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Editorial Note

The present selection of A. Mostowski's work comprises within one edition the most important of his papers, written during nearly forty years of scholarly activity and scattered so far over various publications. We hope that it will gain the approval of all readers who concern themselves with the foundations of mathematics.

In selecting the papers, the Editorial Committee aimed at including, on the one hand, the most important of A. Mostowski's contributions to mathematics and, on the other hand, those papers which have preserved their topical interest and are the most frequently quoted ones.

In the five introductory articles (pp. XIX-XLIV, vol. I) which discuss the main trends in A. Mostowski's research, the numbers of bibliographical reference marks refer to the full bibliography of A. Mostowski's works included in volume one.

Most of A. Mostowski's papers, which were originally published in English, have been reproduced photographically without any changes. In those papers, a double pagination occurs: at the outer corners the running pagination of the volume, and at the inner corner, in square brackets, the reference to the complete bibliography and the pagination of the original version. It is intended to facilitate the use of the international reference marks within each paper which refer to the original page number.

Only a few of the papers included in this edition (namely those marked in the Bibliography with the numbers [1], [2], [3], [5], [6], [7], [51], [79]) have been translated from Polish, French or German, so at the inner corners in those papers the references to the complete bibliography are given only.

We are grateful to all publishers of the original papers here included for their consent to the reproduction of those papers, which has enabled us to prepare this edition of A. Mostowski's work and present it to the readers. In particular the following permissions to reprint Mostowski's papers were given: *Thirty years of foundational studies*, Lectures on the development of mathematical logic and the study of the foundations of mathematics in 1930–1964, Acta Philosophica Fennica XVII (1965), pp. 1–180; *Models of set theory*, Lectures delivered in Varenna, September

1968, C.I.M.E., Italy 1968, pp. 67–179; *The classical and ω-complete arithmetic*, Reprinted with permission of the publisher American Mathematical Society from Journal of Symbolic Logic, copyright © 1958, Volume 23, No. 2, pp. 188–206; *An exposition of forcing*, in: *Algebra and logic*, Lecture Notes in Mathematics 450, Springer-Verlag, Berlin–Heidelberg–New York, © by Springer-Verlag, Berlin–Heidelberg 1975, pp. 220–282; *Observations concerning elementary extensions of ω-models I*, reprinted with permission of the publisher American Mathematical Society from Proceedings of Symposia in Pure Mathematics, copyright © 1974, Volume 25, pp. 349–355; *On extendability of models of ZF set theory to the models of Kelley–Morse theory of classes*, in: *Logic Conference, Kiel 1974*, Lecture Notes in Mathematics 499, © by Springer-Verlag, Berlin–Heidelberg 1975, pp. 460–542.

Andrzej Mostowski (1913-1975)

Professor Andrzej Mostowski was born in Lwów on November 1st, 1913. He studied mathematics in Warsaw in the years 1931–1936. On receiving his M.A. degree he went to Vienna, where he studied under Kurt Gödel, and then to Zürich. In 1938 Mostowski defended his doctoral dissertation, written under the supervision of Alfred Tarski and devoted to the interrelationship of various definitions of the notion of infinite set. He then took up a post at the Meteorological Institute in Warsaw. During the Nazi occupation of Poland he worked as an accountant at a tile factory. In the years 1942–1944 he taught at the Underground University of Warsaw, where he was unofficially appointed a "docent". After the war Professor Mostowski was for a short time on the staff of the Jagiellonian University in Cracow, where—in 1945—he presented his "habilitation" thesis, devoted to the axioms of choice for finite sets. In 1946 he returned to Warsaw and was made assistant professor at the University of Warsaw. He became associate professor in 1947 and full professor in 1951. For the next sixteen years he occupied first the Chair of Algebra and then the Chair of the Foundations of Mathematics at the University of Warsaw, and in 1952 he was Dean of the Faculty of Mathematics and Physics. In the years 1948–1968 he worked also at the Institute of Mathematics of the Polish Academy of Sciences, acting as head of the Division of the Foundations of Mathematics.

In 1956 Professor Mostowski was elected an associate member and in 1963 a full member of the Polish Academy of Sciences. He received Polish State Prizes in 1953 and 1966 and the Jurzykowski Foundation Prize in 1972. In 1973 he was elected a member of the Finnish Academy of Sciences. He was on the editorial boards of several learned journals, including the Fundamenta Mathematicae, the Journal of Symbolic Logic, and the Annals of Mathematical Logic; he was also a co-editor of the Series for Mathematics, Physics and Astronomy of the Bulletin of the Polish Academy of Sciences. His long association with the North-Holland Publishing Company helped to raise the "Studies in Logic and the Foundations of Mathematics" series to its present-day importance.

The academic year 1948–1949 Mostowski spent at the Institute for Advanced Study at Princeton, and in 1969–1970 he was a fellow of the

All Souls College at Oxford. He took part in numerous congresses and conferences all over the world.

In 1972 Professor Mostowski was elected President of the Section of Logic, Methodology and Philosophy of Science of the International Union for the History and Philosophy of Science. He died in Vancouver B.C. on August 22nd, 1975.

A Bibliography of works of Andrzej Mostowski

(compiled by W. Marek) (*)

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(*) Unfortunately, dealing with the bibliography of a scholar as prolific as A. Mostowski one cannot be sure that all the papers have been found and introduced. Still we hope that all the major papers have been included.

We notice that the books are additionally marked with an asterisk.

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C. RAUSZER

Research work of A. Mostowski in logical calculi

(Translated into English by M. M. Węglińska)

Problems concerning logical calculi were not in the sphere of A. Mostowski's main interests. Nevertheless, he always followed very closely the developments of mathematical research not necessarily related to the field he was occupied with at a particular moment and very often dealt with those problems in his papers. Therefore, in summing up Mostowski's works, one should not omit those in which he discussed questions arising, as it were, on the peripheries of his basic research, the more so as they often initiated new trends of investigation. Here belong papers [16], [19], [46] [59], [74], [85], [89] (according to the enclosed list of publications).

Algebraic approach plays an important role in the semantical study of non-classical calculi. It was Mostowski who, in paper [16], introduced algebraic modelling for first order calculi.

The main purpose of the above mentioned paper is to outline a general method which permits us to prove the intuitionistic non-deducibility of many formulas. The method consists in utilizing the connections between intuitionistic logic and the so-called Brouwerian lattices, which were introduced and examined by McKinsey and Tarski in 1946. Mostowski proves the following theorem:

If α is intuitionistically deducible, then the corresponding functional is equal to zero for every non-void set I and every complete Brouwerian lattice L.

This theorem initiated the algebraic approach to non-classical logic, which was later so successfully developed by Henkin, Rasiowa and others.

Some of Mostowski's papers are devoted to a generalization of logical quantifiers. In paper [59] he deals with operators which represent a natural generalization of logical quantifiers and formulates problems for those generalized quantifiers which correspond to the classical problems of first order logic.

Most of the discussion in paper [59] centres around the problem whether it is possible to set up a formal calculus which would enable us to prove all true propositions involving new quantifiers.

Dr. Pacholski's paper, enclosed in this book, gives a more detailed explanation of these problems.

Another generalization of logical quantifiers applicable both in two-valued and in many-valued cases has been proposed and discussed by Rosser and Turquette in their book *Many valued logics*. According to their conception a quantifier is a function correlating a truth value with a non-void set of truth values. They examined the problem of axiomatizability of first order calculi with the arbitrary quantifiers under the assumption that the set of truth values v is finite. Then, in 1959, Rosser discussed a similar problem under the assumption that this set coincides with interval [0, 1]. In paper [80] Mostowski takes up the problem of axiomatizability under a more general assumption, namely that the set of truth values is an ordered set which is bicompact in its order topology. The main feature of the results set forth in that paper is their non-effective character. Mostowski proves the existence of complete sets of axioms and rules of inference for some many-valued predicate calculi without exhibiting them explicitly.

The purpose of paper [85] is to discuss the generalization of two fundamental theorems concerning two-valued predicate calculus to the many-valued case. One of them is the compactness problem. It was examined by Chang and Keisler in their paper *Model theories with truth values in uniform space*. Mostowski gives an alternative proof of their result.

Mostowski's solution of this problem consists in conceiving models over a suitably chosen set J as points of a compact space M and then constructing of the function $\mathrm{Val}(\Phi, \mu)$ in such a way as to make it a continuous function of μ.

The construction of M is the key to the solution of the problem. It is easy to construct a space where $\mathrm{Val}(\Phi, \mu)$ is continuous for all formulas without quantifiers. In order to construct a space where $\mathrm{Val}(\Phi, \mu)$ is continuous for arbitrary Φ, we have to pass from the initial language J to another language J_∞, which is obtained from J by adding an infinite number of functors. (Thus functors by means of which the language J_∞ is obtained play the same role as the epsilon operators of Hilbert). Formulas of J containing quantifiers are reducible in a certain way to the symbols of the new language J_∞ which does not contain quantifiers.

The second problem discussed in paper [85] concerns axiomatizability and its reduction. Mostowski considers the following question:

Is the set of satisfiable (valid) formulas the complement of a recursively enumerable set?

First, in order to answer this question, Mostowski proves a theorem which provides necessary and sufficient conditions for a closed formula of J to be satisfiable. In order to establish the axiomatizability of predicate calculi this theorem can be exploited in various ways. The simplest case is where the set of truth values v is a separable space, but the theorem can also be used for the axiomatizability problems in certain cases where v is not separable. In such cases expressions of J_∞ can no longer be enumerated by means of integers and we must take certain abstract entities of those expressions as the "Gödel numbers". This idea is elaborated in Section 6 of paper [79], where the case of the well-ordered set v is discussed. In paper [84] Mostowski uses the above-mentioned theorem to prove conditions which are needed for the set of satisfiable formulas to be a complement of a recursively enumerable set.

Mostowski, who had a great deal of philosophical knowledge, used to say that it would be wrong to deprive logic, however formal it might be, of a certain, possibly subconscious, philosophical base. Conscious choice is more difficult since at the present stage of discussions about the foundations of mathematics one cannot be sure which of the conflicting.theories is the best or even just good.

In paper [46] Mostowski deals with problems which, according to him, are specific for mathematics and do not occur in other branches of science. They are of a philosophical nature and, in the above-mentioned paper, he tries to solve them within the limits of mathematics alone and by applying mathematical methods only. These problems can be formulated as follows:

1. What is the nature of notions considered in mathematics? To what extent are they formed by man and to what extent are they imposed from outside, and whence do we gain knowledge of their properties?

2. What is the nature of mathematical proofs and what are the criteria allowing us to distinguish correct from false proofs?

(Both problems are taken verbatim from paper [46].)

The paper in question does not pretend to be a lecture on mathematics in the strict sense. Mostowski simply tried to show what, from the point of view of mathematics alone, constitutes answers to these questions, who has dealt with them and in what way.

The completeness problem is an interesting example of a question which arose from philosophical investigations concerning the relations between formal calculi and semantics. The problem has found many purely mathematical applications in spite of its philosophical origin. The aim of paper [74] is to prove a generalization of the Gödel incompleteness theorem:

Let us say that a formula Φ with one numerical free variable is free for a system S if the formulas $\Phi(\Delta_1), \Phi(\Delta_2), \ldots, \Phi(\Delta_n)$ are completely independent for every n (i.e. every conjunction formed of some of these formulas and of the negations of the remaining ones is consistent; $\Phi(\Delta_j)$ denotes the formula obtained from Φ by substituting the j-th numeral for the variable of Φ). Mostowski proves that for every given family of extensions of a formal system S satisfying certain very general assumptions there exists a formula which is free for every extension of that family.

Paper [89] is a collection of 16 lectures delivered at the Summer School in Vaasa, Finland, in 1964. In these lectures Mostowski sketched the development of mathematical logic and the progress of research into the foundations of mathematics in the years 1930–1964. It is obvious that one short course did not permit Mostowski to present in detail all the theories that had been formulated during the period in question. He concentrated in these lectures on giving clear explanations of the main problems of mathematical logic, showed their applications and paid special attention to many-valued calculi.

Mostowski's scholary achievements include a book on mathematical logic [19], which was published over 30 years ago. It was the very first university textbook and the very first monograph on the subject in Polish scientific literature. Mostowski did not limit his presentation to the classical problems of mathematical logic but discussed also the results obtained by Gödel and problems which arose in the field of logic after 1930. Although the book has served many generations of Polish mathematicians, it has never been translated into any foreign language. Modesty was characteristic of Mostowski's personality and that is probably the reason why he opposed any suggestions of a translation.

P. ZBIERSKI

The contribution of Mostowski to the foundations of second order arithmetic

Second order arithmetic (called also analysis) is a theory of irrational numbers. Systematic work on the foundations of this theory probably began in mid-fifties, when Mostowski (with collaboration with Grzegorczyk and Ryll-Nardzewski) published paper [62]. Besides a precise description of the system of second order arithmetic and its syntactical rules the paper contains a beautiful topological proof of Orey's theorem. Also some connections with theory of recursive functions are considered and some similarities between particular classes of effective hierarchy are emphasized. The fundamental work on the subject is the *Formal system of analysis based on an infinitistic rule of proof* [72] presented by Mostowski at the Warsaw "Infinitistic Methods" Symposium in 1959. In this paper Mostowski introduced the important notion of models absolute for well orderings of natural numbers (so called β-models). It is shown that the set of sentences true in all β-models is complete (in the sense of Post) for the family of Π_2^1 sets of natural numbers and the existence of ω-models, which are not β-models follows as a corollary. The existence of a minimal β-model as well as finitely axiomatizable extensions of the system of analysis is also stated and proved. Another important result concerns the absoluteness of arithmetical formulas with respect to the constructible universe. The absoluteness of formulas of type Π_2^1 is proved, and this result, strengthened later by the celebrated result of Shoenfield to Σ_2^1-formulas, has many interesting consequences and applications in the foundation of set theory.

The problem of the existence of ω-models which are not β-models is once again discussed in a joint paper by Mostowski and Suzuki [99]. They find that each denumerable β-model of A_2 (with choice) has an elementary extension which is an ω-model but not a β-model. The method of proof is here totally different. It is further generalized and improved in paper [111] and is now called a "definable quantifier". Another kind of questions arise when one considers the structure of the family of all ω-models (or β-models) or models elementarily equivalent to the principal model. In paper [103]. Mostowski imposes a naturally defined partial ordering of "encodable membership" on the family of ω-models and proves the existence of a subfamily ordered in type $\eta \cdot \omega_1$. The paper contains also some applications of the above mentioned theorem to hyperdegrees. Other partial orderings, namely those of inclusion and of elementary inclusion (along with encodable membership), are widely discussed in paper [106]. Problems connected with standard systems are considered in paper [107]. Using a definable ultraproduct construction, the author proves the existence of a nonstandard model such that the corresponding standard system is not a model. Finally, in paper [102], the problem of definability of Skolem functions is discussed. By using the forcing method it is proved that an arbitrary countable ω-model satisfying the principle of dependent choices can be expanded to an ω-model with a definable well ordering of the universe and hence with definable Skolem functions.

The method works for nonstandard models as well, which shows that the addition of Skolem functions for all sets leads to a conservative extension over the scheme of depended choices.

ANDRZEJ GRZEGORCZYK

Andrzej Mostowski's studies of decidability, recursion and hierarchy

(Translated into English by Z. Adamowicz)

Andrzej Mostowski belonged to those investigators of the foundations of mathematics who found inspiration for logical research basically in mathematics. From the very start of his teaching activities at the University of Warsaw almost until he died he gave lectures in various branches of algebra and arithmetic and—at one time—even in analysis.

He was particularly impressed by algebra and used to say—for some time—that logic should be studied in conjunction with other branches of mathematics, for instance with algebra itself.

In his own research in the foundations of mathematics, however, the algebraical trend was not particularly significant. It is in evidence in Mostowski's investigations of products of models or of automorphisms, but the essence of those investigations derives from problems in set theory or metamathematics.

Mostowski belonged to that generation of mathematicians for whom the foundations of mathematics, although strictly bound with other mathematical disciplines, already consisted a separate body of problems, developing on their own lines, independently of other branches of mathematics, and of philosophy either. The separation of logical research from philosophical studies, taking place gradually during the first half of the 20th century, was effected in Poland mainly under the influence of Alfred Tarski, who was Andrzej Mostowski's principal teacher.

Tarski himself still found in philosophy the inspiration for one of his fundamental papers: "On the notion of truth in the languages of formalized theories".

Philosophical problems, however, became at that moment nothing more than a pretext or a secondary motive for formal investigations. Although the main trend in the interests of logicians during the interwar period can be satisfactorily described in philosophical terminology as an investigation of the "cognitive power" pertaining to logical means of proof and to set theory as well, that cognitive power began to be expressed by means of notions specific to logic or metamathematics such as: independence, completeness, undecidability or interpretability by matrices and models.

It is those problems, and particularly the cognitive power of arithmetic and set theory, that soon became the main concern of Andrzej Mostowski and continued to be so to the end.

In order to sum up, somewhat trivially perhaps, the whole of Mostowski's work, it must be said that he was involved all the time with the most profound problems of the foundations of mathematics; he was fully au courrant with all significant investigations and contributed to every branch of metamathematics results of considerable importance for progress in our knowledge.

Actually, from the very start Mostowski took up research in several domains and his investigations in those domains alternated. In these reflections I would like to draw attention to the trend in his research relating to the notion of recursiveness and effectiveness and the logical hierarchy of notions.

Some of Mostowski's work goes back to pre-war days, namely his investigations made in collaboration with A. Tarski, of the types of well-orderings [24], which, however, were only published in a short note in 1949 during Mostowski's first visit to California. They are investigations of the limits of the effectiveness of our cognition from the point of view of decidable problems. Mostowski's results in this field are interwoven with Tarski's and give in effect an image of the decidability of the elementary theory of well-orderings with addition but without multiplication.

The results in question were later supplemented by A. Ehrenfeucht.

As regards investigation of the power of the means of proof from the point of view of undecidable problems, Mostowski was from the start under the influence of Gödel's achievements concerning the undecidability of arithmetic; he had spent one year as a post-graduate student working under Gödel in Vienna. In the difficult years immediately after the war, not wishing to lose touch with those results, he made his own hand-written copy in a separate note-book of the whole of Gödel's dissertation on undecidable sentences in the Peano arithmetic, of which, as far as I remember there had been in Poland only one copy in the Jagiellonian Library in Cracow. That note-book served his pupils for several years as the only means of getting acquainted with the original.

Other techniques of reproducing texts were not available then; many of them were not even known yet (in this respect progress in the last thirty years has been greater than in the preceding 300 years). Mostowski is probably the last mathematician to use that mediaeval method of duplicating a text.

Mostowski's fascination with Gödel's work on the undecidability of arithmetic caused him to write two popular papers in the subject: one in Polish [11] published in 1946 and one in English [35] published in 1952 in the North-Holland Publishing Company series of monographs in Amsterdam and enjoying considerable popularity, although at the time of its publication Mostowski was already in possession of two stronger proofs of his own, both published in 1949, one in Fundamenta Mathematicae [20] and the other [24] in The Journal of Symbolic Logic.

Respect for the work of others compelling one to study their results before constructing one's own version distinguished Mostowski from many other scholars.

As a matter of fact, Mostowski in collaboration with Tarski and Robinson took in 1949 a very important step forward, which, however, they would not have been able to do without Gödel's earlier results. Namely, they reproduced Gödel's reasoning on the basis of a finite system of axioms, which made it possible to prove the undecidability of numerous mathematical theories in the extensions of which a finite system of axioms of arithmetic could be embedded. This method, expounded in another book [41] of the same North-Holland Publishing Company series in 1953, formed for ten years the basis for proofs of undecidability, until it became eclipsed by the slightly different, extremely ingenious method of Rabin–Scott.

Mostowski's keen interest in the existence of undecidable sentences in arithmetic made him seek his own examples of sentences of this kind. Paper [20] gives a proof of undecidability that is set-theoretical in nature and stronger than Gödel's, and although it is not effective, its content is distinctly mathematical and intuitive on the grounds of the arithmetic of real numbers. But even while following the path traced out by Gödel, Mostowski finally observes that there exists a simple method of proving the undecidability of arithmetic by means of proving the universality of the set of theorems on recursively enumerable sets. If we prove in a purely arithmetical manner the existence of a non-recursive recursively enumerable sets, then the undecidability of arithmetic is proved immediately, without the necessity of assuming—as Gödel did—the ω-consistence of arithmetic. Gödel in 1931 could not of course foresee that the situation would be as simple as that. Those far-reaching simplifications were only published in The Journal of Symbolic Logic in 1958 [62] in a paper written jointly by Mostowski, Grzegorczyk and Ryll-Nardzewski and constituting in those days a summing up of all the methodological investigations fundamental for arithmetic. The paper is mainly concerned with a comparison of ordinary arithmetic with a second order arithmetic containing a non-effective generalization rule, called the omega rule. The rule, not suitable of course for practical use, simply describes a certain approximation to the set of true sentences. Now, the set of theorems of second order arithmetic closed with respect to the omega rule shows many similarities to the set of theorems of elementary arithmetic. If we replace the notion of recursive enumerability in the elementary case by the notion of Π_1^1, we shall obtain in the case of the omega rule analogies of the theorems on completeness, on non-decidability, on representability, etc. The paper in question has become a basis for investigating omega models for arithmetic.

The line of research based on Gödel's discovery ran close to another line of research, namely that concerning the logical hierarchy of mathematical notions. Mostowski's investigations in this respect were undoubtedly rooted in the set-theoretical studies of the Warsaw set-theoretical and topological school, whose main representatives were Sierpiński and Kuratowski. In those studies considerable space was devoted to the estimation of the Borel class or projection class of given sets. The assessment was of logical nature, for it had early been noticed that to the operation of projection corresponds a strictly logical operation of the existential quantifier. On the basis of this analogy Mostowski had constructed, during the war, a hierarchy of arithmetical notions commonly known today as the Kleene–Mostowski hierarchy and consisting in increasing the number of quantifiers in a definition. Because of the circumstances due to the war, paper [13] was only published in 1947 in F. M., later than Kleene's paper on the same subject. It is regarded, however, as one of the most important of Mostowski's discoveries. His mathematical turn of mind made him seek a place in his hierarchy for purely mathematical notions. Investigations of this kind are to be found in papers [21] and [48]. Particularly the latter, published in 1955 in F. M., is an interesting and, in a sense, a historical event. Namely, it shows that the notion of the limit of a sequence in whose definition there occur three quantifiers, the universal one, the existential one and the universal one again, cannot be simplified. All three quantifiers are indispensable: the notion is of class Π_3^0. Both in the proofs of undecidability and in the hierarchy Mostowski always searched for theorems and notions whose intuitive content was purely mathematical. He always asked: what are the natural arithmetical notions having more than three numerical quantifiers? or, what natural mathematical notion is further in logical hierarchy than the notion of well-ordering?

Studies of the types of models were an application of notions from logical hierarchy. The strict concept of model having been made precise thanks to the work of Tarski, the question arose how simple the models for given mathematical theories can be. Lindenbaum's method of completing a theory led to an easy conclusion that every theory has a model of the type $\Pi_2^0 \cap \Sigma_2^0$. It was not certain, however, whether there existed simpler models, particularly recursively enumerable ones. The solution of this problem is the subject of papers [41] and [47], in which it is shown that set theory suitably formulated (in the second of the two papers in the form of one sentence, i.e. with a finite system of axioms) is a theory without recursively enumerable models. The proof was later considerably simplified by Rabin.

A little apart from Mostowski's other activities is his research in computable analysis. Computable analysis attempts to reduce all notions of classical analysis to computable notions. In such reduction, however, many definitions that are equivalent on classical grounds turn out to be non-equivalent in the computable sense. Thus, for instance, though for real numbers different method of recursive approximations are equivalent to one another, they cease to be equivalent for sequences of real numbers, as Mostowski points out in [60]. Mostowski's interest in the above field of study belongs to the early period of his activities; it gave way later to other interests. However, the achievements in this field are fundamental; they are, so to speak, classical, although nowadays sometimes obtained in a simpler way.

W. GUZICKI and W. MAREK

The investigations of Andrzej Mostowski in the foundations of set theory

(Translated into English by Z. Adamowicz)

In investigating the foundations of set theory, it was on two out of the many possible problems that Mostowski concentrated his attention. They were the systems of axioms of set theory and the logical relations between various sentences derivable from those systems. Within the range of systems of set theory which are now more or less universally accepted Mostowski's work concerned two essential types of theories: (a) the Zermelo–Fraenkel theories (cumulative theories of types) with or without individuals, (b) second order theories of the Gödel–Bernays type and of the Kelley–Morse type.

It is a characteristic feature of Mostowski's work that he bases mathematical constructions on a suitably chosen and strictly formulated set theory. This form of "logicism" was obvious in his various papers, from the earliest to the very last. Indeed, investigations into the foundations of set theory were conducted by Mostowski more or less continually. Traces of his interest in those problems are discernible in papers written at various stages of his scientific activity.

Mostowski's doctoral dissertation [2] was devoted to various forms of the definition of infinity (of an infinite set). It is characteristic for the period in which the dissertation was written that the independence of various definitions of finiteness was proved by Mostowski with respect to a system of the theory of types. Indeed, at that time—in the thirties—a generally assumed or at least acceptable, system of set theory did not yet exist. The Zermelo–Fraenkel system was fully formed but not yet in general use and the Gödel–Bernays system did not exist. The dissertation contains also consideration regarding individuals.

The set theory with individuals known today as the Fraenkel–Mostowski system was formulated by Mostowski in [6]; for although it was Fraenkel who had given in 1922 the main ideas of the proof that the axiom of choice is independent of the axioms of set theory, the task of developing those ideas into a form acceptable to mathematicians was completed later by Mostowski.

In Mostowski's investigations a particular role is played by the problem of the independence of the axiom of choice of other set-theoretical axioms and the problem of the independence of various forms of the axiom of choice.

Paper [6], published in 1939 (at the same time as Bernays's paper on the predicative theory of classes) contains a system of axioms of set theory with individuals and a definition of a permutational model. The general idea is that the axioms of set theory do not single out any atoms from among other atoms. Consequently a permutation of the set of atoms (Mostowski always assumed atoms to form a set) extends to an automorphism of the whole universe. Mostowski then considered a certain group G of permutations of the set of atoms A and a certain ideal I of subsets of A (in general it was the ideal of finite subsets of A, though in [15] it was the ideal of countable subsets of an uncountable set of atoms). A set x was called symmetric if there existed a set $E \in I$ such that the permutations from G that did not move the elements of E preserved x. The set E was then called the support of the set x. Hereditarily symmetric sets formed an internal model for set theory. In [6] Mostowski proved that the axiom of choice is indeed stronger from the sentence stating that every set can be linearly ordered. The model used for the proof had been obtained from a model with a countable set of atoms ordered into type η with the use of the group of permutations of A preserving the order and the ideal of finite subsets of A. The most important step in the proof was the so-called lemma on the minimal support: every set belonging to the model has a minimal support. An analogous lemma proved for other models was later repeatedly applied by Mostowski's followers to other results of independence.

For a complete solution of the problem of independence of the axiom of choice from the remaining axioms of set theory mathematicians had to wait until 1963, i.e. until Cohen's results.

Immediately after the war Mostowski published paper [10], which was his "habilitation" thesis. He inaugurated with it the study of the dependence of various forms of the axiom of choice for finite sets. By $[n]$ Mostowski denoted the sentence "every family of n-element sets has a function of choice". If Z was a finite set of natural numbers, then by $[Z]$ Mostowski denoted the conjunction of sentences $[n]$ for $n \in Z$. The question considered in the paper was as follows: when is it possible to prove in set theory the implication $[Z] \rightarrow [n]$? Mostowski formulated certain sufficient conditions and certain necessary conditions, but he did not succeed in proving that the sufficient condition which he had formulated was also necessary. The problem was ultimately solved in the late sixties by R. Gauntt and J. Truss, i.e. after Cohen's discovery of the method of forcing. The most important feature of Mostowski's work was the application of group theory in the proof. Namely, Mostowski shows that certain group-theoretical assumptions can serve for the proof of the implication $[Z] \rightarrow [n]$ and that from this implication follow facts concerning the existence of certain groups. The ultimate solution of the problem is as follows: in order that $[Z] \rightarrow [n]$ it is necessary and sufficient that for every subgroup of $G \subset S_n$ without fixed points there should exist a subgroup $H \subset G$ and a sequence $H_1, \ldots, H_k$ of subgroups of H such that $\sum_{i=1}^{k} [H:H_i] \in Z$.

The idea of the proof arises from the fact that the non-existence of certain groups of permutations permits the construction of a choice set while the existence of such groups permits the construction of a permutational model. It is worth stressing that the condition formulated by Mostowski is effective—for its verification it suffices to investigate the properties of the subgroups of a finite group, namely of S_n. Consequently the existential problem "does there exist in set theory a proof of the implication $[Z] \rightarrow [n]$?" is decidable. Mostowski's paper did not solve the problem completely, but aroused considerable interest in permutational models. It is interesting to note that the discovery of forcing, which has stimulate the interest of mathematicians in set theory, has also contributed to a renewal of investigations of permutational models. Generic models applied to the proofs of independence have been found to be quite similar to Mostowski's permutational models. The similarity has proved to be considerable, as has been shown in papers by various authors on the embedding of permutational models into generic models. It should thus be stated that, in addition to solving certain specific problems in set theory, Mostowski created a new method, which has since been applied by numerous followers; what is more, he created a new field of interest for a large number of mathematicians.

Paper [6] was the last of Mostowski's results published before the war. From his reminiscences (*Reminiscences of logicians*, in: Algebra and logic, Springer Lecture Notes 450) we know that during the Nazi occupation Mostowski obtained several results, in particular with regard to the effect of the axiom of constructibility upon the properties of the projective hierarchy (the manuscripts were burnt during the Warsaw Uprising and Mostowski never published them). After the war Mostowski, having published [10] and [15], apparently abandoned the foundations of set theory for recursion theory. However, in the year 1949 he published paper [20], in which under stronger assumptions he gave a nondecidable arithmetical sentence different from Gödel's. The paper contains a proof of what is commonly called the Mostowski contraction lemma. It concerns the fact that every well-founded and extensional relation is isomorphic with the membership relation on a suitable transitive set. The importance of this fact cannot be overestimated: it belongs to the basic results of the foundations of set theory and is extremely useful in present-day research.

In 1951, Mostowski [27] investigated the relations between the Zermelo–Fraenkel set theory and the Gödel–Bernays theory of classes, discovering interesting sentences unprovable on the grounds of the latter theory. In particular, he showed the unprovability—within the Gödel–Bernays theory—of the so-called Σ_1^1-schema of existence of classes and the induction schema for non-predicative formulas, and also discussed the properties of the satisfaction class for the class of all sets. The paper demonstrated the restrictions of the power of the Gödel–Bernays theory and, in this way, prepared the ground for the development of the non-predicative theory of classes.

The crisis which affected the foundations of set theory in the fifties is also evident in the bibliography of Professor Mostowski's papers. Papers which appeared at that time are remote from the problems of the foundations of set theory, only a few short ones were connected with that field.

P. J. Cohen's results of 1963 caused an increased interest in the foundations of set theory and at the same time accelerated the "mathematization" of the whole discipline. Almost simultaneously, Scott's results concerning the incompatibility of the existence of measurable cardinals with the axiom of constructibility increased the interest in combinatorial set theory and in its connections with the foundations of set theory.

Investigations of forcing (i.e. Cohen's method) Mostowski pursued continually to the last days of his life. He devoted four publications to this subject [91], [97], [100], [112]. The fullest presentation of the theory of constructible sets (together with the whole theory of relative constructibility) can be found in the monograph entitled *Constructible sets with applications* [100]. The monograph, which was preceded by the investigation published in [88] and [91], contains an exposition of a unified theory comprising Gödel's theory of constructible sets with its subsequent modifications. Moreover, it gives the fullest existing exposition of the theory of forcing. An interesting feature of [100] is the adoption, as a metatheory, of the non-predicative theory of classes (called the Kelley–Morse theory). This theory has been known since the research of Hao-Wang, but for unknown reasons has not been worked upon further (motives for investigating it can be found in the introduction to [113]). The applications of the non-predicative theory of classes in [100] drew attention to that interesting domain. Also in Mostowski's work we can find results concerning that theory [113], [116]. He inaugurated investigations of the non-predicative theory of classes, which were then continued by his students (e.g. [109]). They were motivated by Mostowski's results of [27], which we have discussed above.

An important contribution to the study of forcing was made by Mostowski in paper [112]. It referred to his other investigations, in particular to those concerning generalizations of the model theory and their applications to the investigations of models of second order arithmetic. This generalization, called the method of generalized quantifiers, has proved to be close to the so-called "omitting types technique" and to forcing. In particular, models obtained by forcing have proved to be a particular case of models obtained by a certain generalized quantifier.

A search for a deeply motivated system of axioms of set theory and a desire to learn the power of such a system were the essential motives of Mostowski's research in the foundations of set theory and the results which he obtained were of enormous importance for the rise and development of that domain.

L. PACHOLSKI

The work of Andrzej Mostowski in model theory

(Translated into English by Z. Adamowicz)

The main field of Andrzej Mostowski's research were the foundations of mathematics. He regarded model theory basically as a tool for other investigations in the foundations of mathematics and did not rate its own cognitive value very highly. Nevertheless, his work had a considerable influence on the development of model theory, and the notions and methods which he introduced have gained a permanent place in textbooks and monographs. To begin with, we should mention the theory of indiscernible elements, created jointly with Andrzej Ehrenfeucht, the theory of products, and generalized quantifiers. The theory of indiscernible elements has provided one of the most attractive and most frequently used methods of constructing models. It has found numerous applications in model theory itself and also in other branches of the foundations of mathematics. Without the notion of indiscernible elements it would have been impossible to obtain many important results. The paper on generalized quantifiers contributed greatly to the formation of abstract model theory—the so-called "soft model theory".

Mostowski published the results of his investigations in model theory in seven papers. They will be discussed here in chronological order with the exception of the paper on the generalization of Craig's interpolation theorem, which is closely connected with the much earlier paper on generalized quantifiers and will be dealt with next to it.

The paper *On direct products of theories* [33], published in 1952, begins with the statement that it concerns decidability questions. Nevertheless the paper should be assigned to model theory, mainly because it was in model theory that the methods worked out in it were later elaborated and applied. Mostowski dealt in it with the operations of direct power and of the finite product of relational structures. He was interested in the question whether the elementary properties of the power $\mathfrak{A}^I$ can be described in the case where the elementary properties of the structure $\mathfrak{A}$ and the cardinality of I are known; similarly, he inquired whether the elementary properties of the direct product $\mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n$ of relational structures can be expressed by means of elementary properties of the factors. The answer to both questions is positive. Mostowski found an effective procedure for reducing the question about the truth of a sentence in a direct product to the question about the truth of certain sentences in the factors of that product.

The existence of such a procedure has several important consequences. First of all, it implies that the direct product and the direct power are elementary operations, which means that elementary properties of the product depend only on elementary properties of the factors. In other words, powers and products of elementarily equivalent structures are elementarily equivalent, which is not necessarily true for other algebraical constructions. Another consequence of the existence of the above-mentioned procedure is the possibility of reducing the question about the decidability of the theory of a product to the questions about the decidability of the theories of the factors. In particular, the theory of a direct product of structures whose theories are decidable is decidable.

Mostowski's method of reducing the question about the truth of a sentence in a product to a similar question for factors was soon extended in a paper by R. L. Vaught [V1] and then in one by S. Feferman and R. L. Vaught [FV]. They introduced the notion of a generalized product, which comprised several operations known before, including the direct product, not necessarily finite. For the operation of the generalized product they defined the notion of "acceptable sequence". This notion was a generalization of Mostowski's reduction procedure. Finally, in 1965, F. Galvin, basing himself on the same reduction principle, defined autonomous systems. With their aid he obtained many interesting new results concerning Horn sentences and invariant sentences with respect to the operations of the direct product and the reduced product. The results of S. Feferman, R. L. Vaught and F. Galvin were further generalized [W], [WW], [CK] and repeatedly applied, e.g. in investigating the saturation of reduced products and limit reduced powers [S1], [P].

At the time of the appearance of Ehrenfeucht's and Mostowski's paper on automorphisms [54] investigations in model theory were concentrated on the characterization of sentences invariant with respect to various operations. The paper in question was very remote from those problems and probably no one guessed how great its future role would be.

The most important notion introduced in the paper is the notion of sets of indiscernible elements. Let $\mathfrak{A}$ be an arbitrary relational structure and let X be an arbitrary subset of the universe of the structure $\mathfrak{A}$ linearly ordered by the relation $<$. The set X is a set of elements indiscernible in $\mathfrak{A}$ if, for any formula $\varphi(v_0, \ldots, v_{n-1})$ of the language of the structure $\mathfrak{A}$ and for any two increasing sequences $(x_0, \ldots, x_{n-1})$, $(y_0, \ldots, y_{n-1})$ of elements of the set X, $\mathfrak{A} \models \varphi[x_0, \ldots, x_{n-1}]$ iff $\mathfrak{A} \models \varphi[y_0, \ldots, y_{n-1}]$. In other words, a linearly ordered set X included in the universe of the structure $\mathfrak{A}$ is the set of elements indiscernible in $\mathfrak{A}$ if all increasing sequences of elements of X having the same length have the same elementary properties. The most frequently quoted result of Mostowski and Ehrenfeucht is the theorem stating that every elementary theory which has at least one infinite model has models with arbitrarily large sets of indiscernible elements. More precisely, if a theory T has infinite models and $\langle X, < \rangle$ is an arbitrary linearly ordered set, then there exists a model $\mathfrak{A}$ of T containing X in which $\langle X, < \rangle$ is a set of indiscernible elements. Another important notion first used in Ehrenfeucht's and Mostowski's paper is the notion of a model generated by indiscernible elements. A relational structure $\mathfrak{A}$ is generated by a set X if every element of the structure $\mathfrak{A}$ is the value of a certain term with parameters from X.

The main problem investigated by Ehrenfeucht and Mostowski was that of the existence of models with a "large" group of automorphisms. The authors succeeded in solving it completely. The full answer is contained in Theorems 4.3 and 5.7 and in the example preceding the formulation of Theorem 5.7. The example shows that the groups of automorphisms of complicated theories cannot be quite arbitrary. Namely, if the theory is sufficiently rich, then the group of automorphisms of any of its models is isomorphic with the group of automorphisms of a certain linearly ordered set. It follows hence that considerations should be restricted to groups of automorphisms of linearly ordered sets. For such groups the answer is positive, which means that for any linearly ordered set $\langle X, < \rangle$ and for any theory having infinite models there is a model whose group of automorphisms contains the group of automorphisms of the set $\langle X, < \rangle$. The structure in which $\langle X, < \rangle$ is a set of indiscernible elements is such a model. It was found later that the restriction on the group of automorphisms is connected with the non-stability of the theory (M. Morley [M], J. T. Baldwin, A. H. Lachlan [BL]).

The Ehrenfeucht–Mostowski theorem had a large number of applications. Obviously it is impossible to mention all of them here even by name. We shall restrict ourselves to a few of the most important and most typical ones.

In Morley's proof of Łoś's hypothesis about categoricity [M] an important step was the demonstration that there is a large model which realizes a small number of elementary types. The construction of such a model consisted in an application of sets of indiscernible elements. Incidentally, this is the only known method of building large models realizing a small number of elementary types.

Among the trends which have recently been developing in a particularly dynamic way we find investigations of stability. Sets of indiscernible elements [S2] are the main tool of such investigations. At the European Summer Meeting of the Association for Symbolic Logic at Clermont-Ferrand in 1975 S. Shelach gave an exhaustive lecture on the modern investigations of stability and devoted a considerable part of his lecture to the generalizations and applications of the notion of sets of indiscernible elements.

It should be added that sets of indiscernible elements have been used in model theory to construct models of large cardinality for infinitary languages [K1]; it had been impossible to do that by the means used for this purpose in classical model theory.

Sets of indiscernible elements have also played an important role outside model theory, especially in problems connected with large cardinals. One of the possible definitions of a Ramsey cardinal can be given in terms of indiscernible elements: m is a Ramsey cardinal if every relational structure of cardinality m and of a countable similarity type contains a set of cardinality m of indiscernible elements. This definition of a Ramsey cardinal is particularly useful in cases where the apparatus of model theory is used; it was applied in the beautiful proof of Silver's theorem [S] stating that the existence of a Ramsey cardinal implies the existence of $O^{\#}$.

Mostowski's paper on generalized quantifiers was precursory. He was the first to introduce an extension of the elementary logic weaker than higher order logics. Subsequently, many such extensions were defined. Their investigation is now one of the main topics in model theory.

It is a serious defect of the first order language that there are quite a number of important mathematical notions and objects which cannot be described with its aid. On the other hand, higher order logics are extremely difficult to investigate. This is because in those logics we use the notions of set and membership and thus get involved in all the difficulties connected with their definition and axiomatization. Hence higher order logics cannot be complete in any sense.

One of the most important suggestions for strengthening the first order logic is contained in Mostowski's paper on generalized quantifiers. Let T

be a function with values in the set $\{0, 1\}$ defined on pairs of cardinal numbers. To every such function Mostowski adjoins the quantifier Q_T. It φ is a formula (for simplicity let it have one free variable) and $\mathfrak{A}$ is a relational structure, then the formula $Q_T x\varphi$ is satisfied in $\mathfrak{A}$ ($\mathfrak{A}\models Q_T x\varphi$) if (1) $T(|\varphi^{\mathfrak{A}}|, |\neg\varphi^{\mathfrak{A}}|) = 1$. The above definition generalizes the classical notion of quantifier. Namely, if $T(\mathfrak{m}, \mathfrak{n}) = 1$ exactly when $\mathfrak{m} \geqslant 1$, then Q_T is the existential quantifier. Similarly the function T for which $T(\mathfrak{m}, \mathfrak{n}) = 1$ if $n = 0$ describes the universal quantifier. Investigations of generalized quantifiers have concentrated on quantifiers denoted by the symbols Q_α and Q_c. If α is an ordinal number, then the quantifier Q_α is described by a function T_α such that $T_\alpha(\mathfrak{m}, \mathfrak{n}) = 1$ iff $\mathfrak{m} \geqslant \aleph_\alpha$. The quantifier Q_c, called the Chang quantifier, is determined by the function T_c which satisfies the condition $T_c(\mathfrak{m}, \mathfrak{n}) = 1 \leftrightarrow \mathfrak{m}+\mathfrak{n} = \mathfrak{m}$. The symbol L_Q denotes the first order language enriched by the quantifier Q.

The above definition of a generalized quantifier and of the quantifiers Q_α comes from the already mentioned paper by Mostowski. He examined in it the basic properties of generalized quantifiers. Undoubtedly the most interesting result is the theorem stating that the logic L_{Q_0} cannot be axiomatized by a recursively enumerable set of axioms. The idea of the proof is based on the fact that in the logic L_{Q_0} it is possible to give a categorical description of the standard model of the arithmetic. The theorem on non-axiomatizability remains true even for the set of sentences true in all countable structures.

It should be observed that Mostowski, in his comment on the theorem on non-axiomatizability, expressed the view that the non-existence of an effective set of rules of inference and axioms for a logic is not a sufficient argument for the rejection of that logic. In those days the tradition of the Hilbert school was still very strong; consequently Mostowski's opinion could give rise to a serious controversy. In accordance with the above methodological standpoint, Mostowski later introduced the operation of β-consequence in second order arithmetic. Non-effective and infinitistic methods are now commonly used in model theory and the traditional concepts of a language and a deductive system have lost their former importance.

The paper on quantifiers contained another theorem worth mentioning, namely the theorem stating that if the first order language is enriched by a generalized quantifier which is not definable by means of classical quantifiers, then the Löwenheim–Skolem–Tarski theorem no longer holds. Ten

(1) $\varphi^{\mathfrak{A}} = \{a \in \mathfrak{A} : \mathfrak{A}\models\varphi[a]\}$.

years later P. Lindström [L2] strengthened this theorem and obtained a very interesting characterization of the first order logic. In spite of the fact that the full version of the Löwenheim–Skolem–Tarski theorem is false for logics with generalized quantifiers, Mostowski proved that every relational structure contains—elementarily in the sense of L_{Q_0}—a countable structure. He also showed that for countable structures there exists only one non-trivial generalized quantifier.

The above paper by Mostowski left several open problems, which were subsequently studied by numerous logicians. Among them were G. Fuhrken [F] and A. Slomson [S1], who studied the compactness of languages with additional quantifiers. G. Fuhrken connected this question with the investigations of two-cardinal models and $\mathfrak{m}$-like models. It is well known now that, in contradistinction to L_{Q_0}, the languages L_{Q_1} and L_{Q_c} can be axiomatized. This was proved by R. L. Vaught for L_{Q_1} [V2] and by H. J. Keisler for L_{Q_c} [K2]. H. J. Keisler gave also a finite system of axioms for L_{Q_1}. For L_{Q_c} partial results of this type were obtained by Y. Yasuhara [Y] and A. Slomson [S1].

To languages with additional quantifiers Mostowski returned in his last paper in model theory [96]. He considered in it the question whether for certain extensions of the first order logic the Craig interpolation lemma and the Beth definability theorem are true. He showed that for a logic with additional quantifiers the interpolation lemma is false and that for L_{Q_0} the definability theorem is false. These results were later extended by M. Yasuhara [Y], L. Lipner [L] and H. Friedman [F]. It should be emphasized that the class of languages studied in the paper in question was very extensive. Mostowski formulated the necessary conditions for the interpolation lemma to be true in a logic from that class. Investigations, conducted by numerous scholars, into the properties of various logics led at the beginning of the present decade to the rise of a new field of research called "soft model theory". Mostowski's paper on the Craig interpolation theorem can be considered as the first paper in this field.

Two of Mostowski's papers are devoted to the study of the topological space of all countable relational structures. In one of those papers, written jointly with Ehrenfeucht [78], there was built a compact space of countable models which is universal, in the sense that every countable model is isomorphic with a model in that space. In the second paper [82] Mostowski studied classes of countable models for which there exists a universal compact space. He showed that if for a class K there exists a compact universal space, then the class of infinite submodels of elements of K is a class of type UC, and if $K \in EC$ or $K \in PC$, then K possesses a compact universal space.

In addition to the papers discussed above, Mostowski wrote a few more papers closely connected with model theory. Here belong the papers in which he constructed a system of axioms [36] and then a single sentence [47] without a recursively enumerable model. In another paper [22], Mostowski, in collaboration with Tarski, gave a description of elementary types of well-orderings.

It is a characteristic feature of Mostowski's work in model theory that his papers are not devoted to solving problems. None of those papers is a direct continuation of investigations initiated by other mathematicians. He himself inspired new investigations and laid out the main lines of their development. He had a perfect sense of what was significant. His publications always left open a great many serious problems, which were subsequently worked upon by numerous mathematicians. The knowledge of some of Mostowski's results is now fundamental for the study of model theory.

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ANDRZEJ MOSTOWSKI

THIRTY YEARS OF FOUNDATIONAL STUDIES

LECTURES ON THE DEVELOPMENT OF MATHEMATICAL LOGIC AND THE STUDY OF THE FOUNDATIONS OF MATHEMATICS IN 1930–1964

Contents

Foreword

In the summer of 1964 I delivered in the Summer School in Vaasa, Finland, a series of lectures on the development of mathematical logic and of the study of foundations of mathematics in the years 1930—1964. The subject was suggested to me by the Rector of the School, Professor Oiva Ketonen.

When preparing these lectures I had to evaluate critically the period to which my whole scientific activity belongs. As this retrospection turned out to be an exciting mental experiment I accepted with gratitude the proposal of Professors Ketonen, Hintikka and von Wright to work out my lectures and to publish them in the series *Acta Philosophica Fennica.*

A review like the present one can be neither entirely impartial nor entirely complete. The choice of the subjects which one wants to take up thus presents considerable difficulties. Further difficulties arise when one tries to give a concise characterisation of various discoveries and of their mutual relations. The task becomes still incomparably more difficult when one has to express oneself in a foreign language. Only after the completion of the work does one see how far it falls short of the image one had in mind when one started it. If in spite of the difficulties I have decided to publish the lectures, I did it in the hope that they may convey to the (rare) reader some of the enthusiasm with which I witnessed the creation of theories reported on in the following pages.

Introduction

Our aim in these lectures is to sketch the development of mathematical logic and of the study of foundations of mathematics in the years 1930—1964. It will not be possible to enter into the details of all the theories that have been created during this period; we shall, rather, content ourselves with brief indications of their contents and of their applications. Thus the presentation will of necessity be somewhat superficial. We believe, however, that it may nevertheless have some interest as it covers a wide field and thus enables one to see the work done in the last few decades in a wide perspective.

It is customary to distinguish three major movements in the philosophy of mathematics: the intuitionism of Brouwer, the logicism of Frege and Russell and the formalism of Hilbert. The first of them views mathematics in isolation from other branches of science and insists on restricting the notions and methods used in mathematics to the most elementary and intuitive ones. For these reasons, few mathematicians have joined the intuitionistic school. The logicism of Frege and Russell tries to reduce mathematics to logic. This seemed to be an excellent program, but when it was put into effect, it turned out that there is simply no logic strong enough to encompass the whole of mathematics. Thus what remained from this program is a reduction of mathematics to set theory. This can hardly be said to be a satisfactory solution of the problem of foundations of mathematics since among all mathematical theories it is just the theory of sets that requires clarification more than any other. Finally, the formalism of Hilbert sets up a program which requires, first that the whole of mathematics be axiomatised and, secondly, that these axiomatic theories be then proved consistent by using very simple combinatorial arguments. As it has turned out, this program is not realizable; and even if it were, it would hardly satisfy philosophically minded mathematicians because of the inevitable arbitrariness of the axioms.

The philosophical aims of the three schools have thus not been achieved, and it seems to us that we are no nearer to a complete understanding of mathematics than the founders of these schools. In spite of this, it cannot be denied that the activity of these schools has brought about a great number of important new insights and discoveries which have deepened our knowledge of mathematics and its relation to logic. As it often happens, these by-products have turned out to be more important than the original aims of the founders of the three schools. It will be our aim to study these results so as to obtain a picture of how the philosophical programs of the three major schools have influenced the formal development of logic and of the foundational study. We will see that the contribution of each of them has been great and that none of them could exist without the others.

The three schools underwent great changes during the years 1930 –1960. Especially striking has been the development of meta-mathematics which originally aimed at a proof of consistency as envisaged by Hilbert but which has later developed into a much more ambitious theory. The most important results of meta-mathematics have their origin in certain studies started in the early thirties. It is true that these results have discredited in part the philosophical program of the formalist school, but it is also true that the meta-mathematical discoveries have revolutionised our knowledge of mathematics and of formal logic.

Intuitionism has not changed its basic philosophy but has begun to change its formal side. Formulae, which were previously banned altogether, have replaced in part the complicated and often incomprehensible verbal expressions used in older publications. In this way intuitionistic theories, which were previously known almost exclusively in the narrow circle of the followers of Brouwer, became intelligible to other philosophers and mathematicians, to a great benefit of both sides. There also appeared other theories not directly connected with intuitionism but sharing with it the tendency to restrict mathematical notions to very simple ones. Collectively, these theories are known as the constructivistic trend in the modern philosophy of mathematics. Intuitionism in the proper sense of the word is probably the most interesting of these constructivistic theories.

Logicism, which dominated foundational studies in the years prior to 1930, did not create essentially new conceptions after 1930.

There were scholars, *e.g.* Leśniewski, who worked essentially along the lines of the old program of the logicists, but their influence has been small. The program of logicism nevertheless survived in the guise of set-theoretical conceptions. This is only natural if we reflect that type theory and other similar systems to which the logicists tried to reduce mathematics were essentially axiomatic systems of the abstract theory of sets.

Thus we shall have to account for the contributions of the following three main schools of thought: the constructivistic, the metamathematical and the set-theoretical. In the early thirties appeared three publications which can be taken to be representative of these three schools and which greatly influenced their further development. These were the well-known works by Heyting [80], Gödel [54] and Tarski [222]. We shall begin our exposition by discussing these papers and certain other works which immediately depend on them.

Lecture I

Formalization of the intuitionistic logic

Intuitionism as invented by Brouwer rests on several general principles, only some of which are relevant to intuitionistic logic. Very important but not relevant to our immediate purpose is the assumption that general set-theoretical notions are not to be admitted into mathematics and that all mathematics is to be reduced to the arithmetic of integers and to a very special intuitionistic theory of the continuum. Another no less important assumption which is very relevant to intuitionistic logic is the intuitionistic contention that logic does not precede mathematics but is a result of the mathematical activity. A law of logic is, according to this thesis, a form of deduction which has been accepted by mathematicians; before mathematicians have used deductions of this form there was no reason to accept it as a law of logic. Finally, a third thesis prescribes certain forms of reasoning which according to intuitionists are the only ones to be admitted into mathematics. Mathematicians, so says the thesis, can only perform certain (mental) constructions; a mathematical theorem is but a report on these constructions.

In order to make this point clearer let us consider the following situation. Suppose that a mathematician tries to prove the existence of an object with some prescribed properties. In order to do this he performs certain constructions, and if he succeeds in obtaining in this way an object with the requisite properties, then he has proved an intuitionistically admissible existential statement. Let us now suppose that our mathematician has failed to construct directly an object as required, but that he can derive a contradiction from the assumption that there are no objects with the requisite properties. A classical logician would still say that an existential statement has been proved by our mathematician, but an intuitionist would deny this. According to the intuitionistic conception, the proof of impossibility which was carried out by our mathematician is a construction showing the impossibility of proving the general statement: "Every x is deprived of the properties in question"; it is not a construction of an object satisfying these properties. Hence the negation of a general statement is not equivalent to an existential statement. We see from this example that the identification of a mathematical theorem with a construction leads us to reject certain laws of classical logic. As it happens, the formulae which are accepted by intuitionists are true under the classical interpretation; thus intuitionistic logic is a proper part of classical logic.

Heyting [80] was the first to undertake a formalization of intuitionistic logic. He divided his system into two parts, one dealing with propositional logic and the other with the logic of quantification. We shall present here the propositional logic of Heyting's in a slightly modified form.

The primitive notions of this system are: alternation (denoted by $\vee$), conjunction (denoted by $\wedge$), implication (denoted by $\rightarrow$) and the constant F denoting a false sentence. The axioms for alternation and conjunction are the same as in classical logic:

$$\text{(A)} \quad p \rightarrow (p \vee q), \quad q \rightarrow (p \vee q), \quad (p \rightarrow r) \rightarrow \{(q \rightarrow r) \rightarrow [(p \vee q) \rightarrow r]\},$$

$$\text{(K)} \quad (p \wedge q) \rightarrow p, \quad (p \wedge q) \rightarrow q, \quad (r \rightarrow p) \rightarrow \{(r \rightarrow q) \rightarrow [r \rightarrow (p \wedge q)]\}.$$

The axioms for the connective $\rightarrow$ and for the constant F constitute only a part of the corresponding classical axioms:

$$\text{(I)} \quad [p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)], \quad p \rightarrow (q \rightarrow p),$$

$$\text{(F)} \qquad F \rightarrow p.$$

The only rule of inference is the classical *modus ponens*. The equivalence $p \equiv q$ is defined as $(p \rightarrow q) \wedge (q \rightarrow p)$ and the negation $\neg p$ as $p \rightarrow F$.

The system characterized by the axioms (I) is known as positive implicational logic; the axioms (A), (K), (I) characterize the full positive logic. Not all classically true formulae which involve only the connective $\rightarrow$ are derivable from (I); they are derivable from (I) together with the axiom known as Peirce's law:

$$\text{(P)} \qquad [(p \rightarrow q) \rightarrow p] \rightarrow p.$$

The axioms (A), (K), (I), (F), (P) are sufficient for the derivation of all classically true propositional formulae. Indeed, by substituting F for q in (P) we obtain the formula

$$(\mathrm{i}) \qquad (\neg p \to p) \to p.$$

From (I) we can derive easily the formula $p \to [(p \to q) \to q]$ which upon substitution $q = F$ gives us $p \to (\neg p \to F)$, whence according to (F)

$$(\mathrm{ii}) \qquad p \to (\neg p \to q).$$

Finally, the law of syllogism

$$(\mathrm{iii}) \qquad (p \to q) \to [(q \to r) \to (p \to r)]$$

is easily derivable from (I). Thus all the axioms of the well known system of Łukasiewicz are derivable from (I), (F), and (P). Since the formulae $(p \vee q) \equiv (\neg p \to q)$, $(p \wedge q) \equiv \neg (p \to \neg q)$ are easily derivable from (A), (K), and the classical laws for → and ¬, we see that (A), (K), (I), (F), (P) are indeed sufficient for the derivation of all the laws of the classical propositional logic.

Peirce's law (P) is far from being intuitively obvious. The intuitionistic axioms (A), (K), (I), (F) are, on the contrary, very clear and intuitive. We conclude that the intuitionistic logic is simpler and more natural than the classical one.

In order to obtain the logic of quantifiers from the propositional logic we proceed exactly as in the classical case: we add the axioms

$$(\mathrm{Q}) \qquad \bigwedge_x Fx \to Fy, \qquad Fy \to \bigvee_x Fx$$

and the following rules of proof:

$$\frac{A \to Fx}{A \to \bigwedge_x Fx} \qquad \frac{Fx \to A}{\bigvee_x Fx \to A}$$

where A is a formula which does not contain x as a free variable.

The above method of formalizing intuitionistic logic does not differ essentially from the one proposed by Heyting. There are other ways, perhaps more elegant ones, of formalizing this logic, *e.g.* the very simple method of Gentzen [50]; we shall not, however, enter into the details of these other methods.

Several important meta-mathematical theorems about the formalized intuitionistic logic were discovered soon after Heyting published his paper. The most interesting one was the discovery made by Gödel [56], based in part on some earlier results of Glivenko [52], that the classical logic can be interpreted in the intuitionistic one. Gödel defined new connectives $\sim$, $+$, $\cdot$, $\supset$ by means of the equations $\sim p = \neg p$, $p \cdot q = p \wedge q$, $p + q = \sim(\sim p \cdot \sim q)$, $p \supset q = \sim(p \cdot \sim q)$, and showed that if we replace in any formula the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ by $\sim$, $.$, $+$, $\supset$, then all the classically true formulae will go over into intuitionistically provable formulae, whereas formulae which are not classically true go over into formulae which are not intuitionistically provable. In order to obtain an analogous result for the logic of quantifiers we may interpret the existential quantifier as $\neg \bigwedge_x \neg$ and leave the general quantifier unchanged. In this way Gödel proved that the classical logic is faithfully representable in the intuitionistic logic. Of course the intuitionistic logic is identically interpretable in the classical logic, but this identical interpretation is not a faithful one.

Gödel showed that the same relationship exists between certain axiomatic theories based on the classical and the intuitionistic logic. Thus *e.g.* Peano's arithmetic based on the classical logic is interpretable in Peano's arithmetic based on the intuitionistic logic. This theorem gives us an intuitionistic consistency proof for Peano's arithmetic based on the classical logic. It is remarkable that this proof should turn out to be so easy while no strictly finitistic consistency proof exists. Peano's arithmetic based on the intuitionistic logic contains thus many non-finitistic elements.

Much effort was devoted by logicians and mathematicians to attempts to obtain a classical interpretation of the intuitionistic logic. Results in this direction evidently do not interest intuitionists who do not have to interpret their own logic in the classical system, which is unintelligible to them. For people who adhere to classical logic an interpretation is the only method which allows them to understand intuitionistic logic.

A very interesting interpretation was given by Kolmogorov, who interpreted the intuitionistic logic as the logic of problems. The connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$ are interpreted as operations of forming a new problem out of given ones. *E.g.* the implication $p \rightarrow q$ is the problem which consists of reducing the problem q to the problem p. Intuitionistic identities built up by means of the connectives and the variables $p, q, r, \ldots$ will then represent schemata of problems which admit solutions independently of the particular choice of $p, q, r, \ldots$ This interpretation is well in keeping with the intuitionistic conception that a mathematical theorem is always identical with a (theoretical) construction.

More influential than this interpretation was another one devised by Tarski [225]. A very similar conception was also published about the same time by Stone [218].

A close connection between the classical propositional calculus and the calculus of classes had been known since the time of Boole. The parallelism between these calculi is best seen when one uses the same symbols for propositional connectives and for operations on sets; we have then *e.g.* the equivalences $[x \in (X \vee Y)] \equiv [(x \in X) \vee (x \in Y)]$, $(x \in \neg X) \equiv \neg (x \in X)$, *etc.* To each propositional formula we can then associate a Boolean polynomial simply by interpreting the propositional variables as set-variables ranging over all the subsets of an arbitrary set V. The classical logical identities are associated with those polynomials whose value is V independently of the values given to the variables. This interpretation is no longer valid for the intuitionistic logic since *e.g.* the intuitionistically unprovable formula $p \vee \neg p$ is associated with a polynomial whose value is V for every p. The problem formulated by Tarski was to find a class $\boldsymbol{G}$ of subsets of V and suitable operations performable on these subsets such that the following be true: If we correlate with a propositional formula a polynomial obtained by interpreting the connectives as these operations on sets and the propositional variables as arbitrary sets in $\boldsymbol{G}$ we obtain a polynomial identically equal to V if and only if the formula we started with is intuitionistically provable. Tarski showed that we can take as V a suitable topological space, *e.g.* a Euclidean plane, as $\boldsymbol{G}$ the class of open subsets of V, and interpret F as the void set and the connectives $\vee$, $\wedge$, $\rightarrow$ as union, intersection and as the operation $Int\ [(V - X) \cup Y]$, respectively. Open sets are of course those subsets of V which together with any point p contain a sufficiently small neighbourhood of p. $Int\ (A)$ is the interior of A, *i.e.* the largest open set contained in A.

Let us discuss an example of a topological space. Let T be a finite directed tree, *i.e.* a set of points called vertices some of which are connected by directed edges in such a way that there are no closed cycles. (For examples of such trees, see diagrams 1 and 2.)

Diagram 1.

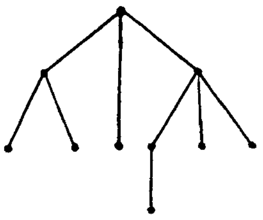
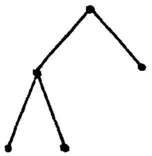


Diagram 2.



Each tree T of the sort just described has one or more primitive vertices, *i.e.* vertices in which no edge ends.[1]

A neighbourhood of a point p is the set of all points q which can be connected with p by a path $pp'p'' \dots p^{(n)} = q$ where the edges pp', $p'p''$, ..., $p^{(n-1)} p^{(n)}$ belong to the tree. Thus an open set has the property that with every point p it contains all points q which can be connected with p by a path.

The following remark, due in principle to Weyl [244], shows the intuitive origin of the topological interpretation. Let us consider sentences of the form $x \in X$ where X is a subset of V. Each such sentence can be either true or false. Let us call it strongly true (or strongly false) if it remains true (false) for all points x' lying sufficiently close to x. Thus the sentence $x \in X$ is strongly true if x lies in the interior of X and strongly false if it lies in the interior of $V-X$. For an x lying on the boundary of X (*i.e.* in the set $\bar{X} \cap \overline{V - X}$ where the bar denotes closure) the sentence is neither strongly true nor strongly false. All this is in accordance with the intuitionistic conception of a set according to which a finite amount of information concerning x should be sufficient to decide whether x belongs or does not belong to the set. Of course this remark, while indicating the connection between topology and the intuitionistic logic, does not by itself suffice to explain the success of Tarski's construction.

Formally speaking Tarski's result says that open subsets of a topological space V form a matrix in which all intuitionistically provable formulae are valid; for a suitable V (*e.g.* if V is a Euclidean

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[1] Formally speaking, a tree is the graph of a one-many relation R such that the relation xR^nx holds for no x and no integer n.

plane) this matrix is adequate for the intuitionistic logic. Of course the general notion of a matrix was known long before Tarski's work on the intuitionistic logic; but never before his work had a matrix with so many elements been actually used nor had any one considered matrices having the structure of a topological space.

Several later studies of the intuitionistic logic and of other many-valued systems drew their inspiration from Tarski's paper.

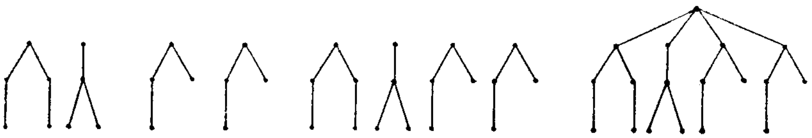
We add some further remarks on the proof of Tarski's result. In one direction it is very easy: it is a routine matter to check that the axioms (A), (K), (I), (F) are valid in matrices consisting of open subsets of an arbitrary topological space V and that the rule of *modus ponens* preserves validity in this matrix. Thus all intuitionistic theorems are valid in these matrices. It is much more difficult to show that exactly those formulae are valid in the matrix that are intuitionistically provable (provided that V satisfies certain conditions). In order to obtain this result Tarski used the following result of Jaśkowski [85]: There exists a denumerable sequence of finite matrices M_n such that a propositional formula A is intuitionistically provable if and only if it is valid in at least one of these matrices.

It is not difficult to describe Jaśkowski's matrices. Each of them consists of open subsets of a finite space determined by a tree. The first matrix corresponds to a tree with but one vertex, and the subsequent ones are obtained from the preceeding ones by two operations which we call O' and O''. The operation O' is simply the operation of joining trees together (without adding new edges to any of them); O'' consists of joining to each other two trees T_1, T_2 so as to add to them one vertex which is joined by edges to the primitive vertices of T_1 and of T_2. This is illustrated by diagram 3.

This clear description of Jaśkowski's matrices is due to Grzegorczyk [70]; proofs of Jaśkowski's theorem are contained in Rose [186] and Scott [197].

Subsequent development brought essential simplifications to

Diagram 3.



T_1 T_2 $O'(T_1, T_2)$ $O''(T_1, T_2)$

Tarski's original proof; see McKinsey and Tarski [142]. Instead of Jaśkowski's matrices Tarski used certain other matrices whose elements are formulae. The idea of using such matrices goes back to Lindenbaum.

The intuitionistic logic is but one of many non-classical logics. Several other such "logics" were defined first by Łukasiewicz and Post and later by other logicians. Some of them were invented for purely formal reason, but several others, *e.g.* modal logic, possess intrinsic philosophical value. It seems to us that the intuitionistic logic occupies a privileged position among these systems: it is the only logic, so far constructed which is actually being used by a relatively large group of actively working scientists. It is also the only one which has been extended beyond propositional logic and the logic of quantifiers and used in the development of certain parts of mathematics. Łukasiewicz, who was the first to conceive an idea of a logic different from the usual one, hoped that one day several logics will emerge which will actually be used, as are for instance the non-Euclidean geometries. Most of the non-classical logics invented so far are not being actually used although several of them are being studied in the meta-mathematical fashion on the basis of the two-valued logic. It looks as if the intuitionistic logic were the only one in the case of which Łukasiewicz's plan has still some chance of realization. At the same time this logic is based on an original and internally coherent view of mathematics. These two circumstances explain the vivid interest which the intuitionistic logic has raised from the moment it was created.

Lecture II

The incompleteness of arithmetic

In this lecture we shall be concerned with another important contribution of the early thirties to the study of the foundations of mathematics, especially with the so called first incompleteness theorem of Gödel [54] which states that the usual axiomatic systems of arithmetic of integers are incomplete. In order to explain the importance of this result we insert some brief historical comments.

Since the publication of the works of Frege and of Russell and Whitehead logicians believed that each intuitively correct deduction can be reconstructed in the classical logical calculus. It was also believed that by adding suitable axioms to the logical calculus we shall be able to construct axiomatic systems in which every intuitively correct mathematical statement will be provable. If there were doubts as to how the axioms (*e.g.* the axioms for set theory) are to be chosen, nobody (except the intuitionists) felt the slightest doubt that the axioms of Peano fully describe the notion of an integer. Hence it was generally believed that all intuitively correct statements of arithmetic are formally derivable from these axioms. This belief was the basis of the philosophical views of the Hilbert school. The representatives of this school were convinced that the notion of truth (the truth of an arithmetical statement) has been defined, since true statements coincide with statements formally derivable from the axioms of Peano. They shifted therefore the emphasis from the problem of defining truth (which was always considered the central problem of philosophy) to the more formal problem of establishing the consistency of Peano's axioms by using very simple combinatorial arguments. We remark parenthetically that such a proof, if it existed, would be interesting independently of the question whether intuitive arithmetic is reducible to Peano's axioms.

Gödel's discovery showed, first, that the identification of true formulae with formulae formally derivable from Peano's axioms is untenable. Secondly, he showed that a consistency proof as required by Hilbert does not exist unless arithmetic is inconsistent. Thus the whole program of the formalist school was dealt a blow from which this school has never really recovered.

Gödel's work became quickly famous. Since it is generally known we can limit our exposition to brief indications.

The main tool invented by Gödel and used by him in the proofs of his theorems was the so-called arithmetization of meta-mathematics. It consists simply of an enumeration of formulae and sequences of formulae with the help of integers. (We shall call the integer correlated with a formula, or a sequence of formulae, under this enumeration the number of this formula or sequence.) The existence of such an enumeration is secured by the fact that the set of all formulae has the same power as the set of integers. Because of this fact we can correlate with each set of formulae and each relation between formulae a set or a relation between integers. In a system in which arithmetical theorems can be proved we can also prove theorems equivalent to certain theorems about formulae. In this way it is theoretically possible to construct within arithmetic statements referring to themselves. Self-referential statements used carelessly can lead to inconsistencies, known as semantic antinomies. Well known examples of these antinomies are the paradox of the liar and Richard's paradox. Gödel's first theorem amounts to showing that these paradoxes would indeed be present in the axiomatic arithmetic if this system were complete. As shown by Wang [242], each of the semantic antinomies known so far can be transformed into an incompleteness proof.

In order to describe Gödel's proof we need some meta-mathematical definitions. Let us assume that $\boldsymbol{T}$ is a consistent system of arithmetic among whose expressions there are the symbols $\bar{0}, \bar{1}, \bar{2}, \ldots,$ called numerals. Let us assume that to each integer n there is associated a numeral $\bar{n}$ and that the formula $\bar{m} \neq \bar{n}$ is provable in $\boldsymbol{T}$ whenever m and n are different integers.

We shall say that a formula F with one free variable is a weak description of a set X of integers if for any integer n the formula $F(\bar{n})$ is provable in $\boldsymbol{T}$ just in case n is an element of X. Sets which possess at least one weak description in $\boldsymbol{T}$ are called weakly representable in $\boldsymbol{T}$. There are only denumerably many such sets because the number of different formulae is denumerable. If F is a weak description of X and $\neg F$ is a weak description of the complement $-X$

of X, then we call F a strong description of X in $\boldsymbol{T}$; sets X which have at least one strong description in $\boldsymbol{T}$ are called strongly representable in $\boldsymbol{T}$.

Weak and strong representability are two of the many possible ways of making precise the vague notion of expressibility of an intuitively given property within a formal system. Consider *e.g.* the set $X = \{0, 2, 4, \ldots\}$ of even numbers; its elements are the integers possessing the intuitively clear property of being divisible by 2. We express this property in the axiomatic arithmetic $\boldsymbol{T}$ by the formula $F(y) = \bigvee_x [y = x + x]$. Obviously $F(\bar{n})$ is provable in $\boldsymbol{T}$ if $n \in X$ and $\neg F(\bar{n})$ is provable in X if it is not the case that $n \in X$; assuming that $\boldsymbol{T}$ is consistent we can say that X is strongly representable in $\boldsymbol{T}$. It is much more difficult to give an example of a set which is weakly but not strongly representable in a system $\boldsymbol{T}$. In fact, the existence of an example of this kind implies the incompleteness of $\boldsymbol{T}$: for if $\boldsymbol{T}$ is complete and F is a weak description of X in $\boldsymbol{T}$, then $n \bar{\in} X$ holds just in case when $F(\bar{n})$ is not provable in $\boldsymbol{T}$, *i.e.* when $\neg F(\bar{n})$ is provable in $\boldsymbol{T}$ and hence $\neg F$ is a weak description of $-X$.

In an analogous way we define the weak and strong representability of a set consisting of pairs or triples or quadruples of integers. Thus we can speak of the representability of binary, ternary, quaternary *etc.* relations between integers. We can also define the representability of functions. *E.g.* if f is a function of one argument ranging over integers whose values are also integers, then we say that a formula $F(x, y)$ with two free variables represents f if (i) F weakly represents the relation $f(n) = m$; (ii) the formula $\bigwedge_x \bigvee!_y F(x,y)$ is provable in $\boldsymbol{T}$. There is no need to distinguish in this case between weak and strong representability.

One can show that if certain formulae which we shall not specify here are provable in $\boldsymbol{T}$, then there is a set U strongly representable in $\boldsymbol{T}$ which consists of triples of integers and which is "universal" in the following sense: whenever a set X of integers is weakly representable in $\boldsymbol{T}$, there is an integer e such that

$$n \in X \equiv \bigvee_p [\langle e, p, n \rangle \in U]. \tag{1}$$

U is simply the set of triples $\langle x, y, z \rangle$ such that x is a number of a formula F with one free variable, and y the number of a finite sequence of formulae whose last term is $F(\bar{z})$ and which represents the formal proof of its last term in $\boldsymbol{T}$. It is of course not obvious that such a set U is strongly representable; at least half of Gödel's paper was devoted to a proof that this is really so.

Another important lemma is the following theorem on projections: under suitable assumptions concerning $\boldsymbol{T}$ it is the case that if X is a set of pairs which is strongly representable in $\boldsymbol{T}$, then the set $\{x : \bigvee_{y} (<x, y> \epsilon X)\}$ is weakly representable.

The theorems on the existence of a universal set and on projections allow us to prove very quickly that $\boldsymbol{T}$ is incomplete. The set $X = \{n : \bigwedge_{q} (< n, q, n > \bar{\epsilon} U)\}$ is easily shown not to be weakly representable; otherwise there would be an integer e for which (1) is true and hence we would obtain

$$\bigwedge_{q} [<n, q, n> \bar{\epsilon}\ U] \equiv \bigvee_{p} [<e, p, n> \epsilon\ U]$$

whence for $n = e$

$$\bigwedge_{q} [<e, q, e> \bar{\epsilon}\ U] \equiv \bigvee_{p} [<e, p, e> \epsilon\ U]$$

which is a contradiction.

On the other hand the complement of X is weakly representable according to the theorem on projections. Hence there exist sets which are weakly but not strongly representable, which implies (as we remarked above) the incompleteness of $\boldsymbol{T}$.

We will now discuss the assumptions which have to be made in order to prove the theorem on projections.

Let X be a set of pairs of integers weakly represented by a formula F. It is natural to expect that the set $P = \{x : \bigvee_{y} [<x, y> \epsilon X]\}$ will be represented by the formula $\bigvee_{w} F(u, w)$. This is indeed the case under certain conditions. Let us first assume that $<x, y> \epsilon X$, *i.e.* that $F(\bar{x}, \bar{y})$ is provable. Using a well-known law of logic we obtain that the formula $\bigvee_{w} F(\bar{x}, w)$ is provable. The only assumption needed in this step is that to every integer y there exists a corresponding numeral. If, as it is often admitted, numerals consist of symbols "1" placed one after another, then our assumption states that the length of each numeral is an integer and that for each integer there exists a numeral of this length. Now let us assume that x is not in the set P, *i.e.* that for every y the pair $<x, y>$ is not in X. Hence — in view of the strong representability of X — we obtain that for every y the formula $\neg F(\bar{x}, \bar{y})$ is provable in $\boldsymbol{T}$. In order to infer that the formula $\bigvee_{w} F(\bar{x}, w)$ is not provable in $\boldsymbol{T}$ Gödel assumed that whenever all formulae of the infinite sequence $\neg A(\bar{0})$, $\neg A(\bar{1}), \ldots$ are provable in $\boldsymbol{T}$ then the existential statement $\bigvee_{y} A(y)$ is not provable in $\boldsymbol{T}$. This assumption is called the ω-consistency of $\boldsymbol{T}$.

The theorem on the universal set requires that certain formulae which we shall not enumerate here be provable in $\boldsymbol{T}$.

Altogether we have 3 assumptions: (A) the one-one correspondence between numerals and integers, (B) the ω-consistency of $\boldsymbol{T}$, (C) the provability in $\boldsymbol{T}$ of certain formulae.

Assumptions (B) and (C) were discussed almost from the beginning of the whole theory. Rosser showed in an important paper [187] that one can replace the assumption of ω-consistency by the much simpler assumption of consistency. The proof of the theorem on projections must then be modified in that a more sophisticated formula must be used to show the weak representability of the set P. It follows from his proof that each consistent extension of Peano's arithmetic is incomplete provided that the set of the numbers of its axioms is weakly representable. Consistent theories with this property are called after Tarski [231] essentially incomplete. We shall see later the importance of these theories for the problem known as the decision problem. Especially important is the fact that there exist essentially incomplete theories based on a finite number of axioms. The simplest such theory (Vaught, unpublished) has one primitive notion R which denotes a binary relation and is based on the axioms

$$\bigvee_{x} \bigwedge_{y} \neg (yRx), \qquad \bigwedge_{xy} \bigvee_{z} \bigwedge_{t} \{tRz \equiv [(t = x) \vee tRy]\}.$$

The assumption (C) can be modified in various ways but of course cannot be dropped altogether. The assumption (A) was noted not long ago by Rieger. Let us discuss it a little more closely. The majority of mathematicians believe that the notion of an integer is uniquely determined and that the integers form a well determined set. If one adheres to this view, then it is obvious that the lengths of formulae are integers and that the assumption (A) is satisfied. It is possible, however, to take a different standpoint and to insist that each set of objects satisfying Peano's axioms can be taken as the set of integers. It is known that there exist many mutually non-isomorphic models of Peano's axioms; hence one can assume that one such set is used in intuitive mathematics and another is used to count symbols in formulae. Then the assumption (A) need not be satisfied. In Rieger's view this remark invalidates the philosophical claims of Gödel's discovery; it is more appropriate to say that it merely discloses one of the assumptions on which Gödel's theorem rests.

Our discussion has so far been limited to one of the many results contained in Gödel's paper [54]. We shall now review briefly its second main result. It is appropriate to remark that Gödel's paper was exceptionally rich in new ideas and that only now, after more than 30 years, the wealth of problems stemming directly from it begins to show signs of exhaustion.

Gödel's second main result is his second undecidability theorem. It was merely sketched in the published paper and was due to appear with a detailed proof in the second part of his paper. This second part was never written, however.

Let Z be the set of pairs $<x, y>$ such that y is the number of a formula of formalized arithmetic $\boldsymbol{T}$ and x the number of its formal proof in $\boldsymbol{T}$. It was shown by Gödel that Z is strongly representable in $\boldsymbol{T}$; his proof indicated how to construct a formula F which strongly represents Z in $\boldsymbol{T}$. Let k be the number of the formula $0 \neq 0$ or of any other formula which is refutable in $\boldsymbol{T}$. The formula $\bigwedge_x [\neg F(x, \bar{k})]$ was denoted by Gödel by *Wid*. Its intuitive content is: there is no formal proof in $\boldsymbol{T}$ of the formula $0 \neq 0$.

The second undecidability theorem asserts that the formula *Wid* is not provable in $\boldsymbol{T}$ provided that $\boldsymbol{T}$ is consistent. If we assume that every combinatorial proof can be formalized within arithmetic, then Gödel's second theorem shows that Hilbert's program of proving in a purely combinatorial way the consistency of arithmetic is not realizable. This assumption is open for discussion, however, as we shall see later when we discuss Gentzen's theorem. Another objection which can be raised against such interpretation of Gödel's second undecidability theorem is this: There are many formulae F strongly representing Z in $\boldsymbol{T}$; Gödel's theorem is valid only for some such formulae. It is not immediately obvious why the theorem proved for just this formula should have a philosophical importance while a similar theorem obtained by a different choice of a formula strongly representing the same set Z is simply false.

Let us consider as an example the formula $F'(x, y) : F(x, y) \wedge \neg F(x, \bar{k})$. If $\boldsymbol{T}$ is consistent, then F' strongly represents Z in $\boldsymbol{T}$ just as F does. Thus the formula $Wid' = \bigwedge_x \neg F'(x, \bar{k})$ can be considered as another formal expression of the consistency of $\boldsymbol{T}$. However, Wid' is obviously provable in $\boldsymbol{T}$.

The reason why the second undecidability theorem holds for some formulae strongly representing Z but fails for others is this: the main step in the proof of this theorem consists of a formalization of an intuitively correct deduction in $\boldsymbol{T}$. (We may add parenthetically that this deduction happens to be the proof of Gödel's first theorem but that this is not essential for the explanation of the phenomenon just described.) The intuitive deduction uses not only premisses of the form $<x, y> \epsilon Z$ and $<x, y> \bar{\epsilon} Z$ for some particular integers x, y but also some properties of Z expressible as general statements. When we formalize the deduction in $\boldsymbol{T}$ we can express the premisses $<x, y> \epsilon Z$, $<x, y> \bar{\epsilon} Z$ by the formulae $F(\bar{x}, \bar{y})$, $\neg F(\bar{x}, \bar{y})$ which are provable in $\boldsymbol{T}$ for all formulae F strongly representing Z. This is no more true for the general statements; for some formulae strongly representing Z these general statements may turn out to be provable in $\boldsymbol{T}$ and for the others not provable.

The first analysis of the second undecidability theorem from this point of view was given by Bernays in [81]. The conditions he imposed on F pertained to the provability of certain general statements in which F (and other related formulae) occur.

A deeper analysis was undertaken by Feferman [36] who introduced the notions of a recursive formula and of a recursively enumerable formula. In order to describe his construction let us assume that ζ is a function taking on values 0 or 1 according as $<x, y> \epsilon Z$ or $<x, y> \bar{\epsilon} Z$. Gödel showed that ζ can be obtained by consecutive substitutions from the function $x + 1$ and a finite number of auxiliary functions τ_j which are defined recursively, *i.e.* which satisfy equations of the form

$$(*) \qquad \tau_j(0, x) = \alpha(x), \qquad \tau_j(n + 1, x) = \beta(\tau_j(n, x), n, x)$$

where β and α involve only the functions $\tau_0, \ldots, \tau_{j-1}$.

If we adjoin to $\boldsymbol{T}$ symbols $\bar{\tau}_j$ for the functions τ_j and add the formulae (*) as new axioms, then we obtain a system which we shall call a primitive recursive extension of $\boldsymbol{T}$. Now a formula F is called primitive recursive if there is a primitive recursive extension

$\boldsymbol{T}'$ of $\boldsymbol{T}$ in which the equivalence $F(x, y) \equiv (\bar{\zeta}(x, y) = 0)$ is provable ($\bar{\zeta}$ is the symbol for the combination of symbols $\bar{\tau}_j$ corresponding to the function ζ). It is easy to show that such a formula F strongly represents Z in $\boldsymbol{T}$. It can also be shown (as was already done in effect by Gödel in [54]) that for every strongly representable set Y there is a primitive recursive formula strongly representing it.

A formula obtained from a primitive recursive formula by prefixing a string of existential quantifiers to it is called recursively enumerable. Feferman's main result states that if $\boldsymbol{T}$ is consistent and F is a recursively enumerable formula strongly representing Z, then the formula $\bigwedge_x \neg F(x, \bar{k})$ is not provable in $\boldsymbol{T}$. The theorem is valid not only for the formal arithmetic based on Peano's axioms but for an arbitrary extension of this system.

The general problem brought up by this analysis can be described as follows: there is given, on the one hand, a set X of integers (or of pairs, triples *etc.*) and, on the other hand, a formal language. We are looking for the best possible definition of X in $\boldsymbol{T}$, *i.e.* for a definition which makes, of all the intuitively true formulae involving X, as many as possible provable in $\boldsymbol{T}$. If we use as definitions formulae strongly representing X, then we can prove in $\boldsymbol{T}$ every formula corresponding to the statements $n \in X$ or $n \bar{\in} X$. If we use primitive recursive formulae as definitions of X we can prove more true statements and still more such statements become provable if we use recursively enumerable formulae. But it is easy to show that no formal definition of the set Z used above will make all *general* arithmetical statements provable in arithmetic. This follows simply from the remark due to Rosser [187] that the set

$$\{p : \bigwedge_x [(<x, p> \bar{\in} Z) \wedge (<x, neg(p)> \bar{\in} Z)]\}$$

is not recursively enumerable (*neg* is here a function such that whenever p is the number of a formula, $neg(p)$ is the number of its negation).

Let us now return to Gödel's second undecidability theorem. For reasons which were set forth above we do not think that this theorem overthrew Hilbert's program although it doubtless showed a weakness in its original formulation. But quite apart from this philosophical claim, Gödel's theorem proved to be a very powerful tool in investigating the relative strength of various axiomatic systems. Whenever we have two systems, both of which contain arithmetic, so that we can prove in one of them the formula *Wid* expressing the consistency of the other, then there is no possibility of interpreting the first system in the second. Such applications were made *e.g.* by Kemeny [93] who compared the relative strength of Russell's theory of types and of Zermelo's axiomatic set theory. Further applications were found by Feferman [37].

In addition to the two undecidability theorems Gödel's paper contained various other results which we shall discuss in the lecture dealing with decision problems. It also contained some deep remarks concerning the way in which the adjunction of variables of higher types modifies the set of provable arithmetical formulae. These remarks were understood only long after the publication of the paper.

As we saw above, Gödel's paper was devoted mainly to problems of consistency and completeness of formal systems. The method invented by Gödel was to compare intuitively true properties of mathematical objects with properties expressible in the formal system under consideration. The sharp division of reasoning into intuitive meta-mathematics and formal mathematics was rejected on principle by the intuitionists; in the hands of Gödel this very division turned out to be an extremely valuable tool for establishing properties of formal systems. This brings once more to light the deep differences between the approach to foundational problems of the intuitionistic school and the approach of the more conservative meta-mathematical school of Hilbert.

Lecture III

Semantics

We shall call the study of relations between mathematical objects and formal expressions naming them "logical semantics". This description of logical semantics is not essentially different from the description of semantics given in linguistics although linguists would not limit themselves to mathematical objects but would replace them by any objects whatsoever and would also replace expressions of a formal language by sentences of the everyday language. Tarski was the first to realize that the basic ideas of semantics can be applied to the study of formalized languages.

When developing semantics we must carefully distinguish between the language in which we speak (the "syntax-language" or the "meta-language") and the language about which we speak (the "object-language"). The meaning of expressions of a language cannot be described in the same language. Thus we have in semantics the same pair of languages which we encountered in our discussion of Gödel's incompleteness theorem. There is a deep difference, however, between the way Tarski developed semantics and the way Gödel discussed certain relations between formulae and sets of integers. Gödel was interested only in very special relations of this kind and tried in every case to reduce these relations to ones which could (via his numbering) be expressed in arithmetic. Tarski on the contrary aims at a general theory of semantic relations; he notes that in some cases these relations are essentially reducible to arithmetic but considers this as a secondary phenomenon. His meta-languages are always very rich and contain sizable parts of set theory.

Tarski showed that all semantic notions can be reduced to one fundamental notion, *viz.* that of a value of a formula. Taken by itself a formula is just a string of symbols and is devoid of any meaning. Thus in order to define the value of a formula we must first fix the values of the simple symbols out of which it is constructed. We shall limit ourselves to the case of first-order formulae, *i.e.* formulae which are constructed from individual variables, predicates, propositional connectives and quantifiers.

It is customary to assume that the propositional connectives denote Boolean operations in a two-element algebra $\{\mathfrak{V}, \mathfrak{F}\}$ whose elements may be called the truth-values. We need, furthermore, three things: (a) an interpretation of the variables; we assume that they denote elements of a set A; (b) an interpretation of predicates; we assume that a predicate with p arguments denotes a relation with p arguments ranging over A, *i.e.* a function which correlates a truth-value with every p-tuple of the elements of A; (c) an interpretation of quantifiers; we assume that the general quantifier denotes a function Q from subsets of A to the set $\{\mathfrak{V}, \mathfrak{F}\}$ such that $Q(A) = \mathfrak{V}$ and $Q(X) = \mathfrak{F}$ for all the other values of X; the interpretation of the existential quantifier is defined by duality.

These conventions determine the interpretation of the language. Since the interpretation of the connectives is the same in all models and the interpretation of quantifiers depends but on A, we see that a model is completely determined by A and by the relations correlated with the predicates.

Once a model $\boldsymbol{M}$ is fixed we can define by induction the value of a formula in $\boldsymbol{M}$ for a given assignment of elements of A to the free variables occurring in the formula. The definition proceeds by induction. The value of an atomic formula $P(x_1, \ldots, x_p)$ is equal to the value of the relation associated with P for the arguments $a_1, \ldots, a_p$ correlated with the variables $x_1, \ldots, x_p$. The value of $F \vee G$ is the Boolean sum of the values of F and of G; and similarly for other connectives. The value of $\bigwedge_{x_i} F$ for a given assignment π of the elements of A to the free variables of $\bigwedge_{x_i} F$ is equal to $Q(X)$ where X is the set of all a in A with the following property: the value of F for an assignment which correlates a with x_i and is otherwise identical with the assignment π is $\mathfrak{V}$.

In what follows we shall use the customary notation $\models_{\boldsymbol{M}} F[a, b, \ldots]$ for the relation defined by the following requirement: the value of F in $\boldsymbol{M}$ for the assignment correlating the elements $a, b, \ldots$ with the consecutive free variables of F is $\mathfrak{V}$. This relation is also read: $a, b, \ldots$ satisfy F in $\boldsymbol{M}$.

The inductive definition of the value of a formula was for the first time formulated explicitly by Tarski, but the notion itself was well understood intuitively and used successfully long before Tarski's paper. Hilbert and his students constantly used the notions of general validity and of satisfiability of formulae which are essentially equivalent to the notion defined by Tarski.

We do not think it essential that Tarski formulated his definition not for a first-order language but for other more comprehensive languages. I am inclined to think that this was on the contrary a somewhat unfortunate circumstance which clouded rather than clarified the problem. From the point of view of syntax there is no essential difference between languages of the first and (say) of the second-order. The language of the second-order can be treated as the language of the first-order with the stipulation that we use special letters for certain variables and place them in a special way in the formulae. What essentially counts is the interpretation (or a model) of the language. Tarski in his first paper chose an interpretation under which the values of second-order variables were arbitrary subsets of A. This is not the only possible model, however, as was shown later by Henkin [77] (*cf.* also Mostowski [149]).

These remarks should not be interpreted as an indication of any doubts about the importance of the progress due to the introduction of semantic notions. Semantics brought order into the various parts of meta-mathematics and allowed one to define and discuss several natural and important notions. While the notions of semantics are extremely simple and natural, the problem of their formalization turned out to be rather deep; it has lead to several new discoveries. Let us discuss these two applications of semantics in turn.

The best known and at the same time the most important notion of semantics is that of logical consequence. Let $F, A_1, \ldots, A_n$ be formulae of the language under consideration. We say then that F is a logical consequence of $A_1, \ldots, A_n$ if, for every model and for every assignment of values to the free variables of these formulae, it is the case that whenever the value of $A_1, \ldots, A_n$ is $\mathfrak{V}$, then so is the value of F. For a first-order language this notion coincides with that of deducibility by means of suitable formal rules of proof. Its advantage as compared with the latter notion lies in its wider range of application. For instance, there exist no adequate rules of proof for higher order languages, and hence the semantic notion of consequence is the only one which we can use in connection with such languages.

Another important notion is that of definability in a given model $\boldsymbol{M}$.

We say that a is definable in $\boldsymbol{M}$ if there is a formula F with one free variable such that a is the unique element x of A for which $\models_{\boldsymbol{M}} F[x]$. A set X of elements of A is definable in $\boldsymbol{M}$ if there is a formula F with one free variable such that for every a in A the equivalence $a \in X \equiv \models_{\boldsymbol{M}} F[a]$ holds. Similar definitions can be formulated for relations with an arbitrary number of arguments.

An interesting and not yet completely solved problem is the following: Is the set of elements definable in a model $\boldsymbol{M}$ itself definable in this model? The answer obviously depends on the model. Tarski investigated this problem for the standard models of higher order arithmetics and found that the set of definable families of sets of integers is not definable. For the set of definable sets of integers, however, the problem has recently been solved by J. W. Addison with the use of the notion of forcing due to Cohen (*cf.* lecture XV).

The notion of definability in a model should not be confused with a completely different but similarly named notion of definability in a theory. This latter notion is not semantic but syntactic in character. Let us explain this for the case of definability of sets. Consider an axiomatic theory $\boldsymbol{T}$ with the primitive notions $R_1, \ldots, R_k$, Y and based on certain axioms. We assume that Y has just one argument. Tarski [223] says that Y is definable in $\boldsymbol{T}$ in terms of $R_1, \ldots, R_k$ if there is a formula F not involving Y which has one free variable and is such that the equivalence $Y(x) \equiv F(x)$ is provable in $\boldsymbol{T}$.

If Y is definable in $\boldsymbol{T}$ in terms of $R_1, \ldots, R_k$ and if all the axioms of $\boldsymbol{T}$ are true in a model $\boldsymbol{M}$, then the set X which interprets Y in $\boldsymbol{M}$ is definable in $\boldsymbol{M}$ and the defining formula can be chosen so that Y does not occur in it. The converse theorem is not true, however; one can easily give trivial examples of theories whose primitive notions R, Y have the following properties: (*i*) Y is not definable in terms of R in $\boldsymbol{T}$; (*ii*) in every model $\boldsymbol{M}$ in which all the axioms of $\boldsymbol{T}$ are true the interpretation of Y is definable in $\boldsymbol{M}$ by a formula not involving Y.

We shall now discuss the properties of the following function: the value of F in $\boldsymbol{M}$. For simplicity's sake we shall assume that no free variables occur in F. We shall also assume that among the relations of the model $\boldsymbol{M}$ there are the following arithmetical relations: x is an integer, $x = y + z$, $x = yz$. It is then possible (using the Gödelian machinery of numbering of formulae) to replace the value-relation (the value of F in $\boldsymbol{M}$ is $\mathfrak{V}$) by the arithmetical relation

(*) f is the number of a formula without free variables which is true in $\boldsymbol{M}$.

Tarski asked the following questions:

(1) Is the relation (*) definable in $\boldsymbol{M}$?

(2) If (*) is not definable in $\boldsymbol{M}$, what new relations should we add to $\boldsymbol{M}$ in order to ensure the definability of (*) in the extended model?

The famous theorem due to Tarski and named after him states that the answer to question (1) is indeed negative. The proof is obtained by a simple application of the diagonal procedure.

Let us enumerate all formulae with one free variable x and denote by F_n the n-th formula; let $D_k(x)$ be a formula which defines k in $\boldsymbol{M}$ (its existence follows easily from the assumption that $\boldsymbol{M}$ contains the basic arithmetical relations). Put $Z_n = \{k : \models_{\boldsymbol{M}} \bigvee_x [F_n(x) \wedge D_k(x)]\}$; by a diagonal argument the set $Z = \{n : n \,\overline{\in}\, Z_n\}$ is different from all sets Z_n. We shall now derive a contradiction from the assumption that (*) is definable in $\boldsymbol{M}$ by showing that if (*) were definable in $\boldsymbol{M}$ by a formula G, then Z would be equal to one of the sets Z_q.

The Gödel number of $\bigvee_x [F_n(x) \wedge D_n(x)]$ is a function of n; it can be shown that this function is definable. Let S be a defining formula. Hence, for every n,

$$\models_{\boldsymbol{M}} \bigvee_{x,\,y} [D_n(x) \wedge S(x, y) \wedge \neg G(y)] \text{ if and only if not } \models_{\boldsymbol{M}} \bigvee_x [F_n(x) \wedge D_n(x)],$$

i.e.

$$\models_{\boldsymbol{M}} \bigvee_x [D_n(x) \wedge F_q(x)] \text{ if and only if not } \models_{\boldsymbol{M}} \bigvee_x [F_n(x) \wedge D_n(x)]$$

where q is the number of the formula $\bigvee_y [S(x, y) \wedge \neg G(y)]$. Replacing here n by q we obtain a contradiction.

Tarski's theorem applied to Peano's arithmetic yields at once the incompleteness theorem. Indeed the set of (numbers of) formulae derivable from Peano's axioms is easily shown to be definable in the (standard) model of arithmetic whereas Tarski's theorem shows that the set of formulae true in this model is not definable in it. Hence the two sets are different which is — in essence — the contention of the incompleteness theorem.

The solution of problem (2) is of course not uniquely determined.

We can obviously add various relations to the model $\boldsymbol{M}$ in such a way that the relation (*) be definable in the new model. The most natural solution is to add to $\boldsymbol{M}$ the relations expressed by "a is a finite sequence of elements of $\boldsymbol{M}$" and "the elements of a sequence a satisfy in $\boldsymbol{M}$ the formula with the number x (*i.e.* the value of this formula is $\mathfrak{W}$)". We shall denote this extended model by $\boldsymbol{M}'$.

The second of these relations can be defined by induction, but it is an induction of a very peculiar form. The usual inductive definition of a function has the form

$$f(0, a) = p(x), \qquad f(n+1, a) = q(f(n, a), n, a)$$

where p and q are given functions. The relation of satisfaction can be defined only by a much more complicated scheme of induction in which the value of $f(n+1, a)$ depends on the values $f(s, b)$ where $s \leq n$ and b is arbitrary. Thus the second inductive equation has the form

$$f(n+1, a) = \Phi\left((\lambda s)_{s \leq n}\, (\lambda b)\, f(s, b), n, a\right)$$

where $(\lambda s)_{s \leq n}\, (\lambda b)\, f(s, b)$ is the function which is defined for $s \leq n$ and for an arbitrary b and whose value is $f(s, b)$, and where Φ is a functional whose values are integers, whose first argument is a function and whose remaining arguments are integers.

In this way Tarski's theory ties up with the theory of inductive definitions and allows us to establish a theorem saying that there exist types of inductive definitions which cannot be reduced to the ordinary inductive definitions in a purely arithmetical way.

Let us finally discuss the problem whether it is possible to develop an axiomatic theory of semantics.

Tarski showed that for any axiomatized theory $\boldsymbol{T}$ we can set up an axiomatic theory $\boldsymbol{T}'$ embodying certain features of the semantics of $\boldsymbol{T}$. The primitive notions of $\boldsymbol{T}'$ are, in addition to those of $\boldsymbol{T}$, the following: the notions of arithmetic (which allow us to speak in $\boldsymbol{T}'$ of numbers correlated with formulae of $\boldsymbol{T}$); the set of all finite sequences of the elements of the universe of $\boldsymbol{T}$; and the satisfaction relation.

The axioms of $\boldsymbol{T}'$ include those of $\boldsymbol{T}$; there are, furthermore axioms corresponding to the inductive clauses of the definition of the satisfaction relation, and axioms characterizing the notion of a finite sequence.

If $\boldsymbol{M}$ is a model of $\boldsymbol{T}$ in which all axioms of $\boldsymbol{T}$ are true and if $\boldsymbol{M}'$ is the extension of $\boldsymbol{M}$ defined above, then all the axioms of $\boldsymbol{T}'$ are true in $\boldsymbol{M}'$. If $\boldsymbol{T}$ is based on a finite number of axioms (or if a suitable form of the rule of mathematical induction is valid in $\boldsymbol{T}'$), then the consistency of $\boldsymbol{T}$ is provable in $\boldsymbol{T}'$. Thus $\boldsymbol{T}'$ is a theory which is essentially stronger than $\boldsymbol{T}$ (*cf.* p. 26). The method of proving the consistency of a given theory $\boldsymbol{T}$ in a stronger theory is often accomplished by interpreting $\boldsymbol{T}'$ in this stronger theory.

An interesting further relationship between $\boldsymbol{T}$ and $\boldsymbol{T}'$ is formulated in a theorem going back to Gödel [58] and developed further by Kreisel and Wang [117]: There exist infinitely many formulae of $\boldsymbol{T}$ which are provable in $\boldsymbol{T}$ (and hence in $\boldsymbol{T}'$) but whose shortest proof in $\boldsymbol{T}$ is k times longer than its shortest proof in $\boldsymbol{T}'$. Here k can be any integer fixed in advance. Thus $\boldsymbol{T}'$ is stronger than $\boldsymbol{T}$ also in the sense that many proofs already existing in $\boldsymbol{T}$ can be essentially shortened in $\boldsymbol{T}'$. Proof of this theorem also uses ideas of semantics but is too involved to be given here.

We thus see that the development of semantics not only allowed one to make precise various intuitively clear meta-mathematical notions but also threw new light on the problems of incompleteness. Semantic ideas simplify Gödel's argument and complement his theorem by showing that axiomatic theories containing arithmetic are not capable of defining their own satisfaction relation nor defining certain other semantic notions. Thus semantics turned out to be useful both in the "constructive" as in the "destructive" parts of meta-mathematical study.

Lecture IV

Computable functions

The need of a systematic study of functions whose values can be calculated by a finite process was felt already in the Hilbert school. Hilbert and his students devoted much attention to the problem known as the decision problem for the first-order logic, *i.e.* to the problem of finding a method which would allow one to decide in a finite number of steps whether any given formula is or is not provable. Many similar problems can be formulated in logic as well as in mathematics. To mention only a few, we have *e.g.* the problem of deciding whether a given string of symbols is a correctly built logical formula, whether a given arithmetical formula is decidable in Peano's arithmetic, whether a given polynomial with integral coefficients is irreducible, *etc.* In each of these examples we have a denumerable class C of objects (formulae, strings of symbols, polynomials) and a subclass C' of this class (the class of provable formulae, of well formed formulae, of irreducible polynomials). We may define a function f on C by putting $f(x) = 0$ for $x \in C'$ and $f(x) = 1$ for $x \in C - C'$ (the characteristic function of C'). The problem is then to find a method allowing us to calculate f in a finite number of steps or to prove that such a method does not exist.

We can obviously solve various problems of this kind without knowing the general definition of a finitistic method. Mathematicians certainly did not feel the need of formulating such a definition when they gave irreducibility criteria for polynomials with integral coefficients. The situation is of course different if we want to show that a specific problem does not admit an algorithmic solution.

Since the class C dealt with in the decision problem is usually denumerable, we can limit ourselves to the case when C is a set of integers. The characteristic function f of C' is then an arithmetical function (*i.e.* a function whose arguments and values are integers), and our problem is to define a class of arithmetical functions whose values can be found in a finitistic way. We are discussing this problem right after the general account of semantics and of the incompleteness theorem partly for historical reasons and partly because there is a far-reaching analogy between the notion of a computable function and the notion of a function definable in a given model. However much we would like to "mathematize" the definition of computability, we can never get completely rid of the semantic aspect of this concept. The process of computation is a linguistic notion (presupposing that our notion of a language is sufficiently general); what we have to do is to delimit a class of those functions (considered as abstract mathematical objects) for which there exists a corresponding linguistic object (a process of computation). In the case of the notion of definability we also encountered the situation that we tried to define a class of functions (or sets or relations) for which there exist certain linguistic objects (formal definitions in the model).

Historically the first example of a class of computable functions was furnished by the primitive recursive functions. They can be defined as the functions that can be obtained from constants and from the successor function $x + 1$ by means of substitutions, of the identification of arguments, of adding superfluous arguments and of applying the schema of primitive recursion (*cf.* p. 32)

$$(*) \qquad f(0, a) = p(a), \qquad f(n + 1, a) = q(f(n, a), n, a).$$

This class became very famous after Gödel showed that these functions suffice for the purpose of numbering all formulae and all sequences of formulae. It is obvious, however, that the class of primitive recursive functions does not include all computable functions. This follows from the existence of a computable function which is universal for the class of the primitive recursive function. A universal function is a function of two variables U such that for every n the function $f(x) = U(n, x)$ is primitive recursive and that each primitive recursive function can be so obtained. Obviously the "diagonal" function $U(n, n) + 1$ is not primitive recursive. It is slightly more cumbersome to prove that U is a computable function. Without going into details we can say that n is a code number which indicates the order of the operations of substitutions and recursions which are used to define a given function as well as the order of the starting functions. Thus $U(n, x)$ is the value at the point x of the function obtained from the starting functions by the use of process with the code number n.

It can be shown similarly that various extensions of the class of primitive recursive functions which are obtained by using a schema of recursion more general than (*) do not exhaust the set of computable functions. This latter class must obviously have the property that no universal function for this class is itself contained in the class.

Several definitions of computable functions and sets were proposed in the decade 1930—1940 and still others appeared later. It is significant that all these definitions, though formally different, proved to be equivalent. We shall not enter into details of these definitions but only give general indications.

1. *Definitions using the notion of representability.* These definitions were among the first to be proposed. The first author who proposed to identify computable functions and sets with the ones which are representable in a formal system was Church [16]. Further definitions of a similar character were proposed by Herbrand and Gödel [57] and by Gödel alone [58]. Church used a very special system called the calculus of λ-conversion, Gödel and Herbrand a weak system whose expressions are exclusively identities. More general was the approach of Gödel in [58]; he noted there that the class of functions which are representable in the formal system of arithmetic of the k-th order is independent of k. This observation was put in a general form by Bernays in [81]; it was shown there that the class of functions representable in any formal system satisfying some very general conditions is one and the same. The main condition to be satisfied is that the relation expressed by: "m is the number of a proof of a formula number n" is primitive recursive.

2. *Arithmetical definitions.* For any relation between integers $R(x, y_1, \ldots, y_n)$ let us denote by $min_x\, R(x, y_1, \ldots, y_n)$ the least x satisfying $R(x, y_1, \ldots, y_n)$ or 0 if such an x does not exist. A class K of functions is closed under the effective min-operation if together with each function $f(x, y_1, \ldots, y_n)$ satisfying the "effectivity-condition" $\bigwedge_{y_1 \cdots y_n} \bigvee_x [f(x, y_1, \ldots, y_n) = 0]$ it contains the function $min_x\, [f(x, y_1, \ldots, y_n) = 0]$.

Kleene [97] developed the theory of computable functions assuming (essentially) the following definition: a function is computable if it belongs to each class which contains the primitive recursive functions and which is closed with respect to the operations of substitution and of the effective minimum. Kleene's original definition was simpler in that he started with an incomparably smaller set of functions.

This arithmetical definition was discussed by J. Robinson [177] who showed how to simplify it still further. In particular she showed that the min-operation may be replaced by the operation of taking the inverse of a function under the assumption that it exists.

A set is computable if its characteristic function is computable. An entirely different definition was proposed not long ago by Kleene [106] in connection with his theory of functionals which will be discussed in lecture VIII; see p. 78.

3. *Canonical systems, Turing machines and algorithms.* Post [160], Turing [234], and Markov [139] proposed still other definitions of computability. A function is computable according to these definitions if it is representable in suitable auxiliary systems. These systems differ, however, from the logical systems dealt with in the definitions given in section 1 in that it is not possible to give any semantic interpretation of them. These auxiliary systems are called canonical systems, Turing machines, and algorithms. Let us describe *e.g.* the algorithms of Markov [139].

For this purpose we consider words in a given finite alphabet A, *i.e.* finite (possibly void) strings of elements of A. If g, g' are two words, then we denote by "$g \to g'$" a rule of transformation which carries a word of the form PgQ into $Pg'Q$; it is assumed that P does not contain the word g. An algorithm is determined by a finite list of such operations:

$$g_1 \to g'_1,\ g_2 \to g'_2, \ldots,\ g_k \to g'_k.$$

Some of these operations are singled out and called the "stop operations". (Markov distinguishes them by placing a dot after them.)

Let P_0 be a word; the algorithm operates in the following way: we look for the least i such that P_0 contains g_i; we represent P_0 in the form Pg_iQ where P does not contain g_i. Now we transform P_0 according to the i-th rule and obtain the word $P_1 = Pg'_iQ$; we repeat the same steps with P_1 and obtain the word P_2; *etc.* If after n steps no operation is applicable or the last operation used was a stop-operation, then P_n is the word obtained from P_0 by the algorithm. If the process never stops, then the algorithm is not applicable to P_0. We thus see that an algorithm determines a function whose domain consists of words (not necessarily of all words) and whose values are words.

Let us now assume that the alphabet contains the symbol 1, and let $\bar{n}$ be the word 11 . . . 1 (n times). A function f is called computable if there is an algorithm such that all numerals $\bar{n}$ belong to its domain and that for each n the algorithm transforms $\bar{n}$ into $\overline{f(n)}$.

As we see, this definition uses notions closely connected with the theory of computers; this connection is still closer in the case of the definition given by Turing, which we shall not reproduce here.

One of the early results of the theory of computable functions was that all the definitions which we enumerated above are equivalent to each other.

Before discussing the properties of computable functions we shall compare the various definitions. None of the definitions is strictly finitistic. In the first group of definitions we require that the representing formulae F be decidable, *i.e.* that for every n there be a proof of either $F(\bar{n})$ or of $\neg F(\bar{n})$; in the second group of definitions we require that the min-operation be applied only to functions satisfying the condition: for every $y_1, \ldots, y_n$ there is an x such that $f(x, y_1, \ldots, y_n) = 0$; in the definition based on the notion of algorithm we require that for each integer n the word $\bar{n}$ belong to the domain of the algorithm. These non-finitistic requirements are needed to exclude the construction of the diagonal function.

In order to eliminate these infinitistic conditions it is necessary to replace computable functions by the more comprehensive class of partial computable functions which differ from the total ones in that they need not be everywhere defined. The simplest possible definition is that given in terms of an algorithm. Every algorithm determines a partial recursive function; its domain consists of integers n such that the algorithm yields a final value when applied to $\bar{n}$ and that this value is a numeral.

For the set of all partial computable functions there exists a universal partial computable function, *i.e.* a function u with two arguments satisfying the following condition: For every partial computable function f there is an integer e such that the functions f and $\lambda x\, u(e, x)$ have the same domain and that $f(x) = u(e, x)$ for every x in the domain of f. We define $u(n, x)$ as an integer y such that the n-th algorithm eventually stops when applied to $\bar{x}$ and yields the value $\bar{y}$. The algorithms are enumerated *e.g.* by the Gödel numbers of the defining sequences $g_i \rightarrow g'_i$.

The diagonal function $u(e, e) + 1$ is still partial computable and hence representable as $u(q, e)$. We are not led to an inconsistency because we have no right to replace e by q here: since u is only a partial function we don't know whether q is in its domain or not.

A set which is void or which is the set of values of a computable (total or partial) function is called recursively enumerable. Other, equivalent definitions of a recursively enumerable set are: the set of values of a primitive recursive function; the domain of a partial computable function; the set weakly representable in *e.g.* Peano's arithmetic. While computable functions serve to make precise the notion of decidability, recursively enumerable sets make precise another notion, *viz.* that of a set generated from given initial elements by a systematic process. For instance, formulae provable in an axiomatic theory based on a finite number of axioms are generated from the axioms by the rules of proof. The domain of a partial recursive function is recursively enumerable. The generating process is in this case the following: we write down one after the other all possible sequences of words and retain only those sequences each term of which (with the exception of the first) is obtained from the previous one by the use of the algorithm. We also reject those sequences whose last term although obtained by a non-stop operation can still be transformed by the basic operations of the algorithm and also those sequences whose first or last term is not a numeral. The required set consists of integers n such that $\bar{n}$ is the first term of one of the remaining sequences.

Post [160] formulated a method of developing the theory of recursively enumerable sets independently of the theory of computable functions. His ideas were very similar to those of Markov. Once the theory of recursively enumerable sets is developed, we can introduce the notions of computable sets and functions. This possibility is a consequence of the following theorems:

(1) The characteristic function of a set X is computable if and only if X and the complement of X are both recursively enumerable.

(2) A partial function f is computable if and only if the set of pairs $\langle x, f(x)\rangle$ is recursively enumerable.

Theorem (1) can be proved as follows: Let us assume that there are two systematic processes which generate X and $-X$. By looking through the elements generated by them we must eventually come to any x given in advance. If x is generated by the first process we put $f(x) = 0$, otherwise $f(x) = 1$. Thus we have a method of calculating the value of f in a finite number of steps.

The general theory of computable functions and of recursively enumerable sets is an elegant and at the same time not very difficult mathematical theory. We shall give below the most important results. For the most part they were first established by Kleene [97].

At the beginning we have a series of elementary theorems stating the closure properties of the class of (total and partial) computable functions under such operations as substitution, recursion, inversion, *etc.* Applied to recursively enumerable sets these theorems yield the result that the class of these sets is closed under cartesian multiplication and under the operations of forming finite unions and intersections and of forming images and counter-images by means of computable functions. All these theorems are obvious if one defines computable functions by means of representability in a sufficiently strong formal system. Next few theorems deal with universal functions. As indicated above, there exists a computable partial function with two arguments which is universal for computable partial functions with one argument. It follows immediately that there exists a set-valued function F such that (a) each recursively enumerable set X is representable as $F(n)$ for a suitable n; (b) the set $\{<m, n> : m \in F(n)\}$ is recursively enumerable. In the language of algorithms this theorem says that there exists a universal algorithm U (or a universal Turing machine), *i.e.* one which is able to generate any algorithmically generable set. Thus if an algorithm A is described by the operations $g_i \to g'_i$ $(i = 1, 2, \ldots, s)$ and if it transforms a word W_0 consecutively into $W_1, W_2, \ldots, W_k$, then U acts on pairs of the form $P_j = <W_j, <<g_1, g'_1>, \ldots, <g_s, g'_s>>>$ and transforms P_j into P_{j+1} for $j = 1, 2, \ldots, k$ and finally P_k into W_k

The existence of a universal function allows us to construct by the diagonal procedure effective examples of sets which are not recursively enumerable and also of recursively enumerable sets whose complements are not recursively enumerable. These examples serve as a starting point for a study of various decision problems. As examples of such problems we may quote the following problems:

Is there an algorithm to decide whether for any two given integers n_1, n_2 the sets $F(n_1)$ and $F(n_2)$ are identical? Is there an algorithm to decide whether two given algorithms define one and the same function?

The answer to both these questions is negative.

Computable functions (total and partial) are important for the general philosophy of mathematics not only because they allow us to express precisely the decision problem and to solve it in many special cases. Their philosophical importance is due to the fact that they can be accepted as a basis of a nominalistic mathematics. This nominalistic trend rejects such abstract notions as sets, functions, *etc.* and admits only those objects which can be named. Computable partial functions are thus evidently acceptable to the nominalists. It is another question whether it is possible to construct reasonable mathematical theories while not accepting any objects besides the computable partial functions. The negative solution of the decision problems formulated above shows that such a mathematical theory would necessarily be non-extensional, which indicates the degree of discrepancy between the nominalistic and the classical approach. These problems will be dealt with in a more detailed way in lecture XI.

There are obviously nominalistic theories of mathematics which admit more objects than just computable functions. Set-theoretically minded mathematicians of the thirties, grouped around Lusin in Moscow and around Sierpiński in Warsaw, developed the theory of Borel sets and of analytical sets which was initiated by the French semi-intuitionists of whom Borel, Baire, and Lebesgue were the best known representatives. Instead of reducing all mathematical notions to integers the semi-intuitionists started from the notion of a real number; sets of such numbers and functions defined on the set of all real numbers were not taken for granted; semi-intuitionists insisted that only nameable sets and functions are admissible. We do not have to enter into the details of what was meant by "nameable" in this context; all that we want to say here is that the theory of sets and of functions acceptable to the semi-intuitionists shows striking analogies to the theory of computable functions. The best explanation of this analogy has been given by Addison [1], [2]. We can thus see that the similarity between the philosophical programs which underlie the theory of computable functions and what is known as the descriptive theory of sets determined to a certain extent the results of these theories.

More technical applications of the theory of computable functions are furnished by various meta-mathematical theorems. We may mention *e.g.* a result of Rosser [187] who strengthened Gödel's incompleteness theorem by showing that the set of (the numbers of) undecidable arithmetical sentences is not recursively enumerable. There have also been attempts to apply the notion of a computable function to make precise the notion of a random sequence occurring in some axiomatic formulations of probability theory (Church [20]). Finally there have been several attempts to construct a computable analogue to the usual set theory and in particular to the theory of well-orderings. These attempts started with the early papers of Church [18] and of Church and Kleene [22]; the subsequent developments will be reported on in lecture XI.

To conclude our account of the early phase of computability theory let us compare once more this theory with semantics. In both these theories we define a class of abstract objects by means of linguistic concepts (we disregard for a moment the purely arithmetical definitions of computable functions). In semantics the language in question could not be purely formal but had to be interpreted. In the theory of computability the situation is different: the language need not be understood; it is sufficient to know how to manipulate its expressions. We can thus say that the theory of computability is purely syntactic. This explains the close ties between this theory and the theory of mathematical machines. The interpretation of a language is defined by means of set-theoretical concepts, which gives rise to the close relations between semantics and the set-theoretical, infinitistic philosophy of mathematics; whereas the theory of computability leans towards a more finitistic nominalistic philosophy.

Lecture V

Theorems of Herbrand and of Gentzen

Herbrand and Gentzen, who worked independently of each other along the lines of Hilbert's program, discovered important theorems which deeply influenced later research in the field of logic. We shall first deal with the work of Herbrand which was somewhat earlier than that of Gentzen. Herbrand's main results are contained in his dissertation [79]. We shall not describe all the results which it contains but shall limit ourselves to what is known as his fundamental theorem. It contains a reduction (in a certain sense) of predicate logic to propositional logic. Of course this reduction cannot be complete because the former theory is undecidable and the latter decidable.

Herband correlates with each formula F of the predicate calculus an infinite sequence of propositional formulae having the form of a disjunction:

$$H_n(F) = H_n = A_1 \vee \ldots \vee A_n;$$

these formulae are called the Herbrand disjunctions. The definition of H_n is effective, *i.e.* H_n can be obtained from F and n by means of a fixed algorithm. The relationship between F and its Herbrand disjunctions is the following: F is provable in the predicate logic if and only if there is an n such that H_n is a theorem of the propositional calculus.

Let us give an example of how the disjunctions H_n are to be formed. Suppose that F is the formula $\bigwedge_x \bigvee_y \bigwedge_z M(x, y, z)$ where M is quantifier-free. Let φ be an arithmetical function such that $\varphi(i) > i$ and $\varphi(i) < \varphi(j)$ for all integers i, j satisfying the inequalities $1 \leq i < j$; we may take as φ *e.g.* the function $(i + 1)^2 + i$. We form the disjunction

$$(*) \qquad \bigvee_{i \leq n} M(x_1, x_i, x_{\varphi(i)})$$

and replace in it every atomic formula by a propositional variable in such a way that different atomic formulae are replaced by different propositional variables. The resulting formula is the n-th Herbrand disjunction.

The formula (*) may be interpreted in the following way (*cf.* Kreisel [112]): Suppose that we are trying to build a counter-example to the formula F. We will then look for an element a and a function f correlating with each p an element $f(p)$ such that $\neg M(a, p, f(p))$ is true. Let us substitute for p arbitrary values $p_1, \ldots, p_n$; thus the conjunction

$$\bigwedge_{i \leq n} \neg M(a, p_i, f(p_i))$$

is true. The formula (*) may thus be interpreted as a statement that whatever our choice of a and of the function f will be, our attempt to build a counter-example to F will fail in the field of at most n elements.

It is very easy to show by induction on n that if (*) is provable in the propositional calculus, then F is provable in the predicate logic. We have only to note that the variable $x_{\varphi(n)}$ occurs exclusively in the last term and that we can therefore use the rule of proof

$$(1) \qquad \frac{A \vee B(x)}{A \vee \bigwedge_{z} B(z)}$$

and obtain the formula $\bigvee_{i \leq n-1} [M(x_1, x_i, x_{\varphi(i)}) \vee \bigwedge_{z} M(x_1, x_i, z)]$. Using the rule

$$(2) \qquad \frac{A \vee B(x)}{A \vee \bigvee_{y} B(y)}$$

we obtain similarly $\bigvee_{i \leq n-1} M(x_1, x_i, x_{\varphi(i)}) \vee \bigvee_{y} \bigwedge_{z} M(x_1, y, z)$. Continuing in this way and using the rule

$$(3) \qquad \frac{q \vee p \vee p}{q \vee p}$$

we obtain finally $\bigvee_{y} \bigwedge_{z} M(x_1, y, z)$, whence by the rule

$$(4) \qquad \frac{C(x_1)}{\bigwedge_{x} C(x)}$$

we obtain F.

It is slightly more difficult to prove, conversely, that if F is provable in the predicate logic, then there is an n such that (*) is provable in the propositional logic. The simplest way of obtaining this result is to use the completeness theorem which will be discussed in the next lecture. Herbrand himself came very close to the discovery of this theorem and probably did not make the decisive step only because he was constrained by the finitistic attitude which he had taken over from Hilbert's school. Instead of this simple and natural way of proving his theorem he used another, more difficult one. It was shown a year ago that his proof was fallacious [28]. A correct, strictly finitistic proof was given by Bernays [81].

The proof sketched above has some remarkable consequences. The only rules used in the derivation of F from (*) are (1)—(4); if (*) is provable in the propositional calculus, then it can be obtained from the axioms of the form $p \vee \neg p$ by the repeated use of the rule (3) and suitable other rules of propositional logic, *e.g.*

$$(5) \qquad \frac{p}{p \vee q}, \quad \frac{p,\, q}{p \wedge q}, \quad \frac{p \wedge (q \vee r)}{(p \wedge q) \vee (p \wedge r)}, \ldots$$

together with rules allowing us to change the order of formulae in disjunctions and conjunctions. In order to obtain a complete set of rules for the predicate calculus we must still add rules allowing us to bring every formula to a prenex normal form. An example of such rules is the following: If the formula $\neg \bigwedge_x A(x)$ is a part of another formula B, then it can be replaced throughout B by $\bigvee_x \neg A(x)$; and conversely.

As we see, Herbrand's theorem allows us to get rid of the classical rule of *modus ponens* altogether. Herbrand wrote: "A cause des difficultés que l'on risque de rencontrer, dans certaines démonstrations par récurrence sur les démonstrations, du fait de la règle d'implication, nous considérons ce résultat comme très important". The importance which Herbrand had in mind can be explained thus. Let us first define by induction the notion of a subformula of a given formula: For any formula A, A is a subformula of A itself; if A is a subformula of B, then it is a subformula of the formulae $B \wedge C$, $B \vee C$, $B \rightarrow C$, $\neg B$ and also of $C \wedge B$, $C \vee B$, and $C \rightarrow B$, and finally of $\bigwedge_x B$ and of $\bigvee_x B$ where x can be replaced by any variable; if $A(x)$ is a subformula of B, then so is $A(y)$ (letters x and y may again be replaced by any variables). Now Herbrand's remark amounts to the following: If F is provable, then there exists a proof of it consisting exclusively of subformulae of F. It is obvious that this result greatly simplifies the study of formal proofs. Because of the clause that $A(y)$ is a subformula of B whenever $A(x)$ is we cannot expect to obtain a solution of the decision problem, but at any case we see that we came very close to it.

Many of Herbrand's results were rediscovered and greatly improved by Gentzen [50] who devised a new logical calculus essentially equivalent to the one used by Hilbert's school but much more flexible.

Gentzen's calculus does not operate with single formulae but with sequences of the form

$$A_1 \ldots A_k \vdash B_1 \ldots B_l \quad (k \geq 0,\ l \geq 0)$$

where the A_t and the B_j are formulae in the usual sense. This sequence is read: At least one of the formulae $B_1, \ldots, B_l$ follows from the assumptions $A_1, \ldots, A_k$. A void string in antecedent means a true assumption and a void string in the consequent means a false conclusion. The only axioms of the Genzen calculus have the form $A \vdash A$. The rules of proof are more numerous than in the usual systems of logic; for every propositional connective we have a pair of rules allowing us to introduce this connective in the antecedent or in the consequent. For instance, the rules for the connective $\neg$ have the form

$$\frac{A_1 \ldots A_k\ X \vdash B_1 \ldots B_l}{A_1 \ldots A_k \vdash B_1 \ldots B_l\ \neg X} \qquad \frac{A_1 \ldots A_k \vdash B_1 \ldots B_l\ X}{A_1 \ldots A_k\ \neg X \vdash B_1 \ldots B_l}$$

Gentzen admitted furthermore a number of rules (called structural rules) which allow us to change the order of formulae in the antecedent and in the consequent and to avoid repetitions of formulae. A final rule called "cut" corresponds to the classical "*modus ponens*" and reads

$$\frac{A_1 \ldots A_k \vdash M\ B_1 \ldots B_l; \quad M\ C_1 \ldots C_p \vdash D_1 \ldots D_q}{A_1 \ldots A_k\ C_1 \ldots C_p \vdash B_1 \ldots B_l\ D_1 \ldots D_q}$$

Now Gentzen's main result says that this last rule is superfluous; every sequence provable by means of it is also provable without it. (A generalization of this theorem was given by Kleene [101].)

For the classical logic Gentzen's theorem yields probably nothing more than Herbrand's fundamental theorem. The flexibility of the Gentzen method is nevertheless obvious from the fact that it is applicable to many non-classical systems and especially to the intuitionistic system. Gentzen's intuitionistic calculus is not as symmetric as the classical calculus since the rule of introduction of the symbol $\neg$ in the antecedent must be dropped and the other rules reformulated in such a way that no sequence with more than one formula occurs as a consequence in any proof. With these limitations Gentzen's theorem on the elimination of cuts is still valid. It is the basis of various meta-mathematical results concerning the intuitionistic logic. *E.g.* it is obvious from this theorem that the intuitionistic propositional calculus is decidable. It may also be shown easily that certain formulae of the intuitionistic predicate logic are not provable intuitionistically. Proofs of this kind proceed by checking all the possible rules that could lead to the formula in question and by showing that none of them is able to yield this result. This is the case *e.g.* with the formula $\bigwedge_x [p \vee F(x)] \to [p \vee \bigwedge_x F(x)]$.

The Gentzen method is applicable to many other non-classical systems, *e.g.* to the minimal logic (*cf.* Ketonen [94]). It is also possible to apply it to systems based on certain infinitistic rules of proof. For instance, Schütte [195] has given a Gentzen style formalization of an arithmetic based on an infinitistic rule which is known as the rule ω and which allows us to obtain the general statement $\bigwedge_x F(x)$ from the infinite set of premisses $F(\bar{0})$, $F(\bar{1}), \ldots$. He has also used it in consistency proofs of certain arithmetics.

We shall now describe some applications of the theorems of Herbrand and of Gentzen.

The effectivity of existential statements. Let $\boldsymbol{T}$ be an axiomatic theory of the first-order based on the usual logic and on an arbitrary number of open axioms, *i.e.* of axioms in which no quantifiers occur. We assume that the primitive notions of $\boldsymbol{T}$ are relations, functions, and constants. Any meaningful combination of constants, variables and functions is called a term. Now let us assume that the formula $\bigwedge_x \bigvee_y F(x, y)$ is provable in $\boldsymbol{T}$. Applying the Herbrand—Gentzen theorem we can show that there are a finite number of terms $t_1, \ldots, t_s$ involving only the variable x such that the disjunction $\bigvee_{i \leq s} F(x, t_i(x))$ is provable in $\boldsymbol{T}$ without using any quantifiers in the proof. We can thus say that all the existential statements provable from open axioms are effective; in other words, that if all existential assumptions in the axioms are made explicit, then all existential theorems will also be explicit. We have a corresponding, although slightly more complicated theorem for formulae with more than two quantifiers, *e.g.* $\bigwedge_x \bigvee_{yz} \bigwedge_{tz} \bigvee_{uv} F(x, \ldots, v)$.

The general consistency theorem (*Bernays* [81]). This theorem allows us to eliminate in certain circumstances all set-theoretical notions from consistency proofs obtained by means of models. Let us assume that we are given an arithmetical model of an axiomatic system in which all the axioms of the system are true. Thus the elements of the model are integers and its relations are arithmetically defined. An axiom, *e.g.* $\bigwedge_x \bigvee_y \bigwedge_z \bigvee_t F(x, y, z, t)$, will be called effectively true in $\boldsymbol{M}$ if there are computable functions f, g satisfying the condition $\models_{\boldsymbol{M}} F(m, f(m), n, g(m, n))$ for all integers m, n. Now Bernays' theorem says that if all the axioms are effectively true in a model $\boldsymbol{M}$, then so are all the consequences of these axioms. It is evident that this theorem is closely related to the previous one dealing with effectivity.

Let us consider as an example Peano's axioms for arithmetic with the schema of induction

$$\bigwedge_y \left\{\bigwedge_x [x < y \to A(x)] \to A(y)\right\} \to \bigwedge_x A(x)$$

limited to formulae $A(x)$ in which no quantifiers occur. It is not difficult to prove that these axioms are effectively true in the usual arithmetical model; hence so are the theorems. Hence the formula $\bigvee_x A(x)$ is provable if and only if there is an integer n such that $A(\bar{n})$ is provable; from this we may infer that the system in question is consistent.

Let us dicuss this result more closely. Once a model of a system is given and once the axioms are shown to be true in the model, it is trivial from the point of view of semantics to infer that the system is consistent. The point of Bernays' consistency theorem is that it does not use semantics and that it dispenses with set-theoretical constructions altogether. It is true that we used the semantic relation $\models_{\boldsymbol{M}} F(m, f(m), n, g(m, n))$ in the formulation of Bernays' theorem. Since F is quantifier-free it is clear, however, that this relation can be defined directly without the whole elaborate semantic theory.

This elimination of set-theoretical constructions from a consistency proof is important not only from the methodological point of view. There are cases (admittedly rather rare ones) where this elimination can yield genuinely new results. A case in point is the problem whether a finitely axiomatizable fragment of Peano's arithmetic can be proved consistent within the full arithmetic. There is no hope of using the general semantic method here, for Peano's arithmetic is far too weak to allow a reconstruction of even the very limited part of semantics which is needed to prove the consistency of a finitely axiomatizable subsystem of arithmetic. (For a profound discussion of these rather intricate matters see Montague [144].) Let us also remark that the possibility of formally proving the consistency of finitely axiomatizable fragments of arithmetic within the whole of arithmetic was the basis of the work of Kreisel and Wang [117] dealing with the lengths of formal proofs.

We shall mention one more result of Gentzen which, although less general, is certainly no less famous than his theorem discussed above. This is his conception of a consistency proof of arithmetic based on transfinite induction (Gentzen [51]).

An adequate formulation of the principle of transfinite induction in its full generality is possible only in set theory. Gentzen, who was working along the lines drawn by Hilbert, used a much more restricted principle which can be expressed in purely arithmetical terms. His principle has the form

$$(*)\qquad \bigwedge_y \Big\{\bigwedge_x [x \prec y \to A(x)] \to A(y)\Big\} \to \bigwedge_x A(x)$$

where A is any arithmetical formula and $\prec$ an arithmetically definable well-ordering of integers. As compared with the set-theoretical transfinite induction this principle is limited in a twofold way: First, we do not speak of sets and we use the principle only to show that all integers satisfy an arithmetical formula A. Secondly, we do not formulate the principle for any well-ordering (which would require a certain amount of set theory) but only for the very special well-orderings that can be defined arithmetically for integers.

For many formulae defining well-orderings of integers the formula (*) is provable in Peano's arithmetic; hence by Gödel's second undecidability theorem such special cases of (*) cannot yield the consistency proof. Gentzen's discovery was that there is a formula $x \prec y$ which defines a well-ordering of integers of the type $\varepsilon_0 = \omega +$

$\omega^{\omega} + \omega^{\omega^{\omega}} + \cdots$ and which has the property that the induction principle (*) for this well-ordering allows us to prove the consistency of arithmetic. It follows that for such a well-ordering $\prec$ the principle (*) is not provable in arithmetic.

The details of Gentzen's proof are too complicated to be given here. The general idea is that to each formal proof there is defined a transfinite number $\alpha < \varepsilon_0$ called the height of the proof; it is shown that if this proof would have as its end formula $0 \neq 0$, then so would also a proof with a lesser height. Hence the existence of a formal inconsistency would violate the induction principle.

A simpler though not essentially different proof was given by Schütte [195]. We shall return to Gentzen's theorem in lecture X on the occasion of Gödel's interpretation of intuitionistic arithmetic.

By way of conclusion, let us try to evaluate the work related to Herbrand's and Gentzen's theorems from a more general point of view. There are undoubtedly two opposing trends in the study of the foundations of mathematics: the infinitistic or set-theoretical and the finitistic or arithmetical. Herbrand's and Gentzen's original discoveries belong of course to the second of these trends but the subsequent work which has been based on these results has borrowed many ideas from the first. This influence of the set-theoretical approach is clearly visible in Bernays' consistency theorem in which semantic notions are consciously imitated in finitistic terms. We may say that Herbrand's and Gentzen's methods allow us to make finitistic certain particular cases of set-theoretical constructions. This intertwining of the two trends and the influence which they have exerted on each other will also be seen clearly in almost all subsequent lectures.

Lecture VI

The completeness problem

The completeness problem was formulated by Hilbert well before the period which interests us here and solved by Gödel in 1931. It is concerned with the relations between the syntactic and the semantic conceptions of logic. The problem can be formulated in two different ways. We can either consider an uninterpreted formal calculus, described in purely syntactic terms, and try to find a semantic interpretation for it; or we can assume as given an interpreted language and look for a formal calculus with purely syntactic rules of proof which would allow us to prove all the true sentences of the language. Both formulations lead to interesting special results. We shall deal first with the former formulation, which is the one used by Gödel.

Gödel solved the problem for the predicate calculus. It is simpler, however, to formulate it for a calculus in which there are no quantifiers. In this case a solution can be obtained in a very natural way by certain algebraic constructions; an extension of these constructions leads to a solution also in the general case.

Let us start with the very simple case in which we are given a number of axioms each of which is an equation, *e.g.* $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $x \cdot y = y \cdot x$ *etc.*; the dot denotes here a binary operation which we shall call "multiplication" in order to have a word for it. The problem is to construct a model in which these axioms are true, *i.e.* to define a set and a binary operation on the elements of the set such that all the axioms are satisfied under this interpretation. In this form the problem is very well known for algebraists, who call it the problem of constructing a free system satisfying given equations. The construction runs as follows: we consider "words" in a (finite or infinite) alphabet A, *i.e.* finite sequences of symbols correlated with the elements a of A. We call two words equivalent if one of them can be obtained from the other by transforming it according to the axioms. For instance, in case of the two axioms mentioned above and of an alphabet containing symbols a, b, the words (ab) $[(ba)b]$ and $[(aa)b](bb)$ are equivalent.

The elements of the model we are looking for are the equivalence classes of words under this relation. The operation $\times$ interpreting the multiplication is defined as $[s] \times [t] = [(s)\,(t)]$ where $[s]$ denotes the equivalence class containing s.

Let us pass to a more complicated situation where the axioms are not necessarily equations but still contain no quantifiers. The axioms are thus propositional combinations of atomic formulae $R(t_1, \ldots, t_k)$ where R_1, $R_2, \ldots$ are predicates and $t_1, \ldots, t_k$ are terms. We assume that the identity predicate occurs among the R_j's and that the appropriate axioms for identity are among the axioms of the theory. A model is now determined by a set (the universe of the model), by the interpretations of the operations and by the interpretations of the predicates. We construct it in the following way: we add the symbols of some alphabet A to the theory and consider only constant terms, *i.e.* terms which contain no variables. We can repeat our previous construction: two constant terms t_1, t_2 are called equivalent if the formula $t_1 = t_2$ is provable in the theory.

The set of all equivalence classes is taken as the universe of the model; the predicate R_j is interpreted as the relation which holds between the classes $[t_1], \ldots, [t_k]$ if and only if the formula $R_j(t_1, \ldots, t_k)$ is provable. Finally, the primitive operations on these classes are defined in the same way as in the case discussed previously.

It is not difficult to show that if the theory is complete with respect to formulae containing neither variables nor quantifiers, then all the axioms of the theory come out true under the interpretation thus obtained. This is no more true if the theory is incomplete, but it is obvious how to rearrange the proof so as to obtain a solution. After adjunction of A we extend the theory to a more comprehensive one which is complete with respect to the formulae in question, and perform our construction for this extended theory.

The extension of a theory to a complete one is a meta-logical problem which possesses a very clear algebraic content. It is closely connected with the theory of representations of Boolean algebras developed by Stone [217]. Stone's representation theorem shows that every Boolean algebra B is isomorphic with a field of sets. The main tool in his theory is the concept of a filter (or an ideal) of B, *i.e.* the concept of a subset F of B such that $x, y \in F \rightarrow x \wedge y \in F$ and $x \in F \rightarrow x \vee y \in F$. A filter which is different from B but which cannot be extended to a filter different from B is called maximal. It follows from Stone's theory that every filter different from B can be extended to a maximal filter. This theorem is in fact equivalent to his representation theorem.

Formulae of a theory can be treated as elements of a Boolean algebra if we identify mutually equivalent formulae. It turns out that the filters and maximal filters of this algebra have a very simple interpretation: a filter consists of formulae provable in an extension of a given theory (obtained by an adjunction of new axioms), and a maximal filter consists of formulae provable in a consistent complete extension of the theory. It is thus seen that the solution of the completeness problem for a theory with quantifier-free axioms follows from Stone's representation theorem for Boolean algebras.

There are several methods to extend the completeness theorem to the case of the full first-order logic. One of them uses what is known as the epsilon-theorem. This theorem shows that every consistent set S of axioms can be replaced by a consistent set of quantifier-free axioms which form a set at least as strong as S. One obtains this new system by adjoining to the given theory certain new primitive operations (called Skolem or Herbrand functions). Each model of the old system can be recovered from a model of the new system simply by dropping the interpretations of the symbols not present in the old theory. The consistency proof for the extended system is not immediately obvious, however, and it is perhaps easier to prove the completeness theorem first and derive from it the ε-theorem.

Another method which does not require any extension of the list of primitive notions of a theory was invented by Henkin [76]. It rests on a deeper analysis of the maximal filters and the relations they bear to models.

Let us consider the Boolean algebra B of sentences (*i.e.* formulae without free variables) of a first-order theory $\boldsymbol{T}$ to which we add constants denoting elements of an infinite alphabet A and the axioms stating that these constants denote different objects.

Every model whose universe I is the set of equivalence classes $[t]$ of terms (without variables) determines a maximal filter F in B. This filter consists of all sentences true in the model. It is not true, however, that every filter determines a model. For instance, if a filter contains all the sentences of the form $\neg A(t)$ and in addition the sentence $\bigvee_x A(x)$, then it is clear that it corresponds to no model. It turns out that this situation is the only one which can prevent a filter to correspond to a model. If F is a maximal filter in B with the property that

(*) for every formula A with one free variable, if the sentence $\bigvee_x A(x)$ belongs to F, then so does at least one formula of the form $A(t)$,

then there is a model (with the universe I) which corresponds to the filter. Hence the completeness problem will be solved when we show the existence of a maximal filter F with the property (*). In case of a denumerable set A several such proofs are known: Rasiowa and Sikorski [172] were the first to prove this theorem by means of topological methods; Tarski (quoted by Feferman in [33]) replaced them by purely Boolean ones, and Rieger [175] used for the same purpose the representation theory of Boolean algebras.

For a non-denumerable set A these methods are not applicable, and we must either use the methods based on the ε-theorems or a (suitably adapted) method of Henkin [76]. We may note that the first hint of a completeness theorem for a non-denumerable A was given by Tarski in an appendix to [222] (*cf.* also Malcev [134]).

A fruitful method of proving the completeness theorem was devised independently by Beth [7], Schütte [195a], [195b], and Hintikka [82]. The essence of their method (which was clearly influenced by Gentzen-type formalizations of logic) is to look systematically for a possible counter-example to a given formula F. Let us sketch this proof:

We shall call a conjunctive a finite sequence of formulae, each provided with one of the symbols $+$ and $-$, *e.g.* $\alpha = \langle F^+, G^-, H^-, K^+ \rangle$. The meaning of such a conjunctive is the same as that of the formula $F \wedge (\neg G) \wedge (\neg H) \wedge K$. The void sequence Λ is also counted as a conjunctive. We shall define certain operations on conjunctives. For a given conjunctive α we first determine whether it contains a pair of formulae F^+, F^-; if this is the case we put $f(\alpha, n) = g(\alpha, n) = \Lambda$ (the void sequence). Otherwise we determine the first formula F in α (*i.e.* the one farthest to the left) which is not an atomic formula. If there is no such formula F we put $f(\alpha, n) = g(\alpha, n) = \alpha$. In the remaining cases we define $f(\alpha, n)$ and $g(\alpha, n)$ as follows:

1. $F = \neg F_1^{\pm}$ $f(\alpha, n) = g(\alpha, n) = < \ldots, F_1^{\mp} >$

(dots represent here formulae different from F present in α);

2. $F = (F_1 \wedge G_1)^+$ $f(\alpha, n) = g(\alpha, n) = < \ldots, F_1^+, G_1^+ >$;

3. $F = (F_1 \vee G_1)^-$ $f(\alpha, n) = g(\alpha, n) = < \ldots, F_1^-, G_1^- >$;

4. $F = (F_1 \vee G_1)^+$ $f(\alpha, n) = < \ldots, F_1^+ >$,
$g(\alpha, n) = < \ldots, G_1^+ >$;

5. $F = (F_1 \wedge G_1)^-$ $f(\alpha, n) = < \ldots, F_1^- >$,
$g(\alpha, n) = < \ldots, G_1^- >$;

6. $F = (\bigvee_x F_1(x))^+$ $f(\alpha, n) = g(\alpha, n) = < \ldots, (F_1(n))^+ >$;

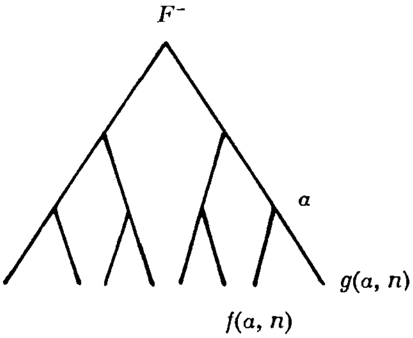
7. $F = (\bigwedge_x F_1(x))^-$ $f(\alpha, n) = g(\alpha, n) = < \ldots, (F_1(n))^- >$;

8. $F = (\bigwedge_x F_1(x))^+$ $f(\alpha, n) = g(\alpha, n) =$
$= < \ldots, (F_1(0))^+, \ldots, (F_1(n))^+, (\bigwedge_x F_1(x))^+ >$;

9. $F = (\bigvee_x F_1(x))^-$ $f(\alpha, n) = g(\alpha, n) =$
$= < \ldots, (F_1(0))^-, \ldots, (F_1(n))^-, (\bigvee_x F_1(x))^- >$.

Now for any formula F without free variables we construct a refutation tree T of F. To this end we consider the full binary tree as drawn in diagram 4,

Diagram 4.



and place $<F^->$ in the vertex 0. We further agree that whenever a non-void conjunctive α is placed in a vertex V, then $f(\alpha, n)$ and $g(\alpha, n)$ are placed in two vertices immediately following V. The value of n (which matters only in cases 6–9) is determined as follows: in applying rules 6 and 7 we choose as n the smallest possible integer not yet used in the tree; in the rules 8, 9, we take as n the greatest of the integers already used.

Dropping from the tree all vertices not occupied by conjunctives we obtain a refutation tree T of F. The branching points of T are all binary; hence by a well-known theorem due to König the tree is either finite or contains an infinite branch. In the former case F is provable and in the latter there is a model in which F is false. This model is defined by taking as its universe the set of all integers and interpreting a predicate, say P, as the relation R which holds between the integers $n_1, \ldots, n_k$, if and only if the formula $(P(n_1, \ldots, n_k))^+$ eventually appears in the conjunctives lying on the branch.

What is remarkable in proofs of this type is that they allow us to construct immediately a formal proof of F provided that no counter-example to it exists.

The construction sketched above differs but inessentially from the one used by Beth.

The completeness theorem found numerous applications to purely algebraic imbedding problems. It is important for such applications that the theorem be proved for a set A of arbitrarily high power. The possibility of applying the completeness theorem to such problems was first pointed out by Malcev [134] who also published the first proof of the theorem independent of the cardinality of A (Malcev's proof was not entirely correct but his mistake can easily be corrected). A typical example of the applications in question is a theorem which says that each partial ordering of a set can be extended to a complete ordering. More sophisticated applications were given by Malcev [135] and by Henkin [78].

As the most important meta-mathematical application of the completeness theorem we mention the result that an incomplete theory based on the first-order logic always has at least two non-isomorphic models. Indeed, if F is a formula which is neither provable nor refutable in a theory $\boldsymbol{T}$, then theories $\boldsymbol{T}'$ and $\boldsymbol{T}''$ obtained from $\boldsymbol{T}$ by adding F and $\neg F$ to its axioms are consistent and thus have models which of course are non-isomorphic. We can prove similarly that if ***T*** is an essentially incomplete theory, then it has $2^{\aleph_0}$ non-isomorphic denumerable models. In particular, Peano's arithmetic has $2^{\aleph_0}$ non-isomorphic denumerable models. The existence of such models was first proved by Skolem [209]; the proof sketched above was indicated by Gödel in a review of Skolem's paper.

The completeness theorem can also be used to establish the existence of non-isomorphic models of certain *complete* theories. Let us take as an example a consistent extension ***T*** of Peano's arithmetic. By adding Skolem functions we obtain a theory ***T'*** which is based on axioms none of which contains quantifiers. We know already that every model of ***T'*** determines a model of ***T***. Now we add to ***T'*** a new constant a, obtaining a theory ***T'***(a). As was already pointed out above, every complete extension ***T**** of ***T'***(a) determines a model whose universe is the set I of all terms of ***T'***(a). The question is now how many such complete extensions ***T**** of ***T'***(a) there are. In order to answer this question we look at the Boolean algebra B' of all formulae of ***T*** with one free variable x. The set of formulae $F(x)$ such that $F(a)$ belongs to ***T**** is clearly a maximal filter in B', and every such filter determines a complete extension of ***T'***(a). Since, in the case of arithmetic, there are $2^{\aleph_0}$ maximal filters in B', we obtain $2^{\aleph_0}$ models. A model ***M**** corresponding to a complete extension ***T**** of ***T'***(a) can be isomorphic to a model ***M***** corresponding to a different complete extension ***T***** of ***T'***(a), but the number of such exceptional extensions ***T***** is at most denumerable since in the isomorphism between ***M**** and ***M***** the element a can be mapped only on one of denumerably many terms, and the image of a determines already the image of the whole of ***M****. It follows that there are $2^{\aleph_0}$ non-isomorphic denumerable models.

We thus see that the existence of mutually non-isomorphic denumerable models of a complete theory depends not on the structure of the Boolean algebra B of sentences but on the structure of the algebra B' of formulae with free variables. The importance of this algebra was first noted by Ryll-Nardzewski [188]; we shall say more about it later when discussing the theory of models.

Several important problems arise when we consider the effectivity of the methods used in the proof of the completeness theorem. We have here really two problems: one concerning methods used to prove the existence of at most denumerable models, and the other concerning methods used to prove the existence of models of any cardinality. Let us start with the second problem, which clearly belongs to the abstract set theory.

After the discussion above it should be clear that the completeness theorem is a consequence of the theorem known as the fundamental theorem of Boolean filter theory, which states that every proper filter in a Boolean algebra B can be extended to a maximal filter. It is easy to show that this theorem is in fact equivalent to the completeness theorem for systems of logic in which we allow an arbitrary number of individual constants (*cf.* Łoś [125]). The fundamental theorem is known to follow from the axiom of choice; it has also been proved that it is independent of the axioms of set theory without the axiom of choice (*cf.* [123]). The more difficult question whether the axiom of choice is equivalent to the fundamental theorem has recently been answered negatively by Halpern [72].

Let us now consider the construction of denumerable models; we can assume that the elements of our models are integers. The natural question to ask here is whether the models one obtains by using the completeness theorem are always recursive or at least recursively enumerable. To make the question more explicit let us assume that F is a formula with (say) one binary predicate. An interpretation of F in the set N of integers is a binary relation, *i.e.* a set R of ordered pairs. The problem is to find out whether there is a recursively enumerable R such that F is true in the model $\langle N, R \rangle$. The answer to this problem is negative. An example of a formula for which there exists no recursively enumerable model is provided *e.g.* by the conjunction of all the axioms of set theory (in the formulation of Gödel [60]). This example is due to Rabin [165].

Kleene [104] and Hasenjaeger [75] showed that one can always find a model in which R is defined in the form

$$m\,R\,n \equiv \bigwedge_{x} \bigvee_{y} S(m, n, x, y)$$

and in the dual form

$$m\,R\,n \equiv \bigvee_{x} \bigwedge_{y} S'(m, n, x, y)$$

with recursive S and S'. Putnam [163] obtained an even simpler form

$$m\,R\,n \equiv \bigvee_{p=0}^{q} [\bigvee_{x} S_p(m, n, x) \wedge \bigwedge_{y} \neg\, S'_p\,(m, n, y)]$$

with recursive S_p and S'_p.

Putnam's result is the best possible one for formulae without the predicate of identity; for formulae with this predicate we can impose an additional requirement on the model to the effect that the predicate of identity be interpreted as identity. The exact form of the simplest definition of a model is not known in this case.

Worth noting here are some strange features of the recursive models of certain simple axioms. For instance, Peano's arithmetic is known to possess just one (up to isomorphism, of course) recursive model; among all the $2^{\aleph_0}$ denumerable models non-isomorphic to the usual (standard) one there is not a single one which would be recursive or even recursively enumerable (*cf.* Feferman [35], Scott [196]). It is not quite clear what causes this peculiar behaviour of various axiomatic theories and what prevents some of them from admitting recursive models at all and others from admitting more than one such model.

Several attempts have been made in the literature to generalize the completeness theorem to more comprehensive systems of logic. Let us consider *e.g.* a language which differs from the usual system of logic by containing one additional quantifier (Sx) (to be read "for at most finitely many x"), additional axiom schemas

$$\bigwedge_{x} \neg F(x) \to (Sx)\; F(x);$$

$$\bigwedge_{x} \{F(x) \equiv [(x = y_1) \vee \ldots \vee (x = y_n)]\} \to (Sx)\; F(x)$$

and one additional rule of proof which allows us to obtain the formula $\neg\,(Sx)\; F(x)$ from the infinite list of formulae

$$\bigvee_{x} F(x),\ \neg \bigvee_{y_1} \ldots \bigvee_{y_n} \bigwedge_{x} \{F(x) \equiv [(x = y_1) \vee \ldots \vee (x = y_n)]\},$$

$$n = 1, 2, \ldots$$

The completeness theorem for this infinitistic logic can be obtained by the algebraic or by the topological method and also by the Beth—Hintikka—Schütte method. This was stated explicitly by Schütte [195$^{\mathrm{a}}$]. It is significant that the methods based on the ε-theorems do not work in this case. We shall later say more about the reasons why these methods cannot be used here.

If we pass to higher logical systems the situation becomes quite different. Let us take as an example the second-order logic. As was remarked in lecture II, the syntactic structure of this logic does not decide whether it is essentially different from the first-order logic. All depends on what kinds of models of the sentences of this logic we admit. If we do not impose any limitations, the completeness theorem will continue to hold and we obtain a completeness theorem which says that a formula which is not refutable has a model in which the second-order variables are restricted to a certain subclass of the full class of subsets of the universe. This result was first obtained by Henkin [77] whose paper has essentially contributed to a better understanding of what higher-order logic really is.

However, if we admit for our second-order logic only models in which the range of the second-order variables is the full class of all the subsets of the universe, then Gödel's first undecidability theorem shows that the completeness theorem is no more true and cannot be saved unless we decide to strengthen so the axioms and rules of proof that they will no longer be effective.

These considerations lead to a different approach to the completeness problem, which is the second approach mentioned in the beginning of this lecture. We take as a starting point not the formal calculus for which an interpretation is to be found but, conversely, a language and a class of "admissible" interpretations of it. The problem is to construct a formal calculus on the formulae of this language such that precisely those formulae are provable which are true in all the admissible models. If by a calculus one means a finite set of recursive rules, then one can in some cases use results from the recursive function theory to prove that no such calculus exists.

There are many results pertaining to the problem just indicated. It is easy to show *e.g.* that for the logic with the quantifier S interpreted in the way indicated above the solution of the problem is negative: the calculus with the infinitistic rule which we discussed cannot be replaced by a calculus with a finite number of recursive rules. This is the case simply because the set of formulae which are true in all models is not recursively enumerable. We may note that this set would be recursively enumerable if the ε-theorem were true for this logic.

If we now change the interpretation of S to read: "there are at most denumerably many elements such that...", then the problem admits a positive solution: using results of Fuhrken [48], Vaught [239] showed not long ago that there exist recursive rules allowing one to prove exactly the formulae which are true in every model (with this interpretation of S). For many other special cases the problem remains open.

As another example let us consider the following problem:

We leave the language of the predicate logic unchanged but narrow down the class of admissible models. Vaught [238] has investigated the classes of formulae which are true in all recursive or recursively enumerable models and found that none of these classes can be axiomatized by recursive rules. A similar result was found earlier by Trachténbrot [233] who found that the set of formulae true in all *finite* models is not recursively enumerable although it is a complement of such a set.

The problem which we discussed in the last few paragraphs can be formulated for various non-classical logics provided we can give them an interpretation. For instance, we may take the interval [0,1] as the set of truth-values (in the case of ordinary logic there are exactly two truth-values) and interpret the propositional connectives as functions defined in this interval and taking their values from it. Quantifiers are interpreted as functions from the subsets of [0,1] to [0,1]. We can then define the basic semantical notions in the same way as in the two-valued case and ask whether the set of formulae true in every model is axiomatizable by a finite set of recursive rules. Chang [15], Scarpellini [193] and others have achieved some results in this direction; it is hard to say, however, whether investigations of many-valued logics will eventually prove to be more than a mere curiosity.

The completeness problem is an interesting example of a question which arose from philosophical investigations concerning the relations between formal calculi and semantics and which has found many purely mathematical applications in spite of its philosophical origin. One often speaks of the relevance of mathematical logic to algebra; it is chiefly the completeness theorem that allows one to connect these two disciplines in such a way that they can deeply influence one another.

Lecture VII

Further development of the recursive function theory

Full and partial computable functions are very well suited to the needs of constructively minded mathematicians. The development of the theory of these functions lead to new notions and constructions however, which do not any more conform completely to the philosophy of constructivism. Most of these new notions were introduced by Kleene. We shall here describe some of them. They are important for purely mathematical reasons but also because they allow us to study some semi-constructivistic theories.

Relative computability. This important notion due to Kleene is best explained in terms of machines. Let f be a function with one argument defined in the set of all integers and taking on integral values. We can view such a function as a device supplying us with the information that $f(0)$ is the value of f at 0, $f(1)$ at 1, and so on. Let us imagine a machine in which this infinite amount of information is stored and which performs the same steps as an ordinary machine. In the course of calculation the machine is from time to time asked to take from the memory the value of f at a point which has been calculated before. A function g calculated by such a machine is called computable with respect to f. The number of times the machine must draw on the information about the values of f depends in general on the particular argument n which appears in the input of the machine but is always finite. Thus in order to calculate g for a given value n of the argument we need only a finite number of values $f(0)$, $f(1)$, . . ., $f(z)$, where z depends on n. Instead of this sequence we can use the single integer $p_0^{f(0)+1} \ldots p_z^{f(z)+1} = \overline{f(z)}$ which synthetizes so to speak the finite sequence of the exponents (p_j denotes the j-th prime).

Kleene [98] proved that a (partial) function computable relative to f has the form

$$g(n) = U\, min_z\, (R(\overline{f(z)}, n)) \tag{1}$$

where U is a fixed primitive recursive function and R a primitive recursive relation. In order to make this result plausible we may remark that in the course of the computation of $g(n)$ the machine performs a process which can be described arithmetically by means of an integer p. We are here invoking of course the whole machinery of Gödel numbers. The process depends on the integers $f(0), \ldots, f(z)$, *i.e.* p is a function of $\overline{f(z)}$ and of n. This function is a partial one not only because g is not necessarily a full function but also because p is undefined if the value of z is too small; the machine cannot complete its work if a sufficient number of the values $f(j)$ is not at hand. A more detailed analysis shows that the dependence of p on $\overline{f(z)}$ and n is given by a recursive relation, say S, whereas the value $g(n)$ is obtained from p by a recursive operation U_1. Taking all these observations together we come to the equivalence $m = g(n) \equiv \bigvee_{z,p} [S(p, \overline{f(z)}, n) \wedge (m = U_1(p))]$ from which we obtain (1) by easy formal transformations.

A similar relativisation is possible for the notion of a recursively enumerable set: a set A of integers is recursively enumerable with respect to f if A is either void or is the set of values of a function computable with respect to f.

Degrees. (Kleene and Post [109]). If g is a partial function computable with respect to f, then we say that its degree is smaller than or equal to f: $deg(g) \leq deg(f)$. The degree of f is defined as the set of all functions g satisfying both inequalities $deg(g) \leq deg(f)$ and $deg(f) \leq deg(g)$.

The degrees form a very interesting structure and much work has been devoted to them. Let us mention a few facts found in the course of this work.

Degrees form an "upper semilattice", *i.e.* for every two degrees α, β there is a smallest degree γ which satisfies $\alpha \leq \gamma$ and $\beta \leq \gamma$. It is not true, however, that for any two α, β there is the greatest degree γ which satisfies $\alpha \geq \gamma$ and $\beta \geq \gamma$ (*cf.* [109]). The first result is almost obvious but the second requires a rather ingenious proof.

One shows easily that the cardinal number of degrees is that of the continuum. Less obvious is the fact established first by Shoenfield [205] that there exists a set of mutually incomparable degrees whose cardinal number is that of the continuum.

The lowest degree **O** consists of course of computable functions. Higher degrees can be found among the arithmetically definable functions which were first introduced by Kleene [98]. A function f is arithmetically definable if there is an arithmetical formula A which defines f, *i.e.* has the property that for every n the number $f(n)$ is the only integer m such that $A(\bar{n}, \bar{m})$ is true in the standard model of arithmetic. The number of quantifiers in A serves as a measure of the complexity of the definition. Functions which can be defined by a formula with n quantifiers but not by a formula with less than n quantifiers belong to the n-th class of the arithmetical hierarchy. In order to avoid misunderstanding we remark that the atomic formulae of A have the form $y = f(x_1, \ldots, x_n)$ where f is *any* computable function.

The simplest class after **O** consists of the characteristic functions of recursively enumerable sets. From what was said in lecture IV we infer easily that there exists a "universal" recursively enumerable set K, *i.e.* a set K such that every recursively enumerable set X is representable as $\{n : f(n) \in K\}$ with a computable f. It follows that the characteristic function c_X of X has a degree less than or equal to that of the characteristic function of K. Denoting the latter degree by $\mathbf{O}'$ we obtain $deg(c_X) \leq \mathbf{O}'$. Since there are X such that c_X is not computable we obtain $\mathbf{O}' \neq \mathbf{O}$.

Most non-computable recursively enumerable sets one encounters in mathematical logic are of degree $\mathbf{O}'$. It would be more accurate to say that their characteristic functions have this degree; for simplicity's sake we shall nevertheless identify sets with their characteristic functions.

Thus *e.g.* the set of the Gödel numbers of theorems provable in any computable extension of Peano's arithmetic has the degree $\mathbf{O}'$; also the set of (the numbers of) refutable sentences has this degree. Several sets which are not recursively enumerable also have this degree, *e.g.* the set of the Gödel numbers of undecidable sentences of any such theory, and the set of the Gödel numbers of first-order formulae which are true in every finite model etc.

Sets recursively enumerable with respect to K have again a "universal" set; the degree of its characteristic function is denoted by $\mathbf{O}''$. Evidently $\mathbf{O}'' > \mathbf{O}'$. We can define similarly the degrees $\mathbf{O}'''$, $\mathbf{O}''''$ *etc.*

We could start not with **O** but with any degree and form the degrees a', a'', ... In order to obtain $(deg(f))'$ we take the degree of a function which is universal for all the functions g whose degrees are $\leq deg(f)$. Evidently this construction requires a proof that such a function exists and that its degree is determined by *deg(f)*. Thus we see that the jump operation ′ allows us to show the existence of arbitrarily high degrees. The iteration of the jump operation does not exhaust all possible degrees. There are *e.g.* degrees greater than each of the degrees $\mathrm{O}^{(n)}$, $n = 1, 2, \ldots$.

The fine structure of the set of degrees is very complicated. The most detailed account of this difficult subject is given in [190].

The computation of the degree of a given set or — in case this set (or its characteristic function) is arithmetically definable — the determination of its place in the arithmetical hierarchy presents in general a very difficult problem. We saw examples of such problems in the previous lecture where we discussed arithmetically definable models for single formulae of the predicate calculus. Several more mathematical examples can be found in [185]. An interesting example was given by Specker [213] who considered complete extensions of Peano's arithmetic. Such extensions can never be recursively enumerable but one can find them already among sets of degree O''; Specker showed that no Boolean combination of recursively enumerable sets (of formulae) can be a consistent complete extension of arithmetic.

The set of arithmetical formulae which are true in the standard model of arithmetic has a higher degree than any $\mathrm{O}^{(n)}$; this set therefore provides us with a simple example of a set which is not arithmetically definable.

Properties of recursively enumerable sets. These sets were studied very extensively, probably because of their relevance for mathematical logic and the theory of machines. The chief contributor to the theory was Post whose paper [160] marks the beginning of a new era in this field.

1. *Post's theorem.* It was known long ago that a set which is recursively enumerable and which has a recursively enumerable complement is computable, *i.e.* has a computable characteristic function. We proved this theorem on p. 39. Generalizing this result Post showed that if a set X as well as its complement belongs to the same class of the arithmetical hierarchy, say to the n-th one, then X is recursive relative to a set belonging to a lower class of the hierarchy, *i.e.* the degree of X is $\leq \mathrm{O}^{(n-1)}$.

The theorem reminds one of a similar result on analytical sets: a set which is analytical and whose complement is analytical is a

Borel set. The analogy is imperfect, however, as we can see from the following theorem which is true for recursively enumerable sets but false for analytical sets:

2. *Theorem on non-separable sets* (Kleene [100]). There are pairs of disjoint sets X, Y which are recursively enumerable (relative to a function f) but which cannot be separated by a set computable with respect to f. In other words there is no set R computable (relative to f) such that $X \subset R$ and $Y \subset -R$.

A simple example of such a pair is the set X of (the numbers of) arithmetical formulae provable in Peano's arithmetic and the set Y of (the numbers of) arithmetical formulae which are refutable in it. Of course the essential undecidability of Peano's arithmetic is an immediate consequence of the non-separability of X and Y. Several theories much weaker than arithmetic also possess the property of non-separability, *i.e.* the property that the set of theorems and the set of their negations are non-separable.

In another example due to Trachténbrot [233] we take as X the set of (the numbers of) formulae of the predicate calculus which are false in at least one finite model and as Y the set of (the numbers of) provable formulae of this calculus.

Inseparable sets have found numerous applications. For instance, they are used in constructing first-order formulae which have no recursively enumerable models, in a proof that Peano's arithmetic has exactly one recursive model (*cf.* lecture VI), and in several other constructions.

3. *Recursively enumerable sets of special kinds.* Post asked in his paper [160] whether there exist recursively enumerable sets of different degrees. His paper contains an account of various unsuccessful attempts to solve this problem. Post stated that the set K of the degree $\mathbf{O}'$ is in a certain sense small: For every recursively enumerable set X disjoint of K it is possible to determine effectively (from the definition of X) an integer not in $X \cup K$. Post called such a set creative. All creative sets have the degree $\mathbf{O}'$, and thus are unsuitable as possible candidates for the solution of the problem. Looking for a possible example of a recursively enumerable set of a smaller degree, Post constructed "big" recursively enumerable sets. His simple sets S have the property that $-S$ does not contain any infinite recursively enumerable sets. Still bigger are the hypersimple sets H whose characteristic property is that if F_1, $F_2, \ldots$ is any recursively enumerable sequence of finite sets, then only

finitely many F_j' s have non-void intersections with $-H$. The sets constructed by Post proved to have the degree **O**′ and thus did not help him to settle the problem. They were nevertheless subjected to further study partly of a purely mathematical and partly of a metamathematical character. Thus *e.g.* Myhill [154] showed the recursive isomorphism of any two creative sets, and Grzegorczyk [68] showed how to obtain easy proofs of Gödel's undecidability theorem by using simple or hypersimple sets. The literature on various kinds of recursively enumerable sets accumulated so rapidly between 1950 and 1960 that it is not possible to discuss it here in detail.

4. *The Friedberg—Mučnik method.* Post's problem was solved simultaneously by Friedberg [44] and Mučnik [153] in 1956, that is full 12 years after the problem was stated. Their construction is highly technical, and we find it impossible to describe it here. We shall only say that the general plan is to obtain two recursively enumerable sets $A = \bigcup_i A_i$, $B = \bigcup_j B_j$ where A_i and B_j are finite sets and where neither of the sets A, B is recursive in the other. The definition proceeds by stages: at even stages $s = 2i$ we take care of the set A_i and at odd stages $s = 2j + 1$ of the set B_j.

If the characteristic function c_A of A were computable relative to c_B we would have a relation

$$c_A(x) = U\,(\min_z T(\overline{c_B(z)}, x, e)) \tag{2}$$

where $T(m, n, 0)$, $T(m, n, 1), \ldots$ is a computable enumeration of all primitive recursive binary relations. Now the choices of A_i at the stages $s = 2i$ are performed so as to obstruct relation (2) for a value e_i, where e_i ranges over all integers as i increases. Choices of the sets B_j are similar but with the roles of A and B reversed. Of course at no stage s are the sets A, B completed, and we are in fact not obstructing relation (2) but a similar relation between the characteristic functions of the approximating sets $\bigcup_{i \leq s/2} A_i$, $\bigcup_{j \leq s/2} B_j$. Thus a special care must be taken to make sure that no further choices make the relation obstructed at some stage of the approximation to reappear in the limit. The method which assures this is called the priority method.

5. *Various applications of the Friedberg—Mučnik method.* This powerful method made it possible to solve several problems concerning recursively enumerable sets and degrees as well as their relations

to meta-mathematics. We may mention as an example the result of Sacks [190] according to which every countable partially ordered set can be embedded in the semilattice of degrees. A result due to Friedberg [46] says that there exists a "maximal" recursively enumerable set M such that the complement $-M$ is infinite but cannot be divided in two infinite parts of the form $R \cap (-M)$ and $(-R) \cap (-M)$ where R is recursively enumerable, *i.e.* that no recursively enumerable set is able to divide $-M$ into two infinite parts.

An example of a meta-mathematical application is provided by a result of Shoenfield [206] who showed the existence of an axiomatic theory (based on an infinite set of axioms) which has the property that the set of the Gödel numbers of its theorems is a recursively enumerable set of any preassigned degree $\alpha < \mathbf{O}'$. For $\mathbf{O} < \alpha < \mathbf{O}'$ we obtain an example of a theory which is undecidable but such that its undecidability cannot be proved by reduction to arithmetic. As we noted above the set of the Gödel numbers of theorems provable in Peano's arithmetic have the degree $\mathbf{O}'$. Feferman [34] gave some general criteria as to when this degree is $\mathbf{O}'$ in case of an arbitrary theory.

Recursive well-orderings. We have already remarked that the theory of computable functions and its generalizations were created for the purpose of reconstructing some fragments of classical mathematics along constructivistic or nominalistic lines. We shall say more about these attempts in lecture XI. Of course the attempts to reconstruct some parts of the classical set theory in computable terms belong to this program. We include a brief discussion of computable well-orderings already here since they will be needed in the next lecture.

Let R be a primitive recursive relation whose domain and counterdomain consist of integers. If R is a well-ordering of a recursively enumerable subset of the set N of all integers, then we say that R is a recursive well-ordering.

We know that all primitive recursive relations can be arranged in a sequence $R_0, R_1, \ldots$ in such a way that the three-argument relation $R_e(x, y)$ is computable. The set of indices e such that R_e is a recursive well-ordering is denoted by O; it plays an important role in the theory of recursive well-orderings.

The ordinal types of relations R_e, e in O, form an initial segment of the Cantorian second number class. The smallest ordinal not belonging to this segment is denoted by ω_1 and called the first non-constructive ordinal. It is a denumerable ordinal, yet there is no primitive recursive relation of this type. It is possible to show that there is no arithmetically definable well-ordering which would have the type ω_1.

Recursive well-orderings are important chiefly because they allow us to express in purely arithmetical terms definitions which usually are expressed in set theory. We have in mind some particular cases of what are known as the definitions by transfinite induction.

A typical definition by transfinite induction consists of three clauses

(i) $A_0 = A$,

(ii) $A_{\xi+1} = F(A_\xi)$,

(iii) $A_\lambda = \bigcup_{\xi<\lambda} A_\xi$ (λ is a limit number)

where A is a set and F a function whose arguments and values are sets. This definition is expressed in set theory because we use in it the notion of an ordinal. If we are interested only in values of A_ξ for $\xi < \alpha < \omega_1$, however, we can eliminate ordinals in favor of integers ordered by a recursive relation R_e with e chosen so that the type of R_e be α. What we define is a function $A(n)$ from integers to sets; the three clauses of the definition now become

(i′) if n_0 is the first integer with respect to the ordering relation R_e, then $A(n_0) = A$;

(ii′) if n' is the immediate successor of n in the ordering R_e, then $A(n') = F(A(n))$;

(iii′) if n does not have an immediate predecessor in the ordering R_e, then

$$\bigwedge_x \{x \in A(n) \equiv \bigvee_y [(y \neq n) \wedge R_e(y, n) \wedge (x \in A(y))]\}.$$

(The formula in (iii′) says that $A(n)$ is the union of the sets $A(y)$ where y preceeds n in the ordering R_e.)

As we see, the set-theoretical notions have disappeared from the definition unless they are present in F.

We can also formulate an arithmetical counterpart to proofs by transfinite induction. Corresponding to each e in O (such that the field of R_e consists of all integers) and to each arithmetical formula G strongly representing R_e we have an arithmetical axiom

$$(*)\ \bigwedge_m \{\bigwedge_n [(n \neq m) \wedge G(n, m) \rightarrow A(n)] \rightarrow A(m)\} \rightarrow \bigwedge_n A(n)$$

which says that if the truth of $A(m)$ follows from the assumption that $A(n)$ is true for all n preceeding m (in the ordering R_e), then $A(n)$ is true for all n.

For certain indices e in O transfinite definitions (i′)—(iii′) are no more powerful than ordinary explicit definitions; for some e in O and some G the axiom schema (*) is also derivable from Peano's axioms for arithmetic. However, this is not true for all e in O.

The transfinite definitions here described have been used several times to obtain extensions of incomplete systems, especially of systems of arithmetic. First steps in this direction were made by Turing [235] who tried to remove the incompleteness of arithmetic by adding to it an undecidable sentence (constructed, say, by the method of Gödel) and repeating this process indefinitely by transfinite induction. A precise formulation of this construction was given recently by Feferman [37].

Another similar application of the transfinite induction to metalogical problems was made by Shoenfield [204] who investigated what is known as the effective rule ω.

The rule ω is an infinitistic arithmetical rule of proof which says that whenever all sentences $A(\bar{0})$, $A(\bar{1})$, . . ., $A(\bar{n})$, . . . are proved, then the sentence $\bigwedge_x A(x)$ can also be considered as proved. The effective rule ω (first proposed by Novikov) strengthens the assumption: instead of requiring that for every n there is a proof of $A(\bar{n})$ we assume that there is an algorithm which produces a proof of $A(\bar{n})$ whenever $\bar{n}$ is fed into it. Of course every arithmetical statement is "provable" by a finite number of applications of the rule ω. Shoenfield showed that transfinite iterations of the effective rule ω also yield a complete system of arithmetic.

Several authors have investigated the strength of the axiom (*) and its dependence on the choice of the formula G as well as on the choice of e. It turns out that there are indices e in O for which axiom (*) is quite strong and allows one to prove *e.g.* the consistency of arithmetic. We shall not discuss this subject but proceed rather to an extension of the arithmetical hierarchy obtained by means of transfinite induction of the kind just described.

Lecture VIII

Hierarchies and functionals

In lecture VII we discussed a hierarchy of sets of integers, known as the arithmetical hierarchy. In mathematical practice we often encounter sets which do not belong to this hierarchy, and it is therefore natural to extend it.

The hyper-arithmetical hierarchy. One obtains it by extending the arithmetical hierarchy into the constructive transfinite.

Without going into technical details we can describe the process as follows: We first fix two arithmetically definable functions π, σ such that whenever e is in O, then so are $e' = \pi(e)$ and $e_n = \sigma(e, n)$; moreover, if the order type of R_e is $\alpha + 1$, then the order type of $R_{\pi(e)}$ is α; and if the order type of R_e is a limit number λ, then the order types of $R_{\sigma(e, n)}$ form a sequence which converges to λ.

We want to define, for each e in O, a universal function F_e whose values are sets and relations which jointly form the e-th level of the hyper-arithmetical hierarchy. If the order type of R_e is 0, then we take as F_e a universal function for the family of recursively enumerable sets and relations. If the order type of R_e is $\alpha + 1$, then we take as F_e a universal function for the family of sets and relations which can be formed from sets and relations of the e'-th class (*i.e.*, from sets and relations $F_{e'}(n)$ where $n = 1, 2, \ldots$) by means of Boolean operations and quantifiers $\bigvee_x$, $\bigwedge_x$, with x ranging over integers. Finally if the order type of R_e is a limit number λ, then F_e is a universal function for the family of sets and relations of the form $\bigcup_n F_{e_n}(f(n))$, where f is a recursive function.

If X is a set (or a relation) such that there is an e in O and an integer n such that $X = F_e(n)$, then we call X a hyper-arithmetical set (or relation).

The success of this construction is due to the circumstance that the values of e' and e_n are given by computable functions π, σ of e

and n. These functions are fixed from the beginning and constantly used in the construction of the levels.

A detailed exposition of the theory based on a definition very similar to the one given above was given by Kleene [105]. It seems to me that one obtains an easier exposition if one starts from another definition based on the notion of representability (*cf.* [71]).

We consider a formal system (S) of second-order arithmetic based on Peano's axioms for integers with the axiom of induction in the form

$$\bigwedge_X \{(0 \in X) \wedge \bigwedge_x [x \in X \rightarrow x+1 \in X) \rightarrow \bigwedge_x (x \in X)\}$$

and on the following set-theoretical axioms:

$$\bigwedge_x [x \in X \equiv x \in Y] \rightarrow (X = Y) \qquad \text{(extensionality)},$$

$$\bigvee_X \bigwedge_x (x \in X \equiv A(x)] \qquad \text{(comprehension)}.$$

In the axiom of comprehension $A(x)$ stands for an arbitrary formula which may or may not involve set variables but does not involve X.

In addition to the usual rules of proof we admit into (S) the (strong) rule ω (*cf.* lecture VII). We define hyper-arithmetical sets (relations) as the ones which are strongly representable in (S).

A related definition was also proposed by Kreisel [114]; he used certain model-theoretical notions, however, instead of the quasi-syntactical notion of representability in (S).

Our definition reveals the source of the close analogy between hyper-arithmetical and computable sets (relations). Indeed, both these classes are defined as consisting of sets (relations) strongly representable in a formal system; the only difference is that in order to obtain computable sets we take as this system the usual second-order arithmetic whereas the hyper-arithmetical sets are obtained if we adjoin to the second-order arithmetic an additional infinitistic rule of inference.

Of course we can introduce (total) hyper-arithmetical functions exactly in the same way as computable functions.

Several definitions used in the theory of computable sets and functions can now be repeated in the hyper-arithmetical case. We can relativize the notion of a hyper-arithmetical function by introducing sets (relations, functions) hyper-arithmetical with respect to a given set (relation, function) X. To obtain this notion we add to the symbols of (S) a new symbol Ξ denoting a set and axioms $\bar{n} \in \Xi$ for those integers n which belong to X as well as axioms $\neg(\bar{n} \in \Xi)$ for those integers n which do not belong to X. Sets (relations, functions) which are strongly representable in the resulting system (S_X) are called hyper-arithmetical relative to X.

Once we have the relative notion we can define degrees, called here hyper-degrees. The hyper-degree of a set (relation, function) X is the class of sets (relations, functions) Y which are hyper-arithmetical with respect to X and have the property that X is hyper-arithmetical with respect to them.

The notion which corresponds to that of a recursively enumerable set is of course the notion of a set weakly representable in (S_X). One can show that these sets coincide with sets definable in the form $\{n : \bigwedge_X R^X(n)\}$ where $R^X(n)$ is a relation arithmetical with respect to X. Another form in which these sets can always be represented is $\{n : \bigwedge_X \bigvee_x T(\overline{c_X(x)}, n)\}$ where c_X is the characteristic function of X, T is a recursive relation, and $\overline{f(x)}$ is equal to $\prod_{i \leq x} p_i^{f(i)+1}$. Sets definable in this form are called Π^1_1-sets. An example is furnished by the set O of integers which enumerate all primitive recursive well-orderings of recursively enumerable subsets of N. The set O is universal for the family Π^1_1 exactly as the set K defined in the previous lecture was universal for the family of recursively enumerable sets: Every Π^1_1-set is representable as $\{n : f(n) \in O\}$ where f is a primitive recursive function. Thus O has the largest hyper-degree among Π^1_1-sets; we shall say that the hyper-degree of O has been obtained from the hyper-degree of hyper-arithmetical sets by the operation of hyperjump.

The analogue of Post's problem which was so difficult for degrees turns out very easy for hyper-degrees. Spector [215^a] has shown that there are no hyper-degrees strictly lower than the hyper-degree of O but not hyper-arithmetical. He also proved that there are continuum many incomparable hyper-degrees. His methods are based on measure theory and category theory.

The analogue of Post's theorem is valid for Π^1_1-sets: A Π^1_1-set whose complement is also a Π^1_1-set is hyper-arithmetical. The separation theorem holds for complements of Π^1_1-sets: If A and B are disjoint and both are complements of Π^1_1-sets, then they can be separated by means of a hyper-arithmetical set. Students of set theory cannot fail to note that all these results are but for notational variation identical with classical results in what is known as the theory of analytic sets. This theory was initiated by Suslin and developed further by Lusin and Sierpiński. It deals with sets of real numbers or more generally with subsets of certain spaces and in particular with sets definable in the form $\bigvee_X \bigwedge_n T(\overline{c_X(n)}, a)$ where T is a closed set and X ranges over sets of integers. We see thus a close analogy between these theories: computable relations correspond to closed subsets of the space, Π^1_1-sets to analytic complements, and hyper-arithmetical sets to Borel subsets of the space.

As an example of a result obtained by pursuing these analogies let us note the following development of a Π^1_1-set into “constituents”:

For every Π^1_1-set $P = \{n : \bigwedge_X \bigvee_n T(\overline{c_X(x)}, n)\}$ there exists a primitive recursive relation C such that

$$n \in P \equiv \bigvee_{e \in O} C(e, n).$$

This development is an analogue of a theorem of Lusin—Sierpiński according to which an analytic complement is representable as a union of Borel sets.

The analogies between the descriptive set-theory and the hyper-arithmetical hierarchy were formulated satisfactorily for the first time by Addison [2].

The analytic hierarchy. A further extension of the hyper-arithmetical hierarchy is the analytic hierarchy of Kleene [105]. We obtain it by dividing sets into classes according to their definitions just as in the case of the arithmetical hierarchy; however, this time we allow formulae containing not only the quantifiers $\bigwedge_x, \bigvee_y$ whose range consists of integers but also quantifiers $\bigwedge_X, \bigvee_Y$ whose range consists of sets of integers. Thus an analytic class consists of sets defined by a formula in which alternating set-quantifiers $\bigwedge_X$ or $\bigvee_X$ are followed by a formula with no set-quantifiers. *E.g.* the fourth analytic class consists of sets

$$(i) \qquad \{k : \bigvee_X \bigwedge_Y \bigvee_Z \bigwedge_T \bigvee_p M(\overline{c_X(p)}, \overline{c_Y(p)}, \overline{c_Z(p)}, \overline{c_T(p)}, k)\}$$

and of sets

$$(ii) \qquad \{k : \bigwedge_X \bigvee_Y \bigwedge_Z \bigvee_T \bigwedge_p M(\overline{c_X(p)}, \overline{c_Y(p)}, \overline{c_Z(p)}, \overline{c_T(p)}, k)\}$$

where M is a primitive recursive relation.

Sets of the form (*i*) we count to the class Σ_4^1 and those of the form (*ii*) to the class Π_4^1.

In the same way as in the arithmetical hierarchy the classes Π_{s+1}^1 and Σ_{s+1}^1 contain sets of a hyper-degree higher than those of all the sets in Π_s^1 and Σ_s^1. Certain questions which were easily disposed of in case of the arithmetical hierarchy are very difficult in the analytic case, however. It is not known, for instance, whether the separation theorems hold for Π_s^1-sets when $s > 2$. It seems probable that this question cannot be answered at all on the basis of the usual axioms of set theory (*cf.* Addison [2]).

We have restricted our account to sets whose elements are integers. In case of analytic hierarchy it is perhaps more natural to consider sets whose elements are subsets of N since such sets are admitted anyhow as values of bound variables. The theory thus generalized is for all practical purposes identical with the theory of projective sets developed in the twenties by Lusin and his school. It is known that this theory abounds in very difficult problems which cannot probably be solved on the basis of the existing set-theoretical axioms. An excellent account of interrelations between the theory of hierarchies and the descriptive theory of sets is contained in Addison's paper [2].

Meta-mathematical applications of higher analytical sets are rather scarce; Scott (unpublished) found some applications showing that sets representable in languages with infinitely long formulae are analytic.

Primitive recursive functionals. The notions of primitive recursiveness and of computability have been extended to objects of higher types; the first step in this direction was taken by Gödel in 1958 [62]. We can look at his idea as a departure from strictly finitistic conceptions (which proved to be too weak to serve as a basis for Hilbert's program) but in quite a different direction from the one which leads to the various hierarchies which we discussed above. Gödel's idea whose germ can be found in writings of Hilbert is to define the notion of computability not only for sets of integers and for functions from integers to integers but also for objects of higher logical types. These objects are called functionals.

Let us explain what functionals are. To this end we first define by induction certain type-symbols: * is a type symbol; whenever $\tau_1, \ldots, \tau_k, \tau_0$ are type-symbols, then so is $(\tau_1, \ldots, \tau_k : \tau_0)$; no other symbols are type-symbols.

Functionals of type * are integers; if $\tau = (\tau_1, \ldots, \tau_k : \tau_0)$ is a type-symbol, then functionals of type τ are functions with k arguments ranging over functionals of type $\tau_1, \ldots, \tau_k$ respectively and whose values are functionals of type τ_0. Thus *e.g.* functionals of type $(* : *)$ are numerical functions, *i.e.* functions from integers to integers, functionals of type $((* : *) : (* : *))$ are functions whose arguments and values are numerical functions *etc.*

Arbitrary functionals are of course highly infinitistic entities accessible only to set-theoretists. Gödel considered a very narrow class of functionals called primitive recursive functionals which — as he showed — are very useful in meta-mathematical investigations. The definition of primitive recursive functionals is as follows (for later use we correlate with each functional f an index $Ind(f)$ which we give in parentheses):

1. A functional of type $(\tau_1, \ldots, \tau_k : *)$ with a constant value 0 is primitive recursive. (Index : $< 1, (\tau_1, \ldots, \tau_k : *) >$.

2. A functional S of type $(* : *)$ defined by $S(x) = x + 1$ is primitive recursive. (Index : 2.)

3. If f has the type $\tau = (\tau_1, \ldots, \tau_k : \tau_0)$ and is primitive recursive, then the functional obtained from f by interchanging the p-th and the q-th argument and the functional obtained from f by an identification of the p-th and the q-th argument are primitive recursive. (Indices $< 3, p, q, Ind(f) >$ and $< 4, p, q, Ind(f) >$.)

4. If f has the type $\tau = (\tau_1, \ldots, \tau_k : \tau_0)$ and is primitive recursive then the functional g of type $(\tau_1, \ldots, \tau_k, \sigma_1, \ldots, \sigma_p : \tau_0)$ defined by the equation

$$g(v_1, \ldots, v_k, w_1, \ldots, w_p) = f(v_1, \ldots, v_k)$$

is primitive recursive. (Index : $< 5, \sigma_1, \ldots, \sigma_p, Ind(f) >$.)

The operation 4 allows us to add inessential variables to a functional.

