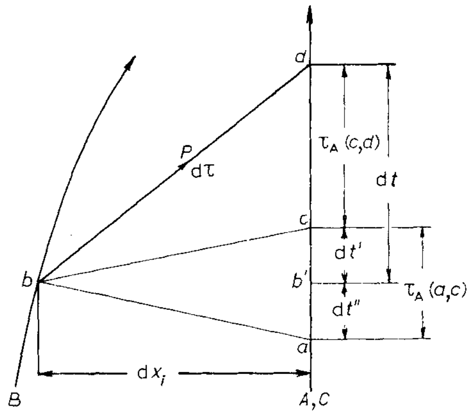

Fig. 20. Space-time relation

The relation between ds, dx^0, and dx^i in GR is given by:

$$ds^2 = -g_{\mu\nu}\,dx^\mu\,dx^\nu \tag{2a}$$

$$= -g_{00}(dx^0)^2 - 2g_{0i}\,dx^i\,dx^0 - g_{ij}\,dx^i\,dx^j, \tag{2b}$$

where the convention of summing over repeated indices is used, Latin indices range over the values 1, 2, 3, and Greek indices over the values 0, 1, 2, 3.

For a clock at rest in the CS, $dx^i = 0$, and

$$ds = \sqrt{(-g_{00})}\,dx^0, \quad \text{or} \quad d\tau = \sqrt{(-g_{00})}\,dt. \tag{3}$$

Since dt is measured by a coordinate clock and $d\tau$ by a standard clock, it can be seen from (3) that $\sqrt{(-g_{00})}$ is the ratio of the rate of a coordinate clock to the rate of a coincident standard clock.

Our goal is to express the time dx^0 measured by synchronized clocks in terms of the lag time $\tau_A(c,d)$, and then eliminate it from (2). To do this, we notice that

$$\tau_A(c,d) = \tau_A(b',d) - \tau_A(b',c),$$
$$= \sqrt{(-g_{00})}(dt - dt'). \tag{4}$$

The interval $\mathrm{d}t'$ can be obtained from the equation of motion, $\mathrm{d}s=0$, of the light signal from B to A. Thus, we get from (2) the quadratic equation:

$$-g_{00}(c\,\mathrm{d}t')^2 - 2(g_{0i}\,\mathrm{d}x^i)\,c\,\mathrm{d}t' - g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = 0,$$

whose solution is:

$$-g_{00}c\,\mathrm{d}t' = g_{0i}\,\mathrm{d}x^i + [(g_{0i}\,\mathrm{d}x^i)^2 - g_{00}g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j]^{\frac{1}{2}}.$$

By means of the definitions:

$$\gamma_i = g_{0i}/\surd(-g_{00}), \qquad \gamma_{ij} = g_{ij} + \gamma_i\gamma_j, \tag{5}$$

this solution can be written in the form:

$$\surd(-g_{00})c\,\mathrm{d}t' = \gamma_i\,\mathrm{d}x^i + (\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j)^{\frac{1}{2}}. \tag{6}$$

If we substitute (6) into (4), solve (4) for $\mathrm{d}t$, and use this result in (2), we get with the help of (1):

$$\tau_P(b,d)^2 = \tau_A(c,d)^2 + 2c^{-1}\tau_A(c,d)\,(\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j)^{\frac{1}{2}}, \tag{7}$$

which is the desired fundamental relation.

We now go one step further, and express the last factor in (7) in terms of the round trip time $\tau_A(a,c)$. If $\mathrm{d}t''$ is the coordinate time interval that the first signal takes to travel from A to B (Fig. 20), then according to (3)

$$\tau_A(a,c) = \surd(-g_{00})(\mathrm{d}t' + \mathrm{d}t'').$$

The expression for $\mathrm{d}t''$ can be obtained from (6) by replacing '$\mathrm{d}x^i$' by '$-\mathrm{d}x^i$'. Thus,

$$\surd(-g_{00})c\,\mathrm{d}t'' = -\gamma_i\,\mathrm{d}x^i + (\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j)^{\frac{1}{2}},$$

and it follows from the last two equations and (6), that

$$\tau_A(a,c) = 2c^{-1}(\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j)^{\frac{1}{2}}. \tag{8}$$

This is the direct bond between space and time geometries.

From (7) and (8) we get (Basri [1965] Eq. 8.8)

$$\tau_P(b,d)^2 = \tau_A(c,d)^2 + \tau_A(c,d)\,\tau_A(a,c), \tag{9}$$

which the fundamental relation expressed purely in terms of proper times (Fig. 20).

3. Neighborhood

Since the rate of a standard clock may vary with both position and time (Sec.1), there is no escape from the formulation of space-time geometry in terms of local relations. The question is: how small should be the space-time intervals between two events, in order that the events can be considered to be in the neighborhood of each other? To answer this question, we seek guidance from the theory of ordinary surfaces.

If a certain experimental error is allowed, then given any point on an ordinary surface, one can always find a region around the point which is small enough that Euclidean geometry is valid in the region within the experimental error. For example, on the surface of a sphere of radius R, the circumference of a circle of radius r is given by (Sommerville [1958] p. 116):

$$C = 2\pi R \sin(r/R).$$

If $r/R \to 0$, then $\sin(r/R) \to r/R$, and $C \to 2\pi r$, which is the usual Euclidean value on a plane. For $r/R = 0.5$, C deviates from $2\pi r$ by about 4%, and for $r/R = 0.1$, the deviation decreases to 0.2%. Similarly, for a right triangle having legs of length a, b, and hypotenuse of length c, we have (Sommerville [1958] p. 119):

$$\cos\frac{c}{R} = \cos\frac{a}{R}\cos\frac{b}{R}.$$

$$\text{As } \frac{c}{R} \to 0,\ \cos\frac{c}{R} \to 1 - \tfrac{1}{2}\left(\frac{c}{R}\right)^2,\ \cos\frac{a}{R}\cos\frac{b}{R} \to 1 - \tfrac{1}{2}\frac{a^2+b^2}{R^2},$$

and this relation reduces to Pythagoras theorem,

$$c^2 = a^2 + b^2,$$

valid in Euclidean geometry.

In analogy with this, a *space-time neighborhood* is defined as a region in which the basic relation (2.9) is satisfied within the experimental error.

Two numbers q, r are said to be equal within the experimental error, if for any p between 0 and 1, the probability that $|q-r|$ is less than the variance of $q-r$ divided by $\sqrt{p}$, is larger than or equal to $1-p(D1)$. This statement is based upon Tchebycheff's inequality (Cramér [1946] p. 182), and does not depend on the probability density function (Eq.VI3.3; see also comments after *T*VI6.1). If the PDF is known, then a more precise statement can be made.

D1. $q \overset{E}{=} r$ *for*

$$(\forall p) . 0 < p < 1 \rightarrow \mathbf{P}\{|q - r| \leq [\mathbf{V}(q) + \mathbf{V}(r)]^{\frac{1}{2}}/\sqrt{p}\} \geq 1 - p .$$

'$q \overset{E}{=} r$' reads 'q and r are *equal within the experimental error*'.

The discussion and relations derived in the previous section were only background material to motivate our theory. We now incorporate the

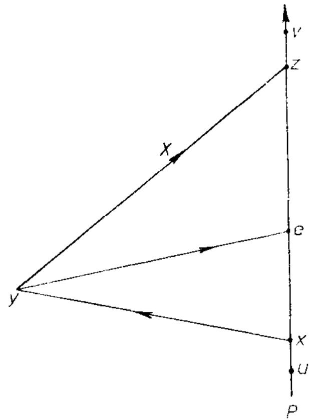

Fig. 21. *P*1

fundamental relation (2.9) into the theory by assuming that given any event e, there exists a space-time neighborhood in which the relation (2.9) is satisfied within the experimental error (**P**1). The size of the neighborhood is determined by the size of the error allowed. This correlation is implicit in **P**1, but is not stated explicitly. More precisely, if P is the output particle of a standard clock, and e is an event on P, then there exist at least two events u, v on P, with u before e and v after e, such that for all events x, y, z and any particle X, if x, z are on P; y, z are on X; $u \preccurlyeq x \prec e \prec z \preccurlyeq v; x\mathscr{F}y \wedge y\mathscr{F}e$ (a first signal starts at x, is reflected at y, and returns at e), $y \prec z$, and P, X are the output particles of equivalent standard clocks, then the following relation is true:

$$\tau_X(y, z)_H{}^2 \overset{E}{=} \tau_P(e, z)_H{}^2 + \tau_P(e, z)_H \, \tau_P(x, e)_H .$$

The symbol $\prec$ is defined in **D**IV4.1, $\tau_P(a, b)$ in **D**VI5.2 and $a\mathscr{F}b$ in **D**V4.1.

P1.

$$(\exists U).(U\mathscr{S}\mathscr{C}P)_H \wedge e \in \mathscr{W}(P)_H \rightarrow (\exists u, v). u, v \in \mathscr{W}(P)_H$$
$$\wedge u \prec_H e \prec_H v \wedge (\forall X, x, y, z). x, z \in \mathscr{W}(P)_H \wedge u \preccurlyeq_H x \prec_H e \prec_H z \preccurlyeq_H v$$
$$\wedge x\mathscr{F}_H y \wedge y\mathscr{F}_H e \wedge y, z \in \mathscr{W}(X)_H \wedge y \prec_H z \wedge (\exists V)(V\mathscr{S}\mathscr{C}X)_H$$
$$\wedge \mathscr{E}\mathscr{S}\mathscr{C}(U, V)_H \rightarrow \tau_X(y, z)_H^2 \overset{E}{=} \tau_P(e, z)_H^2 + \tau_P(e, z)_H \tau_P(x, e)_H .$$

The experimental error in the relation of ***P*1** can be calculated with the help of ***P*VI6.1**. Thus

$$\mathbf{V}\{\tau_X(y,z)^2\} = 2\tau_X(y,z)\,\mathbf{V}\{\tau_X(y,z)\}$$
$$\leq \sigma_T^2 \tau_X(y,z)^2. \tag{1}$$

$$\mathbf{V}\{\tau_P(e,z)^2 + \tau_P(e,z)\tau_P(x,e)\}$$
$$= \mathbf{V}\{\tau_P(e,z)^2\} + \mathbf{V}\{\tau_P(e,z)\tau_P(x,e)\}$$
$$= 2\tau_P(e,z)\,\mathbf{V}\{\tau_P(e,z)\} + \tau_P(x,e)\,\mathbf{V}\{\tau_P(e,z)\}$$
$$+ \tau_P(e,z)\,\mathbf{V}\{\tau_P(x,e)\}$$
$$\leq \sigma_T[\tau_P(e,z)^2 + \tau_P(e,z)\tau_P(x,e)]. \tag{2}$$

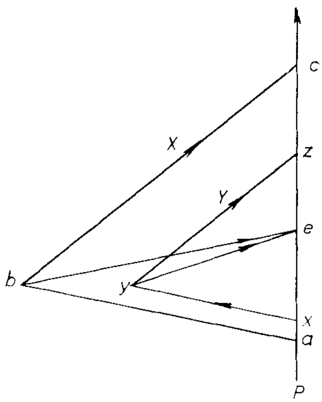

Fig. 22. ***D*2**

The error can now be calculated from (1), (2) and ***D*1**.

Events a,b,c are said to be in the neighborhood of e on P if the following is true (Fig. 22): P is the output particle of a standard clock U, events a,e,c are on P, $a\mathscr{F}b \wedge b\mathscr{F}e$, there exists an output particle X of a standard clock V such that U and V are equivalent, b,c are on X and $b \prec c$, and

$$\tau_X(b,c)^2 \stackrel{E}{=} \tau_P(e,c)^2 + \tau_P(e,c)\,\tau_P(a,e).$$

Moreover, for any events x,y,z and any particle Y and clock W, if x,z are on P, $a \preccurlyeq x \prec e \prec z \preccurlyeq c$, $x\mathscr{F}y \wedge y\mathscr{F}e$, Y is the output particle of W which is equivalent to U, events y,z are on Y, and $y \prec z$, then

$$\tau_Y(y,z)^2 \stackrel{E}{=} \tau_P(e,z)^2 + \tau_P(e,z)\,\tau_P(x,e).$$

In symbols:

D2. $a, b, c\mathscr{N}\ell_H e, P$ *for* $(\exists\, U).(U\mathscr{S}\mathscr{C}P)_H \wedge a, e, c \in \mathscr{W}(P)_H$

$$\begin{aligned}
&\wedge a\mathscr{F}_H b \wedge b\mathscr{F}_H e \wedge (\exists\, V, X)(V\mathscr{S}\mathscr{C}X)_H \wedge \mathscr{E}\mathscr{S}\mathscr{C}(U, V)_H \\
&\wedge b, c \in \mathscr{W}(X)_H \wedge b \prec_H c \\
&\wedge \tau_X(b, c)_H^2 \overset{E}{=} \tau_P(e, c)_H^2 + \tau_P(e, c)_H \tau_P(a, e)_H \\
&\wedge (\forall\, x, y, z, Y, W).\, x, z \in \mathscr{W}(P)_H \wedge a \preccurlyeq_H x \prec_H e \prec_H z \preccurlyeq_H c \\
&\wedge x\mathscr{F}_H y \wedge y\mathscr{F}_H e \wedge (W\mathscr{S}\mathscr{C}Y)_H \wedge \mathscr{E}\mathscr{S}\mathscr{C}(U, W)_H \\
&\wedge y, z \in \mathscr{W}(Y)_H \wedge y \prec_H z \\
&\rightarrow \tau_Y(y, z)_H^2 \overset{E}{=} \tau_P(e, z)_H^2 + \tau_P(e, z)_H \tau_P(x, e)_H.
\end{aligned}$$

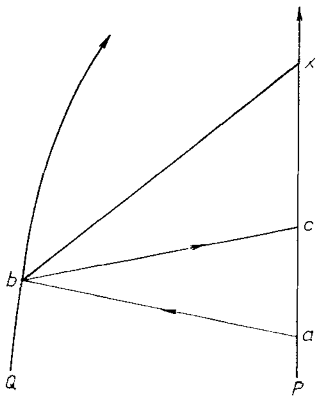

Fig. 23. ***D*4**

Two events a, b are in the space-time neighborhood of each other if one is in the neighborhood of the other on a particle X(**D**3).