5. If f has the type $\tau = (\tau_1, \ldots, \tau_k : \tau_0)$ and g the type $(\sigma_1, \ldots, \sigma_r : \tau_1)$ and both are primitive recursive then the functional h of type $(\sigma_1, \ldots, \sigma_r, \tau_2, \ldots, \tau_k : \tau_0)$ defined by the equation $h(v_1, \ldots, v_r, w_2, \ldots, w_k) = f(g(v_1, \ldots, v_r), w_2, \ldots, w_k)$ is primitive recursive. (Index : $< 6, Ind(f), Ind(g) >$.)

6. If f is a primitive recursive functional of type $(\tau_{i+1}, \ldots, \tau_k : (\tau_1, \ldots, \tau_i : \tau_0))$ then the functional h of type $(\tau_1, \ldots, \tau_k : \tau_0)$ defined by the equation $h(a_1, \ldots, a_k) = [f(a_{i+1}, \ldots, a_k)](a_1, \ldots, a_i)$ is primitive recursive. (Index : $< 7, \mathit{Ind}(f) >$.)

7. If f has the type $(\tau_1, \ldots, \tau_k : \tau_0)$, $i \leq k$, and $g_1, \ldots, g_i$ are functionals of types $\tau_1, \ldots, \tau_i$, and $f, g_1, \ldots, g_i$ are primitive recursive then the functional h of type $(\tau_{i+1}, \ldots, \tau_k : \tau_0)$ defined by the equation

$$h(v_{i+1}, \ldots, v_k) = f(g_1, \ldots, g_i, v_{i+1}, \ldots, v_k)$$

is primitive recursive. (Index: $<8, i, \mathit{Ind}(f), \mathit{Ind}(g_1), \ldots, \mathit{Ind}(g_i)>$.)

8. The functional f of type $((\tau_1, \ldots, \tau_k : \tau_0), \tau_1, \ldots, \tau_k : \tau_0)$ defined by the equation

$$f(v, v_1, \ldots, v_k) = v(v_1, \ldots, v_k)$$

is primitive recursive. (Index: $<9, (\tau_1, \ldots, \tau_k : \tau_0)>$.)

9. If f and g are primitive recursive functionals of types $(\tau_1, \ldots, \tau_k : \tau_0)$ and $(\tau_0{}^*, \tau_1, \ldots, \tau_k : \tau_0)$,
then the functional h defined by recursion

$$h(0, v_1, \ldots, v_k) = f(v_1, \ldots, v_k),$$
$$h(x + 1, v_1, \ldots, v_k) = g(h(x, v_1, \ldots, v_k), x, v_1, \ldots, v_k)$$

is primitive recursive. (Index: $< 10, \mathit{Ind}(f), \mathit{Ind}(g) >$.)

Gödel gave in [62] an axiom system whose smallest model consists of primitive recursive functionals and stated that this system has the same strength as Peano's arithmetic with a strong schema of induction

$$\bigwedge_y \{\bigwedge_x [x \prec y \rightarrow A(x)] \rightarrow A(y)\} \rightarrow \bigwedge_x A(x),$$

where $x \prec y$ is a formula which defines a well-ordering of integers into the type ε_0. No proof of this assertion has been hitherto published. It is obvious from his remark, however, that the theory of primitive recursive functionals while stronger than the ordinary Peano arithmetic is nevertheless still an intuitively clear theory which can serve as a basis for meta-mathematical investigations in the sense of Hilbert. Gödel himself showed such applications; we shall say a few words about them in lecture X.

A very elegant definition of primitive recursive functionals has been given by Grzegorczyk [69].

Computable functionals. Kleene [106] formulated a problem of extending to functionals the notion of computable partial functions.

The definition which he proposed used the machinery of indices which we introduced together with schemata defining the primitive recursive functionals. These indices represent a code in which the process of calculation of primitive recursive functionals is noted. We consider now a new schema: Let $\tau = (\tau_1, \ldots, \tau_k : \tau_0)$ be a type symbol. F_τ is a functional of type $(*, \tau_1, \ldots, \tau_k : \tau_0)$ such that if z is the index of a functional f of type τ and $a_1, \ldots, a_k$ are functionals of types $\tau_1, \ldots, \tau_k$, then $F_\tau(z, a_1, \ldots, a_k) = f(a_1, \ldots, a_k)$. We give to F_τ the index $< 11, \tau >$.

The class of functionals to which an index can be correlated according to these definitions is just the class of (partial) computable functionals in the sense of Kleene.

Kleene gave several definitions equivalent to his original one (*cf. e.g.* [107]). The whole theory is still in its first stages and it is not immediately obvious whether it will find applications to logic.

The actual computation of a Kleene's functional takes on the form of a tree. If we want to compute the value of a functional with the index z for the arguments $a_1, \ldots, a_k$ (of the appriopriate types) we investigate the form of z and reduce the computation to the computation of certain other computable functionals. *E.g.*, if $z = < 11, \tau >$ and $a_1 = q$, then we reduce the problem to finding the value of $f(a_2, \ldots, a_k)$ where f is the functional with the index q. In more complicated situation we may reduce the problem to that of finding, say, $f(a_2, \ldots, a_k)$ where f is a functional with the index $a_1(h(a_2))$ where again h is a functional with a given index r. We have then to compute first $h(a_2)$ (which need not be a number but a function or even a functional), then find the value of $a_1(h(a_2))$ which we consider as given once the functional a_1 is given and then proceed to evaluate $f(a_2, \ldots, a_n)$.

Thus we see that the computation tree of $f(a_1, \ldots, a_k)$ is in general infinite and its cardinality is very big. Though there are only denumerably many computable functionals, their class can be considered only by means of a very strong set theory.

Computable functionals of type $(* : *)$ are just the computable functions which we discussed in lecture IV. Thus Kleene's general definition of computable functionals gives us still another definition of computable functions. One can find in Kleene's [106] a detailed proof that his new definition is equivalent to the usual one.

Bar-recursive functionals. Spector [216] defined a less infinitistic extension of primitive recursive functionals. The new principle he used to define his functionals was called by him the "bar-recursion". Its simplest example can be described as follows: Let s range over finite sequences of integers: $s = < s_0, \ldots, s_{k-1} >$; we call k the length of s and denote it by $lh(s)$. We can identify s with the integer $\Pi_{i<k} p_i^{s_i+1}$. Let Y, G, and H be functionals with types ((* : *) : *), (* : *) and ((* : *), * : *). We assume that $Y(f)$ depends only on a finite number of terms of f, *i.e.* that there is a p such that $Y(f_1) = Y(f_2)$ whenever f_1 and f_2 coincide in their first p terms. We now define the functional F of type (* : *) by the following rule in which s' denotes the infinite sequence $< s_0, \ldots, s_{k-1}, 0, 0, 0, \ldots >$ and $s^\frown a$ the sequence $< s_0, \ldots, s_{k-1}, a >$:

$$F(s) = G(s) \qquad \text{if } Y(s') < lh(s),$$

$$F(s) = H(\lambda\, a\, F(s^\frown a), s) \qquad \text{if } Y(s') \geq lh(s).$$

The computation of F can be described thus: if $Y(s') < lh(s)$ the value of $F(s)$ is explicitly given. Otherwise we reduce our problem to that of finding the function $\varphi(a) = F(s^\frown a)$. If $Y((s^\frown a)') < lh(s^\frown a) = lh(s) + 1$, then $\varphi(a) = G(s^\frown a)$ and our computation is accomplished. Otherwise we look for the function $\psi(a, b) = F(s^\frown a^\frown b)$ *etc.* Because of the assumption which we made the value of Y eventually ceases to increase as we take larger and larger extensions of s. Hence after finite numbers of steps we find the functions $\varphi(a)$, $\psi(a, b), \ldots$ and hence the value of $F(s)$.

Spector generalized this idea to arbitrary types and showed that his bar-recursive functionals can be used to obtain consistency proofs for second-order arithmetic.

Let us finish this review of the various notions of computability by taking a look on the tendency which is apparent in the historical development of the subject.

We started with certain very simple notions, close to the intuitive idea of computability. The class of objects thus obtained proved to be too narrow, however. The need of having a round off theory and of finding objects which would help us to fulfil Hilbert's program forced the logicians to depart more and more from the ideal simplicity of computable functions and to introduce more and more infinitistic objects. The tools thus created have an intrinsic value and formal applications (*e.g.* to proofs of consistency). Whether they fit to a philosophical program of finitism or intuitionism is rather dubious.

It looks as if extreme finitism were too barren to allow really fruitful applications; we obtain however important results when we try to approach it. We shall see in the next lecture how a seemingly very modest limitation imposed on the unrestricted set-theoretical notions has led Gödel to solve important consistency questions in set theory. It seems therefore that it is better not to adhere unrestrictedly to the philosophical program of finitism which did not produce all too important results; on the other hand some limitations going into the direction shown by finitists brings extremely interesting and valuable results.

Lecture IX

Consistency of the axiom of choice and of the continuum hypothesis

In 1940 Gödel published a monograph [60] devoted to the problem of consistency of the basic set-theoretical hypotheses. His main proof was widely commented on, and it has exerted a profound influence on the meta-mathematical and philosophical work of the last two decades. In view of the importance of the topics dealt with by Gödel we shall discuss them at some length.

The consistency proof devised by Gödel is closely related to the subject developed in last two lectures. Gödel constructs a model in which the axiom of choice and the continuum hypothesis are valid by extending the arithmetical hierarchy into the transfinite. We saw in the preceding lecture that the extension of the arithmetical hierarchy into the *constructive* transfinite leads to the hyper-arithmetical sets. If we drop the assumption that the transfinite levels are to correspond to recursive well-orderings and allow arbitrary ordinals as labels of the successive levels, we obtain an incomparably larger family of sets. These sets were called by Gödel constructible sets; they form a model for all set-theoretical axioms together with the axiom of choice and the continuum hypothesis.

A constructively minded mathematician cannot understand Gödel's proof, if he is sincere, for he does not accept the notion of an arbitrary ordinal. He can only interpret this proof in a purely formal way; ordinals will then be certain objects described in the axioms of set theory. We shall not take this stand, however, but rather assume that the general notion of an ordinal is intuitively clear to us.

The family of sets defined by Gödel represents a realization of what is known as the predicative foundation of mathematics. The notion of predicativity was introduced by Poincaré at the beginning of this century. It seemed to him that we shall be able to eliminate set-theoretical antinomies by considering only such sets (functions, relations) as can be defined without referring in the *definiens* to any totality involving the object which we want to define. Such definitions are called predicative.

Gödel made these intuitions precise in the following way. Let A, $B, \dots$ be set-theoretical formulae with at least one free variable x (by a set-theoretical formula we mean a first-order formula built from the atomic formulae $x = y$ and $x \in y$). A model for a formula of this kind is furnished by an arbitrary family K of sets; in such a model the symbol ϵ is interpreted as the relation "being an element of". Every formula A with exactly one free variable determines a subset of K consisting of those elements X in K which satisfy the condition $\models_K A[X]$. In case A has $k+1$ free variables we can say that every choice of values $Y_1, \dots, Y_k$ in K for any k of these variables determines together with A a subset of K:

$$\{X \in K : \models_K A\,[X, Y_1, \dots, Y_k]\}.$$

The family of all sets thus obtained is denoted by $D(K)$ and called the family of set-theoretically definable subsets of K.

Gödel takes now as K_0 the void family and defines a transfinite sequence of sets by induction as follows:

$$K_{\xi+1} = D(K_\xi),$$

$$K_\lambda = \bigcup_{\xi<\lambda} K_\xi$$

(In the second formula λ is a limit number.) Every set in $K_{\xi+1} - K_\xi$ is defined without reference to the totality $K_{\xi+1}$ to which it belongs and only with reference to a smaller totality. In this sense we can say that sets which belong to any K_ξ are admissible from the predicative point of view.

The infinitistic element in this definition lies in the use of arbitrary ordinals.

Sets which belong to $K_{\xi+1} - K_\xi$ are called constructible at level $\xi + 1$. Sets constructible at any level are called constructible.

Constructible sets form again a hierarchy. Unlike the hierarchies discussed in the previous lectures this hierarchy extends into the Cantorian transfinite; we refrain on purpose from imposing any limitation on the ordinals ξ used for labelling the levels.

The following hypothesis has been called by Gödel the axiom of constructibility: Every set is constructible. One could object that this axiom is evidently false since *e.g.* the set $\{a, b\}$ consisting of two objects a, b which are not sets is certainly not an element of K_ξ for any ξ. For Gödel the word "set" has a special meaning, however: A set is a collection a_0 of objects whose members are again sets and which has the property that there are no infinite decreasing sequences $\ldots a_n \in a_{n-1} \in \ldots \in a_1 \in a_0$. The word "set" in the axiom of constructibility has this narrow technical sense.

Even with this limitation the axiom of constructibility is a highly dubious statement. An intuitionist would reject it outright not only because it contains various infinitistic terms but also because it states the existence of a law defining an arbitrary set while it seems more probable that there exist sets which cannot be defined by any law. Take *e.g.* a sequence of sets

$$0, \{0\}, \{0, \{0\}\}, \ldots$$

which we shall denote for short by $\bar{n}_0, \bar{n}_1, \bar{n}_2, \ldots$. An intuitionist would say that we can form a set Z by casting dice and including $\bar{n}_p$ to Z if and only if in the p-th cast we obtained an even number. Such a set Z is certainly non-constructible.

In spite of these doubts it must be admitted that the notion of an arbitrary subset of a given infinite set is not sharply defined and that different interpretations of this notion seem to exist which are all compatible with our common intuition. The axiom of constructibility represents a very definite limitation of this notion; thus various problems whose solution seems hopeless for the unlimited notion of a set can very well become solvable if we accept the new axiom.

It is one of the most difficult tasks for a mathematician to decide whether he has to accept or to reject a new axiom. If sets were real objects existing in the world in the same sense as physical bodies we could leave the decision to experiments of some kind. Since nothing supports this Platonistic assumption, we are left without any criterion of truth if we do not consider as such the formal criteria of consistency and our very unclear "mathematical intuition"

At present we must resign ourselves to the possibility that there exist two equally acceptable set theories: one which accepts the axiom of constructibility and another which rejects it. However unpleasant this situation may be for those (rare) mathematicians who maintain that mathematics discovers truth, we must say that we see no way of deciding which of these two set theories is superior to the other.

Independently of what our attitude to these philosophical questions may be, we can state various formal consequences and properties of the axiom of constructibility.

The most important result is that the axiom is consistent relative to the other axioms of set theory. Gödel proved this theorem by showing that the axiom of constructibility is true in the domain of constructible sets (which can be defined in the usual set theory), and that all set-theoretical axioms are true in this domain. The first fact is proved by an analysis of the notion of constructibility; we have to show that the definition of constructibility is not affected by a relativization of the fundamental notions of set theory to the class of constructible sets. The second fact is proved by showing that there are arbitrarily great ordinals ξ such that all set-theoretical axioms are true in K_ξ. Thus the axioms are not able to distinguish between the whole universe and certain sufficiently big sets. This is in effect a well-known principle of set theory known as the reflection principle.

It is almost obvious that the axiom of constructibility implies the well-ordering theorem. The set K_0 is obviously well-ordered, and the well-ordering of any K_ξ extends in a natural way to a well-ordering of $K_{\xi+1}$. Thus every K_ξ can be well-ordered and hence — by the axiom of constructibility — every set whatsoever can be well-ordered.

Much less obvious is the fact that the axiom of constructibility implies the generalized continuum hypothesis. In order to show *e.g.* that there are only $\aleph_1$ subsets of the continuum Gödel shows that every constructible set X of integers is constructible already at a denumerable level ξ. He thus obtains an enumeration of constructible sets of integers by means of denumerable ordinals, which is precisely the content of the continuum hypothesis.

The famous contraction lemma of Gödel which he used to prove this fact is in effect a form of the Skolem—Löwenheim theorem. According to this theorem there is a denumerable family of sets containing X and forming a model for set theory (with the axiom of constructibility). This family is then "contracted" in order to obtain a transitive family, *i.e.* a family whose all elements are among its subset. The way in which this contraction is executed is best seen on an example: If the given family is $\{A, \{A, \{A\}\}, \{A\}\}$, then the contraction yields $\{0, \{0, \{0\}\}, \{0\}\}$. The minimal element A of the given family is contracted to 0 (which is the absolutely minimal element), and other elements $\{A\}$, $\{A, \{A\}\}$ are accordingly contracted to $\{0\}$ and $\{0, \{0\}\}$.

The general principle is that if $m_1, m_2, \ldots$ are contracted to $m'_1, m'_2, \ldots$, then the set $\{m_1, m_2, \ldots\}$ is contracted to $\{m'_1, m'_2, \ldots\}$.

The resulting denumerable transitive family is isomorphic with the given one and is therefore a model for the set-theoretical axioms including the axiom of constructibility. This implies (as we shall see) that this family is one of the sets K_ξ with a denumerable ξ. Since X is not affected by contraction we obtain $X \epsilon K_\xi$.

In order to show that the sets K_ξ are the only possible transitive families A which are models for the set-theoretical axioms including the axiom of constructibility we analyze the axioms and look at the statements concerning A which express the fact that the axioms are true in A. It turns out that most set-theoretical properties do not change their content when relativized to A. If *e.g.* P and Q are elements of A which satisfy in A the property "P is a subset of Q", then P is a subset of Q. The definitions of ordinals and of the classes K_ξ have the same property of "absoluteness". Hence if the axiom of constructibility $\bigwedge_x \bigvee_\xi (x \epsilon K_\xi)$ is true in A, then for every x in A there is a ξ in A such that $K_\xi \epsilon A$ and $x \epsilon K_\xi$. It follows easily that A is the union of all the sets K_ξ which it contains, whence the theorem easily follows.

The exact proof of Gödel's theorem requires thus a meticulous discussion of the question as to which definitions are absolute. After this very tiresome discussion the proof goes through rather smoothly.

The existence of a definable well-ordering and the realization of the generalized continuum hypothesis are two properties of constructible sets which cannot be established for arbitrary sets. Another such property was found by Scott [198]. He showed that there is no set X in which there exists a constructible denumerably additive non-trivial two-valued measure which is 0 on finite subsets of X. It is still an open and apparently very difficult question whether one can assume without inconsistency that there is a set on which such a non-constructible measure exists.

The axiom of constructibility was used by Gödel already in 1939 to solve several outstanding questions in the theory of projective sets. Meanwhile similar questions arose in the theory of analytic hierarchies, and their solutions have been derived from the axiom of constructibility. Gödel did not publish his results which he merely announced in [59]; full proofs have been published by Novikov [157] and Addison [3].

The basis of these applications is the following result: If the axiom of constructibility is true, then there exists a definable well-ordering of all the sets of integers. This well-ordering can be found already in the second analytic class, *i.e.* it can be defined by either of the two formulae

$$(*) \qquad \begin{array}{l} \bigvee_{Z} \bigwedge_{T} \bigvee_{n} R(X, Y, \overline{c_Z(n)}, \overline{c_T(n)}), \\ \bigwedge_{Z} \bigvee_{T} \bigwedge_{n} S(X, Y, \overline{c_Z(n)}, \overline{c_T(n)}) \end{array}$$

with recursive functionals R, S.

It is easy to prove that no simpler definition of a well-ordering of sets of integers is possible.

The construction of these formulae is not difficult. We saw that every constructible set of integer is an element of K_Ω and that there exists a well-ordering of K_Ω. The elements of K_Ω are denumerable and their elements as well as the elements of their elements *etc.* are denumerable and it is possible to map constructible sets of integers in a one-one way into the elements of K_Ω. The well-ordering of K_Ω goes then over into a well-ordering of the constructible sets of integers. Analyzing the definition of this relation we find that it can be reduced to either of the two forms (*) while the axiom of constructibility implies that we obtain in this way a well-ordering of all the sets of integers.

We can say, in short, that the existence of a definable well-ordering has been proved by expressing the theory of the set K_Ω in the language of the second-order arithmetic. This arithmetization is presented in an especially clear way in [3].

The existence of a definable well-ordering allows us to solve many problems concerning the analytic hierarchy. Addison [2] has discussed the problem of separability for the analytic hierarchy and found that any two disjoint sets which belong to the n-th class Π_n^1 of the analytic hierarchy ($n > 1$) are separable by means of sets which belong to the n-th class together with their complements. For sets whose definitions begin with an existential quantifier the theorem is false. The situation is thus exactly the reverse to what we find in the first analytic class.

Several other mathematical applications have been discovered by Kuratowski [118]. Machover [133] used the axiom of constructibility in discussing an extension of the notion of computability to the theory of functions of ordinal variables. His computable functions were defined by an infinite system of equations. In order to repeat the diagonal construction he had to construct a computable correspondence between ordinals and systems of equations. The existence of such a correspondence follows from the axiom of constructibility.

All these applications suggest of course the question whether the axiom of constructibility is true. The problem would of course be solved if it were possible to derive the axiom of constructibility from the other axioms of set theory. It was shown quite recently that no such derivation exists.

There exist therefore two mutually contradictory systems of set-theoretical axioms: one accepts the axiom of constructibility, the other rejects it. It is a highly pertinent question whether the choice between these two systems is just a matter of taste or whether there are compelling reasons to accept one of them as a basis for mathematics.

Gödel declared himself very strongly in favour of the set theory which rejects the axiom of constructibility, but his reasons are not quite clear [61]. The need to answer the fundamental philosophical question whether there are objective criteria of truth in mathematics has never been felt as strongly as in connection with the axiom of constructibility.

We mention still some formal results somewhat related to the question whether the axiom of constructibility is true or false. If the axiom is false, then there should exist formulae which are valid in the domain of constructible sets but invalid in the whole domain of sets. Shoenfield [207] discussed simple formulae of the form

$$(*)\quad \bigvee_{X}\bigwedge_{n} R\,(\overline{c_X(n)}) \qquad \bigwedge_{X}\bigvee_{n} R\,(\overline{c_X(n)}) \qquad (R \text{ is computable})$$

and of similar form with two set-quantifiers. He found that in the domain of these formulae there is no difference between constructible sets and arbitrary sets. If any of the formulae (*) is true in the domain of constructible sets of integers, it is also true in the domain of all sets of integers, and conversely. Formulae with three or more set quantifiers do not behave in this way, however; there are formulae true in the domain of constructible sets but false in the domain of all sets, provided that non-constructible sets exist.

The notion of constructibility introduced by Gödel has been transformed in various ways, and these transformed notions can be used for several purposes. An interesting modification has been proposed by Cohen [23] who used it to obtain a minimal model for the Zermelo—Fraenkel set theory.

Cohen takes as T_0 the empty set and defines the sets T_a by induction; if $C_a = \bigcup_{\beta<a} T_\beta$, then he includes in T_a all sets of the following form:

$$\{x, y\} \text{ where } x, y \in C_a, \tag{1}$$

$$\bigcup_{y \in x} y \text{ where } x \in C_a, \tag{2}$$

$$\{y : (y \subset x) \wedge (y \in C_a)\} \text{ where } x \in C_a, \tag{3}$$

$$\{z : (z \in C_a) \wedge \bigvee_y [(y \in x) \wedge \models C_a F[y, z, u_1, \ldots, u_k]]\} \tag{4}$$

where F is a first-order formula, $x, u_1, \ldots, u_k \in C_a$ and F satisfies the condition that for every y in x there is exactly one z in C_a such that $\models C_a F[y, z, u_1, \ldots, u_k]$.

He shows that there is a denumerable ordinal a such that $T_{a+1} = T_a$ and that T_a is a model for the Zermelo—Fraenkel set theory. This model is minimal in the sense that it is contained in every transitive family of sets in which all the axioms of set theory are satisfied.

From the existence of the minimal model Cohen drew the following curious consequence which in the case of the Bernays—Gödel axioms was noted already by Shepherdson [201]: There exists no formula R such that one can prove in set theory that the elements x satisfying $R(x)$ form a model for a set theory with the negation of the axiom of constructibility.

The situation is completely different in the case of models satisfying the axiom of constructibility. There exists a formula (obtained by formalizing the definition of a constructible set) such that it is provable in set theory that the totality of elements satisfying this formula is a model for set theory with the axiom of constructibility. Hence we can obtain using this formula a proof of relative consistency of the axiom of constructibility. The result of Shepherdson and Cohen shows that no such method is available if one wants to prove the independence of the axiom of constructibility. Hence the independence problem is more difficult than the problem of consistency.

Let us still mention that Scott proposed other modifications of the notion of constructibility by allowing higher-order definitions in the construction of the set $D(K)$. Scott obtained in this way a very elegant new proof of the relative consistency of the axiom of choice.

Let us summarize the essential steps of our discussion. The problem of the consistency of set-theoretical hypotheses, such as the generalized continuum hypothesis, is a formal problem. The solution which Gödel gave to it was obtained by extending and modifying the nominalistic (predicative) approach to mathematics. There are close connections between the theory of hierarchies and the theory of constructible sets: the latter theory is an extension of the former into the Cantorian transfinite. The actual proof of the generalized continuum hypothesis from the axiom of constructibility uses also the Skolem – Löwenheim theorem and thus ties the theory of constructible sets with the theory of models.

The new axiom helps to solve various problems; yet it must be considered as a very dubious hypothesis. The problem of its independence proved to be very difficult.

Although the theory started with a formal problem, it touches the deep and fundamental problem of truth of set-theoretical hypotheses. We see no way of deciding the question whether the axiom of constructibility is true or false; what is worse, even an exact formulation of the problem does not seem to be possible.

All these facts are highly significant for everyone who is seriously interested in the mutual relations of mathematics and philosophy.

Lecture X

Various interpretations of the intuitionistic logic

We devote this lecture to a review of the interpretations proposed for intuitionistic logic and arithmetic. These interpretations are formulated in the language of classical logic; they were formulated not by intuitionists but by representatives of classical mathematics who wanted to make intuitionistic conceptions accessible to non-intuitionists.

By intuitionistic logic we mean here a predicate calculus without identity but with two quantifiers $\bigvee$, $\bigwedge$. We assume that an infinite number of constants $c_1, c_2, \ldots$ are available in the calculus. The logical axioms are those of the intuitionistic propositional logic plus the following two schemata for quantifiers:

$$\bigwedge_x Fx \to Fa \qquad Fa \to \bigvee_x Fx.$$

The rules of proof are the following: the *modus ponens* and the two rules

$$\frac{A \to Fx}{A \to \bigwedge_x Fx} \qquad \frac{Fx \to A}{\bigvee_x Fx \to A}$$

where A does not contain the free variable x.

In the intuitionistic arithmetic we assume that all formulae $F, G, \ldots$ are built by means of propositional connectives and quantifiers from equations between terms; terms in turn are built from constants and variables by means of symbols for arithmetical functions. Intuitionistic arithmetic is based on all the logical axioms and the usual axioms of Peano.

Topological and algebraic interpretations. This interpretation is a natural extension of the topological interpretation of the propositional calculus which we sketched in lecture I. The underlying idea is to choose an appropriate algebraic structure S as the set of truth-values. In the case of classical logic we choose as S the two-element Boolean algebra. In the case of intuitionistic logic we choose as S a (partially) ordered set which is complete in the sense that every non void subset $S_1 \subset S$ has the greatest lower bound (g.l.b.) and and the least upper bound (l.u.b.) in S. The g.l.b. of S_1 is the greatest element which stands in the relation $\leq$ to every element of S_1 and the l.u.b. of S_1 is the smallest element which stands in the relation $\geq$ to every element of S_1. These elements are denoted by $\bigwedge_{x \epsilon S_1} x$ and $\bigvee_{x \epsilon S_1} x$, respectively, or in case S_1 is a finite set $\{a, b, \ldots, m\}$ simply by $a \wedge b \wedge \ldots \wedge m$ and $a \vee b \vee \ldots \vee m$, respectively.

For arbitrary a, b in S we denote by $a \rightarrow b$ the l.u.b. of the elements x such that $a \wedge x \leq b$: $a \rightarrow b = \bigvee_{x \epsilon S_1} x$ where $S_1 = \{x : a \wedge x \leq b\}$.

A model of intuitionistic logic over S is determined by a set $I \neq 0$ which we interpret as a range of individual variables. Predicates are interpreted as functions with arguments in I and values in S. A valuation is a mapping g of the set of all variables and constants into I and of the set of predicates with k arguments ($k = 0, 1, 2, \ldots$) into the set of functions with k arguments such that the arguments range over I and values over S. We define in the same way as in the classical case the value of a formula A for the valuation g (in symbols: $Val(A, g)$). The definition proceeds by induction: If A is an atomic formula $F(x_1, \ldots, x_k)$ where F is a k-place predicate and the x_i are variables or constants, then $Val(A, g) = f(g(x_1), \ldots, g(x_k))$ where f is the mapping of I^k into S correlated with F. If A is one of the formulae $B \vee C$, $B \wedge C$, $B \rightarrow C$, and $\neg B$, then $Val(A, g)$ is $Val(B, g) \vee Val(C, g)$, $Val(B, g) \wedge Val(C, g)$, $Val(B, g) \rightarrow Val(C, g)$, $Val(B, g) \rightarrow O$, respectively. It should be noted here that the symbols $\wedge$, $\vee$, $\rightarrow$ between formulae denote propositional connectives and the same symbols between expressions $Val(B, g)$ and $Val(C, g)$ denote operations on elements of S; O denotes the minimal element of S, *i.e.* $\bigwedge_{x \epsilon S} x$.

Finally we assume $Val\,(\bigwedge_x F, g) = \bigwedge_{i \epsilon I} Val(F, g_i)$, $Val(\bigvee_x F, g) = \bigvee_{i \epsilon I} Val(F, g_i)$, where g_i is a valuation which correlates the element i to the variable x and coincides on all other places with g.

This construction is an immediate generalization of the usual definition of satisfaction for classical logic. It is applicable with minor changes to other non-classical systems of logic, *e.g.* to the modal logics of Lewis and Langford. The first to give an explicit definition of the function *Val* for intuitionistic logic was Chandrasekharan [13]; since then it has been widely used by various authors. A most comprehensive account of the subject is given in the recent book of Rasiowa and Sikorski [173].

It is easy to prove that for arbitrary S and $I \neq 0$ (satisfying the conditions specified above) we have $Val(F, g) = 1$ $(= 0 \rightarrow 0)$ for all intuitionistically provable formulae F. This result is a basis for various independence proofs in the intuitionistic logic. We can show for instance that for some S, I and g we have $Val(F, g) \neq 1$ where F is the formula

$$\neg \bigwedge_x Hx \rightarrow \bigvee_x \neg Hx.$$

Hence this formula is not provable intuitionistically. The structure S used in this case as well as in many similar cases consists of open subsets of a topological space.

These independence proofs suggest the problem of completeness: Let X be a fixed topological space, let S be the family of its open subsets and let I be a fixed infinite set. Does the set of formulae which identically satisfy the equation $Val(F, g) = 1$ coincide with the set of intuitionistically provable formulae?

Rasiowa [171] solved the completeness problem by showing that there exists a space X satisfying these conditions. Sikorski [210] strengthened her result by proving that there exists a closed subset X of the Cantor discontinuous set which possesses the same property. For many spaces the question is still open.

The intuitionistic models of Beth. Beth [8] proposed another modification of the classical notion of a model and obtained in this way an adequate interpretation of intuitionistic logic. His construction was discussed by Kreisel and Dyson [29] who supplied various details omitted by Beth and corrected several minor inaccuracies.

Let F be a formula containing only the predicates $P_1, \ldots, P_k$. Beth defines his models by considering trees, *i.e.* figures consisting of points joined by oriented edges. It is assumed that there are only finitely many edges starting in a given point and that there is exactly one "initial" point which is reached by no edge. A branch consists of a finite or infinite sequence of edges such that the endpoint of each edge is the beginning of the next. We assume that no branch forms a closed curve. A sub-tree of a given tree consists of points which can be reached from a given point p as well as from all points lying on the branch which connects the initial point with p. This sub-tree is said to be determined by p. A tree T is decomposed into sub-trees $T_{p_1}, \ldots, T_{p_n}$ if each infinite branch starting in the initial point passes through one of the points p_i.

Let now T be a tree and let us assume that with each point p there is correlated a finite sequence of formulae each containing no free variables and no predicate different from $P_1, \ldots, P_k$. These formulae are said to be connected with p.

The inductive definition of satisfaction is as follows: An atomic formula A is true on the tree T if T can be decomposed in a finite number of sub-trees $T_{p_1}, \ldots, T_{p_n}$ such that the formula A is connected with each p_i.

$\neg A$ is true on T if the formula A is true on T' for no sub-tree of T'.

$A \vee B$ is true on T if T can be decomposed in a finite number of trees such that on each of them either A or B is true.

$A \wedge B$ is true on T if A and B are true on T.

$A \rightarrow B$ is true on T if for every sub-tree T' of T it is the case that if A is true on T', then B is true on T'.

$\bigwedge_x Fx$ is true on T if Fc_n is true on T for each n.

$\bigvee_x Fx$ is true on T if T can be decomposed in a finite number of sub-trees $T_{p_1}, \ldots, T_{p_k}$ and if there exist k constants $c_{j_1}, \ldots, c_{j_k}$ such that each Fc_{j_i} is true on T_{p_i}.

The completeness theorem proved by Beth says that intuitionistically provable formulae coincide with formulae which are true on every tree.

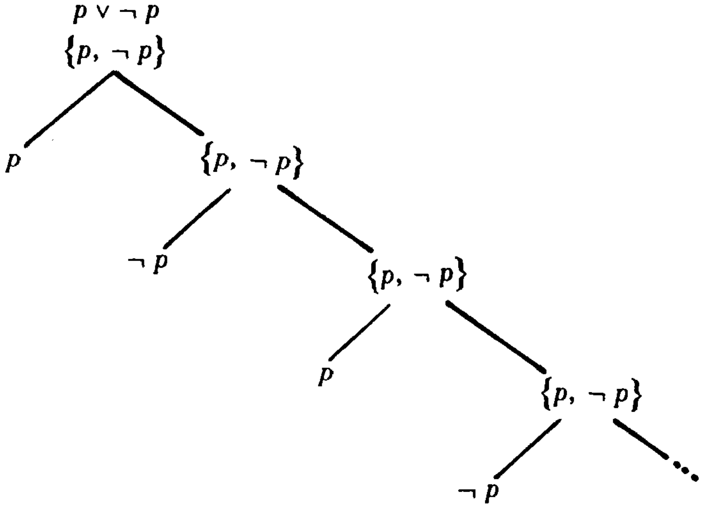
Let us take as an example the formula $p \vee \neg p$ with an atomic p. The tree of diagram 5 (p. 94) shows that this formula is not an intuitionistic theorem. Indeed, it is easy to see that there is no decomposition of this tree in a finite number of sub-trees such that on each sub-tree either p or $\neg p$ be valid.

A related interpretation of intuitionistic logic was given by Lorenzen [124] who used the notion of a "dialogue".

There exists a close connection between the topological interpretation and Beth's construction. Each tree determines a topological space consisting of all its branches. In this space every point p determines a neighbourhood consisting of all branches going through p. We shall call this neighbourhood simply p.

Let us correlate with each predicate P_i (with, say, k arguments) the following function $f_i : f_i(a_1, \ldots, a_k)$ is the union of those neigh-

Diagram 5.



bourhoods p for which $P_i(c_{a_1}, \ldots, c_{a_k})$ is true on T_p. Take as I the set of integers and as S the ordered set of all open subsets of the space determined by the tree. To each P_i we have then correlated a mapping of I^k into S, and we have thus obtained a topological model of the sort discussed above. Kreisel and Dyson [29] showed that — under suitable assumptions — a formula is true on the tree if and only if its value in the corresponding topological model is 1. The assumption which we must make is that an atomic formula true on a sub-tree T' of T must be true on every sub-tree of T'.

Trees and topological models are very convenient in discussing pure logic.[1] Perhaps they could also be used for interpreting Brouwer's theory of "free choice sequences" and their species but no such applications have ever been made. It is less probable that interpretations of this kind can be found for intuitionistic arithmetic.

Realizability. Kleene [99][2] proposed an interpretation of intuitionistic arithmetic. Before describing the details of his construction we insert a few general remarks.

The purpose of an interpretation of a system is to give a precise meaning to notions which are either incompletely explained or taken as

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[1] Kripke's paper [117^{a}], which appeared when this book was in the press, contains further information about this subject.

[2] More recent exposition of this theory is given in [109^{a}], pp. 90—132.

primitives in the system under consideration. In the case of predicate logic the notions to be interpreted are the connectives and the quantifiers. The intuitionists have given in their writings some explanations as to how they understand the connectives. We saw in lecture I that the fundamental notion to which all intuitionistic notions are reducible is that of a construction. This fundamental notion is only implicitly used by intuitionists. Kleene's proposal amounts to making the reduction explicit and moreover to identify constructions with partial computable functions. Since these functions can be enumerated we may formally identify a construction with an integer.

The interpretation of the universal quantifier proposed by Kleene is as follows: a construction whose number is e establishes the truth of the formula $\bigwedge_x Ax$ if the partial computable function f whose number is e has the property that for each n the integer $f(n)$ is the number of a construction which establishes the truth of $A(\bar{n})$. The meaning of an existential formula $\bigvee_x Ax$ is similar. A construction whose number is e establishes the truth of this formula if we can read off from e an integer n and a number e' such that e' is the number of a construction which establishes $A(\bar{n})$.

The details of Kleene's interpretation are as follows: let $K(e)$ and $L(e)$ be two functions such that the mapping $e \rightleftarrows (Ke, Le)$ establishes a one-one correspondence between integers and pairs of integers. Let F be a partial computable function such that $F(0, x), F(1, x), \ldots$ is an enumeration of all partial computable functions. We shall write $F_e(x)$ instead of $F(e, x)$. With these notations we are going to define a relation "e realizes A", where A is an arithmetical formula without free variables.

Case 1. A is an atomic formula; in this case e realizes A if and only if A is true. We use of course the fact that primitive arithmetical predicates are all decidable and that the truth and falsity of an atomic formula is therefore a well-defined notion.

Case 2. A is the formula $B \wedge C$. In this case e realizes A if and only if $K(e)$ realizes B and $L(e)$ realizes C.

Case 3. A is the formula $B \vee C$. In this case e realizes A if and only if either $L(e) = 0$ and $K(e)$ realizes B or $L(e) = 1$ and $K(e)$ realizes C.

Case 4. A is the formula $B \rightarrow C$. In this case e realizes A if and only if for each n either n does not realize B or $F_e(n)$ exists and realizes C.

Case 5. A is the formula $\neg B$. This case is reducible to the previous one because $\neg B$ is the same as $B \rightarrow 0 \neq 0$.

Case 6. A is the formula $\bigwedge\limits_{x} B(x)$. In this case e realizes A if and only if for each n the number $F_e(n)$ exists and realizes $B(\bar{n})$.

Case 7. A is the formula $\bigvee\limits_{x} B(x)$. In this case e realizes A if and only if $K(e)$ realizes $B(\overline{L(e)})$.

The relation "e realizes A" is thus defined. Kleene has shown that all formulae which are provable in intuitionistic arithmetic are realizable by an arbitrary e. The hypothesis that only such formulae have this property has been disproved by Rose [186].

Kleene's notion of realizability hence does not give us an adequate interpretation. This shows that the identification of constructions with computable partial functions is unjustified: There must exist intuitionistically acceptable "constructions" which are not reducible to such functions.

Gödel's interpretation by means of functionals. The principle of this interpretation which was proposed by Gödel in [62] is similar to that of the realizability interpretation but the class of admissible constructions is much wider. These constructions are identified not with functions but with functionals.

In order to see how the functionals appear in the interpretations of formulae let us consider an example. Let A be the formula $\neg \bigwedge\limits_{x} \bigvee\limits_{y} Bxy$ where x and y are numerical variables. Using the idea of the no-counter-example interpretation (*cf.* lecture IV) we interpret this formula in the following way: Whichever function f we choose there exists a counter-example to the formula $\bigwedge\limits_{x} B(x, f(x))$. In other words, there exists a functional Φ of type $((* : *) : *)$ such that we have $\neg F(\Phi(f), f(\Phi(f)))$. The interpretation of the formula $\neg \bigwedge\limits_{x} \bigvee\limits_{y} Fxy$ is thus the formula $\bigvee\limits_{\Phi} \bigwedge\limits_{f} \neg F(\Phi(f), f(\Phi(f)))$.

Gödel defines for every arithmetical formula F (with or without free variables) its translation F^*; the translation has always the same free variables as F and has the form $\bigvee\limits_{p} \bigwedge\limits_{q} F'(p, q)$ where p and q are finite sequences of variables whose values are functionals. The exact definition of the translation is inductive:

If A is an atomic formula, then $A^* = A' = A$;

If A is the formula $B \wedge C$, then A^* is the formula

$$\bigvee_{p_1, p_2} \bigwedge_{q_1, q_2} [B'(p_1, q_1) \wedge C'(p_2, q_2)];$$

If A is the formula $B \vee C$, then A^* is the formula

$$\bigvee_{r, p_1, p_2} \bigwedge_{q_1, q_2} \{[(r = 0) \wedge B'(p_1, q_1)] \vee [(r = 1) \vee C'(p_2, q_2)]\};$$

If A is the formula $B \rightarrow C$, then A^* is the formula [1]

$$\bigvee_{\Phi, \Psi} \bigwedge_{p_1, q_2} [B'(p_1, \Psi(p_1, q_2)) \rightarrow C'(\Phi(p_1), q_2)];$$

(The case when A is $\neg B$ is reducible to the former ones since $\neg B$ is equivalent to $B \rightarrow (0 \neq 0)$);

If A is the formula $\bigvee_{x} B(x)$, then A^* is the formula

$$\bigvee_{x, p} \bigwedge_{q} B'(x, p, q);$$

If A is the formula $\bigwedge_{x} B(x)$, then A^* is the formula

$$\bigvee_{\Phi} \bigwedge_{x, q} B'(\Phi(x), q).$$

The most complicated rule for implication is explained by Gödel as follows: We have to correlate with every *example* of a p_1 satisfying the condition $\bigwedge_{q_1} B'(p_1, q_1)$ an example $p_2 = \Phi(p_1)$ satisfying $\bigwedge_{q_2} C'(\Phi(p_1), q_2)$ and with every *counter-example* q_2 satisfying $\neg C'(\Phi(p_1), q_2)$ a counter-example $q_1 = \Psi(p_1, q_2)$ satisfying $\neg B'(p_1, q_1)$.

The definition of F^* thus completed, Gödel limits the variability of functionals to the class of primitive recursive functionals. Translations of all formulae provable in the intuitionistic arithmetic become then intuitively true statements concerning these functionals. Moreover, these statements are provable in the axiomatic theory $\boldsymbol{T}$ of primitive recursive functionals which we mentioned in lecture VIII. This shows the consistency of the intuitionistic (and hence of the classical) arithmetic relative to $\boldsymbol{T}$.

It is not known whether the property of having the translation provable in $\boldsymbol{T}$ is characteristic for intuitionistically provable arithmetical theorems.

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[1] In this formula Φ and Ψ are finite sequences of variables. If *e.g.* Φ consists of $\varphi, \psi, \ldots$, then $\Phi(p)$ denotes the sequence $\varphi(p), \psi(p), \ldots$ and similarly in other cases.

An extension of Gödel's ideas to second-order arithmetic has been carried out by Spector [216].

All the interpretations we have discussed in this lecture try to explain intuitionistic notions in classical terms. An intuitionist might ask: How can one explain these classical notions in intuitionistic terms? This problem, which is accessible only to intuitionists or people who can think in terms of the intuitionistic logic, has been discussed by Kreisel [113].

It seems to me that the study of mutual interpretations of the classical and intuitionistic systems is extremely useful. By developing them we can hope to reach at least a partial understanding between these two schools.

Lecture XI

Constructive foundations of mathematics

After the discovery of set-theoretical antinomies several mathematicians decided that the only radical solution of the problem raised by these antinomies is to exclude all general set-theoretical notions from mathematics and to limit oneself to the study of those objects that can be effectively defined or constructed. We have already discussed the ideas of the intuitionists and seen that the limitation to constructible objects is an essential feature of their program. There are several other constructive trends less extreme than intuitionism; their program is to limit the domain of admissible mathematical objects to a more or less arbitrarily chosen class without challenging (as the intuitionists do) the classical rules of proof. Since the class of admissible objects is not uniquely determined we cannot speak of a unique constructive trend; there are, on the contrary, many mutually conflicting constructive programs which differ from each other in many details, sometimes important ones, although their general tendencies are similar.

We shall first discuss works whose aim is to examine constructive objects by quite arbitrary means. Since these means are not necessarily admissible from the constructive point of view, it is clear that the results obtained in this way cannot claim philosophical importance. They are sometimes interesting from a purely mathematical point of view, however.

Computable analysis. This theory restricts all mathematical notions and in particular those which occur in mathematical analysis to computable functions. The notion of integer is taken over from classical arithmetic and not analyzed any further. The notion of a real number and all other mathematical notions undergo limitations which aim at an elimination of all non-computable notions.

Specker [211] considered a very narrow class of real numbers which he called primitive recursive. These numbers can be approximated with the accuracy $(^1/_2)^n$ by fractions which have the form $[f'(n) - f''(n)] / g(n)$ where f', f'', g are primitive recursive functions:

$$(*) \qquad | \alpha - [f'(n) - f''(n)] / g(n) | < (^1/_2)^n.$$

He also considered various other types of approximations, *e.g.* by partial sums of the series $\Sigma f(n) / g^n$ where f is a primitive recursive function which satisfies the condition $f(n) < g$ for all n. Still other approximations make use of the notion of a primitive recursive cut. Specker discovered various singularities in the behaviour of these numbers. He proved for instance that a number α may have a primitive recursive decimal expansion whereas 3α fails to have such an expansion. Specker's work was continued among others by Péter [159].

A more comprehensive class consists of the numbers known as computable real numbers. These numbers satisfy for each n the inequality (*) in which f', f'', g are computable functions. They were defined by various authors (Rice [174], Robinson [178], Mazur [141]) who showed that the singularities discovered by Specker in case of primitive recursive numbers do not hold for computable numbers. They proved *e.g.* that a real number α is computable if and only if the sequence of digits in its decimal expansion is computable.

One proves easily that computable numbers form a real closed field. Hence the usual algebraic operations can be performed on computable numbers and yield computable results.

In order to develop further parts of analysis one introduces computable sequences and computable functions of a real variable.

A sequence $\{\alpha_m\}$ of real numbers is computable if there exist computable functions f, f', g of two variables such that the following inequality holds for arbitrary m and n:

$$| \alpha_m - [f(m, n) - f'(m, n)] / g(m, n) | < (^1/_2)^n.$$

In other words, we require that for each n and m we can fix (in a computable way) an interval of length $(^1/_2)^n$ which contains α_m. There is of course a certain degree of arbitrariness in this definition. Instead of using an approximation by rationals we could start from any other means of appproximation, *e.g.* we could require that the n-th decimal digit in the expansion of α_m be given by a computable function of m and n. There are several rather difficult arithmetical questions which arise in connection with these definitions, for instance:

Is there a computable sequence $\{\alpha_m\}$ such that the sequence of the digits of the expansion $\alpha_m = \Sigma f(m, n, g) / g^n$ is computable for no g? Questions of this kind only quite recently have received answers (*cf.* Lachlan [119]).

Computable sequences of reals do not have all the properties one would like them to have. Mazur showed, for example, that there is a computable sequence α_m with all terms different from 0 such that the sequence $1/\alpha_m$ is not computable.

The most important and at the same time the most difficult notion is that of a computable function of a real variable. We shall limit ourselves to functions of a non-negative variable with non-negative real values.

To each such function φ we associate a functional Φ as follows: Let α be approximated by a fraction $f(n) / 2^n$: $| \alpha - f(n) / 2^n | < (^1/_2)^n$. We find a similar approximation of $\beta_n = \varphi(f(n) / 2^n)$:

$$| \beta_n - g(n, m) / 2^m | < (^1/_2)^m$$

Thus $g(n, m)$ is the value of a functional $\Phi(f, n, m)$ whose type is $(((* : *), *, *) : *)$; this functional allows us to find an approximation of $\varphi(\alpha)$ for a given approximation of α. Most definitions of computable real functions make use of this functional Φ.

Banach and Mazur [141] called a function φ computable if it carries any computable sequence α_n again into a computable sequence. The corresponding functional Φ carries then each function $\lambda x\, f(n, x)$ (where f is a computable function) into a computable function.

Functionals with this property are called Banach—Mazur functionals. Such a functional remains a Banach-Mazur functional after arbitrary changes of its values at non-computable arguments. Hence there is no point to consider the Banach—Mazur functionals for non-computable values of arguments.

The Banach—Mazur real functions are continuous at every computable point. Mazur established various other properties of these functions, *e.g.* the so called property of Darboux which says that if $\varphi(\alpha) < 0$, $\varphi(\beta) > 0$ where α and β are computable and $\alpha < \beta$, then there is a computable γ such that $\alpha < \gamma < \beta$ and $\varphi(\gamma) = 0$.

The class of Banach—Mazur functions and functionals is rather wide. Several narrower classes of functions were investigated by various authors. Thus *e.g.* Myhill and Shepherdson [155] and later

Kreisel—Lacombe—Shoenfield [116] considered partial recursive functionals. If $F_0, F_1, \ldots$ is a standard enumeration of partial computable functions and f a fixed partial computable function then the functional $\Phi(F_e) = f(e)$ is a partial recursive functional. Thus the domain of such a functional consists of partial recursive functions and the computation of its value consists of a computation performed on a number e which the argument has in the standard enumeration. We have to assume that f is chosen so that if $F_e = F_e'$ then $f(e) = f(e')$. Partial recursive functionals give rise to a class of real functions, called partial recursive.

Another possibility is to admit partial computable functionals. These functionals correlate with each function f a partial function computable relative to f (*cf.* lecture VII). We shall call them the Kleene functionals. The range of the arguments of a full Kleene functional consists of all functions not only of computable ones. Hence the corresponding real functions are insofar different from the Banach—Mazur functions as they are defined everywhere and not only for computable values of the arguments.

Still another class of functionals was proposed by Grzegorczyk [66] who defined it as the smallest class closed under some operations among whom the operation of effective minimum was the most characteristic. Kleene [106] called these functionals μ-recursive but we prefer the name Grzegorczyk-functionals.

We have thus four notions of functionals and of real functions: Banach—Mazur functionals, partial recursive functionals, Kleene functionals and Grzegorczyk functionals. The mutual relations of these various classes were discussed in several papers.

Friedberg [45] showed that Banach—Mazur functionals form an essentially wider class than partial recursive functionals; this result is very deep. Myhill and Shepherdson [155] proved a much easier result that every partial recursive functional can be extended to a Kleene functional. Kreisel—Lacombe—Shoenfield [116] sharpened this result by showing that the same property is also possessed by functionals defined only on total computable functions.

From standard results on computable functions and functionals (*cf.* Kleene [104]) it follows that Grzegorczyk functionals coincide with total Kleene functionals.

It follows from these results that positive theorems (*i.e.* theorems stating that each functional has a property) valid for the Banach—Mazur class are true for all other classes. Thus all real functions corresponding to partial recursive, partial computable and μ-recursive functionals are continuous and possess the property of Darboux.

Grzegorczyk showed that his functionals and hence the corresponding real functions are uniformly continuous. Lacombe [120] (*cf.* also Specker [212]) showed that real functions of Grzegorczyk not always attain the maximum in a computable point. This negative result severely restricts the possibility of repeating classical proofs in the computable analysis. Such reconstructions were attempted in various papers, for instance in Klaua [96].

A very specific notion of a real function was proposed also by Brouwer. His definition makes use of terms and notions accessible only to intuitionists. Kleene [102] has shown that one obtains a persuasive interpretation if one explains these notions in terms of partial computable functionals. Under his interpretation free choice sequences of Brouwer are simply arbitrary sequences; Brouwer's "functions" are partial computable functionals. Assuming that this interpretation represents faithfully the intuitionistic notions we come to the conclusion that Brouwer's ideas on foundations of analysis were pretty far from constructive ideas in the orthodox constructivism which does not accept arbitrary sequences of integers.

Another branch of computable mathematics is the theory of recursive equivalence of sets of integers created and developed by Dekker and Myhill [27]. This theory examines notions obtained from set-theoretical ones by replacing arbitrary sets by sets of integers and arbitrary mappings of sets by partial recursive ones.

Extensions of computable mathematics. In computable mathematics we reduce all notions to computable ones. Various authors examined other possibilities. Thus *e.g.* Grzegorczyk [67] studied a system which he called the "elementarily definable analysis". In this system all notions are reduced to such as can be defined in terms of integers and their first-order theory. It was Weyl who already in the early twenties developed such a theory for the first time, of course without using the much more modern notion of definability.

Another possibility is to use the class of hyper-arithmetic sets and functions although it is a debatable question whether a theory based on these notions can claim to be constructive.

Some results in hyper-arithmetic analysis were obtained by Kreisel [115] who investigated the possibility of proving in it an analogue of the classical Cantor—Bendixon theorem.

Whatever the mathematical interest of such theories may be it is certain that strictly finitistic theories are much more satisfactory from the philosophical point of view.

Strictly finitistic theories. By strictly finitistic we mean theories which limit not only the class of objects but also the class of admissible methods of proof. According to this terminology even the computable analysis as described above is not strictly finitistic since it operates with classical mathematical notions without restriction and takes no care which laws of logic are used. Even the notion of computable function is not unobjectionable from the strictly finitistic point of view because in all definitions of this notion occur some clauses which cannot be verified in a finite number of steps.

Strictly finitistic attitude was represented since long by Skolem who formulated the concept of recursive arithmetic. His idea was taken up by Goodstein in two books [63], [64] published in 1957 and 1961. The main idea of recursive arithmetic is to develop mathematics as a formal system which operates exclusively with equations. The number of functional constants is not limited, new constants being added either by explicit or inductive definitions. The rules of proof are just the rule of substitution (of terms for variables throughout a proven equation), the rule of "replacing equals by equals"

$$\frac{F = G}{A(F) = A(G)}$$

and a rule which says that $F = G$ whenever both F and G satisfy equations of a recursive definition.

There are no quantifiers in this theory nor are there propositional connectives. We do not assume, in recursive arithmetic the existence of a set of all integers. Also it is irrelevant for this system which kind of logic do we admit since no logical notions occur in it.

Several theorems of analysis can be proved in recursive arithmetic. This is true for theorems expressible by means of approximations of real numbers by rational numbers. For instance if f is a real function and if we can define a primitive recursive operation which from an approximation of x by means of a rational number produces a rational approximation of $f(x)$, then this operation can be taken as a definition of a sort of the function f. Goodstein succeeded to establish a series of theorems which are analogues in recursive arithmetic of the classical theorems of analysis.

The idea of recursive arithmetic found a rather unexpected application. Church [21] constructed a system very similar to this arithmetic in which arithmetical operations are replaced by Boolean ones and applied it to a description of electric circuits with retarding elements.

Another extremely consistent system of constructive mathematics was created by Markov and his collaborators. The basic notion to which all other notions are reduced by the representatives of this school is the notion of an algorithm. In Markov's school all definitions are expressed in everyday language and all references to actual infinity are strictly avoided. Although Markov and his followers consciously refrain from formulating the logic which they admit, it is clear that they accept the intuitionistic logic. Thus we see that there are essential differences between Markov's conceptions and recursive arithmetic: the former school accepts all algorithms, the latter only those which correspond to primitive recursive functions; the former uses (informally) the intuitionistic logic, the latter avoids using logic altogether.

Real numbers and their sequences are defined in Markov's theory as in computable analysis, the only difference being that numbers are replaced everywhere by algorithms which define their successive approximations. One consequence of this is that the relation of identity (for real numbers) is no more decidable, since there is no algorithm which would allow us to decide whether any two given algorithms define approximations of one and the same real number.

The notion of a real function is again defined by Markov with the help of algorithms. He identifies a real function with an algorithm which correlates with each algorithm A another algorithm A' in such a way that if A and A_1 define two approximations to one and the same real number then so do A' and A_1'. We thus see that Markov chose for his real functions the notion equivalent to that of a partial computable functional.

Markov's theory is exposed in [139]. His work was continued by Šanin [192] who investigated analogues of various classical theories in the constructive mathematics of Markov.

He was able to develop even as advanced parts of analysis as the theories of Hilbert space and Lebesgue integral. Because of constructivistic limitations these theories do not behave as their classical models and are usually much less elegant. Šanin [191] found for instance that in the constructive theory of the Lebesgue integral it is not permissible in general to interchange the operation of integration with that of a passage to a limit. Yet it was precisely to obtain this theorem that Lebesgue formulated his definition of the integral!

Personally I do not believe that it is worth-while to reconstruct classical theories in constructive terms. No particularly interesting results have been obtained, and hardly anybody believes that the cumbersome theories obtained in this way will really replace the elegant classical theories. I am inclined to believe that there are branches of mathematics which simply are not susceptible to finitistic treatment.

There are of course branches of mathematics which can be treated in a finitistic way. Abstract algebra is an excellent example of such a domain. Methods and results of the recursive function theory can lead and have in fact led to many important and interesting results in this theory. We are often dealing in algebra with problems of pronounced algorithmic character; for instance, all questions concerning elementary transformations of polynomials taught in school belong to this group. Van der Waerden [240] and in a broader context Shepherdson and Fröhlich [47] discussed the problem which questions of the elementary theory of fields can be answered by using algorithms. To this end they considered fields whose operations are defined by means of computable functions. Thus they used the same device which is constantly used in computable arithmetic and analysis though with a completely different aim in mind. Further work along the same lines was also done by Rabin [168] and the general setting of the problem was given by Malcev [136]. Malcev did not limit himself to special systems like fields but considered arbitrary abstract algebras and the numerations of their elements such as to represent the basic operations of the algebra by means of computable functions. Although the aims which these authors pursue are incomparably more modest than the reconstruction of mathematics in the finitistic theories it is probable that the results of their works will last longer than the more ambitious but less fruitful conceptions of the finitistic school.

Lecture XII

Decision problems

The decision problem as formulated by Hilbert consists in finding criteria which would allow us to check in a finite number of steps whether any given formula of the first-order logic is or is not provable. In the period 1930—1964 this general problem was given an essentially negative answer, *i.e.* it was shown that no such criteria exist. Several partial problems nevertheless admit positive solutions; these positive solutions have found various applications.

Positive results. Let us first discuss a type of problem which was formulated already in Hilbert's school. We consider the semantically defined property of satisfiability of first-order formulae in some domain and ask whether there are criteria which allow us to decide effectively when a formula has this property. It has been shown that such criteria exist for certain classes of formulae.

Let F be a formula in which exactly one predicate P occur. Let us assume that P is binary and that F has the form $\bigvee_x \bigwedge_y M(x, y)$, where M contains no quantifiers, and let H_n be its n-th Herbrand disjunction:

$$H_n : M(x_1, x_2) \vee M(x_2, x_3) \vee \ldots \vee M(x_{n-1}, x_n).$$

We know from lecture V that F is provable if and only if there is an n such that H_n is provable.

This is evidently the case when H_2 is provable in the propositional calculus. We shall show that if H_2 is not provable in the propositional calculus, then no H_n can be provable in it. Indeed, there are the following atomic formulae in $M(x_1, x_2)$:

(i) $$P(x_1, x_1),\ P(x_1, x_2),\ P(x_2, x_1),\ P(x_2, x_2)$$

and hence the following formulae in $M(x_p, x_{p+1})$:

(ii) $P(x_p, x_p), P(x_p, x_{p+1}), P(x_{p+1}, x_p), P(x_{p+1}, x_{p+1})$.

According to our assumption we can assign truth-values t_{11}, t_{12}, t_{21}, t_{22} to the atomic formulae (i) in such a way that M becomes false. Two cases are now possible:

(A) the truth-values t_{ij} can be chosen so that $t_{11} = t_{22}$. In this case we can assign the truth-values t_{11}, t_{12}, t_{21}, t_{22} to the atomic formulae (ii) for $p = 2, 3, \ldots$. These assignments are consistent with each other and give the truth-value "false" to $M(x_p, x_{p+1})$. Hence H_n is not a theorem.

(B) $t_{11} \neq t_{22}$ for every assignment of truth-values t_{ij} to the formulae (i) which make $M(x_1, x_2)$ false. In this case we use the assumption that H_2 is not a theorem and infer that there is an assignment of truth-values t_{ij} to the atomic formulae

(iii) $P(x_1, x_1), P(x_1, x_2), P(x_2, x_1), P(x_2, x_2), P(x_2, x_3),$
$P(x_3, x_2), P(x_3, x_3)$

which makes H_2 false. In this assignment $t_{11} = t_{33}$. Otherwise we would have $t_{22} = t_{33}$ (since $t_{11} \neq t_{22}$ by the definition of Case B), and by substituting x_1 for x_2 and x_2 for x_3 we would obtain an assignment t_{22}, t_{23}, t_{32}, t_{33} of truth-values to the atomic formulae (i) which makes $M(x_1, x_2)$ false and has the property that $P(x_1, x_1)$ and $P(x_2, x_2)$ are assigned the same truth-value. This would contradict the definition of Case B. Hence $t_{11} = t_{33}$, and we see that we can assign to the atomic formulae

$$P(x_{2p+1}, x_{2p+1}), P(x_{2p+1}, x_{2p+2}), P(x_{2p+2}, x_{2p+1}), P(x_{2p+2}, x_{2p+2}),$$
$$P(x_{2p+2}, x_{2p+3}), P(x_{2p+3}, x_{2p+2}), P(x_{2p+3}, x_{2p+3})$$

the same truth-values as to the formulae (iii) and that these assignments are consistent. Thus H_n is provable for no n. In this way we obtain a solution of the decision problem for the class of formulae we are considering. This solution was first given by Bernays and Schoenfinkel in 1928.

Similar combinatorial arguments can be applied to more complicated classes of formulae. The strongest result in this direction is due to Gödel [55] who solved the decision problem for first-order formulae of the form $\bigwedge_{x_1 \ldots x_k} \bigvee_{yz} \bigwedge_{t_1 \ldots t_l} M$, where M has no quantifiers.

Another rather general result is due to Herbrand [79] who solved the decision problem for formulae with an arbitrary arrangement of initial quantifiers followed by a matrix in which the only connectives are disjunction and negation.

Herbrand's theorem discussed in lecture V provides a unifying principle for proofs of this sort.

A different type of a decision problem originated with Skolem [208]; it was developed mainly by Tarski in a number of papers of which the most important is [226]. The general character of these problems can be described as follows: Let us consider an axiomatic theory $\boldsymbol{T}$ based on the first-order logic. We ask for criteria for a given formula F to be provable in $\boldsymbol{T}$. This is the decision problem for the theory $\boldsymbol{T}$. If the general decision problem were solvable, then so would be the decision problem for every finitely axiomatizable theory $\boldsymbol{T}$. Indeed, if $A_1, \ldots, A_n$ are all the axioms of $\boldsymbol{T}$, then F is provable in $\boldsymbol{T}$ if and only if the formula $A_1 \wedge \ldots \wedge A_n \rightarrow F$ is provable in (pure) logic. Hence the decision problem for $\boldsymbol{T}$ is reduced to that for logic. This remark, interesting though it is, does not help us very much since the decision problem for logic is not solvable; moreover, the implication formulae just mentioned are not usually reducible to any of the forms for which the decision problem has been solved.

The method devised by Skolem and developed by Tarski is called the method of elimination. The scheme of this method is as follows: Let $\boldsymbol{T}$ be a first-order theory and

$$(1) \qquad P_i,\ P_i'(x),\ P_i''(x, y),\ P_i'''(x, y, z), \ldots \ (i = 0, 1, \ldots)$$

a sequence of formulae with 0,1, 2, . . . free variables. The number of these formulae may be finite or infinite. Let us assume that (i) each formula W of $\boldsymbol{T}$ is equivalent to a Boolean combination C of formulae (1) and that C can be found effectively for any given W. Let us assume, furthermore, that (ii) if W has at most n free variables, then C is built from those formulae (1) that have at most n free variables. Under these assumptions each W without free variables is equivalent to a Boolean combination of the formulae P_1, $P_2, \ldots$. Hence if we can decide when a Boolean combination of these formulae is provable in $\boldsymbol{T}$ we can decide when an arbitrary formula W is provable.

Assumptions (i) and (ii) are satisfied if formulae (1) satisfy the following conditions: (iii) each atomic formula of $\boldsymbol{T}$ occurs among the formulae (1); (iv) if K is a conjunction of formulae (1) and of their negations, then the formula $W = \bigvee\limits_{x} K$ is equivalent to a Boolean combination of formulae (1) built from those formulae whose free variables occur in W.

The reduction of conditions (i) and (ii) to (iii), (iv) is based on a simple reduction of formulae to a standard form known as the prenex normal form.

The essential property of formulae (1) is (iv). It can be expressed by saying that a necessary and sufficient condition for the existence of an x satisfying K is expressible as a Boolean combination of formulae (1). The name "method of elimination" is borrowed from algebra where we often eliminate an unknown and express by certain equations and inequalities the necessary and sufficient condition for the solvability of an equation.

Let us illustrate the method on a simple example. Let us take as $\boldsymbol{T}$ the theory whose unique non-logical primitive notion is a binary relation R (the identity relation is treated as a logical notion). The axioms of the theory state that the universe is ordered by R and that every element has a predecessor and a successor:

$$Rxx,\ Rxy \wedge Ryx \rightarrow x = y,\ Rxy \wedge Ryz \rightarrow Rxz,$$

$$Rxy \vee x = y \vee Ryx,$$

$$\bigwedge_{x} \bigvee_{yz} (Rxy \wedge Rzx \wedge \bigwedge_{s} \{[Rxs \wedge x \neq s \rightarrow Rys] \wedge [Rsx \wedge s \neq x \rightarrow Rsz]\}.$$

We take as P_0 the formula $\bigwedge\limits_{x} (x = x)$ and as $P_n(x, y)$ the formula

$$Rxy \wedge \bigvee_{x_1 \ldots x_{n+2}} \bigwedge_{0<i<j\leq n+2} [(x_i \neq x_j) \wedge Rx_ix_j \wedge (x = x_1) \wedge (x_{n+2} = y)]$$

$$(n = 0, 1, 2, \ldots).$$

The formula $P_n(x,y)$ says that x precedes y and that there are at least n elements between x and y. It can be shown without much trouble that conditions (iii) and (iv) are satisfied in this example. Hence for each formula W without free variables we can find an equivalent Boolean combination of P_0 alone, whence it follows that the decision problem for $\boldsymbol{T}$ is solvable.

The elimination method was successfully used to solve the decision problem for various theories. The strongest result is due to Tarski [226] who established the decidability of the theory of real closed fields. Axioms of this theory consist of two groups. Axioms of the first group state that the universe of the theory is an ordered field; the number of these axioms is finite. Axioms of the second group form an infinite sequence and state that each equation of degree 3, 5, 7, . . . has at least one root. (If we could define in the theory the general notion of a polynomial of an odd degree, we could replace this infinite sequence of axioms by a single sentence; however, this general notion is not definable.)

Another important example of a decidable theory is the theory of Abelian groups whose decidability has been proved by Szmielew [220]. This theory is of course incomplete; one can even show that it admits $2^{\aleph_0}$ complete extensions. The elimination method is, as we see, applicable to essentially different kinds of theories.

At the present moment the applications of the elimination method seem to be exhausted. With the exception of relatively simple cases familiar in the existing literature, the method leads to forbidding calculations which can hardly be undertaken by anybody. In the last few years new methods have appeared which have made it possible to solve the decision problem for several theories.

One of these new methods rests on the simple remark that complete theories based on a recursively enumerable set of axioms are always decidable. Since various methods of establishing the completeness of theories are known at present, we can in this way obtain solutions of the decision problem for complete theories. We shall say more about such proofs in lecture XIII.

Büchi [10], [11] used the theory of finite automata to obtain a solution of the decision problem of some fragments of the second-order arithmetic. The constants of this theory are 0 (zero) and (successor); there are two types of variables: lower case variables for integers and upper case variables for sets of integers.[1] According to whether we admit arbitrary sets or only finite sets as values of the set variables we distinguish the strong and the weak second-order arithmetic.

Both the weak and the strong second-order arithmetic are interpreted systems. The notions of truth, definability *etc.* in these systems are thus understood in the semantical sense.

In what follows we shall give an account of the work of Büchi concerned with the weak theory.

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[1] We shall often identify a set with its characteristic function.

Büchi showed that each formula of the weak second-order arithmetic (without free variables) is equivalent to a formula of the form

$$(2)\quad \bigvee_{Y_1,\ldots,Y_n} [K(Y_1(0),\ldots, Y_n(0)) \wedge \bigwedge_t B(Y_1(t),\ldots, Y_n(t), Y_1(t'),\ldots, Y_n(t'))]$$

where K and B contain but propositional connectives.

The truth of formula (2) can easily be checked. Let us assume for instance that $n = 1$ and that $B(\mathfrak{F}, \mathfrak{F})$ is true (where $\mathfrak{F}$ is the truth-value "false"). Since Y_1 has to be a finite set, formula (2) is true if and only if there exists a finite sequence $c_0, c_1, \ldots, c_{p-1} = \mathfrak{F}$ of truth-values such that $K(c_0)$ and $B(c_j, c_{j+1})$ are true for each $j < p$. The terms of this sequence are simply the truth-values of $Y_1(j)$, and p is the least integer such that no $q \geq p - 1$ is an element of Y_1. If there are two identical consecutive terms in the sequence $c_0, \ldots, c_{p-1}$ we can drop one of them without altering the properties of the sequence. Furthermore, the sequence $\mathfrak{F}\mathfrak{V}\mathfrak{F}\mathfrak{V}$ can be replaced by $\mathfrak{F}\mathfrak{V}$ and the sequence $\mathfrak{V}\mathfrak{F}\mathfrak{V}\mathfrak{F}$ by $\mathfrak{V}\mathfrak{F}$. Thus we see that it is sufficient to check whether the sequences $\mathfrak{F}$, $\mathfrak{F}\mathfrak{V}\mathfrak{F}$, $\mathfrak{V}\mathfrak{F}$ satisfy the conditions imposed on $c_0, \ldots, c_{p-1}$ and this can obviously be done in a finite number of steps.

The reduction to the form (2) is far from obvious. Büchi obtained it by using certain concepts from the theory of finite automata. Let us define this notion:

A finite automaton is determined by (i) its initial configuration, (ii) its transition functions, and (iii) its output function. The initial configuration is a string $E_1, \ldots, E_m$ of truth-values. The transition functions are propositional formulae $H_j(p_1, \ldots, p_m;\ q_1, \ldots, q_n)$ where $j = 1, 2, \ldots, m$. The output function is a propositional formula $U(p_1, \ldots, p_m)$. The functioning of an automaton can be described as follows: We first fix arbitrarily the values of the parameters $q_1, \ldots, q_n$ in the transition functions by giving them values $X_1, \ldots, X_n$, where each X_s is a function of t ultimately equal $\mathfrak{F}$. At each moment t the automaton is in a "stage" described by a string $r_1(t), \ldots, r_m(t)$ of truth-values. These "stage functions" are defined by induction:

$$r_k(0) = E_k,\quad r_k(t') = H_k(r_1(t), \ldots, r_m(t),\ X_1(t), \ldots, X_n(t)).$$

The output of the automaton at the moment t is defined as $u(t) = U(r_1(t), \ldots, r_m(t))$.

Thus an automaton presents us (for every choice of the parameters $X_1, \ldots, X_n$) with an infinite sequence $u(t)$ of truth-values. The set of values of the parameters for which the terms of this sequence are ultimately $\mathfrak{V}$ is called by Büchi the "behaviour" of the automaton. The main result of Büchi obtained by analyzing the formulae of the weak second-order arithmetic is that each set S of n-tuples $X_1, \ldots, X_n$ (where each X_s is a function of t ultimately equal $\mathfrak{F}$) which is definable in the weak second-order arithmetic is the behaviour of a finite automaton, and conversely.

Since the behaviour of an automaton can be described by a formula of form (2), the desired reduction of formulae to form (2) follows.

The principle of Büchi's result for the strong second-order arithmetic is similar, but the reduction to form (2) is much more involved, chiefly because we cannot assume that the functions X_s are ultimately equal $\mathfrak{F}$.

Let us mention here that the recursive arithmetic of Church [21] coincides with the formal theory of finite automata in Büchi's sense.

Negative results. The basic method used in proofs of undecidability is the reduction of the decision problem for a class K to the decision problem of another class K_0 for which the solution of the decision problem is known to be negative. It is obvious that if the characteristic function of K_0 is computable relative to the characteristic function of K and if the former function is not computable, then the latter is not computable, either. Hence the set K is not computable. Identifying the (intuitive) notion of decidability with the (formal) notion of computability, we obtain in this way a negative solution of the decision problem for the set K. In lectures IV and VII we gave several examples of sets whose characteristic functions are not computable. By means of the reduction procedure it is possible to obtain various proofs of undecidability. We shall mention a few of them.

The first result of this kind is due to Church [17] who proved the undecidability of the full predicate logic, thus solving Hilbert's original problem. Church's result showed at the same time the undecidability of various subclasses of the full predicate logic. It has been shown in a number of works, which started appearing well before 1930, that the decision problem for the set of all formulae of the predicate logic is reducible to the decision problem for various subclasses of this set, each consisting of formulae in the prenex normal form with certain simple prefixes. The best known example of such a subclass consists of formulae with the Skolem prefix $\bigvee_{x_1} \ldots \bigvee_{x_n} \bigwedge_{y_1} \ldots \bigwedge_{y_m} M$, where M has no quantifiers. It may indeed be shown quite easily that to each formula A of the predicate logic there is a formula B of the form $\bigwedge_{x_1} \ldots \bigwedge_{x_n} \bigvee_{y_1} \ldots \bigvee_{y_m} M$ such that B is satisfiable if and only if A is satisfiable (*cf.* Gödel [53]). Many other classes of formulae with the same property have been found; they are sometimes called "reduction types". An account of them can be found in Surányi [219].

The reduction to the Skolem type of formulae as well as most of the other reductions have been obtained by means of combinatorial methods: one expresses the fact that a given formula A has a model and tries to give to the resulting expression as simple a form as possible.

A reduction of a different kind was found recently by Büchi [12] and somewhat later by Moore, Wang and Kahr [86]. Instead of expressing in a simple form the satisfiability of a formula A they used a reduction to problems connected with Turing machines (or, equivalently, with Markov's algorithms). Büchi showed that with each Turing machine one can correlate a formula $F = \bigwedge_{x} A(x) \wedge \bigwedge_{x} \bigvee_{y} \bigwedge_{z} B(x, y, z)$ such that F is satisfiable if and only if the corresponding machine ultimately stops. It follows that the class of provable formulae $\bigvee_{x} P(x) \vee \bigvee_{x} \bigwedge_{y} \bigvee_{z} Q(x, y, z)$ (where P and Q do not contain quantifiers) is not computable. Moore, Wang and Kahr improved this result by showing that the class of true formulae $\bigvee_{x} \bigwedge_{y} \bigvee_{z} Q(x, y, z)$ is not computable. It is worth while to mention that the problem whether a formula of this form (or of Büchi's form above) is satisfiable is algorithmically decidable (*cf.* lecture V).

The prefix problem which we have so far considered is interesting in itself but seems rather artificial. Deeper problems arise when we consider axiomatic theories and ask the question whether the sets of their theorems are computable or not.

The first result of this kind is due to Church who proved (in effect) that the set of theorems provable in Peano's arithmetic is not computable. Let us sketch a modern proof of this result.

Let $\boldsymbol{T}$ be a theory satisfying the following conditions:

1°. There is an infinite sequence of symbols $\Delta_0, \Delta_1, \ldots$ such that each formula of the form $\neg (\Delta_i = \Delta_j)$ is provable in $\boldsymbol{T}$ for $i \neq j$.

2°. Each primitive recursive relation is strongly representable in $\boldsymbol{T}$.

3°. There is a formula $M(x, y)$ which strongly represents the relation $\leq$ in $\boldsymbol{T}$ and has the property that the formulae $M(x, y) \vee M(y, x)$ and $M(x, \Delta_n) \equiv [(x = \Delta_0) \vee \ldots \vee (x = \Delta_n)]$ are provable in $\boldsymbol{T}$ for each n.

Under these assumptions the set of provable formulae of $\boldsymbol{T}$ is not computable.

Indeed, let A and B be two recursively enumerable sets of integers which cannot be separated by computable sets (see p. 66). Since they are recursively enumerable, they can be defined in the form $A = \{n : \bigvee_p R(n, p)\}$, $B = \{m : \bigvee_q S(m, q)\}$ where R and S are primitive recursive relations. Let F and G be formulae which strongly represent R and S in $\boldsymbol{T}$. Then the formulae

$$F' : \bigvee_y \{F(x, y) \wedge \bigwedge_z [M(z, y) \rightarrow \neg G(x, z)]\},$$

$$G' : \bigvee_y \{[G(x, y) \wedge \bigwedge_z [M(z, y) \rightarrow \neg F(x, z)]\}$$

weakly represent A and B and satisfy the condition that the formula $\neg [F'(x) \vee G'(x)]$ is provable in $\boldsymbol{T}$.

Let us now assume that the set of provable formulae of $\boldsymbol{T}$ is computable and let $C = \{n : F'(\Delta_n) \text{ is provable in } \boldsymbol{T}\}$. Hence C is computable, $A \subset C$ and $C \cap B = 0$, which contradicts the assumption of the inseparability of A and B.

The idea of this proof is due to Rosser [187].

As we see we proved not only the undecidability of $\boldsymbol{T}$ but also the undecidability of an arbitrary consistent extension of $\boldsymbol{T}$. Theories each consistent extension of which is undecidable are called essentially undecidable.[1] Many examples of such theories are known. An example of a finitely axiomatizable and essentially undecidable theory was given on p. 22. Other examples can be found in [231].

Putnam [162] has shown the essential undecidability of an arbitrary theory $\boldsymbol{T}$ based on a recursively enumerable set of axioms in which every computable set is strongly representable. In order to obtain this result he uses the Gödel substitution function and constructs a formula F such that F is provable in $\boldsymbol{T}$ if and only if its Gödel number n belongs to a preassigned set strongly representable in $\boldsymbol{T}$. It follows at once that this set cannot coincide with the set of Gödel numbers of the theorems of $\boldsymbol{T}$; thus the latter set is not computable.

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[1] The notion of essential undecidability coincides with that of essential incompleteness introduced on p. 22.

For further discussion of the question which theories are essentially undecidable, see Shoenfield [206]. It is proved in his paper that weak representability of all computable sets in $\boldsymbol{T}$ does not entail the essential undecidability of $\boldsymbol{T}$.

Theories which are essentially undecidable and finitely axiomatizable can be used to establish undecidability of various theories. This method was devised by Tarski [231] who formulated the following test: If τ is an essentially undecidable and finitely axiomatizable theory, if $\boldsymbol{T}$ is a consistent theory and if $\boldsymbol{T}$ and τ have a common consistent extension, then $\boldsymbol{T}$ is undecidable. Using this test Tarski proved, among other things, the undecidability of the theory of groups and of the theory of lattices. It is remarkable that surprisingly weak theories prove to be undecidable; for instance, the theory whose primitive terms are two binary relations and whose axioms state that these relations are reflexive, symmetric, and transitive is undecidable. (If only one equivalence relation is considered, the theory is decidable.) This theory was discussed by Janiczak [84] and H. Rogers [184].

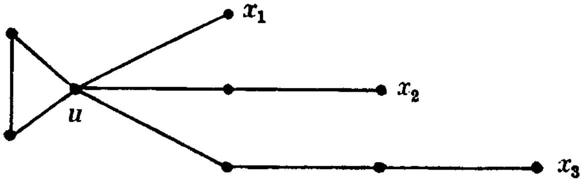
Another highly interesting method was found not long ago by Rabin [170] who showed that one can in many cases dispense with the use of essentially undecidable theories in proofs of undecidability.

Let $\boldsymbol{T}$ and $\boldsymbol{T}_1$ be two theories whose primitive terms are R and R_1; we assume that R and R_1 denote binary relations. Let $\boldsymbol{T}_1$ be undecidable. We require that $\boldsymbol{T}_1$ be interpretable in $\boldsymbol{T}$ in the following sense: There are formulae $D(R, x)$ and $A(R, x, y)$ of $\boldsymbol{T}$ such that (i) all axioms of $\boldsymbol{T}_1$ go over into theorems of $\boldsymbol{T}$ if the universe of $\boldsymbol{T}_1$ is interpreted as the set of elements satisfying $D(R, x)$ and R_1 as the relation defined by $A(R, x, y)$; (ii) every model $\boldsymbol{M}_1$ of $\boldsymbol{T}_1$ can be obtained from a suitable model $\boldsymbol{M}$ of $\boldsymbol{T}$ by taking as the universe of $\boldsymbol{M}_1$ the set $\{a : \models_{\boldsymbol{M}} D[R_M, a]\}$ and as the interpretation of R_1 the relation $\{<a, b> : \models_{\boldsymbol{M}} A[R_M, a, b]\}$. If these assumptions are satisfied, then $\boldsymbol{T}$ is undecidable.

In order to see this, let us denote, for any formula $F_1(R_1)$ of $\boldsymbol{T}_1$, by $F(R)$ the formula of $\boldsymbol{T}$ obtained from F_1 by the process described in (i). Since it can be decided, for each formula of $\boldsymbol{T}$, whether it does or does not correspond to a formula of $\boldsymbol{T}_1$, it suffices to show that F is provable in $\boldsymbol{T}$ if and only if F_1 is provable in $\boldsymbol{T}_1$. In one direction this follows from (i). Now assume that F_1 is not provable in $\boldsymbol{T}_1$; by completeness theorem there is then a model $\boldsymbol{M}_1$ of $\boldsymbol{T}_1$ in which F_1 is false. Using (ii), we infer that there is a model $\boldsymbol{M}$ of $\boldsymbol{T}$

in which F is false; whence Rabin's result follows. Obviously the assumption that R and R_1 are binary is not essential in this proof.

The following example (due to Rabin) will illustrate his method. Let $\boldsymbol{T}_1$ be the theory whose unique primitive term is a k-ary relation R_1 and $\boldsymbol{T}$ the theory whose only primitive term is a binary relation R. No extra-logical axioms are assumed in either theory. Take as D the formula $\neg \bigvee_{y} (xRy)$ and as $A(R, x_1, \ldots, x_k)$ the formula $\bigvee_{u} [(uR^k u) \wedge (uR^1 x_1) \wedge (uR^2 x_2) \wedge \ldots \wedge (uR^k x_k)$, where uR^1x means the same as uRx and where $uR^n x$ is defined (by induction) as $\bigvee_{v} [(uRv) \wedge (vR^{n-1}x)]$. *E.g.* if $k = 3$, the formula $A(R, x_1, x_2, x_3)$ is true just in case when the diagram of R looks as follows:



In this diagram points other than x_1, x_2, x_2 denote elements not satisfying the formula D.

This simple method of Rabin's is in many cases surprisingly efficient.

Slightly different from the problem of decidability of theories is the problem of decidability of models. For every $\boldsymbol{M}$ we may consider the set $T(\boldsymbol{M})$ of those formulae which are true in $\boldsymbol{M}$ and ask whether this set is computable. This problem is obviously equivalent with the decision problem for a theory $\boldsymbol{T}$ whose axioms are all the formulae of $T(\boldsymbol{M})$; thus we do not know *a priori* whether the axioms of $\boldsymbol{T}$ form a recursive set. For this reason not all methods mentioned in connection with the decision problem for theories can be applied to the decision problem of models.

In most cases we may establish the undecidability of a model $\boldsymbol{M}$ by showing that integers and the usual arithmetical operations on them are definable in $\boldsymbol{M}$; *cf.* Robinson [179] and J. Robinson [176]. These investigations often use rather deep results of purely mathematical character.

An interesting open problem is the decision problem for the model consisting of all the subsets of the real line and of the Boolean relations together with the additional relation: X is the closure of Y. This problem was formulated by Grzegorczyk [65] who established the undecidability of various models of similar character but was not able to solve the seemingly simplest case of a one dimensional space.

Many other decision problems are known in mathematics and especially in algebra. Let us mention the problem known as the word problem for semigroups: we consider a finite alphabet $\{a, b, \ldots, k\}$ and "words" in this alphabet, *i.e.* arbitrary finite (possibly void) sequences of these letters. Let $(S'_1, S''_1), \ldots, (S'_n, S''_n)$ be a finite list consisting of pairs of words. We call two words (S', S'') equivalent if S'' can be obtained from S' by a finite number of transformations each of which consists of a replacement in a given word w of a segment identical with S'_i by the word S''_i or conversely $(i = 1, 2, \ldots, n)$. Let E be the set of pairs (S', S'') such that S' is equivalent to S''. Is E a computable set? This is the word problem for semigroups. We can formulate a similar word problem for groups and other decision problems of algebraical character.

Markov [138] and Post [161] reduced the word problem for semigroups to the problem whether any given Turing machine will eventually stop. This yields a negative solution to the word problem. The word problem for groups, which is much more difficult, was solved by Novikov [158]; simpler solutions were found by Boone [9] and other algebraists. We mention these results only briefly since in spite of their importance for algebra they have been used neither in logic nor in the study of foundations of mathematics. It should be stressed, however, that the theory of computable functions created by logicians in order to discuss philosophical and metamathematical problems proved decisive during the early phases of the study of algebraical decision problems. It can safely be said that these algebraical problems would have remained unsettled had not logicians developed the theory of decision problems for uninterpreted and syntactically described logical calculi.

Lecture XIII

The theory of models

The modern form of semantics is the theory of models. Some of its results are quite old. For instance, the Skolem—Löwenheim theorem dating back to 1917 is of basic importance for this theory. The systematic development of model theory was initiated by Tarski in the early fifties [227]. His ideas proved so fruitful that model theory is at present one of the most important parts of the foundational study. The theory has also close ties with abstract algebra, and it has found numerous applications.

The abstract scheme of model theory is as follows: We are given a language L and a class C of objects called models (or realizations) of sentences of L. There is also given a relation between sentences and models which we shall call the satisfaction relation and express by words as follows: "Model $\boldsymbol{M}$ satisfies the sentence F" (or "The sentence F is true in $\boldsymbol{M}$"). We do not assume that sentences are necessarily finite sequences of letters; they may be abstract objects of quite arbitrary character. The language is determined not by the nature of these objects but by the operations performable on them. Thus in various applications we consider languages in which the class of sentences is not denumerable or in which we are allowed to form infinite conjunctions or disjunctions or in which each sentence is an infinite (perhaps even transfinite) sequence of symbols. Also the nature of models is quite arbitrary. In most cases they consist of a set (which we call the universe of the model) and a sequence of relations which correspond to the predicates of the language. This case is by far not the only possible one, however. The great flexibility of model theory is largely due to this freedom in the choice of the language and of the class of models.

We shall write $\boldsymbol{M} \in E(F)$ or $\models_{\boldsymbol{M}} F$ if the sentence F is true in the model $\boldsymbol{M}$. For a set X of sentences we write $\boldsymbol{M} \in E(X)$ if each formula F in X is true in $\boldsymbol{M}$. $\boldsymbol{M}$ is then called a model of X. If X is the set of axioms of a theory, then we also say that $\boldsymbol{M}$ is a model of this theory. We say that a sentence F is a consequence of X if $E(X) \subset E(F)$; the set of all the consequences of X is denoted by $Cn(X)$. The operation Cn has the well known properties of a closure operation: $X \subset Cn(X)$ and $Cn(Cn(X)) = Cn(X)$.

The theory of models is best known in the case in which L is the first-order predicate logic with identity. Sentences of L are then first-order formulae without free variables. This case will be treated in the present lecture. We shall impose no limitations on the number of predicates in L, however, and we shall assume that there are arbitrarily many individual constants in L.

A model is defined as a sequence consisting of a set A (the universe of the model) and of a family of relations in A and of elements of A. This family is indexed by the predicates and the individual constants of L. In this way a relation between the elements of A is correlated with each predicate P of L and an element of A is correlated with each constant of L. We assume that the number of arguments of the relation correlated with P is the same as the number of arguments of P. The satisfaction relation is defined in the usual way explained in lecture III.

In order to simplify our terminology we shall sometimes identify a model with its universe. In this sense we speak of the cardinal number of a model or of an object being an element of the model.

Three relations between models are of importance for us:

1. We say that a model $\boldsymbol{M}_1$ is a submodel of $\boldsymbol{M}_2$ or that $\boldsymbol{M}_2$ is an extension of $\boldsymbol{M}_1$ (in symbols $\boldsymbol{M}_1 \subset \boldsymbol{M}_2$) if (i) the universe A_1 of $\boldsymbol{M}_1$ is a subset of the universe A_2 of $\boldsymbol{M}_2$; (ii) the relations of $\boldsymbol{M}_1$ are obtained from those of $\boldsymbol{M}_2$ by restricting them to A_1; (iii) the interpretations of individual constants are the same in both models.

2. A model $\boldsymbol{M}_1$ is an elementary submodel of $\boldsymbol{M}_2$ (or $\boldsymbol{M}_2$ is an elementary extension of $\boldsymbol{M}_1$, in symbols $\boldsymbol{M}_1 < \boldsymbol{M}_2$) if $\boldsymbol{M}_1 \subset \boldsymbol{M}_2$ and the following condition is satisfied: whenever F is a formula and $a, b, \ldots$ are elements of $\boldsymbol{M}_1$ then the conditions $\models_{\boldsymbol{M}_1} F[a, b, \ldots]$ and $\models_{\boldsymbol{M}_2} F[a, b, \ldots]$ are equivalent.

3. Two models $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are elementarily equivalent (in symbols $\boldsymbol{M}_1 \equiv \boldsymbol{M}_2$) if $\boldsymbol{M}_1 \in E(F) \equiv \boldsymbol{M}_2 \in E(F)$ for each sentence F.

Note that these three relations are meaningful not only in the case when L is the first-order logic but in all cases when we have defined the relation of satisfaction. In order for definitions 1 and 3 to be meaningful it is not even necessary to have defined the relation of satisfaction for formulae with free variables.

The relation $\boldsymbol{M}_1 < \boldsymbol{M}_2$ evidently implies $\boldsymbol{M}_1 \subset \boldsymbol{M}_2$ but not conversely. A connection between the notions of elementary extension and elementary equivalence has been established by Keisler [90]; see p. 130.

Relations 1 and 3 were introduced by Tarski [227], relation 2 by Tarski and Vaught [232].

We shall now discuss the Skolem—Löwenheim theorem.

Analyzing the original proof due to Skolem one obtains the following result (Tarski—Vaught [232]): If m is an infinite cardinal not less than the cardinal m_0 of the constants and predicates of L, and if $\boldsymbol{M}$ is a model of a power $k > m$, then for every cardinal p such that $k \geqq p \geqq m$ there is a model $\boldsymbol{M}_1$ of power p such that $\boldsymbol{M}_1 < \boldsymbol{M}$. This theorem is called the downward Skolem—Löwenheim theorem.

A completely different theorem called in the recent literature the upward Skolem—Löwenheim theorem was proved for the first time by Tarski in a note to [209] (*cf.* Tarski—Vaught [232]): If m is a cardinal satisfying the conditions of the previous theorem and $\boldsymbol{M}$ is a model of power m, then for every cardinal $q \geqq m$ there is a model $\boldsymbol{M}_2$ of cardinality q such that $\boldsymbol{M}_2 > \boldsymbol{M}$.

The upward Skolem—Löwenheim theorem results easily from the compactness theorem:

If $E(X_1) \neq 0$ for every finite subset of a set X, then $E(X) \neq 0$.

For a language L with at most denumerably many constants and predicates this theorem was first proved by Gödel [53] (*cf.* lecture V). The general result is due to Malcev [134] who also first applied this theorem to algebra. A modern proof of the compactness theorem will be sketched in lecture XVI.

The different Skolem—Löwenheim theorems belong to the most important results in the theory of models. We shall show how they can be applied in order to characterize what are known as the spectra of sets of sentences.

Let X be a set of sentences of L. We shall call the spectrum of X the class of cardinals m such that $E(X)$ contains models of power m. It follows from the Skolem—Löwenheim theorems that if the spectrum of X contains at least one cardinal $\geqq m_0$, then it contains all cardinals $\geqq m_0$.

The part of the spectrum consisting of cardinals $< m_0$ is not very well understood even in the simplest case when $m_0 = \aleph_0$. Let us call a set K of integers the finite part of the spectrum of X (or of a single sentence F) if K is the intersection of the spectrum of X (or of F) and of the set of all integers. Scholz [194] asked whether all computable sets are finite parts of the spectra of single formulae. This question has been answered in the negative by Asser [4]. The finite parts of spectra thus form a subclass of the class of computable sets, but the structure of this class is still unknown. For instance, one does not know whether the intersection of two sets in this class is also a member of the class. Some partial results on the finite parts of spectra were given in [151].

Rabin [166] has shown that the upward Skolem—Löwenheim theorem is in general not true if one drops the assumption that the power of $\boldsymbol{M}$ is not less than m_0. He gave a counter-example with $m_0 = 2^{\aleph_0}$. Rabin's paper illustrates the difficulties which we may expect in studying the initial parts of spectra.

Another simple but very fruitful application of Skolem—Löwenheim theorems is provided by new methods of completeness proofs. Since a complete theory with a recursively enumerable set of axioms is decidable, these methods often enable us to establish the decidability of a theory without the cumbersome calculations which are unavoidable in the elimination method discussed in lecture XII.

The first method to be discussed is based on the following lemma which immediately results from the definitions of the notions involved: A theory $\boldsymbol{T}$ is complete if and only if any two of its models are elementarily equivalent.

Let now $\boldsymbol{T}$ be a theory with at most denumerably many constants and predicates ($m_0 = \aleph_0$) and assume that there are no finite models $\boldsymbol{M}$ of $\boldsymbol{T}$. We shall say that $\boldsymbol{T}$ is categorical in power m (Łoś [126], Vaught [236]) if any two models of $\boldsymbol{T}$ with the cardinal number m are isomorphic. Using the Skolem—Löwenheim theorems and the lemma given above we obtain the theorem: If a theory $\boldsymbol{T}$ satisfying the assumptions given above is categorical in an infinite power m, then it is complete (Vaught [236]). The following examples show the efficacy of this theorem:

(1) According to a classical theorem due to Cantor any two linearly ordered sets which are denumerable, dense and have no first or last element are isomorphic. Let $\boldsymbol{T}_1$ be a theory with one binary predicate P based on axioms which state that the universe is densely ordered by P and has no first or last element. By Vaught's theorem $\boldsymbol{T}_1$ is complete and hence decidable. We call $\boldsymbol{T}_1$ the theory of dense ordering. Its decidability can be proved by the elimination method but this proof, while not very difficult, is incomparably more involved than the above semantical proof.

(2) In a similar way we show that the theory $\boldsymbol{T}_2$ of atomless Boolean algebras is complete.

(3) Let $\boldsymbol{T}_3$ be a theory with two ternary predicates S, P and axioms stating that for every two elements x, y of the universe there is exactly one $s = x + y$ such that $S(x, y, s)$ and exactly one $p = xy$ such that $P(x, y, p)$ and further that the universe is an algebraically closed field with characteristic zero.

Since one proves in algebra that any two algebraically closed fields with the same characteristic and the same power $m > \aleph_0$ are isomorphic, we obtain the result that the theory $\boldsymbol{T}_3$ is complete and hence decidable.

The same result is also true for the theory $\boldsymbol{T}_3(p)$ which we obtain from $\boldsymbol{T}_3$ by dropping the axioms concerning the characteristic 0 and assuming instead of them an axiom stating that the characteristic of the field is p.

In order to avoid possible misunderstandings we add a few more comments on the axioms of $\boldsymbol{T}_3$ and $\boldsymbol{T}_3(p)$. Both these theories are based on an infinite number of axioms. Indeed, in order to express in the language L that a field is algebraically closed we must admit axioms which state that all quadratic equations, all equations of degree 3, all equations of degree 4 *etc.* have roots. For each degree we have thus a separate axiom. The fact that the field has characteristic 0 is expressed by the axioms

$$(x + x = 0) \rightarrow (x = 0), (x + x + x) \rightarrow (x = 0), (x + x + x + x = 0) \rightarrow (x = 0), \ldots .$$

which again form an infinite sequence.

In the usual expositions of the field theory all these axioms would be replaced by a finite number of sentences. These sentences would however involve the notions of polynomial and integer and hence would not belong to the first-order language on which theories $\boldsymbol{T}_3$ and $\boldsymbol{T}_3(p)$ are based. A theory based on this less elementary language, while mathematically more convenient, is much less interesting for a logician. No completeness result holds for this less elementary theory.

(4) Kochen [110] has obtained by the method described above Tarski's result that the theory of addition and multiplication of real numbers is decidable.

In view of these applications it would be desirable to have criteria of the categoricity of theories in a given infinite power. No such criteria are known in the general case. For $m = \aleph_0$ the following beautiful result was found by Ryll—Nardzewski [188] and (somewhat later but independently) by Svenonius [219^a] and Engeler [32]: If a theory $\boldsymbol{T}$ with at most denumerably many predicates and constants is complete, then it is categorical in power $\aleph_0$ if and only if for each integer n the Boolean algebra of its formulae with at most n free variables is finite.

Partial results for the case $m > \aleph_0$ can be found in Vaught [237] and [239^a]; the latter paper contains a survey of all results obtained so far in the study of this problem as well as a complete bibliography.

Łoś asked in [126] whether a theory categorical in a power $m > \aleph_0$ is categorical in every non-denumerable power. This very difficult problem was solved (positively) by Morley [147].

The work on the notion of categoricity in a given power is still in progress.

We shall now discuss another semantical method of establishing completeness of theories.

Let $\boldsymbol{M}$ be a model with the universe A; for simplicity we assume that there is just one relation R in $\boldsymbol{M}$ and that it is a binary one. We adjoin to the language L constants for all the elements of A, and denote by t_a the constant denoting a. The set consisting of all formulae $P(t_a, t_b)$ where a and b are elements of A such that aRb, of all formulae $t_a \neq t_b$ such that a and b are different elements of A and of all formulae $\neg P(t_a, t_b)$ where a and b are elements of A such that it is not the case that $a R b$ is called the diagram of $\boldsymbol{M}$ and is denoted by $\boldsymbol{D}(\boldsymbol{M})$.

Let now $\boldsymbol{T}$ be a theory and X its set of axioms. We call $\boldsymbol{T}$ a model-complete theory if for every model $\boldsymbol{M} \in E(X)$ the set $X \cup \boldsymbol{D}(\boldsymbol{M})$ is complete. This notion was introduced by A. Robinson [180]. Before we show how to use it in proofs of completeness we give a few examples which show that the notions of completeness and of model-completeness are different from each other:

The theory $\boldsymbol{T}_1$ is complete and model-complete.

The theory of linear order of type $\omega^* + \omega$ (*cf.* page 110) is complete but not model-complete. Indeed, if $\boldsymbol{M}_0$ is a model of this theory consisting of integers (positive and negative) ordered by the relation $\leq$, then the sentence $\neg \bigvee_x [P(t_0, x) \wedge P(x, t_1) \wedge (x \neq t_0) \wedge (x \neq t_1)]$ is neither provable nor disprovable in $X \cup \boldsymbol{D}(\boldsymbol{M}_0)$.

The theory of algebraically closed fields obtained from the theory $\boldsymbol{T}_3$ by dropping all the axioms referring to the characteristic of the field is incomplete but model-complete. Indeed, the sentence $\bigwedge_{x,y} [S(x,x,y) \rightarrow S(y,y,y)]$ which states that the characteristic of the field is 2 is independent of the axioms. On the other hand if $\boldsymbol{M}$ is a model of the theory then the set $X \cup \boldsymbol{D}(\boldsymbol{M})$ is complete. If the characteristic of $\boldsymbol{M}$ is 0, then the set $X \cup \boldsymbol{D}(\boldsymbol{M})$ is equivalent to the axioms of $\boldsymbol{T}_3$; if the characteristic of $\boldsymbol{M}$ is p, then this set is equivalent to the set of axioms of $\boldsymbol{T}_3(p)$; hence the set is always complete.

Robinson [180] has established a necessary and sufficient condition for model-completeness of a theory which can be tested rather easily. The condition says that for every model $\boldsymbol{M}$ of $\boldsymbol{T}$, for every extension $\boldsymbol{M}'$ of $\boldsymbol{M}$ which is also a model of $\boldsymbol{T}$, and for every sentence B of the form $\bigvee_{x_1,\ldots,x_k} Y$ where Y is a conjunction of atomic formulae or of their negations, the conditions $\boldsymbol{M} \in E(B)$ and $\boldsymbol{M}' \in E(B)$ are equivalent. Using the expression of Robinson: existential sentences B are persistent, *i.e.* their validity in $\boldsymbol{M}$ is preserved by extensions of $\boldsymbol{M}$ to any larger model $\boldsymbol{M}'$ (under the assumption that the axioms X of $\boldsymbol{T}$ are true both in $\boldsymbol{M}$ and in $\boldsymbol{M}'$).

Let us sketch briefly a proof of Robinson's result.

We first assume that $\boldsymbol{T}$ is a model-complete theory. If an existential formula B is true in $\boldsymbol{M}$, then it is obviously true in an extension $\boldsymbol{M}'$ of $\boldsymbol{M}$. If B is false in $\boldsymbol{M}$, then by the completeness of $X \cup \boldsymbol{D}(\boldsymbol{M})$, the sentence $\neg B$ is a consequence of $X \cup \boldsymbol{D}(\boldsymbol{M})$, since the consequences of $X \cup \boldsymbol{D}(\boldsymbol{M})$ are true in $\boldsymbol{M}$. Since the diagram of $\boldsymbol{M}$ is contained in the diagram of $\boldsymbol{M}'$, we obtain that $\neg B$ is a consequence of $X \cup \boldsymbol{D}(\boldsymbol{M}')$ and hence that $\neg B$ is true in $\boldsymbol{M}'$, *i.e.* B is false in $\boldsymbol{M}$.

Now we assume that $\boldsymbol{T}$ is not model-complete. Let $\boldsymbol{M}$ be a model such that $\boldsymbol{M} \in E(X)$ and Z a sentence independent of $X \cup \boldsymbol{D}(\boldsymbol{M})$. We can assume that Z is in the prenex normal form beginning with an existential quantifier and that no sentence in such form with fewer quantifiers is independent of $X \cup \boldsymbol{D}(\boldsymbol{M}')$ for any model $\boldsymbol{M}'$ of $\boldsymbol{T}$. Put $Z = \bigvee_x Q(x)$.

Since $X \cup \boldsymbol{D}(\boldsymbol{M}) \cup \{Z\}$ is consistent there is an extension $\boldsymbol{M}'$ of $\boldsymbol{M}$ such that $\boldsymbol{M}' \in E\ (X \cup \{Z\})$. Hence $\boldsymbol{M}'$ contains an element a such that $Q(t_a)$ is true in $\boldsymbol{M}'$. Since $Q(t_a)$ has less quantifiers than Z we obtain $Q(t_a) \in Cn(X \cup \boldsymbol{D}(\boldsymbol{M}'))$ whence it results easily that $Q(t_a) \in Cn(X \cup \{Y(t_{a_1}, \ldots, t_{a_n})\})$ where Y is a conjunction of finitely many sentences in $\boldsymbol{D}(\boldsymbol{M}')$.

It follows easily that $\left[\bigvee_{x_1,\ldots,x_n} Y(x_1, \ldots, x_n) \rightarrow Z\right] \in Cn(X)$ and hence the sentence $B = \bigvee_{x_1,\ldots,x_n} Y(x_1, \ldots, x_n)$ is not provable from $X \cup \boldsymbol{D}(\boldsymbol{M})$. It follows that B is false in $\boldsymbol{D}(\boldsymbol{M})$. On the other hand B is true in $\boldsymbol{M}'$. Thus the incompleteness of $X \cup \boldsymbol{D}(\boldsymbol{M})$ contradicts the Robinson's condition.

We shall now show how Robinson uses his theorem in the study of completeness. Let us call $\boldsymbol{M}$ a prime model for a theory $\boldsymbol{T}$ if every model $\boldsymbol{M}'$ in which the axioms of $\boldsymbol{T}$ are true contains a sub-model of $\boldsymbol{T}$ isomorphic with $\boldsymbol{M}$. Robinson's main theorem states:

If a model-complete theory $\boldsymbol{T}$ admits a prime model $\boldsymbol{M}$, then it is complete.

Indeed, we shall show that theorems of $\boldsymbol{T}$ coincide with sentences true in $\boldsymbol{M}$. It is sufficient to show that if a sentence Z is true in $\boldsymbol{M}$, it is provable. Otherwise there would exist an extension $\boldsymbol{M}'$ of $\boldsymbol{M}$ such that $\boldsymbol{M}' \in E(X)$ and Z would be false in $\boldsymbol{M}'$. Hence, by the model-completeness $\neg Z$ would be a consequence of $X \cup \boldsymbol{D}(\boldsymbol{M}')$. But this is clearly impossible since Z is a consequence of $X \cup \boldsymbol{D}(\boldsymbol{M})$ and $\boldsymbol{D}(\boldsymbol{M})$ is a subset of $\boldsymbol{D}(\boldsymbol{M}')$.

Because of its assumptions Robinson's theorem is applicable only to a limited class of theories. One of the most important applications was Robinson's result [181] showing the decidability of the set of those first-order sentences involving the predicates $x = y$, $x \leqq y$, $x = y + z$, $x = y.z$ and "x is an algebraic number" which are true for the real numbers. This extension of Tarski's decidability result, which we discussed in lecture XII, could hardly be obtained by the method of elimination of quantifiers.

Further indications concerning completeness proofs using model-theoretical notions can be found in Robinson's book [183].

Another important part of model theory depends on a theorem known as Craig's interpolation lemma [25]. This result is a purely syntactic theorem dealing with the provability of formulae in the first-order logic without identity. It says that if $F(P, Q_1, \ldots, Q_k, x_1, \ldots, x_m)$ and $G(P', Q_1, \ldots, Q_k, x_1, \ldots, x_m)$ are two formulae with the free (predicate and individual) variables indicated and if the implication $F \rightarrow G$ is provable in logic then there is a formula $H = H(Q_1, \ldots, Q_k, x_1, \ldots, x_m)$ such that both implications $F \rightarrow H$ and $H \rightarrow G$ are provable in logic. The essential point is of course that the interpolation formula H contains freely only those variables which are free in F and in G.

We shall reformulate Craig's lemma using semantic notions. In order to simplify the notations we shall assume that $m = 0$ and $k = 1$ and write Q instead of Q_1. Furthermore we assume that P and Q are both binary predicates.

Let A be a fixed infinite set and let M be the family of all models of the form $<A, R>$ where $R \subset A \times A$. If H is a formula with the free variable Q, then we denote by U_H the set of all models $\boldsymbol{M}$ in M such that $\models_{\boldsymbol{M}} H[R]$, *i.e.* H is satisfied in $\boldsymbol{M}$ under the interpretation of Q as R. If F has the free variables P, Q, then we denote by V_F the set of models $\boldsymbol{M}$ in M for which there exists a relation $S \subset A \times A$ such that F is satisfied in the model $\boldsymbol{M}^* = <A, S, R>$ obtained from $\boldsymbol{M}$ by adjunction of the new relation S.

Craigs lemma is equivalent to the following separation principle: If $F = F(P, Q)$ and $G = G(P, Q)$ are formulae such that $V_F \cap V_G = 0$, then there are formulae $H = H(Q)$, $K = K(Q)$ such that $V_F \subset U_H$, $V_G \subset U_K$ and $U_H \cap U_K = 0$.

This formulation of Craig's theorem was a starting point of an extensive work undertaken by Addison in which he tried to establish a common basis for set-theoretical and logical separation principles. (See [3 a].)

The equivalence of this statement with Craig's lemma is a simple corollary to the completeness theorem.

Proofs of Craig's lemma were given, besides by Craig himself, by Lyndon and other authors; see literature quoted in [131].

In order to illustrate the uses of Craig's lemma we shall give a proof of a theorem due to Beth concerning the theory of definitions [6]. This theorem was proved by Beth before Craig in a more complicated way.

We mentioned already in lecture III a method (due in principle to Padoa and stated precisely by Tarski) of proving independence of a primitive notion P of a theory $\boldsymbol{T}$ from other primitive notions $Q_1, \ldots, Q_k$ of $\boldsymbol{T}$. The Padoa–Tarski theorem states that if there are two models $\boldsymbol{M} = <A, R, S_1, \ldots, S_k>$, $\boldsymbol{M}' = <A, R', S_1, \ldots, S_k>$ of $\boldsymbol{T}$ such that $R \neq R'$ then P is independent of $Q_1, \ldots, Q_k$ in $\boldsymbol{T}$.

Beth's theorem says now that this method is always applicable: If there are no models $\boldsymbol{M}$, $\boldsymbol{M}'$ with the properties mentioned above, then P can be defined in $\boldsymbol{T}$ with the help of $Q_1, \ldots, Q_k$ alone.

Proof: Using the completeness theorem we obtain, that if there are no models $\boldsymbol{M}$, $\boldsymbol{M}'$ required in Padoa's method then there is a finite conjunction K of axioms of $\boldsymbol{T}$ such that the formula

$$K(R,\ Q_1, \ldots, Q_k) \to \{K(R',\ Q_1, \ldots, Q_k) \to [R(x_1, \ldots, x_p) \to R'(x_1, \ldots, x_p)]\}$$

is provable in logic. If $L(Q_1, \ldots, Q_k, x_1, \ldots, x_p)$ is an interpolating formula for the formulae

$$F : K(R,\ Q_1, \ldots, Q_k) \wedge R(x_1, \ldots, x_p),$$

$$G : K(R',\ Q_1, \ldots, Q_k) \to R'(x_1, \ldots, x_p),$$

then $R(x_1, \ldots, x_p) \equiv L(Q_1, \ldots, Q_k,\ x_1, \ldots, x_p)$ is provable in $\boldsymbol{T}$ and hence R is definable in $\boldsymbol{T}$ by means of $Q_1, \ldots, Q_k$ alone.

Important extensions of Craig's lemma are due to Lyndon. We shall discuss only one of his results; a full account is given in [131].

First we define by induction the phrase: "An atomic formula $M(t_1, \ldots, t_k)$ (where $t_1, \ldots, t_k$ are terms) occurs positively (negatively) in a formula F". If F and M coincide, then M occurs positively in F. If M occurs positively (negatively) in F, then it occurs positively (negatively) in $F \vee G$, $G \vee F$, $F \wedge G$, $G \vee F$, $G \to F$ $\bigvee_x F$, $\bigwedge_x F$ and negatively (positively) in $\neg F$ and in $F \to G$.

If all the atomic formulae involving R occur positively in F, then we say that R occurs positively in F.

Lyndon's generalization of Craig's lemma is now this: Let F and G be formulae as in Craig's lemma and assume that Q_j occurs positively (negatively) in F and in G; then there exists an interpolation formula H in which Q_j also occurs positively (negatively). It follows from Lyndon's theorem that if a formula F containing a predicate Q has the property that its validity is preserved under extensions of Q, then F is equivalent to a formula in which Q occurs positively. More precisely, the assumption means that the condition $\boldsymbol{M} \in E(F)$ implies $\boldsymbol{M}' \in E(F)$ for each $\boldsymbol{M}'$ obtained from $\boldsymbol{M}$ by replacing the relation S which interprets in $\boldsymbol{M}$ the predicate Q by a relation $S' \supset S$.

This result solved a problem proposed by Marczewski in [137]. It is one of the many results in model theory which relate the set-theoretical properties of models with the syntactic properties of sentences true in these models. We shall consider below some other results of this sort.

Let us consider a class F of models and assume that F contains with each $\boldsymbol{M}$ every model isomorphic to $\boldsymbol{M}$ (we disregard at present the set-theoretical difficulties involved in this definition). We call F an elementary class if there is a set X of first-order sentences such that $F = E(X)$. This terminology is due to Tarski [227]. Our program is to characterize the elementary classes among all possible classes F and also to study relations between the form of sentences which belong to X and the set-theoretical properties of F.

Lyndon's theorem fits in this program since it can be rephrased as follows: If an elementary class F has the property that it contains with each $\boldsymbol{M}$ any model $\boldsymbol{M}'$ obtained by replacing S by a relation $S' \supset S$, then $F = E(X)$ where each formula A in X contains the predicate Q positively.

Historically the first (and simplest) result of this kind was obtained by Tarski [228] and, independently, by Łoś [127]. It says that if a class F contains with each $\boldsymbol{M}$ all the submodels of $\boldsymbol{M}$ and if F is an elementary class, then there is a set X of sentences such that $F = E(X)$ and that all the elements of X are general sentences (*i.e.* have the form $\bigwedge_{xy\ldots} H$ where H does not contain quantifiers).

Another result of this kind due to Łoś—Suszko [130], and also obtained by Chang [14], characterizes elementary classes closed with respect to the operation of forming the union of an increasing sequence of models.

All these results have been essentially generalized by Keisler [88].

Many authors have given various set-theoretical characterizations of elementary classes. Thus *e.g.* Tajmanov [221] obtained a simple characterization in topological terms. He introduced a topology in the class of all models by taking as neighbourhoods of a model $\boldsymbol{M}$ the classes $E(A)$ where A is any formula true in $\boldsymbol{M}$. Elementary classes are just closed subsets of this space. Many other characterizations have been shown by Tajmanov to follow from this simple result.

The deepest result in this direction is due to Keisler [89] who proved (using the generalized continuum hypothesis) that a class F is elementary if and only if it is closed with respect to the operation of forming reduced Cartesian powers and its complement $-F$ is closed with respect to the operation of forming reduced Cartesian products (these notions will be explained in lecture XVI). Less deep but independent of the continuum hypothesis are characterizations given by Kochen [110] who used other operations than those of forming reduced products and powers.

It is remarkable that in spite of their very abstract form these characterizations can be effectively used. For instance, Rabin [169] proved, using these criteria, that the class of groups G which are (isomorphic to) groups of automorphism of models $\boldsymbol{M}$ in $E(X)$ is always an elementary class and thus has the form $E(Y)$. Rabin's theorem states the existence of a set Y for any given X but his proof does not provide any means of actually constructing such a set, and is thus of great interest for evaluating non-effective methods in logic and set theory.

Another problem which has been studied extensively is that of giving set-theoretical criteria for the elementary equivalence of two models.

Keisler [90] proved that $\boldsymbol{M}_1 \equiv \boldsymbol{M}_2$ if and only if some reduced powers of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are isomorphic to each other. It follows that $\boldsymbol{M}_1 \equiv \boldsymbol{M}_2$ if and only if there are two isomorphic models $\boldsymbol{M}_1'$ and $\boldsymbol{M}_2'$ such that $\boldsymbol{M}_1 < \boldsymbol{M}_1'$ and $\boldsymbol{M}_2 < \boldsymbol{M}_2'$. Kochen [110] gave a similar characterization of $\equiv$ in terms of other operations. Less sophisticated but very useful was a characterization given by Ehrenfeucht and Fraïssé.

Fraïssé's [41] definition uses a sequence of equivalence relations $\sim_n$. Let X_1, X_2 be finite subsets of the universes of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$. We write $X_1 \sim_0 X_2$ if the relations of both models restricted to X_1 and X_2 are isomorphic. Now assume that an equivalence relation $\sim_n$ between finite subsets of $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ has already been defined. We define the relation $\sim_{n+1}$ as follows: $X_1 \sim_{n+1} X_2$ if for every x_1 in the universe of $\boldsymbol{M}_1$ there is an x_2 in the universe of $\boldsymbol{M}_2$ such that $X_1 \cup \{x_1\} \sim_n X_2 \cup \{x_2\}$ and conversely.

Fraïssé's theorem says that $\boldsymbol{M}_1 \equiv \boldsymbol{M}_2$ if and only if $0 \sim_n 0$ for each n.

Let us consider as an example the set N of all integers ordered by the $\leq$ relation (its order type is $\omega^* + \omega$) and an extension N' of N ordered by an extension $\leq '$ of the $\leq$-relation in type $\omega^* + \omega + \omega^* + \omega$. We assume that N is an initial segment of N'. Let $\boldsymbol{M}_1 = <N, \leq>$, $\boldsymbol{M}_2 = <N', \leq '>$ and let $X_1 \subset N$, $X_2 \subset N'$ be two finite sets with the same number of elements. We can decompose X_2 into a disjoint union $X_2 = X_2' \cup X_2''$ where $X_2' = X_2 \cap N$, $X_2'' = X_2 - N$. Let X_2' and X_2'' have p' and p'' elements. It is then easy to prove that $X_1 \sim_n X_2$ if and only if there are at least n

elements between the first p' and the last p'' elements of X_1. This implies, in particular, that $\boldsymbol{M}_1 \equiv \boldsymbol{M}_2$.

One could obtain the same result by using the elimination of quantifiers but the method based on Fraïssé's construction is much more conspicuous.

Ehrenfeucht obtained (independently) the same characterization as Fraïssé and expressed it in a very suggestive language of games [30]. He also showed how this characterization can be applied to solve various problems concerning definability of elements in models whose universes consist of ordinals.

We shall see, in the next lecture, that the Ehrenfeucht—Fraïssé method can be extended to certain languages different from the first-order language considered here.

The results of model theory presented here do not exhaust all which have been dealt with in the existing literature. The theory is still in the stage of very rapid development and will certainly find many new applications.

Lecture XIV

Theory of models for non-elementary languages

We mentioned already in lecture XIII that the scheme of the model theory is very general and applicable to various kinds of languages. Several attempts were made to apply this scheme to languages different from the language L of the first-order logic.

We shall report on results obtained for the following languages:

1. The language Q_α. This language differs from L by containing in addition to the symbols of L one new quantifier Q. The sentence $QxFx$ is true in a model $\boldsymbol{M}$ if and only if there are in $\boldsymbol{M}$ at least $\aleph_\alpha$ elements which satisfy F in $\boldsymbol{M}$. We have thus one language with many different interpretations of the constant Q. The notion of a model is the same as in the case of the language L.

2. The language L_α^{II}. This language differs from L by containing variables $X, Y, \ldots$ for sets. A sentence $\bigvee_X F(X)$ is true in a model $\boldsymbol{M}$ if there is a subset of $\boldsymbol{M}$ whose power is $< \aleph_\alpha$ which satisfies F in $\boldsymbol{M}$. Again we see that there is one syntactic structure of the language but a multitude of interpretations. The languages L_α^{II} are said to be of weak second-order. Again models are defined in the same way as in L.

3. The strong second-order language L^{II} has the same syntactic structure as L_α^{II} but a different interpretation of the set variables: the formula $\bigvee_X F(X)$ is true in a model $\boldsymbol{M}$ if there is a subset of $\boldsymbol{M}$ of any cardinality which satisfies $F(X)$ in $\boldsymbol{M}$.

4. The sequential second-order language L_0^s contains variables not for finite sets of elements but for finite sequences of them and also symbols for concatenation of two sequences and for forming a one-term sequence $<a>$ out of a given element a.

5. Higher order languages $L_\alpha^{(n)}$ and $L^{(n)}$ are defined similarly as the languages L_α^{II} and L^{II}. It is possible to combine the methods of construction of these languages and require for instance that arbi-

trary subsets of a model be values of the first-order set-variables X, Y, ... but only finite sets be allowed as values for second-order set-variables *etc*.

6. The infinitistic language L_{ω_1, ω_0}. Let x_i, c_i for $i = 0, 1, 2, \ldots$ be individual variables and constants and let R_i be a predicate with p_i arguments ($i = 0, 1, 2, \ldots$). Atomic formulae of L_{ω_1, ω_0} are expressions $R_i(t_1, \ldots, t_{p_i})$ where each t_i is either a variable or a constant. The rules of formation of more complicated formulae are the same as in L with two additional infinitistic rules: if A_i is an infinite sequence of formulae, then $\sum\limits_i A_i$ and $\prod\limits_i A_i$ are formulae.

Models are just the ordinary models as in the case of L. A sentence $\sum\limits_i A_i$ is true in $\boldsymbol{M}$ if and only if there is an i such that A_i is true in $\boldsymbol{M}$; a sentence $\prod\limits_i A_i$ is true in $\boldsymbol{M}$ if and only if each A_i is true in $\boldsymbol{M}$.

Formulae of L_{ω_1, ω_0} are infinitistic objects; thus even the syntax of this language can be studied only in strong systems of set theory.

In the symbol L_{ω_1, ω_0} the first index ω_1 is the smallest cardinal larger than the cardinal number of terms in any disjunction or conjunction allowed in the language. The second index shows that only a finite number (*i.e.* a number $< \omega_0$) of variables can occur under a quantifier: we can form a sentence $\bigvee\limits_{x_1, \ldots, x_n} F$ (which is, in fact, an abbreviation for $\bigvee\limits_{x_1} \bigvee\limits_{x_2} \ldots \bigvee\limits_{x_n} F$) but we are not allowed to form a sentence $\bigvee\limits_{\{x_1, x_2, \ldots\}} F$ with an infinite sequence of variables under the quantifier.

7. The infinitistic languages $L_{\omega_\mu, \omega_\nu}$ are defined similarly. For instance in L_{ω_2, ω_1} we can form disjunctions and conjunctions of sequences of lengths $< \omega_2$ and also bind by a single quantifier strings of variables whose length is any ordinal $< \omega_1$.

These languages were introduced by Tarski and Scott [200]; the symbolism is due to C. Karp [87] who undertook an extensive study of these languages. The usual first-order logic is contained as a special case among the languages here considered: in fact $L = L_{\omega_0, \omega_0}$.

There exist various relations between the languages we enumerated. Thus *e.g.* Q_a is translatable into L_a^{II} and L_a^{II} into L_{ω_a, ω_a}.

The notions of submodel, elementary extension (with respect to a given language J), elementary equivalence (with respect to J), spectrum, elementary class (with respect to J) can easily be defined in a similar way as in the previous lecture. The same applies to the notion of (logical) consequence. However, not all results can be carried over to the more general theory which we consider here.

The main difference between the model theory for the language L and the model theories of the languages 1—7 above is the failure of the compactness theorem in most of the latter. This theorem is false for the languages Q_0, L_a^{II}, L^{II}, L_0^s and most of the languages $L_{\omega_\mu},\ {}_{\omega_\nu}$. For the languages Q_{a+1} Fuhrken [48] proved a remarkable theorem which implies that the compactness theorem is true in the model theory of these languages. It is not yet known whether this is also true for languages Q_a with a limit index.

The downward Skolem—Löwenheim theorem is valid (with some modifications concerning the minimal power of a model) for all languages defined above. This is no more true for the upward Skolem—Löwenheim theorem which fails for almost all of these languages (notice that we used the compactness theorem in the proof of the upward Skolem—Löwenheim theorem for the language L). Because of the failure of this theorem the structure of spectra in these languages is incomparably more involved than in the case of language L.

In this connection Hanf [73] proved a simple but interesting theorem valid for any language J in which the downward Skolem—Löwenheim theorem is true: for any such language there exists a cardinal $\mathfrak{h}$ (the "Hanf number of J") with the property that if a formula F has a model of power $\mathfrak{h}$, it also has a model of any power $> \mathfrak{h}$. Hanf's proof is not constructive and the actual determination of $\mathfrak{h}$ even for very simple languages presents great difficulties. From results of Morley [148] it follows *e.g.* that for Q_0 the Hanf number is $\beth_{\omega_1}$ where the transfinite sequence $\beth_\xi$ is defined as follows: $\beth_0 = \aleph_0$, $\beth_{\xi+1} = 2^{\beth_\xi}$, $\beth_\lambda = \Sigma_{\xi<\lambda}\, \beth_\xi$ for limit numbers λ.

Montague [146] investigated spectra in higher-order languages $L^{(n)}$ and proved using essentially results of Hintikka [83] that with each formula of $L^{(n)}$ one can correlate a formula of $L^{(\mathrm{II})}$ with the same spectrum. This illustrates the difficulty of the spectrum problem for the language $L^{(\mathrm{II})}$.

The Fraïssé—Ehrenfeucht formulation of the equivalence of models can be carried over to several non-elementary languages. Let us sketch (after Fraïssé [42]) the relevant definitions for the language

Let $\boldsymbol{M}$ be a model with the universe A and A^n the set of all sequences $\langle x_1, \ldots, x_n\rangle$ with $x_j \in A$ for $j = 1, 2, \ldots, n$. Let further $S_\alpha(A)$ be the family of subsets of A of a power $< \aleph_\alpha$ and $S_\alpha(A)^n$ the set of all sequences $\langle X_1, \ldots, X_n\rangle$ with $X_i \in S_\alpha(A)$. We define a sequence of equivalence relations $\sim_k$ whose fields are triples $\langle \boldsymbol{M}, x, X\rangle$ where $\boldsymbol{M}$ is a model, $x \in A^n$, $X \in S_\alpha(A)^n$ and n is an integer.

The relation $\langle \boldsymbol{M}_1, x_1, X_1\rangle \sim_0 \langle \boldsymbol{M}_2, x_2, X_2\rangle$ holds if and only if 1° the function correlating the consecutive terms $x_{11}, x_{12}, \ldots, x_{1n}$ of x_1 to the corresponding terms $x_{21}, x_{22}, \ldots, x_{2n}$ of x_2 is an isomorphism with respect to the relations of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$; 2° $x_{i1} \in X_{1j} \equiv x_{2i} \in X_{2j}$ for all $i, j \leq n$.

If $\sim_k$ is already defined, then we define $\sim_{k+1}$ as follows: the relation $\langle \boldsymbol{M}_1, x_1, X_1\rangle \sim_{k+1} \langle \boldsymbol{M}_2, x_2, X_2\rangle$ holds if and only if for arbitrary $z_1 \in A_1$ and $Z_1 \in S_\alpha(A_1)$ there are $z_2 \in A_2$ and $Z_1 \in S_\alpha(A_2)$ such that

$$\langle \boldsymbol{M}_1, x_1{}^\frown\langle z_1\rangle, X_1{}^\frown\langle Z_1\rangle\rangle \sim_k \langle \boldsymbol{M}_2, x_2{}^\frown\langle z_2\rangle, X_2{}^\frown\langle Z_2\rangle\rangle$$

and conversely.

With these definitions it is easy to show that two models $\boldsymbol{M}_1, \boldsymbol{M}_2$ are equivalent with respect to the language L_α^{II} if and only if $\boldsymbol{M}_1 \sim_k \boldsymbol{M}_2$ for each k.

For infinitistic languages L_{ω_μ, ω_0} the sequence of relations $\sim_k$ can be extended into transfinite but even this transfinite sequence does not yield the full characterization of equivalence with respect to L_{ω_μ, ω_0}. These problems were investigated by Scott who found applications for them in the descriptive set theory [199]. See also [123 a].

Interesting and mostly not yet solved problems result when one investigates analogues of the completeness theorem for various generalizations of the language L. Let us call a sentence F of a language J true if it is true in all models. In case of the language L the set of true sentences is as we know recursively enumerable. For the language Q_0 one shows easily that this set is a Π^1_1-set which is not hyper-arithmetical and the same is true for the languages L_0^{II} and L_0^s. For Q_1 Vaught [239] established the unexpected fact (based on results of Fuhrken [48]) that the set of true sentences of this language is recursively enumerable. The problem has not yet been solved for all languages Q_α. For languages L_α^{II} ($\alpha > 0$) and L^{II} only negative results are known; *e.g.* one knows that the set of true sentences of these languages are not analytic. The problem whether they are constructible in Gödel's sense is open. Still less is known in case of infinitistic languages $L_{\omega_\mu, \omega_\mu}$ because we are lacking suitable hierarchies which would allow us to express the analogues of the completeness theorem.

We shall devote the rest of this lecture to applications of model theories of generalized languages. Whereas there are as we saw in lecture XIII many applications of the usual model theory, the applications of the generalized one are rather scarce. Some of them are worth noticing, however.

1. Axiomatic theories based on the language L_0^s. Axiomatic theories usually considered in meta-mathematics are based on the first-order logic L and the notion of consequence used in them is the (syntactically defined) notion of consequence of L. It follows from the Skolem—Löwenheim theorem for L that such theories are never categorical unless the cardinalities of all their models do not surpass a fixed integer.

Tarski [229] was the first to realize that one often obtains interesting theories, if one bases them on other languages and appropriately changes the notion of consequence. Suppose for instance that we base a theory on the language L_0^s (the theory is then said to be of a weak second-order). We admit then as axioms certain sentences of L_0^s and call a sentence of L_0^s a theorem if it is a consequence of axioms *i.e.* if it is true in every model of the axioms. Such a theory is decidable only if all its models have just one element. Unless this assumption is satisfied, the set of theorems is not recursively enumerable. On the other hand many weak second order theories are categorical and hence complete. Moreover they are often more natural from the mathematical point of view, since we can define in them various notions not definable in the ordinary theories.

Let us take as an example the weak second-order theory of algebraically closed fields (Tarski [229]).

The notion of a polynomial is definable in such a theory (we can identify a polynomial with the sequence of its coefficients). We can also define the value of a polynomial for a given argument and thus express in a single sentence that a field is algebraically closed. Also the notion of characteristic is definable in the theory.

We saw in lecture XIII that the first-order theory of algebraically closed fields requires an infinite number of axioms; on the contrary the weak second-order theory is finitely axiomatizable.

Tarski (*l.c.*) was able to classify all equivalence types (with respect to L_0^s) of algebraically closed fields of a given characteristic. It turned out that fields with a given finite degree of transcendence form such a class; besides these classes there is still one equivalence class containing all algebraically closed fields with infinite degrees of transcendence. As an immediate corollary Tarski obtained the result that if a sentence of L_0^s is valid in the field of complex numbers it is valid in all algebraically closed fields with characteristic 0 and infinite degree of transcendence.

A similar "transfer principle" was obtained earlier by Tarski [226] for sentences of L as a simple corollary to the decidability of the elementary theory of addition and multiplication of complex numbers. Since the theory based on the language L_0^s is incomparably richer than the elementary theory, the new principle represents a much stronger result than the old one.

It follows from the result discussed above that the weak second-order theory of algebraically closed fields becomes complete, upon addition of axioms which determine the characteristic and the transcendence-degree of the field. Other examples of complete weak second-order theories are provided by the arithmetic of integers or of rationals. These theories are even categorical and, in addition, finitely axiomatizable of course with respect to the notion of consequence in L_0^s. The set of sentences of L_0^s which are valid in the field of real numbers is not finitely (and even not hyper-arithmetically axiomatizable (*cf.* [152]). It seems to us that the study of axiomatizability of weak second-order theories deserves a further study.

2. Applications of infinitistic logics $L_{\omega_\mu, \omega_\mu}$ to abstract set theory. We mentioned in lecture VI that the compactness theorem for the first-order logic follows from the existence of a maximal filter in an arbitrary Boolean algebra. A similar connection between the compactness theorem and the existence of certain filters subsists for infinitistic logics $L_{\omega_\mu, \omega_\mu}$.

In order to state this result we must first explain some set-theoretical notions. An ordinal μ is called regular if ω_μ cannot be represented as a limit of a transfinite sequence of a type $< \omega_\mu$ whose terms are $< \omega_\mu$. An ordinal μ is called inaccessible if it is regular and $m < \aleph_\mu \rightarrow 2^m < \aleph_\mu$.

Tarski [230] calls an ordinal μ strongly compact, if every set X of sentences of $L_{\omega_\mu, \omega_\mu}$ with cardinality $\aleph_\mu$ has the property: if every subset of X with a smaller cardinality is satisfiable, then so is X. The connection with the theory of filters established by Tarski is now this: a regular ordinal μ is strongly compact provided that the Boolean algebra B_μ of all subsets of a set of power $\aleph_\mu$ contains a maximal non-principal $\aleph_\mu$-multiplicative filter (*i.e.*, a filter F such that if $R \subset F$ and R has a power $< \aleph_\mu$, then the intersection of R belongs to F). This result reduces the question of existence of filters with the properties just mentioned to a meta-mathematical problem concerning the language $L_{\omega_\mu, \omega_\mu}$. Mathematicians interested in abstract set theory once tried for a long time to settle the question whether for the first inaccessible ordinal $\mu > 0$ the Boolean algebra B_μ contains a nonprincipal, $\aleph_\mu$-multiplicative maximal filter (for all smaller numbers the problem was solved long ago by Tarski and Ulam). The interest in this question is due to the fact that it is closely connected with the abstract measure theory.

Solving this problem proposed by Tarski Hanf [73] showed that the first inaccessible number $\mu > 0$ is strongly incompact whence in particular it follows that there is no filter of the kind indicated above. The measure-theoretic problem was thus solved by the use of the model theory of the language $L_{\omega_\mu, \omega_\mu}$. Hanf's construction showed, moreover, that many other regular ordinals are strongly incompact. The question whether one can assume without inconsistency that there are strongly compact ordinals is open and seems to be very difficult.

We shall outline Hanf's proof of incompactness of $L_{\omega_\mu, \omega_\mu}$ for the first inaccessible ordinal > 0. To obtain his result Hanf considered a set X of sentences of $L_{\omega_\mu, \omega_\mu}$ which describes an axiomatic theory of ordinals. All sentences of X contain but one predicate, *viz.* ϵ. We include in X the axiom of extensionality and the axiom of regularity in the form

$$\neg \bigvee_{v_0, v_1, \ldots} \Pi_n (v_{n+1} \epsilon v_n).$$

Next we add to X sentences which state the existence of all ordinals $\xi < \omega_\mu$. Since each ordinal is equal to the set of its predecessors, it can be described by a formula $S_\xi(x)$ of $L_{\omega_\mu, \omega_\mu}$. *E.g.* for $\xi = \omega_1$, we have as the description of ξ the formula

$$\bigvee_v \{\Pi_{\xi<a} \bigwedge_u [(u \epsilon v_\xi \equiv \Sigma_{\tau<\xi} (u = v_\tau)] \wedge (v_{\omega_1} = x)\}$$

where v stands for the sequence $\{v_a\}$ of type $\omega_1 + 1$.

The sentences which we add to X have the form $\bigvee_x S_\xi(x)$.

Finally we add to X sentences which state (I) that all ordinals are smaller than the first inaccessible ordinal $> \omega$ and (II) that there exists a largest ordinal. While it is clear how to express (II), the sentence (I) is rather involved and cannot be given here. One obtains it expressing in our formal language the definition of various set-theoretical notions.

The cardinal number of X is $\aleph_\mu$. Each set $X_1 \subset X$ with a cardinality $< \aleph_\mu$ is satisfiable in a model of set-theoretical axioms containing only ordinals smaller than a fixed ordinal $< \omega_\mu$. The whole set X is not satisfiable since the largest ordinal Ω of the model (existing on account of (II)) would be smaller than μ (by (I)) but also would be $\geq \xi$ for every $\xi < \mu$ (since we have in X the axioms stating the existence of ξ and $\xi + 1$ for each individually given ξ). We thus obtain $\Omega \geq \mu$ and $\Omega < \mu$ which is a contradiction.

Tarski and Keisler [92] have shown that the results concerning filters in B_μ can be obtained directly without a detour via the metamathematics of the logic $L_{\omega_\mu, \omega_\mu}$. It remains a fact, however, that it was the model theory of infinitistic languages which has led Tarski and Hanf to discoveries in the abstract set theory.

Infinitistic languages are obviously not suitable as a basis for mathematics since even their syntax requires a strong set theory. The above examples show, however, that they are not just idle generalizations but valuable tools for obtaining new results.

Lecture XV

Problems in the foundations of set theory

The abstract set theory has contributed more than any other branch of mathematics to the development of foundational studies. The reasons for this phenomenon are numerous.

One of the basic assumptions of set theory is the axiom of infinity which says that there exist infinite sets. This assumption implies that the scale of infinite cardinals is itself infinite. Thus the axiom of infinity leads us out of the mathematical domains which are close to everyday practice and even to scientific experience. We are thus faced at the very beginning of set theory with the fundamental question of the philosophy of mathematics: which mathematical objects are admissible and why?

The same question arises in connection with sets of small powers, *e.g.*, with sets of integers. One could accept the "Platonistic" attitude and declare that sets (of integers) exist in the same sense as any other objects and that there is thus no arbitrariness in this notion. But even a Platonist must make it clear how he discovers properties of these allegedly existing objects. The adversaries of Platonism seek a solution by accepting one or another form of constructivism. This attitude is often more satisfactory philosophically than Platonism but unfortunately usually destructive for mathematics and even for those parts of it which are well established by the scientific praxis.

Most mathematicians do not perceive the problem which is posed by the abstractness of set theory. They prefer to take an aloof attitude and pretend not to be interested in philosophical (as opposed to purely mathematical) questions. In practice this simply means that they limit themselves to deducing theorems from axioms which were proposed by some authorities.

Interesting though the philosophical questions of foundations of set theory undoubtedly are, we shall not deal with them any further here since, for the reason just explained, the writings of contemporary set theorists and logicians do not offer very much which could help us in solving these problems. The writings of the period 1930—1964 which we are analyzing do not contain much philosophical discussion. These writings contain however great wealth of formal metamathematical results which we will try to summarize.

The best known results of this kind are connected with the axiom of choice and the continuum hypothesis. Many less famous but not less important problems are connected with other axioms or hypotheses of set theory *e.g.* with Suslin's hypothesis.

In this lecture we shall report on some of these problems and their solutions.

The Zermelo—Fraenkel and Bernays—Gödel axioms of set theory. The first axiomatic system of set theory was due to Zermelo. It was perfected soon afterwards by Fraenkel. In the period between 1930—1960 various new axiomatic systems of set theory were proposed. The best known is the system formulated by Bernays [5] and used by Gödel in his famous book [59]. The Bernays—Gödel system has three primitive notions: set, class, and ϵ, whereas Zermelo and Fraenkel used but two: set and ϵ. The distinction between sets and classes goes back to the writings of Cantor who distinguished between "consistent" and "inconsistent" sets.

The introduction of the new primitive notion allowed Bernays and Gödel to present the system of set theory in the form of a finitely axiomatizable system (the basic idea of this reduction was due to von Neumann); the previous systems and in particular the system of Zermelo required an infinite number of axioms.

The comparison of the Bernays—Gödel and Zermelo—Fraenkel set theories led to various discoveries.

Novak [156] showed that sentences provable in the Gödel—Bernays system and not containing the predicate "class" coincide with sentences provable in the Zermelo—Fraenkel system. We express this by saying that the Gödel—Bernays system is an inessential extension of the Zermelo—Fraenkel system. Also the consistency of the Gödel—Bernays system is reducible (in a finitistic way) to that of the Zermelo—Fraenkel system (Novak [156], Shoenfield [202]). Kleene [103] showed that each axiomatic system with a recursively enumerable set of axioms can be extended in an inessential way to a finitely axiomatizable system by adding one new primitive predicate (*cf.* also Vaught—Craig [26]). All these results show that the difference between both systems is not very great as far as their mathematical contents is concerned. The Zermelo—Fraenkel system was shown not to be finitely axiomatizable by Montague [144], [145]. Earlier attempts to obtain this result which were undertaken by Wang [241] and Mostowski [150] contained mistakes. The impossibility of a finite axiomatization of Peano's arithmetic and some other theories was proved (by using non-standard models) by Ryll—Nardzewski [189]; *cf.* also Hauschild [74].

Various extensions of both the Bernays—Gödel and the Zermelo—Fraenkel set theories were proposed in order to secure the existence of high cardinalities. The first step was made by Tarski [224] who formulated an axiom which secures the existence of inaccessible cardinals. Lévy [122] made the next step and formulated axiom schemata which secure the existence of various kinds of inaccessible cardinals. Lévy's schemata have the form of reflexion-principles and state, roughly speaking, that if the universe possesses a property expressed by a set theoretical formula, then there is a set in which this formula is also satisfied and which is closed with respect to operations described in the set-theoretical axioms.

These extensions of Zermelo—Fraenkel (or Bernays—Gödel) set theory are essential; adding Lévy's schemata we are able to prove statements which were formerly not provable. Such statements can even be found among statements concerning integers.

No relative consistency proof for the new axioms exists; its existence is excluded by Gödel's second undecidability theorem. Nevertheless set theoreticians believe that these axioms are consistent and this belief is strengthened by the fact that none of the known antinomies have appeared in the extended systems.

A very strong form of the axiom of infinity states that there are compact regular ordinals $\mu > 0$. This axiom is much stronger than Lévy's schemata. Nothing can be said as yet with regard to its consistency.

Other axiomatic systems of set theory. The Zermelo—Fraenkel (and Bernays—Gödel) set theory arose from attempts to formulate in a consistent way the intuitive assumptions of the naive (Cantorian) set theory. Cantor in the early phase of his work used implicitly the following axiom (axiom schema) of set existence: Whenever F is a formula (with one free variable x), there is a set S consisting of all elements a satisfying F. The same schema was explicitly used by Frege. Since the schema is known to be inconsistent one tried to modify it. The axioms of set theory represent an outcome of these endeavours.

Three ways of modifying the inconsistent principle of set existence were proposed:

(i) One does not accept the principle for all formulae F;

(ii) One restricts the variability of a;

(iii) One imposes simultaneously both limitations (i) and (ii).

The simple type theory as first formulated by Chwistek and Ramsay accepts (iii). In this theory we admit only formulae which conform to the rules of formation prescribed by the theory. Moreover each variable has its prescribed domain (*cf.* Church [19] for an exact presentation of the syntax of the simple type theory). The Zermelo—Fraenkel and Bernays—Gödel systems accept (ii) because the schema of set existence accepted in these theories is the following: If S is a set, then so is $\{a : (a \in S) \wedge F(a)\}$. The restriction (i) is represented by axiomatic systems due to Quine [164] in which the set existence schema is assumed only for so-called stratified formulae whereas no limitation for the domains of variables is assumed. Quine's theory was extensively studied by a number of logicians (*e.g.* Wang [243]) but does not seem to have influenced the work of mathematicians.

It is more than probable that the axioms of set theory have not yet reached their definitive form. In connection with this question one should read the profound article of Gödel [61].

Axiom of choice. This axiom was from the start treated with distrust by many outstanding mathematicians. The philosophical discussion concerning its acceptability was closed well before 1930. In the period between 1930 and 1960 one obtained far-reaching formal results concerning the (relative) consistency and independence of the axiom. The first problem was dealt with in lecture IX. The problem of independence was essentially solved already in 1920 by Fraenkel though not in an entirely precise way. His method, usually called the permutation method, is applicable only to some systems of axiomatic set theory, *e.g.* to the Zermelo—Fraenkel system with the axiom of extensionality in the form

$$Z(x) \wedge Z(y) \wedge \bigwedge_s [(s \in x) \equiv (s \in y)] \rightarrow (x = y),$$

(where $Z(x)$ means: x is a set)

but is not applicable to the Zermelo—Fraenkel system with the additional axiom $\bigwedge_x Z(x)$ or any other axiom fixing the number of objects which are not sets. This shows how limited is the field of applications of the permutation method.

The general idea of the permutation method is this: we start from an infinite set K_0 whose elements are not sets and form new sets by repeating the operations of summation and of forming the power set. In this way we construct sets K_0, $K_1 = K_0 \cup P(K_0)$, $K_2 = K_0 \cup P(K_0) \cup P(K_0 \cup P(K_0))$ *etc.* This sequence can be extended into the transfinite. Each permutation π of K_0 acts on sets in each K_ξ and determines a permutation of this set. Let G be a group of permutations of K_0 and L a lattice of subsets K_0 containing all finite subsets of K_0 and consider only those elements x of K_ξ which have the following property (P): there is a set $X \in L$ such that each permutation π of K_0 which belongs to G and is constant on X leaves x invariant. The class of x which hereditarily have the property (P) is a model for Zermelo—Fraenkel axioms, with the possible exception of the axiom of choice.

Choosing a suitable group G and a suitable lattice L one obtains models by means of which it is possible to obtain independence proofs of various set-theoretical statements and in particular to prove the independence of the axiom of choice.

The literature dealing with these proofs is rather extensive. We refer to the synthetic paper of Lévy [123] as well as the newest additions to the theory obtained by Halpern [72] and Läuchli [121].

Specker [214], Mendelson [143] and Shoenfield [203] have modified the permutation method by considering instead of K_0 a family of sets x with the property $x \in x$. Their method is again applicable only to such set theories in which one can assume without inconsistency that there is an infinite set of such sets.

Cohen's notion of forcing. In 1963 Cohen [24] found a new method which allowed him to establish the independence of the axiom of choice and of the generalized continuum hypothesis from practically every system of set theory built along the Zermelo—Fraenkel lines. The success of his method is due to a new meta-mathematical notion of forcing.

We shall describe briefly this notion.

Let J be a first-order language with finitely many predicates and with infinitely many constants. We do not exclude the possibility that the variables and constants are divided in mutually exclusive types so that only such substitutions are admissible in which a variable is replaced by a constant of the same type. For the present we assume that J does not contain symbols for functions.

Forcing is a relation between a finite consistent sequence P of atomic sentences (*i.e.* atomic formulae without free variables) or negations of such sentences and a sentence F. We call P an "information". An information Q obtained from P by adding new terms to it is called an extension of P. We write then $P \prec Q$.

The definition proceeds by induction on the number of logical constants of F. In order to abbreviate our formulae we shall write $P \Vdash F$ instead of "P forces F"

1. If F is an atomic sentence, then $P \Vdash F$ if and only if F is a term of P.
2. If F is the sentence $F_1 \wedge F_2$ (or $F_1 \vee F_2$) then $P \Vdash F$ if and only if $P \Vdash F_1$ and (or) $P \Vdash F_2$.
3. If F is the sentence $F_1 \rightarrow F_2$, then $P \Vdash F$ if and only if every extension Q of P which satisfies $Q \Vdash F_1$ satisfies also $Q \Vdash F_2$.
4. If F is the sentence $\neg F_1$, then $P \Vdash F$ if and only if no extension Q of P satisfies $Q \Vdash F_1$.
5. If F is the sentence $\bigvee_x F_1$, then $P \Vdash F$ if and only if there is a constant a (of the same type as x) such that $P \Vdash F(a)$.
6. If F is the sentence $\bigwedge_x F_1$, then $P \Vdash F$ if and only if no extension Q of P and no constant a (of the same type as x) satisfies the condition $Q \Vdash \neg F_1(a)$.

Condition 6 can be rephrased thus: for every constant a (of the appropriate type) and every extension Q of P an extension R of Q can be found such that $R \Vdash F_1(a)$.

Most characteristic are conditions 4 and 6. Condition 4 secures that once we have the relation $P \Vdash \neg F_1$, then no matter how we extend P to Q we shall never encounter the situation when we would have to assume $Q \Vdash F_1$. Similarly condition 6 secures that once we have $P \Vdash \bigwedge_x F_1$, we shall never have the relation $Q \Vdash \neg F_1(a)$ whatever a and whatever extension Q of P we choose. Thus if we imagine the sequence P growing as time goes on, then the fact that a relation $P \Vdash \neg F_1$ (or $P \Vdash \bigwedge_x F_1(x)$) holds at a given moment prevents relations of the form $Q \Vdash F_1$ (or $Q \Vdash \neg F_1(a)$) to hold in the future (independently of the way we extended P).

The definition of forcing is interesting quite apart from its applications. Feferman [38] investigated it in the case of a formal system of Peano arithmetic with an additional one place predicate and showed that it is hyper-arithmetical. Grzegorczyk [70] showed that if one considers only sentences without quantifiers, then the sentences F

which are forced by every information P coincide with theorems of the intuitionistic propositional calculus. No such simple characterization is known for arbitrary sentences.

In the case considered by Cohen the language J contained not only constants and variables but also symbols of functions. In order to describe his construction we must go back to the construction of Gödel [60]. Modifying inessentially our definitions from lecture IX we define a transfinite sequence K_μ of sets such that $K_0 = \omega =$ set of all integers, $K_\lambda = \bigcup_{\xi<\lambda} K_\xi$ for limit numbers λ and $K_{\alpha+1}$ is the family of all sets of the form $\{a : (a \in K_\alpha) \wedge \models_{M_\alpha} F[a ; b_1, \ldots, b_k]\}$ where F is a formula with $k+1$ free variables, $b_1, \ldots, b_k \in K_\alpha$ and $M_\alpha = <K_\alpha, \epsilon_\alpha>$ is the model with the universe K_α and with the ϵ-relation limited to K_α.

It follows from Gödel's proof that there is a denumerable ordinal α_0 such that M_{α_0} is a model of set theory (with the axiom of constructibility and hence with the axiom of choice and the generalized continuum hypothesis). Cohen's plan was now to add to M_{α_0} a new set C of integers and close $K_{\alpha_0} \cup \{C\}$ with respect to all operations φ used in the construction of K_{α_0}. He hoped to achieve by a suitable choice of C that C will not be a constructible set in the new model. In order to obtain a model in which the continuum hypothesis does not hold he adjoined to M_{α_0} a sequence $C = \{C_\delta\}$ of new sets of integers such that no two terms of C are identical and δ ranges over the ordinal ω_2 (or ω_3, or $\omega_4, \ldots$) of the model M_{α_0}. Of course the ordinal ω_2 of the model M_{α_0} is denumerable (although this ordinal satisfies in the model the formula "x is non-denumerable"). It is not immediately obvious that a choice of C (or $\{C_\delta\}$) is possible: If we add to M_{α_0} an arbitrary C and close it with respect to the operations φ we will usually not obtain a model for set theory. Even if we are lucky enough and obtain such a model it will usually not have the property (needed in the proof of independence of the continuum hypothesis) which says that if an element ω_2 satisfies in M_{α_0} the formula: "x is the second uncountable ordinal" then the same element ω_2 satisfies this formula in the new model. Thus the appropriate choice of C is the essential point of the whole proof. Cohen achieved it by his theory of forcing. He considered a language in which there are variables and constants of types τ where τ ranges over ordinals $< \alpha_0$. To each element a in K_{α_0} there is a constant of type τ denoting a. In addition there is a constant c denoting a set of integers (or constants c_δ denoting sets of integers and a constant c

for the whole sequence $\{c_\delta\}$) and symbols for the operations φ. Besides these symbols we admit unrestricted variables without any limitations of range. Sentences in which no unrestricted variables occur are called limited, other unlimited. A restricted sentence F has a rank which is defined as the least ordinal ξ such that no constant and no variable of type $\geq \xi$ occurs in F. Informations P contain only sentences of the form $n \in c$ or $n \bar{\in} c$ (and $n \in c_\delta$ or $n \bar{\in} c_\delta$ respectively).

We first define the relation of forcing for limited sentences. Conditions 2—6 remain unchanged but 1 is modified unless F is an atomic formula of the form allowed in P. In other cases the atomic formula $a \in b$ possesses a structure which allows us to reduce the relation $P \Vdash a \in b$ to simpler cases.

We shall illustrate the definition in two cases:

Let b be the constant $\{x : (x \in K_\alpha) \wedge \models_{M_\alpha} F(x, b_1, \ldots, b_k)\}$ and a a constant for an element of K_α. In this case the relation $P \Vdash a \in b$ is defined as $P \Vdash F_\alpha (a, b_1, \ldots, b_k)$ where F_α is a limited statement obtained from F by replacing all unrestricted variables by variables of type α. If b is a constant for an element of K_α and a the constant $\{x : (x \in K_\alpha) \wedge \models_{M_\alpha} F(x, b_1, \ldots, b_k)\}$ then $P \Vdash a \in b$ holds if and only if $P \Vdash \bigwedge_{x_\alpha} [(a = x_\alpha) \wedge (x_\alpha \in b)]$ which in turn can be reduced by the use of rule 5 and by the definition of equation to relations of the form $P \Vdash G$ where G has a rank less than α.

The remarks we made should be sufficient to illustrate how we can define the relation of forcing for limited sentences by the use of transfinite induction.

This definition being completed we define forcing for unlimited sentences by the rules 1—6. No complication arises in case of atomic sentences since they are all limited sentences and hence forcing is defined for them.

Starting from the definition of forcing it is easy to prove (nonconstructively) that there is an infinite increasing sequence $P_1 \prec P_2 \prec \ldots$ of informations with the following properties:

(1) for each sentence F either F or $\neg F$ is eventually forced;

(2) for each n either $n \in c$ or $\neg (n \in c)$ (and for each n and δ either $n \in c_\delta$ or $\neg (n \in c_\delta)$) eventually occurs in the sequence.

Each sequence with the properties (1) and (2) determines a set C of integers (or a sequence $\{C_\delta\}$ of such sets). We define C as the set of such n that the formula $n \in c$ eventually occurs in the sequence $P_1, P_2, \ldots$ (similarly C_δ is the set of integers n such that the formula $n \in c_\delta$ eventually occurs in the sequence $P_1, P_2, \ldots$).

Sets C (or C_δ) defined in this way are called generic.

Cohen showed in his independence proofs that if one adjoins to M_{α_0} any generic set C (or any sequence of such sets) and closes the resulting set with respect to the operations φ, then one obtains a model in which the axiom of constructibility (or the continuum hypothesis) is not valid. The details of this proof are too involved to be given here.

Similar construction allowed Cohen and other mathematicians working with the method of forcing to obtain other proofs of independence. *E.g.* Cohen showed that if the system of Zermelo—Fraenkel without the axiom of choice is consistent, then it remains consistent after adjunction of an axiom stating that there is a denumerable set S of pairs each element of which is a set of real numbers and such that there is no choice set for S.

The main advantage of Cohen's method as compared with the permutation method is that it is applicable not only to the Zermelo—Fraenkel (or Gödel—Bernays) set theory but also to most theories obtained from them by the adjunction of axioms of infinity which we discussed earlier and of axioms which determine the number of nonsets. Thus only now, after this method was created, can we consider the independence problems as solved.

Properties of generic sets. Besides solving the independence problem, Cohen's method suggested many new problems. The most interesting is the study of generic sets. These sets seem to satisfy (with respect to a given model M_{α_0}) intuitions which underly Brouwer's intuitionistic conception of a set (of integers). There is no way to define (in a given language) any such set individually; we can only give information concerning any finite number of individual members of such a set. The work on these sets is in full progress. We shall quote only one result obtained by Cohen: he showed that the family of generic sets is very big. In fact this family is of second category in the Baire space of all sets of integers.

The notion of a generic set can be defined not only for set theory but also for arithmetic and other theories. Ryll—Nardzewski (in an unpublished paper) considered topological properties of generic sets. He proved that if F is an arithmetical formula with a parameter Z ranging over generic sets, then the truth-value of F is a continuous function of Z. This theorem allowed him to characterize the family of generic sets and to prove that it is of second category without recourse to the notion of forcing.

The same results can probably be obtained for set theory but no definite proof exists as yet.

Final remarks. The philosophical importance of set theory is obvious: the fundamental epistemological questions connected with mathematics can be best illustrated on set-theoretical problems.The most fundamental question is, of course, how to decide between the formalistic and "Platonistic" conceptions of the abstract parts of mathematics. (In the case of the less abstract parts the problem is not as acute, because we understand the way in which needs of practical life and of science formed the mathematical theories.)

Cohen's work did not create these philosophical questions. They existed long before meta-mathematics was created. Still the rigorous proof that there are two consistent and mutually incompatible set theories stirred the imagination of many mathematicians who were formerly indifferent to these questions. It is at present hard to tell whether mathematics will accept the existence of these two incompatible set theories or will try to find new axioms which will eliminate one of them or finally will try to limit mathematics to more finitistic domains. We see that the issue between Platonists, formalists and intuitionists is as undecided to-day as it was fifty years ago.

Lecture XVI

On direct and reduced products

We devote this lecture to a rather special algebraic construction which proved of great value in solving various problems of mathematical logic.

The direct product. Let I be a set $\neq 0$ and let $\boldsymbol{M}_i = \langle A_i, R_i \rangle$ for i in I be a model in which A_i is a non-void set and R_i a binary relation. The restriction to models of this special type is not essential: the models $\boldsymbol{M}_i$ could be *any* models as long as they all are of the same type, *i.e.* the number of relations must be the same in all the models $\boldsymbol{M}_i$ and if $R_{i1}, R_{i2}, \ldots$ are relations of $\boldsymbol{M}_i$ then the number of arguments in R_{ik} must be equal to the number of arguments of R_{jk} for arbitrary i, j in I.

The direct product of the models $\boldsymbol{M}_i$ is the model $\boldsymbol{P} = \boldsymbol{P}_{i \in I} \boldsymbol{M}_i = \langle A, R \rangle$ where $A = \boldsymbol{P}_{i \in I} A_i$ is the set of all functions f with domain I satisfying the condition $f(i) \in A_i$ for $i \in I$ and R is a binary relation defined thus:

$$f R g = \bigwedge_{i \in I} [f(i) R_i g(i)]. \tag{1}$$

In the case when all $\boldsymbol{M}_i$ are equal to a model $\boldsymbol{M}$ we denote $\boldsymbol{P}_{i \in I} \boldsymbol{M}_i$ by $\boldsymbol{M}^I$ and call it the direct power of $\boldsymbol{M}$.

This standard algebraic construction has been generalized by Feferman and Vaught [40] and applied to several decision problems. In the generalized notion which they introduced the relation R depends not only of the relations R_i but of some auxiliary relations defined in I.

The reduced direct product. This product is a special case of the notion introduced by Feferman and Vaught but we shall define it directly. Let F be a maximal filter in the Boolean algebra of the family of all subsets of I. This filter determines an equivalence relation $\sim$ in the set A:

$$f \sim g \equiv \{i \in I : f(i) = g(i)\} \in F.$$

The relation $\sim$ is compatible with R, *i.e.*

$$(f \sim g) \wedge (f' \sim g') \rightarrow [fRg \equiv f'Rg'],$$

hence we can form the quotient system $P/\sim\; = \;<A/\sim, R/\sim>$ whose universe is the set $A/\sim$ of all equivalence classes $f/\sim\; = \{g : g \sim f\}$ and whose relation $R/\sim$ is defined by the equivalence

$$(f/\sim)\ R/\sim\ (g/\sim) \equiv fRg$$

The model $<A/\sim, R/\sim>$ is called the reduced product of the $\boldsymbol{M}_i$. It depends of course on the filter F.

The form of the definition given above is due to Tarski (*cf.* [43]) who obtained it by simplifying a much more involved definition proposed by Łoś [128]. Łoś's construction, though no more in use at present, is worth mentioning if only for its connections with many-valued logics. Łoś considered the set $\{i \in I : f(i) = g(i)\}$ as a kind of "distance" of two functions f and g and considered the direct product $\boldsymbol{P}_i\boldsymbol{M}_i$ as a special case of a "logical space" which insofar differs from the usual models as the values of atomic formulae aRb are not merely the truth values $\mathfrak{F}$, $\mathfrak{V}$ but elements of an arbitrary Boolean algebra. Identifying suitably elements of a "logical space" Łoś arrived at what we call now reduced products.

The essential property of reduced products (discovered already by Łoś [128]) is expressed by the equivalence.

(2) $\models_{\boldsymbol{P}/\sim} H[f_1/\sim, \ldots, f_k/\sim] \equiv \{i \in I : \models_{\boldsymbol{M}_i} H[f_1(i), \ldots, f_k(i)]\} \in F$

in which H is an arbitrary first-order formula with one binary predicate and possibly the identity symbol and with k free variables.

The proof of (2) is very easy and uses induction with respect to the number of logical connectives in H.

It follows from (2) that

(3) $\boldsymbol{M}^I/\sim$ is elementarily equivalent with $\boldsymbol{M}$.

The model $\boldsymbol{M}^I/\sim$ contains a sub-model isomorphic with $\boldsymbol{M}$ namely the sub-model consisting of equivalence classes $f/\sim$ where f is a constant function. Identifying $\boldsymbol{M}$ with this sub-model we obtain from (2)

(4) $\boldsymbol{M}^I/\sim$ is an elementary extension of $\boldsymbol{M}$.

If F is a principal filter $F = \{X \subset I : i_0 \in X\}$ where i_0 is a fixed element of I, then the reduced product $\boldsymbol{P}_{i \in I}\, \boldsymbol{M}_i/\sim$ is isomorphic with $\boldsymbol{M}_{i_0}$ and the whole construction is trivial. Thus the only interesting case is when the filter F is non-principal.

Compactness theorem. The first striking application of the reduced direct products was the following simple proof of the compactness theorem (see Frayne—Morel—Scott [43]):

Let X be a set of sentences such that each finite subset of X is satisfiable. We want to prove that there is a model in which all sentences of X are true.

In order to obtain this result by the use of reduced products let I be the family of all finite subsets of X and denote, for $i \in I$ by $\boldsymbol{M}_i$ a model in which all sentences belonging to i are true. For arbitrary sentences $S_1, \ldots, S_k$ in X we denote by $J(S_1, \ldots, S_k)$ the family of sets $i \in I$ such that $(S_1 \in i) \wedge \ldots \wedge (S_k \in i)$. It is obvious that all the families $J(S_1, \ldots, S_k)$ (where $S_1, \ldots, S_k$ range over X and k over arbitrary positive integers) form a filter F_0 in the Boolean algebra of all subsets of I and $0 \bar{\in} F_0$. We extend this filter to a maximal filter F not containing 0. The required model in which all the sentences of X are valid is the reduced product $\boldsymbol{P} = \boldsymbol{P}_{i \in I} \boldsymbol{M}_i/\sim$.

Indeed if $S \in X$, then the set $\{i \in I : S \in i\}$ belongs to F_0 and hence to F whence, by (2), S is true in P.

Another theorem which can be proved in the same way is this: if every finite subset j of a model $\boldsymbol{M}$ can be extended to a model $\boldsymbol{M}_j$ which belongs to a given elementary class K, then the whole $\boldsymbol{M}$ can be so extended.

Both these results can be obtained by an application of Gödel's completeness theorem. The proofs using reduced products are more direct and hence give a better insight in the structure of the models whose existence is stated in the theorems.

Other, deeper applications of the reduced products were mentioned in lecture XIII.

Applications to the abstract set theory. Let us assume that for each i in I the relation R_i orders the universe A_i of $\boldsymbol{M}_i$. It follows from (2) that the universe of the reduced product $\boldsymbol{P} = \boldsymbol{P}_{i \in I} \boldsymbol{M}_i/\sim$ is ordered by the relation $R/\sim$. If the R_i are well-orderings, then, as simple examples show, $\boldsymbol{P}$ need not be well-ordered. However, the following theorem is true:

(5) If F is a σ-multiplicative filter (*i.e.* if $X_n \in F$ for $n = 0, 1, 2, \ldots$ implies $\bigcap_n X_n \in F$), then the reduced product of well-ordered models is itself well-ordered.

Theorem (5) is a basis of all results concerning denumerably additive filters in Boolean algebras of all subsets of a set. We shall show this by sketching the proof (due to Keisler [91]) that the first inaccessible non-denumerable cardinal $\aleph_a$ is not measurable, *i.e.* that there is no non-trivial denumerably multiplicative maximal filter F in the Boolean algebra of all subsets of a set X of power $\aleph_a$.

Let us assume that such a filter F exists and that X is the set of all ordinal $< \omega_a$ ordered by the $\leq$ relation. It can then be shown that F is $\aleph_\lambda$-multiplicative for each $\lambda < a$. According to (5) the reduced power $X^X/\sim$ determined by F is well-ordered. The set $X^X/\sim$ contains a subset similar to X consisting of equivalence classes $f/\sim$ where f is constant. This subset which we shall identify with X can be shown to be a segment of $X^X/\sim$. If F were a principal filter then $X^X/\sim$ and X would be isomorphic. Otherwise the order type of $X^X/\sim$ is greater than X since *e.g.* the equivalence class of the identity function $\delta(x) = x$ does not belong to X. Let $\varphi/\sim$ be the first equivalence class in $X^X/\sim$ after X and consider the sets:

$$A = \{x : \varphi(x) \text{ has an immediate predecessor } \varphi^*(x)\},$$

$$B = \{x : \varphi(x) \text{ has no predecessor and there is an ordinal } \xi < \varphi(x) \text{ such that } \overline{\varphi(x)} \leq 2^{\bar{\xi}}\},$$

$$C = \{x : (\varphi(x) \text{ has no predecessor}) \wedge \bigwedge_{\xi < \varphi(x)} (\overline{\varphi(x)} > 2^{\bar{\xi}})\}$$

(we denote as usual by $\bar{\xi}$ the cardinal number of the set $\{\eta : \eta < \xi\}$). Since $X = A \cup B \cup C$ one at least of the sets A, B, C is in F. If $A \in F$, then putting $\psi(x) = \varphi^*(x)$ for $x \in A$ and $\psi(x) = 0$ otherwise we obtain a function ψ such that $\psi/\sim$ precedes $\varphi/\sim$ in $X^X/\sim$. Hence ψ is a constant and so is φ. This contradiction shows that $A \mathrel{\bar{\in}} F$.

If $B \in F$, then we denote by $\psi(x)$ the smallest $\xi < \varphi(x)$ such that $\overline{\varphi(x)} \leq 2^{\bar{\xi}}$ and infer similarly that φ is a constant, which is impossible.

Finally if $C \in F$, then for each x in C there must exist an ordinal $\lambda(x) < \varphi(x)$ such that $\varphi(x)$ is co-final with $\lambda(x)$, since $\varphi(x)$ is smaller than the first inaccessible cardinal. Again we show that λ is a constant function, $\lambda(x) = \lambda_0$. Hence $\varphi(x) = lim_{\xi < \lambda_0} \vartheta(x, \xi)$ for all $x \in C$.

For each fixed $\xi < \lambda_0$ the function ϑ considered as a function of

x alone has values $< \varphi(x)$, whence it is a constant, $\vartheta(x, \xi) = \gamma(\xi)$ on a set Q_ξ which belongs to F. The intersection $D = \bigcap_{\xi<\lambda_0} Q_\xi \cap C$ belongs to F and for x in D we have $\vartheta(x, \xi) = \gamma(\xi)$. It follows that for x in D the equation $\varphi(x) = lim_{\xi<\lambda_0} \vartheta(x, \xi) = lim_{\xi<\lambda_0} \gamma(\xi)$ is true whence we infer that φ is equal to a constant on a set which belongs to F. This however contradicts the definition of φ.

Hence we obtain a contradiction in all cases. This shows that the filter F with the properties we enumerated cannot exist.

By using a similar argument Scott [198] proved that if there is a non-trivial maximal denumerably multiplicative filter F in the Boolean algebra of a set X, then F is non-constructible. The most recent results (Gaifman [49]) show that under the same assumption for any infinite λ there are no more than $\bar{\lambda}$ constructible subsets of $\{\xi : \xi < \lambda\}$. In particular there are only denumerably many constructible sets of integers provided that measurable cardinals exist! This result was obtained before Gaifman by Rawbotten.

Applications of reduced products to non-standard models of arithmetic. Let $\boldsymbol{M}_0$ be the model $<\omega, \Sigma, \Pi>$ where ω is the set of integers and Σ, Π are the relations $x = y + z$, $x = y \cdot z$. We call $\boldsymbol{M}_0$ the standard model of arithmetic.

For every infinite set I and every non-trivial maximal filter F the reduced power $\boldsymbol{M}_0^I/\sim$ is by (2) elementarily equivalent to $\boldsymbol{M}_0$ but is not isomorphic to $\boldsymbol{M}_0$ since the equivalence class of the diagonal function is different from the values of all numerals in $\boldsymbol{M}_0^I/\sim$. Using the downward Skolem—Löwenheim theorem we can obtain from $\boldsymbol{M}_0^I/\sim$ a denumerable non-standard model elementarily equivalent to $\boldsymbol{M}_0$. In this way we have a simple method of constructing non-standard models.

Scott [196] modified this method by showing that one obtains a non-standard denumerable model by limiting from the start the family $\boldsymbol{P} A_t$. He considered the set D of functions definable in $\boldsymbol{M}_0$ and a maximal non-trivial filter F in the Boolean algebra of definable subsets of $\boldsymbol{M}_0$ (the existence of this filter can be established without the axiom of choice which is necessary when one wants to prove the existence of a maximal non-trivial filter in the Boolean algebra of all subsets of an infinite set). Repeating the construction of the reduced power but including in it only equivalence classes of definable functions, Scott obtained a non-standard denumerable model elementarily equivalent to $\boldsymbol{M}_0$. He also showed that this model is isomorphic with a model constructed by Skolem [209]. The method used by Skolem, while formally different from the method of reduced powers, is thus essentially equivalent to the latter.

Scott's analysis showed that one does not obtain a model if one replaces D by a more restricted set, *e.g.* the set of polynomials or of recursive functions. These negative results showed that it is not easy to construct effective examples of non-standard models. We should mention here results (obtained among others by Feferman [35], Scott [196], these authors jointly with Tennenbaum [39], and Rabin [167]) which show that there are no recursive non-standard models satisfying the axioms of Peano. All these results explain why it is so difficult to establish the independence of number-theoretical sentences from the arithmetical axioms by the use of non-standard models: In order to find such applications one should have a much better knowledge of the structure of these models and more direct methods of their constructions than we have at present.

Of other works on non-standard models we here mention an important paper by Specker—MacDowell [132] who used the method of Skolem to establish the following theorem: every model $\boldsymbol{M}$ of Peano's axioms can be extended to a model $\boldsymbol{M}'$ in such a way that $\boldsymbol{M}'$ is an elementary extension of $\boldsymbol{M}$ and all elements of $\boldsymbol{M}'-\boldsymbol{M}$ are greater than the elements of $\boldsymbol{M}$. Another promising direction of the studies of non-standard models seems to be one concerned with automorphisms of such models (*cf.* Ehrenfeucht—Mostowski [31]).

Non-standard models for analysis. The method of reduced products can be used to obtain non-standard models for an arbitrary theory. A. Robinson investigated such models in a series of papers (see for instance [182], [183]). The theory which he considered is very rich and contains symbols for all functions and relations defined in the set of real numbers. Axioms of the theory are all sentences true for real numbers. Robinson showed that this theory admits non-standard models. He himself constructed these models by the use of the extended completeness theorem. Other writers prefer to construct them by the use of reduced products.

Each non-standard model for Robinson's theory is a non-archimedean field R^* which is an elementary extension of the field R of real numbers. Each function f defined on R has an extension f^* which is defined on R^* and each first-order property of f is preserved by the extension.

The field R^* contains actually infinitely small and actually infinitely great elements, *i.e.* elements a, A such that $0 < |a| < 1/n$

and $|A| > n$ for each integer n. It is thus possible to derive in R^* the basic theorems of analysis using the Leibnizian ideas of infinitesimals. Thus *e.g.* we call a function f continuous at the point x if $f(x + a) - f(x)$ is infinitely small for each infinitely small number a. A derivative of f at the point x is a number d such that the difference $\{[f(x + a) - f(x)]/a\} - d$ is infinitely small for each infinitesimal a. Robinson showed that one obtains in this way a completely rigourous theory which is formally identical with the classical analysis. It is at present not clear, whether the non-standard analysis will bring essentially new results. It is nevertheless remarkable that we can give a clear and precise presentation of ideas which were considered obscure for almost 300 years.

* * *

We stop here our presentation of what we consider as the most important results in the recent development of logic and the foundations of mathematics. The rate of development of these domains is presently so rapid that many new excellent results will certainly appear before these lectures will come to the hands of prospective readers. Let us hope that these new results will not only bring new interesting insights into the details but also allow us to form a sound judgment about the outstanding problems in the philosophy of mathematics which have been waiting so long for a final solution.

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MODELS OF SET THEORY

by

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Lecture I.

Aim of the lectures: to outline various methods used recently in construction of models for axioms of set theory. No completeness in pursuing this aim is attempted.

In the introductory lecture I we describe three systems of axioms for abstract set theory. In all these systems there are two primitive notions: "class" and "membership". We define sets as classes which are capable of being members of other classes: x is a set if and only if there is a class y such that $x \in y$. We also define atoms as objects which have no elements.

The distinction between sets and classes was noted already by Cantor who distinguished between the "consistent sets" and "inconsistent sets" i.e. sets and (proper) classes in the modern terminology. As we shall see there are axiomatic systems in which the existence of proper classes is assumed and other systems in which their existence is excluded.

Set - theoretic formulae: Let $x_1, x_2, \ldots$ be a sequence of different letters (the variables). We shall often replace "x" by any other letter and omit subscripts.

(i) Expressions $Cl(x_i)$, $x_i \in x_j$, $x_i = x_j$ are formulae; the variable x_i is the unique free variable in the first of them and the variables x_i, x_j are the unique free variables of the remaining two.

(ii) If F and G are formulae, then so is the expression $(F) \mid (G)$; the variable x_i is free in this formula if and only if it is free in F or in G.

(iii) If F is a formula, then so is $(x_j)F$; the variable x_i is free in this formula if and only if $i \neq j$ and x_i is free in F.

(The symbol $\mid$ is the Sheffer's stroke and (x_i) is the general quantifier. We define the propositional connectives different from $\mid$ in the usual way; also the existential quantifier is defined as $\neg (x_i) \neg F.$).

We denote by Fr(F) the set of those i for which x_i is a free variable of F.

Predicative formulae are those formulae in which all quantifiers are limited to sets or atomes. We obtain a precise definition of this class of formulae by replacing the rule (iii) by the following:

(iii') If F is a formula, then so is $(x_j)(x_k)\left[(x_j \in x_k) \rightarrow F\right]$ where k is not in Fr(F) and $k \neq j$.

The above expression can also be written as $(x_j)(x_k)\left[(x_j \in x_k) \mid (F \mid F)\right]$

Definitions: x is a set (in symbols S(x)): $Cl(x) \& (Ey)(x \in y)$;

x is a proper class (Pcl(x)): $Cl(x) \& \neg S(x)$;

x is an atom; (At(x)): $(y)(y \notin x)$

We now list axioms of the different systems which we shall consider. The first 9 axioms are common to them all:

Axioms for classes:

Ext. $Cl(x) \& Cl(y) \& (z)\left[(z \in x) \equiv (z \in y)\right] \rightarrow (x = y)$;

Cl_1. $Pcl(x) \rightarrow (Ey)(y \in x)$;

Cl_2. $(y \in x) \rightarrow Cl(x)$;

Fund. $(y \in x) \rightarrow (Ez) \{ (z \in x) \& (t) [(t \in z) \rightarrow (t \notin x)] \}$.

Remarks. Ext is the familiar axiom of extensionality. Cl_1 says that proper classes have elements, i.e., are not atoms. Cl_2 says that whatever has elements is a class. Fund is the axiom of foundation and says that each class which has elements contains at least one minimal element, i.e., one whose elements are not elements of the given class; of course the minimal element may be an atom.

Axioms for sets.

Pair. $[S(x) \vee At(x)] \ \& \ [S(y) \vee At(y)] \rightarrow (Ez) \Big((S(z) \ \& \ (t) \{ (t \in z) \equiv [(t = x) \vee (t = y)] \} \Big)$;

Sum. $S(x) \rightarrow (Eu) \Big(S(u) \ \& \ (z) \{ (z \in u) \equiv (Et) [(z \in t) \& (t \in x)] \} \Big)$

Pot. $S(x) \rightarrow (Ep) [S(p) \& (z) \{ (z \in p) \equiv (Cl(z) \ \& \ (t) [(t \in z) \rightarrow (t \in x)]) \}]$

Emp. $(Ex) [S(x) \ \& (y) \ (y \notin x)]$

Inf. $(Ex) \{ S(x) \ \& \ (Eu)(u \in x) \& (v) \ [(v \in x) \rightarrow (Ew) \ \Big((w \in x) \ \& \ (t) \ \{ (t \in w) \equiv [(t \in v) \vee (t = v)] \} \Big)] \}$

Remarks. These axioms state the existence of the unordered pair of any two objects (which may be sets or atoms), of a union of sets which belong to a given set, of the power set of a given set, of the empty set and of at least one infinite set. The notation for these sets will be the usual one: $\{ x, y \}$ for the unordered pair, $\cup x$ for the union, $P(x)$ for the power set and 0 for the empty set. The uniqueness of these sets follows from Ext.

Further axioms will fulfill a twofold role: first they will determine the number of atoms; secondly they will express the idea that an image of a set under an arbitrary mapping is again a set.

The first question does not seem to be very important for the abstract set theory and it is nowadays customary to dismiss it by simply assuming that with the exception of 0 there are no atoms altogether. We shall follow this custom and adopt the following axiom:

Noatoms: $(x \in y)\&(x \neq 0) \rightarrow S(x)$.

The second aim is much more important. It is achieved in many different ways according to the system of axioms which one adopts.

The system of Zermelo-Fraenkel. In this system we first of all assume that there are no proper classes:

Nopcl: $Cl(x) \rightarrow S(x)$.

This axiom together with Noatoms allows us to simplify the axioms previously given by omitting everywhere the expressions "$S(x)$".

The idea that an image of a set is again a set is expressed in ZF by the following axiom schema:

Subst_{ZF}. $(x)(E!y)F \rightarrow (a)(Eb)(y)\{(y \in b) \equiv (Ex)[(x \in a)\&F]\}$

in which F is an arbitrary formula such that b is not its free variable.

System of Gödel-Bernays. In this system we assume that every predicative formula determines a class and express the idea that the image of a set is again a set not by means of a schema but by a single axiom involving the notion of a class.

First we define the ordered pairs: $\langle x_i, x_j \rangle = \{x_i, \{x_i, x_j\}\}$.

A class whose elements are ordered pairs is called a relation; it is called a function (in symbols $Fn(x)$) if it satisfies the condition:

$(w)(u)(v)\{[(\langle u, v\rangle \in x)\&(\langle u, w\rangle \in x)] \rightarrow (v = w)\}$.

We denote by $x''a$ the class t such that $(y)\{(y \in t) \equiv (Ez)[(z \in a)\&(\langle z, y\rangle \in x)]\}$ provided that such a class exists.

The axioms which we admit in the system GB are now as follows:

Clex_{GB}. $(Ex)(Cl(x) \& (u)\{S(u) \rightarrow [(u \in x) \equiv F]\})$
Subst_{GB}. $Fn(x) \& S(a) \rightarrow S(x''a)$.

Remarks. In Clex_{GB} (class existence scheme) F is a predicative formula in which the variable x is not free. The class whose existence is stated in Clex_{GB} is denoted by $\{u:F\}$; e.g. $\{u: u = u\}$ is the universal class consisting of all sets.

Subst_{GB} is called the axiom of substitution; note that the class $x''a$ exists by virtue of the class existence scheme.

System of Morse. In this system we leave the axiom of substitution unchanged and extend the class existence scheme by allowing arbitrary formulae F. We denote these axioms by Clex_M and Subst_M.

Comparison of the systems ZF, GB and M.

The following theorems are easily established:

I. ZF is interpretable in GB.

Proof. We interpret classes of ZF as sets of **GB.**

II. GB is a subtheory of M (obvious).

III. GB is an inessential extension of ZF. By this we mean that each predicative formula provable in GB is also provable in ZF.

Proof. If F is predicative and provable in ZF, then by I (and its proof), F is provable in GB. Let F be provable in GB. In order to establish its provability in ZF it will be sufficient to show that every model of ZF can be extended to a model of GB in such a way that sets of the new model be identical with sets of the old model. Let therefore M be a model of ZF and let M' be the family of sets $\{t \in M: (t, a_1, \ldots, a_n) \text{ satisfies } F \text{ in } M\}$; here F is a predicative formula with n+1 free variables and $a_1, \ldots, a_n$ are elements of M. Interpret classes as elements of M' and the membership relation as $\in$. The resulting model satisfies all the axioms of GB and contains a submodel isomorphic with M and consisting of sets $\{t \in M: t \in x\}$. All the sets of M' have this form which proves the theorem.

We shall show later that M is an essential extension of GB.

Relativisation. Let A(x) be formula in which x is a free variable; A may contain free variables other than x . From every formula F we may obtain a new formula by replacing each quantifier (x_i) by $(x_i)\left[A(x_i) \rightarrow \ldots\right]$. The new formula is denoted by $F^{(A)}$ and called the formula F relativised to A. We assume that formulae F and A do not have any variable in common.

It is obvious that $(F \mid G)^{(A)} \equiv (F^{(A)} \mid G^{(A)})$, $((x_i)F)^{(A)} \equiv (x_i)F^{(A)}$ if $i \notin Fr(F)$ and $((x_i)F)^{(A)} \equiv (x_i)\left[A(x_i) \rightarrow F^{(A)}\right]$ if $i \in Fr(F)$. The symbol $\equiv$ denotes here the logical equivalence relation (not the connective).

Functions whose values are classes. We insert here a note about functions whose arguments are sets and values are classes. Of course this notion is important only for systems GB and M.

A function all of whose values are non void classes and whose arguments are sets can be defined as a relation. The value of such a function f for the argument x is simply $f_x = \{y: \langle x, y\rangle \in f\}$. If we want to admit 0 as a possible value of a function, then we define f as a pair a, r where r is a relation and a is a class which contains Dom(r) as a subclass. The value of f for the argument x is r_x or 0 according as x is an element of Dom(r) or an element of a-Dom(r).

We write "$f:a\to y$" instead of "f is a function with domain a whose values are subclasses of y". The elementary operations on functions such as superposition or restriction of the domain to a subclass can easily be defined for such functions. A function with domain a whose values for i in a are explicity given will be denoted by $F_{i\in a} f_i$. If $f:a\to z$, $i\notin a$ and $u\subseteq z$, then we denote by $f+(i/u)$ the uniquely determined function with domain $a\cup\{i\}$ whose restriction to a is f and whose value for the argument i is u.

Semantical notions defined in set theory. The basic semantical notion is that of satisfaction: a sequence f of subclasses of a class x satisfy a formula F in x. If we want to discuss this relation in set theory, we have to replace formulae by certain sets. The following theorems are provable in ZF:

IV. There is a smallest set Frm such that $\langle 0,i\rangle$, $\langle 1,\langle i,j\rangle\rangle$, $\langle 2,\langle i,j\rangle\rangle$ are elements of Frm for arbitrary integers i, j and is such that whenever a, b belong to Frm, then so do $\langle 3,\langle a,b\rangle\rangle$ and $\langle 4,\langle i,a\rangle\rangle$ for arbitrary i in ω (the set of integers).

We write $\ulcorner x_i \in x_j \urcorner$ for $\langle 1,\langle i,j\rangle\rangle$ and similarly $\ulcorner Cl(x_i)\urcorner$ for $\langle 0,i\rangle$ and $\ulcorner x_i = x_j\urcorner$ for $\langle 2,\langle i,j\rangle\rangle$. We also write $\ulcorner a \mid b\urcorner$ for $\langle 3,\langle a,b\rangle\rangle$ and $\ulcorner (x_i)a\urcorner$ for $\langle 4,\langle i,a\rangle\rangle$. We call Frm the set of "Gödel sets" of formulae.

By Frm_{pr} we denote the subset of Frm consisting of Gödel sets of predicative formulae.

V. There is a function Fr defined on Frm such that $Fr(\ulcorner Cl(x_i)\urcorner) = \{i\}$, $Fr(\ulcorner x_i \in x_j\urcorner) = Fr(\ulcorner x_i = x_j\urcorner) = \{i, j\}$ and $Fr(\ulcorner a|b\urcorner) = Fr(a) \cup Fr(b)$, $Fr(\ulcorner (x_i)a\urcorner) = Fr(a) - \{i\}$. We call Fr(a) the set of free variables of a.

We call a formula S with 3 free variables a predicative satisfaction formula for one of our three systems ZF, GB, M if the following formulae are provable in these systems:

(i) $S(x, y, z) \rightarrow (x \in Frm_{pr})\&Cl(z)\&(y \in z^{Fr(x)})$;

(ii) $S(\ulcorner Cl(x_i)\urcorner, \{<i, y>\}, z) \equiv (y \in z)$;

$S(\ulcorner x_i \in x_j\urcorner, \{<i, y'>, <j, y''>\}, z) \equiv (y' \in y'')$;

$S(\ulcorner x_i = x_j\urcorner, \{<i, y'>, <j, y''>\}, z) \equiv (y! = y'')$;

(iii) $S(\ulcorner a|b\urcorner, y, z) \equiv [\neg S(a, y \mid Fr(a), z) \vee \neg S(b, y \mid Fr(b), z)]$;

(iv) $S(\ulcorner (x_i)a\urcorner, y, z) \equiv S(a, y, z)$ if $i \notin Fr(a)$; otherwise:

(v) $S(\ulcorner (x_i)a\urcorner, y, z) \equiv (u)[(u \in z) \rightarrow S(a, y \cup \{<i, u>\}, z)]$.