D3. $a\mathscr{T}\mathscr{N}_H b$ *for*
$(\exists\, X, u, v, w)\colon u, v, w\mathscr{N}\ell_H b, X \wedge a \in \{u, v, w\} \,.\vee.\, u, v, w\mathscr{N}\ell_H a, X \wedge b \in \{u, v, w\}.$

T1. $a\mathscr{T}\mathscr{N}_H b \leftrightarrow b\mathscr{T}\mathscr{N}_H a.$
Proof. **D**3.

An *event b is in the space neighborhood of events a, c on P*, if there exists an event x such that a, b, x are in the neighborhood of c on P(**D**4, Fig. 23).

D4. $b\mathscr{S}\mathscr{N}\ell_H P(a, c)$ *for* $(\exists\, x)\, a, b, x\mathscr{N}\ell_H c, P.$

Two particles P, Q are in the space neighborhood of each other, if they are the output particles of equivalent clocks, and any event on one of them is in the space neighborhood of two events on the other (**D**5).

***D*5.** $P\mathscr{SN}_H Q$ *for*
$(\exists U, V).(U\mathscr{SC}P)_H \wedge (V\mathscr{SC}Q)_H \wedge \mathscr{ESC}(U, V)_H.$
$\wedge (\forall u).u \in \mathscr{W}(P)_H \rightarrow (\exists v, w) u\mathscr{SN}\ell_H Q(v, w):$
$\wedge (\forall x).x \in \mathscr{W}(Q)_H \rightarrow (\exists y, z) x\mathscr{SN}\ell_H P(y, z).$

***T*2.** $P\mathscr{SN}_H Q \leftrightarrow Q\mathscr{SN}_H P.$
Proof. ***D*5.**

4. First-signal speed

Referring to Fig. 23, if the round trip time $\tau_P(a, c)$ is measured by a standard clock and the distance of event b from particle P is measured by a linear length measuring instrument (Sec.VII7), the ratio of this distance to this time constitutes a measurement of the local average speed of light during the time interval between a and c. This speed is always found to have the same value, 3×10^8 m/sec, regardless of the gravitational field or state of motion of P (Basri [1965] Sec.7A).

To formulate this, we must first define the distance between an event a and a particle P [see **D**VII(3.2, 7.1)].

***D*1.** $\lambda_{\mathfrak{l}}(a; P)_H$ *for*
$(\iota r)(\exists X)\mu_{\mathfrak{l}}[X(a, a), P; r]_H \wedge \mathscr{LLI}(\mathfrak{l})_H.$

$\lambda_{\mathfrak{l}}(a;P)_H$ is the distance between event a and particle P.

We now assume (***P*1**) that if b is in the space neighborhood of a, c on P (Fig. 23), and $\mathfrak{l}$ is a linear length measuring instrument, then there exists exactly one number r such that $2\lambda_{\mathfrak{l}}(b;P) = r\,\tau_P(a, c)$.

***P*1.** $b\mathscr{SN}\ell_H P(a, c) \wedge \mathscr{LLI}(\mathfrak{l})_H$
$\rightarrow (\exists ! r) 2\lambda_{\mathfrak{l}}(b; P)_H = r\tau_P(a, c)_H.$

The local speed $\mathfrak{c}_{\mathfrak{l}P}$ of a first signal is the number r measured by the $\mathscr{LLI}$, $\mathfrak{l}$, and the standard clock whose output particle is P, such that for any events x, y, z, if y is in the space neighborhood of x, z on P, then $2\lambda_{\mathfrak{l}}(y;P) = r\tau_P(x, z)$.

***D*2.** $\mathfrak{c}_{\mathfrak{l}P}$ *for* $(\iota r).(\exists U)(U\mathscr{SC}P)_H \wedge \mathscr{LLI}(\mathfrak{l})_H$
$\wedge (\forall x, y, z).y\mathscr{SN}\ell_H P(x, z)_H \rightarrow 2\lambda_{\mathfrak{l}}(y; P)_H = r\tau_P(x, z)_H.$

It follows from ***P*1** and ***D*2** that:

***T*1.** $b\mathscr{SN}\ell_H P(a, c) \wedge \mathscr{LLI}(\mathfrak{l})_H$
$\rightarrow 2\lambda_{\mathfrak{l}}(b; P)_H = \mathfrak{c}_{\mathfrak{l}P}\tau_P(a, c)_H.$
Proof. ***P*1, *D*2.**

5. Metric coefficients

If we write:

$$\mathrm{d}l = \lambda_l(b; P),$$

and combine *T*4.1 with Eq.(2.8), we get:

$$\mathrm{d}l^2 = \gamma_{ij}\,\mathrm{d}\mathfrak{x}^i\,\mathrm{d}\mathfrak{x}^j. \tag{1}$$

Thus γ_{ij} are the metric coefficients of the space geometry, since they help express the element of length $\mathrm{d}l$ in terms of the space coordinate differences $\mathrm{d}\mathfrak{x}^i$. Moreover, the form of (1) shows that the space geometry is Riemannian.

To incorporate this result into the theory, we can either assume (2.8) and derive (1), or assume (1) and derive (2.8). We choose the latter alternative, since it is more direct (***P***1).

D1. $\mathfrak{x}^i_{PQ}$ *for* $\mathfrak{x}^i_Q - \mathfrak{x}^i_P$.

P1. $\{P(\mathfrak{x}_P), b[Q(\mathfrak{x}_Q)] - \mathscr{RSC}(O, A, B, C)_\mathrm{I}\}_H$
$\wedge\, b\mathscr{SN}\mathscr{C}_H P(a, c) \to (\exists!\, r_{11}, r_{12}, r_{13}, r_{21}, \ldots, r_{33}).$
$r_{11}, \ldots, r_{33} \in \mathscr{R} \wedge r_{12} = r_{21} \wedge r_{13} = r_{31} \wedge r_{23} = r_{32}$
$\wedge\, \lambda_l(b; P)_H = r_{ij}\mathfrak{x}^i_{PQ}\mathfrak{x}^j_{PQ}.$

The meaning of ***P***1 becomes clear from Fig. 23, (1), ***D***1, and the above remarks.

Nine numbers, r_{ij}, are introduced in ***P***1, but by the assumed symmetry ($r_{ij} = r_{ji}$), only six are independent. There is no loss of generality in the symmetry assumption, because if r_{ij} is written as the sum of a symmetric and antisymmetric parts, the antisymmetric part contributes nothing. To see this, let

$$r_{ij} = \tfrac{1}{2}(r_{ij} + r_{ji}) + \tfrac{1}{2}(r_{ij} - r_{ji}) = s_{ij} + a_{ij},$$

where

$$s_{ij} = \tfrac{1}{2}(r_{ij} + r_{ji}) = s_{ji}, \qquad a_{ij} = \tfrac{1}{2}(r_{ij} - r_{ji}) = -a_{ji}.$$

Then

$$\begin{aligned} 2a_{ij}\mathfrak{x}^i\mathfrak{x}^j &= (a_{ij}\mathfrak{x}^i\mathfrak{x}^j + a_{ji}\mathfrak{x}^j\mathfrak{x}^i) \\ &= (a_{ij} + a_{ji})\mathfrak{x}^i\mathfrak{x}^j \\ &= (a_{ij} - a_{ij})\mathfrak{x}^i\mathfrak{x}^j \\ &= 0. \end{aligned}$$

The metric coefficients introduced in ***P***1 are defined explicitly in:

***D*2.** $\gamma_{ij}(P,a;O,A,B,C,\mathfrak{l})_H$ *for*
$(\imath r).(\exists X,u,v,r_{11},\dots,r_{33},\mathfrak{x}^1,\mathfrak{x}^2,\mathfrak{x}^3,\mathfrak{y}^1,\mathfrak{y}^2,\mathfrak{y}^3).$
$\{X(\mathfrak{x}),a[P(\mathfrak{y})]-\mathscr{RCS}(O,A,B,C)\}_H \wedge a\mathscr{SNE}_H X(u,v)$
$\wedge\,\lambda_{\mathfrak{l}}(a;X)_H=r_{kl}(\mathfrak{x}^k-\mathfrak{y}^k)(\mathfrak{x}^l-\mathfrak{y}^l)\wedge r=r_{ij}.$

γ_{ij} are called the *space metric coefficients*. We can now derive the relation (2.8) in our theory (Fig. 20 with 'P' replaced by 'Q' and 'A' by 'P').

***T*1.** $a,b,d\mathscr{NE}_H c,P\wedge b,d\in\mathscr{W}(Q)_H\wedge(\exists U)(U\mathscr{SC}P)_H$
$\wedge\{P(\mathfrak{x})-\mathscr{RCS}(O,A,B,C)_{\mathfrak{l}}\}_H\rightarrow$
$\tau_Q(b,d)_H^2\overset{E}{=}\tau_P(c,d)_H^2+2\mathfrak{c}_{\mathfrak{l}P}^{-1}\tau_P(c,d)_H[\gamma_{ij}(P,b)\mathfrak{x}^i_{PQ}\mathfrak{x}^j_{PQ}]^{\frac{1}{2}}.$

Proof. ***D*3.2, *T*4.1, *P*1, *D*2.**

To compare time intervals on different world lines, it is necessary to send first signals from one world line to another and compare the interval between departures with the interval between arrivals. Since the coefficients γ_{ij} do not help in relating these intervals, other coefficients are necessary, which are derived now by means of GR.

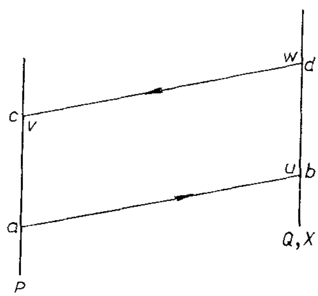

Fig. 24. Comparison of time intervals

Let P,Q be two neighboring particles, a,c two neighboring events on P, and b,d the events of arrival at Q of first signals that left P at a and c, respec-
According to (2.6) and (1), the time interval measured by synchronized
tively (Fig. 24).
standard clocks, that light takes to go from a to b is:

$$\mathfrak{c}(\tau_b-\tau_a)=\lambda(b;P)+\gamma_i(P,a)\,\mathfrak{x}^i_{PQ}.$$

Similarly, the time of travel of the signal from c to d is:

$$\mathfrak{c}(\tau_d-\tau_c)=\lambda(d;P)+\gamma_i(P,c)\,\mathfrak{x}^i_{PQ}.$$

Subtracting these two equations, we get:

$$\begin{aligned}\mathfrak{c}\left[(\tau_d-\tau_b)-(\tau_c-\tau_a)\right] &= \mathfrak{c}\left[\tau_Q(b,d)-\tau_P(a,c)\right]\\ &= \left[\lambda(d;P)-\lambda(b;P)\right]+\left[\gamma_i(P,c)-\gamma_i(P,a)\right]\mathfrak{x}^i_{PQ}\\ &= \left[\lambda(d;P)-\lambda(b;P)\right]+\alpha_i(P,a)\,\mathfrak{c}\tau_P(a,c)\,\mathfrak{x}^i_{PQ},\end{aligned}$$

where

$$\alpha_i(P,a)=\left[\gamma_i(P,c)-\gamma_i(P,a)\right]/\mathfrak{c}\tau_P(a,c), \tag{2}$$

is essentially the time partial derivative of γ_i. Thus,

$$\begin{aligned}\mathfrak{c}\tau_Q(b,d) &= \left[1+\alpha_i(P,a)\,\mathfrak{x}^i_{PQ}\right]\mathfrak{c}\tau_P(a,c)\\ &\quad+\lambda(d;P)-\lambda(b;P).\end{aligned} \tag{3}$$

Consequently, the coefficients that relate the proper time intervals $\tau_P(a,c)$ and $\tau_Q(b,d)$ are α_i, defined in (2). Although the values of γ_i depend upon the synchronization convention and method, α_i do not, and are thus more significant operationally. We, therefore, adopt α_i as the remaining metric coefficients. This gives us nine independent coefficients in all, 6γ's and 3α's. In GR, there are 10 independent metric coefficients. The tenth coefficient, g_{00}, is set equal to -1 in our theory because standard clocks are used exclusively.

Application of (3) in GR is illustrated in Basri [1965] end of Sec.11.

The following postulate is based upon (3):

***P*2.**
$$\begin{aligned}&\{P(\mathfrak{x}_P),Q(\mathfrak{x}_Q)-\mathscr{RCS}(O,A,B,C)_{\mathfrak{l}}\}_H\wedge P\mathscr{SN}_HQ\\ &\wedge a\mathscr{TN}_Hc\wedge a,c\in\mathscr{W}(P)_H\wedge b,d\in\mathscr{W}(Q)_H\wedge a\prec_Hc\\ &\wedge a\mathscr{F}_Hb\wedge c\mathscr{F}_Hd\\ &\rightarrow(\exists!\,r_1,r_2,r_3).\mathfrak{c}_{\mathfrak{l}Q}\tau_Q(b,d)_H=(1+r_i\mathfrak{x}^i_{PQ})\mathfrak{c}_{\mathfrak{l}P}\tau_P(a,c)_H\\ &\qquad+\lambda_{\mathfrak{l}}(d;P)_H-\lambda_{\mathfrak{l}}(b;P)_H.\end{aligned}$$

The coefficients introduced in ***P*2** are defined explicitely in (Fig. 24):

***D*3.**
$$\begin{aligned}&\alpha_i(P,a;O,A,B,C,\mathfrak{l})_H\quad for\\ &(\iota r).(\exists\,X,u,v,w,r_1,r_2,r_3,\mathfrak{x}^1,\mathfrak{x}^2,\mathfrak{x}^3,\mathfrak{y}^1,\mathfrak{y}^2,\mathfrak{y}^3).\,r=r_i\\ &\wedge\{P(\mathfrak{x}),X(\mathfrak{y})-\mathscr{RCS}(O,A,B,C)_{\mathfrak{l}}\}_H\wedge X\mathscr{SN}_HP\wedge v\mathscr{TN}_Ha\\ &\wedge a,v\in\mathscr{W}(P)_H\wedge u,w\in\mathscr{W}(X)_H\wedge a\prec_Hv\wedge a\mathscr{F}_Hu\wedge v\mathscr{F}_Hw\\ &\wedge\mathfrak{c}_{\mathfrak{l}X}\tau_X(u,w)_H=\left[1+r_j(\mathfrak{y}^j-\mathfrak{x}^j)\right]\mathfrak{c}_{\mathfrak{l}P}\tau_P(a,v)_H\\ &\qquad+\lambda_{\mathfrak{l}}(w;P)_H-\lambda\,(u;P)_H.\end{aligned}$$

In the rest of this section postulates are introduced about the existence and uniqueness of the first differences of the metric coefficients. These differences are necessary for the derivation of the equations of motion.