A formula S(x, y, z) with three free variables will be called a satisfaction formula for ZF, GB or M if the following formulae are provable in the corresponding system:

(i') $S(x, y, z) \rightarrow (x \in Frm) \& Cl(z) \& (y: Fr(x) \rightarrow z)$,

(ii') $S(\ulcorner Cl(x_i)\urcorner, y, z) \equiv y_i \subseteq z$,

$S(\ulcorner x_i \in x_j\urcorner, y, z) \equiv y_i \in y_j$;

$S(\ulcorner x_i = x_j\urcorner, y, z) \equiv y_i = y_j$,

(iii') and (iv') as (iii) and (iv),

(v') $S(\ulcorner (x_i)a\urcorner, y, z) \equiv (u)[(u \subseteq z) \rightarrow S(a, y + (i/u), z)]$.

In case of the system ZF there is no need to distinguish between the satisfaction and predicative satisfaction formulae because sequences of classes can be identified with sequences of sets.

The following theorems exhibit essential differences between systems .ZF, GB and M:

VI. There is a satisfaction formula for ZF.

VII. There is a predicative satisfaction formula for M but - provided that M is consistent - no satisfaction formula for M.

VIII. If GB is consistent, then there is no predicative satisfaction formula for GB, and also no satisfaction formula for GB.

We shall sketch the proofs of these theorems.

In order to prove VI we define (in ZF) two operations on sets of finite sequences:

$$\mathrm{Str}(z, k_1, k_2, d_1, d_2) = \{ y \in z^{d_1 \cup d_2} : (y \mid d_1 \notin k_1) \vee (y \mid d_2 \notin k_2) ,$$

$$\mathrm{Qu}(z, k, i, d) = k \text{ if } i \notin d; \text{ otherwise}$$

$$= \{ y \in z^{d - \{i\}} : (u) \left[(u \in z) \rightarrow y \cup \{ \langle i, u \rangle \} \in k \right] \}$$

We obtain a predicative satisfaction formula for ZF by expressing in the language of ZF the conjuction of the following relations: z is a set, $x \in \mathrm{Frm}$, $y \in z^{\mathrm{Fr}(x)}$; there is a finite sequence f such that for each i in Dom(f) either f_i is $\ulcorner \mathrm{Cl}(x_p) \urcorner$ or it is $\ulcorner x_p \in x_q \urcorner$ or it is $\ulcorner x_p = x_q \urcorner$ for some p, q or there exist j, k both smaller than i such that $f_i = \ulcorner f_j \mid f_k \urcorner$ or finally there is a $j < i$ and a p such that $f_i = \ulcorner (x_p)\, f_j \urcorner$; the last term of f is x; there is a finite sequence k with Dom(k) = Dom(f) such that for each i in Dom(f):

if f_i is $\ulcorner \mathrm{Cl}(x_p) \urcorner$, then $k_i = \{ \{ \langle p, u \rangle \} : u \in z \}$;

if f_i is $\ulcorner x_p \in x_q \urcorner$, then $k_i = \{ \{ \langle p, u \rangle, \langle q, v \rangle \} : (u \in v) \;\&\; (u \in z) \;\&\; (v \in z)$;

if $f_i = \ulcorner x_p = x_q \urcorner$, then $k_i = \{ \{ \langle p, u \rangle, \langle q, u \rangle \} : u \in z \}$,

if $f_i = \ulcorner f_j | f_\ell \urcorner$, then $k_i = Str(z, k_j, k_\ell, Fr(f_j), Fr(f_\ell))$,
if $f_i = \ulcorner (x_p)\ f_j \urcorner$, then $k_i = Qu(z, k_j, p, Fr(f_j))$;
y is an element of the last term of k.

We proceed similarly in order to prove the positive part of VII.

The negative part of theorem VII is proved by using the well known technique of Gödel. We assume that there exists a satisfaction formula S for M and denote by s a function such that for every n in Frm with $Fr(n) = \{0\}$ the set $s(n)$ is the Gödel set of the formula resulting from the formula with the Gödel set n by substituting n for its unique free variable. Let n_0 be the Gödel set of the formula $\neg S(s(x_0), 0, V)$ where V is the universal class; we rely here of course on the fact that the function s can be defined in M. Then $\neg S(s(n_0), 0, V)$ has the Gödel set $s(n_0)$. We prove by induction on the length of G that if G is a formula with p free variables $x_1, \dots, x_p$ and with the Gödel set g, then the following equivalence is provable in M:

$$G(x_1, \dots, x_p) \equiv S(g, F_{i \leq p} x_i, V).$$

Using this equivalence for $G = \neg S(s(n_0), 0, V)$ we obtain the result that the equivalence $\neg G \equiv G$ would be provable in M and hence M would be inconsistent.

The detailed proof of theorem VIII will be given later. The main idea is this: we assume the existence of a predicative satisfaction formula for GB and prove in GB the existence of a set which is a model for ZF. From this we derive (always in GB) the existence of a set which is a model for GB and thus infer that

GB is consistent. Thus the consistency of GB would be provable in GB which as is well known entails the inconsistency of GB.

We conclude with a theorem which we shall need later and which can be proved by using the same technique as the one used in the proof of theorem VII.

IX. For each predicative formula T the conjuction of the following formulae is refutable in ZF :

(i'') $(x)(y)\ \Big[T(x,y) \longrightarrow (x \in Frm_{pr})\ \&\ (y$ is a sequence of sets) $\&\ (Dom(y) = Fr(x))\Big]$,

(ii'') $(i)_{\omega}(u)\ \Big[\ T(\ulcorner Cl(x_i)\urcorner,\ \{<i,u>\}) \equiv Cl(u)\Big]$,

$(i)_{\omega}(j)_{\omega}\ (u)(v)\ \Big[\ T(\ulcorner x_i \in x_j\urcorner\ ,\ \{<i,u>, <j,v>\}) \equiv (u \in v)$,

similarly as above with " $\in$ " replaced by " $=$ " ;

(iii'') $(a)(b)(y)\ \Big((a \in Frm_{pr})\&\ (b \in Frm_{pr}) \longrightarrow \Big\{T(\ulcorner\overline{a} \mid \overline{b}\urcorner, y) \equiv \Big[\neg T(a,\ y \mid Fr(a)) \vee \neg T(b,\ y \mid Fr(b))\Big]\Big\}\Big)$

(iv'') $(a)(i)_{\omega}\ (y)\Big\{(a \in Frm_{pr})\&(i \notin Fr(a)) \longrightarrow \Big[T(\ulcorner(x_i)a\urcorner, y) \equiv T(a,y)\Big]\Big\}$

(v'') $(a)(i)_{\omega}\ (y)\Big\{(a \in Frm_{pr})\&(i \in Fr(a)) \rightarrow \Big[T(\ulcorner\overline{(x_i)}\overline{a}\urcorner,\ y) \equiv (u)\ T(a, y \cup \{<i,u>\})\Big]\Big\}$.

Proof. Let t be the Gödel set of the formula $\neg T(s(x_0), 0)$. Hence the Gödel set of the formula $\neg T(s(t), 0)$ is $s(t) = t_0$. Using (i'') - (v'') we infer by induction that if h is the Gödel set of a predicative formula H whose free variables are $x_{i_j}, (j = 0, 1, \ldots, k-1)$ and if f is a sequence with domain $Fr(H)$, then $T(h,f) \equiv H(f_{i_0}, \ldots, f_{i_{k-1}})$. If we take in particular $H = \neg T(t_0, 0)$, we obtain $H \equiv \neg H$. Hence the conjuction of (i'') - (v'') leads to a contradiction.

Terminological remarks. If S is a satisfaction formula (or a predicative satisfaction formula), then we shall write $z \models x[y]$ instead of $S(x, y, z)$. If x is the Gödel set of a formula F, then we shall replace x by F (although it would be more exact to write $\ulcorner F \urcorner$ and not simply F in these formulae).

Let F be a predicative formula with the free variables $x_1, \ldots, x_n$ and let $F^{(z)}$ be its relativisation to the formula $x \in z$. Then the formula

$$(y_1 \in z) \& \ldots \& (y_n \in z) \longrightarrow \left\{ z \models F\left[\{\langle 1, y_1\rangle, \ldots, \langle n, y_n\rangle\}\right] \right.$$
$$\left. \equiv F^{(z)}(y_1, \ldots, y_n) \right\}$$

is provable in each of the considered systems of set theory.

Let z_1, z_2 be classes, $z_1 \subseteq z_2$. If $(x)(y)\left[(x \in Frm_{pr}) \& (y \in z_1^{Fr(x)}) \& (z_1 \models x[y] \rightarrow (z_2 \models x[y]))\right]$, then we say that z_1 is an elementary subclass of z_2 and write $z_1 \prec_{pr} z_2$. A similar definition can be given for the notion of extension in case when x ranges over arbitrary formulae not just the predicative ones. We shall not use this notion however.

LECTURE II

The backbone of the whole set theory is the stratification of the universe into levels (simple theory of types). In discussing this phenomenon we shall assume as known the notion of ordinals and the theorem on definitions by transfinite induction. We denote by On the class of all ordinals; in the case of the system ZF where there are no classes we think of On as of a formula so that On(x) means the same as "x is an ordinal". Sometimes we shall use the expression $x \in$ On even in ZF treating it as equivalent with On(x).

We define by transfinite induction sets R_τ :

$$R_0 = 0, \quad R_{\tau+1} = P(R_\tau), \quad R_\tau = \bigcup\{R_{\tau'} : \tau' \in \tau\} \text{ (}\tau\text{ is a limit number).}$$

Note: What we define is in GB (or any stronger system) a function which correlates with each ordinal τ a set R_τ . in the case of ZF we have a formula R(x, y) such that it is provable in ZF that $(x)\left[On(x) \rightarrow (E!y)R(x,y)\right]$. This unique y is denoted by R_x; it satisfies the equations given above.

I. For every set x there is an ordinal a such that $x \subseteq R_a$.

This theorem is provable in each of our three systems. The proof is obtained easily from the axiom Fund.

The least ordinal a such that $x \subseteq R_a$ is called the rank of x.

We want to discuss the problem whether some of the R_a's are models of ZF. In order to answer this question we shall formulate and prove in M the following theorem:

II (Scott-Scarpellini). Let $A = \bigcup\{A_\tau : \tau \in On\}$ where for each ordinal τ, A_τ is a set, $A_\tau \subseteq A_{\tau+1}$ and $A_\tau = \bigcup\{A_{\tau'} : \tau' \in \tau\}$ for each limit ordinal τ . Under these assumptions:

(a). For each F in Frm_{pr} there is an increasing and continuous mapping $f_F: \mathrm{On} \to \mathrm{On}$ such that if $f_F(a) = a$ and $x \in A_a^{\mathrm{Fr}(F)}$, then $A_a \models F[x] \equiv A \models F[x]$.

(b). There is a function $f: \mathrm{Frm}_{pr} \times \mathrm{On} \to \mathrm{On}$ such that for each F in Frm_{pr} the mapping $a \to f(F, a)$ satisfies the conditions formulated in (a).

(c) There is an increasing and continuous mapping $t: \mathrm{On} \to \mathrm{On}$ such that if $t(a) = a$, then $A_a \prec_{pr} A$.

Proof. (a). We use induction on the number of connectives which occur in F. If F has no connectives, then we take as f_F the identity map. If F is $\ulcorner(G \mid H)\urcorner$ then we take as f_F the composition of f_G and f_H. If F is $\ulcorner(x_i)G\urcorner$ and $i \notin \mathrm{Fr}(G)$, then we take $f_F = f_G$. The only case which requires a more elaborate proof is one in which F has the form $\ulcorner(x_i)H\urcorner$ and $i \in \mathrm{Fr}(H)$.

We first define auxiliary functions $a: A^{\mathrm{Fr}(F)} \to \mathrm{On}$, and $b: \mathrm{On} \to \mathrm{On}$ as follows:

$$a(x) = \min \{ r \in \mathrm{On} : (Eu) [(u \in A_r) \ \& \ (A \models \neg H [x \cup \{<i, u>\}])] \}$$

$$b(r) = \sup \{ a(x) : x \in A_r^{\mathrm{Fr}(F)} \}$$

It follows from these definitions that if $x \in A_r^{\mathrm{Fr}}$, then (1).

$$(u)_A [(A \models H [x \cup \{<i, u>\}]) \equiv (u)_{A_{b(r)}} (A \models H [x \cup \{<i, u>\}])].$$

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(1) The subscripts after quantifiers denote their relativisation to the formula $x \in a$. Thus $(x)_a$ means the same as $(x) [(x \in a) \to \ldots]$.

Now we define $c(0) = 0, c(r+1) = \max(c(r), b(r+1))$ and $c(r) = \sup \{ c(r') : r' < r \}$ for limit numbers r. Finally we put $f_F = f_H \circ c$. This function is obviously increasing and continuous. If $f_F(x) = r$, then $f_H(r) = c(r) = r$ and for every x in $A_r^{Fr(F)}$ we have $c(r) \geq b(r)$. It follows

$$A_r \models F[x] \equiv (u)_{A_r} (A_r \models H [x \cup \{<i, u>\}]) \equiv (u)_{A_r} (A \models H [x \cup \{<i, u>\}])$$

$$\longrightarrow (u)_{A_{b(r)}} (A \models H [x \cup \{<i, u>\}]) \equiv (u)_A (A \models H [x \cup \{<i, u>\}])$$

$$\equiv A \models F [x]$$

and similarly

$$A \models F [x] \equiv (u)_A (A \models H [x \cup \{<i, u>\}]) \longrightarrow (u)_{A_r} (A \models H [x \cup \{<i, u>\}]$$

$$\equiv (u)_{A_r} (A_r \models H [x \cup \{<i, u>\}]) \equiv (A_r \models F [x]).$$

(b). We define $f(F, x) = x$ if F is an atomic formula (without logical operators), $f(\ulcorner F_1 \mid F_2 \urcorner, x) = f(F_1, f(F_2, x))$ and $f(\ulcorner (x_i)H \urcorner, x) = f(H, x)$ if $i \notin Fr(H)$. If $i \in Fr(H)$, then we put $f(\ulcorner (x_i)H \urcorner, x) = f(H, c(x))$ where c is defined similarly as in part (a). The proof is identical as in the case (a).

(c). We define $t(0) = 0, t(r+1) = \max (t(r), \sup \{ f(F, r+1) : F \in Frm_{pr} \})$, $t(r) = \sup \{ t(r') : r' < r \}$ if r is a limit number. If $t(r) = r$, then $f(F, r) = r$ for each predicative formula F and hence, in view of (b), $A_r \prec_{pr} A$.

Remarks on the Scott-Scarpellini theorem. We want to discuss the question whether theorem II can be so reformulated as to become provable in ZF or in GB.

The case of the system ZF. Since there are no proper classes in ZF, we must replace the class A and the relation $x \in A_r$ by formulae. If we do this, then we cannot use the satisfaction formula since there is no formula describing the satisfaction of an arbitrary x in Frm in the domain of all sets satisfying a given formula A. We come around this difficulty by remarking that the satisfiability of an explicitly given formula F in the indicated domain can be expressed by the relativised formula $F^{(A)}$. Thus we shall have not a single theorem starting with the general quantifier "for each F in Frm" but a theorem schema which can be proved separately for each explicitely given formula F. A final change in the wording of the theorem concerns the function f_F: since there are no mappings of On into On in the system ZF we must replace the mapping f_F by a formula which describes it

The Scott-Scarpellini theorem takes thus in ZF the following form:

Let A(x) and B(x, y) be two formulae with the free variables x and x, y respectively; let C be the conjunction of the following formulae:

$$(x)\ \{A(x) \equiv (Ey)\,[On(y)\ \&\ B(x,y)]\},$$
$$(y)\ \{On(y) \rightarrow (Ez)(x)\,[(x \in z) \equiv B(x,y)]\},$$
$$(y)(y')\ \{On(y)\ \&\ On(y')\ \&\ (y' = y+1) \longrightarrow (x)\,[B(x,y) \longrightarrow B(x,y')]\},$$
$$(x)(y)\ \left(Lim(y) \longrightarrow \{B(x,y) \equiv (Ez)\,[(z \in y)\ \&\ B(x,z)]\}\right)$$

For any formula F with two free variables x, y let D_F be the conjunction of the formulae:

$$(x)\ [On(x) \longrightarrow (E!y)\ F(x,y)]$$

$$(x)\,(y)\left[F(x,y) \rightarrow On(x)\ \&\ On(y)\right]$$
$$(x)(y)(z)(t)\ \left[F(x,y)\ \&\ F(z,t)\ \&\ (x \in z) \rightarrow (y \in t)\right]$$
$$(x)(y)\ (Lim(x)\ \&\ F(x,y) \rightarrow (u)\left\{(u \in y) \equiv (Et)\left[(t \in x)\ \&F(t,u)\right]\right\}\ .$$

With these notations the following holds: For arbitrary formulae $A(x)$, $B(x,y)$ and $H(x_1,\dots,x_n)$ there is a formula $F(x,y)$ such that the implications

$$C \rightarrow D_F,$$
$$C\&F(r,r)\&B(x_1,r)\ \&\dots\&B(x_n,r) \rightarrow \left[F^{(A(\cdot))} \equiv F^{(B(\cdot\ ,r))}\right]$$

are provable in ZF.

Notice that the formula C says that the domain of all x's satisfying A is the union of sets $\{x{:}B(x,r)\}$ with r ranging over ordinals and that these sets form an increasing and continuous family while D_F says that F defines an increasing and continuous mapping of ordinals into ordinals.

A similar reformulation of the Scott - Scarpellini theorem is also possible in the case of the system GB. Again we must express the theorem as a scheme because there is no satisfaction formula for GB. However the assumptions of the theorem and the statement concerning the existence of a mapping can be expressed as in system M.

Finally we notice that a theorem similar to the Scott-Scarpellini theorem can be proved (in any of our three systems) for unions of the form $\bigcup\{A_r : r < s\}$ where s is an inaccessible cardinal.

Applications of the Scott-Scarpellini theorem. The following theorem is proved in M; it can also be proved in any system obtained from GB by adding to it new axioms which secure the existence of a a predicative satisfaction formula:

III. There is an increasing and continuous function f: On $\rightarrow$On such that whenever $f(r) = r$, then $R_r \prec_{pr} V$; in particular, R_r is a model of ZF.

Proof. The decomposition $V = \bigcup\{R_r : r \in On\}$ satisfies the assumptions of the Scott - Scarpellini theorem. Since all axiom of ZF are equivalent to predicative sentences and are valid in V, we infer that if R_r is an elementary subset of V, then R_r is a model of ZF.

We can now supply proofs of two theorems which we announced in Lecture I:

IV. There is no predicative satisfaction formula for GB provided that GB is consistent.

Otherwise we could repeat in GB the proof of theorem III and infer that there is a set a which is a model of ZF. Hence by adding to a its definable subsets we would obtain a model for GB. Since this proof would be formalizable in GB we would have a proof in GB that GB is consistent which would entail the inconsistency of GB.

V. M is essentially stronger than GB, provided that GB is consistent.

Proof. There is a predicative satisfaction formula for M but none for GB.

VI. ZF is not finitely axiomatisable, provided that it is consistent.

Proof. We denote by K the conjunction of any finite number of axioms of ZF and show using the version of the Scott - Scarpellini theorem which is provable in ZF that there is an ordinal r such that

$R_r \models K$. Hence the consistency of K is provable in ZF while the consistency of the whole ZF is not so provable.

VII. GB is finitely axiomatisable.

We merely sketch the proof of this theorem. Call F^* the particular instance of the axiom (scheme) Subst_{GB} which corresponds to the formula F. It can then be shown that all the particular instances of Subst_{GB} can be derived from the axioms $F_1^*, \ldots F_9^*$ where $F_1 \ldots F_9$ are the formulae: $(Eu_1)(Eu_2)\left[(u = \langle u_1, u_2\rangle) \& (u_1 \in u_2)\right]$, $(Eu_1)(u = \langle u_1, u_1\rangle)$, $\neg(u \in y)$, $(u \in y) \& (u \in z)$, $(Et)(\langle u, t\rangle \in y)$, $(Eu_1)(Eu_2)\left[(u = \langle u_1, u_2\rangle) \& (u_1 \in y)\right.$, $(Eu)(u = \langle u_1, u_2\rangle) \& (\langle u_2, u_1\rangle \in y)$, $(Eu_1)(Eu_2)(Eu_3)\left[(u = \langle u_1, \langle u_2, u_3\rangle\rangle) \& (\langle u_2, \langle u_3, u_1\rangle\rangle \in y)\right.$, $(Eu_1)(Eu_2)(Eu_3)\left[(u = \langle u_1, \langle u_2, u_3\rangle\rangle) \& (\langle u_2, \langle u_1, u_3\rangle\rangle \in y)\right]$.

These axioms state the existence of the following classes: $E = \{\langle u, v\rangle : u \in v \in V\}$, $I = \{\langle u, u\rangle : u \in V\}$, $V - Y$, $Y \cap Z$, $\mathrm{Dom}(Y)$, $Y \times V$, $\breve{Y}$, $\mathrm{Cnv}_1(Y)$, $\mathrm{Cnv}_2(Y)$ and our theorem reduces to the statement that for every predicative formula H there is a class Y constructed from E and I by means of the operations enumerated above and consisting of all u for which H(u). Details of this proof can be found in Gödel's monograph.

Natural models. If r is an ordinal and R_r is a model of ZF, GB, or M, then we call it a natural model of the corresponding system In theorem III we established (in M) the existence of a "tower" of natural models for ZF which is ordered by the relation $\prec_{pr}$ We now show that the existence even of a single pair R_r, R_s such that $R_r \prec_{pr} R_s$ cannot be established in ZF provided that ZF is consistent. We do this by proving in ZF the following:

VIII. (Montague - Vaught). If $R_r \prec_{pr} R_s$; then R_r is a model of ZF.

Proof. We first show that r is a limit number $\neq 0$. Since the formula $(Ex)(x=x)$ is true in R_s, it is true in R_r and hence $R_r \neq 0$ and $r \neq 0$. If $t \in r$, then $R_t \in R_r$ and hence there is an x in R_s such that $R_t \in x$. Thus the formula $(Ex_1)(x_0 \in x_1)$ is satisfied by R_t in R_s and hence it must also be satisfied in R_r. Thus there is an x in R_r such that $R_t \in x$ and therefore $t+1 \neq r$.

Using the fact that r is a positive limit number we easily check that all axioms of ZF are valid in R_r. The verification is evident in all cases with the exception of the axiom scheme Subst_{ZF} which requires a separate treatement.

Let F be a formula with $k+2$ free variable and let $\bar{p} \in R_r^{Fr(F)-\{i,j\}}$ be such that $R_r \models (x_i)E!x_j\ F[\bar{p}]$. For x in R_r let $f(x)$ be the unique y in R_r such that $R_r \models F[\bar{p} \cup \{\langle i, x\rangle, \langle j, y\rangle\}]$. Let x_p and x_q be variables which do not occur in F and let a be an element of R_r. The set $\mathrm{Im}(f, a) = b$ belongs to R_s and satisfies the condition $R_s \models H[\bar{p} \cup \{\langle p, a\rangle, \langle q, b\rangle\}]$ where H is the formula

$$(x_j)\{(x_j \in x_q) \equiv (Ex_i)[(x_i \in x_p) \mathbin{\&} F]\}.$$

It follows that there is a set b' in R_r such that $R_r \models H[\bar{p} \cup \{\langle p, a\rangle, \langle q, b'\rangle\}]$ and we easily prove that $b' = \mathrm{Im}(f, a)$. Thus axiom Subst_{ZF} is valid in R_r. Theorem VIII can obviously be also proved in M.

In order to obtain more information about the relation $\prec_{pr}$ in the class of all R_r's we introduce the following definition:

We call an ordinal r extendable if for any ordinal s there is a sequence f of order type s such that $f(0) = r$ and $R_{f(t)} \prec_{pr} R_{f(t')}$ for each pair t, t' such that $t < t' < s$.

From part (c) of the Scott-Scarpellini theorem it follows that there are extendable ordinals. Moreover there are ordinals r such that there is a sequence of order type On consisting of R_x's which are elementary extensions of R_r and well ordered by the relation $\prec_{pr}$.

The next theorem is provable in M.

IX. (Ryll Nardzewski). For each extendable ordinal r there are arbitrarily high ordinals $s > r$ such that $R_r \prec_{pr} R_s$ but for no $t > s$ does the relation $R_s \prec_{pr} R_t$ hold.

Proof. Let us assume that there is a (least) extendable r such that there exists an ordinal $s_o > r$ with the properties: whenever $s > s_o$ and $R_r \prec_{pr} R_s$ then $R_s \prec_{pr} R_t$ for some $t > s$. We notice that because of the extendability of r there are arbitrarily high s such that $R_r \prec_{pr} R_s$ and our assumption says that each such s from a certain s_o on can further be extended to an R_t.

For arbitrary finite sequence f of sets we denote by $m(f)$ the least ordinal $> s_o$ such that all terms of f belong to $R_{m(f)}$ and $R_r \prec_{pr} R_{m(f)}$. Consider the formula $T(x, f)$ defined as follows:

$$(x \in Frm_{pr}) \ \& \ Fn(f) \ \& \ (Dom(f) = Fr(x)) \ \& \ (R_{m(f)} \models x[f]) .$$

We shall show that this formula has the charateristic properties (i'')-(v'') of the truth predicate (cf. Lecture I, theorem IX). Since we know that we can refute the conjunction of (i'') - (v'') we shall have the proof that our assumption leads to a contradiction as soon as we verify that T has the properties (i'') - (v''). Of these, (i''), (ii'') and (iv'') are obvious. In order to verify the remaining two we prove a lemma:

If $s_o < s < t$ and $R_r \prec_{pr} R_s$, $R_r \prec_{pr} R_t$, then $R_s \prec_{pr} R_t$.

Proof of the lemma. For each $s > s_o$ such that R_s is an elementary extension of R_r we denote by s' the least ordinal $> s$ such that $R_{s'}$ is an elementary extension of R_s. The existence of s' follows from our assumptions .Now we start from given ordinals s, t and construct two sequence f and g satisfying the inductive equations $f(0)=s$, $f(n+1)= (f(n))'$, $f(u)= \sup\{f(n): n<u\}$, $g(0) = t$, $g(n+1) = (g(n))'$, $g(u)= \sup\{g(n): n<u\}$. We obviously have $R_{f(n)} \prec_{pr} R_{f(m)}$ and $R_{g(n)} \prec_{pr} R_{g(m)}$ for $n < m$; moreover $V = \bigcup\{R_{f(n)} : n \in On\} = \bigcup\{R_{g(n)} : n \in On\}$. Let $x \in Frm_{pr}$ and $y \in R_s^{Fr(x)}$.

Since $R_s \prec_{pr} V$ we infer that $R_s \models x[y] \equiv V \models x[y]$; we prove similary that $R_t \prec_{pr} V$ and therefore $R_t \models x[y] \equiv V \models x[y]$ whence $(R_s \models x[y]) \equiv (R_t \models x[y])$.

The lemma is thus proved.

Verification of condition (iii''). Let us assume that

$T(\ulcorner x_1 \mid x_2 \urcorner, f)$.

This assumption is equivalent to the statement that either $R_{m(f)} \models \ulcorner \neg x_1 \urcorner [f \mid Fr(x_1)]$ or $R_{m(f)} \models \ulcorner \neg x_2 \urcorner [f \mid Fr(x_2)]$. Since by the lemma $R_{m(f)} \prec_{pr} R_{m(f \mid Fr(x_i)}$ for $i = 1, 2$ we infer that the previous statement is equivalent to the disjunction of the statements $R_{m(f \mid Fr(x_i))} \models \ulcorner \neg x_i \urcorner [f \mid Fr(x_i)]$ for $i = 1, 2$ and this disjunction is precisely the right hand side of (iii'').

Verification of condition (v''). We assume that $x \in Frm_{pr}$ and $i \in Fr(x)$. Let us consider the statement $T(\ulcorner (x_i)\, x \urcorner, f)$; this statement says that f is a sequence with domain $Fr(x)$ and that $R_{m(f)} \models x\, [f \cup \{<i, u>\}]$ for every u in $R_{m(f)}$. We have to prove that this statement is equivalent to the following:

for every u, $R_{m(f \cup \{<i, u>\})} \models x\, [f \cup \{<i, u>\}]$

Obviously the first statement is implied by the second because $m(f \cup \{<i, u>\}) = m(f)$ whenever u is in $R_{m(f)}$. Now we assume the first statement and choose an arbitrary u. By the lemma $R_{m(f)} \prec_{pr} R_{m(f \cup \{<i, u>\})}$ and hence the first statement implies $R_{m(f \cup \{<i, u>\})} \models (x_i)x\, [f]$ whence we infer that the second statement is valid. Theorem IX is thus proved

A theorem similar to IX can also be proved for other transfinite sequences of sets, for instance for sets L_r which we shall discuss later.

In connection with theorem IX we discuss briefly the ordinals which are not extendable. Let us call a function f whose domain is an ordinal r and which satisfies the condition $R_{f(n)} \prec_{pr} R_{f(m)}$ for arbitrary $m \in n \in r$ a chain of length r starting in f(0). An ordinal s is not extendable if and only if there is an ordinal r such that there is no chain of length r starting at s. The least such r is called the height of s. The height of an extendable ordinal could additionally be defined as On.

We don't have an exact characterisation of ordinals which are heights of non extendable ordinals. However we can exhibit a rather large number of examples of such ordinals.

We call an ordinal $r > 0$ an R- definable ordinal if there is a predicative formula F with $Fr(F) = \{0\}$ such that whenever $s \in On$ and $r \in R_s$, then r is the unique element of R_s such that $R_s \models F\left[\{<0, x>\}\right]$ while 0 is a unique such x if $r \notin R_s$. We call F a definition of r. We now prove in M:

X. (Wilmers). For every R-definable ordinal r there is an ordinal whose height is r+1.

Proof. Using part (c) of the Scott - Scarpellini theorem we easily prove that there are ordinals x such that R_x contains as element a chain of length r + 1. Let a be a smallest such ordinal and le f be a chain of length r + 1 which belongs to R_a. We put b = f(0) and claim that the height of b is r + 1.

Since f is a chain of length r + 1 starting at b, it is obvious that the height of b is $\geq$ r + 1. Now we assume that there is a chain g of length r + 2 starting at b and derive from this assumption a contradiction.

There exists a predicative formula H with

$Fr(H) = \{0, 1\}$ such that for an arbitrary ordinal p and arbitrary u, v in R_p

(1) $R_p \models H[\{<0, u>, <1, v>\}] \equiv (v \in On) \& (u$ is a chain of length $v)$.

We shall indicate below how to construct such a formula. Assuming that we carried out the construction we proceed as follows.

We easily show that $n \in R_{n+1} \subseteq R_{g(n+1)}$ for arbitrary n in r + 1; since $g(n) \in R_{g(r+1)}$ we see that $g | (r+1) \in R_{g(r+1)}$. Now $g | (r+1)$ is a chainof length r + 1 and hence by (1)

$$R_{g(r+1)} \models H[\{<0, g(r+1)>, <1, r+1>\}] .$$

Denoting by F a definition of r we obtain from the last formula

$$R_{g(r+1)} \models (Ex_0)(Ex_1)(Ex_2)[F(x_2) \& (x_2 \neq 0) \& \& (x_1 = x_2 + 1) \& H] .$$

Since $R_{g(0)}$ is an elementary subset of $R_{g(r+1)}$ we can replace in this formula g(r + 1) by g(0), i.e. by b. We thus infer that there are elements c, d of R_b such that $c \neq 0$, $R_b \models F[\{<2, c>\}]$ and $R_b \models H[\{<0, d>, <1, c+1>\}]$.

The first of these formulae proves that $c = r$ and the second that d is a chain of length $r+1$. Since $d \in R_b$ and $b < a$, we obtain a contradiction with the definition of a.

It remains to construct the formula H. Let $R'(x, y)$ be a formula which defines the relation $x = R_y$ (cf. p 79) and let $G(z, t)$ be the formula

$$(f)\,(x)\Big[(x \in Frm_{pr})\ \&\ (f \in z^{Fr(x)})\ \&\ (z \models x[f]) \rightarrow$$

$$\rightarrow (t \models x[f])\Big].$$

The required formula is

$$On(x_1)\ \&\ Fn(x_0)\ \&\ (Dom(x_0) = x_1)\ \&\ (i)(j)(z)(t)$$

$$\Big[(i \in j)\ \&\ (j \in x_0)\ \&\ (\langle i, z\rangle \in x_0)\ \&\ (\langle j, t\rangle \in x_0) \rightarrow$$

$$\rightarrow G(z, t)\Big].$$

Natural models of GB and of M. These models are situated much less densely than the natural models of ZF: we shall prove the following result in the system ZF + AC resulting from ZF by adjunction of the axiom of choice:

XI. (Shepherdson). R_r is a natural model of GB if and anly if r is of the form $s + 1$ and s is a strongly inaccessible cardinal; this condition is also necessary and sufficient for R_r to be a model of M.

Proof. If R_r is a model of GB or of M, then r is a successor, $r = s + 1$ because there exists a universal class of the model. From the satisfiability of the axiom Subst it follows that if t is an ordinal $< s$ and f a mapping of t

into s, then $\mathrm{Im}(f, t) \in R_s$ and hence there is an ordinal $t' < s$ such that $\mathrm{Im}(f, t) \subseteq t'$. This proves that s is weakly inaccessible.

In order to show that s is strongly inaccessible we have to show that if $t < s$, then the cardinal number of $P(t)$ is less than s. By our assumption $t \in R_s$ and hence $p(t) \in R_s$. Since we assume that R_{s+1} is a model of GB and since there is a well ordered set which belongs to R_s and has the same power as $P(t)$, we infer that there exists an ordinal number in R_s which has the same power as $P(t)$. Thus this cardinal number is smaller than s.

If s is strongly inaccessible, then we easily check that all the axiom of M are true in R_{s+1}. In particular the axiom Subst is satisfied because if f is any mapping of R_s into R_s and a belogs to R_s, then $\mathrm{Im}(f, a)$ belogs to R_s.

A comparison with theorem VIII suggests the following problem: how to characterize ordinals r, s $(r < s)$ satisfying the relation $R_{r+1} \prec_{pr} R_{s+1}$? It is easy to see that this problem is wrongly expressed because the relation in question never holds (R_r is the largest element of R_{r+1} but not of R_{s+1}). Therefore we modify the problem and discuss not the relation of elementary extension but a closely connected relation of elementary embeddability.

Definition. R_r is elementarily embeddable in R_s if there is a function f whith maps R_r isomorphically (with respect to the relation $\in$) onto an elementary subset of R_s .

XII. (Reinhardt). If r is a strongly inaccessible ordinal and R_{r+1} is elementarily embeddable into $R_{r'+p}$ where $r' > r$, then r is measurable.

Proof. Let f be a function which embeds R_{r+1} into $R_{r'+p}$. Since R_{r+1} is a model of GB, the same is true of $R_{r'+p}$ and hence $r' + p = s + 1$, where s is strongly inaccessible. Since R_r is the largest element of R_{r+1} and R_s the largest element of R_{s+1} it follow that f maps R_r onto R_s (i.e. $f(R_r) = R_s$).

Similary $f(r) = s$ since r is the largest ordinal of R_{r+1} and s the largest ordinal of R_{s+1} . On the other hand the range of $f \mid (r + 1)$ is strictly contained in $s + 1$ for reason of cardinality. Hence there is an ordinal $p \in s$ such that $p \notin \text{Rg} (f \mid (r + 1))$. Obviously $p \notin \text{Rg} (f)$ because an ordinal in $\text{Rg} (f)$ must be the value of f for an argument in $r + 1$.

From the properties of f it easily follows that $f(x \cap y) = f(x) \cap f(y)$, $f(x \cup y) = f(x) \cup f(y)$ and $f(x - y) = f(x) - f (y)$ for arbitrary x, y in R_{r+1} . For instance the first equation is established by noticing that $x \cap y$ is characterised as the unique element z which together with x and y satisfies in R_{r+1} the formula $(t) \ (t \in x_0) \equiv [(t \in x_1) \ \& \ (t \in x_2)]$. Hence $f (z)$ together with $f (x)$ and $f (y)$ satisfies the same formula in R_{s+1} .

Now we put $F = \{ x \subseteq r : p \in f(x) \}$ and claim that F is a filter of subsets of r which is prime and t-multiplicative for each $t < r$.

If $x, y \in F$, then $p \in f(x) \cap f(y) = f(x \cap y)$ and hence $x \cap y \in F$.
If $x \subseteq y \subseteq r$ and $x \in F$, then $f(x) = f(x) \cap f(y) \subseteq f(y)$ and hence $p \in f(y)$, i.e., $y \in F$. From $p \in s = f(r)$ it follows that $r \in F$ and hence F is not void.

Hence F is a filter of subsets of r.
If $x \subseteq r$, then $r = x \cup (r - x)$ and we infer $f(r) = s = f(r - x) \cup f(x)$. Since $p \in s$, it follows that either $p \in f(x)$ or $p \in f(r - x)$; both these formulae cannot be true because $f(x) \cap f(r - x) = 0$. Hence either x or $r - x$ belongs to F and F is prime.

Now we show that if $t \in r$, then F is t-multiplicative. Let g be a sequence of type t consisting of elements of F. Since $g \in R_{r+1}$ and g satisfies in R_{r+1} a formula which says that g is a function with domain t, we infer that $f(g)$ satisfies in R_{s+1} the formula saying that it is a function with domain $f(t)$.

The intersection $\bigcap Rg(g)$ can be characterised as the unique element of R_{r+1} which together with g satisfies in R_{r+1} the formula

$$(t \in x_0) \equiv (w)(v)\left[(\langle w, v\rangle \in x_1) \rightarrow (t \in v)\right]$$

Hence the same formula is satisfied in R_{s+1} by $f(\bigcap Rg(g)$ and $f(g))$.

Now we notice that the same formula is also satisfied in R_{s+1} by $\bigcap Rg(f(g))$ and $f(g)$ and hence $\bigcap Rg(f(g)) = f(\bigcap Rg(g))$. Since $Rg(g) \subseteq F$, we see that $p \in f(x)$ for every x in $Rg(g)$. Now notice that $x \in Rg(g)$ is equivalent to

$R_{r+1} \models (Ey)(<y, x_0> \in x_1)\left[\{<0, x>, <1, g>\}\right]$ and hence to $R_{s+1} \models (Ey)(<y, x_0> \in x_1)\left[\{<0, f(x)>, <1, f(g)>\}\right]$; i.e., to $f(x) \in R_g(f(g))$. Thus p is an element of every member of $Rg(f(g))$, i.e., $p \in \bigcap Rg(f(g)) = f(\bigcap Rg(g))$. Hence $\bigcap Rg(g) \in F$.

This proves theorem XII. Notice that this proof like the proof of theorem XI was carried out in the system ZF + AC.

Lecture III

In this lecture we shall apply the Scott - Scarpellini theorem to obtain various families of sets which form models of ZF .

We call a set A predicatively closed if for every a in A , every $F \in Frm_{pr}$ with $0 \in Fr(F)$ and for every f in $a^{Fr(F)-\{0\}}$ the set

$$S_F(a, f) = \{x \in a : a \models F[f \cup \{\langle 0, x\rangle\}]\}$$

belongs to A. The set $S_F(a, f)$ will be called the section of a determined by F and f.

We shall show that all sections of elements a of A can be obtained by iterating finitely many operations Let us consider the following operations:

$A_1(x, y) = \{x, y\}$, $A_2(x, y) = x \cup y$, $A_3(x, y) = x - y$,

$A_4(x, y) = x \circ y$, $A_5(x, y) = \{A_2(u, v) : u, v \in x\}$,

$A_6(x, y) = \{A_3(u, v) : u, v \in x\}$, $A_7(x, y) = \{A_4(u, v) : u, v \in x\}$

$A_8(x, y) = \bigcup x$, $A_9(x, y) = \{\langle y, u\rangle : u \in x\}$,

$A_{10}(x, y) = \{\{\langle 0, u\rangle , \langle 1, v\rangle\} : (u \in v) \ \& \ (u, v \in x)\}$

$A_{11}(x, y) = \{\{\langle 0, u\rangle , \langle 1, u\rangle\} : u \in x\}$,

$A_{12}(x, y) = \{u : \{\langle 0, u\rangle\} \cup y \in x\}$.

I. If $0 \in A$ and A is closed with respect to the operations $A_1 - A_{12}$, then A is predicatively closed.

We shall only indicate the essential steps of the proof.

First we notice that an arbitrary finite sequence whose terms are integers belongsto A. Hence if a set s of finite sequences with a common finite domain $d \subseteq \omega$ belongs to A, then so does the set of all sequences $f \circ x$ where x ranges over s and f is a fixed one one mapping of a set d' onto d.

Hence if the set $\{x \in a^{Fr(F)} : a \models F[x]\}$

belongs to A, then so does the set $\{x \models a^{Fr(F')} : a \models F'[x]\}$

where F' arises from F by a permutation of variables.

By iterating suitably the operations $A_1 - A_{12}$ we

prove that if $d \subseteq \omega$ is a finite set of integers and $a \in A$, then

$a^d \in A$. From this we easily infer that if $s \in A$ and s is a set of finite sequences with a common finite domain d and with ranges contained in a where a is in A, then for every finite set d' such that $d \subseteq d' \subset \omega$ the set $\{f \in a^{d'} : f \mid d \in s\}$ belongs to A.

Finally we show that if $a \in A$, d is a finite set of integers and $n \notin d$, $s \subseteq a^{d \cup \{n\}}$, then the set

$f \in a^d : (Eu)\ [f \cup \{<n, u>\} \in s]\}$ belongs to A along with s. We show this by noticing that in order to obtain this set from s it is sufficient to subtract from every member of s the set of all sets of the form $\{\langle n, t\rangle\}$ where $t \in a$.

Let us now consider the set $D_F(a) = \{f \in a^{Fr(F)} : a \models F[f]\}$. If $a \in A$ and F is one of the formulae $\ulcorner Cl(x_0)\urcorner$, $\ulcorner x_0 \in x_1\urcorner$, $\ulcorner x_0 = x_1\urcorner$, then $D_F(a)$ belongs to A as we easily see inspecting the operations A_{10} - A_{12} Using the operation A_7 we extend this result to the case when F has arbitrary variables. Using the operations A_2, A_3 as well as the remarks established above we show that if $D_{F_i}(a) \in A$ for $i = 1, 2$, then the same result is true for the cases when $F = \ulcorner F_1 \mid F_2\urcorner$ and $F = \ulcorner (x_i) F_1\urcorner$. Finally we use A_{12} in order to construct $S_F(a, f)$ from $D_F(a)$.

The main result of the present lecture is as follows:

II. Let A_r be a family of sets (indexed by ordinals) which satisfies the assumptions of the Scott - Scarpellini theorem. If in addition $A_r \in a = \bigcup\{A_s : s \in On\}$ and A_r is transitive and predicatively closed for every r in On, then A is a model of ZF.

Proof. The verification of most of the axiom is immediate We discuss only the axioms Inf, Pot and Subst which are slightly more difficult to verify.

Axiom of infinity . We define by induction a sequence r_n of ordinals: $r_0 = 0$, $r_{n+1} = \min\{s : A_{r_n} \in A_s\}$ and put

$r = \sup r_n$. then A_r satisfies the conditions stated in the axiom Inf.

Axiom of power set. For $a \in A_r$ and x in $P(a) \cap A$ let $s(x) = \min\{s : x \in A_s\}$, $t = \max(r, \sup\{s(x) : : x \in P(a) \cap A\}$. Then $x \in A \longrightarrow (x \in P(a) \equiv (x \subseteq a)$ & & $(x \in A_t))$. Now take an ordinal u such that $A_t \in A_u$ and determine the section $S_F(A_t, \{<1, a>\})$ of A_u where F is the formula $(v)[(v \in x_0) \longrightarrow (v \in x_1)]$.

This section is equal to $P(a) \cap A$ and since A_u is predicatively closed, it belongs to A_u and hence to A.

Axiom of substitution. Let $a \in A_r$ and let F be a formula with $Fr(F) \geq \{0, 1\}$. Furthermore let p be a sequence in $A^{Fr(F) - \{0, 1\}}$ such that $A \models (x_0)(E!x_1) F[p]$. For x in A_r let $f(x) = \min\{s : (Ey)[(y \in A_s) \& (A \models F[\{<0, x>, <1, y>\} \cup p])]\}$ and put $t = \max(r, \sup\{f(x) : (x \in A_r)\})$. Using the Scott - Scarpellini theorem we determine an ordinal u such that $u > t$ and for arbitrary x, y in A_u the following equivalence holds :

$$(A \models F[\{<0, x>, <1, y>\} \cup p]) \equiv (A_u \models F[\{<0, x>, <1, y>\} \cup p]).$$

Finally we choose an ordinal v such that $A_u \in A_v$ and determine the section $S_G(A_u, \{<j, a> \cup p)$ where j is an integer such that x_j is not free in F and G is the formula $(x_1)\{(x_1 \in x_0) \equiv (Ex_0)[(x_0 \in x_j) \& F]\}$.

This section which we denote by b belongs to A_v and satisfies the condition

$$y \in b \equiv (Ex) \Big[(x \in a) \,\&\, \big(A \models F \big[\{ <0, x> , <1, y> \} \cup p \big] \big) \Big]$$

Remark. Theorem II was proved above on the basis of the system M . There is a version of this theorem which can be established in ZF. Similary as on p. 82 we consider a formula B with two free variables and define a formula A by $(Er) \big[On(r) \,\&\, B(x, r) \big]$. Let C^* be the conjunction of the 4 formulae listed on p. 82 and of the following formulae:

$$(x)\,(y)\,(r) \ \big[On(r) \,\&\, B(x, r) \,\&\, (y \in x) \longrightarrow B(y, r) \big]$$

$$(x)\,(y)\,(r) \ \Big\{ On(r) \,\&\, B(x,r) \,\&\, B(y, r) \longrightarrow \bigwedge_{i \leq 12} (Ez) \big[B(z, r) \,\&\, (z = A_i(x, y)) \big] \Big\}$$

(The first of the above formulae expresses the fact that the set $\{ x : B(x, r) \}$ is transitive and the second that this set is closed with respect to the operations $A_1 - A_{12}$; obviously the second formula should be expressed in the language of ZF which can easily be done by writing down the definitions of $A_i(x, y)$ as set theoretic formulae).

Imitating the proof of theorem II we can derive from C^* in ZF all formulae obtained from the axiom of ZF by relativising all quantifiers to the formula A. In other words the formula A defines an interpretation of ZF in ZF + C^* .

Examples of classes which determine models of ZF.

Example 1. $V = \bigcup \{ R_r : r \in On \}$.

Example 2: constructible sets. We define for an arbitrary set a

$$a' = \{ S_F(a, f) : (F \in Frm_{pr}) \,\&\, (0 \in Fr(F)) \,\&\, (f \in a^{Fr(F) - \{0\}}) \}$$

a' is the family of all sections of a determined by an arbitrary formula F in Frm_{pr} and an arbitrary sequence f with terms in a. We now put

$L_0 = 0$; $L_{r+1} = L'_r$; $L_s = \bigcup \{ L_r : r \in s \}$ (s is a limit number)

The union $L = \bigcup \{ L_r : r \in On \}$ is called the class of constructible sets.

It is easy to construct a relation which well orders the class L. We define it as the union $\bigcup \{ X_r : r \in On \}$ where $X_0 = 0$, $X_s = \bigcup \{ X_r : r < s \}$ if s is a limit number and where X_{r+1} is obtained from X_r by the following construction. For every u in $L_{r+1} - L_r$ we denote by F_u the earliest element of Frm_{pr} such that for some f in $L_r^{Fr(F)} - \{ 0 \}$

$$u = S_F (L_r, f)$$

The term "earliest" refers to a fixed well ordering of the denumerable set Frm_{pr} which we think of as fixed in advance.

If there are many sequences f for which the above equation is true then we denote by f_u the earliest of them in the lexicographical ordering $\ll$ of finite sequences induced by the relation X_r.

Now we define X_{r+1} as the union of X_r, of the set $\{ \langle u, v \rangle : (u \in L_r) \& (v \in L_{r+1} - L_r) \}$ and of the set of pairs $\langle u, v \rangle$ where u and v both belong to $L_{r+1} - L_r$

and either F_u precedes F_v in the well ordering W or $F_u = F_v$ and $f_u \ll f_v$.

In order to prove that L is a model of ZF we establish four simple lemmas:

1. $L_r \in L_{r+1}$ for each r in On.

Proof. $L_r = S_{\ulcorner x_0 = x_0 \urcorner}(L_r, 0)$ and hence L_r is its own section.

2. (i) Each L_r is transitive; (ii) If $s \in r$, then $L_s \subseteq L_r$.

We prove both parts simultaneously by induction on r. For $r = 0$ the lemma is trivial; if it holds for all $r < r_0$ and r_0 is a limit number, then it is obvious that the lemma is also true for r_0. Now assume that $r_0 = r + 1$. It will be sufficient to show that $L_r \subseteq L_{r+1}$ and L_{r+1} is transitive. Thus assume $a \in L_r$; hence $a = a \cap L_r = S_{\ulcorner x_0 \in x_1 \urcorner}(L_r, \{\langle 1, a\rangle\}) \in L_{r+1}$. Transitivity of L_{r+1} follows now from the remark that if x is an element of L_{r+1}, then x is a section of L_r and thus $x \subseteq L_r \subseteq L_{r+1}$.

3. If k is a set of ordinals and $r = \sup k$, then $L_r = \bigcup\{L_s : s \in k\}$.

Proof. If $r \in k$, then the lemma results from the monotonicity of the sequence L_r; if $r \notin k$, then it results from the last inductive equation for the L_x's and the lemma 2(ii).

4. If $a \in L_r$, then each section of a belongs to L_{r+1}.

Proof. We use the following simple fact which can easily be established by induction on the number of connectives in a formula:

Let $F \in Frm_{pr}$ and let F^{*} be obtained from F by relativising all quantifiers to the formula $x \in x_i$ where the variable x_i does not occur in F. If X is a transitive set, $a \in X$ and $f \in a^{Fr(F)}$, then

$$X \models F^{*}\left[f \cup \{<i, a>\}\right] \equiv a \models F\,[f].$$

We take now $X = L_r$, and assume that $0 \in Fr(F)$ and $f \in a^{Fr(F) - \{0\}}$, where $a \in L_r$. We obtain then $S_F(a, f) = S_{F^{*} \,\&\, (x_0 \in x_i)}(L_r, f \cup \{<i, a>\})$ which proves that the section $S_F(a, f)$ belongs to L_{r+1}.

III. L is a model of ZF.

Proof. In view of lemma 2(ii) we can represent L as the union $\bigcup\{L_{\omega \cdot r} : r \in On\}$ where the indices range over limit numbers. Lemmas 1 - 4 show that all assumptions of theorem II are satisfied by the family $L_{\omega \cdot r}$, in particular the predicative closure of $L_{\omega \cdot r}$ results from lemma 4 because each element of $L_{\omega \cdot r}$ belongs to some L_x with $x < \omega \cdot r$ and hence its sections are elements of L_{x+1} which is contained in $L_{\omega \cdot r}$.

Example 3. Relatively constructible sets. We start with a transitive set a and put $L_0(a) = a$, $L_{r+1}(a) = L'_r(a)$, $L_s(a) = \bigcup\{L_r(a) : r < s\}$ (s is a limit number). The union $\bigcup\{L_r(a) : r \in On\} = L(a)$ is called the class of sets constructible in a. We can show similary as in theorem III that L(a) is a model of ZF.

More general notions of relative constructibility are possible. We shall sketch a definition due to Solovay.

We consider a wider class of formulae. We adjoin to the atomic formulae $Cl(x_i)$, $x_i \in x_j$ and $x_i = x_j$ which were used thus far still one type of atomic formulae $U(x_i)$. We call the smallest class of formulae which contain all atomic formulae (including the new ones) and is closed with respect to the operations $F|G$ and $(x_i)F$ the class of generalised formulae. The notion of satisfaction can be defined in a similar way as for the class of ordinary formulae. Whereas formerly we defined the notion of satisfaction of a formula F in a set, we must now use a more general relational system. Let x, y be sets such that $x \supseteq y$. The atomic formula $U(x_i)$ is satisfied in this system by a sequence f if and only if $f_i \in y$; other atomic formulae are satisfied in the system $\langle x, y \rangle$ just in case they are satisfied in x. The inductive clauses of the definition of satisfaction remain the same as before. The notion of section in a relational system $\langle x, y \rangle$ is defined as follows: $S_F(\langle x, y \rangle, f)$ is the set consisting of all a in x such that the sequence $f \cup \{\langle 0, a \rangle\}$ satisfies F in $\langle x, y \rangle$. Let $\langle x, y \rangle'$ be the set of all sections of $\langle x, y \rangle$ in the generalised sense thus explained.

For an arbitrary transitive class A we put now $L_0(A) = 0$, $L_{r+1} = \langle L_r, L_r \cap A \rangle'$ and $L_s = \bigcup\{L_r : r \in s\}$ if s is a limit number.

It can be shown similary as in the proof of theorem III that L(A) is a model of ZF.

Example 4. Various generalisations of constructible sets. The class L can be defined in still another way which is sometimes technically more convenient and moreover susceptible to various generalisations.

Let n be an integer $\geqslant 12$. We define by transfinite induction a mapping $d_n : On \rightarrow On$ in such a way that $d_n(0) = 0$ and the order type of the set $Z_n(x) = \{ r \in On : d_n(x) < r < d_n(x+1) \}$ be $n.(d_n(x))^3$. For limit numbers x we require that $d_n(x) = \sup \{ d_n(r) : r < x \}$.

Let I_n, K_n, L_n, M_n be functions such that the correspondence $x \rightleftarrows (I_n(x), K_n(x), L_n(x), M_n(x))$ determines a one - one mapping of the set $Z_n(x)$ onto $n \times d_n(x) \times d_n(x) \times d_n(x)$. For $y = d_n(x)$ we put $I(y) = K(y) = L(y) = M(y) = 0$.

We now define a transfinite sequence $C : On \rightarrow V$ by induction: $C_0 = 0$, $C_{d_n(x)} = \{ C_y : y \in d_n(x) \}$, $C_x = A_{I_n(x)+1}(C_{K_n(x)}, C_{L_n(x)}) \cap \cap C_{M_n(x)}$ provided that x does not have the form $d_n(t)$ and $I_n(x) < 12$; if $I_n(x) \geqslant 12$, then $C_x = B_{I_n(x)+1}(C \mid x)$.

In the last part of the inductive definition $B_{13}, \ldots, B_n$ are operations on sequences of sets which satisfy the condition $B(f) \subseteq Rg(f) \cup \bigcup Rg(f)$ for each transfinite sequence f.

The sequence $C_{d_n(r)} = C^{*}_r$ is increasing and satisfies the conditions $C^{*}_r \in C^{*}_{r+1}$ and $C^{*}_s = \bigcup \{ C^{*}_r : r < s \}$ where r, $s \in On$ and s is a limit number.

We easly show by induction that each C_r is a transitive set (the factors $C_{M_n(x)}$ were added in the third inductive equation for C_r just in order to achieve the transitivity of the sets C_r). Finally it can be shown that if s is a limit number, then C^*_s is closed with respect to the operations A_1 - A_{12} and hence C^*_s is predicatively closed. From theorem II we thus obtain

IV. The class $C = \bigcup\{C_r : r \in On\}$ is a model of ZF.

Remarks. It should be stressed that the class C is a model of ZF independently of how we choose the operations $B_{13}, \dots B_n$ in the last inductive equation. This shows that we may construct many different models of ZF using the above method.

In case when we do not have any additional operations (i.e., when n = 12) the resulting class is identical with L; the proof of this is rather complicated (see Linden, Sets models and recursion theory, North Holland 1967).

We shall use the symbol C^{min}_x for the x-th term of the sequence C in case when n = 12; the union of all C^{min}_x will be denoted by C^{min}; by the result quoted above, $C^{min} = L$.

We now give an example where we use additional operations. Let n = 15. and let a be a fixed transfinite sequence of type r_0 satisfying the condition

$$a_s \subseteq \{a_r : r < s\} \qquad \text{for each } s < r_0.$$

We define the additional operations B_{13}, B_{14} and B_{15} as follows: if f is not a function or $\mathrm{Dom}(f) \notin On$, then we take $B_i(f) = 0$ for i = 13, 14, 15. Otherwise we denote Dom(f) by r and put

$$B_{13}(f) = a_r \text{ if } r < r_0 \text{ and } B_{13}(f) = 0 \text{ if } r \geqslant r_0,$$

$$B_{14}(f) = \left\{ f_x : (x < r) \;\&\; (I_{15}(x) = 14) \right\} ,$$

$$B_{15}(f) = \left(f \mid K_{15}(r) \right) \cap f_{M_{15}(r)} \quad .$$

The class C obtained in this case depends on a and will be denoted by $C^Z(a)$. The effect of the operation B_{13} is that all terms of a will eventually appear in the sequence $C^Z(a)$. They occupy places x for which $I_{15}(x) = 13$. In view of the definition of B_{14} the terms $C^Z_x(a)$ where $I_{15}(x) = 14$ are consecutive ordinals. Finally terms $C^Z_x(a)$ with $I_{15}(x) = 15$ are equal to the common part of $C^Z_{M_{15}(x)}$ and a segment of the sequence $C^Z_u(a)$ consisting of terms with indices $< K_{15}(x)$.

The effect of the operation B_{15} is this:

Lemma. For each ordinal r the sequence $\left\{ \langle x,\ C^Z_x(a) \rangle : x \in r \right\}$ is an element of $C^Z(a)$.

Proof. Let s be a limit number $> r$ and let t be an ordinal such that $I_{15}(t) = 15$, $K_{15}(t) = r$ and $M_{15}(t) = d_{15}(s)$. In view of the inductive definition of the sequence $C^Z_x(a)$ we have.

$$C^Z_t(a) = \left(C^Z \mid r \right) \cap C^Z_{d_{15}(s)}(a) = \left\{ \langle x,\ C^Z_x(a) \rangle : x \in r \right\} \cap C^Z_{d_{15}(s)}(a) \ .$$

All ordinals $x < r$ are elements of $C^Z_{d_{15}(s)}(a)$; this follows from the remark that there is a sequence of type s of ordinal $< d_{15}(s)$ satisfying the equation $I_{15}(x) = 14$ Since $C^Z_x(a)$ is an element of $C^Z_{d_{15}(s)}(a)$ for every $x < r$, we infer that the pairs $\langle x,\ C^Z_x(a) \rangle$ with

$x < r$ belong to $C^Z_{d_{15}(s)}(a)$ for each $x < r$. Hence $C^Z_t(a)$ is equal to the sequence $\{ \langle x, C^Z_x(a) \rangle : x \in r \}$.

As an application of the above remark we prove the following theorem:

V. The axiom of choice is consistent with ZF.

Proof. Since $C^Z(0)$ is a model of ZF, it will be sufficient to show that the axiom of choice is true in this class. Now it is easy to show that if a transitive class is a model of ZF, then the axiom of choice is true in this class if and only if this class contains with every element x a function which maps ordinals onto a set $y \supseteq x$. In view of the lemma the model $C^Z(0)$ has this property which proves the theorem.

Definable well orderings of the universe. For x in $C^{\min}$ we define $Od(x) = \min \{ r : x = C^{\min}_r \}$. The relation $R^{\min} = \{ \langle x, y \rangle \in C^{\min} \times C^{\min} : Od(x) < Od(y) \}$ is obviously a well ordering of $C^{\min}$.

It can be shown that the relation $R^{\min}$ is definable in $C^{\min}$. Even a stronger theorem is true:

VI. There are finitely many sentences $K_1, \ldots, K_n$ which belong to the set of axioms of ZF and formulae F, G in Frm_{pr} with $Fr(F) = Fr(G) = \{0, 1\}$ such that whenever m is a transitive class in which the axioms $K_1, \ldots, K_n$ are valid, then

$(x \in On \cap m) \rightarrow (m \models (E!x_1)F[\{\langle 0, x\rangle\}])$,

$(x \in On \cap m)$ & $(y \in m) \rightarrow (m \models F[\{\langle 0, x\rangle, \langle 1, y\rangle\}]) \equiv (y = C_x^{min})$,

$(x \in m)$ & $(y \in m) \rightarrow [(m \models G[\{\langle 0, x\rangle, \langle 1, y\rangle\}]) \equiv (\langle x, y\rangle \in R^{min})]$

The formula G is defined from F as follows:

$(Ex_2)(Ex_3)\{On(x_2)$ & $On(x_3)$ & $(x_2 \in x_3)$ & $F(x_2, x_o)$ & $F(x_3, x_1)$ &

$(x_4)[(x_4 \in x_2) \rightarrow \neg F(x_4, x_o)]$ & $(x_5)[(x_5 \in x_3) \rightarrow \neg F(x_5; x_1)]\}$

The construction of F is much more complicated and cannot be given here. However it does not require any new idea: we simply write down in the formal language of ZF the inductive definition of the set C_t^{min}

A theorem similar to VI can also be proved for sets L_r. We shall use this fact later. The formula F can be called and absolute definition of C_r^{min} (or of L_r).

It follows from theorem VI that the class C^{min} possesses a well ordering which is definable in C^{min} Hence the existence of a definable well ordering of the universe is consistent with ZF.

These ideas were further exploited by Gödel, Kuratowski, Addison and others who discussed the well ordering of $C^{min} \cap P(\omega)$ induced by the relation R^{min} and proved that it is projective of the class PCA$\cap$CPCA. This result has numerous applications in proofs that various hypotheses of the descriptive set theory are consistent with ZF.

Example 5. Ordinal definable sets. This class was first discovered by Gödel who did not publish his results and then rediscovered by Scott and Myhill and some years afterwards, independently, by Vopenka and Hajek.

We call a set $x \in R_r$ definable in R_r if there is a formula F in Frm_{pr} with exactly one free variable x_o such that for every t in R_r

the conditions $t \in x$ and $R_r \models F[\{\langle 0, t\rangle\}]$ are equivalent. A set x is ordinal definable if there is an ordinal r such that x is definable in R_r. A set x is hereditarily ordinal definable if for every finite sequence s such that $s_n \in s_{n-1} \in \ldots s_1 \in s_o = x$ all the sets s_j are ordinal definable.

We put $D_o = 0$, $D_{r+1} = \{x \in R_r :$ x is hereditarily ordinal definable$\}$, $D_s = \bigcup\{D_r : r < s\}$ (s is a limit number).

It is obvious that $D_r \subseteq D_s$ for $r < s$, D_r is transitive and $D_s = \bigcup\{D_r : r < s\}$ for limit numbers s. Since the operations $A_1 - A_{12}$ lead from ordinal definable sets again to such sets, we easily infer, using theorem I that every set D_r is predicatively closed.

Lemma . There are arbitrarily great ordinals such that $D_r \in D_{r+1}$.

Proof. It is clear that each $D_r \in R_{r+1}$ and that each element of D_r is hereditarily ordinal definable. Thus it remains to show that there are arbitrarily great ordinals such that D_r is definable in R_{r+1}. To achieve this we first construct a formula which "says" that x is definable in R_y.

Let $B = B(x_o, x_2, x_3,)$ be the conjunction of the following formulae:

$\mathrm{Ord}(x_o)$, $\mathrm{Fnc}(x_2)$, $\mathrm{Dom}(x_2) = x_o + 1$, $\langle 0, 0\rangle \in x_2$, $\langle x_o, x_3\rangle \in x_2$,

$(t)\big[(t \in x_o) \rightarrow (u)\{(\langle t+1, u\rangle \in x_2) \equiv (v)[(v \in u) \rightarrow (\langle v, t\rangle \in x_2)]\}\big]$

$(t)\big[(t \in x_o + 1)\ \&\ \mathrm{Lim}(t) \rightarrow (u)\{(\langle t, u\rangle \in x_2) \equiv (Es)(Ev)[(s \in t)\ \&\ (\langle s, v\rangle \in x_2)\ \&\ (u \in v)]\}\big]$

Let C be the formula (with the free variables x_1, x_3, x_4)

$(x_4 \in \mathrm{Frm}_{pr})\ \&\ (\mathrm{Fr}(x_4) = \{0\})\ \&\ (x_5)\big[(x_5 \in x_1) \equiv (x_3 \models x_4[\{\langle 0, x_5\rangle\}])\ \&\ \&(x_5 \in x_5)\big]$

It is not hard to show that if $r \in 0n$, $r \neq 0$ and r is a limit number, then for arbitrary x, y, z, t, F in R_r the following equivalence are true

$R_r \models B[\{\langle 0, x\rangle, \langle 2, y\rangle, \langle 3, z\rangle\}] \equiv (x \in On)$ & (y is a function with domain x + 1) & (s) $(s \in x) \to (y(s) = R_s)$ & $(y(x) = z)$

If $s < r$, then $R_r \models (E!\, x_2)(E!\, x_3)B[\{\langle 0, s\rangle\}]$,

$R_r \models C[\{\langle 1, t\rangle, \langle 3, z\rangle, \langle 4, F\rangle\}] \equiv (F \in Frm_{pr})$ & $(Fr(F) = \{0\})$ & $(t = \{u \in z : z \models F[\{\langle 0, u\rangle\}]\})$

It follows that if we put $A = (Ex_2)(Ex_3)(Ex_4)[B \,\&\, C]$, then for arbitrary positive limit number r and arbitrary x, t in R_r the equivalence

$$R_r \models A[\{\langle 0, x\rangle, \langle 1, t\rangle\}] \equiv (x \in On) \,\&\, (t \in R_x) \,\&\, (t \text{ is definable in } R_x) .$$

For each ordinal r we denote by f(r) the supremum $\sup \{g(x) : x \in R_r\}$ where g(x) is the least ordinal such that x is definable in R_x or 0 if such an ordinal does not exist. It is obvious that the function f is continuous and non decreasing. It is even strictly increasing because each ordinal r is definable in R_{r+1} but does not even belong to R_r. Hence there are arbitrarily great critical numbers of f, i.e., ordinals r which satisfy the equation $f(r) = r$. We claim that if r is such a number, then D_r is definable in R_{r+1}.

First of all, $D_r \in R_{r+1}$ because $D_r \subseteq R_r$; this is proved by remarking that $D_{x+1} \subseteq R_x$ by the definition of D_{x+1} and then summing over $x \in r$. We have still to exhibit a formula G such that, for every x, $(x \in D_r) \equiv R_{r+1} \models G[\{\langle 0, x\rangle\}]$.

To establish the existence of this formula we notice that r and R_r are obviously definable in R_{r+1}. Furthermore from the definition of D_r we obtain the equivalences

$$\begin{aligned}(x \in D_r) \equiv\ & (Es)\Big[(s < r) \;\&\; (x \in D_{s+1})\Big]\\ & (Es)\Big[(s \in r) \;\&\; (x \in R_s) \;\&\; (x \text{ is hereditarily ordinal definable})\Big]\\ & (Es)\Big[(s \in r) \;\&\; (x \in R_s) \;\&\; (u)(m)\Big\{Fn(u) \;\&\; (m \in \omega) \;\&\; (Dom(u) =\\ & m + 1) \;\&\; (u(0) = x) \;\&\; (i)\big[(i < m) \rightarrow (u(i+1) \in u(i))\big] \longrightarrow\\ & (u(m) \text{ is ordinal definable})\Big\}\;.\end{aligned}$$

We now notice that the quantifier (u) in the part of the equivalence can obviously be limited to R_r, because a finite sequence whose terms belong to R_r is itself an element of R_r (the terms of u belong to the "transitive closure" of x and hence to R_r since we assume that $u(m) \in u(m-1) \in \ldots \in u(0) = x$). It follows that we can replace in the least part of the equivalence the expression (u(m) is ordinal definable) by

$$(u(m) \in R_r) \;\&\; (u(m) \text{ is ordinal definable}).$$

As we know 'ordinal definable' means $(Et)\big[(t \in On) \;\&\; (u(m) \text{ is definable in } R_t)\big]$. However, since $u(m) \in R_r$ and r is a critical number of f we see that the expression '$u(m)$ is definable in R_r' can be replaced by '$(t \in r) \;\&\; (u(m)$ is definable in $R_t)$' and thus by

$$(t \in r) \;\&\; (R_r \models A\big[\{\langle 0, t\rangle, \langle 1, u(m)\rangle\}\big]\;).$$

Thus we finally obtain

$$\begin{aligned}(x \in D_r) \equiv (Es)\ & (s \in r) \;\&\; (x \in R_s) \;\&\; (u)(m)\Big[(u \in R_r) \;\&\; (m \in \omega) \;\&\\ & Fn(u) \;\&\; (Dom(u) = m+1) \;\&\; (\langle 0, x\rangle \in u) \;\&\; (i)\Big\{(i \in m) \longrightarrow\\ & (v)(w)\big[(v \in R_r) \;\&\; (w \in R_r) \;\&\; (\langle i, v\rangle \in u) \;\&\; (\langle i+1, w\rangle \in u)\\ & \longrightarrow (w \in v)\big]\Big\}\Big] \quad \&\; (y)((y \in R_r) \;\&\; (\langle m, y\rangle \in u) \longrightarrow\\ & (Et)\Big\{(t \in R_r) \;\&\; On(t) \;\&\; (R_r \models A\big[\{\langle 0, t\rangle, \langle 1, y\rangle\}\big]\Big\}\Big)\Big]\end{aligned}$$

This formula obviously entails the definability of D_r in R_{r+1}. The lemma is thus proved.

As a corollary to the above lemma we obtain

VII. (Scott - Myhill). The class $D = \bigcup\{D_r : r \in On\}$ is a model of ZF.

Proof. In view of the lemma we can represent D as a union of sets $D'_r = D_{h(r)}$ which satisfy the condition $D'_r \in D'_{r+1}$. As we remarked above, the sets D_x, and hence the sets D'_r, are predicatively closed, transitive and form an increasing sequence. The condition $D'_s = \bigcup \{ D_r : r < s \}$ is satisfied if s is a limit number since the supremum of an increasing sequence of critical numbers for the function f is itself a critical number for f. Thus all assumption of theorem II are satisfied in this case.

If $x \in D$, then there is a smallest ordinal $r = r_x$ such that x is definable in R_r. Among formulae which define x in R_r there is one, call it F_x, which occurs earliest in a standard enumeration of the set Frm_{pr} which we must think of as fixed at the beginning of the whole proof. Thus we have a one-one mapping $x \rightarrow (r_x, F_x)$ of D into $On \times Frm_{pr}$. It is not difficult to show that this mapping restricted to a set $a \in D$ is itself and element of D. From this we infer

VIII. Axiom of choice is valid in D.

It is obvious that $L \subseteq D \subseteq V$; none of the equation $L = D$, $D = V$ can be proved or disproved in ZF.

Lecture IV

This lecture will be based on axioms of M but will be devoted to models of ZF. We shall introduce the notion of height and width of a model and shall compare various models as to their height and width.

By a model of ZF we mean in this lecture a transitive set of sets in which all axiom of ZF are true. It is obvious that the class of models of ZF can be defined by a predicative formula.

Instead of transitive families of sets we could equally well use arbitrary well founded relations. This results from the following lemma provable in ZF.

I ("contraction lemma"). If R is a well founded relation which satisfies the condition

$$(u = v) \equiv (t)((tRu) \equiv (tRv))$$

for arbitrary u, v in the field of R, then R is isomorphic with the relation $\in$ in a transitive family of sets.

The proof of this lemma is easy and will not be given here.

The existence of models follows from the Scott - Scarpellini theorem; this theorem shows for instance that there are ordinals r, s , such that $L_r \prec L$, $D_s \prec D$ and $R_t \prec R$. Hence there are models of ZF of the form L_r, D_s, R_t.

The question arises: are there models of ZF of any given cardinality? The answer results easily from the downward Skolem - Löwenheim theorem:

II. For every ordinal $r \geq \omega$ there is a model of power $|r|$ elementarily equivalent with L.

Proof. We start with a set of power $|r|$, e. g., with r itself and consider an ordinal s (of any power) such that $r \subset L_s \prec L$. For each F in Frm_{pr} with $0 \in Fr(F)$, each infinite set $a \in L_s$ and each sequence f in $a^{Fr(F) - \{0\}}$ we denote by $e_F(f)$ the earliest element x in L_s such that $L_s \models F\,[\{<0, x>\} \cup f]$ or 0 if there is no such element. Furthermore we put $a' = \{e_F(f) : (F \in Frm_{pr})\ \&\ (0 \in Fr(F))\ \&\ (f \in a^{Fr(F) - \{0\}})\}$.

It is obvious that $a \subseteq a'$ (consider the formula $x_0 = x_1$!) and that a and a' have the same power (Frm_{pr} is denumerable and f ranges over the set of all finite sequences with terms in a). Now we form the union $a = \bigcup a_n$ where $a_0 = r$ and $a_{n+1} = a'_n$; it is obvious that the power of a is $|r|$ and that $a \prec L$. Finally we apply the contraction lemma I to obtain the desired transitive set.

A theorem similar to II can be proved for the class D and generally for every class which can be proved to be well orderable.

There are other proofs of the Skolem-Löwenheim theorem which allow us to prove the existence of denumerable models which are elementarily equivalent with any given model whether well orderable or not.

III. For any model of ZF there exists an elementarily equivalent denumerable model.

Proof. Let m be the given model. We shall say that a formula $F \in Frm_{pr}$ with exactly one free variable x_0 describes an ordinal if the set $S_F(m, \{<0, x>\})$ is an ordinal. For each F with this property we consider a costant 0_F; moreover we consider denumerably many constants c_j, $j = 0, 1, 2, \ldots$ These constants are added to the language of ZF; the formulas of the extended language are obtained from the formulae of the old language (i.e., essentially from the elements of Frm_{pr}) by substituting constants for some or all free variables of the formulae.

Let T be the set $\{F \in Frm_{pr} : (Fr(F) = 0) \& (m \models F)\}$; We may call T "the theory of m". Now we denote by T^* the set of sentences of the extended language which can be derived from T and the following sentences:

$\neg(c_j = c_k)$ where $j \neq k$, $(x_0)\left[(x_0 \in 0_F) \equiv F\right]$ where F is a formula which describes an ordinal.

We claimthat if F is a formula (of the extended language) with $Fr(F) = \{0\}$, then $(*)$: $(x_0)F(x_0) \in T^* \equiv (j)_\omega(F(c_j) \in T^*)$,
$(**)$: $(x_0)\left[On(x_0) \rightarrow F(x_0)\right] \in T^* \equiv (G)\left[F(0_G) \in T^*\right]$;
in the formula $(**)$ the letter 'G' ranges over formulae (of the primitive language) which describe ordinals.

The implications from left to right in both $(*)$ and $(**)$ are obvious. The converse implication in the case $(*)$ can be proved by noticing that all the steps in a deduction of $F(c_i)$ from T can be repeated when c_j is replaced by a variable not occurring in the given deduction.

The implication from right to left in $(**)$ can be proved as follows. Write $F(0_G)$ as $F'(0_G, c_1, \dots, c_n)$ where F' is a formula of the primitive language. The constants 0_X different from 0_G can obviously be eliminated by replacing then by terms $\{x : X\}$ which in turn can be eliminated because of their definability in the primitive language. For simplicity we assume that the variables x_i, $i = 1, 2, \dots, n$ do not appear in F'. We denote by F'' the formula $F'(x_o, x_1, \dots, x_n)$ and by K the conjunction of formulae $\neg(x_p = x_q)$ where $1 \leq p < q \leq n$.

Using the deduction theorem we can show that from a proof of the formula $F'(0_G, c_1, \dots, c_n)$ from the assumption T we can obtain a proof of

(i) $(x_o)(x_1)\dots(x_n)\left[(x_o = 0_G) \& K \rightarrow F''\right]$

from the same assumption. To see this it is sufficient to replace in the given proof all the constants c_j by new variables and prefix each formula which appears in the proof with the conjunction of inequalities $\neg(x_p = x_q)$ where $p \neq q$ and x_p and x_q range over the variables which were used to replace the constants c_j.

It follows now that for each formula G which describes an ordinal the formula (of the primitive language)

$$(x_0)(x_1)\ldots(x_n)\quad ((t)[(t \in x_0) \equiv G(t)] \;\&\; K \rightarrow F'')$$

is true in m and hence belongs to T.

We choose for G the formula G_0 which describes the following ordinal r: if there is an ordinal s such that $m \models (Ex_1)\ldots(Ex_n)K \;\&\; \neg F''[\{<0, s>\}]$, then $r = s_0+1$ where s_0 is the least such ordinal; otherwise $r = 0$. The formula G_0 can be written explicitely:

$$(Ey)\Big\{(Ex_0)\Big[(On(x_0) \;\&\; K \;\&\; \neg F'' \;\&\; (v)\{(v \in x_0) \rightarrow (x_0)(x_1)\ldots(x_n)[(v = x_0) \;\&\; K \rightarrow F'']\}) \;\&\; (y \in x_0+1)\Big] \;\&\; (y = x_0)\Big\}.$$

(The formula inside square brackets has the free variable 'y'; since we must insist that the free variable should be 'x_0' we had to add the outer quantifier (Ey) and the equation $(y = x_0)$ at the end in order to have the free variable x_0).

The formula (i) with G replaced by G_0 is thus provable. Since this formula implies the formula

$(x_0)\,[On(x_0) \rightarrow (x_1)\ldots(x_n)(K \rightarrow F'')]$, we obtain the left hand side of (**).

In the further course of the proof we use ideas which underlie the proof of the completeness theorem. We consider the Boolean algebra of formulae of the extended language and denote by B the quotient algebra obtained by the division by the filter T^*. The formulae (*) and (**) prove that the element $(Ex_0)F/T^*$ is the Boolean join of the element $F(c_j)/T^*$ and the element $(Ex_0)\left[On(x_0) \;\&\; F\right]$ is the Boolean join of the element $F(0_G)/T^*$.

By the Rasiowa - Sikorski lemma there is a maximal filter $T^{**} \supset T^*$ which preserves these joins. We define a relational system $\langle C, R\rangle$ where C is the set of all the constants c_j and R is the relation which holds between c_j and c_k if and only if the formula $c_j \in c_k$ is an element of T^{**}. It is well known that all sentences of T^{**} and hence those of T are true in this relational system. We now prove that the relation R is well founded.

Let us denote by H the formula which defines the relation "r is the rank of x." Formally we express H as follows:

$$(Ex_2)(Ex_3)\Big\{B(x_0,x_2,x_3) \;\&\; (x_1 \subseteq x_3) \;\&\; (y)(z)(t)\Big[(y \in x_0) \;\&\; B(y,z,t) \longrightarrow \neg(x_1 \subseteq t)\Big]\Big\}$$

where B is the formula defined on p. 111 . The formulae

$$H(v,x) \longrightarrow On(v), \qquad (x)(E!v)\; H(v,x),$$

$$(x \in y) \;\&\; H(v,x) \;\&\; H(w,y) \longrightarrow (v \in w)$$

are provable in ZF. It follows that for every j the formula

$$(Ev)\left[H(v,c_j) \;\&\; On(v)\right]$$

belongs to T^{**} and hence there is a formula G_j which describes an ordinal and which has the properties that the formulae $H(0_{G_j}, c_j)$ and

$$(***) \quad (c_j \in c_k) \longrightarrow (0_{G_j} \in 0_{G_k})$$

belong to T^{**}

Let r_j be the ordinal described by G_j and let j, k be such

that $c_j R c_k$.
In order to prove that R is well founded it will be sufficient to prove that $r_j < r_k$. Assume the contrary. It follows that $m \models \{x:G_k\} \geq \{x:G_j\}$ and hence the formula $0_{G_k} \geqslant 0_{G_j}$ is in T^*. This however is impossible because the formula $(***)$ clearly implies that $\neg(0_{G_k} \geqslant 0_{G_j})$ belongs to T^{**}.

The well foundedness of the relation R being established we may use theorem I and obtain the desired model.

The above proof does not work in the non denumerable case because no analogue of the Rasiowa - Sikorski lemma is then available.

Definition. We call a height of a model m the least ordinal not in m; we say that a model m_1 is broader than m_2 at the level r where r is an ordinal if $m_1 \cap R_r \supset m_2 \cap R_r$

Example. The natural models have the largest possible breadth

The first remark is obvious and the second follows from the existence of the formula F alluded to in theorem VI of lecture III.
It follows from the properties of this formula that whenever m is a model and $r \in On \cap m$, then $C_r^{min} \in m$. Hence if h is the height of m, then $C_h^{min} \subseteq m$; it can be shown that $C_h^{min} = L_h$.

IV (Cohen - Shepherdson). There is a minimal model (i.e., one which is contained in any model).

This follows from the example above. The minimal model is equal to the first L_r which is a model of ZF.

V. If m is a model, then so is $m \cap L$.

We omit the detailed proof of this theorem. Essentially it can be established as follows: The proof that L is a model of ZF is based on the fact that all axiom of ZF are valid in the universe. Relativising this proof to m we obtain that all these axioms are valid in the domain of constructible elements of m. In view of the example above this domain is $m \cap L$.

Theorem V shows that the height of an arbitrary model is at the same time the height of a constructible model. The converse is not true. E.g., the height of a natural model is always non - denumerable because every such model contains non denumerable well orderings and each such ordering is similar to an ordinal which belongs to the model. On the other hand there are denumerable constructible models.

In lectureVII we shall construct examples of denumerable models of equal heights but of different breadths. Such models are of paramount importance for various independence proofs. Their existence can best be established by means of the notion of forcing introduced by Cohen. However we shall not deal with this notion here. Instead we shall discuss another problem which is much more special but interesting and, as we shall see, far from trivial.

We put $L_r^* = L_r \cap R_{\omega+1}$. It is obvious that $L_r^* \subseteq L_s^*$ for $r \leqslant s \in On$ and hence there is a smallest ordinal c such that L_r^* is constant from c on . We want estimate the size of c.

We note in passing that the same problem can also be formulated for the families $L_r \cap R_k$ where $k > \omega+1$. Smaller values of k are not interesting because $L_r \cap R_n$ is certainly constant from $r = \omega$ on if $n \leqslant \omega$.

Estimates of c from below. Let us call a model m constructible if there is an ordinal r such that $L_r = m$; the ordinal r is called the index of m. It can be shown that the index of a constructible model is equal to its height.

VI. c is greater than the index of the minimal model.

Proof. Let the index of the minimal model be i and the index of the next model in the sequence L_x be j. Then the sentence

'there is a constructible model of ZF'

is true in L_j. Since the Skolem-Löwenheim theorem is provable in ZF (cf. the proof of theorem III above) we infer that the sentence

'there is a denumerable constructible model of ZF'

is true in L_j. Hence there is a set x in L_j which satisfies in L_j the formula 'x is a denumerable model of ZF'. Such a set must be a model of ZF because the relation of satisfaction and the class of axioms of ZF are absolutely definable. Since L_j contain just one model of ZF, x must be equal to this unique model i.e., to L_i. Hence L_j contains a function which maps L_i onto ω and is one-one. Since each ordinal $< i$ is an element of L_i we infer that i is denumerable in L_j and hence there is a set X of integers such that $X \in L_j$ and the relation $\{< m, n > : 2^m(2n - 1) \in X\}$ is of the order type i. It follows that $X \in L_j^*$ but $X \notin L_i^*$ because otherwise i would be an element of i.

In order to obtain a stronger estimate for c we introduce the

Definition. A positive ordinal r is called L-definable if there is a predicative formula F with $Fr(F) = \{0\}$ such that whenever $r \in L_s$ and L_s is a model of ZF, then r is the unique element of L_s such that $L_s \models F[\{<0, x>\}]$ while 0 is a unique such element if $r \notin L_s$ and L_s is a model of ZF (compare a similar notion of R - definability which we introduced in lecture II).

VII. If r is L - definable and L_s is the r-th term of the sequence of the constructible models , then $c > s$.

Proof. Let F be a formula which L - defines r and let L_i and L_j be the $r + 1$st and $r+2$ nd terms of the transfinite sequence which contains all constructible models. We consider the following sentence H:

$$(Ex_0)(Ez)((x_0 \neq 0) \mathbin{\&} On(x_0) \mathbin{\&} (x_0 \in x_1) \mathbin{\&} (z \in x_1) \mathbin{\&} (x_1 \models F[\{<0, x_0>\}]) \mathbin{\&} Fn(z) \mathbin{\&} (Dom(z) = x_0) \mathbin{\&} (t)(u) \{(<t, u> \in z) \rightarrow (u \text{ is a constructible model of ZF}) \mathbin{\&} (v)(w) [\{<v, w> \in z) \mathbin{\&} (t \in v) \rightarrow (w \in u)]\}) .$$

H "says" that x_1 contains a sequence of constructible models of ZF ordered in type r by the relation $\in$. Of course the formalization of H given above is not complete: expressions 'is a constructible model', 'Fn(z)' etc. have to be written exclusively in terms of the primitive notions of ZF. Such a complete formalization of H is easily obtainable.

Since $r \in L_i \in L_j$ and the sequence $\{<x, L_x> : x \in r\}$ belongs to L_i we easily see that $L_j \models (Ex_1)\ [(x_1$ is a model of ZF) & H$]$ We now use the Skolem-Löwenheim theorem which as we know is provable in ZF and hence valid in L_j. We obtain the result that L_j contains an element y which is denumerable in L_j (i.e., a mapping of y onto integers exists in L_j) and which has the property that the formula 'x_1 is a constructible model of ZF' & H is satisfied in L_j by y (more exactly: by the sequence $\{<1, y>\}$). It follows that y is a constructible model of ZF, $y = L_t$ for some t. Hence $j > t$. Since L_t satisfies H in L_j we infer using the definition of L - definability that L_t contains as element a sequence of type r of constructible models of ZF. Hence $t \geqslant i$.

From the inequalities $j > t \geqslant i$ it obviously follows that

$t = i$. Hence L_i is denumerable in L_j and therefore the same is true of r. Now the proof can be brought to an end in the same way as in VI.

The notion of L - definability is closely connected with the notion of strong definability which was discussed in a paper by the present writer. It follows from the result of this paper that L - definable ordinals are $< \omega_1$. Thus if r is the first ordinal which is not strongly definable, then $c \geq r$. Most probably still stronger evaluations of c from below are possible.

Estimate of c from above. We shall show that $c \leq \omega_1$. The proof is based on a device invented by Gödel in his proof that the continuum hypothesis is valid in the model L.

VIII. (Gödel's lemma). If $x \subseteq R_\omega$ and $x \in L$, then $x \in L_{\omega_1}$.

Proof. Let r be such that $x \in L_r$ and L_r is a model of ZF. We consider the relational system $\langle L_r, \in \rangle$. By an application of the Skolem-Löwenheim theorem we obtain a structure $\langle m, \in \rangle$ such that m is denumerable, x and R_ω belong to m and $\langle m, \in \rangle \prec \langle L_r, \in \rangle$. From theorem VI of lecture III we know that there exists a formula $F \in \mathrm{Frm}_{pr}$ with two free variables x_0, x_1 such that for an arbitrary model N of ZF and every ordinal s in $\mathrm{On} \cap N$, and every $u \in N$

$$N \models (E!x_1) F [\{\langle 0, s \rangle\}] ,$$
$$N \models F [\{\langle 0, s \rangle, \langle 1, u \rangle\}] \equiv (u = L_s).$$

The following sentence which expresses the fact that every set belongs to one of the sets L_y is evidently true in the structure $\langle L_r, \in \rangle$: (i) $(x_2)(Ex_0) [\mathrm{Ord}(x_0) \,\&\, (Ex_1) F \,\&\, (x_2 \in x_1)$.

Hence this sentence is true in the structure $\langle m, \in \rangle$ whence we infer that for every y in m there is an s' and an x' such that s' and x' belong to m and

$$m \models \mathrm{Ord}\left[\{\langle 0, s'\rangle\}\right] ; m \models F\left[\{\langle 0, s'\rangle , \langle 1, x'\rangle\}\right], \ y \in x'.$$

The element s' is not necessarily an ordinal since m is not necessarily transitive; similary x' need not be equal to an L_v. Contracting m to a transitive set m^* we obtain a model of ZF; the element s' and x' are contracted to elements s^* and x^* which satisfy formulae similar to those above but with m replaced by m^*. It follows that s is an ordinal and $x^* = L_{s^*}$. Hence $y \in L_{s^*}$ and therefore $m^* \subseteq \bigcup\{L_{s^*}: s^* \in m^* \cap On\} = L_t$. The ordinal t is denumerable because m^* is denumerable. Since x is transformed into itself by the contracting function we infer that $x \in L_t \subseteq L_{\omega_1}$.

From Gödel's lemma we immediately obtain

IX. $c \leq \omega_1$.

Remark. If the formula (i) were true in V, we would obviously have $L^*_{\omega_1} = R_{\omega+1}$ and hence c would be equal to ω_1. Since the assumption that (i) is true in V is consistent with the axioms of ZF we infer that so is the assumption $c = \omega_1$.

A construction of Rowbottom. Scott was the first to prove that the existence of very large cardinals implies the existence of sets which are not constructible. Gaifman improved his result by showing that the existence of measurable cardinals implies the denumerability of $L \cap R_{\omega+1}$, i.e., the inequality $c < \omega_1$. An independent proof of this result was also obtained by Rowbottom and we shall reproduce it below. Rowbottom's result is even stronger than that of Gaifman because he does not assume the existence of measurable cardinals but makes a much weaker assumption.

We denote by $\{A\}^n$ the family $\{Z \subseteq A : |Z| = n\}$.

Definition. We say that a cardinal r satisfies the partition property $r \rightarrow (\omega_1)^{<\omega}_{\omega_1}$ (or for short the property (R)) if the following is true: for every set U of power r and every denumerable family f_n of mappings:

$$f_n : \{U\}^n \rightarrow \omega_1 \qquad (n = 1; 2, \ldots)$$

there is a set $X \subseteq U$ of power ω_1 and a function g: $\omega \rightarrow \omega_1$ such that $f_n(a) = g(n)$ for every a in $\{X\}^n$ and every integer $n \geqslant 1$.

We call X the set of indiscernibles for the family f_n.

Before we define a particular family which shall be used in the proof we establish the

Lemma1. If $s \in On$ and a is a finite sequence whose elements are constructible sets, then the set

$$t(a) = \{ F \in Frm_{pr} : (Fr(F) = Dom(a) \ \& \ (L_s \models F[a])) \}$$

is constructible.

Proof. Since L is a model of ZF there exists in L a set of all F's which satisfy in L the formula obtained by expressing in the language of ZF the condition $(Fr(F) = Dom(a)) \ \& \ (L_s \models F[a])$. Because of the absoluteness of this formula we infer that this set coincides with t(a).

We call t(a) the type of a. Lemma 1 implies that the type of a sequence whose terms are constructible sets is itself constructible. We shall call types t(a) of sequences with constructible elements c - type.

For each finite set a with constructible elements we denote by t'(a) the index (in the transfinite sequence of all elements of L) of the type t(a') where a' is the sequence with the range a and with terms arranged in an increasing order; the ordering relation is that of the natural ordering of L.

We assume the existence of a cardinal r_0 with the property (R) and put $f_n(a) = t'(a)$ for each a in $\{L_{r_0}\}^n$. Hence f_n maps $\{L_{r_0}\}^n$ into ω_1 and since L_{r_0} has power r_0, we obtain a set X of power ω_1 and a function g: $\omega \to \omega_1$ which satisfy the equation $t'(a) = g(n)$ for every a in $\{X\}^n$.

Lemma 2. If a class K has a well ordering which is definable in K and if Y is a subclass of K, then elements definable in the structure $\langle K, \in, y\rangle_{y \in Y}$ form a class D which satisfies the relation $D \prec K$.

Proof. All we need to show is the following: if $F \in Frm_{pr}$, $0 \in Fr(F)$ and $a \in D^{Fr(F) - \{0\}}$, then from $K \models (Ex_0)\, F\,[a]$ it follows that there is an element b in D such that $K \models F\,[\{\langle 0, b\rangle\} \cup a]$. By assumption there is a b in K which satisfies this condition and since a well ordering of K is definable in K, the first element of K which satisfies the condition stated above is definable in K and hence belongs to D.

We apply the lemma to the case where $K = L_{r_0}$ and $Y = X$. The set D is a non - denumerable model of ZF since so is L_{r_0}. By contraction we obtain a transitive set B which is a model of ZF.

Let $u \to u'$ be the contracting function. If a and a^* are two increasing sequences whose common domain is $\{1, 2, \ldots, n\}$ and whose term have the form u' where $u \in X$, then obviously $t'(a) = t'(a^*)$ whence $L_{r_0} \models F\,[a] \equiv L_{r_0} \models F\,[a^*]$ for each F in Frm_{pr} with $Fr(F) = \{1, 2, \ldots, n\}$.

We shall now prove that B has the form L_s with $s \geq \omega_1$. To prove this we denote by F the absolute definition of L (cf. theorem VI in lecture III) and notice that for every x in L_{r_0} there is an ordinal r in $On \cap L_{r_0}$ such that $x \in L_r$, i.e., $L_{r_0} \models F\left[\{<0, r>^0, <1, x>\}\right]$. Hence $L_{r_0} \models (x_2)(Ex_0)(Ex_1)\left[Ord(x_0) \,\&\, F \,\&\, (x_2 \in x_1)\right]$. The same formula is true in B which implies that for every x in B there is an ordinal r in B satisfying the condition $x \in L_r$. Hence $B = \bigcup\{L_r : r \in On \cap B\} = L_s$ where s is the height of B. Since B is not denumerable, we obtain $s \geqslant \omega_1$.

Now we notice that $x \in L \cap R_{\omega+1}$ implies $x \in L_{\omega_1}$ (cf. VIII) and hence $x \in L_s$. Thus if $x \in L^*_{\omega_1}$, then $x = u'$ for some u in D. Since u is definable in the structure $<L_{r_0}, \in, y>_{y \in X}$, the element $u' = x$ is definable in the structure $<B, \in, y'>_{y \in X}$. Denoting by G the formula which defines x we infer that x is the unique element of B which satisfies the condition $(m \in x) \equiv B \models G\left[\{<0, m>, <1, y'_1>, \ldots, <n, y'_n>\}\right]$; here y'_i are all elements of $\{u' : u \in X\}$ which occur in the definition of x. Thus an integer m belongs to x if and only if m together with the $y'_i (i = 1, 2, \ldots, n)$ satisfy G in B.

We can write this result in a more conspicuous way if we denote by $C_m(x_0)$ a formula which says that the m-th integer belongs to x_0 (e.g. $C_0(x_0)$ is the formula $0 \in x_0$, $C_1(x_0)$ is the formula $\{0\} \in x_0$ etc.). We obtain then

$$m \in x \equiv B \models (Ex_0)\, G \,\&\, C_m(x_0)\left[\{<1, y'_1>, \ldots, <n, y'_n>\}\right].$$

Since the elements y'_i are indiscernibles and since we can obviously assume that they form an increasing sequence we infer that the right hand side of this equivalence does not depend on the particular choice of these indiscernibles but only on G and m. Thus each x in $L^*_{\omega_1}$ is determined by a formula and consequently there are only denumerably many elements in $L^*_{\omega_1}$ which proves that $c < \omega_1$.

Lecture V

In order to establish the independence of various set-theoretical hypotheses from the axioms of ZF Scott introduced a new kind of models which are completely different from the ones considered thus far. His construction is closely connected with ideas due to Cohen who first established these independence result. Cohen used for this purpose the notion of forcing. Scott's methods are much easier to deal with. A construction equivalent to that of Scott was also developed by Vopenka. The present lecture as well as the three lectures which follow are based on lectures given by Scott in the Summer School on Set Theory in Los Angeles (1967).

The main idea of Scott /and Vopenka/ is the use of many valued logic. Instead of the Boolean algebra $\{0, 1\}$ of truth values we shall consider an arbitrary complete Boolean algebra B. Instead of sets we shall consider functions with values in B; sets of sets will be replaced by functions with values in B whose arguments are functions of the same character. By induction we define a class of these functions which is stratified in a similar way as the universal class V. Elements of this class are objects which will be used to interpret the basic notion of "set". Under this interpretation each set-theoretic formula has a value which is an element of B. We shall show that all theorems of ZF have the value 1. Thus a formula whose value for some algebra B is not 1 is independent from the axioms.

We denote the basic operation of B by $+, \cdot, -$ with subsript B if necessary. Infinite meets and joins are denoted by $\prod$ and $\sum$ possibly with the index B. The elements $\prod_{x \in B} x$ and $\sum_{x \in B} x$ are deno =

ted by 0 and 1. We also put $x \rightarrow y = -x+y$ and $x \ast y = (x \rightarrow y).(y \rightarrow x)$.

We always assume that B is complete.

The axiomatic basis for all what follows is the set theory M together with the axiom of choice although most of the theorems can be proved already in ZF provided that we add to it the axiom of choice.

We denote by B^V the class of functions f in V whose range is $\subseteq B$. We call Dom(f) the support of f. We now define "partial universes" V_r^B which correspond to our former R_r. The definition is by induction: V_r^B is the class of all B-valued functions whose domains are subsets of $\bigcup\{V_s^B : s < r\}$. For $r = 1$ the only element of V_1^B is the "void" function 0. The union of all partial universes will be denoted by V^B.

The basic semantical notions. We introduce, in analogy to what we did in lecture I the notion of satisfaction. Since we are now dealing with many valued logic /elements of B playing the role of truth values/ we shall have not the division of formulae in those which are satisfied by a sequence and those which are not satisfied by this sequence but a more complicated partition of formulae into sets of formulae which are satisfied by a given sequence with a degree b where $b \in B$. In other words we shall define a B-valued function S_B whose arguments range over the class

$\{\langle F, a\rangle : (F \in Frm) \;\&\; (a \in (V^B)^{Fr(F)})\}$. Scott writes $[\![F(a)]\!]$ for $S_B(F, a)$. If Fr(F) consists of one integer n, then we shall often write $S_B(F, a)$ instead of $S_B(F, \{\langle n, a\rangle\})$.

The definition of S_B proceeds by induction and consists of two parts, one dealing with atomic formulae and another with formulae involving logical operators.

Definition of $S_B(F, a)$ for the case where F contains logical operations. We put

$$S_B(F_1 | F_2, a) = (-S_B(F_1, a | Fr(F_1))) + (-S_B(F_2, a | Fr(F_2))),$$

$$S_B((x_i)F, a) = S_B(F, a) \text{ if } i \notin Fr\, F,$$

$$S_B((x_i)F, a) = \prod_{x \in V^B} (S_B(F, a \cup \{<i, x>\}) \text{ if } i \in Fr(F).$$

Definition of $S_B(F, a)$ for atomic formulae. We first define two auxiliary functions E, I which map $V^B \times V^B$ into B. Let us assume that these functions are already defined on the set $(\{\bigcup V^B_s : s < r\})^2$ and let a, bea pair which belongs to $(\{V^B_s : s \leq r\})^2$ but not to the former set. We put

$$I(a, b) = \prod_{x \in Dom(a)} \{a(x) \rightarrow \sum_{y \in Dom(b)} [b(y). \; I(x, y)]\} \; .$$

$$\prod_{y \in Dom(b)} \{b(y) \rightarrow \sum_{x \in Dom(a)} [a(x). \; I(y, x)]\}$$

$$E(a, b) = \sum_{y \in Dom(b)} [b(y). \; I(a, y)] \; .$$

This is clearly an inductive definition of the sort which can be formalised on the basis of axioms of M. Hence we can assume that there exist functions E, I which satisfy the above equations.

We now put

$$S_B(x_i = x_j, \{<i, a>, <j, b>\}) = I(a, b) \text{ for } i \neq j,$$

$$S_B(x_i = x_i, \{<i, a>\}) = S_B(\, C1(x_i), \{<i, a>\}) = 1,$$

$$S_B(x_i \in x_j, \{<i, a>, <j, b>\}) = E(a, b) \text{ for } i \neq j,$$

$$S_B(x_i \in x_i, \{<i, a>\}) = 0.$$

Remark. If we formalise the construction in the system ZF we obtain two formulae $E'(a, b, x)$ and $I'(a, b, x)$ such that it is provable in ZF that for arbitrary a, b in V^B there is exactly one x such that E'(a, b, x) and exactly one y such that I'(a, b, y). Denoting these unique elements by E(a, b) and I(a, b) we can prove inductive equations for E and I. In this case we cannot define a function which could play the role of S_B; we even cannot define a formula $S'_B(F, a, x)$ for which it would be provable that for arbitrary F in Frm and an arbitrary a in $(V^B)^{Fr(F)}$ there is exactly one x in B and which would have the property that if this unique x is denoted by $S_B(F, x)$, then the inductive equation given in the first part of the definition will be provable. However we can define $S_B(F, x)$ for each explicitly given formula F.

If we work in the system GB, then E and I can be defined but again there is no possibility of defining the function S_B generally.

Validity. A formula F will be called B-valid if $S_B(F, a) = 1$ for every a in $(V^B)^{Fr(F)}$.

Submodels. Any class $W \subseteq V^B$ is called a Boolean submodel. The satisfaction function S_B^W of a submodel is defined as follows. If F is an atomic formula, then $S_B^W(F, a) = S_B(F, a)$ for every a in $W^{Fr(F)}$. If F is not an atomic formula, then the value of $S_B^W(F, a)$ is defined by induction in the same way as the function S_B with the only change that in the case of the formula $(x_i)F$ the domain of variability of "x" is restricted to W.

Elementary submodels. We call W' an elementary submodel of W if $S_B^{W'}(F, a) = S_B^W(F, a)$ for an arbitrary F in Frm and a in $W'^{Fr(F)}$.

Tarski's test. If for every F with $0 \in Fr(F)$ and for an arbitrary sequence a in $W'^{Fr(F) - \{0\}}$ the following equation holds:

$\sum_{x \in W'} S_B^W(F, a \cup \{<0, x>\}) = \sum_{x \in W} S_B^W(F, a \cup \{<0, x>\}),$

then W' is an elementary submodel of W.

In particular, if W' is a submodel of W and for every F in Frm such that $0 \in Fr(F)$ and every a in $W'^{Fr(F) - \{0\}}$ there is an x in W' satisfying the equation $S_B^W(F, a \cup \{<0, x>\}) = \sum_{x \in W} S_B^W(F, a \cup \{<0, x>\})$, then W' is an elementary submodel of W.

Proof. is practically the same as in the two-valued case.

We shall later develop the sematince of Boolean models but must first establish some obvious properties of the function S_B. Theorems 1 - 4 below are so obvious that no proof is needed.

1. If $F \in Frm$ and F' results from F by a correct substitution, then the validity of F implies that of F'.

2. If $F, G \in Frm$ and if the formulae F, $F \rightarrow G$ are valid, then so is G.

3. If $F, G \in Frm$, $i \notin Fr(F)$ and the formula $F \rightarrow G$ is valid; then so is $F \rightarrow (x_i)G$.

4. Axioms of the propositional logic and of quantifier logic are valid.

5. $I(a, a) = 1$.

Proof by induction. Let us assume that the theorem is true for $a \in \bigcup\{V_r^B : r < s\}$ and let $a \in V_s^B - \bigcup\{V_r^B : r < s\}$. The element $I(a, a)$ is the Boolean product of two identical elements $\prod_{x \in Dom(a)} \{a(x) \rightarrow \sum_{y \in Dom(a)} [a(y). I(x, y)]\}$. This latter element is clearly $\leq \prod_{x \in Dom(a)} \{a(x) \rightarrow [a(x). I(x, x)]\}$ whence by using the inductive assumption we obtain that this element is $\prod_{x \in Dom(a)} [a(x) \rightarrow a(x)] = 1$

6. $I(a, b) = I(b, a)$. Proof obvious.

7. $I(a, b).\ I(b, c) \leq I(a, c)$.

Proof. Obvious Boolean calculation reduce the statement to $I(a, b).\ I(b, c).\ a(x) \leq \sum_{y \in Dom(c)} [c(y).\ I(x, y)]$ for $x \in Dom(a)$.

Write this formula for short $H \leq C$. It is clear that $H \leq \sum_{z \in Dom(b)} [b(z).\ I(x, z)]$ whence by multiplying both sides by $I(b, c)$ and noticing that $I(b, c).\ b(z) \leq \sum_{y \in Dom(c)} [c(y).\ I(y, z)]$ we obtain $H \leq \sum_{z \in Dom(b)} \sum_{y \in Dom(c)} [c(y).\ I(x, z).\ I(y, z)]$.

Now we use 6 and obtain $H \leq \sum_{z \in Dom(b)} \sum_{y \in Dom(c)} [c(y).\ I(x, z).\ I(z, y)]$.

The rest of the proof. follows by induction: if the theorem is true for $\langle x, y, z\rangle \in \bigcup\{V_r^B : r < s\}^3$ and $\langle a, b, c\rangle \in \left(\bigcup\{V_r^B : r \leq s\}\right)^3$, then the above inequality implies

$$H \leq \sum_{z \in Dom(b)} \sum_{y \in Dom(c)} [c(y).\ I(x, y)] = C.$$

8. $I(a, b).\ E(a, x) \leq E(b, x)$.

Proof. The left-hand side is $\leq \sum_{z \in Dom(x)} [x(z).\ I(z, a).\ I(a, b)]$ whence by 7 the left-hand side is $\leq \sum_{z \in Dom(x)} [x(z).\ I(z, b)] = E(b, x)$.

9. $I(a, b).\ E(x, a) \leq E(x, b)$.

Proof. The left-hand side is $= I(a, b).\ \sum_{u \in Dom(a)} [a(u).\ I(x, u)] \leq \sum_{u \in Dom(a)} \left(a(u).\ I(x, u).\ \{a(u) \rightarrow \sum_{v \in Dom(b)} [b(v).\ I(u, v)]\}\right) \leq \sum_{u \in Dom(a)} \sum_{v \in Dom(b)} [a(u).\ b(v).\ I(x, u).\ I(u, v)] \leq \sum_{v \in Dom(b)} [b(v).\ I(x, v)] = E(x, b)$.

10. $E(a, a) = 0$

Proof by induction. We assume the theorem for elements in $\bigcup\{V_r^B : r < s\}$ and let $a \in V_s^B$. From the definition we obtain $E(a,\ a) = \sum_{x \in Dom(a)} [a(x).\ I(x,\ a)] \leq \sum_{x \in Dom(a)} [E(x,\ a).\ I(x,\ a)]$ because $a(x) \leq E(x,\ a)$ for each $x \in Dom(a)$. Hence $E(a,\ a) \leq \sum_{x \in Dom(a)} E(x,\ x)$ and by the inductive assumption we obtain $E(a,\ a) = 0$.

11. Lemma on extensionality. If $F \in Frm$, $i \in Fr(F)$, $a \in (V^B)^{Fr(F) - \{i\}}$, then

$$I(x,\ y).\ S_B(F,\ a \cup \{<i,\ x>\}) \leq S_B(F,\ a \cup \{<i,\ y>\}) .$$

Proof. For atomic formulae the lemma follows from 5 - 10. For compound formulae we obtain it immediately using induction on the number of logical operators.

12. Lemma on bounded quantifiers. If $F \in Frm_{pr}$, $i \in Fr(F)$, $j \notin Fr(F)$, $a \in (V^B)^{Fr(F) - \{i\}}$, then

$$S_B((Ex_i)[(x_i \in x_j)\ \&\ F],\ a \cup \{<j,\ x>\}) = \sum_{u \in Dom(x)} [S_B(F,\ a \cup \{<i,\ u>\}).\ x(u)] .$$

Proof. The left-hand side is =

$\sum_{u \in V^B} [S_B(F,\ a \cup \{<i,\ u>\}).\ E(u,\ x)]$ and hence is $\geq$ the right-hand side. On the other hand, the left-hand side is equal to

$$\sum_{u \in V^B} \sum_{z \in Dom(x)} [x(z).\ I(u,\ z).\ S_B(F,\ a \cup \{<i,\ z>\})] \leq$$

$$\sum_{u \in V^B} \sum_{z \in Dom(x)} [x(z).\ S_B(F,\ a \cup \{<i,\ z>\})] = RHS$$

Embedding of V into V^B. If B' is a complete subalgebra

of B then, clearly, $V^{B'}$ is a submodel of V^{B}. In particular we can select for B' a two-element algebra. We shall define a map of V into V^{B} : Let $a \in R_r - \bigcup_{s \in r} R_s$ and assume that $\check{x}$ is defined for x in $\bigcup_{s<r} R_s$. We denote by $\check{a}$ a function with domain $\{\check{x} : x \in a\}$ and with value 1.

We shall prove by induction the following

Theorem 12. The following implications hold for arbitrary a, b in V:

(i) $(a = b) \rightarrow I(\check{a}, \check{b}) = 1$,

(ii) $(a \in b) \rightarrow E(\check{a}, \check{b}) = 1$,

(iii) $(a \neq b) \rightarrow I(\check{a}, \check{b}) = 0$

(iv) $(a \notin b) \rightarrow E(\check{a}, \check{b}) = 0$.

Assume that these formulas are valid for pairs in $(\bigcup\{V^{B}_{r} : r < s\})^2$ and let $\langle a, b\rangle$ be a pair in $(\bigcup\{V^{B}_{r} : r \leq s\})^2$ which does not belong to the previous set.

Formula (i) is obvious in view of 5. If $a \in b$, then $\check{a} \in \mathrm{Dom}(\check{b})$ and $\check{b}(\check{a}) = 1$ whence we obtain (ii). Assume that $a \neq b$ and $x \in a - b$. Hence $\check{x} \in \mathrm{Dom}(\check{a})$ and $\check{a}(\check{x}) = 1$ and for every $\check{y}$ in $\mathrm{Dom}(\check{b})$ we have $y \neq x$ whence by inductive assumption $I(\check{y}, \check{x}) = 0$. Hence $\prod_{z \in \mathrm{Dom}(\check{b})} \check{b}(z).\ I(z, \check{x}) = 0$ whence $I(\check{a}, \check{b}) = 0$. If $a \notin b$, then $a \neq x$ for every x in b whence by (iii) $I(\check{a}, \check{x}) = 0$ for every $\check{x}$ in $\mathrm{Dom}(\check{b})$ and therefore $E(\check{a}, \check{b}) = 0$.

Note: the mapping $a \rightarrow \check{a}$ is not a mapping onto because functions which are not constant do not belong to its range.

From (i) - (iv) and the lemma on bounded quantifiers we obtain by induction the following

Theorem 13. If $F \in Frm_{pr}$ and $a \in V^{Fr(F)}$, then $S_B(F, \check{a}) = 1$ if $V \models F[a]$ and $S_B(F, \check{a}) = 0$ if $V \models \neg F[a]$.

Note. The assumption that $F \in Frm_{pr}$ is essential for the validity of this theorem; it is not true, in general, for arbitrary formulae F in Frm.

Complete homomorphism. Let h map B onto a Boolean algebra B' and preserve all finite and infinite meets and joins. We call such a mapping a complete homomorphism.

A complete homomorphism $h : B \to B'$ determines a mapping $f : V^B \to V^{B'}$ which is defined by induction. Let us assume that f is already defined on $\bigcup\{V_r^B : r < s\}$ and let $a \in V_s^B$ but $a \notin \bigcup\{V_r^B : r < s\}$. Hence s is a successor, $s = t + 1$ and $Dom(a) \subseteq \bigcup\{V_r : r \leq t\}$. We define f(a) as a function whose domain is $\{f(x) : x \in Dom(a)\}$ and whose value for the argument $z \in Dom(f(a))$ is $\sum_{(x \in Dom(a))\ \&\ (f(x) = z)} h(a(x))$.

We shall prove the following result:

14 (Lemma on complete homomorphism). If $F \in Frm$ and $a \in (V^B)^{Fr(F)}$, then

$$h(S_B(F, a)) = S_{B'}(F, f \circ a)$$

Proof. We first prove the lemma for the case where F is an atomic formula and then for the case where F has logical operators.

Case I. It will be sufficient to prove the equation:

$$h(E_B(a, b)) = E_{B'}(f(a), f(b)), \quad h(I_B(a, b)) = I_{B'}(f(a), f(b)).$$

We use transfinite induction and assume that $\langle a, b\rangle \in (\bigcup\{V_r^B : r \leq s\})^2 - (\bigcup\{V_r^B : r < s\})^2$ and also that the lemma is true for pairs in $(\bigcup\{V_r^B : r < s\})^2$. From the definitions and the assumption that h is complete we obtain

$$h(I_B(a,\ b)) = \prod_{x \in \mathrm{Dom}(a)} \Big\{ h(a(x)) \rightarrow \sum_{y \in \mathrm{Dom}(b)} \Big[h(b(y)).\ h(I_B(x,\ y)) \Big] \Big\} \ \ldots$$

(where the dots stand for an expression resulting from the one written above by transposing a with b and x with y). We can replace the product $\prod_{x \in \mathrm{Dom}(a)}$ by $\prod_{z \in \mathrm{Dom}(f(a))} \prod_{(x \in \mathrm{Dom}(a))\ \&\ (f(x) = z)}$ and similary the sum $\sum_{y \in \mathrm{Dom}(b)}$ by $\sum_{t \in \mathrm{Dom}(f(b))} \sum_{(y \in \mathrm{Dom}(b)\ \&\ (f(y) = t)}$. In this way we obtain

$$h(I_B(a,\ b)) = \prod_{z \in \mathrm{Dom}(f(a))} \Big\{ (f(a))\ (z) \longrightarrow \sum_{t \in \mathrm{Dom}(f(b))} \Big[(f(b))(t).\ I_{B'}(f(a),\ f(b)) \Big] \Big\} \ .\ \ldots = I_{B'}(f(a),\ f(b)).$$

The proof for E is similar;

$$h(E(a,\ b)) = \sum_{x \in \mathrm{Dom}(b)} \Big[h(b(x)) \cdot h(I_B(a,\ x)) \Big] = \sum_{z \in \mathrm{Dom}(f(b))} \sum_{(x \in \mathrm{Dom}(b))\ \&\ (f(x) = z)} \Big[(f(b))(z) \cdot I_{B'}(f(a),\ z) \Big] = E_{B'}(f(a),\ f(b)).$$

Case II . it will be sufficient to discuss only the case where $F = (x_i)\ G$ and $i \in Fr(G)$. By definition

$$h(S_B(F, a)) = \prod_{x \in V^B} h(S_B\ (G,\ a \cup \{<i,\ x>\}\ )) = \prod_{x \in V^B} S_{B'}(G, f \circ a \cup \{<i,\ f(x)>\}\ ) \leq \prod_{z \in V^{B'}} S_{B'}(G,\ f \circ a \cup \{<i,\ z>\}) = S_{B'}(F,\ f \circ a).$$

In order to establish the inequality $\geqslant$ we must show that f maps V^B onto $V^{B'}$. This we do again by induction. Let $a' \in V^{B'}_{r+1}$. For every z' in Dom(a') there is a z in V^B_r such that $f(z) = z'$. Let a have the domain $\{z \in V^B_r : f(z) \in \mathrm{Dom}(a')\}$ and let a(z) be an element of B such that $h(a(z)) = a'(f(z))$. We easily prove that $f(a) = a'$.

Automorphisms. A special case of the lemma on complete homomorphism is the following

Theorem 15. If h is an automorphism of B, $F \in Frm$, $a \in (V^B)^{Fr(F)}$, then

$$h(S_B(F, a)) = S_B(F, f \circ a)$$

where f is determined by h as above.

Corollary 16. If $Fr(F) = 0$, then $S_B(F)$ is invariant with respect to all automorphisms of B.

If h is an automorphism, then f is one-to-one. Hence the definition f(x) can be simplified: $(f(x))(z) = h(x(f^{-1}(z)))$.

Theorem 17 (The maximum principle). If $F \in Frm$, $0 \in Fr(F)$, $a \in (V^B)^{Fr(F)}$; then there is an element x in V^B such that $S_B((Ex_0)F, a) = S_B(F, a \cup \{<0, x>\})$.

Proof. Put $S_B(F, a \cup \{<0, u>\}) = f(u)$. From the lemma on extensionality we infer that $I(x, u) . f(u) \leq f(x)$ for arbitrary u and x. For each a in the range of f we denote by r(a) the least ordinal r such that there are u in V_r^B for which $f(u) = a$. Let Q_a be the set of these u. We can assume that for all a in Rg(f) and all u in Q_a the domain Dom(u) is one and the same. This follows from the remark that if $Dom(u) = c$ and $c \subset c'$, then the element u' defined by the equations $Dom(u') = c'$, $u'(x) = u(x)$ for $x \in c$ and $u'(x) = 0$ for $x \in c' - c$, satisfies the equation $I(u, u') = 1$. Thus in all Boolean equations u can be replaced by u'. In the situation which we are now considering we put $d = \bigcup \{ Dom(u): (u \in Q_a) \& (a \in Rg(f))$ and replace each u in Q_a by u' whose domain is d and whose values on Dom(u) coincide with those of u and are 0 outside Dom(u). The set of modified elements we call again Q_a and we put $Q = \bigcup\{Q_a : a \in Rg(f)\}$. The range of the function f restricted to Q is equal to Rg(f).

Now we use axiom of choice and find a function $g : Q \to B$ such that $\sum_{u \in Q} g(u) = \sum_{u \in Q} f(u) = \sum_{x \in Rg(f)} x$ such that $g(u') \cdot g(u'') = 0$ for $u' \neq u''$ and $g(u) \leq f(u)$ for $u \in Q$. (E. g. we can well order Q and put $g(u) = f(u) - \sum f(v)$ where v ranges over elements which precede u.)

We define an element x of V^B by

$Dom(x) = d, \quad x(t) = \sum_{u \in Q} [g(u)\ u(t)]$ for $t \in d$

and claim that this is the required element.

In order to show this it is sufficient to prove that

(✱) $g(u) \leq I(x, u)$ for each a in Rg(f) and u in Q_a.

Assume for a moment that (✱) has been proved. On the one hand it is immediate that $f(x) \leq S_B((Ex_0)\ F,\ a)$. On the other it follows from (✱) that

$$g(u) = g(u) \cdot f(u) \leq I(x,\ u) \cdot f(u) \leq f(x)$$

and hence $f(x) \geq \sum_{u \in Q} g(u) = \sum_{u \in Q} f(u) = \sum_{u \in Rg(f)} u = S_B((Ex_0)F,\ a)$.

The inclusion (✱) is proved by the following calculations: Let $a \in Rg(f)$ and $u \in Q_a$; from the definition of E it follows

$$I(x,\ u) \geq \prod_{t \in d} [-x(t) + g(u)]$$

and from the definition of x

$$-x(t) + u(t) = \prod_{r \in Q} [-g(v) + -v(t)] + u(t) = \prod_{v \in Q} [-v(t) + u(t) + (-g(v))] .$$

Now we notice that $-g(v) \geqslant g(u)$ whenever $v \in Q$ and $v \neq u$. Hence all the terms of the product corresponding to values $v \neq u$ are $\geqslant g(u)$ and the product is $\geqslant g(u) \cdot [(-u(t)) + (-g(u))] = g(u)$.

Elementary subsets of V^B. We shall now prove that there exist ordinals r such that $V_r^B \prec V^B$. Let us put $S_B^*(F, a) = S_B^W(F, a)$ where $W = V_r^B$. We first prove the following

Theorem 18 (Principle of reflection). There is a function $f : Frm \times On \rightarrow On$ such that for each F in Frm

(i) the function $f_F(r) = f(F, r)$ is an increasing and continuous mapping of 0n into 0n;

(ii) if $k = f_F(k)$ and $a \in (V_k^B)^{Fr(F)}$, then $S_B(F, a) = S_B^k(F, a)$.

Proof. For atomic F we put $f(F, r) = r$. If F is the formula $\neg G$ or the formula $(x_n)G$ with $n \notin Fr(G)$, we put $f(F, r) = f(G, r)$. If F is the formula G & H, then we put $f(F, r) = f(G, f(H, r))$. Now let F be the formula $(Ex_n)G$ and let $n \in Fr(G)$.

For each $a \in (V^B)^{Fr(F)}$ we denote by s(a) the least ordinal s such that V_s^B contains an element x satisfying the equation $S_B((Ex_n)G, a) = S_B(G, \{<n, x>\} \cup a)$; if there is no x with this property, then we put $s(a) = 0$. Now we define by induction a function g:

$$g(0) = 0, \quad g(r+1) = \max(g(r), \sup\{s(a) : a \in (V_B^r)^{Fr(F)}\}) + 1$$
$$g(t) = \sup\{g(r) : r < t\} \text{ if } t \text{ is a limit number.}$$

Finally we put $f(F, r) = g(f(G, r))$. The proof that f satisfies the required conditions is similar to the proof of the Scott-Scarpellini theorem.

From the principle of reflection we obtain

Theorem 19. There is an increasing continuous mapping $f : 0n \rightarrow 0n$ such that $(f(k) = k) \rightarrow (V_k^B \prec V^B)$.

Proof. The required function is defined by induction:

$f(0) = 0$, $f(r) = \sup\{f(t) : t < r\}$ if r is a limit number,

$f(r + 1) = \sup\{f(F, r) : F \in \mathrm{Frm}\} + 1$

It is easy to show that $f(k) = k$ implies $f(F, k) = k$ for each F and hence that $S_B(F, a) = S_B^k(F, a)$ for every a in $(V_k^B)^{\mathrm{Fr}(F)}$.

This proves the theorem.

We shall now use the axiom of choice and obtain elementary subsets of V^B with an arbitrary infinite power. To obtain this result we start with an ordinal k of power greater than a given power p and consider the set V_k^B. Using the axiom of choice we correlate with an arbitrary F in Frm such that $0 \in \mathrm{Fr}(F)$ and an arbitrary a in $(V_k^B)^{\mathrm{Fr}(F) - \{0\}}$ an element $x = H(F, a)$ of V_k^B satisfying the conditions set forth in the maximum principle. We can call H the universal Skolem function. Closing an arbitrarily given infinite set $W \subseteq V_k^B$ under the Skolem function $H(F, a)$ we obtain a set W' of the same power as W such that $W' \prec V_k^B$ and hence $W' \prec V^B$. Thus we have proved

Theorem 20 (The theorem of Skolem-Löwenheim). There are elementary submodels of V^B of any infinite power.

Boolean models versus ordinary models. Let W be a subset of V^B. A model $M = \langle A, R\rangle$ where A is a set and R is a binary relation $\subseteq A \times A$ will be said to be elementarily equivalent with W if for any sentence F (i.e. a formula without free variables) $(S_B^W(F) = 1) \equiv (M \models F)$.

The existence of such a model for an arbitrary W is doubtful but we shall show that the following condition secures its existence:

(A) There is a maximal filter $\mathfrak{F}$ of B which preserves all the sums $\sum_{x \in W} S_B^W(F, a \cup \{\langle n, x\rangle\})$ where $F \in \mathrm{Frm}$, $n \in \mathrm{Fr}(F)$

and $a \in W^{Fr(F)} - \{n\}$.

Remark. We say that a maximal filter $\mathfrak{F}$ preserves a sum $b = \sum b_x$ if $b \in \mathfrak{F}$ implies that at least one b_x is in $\mathfrak{F}$. A lemma due to Rasiowa and Sikorski states that in every algebra B there is a maximal filter which preserves a given denumerable number of sums.

For a set $W \subseteq V^B$ and a filter $\mathfrak{F} \subseteq B$ we put
$\bar{a} = \{b \in W : I(a, b) \in \mathfrak{F}\}$ for any a in W.
Thus $\bar{a}$ is an equivalence class of the relation $I(a, b) \in \mathfrak{F}$ which is easily seen to be reflexive, symmetric and transitive (cf. lemmas 5, 6, 7). Let $A = \{\bar{a} : a \in W\}$ and $R = \{\langle \bar{a}, \bar{b} \rangle \in A \times A : E(a, b) \in \mathfrak{F}\}$
The definition of R is correct because $I(a, a') \cdot I(b, b') \cdot E(a, b) \leq E(a', b')$ and hence $E(a, b) \in \mathfrak{F}$ implies $E(a', b') \in \mathfrak{F}$ for any a' in $\bar{a}$ and b' in $\bar{b}$. The model $\langle A, R \rangle$ depends on W, B and $\mathfrak{F}$ and can be denoted by $M(W, B, \mathfrak{F})$. As long as one or more of the parameters W, B, $\mathfrak{F}$ is fixed we shall simplify the notation by omitting symbols whose values are fixed.

Lemma 21. If $\mathfrak{F}$ satisfies (A), $F \in Frm$ and $a \in W^{Fr(F)}$, then

(*) $S^W_B(F, a) \in \mathfrak{F} \equiv M(W, B, \mathfrak{F}) \models F[\bar{a}]$

(in (*) the symbol $\bar{a}$ means the sequence $\{\langle i, \bar{a}_i \rangle\} : i \in Dom(a)\}$).

Proof. If F is one of teh atomic formulae $x_i = x_i$ or $x_j \in x_j$, then both sides of (*) have the same values. If F is the formula $x_i = x_j$, then the left-hand side of (*) is $I(a_i, a_j) \in \mathfrak{F}$ and the right-hand side is $\bar{a}_i = \bar{a}_j$. Thus (*) is true in view of the definition of $\bar{a}$. If F is the formula $x_i \in x_j$, then the left-hand side of (*) is $E(a_i, a_j) \in \mathfrak{F}$ and the right-hand side is $\langle \bar{a}_i, \bar{a}_j \rangle \in R$. Again both sides are equivalent because of the definition of R.

We omit the trivial discussion of the cases where F is one of the formulae $\neg G, G \,\&\, H$ and discuss only the case where F is the formula $(x_n)G$ and $n \in Fr(G)$. The left-hand side of $(*)$ implies in this case

$(**)$ $S_B^W(G, a \cup \{\langle n, x\rangle\}) \in \mathfrak{F}$ for each x in W.

Using the inductive assumption we obtain immediately the right-hand side od $(*)$. If the left-hand side of $(*)$ is false, then we use the maximality of $\mathfrak{F}$ and obtain $\sum_{x \in W} S_B^W(\neg G, a \cup \{\langle n, a\rangle\}) \in \mathfrak{F}$ whence we infer by (A) that at least one term of this sum belongs to $\mathfrak{F}$. Hence $(**)$ is false for at least one x in W and using the inductive assumption we obtain the negation of the right-hand side of $(*)$.

Taking in the lemma $W = V_k^B$ where k is a critical number od the function f defined on p. 143 we can prove the

Theorem 22. There are models elementarily equivalent with V^B.

Proof. The required model is $M(V_k^B, B, \mathfrak{F})$ where $\mathfrak{F}$ is any maximal filter of B. The condition (A) is satisfied because in view of the definition of f each sum mentioned in (A) is equal to one of its terms.

Using the elementary submodel of V^B which is closed with respect to the Skolem functions H(F, a) (see the proof of the Skolem-Löwenheim theorem) we obtain in the same way the

Theorem 23. For any infinite power there exist models of this power elementarily equivalent with V^B.

The well-founded case. The models constructed in the above theorem need not be well-founded. We shall discuss the existence of well-founded models elementarily equivalent to V^B. First we notice the following

Lemma 24. If $\langle A, R\rangle = N$ is a model of ZF and the relation R is well-founded on the set $On_N = \{ a \in A : N \models Ord\,[a] \}$, then N is well-founded.

Proof. Formalizing the definition of sets R_r we obtain a formula such that $Fr(F) = \{0, 1\}$ and the formulae $(x_0)(E!x_1)F$, $F \rightarrow Ord(x_1)$, $F(x_0, x_1)$ & $F(x_2, x_3)$ & $(x_0 \in x_2) \rightarrow (x_1 \in x_3)$ are provable in ZF.

Hence if there were an infinite sequence a_n of elements of N which would be decreasing in the sense that $\langle a_{n+1}, a_n\rangle \in R$ for each n, there would also exist a decreasing sequence of elements of On_N.

Next we establish the important

Theorem 25 (Behaviour of ordinals in V^B). If $u \in V^B$, then there is an ordinal r such that $S_B(Ord, u) = \sum_{s<r} I(u, \check{s})$.

Proof. If $s \in On$, then it is easy to verify that $S_B(Ord, \check{s}) = 1$ and hence $I(u, \check{s}) \leq S_B(Ord, u)$ which proves the inclusion $\geqslant$.

In order to prove the converse inclusion we notice that the formula

$Ord(x_0)$ & $Ord(x_i) \rightarrow (x_0 \in x_1) \vee (x_0 = x_1) \vee (x_1 \in x_0)$

is provable in ZF and hence valid. It follows that for any u in V^B and r in On

$$S_B(Ord, u) \leq E(u, \check{r}) + I(u, \check{r}) + E(\check{r}, u).$$

Since $E(u, \check{r}) = \sum_{x \in Dom(\check{r})} [I(u, x) \cdot E(x, \check{r})] = \sum_{s<r} I(u, \check{s})$ and $E(\check{r}, u) = \sum_{v \in Dom(u)} [u(v) \cdot I(\check{r}, v)]$ we obtain

$$S_B(Ord, u) \leq \sum_{s \leq r} I(u, \check{s}) + \sum_{v \in Dom(u)} I(\check{r}, v).$$

Now we notice that if $r_1 \neq r_2$ and $r_1, r_2 \in On$, then .

$I(\check{r}_1, v) . I(\check{r}_2, v) = 0$ for each v. Since B is a set there cannot exist arbitrarily long sequences of mutually disjoint elements of $B - \{0\}$.

Hence for each v there is an ordinal r_v such that $I(\check{r}, v) = 0$ for every ordinal $r \geqslant r_v$. Choosing in $(*)$ $r \geqslant \sup\{r_v : v \in \mathrm{Dom}(u)\}$ we obtain $S_B(\mathrm{Ord}, u) \leq \sum_{s < r} I(u, \check{s})$.

With the help of this theorem we can now establish the

Theorem 26. there exist well-founded models elementarily equivalent with V^B.

Proof. Let W be a denumerable elementary subset of V^B. For each u in W let r(u) be the least ordinal such that

$$(*) \quad S_B(\mathrm{Ord}, u) = \sum_{s < r(u)} I(u, \check{s}).$$

(The elements $\check{s}$ with $s < r(u)$ do not, in general, belong to W but this has no bearing on the proof).

According to the Rasiowa-Sikorski theorem there is a maximal filter F which preserves all the sums $(*)$. We now claim that the model $M = M(W, B, \mathfrak{F})$ which according to previous theorems is elementarily equivalent with V^B, is well founded. Let X be the set $\{u \in W : S_B(\mathrm{Ord}, u) \in \mathfrak{F}\}$ and notice that

$\mathrm{On}_M = \{\bar{u} : u \in X\}$. For each $u \in X$, there is an ordinal $s < r(u)$ such that $I(u, \check{s}) \in \mathfrak{F}$ because $\mathfrak{F}$ preserves the sum $(*)$. Since $I(u, s_1).I(u, s_2) = 0$ for $s_1 \neq s_2$ we infer that there is just one such ordinal $s = s(u)$. If $u, v \in X$, then $E(u, v) \in \mathfrak{F}$ implies $s(u) \in s(v)$ because $E(u, v).I(u, \check{s}(u)) . I(v, \check{s}(v)) \leq E(\check{s}(u), \check{s}(v))$ and the right-hand side would be zero if $\check{s}(v)$ were smaller than or equal to $\check{s}(u)$. Finally we notice that if $v \in \bar{u}$, then $s(v) = s(u)$ because $I(u, v) \cdot I(u, \check{s}(u)) \leq I(v, \check{s}(u))$.

The function $h(\bar{u}) = s(u)$ is therefore well defined for

$\bar{u}$ in $0n_M$ and has the properties: $Rg(h) \subseteq 0n$, $\langle u, v\rangle \in R \rightarrow h(\bar{u}) < h(\bar{v})$. This proves that the set $0n_M$ is well ordered by R. Hence the model M is well founded.

Lecture VI

In this lecture we shall prove the

Theorem (a). All formulae provable in ZF are valid in an arbitrary Boolean model; (b) The axiom of choice is valid in an arbitrary Boolean model.

Since the rules of proof preserve validity (cf. theorems 1-3 of Lecture V) it will be sufficient to consider only the axioms Now the axioms of (propositional and predicate) logic have been dealt with in theorem 4 of Lecture V and axioms of identity in theorems 7-10 of that Lecture. It remains to consider only the proper set theoretic axioms

(i). The case of the axiom Ext. Since the value of the formula $Cl(x_1)$ is always 1 (see the definition of S_B on p. 132) it is sufficient to show that the value of the formula

$$(x_2)\left[(x_2 \in x_0) \equiv (x_2 \in x_1)\right] \rightarrow (x_0 = x_1)$$

is 1 for an arbitrary sequence $\{\langle 0, a\rangle, \langle 1, b\rangle\}$. Using the lemma on bounded quantifiers (theorem 12 of Lecture V) we prove that the value of the antecedent is

$$\prod_{t \in Dom(a)} \left[-a(t) + E(t, b)\right] \cdot \prod_{t \in Dom(b)} \left[-b(t) + E(t, a)\right] =$$

$$= I(a, b)$$

which proves the theorem.

(ii) The case of the axiom Nopcl. We have to show that $\prod_a \sum_b E(a, b) = 1$. Let $a \in V^B$ and let b be a function with domain $\{a\}$ with the value 1. Obviously $E(a, b) = 1$ whence $\sum_b E(a, b) = 1$ and, since a was arbitrary,

$\prod_a \sum_b E(a, b) = 1$.

(iii) The case of the axiom Cl_1. Since the antecedent of this axiom has value 0 according to (ii), its value is 1.

(iv) The cases of the axiom Cl_2 and Noat. In these axioms the formula Cl(x) forms the postcedent and, since the value of the formula Cl(x) is 1, we immediately obtain the result.

(v) The case of the axiom Emp. Since $E(a, \check{0}) = 0$ we obtain $S_B((x_1)\neg(x_1 \in x_0), \{\langle 0, \check{0}\rangle\}) = 1$ and hence $S_B((Ex_0)(x_1)\neg(x_1 \in x_0), 0) = 1$ This is the desired result because the axiom Nopcl has value 1 and hence the value of $(Ex_1)(x_0 \in x_1)$ is 1 for any sequence $\{\langle 0, a\rangle\}$ (in the present case: for the sequence $\{\langle 0, \check{0}\rangle\}$).

(vi) The case of the axiom Pair. Let a, $b \in V^B$ and let c be a function with domain $\{a, b\}$ identically equal 1. We easily verify that

$$E(x, c) = I(x, a) + I(x, b)$$

which proves the theorem in view of the result (ii) above.

(vii) The case of the axiom Sum. Since we have verified the axiom Nopcl we can reformulate Sum as follows:

$$(Es)((x)\{(x \in s) \rightarrow (Ey)[(y \in a) \ \& \ (x \in y)]\} \ \& \ (y)\{(y \in a) \rightarrow$$
$$\rightarrow (x)[(x \in y) \rightarrow (x \in s)]\}).$$

In order to verify the validity of this formula we select an arbitrary a in V^B and seek an s in V^B such that

$$(*)\ \prod_{x \in Dom(s)} \{-s(x) + \sum_{y \in Dom(a)} [a(y).E(x, y)]\} = 1,$$

$$(**)\ \prod_{y \in Dom(a)}\} \prod_{x \in Dom(y)} [(-a(y))+(-y(x))+ E(x, s)] = 1$$

We select s in such a way that $Dom(s) = \bigcup\{Dom(y) : y \in Dom(a)\}$

and

$s(x) = \sum_{y\in \mathrm{Dom}(a)}[E(x, y)\cdot a(y)]$ for x in Dom(s).

Equation (*) is then evident Since $E(x, s) = \sum_{t\in \mathrm{Dom}(s)}[s(t)\cdot I(x, t)$ we further infer that for z in Dom(a) and x in Dom(z)

$$(-a(z))+(-z(x)) + E(x, s) = (-a(z)) + (-z(x)) + \sum_{y\in \mathrm{Dom}(a)}\sum_{t\in \mathrm{Dom}(y)}[a(y)\cdot E(t, y)\cdot I(x, t)] \geqslant$$

$$(-a(z))+(-z(x)) + (a(z)\cdot E(x, z)\cdot I(x, x)) =$$

$$=(-a(z))+(-z(x)) + E(x, z).$$

Since $z(x)\leq E(x, z)$, the right-hand side is = 1 and equation (**) is proved.

(viii) The case of the axiom Pot. This axiom can be taken in the form (cf. (ii) above)

(*) $(Ex_2)(x_0)[(x_0\in x_2)\equiv F]$ where F is the formula

$(x_2)[(x_2\in x_0)\rightarrow(x_2\in x_1)]$.

The meaning of F is, of course, $x_0\subseteq x_1$. Let a be an element of V^B In order to abbreviate our formulae we put $v(x) = S_B(F, \{\langle 0, x\rangle, \langle 1, a\rangle\})$, thus $v(x)$ is the truth-value of the statement "x is a sub-set of a".

If $f\in B^{\mathrm{Dom}(a)}$ and $f(x)\leq a(x)$ for x in Dom(a), then evidently $v(f) = 1$. Let P be the set of all functions with domain Dom(a) and values $f(x)\leq a(x)$. We shall show that the function p defined by the equations

Dom(p) = P, $p(f) = 1$ for all f in P

satisfies for each s in V^B the equation

(**) $E(s, p) = v(s)$.

We first prove the inclusion $\leq$. By definition $E(s, p) = \sum_{f\in P}[I(s, f)\cdot p(f)] = \sum_{f\in P} I(s, f)$ because $p(f) = 1$. But $v(f) = 1$

for f in P and hence $E(s, p) = \sum_{f \in P}[I(s, f)\cdot v(f)] \le v(s)$ in view of the lemma on extensionality.

Now we prove the inclusion $\geq$. For an arbitrary s we define s^* by means of the equations $Dom(s^*) = Dom(a)$, $s^*(x) = E(x, s)\cdot a(x)$. Hence $s^* \in P$ and $v(s) \le 1 = E(s^*, p)$. Thus in order to finish our proof it is sufficient to show that $v(s) \le I(s, s^*)$ i.e. to prove the two inclusions

$$v(s)\cdot s^*(x) \le E(x, s) \text{ for } x \in Dom(s^*) = Dom(a),$$

$$v(s)\cdot s(y) \le E(y, s^*) \text{ for } y \in Dom(s).$$

The first inclusion results immediately from the definition of s^*; The second is established as follows: since $s(y) \le E(y, s)$ and $v(s)\cdot E(y, s) \le E(y, a)$ we infer

$$v(s)\cdot s(y) \le E(y, s)\cdot E(y, a) = \sum_{x \in Dom(a)}[E(y, s)\cdot I(y, x)\cdot a(x)] \le$$
$$\sum_{x \in Dom(s^*)}[I(y, x)\cdot E(x, s)\cdot a(x)] =$$
$$\sum_{x \in Dom(s^*)}[I(y, x)\cdot s^*(x)] \le$$
$$\sum_{x \in Dom(s^*)}[I(y, x)\cdot E(x, s^*)] \le E(y, s^*).$$

Equation $(**)$ is thus proved. It follows from this equation that the Boolean value of the formula $(*)$ for the argument $\{<1, a>\}$ is 1.

(ix) The case of the axiom Inf. Since the axiom Nopcl has been verified we can write this axiom in the form $(Ex_0)F$ where F is the conjunction of

$$G : (Ex_1)[(x_1 \in x_0) \ \& \ (x_2)\neg(x_2 \in x_1)],$$

$$H : (x_1)[(x_1 \in x_0) \rightarrow (Ex_2)((x_2 \in x_0) \ \& \ (x_3)\{(x_3 \in x_2) \equiv [(x_3 \in x_1) \vee (x_3 = x_1)]\})].$$

In order to verify that the value of this axiom is 1 it will be sufficient to show that $S_B(G, \{\langle 0, \check{\omega}\rangle\}) = S_B(H, \{\langle 0, \check{\omega}\rangle\}) = 1$

The value of G is equal to $\sum_{x \in \mathrm{Dom}(\check{\omega})} [\check{\omega}(x) . \prod_t -E(t, x)]$

This sum is 1 since $\check{\omega}(\check{0}) = \prod_t - E(t, \check{0}) = 1$.

The value of H is (cf. the lemma on bounded quantifiers)

$$\prod_{x \in \mathrm{Dom}(\check{\omega})} (-\check{\omega}(x) + \sum_{y \in \mathrm{Dom}(\check{\omega})} \{\check{\omega}(y) \cdot \prod_{z \in \mathrm{Dom}(y)} [(-y(z)) + (E(x, z) + I(x, z))] \cdot \prod_{z \in \mathrm{Dom}(x)} [-x(z) + E(z, y)] \cdot \prod_z [-I(z, x) + E(z, y)]\}).$$

Since $\mathrm{Dom}(\check{\omega})$ consists of functions $\check{n}$ where $n \in \omega$ and $\check{\omega}(\check{n}) = 1$ we can simplify this expression to

$$\prod_{m \in \omega} \sum_{n \in \omega} \prod_{p < n} [E(\check{m}, \check{p}) + I(\check{m}, \check{p})] \cdot \prod_{q < m} E(\check{q}, \check{n}) \prod_z [-I(z, \check{m}) + E(z, \check{n})] .$$

Using formulae (i)-(iv) established in Theorem 12 (Lecture V) we simplify this to

$$\prod_{m \in \omega} \sum_{n > m} \prod_z [-I(z, \check{m}) + E(z, \check{n})]$$

In order to show that this element is = 1 it will be sufficient to show that for any m in ω $\prod_z [-I(z, \check{m}) + E(z, \check{m}')]$ where $m' = m + 1$. This results easily from the observation that $E(z, \check{m}') = \sum_{p < m'} [\check{m}'(\check{p}) \cdot I(z, \check{p})] \geqslant \check{m}'(\check{m}) \cdot I(z, \check{m}) = I(z, \check{m})$.

(x) The case of the axiom Fund. Similary as in the previous cases we can simplify this axiom by omitting everywhere the clauses that elements to be considered are sets. Thus we can take Fund in the form

$$(x_0)[(Ex_3)(x_3 \in x_0) \rightarrow (Ex_1)\{(x_1 \in x_0) \ \& \ (x_2)[(x_2 \in x_1) \rightarrow \neg(x_2 \in x_0)]\}] .$$

or equivalently

$$(x_0)(x_3)\neg((x_1)\{(x_1 \in x_0) \rightarrow (Ex_2)[(x_2 \in x_1) \ \& \ (x_2 \in x_0)]\} \ \& \ (x_3 \in x_0)).$$

We write this formula briefly as $(x_0)(x_3)\neg F$. Let x, y_0 be arbitrary elements of V^B and assume that the element $b_0 = S_B(f, \{<0, x>, <3, y_0>\})$ is $\neq 0$. We shall show that this assumption results in a contradiction.

Obviously $(*) \quad E(y_0, x) \geqslant b_0$

and

$$S_B((x_1)\{(x_1 \in x_0) \rightarrow (Ex_2)[(x_2 \in x_1) \ \& \ (x_2 \in x_0)]\}, \{<0, x>\}) \geqslant b_0$$

Performing the calculation of S_B we obtain by the use of the lemma on bounded quantifiers

$$(**) \quad \prod_z \left[-E(z, x) + \sum_{t \in Dom(z)} E(t, x)\right] \geqslant b_0.$$

Using $(*)$ we obtain

$$\sum_{t \in Dom(y_0)} \left[b_0 \cdot E(t, x) \cdot E(y_0, x)\right] = b_0 > 0, \text{ i.e.}$$

$$\sum_{t \in Dom(y_0)} b_0 \cdot E(t, x) > 0.$$

Hence we infer that there is a y_1 in $Dom(y_0)$ such that $b_0 \cdot E(y_1, x) = b_1 > 0$. Let us select a y_1 of this kind. Applying $(**)$ again we obtain

$$E(y_1, x) + \sum_{t \in Dom(y_1)} E(t, x) \geqslant b_0$$

whence there exists a y_2 in $Dom(y_1)$ such that $b_1 \cdot E(y_2, x) = b_2 > 0$. Continuing this process we construct (using the axiom of choice) an infinite sequence $y_0, y_1, y_2, \ldots$ such that $y_{n+1} \in Dom(y_n)$. This implies the existence of an infinite descending sequence of ordinals which is impossible.

(xi) The case of the axiom $Subst_{ZF}$. We formulate this axiom as follows:

$$(1) \quad (x_o)(Ex_p)(x_1)\left[F \equiv (x_1 = x_p)\right] \rightarrow (x_m)(Ex_n)(x_1)\left\{(x_1 \in x_n) \equiv (Ex_o)\left[(x_o \in x_m) \;\&\; F\right]\right\}$$

where F is a formula, $1 < p < m < n$, x_m, x_n. x_p do not occur in F and $0, 1 \in Fr(F)$. Whenever $a \in (V^B)^{Fr(F) - \{0, 1\}}$, $x, y \in V^B$, we shall denote the element $S_B(F, \{<0, x>, <1, y>\} \cup a)$ by $f_a(x, y)$. Moreover we shall denote by $a * b$ the element $a.b + (-a)(-b)$ of B; thus if G, H are formulae and $g \in (V^B)^{Fr(G) \cup Fr(H)}$, then $S_B(G \equiv H, g) = S_B(G, g \mid Fr(G)) * S_B(H, g \mid Fr(H))$.

Let $a \in (V^B)^{Fr(F) - \{0, 1\}}$. The value of the formula (1) is $-h(a) + k(a)$ where $h(a) = \prod_x \sum_y \prod_z \left[f_a(x, y) * I(y, z)\right]$ and $k(a) = \prod_u \sum_v \prod_y \left\{E(y, z) * \sum_x \left[E(x, u) \cdot f_a(x, y)\right]\right\}$. We have to prove that $h(a) \leq k(a)$.

First we notice that according to the maximum principle (and the axiom of choice) we can correlate with each x in V^B an $y(x)$ in V^B such that

$$(2) \quad \sum_y \prod_z \left[f_a(x, z) * I(y, z)\right] = \prod_z \left[f_a(x, z) * I(y(x), z)\right].$$

From this we easily obtain the following two lemmas:

$$(3) \qquad h(a) \cdot f_a(x, y') \cdot f_a(x, y'') \leq I(y', y'').$$

Proof. According to (2) $h(a) \cdot f_a(x, y') \leq \left[f_a(x, y') * I(y(x), y')\right]. f_a(x, y') . I(y(x), y')$ and similary $h(a) \cdot f_a(x, y'') \leq I(y(x), y'') \cdot f_a(x, y'')$ whence $h(a) \cdot f_a(x, y') \cdot f_a(x, y'') \leq I(y(x), y') \cdot I(y(x), y'') \leq I(y', y'')$.

$$(4) \qquad h(a) \leq f_a(x, y(x)).$$

Proof. From (2) we see that $h(a) \leq f_a(x, z) * I(y(x), z)$; now we put $z = y(x)$ and use the formula $a * 1 = a$.

From (3) and (4) we obtain

(5) $$h(a)\cdot f_a(x,\ y) \leq I(y,\ y(x)).$$

Let now u be an arbitrary element of V^B and let v be determined by the conditions:

(6) $$\mathrm{Dom}(v) = \{\, y(x) \ :\ x \in \mathrm{Dom}(u) \}\ ,\quad v(y) = \sum\nolimits_{x \in \mathrm{Dom}(u)} [u(x)\cdot f_a(x, y)]$$

We shall consider the value of E(y, v) for an arbitrary y (for y in Dom(v) it is given in (6)). First we notice that

(7) $$-E(y,\ v) + \sum\nolimits_{x \in \mathrm{Dom}(u)} [u(x)\cdot f_a(x,\ y)] = 1,$$

i.e., $E(y,\ v) \leq \sum\nolimits_{x \in \mathrm{Dom}(u)} [u(x)\cdot f_a(x,\ y)]$.

Proof. By definition $E(y,\ v) = \sum\nolimits_{z \in \mathrm{Dom}(v)} v(z)\cdot I(z,\ y)$ whence by (6)

$$E(y,\ v) = \sum\nolimits_{x \in \mathrm{Dom}(u)} \sum\nolimits_{z \in \mathrm{Dom}(v)} [u(x)\cdot f_a(x,\ z)\cdot I(z,\ y)]$$
$$\leq \sum\nolimits_{x \in \mathrm{Dom}(u)} \sum\nolimits_{z \in \mathrm{Dom}(v)} [u(x)\cdot f_a(x,\ y)] = \sum\nolimits_{x \in \mathrm{Dom}(u)} [u(x)\cdot f_a(x,\ y)]$$

The inclusion converse to that given in (7) cannot be proved; but we shall show

(8) $$h(a) \leq -\sum\nolimits_{x \in \mathrm{Dom}(u)} [u(x)\cdot f_a(x,\ y)] + E(y,\ v)$$

Proof. Let $x \in \mathrm{Dom}(u)$. By (5), $h(a)\cdot u(x)\cdot f_a(x,\ y) \leq$ $I(y,\ y(x))\cdot u(x)\cdot f_a(x,\ y) \leq I(y,\ y(x))\cdot u(x)\cdot f_a(x,\ y(x) \leq I(y,\ y(x)).$ $\sum\nolimits_{t \in \mathrm{Dom}(u)} [u(t)\cdot f_a(t,\ y(x))] = v(y(x))\cdot I(y,\ y(x))$ (here we use the fact that $y(x) \in \mathrm{Dom}(v)$). From this inclusion we infer

$$h(a)\cdot u(x)\cdot f_a(x,\ y) \leq \sum\nolimits_{z \in \mathrm{Dom}(v)} [v(z)\cdot I(y,\ z)] = E(y,\ v)$$

Summing over x we obtain (8).

(7) and (8) jointly give

(9) $$h(a) \leq E(y,\ v) \ast \sum\nolimits_{x} [E(x,\ u)\cdot f_a(x,\ y)]$$

because (see the lemma on bounded quantifiers) $\sum\nolimits_{x \in \mathrm{Dom}(u)} [u(x)\cdot$

$f_a(x, y)\Big] = \sum_x \Big[E(x, u)\cdot f_a(x, y)\Big]$. Now we take the product $\prod_y$ on the right-hand side of (9), then the sum $\sum_v$ and finally the product $\prod_u$. In this way we obtain the desired formula $h(a) \leq k(a)$.

Theorems which we have proved show that all the theorems of ZF are valid in an arbitrary Boolean model. We shall now show that the axiom of choice is valid in these models.

(xii) The case of the axiom of choice. This axiom written in full takes the following shape:

$$(x_o)\Big[(x_1)(x_2)(x_3)\Big\{\big[(x_1 \in x_o) \ \& \ (x_2 \in x_o) \ \& \ (x_3 \in x_1) \ \& \ (x_3 \in x_2)\big] \to (x_1 = x_2)\Big\} \to$$
$$(Ex_1)\Big((x_2)(x_3)(x_4)\Big\{\big[(x_2 \in x_o) \ \& \ (x_3 \in x_2) \ \& \ (x_4 \in x_2) \ \& \ (x_3 \in x_1) \ \& \ (x_4 \in x_1)\big] \to$$
$$(x_3 = x_4)\Big\} \ \& \ (x_2)(x_3)(Ex_4)\Big\{\big[(x_2 \in x_o) \ \& \ (x_3 \in x_2)\big] \to \big[(x_4 \in x_2) \ \& \ (x_4 \in x_1)\big]\Big\}\Big)\Big]$$

This formula can be written in the form

$(x_o)\{H(x_o) \to (Ex_1)\,[K(x_o, x_1) \ \& \ L(x_o, x_1)]\}$ where H "says" that x_o is a family of mutually disjoint sets, $K(x_o, x_1)$ "says" that x_1 has at most one element in common with each set which belongs to x_o and $L(x_o, x_1)$ "says" that x_1 has at least one element in common with each non void set which belongs to x_o.

In order to prove that the value of this formula is 1 in each Boolean model we select an arbitrary a in V^B and denote by h(a) the element $S_B(H(x_o), \{<0, a>\})$; we shall exhibit a b in V^B such that $h(a) \leq k(a, b)$ and $h(a) \leq l(a, b)$ where k and l are values of $K(x_o, x_1)$ and $L(x_o, x_1)$.

It is intuitively obvious that the domain of b should be $D(a) = \bigcup \{ \mathrm{Dom}(y) : y \in \mathrm{Dom}(a) \}$ because each choice set for a set consists of elements of the elements of this set. We shall define b(x) for x in D(a) in such a way that for any u, x, y the Boolean product

$E(u, a)\cdot E(x, b)\cdot E(y, b)\cdot E(x, u)\cdot E(y, u)$ be 0 whenever $I(x, y) = 0$ This corresponds to the requirement that the choice set for a set should contain at most one element common with any element of this set. Moreover we shall formulate the definition so as to secure the inclusions $b(x) \leq \sum_u [E(u, a)\cdot E(x, u)]$ and $E(u, a)\cdot E(z, u) \leq \sum_{x \in Dom(b)} [E(x, u)\cdot b(x)]$ which correspond to the requirements that the choice set for a set s consists of the elements of the elements of s and has at least one element in common with each non-void element of s

Let us arrange the elements of D(a) in a transfinite sequence d_r with $r < s$, $s \in On$. The following definition meets all the requirements:

$$Dom(b) = \{d_r : r < s\}, \quad b(d_r) = \sum_u [E(u, a).\ E^*(d_r, u)]$$

where

$$E^*(d_r, u) = E(d_r, u).\ \prod_{t<r} -E(d_t, u)$$

Before calculating k(a, b) we notice that from the definition of h it directly follows that

(1) $h(a).\ E(u, a).\ E(v, a).\ E(d_r, u).\ E(d_r, v) \leq I(u, v)$

Next we prove for arbitrary $r, r' < s$

(2) $h(a).\ E(u, a).\ E(d_r, u)\cdot E(d_{r'}, u).\ b(d_r).\ b(d_{r'}) \leq I(d_r, d_{r'})$.

In order to prove this we merely replace $b(d_r)$, $b(d_{r'})$ by their values and represent the left-hand side as the union of terms

$$h(a).[E(u, a).\ E(d_r, u).\ E(v, a).\ E^*(d_r, v)]\ [E(u, a).\ E(d_{r'}, u).\ E(v', a).\ E^*(d_{r'}, v')]$$

extended over arbitrary v, v'. According to (1) the expression included in the first square bracket is $\le I(u, v)$ and the expression included in the second square brackets is $\le I(u, v')$. Hence according to the lemma on extensionality the left-hand side of (2) is

$$E^{*}(d_r, u)\cdot E^{*}(d_{r'}, u)$$

This product is 0 if $r \neq r'$; thus (2) is evident in this case. If $r \neq r'$ then (2) is also true because its right-hand side is then 1.

We shall now prove

(3) $h(a).\ E(u, a).\ E(x, u).\ E(x, b).\ E(y, u).\ E(y, b) \le I(x, y)$

Notice that this inclusion is very similar to (2); the only difference in that d_r, $d_{r'}$ are replaced by x, y and that instead of $b(d_r)$, $b(d_{r'})$ we now have $E(x, b)$, $E(y, b)$.

In order to prove (3) we denote its left-hand side by L and expand $E(x, b)$, $E(y, b)$ according to the definition of E. We then obtain

$$h(a).\sum_{r,\ r'<s}\left[E(u, a).\ E(x, u).\ E(y, u).\ I(x, d_r).\ b(d_r).\ I(y, d_{r'}).\ b(d_{r'})\right]$$

which by extensionality is

$$h(a)\cdot\sum_{r,\ r'<s}\left[E(u, a).\ E(d_r, u).E(d_{r'}, u).b(d_r).\ b(d_{r'}).\ I(x, d_r).\ I(y, d_{r'})\right].$$

The expression in the square brackets is $\le I(d_r, d_{r'})$(cf(2)), whence

$$L \le \sum_{r,\ r'}\left[I(d_r, d_{r'}).\ I(x, d_r).\ I(y, d_{r'})\right] \le I(x, y).$$

If now we represent (3) as $h(a).\ P \le I(x, y)$ and transform (3) to

$$h(a) \le \prod_{x,\ y,\ u}\left[-P + I(x, y)\right]$$

we obtain $h(a) \le k(a, b)$.

We now pass to the proof of the inclusion $h(a) \le I(a, b)$ and first establish the inclusion

(4) $E(u, a)\ E(d_t, u) \leq \sum_{r<s} [E(d_r, u).\ b(d_r)]$

This is shown as follows: it is obvious that $E(d_t, u) \leq \sum_{r<s} E(d_r, u) = \sum_{r\leq s} E^*(d_r, u)$. The last step is based on a well known Boolean law $\sum_{r<s} x_r = \sum_{r<s} [x_r.\ \prod_{t<r}(-x_t)]$ which allows us to represent each Boolean sum as sum of mutually disjoint elements. Since $E^*(d_r, u) \leq E(d_r, u)$ we can also write $E(d_t, u) \leq \sum_{r<s} [E(d_r, u).\ E^*(d_r, u)]$. Now we multiply both sides by $E(u, a)$ and notice that $E(u, a).\ E^*(d_r, u) \leq \sum_v [E(v, a).\ E^*(d_r, v)] = b(d_r)$. In this way we obtain (4).

We want now to replace in (4) the element d_t by an arbitrary z. To achieve this we use the definition of E and the lemma on extensionality obtaining

$$E(u, a).E(z, u) = \sum_{v \in \mathrm{Dom}(a)} [I(u, v).\ a(v).\ E(z, u)] \leq$$
$$\sum_{v \in \mathrm{Dom}(a)} [I(u, v),\ a(v).\ E(z, v)].$$

Applying again the definition of E we obtain

$$E(u, a).\ E(z, u) \leq \sum_{v \in \mathrm{Dom}(a)} \sum_{w \in \mathrm{Dom}(v)} [I(u, v) a(v).\ I(z, w).\ v(w)]$$

Now we notice that if $v \in \mathrm{Dom}(a)$, then $\mathrm{Dom}(v) \subseteq D(a) = \mathrm{Dom}(b)$ and hence the sum $\sum_{w \in \mathrm{Dom}(v)}$ is $\leq \sum_{t<s}$. Thus we obtain

$$E(u, a).\ E(z, u) \leq \sum_{v \in \mathrm{Dom}(a)} \sum_{t<s} [I(u, v).\ a(v).\ I(z, d_t).\ v(d_t)]$$

Since $I(u, v).\ v(d_t) \leq I(u, v).\ E(d_t, v) \leq E(d_t, u)$ we further obtain $\sum_{t<s} [E(d_t, u). \sum_{v \in \mathrm{Dom}(a)} (I(u, v).\ a(v))] = \sum_{t<s} [E(u, a).\ E(d_t, u)]$ which according to (4) is $\leq \sum_{r<s} [E(d_r, u).\ b(d_r)]$. Since the last sum is obviously $\leq \sum_x [E(x, u).\ E(x, b)]$ we finally infer that

$$E(u, a) . E(z, u) \leq \sum_x \left[E(x, u) . E(x, b) \right] .$$

Performing obvious Boolean transformations we obtain from this inclusion the identity

$$\prod_u \prod_z \sum_x \left\{ - \left[E(u, a) . E(z, u) \right] + \left[E(x, u) . E(x, b) \right] \right\} = 1$$

which is the same as $I(a, b) = 1$. Thus $h(a) \leq I(a, b)$ and the theorem is proved.

Lecture VII

In this lecture we shall construct (after Scott) a model in which the axiom of constructibility is not valid. In connection with this result we shall construct two transitive families of sets which are models of ZF and have equal heights but are not elementarily equivalent.

We consider the Cantor set 2^{ω}, i.e. the set of functions with domain ω and with values in the set $\{0, 1\}$. We introduce in 2^{ω} the usual product topology and denote by B the Boolean algebra of regular closed domains, i.e. of sets which can be represented as closures of open sets. The Boolean operations in B have the following meaning: the sum is the closure of the set theoretical union $\sum_x b_x = \overline{\bigcup_x b_x}$, the product is the closure of the interior of the intersection $\prod_x b_x = \overline{\mathrm{Int}\bigcap_x b_x}$ and $-b$ is the closure of the complement of b. The sets $C_n^i = \{f : f(n) = i\}$ form a sub-base of the space and belong to B.

Another property of B which we shall need is concerned with its automorphisms. Let p be a permutation of ω and let F be a function defined on 2^{ω} by the equation $F(f) = f \circ p$. It can be shown that F is an autohomeomorphism of 2^{ω} and hence the function $H : b \longrightarrow \mathrm{Im}(F, b)$ is an automorphism of B.

Lemma 1. If $0 \neq b \neq 1$, then there is an automorphism of B such that $H(b) \neq b$.

Proof. Since b and $-b$ contain non-void open sets, there are two neighbourhoods U, V the first of which is contained in b and the other in $-b$. We can assume that $U = \bigcap_{j \leq k} C_{m(j)}^{\varepsilon(j)}$. We now determine k integers $m(j)$, $j \leq k$, such that the intersection $V_1 = V \cap \bigcap_{j \leq k} C_{m(j)}^{\varepsilon(j)}$

be non-void. Such integers exist because the requirement $f \in V$ imposes only finitely many conditions on a fixed number, say l, of initial coordinates of f. Thus we can select k coordinates $m(1), \dots, m(k) > l$ and impose on them the condition $f(m(j)) = \mathcal{E}(j)$. The resulting neighbourood W has points in common with V because W contains points with arbitrary initial l coordinates.

Let p be a permutation of ω which maps n(j) onto m(j). If $f \in (-b) \cap W \cap V$ then $F(f) \in U$ because the value of $F(f) = f \circ p$ for the argument n(j) is equal to $\varepsilon(j)$. This proves that $H(-b) \cap b \neq 0$ and hence $-b \cap H^{-1}(b) \neq 0$ whence $H^{-1}(b) \neq b$.

Definition. Let d be a function with domain $\check{\omega}$ such that $d(\check{n}) = C_n^0$.

Remark. We can look upon elements of V^B as a kind of "multivalued sets" such that logical value of a formula "x is in the set" is an element of B not necessarily 0 or 1. Thus we can visualize d as a "multivalued set" such that the truth value of a sentence "n is an element of d" is a union of n + 1 intervals of the Cantor set C consisting of those reals in C which in the ternary scale have the n-th digit 0. It is obvious from this picture and it will be proved formally in lemma 3 below that d is different from all ordinary two-valued sets.

Lemma 2. If F is the formula $(x_2)\left[(x_2 \in x_o) \rightarrow (x_2 \in x_1)\right]$ (i.e. $x_o \subseteq x_1$), then $S_B(F, \{<0, d>, <1, \check{\omega}>\}) = 1$.

Proof. The value in question is $\prod_{x \in \mathrm{Dom}(d)} \left[-d(x) + \check{\omega}(x)\right]$. Since $\mathrm{Dom}(d) = \check{\omega} = \{\check{n} : n \in \omega\}$ and $\check{\omega}(\check{n}) = 1$ for $a \in \omega$ we see that the element $-d(x) + \check{\omega}(x)$ is 1 for each x in Dom(d).

Lemma 3. If $a \subseteq \omega$, then $I(d, \check{a}) = 0$.

Proof. From the definition of I we obtain

$$I(d, \check{a}) = \prod_{x \in \mathrm{Dom}(d)} -d(x) + \sum_{y \in \mathrm{Dom}(\check{a})} [\check{a}(y).\ I(x, y)])$$

$$\prod_{y \in \mathrm{Dom}(\check{a})} (-\check{a}(y) + \sum_{x \in \mathrm{Dom}(d)} [d(x).\ I(x, y)]\ ).$$

In both factors "x" can be replaced by "$\check{n}$" with n ranging over ω and "y" by "$\check{m}$" with m ranging over a. Since $d(\check{n}) = C^{o}_{n}$, $-d(\check{n}) = C^{1}_{n}$, $\check{a}(\check{m}) = 1$ and $I(\check{n}, \check{m})$ is 0 or 1 according as $n \neq m$ or $n = m$, we can simplify the above expression and obtain

$$\prod_{n \in \omega} [C^{1}_{n} + \varepsilon(n, a)] \cdot \prod_{m \in a} C^{o}_{m},$$

where $\varepsilon(n, a) = 1$ if $n \in a$ and 0 otherwise. Thus $I(d, \check{a}) =$

$$\prod_{n \notin a} C^{1}_{n} \cdot \prod_{m \in a} C^{o}_{m} = \prod_{n \notin a,\ m \in a} C^{o}_{m} \cdot C^{1}_{n} = \prod_{p \in \omega} C_{p}^{\varepsilon(p)}$$

where $\varepsilon(p)$ is 0 for $p \in a'$ and 1 otherwise.

In view of the definition of B we further obtain

$$I(d, \check{a}) = \mathrm{Int}\ \overline{\bigcap_{p \in \omega} C_{p}^{\varepsilon(p)}} = \mathrm{Int}\ \overline{\{\mathcal{E}\}} = 0\cdot$$

and the lemma is proved.

We shall now discuss the problem of an effective choice for the set $P(P(\omega))$. Thus we shall investigate the question whether there exists an effectively defined function which correlates with each non-void subset of $P(\omega)$ an element of this set. In order to make precise the notion of an effectively defined function we reformulate the problem as follows: does there exist a formula F with $Fr(F) = \{0, 1\}$ such that the sentence

$$(*)\ \ ([(Ex_1)(x_1 \in x_o)] \ \&\ (x_1)(x_2) \{ [(x_1 \in x_o) \ \&\ (x_2 \in x_1)] \rightarrow (x_2 \in \omega) \}) \rightarrow (E!\ x_1) [(x_1 \in x_o) \ \&\ F]$$

be true. In this sentence the antecedent means that x_o is a non-void set of subsets of ω and the conclusion says that there is just one element of x_o which stands to x_o in the relation defined by F.

If we wish to have the sentence written out exclusively by means of the primitive notions of set theory, we can eliminate the constant "ω" by replacing the expression "$x_2 \in \omega$" by

$$(**)\quad \mathrm{Ord}(x_2) \,\&\, \neg\mathrm{Lim}(x_2) \,\&\, (x_3)\left[(x_3 \in x_2) \to \neg\mathrm{Lim}(x_3)\right]$$

where $\mathrm{Ord}(x_i)$ is the conjunction of the formulae

$$(u)(v)\ \left\{\left[(u \in v) \,\&\, (v \in x_i)\right] \to (u \in x_i)\right\} \qquad \left[x_i \text{ is transitive,}\right]$$

$$(u)(v)\ \left\{\left[u \in x_i) \,\&\, (v \in x_i)\right] \to \left[(u \in v) \vee (u = v) \vee (v \in u)\right]\right\}$$

$$\left[\text{the } \varepsilon\text{-relation is connected in } x_i\right],$$

and $\mathrm{Lim}(x_i)$ is the formula

$$\mathrm{Ord}(x_i) \,\&\, \left[(Eu)(u \in x_i)\right] \,\&\, (v)\left\{(v \in x_i) \to (Fw)\left[(w \in x_i) \,\&\, (v \in w)\right]\right\}$$

$$\left[x_i \text{ is a limit ordinal } > 0\right].$$

The formulation of the problem is still imperfect because the word "true" is unclear. We therefore replace the problem by a relative one. Let M be a (Boolean or ordinary) model for ZF. We say that F determines a choice function for $P(P(\omega))$ in M if ($*$) is valid in M.

From Remarks contained in Lecture III it follows:

Theorem 1. If M is a transitive family of sets which is a model for ZF and if the axiom of constructibility is valid in M, then there is a formula which determines a choice function for $P(P(\omega))$ in M.

We shal now prove

Theorem 2. If B is the Boolean algebra of regular closed sets in $\{0, 1\}^{\omega}$, then there is no formula which determines a choice function for $P(P(\omega))$ in V^B.

Proof. Let us assume that ($*$) is valid in V^B. The main idea is to consider the "multivalued set" s such that $s(d) = 1$ but $s(\check{a}) = 0$ for each $a \subseteq \omega$. Formula F correlates with s one of its elements. The contradiction arises by showing that this set is invariant with respect to all automorphisms of B and thus is a two-valued set.

We define s by equation

$\mathrm{Dom}(s) = \{x \in V^{B}_{\omega+1} : \mathrm{Dom}(x) = \check{\omega}\}$ $s(x) = \prod_{a \leq \omega} [\ I(x, \check{a})]$

In view of the Lemma 3 $s(d) = 1$. Consider now the formula $(x_4)(x_5)\ \{[(x_4 \in x_o) \ \&\ (x_5 \in x_4)] \longrightarrow (x_5 \in \omega)\}$ whose exact meaning is explained in (**) . We claim that the value of this formula for the element s is 1 . This can be verified as follows . In view of the lemma on bounded quantifiers this value is $= \prod_{x \in \mathrm{Dom}(s)}$

$\prod_{y \in \mathrm{Dom}(x)} \{ - [s(x).\ x(y)] + A(y)\}$ where $A(y)$ is the value of formula (**) for the argument y. Since $y \in \check{\omega}$ it can be represented as $\check{a}$ with $a \in \omega$ and we verify immediately that (**) has the value 1 for the argument $\check{n}$.

Thus both the formulae in the antecedent of (*) have the value 1 for the argument $\{<0, s>, <3, d>\}$ and it follows that

(i) $S_B\ ((E!\ x_1)[(x_1 \in x_o) \ \&\ F], s) = 1.$

We define a function a_o which-intuitively speaking-describes the element selected from s:

$\mathrm{Dom}(a_o) = \check{\omega}$, $a_o(\check{n}) = \sum_{x \in \mathrm{Dom}(s)} [s(x).\ x(\check{n}).\ f(x)]$ where

$f(x) = S_B(F, \{<0, s>, <1, x>\})$ and want to prove that $E(a_o, s) = 1$

The following implication is of course provable in ZF:

(ii) $((E!\ x_1)[(x_1 \in x_o) \ \&\ F] \ \&\ (x_2)\{(x_2 \in x_3) \equiv (Ex_1)[(x_1 \in x_o) \ \&\ (x_2 \in x_1) \ \&\ F]\}) \longrightarrow (x_3 \in x_o)$.

This formula results from an obvious theorem of ZF which says that if there is just one x_1 satisfying a condition and x_3 consists of exactly those elements which belong to a set satisfying this condition, then x_3 too satisfies the condition in question.

The first term of the antecedent of (ii) has the value 1 for the argument $\{\langle 0, s\rangle\}$, see (i). We shall show that the value of the second term in the antecedent of (ii) has also the value 1 for the argument $\{\langle 0, s\rangle, \langle 3, a_0\rangle\}$. This term is logically equivalent to the conjunction of

(iii) $(x_2)\{(x_2 \in x_3) \rightarrow (Ex_1) \ [(x_1 \in x_0) \ \& \ F \ \& \ (x_2 \in x_1)]\}$

(iv) $(x_2)(x_1)\{[(x_1 \in x_0) \ \& \ (x_2 \in x_1) \ \& \ F] \rightarrow (x_2 \in x_3)\}$.

The value of (iii) can be calculated using the lemma on bounded quantifiers. The result is

$$\prod_{n\in\omega} \{-a_0(\check{n}) + \sum_{x\in \mathrm{Dom}(s)} [s(x).\ x(\check{n}).\ f(x)]\}$$

and this product is 1 according to the definition of a_0. The value of (iv) is

$$\sum_{x\in \mathrm{Dom}(s)} \sum_{y\in \mathrm{Dom}(x)} [(-s(x)) + (-x(y)) + (-f(x)) + E(y, a_0)] \ .$$

We can replace y by $\check{n}$ since $\mathrm{Dom}(x) = \check{\omega}$. After obvious Boolean calculations we see that the value of (iv) is

$$\prod_{n\in\omega} (- \sum_{x\in \mathrm{Dom}(s)} [s(x).\ f(x).\ x(\check{n})] + a_0(\check{n}))$$

because $E(\check{n}, a_0) = a_0(\check{n})$. Thus in view of the definition of a_0 this product is 1. Thus the whole antecedent of (ii) has the value 1 and we obtain $E(a_0, s) = 1$, i.e.

(v) $$\sum_{x\in \mathrm{Dom}(s)} [s(x).\ I(a_0, x)] = 1$$

We shall deduce a contradiction from this formula. As remarked above we shall obtain it by showing that a_0 is an ordinary two-valued set.

First we show that s is symmetric in the following sense:

If h is an automorphism of B and $f = f_h$ the mapping of V^B onto V^B determined by h (see p. 138), then $f(s) = s$. To see this we notice that if $x \in \mathrm{Dom}(s)$, then $\mathrm{Dom}(x) = \check{\omega}$ and hence $\mathrm{Dom}(f(x)) = \{f(u) : u \in \mathrm{Dom}(x)\} = \check{\omega}$ because elements of $\check{\omega}$ are invariant under f.

Further we calculate the value of $f(s)$ for the argument $s = f(x)$ where $x \in \mathrm{Dom}(s)$:

$$f(s)(z) = h(s(x)) = h(\prod_{a \subseteq \omega} - I(x, \check{a})) = \prod_{a \subseteq \omega} - h(I(x, \check{a})) =$$
$$\prod_{a \subseteq \omega} - I(f(x), f(\check{a})) = \prod_{a \subseteq \omega} - I(z, \check{a}) = s(z).$$

(In the last but one equation we used the obvious equation $f(\check{a}) = \check{a}$). The symmetry of s is thus established.

Using the symmetry of s we obtain by theorem on p. 140

$$h(a_o(\check{n})) = S_B((Ex_1)\left[(x_1 \in x_o) \ \& \ F \ \& \ (x_2 \in x_1)\right], \{<0, f(s)> ,$$
$$<2, f(\check{n})>\}) = a_o(\check{n}) ;$$

hence $a_o(\check{n})$ is invariant with respect to all automorphisms and hence $a_o(\check{n})$ is either 0 or 1 (see lemma 1). Thus $I(a_o, \check{a}_1) = 1$ where $a_1 = \{n : a_o(\check{n}) = 1\}$. Thus - intuitively speaking - the element selected from s is a two-valued set. Using (v), the definition of $s(x)$ and the equation $I(a_o, \check{a}_1) = 1$ we derive

$$1 = \sum_{x \in \mathrm{Dom}(s)} \left[s(x). \ I(\check{a}_1, x)\right] = \sum_{x \in \mathrm{Dom}(s)} \Big[I(\check{a}_1, x).$$
$$\prod_{a \subseteq \omega} (- I(\check{a}, x))\Big]$$

and the right-hand side is obviously 0. Thus we obtained the desired contradiction and theorem 2 is proved.

From theorems 1 and 2 we infer

Theorem 3. No transitive model for ZF in which the axiom of constructibility is valid can be elementarily equivalent with V^B.

In particular we see that the axiom of constructibility is not provable in ZF even if we adjoin to it the axiom of choice. We shall now prove

Theorem 4. There are two (two-valued) transitive models for ZF whose heights are equal but which are not elementarily equivalent.

Proof. Let M be a transitive model elementarily equivalent with V^B and M' the family of constructible elements of M. Then M' has the same height as M and is transitive but M' is not elementarily equivalent with M because there is a formula which determines a choice function for $P(P(\omega))$ in M' whereas no such formula esists for M.

It would be interesting to know whether there exists a formula which determines the choice functions for $P(P(\omega))$ in a natural model.

The answer to this question cannot be given, however, because it essentially depends on the axioms for set theory accepted in metamathematics.

Lecture VIII

In this lecture we shall construct a model in which the continuum hypothesis is false. Our first task will be to express this hypothesis as a sentence of our formalised language . In order to achieve some economy in our notation we shall introduce some abbreviations .

We shall write x, y, z for x_0, x_1, x_2; furthermore we shall use other small Roman letters instead of the variables x_i and shall assume that their choice has been made in such a way that no collision of variables occurs. We shall also make extensive use of limited quantifiers: the formulae $(Ey)_x[-]$, $(Ey)_{\dot{x}}[-]$, $(Ey)_{\ddot{x}}[-]$ will mean : $(Ey)\{(y \in x) \& [-]\}$, $(Ez)_x (Ey)_z [-]$, $(Es)_x (Ey)_{\dot{s}} [-]$ where z, s are now variables not before present in the formulae. The quantifiers $(y)_x$, $(y)_{\dot{x}}$, $(y)_{\ddot{x}}$ are defined in a dual way. Prefixing a formula F which belongs to Frm_{pr} by a limited quantifier of whatever sort we obtain again a formula which belongs to Frm_{pr}. As a final abbreviation we shall use the symbol (Eu, v, . . .) instead of (Eu) (Ev)... and similarly for the general quantifiers and limited existential and general quantifiers.

We now shall list several auxiliary formulae; we add (in square brackets) the intuitive meaning of each formula.

$x \subseteq y : (u)_x(u \in y)$ [inclusion] ;

$P'(x, y, z)$: $(y \in x) \& (z \in x) \& (u)_x [(u = y) \vee (u = z)]$

[x is an unordered pair whose elements are y and z] ;

$P(x, y, z) : (Eu, v)_x [P'(x, u, v) \& P'(u, y, z) \& P''(v, y, y)]$

[x is an ordered pair with the first member y and the second member z] ;

$(x, y, z) : (Eu)_y \, P(u, x, z)$ [the ordered pair with members x, z belongs to y] ;

$\mathrm{Rel}(x) : (s)_x (Eu, v)_s \, P(s, u, v)$ [x is a relation] ;

$\mathrm{Dom}(x, y) : (u)_y (Ev)_x (u, x, v) \,\&\, (u, v)_x [(u, x, v) \to (u \in y)]$

[y is the domain of x] ;

$\mathrm{Rg}(x, y)$: similarly as above but with (u, x, v) replaced by

(v, x, u) [y is the range of x] ;

$\mathrm{Fn}(x) : \mathrm{Rel}(x) \,\&\, (u, v, w, t)_x \{[(u, x, v) \,\&\, (w, x, t)] \to [(u = w) \equiv (v = t)]\}$

[x is a one-one function] ;

$[x, y, z] : \mathrm{Fn}(y) \,\&\, \mathrm{Dom}(y, x) \,\&\, \mathrm{Rg}(y, z)$ [y maps x onto z in a one-one way] ;

$\mathrm{Ord}(x)$: see p. 165 [x is an ordinal] ;

$\mathrm{Lim}(x)$: see p. 165 [x is a limit ordinal > 0] ;

$\mathrm{om}_o(x) : \mathrm{Ord}(x) \,\&\, \mathrm{Lim}(x) \,\&\, (y)_x \neg \mathrm{Lim}(y)$ [x is the ordinal ω] .

Lemma 1. The following formulae are provable in ZF :

(i) $[x, y, z] \,\&\, (u \in z) \to (Ev)_x \, (v, y, u)$;

(ii) $([x, y, z] \,\&\, (v, y, u') \,\&\, (v, y, u'')) \to (u' = u'')$.

This lemma is evident and needs no proof.

Since the formulas listed above are all predicative, we can apply to them theorem 13 from lecture V. In this way we can immediately obtain the value of each of the above formulae for the argument of the form $\{<0, \check{a}>, <1, \check{b}>, <2, \check{c}>\}$. For instance $S_B(x \subseteq y, \{<0, \check{a}>, <1, \check{b}>\})$ is 1 if $a \subseteq b$ and 0 otherwise, similary $S_B(\mathrm{om}_o(x), \{<0, a>\})$ is 1 if $a = \omega$ and 0 otherwise. We notice the result explicitly for the formula $[x, y, z]$:

Lemma 2. $S_B([x\ y,\ z], \{\langle 0, \check{a}\rangle, \langle 1, \check{f}\rangle, \langle 2, \check{b}\rangle\})$ is 1 if f is a one-one mapping of a onto b; otherwise this value is 0.

In order to express the continuum hypothesis we need still one formula which, however, is not an element of Frm_{pr} :

$$om_1(x) : \mathrm{Ord}\,(x) \,\&\, (Ey)_x \{om_o(y) \,\&\, \neg(Ez)\,[y, z, x] \,\&\, (z)_x\,[(y \in z)) \rightarrow (Et)\,[y, t, z]]\} \text{ [x is the first uncountable ordinal] }.$$

The continuum hypothesis abbreviated CH can now be expressed as follows :

$$CH : (x, y, z)\,(\{om_o(x) \,\&\, om_1(y) \,\&\, (t)\,[(t \in z) \equiv (t \subseteq x)]\} \rightarrow (Eu)\,[y, u, z]).$$

Before exhibiting a model V^B in which CH has the value 0 we want to explain the underlying idea. The model V^B will contain 3 elements a, b, c for which the value of the antecedent in CH will be 1 and the value of the consequent will be 0. There is little doubt how to choose a: the only natural choice is $a = \check{\omega}$. The natural choice for c is the element which we constructed in lecture VI when we verified the validity of axiom Pot: this element together with $\check{\omega}$ gives the value 1 for the last formula in the antecedent of CH. Hence we take as b the function with domain $B^{\mathrm{Dom}(\check{\omega})}$ whose value is identically 1 .

Can we take for b the element $\check{\omega}_1$? The answer depends of course on whether the value of the formula $om_1(x_1)$ for the argument $\check{\omega}_1$ is 1. Looking at the formula om_1 we can easily convince ourselves that this is the case provided that the value of the formula $[y, z, x]$ for the arguments $\check{q}(f) : \{\langle 1, \check{\omega}\rangle, \langle 2, f\rangle, \langle 0, \check{\omega}_1\rangle\}$ is 0 for each f in V^B. If this condition is satisfied, then we shall say that $\check{\omega}_1$ is a cardinal in V^B. It can be shown that there are algebras B such that $\check{\omega}_1$ is not a cardinal of V^B .

It is instructive to discuss this phenomenon. In the ordinary model V the ordinals ω and ω_1 are of different powers and thus for each g in V the truth value of the formula $[y, z, x]$ for the argument $\{<1, \omega>, <2, g>, <0, \omega_1>\} = q(g)$ is 0. It follows from lemma 2 that $S_B([y, z, x], \check{q}(\check{g})) = 0$. However the model V^B contains many elements not of the form $\check{g}$, e.g. all functions whose values are not only 1. Hence it may very well happen that there will be in V^B an element f not of the form $\check{g}$ for which $S_B([y, z, x], \check{q}(f)) = 1$.

In next theorem we shall formulate a sufficient condition for B which ensures that V^B does not contain elements f with this property. Hence if B satisfies this condition we can take $b = \check{\omega}_1$.

The problem whether the continuum hypothesis is not valid in V^B depends now on the values of $S_B([x_2, x_0, x_1], \{<0, f>, <1, \check{\omega}_1>, <2, c>\}) = v(f)$. If there is no f such that $v(f) \neq 0$, then the consequent in CH has the value 0 and so has the whole formula CH. Again the problem what is the value of $v(f)$ cannot be answered in a straightforward way. In the ordinary model V the sets ω_1 and $B^{\mathrm{Dom}(\check{\omega})}$ have different powers provided that the cardinal number of B is sufficiently big. However this by itself does not preclude the existence of an f in V^B for which $v(f)$ would be 1.

In next theorem we shall formulate a condition on B which suffices for the equation $v(f) = 0$ to be true throughout V^B. Afterwards we shall construct an algebra B for which this sufficient condition is satisfied. In this way the independence of CH will be proved.

In order to have shorter formulae we introduce the following definitions:

Definition 1. $[a, f, b]_B = S_B([x_j, x_i, x_k], \{<i, f>, <j, a>, <k, b>\})$,

$(x, f, y)_B = S_B((x_j, x_i, x_k), \{<i, f>, <j, x>, <k, y>\})$.

We shall usually omit the index B.

Definition 2. We say that a Boolean algebra satisfies the countable chain condition (abbreviated ccc) if every set $X \subseteq B$ consisting of mutually disjoint elements is at most countable.

Remark: elements b', b'' of B are disjoint if $b' . b'' = 0$.

Theorem 3. Let B satisfy ccc, let b, c be elements of V^B and A a subset of $\mathrm{Dom}(c)$ and let the following assumptions hold :

(1) $$\max(\aleph_0, \overline{\overline{\mathrm{Dom}(b)}}) < \overline{\overline{A}},$$

(2) $$c(y) = 1 \quad \text{for each } y \text{ in } \mathrm{Dom}(c) ;$$

(3) $$I(y, y') = 0 \quad \text{for any two (different) elements of } A .$$

Under these assumptions $\left[b, f, c\right] = 0$ for each f in V^B.

Proof. We put for z in $\mathrm{Dom}(b)$

$$Z(z) = \left\{ y \in A : \left[b, f, c\right] . (z, f, y) \right\} > 0$$

If y, y' belong to $Z(z)$, then by lemma 1 (ii)

$$\left[b, f, c\right] . (z, f, y) . (z, f, y') \leq I(y, y')$$

and hence, if $y \neq y'$, the product $\left[b, f, c\right] . (z, f, y) . \left[b, f, c\right] . (z, f, y') = 0$ according to (3) . Thus $Z(z)$ has at most $\aleph_0$ elements because otherwise B would not satisfy ccc. The union $\bigcup_{z \in \mathrm{Dom}(b)} Z(z)$ has power at most $\max(\aleph_0, \overline{\overline{\mathrm{Dom}(b)}})$ whence, by (1) , there is a y in A which does not belong to any $Z(z)$. Since $y \in \mathrm{Dom}(c)$ we obtain, by (2) , $c(y) = E(y, c) = 1$. Now we use lemma 1(i) and infer that $\left[b, f, c\right] . E(y, c) \leq \sum_v \left[E(v, b) . (v, f, y)\right]$. The left-hand side of this formula is simply $\left[b, f, c\right]$: the right-hand side can be transformed according to the lemma on bounded quantifiers. Thus we obtain

$\left[b, f, c\right] \leq \sum_{z \in \mathrm{Dom}(b)} \left[b(z) . (z, f, y)\right]$. But $\left[b, f, c\right] . (z, f, y) = 0$

because y is not in $Z(z)$ for any z in $\mathrm{Dom}(b)$. Hence we obtain $[b, f, c] = 0$.

We note two corollaries from the theorem proved above :

Corollary 4. If B satisfies ccc and $r \in \omega_1$ then $[\check{r}, f, \check{\omega}_1] = 0$ for each f in V^B.

Proof. Put $b = \check{r}$, $A = \mathrm{Dom}(\check{\omega}_1)$, $c = \check{\omega}_1$ in the previous theorem.

Corollary 5. If B satisfies ccc, then $S_B(om_1(x), \check{\omega}_1) = 1$.

Remark. Strictly speaking we should have taken $\{<0, \check{\omega}_1>\}$ and not $\check{\omega}_1$ as the argument in S_B ; we shall however use the simpler though less accurate notation in order to abbreviate our formulae.

Proof. We can write $om_1(x)$ in the form $\mathrm{Ord}(x) \& (Ey) K$ where K is the conjunction of 4 formulae

$$(*) \qquad om_0(y), \quad y \in x, \quad (z) \neg [y, z, x], \quad (z)([(y \in z) \& (z \in x)] \rightarrow (Et)[y, t, x]) .$$

We shall now calculate the values of these formulae for the argument $\{<0, \check{\omega}_1>, <1, \check{\omega}>\}$ (notice that x is the same as x_0, and y the same as x_1 according to our convention) . Obviously the values of $\mathrm{Ord}(x) \& om_0(y) \& (y \in x)$ is 1, the value of the third formula mentioned in (*) is 1 according to Corollary 4 . In order to establish this result for the last formula (*) it is sufficient to consider an arbitrary ordinal r such that $\omega < r < \omega_1$ and show that there is an f in V^B such that $1 = [\check{\omega}, f, \check{r}]$. Now ω and r have both the same power $\aleph_0$ and thus there is a one-one mapping g of ω onto r. By lemma 2 we obtain $[\check{\omega}, \check{g}, \check{r}] = 1$ which proves our corollary.

Our next theorem reduces the problem of independence of CH to a construction of a suitable algebra:

Theorem 6. Let B be a complete algebra satisfying ccc. Let there be a set J of power $> \aleph_1$ and a function $a: J \times \omega \rightarrow B$ such that for any two $i, j \in J$

$$\prod_{n \in \omega} [a(i, n) \ast a(j, n)] = 0 .$$

Then CH has the value 0 in V^B .

Remark. If a', a'' are elements of B, then $a' \ast a''$ is the element $a'. a'' + (-a').(-a'')$.

Proof . Put $P = B^{\mathrm{Dom}(\check{\omega})}$ and let c be a function such that $\mathrm{Dom}(c) = P$ and $c(x) = 1$ for all x in P . As we have explained on pp. 172 the formulae $om_0(x)$ and $(t)\,[(t \in z) \equiv (t \subseteq x)]$ which appear in the antecedent of CH have the value 1 for the argument $\{<0, \check{\omega}>, <2, c>\}$ (remember that "z" is the same as "x_2" and "x" the same as "x_0") . The last factor in the antecedent of CH is $om_1(y)$; according to Corollary 5 this formula has value 1 for the argument $\{<1, \check{\omega}_1>\}$. Thus the whole antecedent of CH has the value 1 . The theorem will be proved if we show that the value of the consequent is 0 i.e. that $[\check{\omega}_1, f, c] = 0$ for each f in V^B . Let us put $h_i(\check{n}) = a(i, n)$ for $i \in J$, $n \in \omega$. Thus $h_i \in B^{\mathrm{Dom}(\check{\omega})} = \mathrm{Dom}\,(c)$. Denote by A the family of all the functions h_i, $i \in J$. We shall show that with this notation and with $b = \check{\omega}_1$ all the assumptions of theorem 3 are satisfied . Assumption (1) is true because $A = \bar{\bar{J}} > \aleph_1$ and $\mathrm{Dom}(b)$ has power $\aleph_1$. Assumption (2) holds in view of the definition of c. Finally let $y = h_i \neq h_j = y'$. Then $I(y, y')$ is the product of the element

$$\prod_{n \in \omega} \left\{ -h_i(\check{n}) + \sum_{m \in \omega} [I(\check{n}, \check{m}).\ h_j(\check{m})] \right\}$$

and a similar element obtained by interchanging i and j. The first element is $= \prod_{n \in \omega} [-h_i(\check{n}) + h_j(\check{n})] = \prod_{n \in \omega} [-a(i, n) + a(j, n)]$ and thus

$I(y, y') = \prod_{n\in\omega}([-a(i, n) + a(j, n)] \cdot [-a(j, n) + a(i, n)]) = \prod_{n\in\omega}[a(i, n) \ast a(j, n)] = 0$. Thus assumption (3) is satisfied and $[\check{\omega}_1, f, c] = 0$ Theorem 6 is thus proved.

In order to settle the independence of CH we prove finally :

Theorem 7. There exists an algebra satisfying all assumptions of theorem 6 .

Proof. Let J be a set of power $> \aleph_1$ and B the Boolean algebra of regular closed domains in the space P(J) with the usual product topology. It is well known that this algebra is complete. A basis of neighbourhoods in P(J) is furnished by the family of sets $U(X, Y) = \{Z \subseteq J : (X \subseteq Z) \& (Z \cap Y = 0)\}$ where X and Y are disjoint finite subsets of J. These neighbourhoods are open and closed in P(J) and hence belong to B;

First we construct the mapping $a : J\times\omega \rightarrow J$ with properties required in theorem 6. Since $J\times\omega$ and J have equal powers there is a one-one mapping g of $J\times\omega$ onto J. We put $a(i, n) = U(\{g(i, n)\}, 0) = \{Z \subseteq J : g(i, n) \in Z\}$; obviously $a(i, n) \in B$. Let $i \neq j$ and $b(i, j, n) = a(i, n) \ast a(j, n)$. Since a(i, n) is open and closed, its complement in the sense of the algebra B coincides with the set-theoretical complement. Thus we obtain $a(i, n) \cdot a(j, n) = \{X \subseteq J : (g(i, n) \in X) \& (g(j, n) \in X)$ and $-a(i, n) \cdot -a(j, n) = \{X \subseteq J : (g(i, n) \notin X) \& (g(j, n) \notin X)\}$ and hence $b(i, j, n) = X \subsetneq J : (g(i, n) \in X) \& (j, n) \in X)\}$. It follows now that $\prod_n b(i, j, n)$ is the closure of the interior of the set $Z = \bigcap_n b(i, j, n)$. In order to show that the product is 0 it will be sufficient to prove that the interior of Z is void.

Let us assume that Z contains a non-void open set. Hence Z contains a neighbourhood U(X, Y) . Take n so that $g(i, n) \notin X \cup Y$ and $g(j, n) \notin X \cup Y$. Since $g(i, n) \neq g(j, n)$ the intersection $a(i, n) \cap -a(j, n) \cap U(X, Y)$ is non void. This is impossible because this intersection is disjoint from Z and $U(X, Y) \subseteq Z$.

In order to show that B satisfies ccc it is sufficient to show that there is no uncountable family of mutually disjoint sets of the form $U(X, Y)$. This results immediately from a well-known theorem of Marczewski (Fundamenta Mathematicae 34) : for completeness' sake we give here a proof (due to Cohen) of the special case of Marczewski's theorem.

We notice first that $(U(X, Y) \cap U(X', Y') = 0) \equiv [(X \cap Y') \cup (X' \cap Y) \neq 0]$ For if the left-hand side is true, then any set Z contained in J and containing $X \cup X'$ cannot be disjoint from $Y \cup Y'$ and in particular $(X \cup X') \cap (Y \cup Y') \neq 0$ which is equivalent to the righ-hand side because $X \cap Y = X' \cap Y' = 0$. Conversely, if $i \in (X \cap Y') \cup (X' \cap Y)$ e.g. $i \in X \cap Y'$, then each set in $U(X, Y)$ contains i but none set in $U(X', Y')$ does so.

Now let us assume that there is an uncountable family $U(X_r, Y_r)$ where $r \in R$ of mutually disjoint neighbourhoods. Since $X_r \cup Y_r$ is finite, we can at once assume that all these sets $X_r \cup Y_r$ have the same number, say n, of elements. This follows by the observation that for at least n the set $R' = \{r \in R: X_r \cup Y_r \text{ has } n \text{ elements}\}$ must be uncountable and hence we can replace R by the set R'.

We shall obtain a contradiction by showing that for each k there is an uncountable set $R_k \subseteq R$ and a set P_k with exactly k elements such that $P_k \subseteq X_r \cup Y_r$ for each r in R_k. For $k = 0$ we set $P_0 = 0$, $R_k = R$. Reasoning by induction we assume that $k \geqslant 0$ and that sets $P_0 \subseteq \ldots \subseteq P_k$, $R_0 \supseteq \ldots \supseteq R_k$ have already been defined in such a way that P_k has k elements, R_k is uncountable and

(*) $$P_k \subseteq X_r \cup Y_r, \quad P_k \cap X_r = P_k \cap X_s, \quad P_k \cap Y_r = P_k \cap Y_s$$

for arbitrary r, s in R_k.

Since $< X_r, Y_r > \neq < X_s, Y_s >$ whenever $r \neq s$ and since there are only finitely many pairs of subsets of P_k, it is clear that there exists an r_o in R_k such that $(X_{r_o} \cup Y_{r_o}) - P_k \neq 0$. For $s \neq r_o$ the neighbourhoods $U(X_{r_o}, Y_{r_o})$ and $U(X_s, Y_s)$ are disjoint; hence for each s in $R_k - \{r_o\}$ there is an element $i = i(s)$ such that

(* *) $$i \in (X_{r_o} \cap Y_s) \cup (X_s \cap Y_{r_o})$$

Each such i belongs to $X_{r_o} \cup Y_{r_o}$ but none to P_k. Otherwise i would either belong to $P_k \cap X_{r_o} \cap Y_s$ or to $P_k \cap X_s \cap Y_{r_o}$ and hence according to (*) either to $P_k \cap X_s \cap Y_s$ or to $P_k \cap X_{r_o} \cap Y_{r_o}$ which is impossible because these intersections are void.

Now notice that $(X_{r_o} \cup Y_{r_o}) - P_k$ is a finite set and hence there is an i_o in this set such that $i(s) = i_o$ for uncountably many s. Let $P_{k+1} = P_k \cup \{i_o\}$ and let $R_{k+1} = \{ s \in R_k : i(s) = i_o \}$. We have to verify (*) for the sets P_{k+1} and R_{k+1}. First condition is satisfied because $i_o \in X_s \cup Y_s$ for an arbitrary s in R_{k+1}. In order to verify the second condition we have to prove that $(i_o \in X_r) \equiv (i_o \in X_s)$ for arbitrary r, s in R_{k+1}. Thus assume $i_o \in X_s$; by (**) $i_o \in Y_{r_o}$. But (**) is also valid for $i = i_o$ if we replace s by r; hence $i_o \in X_r$ because otherwise i_o would belong to X_{r_o} which is impossible. We show similarly that if $i_o \in X_r$, then $i_o \in X_s$. The verification of the last condition (*) is analogous.

Letting $k = n + 1$ we obtain that $X_r \cup Y_r$ has more than n elements which is a contradiction. Theorem 7 is thus proved.

On the independence of the well-ordering theorem from the ordering principle

by

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(Translated from German original *Über die Unabhängigkeit des Wohlordnungssatzes on Ordnungsprinzip* by M. J. Mączyński)

In the present paper we consider the problem (formulated by Fraenkel) of the independence of the axiom of choice from the so-called ordering principle.(¹)

We precede the formal proof by some introductory remarks in order to facilitate the understanding of the subsequent considerations.

It is known that among the axiomatic systems which represent a formalization and extension of Zermelo's set theory two kinds of systems may be distinguished. To the first kind belong systems which exclude the existence of so-called primitive elements (*Urelemente*), i.e. things that are not sets. The second kind comprises systems in which the existence of infinitely many such primitive elements is provable or at least can be shown to be non-contradictory.(²) Both groups contain systems which permit a formal development of various parts of set theory: for example in Zermelo's system(³) the theory of sets of power $< \aleph_0 + 2^{\aleph_0} + 2^{2^{\aleph_0}} + \ldots$ can be evolved, the framework of Fraenkel's(⁴) or von Neumann's(⁵) system allows one to develop the theory of power below the first inaccessible aleph, and recently some even richer systems have been considered.(⁶)

From the metamathematical point of view, the systems of the first kind are undoubtedly the most interesting ones. But those systems will not be considered here: the method of proof to be used in this paper openly de-

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(¹) See A. Fraenkel, *Einleitung in die Mengenlehre*, 3rd ed., Berlin 1298, p. 320.

(²) See E. Zermelo, *Über Grenzzahlen und Mengenbereiche*, Fund. Math. 16 (1929), pp. 29–47.

(³) See E. Zermelo, *Untersuchungen über die Grundlagen der Mengenlehre*, Math. Ann. 65 (1908), pp. 261–281.

(⁴) See the book of A. Fraenkel cited in (¹), § 16.

(⁵) See J. v. Neumann, *Die Axiomatisierung der Mengenlehre*, Math. Zeitschrift 27 (1928), p. 669 f.

(⁶) See A. Tarski, *Über unerreichbare Kardinalzahlen*, Fund. Math. 30 (1938), pp. 68–89. In particular see § 2.

pends on the existence of primitive elements. On the other hand, our considerations can be applied to all systems of the second kind. However, we will carry out the proof only for a certain fixed system $\mathfrak{S}$, which will be described in § 1. This system was actually introduced by Bernays(7) we only give another formulation for the axiom of definiteness. This slight modification aims at ensuring the possibility of introducing primitive elements.

In the system $\mathfrak{S}$ a considerable part of the "naïve" set theory can be reconstructed, namely that part in which only sets of power below the first unattainable aleph are considered. In particular, the whole theory of order for such sets can be developed and in connection with this the well-ordering theorem and the ordering principle (i.e. the assertion that every set can be ordered) can be formulated. It should only be shown that the well-ordering theorem is independent of all the axioms of $\mathfrak{S}$ and of the ordering principle.

This is carried out by an application of the classical method of interpretation:(8) namely we take from general methodology the fact that to obtain the proof it suffices to give an interpretation of the basic notions of the system $\mathfrak{S}$ in some consistent system $\mathfrak{S}'$ such that all axioms of $\mathfrak{S}$ and the ordering principle change into provable theorems but the well-ordering theorem changes into the negation of a provable theorem. Hence in the sequel we shall restrict ourselves to the derivation of the model and to the proof that in this model the axioms of $\mathfrak{S}$ and the ordering principle are satisfied but the well-ordering theorem is not. The question of consistency of the system $\mathfrak{S}$, since it does not belong to these problems, will be left outside our considerations.(9) It will be a (tacit) assumption in all our theorems that $\mathfrak{S}'$ is a consistent system.

Now how should the system $\mathfrak{S}'$ be selected? It turns out that this choice can be made to a great degree arbitrarily: as $\mathfrak{S}'$ we can take every system of set theory which permits forming powers as large as in $\mathfrak{S}$ and contains the axiom of choice among provable statements; whether $\mathfrak{S}'$ belongs to the first or to the second kind considered above is immaterial. For definiteness we select as $\mathfrak{S}'$ the axiomatic system of von Neumann,(10) which clearly satisfies both requirements. Since it would be very cumbersome to express the proof in the formalism of von Neumann's system, we hold to

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(7) See P. Bernays, *A system of axomatic set theory, Part I*, Journal of Symbolic Logic 2 (1937), pp. 65–77.

(8) See the book of Fraenkel cited in (1);. pp. 340–343.

(9) For this question see A. Tarski, *Der Wahrheitsbegriff in formalisierten Sprachen*, Studia Philosophica 1 (1936), pp. 261–405. In particular, see "*Nachwort*", p. 393 ff.

(10) See footnote (5).

the language of naïve set theory, familiar to all set theoreticians. The complete translation of the proof into the framework of von Neumann's system presents no essential difficulties.

This explains the fact why in our exposition we can afford to use several notions which are of imprecise or even contradictory character in naïve set theory but are admissible in von Neumann's system. To such notions belongs, for example, the notion of the domain of all sets or of all ordinal numbers. The notion of ordinal number, which is often not sufficiently precise in naïve set theory, should be understood in the sense given by von Neumann. Also we can make unlimited use of inductive definitions, which are perfectly admissible in the system of von Neumann.(11)

At the end of this paper we will return to the question of constructing the model in other axiomatic systems.

The basic idea of the proof is closely related to ideas that have been developed in several of Fraenkel's papers.(12) Also the investigations of Lindenbaum and myself concerning the independence of the axiom of choice(13) have been conducive to the writing of this paper. In particular, the construction of the sets A_n (see **15** and footnote (20)), basic for the whole proof has been taken from the joint work of Lindenbaum and myself, soon to be published, where the results of this investigation will be presented in detail.

Concerning the outward form of the proof, I would like to remark that I proved the theorem discussed here earlier, by a formally quite different method: the construction of the model was replaced by certain operations with formulas in some formal language.(14) From the logical point of view

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(11) Besides the paper cited in (5) see also the following works of von Neumann: *Zur Einführung der transfiniten Zahlen*, Acta Litt. ac scientiarum univ. Hung. Franc. Joseph. Sectio sc. math. 1 (1923), pp. 199–208; *Über Definitionen durch transfinite Induktion und verwandte Fragen der allgemeinen Mengenlehre*, Math. Ann. 99 (1928), pp. 373–391.

(12) See *Über den Begriff "definit" und die Unabhängigkeit des Auswahlaxioms*, Sitzungsb. d. Preuss. Akad. d. Wiss. Phys. Math. Klasse (1922), pp. 253–267; *Sur l'axiome du choix*, L'Enseignement Math. 34 (1935), pp. 32–51; *Über eine abgeschwächte Fassung des Auswahlaxioms*, The Journal of Symbolic Logic 2 (1937), pp. 1–25.

(13) See the communiqué of A. Lindenbaum and mine: *Über die Unabhängigkeit des Auswahlaxioms und einiger seiner Folgerungen*, C. R. de la Soc. des Sci. et des Lettres de Varsovie 31 (1938), pp. 27–32.

(14) This so-called method of relativization is due to Tarski and has been presented in my paper entitled: *O niezależności definicji skończoności w systemie logiki* (On the independence of the definition of finitness in the system of logic). This paper appeared as an appendix to the 16th volume of the annual report of the Polish Mathematical Society.

this method has perhaps certain advantages, but it is much more complicated than the one used here. The knowledge of the present method, consisting in constructing the model, I owe to Gödel's lectures on the system of axioms of set theory,(¹⁵) where Gödel used this method to prove the consistence of the axiom of choice. At the end of this paper, I will have another opportunity to return to this important and interesting result of Gödel.

§ 1. The axiom system $\mathfrak{S}$

BASIC NOTIONS: *Individuum, Class,* ε, A.

The logical character of these basic notions is the following: "*individuum*" and "*class*" are predicate names with one free variable; ε is the name of a relation which can hold between two individua or between an individuum and a class; A is an individual constant. As a logical basis for the subsequent system of axioms it is quite sufficient to take the narrower functional calculus (without identity).

DEFINITION I. An individuum x is *identical* with an individuum y (in symbols $x \mathrm{Id} y$) if and only if for every class A the formulas $x \varepsilon A$ and $y \varepsilon A$ are equivalent.

AXIOM 1. *A is an individuum.*

DEFINITION II. x is a *set* if and only if x is an individuum and either $x \mathrm{Id} A$ holds or there exists an individuum y such that $y \varepsilon x$.

AXIOM 2 (the axiom of definiteness). *If x, y are sets such that for every individuum z the formulas $z \varepsilon x$ and $z \varepsilon y$ are equivalent, then $x \mathrm{Id} y$ holds.*

AXIOM 3. *If x is a set and y an individuum such that $y \operatorname{non} \varepsilon x$, then there exists a set z such that for every individuum t the formula $t \varepsilon z$ is equivalent to the disjunction $t \varepsilon x$ or $t \mathrm{Id} y$.*

AXIOM 4 (the axiom of union). *For every set x there exists a set y such that for every individuum z the formula $z \varepsilon y$ holds if and only if there exists an individuum t for which $z \varepsilon t$ and $t \varepsilon x$.*

AXIOM 5 (the axiom of power set). *For every set x there exists a set y such that for every individuum z the formula $z \varepsilon y$ holds if and only if z is a set and for every individuum t the formula $t \varepsilon z$ implies the formula $t \varepsilon x$.*

(¹⁵) At the University of Vienna in the summer semester 1936/37.

Definition III. For arbitrary individua u, v we denote by (u, v) that set x which satisfies the condition: for every individuum t we have $t \varepsilon x$ if and only if $t \mathrm{Id} u$ or $t \mathrm{Id} v$.

Definition IV. For arbitrary individua u, v we denote by $[u, v]$ that set x which satisfies the condition: for every individuum t we have $t \varepsilon x$ if and only if $t \mathrm{Id}(u, u)$ or $t \mathrm{Id}(u, v)$.

Axiom 6 (the axiom of infinity). *There exists a set* x *such that* $\Lambda \varepsilon x$ *and for every individuum* y *the formula* $y \varepsilon x$ *implies the formula* $(y, y) \varepsilon x$.

Axiom 7 (the axiom of empty set). *We do not have* $x \varepsilon \Lambda$ *for any individuum.*

Axiom 8 (the axiom of foundation). *If* A *is a class and there exists an individuum* x *such that* $x \varepsilon A$, *then there exists an individuum* y *such that* $y \varepsilon A$ *and the formulas* $t \varepsilon y$ *and* $t \varepsilon A$ *do not simultaneously hold for any individuum.*

Axiom 9. *If* x *is an individuum, there exists a class* A *such that for every individuum* t *the formulas* $t \varepsilon A$ *and* $t \mathrm{Id} x$ *are equivalent.*

Axiom 10. *If* A *is a class, there exists a class* B *such that for every individuum* t *the formulas* $t \varepsilon B$ *and* $t \text{ non } \varepsilon A$ *are equivalent.*

Axiom 11. *If* A, B *are classes, there exists a class* C *such that for every individuum* t *the formula* $t \varepsilon C$ *is equivalent to the conjunction:* $t \varepsilon A$ *and* $t \varepsilon B$.

Axiom 12. *There exists a class* A *which satisfies the following condition: for every individuum* t *we have* $t \varepsilon A$ *if and only if* t *is a set,* $t \neq \Lambda$, *and from* $x \varepsilon t$, $y \varepsilon t$ *for any individua* x, y *it always follows that* $x \mathrm{Id} y$.

Axiom 13. *There exists a class* A *which satisfies the following condition: for any individuum* t *we have* $t \varepsilon A$ *if and only if there exist individua* u, v *such that* $u \varepsilon v$ *and* $t \mathrm{Id}[u, v]$.

Axiom 14. *For every class* A *there exists a class* B *with the following property: for any individuum* t *we have* $t \varepsilon B$ *if and only if there exist individua* u, v *such that* $u \varepsilon A$ *and* $t \mathrm{Id}[u, v]$.

Axiom 15. *For every class* A *there exists a class* B *such that for any individuum the formula* $t \varepsilon B$ *holds if and only if there exists an individuum* u *such that* $[t, u] \varepsilon A$.

Axiom 16. *For every class* A *there exists a class* B *such that for any individuum* t *the formula* $t \varepsilon B$ *holds if and only if there exist individua* u, v *for which* $[v, u] \varepsilon A$ *and* $t \mathrm{Id}[u, v]$.

AXIOM 17. *For every class A there exists a class B such that for any individuum t the formula $t\ \varepsilon\ B$ holds if and only if there are individua u, v, w for which* $[u, [v, w]]\ \varepsilon\ A$ *and* $t\,\mathrm{Id}\,[[u, v], w]$.

AXIOM 18 (the axiom of replacement). *If x is a set and A a class with the following property:*

for any individua u, v, w, from $[u, v]\ \varepsilon\ A$ *and* $[u, w]\ \varepsilon\ A$ *it follows that* $v\,\mathrm{Id}\,w$

then there exists a set y such that for any individuum z the formula $z\ \varepsilon\ y$ holds if and only if there exists an individuum t for which $t\ \varepsilon\ x$ and $[t, z] \in A$.

From the above axioms the whole "classical" set theory can be derived. Here we do not need to develop the complete theory and we limit ourselves to two definitions concerning the notions of ordering and well-ordering.

DEFINITION V. A set y *orders* a set x whenever $x \neq \Lambda$ and for any individua $u, v, w\ \varepsilon\ x$ the following conditions are satisfied:

$$[u, v]\ \varepsilon\ y \text{ or } [v, u]\ \varepsilon\ y;$$

$$\text{if } [u, v]\ \varepsilon\ y \text{ and } u \text{ non Id } v, \text{ then } [v, u] \text{ non } \varepsilon\ y;$$

$$\text{if } [u, v]\ \varepsilon\ y \text{ and } [v, w]\ \varepsilon\ y, \text{ then } [u, w]\ \varepsilon\ y;$$

$$[u, u]\ \varepsilon\ y.$$

DEFINITION VI. A set y *well-orders* a set x if it orders x and the following condition holds: if z is a set such that z non $\mathrm{Id}\,\Lambda$ and for every individuum t the formula $t\ \varepsilon\ z$ implies the formula $t\ \varepsilon\ x$, then there exists an individuum $u\ \varepsilon\ z$ such that for every individuum $v\ \varepsilon\ z$ holds.

With the aid of these definitions, the ordering principle and the well-ordering theorem can be expressed as follows:

ORDERING PRINCIPLE. *For every set $x \neq \Lambda$ there exists a set y that orders x.*

WELL-ORDERING TTHEOREM. *For every set $x \neq \Lambda$ there exists a set y that well orders x.*

As we have already mentioned, the system $\mathfrak{S}$ represents a modification of the system of Bernays. The difference between the two systems consists in: 1° a different formulation of the axiom of definiteness and 2° a different choice of the basic notions. In the system of Bernays there appear four basic notions: "*Set*", "*Class*", "η" (the relation of membership of a set to a class), "ε" (the relation of membership of a set to a set). The first two notions can in fact be defined with the help of ε and η. We have chosen different basic notions in order to be able to come out with the relation of inclusion, which in our opinion is more in agreement with mathematical usage. The constant Λ has been included in order to construct an axiomatic system which does not exclude the existence of individua that are sets.

In our opinion the "natural" systems of set theory are those which include only two basic notions: "*set*" and "ε". The several formal systems of this kind which have been published are unsatisfactory in that, in addition to proper axioms, they contain "axiom schemes" which are based in fact upon infinitely many axioms. As Mr. Tarski and I have jointly observed, one can give thoroughly finite systems of axioms of this kind which are at least as rich as the system of Bernays. We intend to present such a system of axioms in another paper.

§ 2. The construction of the model

Starting with this section we will work within the framework of von Neumann's system of set theory. We use Latin letters to denote sets, and Gothic letters to denote domains(*) of sets, which in general need not be encompassable by any set. The letters ξ, η, ζ, τ denote ordinal numbers and the letters φ, ψ, $\aleph$—certain functions. Moreover, the usual notation of set theory will be used.

1. For every set x we denote by $\mathfrak{P}(x)$ the system of all non-empty subsets of x.

2 (Inductive definition for $\Sigma_\xi(x)$). For every set x we put: a) $\Sigma_0(x) = x$; b) $\Sigma_\xi(x)$ for $\xi > 0$ is the set of those t for which either $t \in x$ or there are z and $\eta < \xi$ such that $t \in z \in \Sigma_\eta(x)$.

3 (Inductive definition for $\Sigma_\xi(\mathfrak{A})$). For every domain $\mathfrak{A}$ we put: a) $\Sigma_0(\mathfrak{A}) = \mathfrak{A}$; b) $\Sigma_\xi(\mathfrak{A})$ for $\xi > 0$ is the domain of all t for which either $t \in \mathfrak{A}$ or there are $\eta < \xi$ and $y \in \Sigma_\eta(\mathfrak{A})$ such that $x \in y$.

4. *If x, y are sets and $y \in x$, we have $\Sigma_\xi(y) \subset \Sigma_{\xi+1}(x)$ for every ordinal number ξ.*

Proof. If $\xi = 0$ and $t \in \Sigma_0(y) = y$, then there is a $z \in x$ (and clearly we have $z = y$) for which $t \in z \in x = \Sigma_0(x)$. Thus we have $t \in \Sigma_1(x)$, i.e. $\Sigma_0(y) \subset \Sigma_1(x)$. Let us assume that the theorem holds for all numbers

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(*) In the German original, the author uses the term "*Bereich*". In contemporary English mathematical text, the words "class" or "aggregate" would be probably employed. Since the author uses the word "*Klasse*" in another meaning, we have decided to translate "*Bereich*" literally by "domain". (*Translator's remark*)

$\eta < \xi$. If $t \in \Sigma_\xi(y)$, then we have either $t \in y$ or $t \in z \in \Sigma_\eta(y)$ for some z and some $\eta < \xi$. In the first case we clearly have $t \in \Sigma_{\xi+1}(x)$ as $t \in y \in x = \Sigma_0(x)$; in the second case we infer from the induction hypothesis that $t \in z \in \Sigma_{\eta+1}(x)$ and, since $\eta+1 \leq \xi+1$, we obtain $t \in \Sigma_{\xi+1}(x)$. Hence we have $\Sigma_\xi(y) \subset \Sigma_{\xi+1}(x)$, which was to be proved.

5. *If ξ, η are ordinal numbers and $\xi < \eta$, we have $\Sigma_\xi(x) \subset \Sigma_\eta(x)$ for every set x and $\Sigma_\xi(\mathfrak{A}) \subset \Sigma_\eta(\mathfrak{A})$ for every domain $\mathfrak{A}$.*

Proof. If $t \in \Sigma_\xi(x)$, we have $t \in x$ or $t \in z$, where $z \in \Sigma_\zeta(x)$, $\zeta < \xi < \eta$. In both cases in view of **2** we have $t \in \Sigma_\eta(x)$. The proof for domains is quite analogous.

6. For every ordinal number ξ we denote by $\xi - 1$ the number which immediately precedes ξ, if such a number exists. If ξ is a limit number or 0, we put $\xi - 1 = \xi$.

7. *If u, v are sets, we have*

$$\Sigma_0(\{u, v\}) = \{u, v\}$$

and

$$\Sigma_\xi(\{u, v\}) = \{u, v\} + \Sigma_{\xi-1}(u) + \Sigma_{\xi-1}(v) \quad \textit{for all } \xi > 0.$$

Proof. For $\Sigma_0(\{u, v\})$ the formula is evident. Further $t \in \Sigma_1(\{u, v\})$ if and only if $t \in \{u, v\}$ or there is a $z \in \{u, v\}$ such that $t \in z$. But this is equivalent to $t \in \{u, v\} + u + v = \{u, v\} + \Sigma_0(u) + \Sigma_0(v)$. Hence the theorem holds for $\xi = 1$. Let us suppose that it holds for all $\eta < \xi$ where $\xi > 1$. By **2** we have $t \in \Sigma_\xi(\{u, v\})$ if and only if $t \in \{u, v\}$, or if $t \in z \in \Sigma_\eta(\{u, v\})$ for some z and $\eta < \xi$; here one can assume $\eta - 1 < \eta$. By the induction hypothesis the second of these conditions yields either $t \in z \in \{u, v\}$ or $t \in z \in \Sigma_{\eta-1}(u)$, or finally $t \in z \in \Sigma_{\eta-1}(v)$. From $\eta < \xi$ we obtain $\eta - 1 < \xi - 1$, and from **5** it follows that $u \subset \Sigma_{\xi-1}(u)$ and $v \subset \Sigma_{\xi-1}(v)$. Hence we have either $t \in u + v \subset \Sigma_{\xi-1}(u) + \Sigma_{\xi-1}(v)$ or $t \in \Sigma_{\xi-1}(u)$, or $t \in \Sigma_{\xi-1}(v)$ that is $t \in \{u, v\} + \Sigma_{\xi-1}(u) + \Sigma_{\xi-1}(v)$. Conseqnently, we obtain

$$\Sigma_\xi(\{u, v\}) \subset \{u, v\} + \Sigma_{\xi-1}(u) + \Sigma_{\xi-1}(v). \tag{1}$$

If $t \in \Sigma_{\xi-1}(u) + \Sigma_{\xi-1}(v)$ and $\xi \neq 0$ is not a limit number, then by **5** we have $t \in \Sigma_\xi(u) + \Sigma_\xi(v)$. On the other hand, if ξ is a limit number or 0, then by **6** the formulas $t \in \Sigma_{\xi-1}(u) + \Sigma_{\xi-1}(v)$ and $t \in \Sigma_\xi(u) + \Sigma_\xi(v)$ are equivalent. Hence the inclusion $\Sigma_{\xi-1}(u) + \Sigma_{\xi-1}(v) \subset \Sigma_\xi(\{u, v\})$ is proved. Since also $\{u, v\} \subset \Sigma_\xi(\{u, v\})$ clearly holds for $\xi > 0$, we have $\{u, v\} + \Sigma_{\xi-1}(u) + \Sigma_{\xi-1}(v) \subset \Sigma_\xi(\{u, v\})$, which by (1) yields the desired equality.

8. *For every set x and every ordinal number ξ we have*

$$\Sigma_\xi(\Sigma_1(x)) = \Sigma_{\xi+1}(x) = \Sigma_1(\Sigma_\xi(x)).$$

Proof. The formula is evident for $\xi = 0$. We suppose that it holds for all $\eta < \xi$. If $t \in \Sigma_\xi(\Sigma_\xi(x))$, we have either $t \in \Sigma_1(x)$ or $t \in z \in \Sigma_\eta(\Sigma_1(x))$, where $\eta < \xi$. In the first case we have by **5** $t \in \Sigma_{\xi+1}(x)$. In the second case the induction hypothesis gives $t \in z \in \Sigma_{\eta+1}(x)$, which in view of **2**, **5** and $\eta+1 < \xi+1$ implies the formula $t \in \Sigma_{\xi+1}(x)$. Thus we have proved that

$$\Sigma_\xi(\Sigma_1(x)) \subset \Sigma_{\xi+1}(x). \tag{1}$$

Now if $t \in \Sigma_{\xi+1}(x)$, we have either $t \in x$ or $t \in z \in \Sigma_\eta(x)$ for some z and $\eta < \xi+1$. In the first case in view of **2**, **5** and $\eta < \xi$ we have $t \in z \in \Sigma_\xi(x)$, i.e. $t \in \Sigma_1(\Sigma_\xi(x))$. Hence we have

$$\Sigma_{\xi+1}(x) \subset (\Sigma_1(\Sigma_\xi(x)). \tag{2}$$

Finally, if $t \in \Sigma_1(\Sigma_\xi(x))$ we have either $t \in \Sigma_\xi(x)$ or $t \in z \in \Sigma_\xi(x)$ for some z. In the first case we have either $t \in x \subset \Sigma_1(x) \subset \Sigma_\xi(\Sigma(x))$, or $t \in u \in \Sigma_\eta(x)$, where $\eta < \xi$. If the latter holds, we obtain $t \in \Sigma_\eta(\Sigma_1(x))$; hence by induction $t \in \Sigma_\eta(\Sigma_1(x))$ and consequently by **5** $t \in \Sigma_\xi(\Sigma_1(x))$, as we have $\eta \leqslant \xi$. Thus in the first case $t \in \Sigma_\xi(\Sigma_1(x))$ always holds. Similarly we consider the second case: namely we have either $t \in z \in x$ or $t \in z \in v \in \Sigma_\eta(x)$, where $\eta < \xi$. In the first case we obtain $t \in \Sigma_1(x) \subset \Sigma_\xi(\Sigma_1(x))$, and in the second case $t \in z \in \Sigma_1(\Sigma(x)) = \Sigma_\eta(\Sigma_1(x))$, i.e. $t \in \Sigma_{\eta+1}(\Sigma_1(x))$; since we have $\eta+1 \leqslant \xi$, it follows by **5** that $t \in \Sigma_\xi(\Sigma_1(x))$. Hence we obtain the inclusion

$$\Sigma_1(\Sigma_\xi(x)) \subset \Sigma_\xi(\Sigma_1(x)). \tag{3}$$

Now our theorem follows immediately from (1), (2) and (3).

9. *If u, v are sets and $\mathfrak{A}$, $\mathfrak{B}$ domains, we have*

$$\Sigma_\xi(u+v) = \Sigma_\xi(u)+\Sigma_\xi(v)$$

and

$$\Sigma_\xi(\mathfrak{A}+\mathfrak{B}) = \Sigma_\xi(\mathfrak{A})+\Sigma_\xi(\mathfrak{B})$$

for every ordinal number ξ.

Proof. It suffices to consider only the first formula; the proof of the second is quite analogous.

For $\xi = 0$ the formula clearly holds. Now we assume its validity for all $\eta < \xi$. If $t \in \Sigma_\xi(u+v)$, we have either $t \in (u+v)$ or $t \in z \in \Sigma_\eta(u+v)$ where $\eta < \xi$. In the first case we obtain by **5** $t \in \Sigma_0(u)+\Sigma_0(v) \subset \Sigma_\xi(u)+$

$+\Sigma_\xi(v)$; in the second case according to the induction assumption we have $t \in z \in \Sigma_\eta(u)+\Sigma_\eta(v)$; hence $t \in \Sigma_\xi(u)+\Sigma_\xi(v)$. Conversely, if we have $t \in \Sigma_\xi(u)+\Sigma_\xi(v)$ then either $t \in u$ or $t \in v$ or $t \in x \in \Sigma_\eta(u)$ or $t \in y \in \Sigma_\xi(v)$, where $\eta, \zeta < \xi$. In the first two cases we have $t \in u+v \in \Sigma_\xi(u+v)$; in the third case, by the induction assumption, we have $t \in y \in \Sigma_\eta(u)+ +\Sigma_\eta(v) \subset \Sigma_\eta(u+v)$; hence in view of **2** $t \in \Sigma_\xi(u+v)$. Similarly we show in the fourth case that $t \in \Sigma_\xi(u+v)$. This formula generally holds for $t \in \Sigma_\xi(u)+\Sigma_\xi(v)$. Hence we have proved the equivalence of the formulas $t \in \Sigma_\xi(u+v)$ and $t \in \Sigma_\xi(u)+\Sigma_\xi(v)$.

10. *If u, v are sets, $\mathfrak{A}$, $\mathfrak{B}$ domains, and $u \subset v$, $\mathfrak{A} \subset \mathfrak{B}$, then we have $\Sigma_\xi(u) \subset \Sigma_\xi(v)$ and $\Sigma_\xi(\mathfrak{A}) \subset \Sigma_\xi(\mathfrak{B})$ for every ordinal number ξ.*

This follows directly from **9**.

11. *For every ordinal number ξ and every set of sets x we have*

$$\Sigma_\xi\Big(\sum_{y \in x} y\Big) \subset \Sigma_{\xi+1}(x).$$

Proof. By **2** we clearly have $\sum\limits_{y \in x} y \subset \Sigma_1(x)$. By applying **10** and **8** we conclude that

$$\Sigma_\xi\Big(\sum_{y \in x} y\Big) \subset \Sigma_\xi(\Sigma_1(x)) = \Sigma_{\xi+1}(x),$$

which was to be proved.

12. *If x is a set, $\mathfrak{A}$ a domain, and ξ an ordinal number, then we have $\Sigma_\xi(\mathfrak{P}(x)\cdot\mathfrak{A})-\mathfrak{P}(x)\cdot\mathfrak{A} \subset \Sigma_\xi(x)$.*

Proof. The formula is evident for $\xi = 0$. We assume its validity for all $\eta < \xi$. If $t \in \Sigma_\xi(\mathfrak{P}(x)\cdot\mathfrak{A})-\mathfrak{P}(x)\cdot\mathfrak{A}$, then there are $\eta < \xi$ and z such that $t \in z \in \Sigma_\eta(\mathfrak{P}(x)\cdot\mathfrak{A})$. If $z \in \mathfrak{P}(x)\cdot\mathfrak{A}$, we have $t \in z \subset x = \Sigma_0(x) \subset \Sigma_\xi(x)$. If z non $\in \mathfrak{P}(x)\mathfrak{A}$, then by the induction assumption we have $t \in z \in \Sigma_\eta(x)$; thus we have $t \in \Sigma_\xi(x)$. Hence the formula $t \in \Sigma_\xi(x)$ holds in all the cases, which was to be proved.

13 (Inductive definition for a_n). a) $a_0 = 0$; b) $a_{n+1} = \{a_n\}$.

14. $Z = \underset{a_n}{\mathrm{E}}\,[n = 0, 1, 2, \ldots]$.

15. $A_n = Z - \{a_n\}$ *for* $n = 0, 1, 2, \ldots$

16. $K = \underset{A_n}{\mathrm{E}}\,[n = 1, 2, \ldots]$.

17 (Inductive definition for K_ξ).

a) $K_0 = K + \{A_0\}$;

b) $K_\xi = \sum\limits_{\eta<\xi} K_\eta + \mathfrak{P}\big(\sum\limits_{\eta<\xi} K_\eta\big)$ for $\xi > 0$.

18. *If ξ, η are ordinal numbers and $\xi < \eta$, we have $K_\xi \subset K_\eta$.*

The proof by induction presents no difficulties.

19. *For arbitrary ordinal numbers ξ, η we have $\Sigma_\xi(K_\eta) \subset K_\eta + Z$.*

Proof. The theorem holds for $\xi = 0$, since $\Sigma_0(K_\eta) = K_\eta$. We assume that the theorem holds for all $\zeta < \xi$. If $t \in \Sigma_\xi(K_\eta)$, then either we have $t \in K_\eta$ or there are Z and $\zeta < \xi$ such that $t \in z \in \Sigma_\xi\,(K_\eta)$. In the first case we have $t \in K_\eta + Z$. In the second case we conclude from the induction assumption that $t \in z \in K_\eta + Z$. Now if we have $z \in Z$, then by **14** we have $z = a_n$ for some n, and consequently by **13** $t = a_{n-1} \in Z$. But if we have $z \in K_\eta$, there is a smallest number $\tau \leqslant \eta$ for which $z \in K_\tau$. If $\tau = 0$, then by **17** we have $z = \Lambda_m$ for some m, and by **15** $t \in \Lambda_m \subset Z$, i.e. $t \in Z$. If $\tau > 0$, then by **17** we have $K_t = \sum\limits_{\nu<t} + \mathfrak{P}\big(\sum\limits_{\nu<t} K_\nu\big)$ and since t non $\in K_\nu$ for $\nu < \tau$, we have $z \in \mathfrak{P}\big(\sum\limits_{r<\tau} K_\nu\big)$, i.e. by **1** $0 \neq Z \subset \sum\limits_{\nu<\tau} K_\nu$. From this we obtain $t \in \sum\limits_{\nu<\tau} K_\nu$, hence by **18** $t \in K_\eta$. In any case we have $t \in K_\eta + Z$, which was to be proved.

20. By $\mathfrak{M}$ we denote the domain of all x for which there is a ξ such that $x \in K_\xi$, If $x \in \mathfrak{M}$, we denote by $\tau(x)$ the smallest ordinal number ξ such that $x \in K_\xi$.

21. *If $x \in \mathfrak{M}$ and $\tau(x) > 0$, then we have $0 \neq x \subset \sum\limits_{\eta<\tau(x)} K_\eta$ and $\tau(y) < \tau(x)$ for every $y \in x$.*

The proof follows immediately from **17** and **20.**

22. *If x is a set and $0 \neq x \subset K_\xi$, then $x \in K_{\xi+1}$ and consequently $x \in \mathfrak{M}$.*

Proof. From $0 \neq x \subset K_\xi$ it follows that $0 \neq x \subset \sum\limits_{\eta<\xi+1} K_\eta$, *i.e.* $x \in \mathfrak{P}\big(\sum\limits_{\eta<\xi+1} K_\eta\big) \subset K_{\xi+1}$.

23. *If $u \in \mathfrak{M} - K_0$, then $\mathfrak{P}(u) \in \mathfrak{M}$.*

Proof. By **21** we have $0 \neq \mathfrak{P}(u) \subset \mathfrak{P}\big(\sum\limits_{\eta<\tau(u)} K_\eta\big) \subset K_{\tau(u)}$, from which in view of **22** it follows that $\mathfrak{P}(u) \in \mathfrak{M}$.

24. *If u is a set and $0 \neq u \subset \mathfrak{M} \subset K_0$, then $\sum\limits_{x\in u} x \in \mathfrak{M}$.*

Proof. For $x \in u$ we have $\tau(x) > 0$; hence by **21** $0 \neq x \subset \sum\limits_{\eta<\tau(x)} K_\eta$. If ζ is an ordinal number which exceeds all $\tau(x)$ with $x \in u$, then it follows that $x \subset \sum_{\eta<\xi} K_\eta$ for all $x \in u$, and in view of **18** $0 \neq \sum_{x \in u} x \subset K$. By **22** this yields $\sum_{x \in u} x \in \mathfrak{M}$, which was to be proved.

25. 0, $\{0\}$ non $\in \mathfrak{M}$.

Proof. It is obvious that 0, $\{0\}$non $\in K_0$. If $\xi > 0$ and 0non $\in K_\eta$ holds for all $\eta < \xi$, then 0non $\in \sum_{\eta<\xi} K_\eta$ and by **1** we have 0non $\in \mathfrak{P}\big(\sum_{\eta<\xi} K_\eta\big)$. By **17** this implies that 0non $\in K_\xi$; hence by induction 0non $\in \mathfrak{M}$.

If we had $\{0\} \in \mathfrak{M}$, then by **21** we would have $\{0\} \subset \sum_{\eta<\xi} K_\eta$ for $\xi = \tau(\{0\})$. But this is impossible, since 0 non $\in \sum_{\eta<\xi} K_\eta$. This ends the proof.

26. *If* $x \in K_0$ *and* $y \in x$, *then* $y$non $\in \mathfrak{M}$.

Proof. If the theorem were false, then by **14, 15, 17** and **20** we would have $a_n \in K_\xi$ for some n and ξ. By **13** and **21**, this would imply $\{0\} \in K_\xi$, which contradicts **25**.

27. *If* ξ *is an ordinal number and* $x \in \mathfrak{M}$, *then we have* a) $\Sigma_\xi(x) - \mathfrak{M} \subset Z$; b) $\Sigma_\xi(\mathfrak{M}) \subset \mathfrak{M} + Z$.

Proof. a) If $\tau(x) = 0$, then $x = \Lambda_m$ for a certain m and our assertion is evident. If $\tau(x) > 0$, then by **18** and **21** we have $x \subset K_{\tau(x)}$; thus in view of **10** and **19** $\Sigma_\xi \subset K_{\tau(x)} + Z$ and consequently $\Sigma_\xi(x) - K_{\tau(x)} \subset Z$. Since $K_{\tau(x)} \subset \mathfrak{M}$, it follows that $\Sigma_\xi - \mathfrak{M} \subset Z$, which was to be proved.

b) The formula clearly holds for $\xi = 0$. Now if $\xi > 0$ and $\Sigma_\eta(\mathfrak{M}) \subset \mathfrak{M} + Z$ for all $\eta < \xi$, then every $x \in \Sigma_\xi(M)$ satisfies either the formula $x \in \mathfrak{M}$ or the formula $x \in y \in \Sigma_\eta(\mathfrak{M})$, where $\eta < \xi$. In the second case we infer from the induction assumption that $y \in \mathfrak{M} + Z$. If $y \in Z$, then by **13** and **14** we have $x \in Z$. If $y \in \mathfrak{M}$ and $\tau(y) = 0$, then by **15, 16, 17** we have $x \in Z$. Finally, if $\tau(y) > 0$, then by **18** and **21** we have $x \in y \subset K_{\tau(y)} \subset \mathfrak{M}$. Hence in all these cases we obtain $x \in \mathfrak{M} + Z$, which was to be proved.

28. By $\mathfrak{G}_0$ we denote the system of all one-to-one maps of K onto itself. The map arising by the composition of two maps $\varphi, \psi \in \mathfrak{G}_0$ is denoted by $\varphi\psi$ (16); by φ^{-1} we denote the map inverse to φ.

29. For $x \in K$ and $\varphi \in \mathfrak{G}_0$ we denote by $\varphi(x)$ that element of K which arises from x under the map φ.

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(16) I.e., the map ψ is performed first, and then φ.

30 (Inductive definition for $|\varphi, x|$). We put for $x \in \mathfrak{M}$ and $\varphi \in \mathfrak{G}_0$: a) $|\varphi, \Lambda_0| = \Lambda_0$; b) $|\varphi, x| = \varphi(x)$ for $x \in K$; c) $|\varphi, x| = \underset{|\varphi, y|}{\mathrm{E}} [y \in x]$ if $\tau(x) > 0$.

31. *If* $x \in \mathfrak{M}$ *and* $\varphi \in \mathfrak{G}_0$, *then* $|\varphi, x| \in \mathfrak{M}$.

Proof. If $\tau(x) = 0$, then the assertion results directly from **30** a), b). We assume the validity of the assertion for all x with $\tau(x) < \xi$ and consider an x for which $\tau(x) = \xi$. From **25**, **30** c) and the induction assumption it follows that $0 \neq |\varphi, x| \subset \mathfrak{M}$, and in view of **22** we obtain $|\varphi, x| \in \mathfrak{M}$, which was to be proved.

32. *If x is a set and* $0 \neq x \subset \mathfrak{M}$, *then* $|\varphi, x| = \underset{|\varphi, y|}{\mathrm{E}} [y \in x]$.

Proof. From **22** we easily obtain the existence of an ordinal number ξ for which $x \in K_\xi$. Consequently we have $x \in \mathfrak{M}$. By **26** we must have $\tau(x) > 0$, and now our assertion follows directly from **30** c).

33. *If* $x \in \mathfrak{M}$ *and* $\varphi \in \mathfrak{G}_0$, *then* $|\varphi^{-1}, |\varphi, x|| = x$.

Proof. We proceed by induction with respect to $\tau(x)$. If $\tau(x) = 0$, our assertion is evident. Now we assume that it holds for all x with $\tau(x) < \xi$, and let x be an element of $\mathfrak{M}$ with $\tau(x) = \xi$. Since $\xi > 0$, we conclude by **21** that $0 \neq x \subset \mathfrak{M}$, and next, in view of **32** and **31**, we get $0 \neq |\varphi, x| = \underset{|\varphi, y|}{\mathrm{E}} [y \in x] \subset \mathfrak{M}$. By another application of **32** we obtain $|\varphi^{-1}, |\varphi, x|| = \underset{|\varphi^{-1}, |\varphi, y||}{\mathrm{E}} [y \in x]$. Since by **21** we have $\tau(y) < \xi$ for $y \in x$, we conclude from the induction assumption that $|\varphi^{-1}, |\varphi, x|| = x$, which ends the proof.

34. *If* $\varphi \in \mathfrak{G}_0$ *and* $x, y \in \mathfrak{M}$, *then the formulas* $x \in y$ *and* $|\varphi, x| \in |\varphi, y|$ *are equivalent.*

Proof. Let $x \in y$. From **26** it follows that y non $\in K_0$, and hence $\tau(y) > 0$; thus, by **21**, $0 \neq y \subset \mathfrak{M}$ and further, in view of **32**, $|\varphi, y| = \underset{|\varphi, t|}{\mathrm{E}} [t \in y]$. Hence from $x \in y$ it follows that $|\varphi, x| \in |\varphi, y|$. Conversely, let $|\varphi, y| \in |\varphi, y|$. By **31** we have $|\varphi, y| \in \mathfrak{M}$; hence by **32** we obtain $|\psi, |\varphi, y|| = \underset{|\psi, t|}{\mathrm{E}} [t \in |\varphi, y|]$ for $\psi \in \mathfrak{G}_0$. If we put here $\psi = \varphi^{-1}$, by **33** we obtain $x \in y$, which was to be proved.

35. *If* $x \in \mathfrak{M}$ *and* $\varphi \in \mathfrak{G}_0$, then $\tau(x) = \tau(|\varphi, x|)$.

Proof. We proceed by induction with respect to $\tau(x)$. For $\tau(x)$ the assertion is evident in view of **30** a), b). Now let $\xi > 0$, $\tau(x) = \xi$ and let us assume that the assertion holds for all y with $\tau(y) < \xi$. By **21** we have $0 \neq x \subset \sum_{\eta<\xi} K_\eta$, from which in view of the induction hypothesis and **32** we obtain $0 \neq |\varphi, x| \subset \sum_{\eta<\xi} K_\eta$, i.e. $|\varphi, x| \in K_\xi$. We also have $\tau(|\varphi, x|) \leqslant \xi$. If we had $\tau(|\varphi, x||) < \xi$, then by the induction assumption we would obtain $\tau(|\varphi^{-1}, |\varphi, x||) < \xi$, i.e., by **33**, $\tau(x) < \xi$. Since this contradicts our assumption, we have $\tau(|\varphi, x|) = \xi = \tau(x)$, which was to be proved.

36. *If* $x \in \mathfrak{M}$ *and* $\varphi, \psi \in \mathfrak{G}_0$, *then we have* $|\varphi, |\psi, x|| = |\varphi, \psi, x|$.

Proof. We again use induction with respect to $\tau(x)$. If $\tau(x) = 0$, then the theorem holds. We assume that $\xi > 0$ and the theorem holds for all $y \in \mathfrak{M}$ with $\tau(y) < \xi$. Let $\tau(x) = \xi$. By **32** we have $|\psi, x| = \underset{|\psi, y|}{\mathsf{E}} [y \in x]$, where, in view of **21** and **35**, $\tau(|\psi, y|) < \xi$ for $y \in x$. Consequently we have $0 \neq |\psi, x| \subset \mathfrak{M}$, from which by **32** we obtain the formula $|\varphi, |\psi, x|| = \underset{|\varphi, t|}{\mathsf{E}} [t \in |\psi, x|] = \underset{|\varphi, |\psi, t||}{\mathsf{E}} [y \in x]$. Now the induction assumption yields $|\varphi, |\psi, x|| = \underset{|\varphi\psi, y|}{\mathsf{E}} [y \in x] = |\varphi\psi, x|$, which was to be proved.

37. *If* $x \in \mathfrak{M}$, $\varphi \in \mathfrak{G}_0$, *and* ξ *is an ordinal number and* $\Sigma_\xi(x)\mathfrak{M} \in \mathfrak{M}$, *then we have* $|\varphi, \Sigma_\xi(x) \cdot \mathfrak{M}| = \Sigma_\xi(|\varphi, x)| \cdot \mathfrak{M}$.

Proof. The assertion holds for $\xi = 0$. In fact, we have $\Sigma_0(x) = x$ and $\Sigma_0(|\varphi, x|) = |\varphi, x|$. Further we have x non $\in K_0$, since otherwise in view of **26** we would have $x \cdot \mathfrak{M} = 0$, and by **25** it is evident that the assumption $\Sigma_\xi(x) \cdot \mathfrak{M} \in \mathfrak{M}$ could not then be satisfied. Hence we have $\tau(x) > 0$, from which by **21** we obtain $0 \neq x \subset \mathfrak{M}$. Now from **32** it follows that

$$|\varphi, x \cdot \mathfrak{M}| = \underset{|\varphi, y|}{\mathsf{E}} [y \in x \cdot \mathfrak{M}]. \tag{1}$$

Hence we have $z \in |\varphi, x \cdot \mathfrak{M}|$, and so $z = |\varphi, y|$, where $y \in x \cdot \mathfrak{M}$. Thus according to **34** we have $|\varphi, y) \in |\varphi, x|$, and according to **31** we have $|\varphi, y| \in \mathfrak{M}$. Consequently, we have $z \in |\varphi, x| \cdot \mathfrak{M}$, and thus

$$|\varphi, x \cdot \mathfrak{M}| \subset |\varphi, x| \cdot \mathfrak{M}. \tag{2}$$

Now if $z \in |\varphi, x| \cdot \mathfrak{M}$, then by **33**, **34** and **31** we have $|\varphi^{-1}, z| \in x \cdot \mathfrak{M}$, from which by (1) it follows that $|\varphi, |\varphi^{-1}, z|| \in |\varphi, x \cdot \mathfrak{M}|$, i.e., in view of **33**, $z \in |\varphi, x \cdot \mathfrak{M}|$. Hence we obtain the inclusion $|\varphi, x| \cdot \mathfrak{M} \subset |\varphi, x \cdot \mathfrak{M}|$, which together with (2) shows that our theorem holds for $\xi = 0$. Now we assume the validity of this assertion for all $\eta < \xi$. Let $\varphi \in \mathfrak{G}_0$, $x \in \mathfrak{M}$, $\Sigma_\xi(x) \cdot \mathfrak{M} \in \mathfrak{M}$. Then we have $\Sigma_\xi(x) \cdot \mathfrak{M}$ non $\in K_0$, since otherwise, in view of **25** and **26**, we would have $\Sigma_\xi(x) \cdot \mathfrak{M} = 0$ non $\in \mathfrak{M}$. Hence the number $\tau(\Sigma_\xi(x)\cdot\mathfrak{M})$ is > 0, from which by **21** we obtain $0 \neq \Sigma_\xi(x)\cdot\mathfrak{M} \subset \mathfrak{M}$, and consequently, in view of **32**,

$$|\varphi, \Sigma_\xi(x)\cdot\mathfrak{M}| = \underset{|\varphi, y|}{\mathrm{E}}\,[y \in \Sigma_\xi(x)\cdot\mathfrak{M}]. \tag{3}$$

Hence if $z \in |\varphi, \Sigma_\xi(x)\cdot\mathfrak{M}|$ then there is an $y \in \Sigma_\xi(x)\cdot\mathfrak{M}$ for which $z = |\varphi, y|$. Now there are two cases to be considered: either $y \in x$ or there is a t and a number $\eta < \xi$ such that $y \in t \in \Sigma_\eta(x)$. In the first case we infer from **31** and **34** that $z \in |\varphi, x|\cdot\mathfrak{M}$, and a fortiori $z \in \Sigma_\xi(|\varphi, x|)\cdot\mathfrak{M}$. In the second case we reason in the following manner: if we had $t \text{ non} \in \mathfrak{M}$, then by **27** we would have $t \in Z$, from which $y \in Z$ and $y \in u \in K_0$ for a certain u. From **26** it would then follow that $y \text{ non} \in \mathfrak{M}$, which contradicts our assumption. Hence we have $y \in t \in \Sigma_\eta(x)\cdot\mathfrak{M}$. By **34** this implies $z = |\varphi, y| \in |\varphi, t| \in |\varphi\Sigma_\eta(x)\cdot\mathfrak{M}|$, from which by an application of the induction assumption it follows that $z \in \Sigma_\eta(|\varphi, x|)\cdot\mathfrak{M}$. Hence we have proved

$$|\varphi, \Sigma_\xi(x)\cdot\mathfrak{M}| \subset \Sigma_\xi(|\varphi, x|)\cdot\mathfrak{M}. \tag{4}$$

Now, conversely, let $y \in \Sigma_\xi(|\varphi, x|)\cdot\mathfrak{M}$. We again have to distinguish two cases: either $y \in |\varphi, x|\cdot\mathfrak{M}$, or there is a t and a number $\eta < \xi$ such that $y \in t \in \Sigma_\eta(|\varphi, x|)$. In the first case, from **31**, **33**, **34** we obtain $|\varphi^{-1}, y| \in x\cdot\mathfrak{M}$, which in view of **33** and **34** implies $y \in |\varphi, \Sigma_\xi(x)\cdot\mathfrak{M}|$. In the second case we conclude, as above, that $t \in \mathfrak{M}$. From $y \in t \in |\varphi, \Sigma_\eta(|\varphi, x|)\cdot\mathfrak{M}$ it follows by using the induction assumption that $y \in t \in |\varphi, \Sigma_\eta(x)\cdot\mathfrak{M}|$, which by **33** and **34** gives $|\varphi^{-1}, y| \in |\varphi^{-1}, t| \in \Sigma_\xi(x)\cdot\mathfrak{M}$. Consequently we have $|\varphi^{-1}, y| \in \Sigma_\xi(x)\cdot\mathfrak{M}$, hence by another application of **32** and **34** we obtain $y \in |\varphi, \Sigma_\xi(x)\cdot\mathfrak{M}|$. Hence we have proved that

$$\Sigma_\xi(|\varphi, x|)\cdot\mathfrak{M} \subset |\varphi, \Sigma_\xi(x)\cdot\mathfrak{M}|, \tag{5}$$

and formulas (4) and (5) show the validity of the theorem for the number ξ, which was to be proved.

38. A set $\mathfrak{G} \subset \mathfrak{G}_0$ is called a *group* if $\varphi, \psi \in \mathfrak{G}_0$ always implies $\varphi\psi^{-1} \in \mathfrak{G}$. The system of all groups will be denoted by $\mathcal{G}$.

39. If $A \subset K$, $\mathfrak{G} \in \mathcal{G}$, then we denote by $\mathfrak{G}(A)$ the subgroup of $\mathfrak{G}$ consisting of all functions $\varphi \in \mathfrak{G}$ which satisfy the condition $\varphi(x) = x$ for $x \in A$.

40. For $A \subset K$ and $G \in \mathcal{G}$ we denote by $\mathfrak{K}_{\mathfrak{G}}(A)$ the domain of all sets $x \in \mathfrak{M}$ such that $|\varphi, x| = x$ for all $\varphi \in \mathfrak{G}(A)$.

41. *If* $A \subset K$, $\mathfrak{G} \in \mathscr{G}$, $x \in \mathfrak{K}_{\mathfrak{G}}(A)$ *and* $\varphi \in \mathfrak{G}_0$, *then we have* $|\varphi, x| \in \mathfrak{K}_{\mathfrak{G}}(\varphi(A))$.(17)

Proof. First, by **31** we have $|\varphi, x| \in \mathfrak{M}$. From the assumption it follows further that

$$|\psi, x| = x \quad \text{for } \psi \in \mathfrak{G}(A). \tag{1}$$

Now if we have $\chi \in \mathfrak{G}(\varphi(A))$, then by **39** the function $\varphi^{-1}\chi\varphi$ belongs to the group $\mathfrak{G}(A)$. By (1) we also have $|\varphi^{-1}\chi\varphi x| = x$, and, in view of **33** and **36**, $|\chi, |\varphi, x|| = |\varphi, x|$. Since this holds for every $\chi \in \mathfrak{G}(\varphi(A))$, by **40** we obtain $|\varphi, x| \in \mathfrak{K}_{\mathfrak{G}}(\varphi(A))$, which was to be shown.

42. A set $M \subset \mathfrak{P}(K)$ will be called a *$\mathfrak{G}$-ring* whenever $\mathfrak{G} \in \mathscr{G}$ and the following conditions hold:

$$\text{from} X, Y \in M \text{ it follows that } X + Y \in M; \tag{1}$$

$$\sum_{X \in M} X = K; \tag{2}$$

$$\text{from } X \in M, \varphi \in \mathfrak{G} \text{ it follows that } \varphi(X) \in M. \tag{3}$$

The system of all $\mathfrak{G}$-rings will be denoted by $R(\mathfrak{G})$.

43 (Definition of the model). Let $\mathfrak{G} \in \mathscr{G}$ and $M \in R(\mathfrak{G})$. An element $X \in \mathfrak{M}$ will be called an M, *$\mathfrak{G}$-distinguished element* provided it has the following two properties:

$$\text{there is an } A \in M \text{ such that } x \in \mathfrak{K}_{\mathfrak{G}}(A); \tag{1}$$

(2) for every ordinal number ξ we have $\Sigma_\xi(x)\mathfrak{M} \subset \sum_{B \in M} \mathfrak{K}_{\mathfrak{G}}(B)$.(18)

The domain of all $M, \mathfrak{G}$-distinguished elements will be denoted by $\mathfrak{W}_{M,\mathfrak{G}}$.

A domain $\mathfrak{A}$ is called M, *$\mathfrak{G}$-distinguished* if it equals Λ_0 or satisfies the following condition:

$$0 \neq \mathfrak{A} \subset \mathfrak{W}_{M,\mathfrak{G}}; \tag{3}$$

(4) there is a set $A \in M$ such that the formulas $x \in \mathfrak{A}$ and $|\varphi, x| \in \mathfrak{A}$ are equivalent for every function $\varphi \in \mathfrak{G}(A)$.

44. *If* $\mathfrak{G} \in \mathscr{G}$ *and* $M \in R(\mathfrak{G})$, then $\mathfrak{W}_{M,\mathfrak{G}} \subset \sum_{A \in M} \mathfrak{K}_{\mathfrak{G}}(A)$.

The proof follows directly from **43**.

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(17) $\varphi(A)$ denotes as usual $\underset{\varphi(x)}{E}[x \in A]$. We have $|\varphi, A| = \varphi(A)$.

(18) This formula is to be read as follows: for every $t \in \Sigma_\xi(x) \cdot \mathfrak{M}$ there exists a set $B \in M$ such that $t \in \mathfrak{K}_{\mathfrak{G}}(B)$.

45. *If* $\mathfrak{G} \in \mathscr{G}$ *and* $M \in R(\mathfrak{G})$, *then we have* $K_0 \subset \mathfrak{W}_{M,\mathfrak{G}}$.

Proof. If $x \in K_0$ then $x \in \mathfrak{M}$. From **42** it follows further that there is a set $A \in M$ such that $x \in A$. Moreover, by **39**, **40** and **30** b), we have $x \in \mathfrak{K}_{\mathfrak{G}}(A)$. Finally, if ξ is an ordinal number and $y \in \Sigma_\xi(x)$, then y non $\in \mathfrak{M}$, because it is easy to see that the elements of $\Sigma_\xi(x)$ are always identical with a certain a_n, and thus by **26** they cannot belong to $\mathfrak{M}$. Hence, we have $\Sigma_\xi(x) \cdot \mathfrak{M} = 0$. From this it follows by **43** that x is an M, $\mathfrak{G}$-distinguished element, which was to be proved.

46. *If* $\mathfrak{G} \in \mathscr{G}$, $M \in R(\mathfrak{G})$ *and* $x \in \mathfrak{W}_{M,\mathfrak{G}} - K_0$, *then* $x \subset \mathfrak{W}_{M,\mathfrak{G}}$.

Proof. From $x \in \mathfrak{W}_{M,\mathfrak{G}} - K_0$ it follows that $x \in M - K_0$, and thus $\tau(x) > 0$. Hence if $y \in x$ then, by **21**, $y \in \mathfrak{M}$. Since x is an M, $\mathfrak{G}$-distinguished element, for every $t \in \Sigma_0(x)$ and in particular for $t = y$ there is a set $A \in M$ such that $y \in \mathfrak{K}_{\mathfrak{G}}(A)$. Further, by **4** for every ordinal number ξ we have $\Sigma_\xi(y) \cdot \mathfrak{M} \subset \Sigma_{\xi+1}(x) \cdot \mathfrak{M}$, which implies $\Sigma_\xi(y) \cdot \mathfrak{M} \subset \sum_{B \in M} \mathfrak{K}_{\mathfrak{G}}(B)$. Hence by **43** we conclude that $y \in \mathfrak{W}_{M,\mathfrak{G}}$, which was to be proved.

47. *If* $\mathfrak{G} \in \mathscr{G}$, $M \in R(\mathfrak{G})$, $x \in \mathfrak{W}_{M,\mathfrak{G}}$ *and* $\varphi \in \mathfrak{G}$, *then* $|\varphi, x| \in \mathfrak{W}_{M,\mathfrak{G}}$.

Proof. According to **31** we have $|\varphi, x| \in \mathfrak{M}$. Since there is a set $A \in M$ such that $x \in \mathfrak{K}_{\mathfrak{G}}(A)$, by **41** we obtain $|\varphi, x| \in \mathfrak{K}_{\mathfrak{G}}(\varphi(A))$, and by **42** $\varphi(A) \in M$. Thus there is a $B \in M$ such that

$$(1) \qquad |\varphi, x| \in \mathfrak{K}_{\mathfrak{G}}(B).$$

Now let ξ be an ordinal number. According to the assumption

$$(2) \qquad \Sigma_\xi(x) \cdot \mathfrak{M} \subset \sum_{B \in M} \mathfrak{K}_{\mathfrak{G}}(B).$$

If $\Sigma_\xi(|\varphi, x|) \cdot \mathfrak{M} = 0$, we obviously have

$$(3) \qquad \Sigma_\xi(|\varphi, x|) \cdot \mathfrak{M} \subset \sum_{B \in M} \mathfrak{K}_{\mathfrak{G}}(B).$$

If $\Sigma_\xi(|\varphi, x|) \cdot \mathfrak{M} \neq 0$, then we have

$$(4) \qquad \Sigma_\xi(|\varphi, x)| \cdot \mathfrak{M} \in \mathfrak{M}.$$

Namely, if we denote by ζ an arbitrary ordinal number that exceeds all $\tau(y)$ with $y \in \Sigma_\xi(|\varphi, x|) \cdot \mathfrak{M}$,(19) then we obtain from **20** $0 \neq \Sigma_\xi(x) \cdot \mathfrak{M} \subset K_\zeta$, which by **22** implies formula (4).

(19) The existence of such a number ζ is ensured by the following theorem, which is provable in v. Neumann's set theory: *for every set of ordinal numbers, there exists an ordinal number which does not belong to this set.*

Now let $y \in \Sigma_\xi(|\varphi, x|) \cdot \mathfrak{M}$. In view of (4) and **37** we obtain $y \in |\varphi, \Sigma_\xi(x) \cdot \mathfrak{M}|$; hence, by **33** and **34**, $|\varphi^{-1}, y| \in \Sigma_\xi(x) \cdot \mathfrak{M}$. From (2) we conclude further that there is a set $C \in M$ such that $|\varphi^{-1}, y| \in \mathfrak{K}_{\mathfrak{G}}(C)$, from which by **33** and **41** we obtain $y \in \mathfrak{K}_{\mathfrak{G}}(\varphi(C))$. Since by **42** we have $\varphi(C) \in M$, it follows that $y \in \sum_{B \in M} \mathfrak{K}_{\mathfrak{G}}(B)$, and since this holds for every $y \in \Sigma_\xi(|\varphi, x|) \cdot \mathfrak{M}$, we see that (3) is satisfied also in this case. From (1) and (3) it follows by **43** that $|\varphi, x| \in \mathfrak{W}_{M,\mathfrak{G}}$, which was to be proved.

48. *If* $\mathfrak{G} \in \mathscr{G}$, $M \in R(\mathfrak{G})$ *and* $u, v \in \mathfrak{W}_{M,\mathfrak{G}}$, *then* $\{u, v\}$, $\langle u, v\rangle \in \mathfrak{W}_{M,\mathfrak{G}}$.

Proof. If $u \in K_\xi$ and $v \in K_\eta$, then by **18** we have $0 \neq \{u, v\} \in K_\zeta$ where $\zeta = \max(\xi, \eta)$; hence by **22**

$$\{u, v\} \in \mathfrak{M}. \tag{1}$$

Since $u, v \in \mathfrak{M}$, we have $0 \neq \{u, v\} \subset \mathfrak{M}$; thus, by **32**, $|\varphi, \{u, v\}| = \{|\varphi, u|, |\varphi, v|\}$ for $\varphi \in \mathfrak{G}_0$. Now by assumption there exist two sets $A, B \in M$ such that $|\varphi, u| = u$ for $\varphi \in \mathfrak{G}(A)$ and $|\psi, v| = v$ for $\psi \in \mathfrak{G}(B)$. Let us put $C = A + B$; then by **42** we have

$$C \in M. \tag{2}$$

If $\varphi \in \mathfrak{G}(C)$, then by **39** we have $\varphi \in \mathfrak{G}(A) \cdot \mathfrak{G}(B)$; hence $|\varphi, u| = u$ and $|\varphi, v| = v$, i.e. $\{|\varphi, u|, |\varphi, v|\} = \{u, v\}$, or

$$|\varphi, \{u, v\}| = \{u, v\} \quad \text{for } \varphi \in \mathfrak{G}(C). \tag{3}$$

Now let ξ be an ordinal number and $y \in \Sigma_\xi(\{u, v\}) \cdot \mathfrak{M}$. If $\xi = 0$, then $y = u$ or $y = v$; hence there exists a set $B \in M$ such that

$$|\varphi, y| = y \quad \text{for } \varphi \in \mathfrak{G}(B). \tag{4}$$

If $\xi > 0$ then in view of **7** we have $y = u$ or $y = v$ or $y \in \Sigma_{\xi-1}(u) \cdot \mathfrak{M}$ or $y \in \Sigma_{\xi-1}(v) \cdot \mathfrak{M}$. The first two cases have been considered above, in the third and fourth the assumptions $u, v \in \mathfrak{W}_{M,\mathfrak{G}}$ imply that there exists a set $B \in M$ for which (4) holds. Hence formula (4) holds in general, and by **40** it follows that

$$\Sigma_\xi(\{u, v\}) \cdot \mathfrak{M} \subset \sum_{B \in M} \mathfrak{K}_{\mathfrak{G}}(B). \tag{5}$$

From (1), (2), (3) and (5) by **43** it follows that $\{u, v\} \in \mathfrak{W}_{M,\mathfrak{G}}$. By a double application of this formula we obtain $\langle u, v\rangle \in \mathfrak{W}_{M,\mathfrak{G}}$, since $\langle u, v\rangle = \big\{\{u, v\}, \{u, v\}\big\}$.

49. *If* $\mathfrak{G} \in \mathscr{G}$ and $M \in R(\mathfrak{G})$, *then every element of* $\mathfrak{W}_{M,\mathfrak{G}} - K$ *is a distinguished domain.*

The proof can immediately be obtained from **32** and from the definition of a distinguished domain.

50. *If* $\mathfrak{G} \in \mathcal{G}$ *and* $M \in R(\mathfrak{G})$, *then we have* $\Sigma_\xi(\mathfrak{W}_{M,\mathfrak{G}}) \cdot \mathfrak{M} = \mathfrak{W}_{M,\mathfrak{G}}$ *for every ordinal number* ξ.

Proof. The formula clearly holds for $\xi = 0$. We assume its validity for all $\eta < \xi$. By **5** we have

$$(1) \qquad \mathfrak{W}_{M,\mathfrak{G}} \subset \Sigma_\xi(\mathfrak{W}_{M,\mathfrak{G}}) \cdot \mathfrak{M}.$$

Now if $x \in \Sigma_\xi(\mathfrak{W}_{M,\mathfrak{G}}) \cdot \mathfrak{M}$, then either $x \in \mathfrak{W}_{M,\mathfrak{G}}$ or, for a certain $\eta < \xi$ and $y, x \in y \in \Sigma_\eta(\mathfrak{W}_{M,\mathfrak{G}})$. Hence by **27** b) and **26** we must have $y \in \mathfrak{M} - K_0$, since $x \in \mathfrak{M}$ cannot hold. In view of the induction assumption it follows that $x \in y \in \mathfrak{W}_{M,\mathfrak{G}} - K_0$; thus, by **46**, $x \in \mathfrak{W}_{M,\mathfrak{G}}$. We thus have $\Sigma_\xi(\mathfrak{W}_{M,\mathfrak{G}}) \cdot \mathfrak{M} \subset \mathfrak{W}_{M,\mathfrak{G}}$, which together with (1) yields the desired equality.

§ 3. The main theorem

The main theorem states:

If $\mathfrak{G} \in \mathcal{G}$ *and* $M \in R(\mathfrak{G})$ *and if in the axioms of the system* $\mathfrak{S}$ *we replace the words "Individuum", "Class", "*ε*", "*Λ*" by the words "*M*,* $\mathfrak{G}$*-distinguished element", "*M*,* $\mathfrak{G}$*-distinguished domain", "*ε*", "*Λ_0*", respectively, then all the axioms of the system* $\mathfrak{S}$ *become true statements.*

Before we give the proof of this theorem, we have to find out what sense the defined notions of the system $\mathfrak{S}$ acquire when we interpret the basic notions of the main theorem in the way described above.

51. *If* $\mathfrak{G} \in \mathcal{G}$, $M \in R(\mathfrak{G})$ *and* $x, y \in \mathfrak{W}_{M,\mathfrak{G}}$ *then* $x = y$ *if and only if the formulas* $x \in \mathfrak{A}$ *and* $y \in \mathfrak{A}$ *are equivalent for every* M, $\mathfrak{G}$*-distinguished domain.*

Proof. If $x = y$, then the formulas $x \in \mathfrak{A}$ and $y \in \mathfrak{A}$ are clearly equivalent for every domain. Now let the formulas $x \in \mathfrak{A}$ and $y \in \mathfrak{A}$ be equivalent for every M, $\mathfrak{G}$-distinguished domain. Let us consider the domain $\mathfrak{A} = \{x\}$, which is also a set. From **26**, **46** and **49** it follows that $\mathfrak{A}$ is a distinguished domain. Now we have $x \in \mathfrak{A}$, and so we also must have $y \in \mathfrak{A}$, i.e. $x = y$, which was to be proved.

From **51** it can be seen that this is the relation of identity which in our model corresponds to the relation Id of the system $\mathfrak{S}$.

52. *Let* $\mathfrak{G} \in \mathcal{G}$, $M \in R(\mathfrak{G})$ *and* $x \in \mathfrak{W}_{M,\mathfrak{G}}$. *The necessary and sufficient* *condition in order that* $x = \Lambda_0$, *or that there exist an* $y \in \mathfrak{W}_{M,\mathfrak{G}}$ *with* $y \in x$, *is* $x \in \mathfrak{W}_{M,\mathfrak{G}} - K$.

The proof can immediately be obtained from **25**, **26**, **45**, **46**, **17** a) and the formula $\Lambda_0 \operatorname{non} \in K$.

Hence to the notion of "set", which has been introduced in Definition II, corresponds in our model the notion "element of $\mathfrak{W}_{M,\mathfrak{G}} - K$".(20)

53. *If* $\mathfrak{G} \in \mathcal{G}$, $M \in R(\mathfrak{G})$ *and* $u, v \in \mathfrak{W}_{M,\mathfrak{G}}$, *then* $\{u, v\}$ *is the unique element of* $\mathfrak{W}_{M,\mathfrak{G}} - K$ *which satisfies the following condition*:

$(*)$ *for any* $t \in \mathfrak{W}_{M,\mathfrak{G}}$, $t \in x$ *if and only if* $t = u$ *or* $t = v$.

Proof. From **48** we conclude that $\{u, v\}$ belongs to the domain $\mathfrak{W}_{M,\mathfrak{G}}$. Further, by **26** we have $\{u, v\} \operatorname{non} \in K$, since for example $u \in \{u, v\} \cdot \mathfrak{M}$. Finally, an arbitrary t belongs to $\{u, v\}$ if and only if $t = u$ or $t = v$, and so we conclude that the pair $x = \{u, v\}$ satisfies the condition $(*)$ and belongs to the domain $\mathfrak{W}_{M,\mathfrak{G}} - K$. Now let

$$(1) \qquad y \in \mathfrak{M}_{M,\mathfrak{G}} - K$$

be an element satisfying condition $(*)$. Thus we have

$$(2) \qquad u, v \in y,$$

from which in view of **26** and (1) we obtain the formula $y \in \mathfrak{W}_{M,\mathfrak{G}} - K_0$. Now from **46** it follows that every $t \in y$ belongs to the domain $\mathfrak{W}_{M,\mathfrak{G}}$; hence by $(*)$ $y \subset \{u, v\}$. In view of (2) obtain $y = \{u, v\}$, which was to be proved.

54. *If* $\mathfrak{G} \in \mathcal{G}$, $M \in R(\mathfrak{G})$ *and* $u, v \in \mathfrak{W}_{M,\mathfrak{G}}$, *then* $\langle u, v\rangle = \big\{\{u, v\}\big\}$, $\big\{\{u\}\big\}$ *is the only element* x *of* $\mathfrak{W}_{M,\mathfrak{G}} - K$ *which satisfies the following condition*:

for an arbitrary $t \in \mathfrak{W}_{M,\mathfrak{G}}$ *we have* $t \in x$ *if and only if* $t = \{u, v\}$ *or* $t = \{u\}$.

For the proof we put in **53** $\{u, v\}$ instead of u, and $\{u\}$ instead of v and apply **48**.

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(20) From **52** and **45** one can easily conclude that in our model some theorems hold which are not provable either in $\mathfrak{S}$ or in the system of von Neumann. To such theorems belongs e.g. the statement: "*there exist infinite many primitive elements (i.e. individua that are not sets)*". From this it follows, among other things, that including this statement into the axiom system $\mathfrak{S}$ cannot lead to a contradiction provided that von Neumann's set theory is consistent. The basic idea of this proof, which consists in the proper choice of the set Λ_n, is due to A. Lindenbaum and was used for the first time for the proof of the independence of the axiom of choice in the paper cited in footnote (13).

As can be seen from **51**, **53** and **54**, to the notions (u, v) and $[u, v]$, which have been introduced in Definitions III and IV of the system $\mathfrak{S}$, correspond in our model the notions of ordered and unordered pairs, respectively.

Now we pass to the proof of the main theorem for the system $\mathfrak{S}$.

55. *If* $\mathfrak{G} \in \mathcal{G}$ *and* $M \in R(\mathfrak{G})$, *then we have* $\Lambda_0 \in \mathfrak{W}_{M,\mathfrak{G}}$.

Theorem **55**, which follows directly from **45** and **17** a) represents the main theorem for Axiom 1.

56. *Let* $\mathfrak{G} \in \mathcal{G}$, $M \in R(\mathfrak{G})$ *and* $x, y \in \mathfrak{W}_{M,\mathfrak{G}} - K$. *If the formula* $z \in x$ *and* $z \in y$ *are equivalent for every* $z \in \mathfrak{W}_{M,\mathfrak{G}}$, *then* $x = y$.

Proof. There are four possible cases:

(1) $$x, y \in \mathfrak{W}_{M,\mathfrak{G}} - K_0,$$

(2) $$x = \Lambda_0 = y,$$

(3) $$x \in \mathfrak{W}_{M,\mathfrak{G}} - K_0,$$

(4) $$x = \Lambda_0 \quad \text{and} \quad y \in \mathfrak{W}_{M,\mathfrak{G}} - K_0.$$

In the first case we have, by **46**, $x, y \in \mathfrak{W}_{M,\mathfrak{G}}$. It follows from the assumption that x and y have the same elements, and hence $x = y$. In the second case there is nothing to prove. Case (3) is impossible, because, by **46** and **25**, x contains an $M, \mathfrak{G}$-distinguished element whereas y does not. Similarly, (4) cannot hold either. This ends the proof. In view of **51**, **56** represents the main theorem for Axiom 2.

57. *If* $\mathfrak{G} \in \mathcal{G}$, $M \in R(\mathfrak{G})$, $x \in \mathfrak{W}_{M,\mathfrak{G}} - K$, $y \in \mathfrak{W}_{M,\mathfrak{G}}$ *and* y non $\in x$, *then there exists a* $z \in \mathfrak{W}_{M,\mathfrak{G}} - K$ *satisfying the following condition*:

(*) *for an arbitrary* $t \in \mathfrak{W}_{M,\mathfrak{G}}$, $t \in z$ *holds if and only if* $t \in x$ *or* $t = y$.

Proof. We distinguish two cases:

(1) $$x = \Lambda_0,$$

(2) $$x \in \mathfrak{W}_{M,\mathfrak{G}} - K_0.$$

In the first case we put $z = \{y\}$ and according to **48** we have $z \in \mathfrak{W}_{M,\mathfrak{G}}$. Hence $y \in z \cdot \mathfrak{M}$, which, by **26**, proves that

(3) $$z \in \mathfrak{W}_{M,\mathfrak{G}} - K_0.$$

If $t \in z$, then we have $t = y$. Now if t is an element of $\mathfrak{W}_{M,\mathfrak{G}}$ and either $t \in \Lambda_0$ or $t = y$, then, by **26**, the first condition can be omitted and there remains $t = y$, i.e. $t \in z$. Hence z satisfies condition (*), which, in view of **57**, proves the validity of case (1).

In case (2) we put $z = x + \{y\}$. Since $x \in \mathfrak{M} - K_0$ and $y \in \mathfrak{M}$, we have by **21** and **22**

$$x + \{y\} \in \mathfrak{M}. \tag{4}$$

From $y \in (x + \{y\}) \cdot \mathfrak{M}$ it follows by **26** that

$$x + \{y\} \text{ non} \in K_0. \tag{5}$$

From the assumption we infer further that there exist sets $A, B \in \mathfrak{M}$ such that

$$|\varphi, x| = x \text{ and } |\psi, y| = y \text{ for } \varphi \in \mathfrak{G}(A) \text{ and } \psi \in \mathfrak{G}(B). \tag{6}$$

We put $C = A + B$ and we have by **39** and **42**

$$C \in M \quad \text{and} \quad \mathfrak{G}(C) \subset \mathfrak{G}(A) \cdot \mathfrak{G}(B). \tag{7}$$

Now we have $0 \neq z \subset \mathfrak{M}$, because, by **46**, $x \subset \mathfrak{W}_{M,\mathfrak{G}} \subset \mathfrak{M}$ and $y \in \mathfrak{M}$. In view of **32** we conclude that

$$|\varphi, z| = \mathop{\mathsf{E}}_{|\varphi, t|} [t \in z] = \mathop{\mathsf{E}}_{|\varphi, t|} [t \in x] + \{|\varphi, y|\} = |\varphi, x| + \{|\varphi, y|\} \tag{8}$$

for $\varphi \in \mathfrak{G}_0$. From (6), (7) and (8) it follows that $|\varphi, z| = z$ for $\varphi \in \mathfrak{G}(C)$; hence

$$|\varphi, z| \in \mathfrak{K}_{\mathfrak{G}}(C). \tag{9}$$

Now let ξ be an ordinal number. By **7** and **9** we have

$$\Sigma_\xi(z) \cdot \mathfrak{M} = \Sigma_\xi(x) \cdot \mathfrak{M} + \Sigma_{\xi-1}(y) \cdot \mathfrak{M}.$$

According to the assumption, $\Sigma_\xi(x) \cdot \mathfrak{M}$ and $\Sigma_{\xi-1}(y) \cdot \mathfrak{M}$ are both contained in $\Sigma_\xi \mathfrak{K}_{\mathfrak{G}}(B)$. Hence we have

$$\Sigma_\xi(z) \cdot \mathfrak{M} \subset \sum_{B \in M} \mathfrak{K}_{\mathfrak{G}}(B). \tag{10}$$

From (4), (5), (9) and (10) it follows by **43** that $z \in \mathfrak{W}_{M,\mathfrak{G}} - K_0$. Since for any t the formula $t \in z$ holds if and only if $t \in x$ or $t = y$, we can see that z satisfies condition (*). Hence **57** holds also in case (2).

From **57**, **51**, **52** we obtain the main theorem for Axiom 3.

58. *If* $\mathfrak{G} \in \mathscr{G}$, $M \in R(\mathfrak{G})$ *and* $x \in \mathfrak{W}_{M,\mathfrak{G}} - K$, *then there exists an* $y \in \mathfrak{W}_{M,\mathfrak{G}} - K$ *such that a* $t \in \mathfrak{W}_{M,\mathfrak{G}}$ *belongs to* y *if and only if there exists a* $z \in \mathfrak{W}_{M,\mathfrak{G}}$ *for which* $t \in z \in y$.

Proof. We distinguish three cases:

$$x = A_0; \tag{1}$$

(2) $$x \subset K_0;$$

(3) $$x \neq \Lambda_0 \quad \text{and} \quad x \text{ non} \subset K_0.$$

In the first two cases we put $y = \Lambda_0$ and, by **45** and **17** a), we have $y \in \mathfrak{W}_{M,\mathfrak{G}} - K$. Since by **26** not only the formula $t \in y$ but also the formula $t \in z \in y \cdot \mathfrak{W}_{M,\mathfrak{G}} - K$ does not hold for any $t \in \mathfrak{W}_{M,\mathfrak{G}}$, we infer that these formulas are equivalent for every $t \in \mathfrak{W}_{M,\mathfrak{G}}$. Hence the assertion of **58** is true in both these cases.

In case (3) we put $y = \sum_{z \in x - K_0} z$. Since $x \in \mathfrak{W}_{M,\mathfrak{G}} - K_0$, by **46** every $z \in x$ belongs to the domain $\mathfrak{W}_{M,\mathfrak{G}}$. Hence if we have $t \in y$, then there exists a $z \in \mathfrak{W}_{M,\mathfrak{G}}$ such that $t \in z \in x$. Conversely, let t be an element of $\mathfrak{W}_{M,\mathfrak{G}}$ such that $t \in z \in x$ holds for a certain $z \in \mathfrak{W}_{M,\mathfrak{G}}$. By **26** we have $z \text{ non} \in K_0$, and hence $t \in z \in x - K_0$, i.e. $t \in y$. Therefore for every $t \in \mathfrak{W}_{M,\mathfrak{G}}$ the condition $t \in y$ is equivalent to the existence of a z for which $t \in z \in x$. Hence to prove **58** it suffices to show that $y \in \mathfrak{W}_{M,\mathfrak{G}} - K$. This can be shown as follows: since $\tau(x) > 0$, by (3) and (21) there exists a $z \in x \cdot \mathfrak{M}$. By **25** we have $z \neq 0$, and hence

(4) $$y \neq 0.$$

From $x \in \mathfrak{W}_{M,\mathfrak{G}} - K$ it follows by **46** that $x \in \mathfrak{W}_{M,\mathfrak{G}}$; hence $x - K_0 \subset \mathfrak{W}_{M,\mathfrak{G}} - K_0$, and we conclude by another application of **46** that

(5) $$y \subset \mathfrak{W}_{M,\mathfrak{G}} \subset \mathfrak{M}.$$

Let us denote by ζ an arbitrary ordinal number which exceeds all $\tau(z)$ with $z \in y$. Then by **18**, **21** and (4) we have $0 \neq y \subset K_\zeta$ and consequently by **22**

(6) $$y \in \mathfrak{M}.$$

From (4) and (5) we obtain $y \cdot \mathfrak{M} \neq 0$, which by **26** proves that

(7) $$y \text{ non} \in K_0.$$

Since $x \in \mathfrak{W}_{M,\mathfrak{G}}$, there exists a set A such that $|\varphi, x| = x$ for all $\varphi \in \mathfrak{G}(A)$. Since, as is easy to show, $|\varphi, K_0| = K_0$ for every function $\varphi \in \mathfrak{G}_0$, we infer with the aid of **32** that

(8) $$|\varphi, x - K_0| = x - K_0 \quad \text{for} \quad \varphi \in \mathfrak{G}(A).$$

Now let $\varphi \in \mathfrak{G}(A)$ and $t \in y$. Hence there exists a z such that $t \in z \in x - K_0$. Applying **34** and (8), we obtain $|\varphi, t| \in |\varphi, z| \in x - K_0$; hence $|\varphi, t| \in y$. Conversely, if we have $|\varphi, t| \in y$, then for a certain z we have $|\varphi, t| \in z \in x - K_0$, from which we conclude by **33**, **34** and (8) that

$t \in |\varphi^{-1}, z| \in x - K_0$, and hence $t \in y$. Thus the formulas $t \in y$ and $|\varphi, t| \in y$ are equivalent, which proves by **32** that $|\varphi, y| = y$ for $\varphi \in \mathfrak{G}(A)$, or

$$(9) \qquad y \in \mathfrak{K}_{\mathfrak{G}}(A)$$

Now let ξ be an ordinal number. From **11** and **10** it follows that $\Sigma_\xi(y) \cdot \mathfrak{M} \subset \Sigma_{\xi+1}(x - K_0) \cdot \mathfrak{M} \subset \Sigma_{\xi+1}(x) \cdot \mathfrak{M}$. Since according to our assumption $\Sigma_{\xi+1}(x) \cdot \mathfrak{M} \subset \sum_{B \in M} \mathfrak{K}_{\mathfrak{G}}(B)$, we obtain

$$(10) \qquad \Sigma_\xi(y) \cdot \mathfrak{M} \subset \sum_{B+M} \mathfrak{K}_{\mathfrak{G}}(B).$$

From (6), (7), (9) and (10) it follows that $y \in \mathfrak{W}_{M,\mathfrak{G}} - K_0$, which was to be proved.

In view of **52**, Theorem **58** represents the main theorem for Axiom 4.

59. *If* $\mathfrak{G} \in \mathscr{G}$, $M \in R(\mathfrak{G})$ *and* $x \in \mathfrak{W}_{M,\mathfrak{G}} - K$, *then there exists an* $y \in \mathfrak{W}_{M,\mathfrak{G}} - K$ *such that the following condition holds:*

(*) *an arbitrary* $z \in \mathfrak{W}_{M,\mathfrak{G}}$ *belongs to the set* y *if and only if* $z \in \mathfrak{W}_{M,\mathfrak{G}} - K$ *and the formula* $t \in z$ *for every* t *implies the formula* $t \in x$.

Proof. We put

$$(1) \qquad y = \mathfrak{P}(x) \cdot \mathfrak{W}_{M,\mathfrak{G}} + \{A_0\}.$$

First, we wish to show that (*) holds. Thus let $z \in \mathfrak{W}_{M,\mathfrak{G}}$, and $z \in y$. By (1) this is equivalent to the disjunction: either $z = A_0$ or $0 \neq z \subset x$ and $z \in \mathfrak{W}_{M,\mathfrak{G}}$. In the first case we have $z \in \mathfrak{W}_{M,\mathfrak{G}} - K$ by **45** and $z \cdot \mathfrak{W}_{M,\mathfrak{G}} = 0 \subset x$ by **26**. Thus the formulas $t \in z$ and $t \in \mathfrak{W}_{M,\mathfrak{G}}$ imply the formula $t \in x$. In the second case we reason as follows: if $\tau(x) > 0$, then by **21** $0 \neq z \subset y \subset \mathfrak{M}$ and consequently we have $z \cdot \mathfrak{M} = z \in \mathfrak{W}_{M,\mathfrak{G}} \subset \mathfrak{M}$, from which by **26** it follows that z cannot belong to K_0. Hence we have $z \in \mathfrak{W}_{M,\mathfrak{G}} - K$ and $\tau(z) > 0$. Next, by **46**, we have $z \cdot \mathfrak{W}_{M,\mathfrak{G}} = z \subset x$ and the formulas $t \in z$ and $t \in \mathfrak{W}_{M,\mathfrak{G}}$ imply $t \in x$. Thus if $\tau(x) = 0$, then we must have $x = A_0$, because x non $\in K$. We want to show the following:

$$(2) \qquad \text{if } 0 \neq z \subset A_0 \text{ and } z \in \mathfrak{M}, \text{ then } z = A_0.$$

Namely, if we had $z \in \mathfrak{M} - \{A_0\}$, then as can be seen from **15**, **16** and **17** a), we would have $\tau(z) > 0$, and by **21** $z \subset \mathfrak{M}$, which, in view of **26** is clearly impossible. From (2) it follows that if $\tau(x) = 0$ then we must have $z = A_0$, and thus this case has been reduced to the one considered previously.

Now let us assume that z belongs to the domain $\mathfrak{W}_{M,\mathfrak{G}} - K$ and satisfies the following condition: $t \in \mathfrak{W}_{M,\mathfrak{G}}$ and $t \in z$ imply that $t \in x$. Hence we have $z \cdot \mathfrak{W}_{M,\mathfrak{G}} \subset x$. If $z \neq \Lambda_0$, then $z \in \mathfrak{W}_{M,\mathfrak{G}} - K_0$; hence, by **46** and **25**, $0 \neq z = z \cdot \mathfrak{W}_{M,\mathfrak{G}} \subset x$, i.e. $z \in \mathfrak{P}(x) \cdot \mathfrak{W}_{M,\mathfrak{G}}$. Thus the assertion of (*) has been proved.

It remains to show that $y \in \mathfrak{W}_{M,\mathfrak{G}} - K$. This we show as follows: if $x = \Lambda_0$, then by (1) and (2) we have $y = \{\Lambda_0\}$; hence by **45**, **48**, **26**

$$(3) \qquad \{\Lambda_0\} = y \in \mathfrak{W}_{M,\mathfrak{G}} - K.$$

Thus we can assume that $x \neq \Lambda_0$, i.e. $\tau(x) > 0$. By **23** we have $\mathfrak{P}(x) \in \mathfrak{M}$, and hence by **26** $\xi = \tau(\mathfrak{P}(x)) > 0$. From **21** and **18** it follows that every element of y belongs to the set K_ξ. Hence, by **22**, we have

$$(4) \qquad y \in \mathfrak{M}.$$

Since $\Lambda_0 \in y$, we have by **26**

$$(5) \qquad y \,\text{non} \in K_0.$$

From $x \in \mathfrak{W}_{M,\mathfrak{G}}$ it follows that there exists a set $A \in M$ such that $|\varphi, x| = x$ for $\varphi \in \mathfrak{G}(A)$. If $z \in y$ then we have either $z = \Lambda_0$, and hence by **30** a) $|\varphi, z| = z$ for all $\varphi \in \mathfrak{G}_0$, or $0 \neq z \subset x$, $z \in \mathfrak{W}_{M,\mathfrak{G}}$ and $z \neq \Lambda_0$. In the latter case we obtain, in virtue of **32**, **47** and (*), $0 \neq |\varphi, z| \subset x$ and $|\varphi, z| \in \mathfrak{W}_{M,\mathfrak{G}}$ for $\varphi \in \mathfrak{G}(A)$, i.e. $|\varphi, z| \in y$. Similarly one shows with the aid of **33** that, conversely, $|\varphi, z| \in y$ always implies that $z \in y$ for $\varphi \in \mathfrak{G}(A)$. In view of **32**, (4), (5) and **21** this proves that $(\varphi, y) = y$ for $\varphi \in \mathfrak{G}(A)$,

$$(6) \qquad y \in \mathfrak{K}_{\mathfrak{G}}(A).$$

Now let ξ be an ordinal number and let $z \in \Sigma_\xi(y) \cdot \mathfrak{M} - y$. Since, as can easily be seen, $\Sigma_\xi(\{\Lambda_0\}) \cdot \mathfrak{M} = \{\Lambda_0\}$, on the basis of (1), **9**, **12** and the formula $x \in \mathfrak{W}_{M,\mathfrak{G}}$ we obtain

$$z \in \cdot\, \mathfrak{M}\Big(\Sigma_\xi(\mathfrak{P}(x) \cdot \mathfrak{W}_{M,\mathfrak{G}}) - \mathfrak{P}(x) \cdot \mathfrak{W}_{M,\mathfrak{G}}\Big) \subset \Sigma_\xi(x) \subset \sum_{b \in M} \mathfrak{K}_{\mathfrak{G}}(B).$$

Further we have

$$(7) \qquad \Sigma_\xi(y) \cdot \mathfrak{M} - y \subset \sum_{B \in M} \mathfrak{K}_{\mathfrak{G}}(B).$$

If $z \in y$, then by (1), (3) and **44** we have $z \in \sum_{B \in M} \mathfrak{K}_{\mathfrak{G}}(B)$, which, by (7), yields $\Sigma_\xi(y) \cdot \mathfrak{M} \subset \sum_{B \in M} \mathfrak{K}_{\mathfrak{G}}(B)$. Now in view of (4), (5) and (6) it follows that $y \in \mathfrak{W}_{M,\mathfrak{G}} - K$, which was to be proved.

From **59** and **52** we obtain the main theorem for Axiom 5.

60. *If $\mathfrak{G} \in \mathcal{G}$ and $M \in R(\mathfrak{G})$, then there exists an $x \in \mathfrak{W}_{M,\mathfrak{G}} - K$ such* *that* $\Lambda_0 \in x$ *and the formula* $y \in x$ *for every* $y \in \mathfrak{W}_{M,\mathfrak{G}}$ *implies the formula* $\{y\} \in x$.

Proof. We put $x_0 = \Lambda_0$, $x_{n+1} = \{x_n\}$ and denote by x the set of all x_n. It is evident that $\Lambda_0 \in x$ and the formula $y \in x$ implies $y \in x$. On the basis of **22** one proves by induction that $x_n \in K_n$ for $n = 0, 1, 2 \ldots$, from which. by **18** and **21**, we obtain $x \subset K_\omega$, $x \in K_{\omega+1}$, and consequently

$$x \in \mathfrak{M}. \tag{1}$$

Further, we have by (26)

$$x \text{ non} \in K_0. \tag{2}$$

From **30** a) we obtain $|\varphi, \Lambda_0| = \Lambda_0$ for $\varphi \in \mathfrak{G}_0$, from which it is easy to conclude, by a simple induction, that $|\varphi, x_n| = x_n$ for $n = 0, 1, 2, \ldots$ Hence, by **32**, we obtain for an arbitrary set $A \in M$.

$$x \in \mathfrak{R}_{\mathfrak{G}}(A). \tag{3}$$

On the basis of **26** we can easily see that $\Sigma_\xi(x) \cdot \mathfrak{M} = x$ for all ξ. Since every element x_n of x satisfies the condition $|\varphi, x_n| = x_n$ for $\varphi \in \mathfrak{G}_0$, we conclude that $\Sigma_\xi(x) \cdot \mathfrak{M} \subset \mathfrak{R}_{\mathfrak{G}}(A)$ for an arbitrary set $A \in M$. Hence we have

$$\Sigma_\xi(x) \cdot \mathfrak{M} \subset \sum_{B \in M} \mathfrak{R}_{\mathfrak{G}}(B). \tag{4}$$

From (1)–(4) it follows that $x \in \mathfrak{W}_{M,\mathfrak{G}} - K$, which was to be proved. In view of **53** and **52** we obtain from **60** the main theorem for Axiom 6.

61. *If* $\mathfrak{G} \in \mathcal{G}$ *and* $M \in R(\mathfrak{G})$, *then* $\Lambda_0 \cdot \mathfrak{W}_{M,\mathfrak{G}} = 0$.

This theorem, which results directly from **26**, represents the main theorem for Axiom 7.

62. *If* $\mathfrak{G} \in \mathcal{G}$ *and* $M \in R(\mathfrak{G})$ *and if* $\mathfrak{A}$ *is an* $M, \mathfrak{G}$*-distinguished domain and there is an* $x \in \mathfrak{W}_{M,\mathfrak{G}}$ *belonging to the domain* $\mathfrak{A}$, *then there exists an* $y \in \mathfrak{W}_{M,\mathfrak{G}}$ *such that* $y \in \mathfrak{A}$ *and the formulas* $t \in \mathfrak{A}$ *and* $t \in y$ *do not hold simultaneously for any* $t \in \mathfrak{W}_{M,\mathfrak{G}}$.

Proof. Since $\mathfrak{A} \neq \Lambda_0$, we have $0 \neq \mathfrak{A} \subset \mathfrak{W}_{M,\mathfrak{G}} \subset \mathfrak{M}$. Further, there are numbers ξ for which $\mathfrak{A} \cdot \mathfrak{W}_{M,\mathfrak{G}} \cdot K \neq 0$. Now if y is any element of $\mathfrak{A} \cdot \mathfrak{W}_{M,\mathfrak{G}} K_{\xi_0}$, then we have $y \in \mathfrak{A} \cdot \mathfrak{W}_{M,\mathfrak{G}}$. If $\xi_0 = 0$, then, by **26**, we have $\mathfrak{W}_{M,\mathfrak{G}} \cdot \mathfrak{A} \cdot y = 0$; if $\xi_0 > 0$, then we obviously have $\tau(y) = \xi_0$ and consequently, by **21**, $\tau(z) < \xi_0$ for $z \in y$. Thus no element of y can belong to the domain $\mathfrak{A} \cdot \mathfrak{W}_{M,\mathfrak{G}}$, i.e. also in this case we have $\mathfrak{A} \cdot \mathfrak{W}_{M,\mathfrak{G}} \cdot y = 0$, which was to be proved.

62 represents the assertion of the main theorem for Axiom 8.

63. *If* $\mathfrak{G} \in \mathcal{G}$, $M \in R(\mathfrak{G})$ *and* $x \in \mathfrak{W}_{M,\mathfrak{G}}$, *then* $\mathfrak{A} = x$ *is an* M, $\mathfrak{G}$*-distinguished domain which satisfies the following condition*:

(*) *for every* $t \in \mathfrak{W}_{M,\mathfrak{G}}$ *we have* $t \in \mathfrak{A}$ *if and only if* $t = x$.

Proof. By **48** and **26** we have $\{x\} \in \mathfrak{W}_{M,\mathfrak{G}} - K_0$, from which by **49** it follows that $\mathfrak{A}$ is an M, $\mathfrak{G}$-distinguished domain. It is evident that condition (*) is satisfied.

From **63**, **51** and **52** we obtain the theorem for Axiom 9.

64. *If* $\mathfrak{G} \in \mathcal{G}$ *and* $M \in R(\mathfrak{G})$, *and if* $\mathfrak{A}$ *is an* M, $\mathfrak{G}$*-distinguished domain then there is an* M, $\mathfrak{G}$*-distinguished domain* $\mathfrak{B}$ *such that the conditions* $t \in \mathfrak{B}$ *and* t non $\in \mathfrak{A}$ *are equivalent for every* $t \in \mathfrak{W}_{M,\mathfrak{G}}$.

Proof. If $\mathfrak{A} = \mathfrak{W}_{M,\mathfrak{G}}$, then we put $\mathfrak{B} = \Lambda_0$ and easily conclude from **26** that $\mathfrak{B}$ has all the properties required in **64**. If $\mathfrak{A} \neq \mathfrak{W}_{M,\mathfrak{G}}$, then we put $\mathfrak{B} = \mathfrak{W}_{M,\mathfrak{G}} - \mathfrak{A}$. We then have $0 \neq \mathfrak{B} \subset \mathfrak{W}_{M,\mathfrak{G}}$, and the conditions $t \in \mathfrak{B}$ and t non $\in \mathfrak{A}$ are clearly equivalent for every $t \in \mathfrak{W}_{M,\mathfrak{G}}$ and every $\varphi \in \mathfrak{G}(A)$. We conclude therefore that the conditions $x \in \mathfrak{B}$ and $|\varphi, x| \in \mathfrak{B}$ are equivalent for every $x \in \mathfrak{W}_{M,\mathfrak{G}}$ and every $\varphi \in \mathfrak{G}(A)$. Hence the domain $\mathfrak{B}$ is M, $\mathfrak{G}$-distinguished, which was to be proved.

64 represents the assertion of the main theorem for Axiom 10.

65. *If* $\mathfrak{G} \in \mathcal{G}$, $M \in R(\mathfrak{G})$ *and if* $\mathfrak{A}$, $\mathfrak{B}$ *are two* M, $\mathfrak{G}$*-distinguished domains such that* $\mathfrak{A} \cdot \mathfrak{B} \cdot \mathfrak{W}_{M,\mathfrak{G}} \neq 0$, *then* $\mathfrak{A} \cdot \mathfrak{B}$ *is also an* M, $\mathfrak{G}$*-distinguished domain.*

Proof. By **26** we have $\mathfrak{A} \neq \Lambda_0 \neq B$; hence, by **43**, we conclude that $0 \neq \mathfrak{A} \subset \mathfrak{W}_{M,\mathfrak{G}}$ and $0 \neq \mathfrak{B} \subset \mathfrak{W}_{M,\mathfrak{G}}$, which implies

$$0 \neq \mathfrak{A} \cdot \mathfrak{B} \subset \mathfrak{W}_{M,\mathfrak{G}}. \tag{1}$$

By assumption there are sets $A, B \in M$ such that for an arbitrary $x \in \mathfrak{W}_{M,\mathfrak{G}}$ the conditions

$$x \in \mathfrak{A} \quad \text{and} \quad |\varphi, x| \in \mathfrak{A} \quad \text{for} \quad \varphi \in \mathfrak{G}(A) \tag{2}$$

resp. the conditions

$$x \in \mathfrak{B} \quad \text{and} \quad |\varphi, x| \in \mathfrak{B} \quad \text{for} \quad \varphi \in \mathfrak{G}(B) \tag{3}$$

are equivalent. We put $C = A + B$ and we have $C \in M$ by **42**. From (2), (3) and the inclusion $\mathfrak{G}(C) \subset \mathfrak{G}(A) \cdot \mathfrak{G}(B)$ it follows that the conditions $x \in \mathfrak{A} \cdot \mathfrak{B}$ and $|\varphi, x| \in \mathfrak{A} \cdot \mathfrak{B}$ are equivalent for every $x \in \mathfrak{W}_{M,\mathfrak{G}}$ and every $\varphi \in \mathfrak{G}(C)$. In view of (1) we conclude that $\mathfrak{A} \cdot \mathfrak{B}$ is an M, $\mathfrak{G}$-distinguished domain, which was to be proved.

66. *If* $\mathfrak{G} \in \mathcal{G}$, $M \in R(\mathfrak{G})$ *and* $x \in \mathfrak{W}_{M,\mathfrak{G}} - K_0$ *and if* $\mathfrak{A}$ *is an* M, $\mathfrak{G}$*-distinguished domain such that* $\mathfrak{A} \cdot x \neq 0$, *then we have* $\mathfrak{A} \cdot x \in \mathfrak{W}_{M,\mathfrak{G}} - K$.

Proof. By **46** we have $x \subset \mathfrak{W}_{M,\mathfrak{G}}$. Further we have $0 \neq x \cdot \mathfrak{A} \subset \mathfrak{M}$ and by **22**, since $x \cdot \mathfrak{A}$ is a set, $x \cdot \mathfrak{A} \in \mathfrak{M}$. From the assumption $\mathfrak{A} \cdot x \neq 0$ we infer by **26** that $x \cdot \mathfrak{A}$ non $\in K_0$. If we now assume x to be a domain, we conclude by **65** that there exists a set $A \in M$ such that the conditions $t \in \mathfrak{A} \cdot x$ and $|\varphi, t| \in \mathfrak{A} \cdot x$ are equivalent. Hence, by **32** we obtain $|\varphi, \mathfrak{A} \cdot x| = \mathfrak{A} \cdot x$ for $\varphi \in \mathfrak{G}(A)$, i.e.

$$\mathfrak{A} \cdot x \in \mathfrak{K}_{\mathfrak{G}}(A). \tag{1}$$

Now if ξ is an ordinal number, then in view of the assumption and **10** we have

$$\Sigma_\xi(\mathfrak{A} \cdot x) \cdot \mathfrak{M} \subset \Sigma_\xi(x) \cdot \mathfrak{M} \subset \sum_{A \in M} \mathfrak{K}_{\mathfrak{G}}(A). \tag{2}$$

From (1) and (2) it follows that $\mathfrak{A} \cdot x \in \mathfrak{W}_{M,\mathfrak{G}}$; hence we have $\mathfrak{A} \cdot x \in \mathfrak{W}_{M,\mathfrak{G}} - K$, which was to be proved.

67. *If* $\mathfrak{G} \in \mathcal{G}$ *and* $M \in R(\mathfrak{G})$, *and if* $\mathfrak{A}, \mathfrak{B}$ *are two* M, $\mathfrak{G}$*-distinguished domains, then there exists an* M, $\mathfrak{G}$*-distinguished domain* $\mathfrak{C}$ *such that for an arbitrary* $t \in \mathfrak{W}_{M,\mathfrak{G}}$ *the formula* $t \in \mathfrak{C}$ *holds if and only if* $t \in \mathfrak{A}$ *and* $t \in \mathfrak{B}$.

Proof. We put $C = \Lambda_0$ if $\mathfrak{W}_{M,\mathfrak{G}} \cdot \mathfrak{A} \cdot \mathfrak{B} = 0$ and $\mathfrak{C} = \mathfrak{A} \cdot \mathfrak{B}$ if $\mathfrak{W}_{M,\mathfrak{G}} \cdot \mathfrak{A} \cdot \mathfrak{B} \neq 0$. By **65**, in both cases $\mathfrak{C}$ is M, $\mathfrak{G}$-distinguished, and one easily concludes from **26** that this domain satisfies all the conditions of **67**.

67 represents the main theorem for Axiom 11.

68. *Let* $\mathfrak{G} \in \mathcal{G}$ *and* $M \in R(\mathfrak{G})$. *There exists an* M, $\mathfrak{G}$*-distinguished domain* $\mathfrak{A}$ *with the following property*:

(*) *for an arbitrary* $t \in \mathfrak{W}_{M,\mathfrak{G}}$ *we have* $t \in \mathfrak{A}$ *if and only if* $t \in \mathfrak{W}_{M,\mathfrak{G}} - K_0$, *and from* $x \in t$ *and* $y \in t$ *for any* $x, y \in \mathfrak{W}_{M,\mathfrak{G}}$ *it always follows that* $x = y$.

Proof. We denote by $\mathfrak{A}$ the domain of sets $\{x\}$ where $x \in \mathfrak{W}_{M,\mathfrak{G}}$. From **46** we easily conclude that $\mathfrak{A}$ satisfies condition (*). To prove that $\mathfrak{A}$ is an M, $\mathfrak{G}$-distinguished domain, we denote by A an arbitrary set belonging to the ring M. For $\varphi \in \mathfrak{G}(A)$ and $x \in \mathfrak{W}_{M,\mathfrak{G}}$ we have, by **32** and **47**, $|\varphi, \{x\}| = \{|\varphi, x|\}$ and $|\varphi, x| \in \mathfrak{W}_{M,\mathfrak{G}}$. Hence if $y \in \mathfrak{A}$ and $\varphi \in \mathfrak{G}(A)$, then $|\varphi, y|$ also belongs to the domain $\mathfrak{A}$. We put $|\varphi, y|$ for y and φ^{-1} for φ, and we conclude in view of **33** that $|\varphi, y| \in \mathfrak{A}$ implies the formula $y \in \mathfrak{A}$. Hence the formulas $y \in \mathfrak{A}$ and $\varphi \in \mathfrak{G}(A)$ are equivalent. Since according to **45** and **48** we have $0 \neq \mathfrak{A} \subset \mathfrak{W}_{M,\mathfrak{G}}$, we conclude that $\mathfrak{A}$ is an M, $\mathfrak{G}$-distinguished domain.

From **68**, **51** and **52** we obtain the main theorem for Axiom 12.

69. *If* $\mathfrak{G} \in \mathscr{G}$ *and* $M \in R(\mathfrak{G})$, *then there exists an* $M, \mathfrak{G}$*-distinguished domain* $\mathfrak{A}$ *satisfying the following condition*:

(*) *if* $t \in \mathfrak{W}_{M,\mathfrak{G}}$, *then* $t \in \mathfrak{A}$ *holds if and only if there are elements* $u, v \in \mathfrak{W}_{M,\mathfrak{G}}$ *such that* $u \in v$ *and* $t = \langle u, v \rangle$.

Proof. We denote by $\mathfrak{A}$ the domain of all pairs $\langle u, v \rangle$ where $u, v \in \mathfrak{W}_{M,\mathfrak{G}}$ and $u \in v$. It is clear that condition (*) is satisfied. Now let A be an arbitrary set from M, $\varphi \in \mathfrak{G}(A)$, $t \in \mathfrak{A}$. For some $u, v \in \mathfrak{W}_{M,\mathfrak{G}}$ we have $t = \langle u, v \rangle$ and $u \in v$. In view of **32**, **34** and **47** we obtain $|\varphi, t| = \langle |\varphi, u|, |\varphi, v| \rangle$, $|\varphi, u| \in |\varphi, v|$ and $|\varphi, u|, |\varphi, v| \in \mathfrak{W}_{M,\mathfrak{G}}$, which proves that $|\varphi, t| \in \mathfrak{A}$. Conversely, if $|\varphi, t| \in \mathfrak{A}$, then for some $u, v \in \mathfrak{W}_{M,\mathfrak{G}}$ we have $|\varphi, t| = \langle u, v \rangle$ and $u \in v$.

By **32**, **33**, **34** and **47** we conclude that $t = \langle |\varphi^{-1}, u|, |\varphi^{-1}, v| \rangle$, $|\varphi^{-1}, u| \in |\varphi^{-1}, v|$ and $|\varphi^{-1}, u|, |\varphi^{-1}, v| \in \mathfrak{W}_{M,\mathfrak{G}}$; hence $t \in \mathfrak{A}$. Thus the formulas $t \in \mathfrak{A}$ and $|\varphi, t| \in \mathfrak{A}$ are equivalent for $\varphi \in \mathfrak{G}(A)$ and $t \in \mathfrak{W}_{M,\mathfrak{G}}$. By **45** and **48** we have here $0 \neq A \subset \mathfrak{W}_{M,\mathfrak{G}}$. Hence the domain is M, $\mathfrak{G}$-distinguished, which was to be proved.

From **69**, **51** and **54** we obtain the main theorem for Axiom 13.

70. *If* $\mathfrak{G} \in \mathscr{G}$ *and* $M \in R(\mathfrak{G})$, *then for each* $M, \mathfrak{G}$*-distinguished domain* $\mathfrak{A}$ *there exists an* $M, \mathfrak{G}$*-distiguished domain* $\mathfrak{B}$ *with the following property*:

(*) *if* $t \in \mathfrak{W}_{M,\mathfrak{G}}$, *then* $t \in \mathfrak{B}$ *holds if and only if there are elements* $u, v \in \mathfrak{W}_{M,\mathfrak{G}}$ *such that* $u \in \mathfrak{A}$ *and* $t = \langle u, v \rangle$.

Proof. If $\mathfrak{A} = \Lambda_0$, we put $\mathfrak{B} = \Lambda_0$ and easily conclude from **26** that the M, $\mathfrak{G}$-distinguished domain $\mathfrak{B}$ satisfies condition (*). Now if $\mathfrak{A} \neq \Lambda_0$, then we denote by $\mathfrak{B}$ the domain of pairs $\langle u, v \rangle$ where $u \in \mathfrak{A}$ and $v \in \mathfrak{W}_{M,\mathfrak{G}}$. By **43** and **48** we have

$$\mathfrak{B} \subset \mathfrak{W}_{M,\mathfrak{G}}. \tag{1}$$

Since neither $\mathfrak{A}$ nor $\mathfrak{B}$ is empty, we have further

$$0 \neq \mathfrak{B}. \tag{2}$$

By assumption there is a set $A \in M$ such that the conditions $u \in \mathfrak{A}$ and $(\varphi, u) \in \mathfrak{A}$ are equivalent for every $u \in \mathfrak{W}_{M,\mathfrak{G}}$ and every $\varphi \in \mathfrak{G}(A)$. Now let $\varphi \in \mathfrak{G}(A)$ and $t \in \mathfrak{B}$; for some $u \in \mathfrak{A}$ and $v \in \mathfrak{W}_{M,\mathfrak{G}}$ we have $t = \langle u, v \rangle$, from which by **32** and **47** we obtain $|\varphi, t| = \langle |\varphi, u|, |\varphi, v| \rangle$ and $|\varphi, v| \in \mathfrak{W}_{M,\mathfrak{G}}$. Since $|\varphi, u| \in \mathfrak{A}$, it follows that $|\varphi, t| \in \mathfrak{B}$. Quite analogously, it can be shown in view of **32**, **34** and **47** that $|\varphi, t| \in \mathfrak{B}$ implies the formula

$t \in \mathfrak{B}$. Therefore these formulas are equivalent, which proves by (1) and (2) that $\mathfrak{B}$ is an $M, \mathfrak{G}$-distinguished domain. Since it clearly satisfies condition $(*)$, we have established the validity of **70**.

From **70**, **51** and **54** we obtain the main theorem for Axiom 14.

71. Let $\mathfrak{A}$ be a domain. If $\mathfrak{A}$ does not contain an ordered pair, we put $\mathfrak{D}(\mathfrak{A}) = \Lambda_0$; otherwise we denote by $\mathfrak{D}(\mathfrak{A})$ the domain of those x for which there exists an y such that $\langle x, y\rangle \in \mathfrak{A}$.

72. *If* $\mathfrak{G} \in \mathcal{G}$, $M \in R(\mathfrak{G})$ *and if* $\mathfrak{A}$ *is an* $M, \mathfrak{G}$*-distinguished domain, then* $\mathfrak{D}(\mathfrak{A})$ *is an* $M, \mathfrak{G}$*-distinguished domain and*:

$(*)$ *a* $t \in \mathfrak{W}_{M,\mathfrak{G}}$ *belongs to* $\mathfrak{D}(\mathfrak{A})$ *if and only if there is a* $u \in \mathfrak{W}_{M,\mathfrak{G}}$ *with* $\langle t, u\rangle \in \mathfrak{A}$.

Proof. If $\mathfrak{A}$ does not contain an ordered pair, then by **26** and **71** we have neither $\langle t, u\rangle \in \mathfrak{A}$ nor $t \in \mathfrak{D}(\mathfrak{A})$ for $u, t \in \mathfrak{W}_{M,\mathfrak{G}}$. Condition $(*)$ also holds and, by **71** and **43**, $\mathfrak{D}(\mathfrak{A})$ is an $M, \mathfrak{G}$-distinguished domain.

Now we assume that $\mathfrak{A}$ contains an ordered pair. Hence we have $A \neq \Lambda_0$, i.e. by **43**

$$0 \neq \mathfrak{A} \subset \mathfrak{W}_{M,\mathfrak{G}}. \tag{1}$$

If $t \in \mathfrak{D}(\mathfrak{A})$ and $t \in \mathfrak{W}_{M,\mathfrak{G}}$, then by **71** there is a u such that $\langle t, u\rangle \in \mathfrak{A}$. In view of (1) we infer by a double application of **46** and **28** that $u \in \mathfrak{W}_{M,\mathfrak{G}}$. Conversely, if we have $\langle t, u\rangle \in \mathfrak{A}$ where $t, u \in \mathfrak{W}_{M,\mathfrak{G}}$, then, by **71**, $t \in \mathfrak{D}(\mathfrak{A})$. The domain $\mathfrak{D}(\mathfrak{A})$ satisfies condition $(*)$. It remains to show that this domain is $M, \mathfrak{G}$-distinguished. To this aim let us observe that by assumption there exists a set $A \in M$ such that the conditions $x \in \mathfrak{A}$ and $|\varphi, x| \in \mathfrak{A}$ are equivalent for $x \in \mathfrak{W}_{M,\mathfrak{G}}$ and $\varphi \in \mathfrak{G}(A)$. Let $t \in \mathfrak{D}(\mathfrak{A})$ and $\varphi \in \mathfrak{G}(\mathfrak{A})$. Then we have for some $u \in \mathfrak{W}_{M,\mathfrak{G}}$ $\langle t, u\rangle \in \mathfrak{A}$, from which in view of (1) and **46** we obtain $|\varphi, \langle t, u\rangle| \in \mathfrak{A}$. By an application of **32** we get from this $\langle|\varphi, t|, |\varphi, u|\rangle \in \mathfrak{A}$, which proves by **71** that $|\varphi, t| \in \mathfrak{D}(\mathfrak{A})$. Conversely, if $|\varphi, t| \in \mathfrak{D}(\mathfrak{A})$, then according to $(*)$ we have $\langle|\varphi, t|, u\rangle \in \mathfrak{A}$ for a certain $u \in \mathfrak{W}_{M,\mathfrak{G}}$. Since $\varphi^{-1} \in \mathfrak{G}(A)$, by **32** and **33** it follows that $|\varphi^{-1}, \langle|\varphi, t|, u\rangle| = \langle t, |\varphi^{-1}, u|\rangle \in \mathfrak{A}$, and consequently $t \in \mathfrak{D}(\mathfrak{A})$. Hence the formulas $t \in \mathfrak{D}(\mathfrak{A})$ and $|\varphi, t| \in \mathfrak{D}(\mathfrak{A})$ are equivalent for $t \in \mathfrak{W}_{M,\mathfrak{G}}$ and $\varphi \in \mathfrak{G}(A)$, which proves by (1) that $\mathfrak{D}(\mathfrak{A})$ is an $M, \mathfrak{G}$-distinguished domain. This ends the proof of **72**.

From **72** and **54** we obtain the assertion of the main theorem for Axiom 15.

73. *If* $\mathfrak{G} \in \mathcal{G}$ *and* $M \in R(\mathfrak{G})$, *then for each* $M, \mathfrak{G}$*-distinguished domain* $\mathfrak{A}$ *there exists an* $M, \mathfrak{G}$*-distinguished domain* $\mathfrak{B}$ *with the following property*:

(*) *if* $t \in \mathfrak{W}_{M,\mathfrak{G}}$, *then* $t \in \mathfrak{B}$ *holds if and only if there exists an element* $u, v \in \mathfrak{W}_{M,\mathfrak{G}}$ *such that* $t = \langle u, v\rangle$ *and* $\langle u, u\rangle \in \mathfrak{A}$.

Proof. If $\mathfrak{A}$ does not contain an ordered pair, we put $\mathfrak{B} = \Lambda_0$. This domain is $M, \mathfrak{G}$-distinguished and satisfies condition (*), since by assumption $\langle u, v\rangle \in \mathfrak{A}$ does not hold for any u, v and, in view of **26**, $\mathfrak{B}$ does not contain any element of $\mathfrak{W}_{M,\mathfrak{G}}$. Hence we may suppose that $\mathfrak{A}$ contains at least one ordered pair, and thus by **43**, we have

$$(1) \qquad 0 \neq \mathfrak{A} \subset \mathfrak{W}_{M,\mathfrak{G}}.$$

We denote by $\mathfrak{B}$ the domain of pairs for which $\langle u, v\rangle \in \mathfrak{A}$. Hence we have

$$(2) \qquad \mathfrak{B} \neq 0.$$

In view of **26**, **46** and **48** we conclude from (1) that

$$(3) \qquad \mathfrak{B} \subset \mathfrak{W}_{M,\mathfrak{G}}.$$

From the assumption of our theorem we obtain the existence of a set $A \in M$ such that the formulas $x \in \mathfrak{A}$ and $|\varphi, x| \in \mathfrak{A}$ are equivalent for $x \in \mathfrak{W}_{M,\mathfrak{G}}$ and $\varphi \in \mathfrak{G}(A)$. Now let $t \in \mathfrak{B}$ and $\varphi \in \mathfrak{G}(A)$. For some $u, v \in \mathfrak{A}$ we have $t = \langle u, v\rangle$ and $\langle v, u\rangle \in \mathfrak{A}$; moreover by (1), **26** and **46** we have $u, v \in \mathfrak{W}_{M,\mathfrak{G}}$, which, by **32** implies $\langle |\varphi, v|, |\varphi, u|\rangle \in \mathfrak{A}$. Since, according to **32**, $|\varphi, t| = \langle |\varphi, u|, |\varphi, v|\rangle$, we obtain $|\varphi, t| \in \mathfrak{B}$. Quite similarly we can show that $|\varphi, t| \in \mathfrak{B}$ implies the formula $t \in \mathfrak{B}$. Therefore the two formulas are equivalent, which proves by (2) and (3) that the domain $\mathfrak{B}$ is $M, \mathfrak{G}$-distinguished.

Now let $t \in \mathfrak{W}_{M,\mathfrak{G}}$. If $t \in \mathfrak{B}$, then there exist u and v such that $t = \langle u, v\rangle$ and $\langle v, u\rangle \in \mathfrak{A}$. Moreover, $\langle u, v\rangle \in K_0$ cannot hold, because the elements of K_0 are infinite sets. Hence from **46** we obtain $\{u, v\} \in \mathfrak{W}_{M,\mathfrak{G}}$, and another application of the same conclusion shows that $u, v \in \mathfrak{W}_{M,\mathfrak{G}}$. Conversely, if we have $u, v \in \mathfrak{W}_{M,\mathfrak{G}}$ and $\langle v, u\rangle \in \mathfrak{A}$, then $\langle v, u\rangle \in \mathfrak{B}$ holds, according to the definition of $\mathfrak{B}$. This proves that $\mathfrak{B}$ satisfies the condition (*), which was to be proved.

From **73**, **51** and **54** we obtain the assertion of the main theorem for Axiom 16.

74. *If* $\mathfrak{G} \in \mathscr{G}$ *and* $M \in R(\mathfrak{G})$, *then for each* $M, \mathfrak{G}$*-distinguished domain* $\mathfrak{A}$ *there exists an* $M, \mathfrak{G}$*-distinguished domain with the following property*:

(*) *if* $t \in \mathfrak{W}_{M,\mathfrak{G}}$, *then* $t \in \mathfrak{B}$ *holds if and only if there exist elements* $u, v, w \in \mathfrak{W}_{M,\mathfrak{G}}$ *such that* $t = \langle\langle u, v\rangle, w\rangle$ *and* $\langle u, \langle v, w\rangle\rangle \in \mathfrak{A}$.

Proof. We put $\mathfrak{B} = \Lambda_0$ if $\mathfrak{A}$ does not contain any set of the form $\langle x, \langle y, z\rangle\rangle$. Otherwise we denote by $\mathfrak{B}$ the domain of triples $\langle\langle x, y\rangle, z\rangle$ where $\langle x, \langle y, z\rangle\rangle \in \mathfrak{A}$. The remaining part of the proof proceeds exactly as in the proof of **73**.

From **74**, **51** and **54** we obtain the assertion of the main theorem for Axiom 17.

75. *Let $\mathfrak{G} \in \mathscr{G}$, $M \in R(\mathfrak{G})$ and $x \in \mathfrak{W}_{M,\mathfrak{G}} - K$. Further, let $\mathfrak{A}$ be an M, $\mathfrak{G}$-distinguished domain with the following property: for any $u, v, w \in \mathfrak{W}_{M,\mathfrak{G}}$ from $\langle u, v\rangle, \langle u, w\rangle \in \mathfrak{A}$ follows the equation $v = w$. Under these assumptions there exists an $y \in \mathfrak{M}_{M,\mathfrak{G}} - K$ such that for an arbitrary $z \in \mathfrak{W}_{M,\mathfrak{G}}$ the condition $z \in y$ holds if and only if there exists a $t \in \mathfrak{W}_{M,\mathfrak{G}}$ such that $t \in x$ and $\langle t, z\rangle \in \mathfrak{A}$.*

Proof. If there is no $t \in \mathfrak{W}_{M,\mathfrak{G}} \cdot x$ or no $z \in \mathfrak{W}_{M,\mathfrak{G}}$ such that $\langle t, z\rangle \in \mathfrak{A}$, it suffices to put $y = \Lambda_0$. By **45** y is an M, $\mathfrak{G}$-distinguished element which clearly satisfies all the requirements of **75**.

Hence we assume that there is a $t_0 \in \mathfrak{W}_{M,\mathfrak{G}} \cdot x$ and a $z_0 \in \mathfrak{W}_{M,\mathfrak{G}}$ such that $\langle t_0, z_0\rangle \in \mathfrak{A}$. From this it follows by **26**, **48** and **43** that

$$0 \neq \mathfrak{A} \subset \mathfrak{W}_{M,\mathfrak{G}}, \tag{1}$$

$$x \in \mathfrak{W}_{M,\mathfrak{G}} - K_0. \tag{2}$$

We denote by x_1 the set of those $t \in x$ for which there exists a $z \in \mathfrak{W}_{M,\mathfrak{G}}$ such that $\langle t, z\rangle \in \mathfrak{A}$. This set is non-empty since $t_0 \in x_1$ and, taking into account the assumptions of our theorem, we conclude from (2), **46** and **48** that for $t \in x_1$ there exists exactly one $z \in \mathfrak{W}_{M,\mathfrak{G}}$ such that $\langle t, z\rangle \in \mathfrak{A}$. We denote this unique z by t^* and put $y = \mathop{\mathrm{E}}_{t^*}[t \in x_1]$. If $z \in y$, then there is a $t \in x_1$ such that $z = t^*$, from which, in view of (2) and **46**, we obtain $t \in \mathfrak{W}_{M,\mathfrak{G}}$ and $\langle t, z\rangle \in \mathfrak{A}$. Conversely, if $z, t \in \mathfrak{W}_{M,\mathfrak{G}}$ and $t \in x$ with $\langle t, z\rangle \in \mathfrak{A}$, then we have $t \in x_1$ and $t^* = z$, and hence $z \in y$. Thus the formula $z \in y$ holds for $z \in \mathfrak{W}_{M,\mathfrak{G}}$ if and only if there exists a $t \in \mathfrak{W}_{M,\mathfrak{G}}$ such that $t \in x$ and $\langle t, z\rangle \in \mathfrak{A}$. It remains to show that $y \in \mathfrak{W}_{M,\mathfrak{G}} - K$.

For every $t \in x_1$ we have $t^* \in \mathfrak{W}_{M,\mathfrak{G}} \subset \mathfrak{M}$. If ξ is an arbitrary ordinal number which exceeds all $\tau(t^*)$ with $t \in x_1$, then from **18** and **20** it follows that $y \subset K_\xi$. Since $z_0 \in y$, we have $0 \neq y$, and on the basis of **22** we conclude that

$$y \in \mathfrak{M}. \tag{3}$$

Since $z_0 \in y \cdot \mathfrak{M}$, we obtain from **26**

$$y\,\text{non} \in K_0. \tag{4}$$

Clearly we have $x_1 \subset x \cdot \mathfrak{D}(\mathfrak{A})$. Now if $u \in x \cdot \mathfrak{D}(\mathfrak{A})$, then we have $\langle u, v\rangle \in \mathfrak{A}$ for a certain v. By (1) it follows hence that $\langle u, v\rangle \in \mathfrak{W}_{M,\mathfrak{G}}$, where by **26** $\langle u, v\rangle$ does not belong to K_0, since in view of (2) and **48**, $\{u, v\} \in \langle u, v\rangle \cdot \mathfrak{W}_{M,\mathfrak{G}}$. By **46** we have $\langle u, v\rangle \subset \mathfrak{W}_{M,\mathfrak{G}}$, whence $\{u, v\} \in \mathfrak{W}_{M,\mathfrak{G}}$. We again have $\{u, v\}\text{non} \in K_0$, since $u \in \mathfrak{W}_{M,\mathfrak{G}}$ by (2) and **46**. Consequently we have $\{u, v\} \subset \mathfrak{W}_{M,\mathfrak{G}}$, i.e. $v \in \mathfrak{W}_{M,\mathfrak{G}}$. Since $\langle u, v\rangle \in \mathfrak{A}$, it follows that $u \in x_1$. Consequently we have $x \cdot \mathfrak{D}(\mathfrak{A}) \subset x_1$ and thus

$$x_1 = x \cdot \mathfrak{D}(\mathfrak{A}). \tag{5}$$

From (5), (2), **66** and **72** it follows that $x_1 \in \mathfrak{W}_{M,\mathfrak{G}}$, and since $t_0 \in x_1 \cdot \mathfrak{M}$, we have, according to **26**,

$$x_1 \in \mathfrak{W}_{M,\mathfrak{G}} - K_0. \tag{6}$$

In view of **46** and **32** we conclude from (6) that there exists a set $A_1 \in M$ such that the formulas

$$t \in x_1 \text{ and } |\varphi, t| \in x_1 \text{ for } t \in \mathfrak{W}_{M,\mathfrak{G}} \text{ and } \varphi \in \mathfrak{G}(A_1) \tag{7}$$

are equivalent. Further, since $\mathfrak{A}$ is an $M, \mathfrak{G}$-distiguished domain and $\mathfrak{A} \neq \Lambda_0$ there exists a set $A_2 \in M$ such that the formulas

$$t \in \mathfrak{A} \text{ and } |\varphi, t| \in \mathfrak{A} \text{ for } t \in \mathfrak{W}_{M,\mathfrak{G}} \text{ and } \varphi \in \mathfrak{G}(A_2) \tag{8}$$

are equivalent.

Now we put $A = A_1 + A_2$ and according to **42** we have

$$A \in M. \tag{9}$$

Next let $\varphi \in \mathfrak{G}(A) \subset \mathfrak{G}(A_1) \cdot \mathfrak{G}(A_2)$ and $z \in y$. Then there is a $t \in x_1$ such that $\langle t, z\rangle \in \mathfrak{A}$. In view of (7), (8) and **32** it follows that $|\varphi, t| \in x_1$ and $\langle |\varphi, t|, |\varphi, z|\rangle \in \mathfrak{A}$, and thus $|\varphi, z| \in y$. Conversely, if $|\varphi, z| \in y$, then we have $\langle t, |\varphi, z|\rangle \in \mathfrak{A}$ for a $t \in x_1$, from which on the basis of (7), (8), **32** and **33** we conclude that $\langle |\varphi^{-1}, t|, z\rangle \in \mathfrak{A}$ and $|\varphi^{-1}, t| \in x_1$. Consequently we have $z \in y$. Hence for every $\varphi \in \mathfrak{G}(A)$ this condition is equivalent to the condition $|\varphi, z| \in y$. In view of (4), **32** and **40** we obtain hence the formula

$$y \in \mathfrak{K}_{\mathfrak{G}}(A). \tag{10}$$

Now let ξ be an ordinal number. We clearly have $y \subset \Sigma_2(A)$. By (1), **10** and **50** it follows that $y \cdot \mathfrak{M} \subset \Sigma_2(\mathfrak{W}_{M,\mathfrak{G}}) \cdot \mathfrak{M} \subset \mathfrak{W}_{M,\mathfrak{G}}$. Since, as we have seen above, $y \subset \mathfrak{M}$ holds, we infer hence by **50** and **44** that $\Sigma_\xi(y) \subset \Sigma_\xi(\mathfrak{W}_{M,\mathfrak{G}}) \subset \mathfrak{W}_{M,\mathfrak{G}} \subset \sum_{B \in M} \mathfrak{K}_{\mathfrak{G}}(B)$. On the basis of (3), (4), (9), (10) and **43** this proves that $y \in \mathfrak{W}_{M,\mathfrak{G}} - K_0$, which was to be shown.

From **75**, **51** and **54** we obtain the main theorem for Axiom 18. Thus the main theorem has been completely proved.

The main theorem proved above puts us in a position to carry out some independence proofs.

Let us imagine a set μ of statements which can be expressed in the language of the system $\mathfrak{S}$, and let α be a statement of the same kind which does not belong to the set μ. To prove that α is independent of any of the axioms of the system $\mathfrak{S}$ and of the statements from μ, it suffices in view of the main theorem to produce a group $\mathfrak{G} \in \mathcal{G}$ and a ring $M \in R(\mathfrak{G})$ which have the following two properties: (1) if the basic notions of the system $\mathfrak{S}$ are interpreted in the way described in the main theorem, all the statements of μ become true propositions; (2) under the same interpretation α becomes a false statement. An example of this procedure will be provided in the following sections.

§ 4. The independence of the well-ordering theorem from the ordering principle

76. Let $r_1, r_2, \ldots, r_n, \ldots$ be a sequence arbitrarily chosen, but fixed for the sequel, of all rational numbers.

77. If r is a rational number, we put $f(r) = A_n$, where n is the consecutive number of r in the sequence of **76**.

78. *The function f maps in a one-to-one manner the set of all rational numbers onto the set K.*

The proof follows immediately from **77** and **16**.

79. $x \prec y$ holds if and only if $x, y \in K$ and $f^{-1}(x) < f^{-1}(y)$.

80. *The relation $\prec$ is of type η in the set K.*

The proof follows from **78** and **79**.

81. $\mathfrak{G}^+$ denotes the subgroup of $\mathfrak{G}_0$ consisting of functions which preserve the relation $\prec$.

82. M^+ denotes the system of finite subsets of K.

83. $\mathfrak{G}^+ \in \mathcal{G}$ *and* $M^+ \in R(\mathfrak{G}^+)$.

This assertion follows from **81**, **82**, **83** and **42**.

84. For every subset $\mathfrak{T}$ of $\mathfrak{G}_0$ we denote by $|\mathfrak{T}|$ the smallest subgroup of $\mathfrak{G}_0$ which contains $\mathfrak{T}$.

85. *If* $A, B \in M^+$, $A \cdot B = 0$ *and* $a, b \in K$, *then there exists a function* $\varphi \in |\mathfrak{G}^+(A)+\mathfrak{G}^+(B)|$ *such that* $\varphi(a) = b$.

Proof. We can clearly assume that $a \neq b$; hence, say, $a \prec b$. Let us put $\varrho_1 = f^{-1}(a)$, $\varrho_2 = f^{-1}(b)$; so, by **79**, we have $\varrho_1 < \varrho_2$. All elements of $A+B$ are values of the function f for some rational numbers, which we may denote by $\sigma_1, \sigma_2, \ldots, \sigma_n$. The number of σ_i's that satisfy the inequality $\varrho_1 \leqslant \sigma_i \leqslant \varrho_2$ will be denoted by p. We are going to use induction with respect to p.

First let $p = 0$. Then we can determine a rational number $\varepsilon > 0$ such that the closed interval $I = [\varrho_1 - \varepsilon, \varrho_2 + \varepsilon]$ does not contain any σ_i's. Clearly, there exists a one-to-one order-preserving map g of I onto itself which satisfies the conditions $g(\varrho_1 + \varepsilon) = \varrho_1 - \varepsilon, g(\varrho_1) = \varrho_2, g(\varrho_2 + \varepsilon) = \varrho_2 + \varepsilon$. Let us extend g by setting $g(x) = x$ for the rational x' s which do not belong to I. We then obtain a one-to-one order-preserving map of the set of all rational numbers onto itself which preserves all σ_i's and maps ϱ_1 onto ϱ_2. Hence the function $\varphi = fgf^{-1}$ belongs to the system $\mathfrak{G}^+(A)$ and a fortiori to the system $|\mathfrak{G}^+(A)+\mathfrak{G}^+(B)|$ and carries a onto b. Thus our theorem is true for $p = 0$.

Now let us assume that $q > 0$ and that the theorem holds for $p < q$. Let $a, b \in K$ be two elements for which the number p has the value q. If $a \in A$, then a non $\in B$, from which it is easy to conclude that the group $\mathfrak{G}^+(B)$ contains a function φ such that $a \prec \varphi(u) \prec b$. The number of $x \in A+B$ for which we have $\varphi(a) \prec x \prec b$ or $\varphi(a) = x$, or $x = b$ is clearly less than q. By the induction assumption there exists a function $\psi \in |\mathfrak{G}^+(A) + \mathfrak{G}^+(B)|$ such that $\psi(\varphi(a)) = b$. We proceed quite similarly when $a \in B$. Now if a non $\in A+B$, then there is a $c \in A+B$ and a c' non $\in A+B$ such that $a \prec c \prec c' \prec b$, and no $x \in A+B$ satisfies the conditions $a \prec x \prec c$, $c \prec x \prec c'$. We may assume, say, that $c \in A$. Then there exists a function $\varphi \in \mathfrak{G}^+(B)$ such that $\varphi(a) = c'$. According to the induction assumption there exists a function $\psi \in |\mathfrak{G}^+(A)+\mathfrak{G}^+(B)|$ such that $\psi(c') = b$. Hence the function $\varphi\psi$ has all the required properties, which was to be proved.

86. *If* A, $B \in M^+, A \cdot B = 0, n \geqslant 1, a_1, \ldots, a_n, b_1, \ldots, b_n \in K$ *and* $a_1 \prec a_2 \prec \ldots \prec a_n, b_1 \prec b_2 \prec \ldots \prec b_n$, *then there exists a function* $\varphi \in |\mathfrak{G}^+(A)+\mathfrak{G}^+(B)|$ *such that* $\varphi(a_i) = b_i$ *for* $i = 1, 2, \ldots, n$.

Proof. For $n = 1$ our theorem is true in view of **85**. We assume that $p \geqslant 1$ and the theorem holds for $n \leqslant p$. Let $a_1, \ldots, a_{p+1}, b_1, \ldots, b_{p+1}$ be $2p+1$ elements of K such that

$$a_1 \prec a_2 \prec \ldots \prec a_{p+1}, \quad b_1 \prec b_2 \prec \ldots \prec b_{p+1}.$$

By assumption there exists a function $\psi \in |\mathfrak{G}^+(A)+\mathfrak{G}^+(B)|$ for which $\psi(a_i) = b_i$ $(i = 1, 2, \ldots, p)$. We put $\psi(a_{p+1}) = b^*_{p+1}$; since $\psi \in \mathfrak{G}^+$, we have clearly $b_p \prec b^*_{p+1}$. Now we consider the set $I = K \cdot \mathop{\mathrm{E}}_{x} [b_p \prec x]$. From **80** it follows that there exists a function g that maps I in a one-to-one manner onto the whole set K and preserves the relation $\prec$. Further, by **85** there exists a function $\chi \in \left|\mathfrak{G}^+\big(g(A \cdot I)\big)+\mathfrak{G}^+\big(g(B \cdot I)\big)\right|$ such that $\chi\big(g(b_{p+1})\big) = g(b_{p+1})$. Hence the function $h = g^{-1}\chi g$ maps I in a one-to-one manner onto itself, preserves the relation $\prec$ and satisfies the condition $h(b^*_{p+1}) = b_{p+1}$. We extend the function h onto the whole set K by setting $b(x) = x$ for $x \in K-I$ and we put $\varphi = h\psi$. It is evident that this function satisfies the conditions $\varphi(a_i) = b_i$ $(i = 1, 2, \ldots, p+1)$. It remains to show that $\varphi \in |\mathfrak{G}^+(A)+\mathfrak{G}^+(B)|$. Since $\psi \in |\mathfrak{G}^+(A)+\mathfrak{G}^+(B)|$, it suffices to prove that $h \in |\mathfrak{G}^+(A)+\mathfrak{G}^+(B)|$.

As an element of

$$\left|\mathfrak{G}^+\big(g(A \cdot I)\big)+\mathfrak{G}^+\big(g(B \cdot I)\big)\right|$$

the function χ can be represented in the form $\chi_1\chi_2 \ldots \chi_k$, where the χ's with even indices belong, say, to $\mathfrak{G}^+\big(g(A \cdot I)\big)$, and χ_l's with odd indices belong to $\mathfrak{G}^+\big(g(B \cdot I)\big)$. We conclude from this that

$$h(x) = g^{-1}\chi g(x) = (g^{-1}\chi_1 g)(g^{-1}\chi_2 g) \ldots (g^{-1}\chi_k g)(x)$$

for $x \in I$.

The functions $g^{-1}\chi_l g$ $(l = 1, 2, \ldots, k)$ map the set I onto itself and preserve the order relation $\prec$. If $a \in A \cdot I$, then we have $g(a) \in g(A \cdot I)$ and consequently $\chi_l\big(g(a)\big) = g(a)$ for even $l \leqslant k$. Hence we set $\omega_l(x) = g^{-1}\chi_l g(x)$ for $x \in I$ and $\omega_l(x) = x$ for $x \in K-I$ $(l = 1, 2, \ldots, k)$, and we have $l \leqslant k$ for even $l \leqslant k$. Similarly we can show that for odd $l \leqslant k$ we have $\omega_l \in \mathfrak{G}^+(B)$. Therefore we have $\omega_1\omega_2 \ldots \omega_k \in |\mathfrak{G}^+(A)+\mathfrak{G}^+(B)|$. Now we claim that

$$h(x) = \omega_1 \ldots \omega_k(x) \quad \text{for } x \in K. \tag{1}$$

Namely if $x \in I$, then

$$\omega_k(x) = g^{-1}\chi_k g(x) \in I, \quad \omega_{k-1}\omega_k(x) = (g^{-1}\chi_{k-1} g)(g^{-1}\chi_k g)(x) \in I,$$

and so on, so that finally, by induction, we arrive at the formula

$$\omega_1 \ldots \omega_k(x) = (g^{-1}\chi_1 g) \ldots (g^{-1}\chi_k g)(x) = h(x) \in I.$$

Now if $x \in K-I$, we find step by step that

$$\omega_k(x) = x, \quad \omega_{k-1}\omega_k(x) = \omega_k(x) = x, \quad \ldots, \quad \omega_1 \ldots \omega_k(x) = x = h(x).$$

Accordingly, formula (1) has been proved and it follows that

$$h \in |\mathfrak{G}^+(A)+\mathfrak{G}^+(B)|,$$

which was to be proved.

87. *If* $A, B \in M^+$ *and* $A \cdot B = 0$, *then* $\mathfrak{G}^+ = |\mathfrak{G}^+(A)+\mathfrak{G}^+(B)|$.

Proof. If $A = 0$ or $B = 0$, then the assertion is trivial. Hence we assume that neither A nor B is empty; these sets can thus be represented in the form

$$A = \{a_1^{(1)}, \dots, a_{i_1}^{(1)}, a_1^{(2)}, \dots, a_{i_2}^{(2)}, \dots, a_1^{(p)}, \dots, a_{i_p}^{(p)}\},$$
$$B = \{b_1^{(1)}, \dots, b_{j_1}^{(1)}, b_1^{(2)}, \dots, b_{j_2}^{(2)}, \dots, b_1^{(p)}, \dots, b_{j_p}^{(p)}\},$$

where

$$a_1^{(1)} \prec \dots \prec a_{i_1}^{(1)} \prec b_1^{(1)} \prec \dots \prec b_{j_1}^{(1)} \prec \dots \prec a_1^{(p)}$$
$$\prec \dots \prec a_{i_p}^{(p)} \prec b_1^{(p)} \prec \dots \prec b_{j_p}^{(p)}$$

(i_1 or j_p may be equal to 0).

Now let $\varphi \in \mathfrak{G}^+$, we put $c_l^{(k)} = \varphi(a_l^{(k)})$, $d_m^{(k)} = \varphi(b_m^{(k)})$ ($k = 1, 2, \dots, p$, $l = 1, 2, \dots, i_k$, $m = 1, 2, \dots, j_k$) and we then have

$$c_1^{(1)} \prec \dots \prec c_{i_1}^{(1)} \prec d_1^{(1)} \prec \dots \prec d_{j_1}^{(1)} \prec \dots \prec c_1^{(p)}$$
$$\prec \dots \prec c_{i_p}^{(p)} \prec d_1^{(p)} \prec \dots \prec d_{j_p}^{(p)}.$$

By **86** there exists a function $\psi \in |\mathfrak{G}^+(A)+\mathfrak{G}^+(B)|$, such that $\psi(c_l^{(k)}) = a_l^{(k)}$, $\psi(d_m^{(k)}) = b_m^{(k)}$ ($k = 1, 2, \dots, p$, $l = 1, 2, \dots, i_k$, $m = 1, 2, \dots, j_k$).

Hence we have $\psi\varphi(a_l^{(k)}) = a_l^{(k)}$ and $\psi\varphi(b_m^{(k)}) = b_m^{(k)}$ for $k = 1, 2, \dots, p$, $l = 1, 2, \dots, i_k$, $m = 1, 2, \dots, j_k$, which implies $\psi\varphi \in \mathfrak{G}^+(A+B) \subset |\mathfrak{G}^+(A) + \mathfrak{G}^+(B)|$. Since $\varphi = \psi^{-1}(\psi\varphi)$ and $\psi^{-1} \in |\mathfrak{G}^+(A)+\mathfrak{G}^+(B)|$, and consequently $\mathfrak{G}^+ = |\mathfrak{G}^+(A)+\mathfrak{G}^+(B)|$, which was to be proved.

88. *If* $A, B \in M^+$, *then* $\mathfrak{G}^+(A \cdot B) = |\mathfrak{G}^+(A)+\mathfrak{G}^+(B)|$.

Proof. The inclusion

$$\mathfrak{G}^+(A \cdot B) \supset |\mathfrak{G}^+(A)+\mathfrak{G}^+(B)| \tag{1}$$

is evident. To prove the converse inclusion, we reason by induction with respect to the number p of elements of $A \cdot B$. If $p = 0$, then everything follows from **87**. Now we assume that $p \geqslant 1$ and our theorem holds for those $A, B \in M^+$ whose intersection contains less than p elements.

Let A, B be two sets from M^+ whose intersection contains exactly p elements and let a be that element of $A \cdot B$ which is the earliest (with respect to the relation $\prec$). We put $I_1 = K \cdot \underset{x}{\mathrm{E}}\,[x \prec a]$, $I_2 = K \cdot \underset{x}{\mathrm{E}}\,[a \prec x]$. Now let φ be a function in $\mathfrak{G}^+(A \cdot B)$. From $\varphi(a) = a$ it easily follows that φ maps the sets I_1 and I_2 into themselves. We denote by φ_i ($i = 1, 2$) the function, whose values on I_i coincide with those of φ (and which is not defined outside I_i).

By **80** there are two functions f_1 and f_2 which map the sets I_1 and I_2 onto K, respectively, and at the same time preserve the relation $\prec$. The function $f_i \varphi_i f_i^{-1} (i = 1, 2)$ clearly belongs to the group $\mathfrak{G}^+$.

If $x \in f_i(A \cdot I_i) \cdot f_i(B \cdot I_i)$, then we have $f_i^{-1}(x) \in A \cdot B \cdot I_i$. Hence $\varphi_i f_i^{-1}(x) = \varphi f_i^{-1}(x) = f_i^{-1}(x)$, i.e. $f_i \varphi_i f_i^{-1}(x) = x$. Thus we have $f_i \varphi_i f_i^{-1} \in \mathfrak{G}^+(f_i(A \cdot I_i) \cdot f_i(B \cdot I_i))$ for $i = 1, 2$. But since the set $f_i(A \cdot I_i) \cdot f_i(B \cdot I_i)$ contains less than p elements, it follows from the induction assumption that $f_i \varphi_i f_i^{-1} \in |\mathfrak{G}^+(f_i(A \cdot I_i)) + \mathfrak{G}^+(f_i(B \cdot I_i))|$. Hence $f_i \varphi_i f_i^{-1}$ has the form of a composite function

$$f_i \varphi_i f_i^{-1} = \omega_1^{(i)} \ldots \omega_{k_i}^{(i)} \quad (i = 1, 2), \tag{2}$$

where the $\omega_l^{(i)}$'s with odd indices l belong, say, to the group $\mathfrak{G}^+(f_i(A \cdot I_i))$ and the $\omega_l^{(i)}$'s with even indices l belong to the group $\mathfrak{G}^+(f_i(B \cdot I_i))$.

We put for $i = 1, 2$ and $l = 1, 2 \ldots, k$

$$\chi_l^{(i)}(x) = f_i^{-1} \omega_l^{(i)} f_i(x) \quad \text{if } x \in I_i, \tag{3}$$

$$\chi_l^{(i)}(x) = x \qquad \text{if } x \in K - I_i. \tag{4}$$

Further, we put for $i = 1, 2$

$$\psi_i(x) = \varphi_i(x) \quad \text{if } x \in I_i,$$

$$\psi_i(x) = x \qquad \text{if } x \in K - I_i.$$

Then we clearly have $\chi_l^{(i)}$, $\psi_i \in \mathfrak{G}^+$ for $l = 1, 2, \ldots, k_i$ and $i = 1, 2$, and

$$\varphi = \psi_1 \psi_2. \tag{5}$$

Now we claim that

$$\psi_i = \chi_1^{(i)} \cdots \chi_{k_j}^{(i)} \quad (i = 1, 2). \tag{6}$$

Indeed, if $x \in K - I_i$, then we obtain successively $\chi_{k_i}^{(i)}(x) = x \in K - I_i$, $\chi_{k_i-1}^{(i)}(\chi_{k_i-1}^{(i)}(x)) = \chi_{k_i-1}^{(i)}(x) = x \in K - I_i$, and so on, $\chi_1^{(i)} \ldots \chi_{k_i}^{(i)}(x) = x = \psi_i(x)$.

On the other hand, if $x \in I_i$, then we have

$$\chi_{k_i}^{(i)}(x) = f_i^{-1} \omega_{k_i}^{(i)} f_i(x) \in I_i, \qquad \chi_{k_i-1}^{(i)} \chi_{k_i}^{(i)}(x) = f_i^{-1} \omega_{k_i-1}^{(i)} \omega_{k_i}^{(i)}(x) \in I_i,$$

etc.; so that we finally arrive at the formula $\chi_1^{(i)} \ldots \chi_{k_i}^{(i)}(x) = f_i^{-1} \omega_1^{(i)} \ldots \ldots \omega_{k_i}^{(i)} f_i(x)$. By (2) it follows from this that $\chi_1^{(i)} \ldots \chi_{k_i}^{(i)}(x) = \varphi_i(x) = \psi_i(x)$ as $\chi \in I_i$. Accordingly, (6) is established.

Now we show the following:

(7) *for odd* $l \leqslant k_i$ *we have* $\chi_l^{(i)} \in \mathfrak{G}^+(A)$,

(8) *for even* $l \leqslant k_i$ *we have* $\chi_l^{(i)} \in \mathfrak{G}^+(B)$.

It suffices to prove only one of these formulas, e.g. the first. Thus let $x \in A$; if $x \in A \cdot I_i$, then $f_i(x)$ is defined and $f_i(x) \in f_i(A \cdot I_i)$. Since $\omega_l^{(i)} \in \mathfrak{G}^+(f_i(A \cdot I_i))$, it follows that $\omega_l^{(i)} f_i(x) = f_i(x)$, i.e. $f_i^{-1}\omega_l f_i(x) = x = \chi_l^{(i)}(x)$. On the other hand, if $x \in A - I_i$, then we have by (4) $\chi_l^{(i)}(x) = x$. Hence the function $\chi_l^{(i)}$ leaves all elements of A fixed, i.e. we have $\chi_l^{(i)} \in \mathfrak{G}^+(A)$. The proof of (8) is completely analogous.

Now from (6), (7) and (8) it follows that $\psi_i \in |\mathfrak{G}^+(A) + \mathfrak{G}^+(B)|$; hence, in view of (5), $\varphi \in |\mathfrak{G}^+(A) + \mathfrak{G}^+(B)|$. Since φ is an arbitrary function from $\mathfrak{G}^+(A \cdot B)$, it follows that $\mathfrak{G}^+(A \cdot B) \subset |\mathfrak{G}^+(A) + \mathfrak{G}^+(B)|$, and in view of (1) $\mathfrak{G}^+(A \cdot B) = |\mathfrak{G}^+(A) + \mathfrak{G}^+(B)|$, which was to be proved.

89. *If* $0 \neq X \subset M^+$, *then we have* $\mathfrak{G}^+\Big(\prod\limits_{A \in X} A\Big) = \Big|\sum\limits_{A \in X} \mathfrak{G}^+(A)\Big|$.

Proof. If $A \in X$ and $\varphi \in \mathfrak{G}^+(A)$, *then we have* $\varphi(x) = x$ for all $x \in A$, and a fortiori for all $x \in \prod\limits_{A \in X} A$. Hence we have $\varphi \in \mathfrak{G}^+\Big(\prod\limits_{A \in X} A\Big)$, and thus

$$\Big|\sum_{A \in X} \mathfrak{G}^+(A)\Big| \subset \mathfrak{G}^+\Big(\prod_{A \in X} A\Big). \tag{1}$$

Now we denote by A_0 an arbitrary element of X and let $S = \underset{A_0 \cdot A}{\mathrm{E}}\,[A \cdot X]$. The system S consists of finite subsets of A_0 only, and since A_0 is itself finite, S is also finite. From **88** it follows by simple induction that

$$\mathfrak{G}^+\Big(\prod_{A \in S} A\Big) = \Big|\sum_{A \in S} \mathfrak{G}^+(A)\Big|. \tag{2}$$

If $A \in S$, then $A = A_0 \cdot B$ for a certain $B \in X$ and consequently by **88** $\mathfrak{G}^+(A) = |\mathfrak{G}^+(A_0) + \mathfrak{G}^+(B)| \subset \sum\limits_{B \in X} \mathfrak{G}^+(B)$. Hence we obtain the inclusion

$$\Big|\sum_{A \in S} \mathfrak{G}^+(A)\Big| \subset \Big|\sum_{A \in X} \mathfrak{G}^+(A)\Big|. \tag{3}$$

Now we claim that

$$\prod_{A \in S} A = \prod_{A \in X} A. \tag{4}$$

First, the inclusion $\prod\limits_{A \in X} A \subset \prod\limits_{A \in S} A$ is obvious. Conversely, if $x \in \prod\limits_{A \in S} A$ and $B \in X$, then $B \cdot A_0 \in S$; hence $x \in B \cdot A_0 \subset B$. Since B can be chosen arbitrarily, we obtain $x \in \prod_{A \in X} A$, i.e. $\prod_{A \in X} A \subset \prod_{A \in S} A$. Hence (4) has been proved. From (2), (3) and (4) it follows that

$$\mathfrak{G}^+\Big(\prod_{A \in X} A\Big) \subset \Big|\sum_{A \in X} \mathfrak{G}^+(A)\Big|,$$

which shows, in view of (1), the validity of **89**.

90. For $A \in M^+$, we denote by $n(A)$ the number of elements of A.

91. For $A, B \in M^+$, we write $A\varrho B$ if either $n(A) < n(B)$ or $n(A) = n(B)$ and A precedes B in the lexicographical ordering of all $n(A)$-tuples of elements of K.

92. *If* $A, B \in M^+$, $A\varrho B$ *and* $\varphi \in \mathfrak{G}^+$, *then* $\varphi(A)\varrho\varphi(B)$.

Proof. From **91** it follows that $n(\varphi(A)) = n(A)$ and $n(\varphi(B)) = n(B)$. Hence if $n(A) < n(B)$, then we also have $n(\varphi(A)) < n(\varphi(B))$, and consequently, by **91**, $\varphi(A)\varrho\varphi(B)$. On the other hand, if $n(A) = n(B)$, then by **91** the earliest element of $A-B$ remains in the relation $\prec$ to the earliest element of $B-A$. The map φ leaves the relation $\prec$ invariant, whence we conclude that the earliest element of $\varphi(A)-\varphi(B)$ is in the relation $\prec$ to the earliest element of $\varphi(B)-\varphi(A)$. By **91** this shows that $\varphi(A)\varrho\varphi(B)$, which was to be proved.

93. The set of all $A \in M^+$ for which $n(A) = n$ will be denoted by M_n^+ $(n = 0, 1, 2, \ldots)$.

94. We put, for $n = 1, 2, \ldots$: $A_0 = 0$, $A_n = \{f(1), f(2), \ldots, f(n)\}$ (cf. **77**).

95. $A_n \in M_n^+$ *for* $n = 0, 1, 2 \ldots$

The proof follows directly from **93**, **94**, **77**.

96. Let $\sigma_1, \ldots, \sigma_n$ be rational numbers such that $\sigma_1 < \sigma_2 < \ldots < \sigma_n$. We put for a rational x:

$$g_{\sigma_1, \ldots, \sigma_n}(x) = x - \sigma_1 + 1 \quad \text{if } x \leqslant \sigma_1,$$

$$g_{\sigma_1, \ldots, \sigma_n}(x) = \frac{x-\sigma_{i-1}}{\sigma_i - \sigma_{i-1}} + i - 1 \quad \text{if } \sigma_{i-1} < x \leqslant \sigma_i,$$

$$g_{\sigma_1, \ldots, \sigma_n}(x) = x - \sigma_n + n \quad \text{if } x > \sigma_n.$$

97. Let $A \in M_n^\pm$ and $A = \{x_1, \ldots, x_n\}$ where $x_1 \prec \ldots \prec x_n$. We put $\sigma_i = f^{-1}(x_i)$ $(i = 1, 2, \ldots, n)$ and $\varphi_A = fg_{\sigma_1, \ldots, \sigma_n}f^{-1}$ if $n > 0$; for $n = 0$ we denote by φ_A the function satisfying the equation $\varphi_A(x) = x$ for all $x \in K$.

98. *If* $A \in M_n^+$, *then* $\varphi_A \in \mathfrak{G}^+$ *and* $\varphi_A(A) = A_n$.

The proof can be obtained without difficulty from **93**, **94**, **96**, **97** and **77**.

99. For brevity we put $\mathfrak{W}_{M^+,\mathfrak{G}^+} = \mathfrak{W}^+$ and $\mathfrak{K}_{\mathfrak{G}^+}(A) = \mathfrak{K}^+(A)$.

100. For $x \in \mathfrak{W}^+$ we denote by $A(x)$ the intersection of all $A \in M^+$ for which $x \in \mathfrak{K}^+(A)$.

101. *If* $x \in \mathfrak{W}^+$, *then* $x \in \mathfrak{K}^+(A(x))$.

Proof. We denote by X the system of all $A \in M^+$ for which $x \in \mathfrak{K}^+(A)$. By **43** and **99** we have $0 \neq X \subset M^+$, which by **89** and **100** proves that $\mathfrak{G}^+(A(x)) = \left|\sum_{A \in X} \mathfrak{G}^+(A)\right|$. Hence every function $\varphi \in \mathfrak{G}^+(A(x))$ can be represented as a composite function $\varphi = \varphi_1\varphi_2 \dots \varphi_n$, where $\varphi_i \in \mathfrak{G}^+(A_j)$ and $A_i \in X$ $(i = 1, 2, \dots, n)$. In view of **40** we have $|\varphi_i, x| = x$ for $i = 1, 2, \dots, n$. But since by **36** $|\varphi, x| = |\varphi_1, |\varphi_2, \dots, |\varphi_n, x| \dots |$, it follows that $|\varphi, x| = x$. In view of **40** we conclude that $x \in \mathfrak{K}^+(A(x))$, which was to be proved.

102. *If* $x \in \mathfrak{W}^+$ *and* $\varphi \in \mathfrak{G}^+$, *then* $A(|\varphi, x|) = \varphi(A(x))$.

Proof. By **41** and **101** we have $|\varphi, x| \in \mathfrak{K}^+(\varphi(A(x)))$; hence, on the basis of **100**, $A(|\varphi, x|) \in \varphi(A(x))$. Now if we have $B \in M^+$ and

$$(1) \qquad |\varphi, x| \in \mathfrak{K}^+(B),$$

then we have, by **41** and **33**, $x \in \mathfrak{K}^+(\varphi^{-1}(B))$; hence $A(x) \subset \varphi^{-1}(B)$. Since this holds for every B satisfying (1), it follows from **100** that $\varphi(A(x)) \subset A(|\varphi, x|)$, which was to be proved.

In connection with definitions V, VII from § 1, we also introduce the following definition:

103. A set x *orders* a set y if the following conditions hold for arbitrary $u, v, w \in y$:

$$(1) \qquad \langle u, v\rangle \in x \text{ or } \langle v, u\rangle \in x;$$

$$(2) \qquad \text{if } \langle u, v\rangle \in x \text{ and } u \neq v, \text{ then } \langle v, u\rangle \text{ non} \in x;$$

$$(3) \qquad \text{if } \langle u, v\rangle \in x \text{ and } \langle v, w\rangle \in x, \text{ then } \langle u, w\rangle \in x;$$

$$(4) \qquad \langle u, u\rangle \in x.$$

104. A set x *well-orders* a set y provided x orders y and the following condition holds: if $0 \neq z \subset y$, then there exists a $u \in z$ such that $\langle u, v\rangle \in x$ for every $v \in z$.

105. *Let* $\mathfrak{G} \in \mathcal{G}$, $M \in R(\mathfrak{G})$ *and* $y \in \mathfrak{W}_{M,\mathfrak{G}} - K_0$. *In order that an* $x \in \mathfrak{W}_{M,\mathfrak{G}} - K$ *satisfy conditions* (1)–(4) *in* 103 *for arbitrary* $u, v, w \in \mathfrak{W}_{M,\mathfrak{G}} \cdot y$, *it is necessary and sufficient that* x *should order* y *and belong to the domain* $\mathfrak{W}_{M,\mathfrak{G}}$.

Proof. First we assume that x belongs to the domain $\mathfrak{W}_{M,\mathfrak{G}}-K$ and satisfies conditions (1)–(4) in **103** for arbitrary $u, v, w \in \mathfrak{W}_{M,\mathfrak{G}} \cdot y$. Since by **46** $y \subset \mathfrak{W}_{M,\mathfrak{G}}$, these conditions hold for arbitrary $u, v, w \in y$, i.e. x orders y.

Now if $x \in \mathfrak{W}_{M,\mathfrak{G}}$ and x orders y, formulas (1)–(4) in **103** hold for arbitrary $u, v, w \in y$ and a fortiori for arbitrary $u, v, w \in y \cdot \mathfrak{W}_{M,\mathfrak{G}}$. In view of (4) we infer from **25**, **48** and **26** that $x \operatorname{non} \in K_0$, which was to be proved.

Theorem **105** has the following meaning: if one replaces in Definition V, § 1, the words "individuum", "class", "ε", "Λ" by the words "$M, \mathfrak{G}$-distinguished element", "$M, \mathfrak{G}$-distinguished domain", "ε", "Λ_0", respectively, then this definition becomes a statement which defines the notion of an ordered set belonging to the domain $\mathfrak{W}_{M,\mathfrak{G}}$.

106. *Let* $\mathfrak{G} \in \mathscr{G}$, $M \in R(\mathfrak{G})$, $x, z \in \mathfrak{W}_{M,\mathfrak{G}}-K, x \subset y$ *and* $t = z \cdot \underset{\langle u,v\rangle}{\mathsf{E}} [u, v \in x]$. *If z orders the set y, then t orders x and we have* $t \in \mathfrak{W}_{M,\mathfrak{G}}-K$.

Proof. It is evident that t orders the set x. From **26** and **46** it follows that $x \neq \Lambda_0$, since otherwise z could not belong to the domain $\mathfrak{W}_{M,\mathfrak{G}}-K$. Hence from **25**, **26**, **46** we obtain

$$(1) \qquad 0 \neq x \subset \mathfrak{W}_{M,\mathfrak{G}} \quad \text{and} \quad x \operatorname{non} \in K_0.$$

By **48** and **26** we conclude from this that

$$(2) \qquad 0 \neq w = \underset{\langle u,v\rangle}{\mathsf{E}} [u, v \in x] \subset \mathfrak{W}_{M,\mathfrak{G}},$$

$$(3) \qquad w \operatorname{non} \in K_0.$$

From (2) and (3) it follows by **32** that

$$(4) \qquad |\varphi, w| = \underset{\langle |\varphi, u|, |\varphi, v|\rangle}{\mathsf{E}} [u, v \in x] \quad \text{for } \varphi \in \mathfrak{G}_0.$$

By (1) and **32** there exists a set $A \in M$ such that the formulas $u \in x$ and $|\varphi, u| \in x$ are equivalent for $u \in \mathfrak{W}_{M,\mathfrak{G}}$ and $\varphi \in \mathfrak{G}(A)$. Hence by (4) we have

$$(5) \qquad |\varphi, w| = w \quad \text{for } \varphi \in \mathfrak{G}(A).$$

On account of **32**, formula (5) proves that the domain associated with w is $M, \mathfrak{G}$-distinguished. Since $t = z \cdot w$, in view of **66** it follows that $t \in \mathfrak{W}_{M,\mathfrak{G}}$, which was to be proved.

107. *For every ordinal number ξ there exists a set $x \in \mathfrak{W}^+ - K$ which orders the set $K_\xi \cdot \mathfrak{W}^+$.*

Proof. For $A \in M^+$ we denote by $Y(A)$ the set of $y \in K_\xi \cdot \mathfrak{W}^+$ for which $A(y) = A$. Since $A(y)$ is determined uniquely by y, the sets $Y(A)$ are mutually disjoint.

From the axiom of choice we infer the existence of the sets O_n $(n = 0, 1, 2, \ldots)$ which order the sets $Y(A_n)$. With the aid of these sets we establish a relation σ between elements of $K_\xi \cdot \mathfrak{W}^+$ in the following way:

(1) *$u\sigma v$ holds if and only if* 1° $u, v \in K_\xi \cdot \mathfrak{W}^+$ *and* 2° *either* $A(u)\varrho A(v)$ *or* $A(u) = A = A(v)$, *and* $\langle |\varphi_A, u|, |\varphi_A, v| \rangle \in O_{n(A)}$.

Finally we denote by x the set of pairs $\langle u, v \rangle$ for which $u\sigma v$ holds.

Next we wish to show that x orders the set $K_\xi \cdot \mathfrak{W}^+$.

For $u \in K_\xi \cdot \mathfrak{W}^+$ we have by **47**, **35**, **98**, **83** and **102**

$$|\varphi_{A(u)}, u| \in K_\xi \cdot \mathfrak{W}^+ \cdot Y(A_{n(A(u))}).$$

Since $O_{n(A(u))}$ orders the set $Y(A_{n(A(u))})$, we obtain

$$|\varphi_{A(u)}, u|, |\varphi_{A(u)}, u| > \in O_{n(A(u))},$$

i.e. by (1) $u\sigma u$ or

(2) $$\langle u, u \rangle \in x \quad \text{for } u \in K_\xi \cdot \mathfrak{W}^+.$$

Now let $\langle u, v \rangle \in x$ and $u \neq v$. If $A(u)\varrho A(v)$, then by **91** we cannot have either $A(v)\varrho A(u)$ or $A(v) = A(u)$. Hence from (1) it follows that $\langle u, v \rangle$ non $\in x$. If $A(u) = A = A(v)$, then according to (1) we have $\langle |\varphi_A, u|, |\varphi_A, v| \rangle \in O_{n(A)}$. Here we have $|\varphi_A, u| \neq |\varphi_A, v|$, since otherwise we would have $u = v$. In view of the fact that $O_{n(A)}$ orders the set $Y(A_{n(A)})$, we obtain $\langle |\varphi_A, v|, |\varphi_A, u| \rangle$ non $\in O_{n(A)}$. Now the equality $A(u) = A(v)$ excludes the formula $A(v)\varrho A(u)$. We infer from (1) that also in this case $\langle v, u \rangle$ non $\in x$. Hence, in general, the following holds:

(3) *if* $u \neq v$ *and* $\langle u, v \rangle \in x$, *then we have* $\langle v, u \rangle$ non $\in x$.

Now we assume that $\langle u, v \rangle \in x$ and $\langle u, w \rangle \in x$. If $A(u)\varrho A(v)$, we must have $A(u)\varrho A(w)$. Then in any case we have $A(v)\varrho A(w)$ or $A(v) = A(w)$, and the relation ϱ is transitive. Hence in this case we have $\langle u, w \rangle \in x$. Similarly we have $\langle u, w \rangle \in x$ if $A(v)\varrho A(w)$. Now if $A(u) = A(v) = A(w) = A$, then we have by (1) $\langle |\varphi_A, u|, |\varphi_A, w| \rangle \in O_{n(A)}$ and $\langle |\varphi_A, v|, |\varphi_A, w| \rangle \in O_{n(A)}$, whence $\langle |\varphi_A, u|, |\varphi_A, w| \rangle \in O_{n(A)}$, i.e. on account of (1) $u\sigma w$, and hence $\langle u, w \rangle \in x$. Thus we have proved the following:

(4) *from* $\langle u, v \rangle \in x$ *and* $\langle v, w \rangle \in x$ *it follows that* $\langle u, w \rangle \in x$.

If $u, v \in K_\xi \cdot \mathfrak{W}^+$, then we have either $A(u)\varrho A(v)$, or $A(v)\varrho A(u)$, or finally $A(u) = A = A(v)$. In the first case we have by (1) $\langle u, v \rangle \in x$, in the second case $\langle v, u\rangle \in x$. In the third case either $\langle |\varphi_A, u|, |\varphi_A, v|\rangle \in O_{n(A)}$ or $\langle |\varphi_A, v|, |\varphi_A, u|\rangle \in O_{n(A)}$, since by **102** and **93** $|\varphi_A, u|, |\varphi_A, v| \in Y(A_{n(A)})$. Hence also in this case we have $\langle u, v\rangle \in x$ or $\langle v, u\rangle \in x$. Accordingly, the following formula has been proved:

(5) *if* $u, v \in K_\xi \cdot \mathfrak{W}^+$, *then we have either* $\langle u, v\rangle \in x$ *or* $\langle v, u\rangle \in x$.

From (2)–(5) it follows by **103** that x orders the set $K_\xi \cdot \mathfrak{W}^+$.

If $u, v \in K_\xi \cdot \mathfrak{W}^+$, then by **48** and **83** $\langle u, v\rangle \in \mathfrak{W}^+$. Since x is composed of pairs $\langle u, v\rangle$ for which $u, v \in K_\xi \cdot \mathfrak{W}^+$, we have $x \subset \mathfrak{W}^+$, which by **10**, **44**, **50** and **83** implies

(6) $\Sigma_\eta(x) \cdot \mathfrak{M} \subset \mathfrak{W}^+ \subset \sum\limits_{B \in M^+} K^+(B)$ *for every ordinal number* η.

Now let $\langle u, v\rangle \in x$ and $\varphi \in \mathfrak{G}^+(0)$. By **32** we have

$$|\varphi, \langle u, v\rangle| = \langle |\varphi, u|, |\varphi, v|\rangle \tag{7}$$

and by **47** and **35**

$$|\varphi, u|, |\varphi, v| \in K_\xi \cdot \mathfrak{W}^+ . \tag{8}$$

According to (1) two cases are possible: either $A(u) \varrho A(v)$ or

$$A(u) = A = A(v) \quad \text{and} \quad \langle |\varphi_A, u|, |\varphi_A, v|\rangle \in O_{n(A)}. \tag{9}$$

In the first case we obtain from **102** and **92** $A(|\varphi, u|) \varrho A(|\varphi, v|)$, which in view of (1) and (8) gives

$$\langle |\varphi, u|, |\varphi, v|\rangle \in x. \tag{10}$$

Now we assume that the second case holds. We put $B = \varphi(A)$ and $\psi = \varphi_A^{-1} \varphi_B \varphi$. By (9) and **102** we have

$$A(|\varphi, u|) = B = A(|\varphi, v|). \tag{11}$$

Further, we clearly have $\psi \in \mathfrak{G}^+$, since according to **98** $\varphi_A^{-1}, \varphi_B \in \mathfrak{G}^+$. If $x \in A$, then we have $\varphi(x) \in B$; further, by **98**, $\varphi_B \varphi(x) \in A_{n(B)} = A_{n(A)}$, whence by another application of **98** we obtain $\psi(x) = \varphi_A^{-1} \varphi_B \varphi(x) \in A$. The function ψ maps A onto a part of A. Now we claim that

$$\psi(x) = x \quad \text{for } x \in A. \tag{12}$$

Namely, if we had $\psi(x) \neq x$ for a certain $x \in A$, and hence, say, $x \prec \psi(x)$, then we would obtain by iteration

$$x \prec \psi(x) \prec \psi^2(x) \prec \dots \prec \psi^k(x) \prec \dots$$

By the above all $\psi^k(x)$ would belong to the set A. But this is impossible, because A as an element of M^+ is a finite set. Accordingly, (12) has been proved and it follows that $\psi \in \mathfrak{G}^+(A)$. By (9) and **101** this proves that $|\psi, u| = u$ and $|\psi, v| = v$, i.e. on account of **33** and **36** $\langle|\varphi_B, |\varphi, u|| = |\varphi_A, u|$, $||\varphi_B, |\varphi, v|| = |\varphi_A, v|$. Hence by **9** we obtain

$$(13) \qquad \langle|\varphi_B, |\varphi, u||, |\varphi_B, |\varphi, v||\rangle \in O_{n(A)} = O_{n(B)}.$$

From (1), (8), (11) and (13) we again obtain formula (10). Hence we have proved the following:

(14) *if* $\langle u, v\rangle \in x$ *and* $\varphi \in \mathfrak{G}^+(0)$, *then we have* $\langle |\varphi, u|, |\varphi, v|\rangle \in x$.

Putting in (14) φ^{-1} for φ, $|\varphi, u|$ for u and $|\varphi, v|$ for v, we obtain the following

(15) *if* $\langle|\varphi, u|, |\varphi, v|\rangle \in x$ *and* $\varphi \in \mathfrak{G}^+(0)$, *then we have* $\langle u, v\rangle \in x$.

Since we clearly have

$$(16) \qquad 0 \neq x \subset \mathfrak{M},$$

on the basis of **32** it follows from (6), (16) and (17) that $x \in \mathfrak{W}^+$, and from (2) we infer by **26** and **48** that $x \operatorname{non} \in K_0$, which was to be proved.

108. *If* $y \in \mathfrak{W}^+ - K_0$, *then there exists an* $x \in \mathfrak{W}^+$ *which orders* y.

Proof. By **18** and **21** we have $y \subset K_\xi$ for a certain ordinal number ξ, which on account of **46** gives $y \subset K_\xi \mathfrak{W}^+$. By **107** there exists a set from $\mathfrak{W}^+$ which orders $K_\xi \cdot \mathfrak{W}^+$. Hence from **106** we conclude that there exists an $x \in \mathfrak{W}^+$ that orders y, which was to be proved.

109. In view of **105**, **51**, **52** and **54** Theorem **108** can be expressed as follows: when in the ordering principle of § 1 we replace the words "individuum", "class", "ε", "Λ" by the words "M^+, $\mathfrak{G}^+$-distinguished element", "M^+, $\mathfrak{G}^+$-distinguished domain", "$\in$", "Λ_0", respectively, then the ordering principle becomes a true statement.

110. $K \in \mathfrak{W}^+ - K_0$.

Proof. From $0 \neq K \subset K_0$ it follows that $K \subset \mathfrak{P}(K_0) \subset K_1$ from which we obtain

$$(1) \qquad K \in \mathfrak{M}.$$

From **26** it follows further that

$$(2) \qquad K \operatorname{non} \in K_0.$$

If $\varphi \in \mathfrak{G}_0$, we obtain by (1), (2) **25**, **32** and **30** b)

$$|\varphi, K| = \mathop{\mathrm{E}}\limits_{\varphi(\Lambda_n)} [n = 1, 2, \ldots] = K,$$

from which we get

(3) $$K \in \mathfrak{K}^+(0).$$

From **45** we obtain by **10**, **44**, **50** and **83**

(4) $$\Sigma_{\xi}(K)\cdot\mathfrak{M} \subset \Sigma_{\xi}(\mathfrak{W}^+)\cdot\mathfrak{M} \subset \mathfrak{W}^+ \subset \sum_{B\in M^+}\mathfrak{K}^+(B).$$

From (1)–(4) it follows by **43** that $K \in \mathfrak{W}^+ - K_0$, which was to be proved.

111. *There exists no* $x \in \mathfrak{W}^+$ *ordering* K *and satisfying the following condition*:

(*) *if* $z \in \mathfrak{W}^+ - K_0$ *and the formula* $t \in z$ *implies the formula* $t \in K$ *for every* $t \in \mathfrak{W}^+$, *then there exists a* $u \in \mathfrak{W}^+ \cdot z$ *such that* $\langle u, v\rangle \in x$ *for all* $v \in \mathfrak{W}^+ \cdot z$.

Proof. Let us suppose that there is an $x \in \mathfrak{W}^+$ which orders k and satisfies (*). Hence there exists a set $A_1 \in \mathfrak{M}^+$ such that $x \in \mathfrak{K}^+(A_1)$. By **110** there exists a set $A_2 \in M^+$ such that $K \in \mathfrak{K}^+(A_2)$. We put $A = A_1 + A_2$ and we clearly have

(1) $$A \in M^+, \quad x \in \mathfrak{K}^+(A), \quad K \in \mathfrak{K}^+(A).$$

As a finite set, x can be represented in the form $A = \{x_1, \dots, x_n\}$, where $x_1 \prec x_2 \dots \prec x_n$. We put $I_0 = \mathop{\mathrm{E}}_{x}[x \prec x_1]\cdot K$, $I_j = \mathop{\mathrm{E}}_{x}[x_j \prec x \prec x_{j+1}]\cdot K$ for $j = 1, 2, \dots, n-1$ and $I_n = \mathop{\mathrm{E}}_{x}[x_n \prec x]\cdot K$. Clearly, one of the sets I_j must contain at least two elements, since K is infinite. Let $I_{j_0} = z$ be that set. We claim that

(2) $$z \in \mathfrak{W}^+ - K_0.$$

Indeed, we have

(3) $$0 \neq z \subset \mathfrak{M},$$

since $z \subset K_0$. Hence we have according to **26**

(4) $$z \,\mathrm{non} \in K_0.$$

In view of (2) from $z \subset K_0$ it follows that $z \in \mathfrak{P}(K_0) \subset K_1 \subset \mathfrak{M}$ and

(5) $$z \in \mathfrak{M}.$$

By (3), (5) and **32** we have $|\varphi, z| = \mathop{\mathrm{E}}_{\varphi(x)}[x \in z]$ for $\varphi \in \mathfrak{G}^+$. Now let $\varphi \in \mathfrak{G}^+(\{x_{j_0}, x_{j_0+1}\})$.(21) The formulas $x_{j_0} \prec x \prec x_{j_0+1}$ and $x_{j_0} \prec \varphi(x)$

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(21) For $j_0 = 0$ we put $\{x_1\}$ and for $j_0 = n$ we put $\{x_n\}$ in place of $\{x_{j_0}, x_{j_0+1}\}$.