If a is an event on particle P with space coordinates $\mathfrak{x}^1,\mathfrak{x}^2,\mathfrak{x}^3$, and u is any neighboring event on any particle X with space coordinates $\mathfrak{y}^1,\mathfrak{x}^2,\mathfrak{x}^3$, such that a first signal connects a with u, and f is any of the metric coefficients γ_{ij},α_i, then there exists exactly one number r such that

$$[f(X,u)-f(P,a)]/(\mathfrak{y}^1-\mathfrak{x}^1)=r.$$

This number, called the *partial difference of f with respect to $\mathfrak{x}^1$*, is denoted by $f_{,1}(P,a)$. These are the contents of **P**3 and **D**4 below.

P3. $(\forall X,u,\mathfrak{y}^1).a\mathscr{TN}_H u \wedge a\mathscr{F}_H u$
$\wedge \{a[P(\mathfrak{x}^1,\mathfrak{x}^2,\mathfrak{x}^3)], u[X(\mathfrak{y}^1,\mathfrak{x}^2,\mathfrak{x}^3)] - \mathscr{RCS}(O,A,B,C)_{\mathfrak{l}}\}_H$
$\wedge f(P,a)\in\{\gamma_{ij}(P,a;O,A,B,C,\mathfrak{l})_H, \alpha_i(P,a;O,A,B,C,\mathfrak{l})_H\}$
$\wedge f(X,u)\in\{\gamma_{ij}(X,u;O,A,B,C,\mathfrak{l})_H, \alpha_i(X,u;O,A,B,C,\mathfrak{l})_H\}$
$\rightarrow(\exists!r).[f(X,u)-f(P,a)]/(\mathfrak{y}^1-\mathfrak{x}^1)=r.$

D4. $f_{,1}(P,a;O,A,B,C,\mathfrak{l})_H$ *for*
(ιr) r is the unique number in **P**3.

P4. As in **P**3, except that X has the coordinates $\mathfrak{x}^1,\mathfrak{y}^2,\mathfrak{x}^3$, and
$[f(X,u)-f(P,a)]/(\mathfrak{y}^2-\mathfrak{x}^2)=r.$

D5. $f_{,2}(P,a;O,A,B,C,\mathfrak{l})_H$ *for*
(ιr) r is the unique number in **P**4.

P5. As in **P**3, except that X has the coordinates $\mathfrak{x}^1,\mathfrak{x}^2,\mathfrak{y}^3$, and
$[f(X,u)-f(P,a)]/(\mathfrak{y}^3-\mathfrak{x}^3)=r.$

D6. $f_{,3}(P,a;O,A,B,C,\mathfrak{l})_H$ *for*
(ιr) r is the unique number in **P**5.

If a is an event on particle P, and u is any neighboring event on P, then there exists exactly one number r such that

$$[f(P,u)-f(P,a)]/\mathfrak{c}_{\mathfrak{l}P}\tau_P(a,u)=r.$$

This number, called the *time partial difference of f*, is denoted by $f_{,0}(P,a)$.

P6. $(\forall u).a,u\in\mathscr{W}(P)_H \wedge a\mathscr{TN}_H u \wedge \mathscr{LLI}(\mathfrak{l})_H$
$\wedge$ $f(P,a)$ and $f(P,u)$ are defined as in **P**3.
$\rightarrow(\exists!r).[f(P,u)-f(P,a)]/\mathfrak{c}_{\mathfrak{l}P}\tau_P(a,u)_H=r.$

D7. $f_{,0}(P,a;O,A,B,C,\mathfrak{l})_H$ *for*
(ιr) r is the unique number in **P**6.

From now on, the letters denoting the coordinate system are suppressed, with the understanding that there is a specific coordinate system such as that defined in **D**IX6.1, relative to which all coordinates are referred. The subscripts 'I' and '*H*', referring to the LLI and the observer are also suppressed.

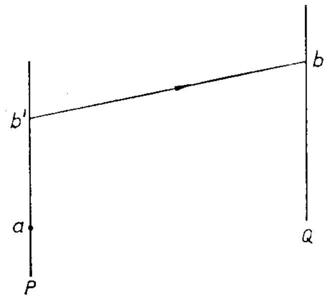

Fig. 25. **P**7

If a is an event on a particle $P(\mathfrak{x}_P)$, b any neighboring event on a particle $Q(\mathfrak{x}_Q)$, and b' an event on P such that $b'\mathscr{F}b$ (Fig. 25), then the total difference of $f(P,a)$ is given by

$$\begin{aligned}\Delta f(P,a) &= f(Q,b) - f(P,a)\\ &= f_{,i}(P,a)\,\mathfrak{x}^i_{PQ} + f_{,0}(P,a)\,c\tau_P(a,b').\end{aligned}$$

P7. $$\begin{gathered}a, b' \in \mathscr{W}(P) \wedge b \in \mathscr{W}(Q) \wedge a\mathscr{T}\mathscr{N}b \wedge b'\mathscr{F}b \rightarrow\\ f(Q,b) - f(P,a) = f_{,i}(P,a)\,x^i_{PQ} + f_{,0}(P,a)\,c\tau_P(a,b').\end{gathered}$$

Since the foundation has already been laid, there is no more virtue in formality, and the material presented in the remainder of this final chapter is accordingly informal.

6. Equations of a space geodesic

In **P**VIII3.1 it was assumed that a geodesic segment is the shortest path between two particles. Consequently, if $P_1(\mathfrak{x}_1), \ldots, P_r(\mathfrak{x}_r)$ lie on a geodesic, and $Q_1(=P_1), Q_2(\mathfrak{y}_2), \ldots, Q_{r-1}(\mathfrak{y}_{r-1}), Q_r(=P_r)$ lie on any path connecting P_1 and P_r such that $\mathscr{R}\mathscr{C}(Q_1, \ldots, Q_r)$(**D**VII4.4), and

$$\mathfrak{y}^i_n = \mathfrak{x}^i_n + \varepsilon^i p^i_n, \qquad p^i_n \geq 0, \varepsilon^i \geq 0 \qquad (1 < n < r), \tag{1}$$

then the distance on the geodesic path is an extremum, i.e.,

$$\frac{\partial}{\partial \varepsilon^i} \sum_{1}^{r-1}{}_n \lambda(Q_n, Q_{n+1})|_0 = 0.^\dagger \tag{2}$$

The subscript '0' stands for '$\varepsilon^1 = \varepsilon^2 = \varepsilon^3 = 0$'. Let

$$\lambda(P_1, P_2) = \ldots = \lambda(P_r, P_{r-1}) = \Delta l,$$
$$\Delta x_n^i = x_{n+1}^i - x_n^i,$$

and assume that γ_{ij} do not vary with time, i.e.,

$$\gamma_{ij,0}(P_n, a) = 0.$$

If for all n, $P_n \mathscr{S}\mathscr{N} P_{n+1}$ and $Q_n \mathscr{S}\mathscr{N} Q_{n+1}$ (***D***3.5), then we get from (2), ***P***5.1 and ***D***5.2:

$$\frac{\partial}{\partial \varepsilon^i} \sum_{1}^{r-1}{}_n [\gamma_{jk}(\mathfrak{y}_n) \Delta \mathfrak{y}_n^j \Delta \mathfrak{y}_n^k]^{\frac{1}{2}}|_0 = 0.$$

Making use of (1) and ***D***5.(4–6), and writing

$$\gamma_{jk}^{(n)} = \gamma_{jk}(x_n), \qquad \gamma_{jk,l}^{(n)} = \gamma_{jk,l}(x_n),$$

we find from this that

$$\frac{\partial}{\partial \varepsilon^i} \sum_{1}^{r-1}{}_n [(\gamma_{jk}^{(n)} + \gamma_{jk,l}^{(n)} \varepsilon^l p_n^l)(\Delta x_n^j + \varepsilon^j \Delta p_n^j)(\Delta x_n^k + \varepsilon^k \Delta p_n^k)]^{\frac{1}{2}}|_0 = 0.$$

This implies

$$\sum_{1}^{r-1}{}_n \left[\gamma_{ij}^{(n)} \frac{\Delta x_n^j}{\Delta l} \Delta p_n^{(i)} + \tfrac{1}{2} \gamma_{jk,i}^{(n)} \frac{\Delta x_n^j}{\Delta l} \frac{\Delta x_n^k}{\Delta l} \Delta l p_n^{(i)}\right] = 0,$$

where the parenthesis around 'i' means there is no summation over i.

Since $p_1^i = p_r^i = 0$, we find

$$\sum_{1}^{r-1}{}_n F_n \Delta p_n = \sum_{1}^{r-1}{}_n F_n(p_{n+1} - p_n) = \sum_{2}^{r-1}{}_n (F_{n-1} - F_n) p_n$$
$$= -\sum_{2}^{r-1}{}_n (\Delta F_{n-1}) p_n = -\sum_{1}^{r-2}{}_n (\Delta F_n) p_{n+1}. \tag{3}$$

† Since continuity is not assumed, the differentiation with respect to ε^i and setting $\varepsilon^i = 0$ may not be justified. Mathematicians could perhaps find a better way of handling this.

Thus, the above equation becomes

$$\sum_{2}^{r-1}{}_{n}\left[-\frac{\Delta}{\Delta l}\left(\gamma_{ij}^{(n-1)}\frac{\Delta \mathfrak{x}_{n-1}^{j}}{\Delta l}\right)+\tfrac{1}{2}\gamma_{jk,i}^{(n)}\frac{\Delta \mathfrak{x}_{n}^{j}}{\Delta l}\frac{\Delta \mathfrak{x}_{n}^{k}}{\Delta l}\right]\Delta l\, p_{n}^{(i)}=0\,.$$

Since p_n^m are independent and arbitrary, we conclude from this that

$$\frac{\Delta}{\Delta l}\left(\gamma_{ij}^{(n-1)}\frac{\Delta \mathfrak{x}_{n-1}^{j}}{\Delta l}\right)=\tfrac{1}{2}\gamma_{j\,k,i}^{(n)}\frac{\Delta \mathfrak{x}_{n}^{j}}{\Delta l}\frac{\Delta \mathfrak{x}_{n}^{k}}{\Delta l}, \qquad 1<n<r, \tag{4}$$

which are the desired equations of a space geodesic.

The simultaneous difference equations (4) determine a set of points having coordinates x_n^i. Each set of subdivisions yields a different path, but eventually Δl gets small enough that the geodesic path obtained by smaller subdivisions cannot be distinguished experimentally from the preceding path (Courant and Hilbert [1953] p. 177).

7. Free trajectories

***D*1.** $a_1[P_1(\mathfrak{x}_1)], \ldots, a_r[P_r(\mathfrak{x}_r)]\mathscr{T}\imath P(O,A,B,C)$ *for*
$\{a_1[P_1(\mathfrak{x}_1)], \ldots, a_r[P_r(\mathfrak{x}_r)] - \mathscr{RCS}(O,A,B,C)\}$
$\wedge P_1\mathscr{SN}P_2 \wedge P_2\mathscr{SN}P_3 \wedge \ldots \wedge P_{r-1}\mathscr{SN}P_r$
$\wedge a_1\mathscr{TN}a_2 \wedge a_2\mathscr{TN}a_3 \wedge \ldots \wedge a_{r-1}\mathscr{TN}a_r \wedge a_1 \prec a_2 \prec \ldots \prec a_r$.

This reads '$a_1[P_1(\mathfrak{x}_1)].\ldots, a_r[P_r(\mathfrak{x}_r)]$ *is a trajectory of P relative to the* CS, $\langle O,A,B,C\rangle$'.

***D*2.** *The trajectory of a particle P is free* if it satisfies the following two conditions: (1) If $a_1[P_1(\mathfrak{x}_1)], u_2[Q_2(\mathfrak{y}_2)], \ldots, u_{r-1}[Q_{r-1}(\mathfrak{y}_{r-1})], a_r[P_r(\mathfrak{x}_r)]$ is a trajectory of a particle Q (Fig. 26), and $a_n\mathscr{TN}u_n, a_n\mathscr{F}b_{n+1}, b_n \in \mathscr{W}(P_n)$, $v_n \in \mathscr{W}(Q_n), u_n\mathscr{F}v_{n+1}, \mathfrak{y}_n^i = \mathfrak{x}_n^i + \varepsilon^i p_n^i$, $\tau_{Q_n}(v_n,u_n) = \tau_{P_n}(b_n,a_n)$ for $1<n<r$, then

$$\frac{\partial}{\partial\varepsilon_i}\sum_{1}^{r-1}{}_{n}\,\tau_Q(u_n,u_{n+1})|_0 = 0\,.^{\dagger} \tag{1}$$

(2) If $a_1[P_1(\mathfrak{x}_1)], u_2[P_2(\mathfrak{x}_2)], \ldots, u_{r-1}[P_{r-1}(\mathfrak{x}_{r-1})], a_r[P_r(\mathfrak{x}_r)]$ is a trajectory of a particle Q (Fig. 27), and $a_n\mathscr{TN}u_n, b_n, v_n \in \mathscr{W}(P_n), a_n\mathscr{F}b_{n+1}, u_n\mathscr{F}v_{n+1}$, $\tau_{P_n}(a_n, u_n) = \varepsilon q_n$ for $1<n<r$, then

$$\frac{\partial}{\partial\varepsilon}\sum_{1}^{r-1}{}_{n}\,\tau_Q(u_n,u_{n+1})|_0 = 0\,.^{\dagger} \tag{2}$$

† See footnote on p. 120.

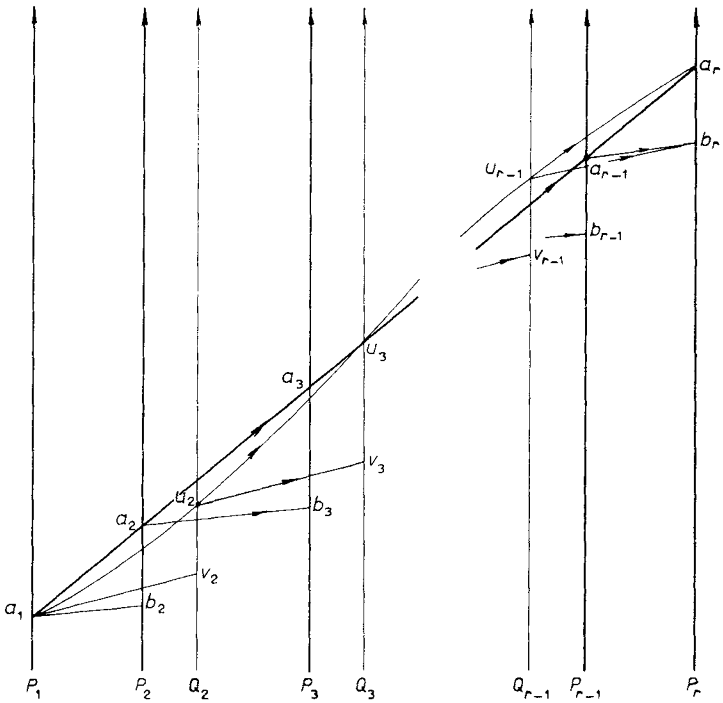

Fig. 26. ***D*2**, condition 1

Condition (1) amounts to variation of the space coordinates, and condition (2) to variation of the time coordinate.

We now derive the equations of motion of a free trajectory. Let

$$c\tau_P(a_1, a_2) = \ldots = c\tau_P(a_{r-1}, a_r) = \Delta s, \tag{3}$$

$$\Delta \mathfrak{x}_n^i = \mathfrak{x}_{n+1}^i - \mathfrak{x}_n^i, \quad \Delta \mathfrak{x}_n^0 = c\tau_{P_n}(b_n, a_n), \tag{4}$$

$$\beta_n^\mu = \Delta \mathfrak{x}_n^\mu / \Delta s, \quad \beta_n = (\gamma_{ij}^{(n)} \beta_n^i \beta_n^j)^{\frac{1}{2}}, \tag{5}$$

where

$$\gamma_{ij}^{(n)} = \gamma_{ij}(P_n, a_n), \quad \gamma_{ij,k}^{(n)} = \gamma_{ij,k}(P_n, a_n). \tag{6}$$

For the trajectory of Fig. 26, we have according to ***T*5.1**,

$$c^2\tau_Q(u_n, u_{n+1})^2 = (\Delta \mathfrak{y}_{n+1}^0)^2 + 2\Delta \mathfrak{y}_{n+1}^0 \left[\gamma_{jk}(Q_n, u_n)\Delta \mathfrak{y}_n^j \Delta \mathfrak{y}_n^k\right]^{\frac{1}{2}}, \tag{7}$$

where

$$\Delta \mathfrak{y}_n^i = \mathfrak{y}_{n+1}^i - \mathfrak{y}_n^i, \quad \text{and} \quad \Delta \mathfrak{y}_n^0 = c\tau_{Q_n}(v_n, u_n).$$

By assumption,

$$\Delta \mathfrak{y}_n^0 = \Delta \mathfrak{x}_n^0, \quad \Delta \mathfrak{y}_n^i + \varepsilon^i \Delta p_n.$$

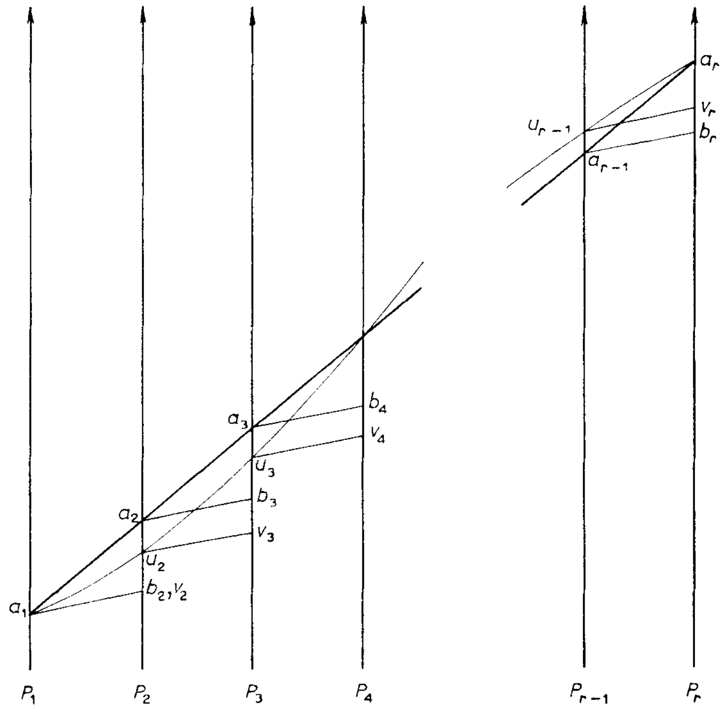

Fig. 27. ***D*2**, condition 2

Moreover, according to ***P*5.7**, we have

$$\begin{aligned}\gamma_{jk}(Q_n, u_n) &= \gamma_{jk}(P_n, a_n) + \gamma_{jk,l}(P_n, a_n)(\mathfrak{y}_n^l - \mathfrak{x}_n^l)\\ &= \gamma_{jk}^{(n)} + \gamma_{jk,l}^{(n)}\varepsilon^l p_n^l.\end{aligned}$$

Substituting these results into (7), (7) into (1), and making use of (3)–(6), we get as in Sec. 6:

$$\sum_1^{r-1}{}_n\, (\beta_{n+1}^0/\beta_n)\big[\gamma_{ij}^{(n)}\beta_n^j \Delta p_n^{(i)} + \tfrac{1}{2}\gamma_{jk,i}^{(n)}\beta_n^j\beta_n^k p_n^{(i)}\Delta s\big] = 0.$$

Notice that $\Delta\mathfrak{y}_{n+1}^0$ is independent of ε_n^i, and only contributes the factor β_{n+1}^0. With the help of (6.3), we get as in (6.4):

$$\frac{\Delta}{\Delta s}\left(\frac{\beta_n^0}{\beta_{n-1}}\gamma_{ij}^{(n-1)}\beta_{n-1}^j\right) = \frac{1}{2}\frac{\beta_{n+1}^0}{\beta_n}\gamma_{jk,i}^{(n)}\beta_n^j\beta_n^k, \qquad 1 < n < r, \tag{8}$$

which are the first three of the equations of motion.

For the trajectory of Fig. 27 and condition 2 of **D**2, we have

$$\begin{aligned}
\Delta \mathfrak{y}_n^i &= \Delta \mathfrak{x}_n^i, \\
\gamma_{ij}(Q_n, u_n) &= \gamma_{ij}(P_n, a_n) + \gamma_{ij,0}(P_n, a_n)\, c\tau_{P_n}(a_n, u_n) \\
&= \gamma_{ij}^{(n)} + \gamma_{ij,0}^{(n)}\, \varepsilon q_n, \\
\Delta \mathfrak{y}_{n+1}^0 &= c\tau(v,u)_{n+1} \\
&= c\left[\tau(b,a) + \tau(a,u) - \tau(b,v)\right]_{n+1}.
\end{aligned}$$

From Fig. 27, we see that $\tau(b,v)_{n+1} \approx \tau(a,u)_n$. Thus

$$\Delta \mathfrak{y}_{n+1}^0 \approx \Delta \mathfrak{x}_{n+1}^0 + \varepsilon(q_{n+1} - q_n)$$

Substituting these results in (7), and (7) into (1), we get:

$$\begin{aligned}
&\sum_{1}^{r-1}{}_n \left[(\beta_{n+1}^0 + \beta_n)(q_{n+1} - q_n) + \frac{\beta_{n+1}^0}{2\beta_n} \gamma_{ij,0}^{(n)} \beta_n^i \beta_n^j \Delta s\, q_n\right] \\
&= \sum_{1}^{r-1}{}_n \left[-\Delta(\beta_n^0 + \beta_{n-1}) + \frac{\beta_{n+1}^0}{2\beta_n} \gamma_{ij,0}^{(n)} \beta_n^i \beta_n^j \Delta s\right] q_n = 0.
\end{aligned}$$

Since q_n are independent and arbitrary, it follows that

$$\frac{\Delta}{\Delta s}(\beta_n^0 + \beta_{n-1}) = \frac{1}{2}\frac{\beta_{n+1}^0}{\beta_n} \gamma_{ij,0}^{(n)} \beta_n^i \beta_n^j, \qquad 1 < n < r, \tag{9}$$

which completes the equations of motion.

According to **T**5.1,

$$\Delta s^2 = (\Delta \mathfrak{x}_{n+1}^0)^2 + 2(\Delta \mathfrak{x}_{n+1}^0)\beta_n \Delta s.$$

Dividing both sides by Δs^2, we find

$$1 = (\beta_{n+1}^0)^2 + 2\beta_{n+1}^0 \beta_n, \qquad 0 < n < r. \tag{10}$$

This equation shows that (8) and (9) are not independent.

8. Inertial systems

***D*1.** An *inertial system* is a set of segments of free trajectories of particles $P_1, \ldots, P_r$. All the particles have at least one space neighbor from the other particles. The trajectory segment associated with P_n is bounded by events a_n, b_n such that $a_n, b_n \in \mathscr{W}(P_n)$ and $a_n \prec b_n$. If P_m and P_n are neighbors ($P_m \mathscr{S}\mathscr{N} P_n$), then so are a_m and $a_n (a_m \mathscr{T}\mathscr{N} a_n)$, and b_m and $b_n (b_m \mathscr{T}\mathscr{N} b_n)$. Moreover, the distance between any two segments remains constant from the beginning to the end of the segments. Roughly speaking, an inertial system defines a space-time region in which the distance between any two free trajectories remains constant.

***P*1.** If a coordinate system such as that defined in ***D*IX6.1** is constructed from four particles of an inertial system, then within the system,

$$\gamma_{ij} = \delta_{ij}, \qquad \alpha_i = 0, \tag{1}$$

where δ_{ij} is the Kronecker delta, which is equal to 1 if $i = j$ and 0 if $i \neq j$.

***T*1.** In an inertial region, the space geometry is Euclidean, all standard clocks have the same relative rate (determined by a sequence of first signals), and synchronization is transitive.

Proof. ***P*1**, ***P*5.(1,2)**.

The steps leading to the definition of an inertial system are as follows: (1) a coordinate system is defined operationally; (2) metric coefficients are introduced; (3) free trajectories are defined in terms of the metric coefficients; (4) an inertial system is defined in terms of the free trajectories. There is no circularity in this definition, every step is operationally meaningful, and the concept of force or interaction is not used.

9. Concluding remarks

We have come a long way from the class of observers. It is hoped that the theory can serve as a foundation to build physics on. At least it is a start of a unified physical theory that can be criticised, improved, or changed. The theory is obviously not complete, since no method is given to calculate the metric coefficients. In GR this is accomplished by means of the field equations and the energy-momentum tensor. In our theory, the field equations would have to be formulated in terms of lag times (see end of Sec. 1).

The place where we stopped is a logical resting place before continuing the long and difficult journey ahead. There are several routes to take. One possibility is to go on with the development of the macroscopic theory. Another possibility is to go into the microscopic domain and find a path back to where we left off. This is perhaps the more fruitful and interesting route, and the author is inclined to go in that direction.

How to extend the theory to the microscopic domain, is a very important problem. There are many interesting questions in this area that are of direct concern to modern physics. For instance, time and space are macroscopic concepts; what do the space-time coordinates in the wave function of quantum mechanics mean? How are the microscopic 'fundamental particles' to be introduced in a macroscopic theory, and how should their interactions be formulated? These, and many other basic questions must be answered before the journey can be resumed.

INTRODUCTION TO THE APPENDICES

Symbolic logic and set theory are presented in the form of one purely deductive theory as described in Chap. I. It is therefore helpful to study that material before going on with this part.

The principal sources for this part are the works of Rosenbloom([1950] Chap. IV), Kleene ([1952] Chap. IV), Nicod [1916], Whitehead and Russell [1925], and Bernays and Fraenkel [1958]. However, the reader who is not familiar with the subject ought to start with the books of Suppes [1957] and Tarski [1959].

This part consists of two chapters. Chap. A lays the complete foundation of symbolic logic and that portion of set theory necessary for the understanding of this work. The remainder of set theory is given in references (Bernays and Fraenkel [1958]). Chap. B provides a classified list of the logical theorems. The proofs of these theorems can be found in Nicod [1916], Whitehead and Russell [1925], and Bernays and Fraenkel [1958].

A. FOUNDATION

1. Concepts

Term concepts (C1–C11): $\mathfrak{T}, \emptyset, \sqcup, (\quad,\quad), {}_{/}, \iota; \mathfrak{V}, \mathfrak{v}, \mathfrak{B}, {}_{|}$.
Class concepts (C12–C13): $\mathfrak{C}, \ni$.
Formula concepts (C14–C19): $\mathfrak{F}, =, \in, |, \forall, \vdash$.

2. Interpretations

(i) *Term Concepts*: –

In the following a *set* means an individual or a definite collection of individuals, whereas a *class* means a collection (possibly indefinite) of individuals that have a common property. A set can be an element of a class or another set, but a class cannot be an element of anything. This distinction between set and class is due to von Neumann and Bernays (Bernays and Fraenkel [1958] pp. 32–33, 56–57).