$\prec x_{j_0+1}$ are clearly equivalent, which shows that $|\varphi, z| = z$. Since $\{x_{j_0}, x_{j_0+1}\} \subset A$, we obtain

$$z \in \mathfrak{K}^+(A). \tag{6}$$

Finally, in view of $z \subset K_0 \subset \mathfrak{W}^+$, we have by **10**, **44**, **50** and **83**

$$\Sigma_\xi(z) \cdot \mathfrak{M} \subset \mathfrak{W}^+ \subset \sum_{B \in M^+} \mathfrak{K}^+(B) \quad \textit{for every ordinal number } \xi. \tag{7}$$

Now (2) follows from (4)–(7) and **43**.

From **46** it follows by (2) that $z \subset \mathfrak{W}^+$. Since $z \subset K$, it follows that for every $t \in \mathfrak{W}^+$ the formula $t \in z$ implies the formula $t \in K$. By (*) there exists a u_0 such that:

$$u_0 \in z, \tag{7}$$

$$\langle u_0, v\rangle \in x \quad \text{for all } v \in z \cdot \mathfrak{W}^+ = z. \tag{8}$$

Now let u_1 be an arbitrary element of z which is different from u_0. Since u_0, u_1 non $\in A$ and by **80** the set z is ordered into type η, we easily infer that there exists a function $\varphi \in \mathfrak{G}^+(A)$ such that $\varphi(u_1) = u_0$. Evidently we have

$$u_0 \neq \varphi(u_0), \tag{9}$$

since otherwise in view of the uniqueness of φ we would have $u_0 = u_1$. Further, we have by (6) $\varphi(u_0) \in z$, and hence by (8)

$$\langle u_0, \varphi(u_0)\rangle \in x. \tag{10}$$

On the other hand, form (8) by the use of (1) and **32** it follows that $\langle \varphi(u_0), \varphi(u_1)\rangle \in x$, i.e.

$$\langle \varphi(u_0), u_0\rangle \in x. \tag{11}$$

But this is a contradiction, because in view of (9) the formulas (10) and (11) are mutually exclusive. Accordingly, our assertion has been proved.

In view of **110**, **105**, **51**, **52** and **54** it follows from **111** that under the interpretation of basic notions given in **109** the well-ordering theorem of § 1 becomes a false statement. Consequently, on the basis of the main theorem we obtain the following theorem.

THEOREM I. *The well-ordering theorem is independent of all axioms of the system $\mathfrak{S}$ and of the ordering principle.*

As an easy application of this we obtain the following corollary.

COROLLARY II. *Let us add to the axioms of the system $\mathfrak{S}$ the following statement*

(*) *for every system of mutually disjoint and finite sets there exists a choice set.*

Then the well-ordering theorem is not derivable from that system.(22)

For the proof it suffices to observe that the statement (*) is derivable from the ordering principle within the framework of the system $\mathfrak{S}$.(23)

As we mentioned in the introduction, the two theorems just proved only hold under the assumption that von Neumann's set theory is consistent. Finally, we would like to mention that this assumption can be replaced by some weaker ones, e.g. by the assumption that the system $\mathfrak{S}$ is without contradiction.

It follows directly from our considerations that the construction of the model can be carried out not only within von Neumann's system but also within some other systems (as e.g. the system of Bernays). Similarly, our main theorem can easily be proved in other systems along the lines of § 3. However, the situation with regard to the theorems of § 4 is quite different. In proving **107** we made essential use of the axiom of choice, and according to all evidence it is impossible to prove Theorem **107** without the use of this axiom.(24) Hence, probably, the proof of **107** cannot be carried out within the framework of the system $\mathfrak{S}$, either. But if we extend the system $\mathfrak{S}$ by adding the axiom of choice, then within this new system $\mathfrak{S}'$ all the derivations of the preceding sections can be exactly reconstructed. Hence Theorem I and Corollary II hold under the assumption that the system is consistent. Now it was proved by Gödel at the end of his previously mentioned lecture(25) that the consistency of $\mathfrak{S}'$ follows from the consistency of $\mathfrak{S}$. Accordingly, the validity of Theorem I and Corollary II is ensured already by the consistency of $\mathfrak{S}$ (or Bernay's system).

Instead of the system $\mathfrak{S}$ one could also consider other systems of set theory and obtain corresponding results. For example, one could take

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(22) Corollary II (in a somewhat different formulation) was put forward by A. Fraenkel in his second and third paper cited in (12). However, Fraenkel's two proofs are not convincing (cf. the communiqué by A. Lindenbaum and myself cited in (13)).

(23) This remark is due to K. Kuratowski. See A. Tarski, *Sur les ensembles finis*, Fund. Math. 6 (1934), pp. 46–95; cf. footnote(2) on the page 82.

(24) This question is loosely connected with the apparently very difficult problem of whether the axiom of choice is also independent in systems that do not allow the existence of primitive elements ("*Urelementen*").

(25) See K. Gödel, *The consistency of the axiom of choice and the generalized continuum hypothesis*, Proc. Nat. Acad. Sci. 24 (1939), pp. 556–557.

the system of Ouine(26) or Skolem,(27) which have fewer basic notions than $\mathfrak{S}$ but are not based upon a finite number of axioms. Consequently, the proof with respect to such systems is by no means an exact repetition of the previous one. To carry out this proof one has to take under consideration several other methods of modern mathematics.(28)

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(26) See W. V. Quine, *Set-theoretic foundation for logic*, Journal of Symb. Logic 1 (1936), pp. 45–57.

(27) See T. Skolem, *Über einige Grundlagenfragen der Mathematik*, Skrifter utgitt av Det Norske Videnskaps Akademi i Oslo. I. Mat.–Nat. Kl. (1929), No. 4.

(28) See my paper introduced in (14) and also the communiqué cited in (13).

On definable sets of positive integers *).

By

Andrzej Mostowski (Warszawa).

The celebrated paper of K. **Gödel** on undecidable statements [1]) had (among others) the effect that several writers began to analyze the notion of functions of natural argument taking on integer values as well as related with them sets of positive integers. The chief purpose of these endeavours was to formulate an exact definition of what may be called „calculable function", i. e. such function $f(n)$ that there exists a method permitting to calculate the value of $f(n)$ for any given n in a finite number of steps. For sets we have the corresponding notion of „calculable sets" for which there is a finite method permitting to decide whether any given integer is in set or not. The solution of this problem given by **Herbrand**, **Gödel**, **Church**, **Kleene** and **Turing** [2]) suggested still other types of sets and of functions. So e. g. **Rosser** and **Kleene** found an interesting class of sets which they called „recursively enumerable" [3]).

The aim of this paper is to show that the two above mentioned classes of sets (and of functions) form the beginning of an infinite sequence of classes whose properties closely resemble those of projective sets [4]). For convenience of readers not acquainted with papers referred to in footnotes [2]) and [3]) I shall develop the theory without using the notion of general recursivity (the final section 6 is the only exception).

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*) See note on the page 112.

[1]) **Gödel** [3]. Numbers in brackets refer to bibliography given at the end of this paper.

[2]) **Gödel** [4], [5], **Church** [2], **Kleene** [9], **Turing** [21]. It is now customary to call calculable functions and sets „general recursive". An excellent exposition of the theory of these functions is to be found in **Hilbert-Bernays** [8], Supplement II, 392-421.

[3]) **Kleene** [9], **Rosser** [14]. Further development will be found in **Post** [12].

[4]) I shall refer to the exposition of the theory of these sets given by **Kuratowski** [10].

It is difficult to predict at present whether the classes of sets and of functions dealt with in this paper will gain the same „right of citizenship" in metamathematics as the class of general recursive sets or functions. I have therefore not so much developed the theory of these classes themselves as tried rather to give some applications and to detect relations between the new classes and notions already known in this field. This explains why proofs of several known theorems are given in this paper (see 4.21, 4.41, 4.43, 5.51, 5.61). I think that owing to the use of methods familiar in the theory of projective sets I obtained not only considerable simplifications of the proofs but also some slight generalisations of the results themselves.

It seems to be possible to develop very extensively the theory of the new classes on the pattern of the theory of projective sets. From this kind of problems only one will be discussed here, to wit the analogue of Souslin's theorem[5]), i. e. the theorem that a recursively enumerable set whose complement is also recursively enumerable must be general recursive[6]). The utility resulting from the analogy with projective sets is thus I think demonstrated.

§ 1. Classes P_n and Q_n.

1.1. Preliminary remarks. Terminology. Metamathematical concepts occuring below (e. g. propositional function, formal proof etc.) refer to a fixed self-consistent logical system S in which the theory of addition and of multiplication of positive integers can be built up. Hence for S may be taken e. g. the system of *Principia Mathematica* of course reformulated so as to render the system more exact[7]). As the subsequent investigations are in high degree independent from the particular choice of the system S I shall give a mere sketch of its structure instead of a detailed description.

In the system S occur variables „x", „y",... of the type of positive integers[8]) as well as signs denoting the numbers 1,2,3,...

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[5]) Kuratowski [10], p. 251, Corollaire 1.

[6]) After having finished the first draft of this paper I became acquainted with the paper Post [12] from which I see that this result has been obtained by E. L. Post already in 1944. From letters I understand that A. Tarski has also found the same theorem.

[7]) Such exact reformulations are given in Gödel [3] and Tarski [17].

[8]) S can contain also other types of variables.

Propositional functions with one, two,...,k variables of the type of positive integers will be denoted by symbols such as „$\varphi(x)$“, „$\psi(x,y)$“ etc. and general „$\varphi(\mathfrak{x})$“, the German letter „$\mathfrak{x}$“ standing for the finite sequence $x_1, x_2, \dots, x_k$ of k variables.

Among the propositional functions occur the arithmetical ones:

$$(1) \qquad x=y, \quad x<y, \quad x=y+z, \quad x=y\cdot z, \quad x=y^z$$

with their usual meaning.

If we substitute in a propositional function, e. g. in $\varphi(x,y)$, for „x“ the sign denoting the number k and for „y“ the sign denoting the number l we get a sentence which will be denoted by „$\varphi(k,l)$“.

The implication and the conjunction of two propositional functions φ and ψ will be written as $\varphi\rightarrow\psi$ and $\varphi\cdot\psi$, the negation of φ as φ'. For quantifiers we use letters „Π“ and „Σ“ with a variable written below.

We admit that the ordinary rules of inference and the ordinary arithmetical axioms are valid in S. The formula $-\varphi$ means that φ is a valid sentence, i. e. that there exists a formal proof of φ.

It will be admitted that it is possible to put variables, propositional functions and formal proofs of the system S in one-to-one correspondence with positive integers[9]. These integers will be called the Gödel-numbers of variables or of propositional functions or of proofs. The correspondence is supposed to be not arbitrary but to fulfill some conditions which will be formulated in 3.1[10].

In the simplest case S contains no other variables than those of the type of positive integers and no other propositional functions than those which can be built up from the propositional functions (1) with the help of quantifiers and logical connectives „$\rightarrow$“, „$\cdot$“ and „$'$“. In this case S will be spoken of as the system of elementary arithmetic and denoted by $\mathfrak{A}$.

The logical symbols: negation „$'$“, implication „$\rightarrow$“, equivalence „$\equiv$“, conjunction „$\cdot$“, alternative „$+$“ and quantifiers occur also (and more frequently) as synonyma of words „not“, „if..., then...“ etc. They are then used not as signs (primitive or defined) of the formal system S but as words of our ordinary language in which we are speaking about the system S. Using Carnap's

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[9] Gödel [3], p. 179.

[10] They represent a generalization of the three conditions of recursivity formulated in Hilbert-Bernays [8], p. 393-394.

terminology we could say that we use the same symbols in object language as in syntax-language[11]. I do not think that this double meaning could cause any misunderstanding[12].

Positive integers will be denoted by letters $m,n,h,k,\ldots$ eventually with subscripts. For any n we put

$$n=2^{s_1(n)}[2s_2(n)-1].$$

Ordered k-ads of positive integers will be called points of k-dimensional space R_k and denoted sometimes by a single German letter $\mathfrak{m},\mathfrak{n},\ldots$ For „$\varphi(n_1,n_2,\ldots,n_k)$" we write then shortly „$\varphi(\mathfrak{m})$".

The set-theoretical notation and terminology is that of Kuratowski [10].

1.2. Decidable functions. A propositional function $\varphi(\mathfrak{x})$ with k free variables will be called decidable, if for any $\mathfrak{n}\,\epsilon\,R_k$ either $\vdash\varphi(\mathfrak{n})$ or $\vdash\varphi'(\mathfrak{n})$. In symbols

$$\prod_{\mathfrak{n}\epsilon R_k}[\vdash\varphi(\mathfrak{n})+\vdash\varphi'(\mathfrak{n})].$$

Here the logical symbols except the negation-sign „$'$" are taken meta-mathematically.

E. g. the propositional functions (1) are decidable.

1.21. *The negation of a decidable propositional function and the logical product of two such functions is again a decidable propositional function.*

For negation the proposition is obvious. Suppose now that $\varphi(\mathfrak{x})$ and $\psi(\mathfrak{y})$ are decidable propositional functions with k and l free variables and let $\mathfrak{m}\,\epsilon\,R_k$, $\mathfrak{n}\,\epsilon\,R_l$. If $\vdash\varphi(\mathfrak{m})$ and $\vdash\psi(\mathfrak{n})$, then $\vdash\varphi(\mathfrak{m})\cdot\psi(\mathfrak{n})$. If either non $\vdash\varphi(\mathfrak{m})$ or non $\vdash\psi(\mathfrak{n})$, then $\vdash\varphi'(\mathfrak{m})$ or $\vdash\psi'(\mathfrak{n})$ since $\varphi(\mathfrak{x})$ and $\psi(\mathfrak{y})$ are both decidable and it follows by the rules of propositional calculus $\vdash[\varphi(\mathfrak{m})\cdot\psi(\mathfrak{n})]'$. Hence $\varphi(\mathfrak{x})\cdot\psi(\mathfrak{y})$ is decidable.

1.22. *If $\varphi(\mathfrak{x},y)$ is a decidable propositional function with $k+1$ free variables, then the propositional functions*

$$\prod_y[(y<x)\rightarrow\varphi(\mathfrak{x},y)] \quad \textit{and} \quad \sum_y[(y<x)\cdot\varphi(\mathfrak{x},y)]$$

are also decidable[13].

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[11] Carnap [1], p. 4.

[12] One could avoid this duality introducing other symbols in the formal system S and other in the meta-system. This is done e. g. in Gödel [3].

[13] Gödel [3], Satz IV, p. 180.

The proof is obvious.

According to 1.21 and 1.22 the connectives of propositional calculus as well as the „limited quantifiers“ $\prod_{y}[(y<x)\rightarrow(\dots)]$ and $\sum_{y}[(y<x)\cdot(\dots)]$ if applied to decidable propositional functions yield again decidable functions. It will be seen later (in 4.21) that the unlimited quantifiers Π_{x} and Σ_{x} may give undecidable propositional functions.

1.23. *Let* $\varphi(\mathfrak{t},\mathfrak{y})$ *and* $\psi(\mathfrak{x},\mathfrak{y})$ *be two decidable propositional functions with* $m+k$ *and* $l+k$ *free variables. Let* $\psi(\mathfrak{x},\mathfrak{y})$ *fulfill the conditions:*

$$\prod_{\mathfrak{x}}\prod_{\mathfrak{y}}\prod_{\mathfrak{z}}\{[\psi(\mathfrak{x},\mathfrak{y})\cdot\psi(\mathfrak{x},\mathfrak{z})]\rightarrow(\mathfrak{y}=\mathfrak{z})\}\,{}^{14)};\tag{2}$$

(3) *For any* $\mathfrak{m}\in R_l$ *there is* $\mathfrak{n}\in R_k$ *such that* $\vdash\psi(\mathfrak{m},\mathfrak{n})$.

Under these assumptions the propositional function

$$\sum_{\mathfrak{y}}[\varphi(\mathfrak{t},\mathfrak{y})\cdot\psi(\mathfrak{x},\mathfrak{y})]\tag{4}$$

is decidable.

Proof. Let us denote by $\vartheta(\mathfrak{t},\mathfrak{x})$ the propositional function (4) and suppose that $\mathfrak{m}\in R_l$, $\mathfrak{p}\in R_m$. Assume that non $\vdash\vartheta(\mathfrak{p},\mathfrak{m})$ and denote by $\mathfrak{n}$ a point of R_k such that $\vdash\psi(\mathfrak{m},\mathfrak{n})$. Hence it cannot be $\vdash\varphi(\mathfrak{p},\mathfrak{n})$, since we would then have $\vdash\vartheta(\mathfrak{p},\mathfrak{m})$ against our assumption. Therefore $\vdash\varphi'(\mathfrak{p},\mathfrak{n})$ and hence

$$\vdash[\psi(\mathfrak{m},\mathfrak{n})\rightarrow\varphi'(\mathfrak{p},\mathfrak{n})].\tag{5}$$

The formula $\vdash\psi(\mathfrak{m},\mathfrak{n})$ yields together with (2)

$$\vdash\prod_{\mathfrak{y}}[(\mathfrak{y}=\mathfrak{n})'\rightarrow\psi'(\mathfrak{m},\mathfrak{y})]$$

and hence by the ordinary rules of propositional calculus

$$\vdash\prod_{\mathfrak{y}}\{(\mathfrak{y}=\mathfrak{n})'\rightarrow[\psi(\mathfrak{m},\mathfrak{y})\rightarrow\varphi'(\mathfrak{p},\mathfrak{y})]\}.$$

Combining this with (5) we get $\vdash\prod_{\mathfrak{y}}[\psi(\mathfrak{m},\mathfrak{y})\rightarrow\varphi'(\mathfrak{p},\mathfrak{y})]$, i. e. $\vdash\vartheta'(\mathfrak{p},\mathfrak{m})$. This proves that $\vartheta(\mathfrak{t},\mathfrak{x})$ is decidable.

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14) If $\mathfrak{y}=(y_1,y_2,\dots,y_k)$ and $\mathfrak{z}=(z_1,z_2,\dots,z_k)$, then $\mathfrak{y}=\mathfrak{z}$ means the conjunction $(y_1=z_1)\cdot(y_2=z_2)\dots(y_k=z_k)$.

1.24. *Under the assumptions of* 1.23

$$\vdash\{\sum_{\mathfrak{y}}[\varphi(\mathfrak{t},\mathfrak{y})\cdot\psi(\mathfrak{x},\mathfrak{y})]\rightarrow\prod_{\mathfrak{z}}[\psi(\mathfrak{x},\mathfrak{z})\rightarrow\varphi(\mathfrak{t},\mathfrak{z})]\}.$$

Proof. From (2) we obtain

$$\vdash\{\varphi(\mathfrak{t},\mathfrak{y})\cdot\psi(\mathfrak{x},\mathfrak{y})\cdot\psi(\mathfrak{x},\mathfrak{z})\rightarrow(\mathfrak{y}=\mathfrak{z})\cdot\varphi(\mathfrak{t},\mathfrak{y})\}$$

and therefore

$$\vdash\{\varphi(\mathfrak{t},\mathfrak{y})\cdot\psi(\mathfrak{x},\mathfrak{y})\cdot\psi(\mathfrak{x},\mathfrak{z})\rightarrow\varphi(\mathfrak{t},\mathfrak{z})\}.$$

This yields

$$\vdash\{[\varphi(\mathfrak{t},\mathfrak{y})\cdot\psi(\mathfrak{x},\mathfrak{y})]\rightarrow[\psi(\mathfrak{x},\mathfrak{z})\rightarrow\varphi(\mathfrak{t},\mathfrak{z})]\}.$$

If we now add the general quantifier in the second term and the particular one in the first, we obtain immediately the desired result.

1.25. *The propositional functions* $\chi(x,y,z)$, $\chi_1(x,y)$ *and* $\chi_2(x,z)$ *defined as*

$$x=2^y(2z+1),\qquad \sum_z\chi(x,y,z),\qquad \sum_y\chi(x,y,z)$$

are decidable and fulfill the formulae:

$$\vdash[\chi(x,y,z)\cdot\chi(x,y',z')\rightarrow(y=y')\cdot(z=z')],$$
$$\vdash[\chi_1(x,y)\cdot\chi_1(x,y')\rightarrow(y=y')],$$
$$\vdash[\chi_2(x,z)\cdot\chi_2(x,z')\rightarrow(z=z')].$$

1.3. Definition of classes $P_n^{(k)}$ and $Q_n^{(k)}$. A set $A\subset R_k$ will be said to belong to the class $P_0^{(k)}$ if there is a decidable propositional function $\varphi(\mathfrak{x})$ with k free variables such that

$$\mathfrak{n}\in A\equiv\vdash\varphi(\mathfrak{n})$$

for any $\mathfrak{n}\in R_k$. We say that $\varphi(\mathfrak{x})$ defines A. For reasons of symmetry we shall denote the class $P_0^{(k)}$ also by $Q_0^{(k)}$.

Let us now suppose that $n\geqslant 0$ and that classes $P_n^{(k)}$ and $Q_n^{(k)}$ $(k=1,2,3,\ldots)$ have already been defined. We then say that a set $A\subset R_k$ belongs to the class $P_{n+1}^{(k)}$ if there is a set $B\in Q_n^{(k+1)}$ such that for any $\mathfrak{n}\in R_k$

$$\mathfrak{n}\in A\equiv\sum_p(\mathfrak{n},p)\in B.$$

A set $A\subset R_k$ belongs to $Q_{n+1}^{(k)}$ if $R_k-A\in P_{n+1}^{(k)}$.

The analogy with the theory of projective sets needs not to be emphasised.

The class $P_0^{(k)} = Q_0^{(k)}$ plays in our theory the same role as the class of Borel-subsets of k-dimensional space plays in the theory of projective sets.

We see from the definition that the rules of inference admitted in the system S permit to decide whether any given „point" $\mathfrak{n}$ belongs to any given set A of the class $P_0^{(k)}$ or not. Hence $P_0^{(k)}$ is the class of general recursive sets mentioned in the introduction. This will be proved formally in 6.31.

From classes $P_n^{(k)}$ and $Q_n^{(k)}$ with $n \geqslant 1$ only one as far as I see is known in the literature. It is the class $P_1^{(1)}$ which was called by Kleene the class of recursively enumerable sets[15]). It will be shown later that $A \in P_1^{(1)}$ if and only if A is the set of values of a general recursive function (see 5.61 and 6.23).

The whole sum $\sum\limits_{n=0}^{\infty} [P_n^{(k)} + Q_n^{(k)}]$ may be characterized as the class of sets $A \subset R_k$ which are definable within the elementary arithmetic. The word „definable" is here used in the following sense[16]): a set $A \subset R_k$ is definable within $\mathfrak{A}$ if there is in $\mathfrak{A}$ a propositional function $\varphi(\mathfrak{x})$ with k free variables such that $\mathfrak{n} \in A$ if and only if $\mathfrak{n}$ fulfills $\varphi(\mathfrak{x})$. The proof of the above theorem presents no difficulty for any one who knows the notion of fulfillness[17]). As its exact definition is rather intricate, we shall omit this proof and content ourselves with the remark that the definability of sets belonging to $P_0^{(k)}$ results from theorem 6.31 given below.

The classes $P_n^{(k)}$ and $Q_n^{(k)}$ such as they were defined depend a priori from the logical system S taken as basis and should properly be denoted by symbols $P_n^{(k)}(S)$ and $Q_n^{(k)}(S)$. As a matter of fact they are independent from the system S provided that this system satisfies some very general conditions as will be shown in 6.3.

§ 2. Elementary properties of classes $P_n^{(k)}$ and $Q_n^{(k)}$.

2.1. Sums, common parts and cartesian products. The most important theorems we intend to establish in this section may be stated as follows: the classes $P_n^{(k)}$ and $Q_n^{(k)}$ are rings of sets for any n (2.17); the property to belong to P_n (or Q_n) is invariant

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[15]) Kleene [9], theorem XI, p. 739.

[16]) Tarski [18] p. 312.

[17]) Tarski [18].

under the cartesian multiplication by an axis (2.14); the sum (or the common part) of an enumerable sequence of sets belonging to $\boldsymbol{P}_n^{(k)}$ (or to $\boldsymbol{Q}_n^{(k)}$) belongs under certain assumptions to the same class (2.16). The remaining theorems are lemmas.

2.11. *If* $A \in \boldsymbol{P}_0^{(k)}$ *and* $B \in \boldsymbol{P}_0^{(k)}$, *then* $A+B$, $A \cdot B$ *and* $R_k - A$ *belong to* $\boldsymbol{P}_0^{(k)}$.

This proposition which follows immediately from 1.21 states that $\boldsymbol{P}_0^{(k)}$ is a field of sets.

2.12. *If* $A \in \boldsymbol{P}_n^{(k)}$, *then* $R_k - A \in \boldsymbol{Q}_n^{(k)}$ *and conversely.*

This follows direct from definition.

2.13. (Change of axes). *If* $\pi(1), \pi(2), \ldots, \pi(k)$ *is any permutation of* $1, 2, \ldots, k$ *and if we denote for any* $A \subset R_k$ *by* A_π *the set of all* $(n_{\pi(1)}, n_{\pi(2)}, \ldots, n_{\pi(k)})$ *for which* $(n_1, n_2, \ldots, n_k) \in A$, *then* $A \in \boldsymbol{P}_n^{(k)} \equiv A_\pi \in \boldsymbol{P}_n^{(k)}$ *and* $A \in \boldsymbol{Q}_n^{(k)} \equiv A_\pi \in \boldsymbol{Q}_n^{(k)}$.

The easy proof proceeding by induction on n will not be given here.

2.14. (Cartesian products). *If* $A \in \boldsymbol{P}_n^{(k)}$ *(or* $A \in \boldsymbol{Q}_n^{(k)}$*), then* $A \times R_1 \in \boldsymbol{P}_n^{(k+1)}$ *(or* $A \in \boldsymbol{Q}_n^{(k+1)}$*).*

Proof. Suppose first that $n=0$ and let $\varphi(\mathfrak{x})$ be a decidable propositional function which defines A. The propositional function $\varphi(\mathfrak{x}) \cdot (x=x)$ is of course decidable and defines $A \times R_1$. Hence $A \times R_1 \in \boldsymbol{P}_0^{(k+1)}$.

Suppose now that 2.14 holds for $n<m$ and let $A \in \boldsymbol{P}_m^{(k)}$. By definition there is a set $B \in \boldsymbol{Q}_{m-1}^{(k+1)}$ such that

$$\mathfrak{n} \in A \equiv \sum_q (\mathfrak{n}, q) \in B.$$

The set $B_1 = \underset{(\mathfrak{n},p,q)}{E} [(\mathfrak{n}, q) \in B]$ arises from $B \times R_1$ by interchanging the two last axes and therefore $B_1 \in \boldsymbol{Q}_{m-1}^{(k+2)}$ by 2.13. Since we have obviously

$$(\mathfrak{n}, p) \in A \times R_1 \equiv \sum_q (\mathfrak{n}, p, q) \in B_1,$$

we infer from the definition that $A \times R_1 \in \boldsymbol{P}_m^{(k+1)}$.

Suppose now that $A \in \boldsymbol{Q}_m^{(k)}$, i. e. $R_k - A \in \boldsymbol{P}_m^{(k)}$. If we repeat the above reasoning taking $R_k - A$ instead of A, we obtain $(R_k - A) \times R_1 \in \boldsymbol{P}_m^{(k+1)}$ or passing to complements and using 2.12

$$R_{k+1} - (R_k - A) \times R_1 \in \boldsymbol{Q}_m^{(k+1)}.$$

The left side is identical with $A \times R_1$ what completes the proof.

Let us put for any $A \subset R_{k+2}$

$$A^* = \underset{(\mathfrak{n},p)}{E}[(\mathfrak{n},s_1(p),s_2(p)) \in A].$$

2.15. $A \in P_n^{(k+2)} \equiv A^* \in P_n^{(k+1)}$ *and* $A \in Q_n^{(k+2)} \equiv A^* \in Q_n^{(k+1)}$.

Proof by induction on n. Suppose first that $n=0$ and that $A \in P_0^{(k+2)}$. Let $\varphi(\mathfrak{t},u,v)$ be a decidable propositional function with $k+2$ free variables which defines A and consider the propositional function

$$\sum_u \sum_v \varphi(\mathfrak{t},u,v)\cdot\chi_1(z,u)\cdot\chi_2(z,v),$$

where $\chi_1(z,u)$ and $\chi_2(z,v)$ have the meaning defined in 1.25. It is obvious that this function defines A^*. It is in addition decidable because it has the form considered 1.23 with „(u,v)" instead of „$\mathfrak{y}$" and with „$\chi_1(z,u)\cdot\chi_2(z,v)$" instead of „$\psi(\mathfrak{x},\mathfrak{y})$". The assumption (2) of 1.23 is satisfied in virtue of 1.25 whereas (3) is obvious. This proves that $A^* \in P_0^{(k+1)}$.

Suppose now conversely that $A^* \in P_0^{(k+1)}$ and that $\varphi(\mathfrak{t},w)$ is a decidable propositional function which defines A^*. By 1.23 and 1.25 the propositional function

$$\sum_w \varphi(\mathfrak{t},w)\cdot\chi(w,u,v)$$

is decidable and defines A. Therefore $A \in P_0^{(k+2)}$.

The theorem is thus proved for $n=0$.

Suppose now that $m>0$ and that the theorem holds for $n<m$. Let $A \in P_m^{(k+2)}$. Hence there is a set $B \in Q_{m-1}^{(k+3)}$ such that

$$(\mathfrak{n},p,q) \in A \equiv \sum_h (\mathfrak{n},p,q,h) \in B.$$

The set

$$B_1 = \underset{(\mathfrak{n},l,h)}{E}[(\mathfrak{n},s_1(l),s_2(l),h) \in B]$$

arises from B^* by interchanging the two last axes; consequently $B_1^* \in Q_{m-1}^{(k+2)}$ by 2.13 and the inductive assumption. Now we see that

$$(\mathfrak{n},l) \in A^* \equiv (\mathfrak{n},s_1(l),s_2(l)) \in A \equiv \sum_h (\mathfrak{n},s_1(l),s_2(l),h) \in B \equiv \sum_h (\mathfrak{n},l,h) \in B_1^*$$

which proves according to the definition 1.3 that $A^* \in P_m^{(k+1)}$. Hence $A \in P_m^{(k+2)} \rightarrow A^* \in P_m^{(k+1)}$.

Suppose now that $A^* \in P_m^{(k+1)}$, i. e. that there is a set $B \in Q_{m-1}^{(k+2)}$ such that

$$(\mathfrak{n},l) \in A^* \equiv \sum_h (\mathfrak{n},l,h) \in B.$$

Let B_1, B_2 and B_3 be defined by the equivalences

$$(\mathfrak{n},k,l)\in B_1 \equiv (\mathfrak{n},l,h)\in B,$$
$$(\mathfrak{n},h,p,q)\in B_2 \equiv (\mathfrak{n},h,2^p(2q+1))\in B_1,$$
$$(\mathfrak{n},p,q,h)\in B_3 \equiv (\mathfrak{n},h,p,q)\in B_2.$$

Obviously $B_2^* = B_1$. Since $B_1 \in Q_{m-1}^{(k+2)}$ by 2.13, we obtain from the inductive assumption $B_2 \in Q_{m-1}^{(k+3)}$ and again by 2.13 $B_3 \in Q_{m-1}^{(k+3)}$. Now observe that

$$(\mathfrak{n},p,q)\in A \equiv (\mathfrak{n},2^p(2q+1))\in A^* \equiv \sum_h (\mathfrak{n},2^p(2q+1),h)\in B$$
$$\equiv \sum_h (\mathfrak{n},h,2^p(2q+1))\in B_1 \equiv \sum_h (\mathfrak{n},h,p,q)\in B_2 \equiv \sum_h (\mathfrak{n},p,q,h)\in E_3.$$

This equivalence proves that $A \in P_m^{(k+2)}$ and hence

$$A \in P_m^{(k+2)} \equiv A^* \in P_m^{(k+1)}.$$

Passing to complements and observing that

$$(R_{k+2}-A)^* = R_{k+1}-A^*,$$

we obtain immediately

$$A \in Q_m^{(k+2)} \equiv A^* \in Q_m^{(k+1)}.$$

The theorem 2.15 is thus proved completely.

We put for any $A \subset R_{k+1}$

$$A_s = \underset{\mathfrak{n}}{E}\,[\sum_q (\mathfrak{n},q)\in A], \qquad A_p = \underset{\mathfrak{n}}{E}\,[\prod_q (\mathfrak{n},q)\in A].$$

2.16. (Infinite sums and products). *If* $n \geqslant 1$, *then*

$$A \in P_n^{(k+1)} \rightarrow A_s \in P_n^{(k)} \quad \textit{and} \quad A \in Q_n^{(k+1)} \rightarrow A_p \in Q_n^{(k)}.$$

Proof. Suppose first that $A \in P_n^{(k+1)}$. For a suitable $B \in Q_{n-1}^{(k+2)}$ we have the equivalence

$$(\mathfrak{n},p)\in A \equiv \sum_q (\mathfrak{n},p,q)\in B.$$

Remembering the definition of the set B^* we obtain

$$\mathfrak{n}\in A_s \equiv \sum_p (\mathfrak{n},p)\in A \equiv \sum_p\sum_q (\mathfrak{n},p,q)\in B \equiv \sum_h (\mathfrak{n},h)\in B^*,$$

for putting $h = 2^p(2q+1)$ we have $(\mathfrak{n},p,q)\in B \equiv (\mathfrak{n},h)\in B^*$. Since $B^* \in Q_{n-1}^{(k+1)}$, the above equivalence proves that $A_s \in P_n^{(k)}$. In order to obtain the second result stated in the theorem it is now sufficient to observe that

$$R_k - A_p = (R_k - A)_s.$$

2.17. (The ring property). *If A and B belong to $\boldsymbol{P}_n^{(k)}$ (or $\boldsymbol{Q}_n^{(k)}$), then $A+B$ and $A\cdot B$ belong to the same class.*

Proof. In view of 2.11 we may suppose that $n \geqslant 1$ and that the theorem holds for lower values of n. From $A\,\epsilon\,\boldsymbol{P}_n^{(k)}$ and $B\,\epsilon\,\boldsymbol{P}_n^{(k)}$ we infer that there are sets C and D belonging both to $\boldsymbol{Q}_{n-1}^{(k+1)}$ such that

$$\mathfrak{n}\,\epsilon\, A \equiv \sum_p (\mathfrak{n},p)\,\epsilon\, C, \qquad \mathfrak{n}\,\epsilon\, B \equiv \sum_p (\mathfrak{n},p)\,\epsilon\, D.$$

From these equivalences we immediately obtain

$$\mathfrak{n}\,\epsilon\, A+B \equiv \sum_p (\mathfrak{n},p)\,\epsilon\, C+D$$

and hence $A+B\,\epsilon\,\boldsymbol{P}_n^{(k)}$ since $C+D\,\epsilon\,\boldsymbol{Q}_{n-1}^{(k+1)}$ by the inductive assumption

Consider now the cartesian products $C\times \boldsymbol{R}_1$ and $D\times \boldsymbol{R}_1$ and denote by $\bar{D}$ the set arising from $D\times\boldsymbol{R}_1$ by interchanging the two last axes. Putting for symmetry $\bar{C}=C\times\boldsymbol{R}_1$, we have

$$(\mathfrak{n},p,q)\,\epsilon\,\bar{C} \equiv (\mathfrak{n},p)\,\epsilon\, C,$$
$$(\mathfrak{n},p,q)\,\epsilon\,\bar{D} \equiv (\mathfrak{n},q)\,\epsilon\, D$$

and $\bar{C},\bar{D}\,\epsilon\,\boldsymbol{Q}_{n-1}^{(k+2)}$ in virtue of 2.13 and 2.14. By the inductive assumption we infer that $\bar{C}\cdot\bar{D}\,\epsilon\,\boldsymbol{Q}_{n-1}^{(k+2)}$ and since

$$\begin{aligned}\mathfrak{n}\,\epsilon\, A\cdot B &\equiv \Big[\sum_p (\mathfrak{n},p)\,\epsilon\, C\Big]\cdot\Big[\sum_q (\mathfrak{n},q)\,\epsilon\, D\Big]\equiv\\ &\equiv \sum_p\sum_q \big[(\mathfrak{n},p)\,\epsilon\, C\big]\cdot\big[(\mathfrak{n},q)\,\epsilon\, D\big]\equiv\sum_p\sum_q (\mathfrak{n},p,q)\,\epsilon\,\bar{C}\cdot\bar{D},\end{aligned}$$

we infer from 2.16 that $A\cdot B\,\epsilon\,\boldsymbol{P}_n^{(k)}$.

Passing to complements and applying 2.12 we obtain the further result that if A and B are in $\boldsymbol{Q}_n^{(k)}$, then $A+B$ and $A\cdot B$ are both in $\boldsymbol{Q}_n^{(k)}$. The theorem 2.17 is thus proved.

It will be proved in 3.32 that neither $\boldsymbol{P}_n^{(k)}$ nor $\boldsymbol{Q}_n^{(k)}$ is a field of sets for $n\geqslant 1$, i. e., that the difference of two sets of the class $\boldsymbol{P}_n^{(k)}$ (or $\boldsymbol{Q}_n^{(k)}$) does not, in general, belong to this class. From 2.17 and 2.12 we obtain however

2.18. *The common part $\boldsymbol{P}_n^{(k)}\cdot\boldsymbol{Q}_n^{(k)}$ is a field of sets for any $n\geqslant 0$.*

The sense of this proposition is that the class $\boldsymbol{P}_n^{(k)}\cdot\boldsymbol{Q}_n^{(k)}$ is closed under the three operations $A+B$, $A\cdot B$ and $A-B$.

2.2. The Kuratowski-Tarski method. This well known method permits to evaluate the Borel class or the projective class of any set provided that its definition has been written down in logical symbols[18]. The method is based on propositions of exactly the same form as our theorems 2.11-2.15 and 2.17[19]. Hence imitating this method we may from the mere form of the definition of any given set $A \subset R_k$ evaluate a n for which $A \epsilon \boldsymbol{P}_n^{(k)}$ or $A \epsilon \boldsymbol{Q}_n^{(k)}$. This illustrates the importance of theorems established in **2.1.**

2.3. Inclusions. Between the classes $\boldsymbol{P}_n^{(k)}$, $\boldsymbol{P}_{n+1}^{(k)}$, $\boldsymbol{Q}_n^{(k)}$ and $\boldsymbol{Q}_{n+1}^{(k)}$ hold following inclusions:

2.31. $\boldsymbol{P}_n^{(k)} \subset \boldsymbol{P}_{n+1}^{(k)} \cdot \boldsymbol{Q}_{n+1}^{(k)}$ *and* $\boldsymbol{Q}_n^{(k)} \subset \boldsymbol{P}_{n+1}^{(k)} \cdot \boldsymbol{Q}_{n+1}^{(k)}$.

Proof. Suppose that $A \epsilon \boldsymbol{P}_n^{(k)}$ and put $A_1 = A \times R_1$. Evidently $\mathfrak{n} \epsilon A \equiv \prod_p (\mathfrak{n}, p) \epsilon A_1 \equiv [\sum_p (\mathfrak{n}, p) \epsilon R_{k+1} - A_1]'$; since $R_{k+1} - A_1 \epsilon \boldsymbol{Q}_n^{(k+1)}$ by 2.12 and 2.14, we infer direct from the definition **1.3** that $A' \epsilon \boldsymbol{Q}_{n+1}^{(k)}$. Hence

$$\boldsymbol{P}_n^{(k)} \subset \boldsymbol{Q}_{n+1}^{(k)}. \tag{6}$$

Passing to complements, we obtain

$$\boldsymbol{Q}_n^{(k)} \subset \boldsymbol{P}_{n+1}^{(k)}. \tag{7}$$

This gives for $n=0$ the inclusions $\boldsymbol{Q}_0^{(k)} \subset \boldsymbol{Q}_1^{(k)}$ and $\boldsymbol{P}_0^{(k)} \subset \boldsymbol{P}_1^{(k)}$. Suppose now that $m > 0$ and that the inclusions

$$\boldsymbol{P}_n^{(k)} \subset \boldsymbol{P}_{n+1}^{(k)}, \qquad \boldsymbol{Q}_n^{(k)} \subset \boldsymbol{Q}_{n+1}^{(k)} \qquad (k=1,2,\ldots) \tag{8}$$

are valid for $n < m$. If $A \epsilon \boldsymbol{P}_m^{(k)}$, then, for a suitable $B \epsilon \boldsymbol{Q}_{m-1}^{(k+1)}$, we have $\mathfrak{n} \epsilon A \equiv \sum_p (\mathfrak{n}, p) \epsilon B$ which proves that $A \epsilon \boldsymbol{P}_{m+1}^{(k)}$ since $B \epsilon \boldsymbol{Q}_m^{(k+1)}$ by the inductive assumption. Hence the first inclusion (8) is true for $n = m$ and passing to complements we obtain the same result for the second one. (8) is thus true for any n. The theorem results now from (6), (7) and (8).

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[18] Kuratowski-Tarski [11], p. 242, Kuratowski [10], p. 168 and 243.

[19] There are, however, no rules in our theory which would correspond to theorems concerning infinite sums or products of Borel (or projective) sets.

[13] J. B. Rosser, *Gödel's theorems for non constructive logics*. The Journal of Symbolic Logic, vol. **2** (1937), pp. 129-137.

[14] J. B. Rosser, *Extensions of some theorems of Gödel and Church*. Ib., vol. **1** (1936), pp. 87-91.

[15] J. B. Rosser, Review of [5]. Ib., vol. 1 (1936), p. 116.

[16] Alfred Tarski, *Fundamentale Begriffe de Methodologie der deduktiven Wissenschaften I*. Monatshefte für Mathematik uud Physik, vol. **37** (1930), pp. 361-404.

[17] Alfred Tarski, *Einige Betrachtungen über die Begriffe der ω-Widerspruchsfreiheit und ω-Vollständigkeit*. Ib., vol. **40** (1933), pp. 97-112.

[18] Alfred Tarski, *Der Wahrheitsbegriff in den formalisierten Sprachen*. Studia Philosophica, vol. **1** (1935), pp. 261-405.

[19] Alfred Tarski, *Sur les ensembles définissables de nombres réels*. Fundamenta Mathematicae, vol. **17** (1931), pp. 210-239.

[20] Alfred Tarski, *On undecidable statements in enlarged systems of logic and the concept of truth*. The Journal of Symbolic Logic, vol. **4** (1939), pp. 105-112.

[21] A. M. Turing, *On computable numbers, with an application to the Entscheidungsproblem*. Proceedings of the London Mathematical Society (2), vol. **42** (1937), pp. 230-265.

Note. This paper was already under press, when an interesting paper of S. C. Kleene, *Recursive predicates and quantifiers* (Transactions of the American Mathematical Society, vol. **53** (1943), pp. 41-83) became available in Poland.

A considerable part of the theory developed above is to be found in the Kleene's paper. It seems me, however, that some of my results are new (e.g. remarks 4.3) and that my presentation of the theory based on analogies with the theory of projective sets may be of some interest for a mathematical reader.

Professor A. Tarski informed me that he also found already in 1942 results very similar to mines.

3.21. *If S fulfills the condition C_s, then there is a universal function $F_0^{(k)}(h)$ for the class $P_0^{(k)}$ such that the set*

$$M_0^{(k)} = \underset{(\mathfrak{n},h)}{E}[\mathfrak{n} \in F_0^{(k)}(h)]$$

belongs to $Q_{s+2}^{(k+1)}$.

Proof.[22]). An integer h is the Gödel-number of a decidable propositional function with k free variables if and only if

$$\Pi_{\mathfrak{n}} \sum_{q} \sum_{l} \sum_{m} \{(h,l,\mathfrak{n}) \in \Gamma_k \cdot (l,m) \in H \cdot [(q,l) \in \Delta + (q,m) \in \Delta]\}.$$

Hence denoting with Θ_k the set of these numbers we infer by the Kuratowski-Tarski method that

$$\Theta_k \in Q_{s+2}^{(1)} \text{ }^{23}). \tag{9}$$

We put now $F_0^{(k)}(h) = 0$ for h non $\in \Theta_k$ and

$$F_0^{(k)}(h) = \underset{\mathfrak{n}}{E}[\sum_{p} \sum_{q} (h,p,\mathfrak{n}) \in \Gamma_k \cdot (q,p) \in \Delta]$$

for $h \in \Theta_k$. The set $M_0^{(k)}$ belongs then to $Q_{s+2}^{(k+1)}$ as we easily see from its definition

$$(\mathfrak{n},h) \in M_0^{(k)} \equiv (h \in \Theta_k) \cdot \sum_{p} \sum_{q} [(h,p,\mathfrak{n}) \in \Gamma_k \cdot (q,p) \in \Delta]$$

using (9), 2.31 and the Kuratowski-Tarski evaluation method.

It remains to prove that $F_0^{(k)}(h)$ is a universal function for the class $P_0^{(k)}$.

Let us suppose that $A \in P_0^{(k)}$ and that $\varphi(\mathfrak{x})$ defines A. If h is the Gödel-number of $\varphi(\mathfrak{x})$, then $h \in \Theta_k$. Let $\mathfrak{n} \in A$ and let p be the Gödel-number of $\varphi(\mathfrak{n})$. Then $(h,p,\mathfrak{n}) \in \Gamma_k$. Since $\vdash \varphi(\mathfrak{n})$, there is a formal proof of $\varphi(\mathfrak{n})$. Denoting with q its Gödel-number, we have $(q,p) \in \Delta$. Now from $h \in \Theta_k$, $(h,p,\mathfrak{n}) \in \Gamma_k$ and $(q,p) \in \Delta$ we obtain $\mathfrak{n} \in F_0^{(k)}(h)$. If, conversely, $\mathfrak{n} \in F_0^{(k)}(h)$, then there are p, q such that $(h,p,\mathfrak{n}) \in \Gamma_k$ and $(q,p) \in \Delta$. Hence p is the Gödel-number of $\varphi(\mathfrak{n})$ and q is the Gödel-number of a formal proof of $\varphi(\mathfrak{n})$. Hence there is at least one formal proof of $\varphi(\mathfrak{n})$ which proves that $\vdash \varphi(\mathfrak{n})$ i. e. $\mathfrak{n} \in A$; therefore $A = F_0^{(k)}(h)$ which proves that

$$A \in P_0^{(k)} \rightarrow \sum_{h} A = F_0^{(k)}(h).$$

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[22]) This proof is essentially due to Kleene [9], theorem IV, p. 736.

[23]) More exactly: for $s=0$ $\Theta_k \in Q_2^{(1)}$ and for $s>0$ $\Theta_k \in Q_{s+1}^{(1)}$. The formula (9) includes both cases.

Suppose, conversely, that $A = \mathbf{F}_0^{(k)}(h)$. If h non $\epsilon\, \Theta_k$, then $A = 0$ and therefore $A \,\epsilon\, \mathbf{P}_0^{(k)}$. If $h \,\epsilon\, \Theta_k$, let $\varphi(x)$ be the decidable propositional function whose Gödel-number is h. We prove similar as above that $\mathfrak{n} \,\epsilon\, A \equiv\, \vdash \varphi(\mathfrak{n})$ and therefore $A \,\epsilon\, \mathbf{P}_0^{(k)}$. Hence

$$\sum_h A = \mathbf{F}_0^{(k)}(h) \rightarrow A \,\epsilon\, \mathbf{P}_0^{(k)}.$$

This completes the proof of 3.21.

3.22 *If S fulfills the condition C_s, then there are functions $\mathbf{F}_n^{(k)}(h)$ and $G_n^{(k)}(h)$ universal for classes $\mathbf{P}_n^{(k)}$ and $Q_n^{(k)}$ and such that the sets*

$$M_n^{(k)} = \underset{(\mathfrak{n},h)}{E}[\mathfrak{n} \,\epsilon\, \mathbf{F}_n^{(k)}(h)] \quad \textit{and} \quad N_n^{(k)} = \underset{(\mathfrak{n},h)}{E}[\mathfrak{n} \,\epsilon\, G_n^{(k)}(h)]$$

belong respectively to $\mathbf{P}_{s+n+2}^{(k+1)}$ or to $Q_{s+n+2}^{(k+1)}$ if $n > 0$ and to $Q_{s+2}^{(k+1)}$ if $n = 0$.

Proof. The theorem was proved in 3.21 for $n = 0$. Suppose that $n \geqslant 0$ and the theorem is true for this value of n and for $k = 1, 2, \ldots$ Put

$$\mathbf{F}_{n+1}^{(k)}(h) = \underset{\mathfrak{n}}{E}\,[\sum_p (\mathfrak{n}, p) \,\epsilon\, G_n^{(k+1)}(h)],$$
$$G_{n+1}^{(k)}(h) = R_k - \mathbf{F}_{n+1}^{(k)}(h).$$

If h is any integer, then $\mathbf{F}_{n+1}^{(k)}(h) \,\epsilon\, \mathbf{P}_{n+1}^{(k)}$, since

$$\mathfrak{n} \,\epsilon\, \mathbf{F}_{n+1}^{(k)}(h) \equiv \sum_p [(\mathfrak{n}, p) \,\epsilon\, G_n^{(k+1)}(h)]$$

and $G_n^{(k+1)}(h) \,\epsilon\, Q_n^{(k+1)}$ by the inductive assumption. Suppose, conversely, that $A \,\epsilon\, \mathbf{P}_{n+1}^{(k)}$, i. e.

$$\mathfrak{n} \,\epsilon\, A \equiv \sum_p (\mathfrak{n}, p) \,\epsilon\, B$$

for a suitable $B \,\epsilon\, Q_n^{(k+1)}$. The function $G_n^{(k+1)}(h)$ being universal for $Q_n^{(k+1)}$, there is an h such that $B \,\epsilon\, G_n^{(k+1)}(h)$ and therefore $A = \mathbf{F}_n^{(k+1)}(h)$. Hence $\mathbf{F}_{n+1}^{(k)}(h)$ is a universal function for $\mathbf{P}_{n+1}^{(k)}$. Passing to complements we immediately see that $G_{n+1}^{(k)}(h)$ is universal for $Q_{n+1}^{(k)}$.

It remains to consider the sets $M_{n+1}^{(k)}$ and $N_{n+1}^{(k)}$. According to their definitions we have

$$(\mathfrak{n}, h) \,\epsilon\, M_{n+1}^{(k)} \equiv \mathfrak{n} \,\epsilon\, \mathbf{F}_{n+1}^{(k)}(h) \equiv \sum_p (\mathfrak{n}, p) \,\epsilon\, G_n^{(k+1)}(h) \equiv \sum_p (\mathfrak{n}, p, h) \,\epsilon\, N_n^{(k+1)}$$

and this proves that $M^{(k)}_{n+1} \epsilon P^{(k+1)}_{s+n+3}$, since $N^{(k+1)}_{n} \epsilon Q^{(k+2)}_{s+n+2}$ by the inductive assumption. Further we have

$$(\mathfrak{n},h) \epsilon N^{(k)}_{n+1} \equiv \mathfrak{n} \epsilon G^{(k)}_{n+1}(h) \equiv \mathfrak{n} \text{ non } \epsilon F^{(k)}_{n+1}(h) \equiv (\mathfrak{n},h) \epsilon R_{k+1} - M^{(k)}_{n+1}$$

and therefore $N^{(k)}_{n+1} \epsilon Q^{(k+1)}_{s+n+3}$. The theorem is thus proved completely.

3.3. Existence-theorems. They follow now easily by the well-known Cantor's diagonal-theorem[24]).

3.31. *If S fulfills the condition C_s, then $P^{(k)}_n \neq P^{(k)}_{n+1}$ and $Q^{(k)}_n \neq Q^{(k)}_{n+1}$ for any $n \geqslant 0$ and $k \geqslant 1$.*

Proof. Let us suppose that S fulfills the condition C_s and that $P^{(k)}_n = P^{(k)}_{n+1}$ for some k and n. We shall show by induction on m that then

$$P^{(k)}_n = P^{(k)}_m = Q^{(k)}_m \qquad \text{for} \qquad m \geqslant n. \tag{10}$$

This holds for $m = n$ since we have $Q^{(k)}_n \subset P^{(k)}_{n+1} = P^{(k)}_n$ which implies $P^{(k)}_n \subset Q^{(k)}_n$ and therefore $P^{(k)}_n = Q^{(k)}_n$. Now suppose that (10) holds for an integer $m \geqslant n$. Obviously $P^{(k)}_n \subset P^{(k)}_{m+1}$. If $A \epsilon P^{(k)}_{m+1}$, then there is a set $B \epsilon Q^{(k+1)}_m$ such that

$$\mathfrak{n} \epsilon A \equiv \sum_p (\mathfrak{n},p) \epsilon B.$$

Let us write $(n_1, n_2, \dots, n_k)$ instead of $\mathfrak{n}$ and consider the set (see 2.15)

$$B^* = \mathop{E}_{(n_1,\dots,n_{k-1},q)} [(n_1,\dots,n_{k-1},s_1(q),s_2(q)) \epsilon B].$$

Since $B^* \epsilon Q^{(k)}_m$, we have $B^* \epsilon P^{(k)}_n$ in virtue of the inductive assumption and the equivalence

$$\mathfrak{n} \epsilon A \equiv \sum_q (s_1(q) = n_k) \cdot [(n_1,\dots,n_{k-1},q) \epsilon B^*]$$

proves that $A \epsilon P^{(k)}_{n+1} = P^{(k)}_n$. Hence $P^{(k)}_{m+1} = P^{(k)}_n$. Passing to complements we obtain $Q^{(k)}_{m+1} = Q^{(k)}_n = P^{(k)}_n$. The formula (10) is thus proved.

Consider now the universal function $F^{(k)}_n(h)$ defined in 3.22 and put

$$A = \mathop{E}_{(n_1,\dots,n_{k-1},h)} [(n_1,\dots,n_{k-1},h) \text{ non } \epsilon F^{(k)}_n(h)].$$

---

[24]) Kuratowski [10], p. 175.

This set does not belong to $P_n^{(k)}$ because otherwise there would be an integer h_0 such that $A = F_n^{(k)}(h_0)$ which is impossible since we would then have

$$(h_0,h_0,\dots,h_0) \in A \equiv (h_0,h_0,\dots,h_0) \text{ non} \in F_n^{(k)}(h_0) \equiv (h_0,h_0,\dots,h_0) \text{ non} \in A.$$

Observe now that

$$(n_1,\dots,n_{k-1},h) \in A \equiv \sum_q (q=h)\cdot[(n_1,\dots,n_{n-1},q) \text{ non} \in F_n^{(k)}(h)] \equiv$$
$$\equiv \sum_q (q=h)\cdot[(n_1,\dots,n_{k-1},q,h) \in R_{k+1} - M_n^{(k)}]$$

which proves according to 3.22 that $A \in P_{s+n+3}^{(k)}$ and consequently $A \in P_n^{(k)}$ according to (10). The assumption $P_n^{(k)} = P_{n+1}^{(k)}$ leads thus to a contradiction.

The inequality $Q_n^{(k)} \neq Q_{n+1}^{(k)}$ will now result if we pass to complements on both sides of $P_n^{(k)} \neq P_{n+1}^{(k)}$.

3.32. *If S fulfills the condition C_s and if $n>0$, then $P_n^{(k)}$ non $\subset Q_n^{(k)}$ and $Q_n^{(k)}$ non $\subset P_n^{(k)}$.*

Proof. From $P_n^{(k)} \subset Q_n^{(k)}$ we obtain $Q_n^{(k)} \subset P_n^{(k)}$ and hence

$$P_n^{(k)} = Q_n^{(k)}. \tag{11}$$

Let us first suppose that $k>1$ and $A \in P_{n+1}^{(k-1)}$. For a suitable $B \in Q_n^{(k)}$ we have

$$n \in A \equiv \sum_p (n,p) \in B.$$

B being in $Q_n^{(k)}$, it is also in $P_n^{(k)}$ in virtue of (11) and hence $A \in P_n^{(k-1)}$ by 2.16. We obtain thus $P_{n+1}^{(k-1)} \subset P_n^{(k-1)}$ against 3.31.

Suppose now that $k=1$. If $A \in P_n^{(1)}$ or $A \in Q_n^{(1)}$, then the set $A_1 = \underset{(p,q)}{E}[2^p(2q+1) \in A]$ belongs to $P_n^{(2)}$ or $Q_n^{(2)}$ since $A_1^* = A$ (comp. 2.15). It is obvious that every set of $P_n^{(2)}$ or $Q_n^{(2)}$ may, for a suitable $A \in P_n^{(1)}$ or $A \in Q_n^{(1)}$, be represented as A_1. Hence $P_n^{(1)} = Q_n^{(1)}$ would lead to the equality $P_n^{(2)} = Q_n^{(2)}$ which we already know to be impossible.

We have thus proved that $P_n^{(k)}$ non $\subset Q_n^{(k)}$ for any k and $n>0$. Passing to complements we obtain $Q_n^{(k)}$ non $\subset P_n^{(k)}$, q. e. d.

We note at last the following result concerning the existence of sets not definable within arithmetic i. e. belonging to no class $P_n^{(k)}$ [25]).

3.33. *The function* $F^{(k)}_{s_1(n)}(s_2(n))$ *is universal for the class* $\sum_{l=0}^{\infty} P_l^{(k)}$ *and the set*

$$A_0 = \underset{(n_1,\dots,n_{k-1},n)}{E} [(n_1,\dots,n_{k-1},n) \text{ non } \epsilon\ F^{(k)}_{s_1(n)}(s_2(n))]$$

does not belong to the sum $\sum_{l=0}^{\infty} P_l^{(k)}$.

Proof. The first part of the theorem results from the equivalence

$$A \,\epsilon \sum_{l=0}^{\infty} P_l^{(k)} \equiv \sum_{l} A \,\epsilon\, P_l^{(k)} \equiv \sum_{l}\sum_{h} A = F_l^{(k)}(h) \equiv \sum_{n} A = F^{(k)}_{s_1(n)}(s_2(n)).$$

The second part is a particular case of the „diagonal theorem" referred to in the foot-note [24]).

§ 4. Applications to theorems of Gödel and Rosser.

4.1. ω-consistency. Let us recall the following definition due to Gödel[26]): A logical system S is called ω-consistent if, for any propositional function $\varphi(x)$ with one free variable, the following implication holds:

$$\prod_{n} \vdash \varphi(n) \rightarrow \text{non} \vdash \sum_{y} \varphi'(y). \tag{12}$$

We could, of course, replace this implication by

$$\vdash \sum_{y} \varphi(y) \rightarrow \sum_{n} \vdash \varphi(n).$$

It is important to observe that the quantifier „$\sum_{n}$" is taken meta-mathematically whereas „$\sum_{y}\varphi(y)$" represents a sentence of the formal system S.

4.2. Gödel's theorem. It states

4.21. *If the system S is ω-consistent and fulfills the condiiton C_s, then there is a sentence ϑ such that neither $\vdash \vartheta$ nor $\vdash \vartheta'$.*

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[25]) The existence of not definable sets has been stated by Tarski [19], p. 221. See also Carnap [1], p. 89.

[26]) Gödel [3], p. 187.

Proof. According to 3.31 the class $P_1^{(1)}-P_0^{(1)}$ is non-empty. Let A by any set of this class and let $B \epsilon P_0^{(2)}$ be a set such that $n \epsilon A \equiv \sum\limits_p (n,p) \epsilon B$. Denoting by $\varphi(x,y)$ any decidable propositional function which defines B we have

$$n \epsilon A \equiv \sum_p \vdash \varphi(n,p). \tag{13}$$

Write $\psi(x)$ instead of $\sum\limits_y \varphi(x,y)$, The formula (13) gives then

$$n \epsilon A \rightarrow \vdash \psi(n)$$

since $\vdash \varphi(n,p) \rightarrow \vdash \sum\limits_y \varphi(n,y)$. If n non ϵA, then $\prod\limits_p$ non $\vdash \varphi(n,p)$ and therefore $\prod\limits_p \vdash \varphi'(n,p)$ because of the decidability of $\varphi(x,y)$. Using (12) we obtain non $\vdash \sum\limits_y \varphi(n,y)$, i.e. non $\vdash \psi(n)$. Hence n non $\epsilon A \rightarrow$ non $\vdash \psi(n)$ and we see that

$$n \epsilon A \equiv \vdash \psi(n). \tag{14}$$

This equivalence would prove that $A \epsilon P_0^{(1)}$ if $\psi(x)$ were decidable. Since A non $\epsilon P_0^{(1)}$, $\psi(x)$ cannot be decidable, i. e. there is an integer n_0 such that neither $\vdash \psi(n_0)$ nor $\vdash \psi'(n_0)$. Denoting $\psi(n_0)$ by ϑ we obtain the desired result.

4.3. Remarks. 4.31. Theorem 4.21 was first established by Gödel for a concrete formal system called P [27]. Rosser [28] generalised this result showing that it holds for any system S in which the Gödel-numbers of valid sentences form a recursively enumerable set. This is essentially the same assumption as our condition C_0. Our proof of 4.21 shows that the theorem holds even under the weaker condition C_s. Hence the Gödel's theorem is valid for all such systems S in which the set of Gödel-numbers of valid sentences is definable in elementary arithmetic.

4.32. If S satisfies the stronger condition C_0, then as shown by Rosser [29] the assumption of ω-consistency can by replaced by the (weaker) assumption of ordinary self-consistency of S. This is in general impossible for systems satisfying the weaker condition C_s ($s>0$) since there exists a logical system S such that its valid sentences form a self-consistent and complete [30] class whereas the set of theirs Gödel-number is definable within $\mathfrak{A}$ (i. e. be-

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[27] Gödel [3]. Satz VI.

[28] Rosser [14], Theorem I A, p. 89.

[29] Rosser [14], Theorem II, p. 89.

[30] I. e. for any ϑ either ϑ or ϑ' is valid.

longs to one of classes $P_s^{(1)}$). Hence S fulfills the condition C_s for an $s > 0$ and there are no undecidable propositions in S. In order to get such a system it is sufficient to apply to system P the procedure with the help of which Lindenbaum has shown that there are complete and self-consistent enlargements of any self-consistent class of sentences[31]).

4.33. It follows from 4.21 that any complete and self-consistent class C of sentences (e. g. of the system P or, more general, of any system which fulfills our condition C_s for an $s \geqslant 0$) must be ω-inconsistent if the set of Gödel-numbers of sentences of this class is definable within the elementary arithmetic $\mathfrak{A}$ (i. e. belongs to $\sum_{n=0}^{\infty} P_n^{(1)}$). It has been stated already by Tarski[32]) that under these conditions C must contain false statements. The ω-inconsistency of C seems to be a new result[32a]).

4.34. The proof of 4.21 remains still valid if we modify the definition 4.1 restricting (12) to decidable propositional functions. The question remains open whether there is an ω-inconsistent system S satisfying this modified definition of ω-consistency.

4.35. Remark 4.31 makes desirable examples of formal systems satisfying C_s for some $s > 0$ but not C_0. One such example will be treated in **4.4** and **4.5** in connection with the so called rule of infinite induction. Another example may be suggested here: Suppose that S contains variables $X, Y, Z, \ldots$ of the type of classes of positive integers and enlarge the system adding to its rules the following one: if $\varphi(A)$ is valid in S for any set A definable within S, then $\prod_X \varphi(X)$ is valid in the enlarged system. The enlarged system is probably self-consistent and fulfills C_s for some $s > 0$ but not for $s = 0$.

4.4. Rule of infinite induction[33]). This rule states that $\prod_x \varphi(x)$ may be admitted as proved if $\prod_p \vdash \varphi(p)$.

4.41. *Under the assumptions of* 4.21 *the system S is not closed under the rule of infinite induction*[34]).

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[31]) See Tarski [16], Satz I. 56, p. 394. The same result is to be found in Gödel [6], pp. 20-21. The theorem in question is also known in the theory of Boolean algebras as the „fundamental theorem of ideal-theory". See e. g. my paper in the previous volume of these Fundamenta, pp. 7-8, footnote [5]).

[32]) Tarski [18], p. 378.

[32a]) This result has been also found independently by A. Tarski in 1942.

[33]) This rule has been introduced by Hilbert [7] which ascribed it a finitary character. The rule was further studied by Tarski [18], pp. 383-387, Carnap [1], p. 26 and Rosser [13], pp. 129-133. This last author calls it „Carnap rule".

[34]) Tarski says: S is ω-incomplete. See [17], p. 105. The theorem 4.41 has been proved by Gödel [3], p. 190 and generalised afterward by Rosser [14], p. 89. Comp. remark 4.31.

Proof. Glancing at the proof of 4.21 we see that the number n_0 which has been defined there does not belong to A since otherwise we would obtain $\vdash \psi(n_0)$ in virtue of (14). Hence (13) gives $\prod_p \vdash \varphi(n_0,p)$ whereas non $\vdash \psi'(n_0)$ yields non $\vdash \prod_x \varphi(n_0,x)$.

We introduce now the concept of an n-valid sentence (in symbols $\vdash_n \varphi$) 35)). For $n=0$ we define $\vdash_0 \varphi$ as $\vdash \varphi$. Suppose now that $n>0$ and that the class of $n-1$-valid sentences has already been defined. We shall write $\vdash_n \varphi$ (read: φ is an n-valid sentence) if φ belongs to every class C satisfying three following conditions:

If $\vdash_{n-1}\psi$, *then* ψ *is in* C;
C *is closed under the rules of inference of* S;
If $\prod_p \vdash_{n-1}\psi(p)$, *then* $\prod_x \psi(x)$ *is in* C.

Speaking less formally, we could say that $\vdash_n \varphi$ holds if and only if φ can be obtained from the axioms of S with the help of rules of inference admitted in S and with the help of the rule of infinite induction, this last rule being used but n times.

A propositional function $\varphi(\mathfrak{x})$ with k free variables will be said to be n-decidable if for any $\mathfrak{n} \in R_k$ either $\vdash_n \varphi(\mathfrak{n})$ or $\vdash_n \varphi'(\mathfrak{n})$.

4.42. *If the class of n-valid sentences is self-consistent and if* $A \in P_n^{(k)} + Q_n^{(k)}$, *then there is an n-decidable propositional function* $\varphi(\mathfrak{x})$ *with k free variables such that*

$$\mathfrak{n} \in A \equiv \vdash_n \varphi(\mathfrak{n})$$

for any $\mathfrak{n} \in R_k$.

Proof by induction on n. For $n=0$ the theorem is obvious. Let us suppose that it holds for an integer $n \geqslant 0$ and for $k=1,2,\ldots$ If $A \in P_{n+1}^{(k)}$, then for a suitable $B \in Q_n^{(k+1)}$ we have

$$\mathfrak{n} \in A \equiv \sum_p (\mathfrak{n},p) \in B$$

for any $\mathfrak{n} \in R_k$. If $n+1$-valid sentences form a self-consistent class, the same holds true for n-valid sentences and the inductive assumption yields the existence of an n-decidable propositional function $\varphi(\mathfrak{x},y)$ with $k+1$ free variables such that

$$(\mathfrak{n},p) \in B \equiv \vdash_n \varphi(\mathfrak{n},p).$$

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35) Rosser [13], pp. 129 130.

Let $\psi(\mathfrak{x})$ denote the propositional function $\sum\limits_{y}\varphi(\mathfrak{x},y)$. From two last equivalences we obtain immediately $\mathfrak{n}\,\epsilon\, A \rightarrow\, \vdash_n \psi(\mathfrak{n})$ and hence

$$(15) \qquad \mathfrak{n}\,\epsilon\, A \rightarrow\, \vdash_{n+1} \psi(\mathfrak{n}).$$

If $\mathfrak{n}$ non $\epsilon\, A$, then $\prod\limits_{p}$ non $\vdash_n \varphi(\mathfrak{n},p)$ and hence $\prod\limits_{p} \vdash_n \varphi'(\mathfrak{n},p)$ which proves accordingly to the definition of $n+1$-valid sentences that $\vdash_{n+1}\psi'(\mathfrak{n})$. Therefore

$$(16) \qquad \mathfrak{n} \text{ non } \epsilon\, A \rightarrow\, \vdash_{n+1} \psi'(\mathfrak{n}).$$

Assuming that the $n+1$-valid sentences form a self-consistent class, we obtain now $\mathfrak{n}$ non $\epsilon\, A \rightarrow$ non $\vdash_{n+1}\psi(\mathfrak{n})$ and finally in virtue of (15)

$$\mathfrak{n}\,\epsilon\, A \equiv\, \vdash_{n+1} \psi(\mathfrak{n}).$$

It remains to prove that $\psi(\mathfrak{x})$ is $n+1$-decidable. We have in fact for any $\mathfrak{n}\,\epsilon\, R_k$

$$\text{either } \mathfrak{n}\,\epsilon\, A \quad \text{or} \quad \mathfrak{n} \text{ non } \epsilon\, A,$$

i. e. with respect to (15) and (16)

$$\text{either } \vdash_{n+1}\psi(\mathfrak{n}) \quad \text{or} \quad \vdash_{n+1}\psi'(\mathfrak{n}).$$

The theorem is thus proved for $A\,\epsilon\, P_{n+1}^{(k)}$. In order to prove it for $A\,\epsilon\, Q_{n+1}^{(k)}$ it is sufficient to observe that (15) and (16) yield the equivalence

$$\mathfrak{n}\,\epsilon\, R_k - A \equiv\, \vdash_{n+1}\psi'(\mathfrak{n})$$

and that the propositional function $\psi'(\mathfrak{x})$ is $n+1$-decidable.

4.43. (Rosser's theorems). *If S fulfills the condition C_s (for an $s \geqslant 0$) and if the n-valid sentences form an ω-consistent class, then: 1⁰ this class is not closed under the rule of infinite induction; 2⁰ there is a ϑ such that neither $\vdash_n \vartheta$ nor $\vdash_n \vartheta'$* [36]).

To obtain the proof of 2⁰ we repeat the proof of 4.21 taking as A any set from $P_{n+1}^{(k)} - P_n^{(k)}$ and replacing „$\vdash$" throughout by „$\vdash_n$". 1⁰ follows then as in 4.41.

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[36]) Rosser [14], theorem VI, p. 132. Similar remarks as in 4.31 apply here.

§ 5. Functions of classes $P_n^{k,l}$ and $Q_n^{k,l}$.

5.1. Definitions. We denote by $R_l^{R_k}$ the class of functions mapping R_k on a subset of R_l 37). A function $f \in R_l^{R_k}$ is said to be of class $P_n^{(k,l)}$ or $Q_n^{(k,l)}$ if the „curve"

$$I_f = \underset{(n,m)}{E}\,[m = f(n)]$$

belongs to $P_n^{(k+l)}$ or $Q_n^{(k+l)}$

Remark. In order to maintain the analogy with the theory of Borel-functions it would be perhaps better to define $P_n^{(k,l)}$ or $Q_n^{(k,l)}$ as the class of functions f such that for any $A \in P_0^{(l)}$ the counter image $f^{-1}(A)$ is of class $P_n^{(k)}$ or $Q_n^{(k)}$ 38). It will be proved in **5.3** that classes $P_n^{(k,l)}$ and $Q_n^{(k,l)}$ defined above possess this property. The converse theorem seems, however, to be false. The analogy with the theory of Borel-sets is here breaking down.

5.2. Images. We put for $f \in R_l^{R_k}$ and $A \subset R_k$

$$f(A) = \underset{m}{E}\,[\underset{n}{\textstyle\sum}(n \in A)\cdot(m = f(n))]$$

and call $f(A)$ the image of A. Obviously

$$m \in f(A) \equiv \sum_{n}(n \in A)\cdot(m = f(n))$$

from what the following theorem immediately results by the Kuratowski-Tarski evaluation method:

5.21. *If* $A \in P_n^{(k)}$ *and* $f \in Q_n^{(k,l)}$ $(n \geqslant 0)$, *then* $f(A) \in P_{n+1}^{(l)}$ *and if* $A \in P_n^{(k)}$ *and* $f \in P_n^{(k,l)}$ $(n \geqslant 1)$, *then* $f(A) \in P_n^{(l)}$.

5.3. Counter-images. If $f \in R_l^{R_k}$ and $A \subset R_l$, then the counter-image of A is defined as

$$f^{-1}(A) = \underset{n}{E}\,[f(n) \in A].$$

Evidently

$$n \in f^{-1}(A) \equiv \sum_{m}[(m = f(n))\cdot(m \in A)] \equiv \prod_{m}[(m = f(n)) \rightarrow (m \in A)]. \tag{17}$$

In virtue of 2.16 we obtain from these equivalences the following theorems:

5.31. *If* $A \in P_n^{(l)}$ *and* $f \in P_n^{(k,l)}$ $(n \geqslant 1)$, *then* $f^{-1}(A) \in P_n^{(k)}$.

5.32. *If* $A \in Q_n^{(l)}$ *and* $f \in P_n^{(k,l)}$ $(n \geqslant 1)$, *then* $f^{-1}(A) \in Q_n^{(k)}$.

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37) Kuratowski [10], p. 199.

38) Kuratowski [10], p. 177.

The evaluation in case $n=0$ is given in the following theorem:

5.33. *If* $A \in P_0^{(l)}$ *and* $f \in P_0^{(k,l)}$, *then* $f^{-1}(A) \in P_0^{(k)}$.

Proof. The assumptions $A \in P_0^{(l)}$ and $f \in P_0^{(k,l)}$ secure the existence of two decidable propositional functions $\varphi(\mathfrak{y})$ and $\psi(\mathfrak{x},\mathfrak{y})$ with l and $k+l$ free variables such that

$$(18) \qquad \mathfrak{m} \in A \equiv \vdash \varphi(\mathfrak{m}) \quad \text{and} \quad [\mathfrak{m}=f(\mathfrak{n})] \equiv \vdash \psi(\mathfrak{n},\mathfrak{m}).$$

Here $\mathfrak{y}$ symbolizes a sequence of l variables $y_1,y_2,\dots,y_l$. Let us denote by $\mathfrak{y}<\bar{\mathfrak{y}}$ the following propositional function

$$(y_1<\bar{y}_1)+(y_1=\bar{y}_1)\cdot(y_2<\bar{y}_2)+\dots+(y_1=\bar{y}_1)\dots(y_{l-1}=\bar{y}_{l-1})\cdot(y_l<\bar{y}_l).$$

$\mathfrak{m}<\bar{\mathfrak{m}}$ says that $\mathfrak{m}$ preceeds $\bar{\mathfrak{m}}$ in the lexicographical ordering of R_l.

Consider now the propositional function $\psi^*(\mathfrak{x},\mathfrak{y})$ defined as

$$\psi(\mathfrak{x},\mathfrak{y}) \cdot \prod_{\bar{\mathfrak{y}}}[(\bar{\mathfrak{y}}<\mathfrak{y}) \rightarrow \psi'(\mathfrak{x},\bar{\mathfrak{y}})].$$

$\psi^*(\mathfrak{n},\mathfrak{m})$ says that $\mathfrak{m}$ is the first point of R_l (with respect to lexicographical ordering) such that $\psi(\mathfrak{n},\mathfrak{m})$. If $\vdash\psi^*(\mathfrak{n},\mathfrak{m})$, then $\mathfrak{m}=f(\mathfrak{n})$ in virtue of (18). If, conversely, $\mathfrak{m}=f(\mathfrak{n})$, then $\mathfrak{p}\neq f(\mathfrak{n})$ for any $\mathfrak{p}$ which preceeds $\mathfrak{m}$ in the lexicographical ordering of R_l and we obtain easily $\vdash\psi^*(\mathfrak{n},\mathfrak{m})$. Hence

$$(19) \qquad [\mathfrak{m}=f(\mathfrak{n})] \equiv \vdash \psi^*(\mathfrak{n},\mathfrak{m}).$$

Further it is plain that

$$\vdash \prod_{\mathfrak{x}}\prod_{\mathfrak{y}}\prod_{\mathfrak{z}}[\psi^*(\mathfrak{x},\mathfrak{y})\cdot\psi^*(\mathfrak{x},\mathfrak{z}) \rightarrow (\mathfrak{y}=\mathfrak{z})].$$

Using 1.22 we see that the propositional function $\psi^*(\mathfrak{x},\mathfrak{y})$ is decidable. From $\vdash \psi(\mathfrak{n},f(\mathfrak{n}))$ we infer at last that for any $\mathfrak{n} \in R_k$ there is an $\mathfrak{m} \in R_l$ such that $\vdash\psi^*(\mathfrak{n},\mathfrak{m})$. Hence all assumptions of 1.23 are fulfilled and we obtain the result that the propositional function $\vartheta(\mathfrak{x})$ defined as

$$\sum_{\mathfrak{y}}\varphi(\mathfrak{y})\cdot\psi^*(\mathfrak{x},\mathfrak{y})$$

is decidable. Denoting by $\zeta(\mathfrak{x})$ the propositional function

$$\prod_{\mathfrak{y}}[\psi^*(\mathfrak{x},\mathfrak{y}) \rightarrow \varphi(\mathfrak{y})],$$

we infer from 1.24 that

$$(20) \qquad \vdash \vartheta(\mathfrak{x}) \rightarrow \zeta(\mathfrak{x}).$$

The first equivalence (17) yields now (with respect to (18) and (19)) the implication:

$$\mathfrak{n} \in f^{-1}(A) \rightarrow \sum_{\mathfrak{m}} (\mathfrak{m} \in A) \cdot (\mathfrak{m} = f(\mathfrak{n})) \rightarrow \sum_{\mathfrak{m}} \vdash \varphi(\mathfrak{m}) \cdot \psi^*(\mathfrak{n}, \mathfrak{m})$$
$$\rightarrow \vdash \sum_{\mathfrak{y}} \varphi(\mathfrak{y}) \cdot \psi^*(\mathfrak{n}, \mathfrak{y}) \rightarrow \vdash \vartheta(\mathfrak{n})$$

whereas the second yields

$$\mathfrak{n} \text{ non} \in f^{-1}(A) \rightarrow \sum_{\mathfrak{m}} (\mathfrak{m} \text{ non} \in A) \cdot (\mathfrak{m} = f(\mathfrak{n})) \rightarrow$$
$$\rightarrow \sum_{\mathfrak{m}} \vdash \varphi'(\mathfrak{m}) \cdot \psi^*(\mathfrak{n}, \mathfrak{m}) \rightarrow \vdash \sum_{\mathfrak{y}} \psi^*(\mathfrak{n}, \mathfrak{y}) \cdot \varphi'(\mathfrak{y}) \rightarrow \vdash \zeta'(\mathfrak{n}),$$

i. e. with respect to (20) $\mathfrak{n} \text{ non} \in f^{-1}(A) \rightarrow \vdash \vartheta'(\mathfrak{n}) \rightarrow \text{non} \vdash \vartheta(\mathfrak{n})$. Hence $\mathfrak{n} \in f^{-1}(A) \equiv \vdash \vartheta(\mathfrak{n})$ and therefore $f^{-1}(A) \in \boldsymbol{P}_0^{(k)}$, q. e. d.

5.4. The function $\min\limits_{p} [(\mathfrak{n}, p) \in A]$. Let us suppose that A is a subset of R_{k+1} such that $\prod\limits_{\mathfrak{n}} \sum\limits_{p} (\mathfrak{n}, p) \in A$ and denote by $\mu_A(\mathfrak{n})$ the smallest integer p such that $(\mathfrak{n}, p) \in A$:

$$[p = \mu_A(\mathfrak{n})] \equiv \{(\mathfrak{n}, p) \in A \cdot \prod_{q} [(q \geqslant p) + (\mathfrak{n}, q) \text{ non} \in A]\}.$$

The Kuratowski-Tarski method leads immediately to the following theorem:

5.41. *If* $A \in \boldsymbol{Q}_n^{(k+1)}$, *then* $\mu_A \in \boldsymbol{Q}_{n+1}^{(k,1)}$ *and if* $A \in \boldsymbol{P}_n^{(k+1)}$ $(n \geqslant 1)$, *then* $\mu_A \in \boldsymbol{P}_{n+1}^{(k,1)} \cdot \boldsymbol{Q}_{n+1}^{(k,1)}$.

For $n = 0$ we have the sharper evaluation:

5.42. *If* $A \in \boldsymbol{P}_0^{(k+1)}$, *then* $\mu_A \in \boldsymbol{P}_0^{(k,1)}$.

Proof. Denote by $\varphi(\mathfrak{x}, x)$ any decidable propositional function which defines A and by $\psi(\mathfrak{x}, x)$ the propositional function

$$\varphi(\mathfrak{x}, x) \cdot \prod_{y} [y < x \rightarrow \varphi'(\mathfrak{x}, y)].$$

$\psi(\mathfrak{x}, x)$ is decidable by 1.22 and it is obvious that it defines the set $\underset{(\mathfrak{n}, p)}{E} [p = \mu_A(\mathfrak{n})]$. Hence $\mu_A \in \boldsymbol{P}_0^{(k,1)}$, q. e. d.

5.5. Post's theorem. This theorem is an exact analogue of the well-known Souslin's theorem concerning sets which are analytical together with their complements. It can be stated as follows:

5.51. $\boldsymbol{P}_1^{(k)} \cdot \boldsymbol{Q}_1^{(k)} = \boldsymbol{P}_0^{(k)}$ [39].

Proof. In virtue of 2.31 we have only to show that if $A \in \boldsymbol{P}_1^{(k)}$ and $R_k - A \in \boldsymbol{P}_1^{(k)}$, then $A \in \boldsymbol{P}_0^{(k)}$. Let B_1 and B_2 be two sets of $\boldsymbol{P}_0^{(k+1)}$ such that

$$\mathfrak{n} \in A \equiv \sum_p (\mathfrak{n}, p) \in B_1, \qquad \mathfrak{n} \in R_k - A \equiv \sum_p (\mathfrak{n}, p) \in B_2.$$

Since $\prod_{\mathfrak{n}} [\mathfrak{n} \in A) + (\mathfrak{n} \in R_k - A)]$, we have $\prod_{\mathfrak{n}} \sum_p [(\mathfrak{n}, p) \in B_1 + B_2]$ which proves accordingly to 5.42 that $\mu_{B_1+B_2} \in \boldsymbol{P}_0^{(k+1,1)}$, the sum $B_1 + B_2$ being of class $\boldsymbol{P}_0^{(k+1)}$ by 2.17. Now define a function $\mathfrak{f} \in R_{k+1}^{R_k}$ putting for any $\mathfrak{n} \in R_k$:

$$\mathfrak{f}(\mathfrak{n}) = (\mathfrak{n}, \mu_{B_1+B_2}(\mathfrak{n})).$$

We have $\mathfrak{f} \in \boldsymbol{P}_0^{(k,k+1)}$, since

$$[(\mathfrak{m}, p) = \mathfrak{f}(\mathfrak{n})] \equiv [(\mathfrak{m} = \mathfrak{n}) \cdot (p = \mu_{B_1+B_2}(\mathfrak{n}))].$$

Evidently

$$\mathfrak{n} \in \mathfrak{f}^{-1}(B_1) \equiv \sum_{(\mathfrak{m},p)} [(\mathfrak{m}, p) = \mathfrak{f}(\mathfrak{n})] \cdot [(\mathfrak{m}, p) \in B_1] \rightarrow$$
$$\rightarrow \sum_{(\mathfrak{m},p)} (\mathfrak{m} = \mathfrak{n}) \cdot [(\mathfrak{m}, p) \in B_1] \rightarrow \sum_p (\mathfrak{n}, p) \in B_1 \rightarrow \mathfrak{n} \in A.$$

If, conversely, $\mathfrak{n} \in A$, then $(\mathfrak{n}, \mu_{B_1+B_2}(\mathfrak{n})) \in B_1$, since otherwise we would obtain $(\mathfrak{n}, \mu_{B_1+B_2}(\mathfrak{n})) \in B_2$ and therefore $\sum_p (\mathfrak{n}, p) \in B_2$ or $\mathfrak{n} \in R_k - A$. Hence $\mathfrak{f}(\mathfrak{n}) \in B_1$ and $\mathfrak{n} \in \mathfrak{f}^{-1}(B_1)$. This proves that $A = \mathfrak{f}^{-1}(B_1)$ and the theorem 5.33 yields the desired result $A \in \boldsymbol{P}_0^{(k)}$.

From 5.51 we obtain two important corollaries:

5.52. *If* $\mathfrak{f} \in \boldsymbol{P}_0^{(k,l)}$ *and* $\mathfrak{g} \in \boldsymbol{P}_0^{(m,k)}$, *then the compounded function* $\mathfrak{f}(\mathfrak{g}(\mathfrak{m}))$ *is of class* $\boldsymbol{P}_0^{(m.l)}$.

Proof. We have

$$[\mathfrak{l} = \mathfrak{f}(\mathfrak{g}(\mathfrak{m}))] \equiv \sum_{\mathfrak{n}} (\mathfrak{n} = \mathfrak{g}(\mathfrak{m})) \cdot (\mathfrak{l} = \mathfrak{f}(\mathfrak{n})) \equiv \prod_{\mathfrak{n}} [(\mathfrak{n} = \mathfrak{g}(\mathfrak{m})) \rightarrow (\mathfrak{l} = \mathfrak{f}(\mathfrak{n}))].$$

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[39] Post [12], p. 290. See footnote [6]) on page 82.

The first equivalence shows that the set $\underset{(\mathfrak{l},\mathfrak{m})}{E}[\mathfrak{l} = f(g(\mathfrak{m}))]$ is of class $\boldsymbol{P}_1^{(m+k)}$ and the second that it is of class $\boldsymbol{Q}_1^{(m+k)}$. Hence by 5.51 it is of class $\boldsymbol{P}_0^{(m+k)}$.

5.53. *A set* $A \subset R_k$ *is in* $\boldsymbol{P}_n^{(k)} \cdot \boldsymbol{Q}_n^{(k)}$ *if and only if its characteristic function* c_A *is in* $\boldsymbol{P}_n^{(k,1)} \cdot \boldsymbol{Q}_n^{(k,1)}$.

Proof. From

$$[c_A(\mathfrak{n}) = p] \equiv [(\mathfrak{n} \in A) \cdot (p = 1) + (\mathfrak{n} \in R_k - A) \cdot (p = 0)]$$

we infer easily that if $A \in \boldsymbol{P}_n^{(k)} \cdot \boldsymbol{Q}_n^{(k)}$, then $c_A \in \boldsymbol{P}_n^{(k,1)} \cdot \boldsymbol{Q}_n^{(k,1)}$ Suppose now that $c_A \in \boldsymbol{P}_n^{(k,1)} \cdot \boldsymbol{Q}_n^{(k,1)}$. From

$$(\mathfrak{n} \in A) \equiv \sum_p [(p = 1) \cdot (c_A(\mathfrak{n}) = p)] \equiv \prod_p [(p = 1) \rightarrow (c_A(\mathfrak{n}) = p)]$$

we see that if $n \geqslant 1$, then $A \in \boldsymbol{P}_n^{(k)} \cdot \boldsymbol{Q}_n^{(k)}$. For $n = 0$ these equivalences yield $A \in \boldsymbol{P}_1^{(k)} \cdot \boldsymbol{Q}_1^{(k)}$ and hence $A \in \boldsymbol{P}_0^{(k)} \cdot \boldsymbol{Q}_0^{(k)}$ in virtue of 5.51.

It is interesting to observe that if $A \in \boldsymbol{P}_0^{(k)}$ and $f \in \boldsymbol{P}_0^{(k,l)}$, then the set $f(A)$ does not necessarily belong to $\boldsymbol{P}_0^{(l)}$, even if f is one-to-one. We see here another discrepance between our theory and the theory of Borel-sets.

It can be shown, however, that if f is an increasing function, i. e., if $\mathfrak{n} < \bar{\mathfrak{n}} \rightarrow f(\mathfrak{n}) < f(\bar{\mathfrak{n}})$, then $f(A) \in \boldsymbol{P}_0^{(k)}$ ($<$ represents here the lexicographical ordering of k-ads or l-ads of integers) [40].

5.6. Sets of the class $\boldsymbol{P}_1^{(k)}$ as values of functions $\boldsymbol{P}_0^{(1,k)}$.

The theorem 5.51 enables us to give a simple proof of the following theorem which discloses the relationship between the concept of the class $\boldsymbol{P}_1^{(k)}$ and that of recursively enumerable sets:

5.61. *If* S *fulfills the condition* C_0, *then the necessary and sufficient condition for a non-empty set* A *be in* $\boldsymbol{P}_1^{(k)}$ *is that there is a function* $f \in \boldsymbol{P}_0^{(1,k)}$ *whose set of values is* A.

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[40] This theorem has been proved by Kleene. See Kleene [9], theorem VII, p. 737, Rosser [14], Corollary I, p. 88, Post [12], p. 291.

Proof[41]. Sufficiency results at once from 5.21. Suppose now that $A \in \boldsymbol{P}_1^{(k)}$ and $\mathfrak{n}_0 \in A$. Let $\varphi(\mathfrak{x},x)$ be a decidable propositional function with $k+1$ free variables such that

$$\mathfrak{n} \in A \equiv \sum_p \varphi(\mathfrak{n},p)$$

for any $\mathfrak{n} \in R_k$.

We shall denote by $\mathfrak{s}(n)$ the sequence of k integers $s_1(s_1(n))$, $(s_1(s_1(s_1(n))))$, ..., $s_1(s_1 ... (s_1(n)) ...)$. An easy induction on k shows that for any $\mathfrak{n} \in R_k$ and $p,q \in R_1$ there is an integer h such that $\mathfrak{n} = \mathfrak{s}(h)$, $p = s_1(h)$ and $q = s_2(h)$.

Let l_0 be the Gödel-number of $\varphi(\mathfrak{x},x)$.

Define now the function $f(n)$ as follows: if $s_1(n)$ is the Gödel-number of a formal proof of $\varphi(\mathfrak{s}(\mathfrak{n}), s_2(n))$, then $f(n) = \mathfrak{s}(n)$; if not, then $f(n) = \mathfrak{n}_0$.

It is obvious that $f(n) \in A$ for any n. Conversely, if $\mathfrak{n} \in A$, then, for a suitable p, $\vdash \varphi(\mathfrak{n},p)$. Denoting by q the Gödel-number of a formal proof of $\varphi(\mathfrak{n},p)$ and by h the integer for which $\mathfrak{s}(h) = \mathfrak{n}$, $s_1(h) = q$, $s_2(h) = p$, we obtain $f(h) = \mathfrak{n}$. Hence A is the set of values of f.

It remains to evaluate the class of f. Remembering the definitions of sets $\varDelta$, E and $\varGamma_k$ given in **3.1** we see that

$$\begin{gathered}[\mathfrak{m} = f(n)] \equiv \{(\mathfrak{m} = \mathfrak{s}(n)) \cdot [(s_1(n) \in E) \cdot \sum_q (s_1(n),q) \in \varDelta \cdot \\ \cdot (l_0,q,\mathfrak{s}(n),s_2(n)) \in \varGamma_{k+1}] + (\mathfrak{m} = \mathfrak{n}_0) \cdot [(s_1(n) \text{ non} \in E) + \\ + \sum_q (s_1(n),q) \in \varDelta \cdot (l_0,q,\mathfrak{s}(n),s_2(n)) \text{ non} \in \varGamma_{k+1}]\}.\end{gathered}$$

This proves the set $\underset{\mathfrak{m},n}{E}[\mathfrak{m} = f(n)]$ to be of class $\boldsymbol{P}_1^{(k+1)}$. Remembering further that if $s_1(n) \in E$, then there is exactly one q such that $(s_1(n),q) \in \varDelta$, we can rewrite the above equivalence in the following form:

$$\begin{gathered}[\mathfrak{m} = f(n)] \equiv \\ \equiv ([\mathfrak{m} = \mathfrak{s}(n)] \cdot \{(s_1(n) \in E) \cdot \prod_q [(s_1(n),q) \in \varDelta \rightarrow (l_0,q,\mathfrak{s}(n),s_2(n)) \in \varGamma_{k+1}]\} + \\ + (\mathfrak{m} = \mathfrak{n}_0) \cdot \{(s_1(n) \text{ non} \in E) + \prod_q [(s_1(n),q) \in \varDelta \rightarrow (l_0,q,\mathfrak{s}(n),s_2(n)) \text{ non} \in \varGamma_{k+1}]\}).\end{gathered}$$

The set $\underset{\mathfrak{m},n}{E}[\mathfrak{m} = f(n)]$ is thus of class $\boldsymbol{Q}_1^{(k+1)}$ and hence by 5.51 it is of class $\boldsymbol{P}_0^{(k+1)}$, q. e. d.

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[41] This proof is essentially due to Kleene [9], theorem III, p. 736.

§ 6. Relations with the theory of general-recursive functions [42].

6.1. Recursivity conditions. We shall suppose that the system S fulfills the following two conditions:

(R_1) *Primitive recursive subsets of* R_k *belong to* $P_0^{(k)}$;

(R_2) *If* φ *is any propositional function with* k *free variables, then the relation* $qB_\varphi\mathfrak{n}$ *which holds between* q *and* n *if and only if* q *is the Gödel-number of a formal proof of* $\varphi(\mathfrak{n})$ *is primitive recursive.*

That these both conditions are fulfilled e. g. for the system P has been proved by Gödel [43].

6.2. Functions of class $P_0^{(k,1)}$ as general recursive functions.

6.21. *If* S *fulfills the condition* R_1, *then any general recursive function* $f(\mathfrak{n})$ *is of class* $P_0^{(k,1)}$.

Proof. If $f(\mathfrak{n})$ is general recursive, then there are: a primitive recursive function $h \in R_1^{R_1}$ and a primitive recursive relation $R(\mathfrak{n},p)$ such that $\prod_{\mathfrak{n}}\sum_{p} R(\mathfrak{n},p)$ and

$$f(\mathfrak{n})=h(\min_p R(\mathfrak{n},p)).$$

According to (R_1) the set $A=\underset{(\mathfrak{n},p)}{E}[R(\mathfrak{n},p)]$ belongs to $P_0^{(k+1)}$ and the function h to $P_0^{(1,1)}$. The function μ_A is of class $P_0^{(k,1)}$ by 5.42 and hence the compounded function $h(\mu_A(\mathfrak{n}))$ is of class $P_0^{(k,1)}$. This compounded function is equal to $f(\mathfrak{n})$ since $\mu_A(\mathfrak{n})=\min_p R(\mathfrak{n},p)$ and hence $f \in P_0^{(k,1)}$.

6.22. *If* S *fulfills the conditions* (R_1) *and* (R_2) *and if* $f \in R_k^{R_1}$ *is a function for which there is a propositional function* $\varphi(\mathfrak{x},x)$ *with* $k+1$ *free variables such that for any* $n \in R_1$ *and* $\mathfrak{n} \in R_k$

$$[n=f(\mathfrak{n})] \equiv \vdash \varphi(\mathfrak{n},n), \tag{21}$$

then $f(\mathfrak{n})$ *is a general recursive function and hence* $f \in P_0^{(k,1)}$ [44].

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[42] In this section we suppose the reader to be acquainted with the theory of general-recursive functions. See footnote [2].

[43] Gödel [3], p. 186.

[44] This theorem has been found by Gödel [5], p. 24. See also Rosser [15], final remark, Kleene [9], theorem VIII, p. 738.

Proof. For any $\mathfrak{n}$ there is an integer q such that $s_1(q)B_\varphi(\mathfrak{n},s_2(q))$, hence by (R_2) the function

$$g(\mathfrak{n})=s_2\{\min_q[s_1(q)B_\varphi(\mathfrak{n},s_2(q))]\}$$

is general recursive[45]. Thus it is sufficient to prove that $f(\mathfrak{n})=g(\mathfrak{n})$. To show this put $q_0=\min_q[s_1(q)B_\varphi(\mathfrak{n},s_2(q))]$. Then $s_1(q_0)$ is the Gödel-number of a formal proof of $\varphi(\mathfrak{n},s_2(q_0))$ which implies the existence of at least one formal proof of $\varphi(\mathfrak{n},s_2(q_0))$, i. e. $\vdash\varphi(\mathfrak{n},s_2(q_0))$ and therefore $f(\mathfrak{n})=s_2(q_0)$ by (21). On the other hand $g(\mathfrak{n})=s_2(q_0)$ in virtue of the definition of $g(\mathfrak{n})$ and hence $f(\mathfrak{n})=g(\mathfrak{n})$, q. e. d.

In order to explain the significance of 6.22 it is well to point out that in virtue of this theorem the existence of any propositional function $\varphi(\mathfrak{x},x)$ with the property (21) implies the existence of (possibly another) *decidable* propositional function $\psi(\mathfrak{x},x)$ with the same property. A simple example will elucidate this state of affairs. Let ϑ be any undecidable sentence, $\varphi(x,y)$ and $\psi(x,y)$ the propositional functions

$$(y=2x)+(y=2x+1)\cdot\vartheta \quad \text{and} \quad y=2x.$$

Then $(m=2n)\equiv\vdash\varphi(m,n)\equiv\vdash\psi(m,n)$, $\varphi(x,y)$ is undecidable and $\psi(x,y)$ decidable.

It is remarkable that no theorem analogous to 6.22 holds for sets. We have seen in the proof of 4.21 (formula (14)) that the equivalence

$$n\in A\equiv\vdash\psi(n)$$

may hold for any n though A does not belong to $P_0^{(1)}$. It is to remark that A must then belong to $P_1^{(1)}$ since

$$n\in A\equiv\sum_q(qB_\psi n).$$

From 6.21 and 6.22 we obtain the following corollary:

6.23. *If S fulfills the conditions (R_1) and (R_2), then $P_0^{(k,1)}$ is the class of general recursive functions with k arguments.*

6.3. Independence of classes $P_n^{(k)}$ and $Q_n^{(k)}$ from S. Subsets of R_k whose characteristic functions are general recursive may be called general recursive k-adic relations. From 6.23 and 5.53 we obtain therefore:

6.31. *If S fulfills the conditions (R_1) and (R_2), then $P_0^{(k)}$ is the class of general recursive k-adic relations.*

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[45] See e. g. Hilbert-Bernays [8], p. 402.

This theorem is important because it shows that the class $P_0^{(k)}$ though defined in 1.3 with the help of notions dependent from the logical system S taken as basis is in reality independent from S, at least if we limit ourselves to consideration of systems which fulfill the recursivity conditions (R_1) and (R_2). In fact, it is known that the class of general recursive relations can be defined without any reference to formalized logical systems [46]. The independence of $P_0^{(k)}$ from S implies of course the independence of other classes $P_n^{(k)}$ and $Q_n^{(k)}$ from S [47].

We note at last the following corollary from 6.31 and 5.61:

6.32. *If S fulfills the conditions (C_0), (R_1) and (R_2), then $P_1^{(1)}$ is the class of recursively enumerable sets.*

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Note. This paper was already under press, when an interesting paper of S. C. Kleene, *Recursive predicates and quantifiers* (Transactions of the American Mathematical Society, vol. **53** (1943), pp. 41-83) became available in Poland.

A considerable part of the theory developed above is to be found in the Kleene's paper. It seems me, however, that some of my results are new (e.g. remarks 4.3) and that my presentation of the theory based on analogies with the theory of projective sets may be of some interest for a mathematical reader.

Professor A. Tarski informed me that he also found already in 1942 results very similar to mines.

THE CLASSICAL AND THE ω-COMPLETE ARITHMETIC[1]

A. GRZEGORCZYK, A. MOSTOWSKI and C. RYLL-NARDZEWSKI

1. The formal systems and auxiliary notions. 1.1. Syntax. We consider two formal systems for the theory of (natural) numbers, both of which are applied second-order functional calculi with equality and the description operator. The two systems have the same primitive symbols, rules of formation, and axioms, differing only in the rules of inference.

The primitive logical symbols of the systems are the improper symbols (,), the propositional connectives $\vee$, &, $\supset$, $\equiv$, $\sim$, the quantifiers ( ), (E), the equality symbol $=$, the description operator ι, infinitely many distinct individual (or number) variables, and for each positive integer k infinitely many distinct k-place function variables. Our systems have in addition the following four primitive nonlogical (or arithmetical) constants: 0, 1, $+$, $\times$.

The classes of "number formulas" (nfs) and "propositional formulas" (pfs) are defined inductively as the least classes of formal expressions (i.e. of concatenations of primitive symbols) satisfying the following conditions:

(1) 0, 1, and the number variables are nfs.

(2) If π, ρ, $\sigma_1, \ldots, \sigma_k$ are nfs, φ, ψ are pfs, x is a number variable, and α^k is a k-place function variable, then $(\pi+\rho)$, $(\pi\times\rho)$, $(\iota x)\varphi$, $\alpha^k(\sigma_1, \ldots, \sigma_k)$

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[1] The present paper is an outgrowth of joint work of the authors carried out in the Seminar on the Foundations of Mathematics in the Mathematical Institute of the Polish Academy of Sciences in the academic year 1955–1956.

The work started with some observations of Mostowski concerning the so-called models absolute for the natural numbers (see [15]). Ryll-Nardzewski then established the connection between the validity of formulas in these models and their provability in formal systems containing the rule ω (his work was done independently of the work of Orey [16], which has meanwhile appeared in print). We then became interested (chiefly on the suggestion of Ryll-Nardzewski) in the problem of carrying over the metamathematical theorems about the system of arithmetic with finitary rules of inference to the system in which the rule ω is also assumed. The way of attacking this problem was suggested by Grzegorczyk and the details were afterwards elaborated and discussed by the three authors jointly.

For the sake of comparison of the finitary system and the system with the rule ω, we thought it useful to present here once again some results concerning the finitary system. In this section our work is for the most part not original and we have suppressed almost all proofs. Also in the final section, in which we are dealing with the system containing the rule ω, we have repeated a number of results due to other authors which we quote in appropriate places.

It will be seen from our paper that the parallelism between the two systems discussed below goes indeed very far. However we leave open the question of the deeper sources of these analogies.

We are much indebted to Dr. J. W. Addison for his helpful suggestions which have enabled us to improve considerably our previous text.

are nfs, and $(\varphi \vee \psi)$, $(\varphi \,\&\, \psi)$, $(\varphi \supset \psi)$, $(\varphi \equiv \psi)$, $\sim\varphi$, $(x)\varphi$, $(Ex)\varphi$, $(\alpha^k)\varphi$, $(E\alpha^k)\varphi$, $(\pi = \rho)$ are pfs.

Together the nfs and pfs are called *formulas*.

We shall use as metamathematical variables $x, y, z, x_1, \ldots$ for number variables, $\alpha^k, \beta^k, \gamma^k, \alpha_1^k, \ldots$ for *k*-placed function variables (with the superscript sometimes omitted as an abbreviation), $\Gamma, \Delta, \Theta, \Gamma_1, \ldots$ for formulas, $\pi, \rho, \sigma, \pi_1, \ldots$ for nfs, and $\varphi, \psi, \vartheta, \varphi_1, \ldots$ for pfs. The result of substituting π for x in Γ is denoted by $\Gamma\binom{x}{\pi}$, or simply by $\Gamma(\pi)$ where there is no danger of misunderstanding.

Parentheses will frequently be omitted or replaced by brackets as an abbreviation. We shall also abbreviate

$$\begin{array}{ll}
\sim(\pi = \rho) & \text{as } \pi \neq \rho, \\
(Ez)[\pi + z = \rho] & \text{as } \pi \leq \rho, \\
(x)[x \leq \pi \supset \varphi] & \text{as } (x)_\pi \varphi, \\
(Ex)[x \leq \pi \,\&\, \varphi] & \text{as } (Ex)_\pi \varphi, \\
(Ex)\varphi \,\&\, (x)(y)[\varphi(x) \,\&\, \varphi(y) \supset x = y] & \text{as } (E!x)\varphi, \\
(\iota x)[\varphi(x) \,\&\, (y)_x[\varphi(y) \supset x = y]] & \text{as } (\mu x)\varphi, \text{ and} \\
(\mu x)[x \leq \pi \,\&\, \varphi] & \text{as } (\mu x)_\pi \varphi.
\end{array}$$

The nfs $1+1$, $1+(1+1)$, ... are abbreviated as 2, 3, ...; these formulas as well as the nfs 0 and 1 are called *numerals*. If *n* is a natural number (i.e. a nonnegative integer), then $\boldsymbol{n}$ is the corresponding numeral.

As usual a pf containing no free variables is called *closed*, and a formula in which no quantifier binding a function variable occurs is called *elementary*. A *polynomial* is an nf containing no occurrences of propositional connectives, quantifiers, or ι. If π and ρ are polynomials, then the pf $\pi = \rho$ is a *polynomial equation* and a conjunction of such pfs is a *system of equations*.

The axioms of our systems are the following:

(3) the axiom schemata of the second-order functional calculus with equality and the description operator,[2] which include:

(3a) the axiom schemata of the first-order predicate calculus;

(3b) axiom schemata for $=$:

$$\pi = \pi,$$
$$\pi = \rho \supset \varphi(\pi) \equiv \varphi(\rho);$$

(3c) an axiom schema for ι:

$$((E!x)\varphi \,\&\, \varphi((\iota x)\varphi)) \vee (\sim(E!x)\varphi \,\&\, (\iota x)\varphi = 0);$$

(3d) the Leśniewski schemata[3]

$$(E\alpha^k)(x_1, \ldots, x_k)[\alpha^k(x_1, \ldots, x_k) = \pi],$$

where α^k does not occur free in π;

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[2] By the *second-order functional calculus* we mean the variation of the second-order predicate calculus in which there are function variables instead of predicate variables. (Cf. Ackermann [1] p. 98 for similar terminology.)

[3] (3d) is a form of the axiom of definability corresponding to Leśniewski's rule of ontological definability. (Cf. Leśniewski [13] p. 123.)

(4) Peano's axioms for 0, 1, $+$, $\times$, including the following formulation of the axiom of induction

$$[\alpha^1(0) = 0 \,\&\, (x)[\alpha^1(x) = \alpha^1(x+1)] \supset (x)[\alpha^1(x) = 0]].$$

We consider the following rules of inference:

(5) the rules for the second-order function calculus (the rule of modus ponens and the quantifier rules);

(6) the (non-finitary) rule ω:

from $\varphi(\pi)$ (for every numeral π) to infer $(x)\varphi(x)$.

Our first formal system, which we denote by (A), has the rules (5) as its rules of inference; the second, which we denote by (A_ω), has one additional rule of inference, the rule ω.

For each set X of pfs, we denote by Cn(X) (by $Cn_\omega(X)$) the least set of pfs containing X and the logical axioms (3) and closed with respect to the rules of inference (5) (with respect to the rules of inference (5)–(6)). Cn(X) is the set of consequences of X; $Cn_\omega(X)$ is the set of ω-consequences of X. The set of consequences of the nonlogical (or arithmetical) axioms (4) is denoted by A and the set of their ω-consequences by A_ω. The system (A) being based on the second-order calculus is evidently much stronger than e.g. the system Z of Hilbert and Bernays.

1.2. Semantics. Before we proceed to a description of the semantics of (A) we make a (metametamathematical) observation that the notions and axioms which we tacitly assume in metamathematics comprise all the ordinary notions and axioms of set theory, so that we can use in our definitions such non-elementary notions as classes and functions of arbitrarily high types. A sequence $\mathfrak{p} = \langle N_\mathfrak{p}, 0_\mathfrak{p}, 1_\mathfrak{p}, +_\mathfrak{p}, \times_\mathfrak{p}, F^1_\mathfrak{p}, F^2_\mathfrak{p}, \ldots\rangle$ is called a *frame* for our formal systems, if and only if

(7) $N_\mathfrak{p}$ is a set,

(8) $0_\mathfrak{p}$, $1_\mathfrak{p} \in N_\mathfrak{p}$,

(9) $+_\mathfrak{p}$, $\times_\mathfrak{p}$ are functions from $N^2_\mathfrak{p}$ into $N_\mathfrak{p}$,[4]

(10) $F^k_\mathfrak{p}$ is a set of functions from $N^k_\mathfrak{p}$ into $N_\mathfrak{p}$.

An example of a frame for our formal systems is the "principal model" $\mathfrak{N}_0 = \langle N_0, 0, 1, +, \times, F^1, F^2, \ldots\rangle$ in which N_0 is the set of natural numbers, F^k is the set of all functions from N^k_0 into N_0, and 0, 1, $+$, $\times$ have their usual meanings.

A *valuation* (with respect to a frame $\mathfrak{p}$) is a mapping of the number variables into $N_\mathfrak{p}$ and of the k-place function variables into $F^k_\mathfrak{p}$. For any frame $\mathfrak{p}$ and valuation f with respect to $\mathfrak{p}$, we can define by induction the value $\mathrm{Val}_{f,\mathfrak{p}}(\Gamma)$ of each formula Γ in the familiar fashion. $\mathrm{Val}_{f,\mathfrak{p}}(\Gamma)$ is an element of $N_\mathfrak{p}$ (if Γ is an nf), or it is one of the truth values V or Λ (if Γ is a pf). $\mathrm{Val}_{f,\mathfrak{p}}(\Gamma)$ depends, of course, only on the values of f on the free

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[4] If X is a set, we denote by X^n the Cartesian product of n copies of X.

variables of Γ. Thus we shall sometimes write

$$\Gamma_{\mathfrak{p}}\begin{pmatrix} \alpha_1, & \ldots, & \alpha_m, & \mathrm{x}_1, & \ldots, & \mathrm{x}_n \\ f(\alpha_1), & \ldots, & f(\alpha_m), & f(\mathrm{x}_1), & \ldots, & f(\mathrm{x}_n) \end{pmatrix}$$

or simply $\Gamma_{\mathfrak{p}}(f(\alpha_1), \ldots, f(\alpha_m), f(\mathrm{x}_1), \ldots, f(\mathrm{x}_n))$ for $\mathrm{Val}_{f,\mathfrak{p}}(\Gamma)$, it being understood that no variable other than those of the upper row is free in Γ.

A formula Γ containing free the distinct variables $\alpha_1^{k_1}, \ldots, \alpha_m^{k_m}, \mathrm{x}_1, \ldots, \mathrm{x}_n$ and no others is said to *define relative to a frame* $\mathfrak{p}$ a function g from $F_{\mathfrak{p}}^{k_1} \times \ldots \times F_{\mathfrak{p}}^{k_m} \times N_{\mathfrak{p}}^{n}$ into $N_{\mathfrak{p}}$ (if Γ is an nf) or into $\{V, \Lambda\}$ (if Γ is a pf), if and only if, for every valuation f,

$$g(f(\alpha_1^{k_1}), \ldots, f(\alpha_m^{k_m}), f(\mathrm{x}_1), \ldots, f(\mathrm{x}_n)) = \mathrm{Val}_{f,\mathfrak{p}}(\Gamma).$$

We identify the functions from a set X into $\{V, \Lambda\}$ with the predicates on X.

It can be shown that the functions and predicates definable relative to $\mathfrak{N}_0$ by elementary formulas (by arbitrary formulas) of our formal systems are exactly the arithmetical functions and predicates (the analytical functions and predicates) in the terminology of Kleene [11].[5]

A pf φ defining relative to a frame $\mathfrak{p}$ the (completely defined) constant predicate V is said to be *valid in* $\mathfrak{p}$. (If φ is closed, we may sometimes say *true in* $\mathfrak{p}$ for *valid in* $\mathfrak{p}$.) If every pf of a set X of pfs is valid in a frame $\mathfrak{p}$, then $\mathfrak{p}$ is called a *(general) model* of X. A model $\mathfrak{p}$ of X in which for each k ($k = 1, 2, \ldots$) $F_{\mathfrak{p}}^{k}$ is the set of all functions from $N_{\mathfrak{p}}^{k}$ into $N_{\mathfrak{p}}$ is called a *standard model of* X; for example, the principal model $\mathfrak{N}_0$ is a standard model of A and of A_ω, which implies that A and A_ω are consistent and even ω-consistent. The models of X which are not standard are called *non-standard models of* X.

It is easily seen that any standard model of A is isomorphic to $\mathfrak{N}_0$ — this is in effect just the theorem that the Peano axioms are "categorical". Moreover, since $\mathrm{A}_\omega \supset \mathrm{A}$, the same is true of standard models of A_ω. However, as we know from Gödel's famous theorem [6], there is a large collection of non-isomorphic non-standard models of A. And by an extension of Gödel's result by Rosser [19], the same is true even for A_ω.

1.3. Hierarchies of sets. We denote by Σ_n^0 and Π_n^0 the classes of sets definable, respectively, in the forms

$$\{x : (-u_1)(+u_2)\ldots(\pm u_n)R(x, u_1, \ldots, u_n)\},$$
$$\{x : (+u_1)(-u_2)\ldots(\mp u_n)R(x, u_1, \ldots, u_n)\},$$

where R is a recursive relation, and $(+u)$ stands for the general quantifier (u), and $(-u)$ for the existential quantifier (Eu), with the range N_0.

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[5] For the case of the arithmetical predicates (when $m = 0$), a detailed proof is given in [9] (cf. pp. 284–285), and indeed the use of "arithmetical" in [12] is vindicated by this proof. The other cases can be handled using a generalization of this argument.

Similarly, we denote by Σ_n^1 and Π_n^1 the classes of sets definable, respectively, in the forms

$$\{x : (-\phi_1)(+\phi_2)\ldots(\pm\phi_n)(\mp u)R(x, \bar{\phi}_1(u), \ldots, \bar{\phi}_n(u))\},$$
$$\{x : (+\phi_1)(-\phi_2)\ldots(\mp\phi_n)(\pm u)R(x, \bar{\phi}_1(u), \ldots, \bar{\phi}_n(u))\},$$

where R is a recursive relation, ϕ_i are variables with the range F^1, and $\bar{\phi}(u) = p_0^{\phi(0)+1} \cdot \ldots \cdot p_{u-1}^{\phi(u-1)+1}$.

We assume familiarity with the papers [10], [11], [12] of Kleene concerning these classes. The class $\Sigma_1^1 \cap \Pi_1^1$ is denoted by HA, and sets of this class are called *hyperarithmetical.* The class $\Sigma_1^0 \cap \Pi_1^0 =$ GR coincides with the class of general recursive sets.

We adopt the usual numbering of formulas and denote by N$^{\underline{o}}$ Γ the Gödel number of Γ. A set X of formulas is said to *belong* to a class X of numbers if the set N$^{\underline{o}}$ X of the Gödel numbers of the formulas in X is in X.

2. The ordinary system of arithmetic. 2.1. The semantical characterization of the rules of proof.

2.1.A. (COMPLETENESS THEOREM [5].) $\varphi \in \mathrm{Cn}(\mathrm{X}) \leftrightarrow \{\varphi$ is true in every model of X$\}$.[6]

COROLLARIES: 2.1.B. {Cn(X) is consistent} ↔ {there exists a model of X}.

2.1.C. (DEDUCTION THEOREM.) $\varphi \in \mathrm{Cn}(\mathrm{X} \cup \{\psi\}) \leftrightarrow \psi \supset \varphi \in \mathrm{Cn}(\mathrm{X})$.

2.1.D. $\varphi \in \mathrm{A} \leftrightarrow \{\varphi$ is true in every model of A$\}$.

Let φ be a pf containing no function variables. We shall call it a *demonstrably recursively enumerable pf* (rec. en. pf), if there are polynomials π_1, π_2, containing number variables x, y, $z_1, \ldots, z_n$ but no function variables, such that $\varphi \equiv (\mathrm{E}x)(y)_x(\mathrm{E}z_1)\ldots(\mathrm{E}z_n)(\pi_1 = \pi_2) \in \mathrm{A}$.[7]

2.1.E. If φ is a closed rec. en. pf, then $\varphi \in \mathrm{A} \leftrightarrow \{\varphi$ is true in $\mathfrak{N}_0\}$.

2.1.F. (INCOMPLETENESS THEOREM [6].) There is a closed rec. en. pf φ such that $\sim\varphi$ is true in $\mathfrak{N}_0$ but does not belong to A.

2.2. Evaluation of the predicate $n \in$ **N$^{\underline{o}}$ Cn(X).** From the definition of Cn(X) we easily obtain:

2.2. A. There is a relation $R(X, n, m)$ recursive uniformly in X such that for every set X of pfs

$$\{n \in \mathrm{N}^{\underline{o}}\,\mathrm{Cn}(\mathrm{X})\} \leftrightarrow \{(Em)R(\mathrm{N}^{\underline{o}}\,\mathrm{X}, n, m)\}.$$

---

[6] The completeness theorem for a calculus of second order seems to be more similar to Henkin's theorem (cf. Henkin [8] p. 85) than to the original version of Gödel's completeness theorem in [5]. However the second-order functional calculus may be considered as a modification of the first-order predicate calculus and the earliest Gödel argument may be applied to it with little modification.

[7] Our definition of a rec.en. pf is motivated by results of Davis [4]. In view of the effective character of Davis' work, every predicate of which we possess a proof that it is recursively enumerable becomes a rec. en. pf after we write it down in symbols of (A).

An immediate corollary to 2.2.A is (cf. the correction at the end):[8]

2.2.B. If $X \in \Sigma^0_n$ or $X \in \Sigma^1_n$ or $X \in \Pi^1_n$, then Cn(X) belongs to the same class; if $X \in \Pi^0_n$, then $\mathrm{Cn}(X) \in \Sigma^0_{n+1}$.

2.2.C. (AXIOMATIZATION THEOREM.) If $\mathrm{Cn}(X) = X \in \Sigma^0_{n+1}$, then there exists a $Y \in \Pi^0_n$ such that $X = \mathrm{Cn}(Y)$.

PROOF.[9] Since $X \in \Sigma^0_{n+1}$, there exists a set S in Π^0_n such that[10]

$$\mathrm{m} \in \mathrm{X} \leftrightarrow (Ek)\langle \mathrm{N^{\underline{o}}}\, \mathrm{m}, k\rangle \in S.$$

We can define the set $Y \in \Pi^0_n$ in such a manner that the equivalence

$$\underbrace{\mathrm{m} \,\&\, \ldots \,\&\, \mathrm{m}}_{k \text{ times}} \in \mathrm{Y} \leftrightarrow \langle \mathrm{N^{\underline{o}}}\, \mathrm{m}, k\rangle \in S$$

is satisfied. Hence $\mathrm{Cn}(Y) = X$.

2.2.D. If $X \in \Sigma^0_{n+1}$ and X is a consistent set of pfs, then there exists a set X_0 in Π^0_{n+1} such that $\mathrm{Cn}(X_0)$ is complete and consistent, $\mathrm{Cn}(X) \subset \mathrm{Cn}(X_0)$, and $\mathrm{Cn}(X_0) \in \Sigma^0_{n+2} \cap \Pi^0_{n+2}$.

PROOF. We represent Cn(X) in the form Cn(Y) where $Y \in \Pi^0_n$, and define a consistent and complete extension Z of Y using Lindenbaum's method [20]: It can be easily shown that $\mathrm{Cn}(Z) = Z \in \Sigma^0_{n+2} \cap \Pi^0_{n+2}$. By 2.2.C we can represent Z in the form $\mathrm{Cn}(X_0)$ where $X_0 \in \Pi^0_{n+1}$.

2.2.E. If $X_0 \in \Sigma^0_n$, then the set $\mathrm{Cn}(X_0)$ is either ω-inconsistent or incomplete (cf. [14] p. 100).

2.3. Characterization of recursive functions by means of the operation Cn. The following characterization of general recursive (GR) functions is known from the work of Kleene [9]:

2.3.A. $\phi \in \mathrm{GR}$, if and only if there is a finite sequence of functions $\phi_1, \ldots, \phi_n = \phi$ and a system of equations E with n function variables $\alpha_1, \ldots, \alpha_n$ such that:

1. α_j has the same number p_j of arguments as ϕ_j.
2. $\phi_1, \ldots, \phi_n$ are the unique functions which satisfy in $\mathfrak{N}_0$ the pf $(x_1, \ldots, x_m)E$, where $x_1, \ldots, x_n$ are all free number variables of E.
3. For every $k_1, \ldots, k_p$ in N_0, there is an l in N_0 such that

$$\alpha_n(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_p) = \boldsymbol{l} \in \mathrm{Cn}(\{(x_1, \ldots, x_m)E\}).$$

Theorem 2.3.A can be formulated in syntactical terms (without use of the notion of satisfaction):

2.3.B. $\phi \in \mathrm{GR}$, if and only if there is a system of equations E with function variables $\alpha_1, \ldots, \alpha_n$ such that:

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[8] Sometimes we shall identify in the continuation the set X of formulas with the set $\mathrm{N^{\underline{o}}}$ X of their numbers.

[9] We have used here the idea of Craig [3].

[10] We adopt here the logical notation of ordered pair for the arithmetical pairing functions. E.g., $\langle m, n\rangle = (m + n)^2 + m$.

1′. ϕ and α_n have the same number p of arguments.

4. For every $k_1, \ldots, k_p, l$, in N_0,

$$l = \phi(k_1, \ldots, k_p) \leftrightarrow (\mathrm{x}_1, \ldots, \mathrm{x}_m)\mathrm{E} \supset \alpha_n(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_p) = \boldsymbol{l} \in \mathrm{A}.$$

2.3.C. Theorems 2.3.A and 2.3.B remain true if we replace E in them by an arbitrary pf.

2.3.D. The function Cn used implicitly in 2.3.A and 2.3.B can be replaced by a more elementary one adapted to drawing consequences exclusively from equations. Details of this can be found in Kleene [9].

2.4. Representability in A. A subset S of N_0 is *representable* by a pf φ in a set X of pfs, if and only if φ has exactly one free number variable x and

$$n \in S \leftrightarrow \varphi(\boldsymbol{n}) \in \mathrm{X} \tag{1}$$

(cf. [7]). S is *strongly representable* by φ in X, if it is representable and its complement is representable by the pf $\sim\varphi$ in X, i.e. if both (1) and the following equivalence hold:

$$n \notin S \leftrightarrow {\sim}\varphi(\boldsymbol{n}) \in \mathrm{X}. \tag{2}$$

Similar definitions can be formulated for the notion of representability of relations.

The following theorem is easily provable. (In the proof of $\leftarrow$, we use (1), (2), and 2.2.B.)

2.4.A. If X is a consistent set of pfs, $\mathrm{A} \subset \mathrm{X}$, and $\mathrm{X} \in \Sigma_1^0$ then $\{S \in \mathrm{GR}\} \leftrightarrow \{S$ is strongly representable in X$\}$. (Cf. [7].)

2.4.B. If X is an ω-consistent set of pfs, $\mathrm{A} \subset \mathrm{X}$, and $\mathrm{X} \in \Sigma_1^0$, then $\{S \in \Sigma_1^0\} \leftrightarrow \{S$ is representable in Cn(X)$\}$.

For the proof see [14]. As a corollary we obtain:

2.4.C. The set Nº A is universal (complete in the sense of Post) for the class Σ_1^0.

Indeed, if φ represents S in A, then according to (1),

$$n \in S \leftrightarrow \theta(n) \in \text{Nº}\,\mathrm{A}$$

where $\theta(n) = \text{Nº}\,\varphi(\boldsymbol{n})$. Note that 2.4.B is applicable to A, since A is ω-consistent.

2.5. Undecidability of A. From 2.4.C it follows immediately that $\mathrm{A} \notin \mathrm{GR}$, i.e. that A is not decidable. The essential undecidability of A can be proved in many different ways (cf. [18], [21], [9], [7]). We present here a proof which we believe to be new. First we establish the following lemma:

2.5.A. If $\mathrm{X}_0 \in \mathrm{GR}$, then there exists a set X in GR such that, for every primitive recursive function θ, $\mathrm{X} \neq \theta^{-1}(\mathrm{X}_0)$.

Proof. Let $\theta_j(n)$ be a general recursive function universal for all primitive recursive functions. The set $X = \{n : \theta_n(n) \notin X_0\}$ is recursive, and has the required property, since from $X = \theta_k^{-1}(X_0)$ we would obtain a contradiction $\theta_k(k) \in X_0 \leftrightarrow k \in X \leftrightarrow \theta_k(k) \notin X_0$.

We shall denote by neg T the set $\{\sim\varphi : \varphi \in T\}$.

2.5.B. (Inseparability theorem.) If T is a set of pfs such that each GR set is strongly representable in T, then the sets N° T and N° neg T are not separable by recursive sets.

Proof. Assume that N° T and N° neg T are separated by a set X_0 in GR:

$$\text{N}^{\underline{\text{o}}}\, \text{T} \subset X_0 \text{ and } \text{N}^{\underline{\text{o}}}\, \text{neg T} \subset -X_0. \tag{3}$$

Choose X as in Lemma 2.5.A, and let φ strongly represent X in T. Putting $\theta(n) = \text{N}^{\underline{\text{o}}}\, \varphi(\boldsymbol{n})$, we obtain, by (1) and (3),

$$n \in X \rightarrow \theta(n) \in \text{N}^{\underline{\text{o}}}\, \text{T} \rightarrow \theta(n) \in X_0. \tag{4}$$

Similarly (2) and (3) imply

$$n \notin X \rightarrow \theta(n) \in \text{N}^{\underline{\text{o}}}\, \text{neg T} \rightarrow \theta(n) \notin X_0. \tag{5}$$

Thus from (4) and (5) we obtain $X = \theta^{-1}(X_0)$, which contradicts 2.5.A.

2.5.C. (Undecidability theorem.) The set A is essentially undecidable.

Proof. By 2.4.A and 2.5.B the sets N° A and N° neg A are not separable by recursive sets. Hence there cannot exist a consistent and decidable extension of A.

We shall still apply Theorem 2.5.B to determine the form of undecidable pfs.

2.5.D. (Incompleteness theorem.) There exists a rec. en. pf Z with two free variables such that for each consistent extension X in Σ_1^0 of A there exist infinitely many integers n for which the pf (Ex)Z($\boldsymbol{n}$, x) is false in $\mathfrak{N}_0$ and undecidable in X.

Proof. Denoting by $G_n(k, m)$ a GR function universal for the primitive recursive functions, we can represent each set S in GR in the form

$$S = \{k : \pi((\mu m)[G_n(k, m) = 0]) = 0\} \tag{6}$$

where π is a fixed primitive recursive function and the effectivity condition

$$(k)(Em)[G_n(k, m) = 0] \tag{7}$$

is satisfied. From (6) and (7) it follows easily that

$$k \in S \leftrightarrow ((Em)\{[G_n(k, m) = 0] \,\&\, [\pi(m) = 0] \,\&\, (l)[l < m \rightarrow G_n(k, l) \neq 0]\}). \tag{8}$$

The ternary relation

$$[G_n(k, m) = 0] \,\&\, [\pi(m) = 0] \,\&\, (l)[l < m \rightarrow G_n(k, l) \neq 0]$$

is recursive and hence strongly represented in A by the formula

$$\Xi(\text{z}, \text{y}, \text{x}) = \Gamma(\text{z}, \text{y}, \text{x}) \,\&\, \psi(\text{x}) \,\&\, (\text{v})[\sim(\text{x} \leq \text{v}) \supset \sim\Gamma(\text{z}, \text{y}, \text{v})]$$

where $\Gamma(z, y, x)$ and $\psi(x)$ are pfs strongly representing in A the relations $G_n(k, m) = 0$ and $\pi(m) = 0$, respectively. From (7), (8), and the definition of representability ((1) and (2)), it follows that

$$k \in S \to (\mathrm{E}x)\Xi(\boldsymbol{n}, \boldsymbol{k}, x) \in \mathrm{A}, \tag{9}$$

$$k \notin S \to \sim(\mathrm{E}x)\Xi(\boldsymbol{n}, \boldsymbol{k}, x) \in \mathrm{A}. \tag{10}$$

From (9) and (10) we conclude that for each set $S \in \mathrm{GR}$ there exists $n \in N_0$ such that S is strongly represented by the formula $(\mathrm{E}x)\Xi(\boldsymbol{n}, y, x)$ in A and hence in every consistent extension of A.

Now let $\mathrm{X} \in \Sigma_1^0$ be a consistent extension of A, and let T be the set of pfs of Cn(X) of the form $(\mathrm{E}x)\Xi(\boldsymbol{n}, \boldsymbol{k}, x)$. By (9) and (10) every GR set is strongly represented in T and hence according to 2.5.B the sets N° T and N° neg T are not separable by recursive sets. If there were but a finite number of sentences of the form $(\mathrm{E}x)\Xi(\boldsymbol{n}, \boldsymbol{k}, x)$ undecidable in X, then the sets N° T and N° neg T would be recursive and hence recursively separable. Thus infinitely many pfs of this form are undecidable in X. Strictly speaking the formulas of the considered form undecidable in X constitute a set of the class $\Pi_1^0 - \Sigma_1^0$.

In order to obtain a pf Z mentioned in the theorem we can join the first two variables of Ξ using the pairing functions.

An alternative proof of Theorem 2.5.D can be obtained by letting Z be a pf strongly representing the primitive recursive predicate W_0 of Kleene [9] p. 308.

3. The ω-complete system of arithmetic.

In this section we shall deal with the system (A_ω) of arithmetic defined in Section 1.1. The so-called models absolute for the natural numbers shall be called more briefly "ω-standard models", or "ω-models". We emphasize that we impose no limitation on the number of the applications of the ω-rule; this number may be an arbitrary transfinite ordinal.

A model $\mathfrak{M}$ is an *ω-standard model* if (under an arbitrary valuation) the values of the numerals exhaust $N_{\mathfrak{M}}$. Thus in order that $\mathfrak{M}$ be an ω-standard model it is necessary and sufficient that the partial models $\overline{\mathfrak{M}} = \langle N_{\mathfrak{M}}, 0_{\mathfrak{M}}, 1_{\mathfrak{M}}, +_{\mathfrak{M}}, \times_{\mathfrak{M}}\rangle$ and $\overline{\mathfrak{N}}_0 = \langle N_0, 0, 1, +, \times\rangle$ be ismorphic.

3.1. The semantical characterization of the rule ω.

3.1.A. (COMPLETENESS THEOREM [16].) If $\mathrm{A} \subset \mathrm{X}$, then $\{\varphi \in \mathrm{Cn}_\omega(\mathrm{X})\} \leftrightarrow \{\varphi$ is true in every ω-standard model of $\mathrm{X}\}$.

For the proof it is sufficient to show that, if $\varphi \notin \mathrm{Cn}_\omega(\mathrm{X})$, then there is a prime ideal in the Lindenbaum algebra L of the system (A_ω) which contains X, $\sim\varphi$ and preserves unions corresponding to quantifiers as well as unions of the form

$$[(\mathsf{E}\mathrm{x})\varphi] = \bigcup_{n=0}^{\infty} [\varphi(n)]$$

where $[\varphi]$ is the element of L containing φ. Using the method of [17], one shows easily that the set of prime ideals satisfying these conditions is residual in the Stone space for L. For a more detailed proof see [16].

Theorems 2.1.B — 2.1.D hold if we replace "Cn" by "Cn_ω", "model" by "ω-standard model", "A" by "A_ω". We denote these theorems by 3.1.B — 3.1.D.

3.1.E. If φ is an elementary pf with n function variables $\alpha_1, \ldots, \alpha_n$, then

$$\{(\alpha_1, \ldots, \alpha_n)\varphi \in \mathrm{A}_\omega\} \leftrightarrow \{(\alpha_1, \ldots, \alpha_n)\varphi \text{ is true in } \mathfrak{N}_0\}.$$

Proof. Let $\mathfrak{M}$ be an ω-standard model and $x^* \to x = t(x^*)$ an isomorphic mapping of $N_{\mathfrak{M}}$ onto N_0. With every $\phi^* \in F^k_{\mathfrak{M}}$ we correlate $\phi = t(\phi^*) \in F^k$ by putting

$$\phi(x_1, \ldots, x_k) = t(\phi^*(t^{-1}(x_1), \ldots, t^{-1}(x_k))).$$

Finally we put $t(V) = V$, $t(\Lambda) = \Lambda$.

If f is a valuation in the model $\mathfrak{M}$, then we denote by tf the valuation $\mathrm{x} \to t(f(\mathrm{x}))$, $\alpha^k \to t(f(\alpha^k))$. We can then prove by induction that, if π is an nf or an elementary pf, then

$$t(\mathrm{Val}_{f,\mathfrak{M}}(\pi)) = \mathrm{Val}_{tf,\mathfrak{N}_0}(\pi).$$

It follows from these formulas that, if φ is not valid in $\mathfrak{M}$, it is not valid in $\mathfrak{N}_0$. Using this and 3.1.D we infer that, if the closure of φ is not provable in A_ω, it is false in $\mathfrak{N}_0$.

The converse implication follows a fortiori from 3.1.D.

3.1.F. (Incompleteness theorem [19].) There is an elementary pf φ with one free function variable α^1 such that $(\mathsf{E}\alpha^1)\varphi$ is true in $\mathfrak{N}_0$ but does not belong to A_ω.

3.2. Evaluation of the predicate $x \in \mathrm{N}^{\underline{o}}\,\mathrm{Cn}_\omega(\mathrm{X})$. 3.2.A. There is a relation $R(\mathrm{X}, k, n)$ recursive uniformly in X such that for every set X of pfs

$$k \in \mathrm{N}^{\underline{o}}\,\mathrm{Cn}_\omega(\mathrm{X}) \leftrightarrow (\phi)(En)R(\mathrm{N}^{\underline{o}}\,\mathrm{X}, k, \bar{\phi}(n)).$$

Proof. In [19] p. 134 it is shown that the predicate $x \in \mathrm{N}^{\underline{o}}\,\mathrm{Cn}_\omega(\mathrm{X})$ is representable in the form $(\phi)P(\phi, x, \mathrm{N}^{\underline{o}}\,\mathrm{X})$, where P does not contain function quantifiers. Using the reduction described in [11], we obtain the desired form.

3.2.B. If $\mathrm{X} \in \Pi^1_n$, then $\mathrm{Cn}_\omega(\mathrm{X}) \in \Pi^1_n$; if $\mathrm{X} \in \Sigma^1_n$, then $\mathrm{Cn}_\omega(\mathrm{X}) \in \Pi^1_{n+1}$.

This is an immediate corollary of 3.2.A (cf. the correction at the end).

3.2.C. It is an open question whether the axiomatization theorem 2.2.C remains true if we replace in it Cn by Cn_ω, Σ^0_{n+1} by Π^1_{n+1}, and Π^0_n by Σ^1_n.

3.2.D. If $X \in \Pi^1_n$ or $X \in \Sigma^1_n$, and $\mathrm{Cn}_\omega(X)$ is a consistent set of pfs, then there is a consistent and complete set $Y \in \Pi^1_{n+2} \cap \Sigma^1_{n+2}$ of pfs such that $X \subset Y = \mathrm{Cn}_\omega(Y)$.

PROOF. Let $\varphi_0, \varphi_1, \ldots$ be a sequence without repetitions containing all closed pfs. We put $\psi_{k,n} = \varphi_k$ if φ_k does not have the form $(\mathrm{E}x_j)\vartheta(x_j)$, and $\psi_{k,n} = \vartheta(n)$ otherwise. Let ϕ, ψ, θ, ξ be functions satisfying the following conditions:

(1a) $\sim\varphi_{\phi(0)} \notin \mathrm{Cn}_\omega(X)$,
(1b) $j < \phi(0) \rightarrow \sim\varphi_j \in \mathrm{Cn}_\omega(X)$,
(2a) $\sim\psi_{\phi(0),\theta(0)} \notin \mathrm{Cn}_\omega(X)$,
(2b) $j < \theta(0) \rightarrow \sim\psi_{\phi(0),\theta(0)} \in \mathrm{Cn}_\omega(X)$,
(3) $\varphi_{\psi(0)} = \psi_{\phi(0),\theta(0)}$,
(4) $\varphi_{\xi(0)} = \varphi_{\psi(0)}$,
(5a) $[\varphi_{\xi(n)} \supset \sim\varphi_{\phi(n+1)} \notin \mathrm{Cn}_\omega(X)]$ and $\phi(n+1) > \phi(n)$.
(5b) $\phi(n) < j < \phi(n+1) \rightarrow [\varphi_{\xi(n)} \supset \sim\varphi_j \in \mathrm{Cn}_\omega(X)]$,
(6a) $\varphi_{\xi(n)} \supset \sim\psi_{\phi(n+1),\theta(n+1)} \notin \mathrm{Cn}_\omega(X)$,
(6b) $j < \theta(n+1) \rightarrow \varphi_{\xi(n)} \supset \sim\psi_{\phi(n+1),j} \in \mathrm{Cn}_\omega(X)$,
(7) $\varphi_{\psi(n+1)} = \psi_{\phi(n+1),\theta(n+1)}$,
(8) $\varphi_{\xi(n+1)} = \varphi_{\xi(n)} \,\&\, \varphi_{\psi(n+1)}$.

These formulas say that $\varphi_{\xi(n)}$ is the conjunction of $\varphi_{\psi(0)}, \ldots, \varphi_{\psi(n)}$, $\phi(n)$ is the least integer satisfying (1a) or (5a) according as $n = 0$ or $n \neq 0$, $\theta(n)$ is the least integer satisfying (2a) or (6a) according as $n = 0$ or $n \neq 0$, and $\varphi_{\psi(n)}$ is $\psi_{\phi(n),\theta(n)}$.

There exists exactly one set of four functions ϕ, ψ, θ, ξ satisfying the above conditions. Denote by $C(\phi, \psi, \theta, \xi)$ the conjunction of (1a) — (8) preceded by quantifiers $(n)(j)$. Since $\varphi_k \in \mathrm{Cn}_\omega(X) \leftrightarrow (\eta)(Ep)Q^X(k, \bar{\eta}(p))$ with Q^X recursive in X (by 3.2.A), we easily see that the predicate $C(\phi, \psi, \xi, \theta)$ can be reduced to the form

(9) $(n)[(E\eta)(x)R^X(\bar{\phi}(n), \bar{\psi}(n), \bar{\theta}(n), \bar{\xi}(n), \bar{\eta}(x))$ &
$(\zeta)(Ex)S^X(\bar{\phi}(n), \bar{\psi}(n), \bar{\theta}(n), \bar{\xi}(n), \bar{\zeta}(x))]$

with R^X and S^X recursive in X. Let $Y = \{\varphi_{\phi(0)}, \varphi_{\phi(1)}, \ldots\}$. Hence, in view of the uniqueness of ϕ, ψ, θ, ξ,

$$\varphi_k \in Y \leftrightarrow (E\phi, \psi, \theta, \xi)[C(\phi, \psi, \theta, \xi) \,\&\, (Ep)(k = \phi(p))] \leftrightarrow$$
$$(\phi, \psi, \theta, \xi)[C(\phi, \psi, \theta, \xi) \rightarrow (Ep)(k = \phi(p))].$$

Formula (9) proves therefore that, if $X \in \Pi^1_n$ or $X \in \Sigma^1_n$, then $Y \in \Pi^1_{n+2} \cap \Sigma^1_{n+2}$.

We are going to prove that Y has the required properties. Let $X_0 = \mathrm{Cn}_\omega(X)$, and let X_{n+1} be the set of φ_j such that $\varphi_{\xi(n)} \supset \varphi_j \in X_0$. From (1), (5), and the deduction theorem 3.1.C, it follows that $X_n = \mathrm{Cn}_\omega(X_n)$ is consistent for each n.

If $\phi(n) < j < \phi(n+1)$, or $j < \phi(0)$, by (1b) and (5b), $\sim\varphi_j \in X_n$ and hence $\varphi_j \notin X$. This proves that $X \subset Y$. We prove similarly that, if there were a j such that $\varphi_j, \sim\varphi_j \notin \mathrm{Y}$, then X_n would be inconsistent from a certain n on. Hence, for each j, $\varphi_j \in \mathrm{Y}$ or $\sim\varphi_j \in \mathrm{Y}$, and we immediately obtain $\mathrm{Cn(Y)} = \mathrm{Y}$. This together with consistency of the sets X_n proves that Y is consistent and complete. For each m, the pf $\varphi_{\psi(m)}$ is in Y, for otherwise X_n would be inconsistent for sufficiently large n. Finally Y is closed with respect to the rule ω. Indeed, if a pf $(\mathrm{x})\sim\vartheta(\mathrm{x})$ is not in Y, then the pf $(\mathrm{Ex})\vartheta(\mathrm{x})$ is in Y and hence is identical with $\varphi_{\phi(n)}$ for an integer n. Thus $\varphi_{\psi(n)}$ belongs to Y, and therefore Y contains a pf of the form $\vartheta(\boldsymbol{s})$.

REMARK 1. The above proof is essentially identical with the proof (due to Tarski) sketched by Feferman in the review quoted by Orey [16] p. 247 for the existence of a prime ideal in the Lindenbaum algebra of A preserving a given sequence of denumerable unions.

REMARK 2. Using the axiom of constructibility and certain results of Addison [2], we can improve our evaluation. If $\mathrm{X} \in \Pi^1_n$ or $\mathrm{X} \in \Sigma^1_n$ and $n > 1$, then a complete and consistent extension $\mathrm{Y} = \mathrm{Cn}_\omega(\mathrm{Y})$ of X can be found already in the class $\Pi^1_n \cap \Sigma^1_n$. We shall not, however, give this proof here. For $n = 1$ the evaluation of the class of Y cannot be improved.

3.3. Characterization of hyperarithmetical functions by means of the operation Cn_ω. A function ϕ is hyperarithmetical (HA) if its graph is in $\mathrm{HA} = \Sigma^1_1 \cap \Pi^1_1$. The following characterization was obtained independently by Addison, Grzegorczyk and Kuznecov.

3.3.A. A function ϕ is HA, if and only if there exists a finite sequence of functions $\phi_1, \ldots, \phi_n = \phi$ and a system of equations E with only the function variables $\alpha_1, \ldots, \alpha_n$ such that:

1. α_j has the same number of arguments as ϕ_j.
2. $\phi_1, \ldots, \phi_n$ are the unique functions which satisfy in $\mathfrak{N}_0$ the pf $(\mathrm{x}_1, \ldots, \mathrm{x}_m)\mathrm{E}$ where $\mathrm{x}_1, \ldots, \mathrm{x}_m$ are all number variables of E.

PROOF. Let π be an nf with the free variables $\alpha_1, \ldots, \alpha_n, \mathrm{x}_1, \ldots, \mathrm{x}_m$ and φ an elementary pf with the same variables. Let f be a valuation such that $f(\alpha_j) = \phi_j$, $f(\mathrm{x}_i) = k_i$. By an easy induction we prove that the function.

$$x_\pi(\phi_1, \ldots, \phi_n, k_1, \ldots, k_m) = \mathrm{Val}_{f,\mathfrak{N}_0}\pi$$

and the predicate

$$R_\varphi(\phi_1, \ldots, \phi_n, k_1, \ldots, k_m) \leftrightarrow \{\mathrm{Val}_{f,\mathfrak{N}_0}\varphi = V\}$$

are arithmetical uniformly in $\phi_1, \ldots, \phi_n$.

Now assume that there exists an E with the properties stated in the theorem. We have then

$$\begin{aligned} &y = \phi(z_1, \ldots, z_q) \leftrightarrow \{(E\phi_1, \ldots, \phi_n)(k_1, \ldots, k_m)R_{\mathrm{E}}(\phi_1, \ldots, \phi_n, \\ &k_1, \ldots, k_m) \,\&\, \phi_n(z_1, \ldots, z_q) = y\}, \\ &y = \phi(z_1, \ldots, z_q) \leftrightarrow \{(\phi_1, \ldots, \phi_n)[(k_1, \ldots, k_m)R_{\mathrm{E}}(\phi_1, \ldots, \phi_n, \\ &k_1, \ldots, k'_m) \rightarrow \phi_n(z_1, \ldots, z_q) = y]\}, \end{aligned} \tag{10}$$

whence we see that the predicate $y = \phi(z_1, \ldots, z_q)$ is HA.

The proof of the converse implication uses Kleene's theorem XXIV of [12] p. 204. We assume for simplicity that ϕ has one argument.

Let the predicate $y = \phi(x)$ be recursive in H_a where $a \in O$. Let $\mathrm{enm}(n, a)$ be a recursive function which enumerates integers $\leq_O a$. Denote by ψ_n the characteristic function of the set $\{z : H_{\mathrm{enm}(n,a)}(z)\}$. We can assume that $\mathrm{enm}(1, a) = a$. From the definitions we obtain that

(11) $\phi(x) = U((\mu y)T_1^{H_a}(e, x, y)) = U((\mu y)T_1^1(\tilde{\psi}_1(y), e, x, y))$

for a fixed integer e and a fixed primitive recursive function U.

The functions $\psi_n(x)$ and $\phi(x)$ can be characterized as the unique functions satisfying (11) and the conditions

$$(12)\qquad \begin{aligned} &\psi_n(x) \leq 1 \text{ for all } n, x \in N_0,\\ &\psi_n(x) = 1 \text{ if } \mathrm{enm}(n, a) = 1,\\ &\psi_p(x) = 1 \leftrightarrow (Ez)T_1^1(\tilde{\psi}_n(z), x, x, z) \text{ if } \mathrm{enm}(p, a) = \mathrm{enm}(n, a) +_O 1_O,\\ &\psi_p(x) = \psi_{\Phi_1((\mathrm{enm}(p,a))_2, (x)_{1,0})}((x)_0) \text{ if } 3 \,|\, \mathrm{enm}(p, a),\end{aligned}$$

where in the last formula Φ_1 is a partial recursive function universal for partial recursive functions as defined in [12] p. 194.

Note that functions and relations occurring in these formulas (with the exception of ϕ and ψ) are recursive and hence arithmetically definable. Hence, if we denote by $R_0(\phi, \psi)$ the predicate obtained from the above formulas by forming their conjunction and prefixing to it general quantifiers binding all free number variables, we infer that R_0 is an elementary definable predicate. Hence there is an elementary pf Ω such that ϕ and ψ are the unique functions which satisfy Ω in $\mathfrak{N}_0$.

We shall now transform Ω so as to obtain a system of equations. First we eliminate the μ-operators and reduce the formula to prenex normal form. Then we eliminate the existential quantifiers by introducing function variables according to the following scheme: If φ has the form
$(\mathsf{x}_1, \ldots, \mathsf{x}_k)(\mathsf{Ey})[\ldots \mathsf{y} \ldots]$, then we replace it by
$(\mathsf{x}_1, \ldots, \mathsf{x}_k, \mathsf{y})([\ldots \alpha_k(\mathsf{x}_1, \ldots, \mathsf{x}_k) \ldots] \,\&\, \{[\ldots \mathsf{y} \ldots] \supset \alpha_k(\mathsf{x}_1, \ldots, \mathsf{x}_k) \leq \mathsf{y}\})$,
where α_k is the new function variable. After a finite number of such steps, we obtain a pf of the form $(\mathsf{x}_1, \ldots, \mathsf{x}_m)\psi$ where ψ is a truth-functional combination of inequalities and equations. We add two more function variables β_1, β_2 denoting the predecessor and subtraction, and denote by Θ their defining equations

$$\beta_1(0) = 0, \quad \beta_1(\mathsf{x}+1) = \mathsf{x},$$
$$\beta_2(\mathsf{x}, 0) = \mathsf{x}, \quad \beta_2(\mathsf{x}, \mathsf{y}+1) = \beta_1(\beta_2(\mathsf{x}, \mathsf{y})).$$

Now the closure (with regard to number variables) of the conjunction of ψ and Θ can be easily transformed to a conjunction of equations accordingly to the following familiar rules of arithmetic:

$$\mathsf{a} = \mathsf{b} \leftrightarrow (\mathsf{a}-\mathsf{b})+(\mathsf{b}-\mathsf{a}) = 0, \quad \mathsf{a} \leq \mathsf{b} \leftrightarrow (\mathsf{a}-\mathsf{b}) = 0,$$
$$\mathsf{a} = 0 \vee \mathsf{b} = 0 \leftrightarrow (\mathsf{a} \times \mathsf{b}) = 0, \quad (\mathsf{a} \neq 0) \leftrightarrow (1-\mathsf{a}) = 0.$$

REMARK 3. Using the pairing functions (which do not belong to the primitives of our system), we can join all function variables in a unique one and present each ϕ in HA in the following normal form

$$\phi = K(\iota\theta)[(\mathrm{x}_1, \ldots, \mathrm{x}_m)\mathrm{E}],$$

where $(\iota\theta)[\varphi]$ denotes the unique function satisfying φ if it exist, E denotes a conjunction of equations, and K is a constant GR function, e.g. $K(x) = x-[\sqrt{x}]^2$.

REMARK 4. Theorem 3.3.A (as well as 2.3.A) remains true if we replace Condition 2 by the following one:

2′. ϕ_n is the unique function which satisfies in $\mathfrak{N}_0$ the pf $(\mathsf{E}\alpha_1, \ldots, \alpha_{n-1})(\mathrm{x}_1, \ldots, \mathrm{x}_m)\mathrm{E}$ where $\mathrm{x}_1, \ldots, \mathrm{x}_m$ are all the number variables of E.

Condition 2′ means that the auxiliary functions $\phi_1, \ldots, \phi_{n-1}$ do not need to be unique.

3.3.B. $\phi \in \mathrm{HA}$, if and only if there is a system of equations E with the function variables $\alpha_1, \ldots, \alpha_n$ such that the following two conditions are satisfied:

1′. ϕ and α_n have the same number q of arguments.

3. For every $k_1, \ldots, k_q$, t in N_0,

$$t = \phi(k_1, \ldots, k_q) \leftrightarrow (\mathrm{x}_1, \ldots, \mathrm{x}_m)\mathrm{E} \supset \alpha_n(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_q) = \boldsymbol{t} \in \mathrm{A}_\omega.$$

PROOF. Let E be chosen as in 3.3.A. The pf

$$(\mathrm{x}_1, \ldots, \mathrm{x}_m)\mathrm{E} \supset \alpha_n(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_q) = \boldsymbol{t} \tag{13}$$

is valid in $\mathfrak{N}_0$ if and only if $t = \phi(k_1, \ldots, k_q)$. By 3.1.E we obtain thus the equivalence 3.

Conversely, if 3 is satisfied, then by 3.1.E we obtain that (13) is valid in $\mathfrak{N}_0$ if and only if $t = \phi(k_1, \ldots, k_q)$. Hence, if $\phi_1, \ldots, \phi_n$ satisfy in $\mathfrak{N}_0$ the pf $(\mathrm{x}_1, \ldots, \mathrm{x}_m)\mathrm{E}$, then $\phi = \phi_n$. Since the pf $(\mathrm{x}_1, \ldots, \mathrm{x}_m)\mathrm{E}$ is satisfiable (otherwise (13) would be valid independently whether or not $t = \phi(k_1, \ldots, k_q)$), we infer that ϕ satisfies Condition 2′ of Remark 4. From this we obtain easily the formulas (10) and hence $\phi \in \mathrm{HA}$.

3.3.C. Theorems 3.3.A and 3.3.B remain valid if we replace in them E by an arbitrary elementary pf.

3.3.D. The function Cn_ω used implicitly in 3.3.B can be replaced by a more elementary one adapted to drawing consequences exclusively from equations.

The rules of proof used in the definition of the restricted function Cn_ω comprise the rules R1, R2 of Kleene [9] p. 264 as well as the following *restricted rule* ω: If π_1, π_2 are polynomials and $\pi_1(\boldsymbol{n}) = \pi_2(\boldsymbol{n})$ is provable for each $n \in N_0$, then so is the equation $\pi_1 = \pi_2$.

Let us abbreviate

$$(\mathsf{E}\alpha_1, \ldots, \alpha_n)\{\varphi \,\&\, (\beta_1, \ldots, \beta_n)[\varphi(\beta_1, \ldots, \beta_n) \supset \alpha_1 = \beta_1 \,\&\, \ldots \,\&\, \alpha_n = \beta_n]\} \text{ as } (\mathsf{E}!\alpha_1, \ldots, \alpha_n)\varphi.$$

3.3.E. If φ is an elementary pf with n function variables $\alpha_1, \ldots, \alpha_n$, then
$$\{(\mathrm{E}!\alpha_1, \ldots, \alpha_n)\varphi \text{ is true in } \mathfrak{N}_0\} \leftrightarrow (\mathrm{E}!\alpha_1, \ldots, \alpha_n)\varphi \in \mathrm{A}_\omega.$$

Proof. In order to simplify the notation we shall consider only the case of $n = 1$. In Kleene [12] p. 208 there is a proof that, for each elementary pf φ with one function variable α, there exists an elementary pf ψ with one function variable β such that the following equivalence is true in $\mathfrak{N}_0$:

$$(14) \qquad (\mathrm{E}\alpha)(\alpha \varepsilon \mathrm{HA}\ \&\ \varphi) \equiv (\beta)\psi.$$

Here $\alpha \varepsilon \mathrm{HA}$ denotes a pf obtained by writing in (A) the definition of the set HA. It can be verified that (14) is not only true in $\mathfrak{N}_0$ but actually provable in (A).[11] Thus from (14) and 3.1.E we infer

$$(15) \qquad \{(\mathrm{E}\alpha)(\alpha \varepsilon \mathrm{HA}\ \&\ \varphi) \text{ is true in } \mathfrak{N}_0\} \leftrightarrow (\mathrm{E}\alpha)(\alpha \varepsilon \mathrm{HA}\ \&\ \varphi) \in \mathrm{A}_\omega.$$

According to 3.3.A, $(\mathrm{E}!\alpha)\varphi$ is true in $\mathfrak{N}_0$ if and only if the following two formulas are true in $\mathfrak{N}_0$:

$$(16) \qquad (\mathrm{E}\alpha)(\alpha \varepsilon \mathrm{HA}\ \&\ \varphi),$$
$$(17) \qquad (\alpha, \beta)[(\varphi(\alpha)\ \&\ \varphi(\beta)) \supset (x)(\alpha(x) = \beta(x)).$$

According to (15) and 3.1.E, both pfs (16) and (17) are true in $\mathfrak{N}_0$ if and only if they belong to A_ω.

In view of the importance of Theorem 3.3.E, we shall sketch another proof of it which is more elementary insofar as it rests on a direct analysis of the proof of 3.3.A. It is of course sufficient to prove that, if $(\mathrm{E}!\alpha)\varphi(\alpha)$ is true in $\mathfrak{N}_0$, then it is provable in A_ω.

Let us consider the formulas (12) above. These represent a definition by transfinite induction on n of the function $\psi_n(x)$. The well-ordering of indices n is given by the recursive relation $n' \prec n'' \leftrightarrow \mathrm{enm}(n', a) <_O \mathrm{enm}(n'', a)$ (cf. Kleene [10] p. 410). The usual scheme of inductive definitions can easily be formalised in (A). In this way we obtain a pf $\Delta(\alpha, \beta)$ such that the unique functions satisfying this pf in $\mathfrak{N}_0$ are $\psi_n(x)$ and $\bar{\psi}_n(x)$. The proot of uniqueness and existence of these functions (both proceeding by transfinite induction on n) can be formalized in (A), and thus we obtain $(\mathrm{E}!\alpha, \beta)\Delta(\alpha, \beta) \in \mathrm{A}$ and therefore also $(\mathrm{E}!\alpha, \beta)\Omega \in \mathrm{A}$ where Ω is the pf. used in the proof of 3.3.A. It follows that the equations E reached in the proof of 3.3.A have the property

$$(18) \qquad (\mathrm{E}!\alpha_1, \ldots, \alpha_n)(x_1, \ldots, x_m)\mathrm{E} \in \mathrm{A}.$$

Indeed E was obtained from Ω by means of transformations which conserve the property (18).

Let us now assume that $(\mathrm{E}!\alpha)\varphi(\alpha)$ is true in $\mathfrak{N}_0$. It follows by 3.1.E that $(\alpha, \alpha')\{\varphi(\alpha)\ \&\ \varphi(\alpha') \supset (x)[\alpha(x) = \alpha'(x)]\} \in \mathrm{A}_\omega$, and hence it remains to prove

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[11] A verification of this proof in the formal system (A) is extremely laborious, as is each formalization of a mathematical reasoning.

that $(E\alpha)\varphi(\alpha) \epsilon A_\omega$. Let ϕ satisfy $\varphi(\alpha)$ in $\mathfrak{N}_0$. By 3.3.C we have $\phi \epsilon$ HA. By 3.3.A there exists a system E of equations satisfying Conditions 1 and 2 of 3.3.A as well as the formula (18). Let ϑ be the pf

$$(\alpha_1, \ldots, \alpha_n)\{(x_1, \ldots, x_m)E \supset \varphi(\alpha_n)\}.$$

Since $(x_1, \ldots, x_m)E$ is satisfied in $\mathfrak{N}_0$ by exactly one system of functions $\phi_1, \ldots, \phi_n = \phi$, and since this set satisfies $\varphi(\alpha_n)$ in $\mathfrak{N}_0$ as well, we infer that ϑ is true in $\mathfrak{N}_0$ and by 3.1.E belongs to A_ω. From this and (18) we finally obtain $(E\alpha)\varphi(\alpha) \epsilon A_\omega$.

3.4. Representability in A_ω. 3.4.A. If X is a consistent set of pfs, $A_\omega \subset X$, and $X \epsilon \Pi^1_1$, then $S \epsilon \text{HA} \leftrightarrow \{S$ is strongly representable in X$\}$.

Proof. If S is strongly representable in X, then, using the conditions (1) and (2) of the definition of representability (in 2.4) and 3.2.A, we infer that S and $-S$ belong to Π^1_1. Hence $S \epsilon$ HA.

Conversely, if $S \epsilon$ HA, then denoting by ϕ_k its characteristic function, we obtain from 3.3.A an elementary pf φ with the free variables $\alpha_1, \ldots, \alpha_k$ such that

(19) $(E!\alpha_1, \ldots, \alpha_k)\varphi$ is true in $\mathfrak{N}_0$,

and ϕ_k is the unique function which jointly with some functions $\phi_1, \ldots, \phi_{k-1}$ satisfies φ in $\mathfrak{N}_0$. Hence we have the equivalences

(20) $n \epsilon S \leftrightarrow \{(\alpha_1, \ldots, \alpha_k)[\varphi \supset \alpha_k(\boldsymbol{n}) = 1]$ is true in $\mathfrak{N}_0\}$,

(21) $n \notin S \leftrightarrow \{(E!\alpha_1, \ldots, \alpha_k)[\varphi \,\&\, \alpha_k(\boldsymbol{n}) \neq 1]$ is true in $\mathfrak{N}_0\}$.

According to 3.1.E and 3.3.E, both pfs on the right hand sides of (20) and (21) are true in $\mathfrak{N}_0$ if and only if they are theorems of A_ω. Hence according to the familar rules of quantifiers, S is strongly represented in A_ω (and so in each consistent extension of A_ω) by the pf

$$(\alpha_1, \ldots, \alpha_k)[\varphi \supset \alpha_k(x) = 1].$$

3.4.B. If $A_\omega \subset X \epsilon \Pi^1_1$, and if X consists exclusively of closed pfs which are true in $\mathfrak{N}_0$, then

$$\{S \epsilon \Pi^1_1\} \leftrightarrow \{S \text{ is representable in } Cn(X)\}.$$

Proof. $S \epsilon \Pi^1_1$, if and only if there exists an elementary pf φ such that

(22) $n \epsilon S \leftrightarrow \{(\alpha)\varphi(\alpha, \boldsymbol{n})$ is true in $\mathfrak{N}_0\}$.

By 3.1.E, the formula $(\alpha)\varphi(\alpha, \boldsymbol{n})$ is true in $\mathfrak{N}_0$ if and only if it belongs to A_ω. Hence $S \epsilon \Pi^1_1$, if and only if S is representable in A_ω. Now let X be an extension of A_ω consisting of true pfs. Thus, if the pf $(\alpha)\varphi(\alpha, \boldsymbol{n})$ belongs to X, it is true and by 3.1.E belongs to A_ω. Hence each set represented by a pf of this form in A_ω is represented by the same formula in X.

The converse implication is evident.

3.4.C. The set Nº A_ω is universal (complete in the sense of Post) for the class Π^1_1.

This follows from 3.4.B in the same way as 2.4.C follows from 2.4.B.

3.5. Hyperarithmetical undecidability of A_ω. From 3.4.C it follows that the set Nº A_ω is not hyperarithmetical. We shall show that no consistent extension of A_ω is hyperarithmetical. The argument is the same as in the case of the classical arithmetic. We establish the lemma:

3.5.A. If $X_0 \epsilon$ HA, then there exists a set X in HA such that $X \neq \theta^{-1}(X_0)$ for every primitive recursive function θ.

The proof is the same as for 2.5.A.

3.5.B. (Inseparability theorem.) If T is a set of pfs such that each HA set is strongly representable in T, then the sets Nº T and Nº neg T are not separable by HA sets.

Proof. From 3.5.A similarly to 2.5.B.

3.5.C. (Hyperarithmetical undecidability of A_ω.) No consistent extension of A_ω is hyperarithmetical.

Proof. From 3.5.B and 3.4.A similarly to 2.5.C.

We can still apply 3.5.B to determine the form of pfs undecidable in A_ω.

3.5.D. For each consistent set X in Π^1_1 of pfs such that $A_\omega \subset X$, there exists a set H_+ in Σ^1_1 of pfs of the form

$$(23) \qquad (\alpha_1, \ldots, \alpha_k)\varphi,$$

where φ is elementary, such that all formulas of the set H_+ are false in $\mathfrak{N}_0$ and undecidable in X.

Proof. Let C be the class of closed pfs of the form

$$(24) \qquad \sim\ldots\sim(\alpha_1, \ldots, \alpha_k)\varphi(\alpha_1, \ldots, \alpha_k, \boldsymbol{n})$$

with zero or more $\sim$'s, where φ is an elementary pf, and let $T = C \cap Cn_\omega(X)$. We put $T^* = \{\varphi : \sim\varphi \epsilon T\}$, and write $\varphi \approx \psi$ if ψ can be obtained from φ by dropping or inserting an even number of signs $\sim$. For $\varphi \epsilon C$,

$$\varphi \epsilon T \cup T^* \leftrightarrow \{\varphi \epsilon Cn_\omega(X) \text{ or } \sim\varphi \epsilon Cn_\omega(X)\},$$

and hence $H = C - (T \cup T^*)$ is the set of pfs (24) undecidable in X.

Since

$$\varphi \epsilon T^* \leftrightarrow \sim\varphi \epsilon T,$$
$$\varphi \epsilon T \leftrightarrow (E\psi)[(\varphi \approx \sim\psi) \,\&\, (\psi \epsilon T^*)],$$

we obtain $T \epsilon \Pi^1_1 \rightarrow T^* \epsilon \Pi^1_1$, $T^* \epsilon \Sigma^1_1 \rightarrow T \epsilon \Sigma^1_1$.

We shall now show that

$$(25) \qquad T \epsilon \Pi^1_1 - \Sigma^1_1, \; T^* \epsilon \Pi^1_1 - \Sigma^1_1.$$

By 3.2.B, $T \epsilon \Pi^1_1$. In view of the above implications we have only to prove that $T \notin \Sigma^1_1$. This we do as follows.

For each set $S \in \mathrm{HA}$, there is by 3.4.A a pf $\psi(\mathrm{x}) = (\alpha_1, \ldots, \alpha_k)\varphi(\alpha_1, \ldots, \alpha_k, \mathrm{x})$ with an elementary φ such that S is strongly represented by $\psi(\mathrm{x})$ in A_ω, i.e.,

$$n \in S \leftrightarrow \psi(\boldsymbol{n}) \in \mathrm{A}_\omega, \quad n \notin S \leftrightarrow \sim\psi(\boldsymbol{n}) \in \mathrm{A}_\omega.$$

Since $\mathrm{A}_\omega \subset \mathrm{Cn}_\omega(\mathrm{X})$, we obtain

$$n \in S \rightarrow \psi(\boldsymbol{n}) \in \mathrm{T}, \quad n \notin S \rightarrow \sim\psi(\boldsymbol{n}) \in \mathrm{T};$$

and since $\mathrm{Cn}_\omega(\mathrm{X})$ is consistent, we obtain

$$n \notin S \rightarrow \psi(\boldsymbol{n}) \notin \mathrm{T}, \quad n \in S \rightarrow \sim\psi(\boldsymbol{n}) \notin \mathrm{T}.$$

By 3.5.B, T and neg T are not separable by means of HA sets, and hence $\mathrm{T} \notin \mathrm{HA}$ and $\mathrm{T} \notin \Sigma^1_1$.

From (25) and the definition of H we obtain $\mathrm{H} \in \Sigma^1_1$.

Since $\mathrm{T} = \mathrm{C}-(\mathrm{HUT}^*)$, we infer that $\mathrm{H} \notin \Pi^1_1$, since otherwise we would have $\mathrm{T} \in \Sigma^1_1$. We thus finally obtain $\mathrm{H} \in \Sigma^1_1 - \Pi^1_1$.

The set H contains a subset H_+ of pfs beginning with a general quantifier. All other pfs of H are obtained from pfs in H_+ by inserting a number of signs $\sim$. Hence H_+ is an intersection of H and a recursive set and therefore $\mathrm{H}_+ \in \Sigma^1_1$. On the other hand, the equivalence

$$\varphi \in \mathrm{H} \leftrightarrow (En, \psi)[(\psi \in \mathrm{H}_+) \,\&\, (\varphi = \underbrace{\sim \ldots \sim}_{n \text{ times}}\psi)]$$

shows that $\mathrm{H}_+ \in \Pi^1_1 \rightarrow \mathrm{H} \in \Pi^1_1$. Hence $\mathrm{H}_+ \in \Sigma^1_1 - \Pi^1_1$. No pf in H_+ is true in $\mathfrak{N}_0$ in view of 3.1.E. Theorem 3.5.D is thus completely proved.

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Correction added in proof May 14, 1958. Theorem 2.2.B, while true, is not a direct corollary of 2.2.A. In order to prove 2.2.B, we note that the relation $R(X, n, m)$ of 2.2.A has the form $(j)_{\mathrm{lh}((m)_1)}[(m)_{1,j} \in X] \mathbin{\&} \mathrm{S}(m, n)$ where S is recursive and $(m)_1$, $(m)_{i,j}$, $\mathrm{lh}(m)$ are recursive functions defined in [9] p. 230.

Similarly, 3.2.B is not a direct corollary of 3.2.A. In order to prove 3.2.B, we notice that by Rosser's proof [19] pp. 133–134 the predicate $n \in \mathrm{Cn}'_{\omega}(X)$ has the form $(C)[(X \subset C) \mathbin{\&} \mathrm{Closed}(C) \supset n \in C]$, and hence can be brought to the form $(\phi)\{(Em)[m \in X \mathbin{\&} \phi(m) = 0] \vee S(\phi, m)\}$ where S is arithmetical.

WARSAW

Formal system of analysis based on an infinitistic rule of proof

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The present paper is a continuation of [5]. In section 3 of [5] we discussed models $\mathfrak{M}$ of analysis (i. e. of second order arithmetic) such that their "integers" were isomorphic to the ordinary integers; we can express this by saying that "integers" of the models $\mathfrak{M}$ were absolute. In the present paper we investigate a still narrower class of models in which not only integers but also well-orderings are absolute. We call these models the β-models. In sections 4, 5, and 6 we discuss properties of a system A_β of analysis whose theorems are just those formulas which are true in all β-models. In section 7 we indicate two applications of our theory. What seems more surprising to us than those applications is the fact that the crucial properties of A_β are entirely different from the corresponding properties of the systems A and A_ω considered in [5]. The difference between those systems is stressed particularly in theorems 6.9 and 8.20. Basically those differences stem from the fact that the separation principles for the family of (weakly) representable sets are true for the system A_β but false for the systems A and A_ω. The basic importance which the separation properties have for the meta-mathematical properties of formal systems is thus once more demonstrated.

1. Auxiliary definitions

1.1. We put $\langle m, n\rangle = 2^m(2n+1)$ and call this integer (¹) the *ordered pair* of m and n. Furthermore we put $s_1(\langle m, n\rangle) = m$ and $s_2(\langle m, n\rangle) = n$.

(¹) Throughout this paper we mean by "integers" non-negative integers 0, 1, 2, ...

1.2. Every integer m determines integers $k, n_0, n_1, \ldots, n_{k-1}$ such that

$$m+1 = 2^{n_0}+2^{n_0+n_1+1}+\ldots+2^{n_0+n_1+\ldots+n_{k-1}+k-1}.$$

We put $k-1 = dl(m)$, $n_j = c(m,j)$ for $j < k$ and $c(m,j) = 0$ for $j \geqslant k$. Functions dl and c are primitive recursive.

1.3. If $dl(m') = dl(m)+1$ and $c(m',j) = c(m,j)$ for $j \leqslant dl(m)$, then we say that m' is an extension of m.

1.4. There is a primitive recursive function $rst(m,j)$ such that if $j \leqslant dl(m)$, then $dl\big(rst(m,j)\big) = j$ and $c\big(rst(m,j),i\big) = c(m,i)$ for $i \leqslant j$. We call $rst(m,j)$ the restriction of m to j.

1.5. There is a primitive recursive function $r(p,n)$ such that if $p < 2^n$, then $p/2^n = \sum_{j=0}^{k} 2^{-n_j}$ where $k = dl\big(r(p,n)\big)$ and $n_j = c\big(r(p,n),j\big)$ for $j \leqslant k$.

1.6. If φ is a function from integers to integers, then we put $\varphi^*(m) = 2^{\varphi(0)}+2^{\varphi(0)+\varphi(1)+1}+\ldots+2^{\varphi(0)+\ldots+\varphi(m)+m}$ (²).

2. Formal definitions, theorems and theorem schemata of A

We abbreviate "lemma" as "L", "definition" as "D". To simplify the formulas we use $\mathrm{x}, \mathrm{y}, \mathrm{z}, \ldots, \alpha^j, \beta^j, \gamma^j, \ldots$ instead of $\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \ldots, \alpha_1^j, \alpha_2^j, \alpha_3^j, \ldots$

L. 2.1. *There is an arithmetic formula* $\mathrm{F}(\mathrm{x},\mathrm{y})$ *such that* $\vdash(\mathrm{x})(\mathbf{E}!\mathrm{y})\mathrm{F}(\mathrm{x},\mathrm{y})$, $\vdash\mathrm{F}(0,1)$, *and* $\vdash\mathrm{F}(\mathrm{x}+1,\mathrm{z})\,\&\,\mathrm{F}(\mathrm{x},\mathrm{y}) \supset (\mathrm{z} = 2\mathrm{y})$.

D. 2.2. $2^{\mathrm{x}} = (\iota\mathrm{y})\mathrm{F}(\mathrm{x},\mathrm{y})$.

D. 2.3. $\langle\mathrm{x},\mathrm{y}\rangle = 2^{\mathrm{x}}(2\mathrm{y}+1)$.

L. 2.4. $\vdash(\langle\mathrm{x},\mathrm{y}\rangle = \langle\mathrm{z},\mathrm{t}\rangle) \supset (\mathrm{x} = \mathrm{z})\,\&\,(\mathrm{y} = \mathrm{t})$.

L. 2.5. $\vdash(\mathbf{E}\mathrm{x},\mathrm{y})[\mathrm{z} = \langle\mathrm{x},\mathrm{y}\rangle] \equiv (\mathrm{z} \neq 0)$.

L. 2.6. $\vdash(\mathrm{z} \neq 0) \supset (\mathbf{E}!\mathrm{x},\mathrm{y})\ (\mathrm{z} = \langle\mathrm{x},\mathrm{y}\rangle)$.

D. 2.7. $\mathrm{s}_1(\mathrm{z}) = (\iota\mathrm{x})(\mathbf{E}\mathrm{y})(\mathrm{z} = \langle\mathrm{x},\mathrm{y}\rangle)$, $\mathrm{s}_2(\mathrm{z}) = (\iota\mathrm{y})(\mathbf{E}\mathrm{x})(\mathrm{z} = \langle\mathrm{x},\mathrm{y}\rangle)$.

Remark. From the axioms for the description operator ([5], p. 189) it follows that $\vdash\mathrm{s}_i(0) = 0$ for $i = 1, 2$.

(²) Instead of the functions dl and φ^* we could use equally well the functions lh and $\bar{\varphi}$ of [6]. The reason why we prefer the functions defined above is that in sections 5 and 6 we shall rely on the theory of sieves in the form developed in [8], in which functions dl and φ^* and not lh and $\bar{\varphi}$ are used.

D. 2.8. $\mathrm{x}\alpha^2\mathrm{y} \equiv \alpha^2(\mathrm{x}, \mathrm{y}) = 0$.

D. 2.9. $\mathrm{Ord}(\alpha^2) \equiv (\mathrm{x}, \mathrm{y}, \mathrm{z})\big(\{[(\mathrm{x}\alpha^2\mathrm{z}) \vee (\mathrm{z}\alpha^2\mathrm{x})] \,\&\, [(\mathrm{y}\alpha^2\mathrm{z}) \vee (\mathrm{z}\alpha^2\mathrm{y})] \supset (\mathrm{x}\alpha^2\mathrm{x}) \,\&\, [(\mathrm{x}\alpha^2\mathrm{y}) \vee (\mathrm{x} = \mathrm{y}) \vee (\mathrm{y}\alpha^2\mathrm{x})]\} \,\&\, [(\mathrm{x}\alpha^2\mathrm{y}) \,\&\, (\mathrm{y}\alpha^2\mathrm{x}) \supset (\mathrm{x} = \mathrm{y})] \,\&\, [(\mathrm{x}\alpha^2\mathrm{y}) \,\&\, (\mathrm{y}\alpha^2\mathrm{z}) \supset (\mathrm{x}\alpha^2\mathrm{z})]\big)$.

D. 2.10. $\mathrm{Fund}(\alpha^2) \equiv (\beta^1)(\mathbf{E}\mathrm{x})\{\sim[\beta^1(\mathrm{x}+1)\,\alpha^2\beta^1(\mathrm{x})] \vee [\beta^1(\mathrm{x}+1) = \beta^1(\mathrm{x})]\}$.

D. 2.11. $\mathrm{Bord}(\alpha^2) \equiv \mathrm{Ord}(\alpha^2) \,\&\, \mathrm{Fund}(\alpha^2)$.

L. 2.12. *For every recursive relation* $R(n_1, \ldots, n_k)$ *and for every recursive function* $f(n_1, \ldots, n_k)$ *there are an elementary formula and an elementary term* $\mathrm{E}_R(\mathrm{x}_1, \ldots, \mathrm{x}_k)$ *and* $\tau_f(\mathrm{x}_1, \ldots, \mathrm{x}_k)$ *such that*

$$R(n_1, \ldots, n_k) \text{ implies } \vdash \mathrm{E}_R(\mathbf{n}_1, \ldots, \mathbf{n}_k)\,,$$

$$non\text{-}R(n_1, \ldots, n_k) \text{ implies } \vdash \sim \mathrm{E}_R(\mathbf{n}_1, \ldots, \mathbf{n}_k)\,,$$

$$m = f(n_1, \ldots, n_k) \text{ is equivalent to } \vdash \mathbf{m} = \tau_f(\mathbf{n}_1, \ldots, \mathbf{n}_k)\,.$$

We say that E_R and τ_f define (or strongly represent) R and f.

D. 2.13. We denote by Ext(y, x), dl(x) and c(x, y) any elementary formula and any elementary terms which define the relation "*m* is an extension of *n*" and the functions *dl* and *c*.

D. 2.14. $$(\beta^1 \text{ is } \alpha^{1*}) \equiv (\mathrm{x})\Big[\big(\mathrm{dl}\big(\beta^1(\mathrm{x})\big) = \mathrm{x}\big)\Big] \,\& \\ \& \,(\mathrm{y})_{\mathrm{x}}\big[\mathrm{c}\big(\beta^1(\mathrm{x}), \mathrm{y}\big) = \alpha^1(\mathrm{y})\big].$$

L. 2.15. $\vdash (\mathbf{E}!\beta^1)(\beta^1 \text{ is } \alpha^{1*})$.

L. 2.16. *For every elementary term* $\tau(\alpha_1^1, \ldots, \alpha_k^1, \mathrm{x}_1, \ldots, \mathrm{x}_{l-1})$ *and every elementary formula* $\mathrm{M}(\alpha_1^1, \ldots, \alpha_k^1, \mathrm{x}_1, \ldots, \mathrm{x}_l)$ *with the free variables indicated there are elementary formulas* $\mathrm{P}_i(\mathrm{z}_1, \ldots, \mathrm{z}_k, \mathrm{y}_1, \ldots, \mathrm{y}_s, \mathrm{x}_1, \ldots, \mathrm{x}_l)$ *and* $\mathrm{Q}_i(\mathrm{z}_1, \ldots, \mathrm{z}_k, \mathrm{y}_1, \ldots, \mathrm{y}_s, \mathrm{x}_1, \ldots, \mathrm{x}_l)$, $i = 1, 2$, *with the free variables indicated such that*

$$\begin{aligned}
&\vdash \tau(\alpha_1, \ldots, \alpha_k^1, \mathrm{x}_1, \ldots, \mathrm{x}_{l-1}) = \mathrm{x}_l \\
&\quad \equiv (\mathbf{E}\beta_1^1, \ldots, \beta_k^1, \gamma_1^1, \ldots, \gamma_s^1, \delta_1^1, \ldots, \delta_s^1)(\mathrm{u})\big[(\delta_1^1 \text{ is } \gamma_1^{1*}) \,\&\, \ldots \,\& \\
&\quad \& \,(\delta_s^1 \text{ is } \gamma_s^{1*}) \,\&\, (\beta_1^1 \text{ is } \alpha_1^{1*}) \,\&\, \ldots \,\&\, (\beta_k^1 \text{ is } \alpha_k^{1*}) \,\& \\
&\quad \& \,\mathrm{P}_1\big(\beta_1^1(\mathrm{u}), \ldots, \beta_k^1(\mathrm{u}), \delta_1^1(\mathrm{u}), \ldots, \delta_s^1(\mathrm{u}), \mathrm{x}_1, \ldots, \mathrm{x}_l\big)\big] \\
&\quad \equiv (\beta_1^1, \ldots, \beta_k^1, \gamma_1^1, \ldots, \gamma_s^1, \delta_1^1, \ldots, \delta_s^1)(\mathbf{E}\mathrm{u})\big[(\delta_1^1 \text{ is } \gamma_1^{1*}) \,\&\, \ldots \,\& \\
&\quad \& \,(\delta_s^1 \text{ is } \gamma_s^{1*}) \,\&\, (\beta_1^1 \text{ is } \alpha_1^{1*}) \,\&\, \ldots \,\&\, (\beta_k^1 \text{ is } \alpha_k^{1*}) \\
&\quad \supset \mathrm{Q}_1\big(\beta_1^1(\mathrm{u}), \ldots, \beta_k^1(\mathrm{u}), \delta_1^1(\mathrm{u}), \ldots, \delta_s^1(\mathrm{u}), \mathrm{x}_1, \ldots, \mathrm{x}_l\big)\big];
\end{aligned}$$

$$\begin{aligned}
&\vdash \mathrm{M}(\alpha_1^1, \ldots, \alpha_k^1, \mathrm{x}_1, \ldots, \mathrm{x}_l) \\
&\quad \equiv (\mathbf{E}\beta_1^1, \ldots, \beta_k^1, \gamma_1^1, \ldots, \gamma_s^1, \delta_1^1, \ldots, \delta_s^1)(\mathrm{u})\big[(\delta_1^1 \text{ is } \gamma_1^{1*}) \,\&\, \ldots \,\& \\
&\quad \& \,(\delta_s^1 \text{ is } \gamma_s^{1*}) \,\&\, (\beta_1^1 \text{ is } \alpha_1^{1*}) \,\&\, \ldots \,\&\, (\beta_k^1 \text{ is } \alpha_k^{1*}) \,\&
\end{aligned}$$

$$\begin{aligned}
&\& \mathrm{P}_2\big(\beta_1^1(\mathrm{u}), \dots, \beta_k^1(\mathrm{u}), \delta_1^1(\mathrm{u}), \dots, \delta_s^1(\mathrm{u}), \mathrm{x}_1, \dots, \mathrm{x}_l\big)\big] \\
&\equiv (\beta_1^1, \dots, \beta_k^1, \gamma_1^1, \dots, \gamma_s^1, \delta_1^1, \dots, \delta_s^1)(\mathbf{E}\mathrm{u})\big[(\delta_1^1 \text{ is } \gamma_1^{1*}) \,\&\, \dots \,\&\\
&\& (\delta_s^1 \text{ is } \gamma_s^{1*}) \,\&\, (\beta_1^1 \text{ is } \alpha_1^{1*}) \,\&\, \dots \,\&\, (\beta_k^1 \text{ is } \alpha_k^{1*})\\
&\supset \mathrm{Q}_2\big(\beta_1^1(\mathrm{u}), \dots, \beta_k^1(\mathrm{u}), \delta_1^1(\mathrm{u}), \dots, \delta_s^1(\mathrm{u}), \mathrm{x}_1, \dots, \mathrm{x}_l\big)\big].
\end{aligned}$$

The proof of this lemma is obtained by formalizing in A the proofs given in [6], p. 316-318.

Finally we note

L. 2.17. $\vdash (\alpha^1)(\mathbf{E}!\beta^k)(\mathrm{x}_1, \dots, \mathrm{x}_k)[\beta^k(\mathrm{x}_1, \dots, \mathrm{x}_k) = \alpha^1(\mathbf{2}^{\mathrm{x}_1} + \mathbf{2}^{\mathrm{x}_1+\mathrm{x}_2+\mathbf{1}} + \dots + \mathbf{2}^{\mathrm{x}_1+\dots+\mathrm{x}_k+\mathbf{k}-\mathbf{1}})]$;

$$\vdash (\beta^k)(\mathbf{E}!\alpha^1)(\mathrm{x})\big[\alpha^1(\mathrm{x}) = \beta^k\big(\mathrm{c}(\mathrm{x}, \mathbf{0}), \dots, \mathrm{c}(\mathrm{x}, \mathbf{k}-\mathbf{1})\big)\big].$$

3. Models absolute for well-orderings

3.1. As in [5] we call a frame an ordered sequence $\mathfrak{p} = \langle N_\mathfrak{p}, 0_\mathfrak{p}, 1_\mathfrak{p}, +_\mathfrak{p}, \times_\mathfrak{p}, F_\mathfrak{p}^1, F_\mathfrak{p}^2, \dots\rangle$ where $N_\mathfrak{p}$ is a set, $0_\mathfrak{p}, 1_\mathfrak{p}$ are elements of $N_\mathfrak{p}$, $+_\mathfrak{p}$, $\times_\mathfrak{p}$ are mappings of $N_\mathfrak{p} \times N_\mathfrak{p}$ into $N_\mathfrak{p}$ and $F_\mathfrak{p}^j$ is a set of mappings of $N_\mathfrak{p}^j = N_\mathfrak{p} \times \dots \times N_\mathfrak{p}$ into $N_\mathfrak{p}$. If f is a valuation with respect to the frame $\mathfrak{p}$ and $\tau(\alpha_1^{i_1}, \dots, \alpha_m^{i_m}, \mathrm{x}_1, \dots, \mathrm{x}_n)$ is a term with the free variables indicated, then we shall sometimes write $\tau^{[\mathfrak{p}]}\big(f(\alpha_1^{i_1}), \dots, f(\alpha_m^{i_m}), f(\mathrm{x}_1), \dots, f(\mathrm{x}_n)\big)$ instead of $Val_{f,\mathfrak{p}}(\tau)$. A similar notation will be used also for formulas, the symbol "$\models_\mathfrak{p}$" replacing the more formal "$= \bigvee$". If $Val_{f,\mathfrak{p}}(\Phi) = \bigvee$ for any f and any axiom Φ of A, then we call $\mathfrak{p}$ a model. Frames which are models will be denoted hereafter by capital German letters.

3.2. For every φ in $F_\mathfrak{M}^2$ we denote by $R_{\mathfrak{M},\varphi}$ a binary relation with the field contained in $N_\mathfrak{M}$ such that $aR_{\mathfrak{M},\varphi}b \equiv \varphi(a, b) = 0_\mathfrak{M}$.

L. 3.3. *If* $R_{\mathfrak{M},\varphi}$ *well-orders* $N_\mathfrak{M}$, *then* $\models_\mathfrak{M}\mathrm{Bord}[\varphi]$.

Proof: obvious.

Theorem converse to 3.3 is not true. This will be proved formally in 5.12. The heuristic reason why the theorem converse to 3.3 is false is this: it may happen that $F_\mathfrak{M}^1$ contains so few functions that no counterexample for $\models_\mathfrak{M}\mathrm{Bord}[\varphi]$ exists in $F_\mathfrak{M}^1$ although it exists outside that set. In connection with this observation we introduce the following definition:

D. 3.4. $\mathfrak{M}$ is a model *absolute for well-orderings* (or shortly a β-model) if for every φ in $F_\mathfrak{M}^2$ the condition $\models_\mathfrak{M}\mathrm{Bord}[\varphi]$ implies that $R_{\mathfrak{M},\varphi}$ well-orders its field.