I1. '$\mathfrak{T}t$' reads 't is a term'. A *term* is an expression which either names or describes a set, or results in a name or description of a set when the variables in the expression are replaced by names or descriptions (Suppes [1957] § 3.2). For example, if m and n are arbitrary numbers, then $2, n, 2m+n$ are all terms. Exactly what constitutes a term is specified in the postulates of Sec. 7. Terms are denoted in the ML by lower case Latin letters.

I2. $\emptyset$ is the *null set*, i.e., the set consisting of no elements.

I3. $a \sqcup b$ is the *term whose elements are the term* b, *and the elements of term* a (Bernays and Fraenkel [1958] p. 65). This concept is used to define the set $\{a_1, \ldots, a_n\}$ and the ordered n-tuple $\langle a_1, \ldots, a_n \rangle$ (see D5.12, 13). Bernays uses ';' instead of '$\sqcup$', but the latter was chosen because it is reminiscent of the set union '$\cup$' and because the semicolon has another meaning.

I4,5. The symbol '(' is called a *left parenthesis* (LP), and ')' is called a right parenthesis (RP). '(A)' means 'A is to be taken as a unit, provided A has an equal number (possibly zero) of LP's and RP's, and such that to every LP there is associated a unique RP to the right of it' (Rosenbloom [1950] pp. 23–24).

I6. The symbol '$_/$' is a *comma* in the OL.

I7. If there exists a unique x that satisfies F, $(\iota x)(F_/a)$ is that x; otherwise, $(\iota x)(F_/a)$ is a. The *description symbol* 'ι' was introduced in this useful form by Bernays (Bernays and Fraenkel [1958] p. 49).

I8. '$\mathfrak{V}x$' means 'x is a set *variable*, i.e., the name of an arbitrary set'. In the ML, variables are denoted by the letters: u, v, w, x, y, z.

I9–I1. '$\mathfrak{B}s$' means 's is a *string of bars*, such as: $_|$, $_{||}$, ...'. These strings are used as subscripts with '$\mathfrak{v}$' to denote variables in the OL, e.g., '$\mathfrak{v}_|$', '$\mathfrak{v}_{||}$', ...

(ii) *Class concepts*: –

I12. '$\mathfrak{C}\mathscr{A}$' means '$\mathscr{A}$ is a *class*'. An example of a class, is the class of numbers; we cannot exhibit all the numbers at once, but we can specify the property of being a number and decide on that basis whether a certain set is a number or not. Another example, is the class of 3-armed humans. Even though there may not be any animal that belongs to this class, the class is well defined. A class of a different kind, is the class of ordered couples of events, one event occurring before the other. Such a class is called a binary relation. In general, a class of ordered n-tuples is called an *n-place relation* (see D5.23). In the ML, classes are denoted by script capital letters.

I13. $x \ni F$ is the *class of all sets x that satisfy F*; it is denoted by '$\{x|F\}$' in some of the literature. This is the only way we have of producing classes, and shows clearly the intimate relationship between classes and properties (predicates).

(iii) *Formula concepts*: –

I14. '$\mathfrak{F}F$' reads 'F is a formula'. A *formula* is either a sentence (name of a proposition) or a sentential function, i.e., an expression which becomes a sentence when certain variables called free variables (defined in Sec. 4), are replaced by names of sets (Tarski [1959] § 2, § 3). Precisely what constitutes a formula is specified in the postulates of Sec. 7. Formulas in the ML are denoted by the letters: F, G, H, I, J, K, L.

I15. '$a = b$' means 'a and b are the same term, i.e., 'a' and 'b' are two names of the same term'. The symbol '$=$' is called the *identity* relation, and applies only between terms. The 'equality' of two classes *defined* in D5.21, means that the two classes have the same elements, and must not be confused with the identity relation.

I16. '$a \in \mathscr{A}$' means 'the term a is an *element* or *member* of the term or class $\mathscr{A}$'. A set can be an element of either another set or a class; for instance,

$1 \in \{1,2\}$ and $1 \in$ (class of integers). But a class cannot be an element of anything. A set having one element is not equal to that element, as can be seen from '$\{\{a,b\}\}$' and '$\{a,b\}$'; $\{\{a,b\}\}$ has the one element $\{a,b\}$, whereas $\{a,b\}$ has the two elements a,b. Moreover, $\in$ is neither symmetric nor transitive, i.e., '$a \in b$' does not imply '$b \in a$', and from '$a \in b$' and '$b \in c$' it does not follow that '$a \in c$'. Since a set is a definite collection of individuals, we can deduce from a sentence such as '$a \in b$' that $b = \{a, x_1, \ldots, x_n\}$, where $x_1, \ldots, x_n$ are terms.

I17. '$F|G$' means 'at least one of the formulas F or G is false', and reads '*F stroke G*'. This concept was introduced by Scheffer [1913] and much economy in both concepts and postulates was achieved with its help. All the usual sentential connectives such as 'not', 'or', 'and', can be defined in terms of it (see D5.1–6).

I18. '$(\forall x)F$' means 'for all x, F is true'. The symbol '$\forall$' is called the *universal* quantifier. The existential quantifier 'there exists at least one x' is defined in terms of '$\forall$' in D5.7.

I19. '$\vdash F$' means '*F* is *true*'. True formulas are produced by means of the postulates of Sec. 8. An example of a true formula is '$x = x$', where x is a variable. Since x is the name of an arbitrary set, '$\vdash x = x$' states that for any arbitrary set it is true that $x = x$. In contrast to this, the formula '$x < y$' is neither true nor false; it acquires a truth value only when 'x' and 'y' are replaced by specific numbers.

3. Punctuation

It is very cumbersome to use parentheses exclusively, and thus points are also used. A *point* is a symbol consisting of one or more *dots*. A point to the right of a logical symbol is called a *right-point* (RPt), and one to the left of a logical symbol, a *left-point* (LPt); the point is said to be *attached* to the symbol in question. Each point in a string indicates a part of the string which would have been enclosed by parentheses in the object language; this part is called the *scope* of the point. If p, q are points in a string, p is said to be *senior* to q if and only if p consists of more dots than q. The scope of any RPt extends to the right up to the first (if any) LPt which is senior to it. Similarly, the scope of any LPt extends to the left up to the first (if any) RPt which is senior to it (Rosenbloom [1950] p. 32).

There are certain symbols that always occur together and cannot be separated such as '$a \in \mathscr{A}$', '$x \ni F$', '$a = b$'. No parentheses or points are used between these symbols. Moreover, in using the symbols defined in D5.1–6,

$\rightarrow$ and $\leftrightarrow$ are considered senior to $|$, $\wedge$, $\vee$, and $\veebar$, all of which are in turn senior to $\sim$. For instance,

$$(((F \wedge (\sim G)) \rightarrow (F \vee (G|G))) \leftrightarrow ((a=b) \veebar (a \in \mathscr{A}))$$

can be written as

$$F \wedge \sim G \rightarrow F \vee .G|G : \leftrightarrow a=b \veebar a \in \mathscr{A}.$$

4. Free and bound variables

A variable 'x' that occurs in a formula in the form '$(\forall x)F$', '$x \ni F$', or '$(\imath x)(F_{/}a)$'; or in any other formula constructed or defined in terms of these formulas is called a *bound variable*; otherwise, it is called a *free variable* (Suppes [1957] § 3.5).

The notation '$F(x,y,\ldots)$' means that the variables 'x','y',..., are free in 'F', and '$F(x,y,\ldots)$' is said to be a *sentential function* of $x,y,\ldots$. If we simply write 'F' for a formula, a variable 'x' may be either free or bound in 'F'.

If we are given '$F(x,y,\ldots)$', then '$F(a,b,\ldots)$' denotes $F(x,y,\ldots)$ with 'x' substituted by 'a', 'y' by 'b',... everywhere in 'F'. The substitution which gives '$F(a)$' must always be performed for the *original* free variable x in the *original* formula '$F(x)$'. For instance, if '$F(x)$' is '$x+a=x$', then '$F(a)$' is '$a+a=a$'. However, '$F(b)$' is '$b+a=b$' not '$b+b=b$', which is obtained by substituting 'b' for 'a' in '$F(a)$'.

The sentence 'x *does not occur free* in F' is denoted by '$x \mathfrak{b} F$' and 'x is *free* in F' by '$x \mathfrak{f} F$'. Notice that '$x \mathfrak{b} F$' and '$x \mathfrak{f} F$' are always in the metalanguage, since '$\mathfrak{b}$' and '$\mathfrak{f}$' are.

5. Definitions

The following set of definitions is restricted to those used in this book.

D1. $\sim F$ *for* $F|F$.

According to I17, '$\sim F$' means 'F is false (F)', and reads '*not* F'.

D2. $F \wedge G$ *for* $\sim(F|G)$.

According to I17 and D1, '$F \wedge G$' means 'F and G are true (T)', and reads '*F and G*'.

D3. $F \vee G$ *for* $\sim F|\sim G$.

'$F \vee G$' means 'at least one of the sentences F or G is true', and reads '*F or G*'.

D4. $G \rightarrow H$ *for* $G | \sim H$.

'$G \rightarrow H$' means 'either G is F; or H is T; or G is F, and H is T'. Since the only logical possibilities are 'G is T', and 'H is T or F'; 'G is F', and 'H is T or F'; it follows that '$G \rightarrow H$' is F only if G is T, and H is F. '$G \rightarrow H$' reads '*G implies H*', or '*if G, then H*', or 'G is *sufficient* for H', or 'H is *necessary* for G'; G is called the *antecedent* and H the *consequent*.

D5. $F \leftrightarrow G$ *for* $F \rightarrow G. \wedge .G \rightarrow F$.

'$F \leftrightarrow G$' means 'either both F and G are T, or both are F'; and reads 'F is *equivalent* to G', or '*F if and only if G*', or 'F is *necessary and sufficient* for G'.

D6. $F \veebar G$ *for* $\sim(F \leftrightarrow G)$.

'$F \veebar G$' means 'either F is T, or G is T, but not both F and G are T', and reads '*F or G, but not F and G*'.

D7. $(\exists x)F$ *for* $\sim(\forall x)\sim F$.

According to I18 and D1, '$(\exists x)F$' means 'there exists *at least one* x which satisfies F'. The symbol '$\exists$' is called the *existential quantifier*.

D8. $(\forall x_1, \ldots, x_n)F$ *for* $(\forall x_1)\ldots(\forall x_n)F$,
$(\exists x_1, \ldots, x_n)F$ *for* $(\exists x_1)\ldots(\exists x_n)F$.

D9. $(\exists^1 x)F(x)$ *for* $(\forall x, y).F(x) \wedge F(y) \rightarrow x = y$.

According to I18,15 and D2,4, '$(\exists^1 x)F(x)$' means 'there exists *at most one* x which satisfies F'.

D10. $(\exists ! x)F$ *for* $(\exists x)F. \wedge (\exists^1 x)F$.

According to D7,9, '$(\exists ! x)F$' means 'there exists *exactly one* x which satisfies F'.

D11. $(\iota x)F$ *for* $(\iota x)(F/\emptyset)$.

According to I7, if $(\exists ! x)F$ then $(\iota x)F$ is this x; otherwise, $(\iota x)F = \emptyset$.

D12. $\{a\}$ *for* $\emptyset \sqcup a$,
$\{a, b\}$ *for* $\{a\} \sqcup b$,
$\{a_1, \ldots, a_{n+1}\}$ *for* $\{a_1, \ldots, a_n\} \sqcup a_{n+1}$.

According to I3, $\{a_1, \ldots, a_n\}$ is the set whose elements are $a_1, \ldots, a_n$.

D13. $\langle a \rangle$ *for* a,
$\langle a, b \rangle$ *for* $\{\{a\}, \{a, b\}\}$,
$\langle a_1, \ldots, a_n \rangle$ *for* $\langle a_1, \langle a_2, \ldots, a_n \rangle \rangle$.

'$\langle a_1, \ldots, a_n \rangle$' is called an *ordered n-tuple*.

D14. V *for* $x \ni x = x$.

Since $x = x$ for any x, it follows from I13 that V is the *universal class*, i.e., the class of all terms.

D15. Λ *for* $x \ni x \neq x$.

Since there is no x for which $x \neq x$, Λ is the *null class*, i.e., the class containing no elements.

In D16–23 below, the symbols 'A' and 'B' denote either classes or terms.

D16. $A^{\sim}$ *for* $x \ni \sim x \in A$.

$A^{\sim}$ is the class of all terms *not* in A, and is called the *complement* of A.

D17. $A \cup B$ *for* $x \ni x \in A \vee x \in B$.

$A \cup B$ is the class of terms that are either in A *or* in B, and is called the *union* of A and B.

D18. $A \cap B$ *for* $x \ni x \in A \wedge x \in B$.

$A \cap B$ is the class of terms that are both in A *and* B, and is called the *intersection* of A and B.

D19. $A \subseteq B$ *for* $(\forall x).x \in A \rightarrow x \in B$.

'$A \subseteq B$' means 'any element of A is an element of B', and reads 'A is a *subclass* (subset) of B'.

D20. $A \subset B$ *for* $A \subseteq B \wedge \sim B \subseteq A$.

'$A \subset B$' means 'A is a subclass of B but not conversely', and reads 'A is a *proper subclass* (subset) of B'.

D21. $A \equiv B$ *for* $A \subseteq B \wedge B \subseteq A$.

'$A \equiv B$' means 'A and B have the same elements', and reads 'A is *equal* to B'. Further remarks about '$\equiv$' are given after P6,7 of Sec. 8 (iii).

D22. $\langle x_1, \ldots, x_n \rangle \ni F$ *for*
$y \ni (\exists x_1, \ldots, x_n).y = \langle x_1, \ldots, x_n \rangle \wedge F$.

'$\langle x_1, \ldots, x_n \rangle \ni F$' denotes the class of all ordered n-tuples y that satisfy F. In some of the literature, this is written as '$\{x_1, \ldots, x_n | F\}$'.

D23. $\Re_n \mathscr{A}$ *for*
$(\forall y). y \in \mathscr{A} \rightarrow (\exists x_1, \ldots, x_n)\, y = \langle x_1, \ldots, x_n \rangle$.