L. 3.5. *β-models are ω-models.*

Proof. Write $\mathrm{M}(\alpha^2)$ for $(\mathrm{x}, \mathrm{y})[(\mathrm{x}\alpha^2\mathrm{y}) \equiv (\mathrm{x} \leqslant \mathrm{y})]$. Obviously $\vdash(\mathbf{E}!\alpha^2)\mathrm{M}(\alpha^2)$ and $\vdash \mathrm{M}(\alpha^2) \supset \mathrm{Bord}(\alpha^2)$, whence it follows that if φ is an element of $F^2_{\mathfrak{M}}$ such that $\models_{\mathfrak{M}} \mathrm{M}[\varphi]$, then $\models_{\mathfrak{M}} \mathrm{Bord}[\varphi]$, and hence, for β-models, that $R_{\mathfrak{M},\varphi}$ well-orders $N_{\mathfrak{M}}$. If $\mathfrak{M}$ is not an ω-model, then $N_{\mathfrak{M}}$ contains an element different from all elements $\mathbf{n}^{[\mathfrak{M}]}$, $n = 0, 1, \ldots$ Let a be the first such element (with respect to the well-ordering $R_{\mathfrak{M},\varphi}$). Since $a \neq 0_{\mathfrak{M}}$, there is a b in $N_{\mathfrak{M}}$ such that $a = b +_{\mathfrak{M}} 1_{\mathfrak{M}}$. Hence $bR_{\mathfrak{M},\varphi}a$ and $b \neq a$, b has the form $\mathbf{n}^{[\mathfrak{M}]}$, and hence so has a: a contradiction.

L. 3.5 shows that dealing with β-models we can consider only models whose "arithmetical part" $\langle N_{\mathfrak{M}}, 0_{\mathfrak{M}}, 1_{\mathfrak{M}}, +_{\mathfrak{M}}, \times_{\mathfrak{M}}\rangle$ coincides with the arithmetical part of the absolute model $\mathfrak{M}_0 = \langle N_0, 0, 1, +, \times, F^1_0, F^2_0, \ldots\rangle$. In view of this we shall abbreviate the notation for β-models to $\langle F^1_{\mathfrak{M}}, F^2_{\mathfrak{M}}, \ldots\rangle$.

Actually the whole model is already determined by $F^1_{\mathfrak{M}}$, as we see from the following lemma:

L. 3.6. *If* $\varphi \in F^1_{\mathfrak{M}}$ *and* $\psi(n_0, \ldots, n_k) = \varphi(2^{n_0} + 2^{n_0+n_1+1} + \ldots + 2^{n_0+\ldots+n_k+k})$, *then* $\psi \in F^{k+1}_{\mathfrak{M}}$; *if* $\psi \in F^{k+1}_{\mathfrak{M}}$ *and* $\varphi(n) = \psi\big(c(n, 0), \ldots, c(n, k)\big)$, *then* $\varphi \in F^1_{\mathfrak{M}}$.

Proof. By L. 2.17.

D. 3.7. If X is a set of formulas, then $Cn_\beta(X)$ is the set of the formulas Φ which are valid in all β-models of X. The set $Cn_\beta(A)$ is denoted by A_β.

L. 3.8. *If* $\mathfrak{M}$ *is an arithmetical extension of* $\mathfrak{M}'$ *and* $\mathfrak{M}$ *is a* β*-model, then so is* $\mathfrak{M}'$.

Proof. If $\varphi \in F^2_{\mathfrak{M}'}$, then conditions $\models_{\mathfrak{M}'} \mathrm{Bord}[\varphi]$ and $\models_{\mathfrak{M}'} \mathrm{Bord}[\varphi]$ are equivalent. Hence if $\mathfrak{M}$ is a β-model, then $\models_{\mathfrak{M}'} \mathrm{Bord}[\varphi]$ implies that $R_{\mathfrak{M},\varphi}$ is a well-ordering. Since $R_{\mathfrak{M},\varphi} = R_{\mathfrak{M}',\varphi}$ this proves the lemma.

L. 3.9. *For every* β*-model* $\mathfrak{M}$ *there is a denumerable* β*-submodel* $\mathfrak{M}_1$ *of* $\mathfrak{M}$ *such that* $\mathfrak{M}$ *is an arithmetical extension of* $\mathfrak{M}_1$. ("Denumerable" means that all sets $F^j_{\mathfrak{M}_1}$ are denumerable.)

Proof. By L. 3.8 and theorem 2.1 of [10].

4. Evaluation of the predicate $\mathrm{N}^0\Phi \in \mathrm{N}^0 Cn_\beta(X)$

4.1. For every function $\mu \in N_0^{N_0}$ we denote by μ^j_k the following mapping of N^j_0 to N_0:

$$\mu^j_k(n_1, \ldots, n_j) = \mu(\langle k, 2^{n_1} + 2^{n_1+n_2+1} + \ldots + 2^{n_1+n_2+\ldots+n_j+j-1}\rangle).$$

The set of all functions μ_k^j, $k = 0, 1, \ldots$, we denote by F_μ^j and the frame $\langle N_0, 0, 1, +, \times, F_\mu^1, F_\mu^2, \ldots\rangle$ by $\mathfrak{M}_\mu$.

L. 4.2. *Let* $\mu, \psi, \vartheta \in N_0^{N_0}$ *and let* $f_{\mu,\psi,\vartheta}$ *be a valuation such that* $f_{\mu,\psi,\vartheta}(\mathrm{x}_k) = \psi(k)$, $f_{\mu,\psi,\vartheta}(a_k^j) = \mu_{\vartheta(k)}^j$, $k, j = 1, 2, \ldots$ *The following predicates* $R(\mu, \psi, \vartheta, \mathrm{N}^0(\tau), r')$ *and* $Q(\mu, \psi, \vartheta, \mathrm{N}^0(\Phi), r'')$

$$(1) \qquad Val_{f_{\mu,\psi,\vartheta},\mathfrak{M}_\mu}(\tau) = r', \qquad Val_{f_{\mu,\psi,\vartheta},\mathfrak{M}_\mu}(\Phi) = r''$$

are hyperarithmetic. (We identify here the truth values $\bigwedge$ and $\bigvee$ with the integers 0 and 1.)

Proof. τ is a term and Φ a formula if and only if there are sequences

$$(2) \qquad \tau_1, \tau_2, \ldots, \tau_l = \tau, \qquad \Phi_1, \Phi_2, \ldots, \Phi_m = \Phi$$

such that

(i) every τ_j is either (i_1) a number variable x_u or has one of the forms (i_2)-(i_5): $0, 1, \tau_t + \tau_s, \tau_t \times \tau_s$ or the form (i_6) $a_v^w(\tau_{t_1}, \ldots, \tau_{t_w})$ where $t, s, t_1, \ldots, t_w < j$ or finally the form (i_7) $(\iota \mathrm{x}_u)\Phi_t$;

(ii) every Φ_h has either the form (ii_1) $\tau_s = \tau_t$ or one of the forms (ii_2)-(ii_5): $\Phi_s \vee \Phi_t$, $\sim\Phi_s$, $(\mathbf{E}\mathrm{x}_u)\Phi_s$, $(\mathbf{E}a_u^k)\Phi_s$ where $s, t < h$.

We symbolize an arbitrary pair of sequences (2) by a single letter P. Let $z(P)$ be the least integer such that no variable with an index greater than $z(P)$ occurs in the formulas (2). For arbitrary integers p, q let $v_{p,q,P}$ be a valuation such that for $k, h \leqslant z(P)$

$$v_{p,q,P}(\mathrm{x}_k) = c(p, k), \qquad v_{p,q,P}(a_h^j) = \mu_{c(q,h)}^j .$$

Finally we denote by $\varphi'_{j,\mu}$ and $\varphi''_{j,\mu}$ the following functions (3):

$$\varphi'_{j,\mu}(n) = Val_{v_{s_1(n),s_2(n),P},\mathfrak{M}_\mu}(\tau_j), \qquad j = 1, 2, \ldots, l,$$

$$\varphi''_{j,\mu}(n) = Val_{v_{s_1(n),s_2(n),P},\mathfrak{M}_\mu}(\Phi_h), \qquad h = 1, 2, \ldots, m.$$

These functions are uniquely determined by the following conditions (we assume that $j = 1, 2, \ldots, l$ and $h = 1, 2, \ldots, m$):

(i_1^*) *If* (i_1), *then* $\varphi'_{j,\mu}(n) = c\big(s_1(n), u\big)$;

(i_2^*)-(i_5^*) *if* (i_2)-(i_5), *then* $\varphi'_{j,\mu}(n) = 0, 1, \varphi'_{s,\mu}(n) + \varphi'_{t,\mu}(n),$
$\varphi'_{s,\mu}(n) \times \varphi'_{t,\mu}(n)$;

(i_6^*) *if* (i_6), *then* $\varphi'_{j,\mu}(n) = \mu_{c(q,v)}^w\big(\varphi'_{t_1,\mu}(n), \ldots, \varphi'_{t_w,\mu}(n)\big)$;

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(3) Strictly speaking $\varphi'_{j,\mu}$ and $\varphi''_{j,\mu}$ depend also on P. We supress the index P to make the formulas more readable.

(i_7^*) *if* (i_7), *then* $\varphi'_{j,\mu}(n)$ *is the unique integer* r *satisfying the condition* $\varphi''_{i,\mu}(\langle\overline{p}, s_2(n)\rangle) = 1$ *where* $\overline{p}$ *is defined by means of the formulas*

$$(*) \qquad c(\overline{p}, i) = c(s_1(n), i) \textit{ for } i \neq u; \qquad c(\overline{p}, u) = r.$$

If no such r *exists or if it is not unique, then* $\varphi'_{j,\mu}(n) = 0$.

(ii_1^*) *If* (ii_1), *then* $\varphi''_{h,\mu}(n) = 1 \equiv [\varphi'_{s,\mu}(n) = \varphi''_{i,\mu}(n)]$;

(ii_2^*) *if* (ii_2), *then* $\varphi''_{h,\mu}(n) = 1-(1-\varphi''_{s,\mu}(n))(1-\varphi''_{i,\mu}(n))$;

(ii_3^*) *if* (ii_3), *then* $\varphi''_{h,\mu}(n) = 1-\varphi''_{s,\mu}(n)$;

(ii_4^*) *if* (ii_4), *then* $\varphi''_{h,\mu}(n) = 1 \equiv$ [*there is an* r *such that* $\varphi''_{s,\mu}(\langle\overline{p}, s_2(n)\rangle) = 1$ *where* $\overline{p}$ *is defined by* $(*)$];

(ii_5^*) *if* (ii_5), *then* $\varphi''_{h,\mu}(n) = 1 \equiv$ [*there is an integer* r *such that* $\varphi''_{s,\mu}(\langle s_1(n), \overline{q}\rangle) = 1$ *where* $\overline{q}$ *is defined by the formulas*

$$(**) \qquad c(\overline{q}, i) = c(s_2(n), i) \textit{ for } i \neq u, \qquad c(\overline{q}, u) = r].$$

Formulas (1) are equivalent to the existence of integers p, q such that

$$(3) \qquad \begin{aligned} c(p, k) &= \psi(k) \textit{ for } k \leqslant z(P), \\ c(q, h) &= \vartheta(h) \textit{ for } h \leqslant z(P). \end{aligned}$$

$$(4) \qquad \varphi'_{l,\mu}(\langle p, q\rangle) = r', \qquad \varphi''_{m,\mu}(\langle p, q\rangle) = r''.$$

Since the functions $\varphi'_{j,\mu}$ and $\varphi''_{h,\mu}$ are determined uniquely, formulas (1) are equivalent to either of the following conditions:

(5) *There are sequences* (2) *with properties* (i), (ii) *and integers* p, q *and functions* $\varphi'_{j,\mu}, \varphi''_{h,\mu}$ $(j \leqslant l, h \leqslant m)$ *satisfying* (i_1^*)-(ii_5^*), (3) *and* (4);

(6) *For arbitrary sequences* (2) *with properties* (i), (ii) *and for arbitrary integers* p, q *and functions* $\varphi'_{j,\mu}, \varphi''_{h,\mu}$ $(j \leqslant l, h \leqslant m)$ *satisfying* (i_1^*)-(ii_5^*) *and* (3), *equations* (4) *are true.*

Quantifiers "*there are sequences* (2)" and "*for arbitrary sequences* (2)" are arithmetic. So are also conditions (i_1^*)-(ii_5^*), (3), and (4). The equivalence of (1), (5), and (6) shows therefore that conditions (1) are hyperarithmetic.

COROLLARY 4.3. *If* $\Phi(\mathrm{x}_1, \ldots, \mathrm{x}_k, a^{j_1}_{l_1}, \ldots, a^{j_m}_{l_m})$ *is a formula with the free variables indicated, then the predicate* $\models_{\mathfrak{M}_\mu} \Phi[n_1, \ldots, n_k, \mu^{j_1}_{l_1}, \ldots, \mu^{j_m}_{l_m}]$ *is hyperarithmetic* (its arguments are $n_1, \ldots, n_k$, $l_1, \ldots, l_m$ and μ).

L. 4.4. *The predicate* "$\mathfrak{M}_\mu$ *is a model of* A" (*abbreviated* $Mod(\mu)$) *is hyperarithmetic.*

The proof follows at once from the equivalence

$$\mathit{Mod}(\mu) \equiv (n)\ [n \textit{ is the Gödel number of a closure of an axiom } \Phi \textit{ of } A \supset\ \models_{\mathfrak{M}_\mu} \Phi].$$

L. 4.5. *The predicate "$\mathfrak{M}_\mu$ is a β-model" (abbreviated as $\mathit{Mod}_\beta(\mu)$) is of class Π^1_1.*

Proof. The predicate "*$R_{\mathfrak{M}_\mu, \mu^2_v}$ well-orders its field*" is obviously of class Π^1_1. The lemma follows therefore from the equivalence:

$$\mathit{Mod}_\beta(\mu) \equiv \mathit{Mod}(\mu) \,\&\, (v)\big[(\models_{\mathfrak{M}_\mu} \mathrm{Bord}[\mu^2_v]) \supset (R_{\mathfrak{M}_\mu, \mu^2_v} \textit{ well-orders its field}\big].$$

L. 4.6. *For every β-model $\mathfrak{M}$ there is a μ such that $\mathit{Mod}_\beta(\mu)$ and $\mathfrak{M}$ is an elementary extension of a model isomorphic with $\mathfrak{M}_\mu$.*

Proof. Define $\mathfrak{M}_1$ as in 3.9 and take an arbitrary μ such that $F^j_\mu = F^j_{\mathfrak{M}_1}$ for $j = 1, 2, \ldots$ A μ of this sort exists in virtue of the denumerability of $\mathfrak{M}_1$.

THEOREM 4.7. *Let X be a set of closed formulas. If $\mathrm{N}^0 X \in \Pi^1_n$, then $\mathrm{N}^0\, \mathit{Cn}_\beta(X) \in \Pi^1_{\max(2,n)}$.*

Proof. By 4.2 the predicate "*$\mathfrak{M}_\mu$ is a model of X*" is hyperarithmetic in X. Since, by 4.6,

$$\Phi \in \mathit{Cn}_\beta(X) \equiv (\mu)[\mathit{Mod}_\beta(\mu) \,\&\, (\mathfrak{M}_\mu \textit{ is a model of } X) \supset\ \models_{\mathfrak{M}_\mu} \Phi],$$

the result follows by 4.5 and 4.2.

COROLLARY 4.8. $\mathrm{N}^0 A_\beta \in \Pi^1_2$.

5. Universality of the set $\mathrm{N}^0 A_\beta$

5.1. Let K be an arbitrary ternary relation between integers and let

$$C^{(K)}_{\varphi,n} = \Big\{a\colon\ s_1(a) < 2^{s_2(a)} \,\&\, (j)_{dl(r(s_1(a),\, s_2(a)))}\; K\Big(\varphi^*(j),\, rst\Big(r\big(s_1(a),\, s_2(a)\big),\, j\Big),\, n\Big)\Big\}.$$

Let $L^{(K)}_{\varphi,n}$ be the binary relation defined by the equivalence

$$(1)\qquad aL^{(K)}_{\varphi,n} b \equiv (a,\, b \in C^{(K)}_{\varphi,n}) \,\&\, \big(s_1(a)/2^{s_2(a)} \geqslant s_1(b)/2^{s_2(b)}\big).$$

In the sequel we shall deal with one fixed relation K and shall therefore drop the upper index K.

L. 5.2. $L_{\varphi,n}$ *orders* $C_{\varphi,n}$.

L. 5.3. $(\mathbf{E}\psi)(m)K(\varphi^*(m),\psi^*(m),n) \equiv \{L_{\varphi,n}$ *does not well-order* $C_{\varphi,n}\}$.

The proof of 5.2 is obvious and the proof of 5.3 is exactly the same as that given in [8], p. 10.

L. 5.4. *Let* ϱ_m *be the following function*:

$$\varrho_m(j) = \begin{cases} c(m,j) & \text{for} \quad j \leqslant dl(m), \\ 0 & \text{for} \quad j > dl(m). \end{cases}$$

If K *is recursive, then* $aL_{\varrho_m,n}b$ *is a recursive relation with* 4 *arguments which we denote by* $P(a,b,m,n)$.

We put $g(a,b) = r(s_1(a),s_2(a)) + r(s_1(b),s_2(b))$.

L. 5.5. *If* $\varrho_m(j)=\varphi(j)$ *for* $j \leqslant g(a,b)$, *then* $P(a,b,m,n) \equiv aL_{\varphi,n}b$.

Indeed in this case $x \in C_{\varrho_m,n} \equiv x \in C_{\varphi,n}$ for $x = a, b$ and the result follows by 5.1 (1).

D. 5.6. Let ξ be a term representing the function $g(a,b)$, and $\mathrm{A}(\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4)$ a formula representing the relation P. Further, let $\Gamma(\gamma^2,\alpha^1,\mathrm{z})$ be the formula

$$(\mathrm{x},\mathrm{y})\{\mathrm{x}\gamma^2\mathrm{y} \equiv (\mathbf{E}\beta^1)[(\beta^1 \text{ is } \alpha^{1*}) \,\&\, \mathrm{A}(\mathrm{x},\mathrm{y},\beta^1(\xi(\mathrm{x},\mathrm{y})),\mathrm{z})]\}$$

and $\Phi(\mathrm{z})$ the formula

$$(\alpha^1)(\mathbf{E}\gamma^2)[\Gamma(\gamma^2,\alpha^1,\mathrm{z}) \,\&\, {\sim}\mathrm{Bord}(\gamma^2)].$$

L. 5.7. $\vdash (\mathbf{E}\gamma^2)\Gamma(\gamma^2,\alpha^1,\mathrm{z})$.

Proof. From the axiom 3d of [5].

L. 5.8. *If* $\mathfrak{M}$ *is an* ω-*model and* $\varphi \in F^1_{\mathfrak{M}}$, $\psi \in F^2_{\mathfrak{M}}$, *then*

$$\models_{\mathfrak{M}} \Gamma[\psi,\varphi,\mathbf{n}] \equiv (a,b)[\psi(a,b)=0 \equiv aL_{\varphi,n}b].$$

Proof. The left-hand side is equivalent to

$$(a,b)\big(\psi(a,b)=0 \equiv (\mathbf{E}\vartheta)\{(\vartheta \in F^1_{\mathfrak{M}}) \,\&\, \models_{\mathfrak{M}}(\vartheta \text{ is } \varphi^*) \,\& \\ \& \models_{\mathfrak{M}}\mathrm{A}[a,b,\vartheta(g(a,b)),n]\}\big).$$

The expression on the right-hand side is equivalent to

$$\models_{\mathfrak{M}}\mathrm{A}[a,b,\varphi^*(g(a,b)),n]$$

(cf. L. 2.15). Since A represents P, this expression is in turn equivalent to $aL_{\varrho_{\varphi^*(g(a,b))},n}b$. Now $\varrho_{\varphi^*(g(a,b))}$ is a function ϱ such that $\varrho(j) = \varphi(j)$ for $j \leqslant dl\big(\varphi^*(g(a,b))\big) = g(a,b)$ and hence by 5.5

$$\models_{\mathfrak{M}} \mathrm{A}\big[a, b, \varphi^*(g(a,b)), n\big] \equiv aL_{\varphi,n}b\,.$$

This proves the lemma.

L. 5.9. *There is a primitive recursive function f such that* $\mathrm{N}^0\,\Phi(\mathbf{n}) = f(n)$.

Proof: obvious.

L. 5.10. $\big\{(\varphi)(\mathsf{E}\psi)(n)K\big(\varphi^*(n), \psi^*(n), c\big)\big\} \equiv \{\Phi(\mathbf{c}) \in A_\beta\}$.

Proof. Assume first that $\Phi(\mathbf{c}) \in A_\beta$ and let $\mathfrak{M}_0$ be the absolute model, i. e., one consisting of all functions. Since $\mathfrak{M}_0$ is a β-model, we obtain $\models_{\mathfrak{M}_0}\Phi(\mathbf{c})$, and hence for every φ in $F^1_{\mathfrak{M}_0}$ there is a function ψ in $F^2_{\mathfrak{M}_0}$ such that $\models_{\mathfrak{M}_0}\Gamma[\psi, \varphi, c]$ and $\models_{\mathfrak{M}_0}\sim\mathrm{Bord}[\psi]$. Using L. 5.8 we infer that $L_{\varphi,c}$ is not a well-ordering, and hence by L. 5.3 that $(\mathsf{E}\psi)(n)K\big(\varphi^*(n), \psi^*(n), c\big)$. Hence the right-hand side of 5.10 implies the left.

Assume now that the left-hand side of 5.10 is true and let $\mathfrak{M}$ be a β-model of A and φ an element of $F^1_{\mathfrak{M}}$. By 5.7 there is a ψ in $F^2_{\mathfrak{M}}$ such that $\models_{\mathfrak{M}}\Gamma[\psi, \varphi, c]$, whence, by lemma 5.8, $\psi(a,b) = 0 \equiv aL_{\varphi,c}b$. Using 5.3 we infer that the relation $\psi(a,b) = 0$ does not well-order its field, and hence that $R_{\mathfrak{M},\psi}$ is not a well-ordering. $\mathfrak{M}$ being a β-model, we further obtain $\models_{\mathfrak{M}}\sim\mathrm{Bord}[\psi]$. Thus the formula $\Phi(\mathbf{c})$ is true in $\mathfrak{M}$. L. 5.10 is thus proved.

THEOREM 5.11. *A_β is universal (i.e. complete in the sense of Post) for the class Π^1_2; in particular A_β does not belong to Σ^1_2.*

Proof. By 5.9 and 5.10.

COROLLARY 5.12. *There are ω-models which are not β-models.*

Proof. A_ω is universal for Π^1_1 and A_β for Π^1_2.

6. Incompleteness and related properties of A_β

6.1. The results of this section depend on a formalization of some of the proofs given in section 5.

Let ζ be a term representing the function $r\big(s_1(a), s_2(a)\big)$, and ξ as before a term representing the function $g(a,b)$. Further let $\mathrm{rst}(\mathrm{x},\mathrm{y})$ be a term representing the function $rst(m,n)$. Finally let $\tau(\mathrm{x},\mathrm{y})$ be a term representing the function $\varrho_m(j)$.

For an arbitrary predicate $M(x, y, z)$ with the free variables indicated define the predicates C, L, Θ, Ξ, and H as follows:

$$\begin{array}{rl} C(\beta^1, x, z): & [s_1(x) < 2^{s_2(x)}] \& (u)_{dl(\zeta(x))} M\big(\beta^1(u), rst\,(\zeta(x), u), z\big); \\ L(\beta^1, x, y, z): & C(\beta^1, x, z) \& C(\beta^1, y, z) \& [s_1(x) \cdot 2^{s_2(y)} \\ & \qquad \geqslant s_2(y) \cdot 2^{s_1(x)}]; \\ \Theta(\beta^1, x, y, z): & (\mathsf{E}\gamma^1)\Big\{(t)\Big[\gamma^1(t) = \tau\Big(\beta^1\big(\xi(x, y)\big), t\Big)\Big] \& \\ & \qquad \& L(\gamma^1, x, y, z)\Big\}; \\ \Xi(\alpha^1, \gamma^2, z): & (\mathsf{E}\beta^1)\{(x, y)[x\gamma^2 y \equiv \Theta(\beta^1, x, y, z)] \& (\beta^1 \text{ is } \alpha^{1*})\}; \\ H(z): & (\alpha^1)(\mathsf{E}\gamma^2)[\Xi(\alpha^1, \gamma^2, z) \& \sim \mathrm{Bord}(\gamma^2)]. \end{array}$$

L. 6.2. $\vdash (\alpha^1)(\mathsf{E}\beta^1, \gamma^1, \delta^1)\big[(\gamma^1 \text{ is } \alpha^{1*}) \& (\delta^1 \text{ is } \beta^{1*}) \&$
$\& (u) M\big(\gamma^1(u), \delta^1(u), z\big)\big] \equiv H(z).$

This lemma represents a formalization of 5.10. It is proved by observing that all the steps of the proof of 5.10 can be carried out in the system A.

THEOREM 6.3. *If* $M(x, y, z)$ *is an elementary formula with the free variables indicated and if the formula*

$$(\alpha^1)(\mathsf{E}\beta^1, \gamma^1, \delta^1)\big[(\gamma^1 \text{ is } \alpha^{1*}) \& (\delta^1 \text{ is } \beta^{1*}) \& (u) M\big(\gamma^1(u), \delta^1(u), \mathbf{n}\big)\big]$$

is true in the absolute model $\mathfrak{M}_0$, *then it is* β*-provable.*

Proof. In view of L. 6.2 it is sufficient to prove that if $\models_{\mathfrak{M}_0} H[n]$, then $\vdash_\beta H(\mathbf{n})$. Let $\mathfrak{M}$ be a β-model and let $\varphi \in F^1_{\mathfrak{M}}$. Because of the axiom (3d) of [5] there is in $F^2_{\mathfrak{M}}$ a zero-one function ψ such that $\psi(a, b) = 0 \equiv \models_{\mathfrak{M}} \Theta[\varphi^*, a, b, n]$. Since M, τ, and ξ are elementary, the right-hand side of this equivalence is independent of $\mathfrak{M}$ (provided of course that $\varphi \in F^1_{\mathfrak{M}}$). Hence $\models_{\mathfrak{M}} \Xi[\varphi, \psi, n]$ and ψ is a unique zero-one function satisfying this condition for given φ and n. This result holds for any $\mathfrak{M}$ and in particular for $\mathfrak{M} = \mathfrak{M}_0$. Since $\models_{\mathfrak{M}_0} H[n]$, it follows that $\models_{\mathfrak{M}_0} \sim \mathrm{Bord}[\psi]$, whence $\models_{\mathfrak{M}} \sim \mathrm{Bord}[\psi]$ since $\mathfrak{M}$ is a β-model. Thus we have $\models_{\mathfrak{M}} \{\Xi[\varphi, \psi, n] \& \sim \mathrm{Bord}[\psi]\}$, whence $\models_{\mathfrak{M}} H[n]$, q. e. d.

A similar theorem holds for $\mathfrak{M}$ depending on a larger number of variables.

THEOREM 6.4. *If* $M(\alpha_1^1, \dots, \alpha_k^1, \beta_1^1, \dots, \beta_l^1, x_1, \dots, x_m)$ *is an elementary formula with the free variables indicated and if the formula*

$$(\alpha_1^1, \dots, \alpha_k^1)(\mathsf{E}\beta_1^1, \dots, \beta_l^1) M(\alpha_1^1, \dots, \alpha_k^1, \beta_1^1, \dots, \beta_l^1, \mathbf{n}_1, \dots, \mathbf{n}_m)$$

is true in $\mathfrak{M}_0$, *then it is provable in* A_β.

Proof. By L. 2.16 and theorem 6.3 generalized to a larger number of variables.

Theorem 6.5. *Π^1_2 coincides with the family of sets weakly representable in A_β.*

Proof. Assume that X is a set such that there is a formula $\Phi(\mathrm{x})$ satisfying the equivalence

$$n \in X \equiv \vdash_\beta \Phi(\mathbf{n}).$$

Then $n \in X \equiv \mathrm{N}^0\, \Phi(\mathbf{n}) \in \mathrm{N}^0 A_\beta$ whence, in view of 5.11, $X \in \Pi^1_2$.

If X is in Π^1_2, then

$$n \in X \equiv (\varphi)(\mathbf{E}\psi)(a)K\big(\varphi^*(a), \psi^*(a), n\big)$$

where K is recursive. If $\mathrm{M}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ is an elementary formula which represents the relation $K(a, b, c)$ in A and $\Phi(\mathrm{z})$ is the formula $(\alpha^1)(\mathbf{E}\beta^1, \gamma^1, \delta^1)\big[(\gamma^1 \text{ is } \alpha^{1*}) \,\&\, (\delta^1 \text{ is } \beta^{1*}) \,\&\, (\mathrm{u})\mathrm{M}\big(\gamma^1(\mathrm{u}), \delta^1(\mathrm{u}), \mathrm{z}\big)\big]$, then $n \in X \equiv \models_{\mathfrak{M}_0} \Phi[n]$, and hence by 6.3 $n \in X \equiv \vdash_\beta \Phi(\mathbf{n})$.

Corollary 6.6. *If X is strongly representable in A_β, then $X \in \Pi^1_2 \cap \Sigma^1_2$.*

Corollary 6.7. *There is an elementary formula* $\mathrm{M}(\mathrm{x}, \mathrm{y})$ *such that the formula*

$$(\mathbf{E}\alpha^1)(\beta^1, \gamma^1, \delta^1)\big[(\gamma^1 \text{ is } \alpha^{1*}) \,\&\, (\delta^1 \text{ is } \beta^{1*}) \supset (\mathbf{E}\mathrm{u})\mathrm{M}\big(\gamma^1(\mathrm{u}), \delta^1(\mathrm{u})\big)\big]$$

is true in $\mathfrak{M}_0$ but not β-provable.

Proof. Let $X \in \Pi^1_2 - \Sigma^1_2$ and let K be a recursive ternary relation such that $n \in X \equiv (\varphi)(\mathbf{E}\psi)(m)K\big(\varphi^*(m), \psi^*(m), n\big)$. If $\overline{\mathrm{M}}$ is an elementary formula which represents K in A, then obviously

$$n \in X \equiv \models_{\mathfrak{M}_0} (\alpha^1)(\mathbf{E}\beta^1, \gamma^1, \delta^1)\big[(\gamma^1 \text{ is } \alpha^{1*}) \,\&\, (\delta^1 \text{ is } \beta^{1*}) \,\& \\ \& \,(\mathrm{u})\overline{\mathrm{M}}\big(\gamma^1(\mathrm{u}), \delta^1(\mathrm{u}), \mathbf{n}\big)\big].$$

Denoting by $\Phi(\mathbf{n})$ the formula on the right-hand side we have, by 6.3,

$$n \in X \equiv \vdash_\beta \Phi(\mathbf{n}).$$

The equivalence n non-$\in X \equiv \vdash_\beta \sim\Phi(\mathbf{n})$ cannot be true for all n since in that case X would be in Σ^1_2. Hence there is an n_0 such that n_0 non-$\in X$ and non $\vdash_\beta \sim\Phi(\mathbf{n}_0)$. It is now sufficient to take $\sim\overline{\mathrm{M}}(\mathrm{x}, \mathrm{y}, \mathbf{n}_0)$ as the formula $\mathrm{M}(\mathrm{x}, \mathrm{y})$.

Corollary 6.8. *A_β is incomplete.*

Thus far the properties of A_β are similar to the corresponding properties of A_ω (cf. [5], theorems 3.1. E, 3.1. F, 3.4. B). We shall now present a theorem whose analogue is not valid for A and A_ω (cf. [5], 2.4. B and 3.4. A).

THEOREM 6.9. *The family of sets strongly representable in A_β is a proper subclass of $\Pi_2^1 \cap \Sigma_2^1$.*

Proof. In [1] a proof was given to the effect that if X, $Y \in \Pi_2^1$ and $X \cap Y = 0$, then there is a set Z in $\Pi_2^1 \cap \Sigma_2^1$ such that $X \subseteq Z$ and $Y \cap Z = 0$. Take $X = \mathrm{N}^0 A_\beta$ and $Y = \{\mathrm{N}^0 \Phi \colon \sim\Phi \in A_\beta\}$. By 4.8 both the set X and the set Y belong to Π_2^1; it is also obvious that they are disjoint. Let Z be a separating set from the class $\Pi_2^1 \cap \Sigma_2^1$ and let $Z_0 = \{n\colon g(n, n) \text{ non-}\in Z\}$ where $g(n, m)$ is a recursive function universal for primitive recursive functions. Obviously $Z_0 \in \Pi_2^1 \cap \Sigma_2^1$. If Z_0 were strongly representable, there would be a formula $\Phi(\mathrm{x})$ such that

$$n \in Z_0 \equiv \vdash_\beta \Phi(\mathbf{n}) \qquad n \text{ non-}\in Z_0 \equiv \vdash_\beta \sim\Phi(\mathbf{n})\,.$$

Since $\mathrm{N}^0\,\Phi(\mathbf{n})$ is a primitive recursive function of n, it would follow that there is an integer k_0 such that

$$n \in Z_0 \equiv g(k_0, n) \in X \qquad n \text{ non-}\in Z_0 \equiv g(k_0, n) \in Y\,,$$

whence we should obtain for $n = k_0$

$$k_0 \in Z_0 \supset g(k_0, k_0) \in Z \supset k_0 \text{ non-}\in Z_0\,,$$

$$k_0 \text{ non-}\in Z_0 \supset g(k_0, k_0) \text{ non-}\in Z \supset k_0 \in Z_0\,.$$

7. Applications to the theory of constructible sets

7.1. Investigations of this and the next section will be based on the Gödel axioms of set theory [4] supplemented by an axiom stating that there is at least one weakly inaccessible ordinal. The least weakly inaccessible ordinal will be denoted by ν_0. We shall use the terminology and notation of [4] with the exception that the nth element of the set $\omega = \Big\{0, \{0\}, \big\{0, \{0\}\big\}, \ldots\Big\}$ will be denoted by Z_n.

7.2. We shall consider a formal system of set theory which we shall call system *ST*. *ST* is a theory with standard formalization with the primitive terms M, Cls, and ε (cf. [4]). The set-theoretical terms and formulas (*ST*-terms and *ST*-formulas) are defined by induction as follows:

1. all variables are ST-terms (we assume that all variables of the system of analysis are variables of ST),

2. if τ_1, τ_2 are ST-terms, then $\tau_1 = \tau_2$ and $\tau_1 \epsilon \tau_2$ are ST-formulas,

3. if Φ_1 and Φ_2 are ST-formulas and x is a variable, then $\sim\Phi_1$, $\Phi_1 \vee \Phi_2$ and $(\mathbf{E}\mathrm{x})\Phi_1$ are ST-formulas and $(\iota \mathrm{x})\Phi_1$ is an ST-term.

Writing the formulas of ST we shall use the convention set forth in [4], p. 3, concerning the use of small and capital letters (set variables and class variables).

Axioms of ST are those of [4].

7.3. Frames of ST are ordered triplets $\mathfrak{S} = \langle A, B, R \rangle$ where A and B are sets and R is a binary relation. If $R = \epsilon_B$ is the ϵ-relation restricted to B, then $\mathfrak{S}$ is called an ϵ-frame. If, in addition, $x \epsilon y \epsilon B$ implies $x \epsilon B$, then $\mathfrak{S}$ is called a transitive frame.

The semantical notions of a value of an ST-term or of an ST-formula are defined in the usual way. Models of ST are frames in which all axioms are true.

7.4. Let F be the function defined in [4], p. 37, and let $G_\xi = F``\xi$ for every ordinal ξ. Every ordinal ξ determines a frame $\mathfrak{S}_\xi = \langle G_\xi, H_\xi, \epsilon_{H_\xi} \rangle$ where H_ξ is the smallest set containing G_ξ as a subset, containing the set $E_\xi = \{\langle x, y \rangle : x, y \epsilon G_\xi \,\&\, x \epsilon y\}$ as an element and closed with respect to the following operations: intersection, complementation with respect to G_ξ, taking the domain, direct multiplication and the operations $\mathfrak{Cnv}_i$, $i = 1, 2, 3$ (cf. [4], 4.4).

L. 7.5. *The frame* $\mathfrak{S}_{\nu_0}$ *is a model of* ST.

Proof of this lemma is implicitly contained in [4], Chapter VI.

7.6. We list below, for later use, some special ST-terms and ST-formulas. Instead of giving their explicit expressions we merely refer to [4] and give the intuitive meaning of the term or formula in question.

0	([4], 2.1, the void set),
$\{\mathrm{x}, \mathrm{y}\}$	([4], 1.1, the unordered pair),
$\langle \mathrm{x}, \mathrm{y} \rangle$	([4], 1.12, the ordered pair),
$\langle \mathrm{x}_1, \ldots, \mathrm{x}_n \rangle$	([4], 1.15, the ordered n-tuple),
$O(\mathrm{x})$	([4], 6.61, x is an ordinal number),

$\omega(\mathrm{x})$ ([4], 8.4, x is an integer),

$\mathfrak{W}\mathrm{ex} \equiv \mathfrak{W}(\mathrm{x})\mathfrak{W}\mathrm{ex}$ ([4], 4.44, 6.2, x is a well-ordering relation),

ω^n ([4], 4.11, 4.12 and 8.4, the direct product of n copies of ω),

$\omega^{(\omega^n)}(\mathrm{x}) \equiv (\mathrm{x}\,\mathfrak{Fn}\,\omega^n) \,\&\, \big(\mathfrak{W}(\mathrm{x}) \subseteq \omega\big)$ ([4], 4.63, 4.44, x is a function of n integral variables with the range contained in ω),

$\mathrm{x} \sim_t \mathrm{y} \equiv \big(\mathrm{x} \in O(\mathrm{n})\big) \,\&\, \big(\mathfrak{Rel}(\mathrm{y})\big) \,\&\, \big(\mathfrak{Un}_2(\mathrm{f})\big) \,\&\, \Big(\mathfrak{D}(\mathrm{f}) = \mathfrak{S}\big(\mathfrak{S}(\mathrm{y})\big)\Big) \,\&$ $\& \big(\mathfrak{W}(\mathrm{f}) = \mathrm{x}\big) \,\&\, (\mathrm{u},\mathrm{v})[\mathrm{uyv} \equiv \mathrm{f}`\mathrm{u} \in \mathrm{f}`\mathrm{v}]$ ([4], 6.62, 4.2, 4.6, 1.5, 4.44, 4.8, 4.211, 4.65, x is an ordinal, y a relation and f establishes an isomorphism between y and ϵ restricted to x),

$\mathrm{x} +_s \mathrm{y} = (\iota \mathrm{z})\Big[(\mathrm{z} \in \omega) \,\&\, \big(\mathrm{z} \simeq \mathrm{x} \times \{0\} + \mathrm{y} \times \big\{\{0\}\big\}\big)\Big]$ ([4], 8.1, z is a cardinal sum of integers x and y),

$\mathrm{x} \times_s \mathrm{y} = (\mathrm{z})[(\iota \mathrm{z} \in \omega) \,\&\, (\mathrm{z} \simeq \mathrm{x} \times \mathrm{y})]$ ([4], 8.1, z is a cardinal product of integers x and y).

Z_n ([4], 7.44, 7.45 etc., the nth element of the set ω).

F ([4], 9.3, function such that $\mathrm{F}`\xi$ is the ξth constructible set).

L. 7.7. *The following formulas are provable in ST:*

$$Z_n +_s Z_m = Z_{n+m}, \qquad Z_n \times_s Z_m = Z_{nm} \text{ for } n, m = 0, 1, 2, \ldots$$

D. 7.8. An ST-term $\tau(\mathrm{x}_1, \ldots, \mathrm{x}_n)$ or an ST-formula $\Phi(\mathrm{x}_1, \ldots, \mathrm{x}_n)$ is *absolute* with respect to a class K of models of ST if for arbitrary models $\mathfrak{S}_i = \langle A_i, B_i, R_i \rangle$, $i = 1, 2$, in K and for arbitrary elements $a_1, \ldots, a_n$ in $A_1 \cap A_2$

$$\tau^{[\mathfrak{S}_1]}[a_1, \ldots, a_n] = \tau^{[\mathfrak{S}_2]}[a_1, \ldots, a_n],$$

$$\models_{\mathfrak{S}_1} \Phi[a_1, \ldots, a_n] \equiv \models_{\mathfrak{S}_2} \Phi[a_1, \ldots, a_n].$$

L. 7.9. *The terms and formulas enumerated in 7.6 are absolute with respect to the transitive ϵ-models.*

We omit the details of the proof, which is straightforward although laborious.

L. 7.10. *If $\mathfrak{S} = \langle A, B, \epsilon_B \rangle$ is a transitive ϵ-model and $x \in B$, then $\models_{\mathfrak{S}} \omega[x]$ if and only if x is one of the sets Z_n; similarly $\models_{\mathfrak{S}} \omega^{(\omega^n)}[x]$ if and only if x is a function with the domain ω^n and with the range contained in ω.*

Proof. It is obvious that $\models_{\mathfrak{S}} \omega[Z_n]$ for $n = 0, 1, 2, \ldots$ Now assume that $x \in B$ and $\models_{\mathfrak{S}} \omega[x]$. Since the ϵ relation is well founded and since $\models_{ST} \omega(\mathrm{x}) \,\&\, (\mathrm{u}, \mathrm{v} \in \mathrm{x}) \supset (\mathrm{u} \in \mathrm{v} \vee \mathrm{u} = \mathrm{v} \vee \mathrm{v} \in \mathrm{u})$, the set x is well-ordered by ϵ. Hence assuming that there are elements in x not of the form Z_n and taking the first of them we immediately arrive at a contradiction.

Proof for the formula $\omega^{(\omega^n)}$ is similar.

7.11. Every transitive ϵ-frame $\mathfrak{S} = \langle A, B, \epsilon_B \rangle$ determines a frame $\mathfrak{M}(\mathfrak{S})$ of the system of analysis. To define it we denote for every mapping ϑ of N_0^k into N_0 by S_ϑ the set of $k+1$-tuples $\langle Z_{n_1}, \ldots, Z_{n_k}, Z_{\vartheta(n_1, \ldots, n_k)} \rangle$ and put $\mathfrak{M}(\mathfrak{S}) = \langle F^1_{\mathfrak{S}}, F^2_{\mathfrak{S}}, \ldots \rangle$ where $F^j_{\mathfrak{S}}$ is the set of ϑ such that $S_\vartheta \in A$.

7.12. We shall now define a mapping $\tau \to \tau'$, $\Phi \to \Phi'$ of terms and formulas of analysis onto ST-terms and ST-formulas:

(i) If τ is one of the terms $0, 1, \mathrm{x}_j$, then τ' is $\mathrm{Z}_0, \mathrm{Z}_1, \mathrm{x}_j$;

(ii) $(\tau_1 + \tau_2)' = \tau_1' +_s \tau_2'$; $(\tau_1 \times \tau_2)' = \tau_1' \times_s \tau_2'$;

(iii) if τ is $a_k^j(\tau_1, \ldots, \tau_j)$, then τ' is

$$(\iota \mathrm{x})[\omega(\mathrm{x}) \,\&\, \langle \tau_1', \ldots, \tau_j', \mathrm{x} \rangle \in a_k^j];$$

(iv) if Φ is $\tau_1 = \tau_2$, then Φ' is $\tau_1' = \tau_2'$;

(v) $(\Phi_1 \vee \Phi_2)' = \Phi_1' \vee \Phi_2'$; $(\sim\Phi)' = \sim\Phi'$;

(vi) $[(\mathbf{E}\mathrm{x}_j)\Phi]' = (\mathbf{E}\mathrm{x}_j)[\omega(\mathrm{x}_j) \,\&\, \Phi']$;

(vii) $[(\mathbf{E}a_k^j)\Phi]' = (\mathbf{E}a_k^j)[\omega^{(\omega^j)}(a_k^j) \,\&\, \Phi']$;

(viii) $[(\iota \mathrm{x}_k)\Phi]' = (\iota \mathrm{x}_k)[\omega(\mathrm{x}_k) \,\&\, \Phi']$.

L. 7.13. *Let f be a valuation of the system of analysis with respect to $\mathfrak{M}(\mathfrak{S})$ and f' a valuation of the system ST with respect to $\mathfrak{S}$ such that the following conditions are satisfied: $f'(\mathrm{x}_j) = Z_{f(\mathrm{x}_j)}$, $f'(a_k^j) = S_{f(a_k^j)}$; $k, j = 1, 2, \ldots$ Then for every term τ and every formula Φ of the system of analysis the following equivalence and equation are true:*

$$n = Val_{f, \mathfrak{M}(\mathfrak{S})}(\tau) \equiv Val_{f', \mathfrak{S}}(\tau') = Z_n,$$

$$Val_{f, \mathfrak{M}(\mathfrak{S})}(\Phi) = Val_{f', \mathfrak{S}}(\Phi').$$

Proof: by an obvious induction. In case (ii) we use L. 7.7, and in cases (vi)-(viii) L. 7.10.

L. 7.14. *If τ is a term of the system of analysis with the free variables $\mathrm{x}_1, \ldots, \mathrm{x}_n$, $a_{k_1}^{j_1}, \ldots, a_{k_m}^{j_m}$, then the following formula is provable in ST:*

$$\omega(\mathrm{x}_1) \,\&\, \ldots \,\&\, \omega(\mathrm{x}_n) \,\&\, \omega^{(\omega^{j_1})}(a_{k_1}^{j_1}) \,\&\, \ldots \,\&\, \omega^{(\omega^{j_m})}(a_{k_m}^{j_m}) \supset \omega(\tau').$$

Proof. In case (i) the lemma is obvious; in case (ii) it follows from the definitions of the terms $\mathrm{x} +_s \mathrm{y}$ and $\mathrm{x} \times_s \mathrm{y}$; in cases (iii) and (viii) it follows from the definitions of terms τ'.

L. 7.15. *If $\mathfrak{S}$ is a model of ST, then $\mathfrak{M}(\mathfrak{S})$ is a model of A.*

Proof. If Φ is one of the axioms (3a), (3b), (3c), (4), then obviously $\models_{\mathfrak{M}(\mathfrak{S})}\Phi$. Now let Φ have the form

$$(\mathbf{E}\alpha^k)(\mathrm{x}_1, \ldots, \mathrm{x}_k)[\alpha^k(\mathrm{x}_1, \ldots, \mathrm{x}_k) = \tau]$$

where $\tau = \tau(\mathrm{x}_1, \ldots, \mathrm{x}_k, \mathrm{y}_1, \ldots, \mathrm{y}_l, \alpha^{j_1}_{m_1}, \ldots, \alpha^{j_p}_{m_p})$ is a term with the free variables indicated and not containing the free variable α^k. The following formula is provable in ST:

$$\begin{aligned}&\omega(\mathrm{y}_1) \mathbin{\&} \ldots \mathbin{\&} \omega(\mathrm{y}_l) \mathbin{\&} \omega^{(\omega^{j_1})}(\alpha^{j_1}_{m_1}) \mathbin{\&} \ldots \mathbin{\&} \omega^{(\omega^{j_p})}(\alpha^{j_p}_{m_p})\\ &\supset (\mathbf{E}\alpha^k)\big(\omega^{(\omega^k)}(\alpha^k) \mathbin{\&} (\mathrm{x}, \mathrm{x}_1, \ldots, \mathrm{x}_k)\{\omega(\mathrm{x}) \mathbin{\&} \omega(\mathrm{x}_1) \mathbin{\&} \ldots \mathbin{\&} \omega(\mathrm{x}_k)\\ &\qquad\qquad\qquad \supset [\langle \mathrm{x}, \mathrm{x}_1, \ldots, \mathrm{x}_k\rangle \in \alpha^k \equiv \mathrm{x} = \tau']\}\big)\,.\end{aligned}$$

It follows that this formula is satisfied in $\mathfrak{S}$. Hence for arbitrary integers $p_1, \ldots, p_l$ and arbitrary functions φ_i in $F^{j_i}_{\mathfrak{S}}$ $(i = 1, 2, \ldots, p)$ there is a function φ in $F^k_{\mathfrak{S}}$ such that for arbitrary integers $q, q_1, \ldots, q_k$

$$q = \varphi(q_1, \ldots, q_k) \equiv Z_q = \tau'^{[\mathfrak{S}]}[q_1, \ldots, q_k, p_1, \ldots, p_l, S_{\vartheta_1}, \ldots, S_{\vartheta_p}]\,.$$

Using L. 7.13 and L. 7.14 we infer that

$$\varphi(q_1, \ldots, q_k) = \tau^{[\mathfrak{M}(\mathfrak{S})]}[q_1, \ldots, q_k, p_1, \ldots, p_l, \vartheta_1, \ldots, \vartheta_p]\,.$$

This proves that axiom (3d) is satisfied in $\mathfrak{M}(\mathfrak{S})$.

L. 7.16. *If $\mathfrak{S}$ is a transitive ϵ-model for ST, then $\mathfrak{M}(\mathfrak{S})$ is a β-model.*

Proof. Let $\varphi \in F^2_{\mathfrak{S}}$ and $\models_{\mathfrak{M}(\mathfrak{S})}\mathrm{Bord}[\varphi]$. It is easy to show that this condition is equivalent to $\models_{\mathfrak{S}}\mathfrak{We}[S^*_\varphi]$ where S^*_φ is the set of pairs $\langle Z_m, Z_n\rangle$ for which $\langle Z_m, Z_n, Z_0\rangle \in S_\varphi$. Obviously S^*_φ belongs to the model $\mathfrak{S}$. Since the formula $\mathfrak{We}(\mathrm{y}) \supset (\mathbf{E}\mathrm{x}, \mathrm{f})[(\mathrm{x} \sim_{\mathrm{f}} \mathrm{y})]$ is provable in ST and hence true in $\mathfrak{S}$, we infer that there are x and f in the field of $\mathfrak{S}$ such that $\models_{\mathfrak{S}}\mathrm{O}[x]$ and $\models_{\mathfrak{S}} x \sim_f S^*_\varphi$. By L. 7.9 the first of these formulas implies that x is an ordinal and the second that S^*_φ is similar to x. Hence S^*_φ is a well-ordering and hence the relation $\varphi(m, n) = 0$ well-orders a subset of N_0.

We come now to the applications of our theory to constructible sets. We denote by $\mathfrak{C}$ the frame $\mathfrak{M}(\mathfrak{S}_{\aleph_0})$ (cf. L. 7.4). The elements of $F^j_{\mathfrak{C}}$ will be called constructible functions.

THEOREM 7.17. *If K is a recursive binary relation such that $(\varphi)(\mathbf{E}\psi)(n)K(\varphi^*(n),\psi^*(n))$, then for every constructible φ there is a constructible ψ such that $(n)K(\varphi^*(n),\psi^*(n))$.*

Proof. Let A be an elementary formula which represents K in the system of analysis. The formula $(\alpha^1)(\mathbf{E}\beta^1,\gamma^1,\delta^1)\big[(\gamma^1 \text{ is } \alpha^{1*}) \,\&\, (\delta^1 \text{ is } \beta^{1*}) \,\&\, (\mathrm{x})\mathrm{A}(\gamma^1(\mathrm{x}),\delta^1(\mathrm{x}))\big]$ is true in $\mathfrak{M}_0$, whence by 6.3 it is true in every β-model; and thus by 7.16 and 7.4 in $\mathfrak{C}$.

The next theorem gives a partial answer to a problem proposed by *Addison:

THEOREM 7.18. *$F^1_{\mathfrak{C}}$ does not coincide with the family of functions whose graphs are in Π^2_2.*

Proof. Let F^* be this family and assume that $F^1_{\mathfrak{C}} = F^*$. Let K be a recursive relation such that every set in Π^1_2 is representable in the form

$$T_n = \{a\colon (\varphi)(\mathbf{E}\psi)(p)K(\varphi^*(p),\psi^*(p),a,n)\}\,.$$

Let A be an elementary predicate which represents K in the system of analysis and let $\Phi(\mathrm{x},\mathrm{y})$ be the formula

$$(\alpha^1)(\mathbf{E}\beta^1,\gamma^1,\delta^1)\big[(\beta^1 \text{ is } \alpha^{1*}) \,\&\, (\delta^1 \text{ is } \gamma^{1*}) \,\& \\ \&\, (\mathrm{z})\mathrm{A}(\beta^1(\mathrm{z}),\delta^1(\mathrm{z}),\mathrm{x},\mathrm{y})\big]\,.$$

If $a \in T_n$, then by 6.3, 7.5 and 7.16 $\models_{\mathfrak{C}}\Phi(\mathbf{a},\mathbf{n})$. If a non-$\in T_n$, then there is a function φ such that $(\psi)(\mathbf{E}p)$ non-$K(\varphi^*(p),\psi^*(p),a,n)$. It has been proved by Addison [3] that if such a function exists there is also a function φ in F^* satisfying the same condition. According to our assumption we obtain $\models_{\mathfrak{C}}\sim\Phi(\mathbf{a},\mathbf{n})$ and hence

$$a \in T_n \equiv \models_{\mathfrak{C}}\Phi(\mathbf{a},\mathbf{n})\,. \tag{1}$$

Since the formula $(\mathbf{E}\alpha)(\mathrm{x})[\alpha(\mathrm{x}) = 0 \equiv \sim\Phi(\mathrm{x},\mathrm{x})]$ is provable in A and $\mathfrak{C}$ is a model of A, we infer that there is a φ in $F^1_{\mathfrak{C}}$ such that

$$\varphi(n) = 0 \equiv \text{non-}\models_{\mathfrak{C}}\Phi(\mathbf{n},\mathbf{n})\,.$$

Observe now that according to our assumption the graph $\{\langle m,n\rangle\colon m = \varphi(n)\}$ is in Π^2_2 and hence the set $\{n\colon \varphi(n) = 0\}$ is in Π^1_2, i. e. for some n_0 coincides with T_{n_0}. Hence

$$a \in T_{n_0} \equiv \text{non-}\models_{\mathfrak{C}}\Phi(\mathbf{a},\mathbf{a})\,,$$

i.e. by (1) $\models_{\mathfrak{C}} \Phi(\mathbf{a}, \mathbf{n}_0) \equiv \text{non} \models_{\mathfrak{C}} \Phi(\mathbf{a}, \mathbf{a})$. If we put here $a = n_0$, we obtain a contradiction.

It is an open problem whether one obtains a contradiction with the axioms of set theory when one assumes that $F^1_{\mathfrak{C}}$ is contained in the family of functions whose graphs are in Π^2_1.

8. Finitely axiomatizable complete extensions of A_β

It has been proved in [5], 2.5. C and 3.5. C that if X is a consistent set of formulas such that $\mathrm{N}^0 X$ is weakly representable in A (or in A_ω), then $Cn(X)$ (or $Cn_\omega(X)$) is incomplete. We shall show in the present section that A_β does not possess a similar property. To obtain this result we need some preparatory lemmas (L. 8.1 and L. 8.2) as well as a detailed discussion of the way in which the theory of constructible functions can be developed in the system of analysis. This discussion which is primarily based on the work of Addison is presented in L. 8.3-L. 8.12.

We say that a formula Φ is demonstrably hyperarithmetic (4) if there are formulas Φ_1, Φ_2 which in addition to the free variables of Φ have one new functional free variable β^p and which satisfy the conditions

$$\vdash \Phi \equiv (\mathbf{E}\beta^p)\Phi_1\,, \qquad \vdash \Phi \equiv (\beta^p)\Phi_2\,.$$

A formula Φ is called *ω-stable* (4) if for every ω-model $\mathfrak{M}$ and for arbitrary φ_s in $F^{t_s}_{\mathfrak{M}}$ $(s = 1, 2, \ldots, k)$ and $n_1, \ldots, n_m$ in N_0

$$\models_{\mathfrak{M}} \Phi[\varphi_1, \ldots, \varphi_k, n_1, \ldots, n_m] \equiv \models_{\mathfrak{M}_0} \Phi[\varphi_1, \ldots, \varphi_k, n_1, \ldots, n_m]\,.$$

L. 8.1. *If Φ_1 and Φ_2 are ω-stable, then so are $\Phi_1 \vee \Phi_2$, $\sim\Phi_1$ and $(\mathrm{E}x_j)\Phi_1$.*

Proof: obvious.

L. 8.2. *Elementary and demonstrably hyperarithmetic formulas are ω-stable.*

Proof. For elementary formulas the proof is obvious. Now let Φ be demonstrably hyperarithmetic. If φ_s is in $F^{t_s}_{\mathfrak{M}}$ for $s = 1, 2, \ldots, k$ and $\models_{\mathfrak{M}} \Phi[\varphi_1, \ldots, \varphi_k, n_1, \ldots, n_m]$, then there is a ψ in $F^p_{\mathfrak{M}}$ such that $\models_{\mathfrak{M}} \Phi_1[\psi, \varphi_1, \ldots, \varphi_k, n_1, \ldots, n_m]$ and hence, Φ_1 being elementary, $\models_{\mathfrak{M}_0} \Phi_1[\psi, \varphi_1, \ldots, \varphi_k, n_1, \ldots, n_m]$, which gives $\models_{\mathfrak{M}_0} \Phi[\varphi_1, \ldots, \varphi_k, n_1, \ldots, n_m]$. If non-$\models_{\mathfrak{M}} \Phi[\varphi_1, \ldots, \varphi_k,$

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(4) This notion has been introduced by Kreisel [7].

$n_1, \ldots, n_m]$, then there is a ψ in $F^{\varphi}_{\mathfrak{M}}$ such that $\models_{\mathfrak{M}} \sim\Phi_x[\psi, \varphi_1, \ldots, \varphi_k, n_1 . \ldots, n_m]$ and we obtain $\models_{\mathfrak{M}_0} \sim\Phi[\varphi_1, \ldots, \varphi_k, n_1, \ldots, n_m]$.

L. 8.3. *There is a demonstrably hyperarithmetic formula* $\mathrm{B}(\alpha^2, \beta^1)$ *and an elementary formula* $\mathrm{A}(\alpha^2, \beta^1, \gamma^1, \mathrm{x})$ *such that*

$$\vartheta \in F^1_{\mathfrak{C}} \equiv (\mathbf{E}\varphi, \psi, i) \models_{\mathfrak{M}_0} \mathrm{Bord}[\varphi] \;\&\; \models_{\mathfrak{M}_0} \mathrm{B}[\varphi, \psi] \;\& \\ \& \models_{\mathfrak{M}_0} \mathrm{A}[\varphi, \psi, \vartheta, i] .$$

Proof. An inspection of the proof given in [2], pp. 341-349 reveals that the predicate $M(\varphi, \psi)$ used there can be written as $\models_{\mathfrak{M}_0}\mathrm{Bord}[\varphi] \;\&\; \models_{\mathfrak{M}_0}\mathrm{B}[\varphi, \psi]$ where B is demonstrably hyperarithmetic. Similarly we obtain formula A from lemma A in [2].

Remark. Formula B is not elementary because of conditions (C. 6) of [2], p. 343. We observe that the formulas given in (C. 5), (C. 6), (C. 7) and (C. 8) are incorrect and should be replaced by the following ones:

C. 5. $Tr^{\varphi}(\alpha, \beta) \equiv (m, m', a, a', b, b')\Big(\big(\beta(m, a, b) = 0\big) \supset N^{\varphi}(m) \;\&\; \Big[S(m', a', b', m, a, b) \supset \big(\beta(m', a', b') = 0\big) \;\& \big(\varphi\big(\alpha(m', a', b'), \alpha(m, a, b)\big) = 0\big)\Big]\Big) \;\& (v)\,(\mathbf{E}m, a, b)\big[\big(\beta(m, a, b) = 0\big) \;\& \big(\alpha(m, a, b) = v\big)\big]$;

C. 6. $J^{\varphi}(m, a, b, u) \equiv (\mathbf{E}\alpha, \beta)\big[Tr^{\varphi}(\alpha, \beta) \;\& \big(\beta(m, a, b) = 0\big) \;\& \;\& \big(\alpha(m, a, b) = u\big)\big] \equiv (\alpha, \beta)\big[Tr^{\varphi}(\alpha, \beta) \supset \big(\beta(m, a, b) = 0\big) \;\& \;\& \big(\alpha(m, a, b) = u\big)\big]$;

C. 7. $J^{\varphi}_h(a, b, u) \equiv (\mathbf{E}l)[(l = \varphi_u) \;\& J^{\varphi}(h, a, b, l)]$;

C. 8. $C_h(\varphi, j) \equiv (\mathbf{E}a, b)J^{\varphi}_h(a, b, j)$.

A further direct corollary from Addison's work is

L. 8.4. *If* $\models_{\mathfrak{M}_0}\mathrm{Bord}[\varphi] \;\&\; \models_{\mathfrak{M}_0}\mathrm{B}[\varphi, \psi] \;\&\; \models_{\mathfrak{M}_0}\mathrm{A}[\varphi, \psi, \vartheta, i]$, *then* $S_{\vartheta} = F`\varphi_i$ *where* φ_i *is the ordinal corresponding to* i *in the well-ordering* $R_{\mathfrak{M}_0,\varphi}$ *and hence less than the order type of* $R_{\mathfrak{M}_0,\varphi}$.

L. 8.5. $\vdash_{\beta} \mathrm{Bord}(\alpha^2) \;\&\; \mathrm{B}(\alpha^2, \beta^1) \supset (\mathbf{E}!\gamma^1)\mathrm{A}(\alpha^2, \beta^1, \gamma^1, \mathrm{x})$.

Proof. We have to show that

$$\vdash_{\beta}(\mathrm{x}, \alpha^2, \beta^1, \gamma^1, \delta^1)[\mathrm{Bord}(\alpha^2) \;\&\; \mathrm{B}(\alpha^2, \beta^1) \;\&\; \mathrm{A}(\alpha^2, \beta^1, \gamma^1, \mathrm{x}) \;\& \\ \& \; \mathrm{A}(\alpha^2, \beta^1, \delta^1, \mathrm{x}) \supset (\gamma^1 = \delta^1)] ,$$

$$\vdash_{\beta}(\mathrm{x}, \alpha^2, \beta^1)[\mathrm{Bord}(\alpha^2) \;\&\; \mathrm{B}(\alpha^2, \beta^1) \supset (\mathbf{E}\gamma^1)\mathrm{A}(\alpha^2, \beta^1, \gamma^1, \mathrm{x})] .$$

By L. 8.4 both formulas after the sign "$\vdash_{\beta}$" are true in $\mathfrak{M}_0$, whence they are β-provable by theorem 6.4.

L. 8.6. $\vdash_\beta \mathrm{Bord}(\alpha^2) \supset (\mathbf{E}\beta^1)\mathrm{B}(\alpha^2, \beta^1)$.

The proof is similar to that of L. 8.5.

D. 8.7. We denote by Φ the axiom of constructivity:

$$(\gamma^1)(\mathbf{E}\alpha^2, \beta^1, \mathrm{x})[\mathrm{Bord}(\alpha^2) \,\&\, \mathrm{B}(\alpha^2, \beta^1) \,\&\, \mathrm{A}(\alpha^2, \beta^1, \gamma^1, \mathrm{x})].$$

L. 8.8. *If ξ is an ordinal such that $\mathfrak{S}_\xi$ is a model of ST, then* $\models_{\mathfrak{M}(\mathfrak{S}_\xi)} \Phi$.

Proof. Let $\vartheta \in F^1_{\mathfrak{M}(\mathfrak{S}_\xi)}$, i.e., let S_ϑ belong to G_ξ (cf. p. 156). Since ξ is obviously a limit number, there is an ordinal $\eta < \xi$ such that $S_\vartheta = F\text{‘}\eta$.

In [4], p. 54-61 there is a proof that if $x \subseteq F\text{“}\omega_\alpha$ and $x \in L$, then $x \in F\text{“}\omega_{\alpha+1}$. Formalizing this proof for $\alpha = 0$ in ST we obtain

$$\vdash_{ST} \mathrm{O}(\mathrm{x}) \,\&\, \omega^\omega(\alpha) \,\&\, (\alpha = \mathrm{F}\text{‘}\mathrm{x}) \supset (\mathbf{E}\mathrm{x}, \mathrm{f}, \mathrm{y})[\mathrm{O}(\mathrm{x}) \,\&$$
$$\&\, (\alpha = \mathrm{F}\text{‘}\mathrm{x}) \,\&\, (\mathfrak{We}\,\mathrm{y}) \,\&\, (\mathrm{y} \subseteq \omega^2) \,\&\, (\mathrm{x} \cup \{\mathrm{x}\} \sim_{\mathrm{f}} \mathrm{y})].$$

Since $\eta \in G_\xi$ and $\models_{\mathfrak{S}_\xi} \mathrm{O}[\eta]$, $\models_{\mathfrak{S}_\xi} \omega^\omega[S_\vartheta]$ and $\models_{\mathfrak{S}_\xi} (S_\vartheta = F\text{‘}\eta)$ (cf. L. 7.9), we infer that $\mathfrak{S}_\xi$ contains an ordinal $\zeta \leqslant \eta$ such that $S_\vartheta = F\text{‘}\zeta$ and elements f and $y \subseteq \omega^2$ such that $\models_{\mathfrak{S}_\xi} (\zeta + 1) \sim_{\mathrm{f}} y$ and $\models_{\mathfrak{S}_\xi} \mathfrak{We}\,y$. Let y' be the set of triples $\langle m, n, i\rangle$ where $i = 0$ or 1 according as $\langle m, n\rangle$ is or is not in y. Obviously y' belongs to the model $\mathfrak{S}_\xi$. Hence y' has the form S_φ. The formula $\models_{\mathfrak{S}_\xi} \mathfrak{We}\,y$ proves that $\models_{\mathfrak{M}(\mathfrak{S}_\xi)} \mathrm{Bord}[\varphi]$. Using L. 8.5 and L. 8.6 we infer that $F^1_{\mathfrak{M}(\mathfrak{S}_\xi)}$ contains functions ψ and ϑ' such that $\models_{\mathfrak{M}(\mathfrak{S}_\xi)} \mathrm{B}[\varphi, \psi]$ and $\models_{\mathfrak{M}(\mathfrak{S}_\xi)} \mathrm{A}[\varphi, \psi, \vartheta', i]$ where we choose i such that the ordinal corresponding to i in the well-ordering determined by $R_{\mathfrak{M}(\mathfrak{S}_\xi), \varphi}$ is ζ. By L. 8.4 we obtain $S_{\vartheta'} = F\text{‘}\zeta$, whence $S_{\vartheta'} = S_\vartheta$, i. e., $\vartheta' = \vartheta$. This proves that

$$\models_{\mathfrak{M}(\mathfrak{S}_\xi)} \mathrm{Bord}[\varphi] \,\&\, \models_{\mathfrak{M}(\mathfrak{S}_\xi)} \mathrm{B}[\varphi, \psi] \,\&\, \models_{\mathfrak{M}(\mathfrak{S}_\xi)} \mathrm{A}[\varphi, \psi, \vartheta, i]$$

and hence that $\models_{\mathfrak{M}(\mathfrak{S}_\xi)} \Phi$.

We shall now show that models $\mathfrak{M}(\mathfrak{S}_\xi)$ are unique β-models in which Φ is true. For that purpose we need

D. 8.9. An ordinal ζ is said to be ***representable in a β-model*** $\mathfrak{M}$ if there is a φ in $F^2_{\mathfrak{M}}$ such that $\models_{\mathfrak{M}} \mathrm{Bord}[\varphi]$ and the order type of $R_{\mathfrak{M}, \varphi}$ is $\geqslant \zeta$.

L. 8.10. *If ζ is representable in a β-model $\mathfrak{M}$ and $\zeta \geqslant \omega$, then there is a ψ in $F^2_{\mathfrak{M}}$ such that the order type of $R_{\mathfrak{M},\psi}$ is ζ.*

Proof. Let $R_{\mathfrak{M},\varphi}$ have the order type $> \zeta$ and let i determine a segment of type ζ. Since

$$\begin{aligned}&\vdash \mathrm{Bord}(\alpha^2) \mathbin{\&} (y)[y\alpha^2 x \supset (\mathbf{E}z)(z\alpha^2 x \mathbin{\&} z > y)]\\ &\quad\supset (\mathbf{E}\beta^2, \gamma^1)\Big[\mathrm{Bord}(\beta^2) \mathbin{\&} (y)\big[y\alpha^2 x \supset (\mathbf{E}z)\big(y = \gamma^1(z)\big)\big] \mathbin{\&}\\ &\qquad\mathbin{\&} (z)\big(\gamma^1(z)\alpha^2 x\big) \mathbin{\&} (y, z)\big[(y\beta^2 z) \supset \big(\gamma^1(y)\alpha^2\gamma^1(z)\big)\big]\Big],\end{aligned}$$

$F^2_{\mathfrak{M}}$ contains a ψ for which $R_{\mathfrak{M},\psi}$ is isomorphic with this segment.

L. 8.11. *If* $\mu \in F^1_{\mathfrak{M}(\mathfrak{S}_\xi)}$ *and* $\mathfrak{M}_\mu$ *is a model, then there is an ordinal representable in* $\mathfrak{M}(\mathfrak{S}_\xi)$ *but not in* $\mathfrak{M}_\mu$.

Proof. Let $\beta^2 < \gamma^2$ be an abbreviation of

$$\begin{aligned}&\mathrm{Bord}(\beta^2) \mathbin{\&} \mathrm{Bord}(\gamma^2) \mathbin{\&} (\mathbf{E}\delta^1, z)(x, y)\{x\beta^2 y \mathbin{\&}\\ &\quad\mathbin{\&} (x \neq y) \supset [\delta^1(x)\gamma^2\delta^1(y)] \mathbin{\&} [\delta^1(x)\gamma^2 z]\} \mathbin{\&} \big(\delta^1(x) \neq \delta^1(y)\big)\\ &\qquad\mathbin{\&} (y)\big[y\gamma^2 z \supset (\mathbf{E}x)(x\beta^2 x) \mathbin{\&} \big(\delta^1(x) = y\big)\big].\end{aligned}$$

Obviously

$$\begin{aligned}&\vdash (\alpha^1)(\mathbf{E}\beta^2)\{\mathrm{Bord}(\beta^2) \mathbin{\&} (\alpha^2, z)\{\mathrm{Bord}(\alpha^2) \mathbin{\&}\\ &\quad\mathbin{\&} (x, y)[\alpha^2(x, y) = \alpha^1(\langle z, 2^x + 2^{x+y+1}\rangle)] \supset (\alpha^2 < \beta^2)\}.\end{aligned}$$

Hence for every μ in $F^1_{\mathfrak{M}(\mathfrak{S}_\xi)}$ there is a φ such that $\models_{\mathfrak{M}(\mathfrak{S}_\xi)} \mathrm{Bord}[\varphi]$ and such that if μ_n is a well-ordering, then the order type ζ of $R_{\mathfrak{M}(\mathfrak{S}_\xi),\varphi}$ is greater than that of $R_{\mathfrak{M}(\mathfrak{S}_\xi),\mu_n}$. Hence ζ is representable in $\mathfrak{M}(\mathfrak{S}_\xi)$ but not in $\mathfrak{M}_\mu$.

L. 8.12. *If* $\mathfrak{M}$ *a* β*-model and satisfies* Φ, *then* $\mathfrak{M} = \mathfrak{M}(\mathfrak{S}_\xi)$ *where* ξ *is the smallest ordinal not representable in* $\mathfrak{M}$.

Proof. Let $\vartheta \in F^1_{\mathfrak{M}}$. Since $\models_{\mathfrak{M}} \Phi$, there are φ, ψ, i such that $\varphi, \psi \in F^1_{\mathfrak{M}}$ and

$$\models_{\mathfrak{M}} \mathrm{Bord}[\varphi], \quad \models_{\mathfrak{M}} \mathrm{B}[\varphi, \psi], \quad \models_{\mathfrak{M}} \mathrm{A}[\varphi, \psi, \vartheta, i].$$

Since $\mathfrak{M}$ is a β-model, A is elementary and B demonstrably hyperarithmetic, we obtain

$$\models_{\mathfrak{M}_0} \mathrm{Bord}[\varphi], \quad \models_{\mathfrak{M}_0} \mathrm{B}[\varphi, \psi], \quad \models_{\mathfrak{M}_0} \mathrm{A}[\varphi, \psi, \vartheta, i].$$

L. 8.4 proves that $S_\vartheta = F`\zeta$ where ζ is an ordinal less than the order type of $R_{\mathfrak{M},\varphi}$. This shows that $\zeta < \xi$ and $\vartheta \in F^1_{\mathfrak{M}(\mathfrak{S}_\xi)}$. Hence $F^1_{\mathfrak{M}} \subseteq F^1_{\mathfrak{M}(\mathfrak{S}_\xi)}$ and by L. 3.6 we obtain $F^j_{\mathfrak{M}} \subseteq F^j_{\mathfrak{M}(\mathfrak{S}_\xi)}$ for $j > 1$. This proves that $\mathfrak{M}$ is a submodel of $\mathfrak{M}(\mathfrak{S}_\xi)$.

Now let $\vartheta \in F^1_{\mathfrak{M}(\mathfrak{S}_\xi)}$. Hence S_ϑ is in the domain of $\mathfrak{S}_\xi$, i.e., $S_\vartheta = F`\zeta$ where $\zeta < \xi$. Let φ be a function in $F^2_{\mathfrak{M}}$ such that the order type of $R_{\mathfrak{M},\varphi}$ is $\zeta + 1$. Using lemma 8.6 we infer that there is a ψ in $F^1_{\mathfrak{M}}$ such that $\models_{\mathfrak{M}} \mathrm{B}[\varphi, \psi]$; by L. 8.5 there is for every i exactly one ϑ_i in $F^1_{\mathfrak{M}}$ such that $\models_{\mathfrak{M}} \mathrm{A}[\varphi, \psi, \vartheta_i, i]$. Now let i be such that i determines a segment of the order type ζ in the well-ordering $R_{\mathfrak{M},\varphi}$. Then by L. 8.4 $\vartheta_i = \vartheta$, whence $\vartheta \in F^1_{\mathfrak{M}}$. This proves that $F^1_{\mathfrak{M}(\mathfrak{S}_\xi)} \subseteq F^1_{\mathfrak{M}}$. Using L. 3.6 we infer that $F^j_{\mathfrak{M}(\mathfrak{S}_\xi)} \subseteq F^j_{\mathfrak{M}}$ for $j > 1$.

L. 8.13. *There are elementary formulas* C′, D′ *such that*

$$Mod_\beta(\mu) \,\&\, \models_{\mathfrak{M}_\mu} \Phi \equiv (\mathbf{E}\psi_1, \psi_2)(n)(\mathbf{E}\psi_3, \psi_4)(\models_{\mathfrak{M}_0} \mathrm{C}'[\mu, \psi_1, \psi_2, \psi_3, n] \vee$$
$$\vee \models_{\mathfrak{M}_0} \mathrm{D}'[\mu, \psi_1, \psi_2, \psi_4, n] \,\&\, \models_{\mathfrak{M}_0} \mathrm{Bord}[\psi_4]) .$$

Proof. The predicate $\models_{\mathfrak{M}_\mu} \Phi$ is hyperarithmetical and therefore there is an elementary formula $\mathrm{C}(\alpha^1, \beta^1)$ such that $\models_{\mathfrak{M}_\mu} \Phi \equiv (\mathbf{E}\psi) \models_{\mathfrak{M}_0} \mathrm{C}[\mu, \psi]$. The predicate $Mod_\beta(\mu)$ can be written in the form $Mod(\mu) \,\&\, (n)\big[\models_{\mathfrak{M}_\mu}(\mathrm{Bord}[\mu^2_n]) \supset \models_{\mathfrak{M}_0} \mathrm{Bord}[\mu^2_n]\big]$. Let $\tau(\alpha^1, \mathrm{x})$ be an elementary term such that $\tau^{[\mathfrak{M}_0]}[\mu, n] = \mu^2_n$ and let D, E be elementary formulas such that $Mod(\mu) \equiv (\mathbf{E}\psi) \models_{\mathfrak{M}_0} \mathrm{D}[\mu, \psi]$ and $\models_{\mathfrak{M}_\mu} \mathrm{Bord}[\mu^2_n] \equiv (\psi) \models_{\mathfrak{M}_0} \mathrm{E}[\mu, \psi, n]$. Hence

$$Mod_\beta(\mu) \,\&\, \models_{\mathfrak{M}_\mu} \Phi \equiv (\mathbf{E}\psi) \models_{\mathfrak{M}_\mu} \mathrm{C}[\mu, \psi] \,\&\, (\mathbf{E}\psi) \models_{\mathfrak{M}_0} \mathrm{D}[\mu, \psi] \,\&$$
$$\& \,(n)\big[(\psi) \models_{\mathfrak{M}_0} \mathrm{E}[\mu, \psi, n] \supset (\mathbf{E}\psi)(\psi = \tau^{[\mathfrak{M}_0]}[\mu, n] \,\&$$
$$\& \models_{\mathfrak{M}_0} \mathrm{Bord}[\psi])\big] .$$

Regrouping the quantifiers, we reduce the right-hand side to the form

$$(\mathbf{E}\psi_1, \psi_2)(n)(\mathbf{E}\psi_3, \psi_4)(\models_{\mathfrak{M}_0} \mathrm{C}[\mu, \psi_1] \,\&\, \models_{\mathfrak{M}_0} \mathrm{D}[\mu, \psi_2] \,\&$$
$$\& \big(\models_{\mathfrak{M}_0} \mathrm{E}[\mu, \psi_3, n] \supset \models_{\mathfrak{M}_0} \mathrm{F}[\mu, \psi_4, n] \,\&\, \models_{\mathfrak{M}_0} \mathrm{Bord}[\psi_4]\big)\big) .$$

Putting $\mathrm{C}'(\alpha^1, \beta^1, \gamma^1, \delta^1, \mathrm{x}) \equiv \mathrm{C}(\alpha^1, \beta^1) \,\&\, \mathrm{D}(\alpha^1, \gamma^1) \sim \mathrm{E}(\alpha^1, \delta^1, \mathrm{x})$ and $\mathrm{D}'(\alpha^1, \beta^1, \gamma^2, \mathrm{x}) \equiv \mathrm{C}(\alpha^1, \beta^1) \,\&\, \mathrm{D}(\alpha^1, \gamma^1) \,\&\, \mathrm{F}(\alpha^1, \gamma^2, \mathrm{x})$ we obtain the form required in the lemma.

L. 8.14. *If there is a set B such that $\langle G_\xi, B, \epsilon_B\rangle$ is a model of ST and Φ, then $\mathfrak{S}_\xi$ is a model of ST.*

Proof. Axioms of groups A and B are obviously satisfied in $\mathfrak{S}_\xi$ (we have only to notice that operations $\cap, \times, \mathfrak{Cnv}_i$, $i = 1, 2, 3$, taking the domain and the complement with respect to G_ξ are all performable in H_ξ and that the set $E = \{\langle a, b\rangle : (a, b \in G_\xi) \,\&\, (a \in b)\}$ belongs to H_ξ). Axioms $C4$ and D are satisfied in $\mathfrak{S}_\xi$ because $H_\xi \subseteq B$ and those axioms have the form of general statements concerning classes and hold in the model $\langle G_\xi, B, \epsilon_B\rangle$. Finally we prove that the axiom E

is satisfied. Since the class theorem (cf. [4], p. 8) is valid in $\mathfrak{S}_\xi$ (that theorem is a consequence of axioms A and B alone), H_ξ contains the set $W = \{\langle a, b\rangle\colon (a, b \in G_\xi) \,\&\, (\mathbf{E}u, v)(u, v \in G_\xi) \,\&$ $\& \models_{\mathfrak{S}_\xi} \mathrm{O}[u] \,\&\, \models_{\mathfrak{S}_\xi} \mathrm{O}[v] \,\&\, (a = F'u) \,\&\, (b = F'v) \,\&\, (u \in v \text{ or } u = v)\}$. By L. 7.9, $W = \{\langle F'u, F'v\rangle\colon (u, v \in G_\xi \cap O) \,\&\, (u \leqslant v)\}$. Hence W is a well-ordering and the field of W is G_ξ; indeed the axiom of constructivity Φ being true in the model $\langle G_\xi, B, \epsilon_B\rangle$, every a in G_ξ can be represented as $F'u$ where u is in $G_\xi \cap O$.

L. 8.15. *There is an ordinal* ξ_0 *such that* $\xi_0 < \Omega$ *and the frame* $\mathfrak{S}_{\xi_0}$ *is a model of* ST.

Proof. It is known [10] that $\mathfrak{S}_{\nu_0}$ contains a denumerable submodel $\Delta = \langle A, B, \epsilon_B\rangle$ such that $\mathfrak{S}_{\nu_0}$ is an arithmetical extension of Δ. We can assume that Δ is transitive because every ϵ-frame satisfying the axiom of extensionality is isomorphic with a transitive ϵ-frame (cf. [9], p. 147). Let ξ_0 be the least ordinal not contained in A as an element. $\xi_0 < \Omega$ since A is denumerable.

The formula $(\mathrm{y})(\mathbf{E}\mathrm{x})[\mathrm{O}(\mathrm{x}) \,\&\, (\mathrm{y} = \mathrm{F'x})]$ is true in $\mathfrak{S}_{\nu_0}$ and hence in Δ. Because of L. 7.9 we infer that if $y \in A$, then there is an ordinal ξ in A such that $y = F'\xi$. This proves that $A \subseteq G_{\xi_0}$. Conversely if $y \in G_{\xi_0}$, then $y = F'\xi$ where $\xi < \xi_0$. Since the formula $(\mathrm{x})[\mathrm{O}(\mathrm{x}) \supset (\mathbf{E}\mathrm{y})(\mathrm{y} = \mathrm{F'x})]$ is true in Δ and since $\models_\Delta \mathrm{O}[\xi]$ (cf. L. 7.9), the set A contains an element y' such that $\models_\Delta y' = \mathrm{F}'\xi$. Applying again L. 7.9 we obtain $y' = y$, i. e., $G_{\xi_0} \subseteq A$. Hence $G_{\xi_0} = A$ and the lemma follows from L. 8.14 and the remark that Φ is true in $\mathfrak{S}_{\nu_0}$ and hence in Δ.

D. 8.16. Let ζ_0 be the least ordinal such that $\mathfrak{S}_{\zeta_0}$ be a model of ST. By L. 8.15, $\zeta_0 < \Omega$.

D. 8.17. We denote by Ψ the formula

$$\Phi \,\&\, (\alpha^1, \beta_1^1, \beta_2^1)(\mathbf{E}\mathrm{x})(\gamma_1^1, \gamma_2^2)\big[\sim \mathrm{C}'(\alpha^1, \beta_1^1, \beta_2^1, \gamma_2^1, \mathrm{x}) \,\&$$
$$\& \big(\mathrm{D}'(\alpha^1, \beta_1^1, \beta_2^1, \gamma_2^2, \mathrm{x}) \supset \sim \mathrm{Bord}(\gamma_2^2)\big)\big].$$

The intuitive meaning of this formula is that every set is constructible but that for no μ is the frame $\mathfrak{M}_\mu$ a model for all axioms and for Φ.

L. 8.18. $\models_{\mathfrak{M}(\mathfrak{S}_{\zeta_0})} \Psi$.

Proof. Assume that $\models_{\mathfrak{M}(\mathfrak{S}_{\zeta_0})} \sim \Psi$. Since Φ is true in $\mathfrak{M}(\mathfrak{S}_{\zeta_0})$ (cf. L. 8.8), this assumption means that there are $\mu, \varphi_1, \varphi_2$ in

$F^1_{\mathfrak{M}(\mathfrak{S}_{\zeta_0})}$ such that for every n there are ψ_3 in $F^1_{\mathfrak{M}(\mathfrak{S}_{\zeta_0})}$ and ψ_4 in $F^2_{\mathfrak{M}(\mathfrak{S}_{\zeta_0})}$ satisfying the following formulas

$$\models_{\mathfrak{M}(\mathfrak{S}_{\zeta_0})} \mathrm{C}'[\mu, \psi_1, \psi_2, \psi_3, n] \vee \models_{\mathfrak{M}(\mathfrak{S}_{\zeta_0})} \mathrm{D}'[\mu, \psi_1, \psi_2, \psi_4, n] \mathbin{\&} \\ \mathbin{\&} \models_{\mathfrak{M}(\mathfrak{S}_{\zeta_0})} \mathrm{Bord}[\psi_4] .$$

Since $\mathfrak{M}(\mathfrak{S}_{\zeta_0})$ is a β-model and formulas C′ and D′ are elementary, it follows that

$$(\mathbf{E}\psi_1, \psi_2)(n)(\mathbf{E}\psi_3, \psi_4)(\models_{\mathfrak{M}_0} \mathrm{C}'[\mu, \psi_1, \psi_2, \psi_3, n] \vee \\ \vee \models_{\mathfrak{M}_0} \mathrm{D}'[\mu, \psi_1, \psi_2, \psi_4, n] \mathbin{\&} \models_{\mathfrak{M}_0} \mathrm{Bord}[\psi_4]) ,$$

i.e., by L. 8.13. $Mod_\beta(\mu)$ and $\models_{\mathfrak{M}_\mu} \Phi$. Thus $\mathfrak{M}_\mu$ is a β-model satisfying Φ. By L. 8.12 we obtain $\mathfrak{M}_\mu = \mathfrak{M}(\mathfrak{S}_\xi)$ where, by L. 8.11, ξ is an ordinal $< \zeta_0$. This is a contradiction.

L. 8.19. *If $\mathfrak{M}$ is a β-model such that $\models_{\mathfrak{M}} \Psi$, then $\mathfrak{M} = \mathfrak{M}(\mathfrak{S}_{\zeta_0})$.*

Proof. From L. 8.12 it follows that $\mathfrak{M} = \mathfrak{M}(\mathfrak{S}_\zeta)$ where ζ is the smallest number not representable in $\mathfrak{M}$. Hence $\zeta \geqslant \zeta_0$. Assume that $\zeta < \zeta_0$. Hence ζ is representable in $\mathfrak{M}(\mathfrak{S}_{\zeta_0})$. Let φ be a function in $F^2_{\mathfrak{M}(\mathfrak{S}_{\zeta_0})}$ such that the order type of $R_{\mathfrak{M}(\mathfrak{S}_{\zeta_0}),\varphi}$ is ζ. Take ψ in $F^1_{\mathfrak{M}(\mathfrak{S}_{\zeta_0})}$ such that $\models_{\mathfrak{M}_0} \mathrm{B}[\varphi, \psi]$ (cf. L. 8.6) and let ν be a function in $F^2_{\mathfrak{M}(\mathfrak{S}_{\zeta_0})}$ such that for every i the function $\nu_i(n) = \nu(i, n)$ satisfies the condition $\models_{\mathfrak{M}_0} \mathrm{A}[\varphi, \psi, \nu_i, i]$. The existence of ν is assured by L. 8.5 and the following theorem

$$\vdash (\mathrm{x})(\mathbf{E}!\beta^1)\Theta(\beta^1, \mathrm{x}) \supset (\mathbf{E}\gamma^2)(\mathrm{x}, \beta^1)\{[(\mathrm{y})(\gamma^2(\mathrm{x}, \mathrm{y}) \\ = \beta^1(\mathrm{y})] \supset \Theta(\beta^1, \mathrm{x}))\}$$

in which $\Theta(\beta^1, \mathrm{x})$ denotes any formula with at least the two free variables indicated.

Now let μ be a function in $F^1_{\mathfrak{M}(\mathfrak{S}_{\zeta_0})}$ such that $\mu(\langle k, 2^n \rangle) = \nu(k, n)$. The existence of such a function is obvious. We claim that $\mathfrak{M}_\mu = \mathfrak{M}(\mathfrak{S}_\zeta)$. This will be the desired contradiction because the existence of a μ of this sort shows by L. 8.13 that Ψ is false in $\mathfrak{M}(\mathfrak{S}_{\zeta_0})$.

First of all we remark that $\mu^1_k(n) = \nu(k, n) = \nu_k(n)$ (cf. 4.1) and so μ^1_k satisfies the condition $\models_{\mathfrak{M}_0} \mathrm{A}[\varphi, \psi, \mu^1_k, k]$. Hence $S_{\mu^1_k} = F^{\backprime}\varphi_k$ and so $\mu^1_k \in F^1_{\mathfrak{M}(\mathfrak{S}_\zeta)}$. Conversely, if $\vartheta \in F^1_{\mathfrak{M}(\mathfrak{S}_\zeta)}$, then for some k $S_\vartheta = F^{\backprime}\varphi_k$ and hence $\models_{\mathfrak{M}_0} \mathrm{A}[\varphi, \psi, \vartheta, k]$, which proves that $\vartheta = \mu^1_k$. This proves that $F^1_{\mathfrak{M}_\mu} = F^1_{\mathfrak{M}(\mathfrak{S}_{\zeta_0})}$.

The identity of the models $\mathfrak{M}_\mu$ and $\mathfrak{M}(\mathfrak{S}_{\zeta_0})$ follows now from the remark made in L. 3.6 that the whole β-model $\mathfrak{M}$ is determined by $F^1_{\mathfrak{M}}$.

THEOREM 8.20. *$Cn_\beta(\Psi)$ coincides with the set of formulas true in $\mathfrak{M}(\mathfrak{S}_{\zeta_0})$ and hence is complete.*

Proof. If Θ is in $Cn_\beta(\Psi)$, then Θ is true in $\mathfrak{M}(\mathfrak{S}_{\zeta_0})$ by L. 8.18. If Θ is true in $\mathfrak{M}(\mathfrak{S}_{\zeta_0})$, then it is true in the unique (and hence every) β-model in which Ψ is true and hence $\Theta \in Cn_\beta(\Psi)$.

In a similar way we can construct denumerably many different complete and finitely axiomatizable extensions of A_β. We do not know, however, whether the number of arbitrary complete extensions of A_β is denumerable.

The rule β treated in this paper has been defined by means of semantical notions. It would be interesting to find an equivalent definition formulated in a syntactical (although infinitistic) manner.

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AN EXPOSITION OF FORCING

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P.J. Cohen invented, in 1963, a method of constructing models for Zermelo-Fraenkel set theory, hereafter abbreviated ZF. In the present lectures we shall describe this method with some modifications (due mainly to Solovay) and apply it to a proof that the continuum hypothesis is independent of ZF. The independence proof is taken over from Cohen [66] without change.

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1. *Logical preliminaries*

The language of ZF is the first order language with identity and with one binary predicate ε. We shall denote this language by L. We write $x \varepsilon y$ instead of $\varepsilon(x,y)$. The logical symbols we use are: $\neg$, $\to$, $\equiv$, $\&$, $\vee$, $\forall x$, $\exists x$. We abbreviate $\forall x[x \varepsilon a \to F]$ by $\forall x \varepsilon a\, F$ and $\exists x\, [x \varepsilon a \,\&\, F]$ by $\exists x \varepsilon a\, F$. Moreover we write $x \subseteq y$ instead of $\forall x \varepsilon z(z \varepsilon y)$ and $x \subset y$ instead of $x \subseteq y \,\&\, x \neq y$. The same abbreviations are used for other variables also.

If ϕ is a formula of L then $Fr(\phi)$ denotes the set of the free variables of ϕ.

The axioms of ZF are as follows:

1. Axiom of extensionality: $\forall x\, \forall y\, [x = y \equiv \forall z\, (z \varepsilon x \equiv z \varepsilon y)]$,
2. Existence of pairs: $\forall x\, \forall y\, \exists z\, \forall t\, [t \varepsilon z \equiv (t = x \vee t = y)]$,
3. Existence of unions: $\forall x\, \exists y\, \forall z\, [z \varepsilon y \equiv \exists t \varepsilon x\, (z \varepsilon t)]$,
4. Existence of power sets: $\forall x\, \exists y\, \forall z\, [z \varepsilon y \equiv z \subseteq x]$,
5. Existence of infinite sets: $\exists x\, \exists y \varepsilon x\, \forall z \varepsilon x\, \exists t \varepsilon x\, [z \subset t]$,
6. Axiom of foundation: $\forall x\, \forall y \varepsilon x\, \exists z \varepsilon x\, \forall t \varepsilon x\, [\neg(t \varepsilon z)]$,

7_ϕ. Axiom scheme of comprehension: $\forall x\, \exists y\, \forall z\, [z \varepsilon y \equiv (\phi \,\&\, z \varepsilon x)]$,

8_ψ. Axiom scheme of replacement: $\forall x\, \exists y\, \forall z \varepsilon x\, \forall t\, [\psi \rightarrow \exists t \varepsilon y\, \psi]$.

In 7_ϕ, ϕ can be any formula of L in which the variable z but not y is free; in 8_ψ, ψ can be any formula of L in which the variables z, t are free but the variable y is not free.

The axiom of choice is the following sentence:

$$\text{AC.}\quad \forall x\, \forall y \varepsilon x\, \forall z \varepsilon x\, \exists s \varepsilon y\, \forall t\, [\neg(t \varepsilon y \,\&\, t \varepsilon z) \vee z = y] \rightarrow \exists w\, \forall y \varepsilon x\, \exists v \varepsilon y\, \forall t \varepsilon y\, (t \varepsilon w \equiv t = v).$$

We shall denote by ZFC the system obtained by adjoining AC to ZF.

Our meta-theory in which we shall study models of ZF will be the set theory ZFC enriched by one additional axiom SM due to Cohen. We shall formulate this axiom below after introducing some definitions. We shall freely use the current set-theoretical notation and shall write set-theoretical formulae using the same logical symbols as in the language L. The membership relation however will be denoted by $\in$.

A family F of sets is called *transitive* if $x \in y \in F$ implies $x \in F$.

We shall assume as known the concept of a relational system and the notion of satisfaction of a formula in such systems. We use the customary notation $M \models \phi[a]$ for: ϕ is satisfied in M by the assignment a of elements of M to the free variables of ϕ. We shall use the same notation also for languages arising from L by adjunction of constants and additional predicates.

Relational systems in which the universe is a transitive family of sets and ε is interpreted as the membership relation will be called *models*.

We shall denote by the same letter a relational system M and its universe. The set of all assignments of elements of M to the free variables of a formula ϕ can then be denoted by $M^{\mathrm{Fr}(\phi)}$.

LEMMA 1.1

If M *is a transitive family of sets such that the set* $\omega = \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\},\ldots\}$ *belongs to* M *and* M *is closed with respect to the formation of pairs and unions then axioms* 1,2,3,5 *and* 6 *are valid in* M.

Proof: routine.

The existence of a model M in which the remaining axioms of ZF are valid cannot be proved on the basis of the usual axioms of set theory even if we assume an additional axiom stating the consistency of ZF.

The additional axiom SM which we mentioned above states:

(SM) There is a denumerable model of ZF.

REMARKS

(i) In view of the Skolem-Löwenheim theorem (SM) follows from a weaker axiom: there is a model of ZF.

(ii) From Gödel [40] it follows that each model M of ZF contains a submodel M' such that all the axioms of ZFC are valid in M'. Thus (SM) can also be replaced by the axiom: there is a model of ZFC.

2. *Algebraic preliminaries*

Let P be a partially ordered set. We denote by $<$ the ordering relation. The elements of P are called "conditions" and are denoted by letters $p, q, r, \ldots$. If $p \leqslant q$ then p is called an *extension* of q; if moreover $p \neq q$ then we write $p < q$ and call p a *proper* extension of q. If $\exists r\,[r < p \;\&\; r < q]$ then p and q are called *compatible*, otherwise incompatible.

Assumptions concerning P.

1. $<$ is a partial ordering.
2. Each condition has a proper extension.
3. If $p \nless q$ then there is an r such that $r < p$ and r is incompatible with q.

Partial orderings satisfying 3 are called *separable*.

Filters.

A set $F \subseteq P$ is called a filter if

(i) $F \neq \emptyset$,

(ii) $p < q$ and $p \in F$ imply $q \in F$,

(iii) $p \in F$ and $q \in F$ imply $\exists r \varepsilon F\, [r < p \;\&\; r < q]$.

LEMMA 2.1

Each filter can be extended to a maximal filter.

Proof. Use Zorn's lemma and the remark that the union of a chain of filters is a filter.

LEMMA 2.2

If F *is a filter and* p *is incompatible with an element of* F *then there is a maximal filter* F' *such that* $p \notin F'$ *and* $F \subseteq F'$.

Proof. Use Zorn's lemma to obtain a maximal element F' of the family of filters containing F as a subset and not containing p as an element.

LEMMA 2.3

If $q \nless p$ *then there is a maximal filter containing* q *but not* p.

Proof. By separability there is an r in P such that $r < q$ and r is incompatible with p. By 2.2 the filter $\{x \in P : x > r\}$ can be extended to a maximal filter and this filter satisfies 2.3.

LEMMA 2.4

If $p_0 > p_1 > \ldots$ *then there is a maximal filter containing all the* p_n.

Proof. Extend the filter $\{x \in P : \exists n\, (x > p_n)\}$ to a maximal one.

We shall now define a topological space. Let $\mathcal{X}$, or more precisely $\mathcal{X}(P,<)$, be the set of all maximal filters of P and take as an open sub-basis of $\mathcal{X}$ the family of all sets

$$[p] = \{F \in \mathcal{X} : p \in F\}.$$

Sets of the form $[p]$ will be called *neighbourhoods*.

THEOREM 2.5

$\mathcal{X}$ is a topological space which satisfies the Baire category theorem.

Proof. Open sets are defined as arbitrary unions of neighbourhoods $[p]$. Hence arbitrary unions of open sets are open.

The intersection of two open sets $\mathcal{O}_1, \mathcal{O}_2$ is open. For if $F \in \mathcal{O}_1 \cap \mathcal{O}_2$ then there are p_1, p_2 such that $F \in [p_1] \subseteq \mathcal{O}_1$ and $F \in [p_2] \subseteq \mathcal{O}_2$. Hence $p_1 \in F$ and $p_2 \in F$ and therefore there is an r in F such that $r < p_1$ and $r < p_2$. It follows that $F \in [r]$ and $[r] \subseteq [p_1] \cap [p_2] \subseteq \mathcal{O}_1 \cap \mathcal{O}_2$. Thus $\mathcal{O}_1 \cap \mathcal{O}_2$ is either void or is a union of neighbourhoods. Hence $\mathcal{X}$ is a topological space.

We now prove that if $\mathcal{O}_n \subseteq \mathcal{X}$ and $\mathcal{O}_n$ is open and dense in $\mathcal{X}$ for $n = 1,2,3,\ldots$ then $\bigcap_n \mathcal{O}_n \neq \emptyset$ (Baire theorem).

Put $G_n = \{p \in P : [p] \subseteq \mathcal{O}_n\}$. From the density of $\mathcal{O}_n$ it follows that for each q in P there is a p in G_n such that $p < q$, i.e. that G_n is dense in P. Let p_0 be arbitrary. From the density of G_1 we infer that $p_1 < p_0$ for some p_1 in G_1. From the density of G_2 we infer that $p_2 < p_1$ for some p_2 in G_2.

Continuing in this way we obtain a decreasing sequence $p_0 > p_1 > \dots$. By 2.4 there is a maximal filter F such that $p_n \in F$ for each n and it follows that $F \in [p_n] \subseteq \mathcal{O}_n$ for each n, i.e. $F \in \bigcap_n \mathcal{O}_n$.

3. *Heuristic remarks about models in which the continuum hypothesis fails*

We assume that we live in a world in which the continuum hypothesis (abbreviated CH) is true and want to construct another world in which CH is false, i.e. in which there is an injection f of ω_α into 2^ω for some $\alpha > 1$. No such injection exists in our world but we can say that if it exists anywhere, we have in our world its finite approximations. These approximations are finite functions p with domains $\subseteq \omega_\alpha \times \omega$ and range $\subseteq \{0,1\}$. Indeed, if f is our hypothetical injection and D is a finite subset of $\omega_\alpha \times \omega$ then putting on D $p(\xi,n) = f(\xi)(n)$ (the value of $f(\xi)$ for the argument n) we obtain a finite approximation of f which, because of the finiteness of D, belongs to our world. Mappings of finite subsets of $\omega_\alpha \times \omega$ into $\{0,1\}$ form a set P which is partially ordered by the relation of inverse inclusion : $p < q$ means that $p \supseteq q$. Each maximal filter of P determines a function defined on the whole set $\omega_\alpha \times \omega$; conversely each mapping f of $\omega_\alpha \times \omega$ into $\{0,1\}$ determines a maximal filter of P consisting of all the finite approximations of f. Maximal filters which belong to our world determine functions which are not one-to-one. It will be our task to find an extension of our world with new maximal filters of P.

Two methods are known at present to construct such extensions. One of them identifies sets with two-valued functions whose values are arbitrary elements of suitable Boolean algebras. Another method identifies the "world" with a denumerable model M of ZF and extends M by adding to it certain subsets of P not previously contained in M.

Below we shall sketch this second method.

4. *Constructible sets*

Let M be a denumerable model of ZF and let $On_M = On \cap M$ be the set of the ordinals of M. We denote by $rk(x)$ the rank of x defined by induction as follows

$$rk(x) = \sup\{rk(y) : y \in x\}.$$

The set $\{x \in M : rk(x) < \alpha\}$ will be denoted by M_α.

Let X be a subset of M whose rank α_0 belongs to On_M and put $\Lambda_\alpha = \emptyset$ or $\Lambda_\alpha = \{X\}$ according as $\alpha < \alpha_0$ or $\alpha > \alpha_0$. We are going to define a family $M[X]$ such that $M \subseteq M[X]$ and $X \in M[X]$. For suitable X this family will be the required model.

For each family B of sets, each formula ϕ of L such that $v \in Fr(\phi)$ and each assignment $\gamma \in B^{Fr(\phi)-\{v\}}$ we define the *extension* of ϕ in B with respect to γ as

$$E_{\phi,\gamma,B} = \{x \in B : B \models \phi[x,\gamma]\}$$

where x,γ is an assignment which correlates x to v and is otherwise identical with γ.

The family of all sets $E_{\phi,\gamma,B}$ where ϕ ranges over formulae of L such that $v \in Fr(\phi)$ and γ ranges over $B^{Fr(\phi)-\{v\}}$ is called the *derived family* and will be denoted by B'.

We define now $M[X]$ as the union $\cup\{B_\alpha[X] : \alpha \in On_M\}$ where $B_0[X] = \emptyset$, $B_\lambda[X] = \cup\{B_\alpha[X] : \alpha < \lambda\}$ if λ is a limit number and $B_{\alpha+1}[X] = B'_\alpha[X] \cup M_\alpha \cup \Lambda_\alpha$

Elements of $M[X]$ will be called *constructible in* X. The following properties of the families $B_\alpha[X]$ and $M[X]$ are easy to prove.

LEMMA 4.1

$M \subseteq M[X]$.

Proof. If $m \in M$, then $m \in M_\alpha$ where $\alpha = rk(m)$ and hence $m \in B_{\alpha+1}[X]$.

LEMMA 4.2

$X \in M[X]$.

Proof. $X \in \Lambda_{\alpha_0}$ and hence $X \in B_{\alpha_0+1}[X]$.

LEMMA 4.3

$B_\alpha[X]$ *is transitive for each* $\alpha \in On_M$.

Proof. We use induction. The lemma is true for $\alpha = 0$. If it is true for $\alpha < \lambda$ where λ is a limit number then it is true for λ. If $x \in y \in B_{\alpha+1}[X]$ then either $y \in B'_\alpha[X]$ or $y \in M_\alpha$ or $y = X$. In the first case y is the extension of a formula in $B_\alpha[X]$ and hence $x \in B_\alpha[X]$. In the second case $rk(y) < \alpha$ and hence $rk(x) < \alpha$ and thus $x \in M_{rk(x)} \subseteq B_{\alpha+1}[X]$. The last case can occur only if $\alpha > \alpha_0$. We have then $x \in X$, $rk(x) < \alpha$ and $x \in M$ because $X \subseteq M$. Hence $x \in M_\alpha$ and $x \in B_{\alpha+1}[X]$.

LEMMA 4.4

$B_\alpha[X] \subseteq B_\beta[X]$ *whenever* $\alpha < \beta$.

Proof. It is sufficient to prove the lemma for the case $\beta = \alpha + 1$. Let $a \in B_\alpha[X]$ and consider the formula $v \in w$ and the assignment γ such that $\gamma(w) = a$. The extension of $v \in w$ in $B_\alpha[X]$ with respect to γ is $\{x \in B_\alpha[X] : x \in a\} = a \cap B_\alpha[X] = a$ because all elements of a belong to $B_\alpha[X]$ in view of the transitivity of $B_\alpha[X]$. Hence $a \in B_{\alpha+1}[X]$.

LEMMA 4.5

$M[X]$ *is transitive.*

Proof. The union of a chain of transitive sets is transitive.

LEMMA 4.6

$B_\alpha[X] \in B_{\alpha+1}[X]$.

Proof. $B_\alpha[X]$ is the extension of the formula $v = v$ in $B_\alpha[X]$ with respect to the void valuation.

LEMMA 4.7

$B_\alpha[X] \in M[X]$ *for each* $\alpha < On_M$.

Proof. By 4.6.

LEMMA 4.8

If $x \in B_\alpha[X]$ *then* $rk(x) < \alpha$.

Proof. For $\alpha = 0$ the lemma is obvious. Assume that $\alpha > 0$ and the lemma holds for $\xi < \alpha$. If α is a limit number there is nothing to prove. If $\alpha = \xi + 1$ and $x \in B_{\xi+1}[X]$ then x is a set of elements which belong to $B_\xi[X]$, hence their ranks are $< \xi$ and the lemma follows.

LEMMA 4.9

$M[X]$ is closed under the formation of pairs and unions.

Proof. Let $a, b \in M[X]$ and let α be an ordinal such that $a, b \in B_\alpha[X]$. The extension of the formula $v = w \vee v = u$ in $B_\alpha[X]$ with respect to the assignment $\gamma(w) = a$, $\gamma(u) = b$ is $\{a,b\}$. The extension of the formula $\exists u\ (u \in w \mathbin{\&} v \in u)$ in $B_\alpha[X]$ with respect to the assignment $\gamma(w) = a$ is $\cup a$. Thus $\{a,b\}$ and $\cup a$ are elements of $B_{\alpha+1}[X]$.

LEMMA 4.10

Axioms 1, 2, 3, 5, 6 of ZF are valid in $M[X]$.

Proof. This follows from Lemmas 1.1, 4.5, 4.9 and the remark that the set ω referred to in Lemma 1.1 is an element of M and hence of $M[X]$.

LEMMA 4.11

The ordinals of $M[X]$ are the same as the ordinals of M.

Proof. x is an ordinal of $M[X]$ if and only if it is a transitive set which belongs to $M[X]$ and whose elements are transitive sets. A transitive set all of whose elements are transitive sets is equal to its rank. By 4.8 the rank of an element of $M[X]$ is $< On_M$. It follows that each ordinal of $M[X]$ is an ordinal of M. The converse implication is obvious.

5. *The ramified language* RL

The important feature of the technique invented by Cohen is that it allows us to speak about the model $M[X]$ remaining so to speak in the model M (we speak in "our world" M about the "fictitious world" which extends M). This is due to the fact that each element of $M[X]$ has a "name" in M. These "names" will be expressions of an auxiliary language which we shall call the ramified language or RL for short. The expressions of RL will be elements of M.

The language RL has an infinite number of variables $v_0, v_1, \ldots,$ two binary predicates ε, $\approx$ called the membership and identity predicates, propositional connectives $\neg$, $\&$ and the universal quantifier $\forall$. Besides these expressions the language RL will have infinitely many constants and infinitely many one-place predicates which will be described a little later.

The rules of formation will be the usual ones. Thus constants and variables are terms of RL. If t_1 and t_2 are terms and V is a one-place predicate then $t_1 \varepsilon t_2$, $t_1 \approx t_2$, Vt_1 are atomic formulae. If ϕ and ψ are formulae, then so are $(\phi) \& (\psi)$, $\neg(\phi)$ and $\forall v\ (\phi)$ for each variable v. The distinction between free and bound occurrences of a variable in a formula is assumed as known; $Fr(\phi)$ denotes the set of all free variables of ϕ. If γ is a sequence of constants and $Dom(\gamma) \subseteq Fr(\phi)$ then $\phi(\gamma)$ denotes the formula resulting from ϕ by substituting $\gamma(v)$ for v throughout ϕ for each v in $Dom(\gamma)$.

Writing formulae of RL we shall often use connectives other than $\&$ and $\neg$ and also the existential quantifier. These symbols are then thought of as abbreviations. Also we shall often use letters v, w, u, etc. instead of $v_0, v_1, v_2, \ldots$.

Since we want to treat expressions of RL as elements of M we identify the primitive symbols of RL with certain elements of M

and agree that if an expression is obtained by writing the symbols $A, B, C, \ldots, H$ one after another, then the whole expression is to be identified with the sequence $\langle A,B,C,\ldots,H\rangle$.

We identify v_j with $\langle 0,j\rangle$ and the symbols $\varepsilon, \approx, \&, \neg$, $($, $)$, $\forall$ with the pairs $\langle 1,j\rangle$, $j = 0,1,\ldots,5,6$. Elements of M which will serve as constants and as one-place predicates will be defined later.

We note that from now on L will be treated as a part of RL.

We describe now the additional predicates and constants of RL. For each ordinal $\alpha \in On_M$ we have in RL a one-place predicate V_α which we shall identify with the pair $\langle 2,\alpha\rangle$. Intuitively V_α denotes the set $B_\alpha[X]$.

We put $\underline{m} = \langle 3,m\rangle$ for each m in M and $\sigma = \langle 4,0\rangle$. For α in On_M, for ϕ a formula of L such that $v \in Fr(\phi)$ and each sequence γ with domain $Dom(\gamma) = Fr(\phi) - \{v\}$ we put

$$c_{\alpha,\phi,\gamma} = \langle 5,\alpha,\phi,\gamma\rangle .$$

Constants of RL can now be defined by transfinite induction:

$$C_0 = \emptyset,\ C_\lambda = \cup\{C_\alpha : \alpha < \lambda\} \text{ for limit numbers } \lambda,$$
$$C_{\alpha+1} = C_\alpha \cup \{\underline{m} : rk(m) < \alpha\} \cup \Gamma_\alpha \cup C'_\alpha$$

where $\Gamma_\alpha = \emptyset$ or $\Gamma_\alpha = \{\sigma\}$ according as $\alpha < \alpha_0$ or $\alpha > \alpha_0$ and C'_α is the set of all $c_{\alpha,\phi,\gamma}$ where ϕ is a formula of L, $v \in Fr(\phi)$ and $\gamma \in C_\alpha^{Fr(\phi)-\{v\}}$.

Let $C = \cup\{C_\alpha : \alpha \in On_M\}$. Elements of C are called constants of RL. For each c in C we define its order $\rho(c)$ as the least α such that $c \in C_\alpha$. Thus $\rho(c)$ is always a successor ordinal.

The intuitive meaning of the constants is as follows: $\underline{m}$ denotes m, o denotes X and $c_{\alpha,\phi,\gamma}$ denotes the extension of ϕ in $B_\alpha[X]$ with respect to the assignment which correlates with each variable $w \in Fr(\phi)$ different from v the object denoted by the constant $\gamma(w)$.

A formula of RL in which each quantifier $\forall x$ (where x is any variable) is followed by an expression of the form $(V_\alpha x \rightarrow \psi)$ is called a limited formula. The index α may differ from quantifier to quantifier within the formula. We shall abbreviate $\forall x[V_\alpha x \rightarrow \psi]$ by $\forall_\alpha x\ \psi$ and $\exists x\ [V_\alpha x\ \&\ \psi]$ as $\exists_\alpha x\ \psi$ The order $\rho(\phi)$ of a limited formula is defined as the larger of the following two ordinals: $\max\{\rho(c) : c$ occurs in $\phi\}$, $\max\{\alpha : V_\alpha$ occurs in $\phi\}$.

Several times we shall have occasion to use an ordering $\prec$ of limited sentences defined as follows: $\phi \prec \psi$ if and only if one of the following conditions is satisfied:

1. $\rho(\phi) < \rho(\psi)$;
2. $\rho(\phi) = \rho(\psi)$ and ϕ contains fewer occurrences of logical operations (i.e. connectives and quantifiers) than ψ;
3. ϕ and ψ are atomic sentences, $\rho(\phi) = \rho(\psi)$ and ϕ has the form $c_1 \varepsilon c_2$ with $\rho(c_1) < \rho(c_2)$ whereas ψ does not have this form;
4. ϕ and ψ are atomic sentences, $\rho(\phi) = \rho(\psi)$, ϕ has the form $c_1 \approx c_2$ and ψ has the form $c_3 \varepsilon c_4$ with $\rho(c_3) > \rho(c_4)$.
5. ϕ and ψ are atomic sentences, $\rho(\phi) = \rho(\psi)$, ϕ has the form $c_1 \approx c_2$ where $\rho(c_1) > \rho(c_2)$ and ψ has the form $V_\alpha c_1$.

If ϕ is a formula, $\alpha \in On_M$, x is a variable and ψ arises from ϕ by replacing an occurrence of the quantifier $\forall x$ by $\forall x\, [V_\alpha x \to V_\alpha c_1]$ then ψ is said to arise from ϕ by a relativization of the considered occurrence of $\forall x$ to V_α. We shall denote by $\phi^{(\alpha)}$ a formula which arises from ϕ by relativizing all occurrences of the quantifier to V_α.

LEMMA 5.1

$C_\alpha \in M$ *for each* $\alpha \in On_M$.

This follows from the remark that the operation which yields C_α from the sequence $\{C_\xi\}_{\xi<\alpha}$ is definable in M.

LEMMA 5.2

For each $\alpha \in On_M$ *the set of all formulae of order* α *belongs to* M.

This is so because such formulae are finite sequences built according to recursive rules from variables and symbols $\neg$, $\&$, $\approx$, V_ξ, $\forall$, ε, c where $\xi < \alpha$ and $c \in C_\alpha$. Since all these symbols form a set which belongs to M, so do all their finite sequences. In view of the recursive character of the formation rules, the formulae themselves also form a set in M.

In connection with these lemmas it is worth noting that neither C nor the set of all formulae of RL belong to M, although of course they are definable subsets of M.

LEMMA 5.3

The relation $\prec$ *is a definable partial well ordering of the limited sentences. Its restriction to the set of sentences of order* $\leqslant \alpha$ *is a set in* M *for each* $\alpha \in On_M$.

Proof. The second part follows from the first and Lemma 5.2. The definability of $\prec$ follows from the remark that the function ρ and clauses 1-4 of the definition of $\prec$ are definable in M. The well-foundedness of $\prec$ is proved as follows: Let A be a non void set of limited sentences and A_0 its subset consisting of sentences of a possibly small order. If A_0 does not contain atomic sentences of the form $c_1 \varepsilon c_2$ or $c_1' \approx c_2'$ then no element of A_0 has a predecessor in A. If A_0 contains at least one sentence of the form $c_1' \approx c_2'$ but no sentence of the form $c_1 \varepsilon c_2$ with $\rho(c_1) < \rho(c_2)$ then each sentence $c_1' \approx c_2'$ which belongs to A_0 is a minimal element of A. Finally if A_0 contains at least one sentence $c_1 \varepsilon c_2$ with $\rho(c_1) < \rho(c_2)$ then each such sentence is a minimal element of A_0.

Limited sentences ϕ which have no predecessors with respect to $\prec$ are $\underline{0} \varepsilon \underline{0}$, $\underline{0} \approx \underline{0}$ and $V_0\underline{0}$.

Values of the constants. Let X be a subset of a set which belongs to M and has rank α_0. For each constant c we define by induction the value of c for the argument X; we denote this value by $c^*[X]$.

Let us assume that $\alpha \in On_M$ and that $c^*[X]$ is already defined for c in C_β with $\beta < \alpha$. Let $c \in C_\alpha$. If α is a limit number there is nothing to define. If $\alpha = \beta + 1$ there are three possibilities:

1) $c \in C'_\beta$, 2) $c = \underline{m}$ where $rk(m) < \beta$, 3) $c = \sigma$.

In cases 2) and 3) we put $c^*[X] = m$ and $c^*[X] = X$ respectively. In case 1) $c = c_{\beta,\phi,\gamma}$ where ϕ is a formula of L, $v \in Fr(\phi)$ and

$\gamma \in C_\beta^{Fr(\phi)-\{v\}}$. In this case we put $c^*[X] = E_{\phi,\gamma^*[X],B_\beta[X]}$ where $\gamma^*[X]$ is a sequence with domain $Fr(\phi) - \{v\}$ whose terms are values of the terms of γ.

LEMMA 5.4

For each $\alpha \in On_M$ *the set* $B_\alpha[X]$ *is identical with the set of all* $c^*[X]$ *where* c *ranges over* C_α.

Proof by an obvious induction.

Lemma 5.4 says that each element of $M[X]$ has a name in C. As a matter of fact each element of $M[X]$ has many names; given an element x in $M[X]$ the set of all its names is a subset of M but not an element of M.

It can be shown that the function which correlates $c^*[X]$ with c is definable in $M[X]$. This will follow from theorems which will be established in the next section and from the following weaker result:

LEMMA 5.5

Let N *be a model of* ZF *such that* $N \supseteq M$ *and let* f *be a function of two arguments the first of which ranges over* C *and the second over subsets belonging to* N *of a fixed set* $P \in M$ *of rank* α_0 *and which is defined by the equation* $f(c,X) = c^*[X]$. *Then* f *is definable in* N.

Proof. f is defined by transfinite induction on the order of c. Since the theorem on definitions by transfinite induction is valid in N, we obtain the desired result.

We close our discussion with a remark on semantics of the language RL. The relational structures which can serve as models for formulae of RL have an infinite type because there are infinitely many constants in RL and also infinitely many one-place predicates in addition to the two binary ones ε and $\approx$. Apart from that the model theory of the language RL does not differ from the model theory of any first order language.

In most cases we shall deal with models whose universe is $M[X]$ and where the predicates ε, $\approx$, V_α are interpreted as $\in$, $=$, $B_\alpha[X]$ and the constants c as $c^*[X]$. Such a model will be denoted briefly by $M[X]$.

6. *Reduction of properties of* M[X] *to* M

The following example shows that, in general, $M[X]$ is not a model of ZF : Let X be a subset of $\omega \times \omega$ such that the relation mXn orders ω similarly to On_M. The existence of X follows from our assumption that M and hence also On_M are denumerable. The family $On \cap M[X]$ is equal to On_M because each element of $M[X]$ has a rank $< On_M$ (see Lemma 4.11). On the other hand $X \in M[X]$ and X has the order type On_M and so the theorem: "for each well ordering there is a similar ordinal" is not valid in $M[X]$. Yet this theorem is provable in ZF. Hence not all axioms of ZF are valid in $M[X]$.

There are no general criteria which would allow us to decide for which sets X the family $M[X]$ is a model of ZF. A sufficient condition which we shall discuss below says that the satisfaction relation

$$M[X] \models \phi[\gamma^*[X]] \tag{1}$$

be expressible in M. More exactly we require that (1) be equivalent to the fact that a formula depending on ϕ be satisfied in M by γ (i.e. the sequence of names of the terms of $\gamma^*[X]$) and by an element of X.

To return to the intuitive picture given in Section 3 where X was a maximal filter in P we may say that we require (1) to be equivalent to a relation definable in M holding between names of terms of $\gamma^*[X]$ and an approximation of an object determined by X.

The exact definition is as follows:

DEFINITION

We say that X is reducible to M if for each formula ϕ of L there is a formula Φ_ϕ with one more free variable such that for each $\gamma \in C^{Fr(\phi)}$

$$M[X] \models \phi[\gamma^*[X]] \equiv \exists x \epsilon X\ [M \models \Phi_\phi[x,\gamma]]. \tag{2}$$

Instead of $M \models \Phi_\phi[x,\gamma]$ we shall say that the condition x establishes ϕ at γ in $M[X]$. (The term in general use is: x forces $\phi(\gamma)$. We selected another term in order to reserve the word "forcing" for the case of a particular formula Φ_ϕ.)

To derive properties of $M[X]$ where X is reducible to M we need the notion of a normal function. Such a function is a strictly increasing and continuous mapping $f : On_M \to On_M$. We call α a critical number for f if $f(\alpha) = \alpha$.

LEMMA 6.1

If f *is a normal function which is definable in* M *then for each* α_0 *in* On_M *there is a critical number* α *of* f *such* *that* $\alpha_0 < \alpha \in On_M$.

Proof. Define a sequence $\{\alpha_n\}$ by induction on $n : \alpha_{n+1} = f(\alpha_n)$. This sequence belongs to M since the theorem on inductive definitions is valid in M. It follows that $\sup\{\alpha_n\} \in M$ and we easily prove that this supremum is the required critical number.

Another important auxiliary result is the following

THEOREM 6.2 (the reflection theorem)

Let X *be reducible to* M *and let* ϕ *be a formula of* L. *There exists a normal function* f_ϕ *definable in* M *such that whenever* α *is a critical number of* f_ϕ *and* $\gamma \in C_\alpha^{Fr(\phi)}$ *then*

$$M[X] \models \phi[\gamma^*[X]] \equiv B_\alpha[X] \models \phi[\gamma^*[X]].$$

REMARK

The reflection theorem is really a theorem scheme: for each formula ϕ we construct separately a definition of a normal function f_ϕ.

Proof. If ϕ has no quantifiers then we take for f_ϕ the identity function $f_\phi(x) = x$. If f_ϕ, f_ψ are already defined and $\theta = \neg\phi$, $\zeta = \phi \mathbin{\&} \psi$ then we take $f_\theta = f_\phi$, $f_\zeta = f_\phi \circ f_\psi$. Using the fact that a superposition $f \circ g$ of two normal definable functions is again such a function and each of its critical numbers is a critical number of the functions f and g we convince ourselves easily that if the reflection theorem is valid for the formulae ϕ, ψ then it is also valid for the formulae $\neg\phi$ and $\phi \mathbin{\&} \psi$. The theorem is also trivially valid for the formula $\exists v\, \phi$ if $v \notin Fr(\phi)$.

We shall now prove the theorem for the formula $\exists v\ \phi$ where $v \in Fr(\phi)$.

For each $\gamma \in C^{Fr(\phi)-\{v\}}$ we put

$$g(\gamma) = \sup_{p\in P}\{\min_{\eta\in On_M}\{\exists c\varepsilon C_\eta\ M \models \Phi_\phi[p,(c,\gamma)]\}\}.$$

Thus $g(\gamma)$ is the least ordinal λ with the property that for each condition p if there is a constant c such that p establishes ϕ at (c,γ) in $M[X]$ then there is a constant c already in C_λ such that the same p establishes ϕ at (c,γ) in M.

Using g we define a function h by transfinite induction:

$h(0) = 0$, $h(\lambda) = \sup\{h(\alpha) : \alpha < \lambda\}$ if λ is a limit number,

$h(\alpha+1) = \max(h(\alpha) + 1, \sup\{g(\gamma) : \gamma \in C_\alpha^{Fr(\phi) - \{v\}}\})$.

h is of course definable in M since the theorem on inductive definitions is valid in M. Moreover h is strictly increasing and continuous i.e. h is normal.

It follows from the last clause of the definition that $h(\alpha+1)$ is an upper bound of the values of $g(\gamma)$ for γ ranging over $C_\alpha^{Fr(\phi)-\{v\}}$. Hence if λ is a limit number and $\gamma \in C_\lambda^{Fr(\phi)-\{v\}}$ then $g(\gamma) < h(\lambda)$ because there are only finitely many terms of γ and each of them belongs to a C_α, with $\alpha < \lambda$; hence there is an $\alpha < \lambda$ such that all terms of γ belong to C_α and so $g(\gamma) < h(\alpha+1) < h(\lambda)$ because h is strictly increasing.

Now let f be the normal function $f_\phi \circ h$ and let α be one of its critical numbers. We shall show that if $\gamma \in C_\alpha^{Fr(\phi)-\{v\}}$ then

(3) $$M[X] \models \exists v\ \phi[\gamma^*[X]] \equiv B_\alpha[X] \models \exists v\ \phi[\gamma^*[X]].$$

First assume the right-hand side of this equivalence. Hence there is an element x of $B_\alpha[X]$ which together with $\gamma^*[X]$ satisfies ϕ in $B_\alpha[X]$. Let $c \in C_\alpha$ be a name of x. Using the inductive assumption and the remark that α is a critical number of f_ϕ we obtain the left-hand side of (3).

Now we assume the left-hand side of (3). There is thus an element of $M[X]$ which, together with $\gamma^*[X]$ satisfies ϕ in $M[X]$. This element can be represented as $c^*[X]$ where c is a constant. Since we assume that X is reducible we obtain a condition p in X which establishes ϕ at (c,γ) in $M[X]$. Using the definition of g we infer that there is a constant c_1 in $C_{g(\gamma)}$ such that p establishes ϕ at (c_1,γ) in $M[X]$. Since α is a critical number of f_ϕ we obtain $M[X] \models \phi[c_1^*[X], \gamma^*[X]]$ and since $g(\gamma) < h(\alpha) = \alpha$ we obtain $(c,\gamma) \in C_\alpha^{Fr(\phi)}$. In view of the inductive assumption we obtain therefore the right-hand side of (3).

We can now verify the validity of Axioms 4, 7 and 8 in $M[X]$.

THEOREM 6.3

If X *is reducible to* M *then the axiom of comprehension is valid in* $M[X]$.

Proof. Let ϕ be a formula of L such that $v \in Fr(\phi)$. Let $a \in M[X]$ and $g \in M[X]^{Fr(\phi)-\{v\}}$. We have to show that there is a b in $M[X]$ such that for each x in $M[X]$

$$(4) \qquad x \in b \equiv x \in a \ \& \ M[X] \models \phi[(x,g)].$$

We can represent a as $c^*[X]$ and g as $\gamma^*[X]$ where $c \in C$ and $\gamma \in C^{Fr(\phi)-\{v\}}$. Let α be an ordinal such that c and all the terms of γ be elements of C_α. Using the reflection theorem we find an ordinal $\beta > \alpha$ such that for each sequence $(c_1,\gamma_1) \in C_\beta{}^{Fr(\phi)}$

$$M[X] \models \phi[c_1^*[X], \gamma_1^*[X]] \equiv B_\beta[X] \models \phi[c_1^*[X], \gamma_1^*[X]].$$

Let us consider the formula $\bar{\Phi} = \phi \,\&\, (v \varepsilon w)$ where w is a new variable and let $\bar{\gamma}$ be an assignment which coincides with γ on the free variables of $\bar{\Phi}$ the variable w excepted and correlates c with w. We claim that the set $b = c^*_{\beta,\bar{\Phi},\bar{\gamma}}[X]$ satisfies (4). By definition b is the extension of $\bar{\Phi}$ in $B_\beta[X]$ with respect to $\bar{\gamma}^*[X]$ i.e.

$$\begin{aligned} x \in b &\equiv (x \in B_\beta[X]) \,\&\, (B_\beta[X] \models \bar{\Phi}[x,\gamma^*[X]]) \\ &\equiv (x \in B_\beta[X]) \,\&\, (B_\beta[X] \models \phi[x,\gamma^*[X]]) \,\&\, (x \in c^*[X]) \\ &\equiv (x \in a) \,\&\, (B_\beta[X] \models \phi[x,\gamma^*[X]]). \end{aligned}$$

The last equivalence is obtained by remarking that $c^*[X] = a$, $a \in B_\beta[X]$ and hence $a \subseteq B_\beta[X]$. Theorem 6.2 is thus proved.

THEOREM 6.4

If X *is reducible to* M *then the axiom of replacement is valid in* $M[X]$.

Proof. Let ψ be a formula of L such that $v, w \in Fr(\psi)$ and let $g \in M[X]^{Fr(\psi)-\{v,w\}}$, $a \in M[X]$. We have to find a set b in $M[X]$ such that whenever x is in a and there is a t in $M[X]$ satisfying $M[X] \models \psi[x,t,g]$, there is a t_1 in b satisfying the same formula.

Similarly as in the previous proof we determine α, c, γ so that $c \in C_\alpha$, $\gamma \in C_\alpha{}^{Fr(\phi)-\{v,w\}}$ and $a = c^*[X]$, $g = \gamma^*[X]$ and put $b = B_\beta[X]$ where

$$\beta = \sup_{p \in \underline{P}} \ \sup_{k \in C_\alpha} \ \min_{\xi \in On_M} \ \exists d \varepsilon C_\xi \ \{M \models \Phi_\psi[p,k,d,\gamma]\}.$$

Note: k and d correspond here to the variables v and w respectively.

Obviously $b \in M[X]$. If x is in a then $x \in B_\alpha[X]$ and hence x has a name k in C_α. If there is a t in $M[X]$ such that $M[X] \models \psi[x,t,g]$ then t has the form $d^*[X]$ for a suitable constant d and hence by the assumption of reducibility there is a $p \in X$ which establishes ψ at (k,d,γ) in $M[X]$. In view of the definition of β there is a constant d_1 in C_β such that p establishes ψ at (k,d_1,γ) in $M[X]$. Thus we obtain $M[X] \models \psi[k^*[X], d_1^*[X], \gamma^*[X]]$, i.e. $M[X] \models \psi[x,d_1^*[X],g]$ and we have proved that there is a $t_1 = d_1^*[X]$ in b satisfying $\psi[x,t_1,g]$ in $M[X]$.

THEOREM 6.5

If X *is reducible to* M *then the axiom of power-sets is valid in* $M[X]$.

Proof. Let $a \in M[X]$, $a = c^*[X]$ where $c \in C_\alpha$. If $x \subseteq a$ and $x \in M[X]$ then x has a name in C. The essential step in the proof is to show that there is an ordinal β such that each subset x of a which has a name (i.e. belongs to $M[X]$) has a name already in C_β.

For any constant d we consider the set

$$S(d) = \{\langle q,k\rangle \in P \times C_\alpha : M \models \Phi_{v \varepsilon w}[q,k,d]\}.$$

This set obviously belongs to M. More exactly $S(d)$

belongs to the power-set of $P \times C_\alpha$, taken in the sense of M. We shall denote this set $P_M(P \times C_\alpha)$. Furthermore let Σ be a family consisting of all pairs $\langle p,s \rangle \in P \times M$ such that there is a d in C satisfying the conditions:

(5) $$M \models \Phi_{v \subseteq w}[p,d,c],$$

(6) $$s = S(d).$$

The family Σ belongs to M because it is a definable subset of the set $P \times P_M(P \times C_\alpha)$ which is an element of M. For $(p,s) \in \Sigma$ let $\beta(p,s)$ be the minimal ordinal ξ such that (5) and (6) hold for a d in C_ξ and let $\beta = \sup\{\beta(p,s) : (p,s) \in \Sigma\}$. We claim that each subset x of a which belongs to $M[X]$ has a name in C_β.

Let us assume that d is a name of a set $x \in M[X]$ such that $x \subseteq a$. Since X is reducible to M there is a p satisfying (5). Thus the pair $(p,S(d))$ belongs to Σ. It follows that there is a d_1 in C_β such that

(7) $$M \models \Phi_{v \subseteq w}[p,d_1,c],$$

(8) $$S(d) = S(d_1).$$

We shall show

(9) $$d_1^*[X] = d^*[X].$$

First notice that from (7) we obtain $d_1^*[X] \subseteq c^*[X]$ because $p \in X$ and thus $d_1^*[X] \subseteq a$.

Now assume that $y \in d^*[X] = x$. Since $x \subseteq a \subseteq B_\alpha[X]$ we obtain $y \in B_\alpha[X]$ and y has a name, say k, in C_α. From

$k^*[X] \in d^*[X]$ we infer that there is a q in X which establishes $k \varepsilon d$ in $M[X]$. Hence $\langle q,k \rangle \in S(d)$ and, in view of (8), $\langle q,k \rangle \in S(d_1)$. Thus q establishes $k \varepsilon d_1$ in $M[X]$ and, since $q \in X$, we obtain $y \in d_1^*[X]$.

Let us assume conversely that $y \in d_1^*[X]$. Since $d_1^*[X] \subseteq a \subseteq B_\alpha[X]$ we can put similarly as above, $y = k^*[X]$ where $k \in C_\alpha$ and we obtain a q in X such that $\langle q,k \rangle \in S(d_1)$. From this and from (8) we infer $\langle q,k \rangle \in S(d)$ and hence $k^*[X] \in d^*[X]$, i.e. $y \in d^*[X]$. The formula (9) is thus proved. To finish the proof we denote by b the extension in $B_\beta[X]$ of the formula $v \subseteq w$ with respect to the assignment $\gamma(w) = a$. The set b is an element of $M[X]$ and consists of subsets of a. If $x \subseteq a$ and $x \in M[X]$ then x has a name in C_β and thus belongs to b. Hence b satisfies the formula $\forall x\, [x \in b \equiv x \subseteq a]$ in $M[X]$.

Theorems 4.10 and 6.3 - 6.5 show that if M is a model of ZF then so is $M[X]$ provided that X is reducible to M. For models of ZFC we have the following result:

THEOREM 6.6

If M *is a model of* ZFC *and* X *is reducible to* M, *then* $M[X]$ *is a model of* ZFC.

Proof. Let $a \in M[X]$. We shall exhibit a relation which well-orders a and is an element of $M[X]$. Since each element of $M[X]$ is a subset of a set of the form $B_\alpha[X]$ it is sufficient to exhibit a well-ordering of $B_\alpha[X]$.

Let $<$ be a relation which well-orders C_α ; the existence of $<$ follows from our assumption that the axiom of choice is valid in M. Let f be a function definable in $M[X]$ such that $f(c,X) = c^*[X]$ for each c in C (see Lemma 5.5). Call c the earliest name of an element a of $M[X]$ if $f(c,X) = x$ but $f(c',X) \neq x$ whenever $c' < c$. The following relation R is easily seen to be the required well-ordering of $B_\alpha[X]$: xRy if and only if the earliest name of x precedes (in the sense of the relation $<$) the earliest name of y. The proof that $R \in M[X]$ is immediate.

7. *Heuristic explanation of the construction of reducible filters*

We shall construct a reducible filter $X \subseteq P$ by considering a theory $\mathcal{T}$ formulated in RL. The theory $\mathcal{T}$ describes $M[X]$ in the sense that for each filter F of P the set $M[F]$ is a model of $\mathcal{T}$. A particular model of $\mathcal{T}$ can be constructed by the well-known method due to Henkin. When applying this method we consider a sequence $\{\phi_n\}$ of all sentences of RL and build a complete extension $\mathcal{T}'$ of $\mathcal{T}$ by successive steps: $\mathcal{T}' = \mathcal{T} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \ \ldots$. In the n-th step we decide of the n-th sentence ϕ_n of RL whether ϕ_n or $\neg\phi_n$ will be included in $\mathcal{T}'$. Also several other sentences have to be included in the n-th step. The inductive definition of $\mathcal{T}_n$ is carried out along with an inductive definition of a decreasing sequence $p_0 > p_1 > \ldots$ of conditions, $\mathcal{T}_n$ being the set of sentences which are true in models $M[Y]$ for almost all Y in $[p_n]$. The words "almost all" mean here "all up to a set of first category".

As is always the case with models built by Henkin's method the complete set $\mathcal{T}'$ determines a model $M[X]$ such that for each sentence ϕ of RL the sentence ϕ is true in $M[X]$ if and only if

$\phi \in \mathcal{T}'$. It follows easily that all the conditions p_n belong to X, because the sentence $\underline{p}_n \varepsilon \sigma$ is true in *all* models M[Y] where $Y \in [p_n]$. It turns out that the conditions p_n generate a maximal filter (see Lemma 9.5 below). Hence X is a maximal filter. We show that it is reducible to M. The reason for this is the following: the formula $M[X] \models \phi[\gamma^*[X]]$ is equivalent to $\phi(\gamma) \in \mathcal{T}_n$ for some n and this in turn is equivalent to

(1) $M[Y] \models \phi(\gamma)$ for almost all Y in $[p_n]$.

We shall show that this relation between ϕ, γ and p_n is definable in M, i.e. has the form $M \models \Phi_\phi[p_n,\gamma]$. The proof of definability of (1) is the most important step in the proof.

In this way we not only prove the reducibility of X to M but establish the meaning of the formulae Φ_ϕ. The formula $M \models \Phi_\phi[p,\gamma]$ says that the set of maximal filters Y in [p] for which $M[Y] \models \phi[\gamma^*[Y]]$ is co-meager (in the space $\mathcal{X}$ of all maximal filters of P). This relation is called the forcing relation.

We proceed now to the details of the proof.

8. *The theory* $\mathcal{T}$

The following sentences of RL are called axioms of $\mathcal{T}$:

(1) $\underline{m} \varepsilon \underline{m}'$ for $m, m' \in M$ such that $m \in m'$,

(2) $\neg(\underline{m} \varepsilon \underline{m}')$ for $m, m' \in M$ such that $m \notin m'$,

(3) $V_\alpha c$ for c in C, $\rho(c) < \alpha$,

(4) $c' \varepsilon c_{\alpha,\phi,\gamma} \equiv \phi^{(\alpha)}(\dot{c}',\gamma)$ if ϕ is a formula of L, $v \in Fr(\phi)$, $\gamma \in C_\alpha^{Fr(\phi)-\{v\}}$, $c' \in C_\alpha$,

(5) $\forall v_0\ \forall v_1\ \{(\forall v_2[v_2 \varepsilon v_0 \equiv v_2 \varepsilon v_1] \equiv (v_0 \approx v_1)\}$,

(6) $\forall v_0\ \forall v_1\ \forall v_2\ \{v_0 \approx v_1 \rightarrow [v_0 \varepsilon v_2 \equiv v_1 \varepsilon v_2]\}$,

(7) $\forall v\ \neg(v \varepsilon v)$,

(8) $\forall v_0\ \forall v_1\ \{v_0 \approx v_1 \rightarrow [V_\alpha v_0 \equiv V_\alpha v_1]\}$ for $\alpha \in On_M$,

(9) $\forall v_0[v_0 \varepsilon \sigma \rightarrow v_0 \varepsilon \underline{P}]$,

(10) $\forall v_0\ \forall v_1[(v_0 \leq v_1\ \&\ v_0 \varepsilon \sigma) \rightarrow v_1 \varepsilon \sigma]$,

(11) $\forall v_0\ \forall v_1\{(v_0 \varepsilon \sigma\ \&\ v_1 \varepsilon \sigma) \rightarrow \exists v_2[v_2 \leq v_0\ \&\ v_2 \leq v_1\ \&\ v_2 \varepsilon \sigma]\}$,

(12) $\exists v_0(v_0 \varepsilon \sigma)$.

In the axioms (10) and (11) we used the abbreviation $v_0 \leq v_1$ for a formula of RL which expresses the fact that the ordered pair $\langle v_0, v_1\rangle$ is an element of the set $\leq$. We should remember here that the ordering $\leq$ of P is a set of ordered pairs of which we assumed that it belongs to M. The explicit definition of $v_0 \leq v_1$ is as follows: let $\Pi'(v_m, v_n, v_p)$ be the formula "v_m is the pair v_n, v_p" i.e.

$$\Pi'(v_m,v_n,v_p) : \forall v_q[v_q \varepsilon v_m \equiv (v_q \approx v_n \vee v_q \approx v_p)]$$

where q is any integer different from m, n, p, e.g. = m+n+p+1. Let $\Pi(v_m,v_n,v_p)$ be the formula: v_m is the ordered pair $\langle v_n,v_p\rangle$ i.e.

$$\Pi(v_m,v_n,v_p) : \exists v_q\ \exists v_r\ [\Pi'(v_m,v_q,v_r)\ \&\ \Pi'(v_q,v_n,v_r)\ \&\ \Pi'(v_r,v_n,v_p)]$$

where $q = m + n + p + 1$, $r = m + n + p + 2$. Finally $v_0 \leq v_1$ is the formula

$$\exists v_2\ [\Pi(v_2,v_0,v_1)\ \&\ (v_2 \varepsilon \underline{\leq})].$$

LEMMA 8.1

If X *is a filter in* P *then all the axioms* (1) - (12) *are valid in* M[X].

Proof: obvious.

It is not true that all the models of the axioms have the form M[X]. This follows for instance from the upward Skolem-Löwenheim theorem according to which there are non-denumerable models of axioms (1) - (12) whereas models M[X] are denumerable for any $X \subseteq P$.

Let us say that a pair F, G of sets of sentences of RL satisfies the postulates if

(i) $F \subseteq G$,

(ii) $\exists v_n\ \phi \in F$ where $v_n \in Fr(\phi)$ implies $\phi(c) \in G$ for some $c \in C$,

(iii) $\exists v_n\ [V_\alpha v_n\ \varepsilon\ \phi] \in F$ where $v_n \in Fr(\phi)$ implies $\phi(c) \in G$ for some $c \in C_\alpha$,

(iv) $V_\alpha c \in F$ implies $c \approx c' \in G$ for some $c' \in C_\alpha$,

(v) $c\ \varepsilon\ c' \in F$ where $c' \in C_\alpha$ implies $c \approx c'' \in G$ for some $c'' \in \cup\{C_\beta : \beta < \alpha\}$,

(vi) $c'\ \varepsilon\ \underline{m} \in F$ implies $c' \approx \underline{n} \in G$ for some $n \in m$.

A set F of sentences of RL is closed if it is consistent, complete, contains the axioms (1) - (12) and the pair F, F satisfies the postulates.

LEMMA 8.2

The theory of M[X] *(i.e. the set of sentences of* RL *which are true in* M[X]*) is closed.*

Proof: obvious.

We shall establish some properties of closed sets:

1. If ϕ is logically valid then $\phi \in F$.

Otherwise $\neg\phi$ would be in F; but each set containing $\neg\phi$ is inconsistent.

2. If ϕ, $\phi \to \psi$ are in F then so is ψ.

Otherwise $\neg\psi$ would be in F; but each set containing the sentences ϕ, $\phi \to \psi$, $\neg\psi$ is inconsistent.

3. If ϕ is logically equivalent to ψ then $\phi \in F$ if and only if $\psi \in F$.

This follows from 1 and 2.

4. If v is the unique free variable of ϕ then $\forall v\, \phi \in F$ is equivalent to: $\phi(c) \in F$ for each c in C.

Since $\forall v\, \phi \to \phi(c)$ is logically true we obtain the implication $\to$ from 1 and 2. If $\forall v\, \phi \notin F$ then $\exists v\, \neg\phi \in F$ hence $\neg\phi(c) \in F$ for some c and so it is not true that $\phi(c) \in F$ for each c.

5. If v is the unique free variable of ϕ then $\exists v\, \phi \in F$ if and only if there is a c in C such that $\phi(c) \in F$.

Proof: similar to 4.

6. If ϕ is a formula, $\gamma \in C^{Fr(\phi)-\{v\}}$, then $c_1 \approx c_2 \in F$ implies $\phi(c_1,\gamma) \equiv \phi(c_2,\gamma) \in F$ for each ϕ.

Proof. If $\phi(\gamma)$ is the atomic formula $v \varepsilon c$ or $c \varepsilon v$ then 6. follows from axioms (5), (6) and the above remarks 1 - 4.

If ϕ is the formula $v \varepsilon v$, 6. follows from axiom (7) : $\neg(c_1 \varepsilon c_1) \in F$ hence $c_1 \varepsilon c_1 \to \psi \in F$ for any ψ and we obtain $c_1 \varepsilon c_1 \to c_2 \varepsilon c_2 \in F$. Similarly $c_2 \varepsilon c_2 \to c_1 \varepsilon c_1 \in F$.

If $\phi(\gamma)$ is the formula $v \approx c$ or $c \approx v$ we prove 6. using axiom (6). If ϕ is the formula $v \approx v$, we again use (6). If ϕ is the formula $V_\alpha v$ we use axiom (8).

Let us now assume 6. for two formulae ϕ', ϕ''. Using tautologous formulae of propositional logic we immediately obtain 6. for the formulas $\neg\phi'$ and $\phi' \,\&\, \phi''$. Also using 4. we obtain that $c_1 \approx c_2 \in F$ implies $\forall w\,[\phi'(c_1,\bar{\gamma}) \equiv \phi'(c_2,\bar{\gamma})] \in F$ where $\mathrm{Dom}(\bar{\gamma}) = \mathrm{Fr}(\phi') - \{v,w\}$. Using the fact that the taulogous sentence $\forall w\,(\phi_1 \equiv \phi_2) \to [\forall w\,\phi_1 \equiv \forall w\,\phi_2]$ is in F we obtain that $c_1 \approx c_2 \in F$ implies $[\forall w\,\phi'(c_1,\bar{\gamma})] \equiv [\forall w\,\phi'(c_2,\bar{\gamma})] \in F$.

7. If $m_1, m_2 \in M$, $m = \{m_1,m_2\}$ then $\Pi'(\underline{m},\underline{m}_1,\underline{m}_2) \in F$.

Proof. Using the definition of Π' and 4. above we see that we have to prove for each c in C the following two formulae:

(*) $c \varepsilon \underline{m} \to (c \approx \underline{m}_1 \vee c \approx \underline{m}_2) \in F$,

(**) $(c \approx \underline{m}_1 \vee c \approx \underline{m}_2) \to c \varepsilon \underline{m} \in F$.

To prove the second formula we observe that $\underline{m}_i \varepsilon \underline{m} \in F$, $i = 1,2$, by (1) and hence $c \approx \underline{m}_i \to c \varepsilon \underline{m} \in F$ for $i = 1, 2$. From this we obtain (**) by a propositional tautology. The formula (*) is shown by contradiction. If (*) were false, then, by completeness, the three sentences $c \varepsilon \underline{m}$, $\neg(c \approx \underline{m}_1)$, $\neg(c \approx \underline{m}_2)$ would belong to F. Using postulate (vi) we would obtain from $c \varepsilon \underline{m} \in F$ that $c \approx \underline{m}_1 \in F$ or $c \approx \underline{m}_2 \in F$ and F would be inconsistent.

8. If $m_1, m_2 \in M$, $m = \langle m_1, m_2 \rangle$ then

$$\forall v\ [\Pi(v, \underline{m}_1, \underline{m}_2) \equiv (v \approx \underline{m})] \in F.$$

Proof is similar to 7.

LEMMA 8.3

If F *is closed then the set* $X = \{p \in \underline{P} : \underline{p}\ \varepsilon\ \sigma \in F\}$ *is a filter in* P.

Proof. $X \neq \emptyset$ because by axiom (12) and 5. there is a constant c such that $c\ \varepsilon\ \sigma \in F$ whence, by axiom (9) and properties 2, 4 above, $c\ \varepsilon\ \underline{P} \in F$ whence by postulate (vi) $c \approx \underline{p} \in F$ for some p in P. Thus, by 6, $\underline{p}\ \varepsilon\ \sigma \in F$ and so $p \in X$.

Next, let $p, q \in X$. Hence $\underline{p}\ \varepsilon\ \sigma \in F$ and $\underline{q}\ \varepsilon\ \sigma \in F$. From axiom (11) we infer that the existential sentence

$$\exists v_2\ (v_2 \leq \underline{p}\ \&\ v_2 \leq \underline{q}\ \&\ v_2 \in \sigma)$$

is in F. Hence there is a constant c such that the sentences

$$c \leq \underline{p}, \quad c \leq \underline{q}, \quad c\ \varepsilon\ \sigma$$

belong to F. From $c\ \varepsilon\ \sigma \in F$ and axiom (9) we infer $c\ \varepsilon\ \underline{P} \in F$ and hence, by postulate (vi) $c \approx \underline{r} \in F$ for some r in P. Now we obtain $\underline{r}\ \varepsilon\ \sigma \in F$, hence $r \in X$ and $\underline{r} \leq \underline{p} \in F$ and $\underline{r} \leq \underline{q} \in F$. These formulae prove that if $s = \langle r, p \rangle$, $t = \langle r, q \rangle$ then $\underline{s}\ \varepsilon\ \underline{\leq} \in F$ and $\underline{t}\ \varepsilon\ \underline{\leq} \in F$ whence $s, t \in \leq$. Thus $r \leq p$ and $r \leq q$.

Finally let $p \in X$ and $p \leq q$. We infer similarly as above that $\underline{p} \leq \underline{q} \in F$ whence, in view of $\underline{p}\ \varepsilon\ \sigma \in X$ and axiom (10) $\underline{q}\ \varepsilon\ \sigma \in F$ and so $q \in X$.

LEMMA 8.4

If F *is a closed set of sentences and* $X = \{p \in P : \underline{p} \varepsilon \sigma \in F\}$ *then* M[X] *is a model of* F.

Proof. First we construct a relational system A in which all sentences $\phi \in F$ are true and then prove that after dividing A by a congruence we obtain a relational system isomorphic with M[X]. We obtain A by a standard method, due to Henkin, of constructing relational systems from constants.

The universe of A will be C; for each c in C the interpretation of c will be c itself. The binary predicates $\approx$, ε will be interpreted as the relations I, E defined as follows:

(i) $c_1 I c_2 \equiv (c_1 \approx c_2 \in F) \quad c_1 E c_2 \equiv (c_1 \varepsilon c_2 \in F)$.

Finally the unary predicates V_α will be interpreted as sets

(ii) $$A_\alpha = \{c \in C : V_\alpha c \in F\}.$$

It follows from remark 6 above that I is a congruence in A.

We prove by induction that if ϕ is a formula of RL and $\gamma \in C^{Fr(\phi)}$ then

(iii) $$A \models \phi[\gamma] \equiv \phi(\gamma) \in F.$$

For atomic formulae ϕ this follows directly from the definition. If the equivalence is true for ϕ and ϕ_1 then using completeness of F and 4 we immediately infer that it is also valid for $\neg\phi$, $\phi \,\&\, \phi_1$ and $\forall v\, \phi$ where v is any variable.

Thus A and also A divided by the equivalence I which we shall denote by A/I are relational systems in which all sentences of F are true.

We prove the following statement: for each limited formula ϕ and each $\gamma \in C^{Fr(\phi)}$

(iv) $$M[X] \models \phi[\gamma^*[X]] \equiv A \models \phi[\gamma].$$

Let us write $(\psi,\delta) < (\phi,\gamma)$ if ϕ, ψ are limited formulae, γ, δ sequences of constants and $\psi(\delta) \prec \phi(\gamma)$ where $\prec$ is the relation defined in Section 5. It will be sufficient to prove (iv) under the assumption that it is valid for all pairs $(\psi,\delta) < (\phi,\gamma)$.

If $\phi(\gamma)$ is minimal with respect to the relation $\prec$ then $\phi(\gamma)$ is one of the sentences $\underline{0} \varepsilon \underline{0}$, $\underline{0} \approx \underline{0}$, $V_0\underline{0}$ and the truth of (iv) is easy to verify.

Let us assume that $\phi(\gamma)$ is not minimal. If ϕ contains logical connectives then either (case 1) $\phi(\gamma) = \neg\phi_1(\gamma)$ or (case 2) $\phi(\gamma) = \phi_1(\gamma \mid Fr(\phi_1)) \,\&\, \phi_2(\gamma \mid Fr(\phi_2))$ or (case 3) $\phi(\gamma) = \forall v_j\, [V_\alpha v_j \rightarrow \phi_1(\gamma)]$ and (iv) is true for formulae ϕ_1, ϕ_2 and arbitrary sequences γ_1, γ_2 of constants such that $(\phi_1, \gamma_1) < (\phi,\gamma)$ and $(\phi_2, \gamma_2) < (\phi,\gamma)$. In case 1 $(\phi_1,\gamma) < (\phi,\gamma)$ hence (iv) is valid for the pair (ϕ_1,γ) and taking negations on both sides we obtain (iv) for the pair (ϕ,γ). In case 2 $(\phi_i, \gamma \mid Fr(\phi_i)) < (\phi,\gamma)$ for $i = 1,2$ and so (iv) is valid for the pairs $(\phi_i, \gamma \mid Fr(\phi_i))$, $i = 1,2$. Taking conjunctions on both sides of the resulting equivalences we obtain (iv) for the formula ϕ. In case 3 the left-hand side of (iv) is equivalent to the statement: for each x in $B_\alpha[x]$

$$M[X] \models \phi_1[x,\gamma^*[X]].$$

Since each element of $B_\alpha[X]$ has a name in C_α and since $c^*[X] \in B_\alpha[X]$ for $c \in C_\alpha$ we infer, using the inductive assumption, that this statement is equivalent to: for each c in C_α, $A \models \phi_1 [c,\gamma]$. Now let c' be an arbitrary constant. If $A \not\models V_\alpha[c']$ then c' satisfies $V_\alpha v_j \to \phi_1(\gamma)$ in A. Otherwise by postulate (iv) there is a constant c in C_α such that $c \mathrel{I} c'$ and since, as we proved above, $A \models \phi_1[c,\gamma]$, we obtain $A \models \phi_1[c',\gamma]$. Hence each c' satisfies in A the formula $V_\alpha v_j \to \phi_1(\gamma)$ and we obtain the right-hand side of (iv).

The converse implication is proved similarly.

It remains to prove (iv) in the case when ϕ is an atomic formula. We have several cases to consider.

Case 1. $\phi(\gamma)$ is the sentence $c_1 \varepsilon c_2$. Subcase (a) : $\rho(c_1) \geqslant \rho(c_2)$. In this case the left-hand side of (iv) is equivalent to $c_1^*[X] \in c_2^*[X]$. Let c_3 be a constant of a possibly smaller order such that $c_1^*[X] = c_3^*[X]$. Thus $\rho(c_3) < \rho(c_2)$ and it follows that the sentences $c_1 \approx c_3$ and $c_3 \varepsilon c_2$ precede the sentence $c_1 \varepsilon c_2$ in the ordering $\prec$.

From the inductive assumption we obtain therefore $A \models c_1 \approx c_3$ and $A \models c_3 \varepsilon c_2$ whence $c_1 \approx c_3 \in F$, $c_3 \varepsilon c_2 \in F$ and therefore $c_1 \varepsilon c_2 \in F$ i.e., $A \models c_1 \varepsilon c_2$. The implication can obviously be reversed.

Subcase (b) : $\rho(c_1) < \rho(c_2)$. We denote $\rho(c_2)$ by $\alpha + 1$ and discuss separately the possible forms of c_2:

Sub-subcase (b1) : $c_2 = \underline{m}$ where $m \in M$ and $rk(m) \leqslant \alpha$. In this case the left-hand side of (iv) is equivalent to $c_1^*[X] = n$

where $n \in m$. Putting $c_3 = \underline{n}$ we have $c_1 \approx c_3 \prec c_1 \varepsilon c_2$ and so, by inductive assumption, $A \models c_1 \approx \underline{n}$ whence $c_1 \approx \underline{n} \in F$ and so $c_1 \varepsilon \underline{m} \in F$, i.e., $A \models c_1 \varepsilon c_2$.

Conversely, by postulate (vi), the formula $A \models c_1 \varepsilon c_2$ implies $c_1 \approx \underline{n} \in F$ for some n in m and the implications above can be reversed.

Sub-subcase (b2) : $c_2 = \sigma$. Here the left-hand side of (iv) is equivalent to $c_1^*[X] = p$ where $p \in X$. Since $\rho(\underline{p}) < \rho(\sigma)$ we can repeat the previous proof. Similarly the right-hand side of (iv) is equivalent to $c_1 \varepsilon \sigma \in F$ from which we infer, using axiom (9) and postulate (vi) that $c_1 \approx \underline{p} \in F$ for some p in X. Hence we obtain $c_1^*[X] = p$ because $c_1 \approx \underline{p} \prec c_1 \varepsilon \sigma$.

Sub-subcase (b3) : $c_2 = c_{\alpha,\phi,\gamma}$ where ϕ is a formula of L, $v \in Fr(\phi)$ and $\gamma \in C_\alpha^{Fr(\phi)-\{v\}}$. In this case the left-hand side of (iv) is equivalent to $c_1^*[X] \in E_{\phi,\gamma^*[X],B_\alpha[X]}$ i.e. to $B_\alpha[X] \models \phi[c_1^*[X], \gamma^*[X]]$. We can replace here ϕ by $\phi^{(\alpha)}$ and $B_\alpha[X]$ by $M[X]$ because the satisfaction of a formula in $B_\alpha[X]$ is equivalent to the satisfaction of the relativized formula $\phi^{(\alpha)}$ in $M[X]$. Now $\phi^{(\alpha)}(c_1,\gamma) \prec c_1 \varepsilon c_2$ because orders of the constants occurring in $\phi^{(\alpha)}(c_1,\gamma)$ are $< \alpha$ and all the unary predicates which occur in $\phi^{(\alpha)}$ have indices α whereas $\rho(c_1 \varepsilon c_2) = \rho(c_2) = \alpha + 1$. Hence we can use the inductive assumption and obtain $A \models \phi^{(\alpha)}[c_1,\gamma]$, i.e., $\phi^{(\alpha)}(c_1,\gamma) \in F$. Using axiom (4) we obtain $c_1 \varepsilon c_{\alpha,\phi,\gamma} \in F$ which is the same as $c_1 \varepsilon c_2 \in F$.

All these steps can obviously be reversed.

Formula (iv) is thus proved in case 1.

Case 2. $\phi(\gamma)$ is the formula $c_1 \approx c_2$. We can assume that $\rho(c_1) \leq \rho(c_2) = \alpha + 1$. The left-hand side of (iv) is equivalent to $c_1^*[X] = c_2^*[X]$ i.e., to $\forall x_{B_\alpha[X]} (x \in c_1^*[X] \equiv x \in c_2^*[X])$ which in turn is equivalent to $\forall c_{C_\alpha} (c^*[X] \in c_1^*[X] \equiv c^*[X] \in c_2^*[X])$. Now we notice that if $c \in C_\alpha$ and $\rho(c_1) \leq \alpha$ then $\rho(c \mathrel{\varepsilon} c_1) < \alpha + 1 = \rho(c_1 \approx c_2)$; if $\rho(c_1) = \alpha + 1$ then the formulae $c \mathrel{\varepsilon} c_1$ and $c_1 \approx c_2$ have the same orders but $c \mathrel{\varepsilon} c_1 \prec c_1 \approx c_2$ according to the definition of $\prec$. Thus in both cases $c \mathrel{\varepsilon} c_1 \prec c_1 \approx c_2$. Similarly $c \mathrel{\varepsilon} c_2 \prec c_1 \approx c_2$. It follows now by the inductive assumption that the left-hand side of (iv) is equivalent to $\forall c_{C\alpha} A \models (c \mathrel{\varepsilon} c_1 \equiv c \mathrel{\varepsilon} c_2)$. From axiom (5) we see that $A \models c_1 \approx c_2$ implies the formula $A \models (c \mathrel{\varepsilon} c_1 \equiv c \mathrel{\varepsilon} c_2)$. Hence the right-hand side of (iv) implies the left-hand side.

It remains to prove that if $A \not\models c_1 \approx c_2$ then there is a c in C_α such that $A \not\models c \mathrel{\varepsilon} c_1 \equiv c \mathrel{\varepsilon} c_2$.

Let us assume $A \not\models c_1 \approx c_2$, i.e., $\neg(c_1 \approx c_2) \in F$. Using axiom (5) and property 5 of closed sets we obtain a constant c' such that either $c' \mathrel{\varepsilon} c_1 \in F$ and $\neg(c' \mathrel{\varepsilon} c_2) \in F$ or $\neg(c' \mathrel{\varepsilon} c_1) \in F$ and $c' \mathrel{\varepsilon} c_2 \in F$. We can limit ourselves to the first case only. We use postulate (v) and infer from $c' \mathrel{\varepsilon} c_1 \in F$ that $c' \approx c \in F$ and $c \mathrel{\varepsilon} c_1 \in F$ for some $c \in C_\alpha$. Hence $c \mathrel{\varepsilon} c_1 \in F$ and $\neg(c \mathrel{\varepsilon} c_2) \in F$ and therefore $A \not\models (c \mathrel{\varepsilon} c_1 \equiv c \mathrel{\varepsilon} c_2)$.

Case 3. $\phi(\gamma)$ is the formula $V_\alpha c$. The left-hand side of (iv) is equivalent to $c^*[X] \in B_\alpha[X]$, i.e., to $c^*[X] = c_1^*[X]$ for some c_1 in C_α. Since $c \approx c_1 \prec V_\alpha c$, according to the definition of $\prec$, we obtain $A \models c \approx c_1$, i.e., $c \approx c_1 \in F$. By axiom (3) $V_\alpha c_1 \in F$ whence $V_\alpha c \in F$ and therefore $A \models V_\alpha c$. Conversely, if $A \models V_\alpha c$, then, by postulate (iv), $c \approx c_1 \in F$ for some c_1 in C_α and the previous steps can be reversed.

The proof of (iv) is thus complete.

In order to finish the proof of Lemma 8.4 we remark that (iv) implies the equivalences

$$M[X] \models c_1^*[X] \in c_2^*[X] \equiv A \models c_1 \mathrm{E}\, c_2,$$

$$M[X] \models c_1^*[X] = c_2^*[X] \equiv A \models c_1 \mathrm{I}\, c_2,$$

$$M[X] \models c^*[X] \in B_\alpha[X] \equiv A \models A_\alpha(c).$$

These equivalences show that $M[X]$ is isomorphic to A/I.

Lemma 8.4 shows that we can obtain filters by constructing closed sets of sentences. In the next section we shall construct such a set and then show that the resulting filter X is reducible to M.

9. *Construction of a closed set* F.

We consider a partially ordered set P as described in Section 2 and denote by $\mathcal{X}$ the space of its maximal filters.

Let K be a σ-additive field of subsets of $\mathcal{X}$ and I a σ-additive ideal in K. For each sentence ϕ of RL we put

$$F_\phi = \{X \in \mathcal{X} : M[X] \models \phi\}.$$

We shall assume that K and I have the following properties:

(A) $F_\phi \in K$ for each sentence ϕ of RL,

(B) If $p \in P$ then $[p] \notin I$,

(C) If $H \in K - I$ then there is a p in P such that $[p] - H \in I$,

(D) For each formula ϕ of RL the binary relation $[p]-F_{\phi(\gamma)} \in I$, where $p \in P$, and $\gamma \in C^{Fr(\phi)}$, is definable in M.

We shall show in Sections 10-12 that (A) - (D) are satisfied if K is the field of Borel sets in $\mathcal{X}$ and I the ideal of meager sets.

The binary relation from condition (D) will be written as $p \Vdash \phi(\gamma)$ and read "p forces $\phi(\gamma)$".

We note some simple consequences of the definitions and assumptions (A) - (D).

LEMMA 9.1

If $p,q \in P$ *and* $p \nless q$, *then* $[p] - [q] \notin I$.

Proof. There is r in P such that $r < p$ and r and q are incompatible, hence $[r] \subseteq [p] - [q]$ and thus if $[p] - [q]$ were in I we would have a contradiction with (B).

LEMMA 9.2

$$F_{\neg\phi} = \mathcal{X} - F_{\phi} \ ; \qquad F_{\phi \& \psi} = F_{\phi} \cap F_{\psi} \ ;$$

$$F_{\forall x\ \phi} = \bigcap_{c \in C} F_{\phi(c)} \ , \quad F_{\forall_{\alpha} x\ \phi} = \bigcap_{c \in C_{\alpha}} F_{\phi(c)} .$$

Proof results immediately from the definition of F_{ϕ}.

Let us arrange in an infinite sequence $\{\phi_n\}_{n\in\omega}$ all sentences of RL. Let $\{\psi_n\}_{n\in\omega}$ be a sequence consisting of all axioms (1) - (12). For any finite set S of sentences we denote by $\bigwedge S$

their conjunction and by $\{S,\alpha,\beta,\dots,\gamma\}$ the set $S \cup \{\alpha,\beta,\dots,\gamma\}$. In order to construct a closed set we try to define an increasing sequence $\{F_n\}_{n\in\omega}$ of finite sets of sentences such that the following requirements be met for each $n > 0$:

(R_1) F_n is consistent,

(R_2) $\psi_{n-1} \in F_n$,

(R_3) either ϕ_{n-1} or $\neg\phi_{n-1}$ belongs to F_n,

(R_4) the pair (F_{n-1},F_n) satisfies the postulates.

It is clear that if these requirements are met, the union $\cup F_n$ will be closed.

LEMMA 9.3

There are infinite sequences $\{p_n\}_{n\in\omega},\{F_n\}_{n\in\omega}$ *consisting of conditions and finite sets of sentences respectively such that, for each integer* n, $p_n \Vdash \wedge F_n$ *and* F_n *satisfies the requirements* (R_1) - (R_4).

Proof. For $n = 0$ we take $F_0 = \emptyset$ and define p_0 to be any element of P. Let us assume that p_n and F_n are already constructed. We construct F_{n+1} by adjoining to F_n several sentences. First of all we adjoin ψ_n. Since ψ_n is true in $M[X]$ for each X, the formula $p_n \models \wedge\{F_n,\psi_n\}$ continues to hold.

Next we try to adjoin ϕ_n or $\neg\phi_n$ to $\{F_n,\psi_n\}$. For each X in $\mathfrak{X}$ we either have $M[X] \models \phi_n$ or $M[X] \models \neg\phi_n$. Thus the set $[p_n]$ decomposes into two parts consisting of maximal filters X for which the former or the latter formula holds. Both of these parts belong to K but it cannot be the case that both of them are in I

since otherwise $[p_n]$ itself would belong to I. Hence one of these parts, call it H, is in $K - I$ and hence by (C) there is a $p'_n < p_n$ such that $[p'_n] - H \in I$. We denote by F'_n the set $\{F_n, \psi_n, \phi_n\}$ if ϕ_n is true in the models $M[X]$ where $X \in H$, and the set $\{F_n, \psi_n, \neg\phi_n\}$ if $\neg\phi_n$ is true in these models. Thus we obtain

$$p'_n \Vdash \wedge F'_n$$

and F'_n satisfies the requirements $(R_1) - (R_3)$.

In order to satisfy the requirement (R_4) we have still to add various sentences to F'_n and restrict, if necessary, the condition p'_n.

Let us first add new sentences to F'_n so as to obtain a set which together with F_n satisfies the postulate (ii). To achieve this we enumerate the existential sentences (i.e. sentences beginning with the symbols $\neg\forall v_j$ which belong to F_n. Let these sentences be

$$(*) \qquad \exists w\theta, \exists w'\theta', \ldots, \exists w_k\theta^{(k)}.$$

For each X in $[p'_n]$ the sentence $\exists w\theta$ is true in $M[X]$ and so for each X there exists a constant c_X such that $M[X] \models \theta(c_X)$. The set $[p'_n]$ is thus decomposed into a denumerable union of sets $S_c = \{X \in [p'_n] : M[X] \models \theta(c)\}$. By (A) these sets belong to K but it cannot be the case that they are all elements of I. Hence there is a constant c such that $S_c \notin I$ and hence by (C) there is a condition $p''_n < p'_n$ such that $[p''_n] - S_c \in I$. It follows that $p''_n \Vdash \theta(c)$. Thus adjoining $\theta(c)$ to F'_n we obtain a set F''_n such that $p''_n \Vdash \wedge F''_n$. Repeating this process again k times we finally obtain a set $F_n^{(k)}$ and a condition $p_n^{(k)}$ such that

$p_n^{(k)} \Vdash \wedge F_n^{(k)}$ and $F_n^{(k)}$ contains sentences $\theta(c)$, $\theta'(c')$, ..., $\theta^{(k)}(c^{(k)})$ for each of the formulae (*). Thus the pair $(F_n, F_n^{(k)})$ satisfies postulate (ii).

The procedure for the remaining postulates is very similar. In case of postulate (iii) we consider limited existential sentences, i.e. sentences of the form $\exists_\alpha w\ \zeta$ which belong to F_n and for each such sentence find a constant c in C_α and a condition $p_n''' < p_n^{(k)}$ which forces $\zeta(c)$. In case of postulate (iv) we consider sentences of the form $V_\alpha c$ which belong to F_n and find for each such sentence a constant c' in C_α such that $c \approx c'$ can be adjoined to the sets previously constructed. In case of postulate (v) we consider sentences $c \varepsilon c'$ in F_n and find for each of them a constant c'' of order $< \rho(c')$ such that $c \approx c''$ can be adjoined. Finally in case of postulate (vi) we deal with sentences of the form $c \varepsilon \underline{m}$ which belong to F_n and find for each such sentence an element n' of m such that the sentence $c \approx \underline{n}'$ can be adjoined.

Lemma 9.3 is thus proved.

The sequences $\{p_n\}_{n\in\omega}$ and $\{F_n\}_{n\in\omega}$ constructed in Lemma 9.3 determine two filters: one is the filter X_0 generated by the conditions p_n and the other is the filter $X = \{p : \underline{p} \varepsilon \sigma \in F\}$ where $F = \cup F_n$. We shall show that these filters are identical. First we note the useful

LEMMA 9.4

If $\{p_n\}_{n\in\omega}$, $\{F_n\}_{n\in\omega}$ *are sequences satisfying Lemma 9.3 and* $F = \cup F_n$ *then* $\phi \in F \equiv \exists n\ (p_n \Vdash \phi)$ *for each sentence* ϕ *of* RL.

Proof. $\phi \in F \equiv \exists n\ (\phi \in F_n) \rightarrow \exists n\ (p_n \Vdash \phi)$ because the sentence $\wedge F_n \to \phi$ is logically true whenever $\phi \in F_n$. Conversely, if $\phi \notin F$, then, $\neg\phi \in F$ and hence, by the above proof, $p_n \Vdash \neg\phi$ for some n. Assuming $p_m \Vdash \phi$ and putting $k = \max(m,n)$ we would obtain $p_k \Vdash \neg\phi$ and $p_k \Vdash \phi$ which is impossible by (B).

LEMMA 9.5

If $\{p_n\}_{n\in\omega}$ *and* $\{F_n\}_{n\in\omega}$ *are as in Lemma 9.3 and* $F = \cup F_n$ *then the filter* $X = \{p \in P : \underline{p} \, \varepsilon \, \sigma \in F\}$ *is identical with the filter generated by the sequence* $\{p_n\}$; *moreover* X *is maximal.*

Proof. Let $p \in P$. For each Y in $[p]$ we have $p \in Y$ and so $M[Y] \models \underline{p} \, \varepsilon \, \sigma$. Hence $[p] - F_{p\varepsilon\sigma} = \emptyset$ and $p \Vdash \underline{p} \, \varepsilon \, \sigma$. For $p = p_n$ we obtain $p_n \, \varepsilon \, \sigma \in F$. We have thus shown that the filter generated by the p_n's is contained in X.

Next we show that if $p \in X$ then p belongs to the filter generated by the conditions p_n.

Since $\underline{p} \, \varepsilon \, \sigma \in F$ there is an integer n such that the formula $\wedge F_n \to (\underline{p} \, \varepsilon \, \sigma)$ is logically true and so $p_n \Vdash \underline{p} \, \varepsilon \, \sigma$. We claim that $p_n < p$. Otherwise there would exist a $q < p_n$ such that q and p are incompatible. Hence no filter Y in $[q]$ would satisfy $p \in Y$, i.e., the difference $[q] - \{Y \in \mathcal{X} : M[Y] \models \underline{p} \, \varepsilon \, \sigma\}$ would be equal to $[q]$. From $p_n \Vdash \underline{p} \, \varepsilon \, \sigma$ we obtain however $[p_n] - \{Y \in \mathcal{X} : M[Y] \models \underline{p} \, \varepsilon \, \sigma\} \in I$ and so $[q] - \{Y \in \mathcal{X} : M[Y] \models \underline{p} \, \varepsilon \, \sigma\} \in I$ because $[q] \subseteq [p_n]$. Thus we would obtain the result that $[q] \in I$ which is impossible.

Finally we show that X is maximal. Let Y be a filter in P such that $X \subset Y$ and assume that $p \in Y - X$. Hence $\neg(\underline{p} \, \varepsilon \, \sigma) \in F$ and therefore $p_n \Vdash \neg(\underline{p} \, \varepsilon \, \sigma)$ for an integer n. Since

$p \in Y$, the conditions p and p_n are compatible; let $r < p_n$ and $r < p$. Since $[r] - F_{\neg(\underline{p}\varepsilon\sigma)} \subseteq [p_n] - F_{\neg(\underline{p}\varepsilon\sigma)}$ we obtain $[r] \cap F_{\underline{p}\varepsilon\sigma} \in I$. On the other hand the relation $r < p$ proves that for each Y in $[r]$ the formula $p \in Y$ and hence also the formula $M[Y] \models \underline{p}\ \varepsilon\ o$ is true. Thus $[r] - F_{\underline{p}\varepsilon\sigma} = \emptyset$, $[r] \subseteq F_{\underline{p}\varepsilon\sigma}$ which together with the previous relation shows that $[r] \in I$. Since this contradicts the assumption (B), Lemma 9.5 is proved.

Taking our lemmas together we obtain

THEOREM 9.6

There exists a sequence $\{p_n\}_{n\in\omega}$ *of conditions such that the set* $F = \{\phi : \exists n\ p_n \Vdash \phi\}$ *is closed. The filter* $X = \{p \in P : \underline{p}\ \varepsilon\ \sigma \in F\}$ *is reducible to* M *and maximal. It is identical with the filter generated by the conditions* p_n.

Proof. Let $\{p_n\}_{n\in\omega}$ and $\{F_n\}_{n\in\omega}$ be the sequences constructed in Lemma 9.3 and put $F = \cup F_n$. From 9.3 it follows that F is closed and from 9.4 that it coincides with the set $\{\phi : \exists n\ (p_n \Vdash \phi)\}$. From 8.4 it follows that $M[X] \models \phi[\gamma^*[X]] \equiv \phi(\gamma) \in F \equiv \exists n\ p_n \Vdash \phi(\gamma)$. Since each p_n belongs to X and each element p of X is $> p_n$ for some n we can write this condition as $\exists p \in X\ p \Vdash \phi(\gamma)$. In view of the assumption (D) there is a formula Φ_ϕ such that $p \Vdash \phi(\gamma) \equiv M \models \Phi_\phi[p,\gamma]$ and so X is reducible to M. The remaining two statements follow from 9.5.

10. Verification of assumptions (A), (B) *and* (C)

Let K be the field of Borel subsets of $\mathcal{X}$ and I the ideal of meager sets. We are going to prove that the assumptions

(A) - (D) of Section 9 are satisfied.

(B) follows from Baire category theorem (see 2.5). The proof of (C) is as follows: Each non-meager Borel set H has the form $(G - N) \cup N'$ where N, N' are meager and G is open and not empty (see Kuratowski [66], p.88). Hence if $[p] \subset G$, then $[p] - H \subseteq N$ and therefore $[p] - H$ is meager.

Proof of (A). From Lemma 9.2 it follows that if F_ϕ is Borel for each sentence ϕ containing less than n symbols for logical operations then it is true for the case when ϕ contains n such symbols. Thus it is sufficient to prove (A) for atomic sentences. It is more convenient to prove it more generally for limited sentences. We show that if ϕ is a limited sentence and for each $\psi \prec \phi$ the set F_ψ is Borel then so is F_ϕ.

The case when ϕ has no predecessors with respect to $\prec$ is trivial because ϕ is then one of the sentences $\underline{0} \in \underline{0}$, $\underline{0} \approx \underline{0}$, $V_0\underline{0}$ and F_ϕ is either the void set or the whole space $\mathcal{X}$.

Now let us assume that ϕ has predecessors. The cases when ϕ contains symbols for logical operations can be disposed of as above. Let ϕ now be atomic. We have three cases to consider.

1) $\phi = c_1 \varepsilon c_2$. We distinguish two subcases:

1a) $\rho(c_1) > \rho(c_2)$,

1b) $\rho(c_1) < \rho(c_2)$

Subcase 1a). Put $\rho(c_2) = \alpha$. By definition $X \in F_\phi \equiv c_1{*}[X] \in c_2{*}[X] \equiv \exists c \in C_\alpha[(c_1{*}[X] = c{*}[X]) \;\&\; (c{*}[X] \in c_2{*}[X])]$ whence

$$F_\phi = \bigcup_{c \in C_\alpha} (F_{c \varepsilon c_2} \cap F_{c \approx c_1}).$$

Since $c \varepsilon c_2$ and $c \approx c_1$ precede $c_1 \varepsilon c_2$ in the ordering $\prec$ the result follows by inductive assumption.

Subcase 1b). If $c_2 = \underline{m}$ then we show similarly that $F_\phi = \cup\{F_{c_1 \approx \underline{n}} : n \in m\}$ whence the result follows because $c_1 \approx \underline{n} \prec c_1 \varepsilon c_2$.

If $c_2 = \sigma$ then $X \in F_\phi$ is equivalent to $\exists p \in P\, [X \in [p]\ \&\ (c_1^*[X] = p)]$. Hence $F_\phi = \bigcup_{p \in P} ([p] \cap F_{c_1 \approx \underline{p}})$ and F_ϕ is Borel because $c_1 \approx \underline{p} \prec c_1 \varepsilon \sigma$.

If $c_2 = c_{\alpha,\phi,\gamma}$ then $X \in F_\phi$ is equivalent to $X \in F_{\phi^{(\alpha)}(c_1,\gamma)}$ and again the inductive assumption is applicable because $\phi^{(\alpha)}(c_1,\gamma) \prec c_1 \varepsilon c_2$.

Case 2. $\phi = c_1 \approx c_2$. We can assume that $\rho(c_1) \leq \rho(c_2) = \alpha$. The relation $X \in F_\phi$ is equivalent to

$$X \in \bigcap_{c \in C_\alpha} [F_{c \varepsilon c_1} \cap F_{c \varepsilon c_2}] \cup [(\mathfrak{X} - F_{c \varepsilon c_1}) \cap (\mathfrak{X} - F_{c \varepsilon c_2})]$$

whence we reduce the theorem to the case 1.

Case 3. $\phi = V_\alpha c$. Put $\rho(c) = \alpha$. In this case $X \in F_\phi$ is equivalent to $c^*[X] = c_1^*[X]$ for some $c_1 \in C_\alpha$ and hence $F_\phi = \bigcup_{c_1 \in C_\alpha} F_{c \approx c_1}$ whence the theorem is reduced to Case 2. Assumption (A) is thus verified.

Before verifying assumption (D) we must establish some properties of the forcing relation.

11. Properties of the forcing relation

We denote by ϕ, ψ sentences of RL and by p, q, r elements of P. By $\phi(v,w,\ldots)$ we denote formulae of RL all of whose free variables are among $v, w, \ldots$.

LEMMA 11.1

If $p < q$ *and* $q \Vdash \phi$ *then* $p \Vdash \phi$.

Proof. $[p] - F_\phi \subseteq [q] - F_\phi$; hence if the right-hand side belongs to I then so does the left.

LEMMA 11.2

If $\phi \to \psi$ *is logically valid then* $p \Vdash \phi$ *implies* $p \Vdash \psi$.

Proof. $F_\phi \subseteq F_\psi$ and hence $[p] - F_\psi \subseteq [p] - F_\phi$.

LEMMA 11.3

If ϕ *and* ψ *are logically equivalent then* $p \Vdash \phi$ *is equivalent to* $p \Vdash \psi$.

This follows from 11.2.

LEMMA 11.4

$p \Vdash \phi \,\&\, \psi$ *is equivalent to* $(p \Vdash \phi) \,\&\, (p \Vdash \psi)$.

Proof. $[p] - F_{\phi\&\psi} = [([p] - F_\phi) \cup ([p] - F_\psi)]$; it is now sufficient to note that the union of two sets belongs to I if and only if each of these sets does.

LEMMA 11.5

$p \Vdash \neg\phi$ *is equivalent to* $\forall q_{<p}\ (q \nVdash \phi)$.

Proof. The left-hand side is equivalent to $[p] \cap F_\phi \in I$; hence if the left-hand side is true and $q < p$ then $[q] \cap F_\phi \in I$

and so $[q] - F_\phi \notin I$ by (B). If the left-hand side is false then by (C) there is q such that $[q] - ([p] \cap F_\phi) \in I$ and we obtain $([q] - [p] \cup ([q] - F_\phi) \in I$. Since by (C) $[q] - [p] \in I$ if and only if $q < p$ we finally obtain $q < p$ and $q \nVdash \phi$, i.e., the right-hand side is false.

LEMMA 11.6

$p \Vdash \forall v.\ \phi(v)$ *is equivalent to* $\forall c \in C (p \Vdash \phi(c))$.

Proof. $[p] - F_{\forall v\ \phi(v)} = \bigcup_{c \in C}([p] - F_{\phi(c)})$ by 9.2. Using the σ-additivity of I we infer that this union belongs to I if and only if each of its members does.

LEMMA 11.7

$p \Vdash c \varepsilon \underline{m}$ *is equivalent to* $\forall q < p\ \exists r < q\ \exists n \in m\ (r \Vdash c \approx \underline{n})$.

Proof. The left-hand side is equivalent to $\forall q < p\ (q \Vdash c \varepsilon \underline{m})$ and hence to $\forall q < p\ ([q] - \bigcup_{n \in m} F_{c \approx \underline{n}} \in I)$ because $F_{c\varepsilon\underline{m}} = \bigcup_{n \in m} F_{c \approx \underline{n}}$. It follows that there exists an element n of m such that $[q] \cap F_{c\approx\underline{n}} \notin I$ and hence, by (C), there is a condition r such that $[r] - ([q] \cap F_{c\approx\underline{n}}) \in I$. This proves that $[r] \subseteq [q]$ and $[r] - F_{c\approx\underline{n}} \in I$, i.e., the left-hand side implies the right.

If the left-hand side is false then $[p] - F_{c\varepsilon\underline{m}} \notin I$ and hence, by (C), there is a q such that $[q] - ([p] - F_{c\varepsilon\underline{m}}) \in I$. It follows that $q < p$ and $[q] \cap F_{c\varepsilon\underline{m}} \in I$. Since $F_{c\approx\underline{n}} \subseteq F_{c\varepsilon\underline{m}}$ this proves that $[q] \cap F_{c\approx\underline{n}}$ is in I for each n in m and the same is true for each $r < q$.

LEMMA 11.8

is equivalent to $p \Vdash c \,\varepsilon\, \sigma$ *is equivalent to* $\forall q < p\ \exists r < q\ \exists s > r\ (r \Vdash c \approx \underline{s})$.

Proof. We argue as in the previous proof using the equation $F_{c\varepsilon\sigma} = \bigcup_{s\in\underline{P}} ([s] \cap F_{c\approx\underline{s}})$.

LEMMA 11.9

If ϕ *is a formula of* L, $v \in Fr(\phi)$, $\alpha \in On_M$, $c' \in C_\alpha$ *and* $\gamma \in C_\alpha^{Fr(\phi)-\{v\}}$ *then* $p \Vdash c' \,\varepsilon\, c_{\alpha,\phi,\gamma}$ *is equivalent to* $p \Vdash \phi^{(\alpha)}(c',\gamma)$.

Proof. Putting $c = c_{\alpha,\phi,\gamma}$ we easily show that $F_{c'\varepsilon c} = F_{\phi^{(\alpha)}(c',\gamma)}$.

LEMMA 11.10

$p \Vdash V_\alpha c$ *is equivalent to* $\forall q < p\ \exists r < q\ \exists c' \in C_\alpha\ (r \Vdash c \approx c')$.

Proof uses the same technique as 11.7 and the observation that $F_{V_\alpha c} = \bigcup_{c'\in C_\alpha} F_{c\approx c'}$.

LEMMA 11.11

If $c_1, c_2 \in C_{\alpha+1}$ *then* $p \Vdash c_1 \approx c_2$ *is equivalent to*

$$\forall c \in C_\alpha\{\forall q < p\ [(q \Vdash c \,\varepsilon\, c_1) \to \exists r < q\ (r \Vdash c \,\varepsilon\, c_2)]\ \&$$

$$\forall q < p\ [(q \Vdash c \,\varepsilon\, c_2) \to \exists r < q\ (r \Vdash c \,\varepsilon\, c_1)]\}.$$

Proof. It is immediate that $F_{c_1\approx c_2} = \bigcap_{c\in C} F_{\phi(c)}$ where ϕ is the formula $v \,\varepsilon\, c_1 \equiv v \,\varepsilon\, c_2$. From the σ-additivity of I it follows that $p \Vdash c_1 \approx c_2$ is equivalent to $\forall c \in C_\alpha (p \Vdash \phi(c))$. In order to bring the result to the desired form we express $\phi(c)$ by means of the connectives $\&$ and $\neg$ alone and obtain the sentence

$$\phi'(c) = \neg[\phi_1(c) \mathbin{\&} \neg\phi_2(c)] \mathbin{\&} \neg[\phi_2(c) \mathbin{\&} \neg\phi_1(c)]$$

where $\phi_i(c) = c \varepsilon c_i$ for $i = 1,2$. Since $\phi(c)$ and $\phi'(c)$ are logically equivalent (or more exactly: since $\phi(c)$ is just an abbreviation of $\phi'(c)$), the relations $p \Vdash \phi(c)$ and $p \Vdash \phi'(c)$ are equivalent. We now use Lemmas 11.4 and 11.5 and after easy transformations obtain the desired result.