'$\Re_n \mathscr{A}$' means '$\mathscr{A}$ is a class of ordered n-tuples', and reads '$\mathscr{A}$ is an *n-place relation*'. For the special cases of $n=2$ and $n=3$, $\mathscr{A}$ is said to be a *binary* and a *ternary* relation, respectively.

In all of the remaining definitions, it is assumed that $\mathscr{A}$ is a class and $\mathscr{R}$ is a binary relation.

D24. $a\mathscr{R}b$ *for* $\langle a,b \rangle \in \mathscr{R}$.

D25. ($\mathscr{R}$ is a *reflexive* in $\mathscr{A}$) *for*
$(\forall x).\ x \in \mathscr{A} \rightarrow x\mathscr{R}x$.

D26. ($\mathscr{R}$ is *irreflexive* in $\mathscr{A}$) *for*
$(\forall x).\ x \in \mathscr{A} \rightarrow \sim x\mathscr{R}x$.

D27. ($\mathscr{R}$ is *symmetric* in $\mathscr{A}$) *for*
$(\forall x,y).\ x,y \in \mathscr{A} \wedge x\mathscr{R}y \rightarrow y\mathscr{R}x$.

D28. ($\mathscr{R}$ is *asymmetric* in $\mathscr{A}$) *for*
$(\forall x,y).\ x,y \in \mathscr{A} \wedge x\mathscr{R}y \rightarrow \sim y\mathscr{R}x$.

D29. ($\mathscr{R}$ is *antisymmetric* in $\mathscr{A}$) *for*
$(\forall x,y).\ x,y \in \mathscr{A} \wedge x\mathscr{R}y \wedge y\mathscr{R}x \rightarrow x=y$.

D30. ($\mathscr{R}$ is *transitive* in $\mathscr{A}$) *for*
$(\forall x,y,z).\ x,y,z \in \mathscr{A} \wedge x\mathscr{R}y \wedge y\mathscr{R}z \rightarrow x\mathscr{R}z$.

D31. ($\mathscr{R}$ is *semitransitive* in $\mathscr{A}$) *for*
$(\forall x,y,z).\ x,y,z \in \mathscr{A} \wedge x\mathscr{R}y \wedge x\mathscr{R}z \rightarrow y\mathscr{R}z$.

D32. ($\mathscr{R}$ is an *equivalence* in $\mathscr{A}$) *for*
$\mathscr{R}$ is reflexive, symmetric, and transitive in $\mathscr{A}$.

D33. ($\mathscr{R}$ is a *strict partial ordering* of $\mathscr{A}$) *for*
$\mathscr{R}$ is asymmetric and transitive in $\mathscr{A}$.

D34. ($\mathscr{R}$ is a *partial ordering* of $\mathscr{A}$) *for*
$\mathscr{R}$ is reflexive, antisymmetric, and transitive in $\mathscr{A}$.

D35. $\Re_{M-1}\mathscr{R}$ *for*
$\Re_2 \mathscr{R} \wedge (\forall x,y,z).\ x\mathscr{R}y \wedge x\mathscr{R}z \rightarrow y=z$.

'$\Re_{M-1}\mathscr{R}$' reads '$\mathscr{R}$ is a *many-one relation*, or a *function*'.

D36. $\Re_{1-M}\mathscr{R}$ *for*
$\Re_2 \mathscr{R} \wedge (\forall x,y,z).\ y\mathscr{R}x \wedge z\mathscr{B}x \rightarrow y=z$.

'$\Re_{1-M}\mathscr{R}$' reads '$\mathscr{R}$ is a *one-many relation*'.

D37. $\mathfrak{R}_{1-1}\mathscr{R}$ *for* $\mathfrak{R}_{M-1}\mathscr{R} \wedge \mathfrak{R}_{1-M}\mathscr{R}$.
'$\mathfrak{R}_{1-1}\mathscr{R}$' reads '$\mathscr{R}$ is a *one-one relation,* or a *correspondence*'.

6. Axioms

A1. $\mathfrak{B}_{|}$.

According to I9–11, this axiom states that $_|$ is a string of bars (consisting of a single bar in this case).

A2. $\mathfrak{T}\emptyset$.

In accordance with I1 this means that $\emptyset$ is a term. The fact that it is the *null* set, is expressed in P10 of Sec. 8.

7. Postulates of classification

(i) *Variables*: –

PV1. $\mathfrak{B}x \twoheadrightarrow \mathfrak{B}x_|$.

The meaning of the metalinguistic symbol '$\twoheadrightarrow$' is given in (I2.1). In essence, PV1 states that if the string $\mathfrak{B}x$ is admissible, then $\mathfrak{B}x_|$ is also admissible (a product). According to A1, $\mathfrak{B}_|$ is admissible, and thus $\mathfrak{B}_{||}$ is a product. Repeated application of PV1 yields the products $\mathfrak{B}_{|||}, \mathfrak{B}_{||||}, \ldots$; this means that $_|, _{||}, _{|||}, \ldots$ are strings of bars.

PV2. $\mathfrak{B}x \twoheadrightarrow \mathfrak{V}\mathfrak{v}x$.

Replacing the string variable 'x' by the strings '$_|$', '$_{||}$', ..., we get the products '$\mathfrak{V}\mathfrak{v}_|$', '$\mathfrak{V}\mathfrak{v}_{||}$', ... According to I8, this means that $\mathfrak{v}_|, \mathfrak{v}_{||}, \ldots$ are variables of the OL. We are thus able to produce as many variables in the OL as we wish with the help of only the four concepts: $\mathfrak{B}$, $_|$, $\mathfrak{V}$, and $\mathfrak{v}$ (Rosenbloom [1950] p. 187).

(ii) *Terms*: –

PT1. $\mathfrak{V}x \twoheadrightarrow \mathfrak{T}x$.

With the help of this and the above results we can generate the products: $\mathfrak{T}\mathfrak{v}_|, \mathfrak{T}\mathfrak{v}_{||}, \ldots$.

Any formula which can be constructed in terms of sets can also be constructed by means of PT1 in terms of variables, and that is how sentential functions are produced.

PT2. $\mathfrak{T}a, \mathfrak{T}b \twoheadrightarrow \mathfrak{T}(a \sqcup b)$.

Making use of A2 and D12,13, we can infer from this that if a and b are terms, so are $\{a\}$, $\{a,b\}$, $\langle a \rangle$, and $\langle a,b \rangle$. The generalization to $\{a_1,\ldots,a_n\}$ and $\langle a_1,\ldots,a_n \rangle$ is straightforward.

PT3. $\mathfrak{V}x, \mathfrak{T}a, \mathfrak{F}F \twoheadrightarrow \mathfrak{T}((\imath x)(F_{/}a))$.

If we replace 'a' by '$\emptyset$' in PT3, and make use of A2 and D11, we can derive the theorem:

$$\mathfrak{V}x, \mathfrak{F}F \twoheadrightarrow \mathfrak{T}((\imath x)F). \quad (1)$$

PT4. $\mathfrak{C}\mathscr{A}, \vdash a \in \mathscr{A} \twoheadrightarrow \mathfrak{T}a$.

In our theory and other theories, many of the primitive concepts are classes, i.e., the string '$\mathfrak{C}\mathscr{A}$' is an axiom. By means of PT4 we can assert that any element of a class is a term.

(iii) *Formulas*: –

PF1. $\mathfrak{T}a, \mathfrak{T}b \twoheadrightarrow \mathfrak{F}(a = b)$.
PF2. $\mathfrak{T}a, \mathfrak{T}b \twoheadrightarrow \mathfrak{F}(a \in b)$.
PF3. $\mathfrak{T}a, \mathfrak{C}\mathscr{A} \twoheadrightarrow \mathfrak{F}(a \in \mathscr{A})$.

These are the basic (atomic) formulas from which all other formulas are constructed. Although a term can be an element of either a term or a class, a class cannot be an element of anything, which is one of the main distinctions between a set and a class.

PF4. $\mathfrak{F}F, \mathfrak{F}G \twoheadrightarrow \mathfrak{F}(F|G)$.

Making use of D1–6, we conclude from PF4 that if F and G are formulas, so are: $\sim F$, $F \wedge G$, $F \vee G$, $F \rightarrow G$, $F \leftrightarrow G$, and $F \veebar G$.

PF5. $\mathfrak{V}x, \mathfrak{F}F \twoheadrightarrow \mathfrak{F}((\forall x)F)$.

With the help of this and D7,9,10, we see that if x is a variable and F is a formula, then '$(\exists x)F$', '$(\exists^1 x)F$', and '$(\exists ! x)F$' are also formulas.

(iv) *Classes*: –

PC1. $\mathfrak{V}x, \mathfrak{F}F \twoheadrightarrow \mathfrak{C}(x \ni F)$.

This is the only method we have of producing classes, aside from introducing axioms of the form '$\mathfrak{C}\mathscr{A}$'. The following theorem is a consequence of PC1:

T1. $\mathfrak{T}a \twoheadrightarrow \mathfrak{C}(\mathfrak{v}_{|} \ni \mathfrak{v}_{|} = a)$.

Proof. A1, PV2, PT1 : $\mathfrak{V}\mathfrak{v}_{|} \twoheadrightarrow \mathfrak{T}\mathfrak{v}_{|}$.

PF1 : $\mathfrak{T}\mathfrak{v}_{|}, \mathfrak{T}a \twoheadrightarrow \mathfrak{F}(\mathfrak{v}_{|} = a)$.

PC1 : $\mathfrak{V}\mathfrak{v}_{|}, \mathfrak{F}(\mathfrak{v}_{|} = a) \twoheadrightarrow$ Output.

The procedure of proof is explained in (I4). According to T1, to every term a there corresponds a class, namely the class of terms equal to a. In contrast to this, if we are given a class $\mathscr{A}$, there is no way of producing a term a that represents $\mathscr{A}$, in the sense $a \equiv \mathscr{A}$ (See D21).

8. Postulates about true formulas

(i) *Propositions*: –

P1. $\vdash F, \vdash F \to G \twoheadrightarrow \vdash G$.

This is the most frequently used postulate in deduction, and is called the Principle of Inference, Law of Detachment, or *modus ponens*. It states that if the truth of F and 'F implies G' is established, then we may also accept the truth of G.

P2. $\mathfrak{F}F, \mathfrak{F}G, \mathfrak{F}H, \mathfrak{F}I, \mathfrak{F}J \twoheadrightarrow$
$\vdash F|.G|H:|:I \to I.|.(J|G) \to (F|J)$.

Nicod [1916] proposed this postulate and proved that the five postulates of Principia Mathematica [1925] concerning propositions, and thus all of propositional calculus, can be derived from it and P1. Both P1 and P2 can be written purely in terms of the Sheffer stroke '|', with the help of D4,1.

(ii) *Quantifiers*: –

The following three postulates about the universal quantifier are based upon postulates FII, FVI, FVII of Rosenbloom ([1950] pp. 93–94):

P3. $\mathfrak{V}x, \vdash F \twoheadrightarrow \vdash (\forall x) F$.

P4. $\mathfrak{V}x, \mathfrak{T}a, \mathfrak{F}F(x) \twoheadrightarrow \vdash (\forall x) F(x). \to F(a)$.

P5. $\mathfrak{V}x, \mathfrak{F}F, \mathfrak{F}G, x\mathfrak{b}F \twoheadrightarrow \vdash (\forall x).F \to G: \to .F \to (\forall x) G$.

With the help of these postulates, all propositions about quantifiers assumed or derived by other authors can be proved. Postulate P3 is called the Rule of Universal Generalization, and P4 the Rule of Universal Specification. According to P3, if F is true, one can conclude that for all x, F is true, regardless whether 'x' is bound or free in 'F' (see Sec. 4). If $x\mathfrak{b}F$ then '$(\forall x)F$' is trivially true, but if $x\mathfrak{f}F$ then '$(\forall x)F$' is a generalization of F. Postulate P4 permits us to go in the opposite direction; it states that if F is true for all x, then F is true for any particular set a. Thus all the variables are set variables. P5 means: If the strings $\mathfrak{V}x$, $\mathfrak{F}F$, $\mathfrak{F}G$ are admissible and 'x' does not occur free in 'F' ('$x\mathfrak{b}F$' can never be in the OL because '$\mathfrak{b}$' is not), we can admit the true sentence 'If for all x, F implies G, then F implies for all x, G.'

(iii) *Identity and membership*: –

P6. $\mathfrak{T}a, \mathfrak{T}b, \mathfrak{C}\mathscr{A} \twoheadrightarrow \vdash a=b \rightarrow .a \in \mathscr{A} \rightarrow b \in \mathscr{A}.$

P7. $\mathfrak{T}a, \mathfrak{T}b \twoheadrightarrow \vdash a \equiv b \rightarrow a=b.$

These are called the postulates of *equality* and *extensionality*, respectively (Bernays and Fraenkel [1958] pp. 52–53). P6 states that if $a=b$, then $a\in\mathscr{A}$ implies $b\in\mathscr{A}$; with the help of P12 below, this means that if $a=b$, any proposition which holds for a must also hold for b. According to P7, if two terms a, b have the same elements, i.e., have the same extension (see D21), then $a=b$.

(iv) *Description*: –

P8. $\mathfrak{V}x, \mathfrak{T}a, \mathfrak{T}b, \mathfrak{F}F(x) \twoheadrightarrow$
$\vdash F(a) \wedge (\forall x).F(x) \rightarrow x=a : \rightarrow (\imath x)(F(x)_{/}b)=a.$

P9. $\mathfrak{V}x, \mathfrak{T}b, \mathfrak{F}F(x) \twoheadrightarrow$
$\vdash \sim(\exists! x)F(x). \rightarrow (\imath x)(F(x)_{/}b)=b.$

The preceeding postulates are equivalent to those given by Bernays ([1958] p. 49), and express the properties of $\imath$ stated in I7. According to P8, if the term a is the only one that satisfies F, then $(\imath x)(F(x)_{/}b)=a$; P9 states that if there is no unique x that satisfies F, then $(\imath x)(F(x)_{/}b)=b$.

(v) *Sets and classes*: –

P10. $\mathfrak{T}a \twoheadrightarrow \vdash \sim a \in \emptyset.$

P11. $\mathfrak{T}a, \mathfrak{T}b, \mathfrak{T}c \twoheadrightarrow \vdash c \in (a \sqcup b) \leftrightarrow c \in a \vee c=b.$

P12. $\mathfrak{V}x, \mathfrak{T}a, \mathfrak{F}F(x) \twoheadrightarrow \vdash a \in . x \ni F(x) : \leftrightarrow F(a).$

These three postulates express the meaning of the concepts $\emptyset$, $\sqcup$, and $\ni$ interpreted in I2, 3, 13, and are also due to Bernays ([1958] p. 65). Postulate P10 states that no term is an element of the null set $\emptyset$; P11 that c is an element of $a \sqcup b$ if and only if $c\in a$ or $c=b$; and P12 that the sentence 'a is an element of the class of all x satisfying F' is equivalent to the sentence 'a satisfies F'.

Three more postulates concerning sets and classes are necessary to complete the deductive formulation of set theory (Bernays and Fraenkel [1958]). They are not discussed here because they are not needed in this book.

9. Application of symbolic logic

To use SL in a deductive theory, it is necessary to supplement the foundation in this appendix by additional concepts, axioms, and postulates, depending upon the field of interest. To illustrate, consider the following formulation of a part of the theory of natural numbers:

Concepts: $\mathcal{N}$, 0, $'$.

Interpretations: $\mathcal{N}$ is the class of natural numbers, 0 is the number *zero*, and n' is the *successor* of the number n.

Axioms: $\mathfrak{C}\mathcal{N}$, $\vdash 0 \in \mathcal{N}$. (2)

Postulate: $\vdash n \in \mathcal{N} \rightarrow \vdash n' \in \mathcal{N}$. (3)

First of all, we notice that the concepts $\mathfrak{C}$, $\vdash$, and $\in$ belong to SL. Then by using PT4, we get:

$$\mathfrak{C}\mathcal{N}, \vdash 0 \in \mathcal{N} \rightarrow \mathfrak{T}0.$$

Since according to (2) the input is admissible, $\mathfrak{T}0$ is a product, i.e., 0 is a term. From this result and (3) we can also deduce that $0', 0''$ ... are terms (the usual designations for these numbers are '1', '2',...). This shows how the concepts and postulates of SL play an essential role in both the formulation and development of a deductive theory; a fact which is amply illustrated throughout this book.

B. LOGICAL THEOREMS

In this chapter, a classified list is given of the logical theorems needed in this work. The proof of these theorems can be found in Nicod [1916], Whitehead and Russell [1925] and Bernays and Fraenkel [1958].

As explained in Sec.I4, theorems are productions. But since the input strings can be easily deduced from the form of the output string, only the output of each theorem is presented.

1. Rules of inference

T1. $\vdash F, \vdash F \to G \twoheadrightarrow \vdash G.$

$\vdash F, \vdash F \leftrightarrow G \twoheadrightarrow \vdash G.$

$\vdash F, \vdash G \leftrightarrow F \twoheadrightarrow \vdash G.$

T2. $\vdash F \to G, \vdash G \to H \twoheadrightarrow \vdash F \to H.$

$\vdash F \to G, \vdash G \leftrightarrow H \twoheadrightarrow \vdash F \to H.$

$\vdash F \to G, \vdash H \leftrightarrow G \twoheadrightarrow \vdash F \to H.$

$\vdash F \leftrightarrow G, \vdash G \to H \twoheadrightarrow \vdash F \to H.$

$\vdash G \leftrightarrow F, \vdash G \to H \twoheadrightarrow \vdash F \to H.$

T3. $\vdash F \to G, \vdash G \to F \twoheadrightarrow \vdash F \leftrightarrow G.$

T4. $\vdash F \leftrightarrow G, \vdash G \leftrightarrow H \twoheadrightarrow \vdash F \leftrightarrow H.$

T5. $\vdash F \to .G \to H, \vdash H \to I \twoheadrightarrow \vdash F \to .G \to I.$

T6. $\vdash F \to .G \to H, \vdash I \to G \twoheadrightarrow \vdash F \to .I \to H.$

T7. $\vdash F_1, \ldots, \vdash F_n \twoheadrightarrow \vdash F_1 \wedge \ldots \wedge F_n.$

T8. $\vdash F \to G_1, \ldots, \vdash F \to G_n \twoheadrightarrow \vdash F \to G_1 \wedge \ldots \wedge G_n.$

T9. $\vdash F_1 \to G, \ldots, \vdash F_n \to G \twoheadrightarrow \vdash F_1 \vee \ldots \vee F_n \to G.$

T10. $\vdash F \twoheadrightarrow \vdash F \wedge G \leftrightarrow G.$

T11. $\vdash \sim F \twoheadrightarrow \vdash F \vee G \leftrightarrow G.$

T12. $\vdash (\forall x)F, \vdash (\forall x).F \to G \twoheadrightarrow \vdash (\forall x)G.$

T13. $\vdash (\forall x).F \to G, \vdash (\forall x).G \to H \twoheadrightarrow \vdash (\forall x).F \to H.$

T14. If s,t are any strings, then

$\vdash F \leftrightarrow G, \vdash sFt \twoheadrightarrow \vdash sGt.$

2. Propositions

T1. $\sim .F \wedge \sim F.$

T2. $F \vee \sim F.$

T3. $F \veebar \sim F.$

T4. $F \wedge G. \veebar .F \wedge \sim G. \veebar . \sim F \wedge G. \veebar . \sim F \wedge \sim G \wedge H.$
$\veebar . \sim F \wedge \sim G \wedge \sim H.$

T5. $F \rightarrow .G \rightarrow F.$

T6. $\sim F \rightarrow .F \rightarrow G.$

T7. $F \wedge .F \rightarrow G : \rightarrow G.$

T8. $\sim G \wedge .F \rightarrow G : \rightarrow \sim F.$

T9. $F \rightarrow G \wedge \sim G. \rightarrow \sim F.$

T10. $F \wedge .F \veebar G : \rightarrow \sim G.$

T11. $\sim F \wedge .F \veebar G : \rightarrow G.$

T12. $F \rightarrow G. \rightarrow .F \wedge H \rightarrow G \wedge H.$
$\rightarrow .F \vee H \rightarrow G \vee H.$

T13. $F \rightarrow G. \wedge .H \rightarrow I : \rightarrow .F \vee H \rightarrow G \vee I.$

T14. $F_1 \wedge \ldots \wedge F_n \rightarrow F_i (i = 1, \ldots, n).$

T15. $F_i \rightarrow F_1 \vee \ldots \vee F_n (i = 1, \ldots, n).$

T16. $F \leftrightarrow F.$

T17. $\sim \sim F \leftrightarrow F.$

T18. $F \wedge F \leftrightarrow F.$

T19. $F \vee F \leftrightarrow F.$

T20. $F \rightarrow G. \leftrightarrow . \sim G \rightarrow \sim F.$

T21. $F \rightarrow G. \leftrightarrow . \sim F \vee G.$

T22. $F \rightarrow G. \leftrightarrow . \sim (F \wedge \sim G).$

T23. $F \leftrightarrow G. \leftrightarrow .G \leftrightarrow F.$

T24. $F \leftrightarrow G. \leftrightarrow . \sim G \leftrightarrow \sim F.$

T25. $F \wedge G \leftrightarrow \sim (F \rightarrow \sim G).$

T26. $F \vee G \leftrightarrow . \sim F \rightarrow G.$

T27. $F \veebar G \leftrightarrow : F \vee G. \wedge . \sim (F \wedge G).$

T28. $F_1 \wedge \ldots \wedge F_n \leftrightarrow F_{i_1} \wedge \ldots \wedge F_{i_n},$

T29. $F_1 \vee \ldots \vee F_n \leftrightarrow F_{i_1} \vee \ldots \vee F_{i_n},$

T30. $F_1 \veebar \ldots \veebar F_n \leftrightarrow F_{i_1} \veebar \ldots \veebar F_{i_n},$

where $(i_1, \ldots, i_n)$ is any permutation of $(1, \ldots, n)$.

T31. $F \rightarrow .G \rightarrow H : \leftrightarrow : G \rightarrow .F \rightarrow H.$

T32. $F \wedge G \rightarrow H. \leftrightarrow .F \rightarrow .G \rightarrow H$
$\leftrightarrow .F \wedge \sim H \rightarrow \sim G.$

T33. $F \rightarrow G. \wedge .H \rightarrow I : \leftrightarrow .F \wedge H \rightarrow G \wedge I.$

T34. $F \rightarrow G_1 \wedge \ldots \wedge G_n . \leftrightarrow : F \rightarrow G_1 . \wedge \ldots \wedge .F \rightarrow G_n.$

T35. $F \rightarrow G_1 \vee \ldots \vee G_n . \leftrightarrow : F \rightarrow G_1 . \vee \ldots \vee .F \rightarrow G_n.$

T36. $F_1 \wedge \ldots \wedge F_n \rightarrow G. \leftrightarrow : F_1 \rightarrow G. \vee \ldots \vee .F_n \rightarrow G.$

T37. $F_1 \vee \ldots \vee F_n \rightarrow G. \leftrightarrow : F_1 \rightarrow G. \wedge \ldots \wedge .F_n \rightarrow G.$

T38. $\sim (F_1 \wedge \ldots \wedge F_n) \leftrightarrow \sim F_1 \vee \ldots \vee \sim F_n.$

T39. $\sim (F_1 \vee \ldots \vee F_n) \leftrightarrow \sim F_1 \wedge \ldots \wedge \sim F_n.$

T40. $F \wedge .G_1 \vee \ldots \vee G_n : \leftrightarrow : F \wedge G_1 . \vee \ldots \vee .F \wedge G_n.$

T41. $F \vee .G_1 \wedge \ldots \wedge G_n : \leftrightarrow : F \vee G_1 . \wedge \ldots \wedge .F \vee G_n.$

T42. $F \vee G. \wedge .H \vee I : \leftrightarrow : F \wedge H. \vee .F \wedge I. \vee .G \wedge H. \vee .G \wedge I.$

3. Quantifiers

T1. $(\forall x) F(x). \leftrightarrow (\forall y) F(y).$

T2. $(\forall x)(F \rightarrow G). \leftrightarrow : (\forall x) F. \rightarrow (\forall x) G.$

T3. $(\forall x)(F \leftrightarrow G). \rightarrow : (\forall x) F. \leftrightarrow (\forall x) G.$

T4. $(\forall x) F \wedge G. \leftrightarrow : (\forall x) F. \wedge (\forall x) G.$

T5. $(\forall x) F. \vee (\forall x) G : \rightarrow (\forall x) F \vee G.$

T6. $x \mathfrak{b} F \twoheadrightarrow \vdash (\forall x) F. \leftrightarrow F.$

T7. $x \mathfrak{b} F \twoheadrightarrow \vdash (\forall x) F \wedge G. \leftrightarrow F \wedge (\forall x) G.$

T8. $x \mathfrak{b} F \twoheadrightarrow \vdash (\forall x) F \vee G. \leftrightarrow F \vee (\forall x) G.$

T9. $(\exists x) F(x). \leftrightarrow (\exists y) F(y).$

T10. $(\exists x) F. \rightarrow (\exists x) G : \rightarrow (\exists x).F \rightarrow G.$

T11. $(\exists x) F \wedge G. \rightarrow : (\exists x) F. \wedge (\exists x) G.$

T12. $(\exists x) F \vee G. \leftrightarrow : (\exists x) F. \vee (\exists x) G.$

T13. $x \mathfrak{b} F \twoheadrightarrow \vdash (\exists x) F. \leftrightarrow F.$

T14. $x \mathfrak{b} F \twoheadrightarrow \vdash (\exists x)(F \rightarrow G). \leftrightarrow .F \rightarrow (\exists x) G.$

T15. $x \mathfrak{b} F \twoheadrightarrow \vdash (\exists x) F \wedge G. \leftrightarrow F \wedge (\exists x) G.$

T16. $x \mathfrak{b} F \twoheadrightarrow \vdash (\exists x) F \vee G. \leftrightarrow F \vee (\exists x) G.$

T17. $\sim (\exists x) \sim F. \leftrightarrow (\forall x) F,$
$(\exists x) \sim F. \leftrightarrow \sim (\forall x) F,$
$\sim (\exists x) F. \leftrightarrow (\forall x) \sim F,$
$(\exists x) F. \leftrightarrow \sim (\forall x) \sim F.$

T18. $(\forall x)(F \rightarrow G). \leftrightarrow \sim (\exists x).F \wedge \sim G.$

T19. $x \mathfrak{b} G \twoheadrightarrow \vdash (\forall x)(F \rightarrow G). \leftrightarrow : (\exists x) F. \rightarrow G.$

T20. $(\exists x)(\forall y) F. \rightarrow (\forall y)(\exists x) F.$

T21. $(\exists!\, x)F(x).\leftrightarrow(\exists\, y)(\forall\, x).F(x)\leftrightarrow x=y.$

T22. $(\exists\, x)x\equiv y\ni F(y).\rightarrow(\exists!\, x)x\equiv y\ni F(y).$

4. Descriptions

T1. $a=(\imath\, x)(x=a).$

T2. $(\forall\, x)(F\leftrightarrow G).\rightarrow(\imath\, x)F=(\imath\, x)G.$

T3. $(\exists!\, x)F(x).\rightarrow:a=(\imath\, x)F(x).\leftrightarrow F(a).$

T4. $(\forall\, x).F(x)\leftrightarrow x=a:\rightarrow:a=(\imath\, x)F(x).\leftrightarrow F(a).$

5. Identity

T1. $=$ is an equivalence relation.