LEMMA 11.12

If $c \in C_{\alpha+1}$ *and* $d \in C$ *then* $p \Vdash d \in c$ *is equivalent to* $\forall q_{<p} \exists r < q \exists c' \in C_\alpha[(r \Vdash d \approx c') \mathbin{\&} (r \Vdash c' \varepsilon c)]$.

Proof. Similar to that of 11.7 and uses the decomposition $F_{d\varepsilon c} = \cup[F_{d\approx c'} \cap F_{c'\varepsilon c}]$ where c' ranges over C_α.

12. Definability of the forcing relation.

We shall base our proof on the following theorem scheme on definability by transfinite induction. Let U be a subset of M and R a well founded relation which partially orders U. Let us assume that U and R are definable in M and that for each u in U the set of its R-predecessors $R(u) = \{v \in U : v \neq u \mathbin{\&} vRu\}$ belongs to M. Finally let H be a function definable in M which correlates an element of U with each pair a, A where $a \in U$, $A \in M$ and A is a function with domain $R(u)$. Under these assumptions there is a unique function G with domain U such that

$G(u) = H(u, G \upharpoonright R(u))$ for each u in U and this function is definable in M. (Note: $G \upharpoonright R(u)$ is the restriction of the function G to $R(u)$).

This theorem is but an inessential extension of the theorem on definitions by transfinite induction whose proof can be found in many textbooks of set theory. We shall not enter into details of this proof here.

We shall now prove the definability of the forcing relation. If ϕ is a formula of L which contains logical operators then either $\phi = \neg\psi$ or $\phi = \psi \,\&\, \theta$ or $\phi = \forall v\, \psi$ where v is a variable. If the relations $p \Vdash \psi(\gamma)$ and $p \Vdash \theta(\delta)$ are definable in M then so is the relation $p \Vdash \phi(\gamma)$ in view of Lemmas 11.4 - 11.6. Thus in order to verify assumption (D) it is sufficient to prove it for the case of atomic formulae. We shall establish a slightly stronger result:

LEMMA 12.1

The binary relation $p \Vdash \phi$ *where* $p \in P$ *and* ϕ *is a limited sentence of* RL *is definable in* M.

Proof. Let us consider pairs (p,ϕ) where $p \in P$ and ϕ is a limited sentence of RL. The set U of these pairs is definable in M. We order it partially by the following well founded relation R:

$$(p,\phi)R(q,\psi) \equiv \phi \prec \psi \ .$$

Let us put $G(p,\phi) = 0$ or 1 according as $p \Vdash \phi$ or $p \nVdash \phi$. In order to prove that the forcing relation is definable in M it is sufficient to show that the function G is definable in M

and we achieve this by showing that G satisfies a recursive equation $G(p,\phi) = H(p,\phi,G \restriction R(p,\phi))$ where H is a definable function. The proper choice of H becomes clear when we examine Lemmas 12.4 - 12.12. These lemmas show that the forcing relation $p \Vdash \phi$ can be reduced to some forcing relations between elements of P and limited sentences which precede ϕ with respect to the ordering $\prec$; thus these conditions can be expressed by means of the values of G limited to the set $R(p,\phi)$. E.g., if $\phi = \neg\psi$ then $G(p,\phi) = 0$ if and only if $\forall q < p\ [G(q,\psi) = 1]$. Accordingly we put $H(p, \neg\psi, A) = 0$ if and only if $\forall q < p\ (A(q,\psi) = 1)$. For other forms of ϕ the procedure is similar.

We can now give the exact definition of H. Let $a = (p,\phi)$; then $H(a,A)$ is defined for $a \in U$ and $A \in \{0,1\}^{R(a)} \cap M$.

If a has no R-predecessors then ϕ is one of the sentences $\underline{0} \varepsilon \underline{0}$, $V_0\underline{0}$, $\underline{0} \approx \underline{0}$ and we put $H(a,A) = 1$ in the first two cases and $= 0$ in the third.

If $\phi = \neg\psi$ then $H(a,A) = 0 \equiv \forall q < p\ (A(q,\psi) = 1)$;

If $\phi = \psi \,\&\, \theta$ then $H(a,A) = 0 \equiv A(p,\psi) = A(p,\theta) = 0$;

If $\phi = \forall_\alpha v\ \psi$ then $H(a,A) = 0 \equiv \forall c \in C_\alpha (A(p, \psi(c)) = 0)$;

If $\phi = c_1 \varepsilon c_2$ and $\rho(c_1) > \rho(c_2) = \alpha$ then $H(a,A) = 0 \equiv$

$$\forall q < p\ \exists r < q\ \exists c' \in C_\alpha [A(r,c_1 \approx c') = A(r,c' \varepsilon c_2) = 0];$$

If $\phi = c_1 \varepsilon c_2$ and $\rho(c_1) < \rho(c_2)$ and $c_2 = \underline{m}$ then

$$H(a,A) = 0 \equiv \forall q < p\ \exists r < q\ \exists n \in m\ (A(r,c_1 \approx \underline{n}) = 0);$$

If $\phi = c_1 \varepsilon c_2$ and $\rho(c_1) < \rho(c_2)$ and $c_2 = \sigma$ then

$$H(a,A) = 0 \equiv \forall q < p\ \exists r < q\ \exists s > r\ (A(r,c_1 \approx \underline{s}) = 0);$$

If $\phi = c_1 \varepsilon c_2$ and $\rho(c_1) < \rho(c_2)$ and $c_2 = c_{\alpha,\phi,\gamma}$ then

$$H(a,A) = 0 \equiv A(p,\psi^{(\alpha)}(c_1\gamma)) = 0;$$

If $\phi = c_1 \approx c_2$ and $\max(\rho(c_1), \rho(c_2)) = \alpha + 1$ then

$$H(a,A) = 0 \equiv \forall c \in C_\alpha \{\forall q < p\ [A(q,c\ \varepsilon\ c_1) = 1 \vee \exists r < q\ A(r,c\ \varepsilon\ c_2) = 0]\ \&\ \forall q < p\ [A(q,c\ \varepsilon\ c_2) = 1 \vee \exists r < q\ A(r,c\ \varepsilon\ c_1) = 0]\};$$

If $\phi = V_\alpha c$ then $H(a,A) = 0 \equiv \forall q < p\ \exists r < q\ \exists c' \in C_\alpha$

$$(A(r,c \approx c\ ) = 0).$$

The function H is of course definable in M. Using Lemmas 12.4 - 12.12 we prove that $G(a) = H(a, G \restriction R(a))$ for each a in U. Hence G is definable in M and so is the forcing relation because $p \Vdash \phi \equiv G(p,\phi) = 0$ whenever ϕ is a limited sentence. Thus condition (D) is verified.

13. *Additional remarks*

Let $D \subseteq P$ be a set dense in P i.e., such that for every p in P there is a q in D such that $q < p$. We shall say that D is dense in P under p if $\forall q < p\ \exists r \in D\ r < q$.

In theorem 9.9 we established the existence of a sequence $\{p_n\}$ of conditions which has the property that the set $F = \{\phi : \exists n\ p_n \Vdash \phi\}$ is closed. We want to characterize sequences with this property.

THEOREM 13.1

If $\{p_n\}_{n\in\omega}$ *is a sequence such that the set* $F = \{\phi : \exists n\ p_n \Vdash \phi\}$ *is closed then the filter* X *generated by* $\{p_n\}$ *has common elements with every set* D *which belongs to* M *and is dense in* P.

Proof. Let $D \in M$ be a dense set. If the sentence $\exists v\,[(v \varepsilon \underline{D})\ \&\ (v \varepsilon \sigma)]$ belongs to F then there is an element p of M such that $\underline{p} \varepsilon \underline{D} \in F$ and $\underline{p} \varepsilon \sigma \in F$. It follows that for some integer n, $p_n \Vdash \underline{p} \varepsilon \underline{D}$ and $p_n \Vdash \underline{p} \varepsilon \sigma$. The first relation implies $p \in D$ and the second $p_n < p$ (see 11.8). Hence $p \in D \cap X$.

We shall now show that the assumption $\exists v\,[(v \varepsilon \underline{D})\ \&\ (v \varepsilon \sigma)] \notin F$ leads to a contradiction. This assumption implies that the sentence $\forall v\,\neg[(v \varepsilon \underline{D})\ \&\ (v \varepsilon \sigma)]$ belongs to F and thus is forced by a condition p_n from the initially given sequence. Thus for each constant c the condition p_n forces the sentence $\neg[(c \varepsilon \underline{D})\ \&\ (c \varepsilon \sigma)]$. We choose for c the constant $\underline{q}$ where q is an element of D such that $q < p_n$. Using Lemma 11.5 we obtain $q \nVdash (\underline{q} \varepsilon \underline{D})\ \&\ (\underline{q} \varepsilon \sigma)$, i.e., either $q \nVdash \underline{q} \varepsilon \underline{D}$ or $q \nVdash \underline{q} \varepsilon \sigma$. Both these alternatives are clearly false.

A filter X is called generic if $X \cap D \neq \emptyset$ for each set D which is dense in P and belongs to M (more exactly X is called generic in P over M). Theorem 13.1 can thus be expressed as follows: If $\{p_n\}_{n\in\omega}$ is a sequence such that the set $\{\phi : \exists n\ p_n \Vdash \phi\}$ is closed then the filter generated by the p_n's is generic. We shall prove that also the converse of this theorem is true. First we need a lemma:

LEMMA 13.2

A generic filter intersects every set $D \subseteq P$ *which belongs to* M *and is dense under* p *where* p *is any element of the filter.*

Proof. Put $D' = D \cup \{q \in P : q \text{ is incompatible with } p\}$. In view of the separability of P the set D' is dense in P and hence if X is generic then $X \cap D' \neq \emptyset$. Since no element of X is incompatible with p, we obtain $X \cap D \neq \emptyset$.

THEOREM 13.3

If X *is generic then the set* $F = \{\phi : \exists p \in X\ p \Vdash \phi\}$ *is closed.*

Proof. (1) F is consistent. Otherwise there would be a finite set $\phi_1, \ldots, \phi_n$ of sentences in F such that the conjunction ϕ of these sentences is inconsistent, i.e., has no model. By assumption each ϕ_j is forced by a condition p_j in X. Since X is a filter we obtain a condition p in X such that $p < p_j$ for each $j < n$ and so $p \Vdash \phi$. Thus $[p] - F_\phi \in I$, therefore $[p] \cap F_\phi \neq \emptyset$ and we obtain a contradiction because for each Y in F_ϕ the family $M[Y]$ is a model of ϕ.

(2) F is complete. Let ϕ be a sentence and $D = \{p \in P : p \Vdash \phi \text{ or } p \Vdash \neg\phi\}$. In view of the definability of the forcing relation we have $D \in M$. We shall show that D is dense in P. For let q be any condition. By 11.5 if $q \nVdash \neg\phi$ then there is a condition $p < q$ such that $p \Vdash \phi$. Hence either $q \in D$ or some extension p of q is in D.

Since X is generic we obtain now $X \cap D \neq \emptyset$; if p belongs to this intersection, then either p forces ϕ or p forces

$\neg\phi$ whence either ϕ or $\neg\phi$ belongs to F.

(3) The axioms 1 - 12 given in Section 8 belong to F. This is so because these axioms are true in all models $M[Y]$ where Y is any filter; hence they are forced by *any* conditions.

(4) F satisfies the postulates (ii) - (vi). Since the verification is practically the same for all the postulates we shall give the proof only for the postulate (ii). Thus let us assume that $\exists v_n \phi \in F$, i.e., $p \Vdash \exists v_n \phi$ for some p in X. Since $\exists v_n$ is an abbreviation of $\neg\forall v_n\neg$ we can apply Lemmas 11.5 and 11.6 and obtain $\forall q < p\, \exists r < q\, \exists c \in C\; r \Vdash \phi(c)$. This means that the set $D = \{r \in P : \exists c \in C\; r \Vdash \phi(c)\}$ is dense under p and so, since this set belongs to M, we obtain that there is a condition r in $D \cap X$. Hence there is a constant c such that $r \Vdash \phi(c)$ which proves that $\phi(c) \in F$. The postulate (ii) is thus verified.

Theorems 13.1 and 13.3 suggest an alternative method of constructing models. We start with the definition of forcing and establish first of all Lemmas 11.1 - 11.12. Then we define generic filters and prove their existence essentially as in the proof of the Baire theorem. Next we establish Theorem 13.2 obtaining a closed set. Finally we prove that each closed set determines a reducible filter X as we did in Section 8.

This alternative method is essentially the one which was used by Cohen. Most authors follow Cohen by defining forcing from the start by transfinite induction (in the model M). This allows then to avoid the cumbersome verification of condition (D). The only defect of this method is that it is not easy for the beginner to grasp the intuitive meaning of the forcing relation.

The connection between forcing and the concept of meager sets was discovered by Takeuti and Ryll-Nardzewski and we exploited their ideas in the proofs given above.

14. Preservation of cardinals

If $\alpha \in M$ is an ordinal then we say that α is a cardinal of M if there is no element f of M which is a mapping of a smaller ordinal onto α. One can show by examples that a cardinal of M need not be a cardinal of $M[X]$.

We shall derive a sufficient condition for a cardinal of M to remain a cardinal of $M[X]$.

DEFINITION

We denote by $\theta_M(P)$ the least ordinal α of M such that for each set $Q \subseteq P$ consisting of mutually incompatible conditions and such that $Q \in M$ there is in M a one-one mapping of Q into α.

LEMMA 14.1

If $M \vdash \mathrm{ZFC}$ *then* $\theta_M(P)$ *exists.*

Proof. We can formalize in ZFC the proof that for each partially ordered set there is a least cardinal larger than or equal to the cardinal of any set of mutually incompatible elements of P.

LEMMA 14.2

If X *is a generic filter and* α *is a cardinal of* M, $M \vdash \mathrm{ZFC}$ *and* $\alpha > \theta_M(P)$ *then* α *is a cardinal of* $M[X]$.

Proof. Let us assume that there is in $M[X]$ a function f and an ordinal $\beta < \alpha$ such that f maps β onto α. Expressing these facts in RL we obtain a formula $\phi(f,\underline{\alpha},\underline{\beta})$: $\mathrm{Funct}(f)$ & $(\mathrm{Dom}(f) = \underline{\beta})$ & $(\mathrm{Rg}(f) = \underline{\alpha})$ which is true in $M[X]$. Denoting by c a constant such that $c^*[X] = f$ we infer from the assumption that X is generic that there is a condition p_0 in X satisfying $p_0 \Vdash \phi(c,\underline{\alpha},\underline{\beta})$.

Consider now the set

$$S = \cup\{Z_\xi : \xi < \beta\} \text{ where } Z_\xi = \{\eta : \exists p \leq p_0 \;\; p \Vdash \exists v\, [\Pi(v,\underline{\xi},\underline{\eta}) \;\&\; v \in c]\} \cap \alpha.$$

From the definability of $\Vdash$ we see that $Z_\xi \in M$.

The cardinal number of Z_ξ (calculated in M) is $\leq \theta_M(P)$. To see this we correlate (using the axiom of choice) a condition $p \leq p_0$ to each η in Z_ξ so that $p \Vdash \underline{\langle \xi,\eta \rangle} \in c$. Since $p_0 \Vdash \mathrm{Funct}(c)$ it cannot be the case that two compatible p_1, p_2 be correlated to two different ordinals η_1,η_n. Hence the set of conditions correlated with elements of Z_ξ consists of mutually incompatible conditions and hence its cardinal number (in M) is $\leq \theta_M(P)$. Since $\beta < \alpha$ and $\theta_M(P) < \alpha$ it follows that the cardinal number of S is $< \alpha$. On the other hand $\alpha \subseteq S$ because $\mathrm{Rg}(f) = \alpha$ and thus for each η in α there is a ξ in β and a p in X such that $p \Vdash \underline{\langle \xi,\eta \rangle} \in c$. Lemma 14.2 is thus proved.

15. *The independence of* CH

Let $M \vdash \mathrm{ZFC}$, $\alpha > \omega$, $x \in M$. We take as P the set of finite functions p such that $\mathrm{Dom}(p) \subseteq \alpha \times \omega$, $\mathrm{Rg}(p) \subseteq \{0,1\}$.

P is obviously an element of M. We order P by convention: $p < q$ if and only if $p \supseteq q$. All the assumptions which we made in Section 1 are satisfied by P. In particular p, q are compatible if and only if they coincide on the intersection of their domains.

LEMMA 15.1

$$\theta_M(P) = \omega.$$

The proof is due to Cohen but the theorem was already proved in 1941 by Marczewski. In order to prove the lemma we formalize the following reasoning in ZFC.

We consider finite functions with values in $\{0,1\}$ and with domains $\subseteq A$ where A is infinite (in our case $A = \alpha \times \omega$). Let us assume that there is a non-denumerable set R_1 of mutually incompatible such functions. R_1 can be decomposed into a denumerable union of sets, two functions being included in the same set if their domains have the same number of elements. One of these sets is non-denumerable; to save notation we assume that all the functions in R_1 have domains of power exactly k. Let $p_1 \in R_1$. If $q \in R_1$ and $p_1 \neq q$, then p_1 and q are incompatible and this can happen only if there is an element $t = t_q$ in the domain of p_1 such that $t \in \mathrm{dom}(q)$ and $p_1(t) \neq q(t)$. Now there are only finitely many elements t in $\mathrm{dom}(p_1)$ and non-denumerably many q's. Hence there is a non-denumerable family $R_2 \subseteq R_1$ and an element $t_1 \in \mathrm{dom}(p_1)$ such that for each $q \in R_1$ the element t_1 is in $\mathrm{dom}(q)$ and $p_1(t_1) \neq q(t_1)$. Let $p_2 \in R_2$. Hence $\mathrm{dom}(p_2)$ contains t_1. Again we see that there are non-denumerably many elements $q \in R_2$ such that for some fixed t_2, $t_2 \in \mathrm{dom}(p_2)$, $q(t_2) \neq p_2(t_2)$. It cannot be the case that $t_2 = t_1$ because we would then have $q(t_2) = 1 - p_2(t_2) = 1 - p_2(t_1) = 1 - (1 - p_1(t_1))$ since $p_2(t_1) = 1 - p_1(t_1)$. But this is impossible because $q(t_2)$, i.e. $q(t_1)$, is different from $p_1(t_1)$. Thus we see that $\mathrm{dom}(p_2)$ has at least 2 elements t_1 and t_2. Continuing this reasoning we obtain a p_3 and a non-denumerable subset R_3 of R_2 such that p_3 has at least 3 elements in its domain. After $k + 1$ steps we arrive at a p_{k+1} in $R_k \subseteq R_1$ with $k+1$ elements in its domain which contradicts our assumption.

THEOREM 15.2

There are models in which CH *is false.*

Proof. Let $M \Vdash \mathrm{ZFC}$ be denumerable; take any cardinal α of M which is greater than the first uncountable cardinal of M. Define P as in the lemma above and let X be generic in P. We are going to prove that $M[X] \Vdash \neg\mathrm{CH}$. Since $M[X]$ and M have the same cardinals it is sufficient to show that $M[X]$ has an element f which is a function with domain α, with range $\subseteq 2^{\omega} \cap M[X]$ and which is an injection.

We obtain f by taking the union $\Phi = \cup X$ which is a mapping of $\alpha \times \omega$ into $\{0,1\}$ and putting $f(\xi) = \Phi_{\xi}$ where $\Phi_{\xi}(n) = \Phi(\xi,n)$. Obviously $\Phi \in M[X]$ and so $f \in M[X]$. It remains to prove that f is an injection, i.e. $f(\xi) \neq f(\eta)$ for $\xi \neq \eta$. Let us assume that this is not the case, i.e. that there are $\xi, \eta < \alpha$ such that $\xi \neq \eta$ and $\Phi(\xi,n) = \Phi(\eta,n)$ for each n. The following formula of RL expresses this fact (i.e. is true in $M[X]$):

$$\forall v_0\ \forall v_1\ \{v_0\ \varepsilon\ \sigma \rightarrow [\langle \underline{\xi}, v_1, \underline{0}\rangle\ \varepsilon\ v_0 \equiv \langle \underline{\eta}, v_1, \underline{0}\rangle\ \varepsilon\ v_0]\}$$

(as before we assumed here that ordered pairs and triplets can be defined by formulae of RL). It will be more convenient to write this formula as

(*) $$\forall v_0 \; \forall v_1 \; \neg\{(v_0 \; \varepsilon \; \sigma) \; \& \; \neg\psi\}$$

where ψ is the formula $\langle \underline{\xi}, v_1, \underline{0} \rangle \; \varepsilon \; v_0 \equiv \langle \underline{\eta}, v_1, \underline{0} \rangle \; \varepsilon \; v_0$. Since (*) is assumed to be true in $M[X]$ there is a p in X which forces this formula. Using Lemmas 11.4 and 11.5 we infer that for each q in P and each n in ω

(**) $$p \Vdash \neg\{(\underline{q} \; \varepsilon \; \sigma) \; \& \; \neg\psi(\underline{q}, \underline{n})\}.$$

We now notice that the domain of p is finite and thus there is n such that neither $\langle \xi, n \rangle$ nor $\langle \eta, n \rangle$ are in dom(p). Adding $\langle \xi, n, 1 \rangle$ and $\langle \eta, n, 0 \rangle$ to p we obtain a condition $q < p$. Apply now to (**) the Lemma 11.4. We obtain

$$q \nVdash [\underline{q} \; \varepsilon \; \sigma \; \& \; \neg\psi(\underline{q}, \underline{n})]$$

i.e. $q \nVdash (\underline{q} \; \varepsilon \; \sigma)$ or $q \nVdash \neg\psi(\underline{q}, \underline{n})$ which is equivalent to $q \nVdash (\underline{q} \; \varepsilon \; \sigma)$ or $\exists r < q \;\; r \Vdash \psi(\underline{q}, \underline{n})$. The first part of this disjunction is obviously false. The second is false too because for each $r < q$ and each Y in $[r]$ the truth value of $\langle \underline{\xi}, \underline{n}, \underline{0} \rangle \; \varepsilon \; \underline{q}$ in $M[Y]$ is "false" and that of $\langle \underline{\eta}, \underline{n}, \underline{0} \rangle \; \varepsilon \; \underline{q}$ is "true" and so $r \nVdash \psi(\underline{q}, \underline{n})$.

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Some Impredicative Definitions in the Axiomatic Set-Theory.

By

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Let (S) denote the Zermelo-Fraenkel set-theory based on the following axioms

(A_1) $(x_1, x_2)\,[(x_3)\,(x_3 \in x_1 \equiv x_3 \in x_2) \supset x_1 = x_2]$,

(A_2) $(x_1, x_2)\,(\exists x_3)\,(x_4)\,[x_4 \in x_3 \equiv (x_4 = x_1 \vee x_4 = x_2)]$,

(A_3) $(x_1)\,(\exists x_2)\,(x_3)\,[x_3 \in x_2 \equiv (x_4)\,(x_4 \in x_3 \supset x_4 \in x_1)]$,

(A_4) $(x_1)\,(\exists x_2)\,(x_3)\,[x_3 \in x_2 \equiv (\exists x_4)\,(x_3 \in x_4 \cdot x_4 \in x_1)]$,

(A_5) $(\exists x_1)\,(\exists x_2)\,(x_2 \in x_1 \cdot (x_2)\,\{x_2 \in x_1 \supset (\exists x_3)\,[x_2 \neq x_3 \cdot x_3 \in x_1 \cdot (x_4)\,(x_4 \in x_2 \supset x_4 \in x_3)]\})$,

(A_6) $(x_k)\,(x_{k_1}, \ldots, x_{k_p})\,\{(x_l)\,[x_l \in x_k \supset (\exists x_m)\,(x_n)\,(\Phi \equiv x_n = x_m)] \supset \supset (\exists x_q)\,(x_n)\,[x_n \in x_q \equiv (\exists x_l)\,(x_l \in x_k \cdot \Phi)]\}$,

(A_7) $(x_{k_1}, \ldots, x_{k_p})\,\{(\exists x_k)\Phi \supset (\exists x_k)\,[\Phi \cdot (x_l)\,(x_l \in x_k \supset \sim \Phi')]\}$ 1).

(A_6) and (A_7) are axiom schemata. The letter Φ in (A_6) replaces any expression (with free variables $x_l, x_n, x_{k_1}, \ldots, x_{k_p}$, and x_k 2)) built up according to the following rules: If i and j are integers, then $x_i \in x_j$ and $x_i = x_j$ are formulas; if Θ is a formula and j an integer, then $(\exists x_j)\Theta$ is a formula; if Θ and Z are formulas, then so is $\Theta | Z$ 3). We assume that x_q is not free in Φ.

The letter Φ in (A_7) replaces a formula with free variables $x_k, x_{k_1}, \ldots, x_{k_p}$ and Φ' replaces the formula resulting from Φ by substitution of the letter x_l for x_k on every place where x_k is free in Φ. It is supposed that x_l is not bound in Φ.

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1) (A_1) is the axiom of extensionality, (A_2) — the pair-axiom, (A_3) — the powerset axiom, (A_4) — the sum-set axiom, (A_5) — the axiom of infinity, (A_6) — the axiom of replacement, and (A_7) — the restrictive axiom (the „Axiom der Fundierung" of Zermelo).

2) x_k must not necessarily be a free variable of Φ.

3) Other logical connectives can be defined by the stroke | in the well-known manner.

We assume in (S) the well-known rules of proof, namely the *modus ponens*, the rule of substitution and the rules of omission and of introduction of quantifiers. Furthermore we assume special rules which enable us to prove every tautological formula including the identity-symbol:

R_1. *If Φ, Ψ, and Θ are formulas, then the formulas*

$$(\Phi\supset\Psi)\supset[(\Psi\supset\Theta)\supset(\Psi\supset\Theta)],\quad \Phi\supset(\sim\Phi\supset\Psi),\quad (\sim\Phi\supset\Phi)\supset\Phi$$

are provable.

R_2. *The formula $x_k = x_k$ is provable.*

R_3. *If Φ is a formula, x_l is not a bound variable of Φ and Φ' differs from Φ only by containing free occurrences of x_l on one or several places where Φ contains free occurrences of x_k, then the formula*

$$x_k = x_l \supset (\Phi \equiv \Phi')$$

is provable.

Let (S') be the Bernays-Gödel system of set-theory. We shall not describe the details of this system because it is sufficiently well known from the literature[4]. We remark only that every expression meaningful in (S) is also meaningful in (S') and every axiom of (S) is provable in (S').

It has been proved by Novak[5] that if (S) is consistent, then (S') is also consistent[6]. Since this proof is formalizable in (S'), it follows that the consistency of (S) cannot be proved in (S'). On the other hand (S') arises from (S) by addition of variables of the next higher type and therefore the so-called definition of truth for (S) is formalizable in (S')[7]. Since the „whole theory of truth" makes it possible to prove the consistency of a system for which the notion of satisfaction has been defined[8], we infer that certain properties of the notion of truth for (S) cannot be established in (S').

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[4] See, e. g., Bernays [1] or Gödel [2].

[5] See Novak [3] and Rosser-Wang [5].

[6] It can even be shown that every formula provable in (S') and expressible in (S) must be provable already in (S). We give here a simple proof based on results established by Novak [3]. Suppose that Φ is expressible in (S), provable in (S') but not provable in (S). Let (S_1) be the system got from (S) by addition of $\sim\Phi$ as a new axiom. Then (S_1) is consistent and the corresponding system (S_1') obtained from (S_1) by the method described by Novak must be consistent too. On the other hand (S_1') is at least as strong as (S') and therefore Φ is provable in (S_1') which is a contradiction because $\sim\Phi$ is evidently provable in (S_1').

A more elaborate proof is given in Rosser-Wang [5].

[7] See section 1 below.

[8] See Tarski [6], pp. 359, 392.

An exact analysis of this situation leads to the following three theorems the proofs of which will be sketched in this paper:

Theorem I. *There is an expression $V(x_1)$ of (S') with exactly one free variable x_1 such that if Φ is an arbitrary expression of (S) without free variables and n the Gödel number of Φ, then the equivalence*

$$\Phi \equiv V(n)$$

is provable in (S') [9]. *The formula $V(x_1)$ has the form $(\exists X)\, A(X,x_1)$ where $A(X,x_1)$ is a formula without bound class variables. If Φ is a theorem of (S), then $V(n)$ is provable in (S'), but the general theorem*

(x_1) [*x_1 is the Gödel number of a theorem of* $(S) \supset V(x_1)$]

is not provable in (S') provided that (S') is consistent.

Theorem II. *There is an expression $\Theta(x_1)$ of the form $(\exists X)\, B(X,x_1)$ where $B(X,x_1)$ does not contain bound class variables such that the formulas*

$$\Theta(1) \quad \textit{and} \quad (n)\,[\Theta(n) \supset \Theta(n+1)]$$

are both provable in (S') but $(n)\,\Theta(n)$ [10] *is not provable in (S') provided that (S') is consistent.*

Theorem III. *There is an expression $\Phi(x_1)$ of the form $(\exists X)\, C(X,x_1)$ where $C(X,x_1)$ does not contain bound class variables such that the formula*

$$(\exists X)\,(x_1)\,[x_1 \in X \equiv \Phi(x_1)]$$

is not provable in (S') provided that (S') is consistent [11].

1. We begin with the proof of theorem I. In order to construct the formula $V(x_1)$ with the properties required in the theorem we shall formalize in (S') the definition of satisfaction and of truth for (S). Let us therefore recall briefly these definitions.

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[9] $V(n)$ is the expression resulting from $V(x_1)$ by substitution of the n-th numeral (suitably defined in (S')) for the variable x_1.

[10] The variable n ranges over the set of integers defined in (S').

[11] This theorem shows that the system NQ considered by **Wang** [7] is essentially stronger than the system of **Bernays-Gödel**. This follows also from the fact that the consistency of the **Bernays-Gödel** system is provable in NQ.

For every formula Φ of (S) there exists a sequence of formulas

$$(1) \qquad \Phi_1, \Phi_2, \ldots, \Phi_n = \Phi$$

such that for every $i \leqslant n$ Φ_i is either a formula of the form $x_k = x_l$ or of the form $x_k \epsilon x_l$ or one of the following two cases is satisfied:

(2) *there are integers* j, h *less than* i *such that* $\Phi_i = \Phi_j | \Phi_h$,

(3) *there are integers* j *and* m *such that* $j < i$ *and* $\Phi_i = (\exists x_m)\Phi_j$.

A sequence (1) satisfying these conditions will be called a ***construction-sequence*** or briefly a ***C-sequence*** for Φ.

We denote by s_i the set of integers q such that x_q is free in Φ_i.

A ***finite sequence of sets*** is defined as a finite set f of ordered pairs $\langle u,v\rangle$ such that if $\langle u,v\rangle \epsilon f$ and $\langle u,v_1\rangle \epsilon f$, then $v = v_1$. The u's of the pairs $\langle u,v\rangle$ belonging to f form the ***domain*** $D(f)$ of f. If $\langle u,v\rangle \epsilon f$, we write $v = f(u)$. If $s \subset D(f)$, then $f|s$ is the set of pairs $\langle u,v\rangle$ such that $u \epsilon s$ and $\langle u,v\rangle \epsilon f$.

A ***finite sequence of classes*** is defined as a class F of finite sequences of sets with a common domain D which is at the same time called the ***domain of*** F[12]. If $x \epsilon D$, then the class of all y's such that there is a sequence f with the properties $\langle x,y\rangle \epsilon f \epsilon F$ is called the x-th term of F and denoted by F_x. Note that elements of F_x can be arbitrary sets, in particular arbitrary finite sequences of sets.

To each C-sequence (1) we let correspond a finite sequence of classes F with the domain D consisting of all integers $\leqslant n$. If $i \leqslant n$ and Φ_i is the formula $x_k = x_j$ or $x_k \epsilon x_j$, then F_i is the class of all sequences f such that $D(f) = \{k,j\}$ and $f(k) = f(j)$ or $f(k) \epsilon f(j)$.

If Φ_i satisfies the condition (2), then F_i is the class of all sequences f such that $D(f) = s_i$ and either $f|s_j \operatorname{non} \epsilon F_j$ or $f|s_h \operatorname{non} \epsilon F_h$.

Finally if Φ_i satisfies the condition (3) and x_m is free in Φ_i, then F_i is the class of all sequences f such that $D(f) = s_i$ and there exist an integer m and a set a for which $f + \{\langle m,a\rangle\} \epsilon F_j$. If x_m is not free in Φ_j, then we put $F_i = F_j$.

A sequence of classes F which satisfies the above conditions is called an ***S-sequence*** for Φ corresponding to the C-sequence (1).

This definition says of course nothing about the existence of S-sequences.

We say that a sequence f ***satisfies*** Φ if there is an S-sequence F for Φ such that $f \epsilon F_n$.

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[12] See **Robinson** [4].

If Φ has no free variables, then the only sequence which can possibly satisfy Φ is the void sequence 0. If $0 \epsilon F_n$, we say that Φ is *true*, otherwise that Φ is *false*[13]).

It is clear that these definitions can be expressed in (S'). The only difficulty lies in the presence of the meta-mathematical notions „formula", „quantifier" etc. It is however possible to eliminate all these notions in favour of the purely arithmetical ones, using the well-known technique of the Gödel numbers and identifying the „linguistic" concepts with their arithmetical counterparts.

The definition of satisfaction thus formalized in (S') takes on the form of a formula $Stsf(x_1, x_2) \equiv (\exists X) M(X, x_1, x_2)$ where x_1 runs over the set of the Gödel numbers of formulas and x_2 over the class of finite sequences of sets. The definition of truth takes on the form of a formula $V(x_1)$ of the form $(\exists X) A(X, x_1)$:

$$\begin{aligned} V(x_1) &\equiv (\exists x_2)(x_3)[\sim(x_3 \epsilon x_2) \cdot Stsf(x_1, x_2)] \\ &\equiv (\exists X)\{(\exists x_2)(x_3)[\sim(x_3 \epsilon x_2) \cdot M(X, x_1, x_2)]. \end{aligned}$$

We shall now prove a series of lemmas which will lead to the proof of theorem I. All these lemmas are concerned with properties of formulas of (S) and since we wish to state and to prove them in (S') we must explain in a few words in what way such theorems can be expressed in (S').

There are two different ways to express in (S') meta-mathematical theorems about (S). One of them uses the method of Gödel and identifies expressions with their Gödel numbers Instead to say that every expression Φ of this or other class K possesses a property P we say that every integer which is the Gödel number of an expression from the class K possesses the property P' obtained from P by substitution of the arithmetical notions for the corresponding meta-mathematical ones.

Whether this method is applicable or not depends on the nature of the class K and of the property P and notably on the possibility to define K and P by means formalizable in (S').

A theorem about formulas expressed in this way in the symbolism of (S') becomes a single theorem of (S').

Another possible method is to express theorems about formulas of (S) as theorem schemata of (S'). The general theorem *every Φ has the property P* is then expressed in the form of an in-

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[13]) See Tarski [6], pp. 313-314.

finite sequence of theorems of (S') each formula Φ contributing one theorem to the sequence. This method is sometimes advantageous because the arithmetical counterpart P' of P is not always definable in (S') and even if it is definable in (S'), the general theorem described in the preceeding paragraph does not need to be provable in (S') although each formula of the sequence representing the theorem schema is provable in (S'). We shall see later that both situations can actually occur (cf. lemmas Σ_4 and Σ_5).

In order to facilitate our exposition we shall always identify formulas of (S) with their Gödel numbers and shall avoid as far as possible the use of logical symbols. These simplifications, convenient though they are, obliterate sometimes completely the difference between single theorems and theorem schemata of (S'). We shall therefore denote theorems by the letter „T“ and theorem schemata by the letter „Σ“. We remark that proofs of all theorems and of all particular instances of the schemata are based exclusively on the axioms of (S').

T_1. *For every formula of the form* $x_k = x_j$ *or* $x_k \in x_j$ *there exists an S-sequence*

Proof. It is sufficient to take for this sequence a one term sequence F such that F_1 is the class of all sequences $\{\langle k,a\rangle, \langle j,b\rangle\}$ where $a = b$ or $a \in b$.

T_2. *If* Φ *and* Ψ *are two formulas for which there exist S-sequences, then there exists an S-sequence for the formula* $\Phi|\Psi$.

Proof. Let F and G be the S-sequences for the formulas Φ and Ψ and let F_m and G_n be the last terms of these sequences. Finally let s_1 be the set of integers i for which x_i is free in Φ and s_2 the set of integers i for which x_i is free in Ψ. From the axioms of (S') follows easily the existence of the class Z of all sequences f such that $D(f) = s_1 + s_2$ and either $f|s_1$ does not belong to F_m or $f|s_2$ does not belong to G_n. We obtain now an S-sequence H for the formula $\Phi|\Psi$ putting $H_i = F_i$ for $i \leqslant m$, $H_{m+j} = G_j$ for $j \leqslant n$, and $H_{m+n+1} = Z$.

T_3. *If there exists an S-sequence for an expression* Φ, *then there exists also an S-sequence for the expression* $(\exists x_m)\Phi$.

Proof is similar to that of T_2.

Let now n be one of the integers 1,2,3,... Applying T_1, T_2, and T_3 n times we obtain the following theorem schema:

Σ_4. *If* Φ *is a formula of* (S), *then for every C-sequence* (1) *ending with* Φ *there exists a corresponding S-sequence.*

As indicated, Σ_4 is a theorem schema; the existence of the S-sequences stated in the schema is provable for each formula Φ separately. The general theorem

$$(\Phi)\,(\exists F)\quad [F \textit{ is an } S\textit{-sequence for } \Phi]$$

is expressible in (S') but we see no way to prove it from the axioms of (S').

Σ_5. *If* Φ *is an expression of* (S) *with free variables* $x_{k_1},\dots,x_{k_p}$, (1) *a* C*-sequence for* Φ, *and* F *a corresponding* S*-sequence, then*

$$f=\{\langle k_1,x_{k_1}\rangle,\dots,\langle k_p,x_{k_p}\rangle\}\supset(f\in F_n\equiv\Phi). \tag{4}$$

Proof. We proceed by induction with respect to n, the length of the C-sequence (1). If $n=1$, then Φ has either the form $x_{k_1}=x_{k_2}$ or the form $x_{k_1}\in x_{k_2}$ and F is a one termed sequence such that F_1 is the class of all sequences $\{\langle k_1,a\rangle, \langle k_2,b\rangle\}$, where $a=b$ or $a\in b$. Hence (4) becomes in this case one of the tautological formulas

$$f=\{\langle k_1,x_{k_1}\rangle,\ \langle k_2,x_{k_2}\rangle\}\supset(f\in F_1\equiv x_{k_1}=x_{k_2}),$$
$$f=\{\langle k_1,x_{k_1}\rangle,\ \langle k_2,x_{k_2}\rangle\}\supset(f\in F_1\equiv x_{k_1}\in x_{k_2}).$$

Suppose that (4) is provable for formulas with C-sequences shorter than n and let Φ be a formula with a C-sequence (1) of the length n. We have to consider the two cases (2) and (3).

In case (2) we have $\Phi=\Phi_j|\Phi_h$ with $j<n$ and $h<n$. Let s_j be the common domain of sequences from F_j and s_h the common domain of sequences from F_h. By definition of S-sequences we obtain

$$f\in F_n\equiv(f|s_j\ \mathrm{non}\in F_j\vee f|s_h\ \mathrm{non}\in F_h). \tag{5}$$

The inductive assumption gives the equivalences

$$f|s_j\in F_j\equiv\Phi_j,\qquad f|s_h\in F_h\equiv\Phi_h$$

and we obtain from them and from (5)

$$f\in F_n\equiv(\sim\Phi_j\vee\sim\Phi_h)\equiv\Phi_j|\Phi_h\equiv\Phi.$$

In case (3) we have $\Phi=(\exists x_m)\Phi_j$ with $j<n$. We can suppose that x_m is free in Φ_j since otherwise the theorem is trivial. From the inductive assumption we obtain the equivalence

$$f+\{\langle m,x_m\rangle\}\in F_j\equiv\Phi_j$$

and the definition of satisfaction gives another equivalence

$$f\in F_n\equiv(\exists x_m)\,[f+\{\langle m,x_m\rangle\}\in F_j].$$

Both equivalences together entail the equivalence

$$f \in F_n \equiv (\exists x_m)\Phi_f \equiv \Phi.$$

Theorem Σ_5 is thus proved.

The difference between the schemata Σ_4 and Σ_5 lies in the fact that Σ_5 is not expressible in (S') as a single theorem. If we try to express Σ_5 as a single statement of (S'), we must replace the Φ on the right side of the equivalence by its Gödel number and the theorem evidently looses sense because on both sides of an equivalence must stay formulas and not numbers.

2. Theorem Σ_5 shows that the definition of truth which we adopted for the system (S) satisfies the conditions imposed on that notion by Tarski [6], p. 305. Furthermore this fact can be proved in (S') for each particular formula of (S) We shall now analyse the question why the consistency of (S) cannot be proved in (S') although a „good" definition of truth is formalizable in (S'). We shall show that the real source of this illusionary paradox lies in the fact that although every particular instance of the schema

(Σ) *if Φ is provable in (S), then Φ is true*

can be proved in (S'),yet the general theorem

(T) *if Φ is provable in (S), then Φ is true*

cannot be deduced from the axioms of (S').

T_6. *The axioms (A_1)—(A_5) are true.*

This is merely a restatement of the fact that the axioms (A_1)—(A_5) are at the same time axioms of (S'). For the sake of completeness we indicate the method of proof for the axiom (A_1).

The void sequence satisfies the formula (A_1) if and only if every two termed sequence $f=\{\langle 1,a\rangle, \langle 2,b\rangle\}$ satisfies the formula

$$(6) \qquad (x_3)(x_3 \in x_1 \equiv x_3 \in x_2) \supset x_1 = x_2.$$

Applying the definition of satisfaction we infer that f satisfies the formula (6) if and only if it either does not satisfy the formula $(x_3)(x_3 \in x_1 \equiv x_3 \in x_2)$ or does satisfy the formula $x_1 = x_2$. Applying again the definition of satisfaction we transform this condition into an equivalent one as follows: either there is a set c such that the conditions

the sequence $\{\langle 1,a\rangle, \langle 3,c\rangle\}$ *satisfies the formula* $x_3 \in x_1$,

the sequence $\{\langle 2,b\rangle, \langle 3,c\rangle\}$ *satisfies the formula* $x_3 \in x_2$

are not equivalent or f satisfies the formula $x_1 = x_2$.

Applying still once more the definition of satisfaction we reduce the above conditions to the following: either there exists a set c such that $\sim(c \in a \equiv c \in b)$ or $a = b$.

It follows now directly from the axiom (A_1) which is valid in (S') that these conditions are satisfied.

Σ_7. *For every formula Φ of (S) the formulas (A_6) and (A_7) corresponding to the formula Φ are true.*

It will be sufficient to indicate the method of proof for the axiom schema (A_6). Let Φ be a formula of (S) with the free variables $x_l, x_n, x_{k_1}, \dots, x_{k_p}, x_k$. Applying the definition of satisfaction we prove easily that the assertion of Σ_7 is equivalent to the following statement: *If*

(7) *x_k is a set and for every x_l in x_k there exists exactly one set x_n such that Φ,*

then there exists a set x_q such that $x_n \in x_q$ if and only if there is an x_l in x_k such that Φ.

In order to prove this statement let us assume (7). By theorem Σ_5 there exists a class X such that

$$\{\langle k, x_k\rangle, \langle l, x_l\rangle, \langle n, x_n\rangle, \langle k_1, x_{k_1}\rangle, \dots, \langle k_p, x_{k_p}\rangle\} \in X \equiv \Phi.$$

Let U be the class of pairs $\langle x_l, x_n\rangle$ such that

$$\{\langle k, x_k\rangle, \langle l, x_l\rangle, \langle n, x_n\rangle, \langle k_1, x_{k_1}\rangle, \dots, \langle k_p, x_{k_p}\rangle\} \in X.$$

The existence of U (which depends of course on „parameters" $x_k, x_{k_1}, \dots, x_{k_p}$) follows from the class theorem which is valid in (S'). It follows from (7) that for every x_l in x_k there exists exactly one x_n such that $\langle x_l, x_n\rangle \in U$. We apply now to U and x_k the axiom of replacement and obtain a set x_q with the desired properties.

The above argument would remain valid with the letter „Φ" replaced everywhere by „*the sequence* $f = \{\langle k, x_k\rangle, \langle l, x_l\rangle, \langle n, x_n\rangle, \langle k_1, x_{k_1}\rangle, \dots, \langle k_p, x_{k_p}\rangle\}$ *satisfies* Φ" if we only knew that there exists a class X of sequences which satisfy Φ. Hence applying theorems T_2 and T_3 we obtain the following theorem:

T_8. ***If Φ and Ψ are formulas of (S) such that the instances of the axiom schemata (A_6) and (A_7) corresponding to these formulas are true, then the formulas $\Phi | \Psi$ and $(\exists x_m)\Phi$ have the same property.***

As is well known the rules of proof lead from true formulas again to true formulas. To express conveniently this theorem we introduce the notion of valid formulas.

A formula Φ with free variables $x_{k_1},\dots,x_{k_p}$ is called *valid* if every sequence f with the domain $\{k_1,\dots,k_p\}$ satisfies Φ.

T_9. *If Φ and Ψ are two valid formulas, then all formulas resulting from them by the rules of proof are also valid.*

This theorem follows immediately from the fact that the rules of proof admitted in (S) are also valid in (S'). The method of proving this will be exemplified sufficiently well on the following example.

One of the rules states that if the formula $\Phi \supset \Psi$ is already proved and the variable x_m is not free in Ψ, then the formula $(\exists x_m)\Phi \supset \Psi$ can also be considered as proved. Now let us assume that the formula $\Phi \supset \Psi$ is valid but the formula $(\exists x_m)\Phi \supset \Psi$ is not.

Les s_1 be the set of integers i such that x_i is free in Φ and s_2 the set of integers j such that x_j is free in Ψ. From the definition of satisfaction follows the existence of a sequence f such that $f|s_1$ satisfies $(\exists x_m)\Phi$ but $f|s_2$ does not satisfy Ψ. If x_m is not free in Φ we have already a contradiction since $f|s_1$ satisfies Φ but $f|s_2$ does not satisfy Ψ, hence f does not satisfy the formula $\Phi \supset \Psi$ against our assumption that this formula is valid. If x_m is free in Φ then there exists an a such that the sequence $f|s_1 + \{\langle m,a \rangle\}$ satisfies Φ and hence the sequence $f + \{\langle m,a \rangle\}$ does not satisfy the implication $\Phi \supset \Psi$ contrary to the assumption that this implication is valid.

We arrange now all the formulas falling under the schemata (A_6) and (A_7) into an infinite sequence

$$B_1, B_2, B_3, \dots \tag{8}$$

in such a way that the formulas corresponding to the composite expressions $\Phi|\Psi$ and $(\exists x_m)\Phi$ occur later in the sequence than the formulas corresponding to the simple expressions Φ and Ψ.

We shall say that Φ is a *theorem of at most n-th order* of (S) if Φ is provable from the axioms (A_1)—(A_5) and at most n first terms of the sequence (8) by at most n applications of the rules of proof.

From theorems T_8 and T_9 we obtain

T_{10}. *If every theorem of the n-th order is true, then every theorem of the $n+1$-st order is true.*

Using T_6 and applying successively the theorem T_{10}, we see that the following schema contains exclusively formulas provable in (S'):

Σ_{11}. *Every theorem of the n-th order ($n=1,2,\ldots$) is true.*

It follows from this schema that if m is the Gödel number of an arbitrary theorem of (S), then $V(m)$ is provable in (S'). The general theorem however

$$(n)\,(\Phi)\,[(\Phi \textit{ is a theorem of the n-th order}) \supset (\Phi \textit{ is true})]$$

though expressible in (S') is not provable in (S') provided that (S') is consistent. If this theorem were provable in (S'), then the theorem

$$(\Phi)\,[(\Phi \textit{ is a theorem of } (S)) \supset (\Phi \textit{ is true})]$$

would also be provable in (S') and since the theorem

$$(\Phi \textit{ is true}) \supset \sim(\sim\Phi \textit{ is true})$$

is provable in (S'), we would infer that the consistency of (S) is provable in (S'). This however entails the inconsistency of (S) and hence the inconsistency of (S').

In view of Σ_5 and Σ_{11} the last remarks complete the proof of the theorem I. At the same time we have explained why the consistency of (S) is unprovable in (S') in spite of the fact that a satisfactory definition of „truth“ for the system (S) is formalizable in (S').

3. We shall now deduce from the previous results the theorems II and III mentioned in the introduction.

Let $\Theta(x)$ be the formula which we obtain writing in the symbols of (S') the following statement:

x is an integer and every theorem of the x-th order is true.

It follows from T_6 and T_{10} that the formulas

$$\Theta(1) \quad \text{and} \quad (n)\,[\Theta(n) \supset \Theta(n+1)]$$

are provable in (S') whereas the discussion given at the end of section **2** shows that if (S') is consistent, then the general statement $(n)\,\Theta(n)$ is not provable in (S').

As to the form of the formula $\Theta(x)$ it is immediate that it can be written as

$$(9) \qquad (m)\,[m<\varphi(x)\supset(\exists X)\,A(X,m)]$$ [14])

where $\varphi(x)$ is a number-theoretic function such that $\varphi(x)$ exceeds the Gödel numbers of all theorems of the order x at most. Indeed $\Theta(x)$ says that for every theorem of an order $\leqslant x$ there exists an S-sequence such that the void sequence belongs to its last term.

We can now transform the expression (9) into an equivalent one which says that there is a finite sequence Y with the domain consisting of integers less than $\varphi(x)$ and such that if $m\leqslant\varphi(x)$, then the m-th term Y_m of Y satisfies the condition $A(Y_m,m)$.

In this way we give to $\Theta(x)$ the form $(\exists X)\,B(X.x)$ required in theorem II.

Finally we prove theorem III. Let us take as $\Phi(x)$ the formula

x is an integer and $\sim\Theta(x)$.

Suppose that the formula

$$(\exists X)\,(x)\,[x\in X\equiv\Phi(x)]$$

is provable in (S') and consider the class X such that $x\in X\equiv\Phi(x)$. Using the restrictive axiom[15]) we infer that

$$(10) \qquad X=0\vee(\exists x)\,[(x\in X)\cdot(\textit{no element of } x \textit{ is in } X)].$$

Since $\Theta(1)$ is provable in (S'), we obtain $x\in X\supset x\neq 1$. Hence (10) entails[16]) that

$$X=0\vee(\exists x)\,[(x\in X\cdot(x-1 \text{ non } \in X)]$$

and therefore we obtain

$$X=0\vee(\exists x)\,[\Theta(x-1)\cdot\sim\Theta(x)].$$

Since $\Theta(x-1)\supset\Theta(x)$ is provable in (S') we can simplify this formula to $X=0$. But if this formula were provable in (S'), then the formula $(n)\,\Theta(n)$ would also be provable in (S') and hence (S') would be inconsistent. Theorem III is thus proved.

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[14]) The variable m ranges over the set of Gödel numbers of theorems of (S).

[15]) See Bernays [1], axiom VII or Gödel [2], axiom D.

[16]) We recall that in the Bernays theory of integers the *less-than* relation is identical with $\in$. Cf. Bernays [1], pp. 8-9.

4. We conclude with the following general remark which should clarify the intention of the paper.

People working with **Tarski**'s theory of truth generally believe that if a satisfactory definition of truth for a system (or „language“) (S) can be set up in another system („meta-language“) (S'), then the consistency of (S) is provable in (S'). By a satisfactory definition of truth is meant a definition which satisfies the „convention $\mathfrak{W}$“ given on p. 305 of **Tarski** [6].

It follows from theorem 1 proved above that one should be careful making such general statements. If the meta-language of (S) is very weak (though stronger than (S) itself), then as our theorem 1 shows the statement in question can even be false.

In order to be sure that the consistency of (S) is provable in (S') by the methods used in the theory of „truth“, one has to require that the general theorem

T. Each formula provable in (S) *is true,*

be provable in (S'). This is certainly the case if the following two theorems

T'. Each axiom of (S) *is true,*

T''. If Φ *arises from true formulas by means of a rule of proof admitted in* (S), *then* Φ *is true*

are provable in (S') and if the induction principle

if $\Theta(1)$ *and* $(n)[\Theta(n)\supset\Theta(n+1)]$ *are provable in* (S'), *then so is* $(n)\,\Theta(n)$

holds in (S') for arbitrary formulas.

It is entirely conceivable (although I did not succeed to find a suitable example) that for certain systems (S) and (S') both T' and T'' are provable in (S') and yet the consistency of (S) is not provable in (S') because of the lack of a sufficiently strong induction principle in (S').

On the other hand the consistency of (S) is evidently provable in each ω-complete system (S') in which a satisfactory definition of truth for (S) is formalizable. This remark is however of little practical value, since according to the well-known fundamental theorem of **Gödel** no finitary system containing arithmetic of integers is ω-complete.

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Correction to the paper „Some Impredicative Definitions in the Axiomatic Set-Theory" by Andrzej Mostowski.

Dr Hao Wang from the University of Harvard has called my attention to the fact that a statement made on p. 118 of my cited paper is incorrect.

Contrary to what is said in the quoted place the scheme

(Σ) *if Φ is provable in* (S), *then Φ is true*

contains formulas which are certainly unprovable in (S') provided that the system (S) is consistent. Indeed, if Φ_0 is an arbitrary formula of (S) such that the negation of Φ_0 is provable in (S), then the sentence

if Φ_0 is provable in (S), *then Φ_0 is true*

could be provable in (S') only if (S) were inconsistent.

The correct wording of the scheme (Σ) is as fallows:

(Σ^*) *For every Φ — if Φ can be proved in* (S) *in at most n steps from at most n axioms, then Φ is true* ($n=1,2,3,\ldots$).

It is exactly this scheme which is actually proved in the paper (cf. scheme (Σ_{11}) on p. 121). The scheme (Σ) although stated in the introductory remarks to the section 2 was neither proved nor used anywhere in the paper.

For symmetry one could still reformulate the theorem (T) on p. 118 as follows:

(T) *For every Φ and every n — if Φ can be proved in* (S) *in at most n steps from at most n axioms, then Φ is true.*

Note that in the incorrect scheme (Σ) it is the letter „Φ" which has to be replaced by an arbitrary formula of (S) in order to obtain a sentence of (S') whereas in the correct scheme (Σ^*) the variable „Φ" is bound and a sentence of (S') can be obtained upon substituting an arbitrary numeral 1,2,... for the variable „n".

Models of axiomatic theories admitting automorphisms

by

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The present paper is concerned with models of axiomatic theories based on the first order logic with identity and more specifically with automorphisms of such models. The main results of the paper are contained in section 5 and in particular in theorem 5.7 which says that if a theory possesses at least one infinite model, it also possesses a model with a "very large" automorphism group. It is a corollary to this theorem that axiomatic systems of arithmetic possess models which admit non-trivial automorphisms. This corollary solves a problem formulated by G. Hasenjaeger.

From the point of view of methods it may be interesting to note that the proofs of our fundamental results are not constructive and that for two reasons: First we use a theorem which states that if a theory is consistent, then the set of its axioms can be extended to a consistent and complete set. Secondly we use the so called ordering principle, *i. e.* an axiom stating that every set can be ordered. Since in the whole paper we are dealing with theories containing an arbitrary (not necessarily denumerable) number of constants, we see that the first non-constructive theorem mentioned above is equivalent to the so called fundamental theorem of the ideal theory in Boolean algebras (Henkin [2], especially p. 89 and Łoś [4]). Since the ordering principle is known to follow from that theorem (Łoś and Ryll-Nardzewski [6]), we conclude that the non-constructive tools used in the proofs of our principal theorems are all reducible to the fundamental theorem of the ideal theory in Boolean algebras.

It should also be mentioned that our proofs provide another instance of what has been called by Tarski [10] "the principle of condensation of singularities": The existence of a model admitting a large group of automorphisms is equivalent to the simultaneous satisfiability of an infinite number of sentences. We secure the satisfiability of these sentences by showing that the adjunction of an arbitrary finite number of them to the axioms does not render the theory inconsistent.

In order to make the paper self-contained we have collected in the introductory sections 1-3 all the notions and lemmas which are necessary to an exact formulation of the main theorems and to their proofs. None of these sections contain new results: in sections 1 and 2 we lay down the terminology and recall some well-known facts concerning models. In section 3 we expound the general method (due to Henkin [1], Novak [7], and Rasiowa [9]) of constructing models for arbitrary theories. In section 4 we recall some properties of automorphisms and prove a theorem stating that for each group G there is a theory some models of which possess an automorphism-group isomorphic with G (this is the only theorem in our paper in whose proof the full axiom of choice is used).

It seems to us that the automorphism-groups discussed in the present paper deserve a closer study. We intend publishing some of their applications in subsequent papers.

1. Axiomatic theories and their syntax. We consider axiomatic theories based on the functional calculus of the first order. Every such theory S is determined by three sets: 1° $F(S)$, the set of functors (symbols for functions), 2° $P(S)$, the set of predicates (symbols for relations (*i. e.*, for propositional functions)), 3° $A(S)$, the set of axioms. We make no assumptions as to the cardinal numbers of these sets, which may be finite or denumerable or even non-denumerable. We assume however that $P(S)$ contains at least one symbol, viz. the identity predicate ι. If φ is a functor or a predicate, then we denote by $a(\varphi)$ the number of arguments of φ. We do not exclude the case where $a(\varphi)=0$; in this case φ is called a *constant*. Of course we assume that $a(\iota)=2$. Finally we assume that all the theories which will be considered below contain the same individual variables and we denote these variables[1] by $\xi_1, \xi_2, \xi_3, \ldots$

By $W(S)$ we denote the class of terms of S. Thus $W(S)$ is the smallest class that contains all the variables and contains the expression $\varphi(\omega_1, \ldots, \omega_{a(\varphi)})$ (where $\varphi \epsilon F(S)$) whenever it contains $\omega_1, \omega_2, \ldots, \omega_{a(\varphi)}$.

By $Z(S)$ we denote the class of (sentential) matrices of S. Thus $Z(S)$ is the smallest class satisfying the following conditions: 1° If $\pi \epsilon P(S)$ and $\omega_1, \ldots, \omega_{a(\pi)} \epsilon W(S)$, then $\pi(\omega_1, \ldots, \omega_{a(\pi)}) \epsilon Z(S)$: 2° if $\zeta_1, \zeta_2 \epsilon Z(S)$, then $\sim\zeta_1$, $\zeta_1 \cdot \zeta_2 \epsilon Z(S)$; if $\zeta \epsilon Z(S)$, then $(\exists \xi_n)\zeta \epsilon Z(S)$ for $n=1,2,\ldots$[2].

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[1] The letters ξ_i are not variables but names for them. In a similar way we construe the symbols "$\sim$", "$\exists$" etc. which we shall use below as names for symbols actually occurring in S.

[2] Other logical operations can be defined in an obvious way in terms of negation, conjunction, and existential quantifier.

An expression which results from an expression $a \in W(S) \cup Z(S)$ by a substitution of a term ω_j for a variable ξ_j $(j=1,2,\ldots)$ will be denoted by

$$\mathrm{subst}\, a \begin{pmatrix} \xi_1, \xi_2, \ldots \\ \omega_1, \omega_2, \ldots \end{pmatrix}$$

or, in cases where no misunderstanding is possible, more simply by $\mathrm{subst}\, a(\omega_1, \omega_2, \ldots)$. We omit the explicit formulation of the well-known conditions which must be satisfied in order that the operation subst be performable.

A *matrix* $\zeta \in Z(S)$ is *open* or *closed* according as it contains no bound or no free variables. A *term* $\omega \in W(S)$ is called *constant* if it contains no variables. The set of constant terms will be denoted by $W^*(S)$.

The class of theorems of S will be denoted by $T(S)$. The following matrices are assumed to be contained in $T(S)$ for each S:

$$\xi_k \iota \xi_k, \quad \xi_k \iota \xi_l \supset \xi_l \iota \xi_k, \quad (\xi_k \iota \xi_l)(\xi_l \iota \xi_m) \supset (\xi_k \iota \xi_m),$$
$$\xi_k \iota \xi_l \supset \omega \iota\, \mathrm{subst}\, \omega \begin{pmatrix} \xi_k \\ \xi_l \end{pmatrix}, \quad \xi_k \iota \xi_l \supset \left[\zeta \equiv \mathrm{subst}\, \zeta \begin{pmatrix} \xi_k \\ \xi_l \end{pmatrix}\right],$$
$$(\omega \in W(S),\ \zeta \in Z(S),\ k,l,m=1,2,\ldots).$$

A theory S' is called an *extension* of S if $F(S) \subset F(S')$, $P(S) \subset P(S')$, and $T(S) \subset T(S')$. The extension is called *inessential* if $P(S')=P(S)$ and $T(S') \cap Z(S)=T(S)$.

A *theory* S is called *open* if all matrices that belong to $A(S)$ are open.

From the so called second ε-theorem (Hilbert and Bernays [3], p. 18-33) follows

THEOREM 1.1. *For every theory S there exists an open theory S' which is an inessential extension of S.*

2. Models of axiomatic theories.

Let S be a theory and X a set. We consider a function M with the following properties: 1° M assigns a function M_φ (with $a(\varphi)$ arguments) defined in X and taking on values which are elements of X to each $\varphi \in F(S)$; 2° M assigns a relation M_π (with $a(\pi)$ arguments) defined in X to each $\pi \in P(S)$; 3° M assigns the relation of identity in X to the predicate ι. Every such function M we call a *pseudo-model of S over X*.

Let M be a pseudo-model of S over X. A function

$$f = \begin{pmatrix} \xi_1, \xi_2, \ldots \\ x_1, x_2, \ldots \end{pmatrix}$$

which assigns an element of X to each variable we call a *valuation*. We put $\mathrm{val}_{fM} \xi_j = x_j$ and extend this definition over the whole class $W(S)$ by assuming

$$\mathrm{val}_{fM} \varphi(\omega_1, \ldots, \omega_{a(\varphi)}) = M_\varphi(\mathrm{val}_{fM} \omega_1, \ldots, \mathrm{val}_{fM} \omega_{a(\varphi)}).$$

Instead of $\mathrm{val}_{fM}\omega$ we shall usually write

$$\mathrm{val}_M\omega\begin{pmatrix}\xi_1,\xi_2,\dots\\ x_1,x_2,\dots\end{pmatrix}$$

or simpler $\mathrm{val}_M\omega(x_1,x_2,\dots)$.

Let $\pi\in P(S)$, $\omega_1,\dots,\omega_{a(\pi)}\in W(S)$, and $\zeta=\pi(\omega_1,\dots,\omega_{a(\pi)})$. We define [3]

$$\mathrm{stsf}_{fM}\zeta\equiv M_\pi(\mathrm{val}_{fM}\omega_1,\dots,\mathrm{val}_{fM}\omega_{a(\pi)})$$

and extend this definition over the whole set $Z(S)$ by assuming [4]

$$\mathrm{stsf}_{fM}(\sim\zeta)\equiv\sim\mathrm{stsf}_{fM}\zeta,\qquad \mathrm{stsf}_{fM}(\zeta_1\cdot\zeta_2)\equiv\mathrm{stsf}_{fM}\zeta_1\cdot\mathrm{stsf}_{fM}\zeta_2,$$

$$\mathrm{stsf}_{fM}(\exists\xi_n)\zeta\equiv(\exists f')[(f'\sim_n f)\cdot\mathrm{stsf}_{f'M}\zeta],$$

where the formula $f'\sim_n f$ means that $f'(\xi_j)=f(\xi_j)$ for $j\neq n$.

Instead of $\mathrm{stsf}_{fM}\zeta$ we shall usually write

$$\mathrm{stsf}_M\zeta\begin{pmatrix}\xi_1,\xi_2,\dots\\ x_1,x_2,\dots\end{pmatrix}$$

or simpler $\mathrm{stsf}_M\zeta(x_1,x_2,\dots)$.

We denote by V_M the set of matrices ζ which are valid in M, *i. e.* are such that $\mathrm{stsf}_{fM}\zeta$ holds for all f. If $A(S)\subset V_M$, then we say that M is *a model of* S.

Let S' be an extension of S and let M' and M be pseudo-models of S' and S over the same set X. We call M' an extension [5] of M if $M'_\varphi=M_\varphi$ for $\varphi\in F(S)\cup P(S)$.

The following theorem is an immediate consequence of the above definitions:

THEOREM 2.1. *If S' is an extension of S and M' an extension of M, then* $\mathrm{stsf}_{fM'}\zeta\equiv\mathrm{stsf}_{fM}\zeta$ *for each valuation f and each $\zeta\in Z(S)$.*

3. The construction of models.

Let S be an open theory which possesses a model over an infinite set and let X be an arbitrary set. We assume that there is a one-to-one correspondence between the elements of X and certain symbols which do not occur in S. For simplicity we shall identify the elements of X with the corresponding symbols.

We extend the theory S to a theory $S^*(X)$ by adding the elements of X to the set $F(S)$ and the matrices $\sim(x'\iota x'')$ where $x',x''\in X$, $x'\neq x''$ to the set $A(S)$. We assume that $a(x)=0$ for $x\in X$, *i. e.* that each x is a constant term of the theory $S^*(X)$.

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[3] Here, as in many places below, we use the logical symbols as abbreviations of certain expressions of the informal language.

[4] Note that in these formulas logical symbols have double meanings: they occur as names of symbols of S and as abbreviations of expressions in the informal language.

[5] This meaning of the word "extension" is narrower than the meaning attributed to this word by Łoś. Cf. J. Łoś [5].

Lemma 3.1. *The theory* $S^*(X)$ *is consistent.*

Proof. Let M be a model of S over an infinite set Y. Let us first assume that X is a finite set consisting of the elements $x_1,\dots,x_n$. Let $y_1,\dots,y_n$ be different elements of Y. We extend the model M of S over Y to a pseudo-model M^* of $S^*(X)$ over Y by putting

$$M^*_\varphi = M_\varphi \quad \text{for} \quad \varphi \in F(S)\,, \qquad M^*_{x_j} = y_j \quad \text{for} \quad j=1,2,\dots,n\,,$$
$$M^*_\pi = M_\pi \quad \text{for} \quad \pi \in P(S)\,.$$

From theorem 2.1 it immediately follows that if $\zeta \in A(S)$, then $\zeta \in V_{M^*}$. Since the formula $\sim(x' \iota x'') \in V_{M^*}$ is evident, we conclude that M^* is a model of $S^*(X)$ over Y. Hence $S^*(X)$ is consistent.

The general case can be reduced to the case of a finite X by the observation that an inconsistency of $S^*(X)$ would entail the inconsistency of $S^*(X_1)$ where X_1 is a finite subset of X.

Now let I be an arbitrary consistent and complete subset of $Z\big(S^*(X)\big)$ containing $A\big(S^*(X)\big)$. The existence of I is secured by lemma 3.1. We denote by $S(X,I)$ a theory S' such that $F(S')=F\big(S^*(X)\big)$, $Z(S')=Z\big(S^*(X)\big)$ and $A(S')=I$. Two constant terms ω_1,ω_2 of the theory $S(X,I)$ will be called *equivalent* if $\omega_1 \iota \omega_2 \in I$. We write then $\omega_1 \approx \omega_2$. The following properties of the relation $\approx$ are obvious:

Lemma 3.2. $\approx$ *is an equivalence relation and* x_1 *non* $\approx x_2$ *for* $x_1,x_2 \in X$, $x_1 \neq x_2$.

Lemma 3.3. *If* $\varphi \in F\big(S(X,I)\big)$, $\pi \in P\big(S(X,I)\big)$, $\omega_j,\omega'_j,\tau_k,\tau'_k$ *are constant terms of the theory* $S(X,I)$ $(j \leqslant a(\varphi),\ k \leqslant a(\pi))$, *and if* $\omega_j \approx \omega'_j$, $\tau_k \approx \tau'_k$ *for* $j \leqslant a(\varphi)$, $k \leqslant a(\pi)$, *then*

$$\varphi(\omega_1,\dots,\omega_{a(\varphi)}) \approx \varphi(\omega'_1,\dots,\omega'_{a(\varphi)})\,, \qquad \pi(\tau_1,\dots,\tau_{a(\pi)}) \equiv \pi(\tau'_1,\dots,\tau'_{a(\pi)}) \in I\,.$$

We denote by $\mathfrak{X}_X$ the set of equivalence classes of $W^*\big(S(X,I)\big)$ under the relation $\approx$. The equivalence class containing a constant term ω will be denoted by $[\omega]$.

We assign to a functor $\varphi \in F(S)$ a function $\mathfrak{M}_\varphi$ such that

$$\mathfrak{M}_\varphi([\omega_1],\dots,[\omega_{a(\varphi)}]) = [\varphi(\omega_1,\dots,\omega_{a(\varphi)})]\,,$$

and to a predicate $\pi \in P(S)$ a relation $\mathfrak{M}_\pi$ such that

$$\mathfrak{M}_\pi([\omega_1],\dots,[\omega_{a(\pi)}]) \equiv \pi(\omega_1,\dots,\omega_{a(\pi)}) \in I\,.$$

It follows from 3.3 that the values of $\mathfrak{M}_\varphi$ and of $\mathfrak{M}_\pi$ do not depend on terms ω_j but on the equivalence classes $[\omega_j]$. Since

$$\mathfrak{M}_\iota([\omega_1],[\omega_2]) \equiv \omega_1 \iota \omega_2 \in I \equiv \omega_1 \approx \omega_2 \equiv [\omega_1]=[\omega_2]\,,$$

we obtain

LEMMA 3.4. *The function* $\mathfrak{M}$ *is a pseudo-model of* S *over the set* $\mathfrak{X}_X$.

The pseudo-model $\mathfrak{M}$ depends on the sets X and I and will therefore be denoted by $\mathfrak{M}(X,I)$ if its dependence on X and I will have to be emphasized.

From the definitions we obtain by an easy induction

LEMMA 3.5. *If* $\omega \in W(S)$ and $\tau_1,\tau_2,\ldots$ *are constant terms of* $S(X,I)$, *then* $\mathrm{val}_{\mathfrak{M}}\,\omega([\tau_1],[\tau_2],\ldots)=[\mathrm{subst}\,\omega(\tau_1,\tau_2,\ldots)]$.

LEMMA 3.6. *If* ζ *is an open matrix of* S *and* $\tau_1,\tau_2,\ldots$ *are constant terms of* $S(X,I)$, *then* $\mathrm{stsf}_{\mathfrak{M}}\,\zeta([\tau_1],[\tau_2],\ldots)\equiv \mathrm{subst}\,\zeta(\tau_1,\tau_2,\ldots)\in I$.

Since I contains the axioms of S and these axioms are open matrices, we obtain from lemma 3.6

THEOREM 3.7. $\mathfrak{M}(X,I)$ *is a model of* S *over* $\mathfrak{X}_X$.

Again let S be an arbitrary theory and X an arbitrary set. Let M be a pseudo-model of S over a set Y and let M' be its extension to a pseudo-model of $S^*(X)$ over Y. The following theorem will be needed in section 5:

THEOREM 3.8. *If* $\omega\in W(S)$, $\zeta\in Z(S)$, ζ *is open, and if* $x_1,x_2,\ldots\in X$, *then* $\omega'=\mathrm{subst}\,\omega(x_1,x_2,\ldots)$ *is a constant term of* $S^*(X)$ *and* $\zeta'=\mathrm{subst}\,\zeta(x_1,x_2,\ldots)$ *is a closed matrix of* $S^*(X)$; *moreover*

$$\mathrm{val}_{M'}\,\omega'=\mathrm{val}_M\,\omega(M'_{x_1},M'_{x_2},\ldots)\,, \tag{3.8.1}$$

$$\mathrm{stsf}_{M'}\,\zeta'\equiv\mathrm{stsf}_M\,\zeta(M'_{x_1},M'_{x_2},\ldots)\,. \tag{3.8.2}$$

Proof. If $\omega=\xi_j$, then both the left and the right hand sides of (3.8.1) are equal to M'_{x_j}. If $\omega=\varphi(\omega_1,\ldots,\omega_{a(\varphi)})$, then $\omega'=\varphi(\omega'_1,\ldots,\omega'_{a(\varphi)})$ where the accents denote the operation $\mathrm{subst}(x_1,x_2,\ldots)$. If (3.8.1) holds for the terms ω_j $(j\leqslant a(\varphi))$, then

$$\begin{aligned}\mathrm{val}_{M'}(\omega')&=M'_\varphi(\mathrm{val}_{M'}\,\omega'_1,\ldots,\mathrm{val}_{M'}\,\omega'_{a(\varphi)})\\&=M'_\varphi\big(\mathrm{val}_M\,\omega_1(M'_{x_1},M'_{x_2},\ldots),\ldots,\mathrm{val}_M\,\omega_{a(\varphi)}(M'_{x_1},M'_{x_2},\ldots)\big)\,.\end{aligned}$$

Since $M'_\varphi=M_\varphi$, we obtain (3.8.1) for the term ω.

Proof of (3.8.2) is similar.

4. Automorphisms of models. Let M be a model of S over X. A one-one mapping f of X onto itself is called an *automorphism* of M if the following equations are satisfied for arbitrary $\varphi\in F(S)$, $\pi\in P(S)$ and $x_1,x_2,\ldots\in X$:

$$f\big(M_\varphi(x_1,\ldots,x_{a(\varphi)})\big)=M_\varphi\big(f(x_1),\ldots,f(x_{a(\varphi)})\big)\,,$$

$$M_\pi(x_1,\ldots,x_{a(\pi)})\equiv M_\pi\big(f(x_1),\ldots,f(x_{a(\pi)})\big)\,.$$

The group of automorphisms of M is denoted by G_M.

In the following two lemmas we note some well-known properties of automorphisms:

Lemma 4.1. *If* $f \in G_M$, $\omega \in W(S)$, $\zeta \in Z(S)$, *and* $x_1, x_2, \ldots \in X$, *then*

$$f\big(\mathrm{val}_M\, \omega(x_1, x_2, \ldots)\big) = \mathrm{val}_M\, \omega\big(f(x_1), f(x_2), \ldots\big),$$

$$\mathrm{stsf}_M \zeta(x_1, x_2, \ldots) \equiv \mathrm{stsf}_M \zeta\big(f(x_1), f(x_2), \ldots\big).$$

Lemma 4.2. *If S' is an extension of S, M is a model of S over X and M' a model of S' which is an extension of M, then $G_{M'} \subset G_M$.*

We shall now show that each group can be represented as G_M for a suitably chosen model M of a suitable theory S.

Theorem 4.3. *For each group G there is a theory S and a model M of S such that the groups G_M and G are isomorphic.*

Proof. We call, as usual, a left translation of G a mapping l of G onto itself defined by means of the formula $l(g) = g_0 g$, where g runs over G and g_0 is a fixed element of G.

We take as $F(S)$ the empty set and as $P(S)$ the set consisting of ι and of binary predicates π_f where f runs over one-one mappings of G onto itself that are not left translations of G. The set $A(S)$ is to consist exclusively of the axioms of identity enumerated on p. 52.

If f is a one-one mapping of G onto itself that is not a left translation of G, then there are two elements g_1, g_2 of G such that $f(g_1) \cdot g_1^{-1} \neq f(g_2) \cdot g_2^{-1}$. We select for each f a pair g_{1f}, g_{2f} of elements of G satisfying this condition and denote by M_{π_f} the binary relation defined in G such that

$$M_{\pi_f}(g', g'') \equiv (\exists g)[(g \in G) \cdot (g' = g g_{1f}) \cdot (g'' = g g_{2f})]. \tag{4.3.1}$$

Denoting by M_ι the relation of identity in G, we obtain a model of S over G.

Let l be a left translation of G, $l(g) = g_0 g$ where $g_0 \in G$. From (4.3.1) we immediately obtain

$$M_{\pi_f}(g', g'') \equiv M_{\pi_f}\big(l(g'), l(g'')\big) \quad \text{for} \quad \pi_f \in P(S) \tag{4.3.2}$$

and hence $l \in G_M$.

If l is not a left translation of G, then (4.3.2) does not hold for all $\pi \in P(S)$. Indeed, suppose that (4.3.2) is true for $f = l$. Since $M_{\pi_l}(g_{1l}, g_{2l})$, we obtain $M_{\pi_l}\big(l(g_{1l}), l(g_{2l})\big)$ and hence we infer that there is a $g \in G$ such that $l(g_{1l}) = g \cdot g_{1l}$ and $l(g_{2l}) = g\; g_{2l}$, *i. e.* $l(g_{1l}) \cdot g_{1l}^{-1} = l(g_{2l}) \cdot g_{2l}^{-1}$, which contradicts the choice of the elements g_{1l}, g_{2l}. Hence l is not an automorphism of M.

It follows that G_M is identical with the group of all left translations of G and hence isomorphic with G.

5. Models with non-trivial automorphism groups. The following theorem due to Ramsey ([8], theorem A on p. 384) is basic for all theorems given in this section:

THEOREM 5.1. *Let Y be an infinite set and Y^n the set of subsets of Y having exactly n elements. If $Y^n = C_1 \cup \ldots \cup C_k$ is a partition of Y^n into mutually disjoint sets, then there is a $j \leqslant k$ and an infinite set $Y_1 \subset Y$ such that $Y_1^n \subset C_j$.*

In the sequel we consider an open theory S and a model $\mathcal{M}(X,I)$ of S (cf. section 3).

LEMMA 5.2. *A one-one mapping h of X onto itself determines at most one automorphism f of $\mathcal{M}(X,I)$ satisfying the condition $f([x]) = [h(x)]$* for $x \in X$.

Proof. A constant term τ of $S^*(X)$ has the form $\tau = \mathrm{subst}\,\omega(x_1, x_2, \ldots)$ where $\omega \in W(S)$ and $x_j \in X$ for $j = 1, 2, \ldots$ Hence by 3.5 and 4.1 we obtain the formula $f([\tau]) = \mathrm{val}_{\mathcal{M}}\,\omega\big(f([x_1]), f([x_2]), \ldots\big)$, which shows that the value of $f([\tau])$ is determined by the values of $f([x])$ for $x \in X$. This proves the lemma.

If h is a one-one mapping of X onto itself for which there exists an automorphism f with the properties described in lemma 5.2, then we shall say that h induces an automorphism. The automorphism induced by h will be denoted by f_h.

LEMMA 5.3. *If $h_1 \neq h_2$ and the automorphisms f_{h_1}, f_{h_2} exist, then $f_{h_1} \neq f_{h_2}$.*

Proof follows immediately from lemma 3.2.

LEMMA 5.4. *A one-one mapping h of X onto itself induces an automorphism of $\mathcal{M}(X,I)$ if and only if the following condition is satisfied by each open matrix ζ and each assignment $\begin{pmatrix}\xi_1, \xi_2, \ldots \\ x_1, x_2, \ldots\end{pmatrix}$:*

$$\mathrm{subst}\,\zeta\begin{pmatrix}\xi_1, \xi_2, \ldots \\ x_1, x_2, \ldots\end{pmatrix} \equiv \mathrm{subst}\begin{pmatrix}\xi_1, & \xi_2, \ldots \\ h(x_1), & h(x_2), \ldots\end{pmatrix} \in I\,. \tag{5.4.1}$$

Proof. From the completeness of I it follows that exactly one of the closed matrices $\mathrm{subst}\,\zeta(x_1, x_2, \ldots)$, $\sim\mathrm{subst}\,\zeta(x_1, x_2, \ldots)$ belongs to I. We can assume that it is the first.

By 3.6 we obtain the formula $\mathrm{stsf}_{\mathcal{M}}\zeta([x_1], [x_2], \ldots)$, whence we infer by lemma 4.1 that if h induces an automorphism of $\mathcal{M}(X,I)$, then $\mathrm{stsf}_{\mathcal{M}}\zeta\big([h(x_1)], [h(x_2)], \ldots\big)$, *i. e.*, by 3.6 $\mathrm{subst}\,\zeta\big(h(x_1), h(x_2), \ldots\big) \in I$. This proves the formula (5.4.1).

Let us now assume that (5.4.1) holds for each open matrix ζ and let τ be a constant term of the theory $S^*(X)$. We assign variables of S

to the elements of the set X occurring in τ in such a way that different variables are correlated with different elements. Let $\bar{x}$ denote the variable assigned to x and let $\bar{\tau}$ be a term of S obtained from τ by replacing each x by the corresponding variable $\bar{x}$.

Now let τ_1, τ_2 be two constant terms of the theory $S^*(X)$ and let $x_1, \dots, x_n$ be all the elements of X which occur in τ_1 or in τ_2 or in both of them. We shall show that

(5.4.2) *If* $\tau_1 \approx \tau_2$, *then* $\operatorname{subst}\bar{\tau}_1\begin{pmatrix}\bar{x}_1, \dots, \bar{x}_n\\ h(x_1), \dots, h(x_n)\end{pmatrix} \approx \operatorname{subst}\bar{\tau}_2\begin{pmatrix}\bar{x}_1, \dots, \bar{x}_n\\ h(x_1), \dots, h(x_n)\end{pmatrix}$.

Indeed, $\tau_1 \approx \tau_2$ means that $\tau_1 \iota \tau_2 \in I$, whence

$$\operatorname{subst}\bar{\tau}_1 \iota \bar{\tau}_2 \begin{pmatrix}\bar{x}_1, \dots, \bar{x}_n\\ x_1, \dots, x_n\end{pmatrix} \in I\,.$$

Now we use (5.4.1), in which we take $\zeta = \tau_1 \iota \tau_2$ and replace the variables $\xi_1, \xi_2, \dots$ by $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$. In this way we obtain

$$\operatorname{subst}\bar{\tau}_1 \iota \bar{\tau}_2 \begin{pmatrix}\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\\ h(x_1), h(x_2), \dots, h(x_n)\end{pmatrix} \in I\,,$$

which proves (5.4.2).

From (5.4.2) it follows that defining f_h by means of the formula

$$f_h([\tau]) = \left[\operatorname{subst}\bar{\tau}\begin{pmatrix}\bar{x}_1, \dots, \bar{x}_n\\ h(x_1), \dots, h(x_n)\end{pmatrix}\right]$$

(where $x_1, \dots, x_n$ are all the elements of X that occur in τ), we obtain a function defined on $\mathfrak{X}_X$.

Each element $[\tau]$ of $\mathfrak{X}_X$ is the value of f_h for a suitable argument. Indeed, if

$$\tau' = \operatorname{subst}\bar{\tau}\begin{pmatrix}\bar{x}_1, \dots, \bar{x}_n\\ h^{-1}(x_1), \dots, h^{-1}(x_n)\end{pmatrix},$$

then

$$\bar{\tau}' = \operatorname{subst}\bar{\tau}\begin{pmatrix}\bar{x}_1, \dots, \bar{x}_n\\ \overline{h^{-1}(x_1)}, \dots, \overline{h^{-1}(x_n)}\end{pmatrix},$$

and hence

$$\begin{aligned} f_h([\tau']) &= \left[\operatorname{subst}\bar{\tau}'\begin{pmatrix}\overline{h^{-1}(x_1)}, \dots, \overline{h^{-1}(x_n)}\\ h(h^{-1}(x_1)), \dots, h(h^{-1}(x_n))\end{pmatrix}\right] \\ &= \left[\operatorname{subst}\bar{\tau}\begin{pmatrix}\bar{x}_1, \dots, \bar{x}_n\\ x_1 \dots, x_n\end{pmatrix}\right] = [\tau]\,. \end{aligned}$$

In a similar way we show that the mapping f_h is one-one. Indeed, if $f_h([\tau_1]) = f_h([\tau_2])$ and $x_1, \dots, x_n$ have the same meaning as in (5.4.2), then

$$\operatorname{subst}\bar{\tau}_1\begin{pmatrix}\bar{x}_1, \dots, \bar{x}_n\\ h(x_1), \dots, h(x_n)\end{pmatrix} \approx \operatorname{subst}\bar{\tau}_2\begin{pmatrix}\bar{x}_1, \dots, \bar{x}_n\\ h(x_1), \dots, h(x_n)\end{pmatrix},$$

and hence

$$\operatorname{subst}\bar\tau_1\,\iota\,\bar\tau_2\begin{pmatrix}\bar x_1 & ,\dots, & \bar x_n\\ h(x_1) & ,\dots, & h(x_n)\end{pmatrix}\in I\,.$$

Using (5.4.1) we obtain

$$\operatorname{subst}\bar\tau_1\,\iota\,\bar\tau_2\begin{pmatrix}\bar x_1,\dots,\bar x_n\\ x_1,\dots,x_n\end{pmatrix}\in I$$

whence $\tau_1\,\iota\,\tau_2\in I$, $\tau_1\approx\tau_2$, and $[\tau_1]=[\tau_2]$.

Finally we shall show that f_h is an automorphism of $\mathfrak{M}(X,I)$. For $\varphi\in F(S)$ we have

$$f_h\big(\mathfrak{M}_\varphi([\tau_1],\dots,[\tau_{a(\varphi)}])\big)=f_h\big([\varphi(\tau_1,\dots,\tau_{a(\varphi)})]\big)\,.$$

By putting $\omega=\varphi(\tau_1,\dots,\tau_{a(\varphi)})$ and observing that $\bar\omega=\varphi(\bar\tau_1,\dots,\bar\tau_{a(\varphi)})$ we obtain further

$$\begin{aligned}
f_h\big(\mathfrak{M}_\varphi([\tau_1],\dots,[\tau_{a(\varphi)}])\big)&=f_h([\omega])=\left[\operatorname{subst}\bar\omega\begin{pmatrix}\bar x_1 & ,\dots, & \bar x_n\\ h(x_1) & ,\dots, & h(x_n)\end{pmatrix}\right]\\
&=\left[\varphi\left(\operatorname{subst}\bar\tau_1\begin{pmatrix}\bar x_1 & ,\dots, & \bar x_n\\ h(x_1) & ,\dots, & h(x_n)\end{pmatrix},\dots,\operatorname{subst}\bar\tau_{a(\varphi)}\begin{pmatrix}\bar x_1 & ,\dots, & \bar x_n\\ h(x_1) & ,\dots, & h(x_n)\end{pmatrix}\right)\right]\\
&=\mathfrak{M}_\varphi\left(\left[\operatorname{subst}\bar\tau_1\begin{pmatrix}\bar x_1 & ,\dots, & \bar x_n\\ h(x_1) & ,\dots, & h(x_n)\end{pmatrix}\right],\dots,\left[\operatorname{subst}\bar\tau_{a(\varphi)}\begin{pmatrix}\bar x_1 & ,\dots, & \bar x_n\\ h(x_1) & ,\dots, & h(x_n)\end{pmatrix}\right]\right)\\
&=\mathfrak{M}_\varphi\big(f_h([\tau_1]),\dots,f_h([\tau_{a(\varphi)}])\big)\,.
\end{aligned}$$

This is the required automorphism-property for $\varphi\in F(S)$.

If $\pi\in P(S)$, then

$$\begin{aligned}
\mathfrak{M}_\pi([\tau_1],\dots,[\tau_{a(\pi)}])&\equiv\pi(\tau_1,\dots,\tau_{a(\pi)})\in I\\
&\equiv\operatorname{subst}\pi(\bar\tau_1,\dots,\bar\tau_{a(\pi)})\begin{pmatrix}\bar x_1,\dots,\bar x_n\\ x_1,\dots,x_n\end{pmatrix}\in I\,,
\end{aligned}$$

where $x_1,\dots,x_n$ are all the elements of X that occur in $\pi(\tau_1,\dots,\tau_{a(\pi)})$. Using (5.4.1) for $\zeta=\pi(\bar\tau_1,\dots,\bar\tau_{a(\pi)})$ we obtain therefore

$$\mathfrak{M}_\pi([\tau_1],\dots,[\tau_{a(\pi)}])\equiv\operatorname{subst}\pi(\bar\tau_1,\dots,\bar\tau_{a(\pi)})\begin{pmatrix}\bar x_1 & ,\dots, & \bar x_n\\ h(x_1) & ,\dots, & h(x_n)\end{pmatrix}\in I\,.$$

The right-hand side of this equivalence means precisely the same as $\mathfrak{M}_\pi\big(f_h([\tau_1]),\dots,f_h([\tau_{a(\pi)}])\big)$. Lemma 5.4 is thus proved.

In order to express conveniently the content of lemmas 5.2, 5.3, and 5.4 we shall adopt the following

Definition. A group G_1 of transformations of a set X_1 *strongly contains* a group G of transformations of a set X if $X_1\supset X$ and each $f\in G$ can be extended to at least one function $f_1\in G_1$.

If G is a cyclic group generated by a transformation h, then instead of saying that G_1 strongly contains G we shall say that G_1 strongly contains h.

From lemmas 5.2, 5.3, and 5.4 we obtain

THEOREM 5.5. *Let S be an open theory and X a set. In order that there exist a model M of S over a set $X_1 \supset X$ such that G_M strongly contains a group G of transformations of X it is necessary and sufficient that the theory $S^*(X)$ remain consistent after the adjunction of all equivalences (5.4.1) to its axioms where h is an arbitrary element of G and ζ an arbitrary open matrix of S.*

Proof. If the condition is satisfied, we can extend the set $T(S^*(X))$ to a complete set I satisfying (5.4.1). On using lemma 5.4 we obtain a model $\mathfrak{M}(X,I)$ whose automorphism group strongly contains the group of transformations $[x]\to[h(x)]$ of the set $[X]=\underset{[x]}{E}[x \in X]$. Since there is a one-one correspondence between the elements of X and those of $[X]$, we can exchange the classes $[x]$ for the elements x and obtain thus from $\mathfrak{M}(X,I)$ (which is a model of S over $\mathfrak{X}_X$) a model M of S over a set $X_1 \supset X$ such that G_M strongly contains the group G.

Conversely, if there is a model M of S over a set $X_1 \supset X$ such that G_M strongly contains G, then we use 4.1 and find that formulas (5.4.1) belong to V_M for each open matrix $\zeta \in Z(S)$ and each $h \in G$. Since the axioms of $S^*(X)$ are evidently elements of V_M, we obtain the desired consistency.

THEOREM 5.6 [6]. *Each theory S (not necessarily open), which possesses at least one model over an infinite set, possesses a model M_0 such that the group G_{M_0} strongly contains an infinite cyclic group.*

Proof. Let us first assume that S is open and consider an infinite set

$$X=\{\dots,x_{-n},\dots,x_{-1},x_0,x_1,\dots,x_n,\dots\}$$

where $x_i \neq x_j$ for $i \neq j$. Let h be the transformation $h(x_j)=x_{j+1}$ $(j=0,\pm 1,\pm 2,\dots)$.

In order to prove our theorem we have to show that the adjunction of equivalences (5.4.1) (where $\zeta \in Z(S)$ and $x_1,\dots,x_n$ are to be replaced by arbitrary elements of X) does not render theory $S^*(X)$ inconsistent. It will of course be sufficient to show that no inconsistency occurs if we adjoin an arbitrary finite number of equivalences (5.4.1) to the axioms of $S^*(X)$.

---

[6] Theorem 5.6 is contained as a special case in the theorem 5.7 which follows. Since however the proof of theorem 5.6 is much simpler than the proof of theorem 5.7 we thought it useful to give an independent proof of theorem 5.6.

Let us therefore consider s open matrices $\zeta_1, \ldots, \zeta_s \in Z(S)$ and assume that no variable different from $\xi_1, \xi_2, \ldots, \xi_t$ occurs in any of these matrices. We consider further s sequences of integers each containing exactly t (not necessarily different) terms:

$$i_1, j_1, \ldots, m_1; \quad i_2, j_2, \ldots, m_2; \quad \ldots; \quad i_s, j_s, \ldots, m_s.$$

We may assume that the terms of these s sequences lie in the interval $-n \leqslant x \leqslant n$.

We extend S to a theory S^* such that $P(S^*) = P(S)$, $F(S^*) = F(S) \cup \{x_{-n}, \ldots, x_{n+1}\}$ (where $a(x_j) = 0$ for $-n \leqslant j \leqslant n+1$) and $A(S^*)$ is obtained from $A(S)$ by adjunction of the matrices

$$(5.6.1) \qquad \operatorname{subst} \zeta_l \begin{pmatrix} \xi_1, \xi_2, \ldots, \xi_t \\ x_{i_l}, x_{j_l}, \ldots, x_{m_l} \end{pmatrix} = \operatorname{subst} \zeta_l \begin{pmatrix} \xi_1, & \xi_2 & , \ldots, & \xi_t \\ h(x_{i_l}), & h(x_{j_l}), & \ldots, & h(x_{m_l}) \end{pmatrix}$$
$$(l = 1, 2, \ldots, s),$$

$$(5.6.2) \qquad \sim(x_i \iota x_j) \qquad -n \leqslant i < j \leqslant n+1.$$

Note that the only symbols of S^* that do not occur in S are $x_{-n}, \ldots, x_n, x_{n+1}$.

In order to prove the consistency of S^* we shall construct a model for this theory. To this effect we first assign $2n+1$ different variables of S to the elements $x_{-n}, \ldots, x_{n+1}$ and denote by $\bar{x}_p$ the variable corresponding to x_p. We consider further 2^s matrices

$$(5.6.3) \quad \psi_{\varepsilon_1 \ldots \varepsilon_s} = \operatorname{subst} \zeta_1^{\varepsilon_1} \begin{pmatrix} \xi_1, \xi_2, \ldots, \xi_t \\ \bar{x}_{i_1}, \bar{x}_{j_1}, \ldots, \bar{x}_{m_1} \end{pmatrix} \ldots \operatorname{subst} \zeta_s^{\varepsilon_s} \begin{pmatrix} \xi_1, \xi_2, \ldots, \xi_t \\ \bar{x}_{i_s}, \bar{x}_{j_s}, \ldots, \bar{x}_{m_s} \end{pmatrix}$$

where $\varepsilon_p = \pm 1$ for $p = 1, 2, \ldots, s$ and ζ^ε stands for ζ or $\sim\zeta$ according as $\varepsilon = +1$ or $\varepsilon = -1$. These matrices evidently belong to $Z(S)$; their free variables are $\bar{x}_{-n}, \ldots, \bar{x}_n$ or some of these variables. It is also evident that the matrices (5.6.3) possess the following properties:

$$(5.6.4) \qquad \sim(\psi_{\varepsilon_1 \ldots \varepsilon_s} \cdot \psi_{\eta_1 \ldots \eta_s}) \in T(S) \quad \text{for} \quad (\varepsilon_1 \ldots \varepsilon_s) \neq (\eta_1 \ldots \eta_s),$$

(5 6.5) the alternation of 2^s matrices (5.6.3) belongs to $T(S)$.

Now let M be a model of S over an infinite set Y. The existence of M is secured by the assumptions of the theorem. We assume Y to be ordered by an arbitrary relation $\prec$ which, in general, has nothing in common with relations definable in S. Let Y^{2n+1} be the set consisting of subsets of Y with exactly $2n+1$ elements and let $C_{\varepsilon_1 \ldots \varepsilon_s}$ be the set containing as elements all those sets $\{y_{-n}, \ldots, y_n\} \subset Y$ for which $y_{-n} \prec \ldots \prec y_n$ and

$$(5.6.6) \qquad \operatorname{stsf}_M \psi_{\varepsilon_1 \ldots \varepsilon_s} \begin{pmatrix} \bar{x}_{-n}, \ldots, \bar{x}_n \\ y_{-n}, \ldots, y_n \end{pmatrix},$$

From (5.6.4) and (5.6.5) it is clear that the sets $C_{\varepsilon_1\ldots\varepsilon_s}$ determine a partition of Y^{2n+1}. Applying theorem 5.1 we infer that there is a fixed system of indices $\varepsilon_1,\ldots,\varepsilon_s$ and an infinite set $Y_1 \subset Y$ such that $Y_1^{2n+1} \subset C_{\varepsilon_1\ldots\varepsilon_s}$. We choose from Y_1 $2n+2$ elements $y_{-n},\ldots,y_n,y_{n+1}$ such that $y_{-n} \leqslant \ldots \leqslant y_n \leqslant y_{n+1}$. Hence we have the formula (5.6.6) and also the formula

$$\text{stsf}_M \psi_{\varepsilon_1\ldots\varepsilon_s}\begin{pmatrix}\bar{x}_{-n}, & \ldots, & \bar{x}_n \\ y_{-n+1}, & \ldots, & y_{n+1}\end{pmatrix}. \tag{5.6.7}$$

We now define a pseudo-model M^* of S^* over Y by assuming

$$M^*_\varphi = M_\varphi \quad \text{for} \quad \varphi \in F(S), \qquad M^*_{x_j} = y_j \quad \text{for} \quad j = -n,\ldots,n,n+1,$$
$$M^*_\pi = M_\pi \quad \text{for} \quad \pi \in P(S).$$

If $\zeta \in A(S)$, then $\zeta \in V_M$ and hence $\zeta \in V_{M^*}$ (cf. theorem 2.1). Axioms (5.6.2) of S^* are evidently contained in V_{M^*} because $M^*_{x_i} = y_i \neq y_j = M^*_{x_j}$ for $i \neq j$. Formulas (5.6.6) and (5.6.7) prove that

$$\text{stsf}_M \zeta_l^{\varepsilon_l}\begin{pmatrix}\bar{x}_{i_l}, & \ldots, & \bar{x}_{m_l} \\ y_{i_l}, & \ldots, & y_{m_l}\end{pmatrix} \quad \text{and} \quad \text{stsf}_M \zeta_l^{\varepsilon_l}\begin{pmatrix}\bar{x}_{i_l}, & \ldots, & \bar{x}_{m_l} \\ y_{i_l+1}, & \ldots, & y_{m_l+1}\end{pmatrix}$$

for $l = 1,2,\ldots,s$ and hence, in accordance with theorem 3.8,

$$\text{stsf}_{M^*}\left(\text{subst}\,\zeta_l^{\varepsilon_l}\begin{pmatrix}\xi_1, & \ldots, & \xi_t \\ x_{i_l}, & \ldots, & x_{m_l}\end{pmatrix}\right) \quad \text{and} \quad \text{stsf}_{M^*}\left(\text{subst}\,\zeta_l^{\varepsilon_l}\begin{pmatrix}\xi_1, & \ldots, & \xi_t \\ h(x_{i_l}), & \ldots, & h(x_{m_l})\end{pmatrix}\right).$$

From these two formulas it follows that axioms (5.6.1) are valid in M^*, *i. e.* belong to V_{M^*}. This proves the consistency of S^*.

Theorem 5.6 is thus proved for the case of an open theory. The general case can be reduced to the case of an open theory by means of theorems 1.1 and 4.2.

The following example shows that theorem 5.6 ceases to be true if we replace in it the words "infinite cyclic group" by the words "an arbitrary transformation group".

Assume that S is a consistent theory and that $P(S)$ contains a binary predicate π such that the matrices

$$(\xi_1 \pi \xi_2)\cdot(\xi_2 \pi \xi_3) \supset (\xi_1 \pi \xi_3), \qquad \sim(\xi_1 \pi \xi_1), \qquad (\xi_1 \pi \xi_2) \vee (\xi_1 \iota \xi_2) \vee (\xi_2 \pi \xi_1)$$

belong to $T(S)$.

If M is an arbitrary model of S over an arbitrary set X_1, then G_M does not contain functions which, limited to a subset X of X_1, are transformations of finite order different from identity. For assume that $f \in G_M$, $f(x) \neq x$, and f, limited to a set $X \subset X_1$ containing x, is a transformation of order n. Since X is ordered by the relation M_n, we have either $M_n(x,f(x))$ or $M_n(f(x),x)$. It will be sufficient to consider only the first case. We have evidently $M_n(f^j(x),f^{j+1}(x))$ for $j=0,1,2,\dots,n-1$ because f is an automorphism of M. By the transitivity of M_n we obtain therefore $M_n(x,f^n(x))$, *i. e.* $M_n(x,x)$, which is a contradiction.

In connection with these remarks we shall introduce the following

Definition. For each set X ordered by a relation $\precsim$ we denote by $G(X,\precsim)$ the group of all transformations of X onto itself leaving invariant the relation $\precsim$ (*i. e.* satisfying the condition $x_1 \precsim x_2 \equiv f(x_1) \precsim f(x_2)$ for $x_1,x_2 \in X$).

THEOREM 5.7. *If a theory S has at least one model over an infinite set, then for each ordered set X there is a model M_0 of S such that G_{M_0} strongly contains $G(X,\precsim)$.*

Proof. As in the proof of theorem 5.6 we can limit ourselves to the case of an open theory S. According to theorem 5.5 we have only to show that the theory $S^*(X)$ remains consistent after the adjunction to its axioms of all matrices (5.4.1) where ζ is an open matrix of S and $h \in G(X,\precsim)$. This again can be reduced to the proof that the theory S remains consistent after the adjunction of an arbitrary finite number of axioms of the form (5.4.1) and of a finite number of axioms of $S^*(X)$ which are not already contained in $A(S)$.

Accordingly we consider a finite number of open matrices $\zeta_1,\zeta_2,\dots,\zeta_s$ of S and assume that no variable different from $\xi_1,\xi_2,\dots,\xi_t$ occurs in any of these matrices. We further consider s sequences each containing t elements of X

$$x_{(i-1)t+1},\dots,x_{it}, \qquad i=1,2,\dots,s \tag{5.7.1}$$

(we do not assume that $x_j \neq x_k$ for $j \neq k$). Finally we consider s functions $g_1,\dots,g_s \in G(X,\precsim)$ and denote by X^* the set containing all the elements (5.7.1) and all the elements $g_i(x_j)$ where $i=1,2,\dots,s$ and $j=1,2,\dots,st$. We extend S to a theory S^* assuming that $F(S^*)=F(S) \cup X^*$, $P(S^*)=P(S)$ and letting $A(S^*)$ to consist of $A(S)$ and of matrices

$$\sim(x' \iota x'') \qquad x',x'' \in X^*, \qquad x' \neq x'', \tag{5.7.2}$$

$$\operatorname{subst}\zeta_i\begin{pmatrix}\xi_1 & ,\dots, & \xi_t\\ x_{(i-1)t+1} & ,\dots, & x_{it}\end{pmatrix} \equiv \operatorname{subst}\zeta_i\begin{pmatrix}\xi_1 & ,\dots, & \xi_t\\ g_i(x_{(i-1)t+1}) & ,\dots, & g_i(x_{it})\end{pmatrix}, \tag{5.7.3}$$

$$i=1,2,\dots,s.$$

In order to prove the theorem it will be sufficient to define a model of S^*.

We begin by assigning a variable $\bar{x}$ to each element x of X^* in such a way that $\bar{x}'\neq\bar{x}''$ for $x'\neq x''$. We further put $p=st$ and introduce the following matrices:

$$\psi_i=\operatorname{subst}\zeta_i\begin{pmatrix}\xi_1 & ,\ldots, & \xi_t\\ \bar{x}_{(i-1)t+1} & ,\ldots, & \bar{x}_{it}\end{pmatrix},\qquad i=1,2,\ldots,s,$$

$$\bar{\psi}_i=\operatorname{subst}\zeta_i\left(\frac{\xi_1}{g_i(x_{(i-1)t+1})},\ldots,\frac{\xi_t}{g_i(x_{it})}\right),\qquad i=1,2,\ldots,s.$$

Since the assignment $x\to\bar{x}$ is one-one, we easily see that axioms (5.7.3) can be written in the form

$$\text{(5.7.4)}\qquad \operatorname{subst}\psi_i\begin{pmatrix}\bar{x}_1,\ldots,\bar{x}_p\\ x_1,\ldots,x_p\end{pmatrix}=\operatorname{subst}\psi_i\begin{pmatrix}\bar{x}_1 & ,\ldots, & \bar{x}_p\\ g_i(x_1) & ,\ldots, & g_i(x_p)\end{pmatrix},\qquad i=1,2,\ldots,s.$$

We have noted above that the elements (5.7.1) need not be distinct; let us assume that they form a set with n elements

$$\text{(5.7.5)}\qquad X_0=\{x_1,\ldots,x_p\}=\{x_1^0,\ldots,x_n^0\}$$

where $x_i^0\neq x_j^0$ for $i\neq j$. Each of the sets

$$X_i=\{g_i(x_1^0),\ldots,g_i(x_n^0)\}\qquad i=1,2,\ldots,s$$

has exactly n elements and is ordered similarly to X_0. The set X^* is the union of the sets $X_0,X_1,\ldots,X_s$:

$$X^*=X_0\cup X_1\cup\ldots\cup X_s=\{x_1^0,\ldots,x_n^0,\ldots,x_m^0\}.$$

Let M be a model of S over an infinite set Y. We can assume that the set Y is ordered and denote by $\leqslant$ the ordering relation.

Let U be an element of Y^n, *i. e.* a subset of Y with exactly n elements. A sequence $(u_1,\ldots,u_p)$ with p (not necessarily distinct) terms $u_j\in U$ will be called a distinguished ordering of U if $u_h\leqslant u_j\equiv x_h\precsim x_j$ for $h,j<p$. It is evident that for each $U\in Y^n$ there exists exactly one distinguished ordering.

We now define a partition of Y^n into 2^s sets $C_{\varepsilon_1\ldots\varepsilon_s}$ where $\varepsilon_i=\pm1$ for $i=1,2,\ldots,s$ by including a set $U\in Y^n$ to $C_{\varepsilon_1\ldots\varepsilon_s}$ if the distinguished ordering $(u_1,\ldots,u_p)$ of U satisfies the condition

$$\text{(5.7.6)}\qquad \operatorname{stsf}_M\psi_i^{\varepsilon_i}\begin{pmatrix}\bar{x}_1,\ldots,\bar{x}_p\\ u_1,\ldots,u_p\end{pmatrix}\quad\text{for}\quad i=1,2,\ldots,s.$$

It is evident that the union of all sets $C_{\varepsilon_1\ldots\varepsilon_s}$ is Y^n and that two different sets $C_{\varepsilon_1\ldots\varepsilon_s}$ are disjoint. By theorem 5.1 there is a fixed system $\varepsilon_1,\ldots,\varepsilon_s$ of indices ±1 and an infinite set $Y_1\subset Y$ such that (5.7.6) holds for each $U \in Y_1^n$. We select from Y_1 a set $\{y_1, \ldots, y_m\}$ with m elements ordered (by the relation $\leqslant$) similarly to X^*:

$$(5.7.7) \qquad y_i \leqslant y_j \equiv x_i^0 \lesssim x_j^0 \qquad i,j \leqslant m.$$

We can now define a model M^* of S^* over Y by taking

$$M^*_\varphi = M_\varphi \quad \text{for} \quad \varphi \in F(S), \qquad M^*_{x_j^0} = y_j \quad \text{for} \quad j = 1, 2, \ldots, m,$$
$$M^*_\pi = M_\pi \quad \text{for} \quad \pi \in P(S).$$

It is evident that axioms of S and axioms (5.7.2) are valid in M^*. It remains therefore to prove that axioms (5.7.4) are valid in M^*. We first prove the following auxiliary statements:

(5.7.8) The sequence $(M^*_{x_1}, \ldots, M^*_{x_p})$ is a distinguished ordering of the set $\{M^*_{x_1}, \ldots, M^*_{x_p}\}$,

(5.7.9) The sequence $(M^*_{g_i(x_1)}, \ldots, M^*_{g_i(x_p)})$ is a distinguished ordering of the set $\{M^*_{g_i(x_1)}, \ldots, M^*_{g_i(x_p)}\}$.

(Note that both sets, $\{M^*_{x_1}, \ldots, M^*_{x_p}\}$ and $\{M^*_{g_i(x_1)}, \ldots, M^*_{g_i(x_p)}\}$, have exactly n elements).

Proof of (5.7.8). Each x_h $(h \leqslant p)$ is identical with x_u^0 where $u \leqslant n$ (cf. (5.7.5)). Assume that $h, j \leqslant p$ and $x_h = x_u^0$, $x_j = x_v^0$. Hence we have the equivalence

$$M^*_{x_h} \leqslant M^*_{x_j} \equiv M^*_{x_u^0} \leqslant M^*_{x_v^0} \equiv y_u \leqslant y_v$$

which together with (5.7.7) yields

$$M^*_{x_h} \leqslant M^*_{x_j} \equiv x_u^0 \lesssim x_v^0 \equiv x_h \lesssim x_j, \quad \text{q. e. d.}$$

Proof of (5.7.9). Each $g_i(x_h)$ $(h \leqslant p)$ is an element of X_i and hence identical with an element of the form $g_i(x_u^0)$ where $u \leqslant n$. Assume that $h, j \leqslant p$ and $g_i(x_h) = g_i(x_u^0)$, $g_i(x_j) = g_i(x_v^0)$. Since $g_i(x_u^0)$ and $g_i(x_v^0)$ belong to X^*, they are identical with elements x_w^0, x_z^0 where $w, z \leqslant m$. Hence, on account of (5.7.7), we obtain

$$M^*_{g_i(x_h)} \leqslant M^*_{g_i(x_j)} \equiv M^*_{g_i(x_u^0)} \leqslant M^*_{g_i(x_v^0)} \equiv M^*{}_{\substack{0\\w}} \leqslant M^*_{x_z^0}$$
$$\equiv y_w \leqslant_z y \equiv x_w^0 \lesssim x_z^0 \equiv g_i(x_u^0) \lesssim g_i(x_v^0) \equiv g_i(x_h) \lesssim g_i(x_j).$$

Since $g \in G(X, \lesssim)$, it preserves the ordering relation $\lesssim$ and hence the last part of the above formula is equivalent to $x_h \lesssim x_j$, q. e. d.

We can now prove that axioms (5.7.4) are valid in M^*. From (5.7.6), (5.7.8), and the remark that $M^*_{x_1}, \ldots, M^*_{x_p}$ are elements of Y_1 we obtain the formulas

$$\operatorname{stsf}_M \psi_i^{\varepsilon_i}\begin{pmatrix} \bar{x}_1, \ldots, \bar{x}_p \\ M^*_{x_1}, \ldots, M^*_{x_p} \end{pmatrix}, \qquad i = 1, 2, \ldots, s$$

whence, on account of theorem 3.8, we further obtain

$$\text{(5.7.10)} \qquad \mathrm{stsf}_{M^*}\mathrm{subst}\,\psi_i^{\varepsilon_i}\begin{pmatrix}\bar{x}_1,\dots,\bar{x}_p\\ x_1,\dots,x_p\end{pmatrix}, \qquad i=1,2,\dots,s\,.$$

From (5.7.9) we obtain in the same manner

$$\mathrm{stsf}_M\psi_i^{\varepsilon_i}\begin{pmatrix}\bar{x}_1 & ,\dots, & \bar{x}_p\\ M^*_{g_i(x_1)}, & \dots, & M^*_{g_i(x_p)}\end{pmatrix}, \qquad i=1,2,\dots,s\,.$$

Since ψ_i contains only the variables $\bar{x}_{(i-1)t+1},\dots,\bar{x}_{it}$, the last formula can be written in the form

$$\mathrm{stsf}_M\psi_i^{\varepsilon_i}\begin{pmatrix}\bar{x}_{(i-1)t+1} & ,\dots, & \bar{x}_{it}\\ M^*_{g_i(x_{(i-1)t+1})}, & \dots, & M^*_{g_i(x_{it})}\end{pmatrix}, \qquad i=1,2,\dots,s\,.$$

We now remark that $\bar{\psi}_i$ results from ψ_i by a substitution of variables $\overline{g_i(x_{(i-1)t+1})},\dots,\overline{g_i(x_{it})}$ for the variables $\bar{x}_{(i-1)t+1},\dots,\bar{x}_{it}$. Hence we can write the last formula in the form

$$\mathrm{stsf}_M\bar{\psi}_i^{\varepsilon_i}\begin{pmatrix}\overline{g_i(x_{(i-1)t+1})}, & \dots, & \overline{g_i(x_{it})}\\ M^*_{g_i(x_{(i-1)t+1})}, & \dots, & M^*_{g_i(x_{it})}\end{pmatrix}, \qquad i=1,2,\dots,s\,.$$

We simplify this formula by inserting the "fictitious" variables $\overline{g_i(x_{jt+k})}$ $(j\neq i,\ k=1,2,\dots,t)$ in the upper row. The validity of the formula is unaffected since these variables do not occur in $\bar{\psi}_i$. We thus obtain

$$\mathrm{stsf}_M\bar{\psi}_i^{\varepsilon_i}\begin{pmatrix}\overline{g_i(x_1)}, & \dots, & \overline{g_i(x_p)}\\ M^*_{g_i(x_1)}, & \dots, & M^*_{g_i(x_p)}\end{pmatrix}, \qquad i=1,2,\dots,s$$

or, what amounts to the same,

$$\mathrm{stsf}_M\psi_i^{\varepsilon_i}\begin{pmatrix}\bar{x}_1 & ,\dots, & \bar{x}_p\\ M^*_{g_i(x_1)}, & \dots, & M^*_{g_i(x_p)}\end{pmatrix}, \qquad i=1,2,\dots,s\,.$$

Using orem 3.8 we finally obtain the formula

$$\mathrm{stsf}_{M^*}\mathrm{subst}\,\psi_i^{\varepsilon_i}\begin{pmatrix}\bar{x}_1 & ,\dots, & \bar{x}_p\\ M^*_{g_i(x_1)}, & \dots, & M^*_{g_i(x_p)}\end{pmatrix}, \qquad i=1,2,\dots,s\,,$$

which together with (5.7.10) proves that the matrix (5.7.4) is valid in M^*.

Theorem 5.7 is thus proved.

6. We shall conclude by proving one more theorem, which is not directly connected with the subject-matter of the present paper but which will be needed in one of the subsequent papers mentioned at the end of the introduction. It seems appropriate to include the proof here because the method of proof is very close to that used in the proof of theorem 5.7.

THEOREM 6.1. *Let X be a set ordered by a relation $\precsim$ and S an open theory which possesses a model M over an infinite set Y ordered by a relation $\leqslant$. Further let Y^* be an infinite set contained in Y and η an open matrix of S with the free variables $\xi_1,\ldots,\xi_q$ such that*

$$\mathrm{stsf}_M\,\eta\begin{pmatrix}\xi_1,\ldots,\xi_q\\ y_1,\ldots,y_q\end{pmatrix}$$

for each sequence $(y_1,\ldots,y_q)$ of elements of Y^* satisfying the conditions $y_1\leqslant y_2\leqslant\ldots\leqslant y_q$. Under these assumptions there exists a model M_0 of S over a set $X_1\supset X$ such that

(6.1.1) *G_{M_0} strongly contains the group $G(X,\precsim)$,*

(6.1.2) $\mathrm{stsf}_{M_0}\eta\begin{pmatrix}\xi_1,\ldots,\xi_q\\ x_1,\ldots,x_q\end{pmatrix}$ *holds for each sequence $(x_1,\ldots,x_q)$ such that $x_1\precsim x_2\precsim\ldots\precsim x_q$.*

Proof. We first show that $S^*(X)$ remains consistent if we add to its axioms 1° all formulas (5.4.1) where $h\in G(X,\precsim)$, ζ is an open matrix of S and $x_1,x_2,\ldots$ are arbitrary elements of X, 2° all matrices

$$\mathrm{subst}\,\eta\begin{pmatrix}\xi_1,\ldots,\xi_q\\ x_1,\ldots,x_q\end{pmatrix} \tag{6.1.3}$$

where $x_1,\ldots,x_q\in X$ and $x_1\precsim x_2\precsim\ldots\precsim x_q$. As before it is sufficient to exhibit for each finite subset X^* of X a model of a theory S^* such that $F(S^*)=F(S)\cup X^*$, $P(S^*)=P(S)$, and $A(S^*)$ consists of $A(S)$ and of those matrices (5.7.2), (5.7.3), and (6.1.3) which contain no x from the outside of X^*.

To achieve this result we repeat word for word the construction carried out in the proof of theorem 5.7 with the only change that we construct the partition not of the whole set Y^n but of its part Y^{*n}. In this way we obtain a pseudo-model M^* of S^* over Y in which $M^*_x\in Y^*$ for $x\in X^*$ and in which axioms belonging $A(S)$ as well as the axioms (5.7.2) and (5.7.3) are valid. If $x_1,\ldots,x_q\in X^*$ and $x_1\precsim\ldots\precsim x_q$, then $M^*_{x_1}\leqslant M^*_{x_2}\leqslant\ldots\leqslant M^*_{x_q}$ (cf. (5.7.7)) and, since $M^*_{x_1},\ldots,M^*_{x_q}$ belong to Y^*, the assumptions of the theorem yield

$$\mathrm{stsf}_{M^*}\mathrm{subst}\,\eta\begin{pmatrix}\xi_1,\ldots,\xi_q\\ x_1,\ldots,x_q\end{pmatrix}.$$

The consistency of $S^*(X)$ extended as indicated above is thus proved.

We now select a complete set I which contains $A(S^*(X))$ as well as matrices (5.4.1), and (6.1.3), and consider the model $\mathfrak{M}(X,I)$ of S

over $\mathfrak{X}_X$. Each function $h \in G(X, \preccurlyeq)$ determines an automorphism f_h of $\mathfrak{M}(X,I)$ (cf. lemma 5.4), and the formula

$$\mathrm{stsf}_{\mathfrak{M}(X,I)}\,\eta\begin{pmatrix}\xi_1\ ,\ldots,\ \xi_q\\ [x_1],\ldots,[x_q]\end{pmatrix}$$

holds for each sequence $(x_1,\ldots,x_q)$ such that $x_1 \preccurlyeq x_2 \preccurlyeq \ldots \preccurlyeq x_q$ (cf. lemma 3.6). Owing to the fact that $[x'] \neq [x'']$ for $x' \neq x''$, we can identify the classes $[x]$ where $x \in X$ with the elements x, and obtain thus a model M_0 satisfying (6.1.1) and (6.1.2).

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On ω-models which are not β-models

by

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In this paper we shall prove a theorem which, roughly speaking, says that β-models for the second-order arithmetic (see [1]) cannot be distinguished from ω-models by elementary sentences. Although this result is by no means surprising, the proof of it is not immediately obvious. In section 6 we state a similar result for models of the Zermelo–Fraenkel set theory and give a solution of a problem concerning the existence of models which are $\aleph_\tau$-standard but are not $\aleph_{\tau+1}$-standard. This problem was formulated in [3].

1. Syntax. In our formal language we shall use $\vee$, &, $\rightarrow$, $\neg$, $\equiv$ as propositional connectives, (E), ( ) as quantifiers. Variables will be denoted by Roman letters and the predicate of identity by "$\approx$". We shall use the abbreviation $(\mathrm{E!\,x})F$ for $(\mathrm{Ez})(\mathrm{x})[(\mathrm{x} \approx \mathrm{z}) \equiv F]$. (1)

We shall consider a first order theory T which has the primitive predicates N, S, E, A, P and possibly still other predicates. N, S will have one argument, E two and A, P three. We read N(x) as "x is an integer", S(x) as "x is a set of integers", E(x, y) as "x is an element of y", A(x, y, z) as "x is the sum of y and z" and P(x, y, z) as "x is the product of y and z".

In order to make our formulae more readable we introduce a number of simplifications.

We shall abbreviate $(\mathrm{x})[\mathrm{N}(\mathrm{x}) \rightarrow \ldots]$ as $(\mathrm{x})_{\mathrm{N}} \ldots$ and $(\mathrm{Ex})[\mathrm{N}(\mathrm{x}) \& \ldots]$ as $(\mathrm{Ex})_{\mathrm{N}} \ldots$; we also use similar symbols for quantifiers limited to S. Sometimes even the index N or S can be omitted, because we shall use lower case Roman letters a, b, ..., n as variables "ranging over elements of N" and upper case Roman letters X, Y, ..., F, ... as "variables ranging over elements of S". (Letters x, y, ... will be used whenever the domain

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(1) We use in the meta-language the abbreviations $(\exists x)$, $(\forall x)$, and $\Rightarrow$ for "there is an x", "for every x", and "if..., then...". The symbol "&" will also be used as an abbreviation of "and" and the symbol "ϵ" as an abbreviation of "is an element of"

of variability is unrestricted). Also a formula F in which the variable a occurs will be thought of as an abbreviation of $N(a) \rightarrow F$ and similarly for formulae with other variables b, c, ... Similar remarks apply to formulae with the free variables X, Y, ... Finally we write "$x \epsilon y$" for $E(x, y)$.

The axioms will be interspersed with definitions (numbered D1, D2, ...). At each point when axioms formulated up to this place allow one to derive a theorem of the form $(E!x) F(x, ...)$ we shall allow a definition of the form $f(...) = (\iota x) F(x, ...)$; the symbol f will be allowed to occur in subsequent axioms.

Of course, all these abbreviations and simplifications are really not necessary: with some patience it would be possible to write all axioms in the "official" language of the first order logic.

I. ARITHMETICAL AXIOMS.

1. $[A(x, y, z) \vee P(x, y, z)] \rightarrow N(x) \,\&\, N(y) \,\&\, N(z)$.
2. $(E!a) A(a, b, c) \,\&\, (E!a) P(a, b, c)$.

D1. $b + c = (\iota a) A(a, b, c)$, $b \cdot c = (\iota a) P(a, b, c)$.

3. $(E!a) A(a, a, a)$.

D2. $0 = (\iota a) A(a, a, a)$.

4. $(E!a)[\neg (a \approx 0) \,\&\, P(a, a, a)]$.

D3. $1 = (\iota a)\big(\neg (a \approx 0) \,\&\, P(a, a, a)\big)$.

5. $\neg (a + 1 \approx 0)$.
6. $(a + 1 \approx b + 1) \rightarrow (a \approx b)$.
7. $a + 0 \approx a$.
8. $a + (b + 1) \approx (a + b) + 1$.
9. $a \cdot 0 \approx 0$.
10. $a \cdot (b + 1) \approx (a \cdot b) + a$.

II. SET-THEORETIC AXIOMS.

1. $\neg S(a)$.
2. $(x \epsilon y) \rightarrow N(x) \,\&\, S(y)$.
3. $(a)[(a \epsilon X) \equiv (a \epsilon Y)] \rightarrow (X \approx Y)$.

III. AXIOM OF INDUCTION.

$$(0 \epsilon X) \,\&\, (a)[(a \epsilon X) \rightarrow (a + 1 \epsilon X)] \rightarrow (a \epsilon X) .$$

IV. AXIOM SCHEME OF COMPREHENSION.

$$(EX)(a)[(a \epsilon X) \equiv \Phi];$$

in this axiom Φ may be any formula in which the variable X does not occur freely.

D4. $\{a \colon \Phi\} = (\iota X)(a)[(a \epsilon X) \equiv \Phi]$.

In D4 we assume that Φ does not contain X as a free variable; of course, D4 is not a single definition but a scheme.

From the above axioms one can deduce the theorem

$$(E!c)\big((c+c) \approx (a+b)\cdot[(a+b)+1]\big)\,,$$

and hence we can formulate the definitions

D5. $(a, b) = (\iota c)\{(c+c) \approx (a+b)\cdot[(a+b)+1]\}$.

D6. $X^{(a)} = \{b\colon (a, b) \in X\}$.

V. Axiom scheme of choice.

$$(a)(EX)\Phi \rightarrow (EY)(a)(EX)[(X \approx Y^{(a)}) \,\&\, \Phi].$$

In this scheme Φ is any formula in which the variable Y is not free.

It is known that axiom scheme V implies IV but we shall not use this fact in our considerations.

2. Auxiliary formal theorems and definitions. In this section we collect some further abbreviations and definitions and formulate a few theorems which can be proved in the basis of axioms I-V.

D7. $aXb \equiv (a, b) \in X$.

D8. $\mathrm{Ord}(X) \equiv (a)(aXa) \,\&\, (a)(b)(c)[(aXb) \,\&\, (bXc) \rightarrow$
$\rightarrow (aXc)] \,\&\, (a)(b)[(aXb) \vee (a \approx b) \vee (bXa)] \,\&$
$(a)(b)[(aXb) \,\&\, (bXa) \rightarrow (a \approx b)]$.

D9. $\mathrm{Bord}(X) \equiv \mathrm{Ord}(X) \,\&\, (Y)(a)\big((a \in Y) \rightarrow (Eb)\{(b \in Y) \,\&$
$(c)[(c \in Y) \rightarrow (bXc)]\}\big)$.

Obviously Ord defines "orderings of N" and Bord "well-orderings of N".

D10. $\mathrm{Fn}(X) \equiv (a)(E!b)(aXb) \,\&$
$(a)(a')(b)[(aXb) \,\&\, (a'Xb) \rightarrow (a \approx a')]$.

This formula defines "one-one mappings of N into N".

D11. $\mathrm{Imb}(F, X, Y) \equiv \mathrm{Fn}(F) \,\&\, (a)(a')(b)(b')\{aFb \,\&\, a'Fb' \rightarrow$
$\rightarrow [(aXa') \equiv (bYb')]\}$.

This formula defines the notion: F is an isomorphic imbedding of the relation aXa' in the relation bYb'.

D12. $X \precsim Y \equiv (EF)\mathrm{Imb}(F, X, Y)$.

It is very easy to show that the transitivity of $\precsim$ is provable in T:

$$(X \precsim Y) \,\&\, (Y \precsim Z) \rightarrow (X \precsim Z)\,.$$

We mention still that for each integer $n \geqslant 1$ it is possible to define a formula Q_n with $n+1$ free variables $a, a_1, \ldots, a_n$ such that the following theorems are provable:

(*) $(\mathrm{E!a})Q_n(\mathrm{a}, \mathrm{a}_1, \ldots, \mathrm{a}_n)$;

(**) $(\mathrm{E!a}_1, \ldots, \mathrm{a}_n)Q_n(\mathrm{a}, \mathrm{a}_1, \ldots, \mathrm{a}_n)$.

Thus Q_n allows us to define a "one-one mapping of N^n onto N". The definition of Q_n proceeds by induction:

$$Q_1(\mathrm{a}, \mathrm{a}_1) \equiv (\mathrm{a} \approx \mathrm{a}_1);$$

$$Q_{n+1}(\mathrm{a}, \mathrm{a}_1, \ldots, \mathrm{a}_n, \mathrm{a}_{n+1}) \equiv (\mathrm{Eb})\big[Q_n(\mathrm{b}, \mathrm{a}_1, \ldots, \mathrm{a}_n) \,\&\, \big(\mathrm{a} \approx (\mathrm{b}, \mathrm{a}_{n+1})\big)\big].$$

In view of (*) and (**) we can admit for each n and each $i \leqslant n$ the definition

D13. $\mathrm{pr}_i^n(\mathrm{a}) = (\iota \mathrm{a}_i)(\mathrm{Ea}_1, \ldots, \mathrm{a}_{i-1}, \mathrm{a}_{i+1}, \ldots, \mathrm{a}_n)Q_n(\mathrm{a}, \mathrm{a}_1, \ldots, \mathrm{a}_n)$.

3. Relational systems. We shall denote by L the first order language in which formulae of T are written. Since we shall also deal with various extensions of L, we shall recall here some definitions from model theory in case of an arbitrary first order language L^* whose expressions contain not only predicates but individual constants as well.

A relational system $\mathfrak{M}$ of type L^* is an ordered pair $\langle A, \mu\rangle$ where A is a set and μ a function; the domain of μ is the set of all primitive predicates and of individual constants of L^* and $\mu(c) \in A$ if c is an individual constant, $\mu(\varrho) \subseteq A^n$ if ϱ is an n-ary predicate other than $\approx$ and $\mu(\approx) = \{\langle x, x\rangle : x \in A\}$. We use capital German letters to denote relational systems. Instead of $\mu(\mathrm{N})$ we shall write $N_{\mathfrak{M}}$ and similarly for other (primitive or defined) predicates other than $\approx$. The values of various terms in $\mathfrak{M}$ will be denoted by a suffix $\mathfrak{M}$ added to the term; e.g. $(a, b)_{\mathfrak{M}}$ denotes the value of the term (a, b) for the assignment of a to the variable a and of b to the variable b.

The semantical notions of satisfaction, model, elementary extension, reduct, diagram, etc. are defined as usual. The notion of definability will be used in the following sense. A relation $R \subseteq A^n$ is definable in $\mathfrak{M}$ if there are an integer k, a sequence $b_1, \ldots, b_k$ of elements of A and a formula F of L^* with $n+k$ free variables such that $\langle a_1, \ldots, a_n\rangle \in R$ if and only if $\models_{\mathfrak{M}} F[a_1, \ldots, a_n; b_1, \ldots, b_k]$ for arbitrary $a_1, \ldots, a_n$ in A.

If L^* contains the predicates N, A; P of T, then the relational system $\langle N_{\mathfrak{M}}, \mu'\rangle$ where $\mu'(\mathrm{A}) = A_{\mathfrak{M}}$ and $\mu'(\mathrm{P}) = P_{\mathfrak{M}}$ is called the *arithmetical part* of $\mathfrak{M}$.

A model $\mathfrak{M}$ of T is called an *ω-model* if its arithmetical part is isomorphic to the standard model $\mathfrak{A}_0$ of arithmetic. In this case we shall usually identify the arithmetical part of $\mathfrak{M}$ with $\mathfrak{A}_0$ and each X in $S_{\mathfrak{M}}$ with the set of integers n which together with X satisfy the formula $\mathrm{n} \in \mathrm{X}$ in $\mathfrak{M}$.

A model $\mathfrak{M}$ of T is called a *β-model* if for each X in $S_{\mathfrak{M}}$ the condition $\models_{\mathfrak{M}} \mathrm{Bord}[X]$ implies that the relation $\{\langle m, n\rangle \in N_{\mathfrak{M}}^2 : \models_{\mathfrak{M}} m X n\}$ well orders the set $N_{\mathfrak{M}}$. (Strictly speaking, we should have written $\models_{\mathfrak{M}} \mathrm{Bord}(\mathrm{X})[X]$ and $\models_{\mathfrak{M}} (\mathrm{m X n})[m, X, n]$ instead of $\models_{\mathfrak{M}} \mathrm{Bord}[X]$ and $\models_{\mathfrak{M}} m X n$ but we shall use the simplified way of writing whenever possible.) It is known (and easy to prove) that β-models are ω-models but not conversely.

4. The pigeon-hole principle. As is well known this principle says that if many object are put into a small number of drawers, then at least one drawer contains many objects. In our case the objects will be well orderings of integers and the number of drawers will be denumerable

Let Φ be a formula of L in which U is a free variable and Ψ a formula of L in which U and a are free variables. We shall write these formulae as $\Phi(\mathrm{U})$ and $\Psi(\mathrm{U}, \mathrm{a})$ although we do not exclude the possibility that one or both of these formulae contain free variables other that U and a.

Let A be the conjunction of the following formulae:

$$(1) \qquad (\mathrm{X})\{\mathrm{Bord}(\mathrm{X}) \rightarrow (\mathrm{EU})[\Phi(\mathrm{U}) \,\&\, (\mathrm{X} \precsim \mathrm{U})]\};$$

$$(2) \qquad (\mathrm{U})(\mathrm{Ea})[\Phi(\mathrm{U}) \rightarrow \Psi(\mathrm{U}, \mathrm{a})].$$

THEOREM 1. *The following formula is provable in T*:

$$\mathrm{A} \rightarrow (\mathrm{Ea})(\mathrm{X})\{\mathrm{Bord}(\mathrm{X}) \rightarrow (\mathrm{EU})[\Psi(\mathrm{U}, \mathrm{a}) \,\&\, (\mathrm{X} \precsim \mathrm{U})]\}.$$

Instead of carrying out a formal proof using axioms of T and rules of proof formulated in logic we shall sketch it in the everyday's language of the "working mathematician". We shall supply enough details to convince the reader that the proof can be transformed into a formal proof in T.

We assume A and the negation of the formula after the first arrow, i.e. the formula

$$(3) \qquad (\mathrm{a})(\mathrm{EX})\{\mathrm{Bord}(\mathrm{X}) \,\&\, (\mathrm{U})[\Psi(\mathrm{U}, \mathrm{a}) \rightarrow \neg (\mathrm{X} \precsim \mathrm{U})]\}.$$

Our aim is to derive a contradiction from these assumptions.

First we use the axiom of choice and derive from (3)

$$(4) \qquad (\mathrm{EY})(\mathrm{a})\{\mathrm{Bord}(\mathrm{Y}^{(\mathrm{a})}) \,\&\, (\mathrm{U})[\Psi(\mathrm{U}, \mathrm{a}) \rightarrow \neg (\mathrm{Y}^{(\mathrm{a})} \precsim \mathrm{U})]\}.$$

Let Y satisfy the condition stated above. From axiom IV we easily derive that there is a Z such that the following equivalence holds for arbitrary $\mathrm{a}, \mathrm{a}', \mathrm{n}, \mathrm{n}'$:

$$(5) \qquad (\mathrm{a}, \mathrm{n})\mathrm{Z}(\mathrm{a}', \mathrm{n}') \equiv \{(\mathrm{a} < \mathrm{a}') \vee [(\mathrm{a} \approx \mathrm{a}') \,\&\, (\mathrm{n}\,\mathrm{Y}^{(\mathrm{a})}\mathrm{n}')]\}.$$

We want to show that $\mathrm{Y}^{(\mathrm{a})}$ can be imbedded into Z. The imbedding function is obviously the map $\mathrm{n} \rightarrow \langle \mathrm{a}, \mathrm{n} \rangle$. Formally speaking, we define $\mathrm{F}^{(\mathrm{a})}$ as $\{\mathrm{b}\colon (\mathrm{En})(\mathrm{b} \approx \langle \mathrm{n}, (\mathrm{a}, \mathrm{n})\rangle)\}$ and prove using D10 that $\mathrm{Fn}(\mathrm{F}^{(\mathrm{a})})$. Since

$$\mathrm{n}\,\mathrm{Y}^{(\mathrm{a})}\,\mathrm{n}' \equiv (\mathrm{a}, \mathrm{n})\mathrm{Z}(\mathrm{a}, \mathrm{n}'),$$

we infer using D11 that $\mathrm{Imb}(\mathrm{F}^{(a)}, \mathrm{Y}^{(a)}, \mathrm{Z})$. Hence by D12

$$\mathrm{Y}^{(a)} \precsim \mathrm{Z}\,. \tag{6}$$

On the other hand, we can derive from (5) that Bord(Z) and hence, according to (1) and (2) that there is an a and a U such that $\Psi(\mathrm{U}, \mathrm{a})$ and $\mathrm{Z} \precsim \mathrm{U}$. Using (6) we obtain $\mathrm{Y}^{(a)} \precsim \mathrm{U}$ since the transitivity of $\precsim$ is provable in T. But now we have a contradiction since, according to (4) for no U such that $\Psi(\mathrm{U}, \mathrm{a})$ does the formula $\mathrm{Y}^{(a)} \precsim \mathrm{U}$ hold. Our theorem is thus proved.

We do not know whether this theorem remains valid when the axiom scheme V of choice is removed from the axioms of T.

We shall formulate theorem 1 in a semantical way. Let $\mathfrak{M}$ be a model of T and let $K = \{X \in S_{\mathfrak{M}}\colon \models_{\mathfrak{M}} \mathrm{Bord}[X]\}$. We shall say that a set $C \subseteq S_{\mathfrak{M}}$ is *unbounded* if for every X in K there is a U in C such that $X \precsim_{\mathfrak{M}} U$.

We say that a relation $D \subseteq N_{\mathfrak{M}} \times S_{\mathfrak{M}}$ covers C if for every element X in C there is an a in $N_{\mathfrak{M}}$ such that $\langle a, X\rangle \in D$. This can be expressed as $C \subseteq \bigcup \{D_a\colon a \in N_{\mathfrak{M}}\}$ where D_a is the set $\{X \in S_{\mathfrak{M}}\colon \langle a, X\rangle \in D\}$.

The pigeon hole principle in its semantical form is the following result:

THEOREM 2. *If $\mathfrak{M}$ is a model of T and D is a definable relation $\subseteq N_{\mathfrak{M}} \times S_{\mathfrak{M}}$ which covers an unbounded definable set $C \subseteq S_{\mathfrak{M}}$, then at least one D_a is unbounded.*

Proof. It is sufficient to take in theorem 1 for Φ a formula which defines C and for Ψ a formula which defines the relation D.

5. A theorem on β-models.

In this section we shall use the pigeon hole principle in order to establish our main result.

THEOREM 3. *For any denumerable β-model $\mathfrak{M}$, there exists an ω-model which is an elementary extension of $\mathfrak{M}$ and is not a β-model.*

Proof. We introduce, as auxiliary symbols, the constant symbols Δ_m for every element m of $\mathfrak{M}$ and the constant symbol R. The language L augmented by those symbols is denoted by L_1.

The interpretation of the symbols of the language is determined by the structure $\mathfrak{M}$.

The value of R will in most cases be an element of $S_{\mathfrak{M}}$ which satisfies the formula Bord(X) in $\mathfrak{M}$.

In the relational systems of type L_1, which we shall consider, the constant Δ_m will always be interpreted as m. Hence the relational systems are determined by the value R of the constant R and can be denoted by $(\mathfrak{M}, R)$.

We shall assume that the arithmetical part of $\mathfrak{M}$ has been identified with $\mathfrak{A}_0$ (cf. p. 86) and elements of $S_{\mathfrak{M}}$ with sets of integers. We can and will interpret each element X of $S_{\mathfrak{M}}$ as a binary relation $\{\langle m, n\rangle : \models_{\mathfrak{M}} m X n\}$; in case this relation is many-one we can speak of X as being a function.

As in section 4 we denote by K the set of all X in $S_{\mathfrak{M}}$ for which $\models_{\mathfrak{M}} \mathrm{Bord}[X]$.

Let A be the set of all sentences of L_1 which do not contain symbol R and are true in the structure $\mathfrak{M}$. We can represent the set A as the union of an increasing sequence $\langle A_n\rangle_{n\in\omega}$ of finite sets of sentences for which the following condition (A) holds:

$$(\mathrm{Ev})\big(\mathrm{N}(\mathrm{v}) \,\&\, \Psi(\mathrm{v})\big) \in A_n \Rightarrow (\exists i)(\Psi(\Delta_i) \in A_n) .$$

Let us fix an enumeration $\langle \Phi_i\rangle_{i\in\omega}$ of all the sentences of the language L_1.

Let us say that R is in the class $D_S(i_0, \ldots, i_n)$ if the following conditions are satisfied:

I. *$R \in K$ and $S \in K$.*

II. *$i_n R i_{n-1} \ldots i_1 R i_0$ and $i_n \neq i_{n-1} \neq \ldots i_1 \neq i_0$.*

III. ***There is a function in $\mathfrak{M}$ which maps the field of S order-isomorphically into the R-predecessors of i_n.***

It is obvious that D_S is extensional in the following sense: whenever S and S' are in $\mathfrak{M}$ and there is in $\mathfrak{M}$ a function which establishes an isomorphism between S and S', then $D_S(i_0, \ldots, i_n) = D_{S'}(i_0, \ldots, i_n)$. More generally, this equation holds for arbitrary S, S' in $S_{\mathfrak{M}}$ such that $S \precsim_{\mathfrak{M}} S'$ and $S' \precsim_{\mathfrak{M}} S$.

We define by induction a monotonically increasing sequence $\langle B_n\rangle_{n\in\omega}$ of finite sets of sentences of the language L_1 and a sequence of natural numbers $\langle i_n\rangle_{n\in\omega}$. These sequences are required to satisfy the following conditions $\langle C_n\rangle_{n\in\omega}$

(i) Bord(R) *is in* B_n,

(ii) $A_n \subseteq B_n$,

(iii) *if* $n > 0$, *then the sentences* $\Delta_{i_n} \mathrm{R} \Delta_{i_{n-1}}$ *and* $\neg(\Delta_{i_n} \approx \Delta_{n-1})$ *are in* B_n,

(iv) *for* $j < n$, *either* Φ_j *or* $\neg\Phi_j$ *is in* B_n,

(v) *if* $j < n$, Φ_j *is in* B_n *and* Φ_j *has the form* $(\mathrm{Ev})\big(\mathrm{N}(\mathrm{v}) \,\&\, \Psi(\mathrm{v})\big)$, *then* $\Psi(\Delta_i)$ *is in* B_n *for some* i *in* ω,

(vi) *for every* S *in the class* K, *there is an* R *such that* $R \in D_S(i_0, \ldots, i_n)$ *and* $\models_{(\mathfrak{M},R)} B_n$.

Construction of the sequences.

Step 0.

Determination of the number i_0. i_0 can be any natural number, say 0.

Determination of the set B_0. We take $A_0 \cup \{\mathrm{Bord}(\mathrm{R})\}$ as B_0.

Verification of the conditions C_0. Conditions (i) and (ii) are evident. Conditions (iii), (iv) and (v) are true vacuously. Finally, (vi) is satisfied, because for every S in K there is an R in K such that the last element of the field of R is i_0 and there is a function F in $\mathfrak{M}$ which maps the field of S order-isomorphically into the set of R-predecessors of i_0. If e.g. $i_0 = 0$, then it is sufficient to take $F(n) = n+1$ and define R as the set consisting of all pairs $(n, 0)_{\mathfrak{M}}$ and of pairs $(F(n), F(m))_{\mathfrak{M}}$ with nSm.

Step $n+1$. We assume that we have already defined the sequences $\langle B_j\rangle_{0\leqslant j\leqslant n}$ and $\langle i_j\rangle_{0\leqslant j\leqslant n}$ which satisfy conditions $\langle C_j\rangle_{0\leqslant j\leqslant n}$. Let I_ε be the class of all S such that $S \in K$ and

$$(\exists R)\big(R \in D_S(i_0, \ldots, i_n) \;\&\; \models_{(\mathfrak{M},R)} B_n \;\&\; \models_{(\mathfrak{M},R)} \Phi_n^\varepsilon\big)\,,$$

where $\varepsilon \in \{0, 1\}$ and $\Phi_n^0 = \Phi_n$ and $\Phi_n^1 = \neg\Phi_n$. Since the set B_n is finite, the set I_ε is definable in the structure $(\mathfrak{M}, R)$. We shall show that either I_0 or I_1 coincides with the class K of all well-orderings of ω in the structure $\mathfrak{M}$. Let us assume $S \notin I_0$ for some S in the class K. Hence

$$(\forall R)_{\mathfrak{M}}\big(R \in D_S(i_0, \ldots, i_n) \;\&\; \models_{(\mathfrak{M},R)} B_n \Rightarrow \models_{(\mathfrak{M},R)} \neg\, \Phi_n\big)\,.$$

By our inductive assumption (vi), there is an R such that $R \in D_S(i_0, \ldots, i_n)$ and $\models_{(\mathfrak{M},R)} B_n$. S is therefore in I_1. Thus we proved that $I_0 \cup I_1 = K$. The set I_ε is monotone in the sense that if $S' \in I_\varepsilon$ and $S \precsim_{\mathfrak{M}} S'$, then $S \in I_\varepsilon$. Since $I_0 \cup I_1 = K$, either I_0 or I_1 is cofinal with K. The set which is cofinal with K and is monotone must coincide with K. Hence either I_0 or I_1 coincides with K. Let $\bar{\varepsilon}$ be the smallest ε such that $I_\varepsilon = K$.

Determination of the number i_{n+1}. Let S be in the set K_i if and only if

$$S \in K \;\&\; (\exists R)\big(R \in D_S(i_0, \ldots, i_n, i) \;\&$$
$$\models_{(\mathfrak{M},R)} B_n \cup \{\Phi_n^{\bar{\varepsilon}}, \Delta_i \mathrm{R} \Delta_{i_n}, \neg(\Delta_i \approx \Delta_{i_n})\}\big)\,.$$

Let S be in the set K and S^* be an element in K whose order type is the successor of that of S. By our choice of $\bar{\varepsilon}$, there is an $R \in D_{S^*}(i_0, \ldots, i_n)$ such that $\models_{(\mathfrak{M},R)} B_n \cup \{\Phi_n^{\bar{\varepsilon}}\}$. Let i^* be the greatest element in the ordering S^* and let i be the image of i^* by an order-preserving map in $\mathfrak{M}$ of the field of S^* into the R-predecessors of i_n. The conditions $R \in D_S(i_0, \ldots, i_n, i)$ and $\models_{(\mathfrak{M},R)} \{\Delta_i \mathrm{R} \Delta_{i_n}\} \cup \{\neg(\Delta_i \approx \Delta_{i_n})\}$ are satisfied. We have proved, therefore, $(\forall S)\big(S \in K \Rightarrow (\exists i)(S \in K_i)\big)$. Since the relation $S \in K_i$ is definable in $\mathfrak{M}$, we can apply the pigeon hole principle to prove

$$(\exists i)(\forall S)\big(S \in K \Rightarrow (\exists S')(S \precsim_{\mathfrak{M}} S' \;\&\; S' \in K_i)\big)\,.$$

Since the sets K_i are monotone, $(\exists i)(\forall S)(S \in K \Rightarrow S \in K_i)$. We take as i_{n+1} the least such i.

Determination of the set B_{n+1}.

Case 1. *Φ_n^{ε} is already in B_n.* We take $B_n \cup A_{n+1} \cup \{\Delta_{i_{n+1}} \mathrm{R} \Delta_{i_n}\} \cup \{\neg(\Delta_{i_{n+1}} \approx \Delta_{i_n})\}$ as B_{n+1}.

Case 2. *Φ_n^{ε} is not in B_n.*

Subcase 2.1. *$\bar{\varepsilon} = 1$, or $\bar{\varepsilon} = 0$ and Φ_n is not of the form* $(\mathrm{Ev})\big(\mathrm{N}(\mathrm{v}) \,\&\, \Psi(\mathrm{v})\big)$. We take $B_n \cup A_{n+1} \cup \{\Phi_n^{\varepsilon}, \Delta_{i_{n+1}} \mathrm{R} \Delta_{i_n}, \neg(\Delta_{i_{n+1}} \approx \Delta_{i_n})\}$ as B_{n+1}.

Subcase 2.2. *$\bar{\varepsilon} = 0$ and Φ_n is of the form*

$$(\mathrm{Ex}_1)\big(\mathrm{N}(\mathrm{x}_1) \,\&\, \dots \,\&\, (\mathrm{Ex}_s)\big(\mathrm{N}(\mathrm{x}_s) \,\&\, \Psi(\mathrm{x}_1, \dots, \mathrm{x}_s)\big)\dots\big)$$

where $\Psi(\mathrm{x}_1, \dots, \mathrm{x}_s)$ is not of such a form.

Let $\widetilde{\Psi}(\mathrm{a})$ be the formula

$$\Psi\big(\mathrm{pr}_1^s(\mathrm{a}), \dots, \mathrm{pr}_s^s(\mathrm{a})\big)$$

and let $f_i(e)$ be the value of the term $\mathrm{pr}_i^s(\Delta_e)$ in $\mathfrak{M}$. Then we have the equivalence

$$\models_{(\mathfrak{M},R)} \widetilde{\Psi}(\Delta_e) \iff \models_{(\mathfrak{M},R)} \Psi(\Delta_{f_1(e)}, \dots, \Delta_{f_s(e)}) .$$

Let $\mathcal{S}$ in the set C_e if and only if

$$\mathcal{S} \in K \,\&\, (\exists R)(R \in D_{\mathcal{S}}(i_0, \dots, i_{n+1}) \,\&$$
$$\models_{(\mathfrak{M},R)} B_n \cup A_{n+1} \cup \{\widetilde{\Psi}(\Delta_e), \Delta_{i_{n+1}} \mathrm{R} \Delta_{i_n}, \neg(\Delta_{i_{n+1}} \approx \Delta_{i_n})\}.$$

We apply, once again, the pigeon hole principle to the sequence $\langle C_e \rangle_{e \in \omega}$ which is definable in $\mathfrak{M}$. By our choice of the numbers i_{n+1} and $\bar{\varepsilon}$, $(\forall \mathcal{S})\big(\mathcal{S} \in K \Rightarrow (\exists e)(\mathcal{S} \in C_e)\big)$. By applying the pigeon hole principle,

$$(\exists e)(\forall \mathcal{S})\big(\mathcal{S} \in K \Rightarrow (\exists \mathcal{S}')(\mathcal{S} \precsim_{\mathfrak{M}} \mathcal{S}' \,\&\, \mathcal{S}' \in C_e)\big) .$$

Since the sets C_e are monotone, $(\exists e)(\forall \mathcal{S})(\mathcal{S} \in K \Rightarrow \mathcal{S} \in C_e)$. We take as $\bar{e}$ the least such e. We take as B_{n+1}, in this subcase,

$$B_n \cup A_{n+1} \cup \big\{\Psi(\Delta_{f_1(e)}, \dots, \Delta_{f_s(e)}), (\mathrm{Ex}_s)\big(\mathrm{N}(\mathrm{x}_s) \,\&$$
$$\Psi(\Delta_{f_1(e)}, \dots, \Delta_{f_{s-1}(e)}, \mathrm{x}_s)\big), \dots, \Phi_n\big\} \cup \{\Delta_{i_{n+1}} \mathrm{R} \Delta_{i_n}\} \cup \{\neg(\Delta_{i_{n+1}} \approx \Delta_{i_n})\} .$$

Verification of the conditions C_{n+1}. In all cases conditions (i)-(iv) are clearly satisfied. Conditions (v) and (vi) are satisfied in subcase 2.1 because of our choice of the numbers i_{n+1} and $\bar{\varepsilon}$ and because of property (A) of the sequence $\bigcup\limits_n A_n = A$. Conditions (v) and (vi) are satisfied in the subcase 2.2 because of property (A) of the sequence $\bigcup\limits_n A_n = A$ and because of the choice of the numbers $i_{n+1}, \bar{\varepsilon}, \bar{e}$.

Let us consider the set $B = \bigcup\limits_n B_n$. This set is consistent since every finite subset of B has a model by condition (vi). The set B is ω-closed by condition (v). By the Henkin–Orey completeness theorem for ω-closed consistent theories (cf. [2], p. 231) there is an ω-standard model $\mathfrak{M}_1$ for the theory B. The structure $\mathfrak{M}_1$ is an ω-model since $\mathfrak{M}_1$ is ω-standard. The structure $\mathfrak{M}_1$ is an elementary extension of $\mathfrak{M}$ since the set A is included in the set B. Consider the value R_1 of the constant symbol R in $\mathfrak{M}_1$. By condition (i), $\models_{\mathfrak{M}_1} \mathrm{Bord}[R_1]$. By condition (iii), the sequence $\langle i_n \rangle_{n \in \omega}$ forms a descending chain with respect to the ordering R_1. The ω-model $\mathfrak{M}_1$ is not, therefore, a β-model. Thus the L-reduct of $\mathfrak{M}_1$ is the required model of T and our theorem is proved.

Corollary 1. *For any β-model $\mathfrak{M}$ of T, there is an elementarily equivalent ω-model $\mathfrak{M}_1$ which is not a β-model.*

Proof. Every β-model $\mathfrak{M}$ is an elementary extension of a denumerable β-model $\mathfrak{M}_1$ [1].

Corollary 2. *If there is an ω-model $\mathfrak{M}$ for a set of sentences A, then there is an ω-model $\mathfrak{M}_1$ for the set A which is not a β-model.*

6. An application to set theory. Using the construction carried out in section 5, we can construct a new family of non-standard models for set theory.

Let S be a consistent extension of ZF. A formula φ with one free variable is said to *define a cardinal* in S if the sentence $(\mathrm{E}!\mathrm{v})\{\varphi(\mathrm{v}) \,\&\, (\mathrm{v})[\varphi(\mathrm{v}) \to \mathrm{Card}(\mathrm{v})]\}$ is provable in S. We denote by φ^+ the formula

$$\mathrm{Card}(\mathrm{v}_0) \,\&\, (\mathrm{v}_1)\big(\varphi(\mathrm{v}_1) \to \mathrm{v}_1 < \mathrm{v}_0\big) \,\&$$
$$(\mathrm{v}_1)(\mathrm{v}_2)\big((\mathrm{v}_2 < \mathrm{v}_0)\, \mathrm{Card}(v_2) \,\&\, \varphi(\mathrm{v}_1) \to (\mathrm{v}_2 \leqslant \mathrm{v}_1)\big)\,.$$

A model $\mathfrak{M}$ for S is called *φ-standard* if there is no infinite descending chain $a_0 \ni_{\mathfrak{M}} a_1 \ni_{\mathfrak{M}} \ldots$ of ordinals smaller than the cardinal $\aleph(\mathfrak{M}, \varphi)$ where $\aleph(\mathfrak{M}, \varphi)$ is the unique element of $\mathfrak{M}$ which satisfies the formula φ in $\mathfrak{M}$.

The existence of a φ-standard, φ^+-non-standard model is known in the case when φ is a formula defining the first infinite cardinal $\aleph_0$ (cf., [3]).

We shall prove the following

Theorem 4. *For any denumerable φ-standard model $\mathfrak{M}$ for S there is an elementary extension $\mathfrak{M}_1$ or $\mathfrak{M}$ which is φ-standard but is not φ^+-standard.*

Proof. We introduce, as auxiliary symbols, the constant symbols Δ_m for every element m of $\mathfrak{M}$, the constant symbol R and an unary predicate symbol N. The interpretation of the symbols Δ_m of the extended language is the same as in the proof of theorem 3. The symbol N is interpreted as the set of ordinals smaller than $\aleph(\mathfrak{M}, \varphi)$. We can define the sequences $\langle B_n \rangle_{n \in \omega}$ and $\langle i_n \rangle_{n \in \omega}$ in almost the same way as above. The pigeon hole principle which played a crucial role in the previous construction can be used in the present situation. To see this we merely notice that $\aleph(\mathfrak{M}, \varphi)$

and $\varkappa(\mathfrak{M}, \varphi^+)$ are different cardinals of $\mathfrak{M}$ and, since $\mathfrak{M}$ is a model ZF, the pigeon hole principle holds in $\mathfrak{M}$ for these cardinals.

We can also prove the following corollaries:

COROLLARY 3. *For any φ-standard model $\mathfrak{M}$ of S there is an elementarily equivalent structure $\mathfrak{M}_1$ which is φ-standard but not φ^+-standard.*

COROLLARY 4. *For any set of sentences A, if there is a φ-standard model $\mathfrak{M}$ of A, then there is a φ-standard, φ^+-non-standard model of A.*

Note added on June 20, 1968. Several weeks after the present paper was accepted for publication we saw a paper: H. J. Keisler and M. Morley, *Elementary Extensions of Models of Set Theory* (Israel Jour. Math. 6 (1968), pp. 49-65) which appeared in March 1968. From the strictly logical point of view the results contained in Sections1-5 of our paper are independent from results established by Keisler and Morley. However, the methods used by these authors are the same as those which were used by us. The results of our Section 6 are weaker than those established by Keisler and Morley.

After some deliberations we decided not to withdraw our paper because we believe that the readers who will compare both papers will get useful insights into the close relationship which exists between the meta-mathematics of set theory and that of the second order arithmetic.

Note added on March 19, 1969. Results of our section 6 were also obtained by K. Hrbaček who used a completely different method.

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OBSERVATIONS CONCERNING ELEMENTARY EXTENSIONS OF ω-MODELS[1]

ANDRZEJ MOSTOWSKI

Y. Suzuki and the author proved in [3] the following:

THEOREM 1. *Every denumerable ω-model of second order arithmetic admits an elementary extension which is also a denumerable ω-model but is not a β-model.*

Here ω-models are relational systems of the form $\langle N \cup S, N, +, \times, \in\rangle$ where N are the integers, $S \subseteq P(N)$ and $+$, $\times$, $\in$ have their usual meanings. A β-model is an ω-model M with the following property: Whenever F is a formula with all quantifiers restricted to N and $a_1, a_2, \ldots$ are elements of M which satisfy in M the formula $(Y)_S F$, then these elements satisfy this formula in the principal model $\langle N \cup P(N), N, +, \times, \in\rangle$.

By the second order arithmetic (A_2 for short) we mean the system described in [3]. We denote by A_2^- the system obtained from A_2 by dropping the axiom-scheme of choice.

The present paper arose from unsuccessful attempts to generalize Theorem 1 to the case of ω-models of A_2^-. These attempts have led the author to formulate the concept of a quantifier definable in a given system. This concept together with an application to an alternative proof of Theorem 1 will be presented below. The

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[1] In the lecture given at the Symposium I spoke about Tarski's work on metamathematics of set-theory. In particular I stressed the importance of his basic paper devoted to the concept of truth for the formulation of axioms of infinity and on his work on the measurability of the first inaccessible aleph. I also mentioned his paper *Einige Betrachtungen über die Begriffe der ω-Widerspruchsfreiheit und ω-Vollständigkeit*, Monatsh. Math. und Phys. 40 (1933), 97–112, where we find the first application of the now widely used method of proving the consistency of a (not finitely axiomatisable) theory T by finding models for every finitely axiomatisable fragment of T. Results contained in the present paper were mentioned in my lecture as recent applications of this method.

author hopes that a further discussion of definable quantifiers will eventually lead to a solution of the problem of whether Theorem 1 is valid for models of A_2^-, and perhaps of other problems.

1. Although ω-models of A_2 will be our main interest we prefer to formulate definitions in a more general setting. Thus we consider a denumerable first order language L and single out in L a one-place predicate N. We also assume that L contains denumerably many individual constants, called numerals, $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$ and possibly other constants.

If $\mathcal{N}$ is a denumerable set then a model of L will be called an ω-$\mathcal{N}$-model if it is isomorphic with a model in which the denotations of the numerals belong to $\mathcal{N}$ and each element of $\mathcal{N}$ is a denotation of a numeral.

Let F be a letter not occurring in L. We add to the atomic formulae of L expressions of the form $F(x)$ where x is a term of L. We call these expressions "auxiliary atomic formulae" and denote by L^* the new language.

Each sentence of L^* containing the letter F will be called a definable quantifier.

If A is a formula of L, v a variable and Q a definable quantifier, then we denote by $(Qv)A$ a formula of L obtained by means of the following operations:

(1) All bound variables of Q are replaced by variables occurring neither in A nor in Q.

(2) For every variable w each occurrence of an auxiliary atomic formula $F(w)$ in Q is replaced by $Sb(v/w)A$.

EXAMPLES. (1) Let L be the language of A_2 where N is the predicate "to be an integer" and the numerals $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$ are defined as usual. Then, the following sentence of L^*,

$$\Omega: (X)(EY)\{F(Y) \;\&\; \neg(Ex)(y)[y \in Y \equiv J(x, y) \in X]\},$$

is a definable quantifier (see [3] for the definition of symbols used in this definition). The sentence $(\Omega v)A(v)$ means that the family of sets satisfying A is nondenumerable.

(2) With L the same as above the sentence of L^*,

$$\Xi: (X)\{\mathrm{Bord}(X) \rightarrow (EY)[F(Y) \;\&\; \mathrm{Bord}(Y) \;\&\; (X \prec Y)]\},$$

is a definable quantifier. Here $\mathrm{Bord}(X)$ is a formula which says that X is a well-ordering of ω and $X \prec Y$ is a formula which says that the relation

$$\{\langle x, y\rangle : J(x, y) \in X\}$$

is embeddable in $\{\langle x, y\rangle : J(x, y) \in Y\}$; see [3].

The sentence $(\Xi X)A(X)$ "says" that there are arbitrarily large well-orderings of ω which satisfy A.

(3) Let L be the language of the Zermelo-Frankel set theory with additional individual constants $a, \mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$. We define N as the predicate $x \in a$.

The sentence of L^*,

$$\Theta: (x)\{\mathrm{On}(x) \rightarrow (Ey)[F(y) \;\&\; (\mathrm{rk}(y) > x)]\},$$

is a definable quantifier and the sentence $(\Theta x)A(x)$ means that there are elements y of arbitrary high ranks which satisfy A.

All these quantifiers are similar to the ordinary general quantifier insofar as they state the existence of "many" elements satisfying a given condition.

We shall now discuss some properties of quantifiers pertaining to deducibility of sentences involving them. Thus we fix a set Ax of sentences of L and call them axioms.

Definition. A definable quantifier Q is said to be (a) nontrivial, (b) additive, (c) σ-additive, (d) monotone with respect to Ax if for each formulae A, A_1, A_2 of L, with exactly one free variable x, and each formula B of L, with exactly two free variables x, y, the following formulae are provable in Ax:

(a) $$(Qx)A(x) \to (Ex)(Ey)[\neg(x = y) \,\&\, A(x) \,\&\, A(y)],$$

(b) $$(Qx)[A_1(x) \vee A_2(x)] \equiv [(Qx)A_1(x) \vee (Qx)A_2(x)],$$

(c) $$(Qx)(Ey)[N(y) \,\&\, B(x, y)] \equiv (Ey)[N(y) \,\&\, (Qx)B(x, y)],$$

(d) $$(x)[A_1(x) \to A_2(x)] \to (Qx)[A_1(x) \to (Qx)A_2(x)].$$

Examples. The quantifiers Ω and Ξ have the properties (a), (b), (c), (d) with respect to A_2; the quantifier Θ has these properties with respect to ZF.

Proofs of (a), (b) are immediate.

Proof of (c) for the quantifier Ξ is contained in [3, Theorem 1, p. 87]. In the case of quantifier Ω we obtain this proof by formalizing in A_2 the usual proof that if a denumerable union of sets is not denumerable, then at least one member of the union is not denumerable (cf. also [1, p. 156]). In the case of the quantifier Θ we formalize in ZF the proof that if a class $\bigcup_{x \in a} K(x)$ contains elements of an arbitrary high rank then so does at least one class $K(x)$.

The following two lemmas are valid for an arbitrary nontrivial additive monotone quantifier.

Lemma 1. *Whenever F and G are two formulae each with at most one free variable, if $M \models (Qx)F(x)$ and $M \models \neg(Qx) \neg G(x)$, then $M \models (Qx)[F(x) \,\&\, G(x)]$.*

(*Hint.* Use the equivalence $F(x) \equiv [F(x) \,\&\, G(x)] \vee [F(x) \,\&\, \neg G(x)]$.)

Lemma 2. *If $M \models (Qx)F(x)$, then $M \models (Qx)[F(x) \,\&\, \neg(x = a)]$ for each individual constant of L.*

This follows from Lemma 1 applied to the formula $G(x)$: $\neg x = a$.

We shall need in the sequel a more special property of a definable quantifier: Let U be a formula of L with just one free variable x and let R be a formula of L with just two free variables. A definable quantifier Q will be called normal with respect to a set Ax of axioms and to formulae U, R if the formulae

(e) $$(Qx)[U(x) \,\&\, \neg N(x)],$$

(f) $$U(x) \to \neg(Qy) \neg R(x, y),$$

are provable from axioms Ax.

Examples. Ω is normal with respect to A_2 and formulae $S(x)$ and $x \neq y$;

Ξ is normal with respect to A_2 and formulae $\mathrm{Bord}(x)$ and $\mathrm{Bord}(y) \to x \prec y$; Θ is normal with respect to ZF and formulae $x = x$ and $x \neq y$.

THEOREM 2. *If Q is a normal, nontrivial, additive, monotone and σ-additive quantifier with respect to a set Ax of axioms and to formulae U, R and if M is a denumerable ω-$\mathcal{N}$-model of Ax, then there is an elementary extension M^* of M which is an ω-$\mathcal{N}$-model of Ax and which contains an element c such that $c \notin M$, $M^* \models U[c]$ and $M^* \models R[x, c]$ for each element x of M satisfying $M \models U[x]$.*

Before proving Theorem 2 we shall derive from it some corollaries by applying it to the quantifiers Ω, Ξ, Θ.

COROLLARY 1. *Each denumerable ω-model of A_2 has a proper elementary extension which is also an ω-model.*

(See [1, Chapter 28] where this result was proved by a very similar method.)

COROLLARY 2. *Each denumerable ω-model M of A_2 has a proper elementary extension which either is not a β-model or is higher than M (the height of a β-model is the least ordinal not representable in M).*

COROLLARY 3. *If M is a denumerable model M of* ZF *and $a \in M$, then there is a proper elementary extension M' of M such that $a \cap M' = a$, i.e., a has in M' exactly those elements which it has in M.*

This corollary is weaker than a theorem of Keisler-Morley [2].

From Corollary 2 we obtain an alternative proof of Theorem 1. Let M be a denumerable β-model of A_2 and assume that every ω-model M' which is an elementary extension of M is a β-model. From Corollary 2 we derive the existence of an increasing chain $M = M_0 \prec M_1 \prec \cdots \prec M_\xi \prec \ldots$, $\xi < \omega_1$, of denumerable β-models with increasing heights. Hence, for each relation $R \subseteq \omega \times \omega$ which well-orders ω, there is an elementary extension M^* of M which is an ω-model and which contains an element X satisfying the condition $M^* \models \mathrm{Bord}(X)$ such that X and R define similar orderings. Conversely if there exists an ω-model $M^* \succ M$ containing X such that $M^* \models \mathrm{Bord}(X)$ and X and R define similar orderings, then R is a well-ordering. To see this we merely notice that, by our assumption, M^* is a β-model. By replacing denumerable ω-models by their codes we easily see that the predicate

$$\begin{aligned}(Ef, X, M^*)[&(M \prec M^*) \;\&\; (M^* \text{ is a denumerable } \omega\text{-model}) \;\&\; (X \in M^*)\\ &\;\&\; (M^* \models \mathrm{Bord}(X)) \;\&\; (f \text{ maps } \omega \text{ onto } \omega \text{ and establishes a}\\ &\qquad \text{similarity between the order types of } R \text{ and } X)]\end{aligned}$$

is a Σ_1^1-predicate. This is impossible because by the classical results of descriptive set theory the predicate "R is a well-ordering of ω" is not Σ_1^1. Thus M must have at least one elementary extension which is an ω-model but not a β-model.

PROOF OF THEOREM 2. An occurrence of the general quantifier "(x)" is called restricted if it is followed by the formula $N(x)$ and the symbol for implication or by the formula $\neg N(x)$ and the symbol of implication. We define similarly restricted occurrences of an existential quantifier. A formula of L in which all the quantifiers have only restricted occurrences is called restricted. For each formula A of L there exists a restricted formula A' such that $A \equiv A'$ is provable in the first order logic.

We form from L a new language L_1 by adding to L constants $\gamma_{a(n)}$ where $a(n)$, $n = 0, 1, 2, \ldots$, is a sequence consisting of all the elements of the given model M. Constants γ_a with a in $\mathcal{N}$ can be identified with the constants $\mathbf{0}, \mathbf{1}, \ldots$, in some appropriate manner. We also arrange in a sequence $\delta_0, \delta_1, \ldots$, those γ_a for which $M \models U[a]$. From L_1 we form a language L_2 adding to L_1 a new constant γ.

Let $A_0, A_1, \ldots$ be a sequence whose terms are all the restricted sentences of L_1 which are true in M under the interpretation of γ_a as a and let $B_0, B_1, \ldots$ be a sequence consisting of all the restricted sentences of L_2.

We shall construct a sequence $\mathscr{F}_0, \mathscr{F}_1, \ldots$ of finite sets of restricted sentences of L_2 such that the following properties are satisfied for each integer n.

(1) $U(\gamma), \neg N(\gamma) \in \mathscr{F}_0$;

(2) if $n > 0$ then $A_{n-1} \in \mathscr{F}_n$;

(3) if $n > 0$ then $R(\delta_{n-1}, \gamma) \in \mathscr{F}_n$;

(4) if $n > 0$ then $\neg(\gamma = \gamma_{a(n-1)}) \in \mathscr{F}_n$;

(5) if $n > 0$ then either $B_{n-1} \in \mathscr{F}_n$ or $(\neg B_{n-1}) \in \mathscr{F}_n$;

(6) if $n > 0$ then $\mathscr{F}_{n-1} \subseteq \mathscr{F}_n$;

(7) if $n > 0$ and a sentence of the form $(Ex)[N(x) \,\&\, C(x)]$ belongs to $\mathscr{F}_{n-1}$, then $C(\mathbf{p}) \in \mathscr{F}_n$ for a suitable p;

(8) $M \models (Qx) \bigwedge \mathscr{F}_n(x)$.

The symbol $\bigwedge \mathscr{F}_n(x)$ in (8) denotes a formula obtained by taking the conjunction of all sentences which belong to $\mathscr{F}_n$ and replacing γ by a variable x which does not occur in any sentence of $\mathscr{F}_n$.

Let us assume for a moment that the sequence $\mathscr{F}_n$ is constructed. In order to achieve the proof of Theorem 2 we notice that the union $\bigcup \mathscr{F}_n$ has the property (7), is complete by (5) and consistent. The consistency follows from the observation that, by (8), M contains an element c_n such that $M \models \bigwedge \mathscr{F}_n[c_n]$ and hence $\mathscr{F}_n$ has a model. By a theorem of Orey (see [1, p. 54]), $\bigcup \mathscr{F}_n$ has a model M^* in which N is interpreted as the set of all denotations of the constants $\mathbf{0}, \mathbf{1}, \ldots$. Let c be the denotation of γ in M^*.

By (2), M can be embedded in M^*. Thus we can assume that $M \subseteq M^*$ and that the denotation of γ_a in M^* is a. In particular the set of denotations of the constants $\mathbf{p}$ in M^* is the same as in M and thus coincides with $\mathcal{N}$. Hence M^* is an ω-$\mathcal{N}$-model. From (2) it follows that $M \prec M^*$, from (4) that $c \in M^* - M$ and from (1) that $M^* \models U[c]$. Finally if $x \in M$ and $M \models U[x]$, then there is a k such that x is the denotation of δ_k (in M and in M^*) and hence, by (3), $M^* \models R[x, c]$. Thus M^* is the required model.

We now indicate how to construct the sequence $\mathscr{F}_n$. For $n = 0$, we take as $\mathscr{F}_0$ the set consisting of just two sentences $U(\gamma)$ and $\neg N(\gamma)$. Conditions (2)–(7) are satisfied vacuously and condition (8) is satisfied because Q is a normal quantifier (see (e)). Let us now assume that $n \geqq 0$ and $\mathscr{F}_n$ satisfies (2)–(8). We shall construct $\mathscr{F}_{n+1}$. Let $\mathscr{F}'_n$ be the set obtained from $\mathscr{F}_n$ by adding the sentences A_n, $R(\delta_n, \gamma)$ and $\neg(\gamma = \gamma_{a(n)})$. Thus (2), (3), (4) will be satisfied whatever still may be added to $\mathscr{F}'_n$. From Lemmas 1 and 2 and the property (f) of Q we infer that $M \models (Qx) \bigwedge \mathscr{F}'_n(x)$. Now we use the additivity of Q and obtain

$$M \models (Qx)[\bigwedge \mathscr{F}'_n(x) \;\&\; B_n(x)] \vee (Qx)[\bigwedge \mathscr{F}'_n(x) \;\&\; \neg B_n(x)].$$

We put $\mathscr{F}''_n = \mathscr{F}'_n \cup \{B_n\}$ if the first member of the above disjunction is valid in M and $\mathscr{F}''_n = \mathscr{F}'_n \cup \{\neg B_n\}$ if the second but not the first member is valid in M. Hence (5) is true for $\mathscr{F}''_n$ and will remain true if we add to $\mathscr{F}''_n$ any number of sentences. Also (8) is valid for $\mathscr{F}''_n$. In order to get a set satisfying all conditions (1)–(8) we have still to satisfy (7). To achieve this we select those sentences in $\mathscr{F}''_n$ which have the form $(Ex)[N(x) \;\&\; C(x)]$.

Let these sentences be $(Ex_i)[N(x_i) \;\&\; C_i(x_i)]$, $i = 0, 1, \dots, k_n$. We shall find integers p_i such that the set $\mathscr{F}_{n+1} = \mathscr{F}''_n \cup \{C_0(\mathbf{p}_0), \dots, C_{k_n}(\mathbf{p}_{k_n})\}$ satisfies (8). Such a set satisfies then condition (7) and therefore all the conditions (2)–(8).

We shall define by induction integers $p_0, \dots, p_{k_n}$ in such a way that the sets $\mathscr{F}''_{n,i} = \mathscr{F}''_n \cup \{C_0(\mathbf{p}_0), \dots, C_i(\mathbf{p}_i)\}$ satisfy (8). So let us assume that $p_0, \dots, p_{i-1}$ are already defined. We shall indicate how to select p_i.

By assumption, the set $\mathscr{F}''_{n,i-1}$ satisfies (8). Since the existential formula $(Ex_i)[N(x_i) \;\&\; C(x_i)]$ belongs to $\mathscr{F}''_{n,j-1}$ we can transform the conjunction $\bigwedge \mathscr{F}''_{n,i-1}$ into a logically equivalent formula

$$(Ex_i)[(\bigwedge \mathscr{F}''_{n,i-1}) \;\&\; N(x_i) \;\&\; C(x_i)].$$

We can now use the σ-additivity of Q to obtain

$$M \models (Ex_i)(Qx)[\bigwedge \mathscr{F}''_{n,i-1}(x) \;\&\; C_i(x_i, x) \;\&\; N(x_i)];$$

here $\bigwedge \mathscr{F}''_{n,i-1}(x) = Sb(\gamma/x) \bigwedge \mathscr{F}''_{n,i-1}$ and $C_i(x_i, x) = Sb(\gamma/x)C(x_i)$. Remembering that M is an ω-$\mathscr{N}$-model we find an element a of $\mathscr{N}$ which satisfies the formula

$$M \models (Qx)[\bigwedge \mathscr{F}''_{n,i-1}(x) \;\&\; C(\gamma_a, x)].$$

Since a is a denotation of a constant $\mathbf{p}_i$ we obtain the desired integer p_i.

Proof of Theorem 2 is thus complete.

We finish by formulating some problems suggested by our discussion: We assume that L and N have the same meaning as in Example 1.

(1) Are there nontrivial, σ-additive definable quantifiers in the system A_2^-?

(2) Does Corollary 1 hold for A_2^-?

REMARK ADDED IN PROOF. Professor Keisler has kindly drawn the attention of the author to a close similarity of Theorem 2 proved above and Corollary 2.13 of Professor Keisler's paper in Ann. Math. Logic 1 (1970), p. 19. However, these results are not identical.

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An undecidable arithmetical statement.

By

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The purpose of this paper is to give an alternative proof of the existence of formally undecidable sentences. Instead of the arithmetization of syntax and the diagonal process which were used by Gödel in his famous paper of 1931[1], I shall make use of some simple set-theoretic lemmas and of the Skolem-Löwenheim theorem.

My result is in some respect stronger than that of Gödel. The sentence constructed by his method ceases to be undecidable if one enlarges the underlying logic by a new rule of proof, in the simplest case by the rule of infinite induction[2]. The undecidability of the sentence to be constructed here is, on the contrary, independent of whether we accept the absolute notion of integers or the relative (axiomatic) one[2a].

On the other hand the proof of undecidability to be given below is unlike that of Gödel non-finitary. It rests on the axioms of the Zermelo-Fraenkel set-theory including the axiom of choice and an additional axiom ensuring the existence of at least one inaccessible aleph[3]. Finally the method of Gödel gives undecidable sentences expressed in terms of the arithmetic of natural numbers whereas we shall obtain here a sentence from the arithmetic of reals.

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[1] Gödel [4]. Numbers in brackets refer to the bibliography on p. 163.

[2] Tarski [12].

[2a] Other such sentences have been constructed by Rosser [10] and Tarski [15]. My method is different from theirs.

[3] Tarski [14]. Using Tarski's terminology we would have to say that $\aleph_\xi$ is weakly inaccessible.

1. Axioms of set-theory. V. Neumann 4), Bernays 5), and Gödel 6) have shown how to build up an axiomatic system of set theory, presupposing only the restricted functional calculus (with identity) as logical basis, in such a way that it contains only a finite number of axioms. A slight modification of the systems of Bernays and Gödel allows us to reduce the number of primitive notions to one, viz. a binary relation ϵ 7).

We shall state here explicitly the axioms of the resulting system (S).

Elements of the field of ϵ are called classes and elements of the domain of ϵ are called sets. Thus every set is a class but not conversely. The lower case Latin letters will be used as variables for sets and the upper case Latin letters as variables for classes.

The axioms fall into several groups 8):

Group A.

$A1$. $\prod_u(u \epsilon X \equiv u \epsilon Y) \rightarrow X = Y$,

$A2$. $\sum_z \prod_u[u \epsilon z \equiv (u = x + u = y)]$.

The set z whose existence is stated in $A2$ is called the non-ordered pair of x and y and denoted by $\{x,y\}$. The ordered pair of x and y is defined as

$$\langle x,y\rangle = \{\{x,y\}, \{x,x\}\}.$$

Group B.

$B1$. $\sum_A \prod_{xy}(\langle x,y\rangle \epsilon A \equiv x \epsilon y)$,

$B2$. $\sum_C \prod_u[u \epsilon C \equiv (u \epsilon A)\cdot(u \epsilon B)]$,

$B3$. $\sum_B \prod_u[u \epsilon B \equiv (u \epsilon A)']$,

$B4$. $\sum_B \prod_x(x \epsilon B \equiv \sum_y \langle y,x\rangle \epsilon A)$,

$B5$. $\sum_B \prod_{xy}(\langle y,x\rangle \epsilon B \equiv x \epsilon A)$,

$B6$. $\sum_B \prod_{xy}(\langle x,y\rangle \epsilon B \equiv \langle y,x\rangle \epsilon A)$,

$B7$. $\sum_B \prod_{xyz}(\langle x,\langle y,z\rangle\rangle \epsilon B \equiv \langle y,\langle z,x\rangle\rangle \epsilon A)$,

$B8$. $\sum_B \prod_{xyz}(\langle x,\langle y,z\rangle\rangle \epsilon B \equiv \langle x,\langle z,y\rangle\rangle \epsilon A)$.

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4) v. Neumann [8].

5) Bernays [1].

6) Gödel [5].

7) The possibility of this reduction has been realised by A. Tarski. See a remark made on p. 208 of [7].

8) The logical symbols to be used in this paper are as follows: → (if, then), + (or), · (and), $\equiv$ (if and only if), ′ (not), Π (for every), Σ (there is). The axioms should have been preceded by general quantifiers so as to bind all the variables.

From these axioms it follows that there is a uniquely determined class 0 (called the null-class) such that $\prod_x (x \in 0)'$.

We shall use the following abbreviations:

$$X \subset Y \quad \text{for} \quad \prod_u (u \in X \rightarrow u \in Y),$$
$$J(X) \quad \text{for} \quad \prod_{uvw} (\langle v, u \rangle \in X \cdot \langle w, u \rangle \in X \rightarrow v = w).$$

Group C.

$C1$. $\sum_x ((x \neq 0) \cdot \prod_y \{(y \in x) \rightarrow \sum_z [(z \in x) \cdot (z \neq y) \cdot (y \subset z)]\})$,

$C2$. $\sum_y \prod_{uv} [(u \in v) \cdot (v \in x) \rightarrow (u \in y)]$,

$C3$. $\sum_y \prod_u [(u \hat{\subset} x) \rightarrow (u \in y)]$,

$C4$. $J(A) \rightarrow \sum_y \prod_u \{(u \in y) \equiv \sum_v [(v \in x) \cdot (\langle u, v \rangle \in A)]\}$.

Group D.

$D1$. $A \neq 0 \rightarrow \sum_u \{(u \in A) \cdot \prod_v [(v \in u) \rightarrow (v \in A)']\}$.

Group E.

$E1$. $\sum_A (J(A) \cdot \prod_x \{(x \neq 0) \rightarrow \sum_y [(\langle y, x \rangle \in A) \cdot (y \in x)]\})$.

2. Preliminary lemmas. We shall assume as known the derivation of set-theoretical theorems from the above axioms and we shall make free use of them stating them in the current notations and terminology [9]. This applies in particular to theorems concerning transfinite ordinals and to definitions by transfinite induction [10].

We define by transfinite induction the sets t_ξ as follows:

$$t_0 = 0, \quad t_{\xi+1} = E_x [x \subset t_\xi], \quad t_\lambda = \sum_{\eta < \lambda} t_\eta$$

(λ — limit number).

It is easy to see that $t_\eta \subset t_\xi$ for $\eta < \xi$.

If $x \in t_{\xi+1} - t_\xi$, then x is said to be of the *type* ξ: It follows from the axiom $D1$ that for every set x there is an ordinal ξ such that x is of the type ξ.

Indeed, the axioms of group B entail the existence of the class A of those sets which are of no type. If A were non void, there

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[9] It will therefore not always be possible to use small and capital letters in the manner explained on p. 144.

[10] For the treatment of ordinals on the basis of axioms $A1$—$E1$ cf. [1], [5], [8], and [9].

would exist by $D1$ an element u such that $u \epsilon A$ but no element of u is in A. Hence every element v of u would be of a type, say $\xi(v)$, and all the ordinals $\xi(v)$ would form a set (according to the axiom $C4$). Now there exists for every set of ordinals $\xi(v)$ an ordinal ξ surpassing all of them; hence we would obtain $u \subset t_\xi$ i. e. $u \epsilon t_{\xi+1}$, which proves that u is of a type not greater than $\xi+1$ and hence u non ϵA. This contradiction shows that $A=0$.

The sets

$$0, \quad \{0\}, \quad \{0,\{0\}\}, \quad \{0,\{0\},\{0,\{0\}\}\},\ldots$$

will be identified with the integers

$$1, \quad 2, \quad 3, \quad 4,\ldots$$

and their set will be denoted by N. It is known how to define the arithmetical operations, e. g.

$$x+y, \quad x\cdot y, \quad x^y, \quad 2^{x-1}(2y-1),\ldots$$

on elements of N.

Every class of ordered pairs is called (binary) *relation*. If R is a relation, then xRy means the same as $\langle x,y\rangle \epsilon R$. The classes

$$E_x\Sigma_y xRy, \quad E_y\Sigma_x xRy, \quad E_x\Sigma_y(xRy+yRx)$$

are called *domain*, *converse domain*, and *field* of R and denoted by $D_+(R)$, $D_-(R)$, and $C(R)$.

If $B \subset C(R)$, $x \epsilon B$ and no $y \epsilon B$ satisfies the condition yRx, then x is called a *minimum* of R in B. If R has at least one minimum in every non void class $B \subset C(R)$, then R is called *well-founded*[11].

R is *internal* if for every two elements x_1, x_2 of its field the following equivalence holds:

$$\Pi_y(yRx_1 \equiv yRx_2) \equiv (x_1=x_2).$$

An internal relation R has at most one minimum in $C(R)$.

For any set x we denote by ϵ_x the ϵ-relation limited to x, i. e. such that

$$u \,\epsilon_x\, v \equiv (u \epsilon x)\cdot(u \epsilon v)\cdot(v \epsilon x).$$

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[11] Zermelo [17].

Theorem 1. *ϵ_x is an internal relation if and only if x satisfies the following condition $E(x)$:*

$$\prod_{uv}((u \in x)\cdot(u \neq v)\cdot(v \in x) \rightarrow \{[(u-v)+(v-u)]\cdot x \neq 0\}).$$

The proof is obvious.

Theorem 2. *ϵ_x is well-founded for every non-void set x.*

Proof. Evidently there is an ordinal ξ such that $x \subset t_\xi$. Suppose that c is a non-void subset of the field of ϵ_x. Since $c \subset x$, there are ordinals $\eta \leqslant \xi$ such that $c \cdot t_\eta \neq 0$. If ζ is the smallest ordinal of this kind, then every element of $c \cdot t_\zeta$ is a minimum of ϵ_x in c. The existence of these minima shows that ϵ_x is well-founded, q. e. d.

A set s is called *complete*[12]) if every element of s is a part of s: $\prod_y (y \in s \rightarrow y \subset s)$.

We assume as known the notion of *isomorphism* of two relations.

Theorem 3. *For every well-founded and internal relation R whose field is a set there is a set s such that $E(s)$, R is isomorphic with ϵ_s, and the field of ϵ_s is complete.*

Proof. R being internal and well-founded, there is exactly one minimum z_0 of R in $C(R)$. Put

$$\{z_0\} = m_0, \quad f(z_0) = 0$$

and suppose that sets m_α are already defined for $\alpha < \xi$ and that a function f is defined on the sum $\sum_{\alpha<\xi} m_\alpha$.