T2. $a=b\rightarrow.F(a)\leftrightarrow F(b).$

T3. $a=b\wedge F(a)\leftrightarrow a=b\wedge F(b).$

T4. $a=b\leftrightarrow(\forall\, x).x=a\leftrightarrow x=b.$

T5. $F(a)\leftrightarrow(\forall\, x).x=a\rightarrow F(x).$

T6. $F(a)\leftrightarrow(\exists\, x).x=a\wedge F(x).$

6. Set or class relations

T1. $A\subseteq A.$

T2. $A\subseteq B\wedge B\subseteq C\rightarrow A\subseteq C.$

T3. $A\subseteq B\leftrightarrow A\subset B\vee A\equiv B.$

T4. $\sim A\subset A,$

$A\subset B\rightarrow\sim B\subset A,$

$A\subset B\wedge B\subset C\rightarrow A\subset C.$

T5. $a\equiv b\leftrightarrow a=b.$

T6. $\equiv$ is an equivalence relation.

T7. $A\cup A\equiv A.$

T8. $A\cup B\equiv B\cup A.$

T9. $A\cup(B\cup C)\equiv(A\cup B)\cup C.$

T10. $A\cap A\equiv A.$

T11. $A\cap B\equiv B\cap A.$

T12. $A\cap(B\cap C)\equiv(A\cap B)\cap C.$

T13. $(A\cup B)^{\sim}\equiv A^{\sim}\cap B^{\sim}.$

T14. $(A\cap B)^{\sim}\equiv A^{\sim}\cup B^{\sim}.$

T15. $A\cup(B\cap C)\equiv(A\cup B)\cap(A\cup C).$

T16. $A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$.
T17. $A \cap B \subseteq A$.
T18. $A \subseteq A \cup B$.
T19. $A \cap B \subseteq A \cup B$.
T20. $A \cap B \equiv A \leftrightarrow A \subseteq B$.
T21. $A \cup B \equiv B \leftrightarrow A \subseteq B$.
T22. $\emptyset \subseteq a$.
T23. $a \subseteq \emptyset \leftrightarrow a = \emptyset$.
T24. $(\forall x) \sim x \in a . \rightarrow a = \emptyset$.
T25. $a \in \{b\} \leftrightarrow a = b$.
T26. $a \in \{b_1, \ldots, b_n\} \leftrightarrow a = b_1 \vee \ldots \vee a = b_n$.
T27. $\{a_1, \ldots, a_n\} = \{a_{i_1}, \ldots, a_{i_n}\}$,

where $(i_1, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$.

T28. $i \in \{1, \ldots, n\} \rightarrow a_i \in \{a_1, \ldots, a_n\}$.
T29. $\langle a_1, \ldots, a_n \rangle = \langle b_1, \ldots, b_n \rangle \rightarrow a_1 = b_1 \wedge \ldots \wedge a_n = b_n$.

7. Relations

T1. $\mathscr{R}$ is reflexive and semi-transitive in $\mathscr{A}$
$\leftrightarrow \mathscr{R}$ is equivalence in $\mathscr{A}$.
T2. $\mathscr{R}$ is asymmetric in $\mathscr{A} \rightarrow \mathscr{R}$ is irreflexive in $\mathscr{A}$.
T3. $\mathscr{R}$ is strict partial-ordering of $\mathscr{A}$
$\leftrightarrow \mathscr{R}$ is irreflexive, asymmetric, and transitive in $\mathscr{A}$.

REFERENCES

BASRI, S., Rev. Mod. Phys. **37** (1965) 288.

BERNAYS, R. and A. FRAENKEL, *Axiomatic Set Theory* (North-Holland Publishing Co., Amsterdam, 1958).

BLUMENTHAL, L., *Theory and Applications of Distance Geometry* (Oxford University Press, Oxford, 1953).

BORSUK, K. and W. SZMIELEW, *Foundations of Geometry* (North-Holland Publishing Co., Amsterdam, 1960).

BROWNLEE, K., *Statistical Theory and Methodology in Science and Engineering* (Wiley and Sons, New York, 1960).

BUSEMANN, H., *The Geometry of Geodesics* (Academic Press, New York, 1955).

CARNAP, R., *Introduction to Symbolic Logic and its Applications* (Dover Publications, New York, 1958).

CARTAN, E., *Leçons sur la Géométrie des Espaces de Riemann* (Gauthier-Villars, Paris, 1928).

COURANT, R. and D. HILBERT, *Methods of Mathematical Physics*, Vol. 1 (Interscience Publishers, New York, 1953).

CRAMÉR, H., *Mathematical Methods of Statistics* (Princeton University Press, Princeton, 1946).

DICKE, R. and others, *Gravitation and Relativity* (Benjamin, New York, 1964).

EISENHART, L., *Riemannian Geometry* (Princeton University Press, Princeton, 1949).

FINZI, A., Nuovo cimento **20** (1961) 1079.

FORDER, H., *The Foundations of Euclidean Geometry* (Dover Publications, New York, 1958).

FORSYTH, A., *Intrinsic Geometry of Ideal Space*, Vols. I and II (MacMillan and Co., London, 1935).

KLEENE, S. C., *Introduction to Metamathematics* (Van Nostrand Co., Princeton, 1952).

KÜNDIG, W., Phys. Rev. **129** (1963) 2371.

MACEK, W. and W. DAVIS, Appl. Phys. Letters **2** (1963) 67.

MOCKLER, R., Radio Science Journal of Research **68 D** (1964) 523.

MUNK, W. and G. MACDONALD, *The Rotation of the Earth* (Cambridge University Press, Cambridge, 1960).

NICOD, J., Proc. Camb. Phil. Soc. **19** (1916) 32.

POUND, P. and G. REBKA, Phys. Rev. Letters **4** (1960) 337.

REICHENBACH, H., *Axiomatik der Relativistischen Raum-Zeit-Lehre* (Friedrich Wieweg and Son, Braunschweig, 1924).

REICHENBACH, H., *The Philosophy of Space and Time* (Dover Publications, New York, 1958).

ROBB, A., *A Theory of Time and Space* (Cambridge University Press, Cambridge, 1914).

ROSENBLOOM, P., *The Elements of Mathematical Logic* (Dover Publications, New York, 1950).

RUSSELL, B., *The Analysis of Matter* (Dover Publications, New York; Republication of 1927 ed.).

SAGNAC, G., Compt. Rend. **157** (1913) 708, 1410.

SCHEFFER, H., Trans. Amer. Math. Soc. **14** (1913) 481.

SCHNELL, K., *Eine Topologie der Zeit in Logistischer Darstellung* (Munster in Westfallen, 1938).

SOMMERVILLE, D., *The Elements of Non-Euclidean Geometry* (Dover Publications, New York, 1958).

SUPPES, P., *Introduction to Logic* (Van Nostrand Co., Princeton, 1957).

SYNGE, J., *Relativity: The Special Theory* (North-Holland Publishing Co., Amsterdam, 1956).

TARSKI, A., *Introduction to Logic and to the Methodology of Deductive Sciences* (Oxford University Press, New York, 1959).

WHITEHEAD, A. and B. RUSSELL, *Principia Mathematica*, Vol. 1, 2nd ed. (Cambridge University Press, Cambridge, 1925).

SYMBOLS

Symbol	Place introduced	Page
$\mathscr{H}$	***C*** II 2.1	12
$\mathscr{I}$	***D*** VI 2.2	52
$\mathscr{J}$	***C*** II 4.3	14
$R\mathscr{J}_H S$	***D*** II 4.1	14
$\mathscr{J}_H(S)$	***D*** II 4.2	17
$\mathscr{K}$	***C*** II 5.4	19
$_G\mathscr{K}_H$	***D*** II 5.1	19
$\mathscr{K}(A)$	***D*** II 5.3	21
$\mathscr{L}(P,Q,R)_H$	***D*** VIII 4.1	84
$\mathscr{L}(P_1,\ldots,P_n)_H$	***D*** VIII 4.2	85
$\mathscr{L}\mathscr{I}(P,Q,\mathrm{I})_H$	***D*** VII 2.5	69
$\mathscr{L}\mathscr{I}(\mathrm{I})_H$	***D*** VII 2.6	70
$\mathscr{L}\mathscr{L}\mathscr{I}(\mathrm{I})_H$	***D*** VII 7.1	77
$\mathscr{M}$	***D*** VII 3.4	71
$\mathscr{M}_\mathrm{I}$	***D*** VII 3.3	71
$\mathscr{M}id$	***D*** VIII 6.1	87
$\mathscr{M}\mathscr{L}$	***D*** VII 7.5	78
$\mathscr{M}\mathscr{L}_\mathrm{I}$	***D*** VII 7.4	78
$\mathscr{N}$	***D*** IX 1.1	90
$\mathscr{N}b_H$	***D*** X 3.2	113
$\mathscr{O}$	***D*** II 5.6	23
$\mathscr{O}(H)$	***D*** II 5.5	22
$\mathscr{O}(H_1,\ldots,H_n)$	***D*** II 5.7	23
P	Line 6	54
$\mathscr{P}(H)$	***D*** III 3.1	26
$\mathscr{P}(H_1,\ldots,H_n)$	***D*** III 3.2	27
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Symbol	Place introduced	Page
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$\{P(\mathfrak{x}) - \mathscr{RCS}(O,A,B,C)_{\mathfrak{l}}\}_H$	***D*** IX 6.2	105
$\{a[P(\mathfrak{x})] - \mathscr{RCS}(O,A,B,C)_{\mathfrak{l}}\}_H$	***D*** IX 6.3	105
$\mathscr{S}$	***C*** V 3.8	40
$\mathscr{S}_H$	***D*** V 3.1	41
S_α, S	Eq. VI 3.2	54
$\mathscr{SC}$	***D*** VI 4.1	57
$\mathscr{SC}(A)_H$	***D*** VI 4.2	57
$a_1, \ldots, a_n \mathscr{SC}_H A$	***D*** VI 4.3	57
$\mathscr{S}eq$	***D*** VI 2.1	52
$\mathscr{SN}_H$	***D*** X 3.5	114
$\mathscr{SN}b_H$	***D*** X 3.4	113
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$\mathscr{T}_H$	***D*** IV 4.3	35
$\mathscr{TN}_H$	***D*** X 3.3	113
$\mathscr{T}\mathfrak{r}$	***D*** X 7.1	121
$\mathscr{U}$	***D*** VII 2.1	67
$\mathscr{V}$	***D*** VI 3.1	53
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$\mathscr{W}$	***D*** V 2.2	39
$\mathscr{X}_H$	***D*** VIII 5.1	85
$\mathscr{Y}$	***D*** VII 2.4	69
$\mathscr{Z}$	***C*** II 3.2	13
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$\mathscr{Z}_H$	***D*** II 3.2	13

Symbol	Place introduced	Page

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$\mathfrak{r}_L$	***D*** VII 2.5	69
$\mathfrak{T}$	I A 2.1	131
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3. Greek

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$\lambda_{\mathrm{I}}(P,Q)_H$	***D*** VII 7.3	77
$\lambda_{\mathrm{I}}(a;P)_H$	***D*** X 4.1	114
μ, μ_α	Eq. VI 3.6	54
$\mu_{\mathrm{I}}[P(a,b),Q;n]_H$	***D*** VII 3.1	70
$\mu_{\mathrm{I}}[P(a,b),Q]_H$	***D*** VII 3.2	71
$\mu_{\mathrm{I}}(P,Q;r)_H$	***D*** VII 4.1	72
$\mu_{\mathrm{I}}(P,Q)_H$	***D*** VII 4.2	72
ν_α	Eq. VI 3.1	54

Symbol	Place introduced	Page
σ	Eq. VI 3.7	54
σ_{T}	***D*** VI 4.1	57
σ_{L}	***D*** VII 2.5	69
$\tau(A;a,b;n)_H$	***D*** VI 5.1	58
$\tau_A(a,b)_H$	***D*** VI 5.2	59
$\langle\tau_A(a,b)_H\rangle$	***D*** VI 6.1	62
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4. Miscellaneous

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$\subseteq_H$	***D*** III 1.1	25
$\subset_H$	***D*** III 1.2	25
$\equiv_H$	***D*** III 1.3	25
$\ni\in_H$	***D*** III 2.1	26
$\supset\subset_H$	***D*** III 2.2	26
$\asymp$	***D*** IV 3.1	32
$\prec$	***D*** IV 4.1	33
$\preccurlyeq$	***D*** IV 4.2	34
$\circledless$	***D*** V 1.1	36
$\cdot\circledless$	***D*** V 1.2	37
$\circledless\cdot$	***D*** V 1.3	37
$\cdot\circledless\cdot$	***D*** V 1.4	37
$\underset{H}{\rightarrowtail}$	***D*** V 2.1	38
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$[,<,]_{p,H}$	***D*** VI 6.4	63
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$\overline{\underline{\asymp}}$	***D*** VII 5.1	73
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$\upharpoonleft\downharpoonright$	***D*** IX 4.2	98

5. Logical

CONVENTIONS

Below are given the letters used to represent different items in the book.

Events: *a*, *b*, *c*, *d*, *e*.

Formulas: *F*, *G*, *H*, *I*, *J*.

Integers (positive): *i*, *j*, *k*, *l*, *m*, *n*, *p*, *q*, *r*, *s*.

Labels of physical items: ***A***, ***C***, ***D***, ***I***, ***P***, ***T***.

Labels of logical items: A, C, D, I, P, T.

Length instruments scale values: $\mathfrak{k}$, $\mathfrak{l}$, $\mathfrak{m}$, $\mathfrak{n}$.

Objects: *A*, *B*, *C*, *D*.

Observers: *F*, *G*, *H*.

Particles: *P*, *Q*, *R*, *S*, *T*, *U*.

Real numbers: p, q, r, s, t, δ, ε, σ.

Sensations: *R*, *S*, *T*.

Space coordinates: $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$.

Terms: *a*, *b*, *c*, *d*.

Variables for objects, observers, particles and sensations: *U*, *V*, *W*, *X*, *Y*, *Z*.

Variables for events and terms: *u*, *v*, *w*, *x*, *y*, *z*.

Variables for integers, real numbers, and scale values are the same as the letters used to represent them.

ABBREVIATIONS

		Page
CS	Coordinate System	102
DT	Deductive Theory	6
GR	General Relativity (Einstein's Theory)	106
LI	Length Instrument	69
LLI	Linear Length Instrument	77
ML	Meta Language	5
OL	Object Language	5
PDF	Probability Density Function	54
SC	Standard Clock	57
SG	Space Geodesic	76
SL	Symbolic Logic	9

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