Let m_ξ be the set of all $x \in C(R) - \sum_{\alpha<\xi} m_\alpha$ for which the following condition is satisfied

$$\prod_y [yRx \rightarrow y \in \sum_{\alpha<\xi} m_\alpha]$$

and let $f(x)$ be defined on m_ξ by the equation

$$f(x) = E_{f(y)}[yRx]. \tag{1}$$

This definition of $f(x)$ is correct since yRx implies (for $x \in m_\xi$) that $y \in \sum_{\alpha<\xi} m_\alpha$ and therefore $f(y)$ is defined according to the inductive assumption.

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[12]) Gödel [5], p. 23.

It is evident that $m_\xi \cdot m_\eta = 0$ for $\xi \neq \eta$ and that $m_\xi \subset C(R)$ for every ξ. Since $C(R)$ is a set, there must be a (smallest) ordinal ζ such that $m_\zeta = 0$.

We shall show that the difference $D = C(R) - \sum_{\alpha<\zeta} m_\alpha$ is void. Otherwise there would be a minimum z of R in D and z would satisfy the condition

$$yRz \rightarrow y \text{ non } \epsilon\, D \rightarrow y \,\epsilon \sum_{\alpha<\zeta} m_\alpha,$$

i. e. z would be an element of m_ζ contrary to the definition of ζ. Hence $D = 0$ and consequently

$$C(R) = \sum_{\alpha<\zeta} m_\alpha.$$

We now show by induction on ξ that f is a one-one mapping of the sum $\sum_{\alpha<\xi} m_\alpha$ on a subset of t_ξ and satisfies the equivalence

$$xRy \equiv f(x) \,\epsilon\, f(y) \tag{2}$$

for every pair x, y of the elements of $\sum_{\alpha<\xi} m_\alpha$.

This is evident for $\xi = 1$; it is also easy to see, that if β is a limit number and the theorem holds for $\xi < \beta$, it holds also for $\xi = \beta$.

It remains to consider the case $\beta = \gamma + 1$ under the assumption that the proposition holds for $\xi = \gamma$.

If $x \,\epsilon\, m_\gamma$, then by (1) $f(x)$ is a set of elements each of which is of a type $<\gamma$. Hence $f(x)$ is at most of the type γ and therefore $f(x) \,\epsilon\, t_{\gamma+1} = t_\beta$. The function f maps thus the sum $\sum_{\alpha<\beta} m_\alpha$ on a subset of t_β.

In order to prove that this is a one-one mapping let us suppose that x and y are two different elements of the sum $\sum_{\alpha<\beta} m_\alpha$. Since R is internal, there is an element u such that either $(uRx) \cdot (uRy)'$ or $(uRx)' \cdot (uRy)$. It will be sufficient to consider the first case. From uRx it follows that $x \neq z_0$ and hence by (1) $f(u) \,\epsilon\, f(x)$. Furthermore $u \,\epsilon \sum_{\alpha<\gamma} m_\alpha$ since x is an element of at most m_γ. If $f(u)$ were an element of $f(y)$, there would be an element $y_1 \,\epsilon \sum_{\alpha<\gamma} m_\alpha$ such that $f(u) = f(y_1)$ and $y_1 Ry$. Since f is a one-one mapping on $\sum_{\alpha<\gamma} m_\alpha$, it would follow $u = y_1$, and consequently uRy what contradicts the assumption. Hence $f(u)$ non $\epsilon\, f(y)$ which shows that $f(x) \neq f(y)$. Hence

$$x \neq y \rightarrow f(x) \neq f(y),$$

i. e. f is a one-one mapping.

If x and y are elements of $\sum_{\alpha<\beta} m_\alpha$ and xRy, then $f(x) \,\epsilon\, f(y)$ according to (1). If $f(x) \,\epsilon\, f(y)$, then by (1) there is a z such that zRy and $f(x) = f(z)$. f being a one-one mapping, we obtain $x = z$ and xRy. The equivalence (2) is thus proved.

Putting $\xi=\zeta$ we infer that f maps $C(R)$ on a set s and satisfies the equivalence (2) for every pair x,y of the elements of $C(R)$. f being a one-one mapping, it follows that the relations R and ϵ_s are isomorphic. Since R is internal, ϵ_s is internal too and theorem 1 gives $E(s)$.

In order to prove that $C(\epsilon_s)$ is complete let us suppose that $y \,\epsilon\, C(\epsilon_s)$. It follows that $y \,\epsilon\, s$ and hence $f^{-1}(y) \,\epsilon\, C(R)$ and consequently there is an $\alpha<\zeta$ such that $f^{-1}(y) \,\epsilon\, m_\alpha$. If $\alpha=0$, then $f^{-1}(y)=z_0$, $y=0$ and $y\subset C(\epsilon_s)$. If $\alpha\neq 0$, then $y=ff^{-1}(y)$ is by (1) identical with the set of all $f(z)$ for which $zRf^{-1}(y)$. Since $f(z)\,\epsilon_s\,y$, it follows that all these $f(z)$ belong to the field of ϵ_s and therefore $y\subset C(\epsilon_s)$, which completes the proof of theorem 3.

For every set $A\subset N$ we put

$$R_A=E_{\langle m,n\rangle}[2^{m-1}(2n-1)\,\epsilon\,A].$$

In this way a one-one correspondence is established between subsets of N and binary relations whose fields are contained in N.

The following theorem is evident:

Theorem 4. *For every relation R with at most denumerable field there is a set $A\subset N$ such that the relations R and R_A are isomorphic.*

3. Set-theoretical and arithmetical formulae. A *set-theoretical formula* is an expression built up from elementary expressions of the form

$$a=b, \quad a\,\epsilon\, b$$

with the help of the logical connectives and quantifiers $\prod_a$ and $\sum_a$. The letters a, b may be replaced by any other letters.

The formulae included in a given formula are called its constituents.

If one wishes to make general statements about formulae one has to distinguish between the object- and syntax languages and recur to intuitively clear but sometimes clumsy semantical notions. For the readers benefit we may dispense with these complications since we shall never make general statements about formulae: we shall limit ourselves to the consideration of a finite number of formulae. Symbols like $\Phi, \Psi, \ldots$ or $\Phi(\epsilon, a, \ldots, m)$, $\Psi(\epsilon, a, \ldots, m)$ are not names of formulae but abbreviations of them and specifically of those of them in which variables $a,\ldots,m$ are free. We postpone till § 5 the enumeration of formulae to be used in our proof, we remark only that their class contains with every formula its constituents.

Saying that all formulae have a property we wish to say that all formulae of the considered finite class have this property. It will be easily seen that we eliminate in this way all the meta-mathematical notions and that all lemmas to be proved below belong entirely to the object-language.

Suppose that the letter x does not occur in Φ. Replace in this formula the unrestricted quantifiers $\prod_h$, $\sum_h$ (where h is any letter) by the restricted ones

$$\prod_h[h \in x \to \dots], \quad \sum_h[(h \in x)\dots]$$

(which we will sometimes write, more conveniently, as $\prod_{h\in x}$ and $\sum_{h\in x}$).

The resulting formula will be abbreviated as Φ_x and called the *formula Φ relativized to x.*

The symbol $\Phi(R,a,\dots,m)$ will be used as an abbreviation of the formula resulting from $\Phi(\epsilon,a,\dots,m)$ by substituting in it the letter R for the letter ϵ.

Theorem 5. $(\Phi_x)_y \equiv \Phi_{x\cdot y}$.

Proof. The theorem is evident for quantifier-free formulas. Since its validity for the formulae Φ and Ψ entails its validity for the formulae Φ' and $\Phi+\Psi$ it will be sufficient to prove that if it is true for a formula Φ, it is true also for the formula $\sum_h\Phi$. Let Ψ be an abbreviation of the latter formula. Then

$$\Psi_x \equiv \sum_h[(h \in x)\cdot\Phi], \quad (\Psi_x)_y \equiv \sum_{h\in y}[(h \in x)\cdot(\Phi_x)_y]$$

and therefore

$$(\Psi_x)_y \equiv \sum_h[(h \in x)\cdot(h \in y)\cdot(\Phi_x)_y] \equiv \sum_{h\in x\cdot y}(\Phi_x)_y.$$

On the other hand $\Psi_{x\cdot y} \equiv \sum_{h\in x\cdot y}\Phi_{x\cdot y}$ and since $\Phi_{x\cdot y} \equiv (\Phi_x)_y$ by the inductive assumption, we obtain $(\Psi_x)_y \equiv \Psi_{x\cdot y}$ which proves the theorem.

Theorem 6. *If* $s = C(\epsilon_x)$ *and* $a \in s,\dots,m \in s$, *then*

$$\Phi_s(\epsilon,a,\dots,m) \equiv \Phi_s(\epsilon_x,a,\dots,m). \tag{3}$$

Proof. If Φ is an elementary formula $a=b$, then (3) reduces to the tautological equivalence $a=b \equiv a=b$.

If Φ is an elementary formula $a \epsilon b$, then (3) is equivalent to

$$a \epsilon b \equiv [(a \epsilon x)\cdot(a \epsilon b)\cdot(b \epsilon x)]$$

which is true since $a \epsilon s$ and $b \epsilon s$ by the hypothesis and s is evidently a subset of x.

From (2) it follows

$$\Phi'_s(\epsilon, a, \dots, m) \equiv \Phi'_s(\epsilon_x, a, \dots, m)$$

which proves that the validity of (3) for a formula Φ implies its validity for the formula Φ'.

If besides (2) the following equivalence holds

$$\Psi_s(\epsilon, a, \dots, h, n, \dots, q) \equiv \Psi_s(\epsilon_x, a, \dots, h, n, \dots, q),$$

then

$$\Phi_s(\epsilon, a, \dots, m)\cdot\Psi_s(\epsilon, a, \dots, h, n, \dots, q) \equiv \Phi_s(\epsilon_x, a, \dots, m)\cdot\Psi_s(\epsilon, a, \dots, h, n, \dots, q),$$

which shows that if the theorem is true for two formulae, it is true for their conjunction.

Suppose now that (3) holds for a formula Φ and let $\Psi(\epsilon, b, \dots, m)$ be the abbreviation of $\sum_a \Phi(\epsilon, a, b, \dots, m)$. Let $b, \dots, m$ be elements of s. If $\Psi_s(\epsilon, b, \dots, m)$, then there is an $a \epsilon s$ such that $\Phi_s(\epsilon, a, \dots, m)$ which yields according to (3) $\Phi_s(\epsilon_x, a, \dots, m)$ and hence $\sum_{a \epsilon s} \Phi(\epsilon_x, a, \dots, m)$. Thus

$$\Psi_s(\epsilon, b, \dots, m) \rightarrow \Psi_s(\epsilon_x, b, \dots, m).$$

If $\Psi_s(\epsilon_x, b, \dots, m)$, then there is an $a \epsilon s$ such that $\Phi_s(\epsilon_x, a, \dots, m)$ and therefore by (3) $\Phi_s(\epsilon, a, \dots, m)$ which gives $\sum_{a \epsilon s} \Phi_s(\epsilon, a, \dots, m)$. Hence

$$\Psi_s(\epsilon_x, b, \dots, m) \rightarrow \Psi_s(\epsilon, b, \dots, m).$$

Theorem 6 is thus proved for the formula Ψ, q. e. d.

***Theorem* 7**[13]. *If the relations* R_1 *and* R_2 *are isomorphic,* $s_1 = C(R_1)$, $s_2 = C(R_2)$, $a_1, \dots, m_1 \epsilon C(R_1)$ *and* $a_2, \dots, m_2$ *correspond to* $a_1, \dots, m_1$ *in the isomorphism between* R_1 *and* R_2, *then*

$$\Phi_{s_1}(R_1, a_1, \dots, m_1) \equiv \Phi_{s_2}(R_2, a_2, \dots, m_2).$$

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[13] Theorem 7 is a special case of a theorem proved by A. Tarski and A. Lindenbaum [16].

Proof. The theorem is evident if Φ has the form $a=b$ or $a\in b$. We show similarly as in the proof of theorem 6 that if the theorem holds for two formulae, it holds also for their negations and their conjunction.

It remains to consider the formula $\sum_a\Phi(\epsilon,a,\dots,m)$ which we shall abbreviate as $\Psi(\epsilon,b,\dots,m)$.

If $\Psi_{s_1}(R_1,b_1,\dots,m_1)$, then there is an $a_1\in s_1$ such that $\Phi_{s_1}(R_1,a_1,b_1,\dots,m_1)$. If a_2 is the element of s_2 corresponding to a_1 in the isomorphism between R_1 and R_2, then the inductive assumption gives $\Phi_{s_2}(R_2,a_2,b_2,\dots,m_2)$ from which we infer that

$$\sum_{a_2\in s_2}\Phi_{s_2}(R_2,a_2,b_2,\dots,m_2).$$

Hence

$$\Psi_{s_1}(R_1,b_1,\dots,m_1)\rightarrow\Psi_{s_2}(R_2,b_2,\dots,m_2).$$

The converse implication is proved in the same manner.

Besides the set-theoretical formulae we shall consider the *arithmetical formulae*. In order to define them we first explain what is meant by a *term*: the symbols 1,2,3,... and the small Latin letters in italics are terms; if m and n are terms, then $m+n$, $m-n$, $m\cdot n$, m^n are terms.

The simplest arithmetical formulae are equalities $m=n$ and expressions $m\in A$ where m and n are terms and A any upper case Latin letter.

Other arithmetical formulae are built up from the simplest ones with the help of the logical connectives *and*, *or*, *not*, etc. and the quantifiers of the following four forms

$$(*)\qquad \begin{array}{ll} \prod_p[p\in N\rightarrow\dots], & \sum_p[(p\in N)\cdot\dots] \\ \prod_A[A\subset N\rightarrow\dots], & \sum_A[(A\subset N)\cdot\dots]. \end{array}$$

These quantifiers will be written afterwards as

$$\prod_{p\in N},\quad \sum_{p\in N},\quad \prod_{A\subset N},\quad \sum_{A\subset N}.$$

The letters p and A may be replaced here by any other letters.

An arithmetical formula is called *elementary* if it does not contain quantifiers of the form (*).

The following theorem is an immediate consequence of the admitted definitions and of the equivalence $pR_Aq\equiv 2^{p-1}(2q-1)\in A$:

Theorem 8. *If $\Phi(\epsilon, a, \ldots, m)$ is a set-theoretical formula and $A \subset N$, then $\Phi_{C(R_A)}(R_A, a, \ldots, m)$ as well as $\Phi_N(R_A, a, \ldots, m)$ are equivalent to elementary arithmetical formulae.*

It is evident that every proposition concerning only natural or real numbers is expressible as an arithmetical formula. Furthermore those arithmetical formulae which correspond to usual axioms of real number arithmetics are derivable from the axioms $A1$—$E1$ of the system (S). It follows that the whole classical mathematics may be expressed and proved in the system (S).

4. Reduction of certain set-theoretical formulae to the arithmetical formulae. The theorem to be proved in this section is the main result of the whole paper. It is a simple corollary to the well-known theorem of Skolem-Löwenheim which for our purpose may be stated in the following form:

Theorem 9. *If $a \in x, \ldots, m \in x$ and $\Phi_x(\epsilon, a, \ldots, m)$, then there is an at most denumerable subset y of x such that*

$$(4) \quad a \in y, \ldots, m \in y, \qquad (5) \quad E(y),$$

$$(6) \quad C(\epsilon_y) = y, \qquad (7) \quad \Phi_y(\epsilon, a, \ldots, m).$$

Proof[14]. We may evidently assume that Φ has the normal form

$$\Pi x_1 \cdots \Pi x_m \Sigma y_1 \cdots \Sigma y_n \Pi z_1 \cdots \Pi z_p \Sigma t_1 \cdots \Sigma t_q \cdots \Psi,$$

where

$$\Psi = \Psi(\epsilon, x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p, t_1, \ldots, t_q, \ldots)$$

is quantifier-free.

Owing to the axiom of choice the condition $\Phi_x(\epsilon, a, \ldots, m)$ is equivalent to the existence of a set of functions

$$f_i(x_1, \ldots, x_m), \quad g_j(x_1, \ldots, x_m, z_1, \ldots, z_p), \ldots$$

(where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, q, \ldots$) with the following properties:

1^0 *they are defined for all the values of their arguments running through x;*

2^0 *if $x_1, \ldots, x_m, z_1, \ldots, z_p, \ldots$ are elements of x, then*

$$(8) \quad \Psi(\epsilon, x_1, \ldots, x_m, f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m), z_1, \ldots, z_p,$$
$$g_1(x_1, \ldots, x_m, z_1, \ldots, z_p), \ldots, g_q(x_1, \ldots, x_m, z_1, \ldots, z_p), \ldots).$$

---

[14] For this proof compare Skolem [11].

Let $h(z,t)$ be a function defined for $z,t \in x$ and such that $h(z,t) \in (z-t)+(t-z)$ for $z \neq t$. The existence of such a function follows again from the axiom of choice.

Now define y as a smallest set such that

$$a,\dots,m \in y,$$

if $x_1,\dots,x_m,z_1,\dots,z_p,\dots \in y$, *then* $f_i(x_1,\dots,x_m), g_j(x_1,\dots,x_m,z_1,\dots,z_p),\dots \in y$ (*for* $i=1,2,\dots,n,\ j=1,2,\dots,q,\dots$),

if $z,t \in y$ *and* $z \neq t$, *then* $h(z,t) \in y$.

It is evident that y is an at most denumerable set and satisfies the condition (4). If $u,v \in y$ and $u \neq v$, then $h(u,v) \in [(u-v)+(v-u)]\cdot y$, whence $[(u-v)+(v-u)]\cdot y \neq 0$ which proves that y satisfies the condition (5). Since $0 \in C(\epsilon_y)$ and $h(z,0) \in z$ for $z \neq 0$ we see easily that $C(\epsilon_y)=y$ and hence y satisfies the condition (6). Finally y satisfies the condition (7) since (8) holds for every $x_1,\dots,x_m,z_1,\dots,z_p,\dots \in y$ and the values of the functions $f_i,g_j,\dots$ occurring in (8) belong to y. Theorem 9 is thus proved.

It will be convenient to use the abbreviation $[x]$ for $C(\epsilon_x)$. It is evident that

$$(y \in [x]) \equiv (y \in x)\cdot \textstyle\sum_t\{(t \in x)\cdot[(y \in t)+(t \in y)]\}$$

from which it follows that

$$(9) \quad \Phi_{[x]}(\epsilon) \equiv \textstyle\sum_s\{\Phi_s(\epsilon)\cdot \prod_y[(y \in s) \equiv (y \in x)\cdot \sum_t\{(t \in x)\cdot[(y \in t)+(t \in y)]\}]\}$$

for any formula $\Phi(\epsilon)$. This equivalence will be used in section 5.

We shall now prove the following theorem:

Theorem 10. *For every set-theoretical formula* $\Phi(\epsilon)$ *there is an elementary arithmetical formula* $\mathcal{G}(A,B)$ *with two free variables such that*

$$\textstyle\sum_x\{\Phi_{[x]}(\epsilon)\cdot \prod_y[(y \in [x]) \rightarrow (y \subset [x])]\} \equiv \sum_{A\subset N} \prod_{B\subset N} \mathcal{G}(A,B).$$

Proof. By theorem 8 there is an elementary arithmetical formula $\mathcal{H}(A)$ such that

$$\Phi_{C(R_A)}(R_A) \equiv \mathcal{H}(A).$$

Let $\mathcal{K}(A)$ be the elementary arithmetical formula equivalent to

$$\prod_{m,n \in C(R_A)}[m \neq n \rightarrow \sum_{r \in C(R_A)}(r R_A m \equiv r R_A n)'].$$

$\mathcal{K}(A)$ says of course that R_A is internal.

Let $\mathcal{L}(A,B)$ be the elementary arithmetical formula equivalent to

$$\sum_{m \in N}(m \in B) \cdot \prod_{p \in N}\{(p \in B) \rightarrow \sum_{n \in N}[(n R_A p) + (p R_A n)]\} \rightarrow$$
$$\rightarrow \sum_{m \in N}\{(m \in B) \cdot \prod_{n \in N}[(n \in B) \rightarrow (n R_A m)']\}.$$

$\mathcal{L}(A,B)$ says that if B is a non void subset of $C(R_A)$, then R_A has a minimum in B.

Now take as $\mathcal{G}(A,B)$ the conjunction

$$\mathcal{H}(A) \cdot \mathcal{K}(A) \cdot \mathcal{L}(A,B).$$

We shall show that this formula has the desired property.

Suppose first that there is a set x such that $\Phi_{[x]}(\epsilon)$ and $[x]$ is complete.

By theorem 9 there is an at most denumerable set y such that

$$E(y), \quad C(\epsilon_y) = y, \quad \text{and } \Phi_y(\epsilon).$$

The second and third conditions give in virtue of theorem 6 $\Phi_y(\epsilon_y)$. Since the field of ϵ_y is at most denumerable, there is by theorem 4 a set $A \subset N$ such that the relations ϵ_y and R_A are isomorphic. Applying theorem 7 we obtain from $\Phi_y(\epsilon_y)$

$$\Phi_{C(R_A)}(R_A),$$

which proves that $\mathcal{H}(A)$.

We show further that $\mathcal{K}(A)$. For this purpose we assume that $m \neq n$ and $m,n \in C(R_A)$. Let f be a function which establishes an isomorphism between R_A and ϵ_y. Thus $f(m)$ and $f(n)$ are different elements of y. Since $E(y)$, there is a $z \in y$ such that $[z \epsilon f(m) \equiv z \epsilon f(n)]'$. Putting $r = f^{-1}(z)$ we get $r \in C(R_A)$ and $(rR_A m \equiv rR_A n)'$ which proves that $\mathcal{K}(A)$.

We shall finally show that if $B \subset N$, then $\mathcal{L}(A,B)$. This is evident if $B = 0$. Assume now that $B \neq 0$ and that B is a non void subset of the field of R_A. Since R_A is isomorphic with ϵ_y and ϵ_y is well-founded (by theorem 2), R_A is well-founded too. If m is a minimum of R_A in B, then $m \in B$ and

$$\prod_{n \in N}[(n \in B) \rightarrow (n R_A m)'].$$

This proves that $\mathcal{L}(A,B)$.

We have thus proved, that if $\Phi_{[x]}(\epsilon)$, then there is a set $A\subset N$ such that $\mathcal{H}(A)$, $\mathcal{K}(A)$, and (for every $B\subset N$) $\mathcal{L}(A,B)$. Hence

$$\sum_x \Phi_{[x]}(\epsilon)\rightarrow\sum_{A\subset N}\prod_{B\subset N}\mathcal{G}(A,B).$$

Suppose now that there is a set $A\subset N$ such that

$$\mathcal{G}(A,B)$$

for every set $B\subset N$. It follows that

$$\mathcal{H}(A),\quad \mathcal{K}(A),\quad \text{and}\quad \mathcal{L}(A,B)$$

for every $B\subset N$. From $\mathcal{K}(A)$ we see immediately that R_A is internal and from $\prod_{B\subset N}\mathcal{L}(A,B)$ that R_A is well founded. By theorem 3 there is a set x such that ϵ_x is isomorphic with R_A and the field $[x]$ of ϵ_x is complete. From $\mathcal{H}(A)$ we obtain $\Phi_{C(R_A)}(R_A)$ and applying theorem 7 we obtain $\Phi_{[x]}(\epsilon_x)$. This gives in virtue of theorem 6 $\Phi_{[x]}(\epsilon)$, whence, $[x]$ being complete

$$\sum_{A\subset N}\prod_{B\subset N}\mathcal{G}(A,B)\rightarrow\sum_x\{\Phi_{[x]}(\epsilon)\cdot\prod_y[(y\in[x])\rightarrow(y\subset[x])]\}.$$

The proof of theorem 10 is thus complete.

5. Construction of an undecidable sentence. A limit ordinal ξ is called *inaccessible* if it satisfies the following condition:

if $x\subset t_\xi$ and x is not of the same power as t_ξ, then $x\in t_\xi$.

We shall add to the axioms of the system (S) the following axiom

F1. There is at least one inaccessible ordinal $\xi>\omega$

and shall call (S_1) the resulting system of axioms.

In this section we shall prove in (S_1) theorems about the system (S).

Let $\Phi(\epsilon)$ be the conjunction of all the axioms of the system (S) preceded by universal quantifiers so as to render all the variables apparent.

We shall apply the theorems established in sections 3 and 4 taking all constituents of $\Phi(\epsilon)$ as well as their normal forms as elements of the finite class of formulae which were till now left unspecified.

Theorem 11. *If ξ is an inaccessible ordinal, then* $\Phi_{t_{\xi+1}}(\epsilon)$.

We omit the proof of this theorem since it is very easy and essentially known[15].

Let $\Delta(\epsilon)$ be the following formula

$$\sum_x \sum_s \{\Phi_s(\epsilon) \cdot \prod_y [(y \epsilon s) \equiv (y \epsilon x) \cdot \sum_t \{(t \epsilon x) \cdot [(y \epsilon t) + (t \epsilon y)]\}] \cdot \\ \cdot \prod_y \prod_t [(y \epsilon s) \cdot (t \epsilon y) \rightarrow (t \epsilon s)]\}. \tag{10}$$

According to (9) (see p. 154) this formula could have been written as

$$\sum_x \{\Phi_{[x]}(\epsilon) \cdot \prod_y [(y \epsilon [x]) \rightarrow (y \subset [x])]\}, \tag{10*}$$

we prefer however the more complicated expression (10) since it is important for what follows to have the formula $\Delta(\epsilon)$ written without abbreviations such as „$[x]$" and „$\subset$".

Theorem 12. *There is a set a such that* $\Phi_{[a]}(\epsilon) \cdot \Delta_{[a]}(\epsilon)$.

Proof. Let ξ be the first inaccessible ordinal greater than ω and put $a = t_{\xi+1}$. Since $[a] = a$, we obtain from theorem 11

$$\Phi_{[a]}(\epsilon). \tag{11}$$

It follows from theorem 9 that there is an at most denumerable subset y of a such that

(12) $E(y)$, (13) $[y] = C(\epsilon_y) = y$, and (14) $\Phi_y(\epsilon)$.

The relation ϵ_y is well-founded and internal (see (12) and theorems 1 and 2) and therefore (see theorem 3) there is a set s such that

(15) ϵ_y *and* ϵ_s *are isomorphic,*

(16) *the set* $[s] = C(\epsilon_s)$ *is complete.*

It is easy to see that (13) and (15) entail the equality

$$[s] = C(\epsilon_s) = s. \tag{17}$$

s is evidently an at most denumerable subset of a i. e. of $t_{\xi+1}$. Hence if $m \epsilon s$, then $m \epsilon t_{\xi+1}$, i. e. $m \subset t_\xi$. Since $m \epsilon s$ implies $m \subset s$ because of (16) and (17), it follows that every element m of s is at

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[15] Cf. Kuratowski [6].

most denumerable and hence $m \epsilon t_\xi$ according to the definition of inaccessible ordinals. This proves that

$$s \subset t_\xi \tag{18}$$

and we obtain $s \epsilon t_\xi$ and finally

$$s \epsilon a. \tag{19}$$

Applying theorem 7 to the formula $\Phi(\epsilon)$ and using (13), (15), and (17), we obtain the equivalence $\Phi_s(\epsilon_s) \equiv \Phi_y(\epsilon_y)$. Theorem 6 shows that we may omit the subscripts s and y staying by the letter ϵ. In view of (14) we obtain from the modified equivalence $\Phi_s(\epsilon)$ and further (since $s = st_\xi = sa$ according to (18)) $\Phi_{s\cdot a}(\epsilon)$. This gives in virtue of theorem 5

$$[\Phi_s(\epsilon)]_a. \tag{20}$$

It follows further from (17) that

$$\begin{aligned} y \epsilon s &\equiv y \epsilon [s] \\ &\equiv (y \epsilon s) \cdot \textstyle\sum_t\{(t \epsilon s) \cdot [(y \epsilon t) + (t \epsilon y)]\}. \end{aligned}$$

Since $(t \epsilon s) \rightarrow (t \epsilon a)$ we may replace the quantifier $\sum_t$ by $\sum_{t\epsilon a}$ and obtain the equivalence

$$(y \epsilon s) \equiv (y \epsilon s) \cdot \textstyle\sum_{t\epsilon a}\{(t \epsilon s) \cdot [(y \epsilon t) + (t \epsilon y)]\}.$$

This equivalence being valid for every y, we infer that

$$\textstyle\prod_{y\epsilon a}[(y \epsilon s) \equiv (y \epsilon s) \cdot \sum_{t\epsilon a}\{(t \epsilon s)\ [(y \epsilon t) + (t \epsilon y)]\}]. \tag{21}$$

From (16) and (17) we see that $(y \epsilon s) \cdot (t \epsilon y) \rightarrow (t \epsilon s)$ which proves that

$$\textstyle\sum_{y\epsilon a} \prod_{t\epsilon a}[(y \epsilon s) \cdot (t \epsilon y) \rightarrow (t \epsilon s)]. \tag{22}$$

Consider now the conjunction of (19), (20), (21), and (22) and put the quantifier $\sum_s$ before it. We obtain thus

$$\begin{gathered} \textstyle\sum_{s\epsilon a}\{[\Phi_s(\epsilon)]_a \cdot \prod_{y\epsilon a}[(y \epsilon s) \equiv (y \epsilon s) \cdot \sum_{t\epsilon a}\{(t \epsilon s) \cdot [(y \epsilon t) + (t \epsilon y)]\}] \cdot \\ \cdot \textstyle\prod_{y\epsilon a} \prod_{t\epsilon a}[(y \epsilon s) \cdot (t \epsilon y) \rightarrow (t \epsilon s)]\} \end{gathered}$$

from which follows the validity of the weaker formula

$$\begin{gathered} \textstyle\sum_{x\epsilon a} \sum_{s\epsilon a}\{[\Phi_s(\epsilon)]_a \cdot \prod_{g\epsilon a}[(y \epsilon s) \equiv (y \epsilon x) \cdot \sum_{t\epsilon a}\{(t \epsilon x) \cdot [(y \epsilon t) + (t \epsilon y)]\}] \cdot \\ \cdot \textstyle\prod_{g\epsilon a} \prod_{t\epsilon a}[(y \epsilon s) \cdot (t \epsilon y) \rightarrow (t \epsilon s)]\}. \end{gathered}$$

This formula is identical with the formula $\Delta(\epsilon)$ relativized to a, i. e. with $\Delta_a(\epsilon)$. Since $[a]=a$, we obtain $\Delta_{[a]}(\epsilon)$ which together with (11) yields $\Phi_{[a]}(\epsilon)\cdot\Delta_{[a]}(\epsilon)$. Theorem 12 is thus proved.

Theorem 13. *There is a set b such that $\Phi_{[b]}(\epsilon)\cdot[\Delta_{[b]}(\epsilon)]$.*

Proof. Looking at the proof of theorem 12 we see that there are sets $w\,\epsilon\,t_{\xi+2}$ such that

$$\Phi_{[w]}(\epsilon), \tag{23}$$

$$\textit{if}\quad y\,\epsilon\,[w],\quad \textit{then}\quad y\subset[w]. \tag{24}$$

E. g. $t_{\xi+1}$ is such a set. Let k be the set of all these sets w. By axiom $D1$ there is a set b such that $b\,\epsilon\,k$ but no element of b is in k.

From $b\,\epsilon\,k$ it follows in virtue of (23)

$$\Phi_{[b]}(\epsilon). \tag{25}$$

Suppose now that $\Delta_{[b]}(\epsilon)$. It follows according to (10) that there are sets x and s such that

$$x\,\epsilon\,[b]\quad \textit{and}\quad s\,\epsilon\,[b], \tag{26}$$

$$[\Phi_s(\epsilon)]_{[b]}, \tag{27}$$

$$\prod_{y\epsilon[b]}[(y\,\epsilon\,s)\equiv(y\,\epsilon\,x)\cdot\sum_{t\epsilon[b]}\{(t\,\epsilon\,x)\cdot[(y\,\epsilon\,t)+(t\,\epsilon\,y)]\}], \tag{28}$$

$$\prod_{y\epsilon[b]}\prod_{t\epsilon[b]}[(y\,\epsilon\,s)\cdot(t\,\epsilon\,y)\rightarrow(t\,\epsilon\,s)]. \tag{29}$$

(28) and (29) can be rewritten thus

$$\prod_y[(y\,\epsilon\,s\cdot[b])\equiv(y\,\epsilon\,x\cdot[b])\cdot\sum_t\{(t\,\epsilon\,x\cdot[b])\cdot[(y\,\epsilon\,t)+(t\,\epsilon\,y)]\}],$$
$$\prod_y\prod_t[(y\,\epsilon\,s\cdot[b])\cdot(t\,\epsilon\,y\cdot[b])\rightarrow(t\,\epsilon\,s)].$$

According to (24) and (26) we have $x\subset[b]$, $s\subset[b]$, and (for $y\,\epsilon\,[b]$) $y\subset[b]$; we may therefore omit the letter b in square brackets thorought these formulae and obtain thus

$$\prod_y[(y\,\epsilon\,s)\equiv(y\,\epsilon\,x)\cdot\sum_t\{(t\,\epsilon\,x)\cdot[(y\,\epsilon\,t)+(t\,\epsilon\,y)]\}],$$
$$\prod_y\prod_t[(y\,\epsilon\,s)\cdot(t\,\epsilon\,y)\rightarrow(t\,\epsilon\,s)].$$

The first formula shows that

$$s=[x] \tag{30}$$

and therefore the second is equivalent to

$$y \epsilon [x] \rightarrow \dot{y} \subset [x]. \tag{31}$$

Apply now theorem 5 to the formula (27). We obtain $\Phi_{s\cdot[b]}(\epsilon)$ what gives $\Phi_s(\epsilon)$ since $s \subset [b]$. Replace here s by $[x]$ according to (30); it comes

$$\Phi_{[x]}(\epsilon). \tag{32}$$

Comparing (31) and (32) with (23) and (24), we see that $x \epsilon k$. On the other hand (26) proves that $x \epsilon b$. We arrive thus at a contradiction since no element of b is in k.

This contradiction shows that it cannot be $\Delta_{[b]}(\epsilon)$ and the theorem 13 follows from (25).

From theorems 12 and 13 we infer easily the following

Theorem 14. *The formula $\Delta(\epsilon)$ is neither demonstrable nor refutable in (S).*

Proof[16]). If $\Delta(\epsilon)$ were demonstrable in (S), then every relation satisfying the formula $\Phi(\epsilon)$ would satisfy the formula $\Delta(\epsilon)$. Now there is by theorem 13 a set b such that all the axioms of the system (S) remain true if sets and classes are interpreted as elements of $[b]$ and ϵ as ϵ_b; by the same interpretation $\Delta(\epsilon)$ is carried over into a false statement. $\Delta(\epsilon)$ is therefore non-demonstrable in (S).

Using theorem 12 we show in the same manner that $\Delta'(\epsilon)$ is non demonstrable in (S) i. e. that $\Delta(\epsilon)$ is not refutable, q. e. d.

The formula $\Delta(\epsilon)$ yields thus an instance of an undecidable formula. Observe now that according to (9) the formulae (10) and (10*) are equivalent. Applying theorem 10 to the formula (10*) we infer that the formula $\Delta(\epsilon)$ is equivalent to an arithmetical formula of the form $\sum_{A \subset N} \prod_{B \subset N} \mathcal{G}(A,B)$, where $\mathcal{G}(A,B)$ is an elementary arithmetical formula. This equivalence being provable in (S), we obtain

Theorem 15. *There is an arithmetical formula $\mathcal{F}$ of the form $\sum_{A \subset N} \prod_{B \subset N} \mathcal{G}(A,B)$ where $\mathcal{G}(A,B)$ is an elementary arithmetical formula such that $\mathcal{F}$ is undecidable in (S).*

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[16]) In this proof use is made of some notions from the general methodology of deductive systems.

6. Remarks. (1) We compare here the undecidability of the formula $\mathcal{F}$ (cf. Theorem 15) with the undecidability of formulae constructed by the method of Gödel, Rosser, and Tarski[17]. The latter formulae assert their own undecidability, their intuitive truth is therefore evident. They become decidable after the introduction of suitable axioms or of a suitable rule of proof.

In the case of the Gödel's formula it is sufficient to adjoin the rule of infinite induction, i. e. to replace the axiomatic (relative) notion of integers by the absolute one.

To decide formulae constructed by Rosser and Tarski we have to adjoin to the system a number of intuitively obvious axioms stating certain properties of the notion of truth for sentences containing exclusively variables whose types do not surpass a fixed type n (in Rosser's case $n=2$).

The intuitive truth or falsity of the formula $\mathcal{F}$ from the theorem 15 is not evident unless one assumes the existence of inaccessible limit numbers. No „reasonable" rule of proof seems to exist which would be sufficient to decide within (S) whether $\mathcal{F}$ is true of false.

We see thus that the undecidability of $\mathcal{F}$ is caused by other circumstances than the undecidability of formulae constructed by the method of Gödel.

On the other hand, if we define the „absolute" undecidability as the undecidability irrespective of any assumption concerning the existence of sufficiently high cardinals or ordinals[18], we see immediately that $\mathcal{F}$ is not absolutely undecidable since it follows from the axiom $F1$. A mathematical Platonist who believes in the existence of „any" cardinal and „any" ordinal would therefore consider $\mathcal{F}$ as incontestably true.

The surprising property of $\mathcal{F}$ is that its truth cannot be established without presupposing the existence of inaccessible ordinals. Also it can be decided neither within arithmetic nor within the theory of function nor within any theory translatable into a subsystem of (S). Yet $\mathcal{F}$ expresses a fact concerning real numbers: it states that a CA-set is non void.

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[17] Gödel [4], Rosser [10], Tarski [15].

[18] I owe the acquaintance with this notion to conversations with Tarski. The explanation given in the text is of course very vague and it is doubtful whether an exact definition of the notion of absolute undecidability will ever be found. Cf. Tarski [14], p. 87.

We note still one (though unimportant) difference between the undecidable formulae of Gödel and the undecidable formula $\mathcal{S}$. The formulae defined by Gödel are so long that it is practically impossible to write them down explicitly. The formula $\mathcal{S}$ is very long too, but it occupies not more than one or two pages.

(2) The unprovability of $\Delta(\epsilon)$ within (S) could have been proved as follows. The arithmetization of meta-mathematics enables us to express as an elementary arithmetical formula the following meta-mathematical statement: *(S) is a self-consistent system.* Let $\mathcal{W}$ be the arithmetical formula obtained in this way. It is easy to see that $\Delta(\epsilon) \rightarrow \mathcal{W}$ is provable in (S). Indeed $\Delta(\epsilon)$ says that there is a model satisfying all the axioms of (S) and therefore $\Delta(\epsilon)$ implies that (S) is self-consistent.

Now $\mathcal{W}$ has been shown by Gödel[19] to be unprovable within (S) and hence $\Delta(\epsilon)$ is also unprovable.

We remark however that $\mathcal{W}$ is decidable if one adjoins the rule of infinite induction. The above proof gives therefore less than the former proof since it leaves open the possibility that $\Delta(\epsilon)$ may become provable after assuming the rule of infinite induction.

(3) If the existence of inaccessible numbers were provable in (S), then $\Delta(\epsilon)$ would be provable too (cf. theorem 11). Hence it is impossible to prove within (S) the existence of these numbers[20].

(4) The following remark concerns the theorem 9 of Skolem-Löwenheim. It might seem that the hypothesis of theorem 9 is unnecessarily strong. Indeed one can prove[21] the following theorem 9* which we propose to call „theorem of Skolem-Gödel": *if $\Psi(\epsilon)$ is a non-contradictory*[22] *formula, then there is a relation R with at most denumerable field C such that $\Psi_C(R)$.*

Theorem 9* however is neither stronger nor weaker than theorem 9. Its hypothesis is indeed considerably weaker than that of theorem 9 but its conclusion is weaker too since it cannot be ascertained that R is well founded. As a matter of fact it can be proved that for several formulae $\Psi(\epsilon)$ the R of the theorem 9* cannot be well founded.

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[19] Gödel [4], Theorem XI, p. 196.

[20] This has been proved by Kuratowski [6]. Cf. also Firestone and Rosser [2].

[21] Gödel [2].

[22] I. e. such that taking $\Psi(\epsilon)$ as an axiom and applying all the rules of functional calculus one obtains never a contradiction.

In theorem 10 we used theorem 3 which could be proved only for well-founded R. This shows that theorem 9* would be useless for our purpose.

(5) We shall show here that the axiom of choice could have been eliminated from the above proof. We have used this axiom only in the proof of theorem 9. The following theorem is however provable without this axiom:

Theorem 9.** *If x is a well ordered set, $a \in x, \dots, m \in x$ and $\Phi_x(\epsilon, a, \dots, m)$, then there is an at most denumerable set y such that $a \in y, \dots, m \in y$, $E(y)$, $C(\epsilon_y) = y$, and $\Phi_y(\epsilon, a, \dots, m)$.*

Let $\Phi(\epsilon)$ be again the conjunction of the axioms $A1$—$E1$.

Gödel[23]) has defined a transfinite sequence of well ordered sets $m_0, m_1, \dots, m_\xi, \dots$ such that if ξ is a suitable ordinal[24]), then $\Phi_{m_{\xi+1}}(\epsilon)$. Now the proofs of theorems 10-15 can be repeated using theorem 9** instead of 9.

(6) The finiteness of the axiom-system $A1$—$E1$ has enabled us to carry out the proof of undecidability without using any semantical notion. To extend our-construction to systems based on an infinite number of axioms, one has to take the semantical notion of satisfaction[25]) into consideration and the proof becomes much more complicated.

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On extendability of models of ZF set theory to the models of Kelley-Morse theory of classes

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§-1. Introduction

The standard axiomatisation of set theory due to Zermelo, Fraenkel and others was extended by von Neumann, Bernays and Gödel to an axiomatisation in which there appears, apart from the basic notion of a "set", the notion of a "class". Intuitively, classes are properties of sets, it being understood that we identify properties with the same extensions. This intuition derives from Cantor, who spoke of consistent and inconsistent classes.

The introduction of classes also extends the notion of "function": some classes which are not necessarily sets are functions. But the basic intuition of set theory that the image of a set is again a set is preserved.

If we restrict the scheme of class existence:

$$(EX)(x)(x \in X \Longleftrightarrow \Phi(x))$$

(where "X" does not appear in Φ) to formulae in which no quantifier binds class variables, we obtain the so-called "predicative class theory" of Bernays and Gödel (or "GB"). If, however, no restriction of this sort is imposed, i.e. if Φ may contain quantifiers with variables ranging over all classes, the theory which results is called the Kelley-Morse theory of classes (or "KM"). In both cases, GB and KM, we accept the class form of the replacement axiom.

The system GB corresponds to the intuition that classes do not form an acceptable totality, although some operations on classes are acceptable, by the composition of appropriate operations we get classes $\{x : \Phi(x)\}$ for predicative Φ's. The system KM corresponds to the conviction that the aggregate of all classes does form an acceptable totality and is a legitimate mathematical object.

The authors' standpoint is the following. They agree with the opinion that there is no reason to assume the properties of sets form an aggregate which is a legitimate mathematical object; but they think that the extensions of properties do form such an acceptable totality, and therefore that the system KM has as strong an intuitive basis as the system ZF. We claim, in fact, that KM is a formalisation of "second order ZF set theory", and that, in particular, the form of the replacement axiom which is accepted in KM is in accordance with this claim. Thus we believe that KM is a very good system for the formalisation and development of mathematics.

This conviction leads to a comparison of the relationships between Peano arithmetic and second order arithmetic on the one hand and between ZF and KM on the other.

The intermediate systems (i.e. systems lying between GB and KM, for instance KM_n, in which the scheme of class existence is restricted to Σ^1_n formulas) are not considered here, although a substantial number of results (in particular §3) may be extended to that case. An interesting interpretation of the Δ^1_1 class existence scheme is given in [11].

It is well-known that the theory KM is stronger than ZF, and that there are statements in the language of ZF set theory provable in KM but not (under assumption of the consistency of ZF) in ZF. Sentences which assert the existence of transitive models of ZF set theory may serve as examples; another example is a sentence asserting the existence of a model of GB and the Σ^1_n-class existence scheme. Let ZF^{KM} be the set of formulas Φ of the language of ZF set theory such that the relativisation $\bar{\Phi}^V$ of Φ to the universe of sets is provable in KM. The system ZF^{KM} is an extension of ZF, and is axiomatisable, but no axiomatisation (in the language of set theory) is known. Our feeling is that ZF^{KM} consists of sentences as true as those of ZF set theory. (Clearly the consistency of ZF^{KM} is formally equivalent to that of KM).

Consider a model $\langle M,E \rangle$ of ZF^{KM}. For general model-theoretic reasons $\langle M, E \rangle$ is elementarily equivalent to a model $\langle N,E' \rangle$ obtained from a model $\langle R,E'' \rangle$ of KM by restricting it to sets of the model (with restricted membership relation). Thus it seems that it is most natural to begin studies of models of ZF^{KM} by considering models which are restrictions (in the above sense) of the models of KM. Such models are called "extendable".

In the present paper we study extendable models.

The paper is organized as follows: §0 deals with preliminaries. In Part One we give a number of extendable and non-extendable models. We introduce the important notion of β - extendability (corresponding to β - models of second order arithmetic), which is a restriction of the notion of extendability. We show that every elementary class consisting of models of signature $\langle 1, 2 \rangle$ contains an element not extendable to a model of KM. For the ω - models we have a stronger result, which, in view of recent work of Krajewski, is optimal. We show that extendability is a PC and β - extendability is PCPC property. We establish the Σ^1_1 - reflection for the theory KM. We discuss the connections between the extendability phenomenon and the height of the model.

In Part Two we deal mainly with β - extendability. For a given transitive model $\langle M, \in \rangle$ of ZF set theory, we define a class $R.A.^M \subseteq \mathcal{P}(M)$ and show that $\langle M, \in \rangle$ is a β - extendable model iff $\langle R.A.^M, M, \in \rangle$ is a model of KM. Since the class $R.A.^M$ is defined by a constructive (although transfinite) process, this result may be considered as a criterion of β - extendability : to see whether $\langle M, \in \rangle$ is a β - extendable model, we have only to check whether the images of sets in M by functions from $R.A.^M$ belong to M or not. In the former case M is β - extendable, in the latter not. The class $R.A.^M$ is called ramified analysis over M; its construction closely follows work of Gandy and Putnam, who proved similar results in the case of second order arithmetic. (Gandy's results were unfortunately never published.). Transposing other unpublished ideas of Gandy to set theory, we prove that $\langle R.A.^{M}, M, \in \rangle$ is the smallest β - extension of the model M. Considerations used on this proof allow us to get inner interpretations of KM in itself satisfying in addition various forms of choice. In the course of this argument, we discover an interesting difference between second order arithmetic and the theory KM, namely that the latter has minimal transitive models; the former has not, as was shown by Friedman in [3]. This solves a problem in [3].

The results of Part Three were proved by W.Marek. The most important result is a proof that the notions of extendability and of β - extendability of transitive models of ZF are different. (This is not surprising, on analogy with second order arithmetic, but it has to be proved). Using methods of Barwise and Wilmers, we show that the least ordinal α for which $\langle L_{\alpha}, \in \rangle$ is extendable is smaller than the least ordinal α' for which $\langle L_{\alpha'}, \in \rangle$ is β - extendable. Under reasonable assumptions (namely that α' exists), both α and α' are denumerable. Moreover, α is denumerable in $\langle L_{\alpha'}, \in \rangle$, and $\langle L_{\alpha}, \in \rangle$ is extendable in $\langle L_{\alpha'}, \in \rangle$.

We give sufficient conditions of extendability and β - extendability. They are entirely constructive; they appeal to the next admissible set, in the case of extendability, and to the next Σ^{0}_{∞} - admissible set, in the case of β - extendability.

The subject of this paper has been studied by both of us for quite a long time. We started systematic research on it in 1970 and discussed it over many hours; hence our common authorship of the paper. Our earlier related results appeared separately in [12], [6], [9], [8], [10], [14] (arranged in order of appearance).

In our opinion, the notion of extendability merits further research. The most important problem seems to us to be the axiomatisation of the ZF^{KM} set theory. In particular, we are interested in mathematically interesting consequences of ZF^{KM} which are not consequences of ZF.

We are grateful to our colleagues from Warsaw: W.Guzicki, St.Krajewski, M.Srebrny and P.Zbierski for many valuable discussions and remarks. We owe a lot to R.O. Gandy whose ideas are so clearly seen in the part two.

§ 0. Preliminaries

Four basic theories we deal with in the paper are ZFC, ZF^{KM}, KM and KMC. They all are formulated in the same language L_{ST} ZFC is the usual set theory of Zermelo and Fraenkel (with choice). ZF^{KM} was defined in the introduction. In order to introduce KM we proceed as follows: We change the language L_{ST} into two-sorted language defining a predicate $\mathfrak{Z}(X) \Leftrightarrow (EY)(X \in Y)$ and using small Latin letters for variables ranging over sets i.e. classes with the property $\mathfrak{Z}(.)$. KM is the theory based on the following axioms:

1) Extensionality
2) Pairing for sets
3) Sum for sets
4) Powerset axiom
5) Infinity axiom
6) Foundation axiom
7) Choice axiom
8) Class existence scheme: $(EX)(x)(x \in X \Longleftrightarrow \Phi(x))$
   X not free in $\Phi(.)$
9) Replacement axiom (class form)
   If in addition we add the scheme 10)
10) Choice scheme
   $(x)(EY)\,\Phi(x,Y) \Rightarrow (EY)(x)\,\Phi(x, Y^{(x)})$

where $Y^{(x)} = \{ y : \langle x,y \rangle \in Y \}$,

the resulting theory is called KMC.

Models of KM (KM$\mathbb{C}$) may always be represented in the form $\langle \mathcal{T} \cup M, M, E \rangle$ where $\mathcal{T} \subseteq \mathcal{P}(M)$: namely it follows from the axiom of extensionality and the definition of $\mathfrak{z}(\cdot)$ that proper classes may be uniquely represented by subsets of M where M is the set of all sets of the model.

The height of a model $\langle M, E \rangle$ is the supremum of ordinals represented in $\langle M, E \rangle$. In case when $\langle M, E \rangle$ is a transitive model of ZF, the height h(M) is equal to $M \cap On$. But when $\langle M, E \rangle$ is a model of KM then the height of it is usually much larger since there are wellorderings in M of length bigger than On.

If $\langle N, E \rangle$ is a structure and $x \in N$ then we put $x^{*} = \{ y : N \models y \in x \}$. In case when N is transitive and $E = \in \restriction N$ then $x^{*} = x$.

Throughout the paper we use standard model theoretic and set-theoretic terminology. If X is a class then $X^{(x)}$ is called the $x^{\underline{th}}$ section of X. We write $X \eta Y$ instead of $(Ex)(X = Y^{(x)})$. In this way – intuitively – Y codes a collection $\{ Y^{(x)} : x \in \mathrm{Dom}\ Y \}$. In part two we use the following property of the theory KM: wellorderings are comparable i.e. If X and Y are wellorderings then either X is similar to a (unique) initial segment of Y or conversly. This in turn implies that any two non-standard wellorderings which are wellorderings in the sense of a given model have the same initial wellordered type (it is in fact the height of the model). In part three we use-standard by now-facts from the theory of so called admissible sets. In particular we assume the working knowledge of classical results of Barwise on Σ_1 – compactness of denumerable admissible sets. We assume also some knowledge of constructible hierarchy.

§ 1. Extendability

Definition: Let $\langle M,E \rangle$ be a model of ZFC set theory and T a theory. $\langle M,E \rangle$ is called T - extendable iff there is $\mathcal{F} \subsetneq \mathcal{P}(M)$ such that $\langle \mathcal{F} \cup M, M, E' \rangle \models T$ where $E' = E \cup [(M \times \mathcal{F}) \cap \in]$ We shall consider models which are KM- or KMC - extendable. When from the context it is clear what is T we call $\langle M,E \rangle$ just extendable.

In case when M is a transitive set and $E = \in \cap M^2$ then - if $\langle M, \in \rangle$ is extendable - we may find $\mathcal{F}$ such that $M \subseteq \mathcal{F} \subseteq \mathcal{P}(M)$ and $\langle \mathcal{F}, M, \in \rangle \models$ KM. In fact we shall be mainly interested in such structures.

The simplest property of the theory KM is that for every formula of L_{ST}, if $ZFC \vdash \varphi$ then $KM \vdash (\varphi)^V$. It follows in particular that if $\langle M, V^M, E \rangle \models KM$ then $\langle V^M, E \restriction V^M \rangle \models ZFC$. Let $ZF^{KM} = \{\varphi : KM \vdash (\varphi)^V\}$. Then ZF^{KM} is a recursively enumerable set of sentences and therefore it has a recursive axiomatization Moreover it is easy to see that the theory KM is consistent iff the theory ZF^{KM} is consistent Unfortunately we do not know any axiomatization of ZF^{KM}. The theory ZF^{KM} is much richer than ZFC. In particular $ZF^{KM} \vdash$ "There exists a transitive model of ZFC". A much stronger statement provable in ZF^{KM} is the following

(*) "For every ordinal α there exists a sequence f, defined on α and such that f is an elementary tower of natural models of ZFC".

Indeed, using the classical reasoning of Montague Vaught (which we present below under the name "over-and-over-and-over-again") one can prove that V is the union of an elementary tower $\{R_{\xi_\alpha}\}_{\alpha \in On}$

The last statement is not expressible in L_{ST} as a statement about sets since it has the form: "There exists an increasing function $\xi : On \longrightarrow On$ such that $\alpha < \beta$ implies $\langle R_{\xi_\alpha}, \in \rangle \prec \langle R_{\xi_\beta}, \in \rangle$". The quantifier "there exists ξ" binds a class variable and not a set variable. A (very unsatisfactory) interpretation of this statement is the formula (*). It should be noted that the above statement is provable already in quite weak subsystems of KM. Although not provable in GB theory of classes it is derivable already from the Σ^1_1 class existence scheme. Another type of statement provable in KM is the following

"There exists an increasing function $\rho : On \longrightarrow On$ such that $\alpha < \beta$ implies . $\langle R_{\rho_\alpha}, \in \rangle \prec \langle R_{\rho_\beta}, \in \rangle$ and such that $(\alpha)_{On}\,(EX)(X \subseteq R_{\rho_\alpha + 1}$ & "$\langle X, R_{\rho_\alpha}, \in \rangle$ satisfies GB $+ \Sigma^1_n$ - comprehension")".

A typical reasoning which we call "over-and-over-and-over-again" and use at least four times in this paper is the following one:

Theorem: There are arbitrarily large α such that $\langle R_\alpha, \in \rangle \prec \langle V, \in \rangle$.

Proof: Using the scheme 8 we are able to prove full scheme of induction and so we are able to prove that for every class X and in particular for the class V there exists the class $Stsf_X$ consisting of pairs $\langle \varphi, \vec{x} \rangle$ such that all terms of $\vec{x}$ belong to X and $\langle X, \in \rangle \models \varphi[\vec{x}]$. Applying the class form of the Skolem Löwenheim theorem (it is provable in KM, cf [14]) we get a set s_0 such that $\langle s_0, \in \rangle \prec \langle V, \in \rangle$

We may assume that $u \subseteq s_0$ (where u is a fixed set).

Notice that there exists α_0 such that $s_0 \subseteq R_{\alpha_0}$.

Now we define inductively two sequences $\{\alpha_n\}_{n\in\omega}$, $\{s_n\}_{n\in\omega}$ such that: I $\langle s_n, \in\rangle \prec \langle V, \in\rangle$

II $s_n \subseteq R_{\alpha_n} \subseteq s_{n+1}$

It is clear that $\bigcup_{n\in\omega} s_n = \bigcup_{n\in\omega} R_{\alpha_n} = R_{\bigcup_{n\in\omega}\alpha_n}$

Put $\alpha = \bigcup_{n\in\omega}\alpha_n$. Since $\langle \bigcup_{n\in\omega} s_n, \in\rangle \prec \langle V, \in\rangle$ therefore we have

$$\langle R_\alpha, \in\rangle \prec \langle V, \in\rangle$$

By the construction $u \subseteq R_\alpha$ so α may be chosen arbitrarily big. The proof as we presented it needs global form of choice, the Gödel's axiom E. Considering s_n's of least possible rank we are able to use only the local form of choice.

The sequence $\{R_{\xi_\alpha}\}_{\alpha\in On}$ of the consecutive natural elementary submodels of V may be characterized as follows: ξ_0 is the least ordinal majorizing all ordinals definable in V (i.e. definable by formulas of the form $\varphi^V(.)$). Similarly $\xi_{\alpha+1}$ is the least ordinal bigger than all ordinals definable in V by formulas with the parameter R_{ξ_α}.

In the language L_{ST} we are able to express the notion of wellordering, namely :

Definition: $W.O.(X) \Leftrightarrow X \subseteq V^2 \,\&\, (x,y)(x \in \mathrm{Dom}\, X \,\&\, y \in \mathrm{Dom}\, X \,\&\, \langle x,y \rangle \in X$
$\&\, \langle y,x \rangle \in X \Rightarrow y = x) \,\&\, (Z)(Z \subseteq \mathrm{Dom}\, X \,\&\, Z \neq \emptyset \Rightarrow (Ez)(z \in Z \,\&\, (w)(w \in Z \Rightarrow$
$\langle z,w \rangle \in X)))$.

Similarly as in case of models of the second order arithmetic we introduce the notion of β - models.

Definition: The structure $\langle \mathcal{F}, M, \in \rangle$ (where $\mathcal{F} \subseteq \mathcal{P}(M)$) is called β - model (or - equivalently - is said to possess β - property) iff for all $X \in \mathcal{F}$, $\langle \mathcal{F}, M, \in \rangle \models W.O.[X]$ implies that X is a wellordering.

This leads to the following natural definition:

Definition: The model $\langle M, \in \rangle$ is called β - KM - extendable (β - KMC - extendable) iff there is $\mathcal{F} \subseteq \mathcal{P}(M)$ such that $\langle \mathcal{F}, M, \in \rangle$ is a β - model of KM (β - model of KMC).

Transitive models of ZF set theory are necessarily β - models. *) This follows from the fact that
$ZF \vdash (X)(W.O.(X) \Rightarrow (E\alpha)(\mathrm{Ord}(\alpha) \,\&\, X \cong \langle \alpha, \in \rangle))$ and additionally from the fact that the formula "(.) is an ordinal" is an absolute formula with respect to transitive structures. In case of the theory KM the situation is different (we show this in § 3, although earlier it was proved in [9]). A curiosity with respect to this is the following lemma:

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*) Indeed much weaker theories have this property; it is sufficient to assume Δ_0 - collection and Σ_1 - comprehension. The fact that powerset axiom is not used we employ later.

Lemma: Formula W.O.(.) is equivalent (in KM) to a predicative one.

Proof: Consider the following formula w.o.(X): $(x,y)\ (x \in \mathrm{Dom}\, X \ \&\ y \in \mathrm{Dom}\, X \ \&\ \langle x,y\rangle \in X \ \&\ \langle y,x\rangle \in X \Rightarrow y = x)\ \&\ (z)(z \subseteq \mathrm{Dom}\, X \ \&\ z \neq \emptyset \Rightarrow (Eu)(u \in z \ \&\ (v)(v \in z \Rightarrow \langle u,v\rangle \in X)$. Clearly $W.O.(X) \Rightarrow w.o.(X)$. Assume now $\neg W.O.(X)$; Consider x_0 of least rank such that the Class $X \cap \{t : \langle t, x_0\rangle \in X\}^2$ is not wellfounded. Now inductively define x_{n+1} as an element of $\{t : \langle t, x_n\rangle \in X\} - \{x_n\}$ of least possible rank such that $X \cap \{t : \langle t, x_{n+1}\rangle \in X\}^2$ is not wellfounded. $\{x_n\}_{n\in\omega}$ is a set not wellfounded in X. Again we can eliminate global choice from the proof.

The notions of extendability and β - extendability coincide on some classes of models; as shown in [9], if $\langle M, \in\rangle$ is a transitive model of ZFC and $\mathrm{cf}(M \cap \mathrm{On}) \geqslant \omega_1$ then every extension of $\langle M, \in\rangle$ is necessarily a β - extension. Indeed if $\mathcal{T}$ is an extension and $\langle \mathcal{T}, M, \in\rangle \models W.O.[X]$ then, if $\{x_n\}_{n\in\omega}$ is an X - descending sequence then by our assumption on the cofinality character of $\mathrm{On} \cap M$, $\{x_n\}_{n\in\omega} \subseteq R^M_\alpha$ for some $\alpha \in \mathrm{On} \cap M$. But then, $X \restriction R^M_\alpha$ is not a wellordering; since $X \restriction R^M_\alpha \in M$ therefore $\langle M, \in\rangle \models \neg W.O.[X \restriction R^M_\alpha]$ contradicting $\langle \mathcal{T}, M, \in\rangle \models W.O.[X]$.

We investigate now the extendability of some models of ZFC.

Proposition: If α is a strongly inaccessible cardinal then $\langle R_\alpha , \in \rangle$ is β - KMC extendable.

Proof: $\langle R_{\alpha+1} , R_\alpha , \in \rangle$ is a desired extension.

Notice that by Skolem–Löwenhein theorem, $\langle R_\alpha , \in \rangle$ has also other extensions. It has even more than α of them. Under assumption of regularity of 2^α it has even at least 2^α of them.

Proposition: If α is a weakly inaccessible cardinal then $\langle L_\alpha , \in \rangle$ is β - KMC - extendable.

Proof: If α is weakly inaccessible then $\langle L , \in \rangle \models$ "α is a strongly inaccessible cardinal". Moreover $\langle L, \in \rangle \models$ "$L_\alpha = R_\alpha$" thus $\langle L, \in \rangle \models$ "L_α is β - KMC - extendable". However the latter statement is absolute because L is a transitive model of ZF and therefore it is a β - model. Note that one of β - extensions of $\langle L_\alpha , \in \rangle$ is $L_{\alpha^+} \cap \mathcal{P}(L_\alpha)$.

Proposition: The least transitive model of ZFC is not extendable.

Proof: The formula "(.) is a transitive model of ZFC" is Δ^{ZF}_1 and thus absolute with respect to transitive models of ZF. If the least transitive model were extendable then it would satisfy the statement "there exists a transitive model of ZFC". But it does not since it is the smallest one.*)

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*) Once again we do not use the full power of KM; Σ^1_1 class existence scheme is enough.

Let $\{\Theta_\xi\}_{\xi \in On}$ be the consecutive enumeration of heights of transitive models of ZFC.

Proposition: Let φ be a formula such that $ZFC \vdash (E!\,\alpha)\,\varphi$ and φ is absolute with respect to transitive models of ZFC.

Let α_φ be the unique object satisfying φ. Then $\langle L_{\Theta_{\alpha_\varphi}}, \in\rangle$ is not extendable.

Proof: If $\langle M, \in\rangle$ is extendable then, since $\alpha_\varphi \in M$ therefore also $\Theta_{\alpha_\varphi} \in M$ and $L_{\Theta_{\alpha_\varphi}} \in M$. Thus $M \neq L_{\Theta_{\alpha_\varphi}}$.

The above proposition can be generalized to the ordinal numbers definable in theories stronger than ZFC. Indeed it is enough that T is a recursive theory such that $KM \vdash$ "$\langle V, \in\rangle \models T$"

For instance $ZF^{KM_n} = \{\varphi : GB + \Sigma^1_n \text{ class existence scheme} \vdash \varphi\}$

Definition: If Ξ is a strongly inaccessible cardinal then α_Ξ is the least β such that $\langle R_\beta, \in\rangle \prec \langle R_\Xi, \in\rangle$

Proposition: If Ξ is a strongly inaccessible cardinal then $\langle R_{\alpha_\Xi}, \in\rangle$ is a non extendable transitive model of ZF^{KM}.

Proof: Since $\langle R_\Xi, \in\rangle$ is extendable therefore it satisfies ZF^{KM}. Thus $\langle R_{\alpha_\Xi}, \in\rangle$ also satisfies ZF^{KM}.

By absoluteness of the notion of rank with respect to transitive models of ZFC we get, for $\eta < \alpha_\Xi$ $(R_\eta)^{\langle R_{\alpha_\Xi}, \in\rangle} = R_\eta \cap R_{\alpha_\Xi} = R_\eta$.

If $\langle R_{\alpha_\Xi}, \in\rangle$ were extendable then there would exist $\eta < \alpha_\Xi$

such that in the extension $\langle \mathfrak{F}, R_{\alpha_\xi}, \in \rangle$ we would have

$$\langle \mathfrak{F}, R_{\alpha_\xi}, \in \rangle \models "\langle R_\eta, \in \rangle \prec \langle V, \in \rangle"$$

But then $\langle R_\eta^{\langle R_\xi, \in \rangle}, \in \rangle \prec \langle R_{\alpha_\xi}, \in \rangle$ i.e.

$$\langle R_\eta, \in \rangle \prec \langle R_{\alpha_\xi}, \in \rangle$$

contradicting minimality of α_ξ .

The analysis of the proof shows that $\langle R_{\alpha_\xi}, \in \rangle$ is in fact not extendable to a model of GB + Σ_1^1 class existence scheme.

Theorem (Krajewski): If $\langle M, E \rangle$ is a model of ZFC then there is $\langle N, E' \rangle$ such that $\langle M, E \rangle \equiv \langle N, E' \rangle$ and such that $\langle N, E' \rangle$ is not extendable.

Proof: By the main result of [17] there is a model $\langle N, E' \rangle$ such that $\langle N, E' \rangle \equiv \langle M, E \rangle$ and such that all ordinals of $\langle N, E' \rangle$ are definable in $\langle N, E' \rangle$. We claim that $\langle N, E' \rangle$ is the desired model.

Indeed, if it were extendable then, in the extension $\langle \mathfrak{F}, N, E'' \rangle$ we would have $\langle \mathfrak{F}, N, E'' \rangle \models "\langle R_\alpha, \in \rangle \prec \langle V, \in \rangle"$ for some ordinal α of $\langle N, E' \rangle$. Consider the replica, $(R_\alpha^{\langle N, E' \rangle})^*$ of the object $R_\alpha^{\langle N, E' \rangle}$. Then in particular $\langle (R_\alpha^{\langle N, E' \rangle})^*, E' \upharpoonright (R_\alpha^{\langle N, E' \rangle})^* \rangle$ is a proper elementary subsystem of $\langle N, E' \rangle$. Under this condition all definable elements of $\langle N, E' \rangle$ must be in $(R_\alpha^{\langle N, E' \rangle})^*$. But α is not there, which gives the desired contradiction.

Corollary: If $\mathcal{E}$ is the class of all extendable models and $\mathcal{K}$ an arbitrary elementary class in the language of L_{ST} then $\mathcal{K}-\mathcal{E} \neq \emptyset$

In the case of ω - models i.e. models with natural numbers ordered in the type ω , we get a slightly stronger result: *)

Theorem: If $\langle M,E \rangle$ is an ω - model of ZFC then either $\langle M,E \rangle$ is not extendable or there is an ordinal α of the model $\langle M,E \rangle$ such that $\langle (R_\alpha^{\langle M,E \rangle})^*, E \restriction (R_\alpha^{\langle M,E \rangle})^* \rangle \prec \langle M,E \rangle$ and $\langle (R_\alpha^{\langle M,E \rangle})^*, E \restriction (R_\alpha^{\langle M,E \rangle})^* \rangle$ is not extendable.

Proof: The key fact is the following tedious lemma:

Lemma: If x,y are two elements of M such that $\langle M,E \rangle \models$ " x is a pair $\langle x_0, E' \rangle$ and y is a pair $\langle y_0, E'' \rangle$ and $E' \subseteq x_0^2$ and $E'' \subseteq y_0^2$ " then ($\langle M,E \rangle \models$ "$x \prec y$" iff $\langle x_0^*, (E')^0 \rangle \prec \langle y_0^*, (E'')^0 \rangle$).

Proof of the lemma: We show that the satisfaction class for x inside of the model $\langle M,E \rangle$ and the satisfaction class for $\langle x_0^*, E^0 \rangle$ in V are isomorphic. Indeed consider the object z which is the satisfaction class for x in the model $\langle M,E \rangle$. Then z^* consists of objects being pairs $\langle \varphi , \vec{s} \rangle$ (in the sense of M) where $\langle M,E \rangle \models$ " φ is a formula" and

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*) We first define:

Definition: If $x \in M$ then $x^0 = \{ \langle z , t \rangle : \langle M,E \rangle \models$ " $\langle z , t \rangle \in x$" $\}$

$\langle M, E \rangle \models$ " s is a finite sequence of elements of x_0 " and $\langle M, E \rangle \models$ " $x \models \varphi[\vec{s}]$ ". Now we use the fact that $\langle M, E \rangle$ is an ω - model and thus the notion of a formula is absolute with respect to $\langle M, E \rangle$. Also the notion of finiteness is absolute i.e. $\langle M, E \rangle \models$ " s is finite " implies that $(s)^{*}$ is actually finite. Now we show by induction (which is allowed since $\langle M, E \rangle$ is an ω - model) that $\langle M, E \rangle \models$ " $x \models \varphi[\vec{s}]$ " iff $\langle x_0^{*}, E^{o} \rangle \models \varphi[((s)^{*})^{o}]$. Thus we had shown the isomorphism. Finally let us notice that $\langle x_0^{*}, E^{o} \rangle \prec \langle y_0^{*}, (E')^{o} \rangle$ is equivalent to the fact $\mathrm{Stsf}_{\langle (x_0)^{*}, E^{o} \rangle} = \mathrm{Stsf}_{\langle (y_0)^{*}, (E')^{o} \rangle} \cap (\mathrm{Form} \times \bigcup_{n \in \omega} {}^{n}(x_0^{*}))$.

Making the calculations inside and outside of the model and taking into account that $(x \cap y)^{*} = x^{*} \cap y^{*}$. (Where the symbol $\cap$ on the left hand side denotes an operation in the model and on the right hand side a set theorétic operation) an finally using once more absolutness of a finite sequence, we get the result.

With the lemma proved we prove the theorem as follows.

Let α be the least ordinal - in the sense of $\langle M, E \rangle$ - such that in the extension $\langle \mathcal{T}, M, E' \rangle \models$ "$\langle R_\alpha, \in \rangle \prec \langle V, \in \rangle$" (Clearly under the assumption of extendability there must be α with this property).

We claim that $\langle (R_{\alpha}^{\langle M,E\rangle})^{*}, E\restriction(R_{\alpha}^{\langle M,E\rangle})^{*}\rangle$ is not extendable. Suppose it were. Then there must be an ordinal β – in the sense of $\langle (R_{\alpha}^{\langle M,E\rangle})^{*}, E\restriction(R_{\alpha}^{\langle M,E\rangle})^{*}\rangle$ – such that in the extension $\langle \mathfrak{S}, (R_{\alpha}^{\langle M,E\rangle})^{*}, (E\restriction(R_{\alpha}^{\langle M,E\rangle})^{*})'\rangle$ the formula "$\langle R_{\beta}, \in\rangle \prec \langle V, \in\rangle$" holds.

Thus $\langle (R_{\beta}^{\langle (R_{\alpha}^{\langle M,E\rangle})^{*}, E\restriction(R_{\alpha}^{\langle M,E\rangle})^{*}\rangle})^{*}, E\restriction(R_{\beta}^{\langle (R_{\alpha}^{\langle M,E\rangle})^{*}, E\restriction(R_{\alpha}^{\langle M,E\rangle})^{*}\rangle})^{*}\rangle$ is an elementary subsystem of $\langle (R_{\alpha}^{\langle M,E\rangle})^{*}, E\restriction(R_{\alpha}^{\langle M,E\rangle})^{*}\rangle$

But $\langle M,E\rangle$ is an rank extension of $\langle (R_{\alpha}^{\langle M,E\rangle})^{*}, E\restriction(R^{\langle M,E\rangle})^{*}\rangle$ (cf [17]) and therefore $(R_{\beta}^{\langle (R_{\alpha}^{\langle M,E\rangle})^{*}, E\restriction(R_{\alpha}^{\langle M,E\rangle})^{*}\rangle})^{*} =$
$= (R_{\beta}^{\langle M,E\rangle})^{*}$.

Thus $\langle (R_{\beta}^{\langle M,E\rangle})^{*}, E\restriction(R_{\beta}^{\langle M,E\rangle})^{*}\rangle \prec \langle (R_{\alpha}^{\langle M,E\rangle})^{*}, E\restriction(R_{\alpha}^{\langle M,E\rangle})^{*}\rangle$

and so, using the lemma we have $\langle M,E\rangle \models$ "$\langle R_{\beta}, \in\rangle \prec \langle R_{\alpha}, \in\rangle$" contradicting the choice of α. *)

The extendable models always satisfy ZF^{KM}. The compactness theorem implies the following theorem:

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*) As shown by St.Krajewski the assumption that $\langle M,E\rangle$ is an ω – model cannot be omitted. Indeed he shows the following theorem

Theorem: If $\mathfrak{M} = \langle M,E\rangle$ is an extendable model then there exists a cardinal λ and an ultra filter D on λ such that the ultrapower $\mathfrak{N} = \mathfrak{M}^{\lambda}/D$ is extendable and for every ordinal α of $\mathfrak{N}$, if in the extension $\langle \mathfrak{T}, \mathfrak{N}\rangle \models$ "$\langle R_{\alpha}, \in\rangle \prec \langle V, \in\rangle$" holds then $\langle (R_{\alpha}^{\mathfrak{N}})^{*}, E'\restriction(R_{\alpha}^{\mathfrak{N}})^{*}\rangle$ is extendable.

Theorem: $\langle M,E \rangle$ is a model of ZF^{KM} iff there is model $\langle N,E' \rangle$ such that $\langle M,E \rangle \equiv \langle N,E' \rangle$ and $\langle N,E' \rangle$ is extendable model.

Proof: Implication from the right hand side to the left hand side is obvious. Assume $\langle M,E \rangle \models ZF^{KM}$. It is enough to show that $KM + (\mathrm{Th}(\langle M,E \rangle))^{V}$ is consistent. Otherwise $KM \vdash (\neg\varphi)^{V}$ for some $\varphi \in \mathrm{Th}(\langle M,E \rangle)$ thus $\neg\varphi \in ZF^{KM}$. But $ZF^{KM} \subseteq \mathrm{Th}(\langle M,E \rangle)$, contradiction.

The ultrapower $\mathfrak{M}^{\lambda}/D$ of an extendable model is again extendable. Thus applying the theorem of Frayne we get the following result:

Proposition (St.Krajewski): If $\langle M,E \rangle$ is a model of ZF^{KM} then there is an elementary extension of it $\langle N,E' \rangle$ which is an extendable model.

We come back now to the discussion of the ordinal $\alpha_{\boxminus}$

Proposition: $\alpha_{\boxminus}$ is a cardinal.

Proof: Since $\langle V, \in \rangle$ is a rank extension of $\langle R_{\alpha_{\boxminus}}, \in \rangle$ therefore the notion of a cardinal is absolute with respect to $\langle R_{\alpha_{\boxminus}}, \in \rangle$. Since $\langle R_{\alpha_{\boxminus}}, \in \rangle$ is a model of ZFC therefore it is a limit of its own cardinals. Thus $\alpha_{\boxminus}$ is a limit of cardinals and so is itself a cardinal.

Notice that the cofinality character of $\alpha_{\boxminus}$ is always ω. As is well known, if $\mathfrak{M}$ is a natural model of the theory KM (and even of the theory GB) i.e. a model of the form $\langle R_{\alpha+1}, R_{\alpha}, \in \rangle$ then α is a strongly inaccessible cardinal. If however we consider models of the form $\langle \mathcal{F}, R_{\alpha}, \in \rangle$ without stipulation that $\mathfrak{F} = R_{\alpha+1}$, then, under the assumption that inaccessible cardinals exist we may find extendable models of the form $\langle R_\alpha, \in\rangle$ Indeed we have the following theorem:

Theorem: If Ξ is an inaccessible cardinal then there are arbitrarity large $\eta < \Xi$ such that $\langle R_\eta, \in\rangle$ is extendable, $\langle R_\eta, \in\rangle \prec \langle R_\Xi, \in\rangle$, cf $\eta = \omega$.

Proof: We use the "over-and--over -and-over-again" method. Let $u \in R_\Xi$. Consider the system $\langle R_{\Xi+1}, R_\Xi, \in\rangle$ and its subsystem $\langle A^0, A^0_1, \in\rangle$ such that $u \subseteq A^0_1$. The objects in A^0_1 are elements of R_Ξ, objects in $A^0 - A^0_1$ are elements of $R_{\Xi+1} - R_\Xi$. We define as before sequences $\{A^n\}_{n\in\omega}, \{A^n_1\}_{n\in\omega}$ $\{\eta_n\}_{n\in\omega}$ such that:

$$\langle A^j, A^j_1, \in\rangle \prec \langle R_{\Xi+1}, R_\Xi, \in\rangle$$

$$A^j_1 \subseteq R_{\eta_j} \subseteq A^{j+1}_1 ,$$

$$\eta_j < \Xi , \quad \overline{\overline{A^{j+1}}} = \beth_{\eta_j}$$

As before $A^j_1 \subseteq R_\Xi$, $A^j - A^j_1 \subseteq R_{\Xi+1} - R_\Xi$.

Now set $A = \bigcup_{j\in\omega} A^j$, $A_1 = \bigcup_{j\in\omega} A^j_1$, $\eta = \bigcup_{j\in\omega} \eta_j$

Then $\langle A, A_1, \in\rangle \prec \langle R_{\Xi+1}, R_\Xi, \in\rangle$

Let us note firstly that A_1 is transitive (though A is not) and $A_1 = R_\eta$. This follows from the fact that $A^j_1 \subseteq R_{\eta_j} \subseteq A^{j+1}_1$. Thus $\langle A, R_\eta, \in\rangle \prec \langle R_{\Xi+1}, R_\Xi, \in\rangle$. For each $X \in A$ put

$X^+ = X \cap R_\eta$. Let $\mathfrak{F} = \{ X^+ : X \in A \}$. Clearly $\mathfrak{F}$ arises simply from A by the contraction procedure and so $\langle A, R_\eta, \in \rangle \cong \langle \mathfrak{F}, R_\eta, \in \rangle$. Thus $\langle \mathfrak{F}, R_\eta, \in \rangle \models$ KM. By our construction $\langle R_\eta, \in \rangle \prec \langle R_\Xi, \in \rangle$ and of $\eta = \omega$.

Notice that the system $\langle \mathfrak{F}, R_\eta, \in \rangle$ is a β - model. Indeed let $\langle \mathfrak{F}, R_\eta, \in \rangle \models$ W.O.$[X]$. Then for some $Z \in A$, $X = Z^+$. By our construction $\langle A, R_\eta, \in \rangle \models$ W.O.$[Z]$. The analysis of the form of Z shows that X is a restriction of Z to R_η. Now $\langle A, R_\eta, \in \rangle \prec \langle R_{\Xi+1}, R_\Xi, \in \rangle$. Thus $\langle R_{\Xi+1}, R_\Xi, \in \rangle \models$ W.O.$[Z]$ and so Z is a wellordering. Since Z is a wellordering, therefore all its restrictions are and so X is a wellordering.

When we look closely to the proof we find that we did'nt use all power of inaccesability. This leads to the following definition.

Definition: A model $\langle M, \in \rangle$ is called 2-extendable iff there exist two extensions $\mathfrak{F}_1$ and $\mathfrak{F}_2$ of $\langle M, \in \rangle$ such that: $\langle \mathfrak{F}_i, M, \in \rangle \models$ KM, $i = 1,2$ and $(EX)(X \in \mathfrak{F}_2 \,\&\, (Y)(Y \in \mathfrak{F}_1 \Longleftrightarrow Y_\eta X))$ i.e the extension $\mathfrak{F}_1$ is codable within $\mathfrak{F}_2$; in particular $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$.

By virtually the same reasoning as above we have the following theorem:

Theorem: If $\langle M, \in \rangle$ is 2-extendable then there exists $\alpha \in On \cap M$ such that $\langle R^M_\alpha, \in \rangle \prec \langle M, \in \rangle$ and $\langle R^M_\alpha, \in \rangle$

is extendable. If in particular the smaller extension $\mathcal{F}_1$ is a β-extension, then $\langle R^M_\alpha, \in \rangle$ may be chosen β-extendable.

Analogous hierarchies of n-extendability and α-extendability may be introduced, with analogous results. However, we shall not pursue the matter here.

Since we had shown that extendability is not an elementary property of models, it seems reasonable to investigate whether this property is connected with the height of model of ZF.

Theorem: If there exists a strongly inaccessible cardinal and $0^{\#}$ exists then there are transitive models M_1 and M_2 of ZF^{KM} such that $On \cap M_1 = On \cap M_2$, M_1 is extendable (even β-extendable) and M_2 is not. M_1 and M_2 can be chosen denumerable.

Proof: We first produce uncountable models M_1 and M_2 with the desired property as follows. If κ is an inaccessible cardinal then consider α_κ. As we proved before $\langle R_{\alpha_\kappa}, \in \rangle$ is not extendable. But α_κ is a cardinal and since $0^{\#}$ exists it is strongly inaccessible in L. Thus $\langle L, \in \rangle \models$ "$\langle L_{\alpha_\kappa}, \in \rangle$ is β-extendable". Since $\langle L, \in \rangle$ is transitive therefore $\langle L_{\alpha_\kappa}, \in \rangle$ is actually β-extendable. Set $M_1 = \langle L_{\alpha_\kappa}, \in \rangle$ $M_2 = \langle R_{\alpha_\kappa}, \in \rangle$. Clearly both of them satisfy ZF^{KM}. We construct denumerable models M_1 and M_2 with the above properties as follows. Let ζ be the least ordinal β such that $\alpha_\kappa \in \beta$ and $\langle R_\beta, \in \rangle \prec \langle R_\kappa, \in \rangle$. Consider the structure $\langle R_\zeta, \in, \alpha_\kappa \rangle$. Clearly $\langle R_\zeta, \in, \alpha_\kappa \rangle$ satisfies "$\langle R_{\alpha_\kappa}, \in \rangle$ has no elementary submodel of the form $\langle R_\eta, \in \rangle$ & $\langle L_{\alpha_\kappa}, \in \rangle$ is β-extendable".

Pick denumerable transitive model $\langle M, \in, \lambda \rangle$ elementarily equivalent to $\langle R_\zeta, \in, \alpha_\kappa \rangle$. We claim that $\langle R_\lambda^{\langle M,\in\rangle}, \in \rangle$ is not extendable. If $\langle R_\lambda^{\langle M,\in\rangle}, \in \rangle$ were extendable then for some $\eta < \lambda$

$$\langle R_\eta^{\langle R_\lambda^{\langle M,\in\rangle}, \in\rangle}, \in\rangle \prec \langle R^{\langle M,\in\rangle}, \in \rangle$$

By a reasoning we used twice, $R_\eta^{\langle R_\lambda^{\langle M,\in\rangle}, \in\rangle} = R_\lambda^{\langle M,\in\rangle}$

Thus $\langle R_\eta^{\langle M,\in\rangle}, \in\rangle \prec \langle R_\lambda^{\langle M,\in\rangle}, \in \rangle$ and so

$$\langle M, \in \rangle \models \text{"} \langle R_\eta, \in\rangle \prec \langle R_\lambda, \in \rangle \,\&\, \eta \in \lambda \text{"}$$

This however contradicts the fact that $\langle M, \in, \lambda\rangle \equiv \langle R_\zeta, \in, \alpha_\kappa \rangle$

Since $\langle M, \in \rangle$ is transitive and $\langle M, \in\rangle \models$ "L_λ is β - extendable" therefore $\langle L_\lambda^{\langle M,\in\rangle}, \in \rangle$ is indeed β- extendable i.e $\langle L_\lambda, \in \rangle$ is β-extendable.

Thus the height of the model does not determine the extendability property.

There is positive result concerning Cohen extensions of extendable models.

Theorem: If $\langle M, \in \rangle$ is a denumerable transitive extendable model, $\langle P, \leq \rangle \in M$ is a notion of forcing, G any M-generic ultrafilter in $\langle P, \in \rangle$ then $\langle M[G], \in \rangle$ is again extendable.

Proof: Following [2] we find that if $\langle N, V^N, \in \rangle$ is a denumerable transitive model of KM, $\langle P, \leq \rangle \in V^N$ then $\langle N[G], V^{N[G]}, \in \rangle$ is a model of KM (Actually, Chuaqui proves this for a larger class of notions of forcing, some of them being proper classes of N). Thus we only need to show that, if $M = V^N$ then $M[G] = V^{N[G]}$. This follows from the fact that if G is M-generic then (under assumption $\langle P, \in \rangle \in M$) it is necessarily N-generic, and the fact that if $K_G(X) \in V^{N[G]}$ then for some set x, $K_G(X) = K_G(x)$.

We show now a strong form of the reflexion principle for the theory KM.

Let X be a class. We define a relation $\mathrm{Sat}(X, \varphi, \vec{x})$ between formulas of L_{ST} and finite sequences of elements of Dom X which satisfies the following conditions

$$\mathrm{Sat}(X, \ulcorner v_i \in v_j \urcorner, \vec{x}) \iff X^{(x_i)} \in X^{(x_j)}$$

$$\mathrm{Sat}(X, \ulcorner v_i = v_j \urcorner, \vec{x}) \iff X^{(x_i)} = X^{(x_j)}$$

$$\mathrm{Sat}(X, \ulcorner \neg \varphi \urcorner, \vec{x}) \iff \neg\, \mathrm{Sat}(X, \varphi, \vec{x})$$

$$\mathrm{Sat}(X, \ulcorner \varphi \,\&\, \psi \urcorner, x) \iff \mathrm{Sat}(X, \varphi, \vec{x}) \,\&\, \mathrm{Sat}(X, \psi, \vec{x})$$

$$\mathrm{Sat}(X, \ulcorner (E v_i) \varphi \urcorner, \vec{x}) \iff (Ex)_{\mathrm{Dom} X}\, \mathrm{Sat}(X, \varphi, \vec{x}(\tfrac{i}{x}))$$

where $\vec{x}(\tfrac{i}{x}) = (\vec{x} - \{i\} \times V) \cup \{\langle i, x \rangle\}$

We define Sat(.,.,.) as the smallest relation satisfying the above. In case when X is a set, $\mathrm{Sat}(X, \ulcorner\varphi\urcorner, \langle x_1 \ldots, x_k \rangle)$ is equivalent to the following: $\langle \{ y : y \eta X \}, \in \rangle \models \varphi[X^{(x_1)}, \ldots, X^{(x_k)}]$

We have the following lemma:

Lemma: If φ is a predicative formula and X a class such that

$(x)(x \in V \Rightarrow x \eta X)$ and $X_1 = X^{(x_1)}, \ldots, X_k = X^{(x_k)}$ then

$$\varphi(X_1, \ldots, X_k) \iff \mathrm{Sat}(X, \varphi, x)$$

Proof: By induction on the complexity of formulas. For atomic formulas and boolean connectives the proof is obvious. In the case of the existential quantifier we use the fact that $(x)(x \eta X)$.

Lemma: If φ is a Σ^1_1 formula then $(X_1) \ldots (X_n)(\varphi(X_1, \ldots, X_n)$

$\iff (EX)(Ex_1) \ldots (Ex_n) [X_1 = X^{(x_1)} \& \ldots \& X_n = X^{(x_n)} \& \mathrm{Sat}(X, \ulcorner\varphi\urcorner, x)$

$\& (x)(x \eta X)])$

Proof: Let $\ulcorner\varphi\urcorner = \ulcorner(E v_i)\Psi\urcorner$ where Ψ is a predicative formula. Let $X_1, \ldots, X_n, Z$ be given such that $\Psi(Z, X_1, \ldots, X_n)$. We form the class X as follows: $X = \{0\} \times Z \cup \bigcup_{x \in V} \{\{x\}\} \times x \cup \bigcup_{i=1}^{n} \{i+1\} \times X_i$.

Then by the preceding lemma $\mathrm{Sat}(X, \ulcorner\Psi\urcorner, \langle 0, 2, \ldots, n+1 \rangle)$

thus $\mathrm{Sat}(X, \ulcorner\varphi\urcorner, \langle 2, \ldots, n+1 \rangle)$. Since $X^{(2)} = X_1, \ldots, X^{(n+1)} = X_n$

therefore $(Ex_1) \ldots (Ex_n)(X_1 = X^{(x_1)} \& \ldots \& X_n = X^{(x_n)} \& \mathrm{Sat}(X, \ulcorner\varphi\urcorner, x)$.

Conversely, assume $\mathrm{Sat}(X, \ulcorner\varphi\urcorner, \vec{x})$ & $X_1 = X^{(x_1)}$ &... & $X_n = X^{(x_n)}$.

Then $\mathrm{Sat}(X, \ulcorner(Ev_1)\psi\urcorner, \vec{x})$. So for some $z \in \mathrm{Dom}X$, $\mathrm{Sat}(X, \ulcorner\psi\urcorner, \vec{x}(\frac{1}{z}))$. Consider $X^{(z)}$. By the preceding lemma again $\psi(X^{(x)}, X_1, \ldots, X_n)$ and thus $(EZ)\psi(X_1, \ldots, X_n)$

Theorem (Σ^1_1 reflection principle): If $\varphi \in \Sigma^1_1$ then for every $X_1, \ldots, X_n$ there are arbitrarily large α's such that:

(a) $\varphi(X_1, \ldots, X_n) \Rightarrow \langle R_{\alpha+1}, R_\alpha, \in\rangle \models \varphi[X_1 \cap R_\alpha, \ldots, X_n \cap R_\alpha]$

(b) $\langle R_\alpha, \in\rangle \prec \langle V, \in\rangle$.

Proof: Let $X_1, \ldots, X_n$ be given. If $\neg\varphi(X_1, \ldots, X_n)$ then let α be appropriately large such that $\langle R_\alpha, \in\rangle \prec \langle V, \in\rangle$.

Suppose now $\varphi(X_1, \ldots, X_n)$. By the preceding lemma there is a class X and a sequence $\vec{x}$ such that $\mathrm{Sat}(X, \varphi, \vec{x})$ where, for each i, $1 \leqslant i \leqslant n$, $X^{(x_i)} = X_i$. Moreover, $(x)(x \eta X)$. Now we use the "over-and-over-and-over-again" method once more (using Skolem-Löwenheim, class version). We define four sequences $\{z^n\}_{n\in\omega}$, $\{z^n_o\}_{n\in\omega}$, $\{X_n\}_{n\in\omega}$, $\{t_n\}_{n\in\omega}$ inductively as follows: Let u be a given set. There is $w \subseteq \mathrm{Dom}\, X$ such that $u = \{X^{(s)} : s \in w\}$.

z^o is any subset of Dom X such that, for all $\vec{x} \in \bigcup_{n\in\omega}{}^n(z^o)$, for all φ's $\mathrm{Sat}(X_o, \varphi, \vec{x}) \Longleftrightarrow \mathrm{Sat}(X, \varphi, \vec{x})$ where $X_o = X \cap (z^o \times V)$, $z^o_o = \{z \in z^o : X^{(z)} \in V\}$;

t_0 is any set containing z_0^0 such that for some η

$\{X^{(z)} : z \in t_0\} = R_\eta$ (Since z_0^0 is a set, t_0 can be found)

Now assume z^n, z_0^n, X_n, t_n are known.

z^{n+1} is a subset of Dom X such that $t_n \subsetneq z^{n+1}$, $\overline{\overline{z^{n+1}}} = \overline{\overline{t_n}}$

and such that for all $\vec{x} \in \bigcup_{k \in \omega} {}^{k}(z^{n+1})$, all $\varphi \in L_{sr}$

$$Sat(X_{n+1}, \ulcorner\varphi\urcorner, \vec{x}) \Longleftrightarrow Sat(X, \ulcorner\varphi\urcorner, \vec{x})$$

where $X_{n+1} = X \cap (z^{n+1} \times V)$.

$z_0^{n+1} = \{v \in z^{n+1}: X^{(v)} \in V\}$

t_{n+1} is a set containing z_0^{n+1} such that for some η,

$\{X^{(z)} : z \in t_{n+1}\} = R_\eta$. Now form $z^\omega = \bigcup_{n \in \omega} z^n$,

$X_\omega = X \cap (z^\omega \times V)$, $z_0^\omega = \bigcup_{n \in \omega} z_0^n$ and $t_\omega = \bigcup_{n \in \omega} t_n$

We find that $z_0^\omega = \{v \in z^\omega : X^{(v)} \in V\}$

By our construction there is η such that $\{X^{(v)} : v \in t_\omega\} = R_\eta$

The following holds:

(I) $(\vec{z})(\vec{z} \in \bigcup_{n \in \omega}({}^{n}(z^\omega)) \Longrightarrow (Sat(X_\omega, \ulcorner\varphi\urcorner, \vec{z}) \Longleftrightarrow Sat(X, \ulcorner\varphi\urcorner, \vec{z}))$

(II) $z_0^\omega = t_\omega$

Since z^ω is a set we may assume that for $z_1, z_2 \in$ Dom X_ω, $z_1 \neq z_2 \Longrightarrow X_\omega^{(z_1)} \neq X_\omega^{(z_2)}$.

Define now on Dom X_ω a relation E as follows

$$\langle x, y\rangle \in E \iff X_\omega^{(x)} \in X_\omega^{(y)}$$

The relation E is wellfounded and extensional (for the last fact we need, for $z_1 \neq z_2$, $X_\omega^{(z_1)} \neq X_\omega^{(z_2)}$). Thus $\langle \mathrm{Dom}X_\omega, E\rangle \simeq \langle N, \in\rangle$ for some transitive N.

We find that

(III) $(x)(\mathrm{Sat}(X_\omega, v_i \in V, \{\langle i,x\rangle\})) \iff X_\omega^{(x)} \in R_\eta$

Thus $N \subseteq R_{\eta+1}$, $V^N = R_\eta$

The analysis of the isomorphism $i : \langle \mathrm{Dom}X_\omega, E\rangle \longrightarrow \langle N, \in\rangle$ gives the following : $i(x) = X_\omega^{(x)} \cap R_\eta = X^{(x)} \cap R_\eta$

Since $\mathrm{Sat}(X_\omega, \varphi, \vec{x})$ therefore $\langle N, V^N, \in\rangle \models \varphi[X^{(x_1)} \cap R_\eta, \ldots, X^{(x_n)} \cap R_\eta]$

Finally for $\varphi \in \Sigma_1^1$ $\langle N, R_\eta, \in\rangle \models \varphi[z_1, \ldots, z_n]$ implies $\langle R_{\eta+1}, R_\eta, \in\rangle \models \varphi[z_1, \ldots, z_n]$. Thus $\mathrm{Sat}(X, \varphi, \vec{x})$ implies $\langle R_{\eta+1}, R_\eta, \in\rangle \models \varphi[X^{(x_1)} \cap R_\eta, \ldots, X^{(x_n)} \cap R_\eta]$ (whenever $\varphi \in \Sigma_1^1$) Clearly $\langle R_\eta, \in\rangle \prec \langle V, \in\rangle$.

Considering z^n's, and t_n's of least possible rank we may eliminate the usage of the global form of the axiom of choice.

Definition: (a) A class $\mathcal{K}$ of models is a PC class with respect to the class $\mathcal{L}$ iff $\mathcal{K} \subseteq \mathcal{L}$ and there is a set of sentences S in the language $L_{ST}(A)$ (arising from L_{ST} by adding unary predicate A) such that

$$(\mathfrak{M})\ (\mathfrak{M} \in \mathcal{L} \Rightarrow (\mathfrak{M} \in \mathcal{K} \Leftrightarrow (EX)(X \subseteq |\mathfrak{M}| \,\&\, (\varphi)(\varphi \in S \Rightarrow \langle \mathfrak{M}, X \rangle \models \varphi))))$$

(b) A class $\mathcal{K} \subseteq \mathcal{L}$ is CPC with respect to $\mathcal{L}$ iff $\mathcal{L} - \mathcal{K}$ is PC class. Analogously we define PCPC, CPCPC classes etc.

Theorem: The class of extendable models is a PC class with respect to the class of all models of ZFC.

Proof: If $\mathfrak{M}$ is extendable then - by virtue of Skolem-Löwenheim theorem there is $C \subseteq \mathcal{P}(|\mathfrak{M}|)$ which extends $\mathfrak{M}$ and such that $\bar{\bar{C}} = \overline{\overline{|\mathfrak{M}|}}$.

Let f be an enumeration of C with elements of $|\mathfrak{M}|$. Finally let $X = \{\langle x, y \rangle^{\mathfrak{M}} : y \in f(x)\}$.

We have the following lemma:

Lemma: For every formula φ of L_{ST} there is a formula Ψ_{φ} of $L_{ST}(A)$ such that

$$\langle C, M, E' \rangle \models \varphi\,[x_1, \ldots, x_k, f(y_1), \ldots, f(y_k)]$$

$$\langle M, E, X \rangle \models \Psi_{\varphi}\,[x_1 \ldots x_k,\ y_1 \ldots y_k]$$

Moreover the mapping $\varphi \mapsto \Psi_{\varphi}$ is effective.

We leave the proof to the willingful reader.

Now: $\mathfrak{M}$ is extendable $\Longleftrightarrow$

$$(EC)(C \leq \mathcal{P}(|\mathfrak{M}|) \cup M \ \& \ (\varphi)(\varphi \in KM \Rightarrow \langle C, M, E' \rangle \models \varphi))$$

$$\Longleftrightarrow (EX)(X \leq |\mathfrak{M}| \ \& \ (\varphi)(\varphi \in KM \Rightarrow \langle M, E, X \rangle \models \Psi_{\varphi})$$

thus the class of extendable models is a PC class.

One shows (just by appropriate modification of the above definition) that the class of β- extendable models is a PCPC class. By the existence of the least β - extension (cf § 2) it is also a CPCPC class. We do not know whether it is a CPC class (with respect to the class of all models of ZFC).

§ 2. Ramified analysis and β -extendability

We present here a construction of the least β -extension of a model $\langle M,E \rangle$ (provided $\langle M,E \rangle$ is β -extendable). The construction follows closely the one of Gandy used in his proof of existence of the least β -model of analysis. However we have to change some details since not every model has a definable wellordering.

We use instead another interesting property of transitive models of ZF; Every transitive model of ZF has a definable prewellordering (according to the rank of elements) such that every equivalence class of this prewellordering is a set. This fact will be used to show that ramified analysis has a prewellordering such that every equivalence class of it is codable as a class with the domain being a set.

Let M be a transitive set, $U \subseteq \mathcal{P}(M)$ family of subsets of M, we consider a structure $\langle U, M, \in \rangle$.

Definition: $\mathcal{D}(\langle U, M, \in \rangle)$ is the family of subsets of M, parametrically definable over $\langle U, M, \in \rangle$ i.e. of the form

$$\{ x \in M : \langle U, M, \in \rangle \models \varphi[x, \vec{z}] \}$$

where φ is a formula and $\vec{z}$ a sequence of parameters.

Let us note that $\langle U, M, \in \rangle \models \mathfrak{z}[A]$ is equivalent with $A \in M$.

Let $X \subseteq M$.

We define $R.A_0^{M,X} = M \cup \{X\}$

$$R.A_{\alpha+1}^{M,X} = \mathcal{D}(\langle R.A._{\alpha}^{M,X}, M, \in \rangle)$$

$$R.A_{\lambda}^{M,X} = \bigcup_{\xi < \lambda} R.A_{\xi}^{M,X}$$

and finally $R.A.^{M,X} = \bigcup_{\xi \in On} R.A._{\xi}^{M,X}$

Since $R.A._{\alpha}^{M,X} = R.A_{\alpha+1}^{M,X}$ implies that $R.A_{\alpha}^{M,X} = R.A.^{M,X}$ therefore by cardinality argument there must be ρ such that $R.A._{\rho}^{M,X} = R.A.^{M,X}$.

Definition: Let $K \subseteq M$. K is called M-amenable iff either K is not function or K is a function and $(x)(x \in M \Rightarrow K * x \in M)$.

Theorem 2.1. (Ramified analysis theorem). If $\langle M, \in \rangle$ is a transitive model of ZFC and if every element of $R.A.^{M,X}$ is M-amenable then $\langle R.A.^{M,X}, M, \in \rangle$ is a β-model of KM.

Proof. The family $R.A.^{M,X}$ is - by virtue of construction - closed under the scheme of class existence (Indeed, if $R.A.^{M,X} = R.A.^{M,X}_{\eta}$ then $R.A.^{M,X}_{\eta+1} \subseteq R.A.^{M,X} = R.A.^{M,X}_{\eta}$ i.e. $\mathfrak{D}(\langle R.A.^{M,X}_{\eta}, M, \in \rangle) \subseteq R.A.^{M,X}_{\eta}$).

The axiom of substitution holds by the amenability property of $R.A.^{M,X}$. Since $V^{R.A.^{M,X}} = M$ (by our previous remark) therefore the axiom of power set holds too.

The proof of the fact that $R.A.^{M,X}$ is a β - model we defer until we get appropriate technique.

Lemma 2.1: If $\langle R.A.^{M,X}_{\xi}, M, \in \rangle \models KM$ then $R.A.^{M,X} = R.A.^{M,X}_{\xi}$.

Proof: If $\langle R.A.^{M,X}_{\xi}, M, \in \rangle \models KM$ therefore $\mathfrak{D}(\langle R.A.^{M,X}_{\xi}, M, \in \rangle) \subseteq R.A.^{M,X}_{\xi}$. By induction $R.A.^{M,X}_{\eta} = R.A.^{M,X}_{\xi}$ for all $\eta \geqslant \xi$.

Our task - in fact for all the rest of this chapter - will be to prove that $R.A.^{M,X}$ is definable in every β - model $\langle \mathcal{F}, M, \in \rangle$ of KM such that $X \in \mathcal{F}$.

Before we go into the proof we need certain extension of the language. We add to the language of L_{ST} predicates $\mathcal{M}(\cdot)$ and $\mathfrak{X}(\cdot)$ and assume the following axioms:

1) $(x)(\mathcal{M}(x) \longrightarrow \mathfrak{Z}(x))$

2) $\mathrm{Trans}\,(\mathcal{M}(\cdot))$

3) $(Z\,F\,C)^{\mathcal{M}(\cdot)}$

4) $(Y)(\mathfrak{X}(Y) \Rightarrow \mathfrak{M}(Y))$

Since there is no reason to assume that any of objects M,X is definable in KM we have to start at the language level. The theory $KM_{\mathfrak{M},\mathfrak{X}}$ is the theory KM + 1) ... 4) where in the class existence scheme we allow $\mathfrak{M}$ and $\mathfrak{X}$ to appear.

It is clear that $KM_{\mathfrak{M},\mathfrak{X}}$ is a conservative extension of KM. Let us notice that if $\langle \mathfrak{T}, M, \in \rangle \models KM$, $X \in \mathfrak{T}$ then the structure $\langle \mathfrak{T}, M, \in, M, X \rangle$ is a model of $KM_{\mathfrak{M},\mathfrak{X}}$. There are models of $KM_{\mathfrak{M},\mathfrak{X}}$ of different form. For instance $\langle \mathfrak{T}, M, \in, L^M, X \rangle$ where $X \in \mathfrak{T}$ and $X \subseteq L^M$. In the sequel proofs will be done in $KM_{\mathfrak{M},\mathfrak{X}}$.

For a moment we are going to study prewellorderings

Definition: A prewellordering (p.w.o) is any relation reflexive, transitive and satisfying the wellfoundedness condition

$(z)(z \neq \emptyset \ \& \ z \subseteq \mathrm{Dom}(\prec) \Rightarrow (Ez)(z \in z \ \& \ (t)(t \in z \Rightarrow z \prec t)$

If $\prec$ is a p.w.o we define $\sim_{\prec}$ as follows: $x \sim_{\prec} y \Leftrightarrow (x \prec y \ \& \ y \prec x)$ $\sim_{\prec}$ is an equivalence; let $Cl_{\prec}(x)$ be an equivalence class of x in $\sim_{\prec}$

Definition: A p.w.o $\prec$ is a good p.w.o. (gpwo) iff $(x)(x \in \mathrm{Dom} \prec \Rightarrow Cl_{\prec}(x) \in V)$

If $\prec$ is a gpwo then $\prec$ determines a wellordering $\tilde{\prec}$ on the class $\mathrm{Dom} \prec /_{\sim_{\prec}}$ as follows: $Cl(x) \tilde{\prec} Cl(y) \Leftrightarrow x \prec y$.

Conversly if $<$ is a wellordering and $F : \mathrm{Dom}(<) \longrightarrow V$ satisfies conditions: a) $x \neq y \Longrightarrow F(x) \cap F(y) = \emptyset$

b) $F(x) \neq \emptyset$

then $<$ and F determine natural p.w.o $\precsim$ on $\bigcup_{x \in \mathrm{Dom}(<)} F(x)$

namely $x_1 \lesssim x_2 \Longleftrightarrow (Ey_1)(Ey_2)(x_1 \in F(y_1) \,\&\, x_2 \in F(y_2) \,\&\, y_1 < y_2)$

Operations $\tilde{\prec}$ and $\lesssim$ commute (up to isomorphism).

In the sequel we will need one more operation:

Let Y be a class such that Dom Y is set and

a) $(y)(y \in \mathrm{Dom}\, Y \Longrightarrow \mathrm{W.O.}(Y^{(y)}))$ and

b) $(y_1)(y_2)(y_1, y_2 \in \mathrm{Dom}\, Y \Longrightarrow Y^{(y_1)} \cong Y^{(y_2)}$).

We call Y mixable iff it satisfies a) and b).

The ordering Y^{mix} is defined as follows:

$$\mathrm{Dom}\, Y^{\mathrm{mix}} = \{ f \in {}^{(\mathrm{Dom}\, Y)}V : (x_1)(x_2)(x_1, x_2 \in \mathrm{Dom}\, Y \Rightarrow Y^{(x_1)} \upharpoonright f(x_1) \cong Y^{(x_1)} \upharpoonright f(x_2) \}$$

$$f_1 \prec_{Y^{\mathrm{mix}}} f_2 \Longleftrightarrow (Ex)(x \in \mathrm{Dom}\, Y \,\&\, f_1(x) \prec_{Y^{(x)}} f_2(x))$$

Lemma 2.2. If Y is mixable then for all $x \in \mathrm{Dom}\, Y$, $Y^{\mathrm{mix}} \lesssim Y^{(x)}$

Definition: (a) If Y_1 and Y_2 are classes then

$\{0\} \times Y_1 \cup \{1\} \times Y_2$ is called ordered pair of Y_1, Y_2 and is denoted by $\langle Y_1, Y_2 \rangle$

(b) If Y is a class, $Y \subseteq V \times V$. $\prec$ is a gpwo of Dom Y then the pair $\langle Y, \prec \rangle$ is called a gpwo family.

(c) If $\prec$ happens to be a wellordering of Dom Y then $\langle Y, \prec \rangle$ is called a wellordered family.

Definition: A proper formula is a formula Φ such that

(a) $0 \in \mathrm{Fr}\,\Phi$ (b) $i \in \mathrm{Fr}\,\Phi \Longrightarrow i \dot{-} 1 \in \mathrm{Fr}\,\Phi$

($\mathrm{Fr}\,\Phi$ is the set of indices of free variables in Φ)

Since we identify formulas with their Gödel numbers, the set of proper formulas is a set of numbers; we denote it by Pform. Usage of proper formulas allows us not to bother about which are the free variables of the formula, thus simplyfying the formalization of the operation $\mathcal{D}(\cdot)$.

If $\prec$ is a pwo then $\bigcup_{n \in \omega} {}^{n}\mathrm{Dom}(\prec)$ has a natural pwo. We denote it by $\prec_{\mathrm{alex}}$. It is the following ordering:

$$\mathrm{lh}(\vec{x}) < \mathrm{lh}(\vec{y}) \vee \big(\mathrm{lh}(\vec{x}) = \mathrm{lh}(\vec{y}) \ \& \ (\mathrm{E}k)(j)(j < k \Longrightarrow \vec{x}(j) \sim \vec{y}(j) \ \& \ \vec{x}(k) \prec \vec{y}(k) \ \& \ \neg\, \vec{y}(k) \prec \vec{x}(k)\big) \vee \big(\mathrm{lh}(\vec{x}) = \mathrm{lh}(\vec{y}) \ \& \ (j)(j \in \mathrm{Dom}\, x \Rightarrow x(j) \sim y(j))\big)$$

Lemma 2.3. If $\prec$ is a gpwo then $\prec_{\mathrm{alex}}$ is a gpwo.

Proof: Assume $\vec{s}_1 \prec_{\mathrm{alex}} \vec{s}_2 \prec_{\mathrm{alex}} \vec{s}_1$: Then it is obvious that $\mathrm{lh}(\vec{s}_1) = \mathrm{lh}(\vec{s}_2)$. Let $\vec{s}_1 = \langle z_1, \ldots, z_k \rangle$, $\vec{s}_2 = \langle t_1, \ldots, t_k \rangle$. We show by induction that

$z_1 \sim t_1 \quad \ldots \quad z_k \sim t_k$. This however shows how the

classes of ${}^{\sim}\!\prec_{\mathrm{alex}}$ look like:

$\mathrm{Cl}_{\prec_{\mathrm{alex}}}(\langle z_1 \ldots z_k \rangle) = \{ \langle t_1 \ldots t_k \rangle : z_1 \sim t_1 \,\&\ldots\&\, z_k \sim t_k =$

$\mathrm{Cl}\ (z_1) \times \ldots \times \mathrm{Cl}\ (z_k)$. The latter class is a set.

If $\prec$ is a pwo then in the class $\bigcup_{\varphi \in \mathrm{Pform}} \{\varphi\} \times ({}^{(\mathrm{Fr}(\varphi)-\{0\})}\mathrm{Dom}(\prec))$

there is a special pwo called the derived ordering of X and denoted by X' namely

$$\langle \varphi , \vec{x} \rangle \prec' \langle \Psi , \vec{y} \rangle \Leftrightarrow (\varphi < \Psi) \vee (\varphi = \Psi \ \&\ \vec{x} \prec_{\mathrm{alex}} \vec{y})$$

One proves that: If $\prec$ is a gpwo then $\prec'$ is a gpwo and in particular that if $\prec$ is a wellordering then $\prec'$ is also a wellordering.

We recall that the formula $\mathfrak{z}(.)$ served as formalization of the predicate "$x \in V$"

Definition: Let Y be a class such that

(i) $(x)(\mathfrak{M}(x) \Longrightarrow x\ \eta\ Y)$

(ii) $\mathrm{Sat}(Y, \mathfrak{z}(.), t) \Longrightarrow \mathfrak{M}(Y^{(t)})$

then we define $\mathcal{D}_{\mathfrak{M}}(Y) = \{\langle\langle \varphi , \vec{z} \rangle , x \rangle : \varphi \in \mathrm{Pform}\ \&$

$(\vec{z} \in {}^{(\mathrm{Fr}\ - \{0\})}\mathrm{Dom}\ Y)\ \&\ (Et)(t \in \mathrm{Dom}Y\ \&\ Y^{(t)} = x$

$\&\ \mathrm{Sat}(Y, \varphi , \vec{z}(\overset{\circ}{t})))$.

Let Y be a set, then we say that the family of sets $\{ X : X\ \eta\ Y \}$ i.e. $\{ Y^{(t)} : t \in \mathrm{Dom}\ Y \}$ is codable by Y (or – equivalently – that Y is a code for $\{ Y^{(t)} : t \in \mathrm{Dom}\ Y \}$. In order to explain the meaning of the operation $\mathcal{D}_{\mathcal{M}}(\cdot)$, let us remark that if Y,M are sets $Y^{(t)} \subseteq M$ for all $t \in \mathrm{Dom}\ Y$, $\mathcal{M}(x) \Longleftrightarrow x \in M$, then $\mathcal{D}_{\mathcal{M}}(Y)$ is nothing else but a certain code for the family $\mathcal{D}(\langle \{Z : Z \eta Y\}, M, \in \rangle)$ as it was defined on page 491.

Let us note that the operation $\mathcal{D}_{\mathcal{M}}(\cdot)$ makes sense also in case when Y is a proper class and $\mathcal{M}(x) \Longleftrightarrow x \in M$ where M is a proper class; In this case however $\mathcal{D}(\langle \{Z : Z \eta Y\}, M, \in \rangle)$ was not defined.

Lemma 2.4.: If $\langle Y, \prec \rangle$ is a gpwo family (i.e. $\prec$ is a gpwo) then $\langle \mathcal{D}_{\mathcal{M}}(Y), \prec' \rangle$ is also a gpwo family.

Lemma 2.5.: If $\langle Y_1, \prec_1 \rangle$ is a gpwo family then there is a unique gpwo family $\langle Y_2, \prec_2 \rangle$ (called the concentration of $\langle Y_1, \prec_1 \rangle$) satisfying the following conditions

a) $(W)(W \eta Y_1 \Longleftrightarrow W \eta Y_2)$

b) $(x)(x \in \mathrm{Dom}\ Y_2 \Longrightarrow Y_1^{(x)} = Y_2^{(x)})$

c) $\prec_2 = \prec_1 \restriction \mathrm{Dom}\ Y_2$

d) $(x)(y)(x \in \mathrm{Dom}\ Y_1 \,\&\, y \in \mathrm{Dom}\ Y_2 \,\&\, x \prec y \,\&\, \neg\, y \prec x \Longrightarrow Y_1^{(x)} \neq Y_1^{(y)})$

e) $(x)(y)(x \sim_{\prec_1} y \,\&\, Y_1^{(x)} = Y_1^{(y)} \Longrightarrow (x \in \mathrm{Dom} Y_2 \Longleftrightarrow y \in \mathrm{Dom} Y_2))$

Proof: Let us describe how $\langle Y_2, \prec_2 \rangle$ arises from $\langle Y_1, \prec_1 \rangle$.

Let $Z \eta Y_1$, $Z = Y_1^{(z)}$. We pick $\prec_1$-least z_1 with this property. Since $\prec_1$ is a gpwo z_1 needs not to be unique. We leave in the $\mathrm{Dom}(\prec_2)$ all these $\prec_1$ - least z_1's but erase all other z's which have the property that $Y_1^{(z)} = Z$. This procedure determines both Y_2 and $\prec_2$. The conditions a) - e) were determined to give this procedure. ∎

The unique pair constructed in the concentration procedure is called concentration of $\langle Y, \prec \rangle$ and denoted $\langle Y^-, \prec^- \rangle$ Now let M be the class $\{ x : \mathcal{M}(x) \}$, $\prec$ the class $\{ \langle x,y \rangle : x \in M \ \& \ y \in M \ \& \ \rho(x) \leq \rho(y) \}$, $X = \{ x : \mathcal{X}(x) \}$, define $\tilde{E}_M = \{ \langle \{y\}, x \rangle : x \in y \}$, $E'_M = E_M \cup \{ \langle 0, x \rangle : x \in X \}$ In the case when M is a set then $\tilde{E}_M$ is a code for M and E'_M is a code for $M \cup \{X\}$.

Notice that $\mathrm{Dom}\, E'_M$ has a special gpwo $\prec^+$ defined as follows

$$x \prec^+ y \iff (x = 0) \vee (Et)(Eu)(x = \{t\} \ \& \ y = \{u\} \ \& \ t \prec u)$$

Lemma 2.6.: If T is a wellordering then there are unique classes U_T and ϑ_T satisfying the following conditions:

1) $\mathrm{Dom}\, U_T = \mathrm{Dom}\, T = \mathrm{Dom}\, \vartheta_T$

2) $(x)(x \in \mathrm{Dom}\, T \Rightarrow \langle U_T^{(x)}, \vartheta_T^{(x)} \rangle$ is a gpwo family)

3) If x_0 is the first element of T then

$$U_T^{(x_0)} = E_M \qquad \vartheta_T^{(x_0)} = \prec^{+}$$

4) If y is a successor of x in the wellordering T then

$$U_T^{(y)} = \mathcal{D}_{\mathcal{R}}(U_T^{(x)}) \qquad \vartheta_T^{(y)} = (\vartheta_T^{(x)}),$$

5) If y is limit in T then

$$U_T^{(y)} = \bigcup_{x <_T y} \bigcup_{z \in \mathrm{Dom}\, U_T^{(x)}} \{\langle x, z \rangle\} \times (U_T^{(x)})^{(z)}$$

$$\vartheta_T^{(y)} = \{\langle\langle x, z\rangle, \langle x_1, z_1\rangle\rangle : x <_T y \;\&\; x_1 <_T y$$

$$\&\; z \in \mathrm{Dom}\, U_T^{(x)} \;\&\; z_1 \in \mathrm{Dom}\, U_T^{(x_1)} \;\&$$

$$(x <_T x_1 \;\&\; x \neq x_1) \vee (x = x_1 \;\&\; z \prec_{\vartheta_T^{(x)}} z_1)\}$$

Proof: We make use of the theorem on inductive definitions by transfinite induction on elements of wellorderings. We pick inductive clauses to correspond to the construction described on the lemma. One point which needs some explanation is that $\vartheta_T^{(y)}$ is a gpwo when y is limit. Notice however that this is obvious by the method we produce $\vartheta_T^{(y)}$. It is a direct union (according to T) of disjoint copies of $\vartheta_T^{(x)}$ for $x <_T y$.

The reader who had enough patience to come to this point deserves an explication.

Intuitively $U_T^{(x)}$ is a code for $R.A_{\alpha}^{M,X}$ where $\alpha = \overline{T \restriction x}$ ($T \restriction x$ is an initial segment of T determined by x) and U_T is a code for the sequence $\{\langle \alpha, R.A_{\alpha}^{M,X} \rangle : \alpha < \beta\}$ where $\beta = \overline{T}$ $\vartheta_T^{(x)}$ is a code for certain uniformly definable prewellordering of $R.A_{\alpha}^{M,X}$. This prewellordering is "thin" in the following sense: each of its equivalence classes is codable as a subclass Z of M such that $\mathrm{Dom}\ Z \in M$.

The main point of this construction is that U_T and ϑ_T depend very loosely on T. If T_1 and T_2 are similar wellorderings then the unique similarity function F of T_1 and T_2 generates a sort of similarity between U_{T_1} and U_{T_2}. Similarly for ϑ_{T_1} and ϑ_{T_2}. U_T is called a diagram of construction of the ramified analysis along the wellordering or simply a diagram. ϑ_T is called the diagram of prewellordering of the ramified analysis along the wellordering T or simply pwo diagram. Notice that the complications we came into the clause 5 of the preceding lema arose from the fact that in order to avoid use of choice scheme we had to pass the limit points in a uniform vay.

Definition: Let $Z \subseteq M$, $Od\ (Z, T)$ is an abbreviaton of the following formula: $W.O.(T)\ \&\ (E\,t)$ ("t is the last element of T" $\&\ Z \eta U_T^{(t)}\ \&\ (u)(u \in (\mathrm{Dom}(T) - \{t\}) \Rightarrow \neg Z \eta U_T^{(u)}$)

Intuitively Od (Z, T) means that Z belongs to $R.A.^{M,X}_{\alpha} - \bigcup_{\beta<\alpha} R.A^{M,X}_{\beta}$ and $\alpha = \overline{T}\restriction t$ where t is last element of T.

Lemma 2.7. If T_1, T_2 are wellorderings, $T_1 \cong T_2$ and F establishes their isomorphism then: $(x)(x \in \text{Dom } T_1 \Rightarrow (Y)$ $(Y \,\eta\, U^{(x)}_{T_1} \Leftrightarrow Y \,\eta\, U^{(Fx)}_{T_2}))$.

Proof: By induction on the length of the wellordering T_1. For the first elements of T_1 and T_2 the equivalence is obvious. All the rest follows from the following: $(Z)(Z \,\eta\, X_1 \Leftrightarrow Z \,\eta\, X_2) \Rightarrow$ $(Z)(Z \,\eta\, \mathcal{D}_{\Lambda}(X_1) \Leftrightarrow Z \,\eta\, \mathcal{D}_{\Lambda}(X_2))$. Similar fact may be proved for ϑ_T. ∎

Using the lemma 2.7. we show the lemma 2.8., formalising the remark preceding lemma 2.7.

Lemma 2.8. Od (Z, T) & Od $(Z, T_1) \Rightarrow T \cong T_1$

Definition: Let $\langle Y, \prec \rangle$ be a gpwo family and $Z \,\eta\, Y$. We define $\langle Y, \prec \rangle (Z) = \{ s : Y^{(s)} = Z$ & s is $\prec$ - minimal with this property $\}$.

Notice that $\langle Y, \prec \rangle (Z) = \langle Y^-, \prec^- \rangle (Z)$

Definition: (a) r.a. (T,Z) is an abbreviation of the formula:

$$(Et)(t \in \text{Dom } T \ \& \ Z \,\eta\, U^{(t)}_T)$$

(b) r.a. (Z) is an abbreviation of the formula:

$$(ET)(W.O.\ (T) \ \& \ r.a.\ (T,Z))$$

(c) $Z_1 \prec_{r.a.} Z_2$ is an abbreviation of the formula:

$$(T_1)(T_2)\Big[\,\mathrm{Od}\,(Z_1, T_1)\ \&\ \mathrm{Od}\,(Z_2, T_2) \Longrightarrow (Ez)(z \in \mathrm{Dom}(T_2)\ \&$$

$$T_1 \cong T_2 \upharpoonright z)\Big] \vee (ET)\Big\{\ \mathrm{Od}\,(Z_1, T)\ \&\ \mathrm{Od}\,(Z_2, T)\ \&$$

$$(t)(t \in \mathrm{Dom}(T)\ \&\ Z_1 \eta\ U_T^{(t)}\ \&\ Z_2\ \eta\ U_T^{(t)} \Rightarrow$$

$$(w_1)(w_2)(w_1 \in \langle U_T^{(t)},\ \vartheta_T^{(t)} \rangle\ (Z_1)$$

$$\&\ w_2 \in \langle U_T^{(t)},\ \vartheta_T^{(t)} \rangle\ (Z_2) \Rightarrow w_1 \prec_{(\vartheta_T^{(t)})} w_2))$$

Intuitively r.a. (T,Z) means $Z \in R.A_{\underline{T}}^{M,X}$, r.a.$(Z)$ means $Z \in R.A^{M,X}$ and $Z \prec_{r.a.} Y$ means. Z is constructed in the process of construction of $R.A^{M,X}$ earlier than Y i.e. either the order of construction of Z is smaller than that of Y or (if their order is the same then either Z was defined by a formula whose Gödel number was smaller than that used to define Y or alternatively if it is the same formula then the parameters used to define Z are lexicographically earlier than that used to define Y).

Let us note that we could use U_T and ϑ_T as terms since indeed they were unique by the lemma 2.2. Formally we should use formulas $U(\cdot,\cdot)$ and $\vartheta(\cdot,\cdot)$ such that:

(a) $U(T, Y) \Longleftrightarrow W.O.(T)\ \&\ Y = U_T$

(b) $\vartheta(T,Y) \Longleftrightarrow W.O(T)\ \&\ Y = \vartheta_T$.

Thus while speaking on absoluteness of U_T and ϑ_T we mean the absoluteness of the formulas U and ϑ.

Lemma 2.8. : $\prec_{r.a.}$ defines a gpwo of r.a.

Proof: The precise meaning of the lemma is that:

$$(E\,Z)(r.a\,(Z)\,\&\,\Phi(Z)) \Rightarrow (E\,Z)(r.a.(Z)\,\&\,\Phi(Z)\,\&\,(Y)(\Phi(Y)\,\&\,r.a(Y)$$

$$\Rightarrow Z \prec_{r.a} Y))$$

Pick firstly least T such that $(EY)(Od\,(T,Y)\,\&\,\Phi(Y))$. Consider now any $\vartheta_T^{(t)}$ minimal $z \in \mathrm{Dom}\, U_T^{(t)}$ such that $\Phi((U_T^{(t)})^{(z)})$ (where t is the last element of T). As $(\vartheta_T^{(t)})^-$ is an initial segment of $\prec_{r.a.}$ we are done.

Now let M be a β - extendable transitive model of ZFC.

Let $\mathcal{F} \subseteq \mathcal{P}(M)$ be a β - extension of M, $X \in \mathcal{F}$ is fixed subset of M. Then the structure $\langle \mathcal{F}, M, \in, M, X \rangle$ is a β - model of $KM_{\Omega, \mathcal{X}}$.

Recall that $h(\mathcal{F})$ is a supremum of the types of well-orderings in $\mathcal{F}$.

The construction of the relativized ramified analysis was conducted above two times.

In the first definition we constructed a family of subsets of a transitive set, in the second we defined a predicate in the theory KM. . The next lemma connects these two definitions and allows us - while interpreting Ω as M - to use interchangingly $R.A^{M,X}_{\alpha}$ and the family defined by the predicate r.a.(T,.) (where $\overline{T} = \alpha$). $\mathcal{F}$, M, X are fixed for the time being.

Lemma 2.9. If $T \in \mathcal{F}$, $\overline{T} = \alpha$ and $\langle \mathcal{F}, M, \in \rangle$ is a β - model, $X \in \mathcal{F}$ then, for all $Z \in \mathcal{F}$

a) $\langle \mathcal{F}, M, \in, M, X \rangle \models$ r.a. $[T,Z] \Leftrightarrow Z \in R.A.^{M,X}_{\alpha}$

and so, for all $Z \in \mathcal{F}$

$\langle \mathcal{F}, M, \in, M, X \rangle \models$ r.a. $[Z] \Leftrightarrow Z \in R.A^{M,X}_{\eta}$ where $\eta = h(\mathcal{F})$.

b) $R.A^{M,X}_{\eta+1} \subseteq \mathcal{F}$

c) $R.A^{M,X}_{\eta+1}$ does not contain a wellordering of type η

Proof: Clearly a) implies b) and b) implies c).

To prove a) we have to show the absoluteness - with respect to $\langle \mathcal{F}, M, \in, M, X \rangle$ of U_T which follows from the fact that $\mathcal{F}$ is a β - model.

Lemma 2.10. If $\alpha < \eta$ then the structure $\langle R.A^{M,X}_{\alpha}, M, \in \rangle$ has the β - property i.e. $(Y)(Y \in R.A^{M,X}_{\alpha} \Rightarrow (\langle R.A^{M,X}_{\alpha}, M, \in \rangle \models W.O[Y]$ $\Rightarrow Y$ is a wellordering)).

Proof: By lemma 2.9. $Y \in \mathcal{F}$ and so, if Y is not a well-ordering then $\langle \mathcal{F}, M, \in \rangle \models \neg W.O.[Y]$ (here we use β - property of $\langle \mathcal{F}, M, \in \rangle$). Thus there is $x \in M$ such that $\langle \mathcal{F}, M, \in \rangle \models$ "$x \subseteq \mathrm{Dom}\ Y$ & $x \neq \emptyset$ & "x has no Y-first element". The formula in " "'s is predicative and all the parameters are in $R.A^{M,X}_{\alpha}$. Thus $\langle R.A^{M,X}_{\alpha}, M, \in \rangle \models$ "$x \subseteq \mathrm{Dom}\ Y$ & $x \neq \emptyset$ & "x has no Y-first element". So $\langle R.A^{M,X}_{\alpha}, M, \in \rangle \models \neg W.O.[Y]$.

Definition: γ_0 is the first ordinal γ such that $R.A^{M,X}_{\gamma+1}$ does not contain a wellordering of type $\geqslant \gamma$.

From the lemma 2.9. (c) it follows that $\gamma_0 \leqslant \eta$.

We are going to prove that $\langle R.A.^{M,X}_{\gamma_0}, M, \in \rangle$ has a definable prewellordering; one such ordering is $\prec_{r.a.}$ restricted to this set (which is absolute with respect to $\langle R.A^{M,X}_{\gamma_0}, M, \in \rangle$).

Lemma 2.11. γ_0 is a limit ordinal.

Proof: from a wellordering of type γ one can-putting the first element to the end - produce a wellordering of type $\gamma_0 + 1$. This construction does not lead outside of $R.A.^{M,X}_{\gamma_0}$. ∎

Lemma 2.12. If $\alpha < \gamma_0$, $\beta < \gamma_0$ then $\alpha + \beta, \alpha \cdot \beta < \gamma_0$

Proof: If T_1, T_2 are wellorderings of types α and β

respectively and if they belong to some $R.A^{M,X}_{\xi}$ then orderings of type $\alpha + \beta$, $\alpha \cdot \beta$ belong to $R.A^{M,X}_{\xi+1}$ (as they are definable over $R.A^{M,X}_{\xi}$). Since ζ_0 is limit we get the result.

Lemma 2.13. Each of the structures $\langle R.A^{M,X}_{\alpha}, M, \in\rangle, (\alpha \leq \zeta_0)$ is a model of Gödel Bernays theory of classes.

Proof: for $\alpha = 1$ the statement is well known. The union of an ordered family of models of Gödel Bernays theory of classes (with fixed V) is a model of Gödel Bernays theory of classes. This fixes the limit case. So what we need to prove is the successor case.

This is shown as usual by proving the closure under operations corresponding to the axioms of group B. Note that the fact that $R.A.^{M,X}_{\zeta_0} \subseteq \mathcal{F}$ implies the validity of the axiom of replacement.

Generally, in the theory GB we are not able to prove the comparability of wellorderings (this needs Σ^1_1 class existence scheme). But the structure $\langle R.A^{M,X}_{\zeta_0}, M, \in \rangle$ does have the comparability property.

Lemma 2.14. If T_1, T_2 are wellorderings, $T_1, T_2 \in R.A^{M,X}_{\alpha}$, $T_1 \cong T_2$. Then the similarity function may be found in $R.A.^{M,X}_{\alpha + \overline{T}_1 + 1}$.

Proof: We show that the similarity function is definable over

$\langle R.A^{M,X}_{\alpha+T_1}, M, \in\rangle$. We prove it inductively. Let F be a similarity function for T_1 and T_2. Then $F \upharpoonright O_{T_1}(x)$ is a similarity function of $T_1 \upharpoonright x$ and $T_2 \upharpoonright F(x)$. By inductive assumption $F \upharpoonright O_{T_1}(x)$ belongs to $R.A^{M,X}_{\alpha+T_1}$ (for all $x \in \mathrm{Dom}T_1$). By their uniqueness it follows that $G = {}_{x \in \mathrm{Dom}T_1}(F \upharpoonright O_{T_1}(x))$ belongs to $R.A^{M,X}_{\alpha+T_1+1}$. If T_1 has no last element then this union is the desired similarity function. If T_1 has last element-say t_0-then T_2 has also last element-say u_0- and $G \cup \{\langle t_0, u_0\rangle\} \in R.A^{M,X}_{\alpha+T_1+1}$ since the latter is a model of Gödel-Bernays theory of classes. Since $F = G \cup \{\langle t_0, u_0\rangle\}$ we are done.

Lemma 2.15. Relations of similarity and of "less then" for the wellorderings are absolute with respect to $\langle R.A^{M,X}_{\gamma_0}, M, \in\rangle$

Proof: By 2.12. and 2.14.

Lemma 2.16. a) If $Y \in R.A^{M,X}_{\alpha}$ then $\mathcal{D}_{\mathcal{N}}(Y) \in R.A^{M,X}_{\alpha+1}$

b) If $T \in R.A^{M,X}_{\alpha}$ then $T' \in R.A^{M,X}_{\alpha}$

Proof: Since $\langle R.A^{M,X}_{\alpha}, M, \in\rangle$ is a model of Gödel-Bernays theory of classes therefore for each φ and t $\{\langle\langle\varphi, t\rangle, x\rangle : (Ey)(Y^{(y)} = x \ \& \ \mathrm{Sat}(Y, \varphi, y^\frown t))\}$ belongs to $R.A^{M,X}_{\alpha}$. Since $\langle R.A^{M}_{\alpha}, M, \in\rangle$ is an ω - model, the notion of the formula is absolute and thus $\mathcal{D}_{\mathcal{N}}(Y)$ is definable over $\langle R.A._{\alpha}^{M,X}, M, \in \rangle$. Thus $\mathcal{D}_{\mathfrak{A}}(Y)$ belongs to $R.A._{\alpha+1}^{M,X}$.

b) T' is definable from T by predicative formula.

Lemma 2.17. (a) $R.A._{\gamma_0}^{M,X}$ is closed with respect to the operation $\mathcal{D}_{\mathfrak{A}}(.)$

(b) The formula defining the operation $\mathcal{D}_{\mathfrak{A}}(.)$ is absolute with respect to $\langle R.A._{\gamma_0}^{M,X}, M, \in \rangle$.

Proof: a) Follows from 2.16.a and 2.11.

(b) We establish firstly that the formula Sat(.,.,.) is absolute with respect to $\langle R.A._{\gamma_0}^{M,X}, M, \in \rangle$. This in turn implies absoluteness of $\mathcal{D}_{\mathfrak{A}}(.)$. ∎

Lemma 2.18. If $T \in R.A._{\alpha}^{M,X}$, T is a wellordering then U_T and ϑ_T belong to $R.A._{\alpha+\bar{T}+1}^{M,X}$

Proof: Analogous to the proof of the lemma 2.14 using the lemma 2.16. ∎

Lemma 2.19. (a) $R.A._{\gamma_0}^{M,X}$ is closed with respect to the operations U_T and ϑ_T. (b) (Formulas defining) The operations U_T and ϑ_T are absolute with respect to $\langle R.A._{\gamma_0}^{M,X}, M, \in \rangle$

Proof: (a) From lemma 2.18 and 2.11.

(b) As before by the analysis of defining formulas.

Lemma 2.20. (a) The concentration operation $(.)^{-}, (.)^{-}$ is absolute with respect to $\langle R.A^{M,X}_{\zeta_0}, M, \in \rangle$.

(b) The value operation $\langle ., . \rangle (Y)$ is absolute with respect to $\langle R.A^{M,X}_{\zeta_0}, M, \in \rangle$

Lemma 2.21. (Analogue of Gödel's $\langle L, \in \rangle \models V = L$)
$\langle R.A.^{M,X}_{\zeta_0}, M, \in \rangle \models (X)$ r.a. (X)

Proof: We need to prove that in $\langle R.A^{M,X}_{\zeta_0}, M, \in \rangle$ the following formula is satisfied :
$(Y)(ET)(W.O.(T) \ \& \ (Et)(t \in \mathrm{Dom}T \ \& \ Y \eta U_T^{(t)}))$. Let Y be given, $Y \in R.A^{M,X}_{\zeta_0}$. Thus for some $\alpha < \zeta_0$, $Y \in R.A.^{M,X}_{\alpha}$. By the definition of ζ_0 there must be a wellordering T in $R.A.^{M,X}_{\alpha+1}$ such that the type of T, $\bar{T}$ is bigger than or equal to α

Thus $U_T \in R.A^{M,X}_{\alpha+\alpha+2}$ (by the lemma 2.18). Once again by 2.12 $U_T \in R.A^{M,X}_{\zeta_0}$. As order of construction of Y is at most α, we are done.[*)]

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*) It is clear from the reasoning that the consideration of ζ_0 (This trick is due to Gandy) is basic to the success of our construction. Because it may well happen that $\zeta_0 < \eta$ and then we would have inside of $R.A.^{M,X}$ too few wellorderings to reach η (η is $h(\mathfrak{T})$). In case when $\langle \mathfrak{T}, M, \in \rangle \models$ KMC we can show directly - in $\langle \mathfrak{T}, M, \in \rangle$ - that $\langle R.A^{M,X}_{\eta}, M, \in \rangle$ satisfies KM. But this reasoning does not lead to basic lemmas 2.21 and 2.22.

Lemma 2.22. The formula $\prec_{r.a.}$ is absolute with respect to $\langle R.A.^{M,X}_{\gamma_0}, M, \in \rangle$

Proof: Using previous lemmas it is enough to prove absoluteness of the formula $Od(.,.)$ which we leave to the reader.

To prove the reflection principle (and thus that $\langle R.A^{M,X}_{\gamma_0}, M, \in \rangle$ is a model of KM) we follow the classical proof of Levy of the reflection principle ZF.

Theorem 2.2.: (Reflection Principle for $\langle R.A.^{M,X}_{\gamma_0}, M, \in \rangle$

For every formula Φ of L_{ST} there are arbitrarily large $\alpha < \gamma_0$ such that for all $X_1 \dots X_n \in R.A^{M,X}_{\alpha}$

$$\langle R.A.^{M,X}_{\alpha}, M, \in \rangle \models \Phi[X_1,\dots,X_n] \iff$$

$$\iff \langle R.A^{M,X}_{\gamma_0}, M, \in \rangle \models \Phi[X_1,\dots, X_n]$$

To show this we need three facts:

1° The possibility of bounding the places where examples for existential formulas appear.

2° Every definable functional on $\langle R.A.^{M,X}_{\gamma_0}, M, \in \rangle$ which takes as values wellorderings, is invariant under similarity of wellorderings and is continuous is majorized by a functional of the same sort which is in addition increasing.

3° Every definable, increasing, invariant and continuous functional has arbitrarily large critical points.

We show 1° leaving 2° and 3° to the reader. In both cases the idea of proof is similar to that of 1°. Namely in showing that appropriate supremum of wellorderings exist.

Proof of 1° Let $U_T^{(x)}$ be given (i.e. a code for $R.A^{M,X}_{T\upharpoonright x}$). Assume that for every $Z \eta U_T^{(x)}$ there is $W \in R.A^{M,X}_{\gamma_0}$ such that $\langle R.A^{M,X}_{\gamma_0}, M, \in \rangle \models \Phi[Z, W]$. We show that there is $\rho < \gamma_0$ such that for every $Z \in U_T^{(x)}$ there is $W \in R.A^{M,X}_{\rho}$ such that $\langle R.A^{M,X}_{\gamma_0}, M, \in \rangle \models \Phi[Z, W]$.

For every $Z \eta U_T^{(x)}$ i.e. for every $z \in \mathrm{Dom}\, U_T^{(x)}$ we may find $\prec_{r.a.}$ - minimal wellordering T_z such that appropriate W may be found in $R.A.^{M,X}_{T_z}$. Unfortunately T_z is not unique. Consider the shortest T_z's. Still we are not able to claim T_z unique. However we shall find a new wellordering similar to T_z and which is uniquely determined by z.

Definition: $S \subseteq V^2$ is called small class ordinal (s.c.o) iff

a) $\mathrm{Dom}\, S \notin V$

b) $(Y)(Y \eta S \Rightarrow W.O.(Y))$

c) $(Y_1)(Y_2)(Y_1 \eta S \ \& \ Y_2 \eta S \Rightarrow Y_1 \cong Y_2)$

Definition: Classes $Z_1, Z_2 \subseteq V^2$ are almost equal (a.e) iff $(Y)(Y \eta Z_1 \Leftrightarrow Y \eta Z_2)$

Lemma: There is a predicate $\mathrm{Sel}(\cdot,\cdot)$ such that:

a) $\mathrm{Sel}(Z, Y) \Rightarrow \mathrm{W.O.}(Y) \ \& \ \mathrm{s.c.o.}(Z)$

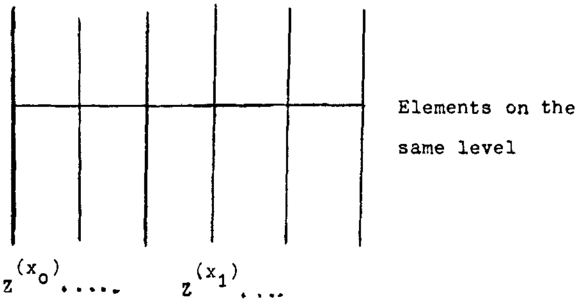
b) $Z_1 \text{ a.e. } Z_2 \Rightarrow (\mathrm{Sel}(Z_1, Y) \Leftrightarrow \mathrm{Sel}(Z_2, Y))$

c) $\mathrm{s.c.o.}(Z) \Rightarrow (E!\,Y)\ \mathrm{Sel}(Z, Y)$

d) $(x)(Y)(x \in \mathrm{Dom}Z \ \& \ \mathrm{Sel}(Z, Y) \Rightarrow Y \cong z^{(x)})$

The most natural idea would be, to consider instead of elements on the same level see the picture just set of those functions on Dom Z taking as values elements. Unfortunately sets of elements on different levels may be identical. Let us notice however that elements f of Dom Z^{mix} such that for given z, $\mathfrak{R}\, f = z$, form necessarily an element of V.

So preceeding formally call a level of Z an $\mathfrak{R}\, f$ for some $f \in \mathrm{Dom}\ Z^{\mathrm{mix}}$



Let us consider an order type of these $f \in \mathrm{Dom}\ Z^{\mathrm{mix}}$ such that $\mathfrak{R}\, f = z$ (z fixed). This type is an ordinal and does not depend at all on Z in the sense that if Z_1 and Z_2 are a.e. then the appropriate types in Z_1^{mix} and Z_2^{mix} are the same.

Define now, for z being the set of all elements on the same level α_z to be the type of the set of all f's such that $\Re f = z$.

Form now the class $H_Z = \bigcup\{\alpha_z \times \{z\} : z$ is the set of all elements on some level of $Z\}$.

Notice again that if Z_1 a.e. Z_2 then $H_{Z_1} = H_{Z_2}$.

Order now H_Z as follows: $\langle \alpha, s\rangle \prec^* \langle \beta, t\rangle$ iff "The initial segment of Z^{mix} determined by the $\beta^{\underline{th}}$ function f(in Z^{mix}) such that $\Re f = t$ contains a subset ordered by Z^{mix} in type $\geq\alpha$ of functions g such that $\Re g = s$".

The predicate Sel(.,.) is a description of $\prec^*$ from Z as constructed above. ∎

Lemma: $\langle R.A^{M.X}_{\gamma_0}, M, \in\rangle$ is closed with respect to the operation determined by Sel, moreover Sel(.,.) is absolute with respect to $\langle R.A^{M,X}_{\gamma_0}, M, \in\rangle$.

Using the above lemma we are able to prove certain uniformization principle for $R.A^{M.X}$

Lemma: Let $\mathcal{H}(.,.)$ be a predicate such that:

$$\langle R.A^{M,X}_{\gamma_0}, M, \in\rangle \models (\mathcal{H}(Z,Y) \Rightarrow W.O.(Y)) \;\&$$

$$(\mathcal{H}(Z,Y) \;\&\; Y \cong Y_1 \Rightarrow \mathcal{H}(Z, Y_1))$$

Then there is a predicate $\mathcal{H}'(.,.)$ such that

$$\langle R.A.^{M,X}_{\daleth c}, M, \in \rangle \models (\mathcal{H}'(Z, Y) \Rightarrow \mathcal{H}(Z,Y)) \,\&\, [(EY)\, \mathcal{H}(Z,Y) \Rightarrow (E!\, Y)\, \mathcal{H}'(Z,Y)$$

Proof: We describe a construction of $\mathcal{H}'$. Given Z consider all $\prec_{r.a}$ minimal and shortest Y's such that $\mathcal{H}(Z,Y)$. This collection may be coded as an s.c.o. Any two s.c.o's coding it are almost equal. Using Sel we get the appropriate wellordering.

Now, to finish the proof of 1°:

Let T_z be the wellordering obtained when the uniformization principle was applied to the predicate:

$$\mathcal{H}((z,S) \Longleftrightarrow (E\, W)(Ey)(y \in \mathrm{Dom}\; S \;\&\; W \eta\; U_S^{(y)} \;\&\; \Phi((U_T^{(x)})^{(z)}, W))$$

Form the class K as follows:

$$K = \bigcup_{z \in \mathrm{Dom}\, \mathcal{U}_T^{(x)}} \{z\} \times \mathrm{Dom}\; T_z$$

Define a relation $\sim$ on K as follows:

$$\langle z_1, x\rangle \sim \langle z_2, y\rangle \Longleftrightarrow T_{z_1} \upharpoonright x \cong T_{z_2} \upharpoonright y$$

From every equivalence class of $\sim$ pick elements of the smallest rank; Let L be a class of these sets. Define now

$$[\langle z_1, x\rangle] \prec^{s} [\langle z_2, y\rangle] \text{ iff } (Et)(t\; T_{z_2}\; y \;\&\; T_{z_1} \upharpoonright x \cong T_{z_2} \upharpoonright t)$$

$\prec^S$ is a wellordering majorizing all T_z's. $\prec^S$ is definable over $\langle R.A._{\gamma_0}^{M,X}, M, \in \rangle$ and so, belongs to $R.A._{\gamma_0+1}^{M,X}$. Thus the type of $\prec^S$ must be less than γ_0 and so we have shown that the appropriate supremum exists below γ_0.

The functional which we adjoin now to the formula Φ is the following (we use – as before – the symbols $R.A._{\prec}^{M,X}$ to make it more readable)

$$R'_{\Phi}(T_1,T_2) \Leftrightarrow (Z)(Z \in R.A._{T_1}^{M,X} \,\&\, (EY)\,\Phi\,(X,Y) \Rightarrow (EY)(Y \in R.A._{T_2}^{M,X} \,\&\, \Phi\,(Z,Y)$$

$\&$ " T_2 is a shortest wellordering with this property")).

The functional R' is definable, continuous and invariant with respect to the similarity of wellorderings. In order to get critical point used to reflect Φ we have to majorize it by a definable functional with the same properties and in addition increasing.

This is the reason why we prove 2° and 3°.

We leave the details to the experienced reader.

Since $\langle R.A._{\gamma_0}^{M,X}, M, \in \rangle$ has the reflection property therefore it is a model of KM. From the existence of a definable gpwo we derive:

Theorem 2.3. $\langle R.A._{\gamma_0}^{M,X}, M, \in \rangle$ satisfies the following collection scheme:

$$(x)(EY)\,\Phi\,(x,Y) \Rightarrow (EY)(x)(Ey)(y \in \mathrm{Dom}\ Y \,\&\, \Phi\,(x, Y^{(y)}))$$

Proof: We pick $\prec_{r.a}$ - minimal Y's good for X and give them together.

Let Coll $_\Phi$ and $\mathbb{C}_\Phi$ be collection scheme and choice scheme instance for Φ respectively

Theorem 2.4. KM + Coll $_\Phi$ + Global Choice $\vdash \mathbb{C}_\Phi$

Proof: Assume Coll $_\Phi$ and global choice i.e. let $\prec$ be a wellordering of the whole class V. Assume $(x)(EY)\Phi(x, Y)$. Then by Coll $_\Phi$, $(EY)(x)(Ey)(y \in \mathrm{Dom}\ Y \ \& \ \Phi(x, Y^{(x)}))$.

Let R_x be a subset of Dom Y consisting of y's such that $\Phi(x, Y^{(y)})$. Let z_x be a $\prec$ - first element of Dom Y. Form $Y_1 = \bigcup_{x \in V} \{x\} \times Y^{(z_x)}$. Y_1 makes $\mathbb{C}_\Phi$ true.

Thus we see that, if $\langle R.A.^{M,X}_{\gamma_0}, M, \in \rangle$ satisfies the global choice then it automatically satisfies the choice scheme. This happens for instance when M has a wellordering definable in $\langle R.A^{M,X}_{\gamma_0}, M, \in \rangle$

We have a much nicer situation when $\langle M, \in \rangle$ has a definable wellordering, say $\prec$. Applying the whole construction to $\prec$ (i.e. letting $\mathcal{W}_T^{(t_0)} = \prec$) we get a definable wellordering of the whole $\langle R.A^{M,X}_{\gamma_0}, M, \in \rangle$

Since the existence of definable wellordering in the presence of choice scheme implies the scheme of dependent choices we sum up the situation as follows:

Theorem 2.5. (a) If $\langle M, \in \rangle$ is a transitive β-extendable model of ZFC then there is the smallest β-extension of $\langle M, \in \rangle$. This extension has a definable without parameters good prewellordering and, apart of the axioms of KM satisfies additionally the collection scheme.

(b) If $\langle M, \in \rangle$ is a transitive β-extendable model of ZFC, $\mathcal{F}$ is any β-extension of $\langle M, \in \rangle$, $X \subseteq M$, $X \in \mathcal{F}$ then there is the smallest β-extension of $\langle M, \in \rangle$ containing X. As before this extension has a good prewellordering definable with the parameter X and satisfies additionally the collection scheme.

(c) If $\langle M, \in \rangle$ is a transitive β-extendable model of ZFC and has a definable wellordering then the smallest β-extension of $\langle M, \in \rangle$ has a wellordering definable without parameters, satisfies the choice scheme and the scheme of dependent choices.

(d) If $\langle M, \in \rangle$ is a transitive β-extendable model of ZFC and has a definable wellordering, and if $\mathcal{F}$ is a β-extension of $\langle M, \in \rangle$, $X \in \mathcal{F}$ then the smallest β-extension of $\langle M, \in \rangle$ containing X has a definable wellordering (with a parameter X) and satisfies the choice scheme and the scheme of dependent choices.

Careful inspection shows that $\prec_{r.a.}$ is Δ^1_1 and r.a. is Σ^1_1.

The reasoning used in the proof of the theorem 2.5. may be applied to a proof that $V = L$ is relatively consistent with KM Indeed when M is interpreted as L the formula r.a. defines an inner model of KM $+ V = L$ in KM. More precisely let $r.a.^{L}(.)$ be this formula (i.e. $\mathcal{M}(x) \iff x \in L$, $X = \emptyset$.)

Definition: If T is a wellordering, $T + 1$ is the class arising from T by putting the first element of T to the end.

Let $r.a_{\Box}^{L}$ be $r.a^{L}$ if there is no wellordering X such that $\neg (EY)\, Y \cong X \,\&\, r.a^{L}(X + 1, Y)$ and let $r.a_{\Box}^{L}(.)$ be r.a. $(Z,.)$ if Z is the shorlest wellordering with this property (Intuitively we consider $R.A.^{L}$ if there is no ξ such that $R.A^{L}_{\xi+1}$ does not contain a wellordering of type ξ or $R.A.^{L}_{\xi_0}$ for the least ξ_0 such that $R.A^{L}_{\xi_0+1}$ does not contain a wellordering of type ξ_0). By similar reasoning as in the proof of the theorem 2.5. we show that the formula $r.a.^{L}_{\Box}$ is an inner interpretation of KM $+ V = L$ in KM (the trick with $\Box$ is again due to Gandy).

There is an important modification. We need to show that the classes satisfying $r.a^{L}_{\Box}$ are L - amenable i.e. that if X is an $r.a^{L}_{\Box}$ - class which is a function, $x \in L$ then $X * x \in L$. This needs a form of the condensation Lemma of Gödel proved as in the ZF case (In fact this was the beginning of investigations of the second author on the problems of this paper). Note that when we knew that M was

β - extendable then the property of M - amenability of $R.A.^M$ classes was automatic.

The syntactic contents of the reasoning leading to the theorem 2.5 may be summed up in the following

Metatheorem: a) There is a formula $\Phi(.)$ such that

1) $KM \vdash \Phi(V)$

2) $KM \vdash (x)(x \in V \Rightarrow \Phi(x))$

3) For every Ψ being an axiom of KM or an instance of the collection scheme

$KM \vdash (\Psi)^{\Phi}$

b) There is a formula $\Theta(.)$ such that

4) $KM \vdash (V = L)^{\Theta}$

5) $KM \vdash \Theta(L)$

6) $KM \vdash (x)(x \in L \Rightarrow \Theta(x))$

7) For every Ψ being an axiom of KM or an instance of the scheme of choice $KM \vdash (\Psi)^{\Theta}$

Proof: In case a) take as Φ the formula $r.a_{\Box}(.)$ with $M(x) \Leftrightarrow \zeta(x)$.

In case b) take as Θ the formula $r.a._{\Box}^{L}$.

Now we are finally able to complete

The proof of theorem 2.1.

Assume that $\langle R.A.^{M,X}, M, \in \rangle$ is not a β - model (though it is a model). By the comparability of wellorderings all false (or as we say below nonstandard) wellorderings are longer than all standard (i.e.true) wellorderings in $R.A.^{M,X}$ and so all these nonstandard wellorderings have the same type of the maximal wellordered initial segment. Call the type of this segment α . Clearly $R.A.^{M,X}$ does not contain a wellordering of type α . Since $R.A^{M,X}_{\alpha+1} \leq R.A.^{MX}$ therefore also $R.A^{M,X}_{\alpha+1}$ does not contain a wellordering of type α . Let γ_0 - as before - be least γ such that $R.A^{M.X}_{\gamma+1}$ does not contain a wellordering of type γ .

Case A: $\gamma_0 < \alpha$. We claim that $\langle R.A^{M,X}_{\gamma_0}, M, \in \rangle$ has the property β .

First we remark that if $\langle R.A^{M,X}_{\gamma_0}, M, \in \rangle \models$ W,O.[T] then $\langle R.A^{M,X}, M, \in \rangle \models$ W.O.[T] . Otherwise, since $\langle R.A.^{M,X}, M, \in \rangle$ is a model of KM, there would be a set (i.e. an element of M) not wellfounded in T . Since $M \subseteq R.A^{M,X}_{\gamma_0}$ therefore $\langle R.A^{M,X}_{\gamma_0}, M, \in \rangle \models \neg$ W.O [T] contrary to the assumption. Now assume again $\langle R.A^{M,X}_{\gamma_0}, M, \in \rangle \models$ W.O. [T] but T is not a wellordering. By the above $\langle R.A.^{M,X}, M, \in \rangle \models$ W.O. [T] and so the initial wellordered segment of T has a type α which is bigger than γ_0 But then there is a initial segment of T of type $> \gamma_0$ in

$R.A^{M,X}_{\gamma_0}$ and thus also in $R.A^{M,X}_{\gamma_0+1}$ contradicting the choice of γ_0.

Moreover every element of $R.A^{M,X}_{\gamma_0}$ is M - amenable (since $R.A^{M,X}_{\gamma_0} \subseteq R.A^{M,X}$).

Now we know that $\langle R.A^{M,X}_{\gamma_0}, M, \in \rangle$ has the β - property and as before - by the property of γ_0 we prove the reflection property of $\langle R.A^{M,X}_{\gamma_0}, M, \in \rangle$. Thus it happens that $\langle R.A^{M,X}_{\gamma_0}, M, \in \rangle \models KM$.

By the lemma 2.1. $R.A^{M,X}_{\gamma_0} = R.A^{M,X}$ and so $\langle R.A^{M,X}, M, \in \rangle$ is a β - model of KM contradicting our assumption.

<u>Case B:</u> $\gamma_0 = \alpha$. As before we show that $\langle R.A^{M,X}_{\alpha}, M, \in \rangle$ is a model of GB theory of classes.

We prove now that:

1) For standard wellorderings $T \in R.A^{M,X}_{\alpha}$, $U_T \in R.A^{M,X}_{\alpha}$

2) For nonstandard wellorderings $T \in R.A^{M,X}_{\alpha}$, $U_T \notin R.A^{M,X}_{\alpha}$

(The point 2) has to be understood as follows: If T is not a standard wellordering then $\langle R.A^{M,X}_{\alpha}, M, \in \rangle \models \neg (EX) U(X,T)$)

Point 1) is proved by the same reasoning as the proof of theorem 2.5.

Point 2) we prove as follows: Since $\langle R.A^{M,X}_{\alpha}, M, \in \rangle$ is a model of GB therefore together with U_T for nonstandard T we get $U_T^{(x)}$ for some x such that $T \upharpoonright x$ is nonstandard.

By 1) $U_T^{(x)}$ contains all the classes belonging to $R.A_\alpha^{M,X}$. Now we construct the diagonal class for $U_T^{(x)}$ (i.e. the class $\{z \in \mathrm{Dom}\, U_T^{(x)} : z \notin (U_T^{(x)})^{(z)}\}$. This class being predicative in U_T belongs to $R.A_\alpha^{M,X}$ but is different from all the $(U_T^{(x)})^{(z)}$ which contradicts the fact that $U_T^{(x)}$ contains all classes of $R.A_\alpha^{M,X}$.

The points 1) and 2) allow us to discern the well-orderings among the objects satisfying in $\langle R.A_\alpha^{M,X}, M, \in\rangle$ the formula W.O. (We still do not know that $\langle R.A_\alpha^{M,X}, M, \in\rangle$ is a β - structure!) namely these are the objects for which U_T exists. As before we check the absoluteness of $Od(\cdot,\cdot)$, $U(\cdot,\cdot)$ and $\mathcal{M}(\cdot,\cdot)$ and $\prec_{r.a.\mathcal{M}}(\cdot,\cdot)$. Now as before we show the reflection principle using instead of all objects satisfying in $\langle R.A_\alpha^{M,X}, M, \in\rangle$ the formula W.O. only those for which U_T exists.

So $\langle R.A_\alpha^{M,X}, M, \in\rangle$ is a model, and since $KM \vdash (T)(W.O.(T) \Rightarrow (EX)(U(X,T)))$ we get the desired contradiction with the presence of nonstandard wellorderings in $R.A_\alpha^{M,X}$. Indeed we proved that for the nonstandard T's there is no U_T in $R.A_\alpha^{M,X}$. Thus also in this case $R.A^{M,X}$ (i.e. $R.A_\alpha^{M,X}$) is a β - model.

Lemma 2.22.: If $\langle \mathfrak{F}, M, \in \rangle$ is a β - model for KM and $\mathfrak{G} \subseteq \mathfrak{F}$ is a family of subsets of M such that $\langle \mathfrak{G}, M, \in \rangle \models$ KM then $\langle \mathfrak{G}, M, \in \rangle$ is also a β - model.

Proof: If $X \in \mathfrak{G}$ is not a wellordering therefore, since $\mathfrak{G} \subseteq \mathfrak{F}$, $\langle \mathfrak{F}, M, \in \rangle \models \neg \mathrm{W.O.}[X]$ ($\mathfrak{F}$ is a β -model). Thus there is $x \in M$ such that x is not wellfounded in X. Thus $\langle \mathfrak{G}, M, \in \rangle \models \neg \mathrm{W.O.}[X]$

Corollary: If there exists a β - model of KM then there is a β - model $\langle \mathfrak{F}, M, \in \rangle$ of KM such that, for all $\mathfrak{G} \subseteq \mathfrak{F}$ if $\langle \mathfrak{G}, M, \in \rangle$ is a model then $\mathfrak{G} = \mathfrak{F}$

Proof: Let $\langle \mathfrak{F}, M, \in \rangle$ be a β - model; consider $\langle \mathrm{R.A.}^{M}, M, \in \rangle$. It is the least β - model for KM (with M as the universe of sets). This together with the lemma 2.22, completes the proof. ∎

The corollary shows an important difference between ω - models of second order arithmetic and transitive models of KM. In the case of the former system there is no minimal ω - model as shown by H.Friedman in [3]. In the case of models of KM there are - under suitable assumptions - minimal transitive models. This answers the question of H.Friedman from the introduction of the aforenamed paper.

If the constructed family $R.A.^{M}$ is not a model (thus not a β - model) then in the process of construction there must appear a class which is not M - amenable. Thus there must be a least α such that $R.A^{M,X}_{\alpha}$ contains non - M - amenable class. Clearly $R.A._{0}^{M,\emptyset}$ does not contain such an animal. Similarly $R.A_{1}^{M,\emptyset}$ - since this class is the least model for GB which has M as the universe of sets.

Theorem 2.7.: Let $\mathcal{N}$ be the least transitive model of ZF. Then $R.A._{2}^{\mathcal{N}}$ contains a class which is not $\mathcal{N}$ - amenable.

Proof: As shown in [13], $\mathcal{N} = L_{\alpha}$ where α is the least β such that $\langle L_{\beta}, \in \rangle$ models ZF. In [7] it is shown that $(L_{\alpha+2} - L_{\alpha}) \cap \mathcal{P}(\omega) \neq \emptyset$. It is easy to show that $L_{\alpha+2} \cap \mathcal{P}(\omega) \subseteq R.A._{2}^{\mathcal{N}}$. Thus there is a subclass of ω which is not a set and so non $\mathcal{N}$- amenable class.

One can however prove the following

Theorem 5: If M is a transitive model of ZFC, and M is extendable, then all classes in $R.A._{h(M)^{+}}^{M,\emptyset}$ are M - amenable (where $h(M)^{+}$ is the least admissible ordinal bigger than $h(M)$)

Sketch of the proof: All the ordinals less than $(h(M))^{+}$ are representable in every extension of the model M (it follows from the fact that the standard part of any possibly nonstandard admissible set containing M has to contain α^{+}, as a subset, for every standard α). Thus also classes U_{T} for

$\overline{T} < \alpha^+$ are in every extension and so the classes constructed before α^+ are M - amenable.

Finally let us notice that in case when $\langle M, \in \rangle$ is not β - extendable but is a model of ZFC set theory we can show by slightly modified reasoning that $\langle R.A.^M, M, \in \rangle$ is a least β - model of KM - $\{$replacement axiom$\}$. This gives the following theorem:

Theorem 2.1'. If $\langle M, \in \rangle$ is a transitive model of ZFC then $\langle R.A.^M, M, \in \rangle$ is the least β - model of (KM - $\{$replacement axiom$\}$) with the class of sets equal M (i.e. it can process semisets in sense of Vopenka & Hajek). It is a model of KM just in case when $\langle M, \in \rangle$ is β - extendable.

§ 3. Extendability vs. β - extendability

We shall deal now with models of KMC. Indeed we remember that every transitive model $\langle \mathcal{T}, M, \in \rangle$ of KM has a transitive submodel $\langle \mathcal{G}, L^M, \in \rangle$ which satisfies KM + scheme of choice.

Thus while considering the heights of transitive extendable models we may restrict ourselves to the models extendable to the models of KMC. (Notice that $h(M) = h(L^M)$ and $L^M = L_{h(M)}$)

Let T denote the following sentence of L_{ST} :

$(Ex)(\omega \in x \ \& \ (z)(z \in x \Rightarrow \mathcal{P}(z) \in x) \ \& \ (f)(y)(y \in x \ \& \ \mathrm{Func}(f) \ \&$

$f \leq x \Rightarrow f * y \in x))$

Thus the sentence T means that "There is an inaccessible family of sets". We notice that T is Σ_2 formula and the formula "(.) is an inaccessible family of sets" is Π_1.

In [6] the following fact is proved:

Proposition 3.1.: The theory $ZFC^- + T$ is interpretable in KMC by means of wellfounded trees.

Let us look more carefully at this interpretation. The inaccessible family is representable by a tree coding V. Trees of rank less than On represent elements of the maximal inaccessible family.

The trees of the rank less than or equal to On have realizations; in case of trees of rank less than On the realization is a set. In case of trees of rank On the realization is a proper class.

The proposition 3.1. Leads to the following: If $\langle \mathfrak{T}, M, \in \rangle \models KMC$ then $\langle \text{Trees}^{\langle \mathfrak{T}, M, \in \rangle}, \text{Eps}, \text{Eq} \rangle \models ZFC + T$ (where Eps and Eq are appropriate relations interpreting $\in$ and $=$). Every tree has a rank which is a wellordering. Assume $\langle \mathfrak{T}, M, \in \rangle \models$ "T is a tree" & "U is a rank of T".

Then T is a tree iff U is wellordering. This immediately implies that the notion of a tree is absolute exactly for β - models.

Putting all these together we get a semantic version of the results from [6] :

Proposition 3.2. $\langle \mathcal{F}, M, \in \rangle$ is a β - model of KMC iff There is a transitive model $\langle \mathcal{N}, \in \rangle$ of $ZFC^- + T$ such that

1) $M \in \mathcal{N}$

2) $\langle N, \in \rangle \models$ " M is an inaccessible family of sets"

3) $\mathcal{F} = \mathcal{P}(M) \cap \mathcal{N}$

The proof $\Rightarrow$ is roughly the following. We take all well-founded trees (without nontrivial automorphisms) and take as Eq and Eps isomorphism and membership of trees relations. Then $\langle \mathrm{Trees}^{\langle \mathcal{F}, M, \in \rangle}, \mathrm{Eq}, \mathrm{Eps} \rangle$ is a model (without absolute equality) of $ZFC^- + T$. Thus the structure $\langle \mathrm{Trees}^{\langle \mathcal{F}, M, \in \rangle}, \mathrm{Eps} \rangle / \mathrm{Eq} \models ZFC^- + T$. We take now realizations of trees from $\mathrm{Trees}^{\langle \mathcal{F}, M, \in \rangle}$. Since $\langle \mathcal{F}, M, \in \rangle$ was a β - model they are really trees and so they indeed have realizations. (The process of realization is similar to contraction procedure). We get an isomorphic model $\langle N, \in \rangle$. The equivalence class of a tree coding M is a desired inaccessible family. By class existence in $\langle \mathcal{F}, M, \in \rangle$ the subsets of M being in $\mathcal{F}$ and only them are in $N \cap \mathcal{P}(M)$. The proof of $\Leftarrow$ is obvious.

Corollary: $\langle M, \in \rangle$ is β - KMC - extendable iff there is transitive model $\langle N, \in \rangle$ of $ZFC^- + T$ such that

1) $M \in N$

2) $\langle N, \in \rangle \models$ "M is an inaccessible family of sets".

If $\langle M, E \rangle$ is a relational structure, $E \subseteq M \times M$ then Sp M is the set of those $m \in M$ which are wellfounded i.e. those for which there is no infinite E - descending sequence beginning with m. If $\langle M, E \rangle$ satisfies extensionality then $\langle \mathrm{Sp}\, M, E \upharpoonright \mathrm{Sp}\, M \rangle$ is isomorphic to a transitive structure $\langle A, \in \rangle$. Thus we may simply assume that Sp M is transitive (when $\langle M, E \rangle$ satisfies extensionality).

Further analysis of the notion of the tree allows us to give an analogue of the proposition 3.2. for extendable but not necessarily β - extendable transitive models.

Proposition 3.3. $\langle \mathfrak{T}, M, \in \rangle$ is a transitive model of EMC iff there is a model $\langle N, E \rangle$ of $ZFC^- + T$ such that

1) $M \in \mathrm{Sp}\, N$

2) $\langle N, E \rangle \models$ "M is an inaccessible family of sets"

3) $\mathfrak{T} = \mathcal{P}(M) \cap \mathcal{N}$

Proof: Again $\Leftarrow$ is obvious (we tacitly assume that the objects in $N - \mathrm{Sp}\, N$ are not subsets of M)

$\Rightarrow$ Once more consider (wellfounded trees) $\langle \mathfrak{T}, M, \in \rangle$ i.e. objects which satisfy in $\langle \mathfrak{T}, M, \in \rangle$ the formula "(.) is a tree". All real trees which are in $\mathfrak{T}$ are there but there may be also some "nonstandard trees".

When we make the model $\mathfrak{M} = \langle (\mathrm{Trees})^{\langle \mathcal{F}, M, \in \rangle}, \mathrm{Eps}^{\langle \mathcal{F}, M, \in \rangle} \rangle / \mathrm{Eq}^{\langle \mathcal{F}, M, \in \rangle}$ then: The standard part of the model $\mathfrak{M}$ will consist of equivalence classes of wellfounded trees. Nonstandard trees (if there are any) give nonstandard elements of $\mathfrak{M}$. But the tree representing M has rank On and so is standard; thus its realization exists and is in Sp $\mathfrak{M}$. By class existence in $\langle \mathcal{F}, M, \in \rangle$ the subsets of M being in $\mathcal{F}$, and only them, are the subsets of M in $\mathfrak{M}$.

Corollary: $\langle M, \in \rangle$ is KMC -extendable iff there is $\langle N, E \rangle$ a model of $ZFC^- + T$ such that:

1) $M \subseteq \mathrm{Sp}\, \mathcal{N}$

2) $\langle N, E \rangle \models$ "M is an inaccessible family of sets".

As we noted, if $\langle M, \in \rangle$ is extendable then $\langle L^M, \in \rangle$ is extendable. In case when $\langle M, \in \rangle$ is β - extendable and we take constructibility interpretation Θ (cf §2) within the β - extension $\langle \mathcal{F}, M, \in \rangle$ we get a structure $\langle \mathcal{G}, L^M, \in \rangle$ which is a β - model. Therefore, if $\langle M, \in \rangle$ is β - extendable then $\langle L^M, \in \rangle$ is also β - extendable. This leads us to the following definition:

Definition: a) α is extendable ordinal iff $\langle L_\alpha, \in \rangle$ is an extendable model

b) α is β - extendable ordinal iff $\langle L_\alpha, \in \rangle$ is a β - extendable model.

Let us notice that - by our results in the §2 - KM and KMC -

extendability of L_α coinoide (since it posesses a definable wellordering).

The same fact holds for β - extendability.

Our criterions of extendability and β - extendability were highly ineffective in the sense that it was not clear where to look for the extensions. For the models of the form L_α and β - extendability we have quite nice criterion;For other models and weaker form of extendability we show later some criterions.

Definition: If α is an ordinal then α^* is the least ordinal β such that: 1) $\alpha \in \beta$
2) $\langle L_\beta, \in \rangle \models ZF^-$.

Theorem 3.1. α is β - extendable iff $\langle L_{\alpha^*}, \in\rangle \models$ "L_α is an inaccessible family".

Proof.: $\Rightarrow$ By the corollary after the proposition 3.1.there is $\langle N, \in \rangle$ such that $\langle N, \in \rangle \models$ "L_α is an inaccessible family of sets", $\langle N, \in\rangle \models ZFC^-$.

Then $\langle L^N, \in\rangle \models ZFC^-$ and since "(.) is an inaccessible family" is a Π_1 formula therefore $\langle L^N, \in \rangle \models$ "L_α is an inaccessible family of sets". Since $L^N = L_{h(N)}$ we have $h(N) \geqslant \alpha^*$ and so using again the fact that "(.) is an inaccessible family of sets" is Π_1 we get $\langle L_{\alpha^*}, \in\rangle \models$ "L_α is an inaccessible family of sets".

$\Leftarrow$ Immediate by the same corollary and the fact that $\langle L_\alpha, \in \rangle \models ZF^-$ implies $\langle L_\alpha, \in \rangle \models ZFC^-$.

Corollary: If α is β-extendable than $R.A.^{L_\alpha} = L_{\alpha^+} \cap \mathcal{P}(L_\alpha)$ *)

Proof: Both $\langle R.A^{L_\alpha}, L_\alpha, \in \rangle$ and $\langle L_{\alpha^+} \cap \mathcal{P}(L_\alpha), L_\alpha, \in \rangle$ are the smallest β-extensions of $\langle L_\alpha, \in \rangle$

For the rest of this paper we assume that there are β-extendable ordinals.

Definition: a) α^0 is the least extendable ordinal

b) α^1 is the least β-extendable ordinal.

Lemma 3.1. Both α^0 and α^1 are denumerable.

Proof: Obvious by Skolem-Löwenheim.

Theorem 3.2. (On difference) a) $\alpha^0 < \alpha^1$

b) $\langle L_{\alpha^1}, \in \rangle \models$ " α^0 is denumerable"

c) $\langle L_{\alpha^1}, \in \rangle \models$ " α^0 is extendable".

We will prove our theorem using Barwise Σ_1 compactness theorem. The basic fact is that Barwise theorem is provable in ZFC and thus valid in every model of ZFC. This fact was first noted by Barwise [1] and then by Wilmers [18]. We assume that the reader is familiar with the theorem of Barwise and some of its standard applications.

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*) As pointed to us by M.Srebrny, R.B.Jensen in his Habilitationschrift (unpublished) proves this equality for all α's.

We need the following lemma:

Lemma 3.2.: α^+, the least admissible ordinal greater than α is smaller than α^*.

Proof: It is known that there is a single sentence Φ such that Φ +"scheme of foundation"is equivalent to KP (we were informed about this by G.Kreisel and C.Smorynski). By the reflection principle in $\langle L_{\alpha^*}, \in \rangle$ there is $\beta < \alpha^*$ such that Φ holds in $\langle L_\beta, \in \rangle$ (Since Φ holds in $\langle L_{\alpha^*}, \in \rangle$). Thus L_β is admissible and since $\alpha^+ \leq \beta$ we are done.

Proof of the theorem 3.2. ("on difference")

Consider the system $\langle L_{(\alpha^1)^+}, \in \rangle$ and the theory $\mathcal{T}$ in the infinitary language $\mathcal{L}_{L_{(\alpha^1)^+}}$ based on 3 groups of axioms:

a) $\bigwedge$ ZFC$^-$

b) $\in$ - diagram of $L_{(\alpha^1)^+}$

c) "L_{α^1} is an inaccessible family of sets"

The theory $\mathcal{T}$ is definable over $\langle L_{(\alpha^1)^+}, \in, \{\alpha^1\} \rangle$ by a Σ_1 formula Θ and is consistent since it has a model (for instance $\langle L_{(\alpha^1)^*}, \in \rangle$ is a model of $\mathcal{T}$). Therefore the structure $\langle L_{(\alpha^1)^+}, \in, \{\alpha^1\} \rangle$ satisfies the formula Consis$_\Theta$, where Consis$_\Theta$ is a finitary sentence of L_{ST} expressing consistency of $\mathcal{T}$.

Since $L_{(\alpha^1)^+} \in L_{(\alpha^1)^*}$ therefore since $\langle L_{(\alpha^1)^*}, \in \rangle$ satisfies the full scheme of choice and has a definable well-ordering we have <u>inside</u> $L_{(\alpha^1)^*}$ a denumerable (within $L_{(\alpha^1)^*}$) elementary substructure $\langle A, \in \restriction A, \{\alpha^1\} \rangle$ of $\langle L_{(\alpha^1)^+}, \in, \{\alpha^1\} \rangle$

The structure $\langle A, \in \restriction A, \{\alpha^1_1\}$ is isomorphic again within $L_{(\alpha^1)^*}$ to a structure $\langle B, \in, \{\delta\} \rangle$ where B is transitive. By standard reasoning $B = L_\gamma$ for some γ. γ is denumerable within $\langle L_{(\alpha^1)^*}, \in \rangle$ and so it is denumerable within $\langle L_{\alpha^1}, \in \rangle$ since $\langle L_{(\alpha^1)^*}, \in \rangle \models$ "L_{α^1} is an inaccessible family of sets". Consider now $\langle L_\gamma, \in, \{\delta\} \rangle$. First of all we notice that $\gamma = \delta^+$.

Moreover $\langle L_\gamma, \in, \{\delta\} \rangle \models \text{Consis}_{\Theta}$

Now let us look what the formula Θ defines over $\langle L_\gamma, \in, \{\delta\} \rangle$

It is clear that it defines the following theory:

a') $\bigwedge$ ZFC$^-$

b') $\in$ diagram of L_γ

c') "L_δ is an inaccessible family of sets"

As this theory is Σ_1 definable and $\langle L_\gamma, \in, \{\delta\} \rangle \models \text{Consis}_{\Theta}$ we apply now Barwise compactness theorem within $\langle L_{\alpha^1}, \in \rangle$. Since γ is denumerable in $\langle L_{\alpha^1}, \in \rangle$, $\langle L_\gamma, \in, \{\delta\} \rangle$ is Σ_1 - complete in $\langle L_{\alpha^1}, \in \rangle$

Thus we get within $\langle L_{\alpha'}, \in \rangle$ a denumerable model of the theory definable over $\langle L_{\gamma}, \in, \{\delta\} \rangle$ by Θ, i.e. of the axiom groups a'), b'), c').

Let $\langle N, E \rangle$ be a model of this theory. By the condiotion b') $\langle N, E \rangle$ is an end extension of $\langle L_{\gamma}, \in \rangle$ (within $\langle L_{\alpha'}, \in \rangle$ but this is an absolute statement). Since $\delta < \gamma$ therefore δ belongs to the standard part of $\langle N, E \rangle$. We apply now the corollary of the proposition 3.3.

So $\langle L_{\delta}, \in \rangle$ is an extendable model. Clearly $\alpha^{\circ} \leqslant \delta$ and so both

a) and b) of the theorem hold.

To show c) we apply within $\langle L_{\alpha'}, \in \rangle$ Skolem Löwenheim result of Nadel [16], since α° is denumerable in $\langle L_{\alpha'}, \in \rangle$ and thus between α° and $\omega_1^{\langle L_{\alpha'}, \in \rangle}$ there are recursively inaccessible ordinals.

Definition: We call an admissible set A $\underset{\sim}{\Sigma}_1$ - complete iff for every $\underset{\sim}{\Sigma}_1$ definable theory $\mathcal{T}$,

$$\langle A, \in \rangle \models \mathrm{Consis}_{\Theta} \text{ iff } \mathcal{T} \text{ has a model}$$

where Θ is a $\underset{\sim}{\Sigma}_1$ formula defining $\mathcal{T}$

By the Barwise compactness theorem together with completeness theorem for languages $\mathcal{L}_M$ (M denumerable) we find that all denumerable admissible sets are $\underset{\sim}{\Sigma}_1$-complete.

Analyzing the proof of the theorem 3.2. we get

Theorem 3.3. There is a formula Φ such that whenever $\langle M^+, \in \rangle$ is $\underset{\sim}{\Sigma}_1$ complete then :

$$\langle M, \in \rangle \text{ is KMC - extendable iff } \langle M^+, \in, M \rangle \models \Phi$$

Proof: Φ is a Π_1 sentence stating the consistency of the following theory $\mathcal{T}$:

a) $\bigwedge \mathrm{ZFC}^-$

b) " $\in$ - diagram of the world (it is called EE in [5])

c) "M is inaccessible family of sets"

Let us notice that b) uniformly defines an $\in$ diagram of admissible set over itself.

We use the following fact :

If $\langle N, E \rangle \models \mathrm{KP}$ then $M \in \mathrm{Sp}\, N$ iff $M^+ \subseteq \mathrm{Sp}\, N$

To prove the theorem assume firstly that $\langle M, \in \rangle$ is KMC extendable. By the corollary to the proposition 3.3. we find that there is a model $\langle N, E \rangle$ of a) and c) and such that $M \in \mathrm{Sp}\, N$. Thus $M^+ \subseteq \mathrm{Sp}\, N$ and so $\langle N, E \rangle$ satisfies an $\in$-diagram of M^+. Thus $\langle N, E \rangle$ is a model of $\mathcal{T}$ (more precisely of the theory defined over $\langle M^+, \in \rangle$ by Θ).

Conversly, if $\langle M^+, \in, M \rangle \models \mathrm{Consis}_{\Theta}$ then, by $\underset{\sim}{\Sigma}_1$ completeness of $\langle M^+, \in \rangle$ and by the fact that $\mathcal{T}$ is $\underset{\sim}{\Sigma}_1$

definable we get a model $\langle N,E \rangle$ of $\mathcal{T}$.

$M^+ \subseteq \mathrm{Sp}\, N$ and so $M \in \mathrm{Sp}\, N$. Using once more the corollary to the proposition 3.3. we are done.

Corollary: If M is denumerable then

$$\langle M, \in \rangle \text{ is extendable iff } \langle M^+, \in , M \rangle \models \Phi$$

We come back to the proof of the theorem 3.2. It was definitely not economic for the following two reasons.

1) Remark that $\langle L_\alpha , \in \rangle$ need not be β - extendable in order to make our reasoning work. What we need is that there is an extension $\mathcal{F}$ of $\langle L_\alpha , \in \rangle$ such that $h(\mathcal{F}) > \alpha^+$.

2) We did not use the following fact: Every ω - model of ZFC which is extendable contains its own theory.

We define: $\alpha^{(0)} = \alpha \qquad \alpha^{(\xi+1)} = (\alpha^{(\xi)})^+$

$\alpha^{(\lambda)} = \bigcup_{\xi<\lambda} \alpha^{(\xi)}$ if this ordinal is admissible or $\left(\bigcup_{\xi<\lambda} \alpha^{(\xi)} \right)^+$ otherwise.

Definition: An extendable model $\langle L_\alpha, \in \rangle$ is " ξ - good" iff it has an extension $\langle \mathfrak{F}, L_\alpha, \in \rangle$ such that $\alpha^{(\xi)} < h(\mathfrak{F})$

Using the reasoning of the proof of the theorem 3.2. we get

Theorem 3.4. a) Every 1 - good model contains as an element 0-good (i.e. transitive extendable) model.

b) If $k \in \omega$ then every $(k + 1)$-good model contains as an element k - good model.

The theorem 3.4. may be extended to all recursive ordinals.

Following the line of 2) we find that in the proof of the theorem 3.2. we could add the following clause d) to the a),b),c): $\left(Th(L_{\alpha^1}, \in)\right)^{L_{\alpha^1}}$ as the latter is L_{α^1} - finite. Therefore we have the following:

Theorem 3.5. If $\langle M, \in \rangle$ is β - KMC - extendable then there is $N \in M$ such that $\langle N, \in \rangle \equiv \langle M, \in \rangle$ and

$\langle M, \in \rangle \models \bar{\bar{N}} = \aleph_0$ & $\langle N, \in \rangle$ is extendable".

(Thus $\langle N, \in \rangle$ is indeed extendable).

The proof of 3.5. needs a subtler considerations of β - extendable models.

Namely in the proposition 3.2. one may add 4) "Every set is equipollent to an ordinal". The model produced from trees satisfies Skolem-Löwenheim theorem and so we work as in 3.2.

Additionally we must prove that $L_{\alpha^+}[M] \in L_{\alpha^*}[M]$ which is again obvious. We close the paper with the informations on the number of extensions of $\langle M, E \rangle$.

In [9] the following is proved:

Proposition 3.4. If $\langle \mathfrak{F}, M, \in \rangle$ is a denumerable model of KM then there is a proper extension $\mathfrak{G}$ of $\mathfrak{F}$ such that:

1) $\langle \mathfrak{F}, M, \in \rangle \prec \langle \mathfrak{G}, M, \in \rangle$

2) $\langle \mathfrak{G}, M, \in \rangle$ is not a β - model

Moreover there is $2^{\aleph_0}$ $\mathfrak{G}$'s of power $\aleph_0$ and $2^{\aleph_1}$ of power $\aleph_1$.

We do not know any necessary and sufficient condition under which a β - model $\langle \mathfrak{F}, M, \in \rangle$ has a proper elementary extension $\langle \mathfrak{G}, M, \in \rangle$ also being a β - model.

There are however some necessary and some sufficient conditions:

Some of them are due to Guzicki [4]

1) If we want to get a model of the same height as $\mathfrak{F}$ then $\langle \mathfrak{F}, M, \in \rangle$ must satisfy the negation of the class form of relative constructibility.

2) Sufficient: The ones given in [4]. They give stronger results than those of our proposition 3.5. (although they go in

different direction)

Guzicki's models are forcing models-quite exeptional fact since they are also elementary extensions.

Under assumption of Martin's axiom Guzicki's construction gives $2^{2^{\aleph_0}}$ β- models of power $2^{\aleph_0}$

Definition: A model $\langle \mathcal{T}, M, \in \rangle$ of KM satisfies condition (B) iff there is a model $\langle N,E \rangle$ of $ZFC^- + T$ such that:

1) $\mathcal{T} \in \mathrm{Sp}\, \mathfrak{N}$

2) $\langle N,E \rangle \models \mathcal{T} = \mathcal{P}(M)$

3) $\langle N,E \rangle \models$ "M is an inaccessible family of sets"

4) $\langle N,E \rangle \models$ " $\overline{\overline{\mathcal{P}(M)}} > \overline{\overline{M}}^{+}$ "

5) $\langle N,E \rangle$ is M^+ - standard

(here M^+ denotes next cardinal in $\langle N,E \rangle$)

Proposition 3.5. ([10]) If $\langle \mathcal{T}, M, \in \rangle$ is a denumerable model of KMC satisfying condition (B) then there are $2^{\aleph_0}$ proper elementary denumerable extensions $\langle \mathcal{S}, M, \in \rangle$ satisfying conditions (B) and $2^{\aleph_1}$ of such extensions of power $\aleph_1$, all these extensions can be chosen to have the same height as $\mathcal{T}$.

Proof: Using the quantifier "there is more than M^+" in $\langle N, E \rangle$ ∎

We have the following lemma:

Lemma 3.2. If $\langle \mathcal{T}, M, \in \rangle$ has a property (β) then it is a β - model.

By this lemma, a countable model satisfying (β) has $2^{\aleph_0}$ proper elementary denumerable extensions each of which is a β- model.

For the non-denumerable models almost nothing is known. If α is a strongly inaccessible cardinal then $\langle R_\alpha, \in \rangle$ has 2^α extensions of power α. There are even 2^α extensions being elementary subsystem of $\langle R_{\alpha+1}, R_\alpha, \in \rangle$

If $V = L$ then the elementary subsystems of $\langle R_{\alpha+1}, R_\alpha, \in \rangle$ are linearly ordered by inclusion. In the same time it is relatively consistent to assume that they are not linearly ordered by inclusion; even under the assumption that $\langle R_\alpha, \in \rangle \models V = L$.

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