STUDIES IN LOGIC

AND

THE FOUNDATIONS OF MATHEMATICS

NORTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM·LONDON

INTUITIONISTIC LOGIC MODEL THEORY AND FORCING

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1969

NORTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM·LONDON

Library of Congress Catalogue Card Number: 79–102718

SBN 7204 2256 6

PUBLISHERS

NORTH HOLLAND PUBLISHING COMPANY – AMSTERDAM

NORTH HOLLAND PUBLISHING COMPANY LTD – LONDON

PRINTED IN THE NETHERLANDS

ACKNOWLEDGMENTS

I would like to express my thanks to Professor Raymond Smullyan for his guidance and encouragement during the preparation of this thesis. I would also like to thank the library staff at the Belfer Graduate School of Science for much friendly unconventional assistance.

This work was supported by National Aeronautics and Space Administration Training Grant NSG(T)144, and by Air Force Office of Scientific Research Grants AFOSR68-1375 and AFOSR433-65.

With the exception of a few minor corrections and changes, this work duplicates the author's Doctoral Dissertation, written at the Belfer Graduate School of Science, Yeshiva University, and submitted June 1968.

to my parents

INTRODUCTION

In 1963 P. Cohen established various fundamental independence results in set theory using a new technique which he called *forcing*. Since then there has been a deluge of new results of various kinds in set theory, proved using forcing techniques. It is a powerful method. It is, however, a method which is not as easy to interpret intuitively as the corresponding method of Gödel which establishes consistency results. Gödel defines an intuitively meaningful transfinite sequence of (domains of) classical models M_α, defines the class L to be the union of the M_α over all ordinals α, and shows L is a classical model for set theory [4; see also 3]. He then shows the axiom of constructability, the generalized continuum hypothesis, and the axiom of choice are true over L, establishing consistency.

In this book we define transfinite sequences of S. Kripke's intuitionistic models [13] in a manner exactly analogous to that of Gödel in the classical case (in fact, the M_α sequence is a particular example). In a reasonable way we define a "*class*" model for each sequence, which is to be a limit model over all ordinals. We show all the axioms of set theory are intuitionistically valid in the class models. Finally we show there are particular such sequences which provide: a class model in which the negation of the axiom of choice is intuitionistically valid; a class model in which the axiom of choice and the negation of the continuum hypothesis are intuitionistically valid; a class model in which the axiom of choice, the generalized continuum hypothesis, and the negation of the axiom of constructability are intuitionistically valid. From this the *classical* independence results are shown to follow.

The definition of the sequences of intuitionistic models will be seen to be essentially the same as the definition of forcing in [3]. The difference is in the point of view. In Cohen's book one begins with a set M which is a *countable* model for set theory and, using forcing, one constructs a second countable model N "on top of" M. Forcing enables one to "discuss" N in M even though N is not a sub-model of M. Various such N are constructed for the different independence results. Cohen points out [3, pp. 147, 148] that actually the proofs can be carried out without the need for a countable model, and without constructing any classical models; this is the point of view we take. It is the forcing relation itself that the center of attention [see 3, page 147], though now it has an intuitive interpretation.

A similar program has been carried out by Vopěnka and others. [See the series of papers 22, 23, 24, 27, 6, 25, 7, 8, 26, 28]. The primary difference is that these use topological intuitionistic model theory while we use Kripke's, which is much closer in form to forcing. Also the Vopěnka series uses Gödel-Bernays set theory and generalizes the F_α sequence, while we use Zermelo-Fraenkel set theory and generalize the M_α sequence. The Vopěnka treatment involves substantial topological considerations which we replace by more "logical" ones.

This book is divided into two parts. In part I we present a thorough treatment of the Kripke intuitionistic model theory. Part II consists of the set theory work discussed above.

Most of the material in Part I is not original but it is collected together and unified for the first time. The treatment is self-contained. Kripke models are defined (in notation different from that of Kripke). Tableau proof systems are defined using *signed* formulas (due to R. Smullyan), a device which simplifies the treatment. Three completeness proofs are presented (one for an axiom system, two for tableau systems), one due to Kripke [13], one due independently to R. Thomason [21] and the author, and one due to the author. We present proofs of compactness and Löwenheim-Skolem theorems. Adapting a method of Cohen, we establish a few connections between classical and intuitionistic logic. In the propositional case we give the relationship between Kripke models and algebraic ones [16] (which provides a fourth completeness proof in the propositional case). Finally we present Schütte's proof of the intuitionistic Craig interpolation lemma [17], adapted to Kleene's tableau system G3 as modified by the use of signed formulas. No attempt is made to use methods of proof acceptable to intuitionists.

Chapter 7 begins part II. In it we define the notion of an *intuitionistic* Zermelo-Fraenkel (ZF) model, and the intuitionistic generalization of the Gödel M_α sequence. Most of the chapter is devoted to showing the class models resulting from the sequences of intuitionistic models are intuitionistic ZF models. This result is demonstrated in rather complete detail, especially sections 8 through 13, not because the work is particularly difficult, but because such models are comparatively unfamiliar.

In chapter 8 the independence of the axiom of choice is shown.

In chapter 9 we show how ordinals and cardinals may be represented in the intuitionistic models, and establish when such representatives exist.

Chapter 10 establishes the independence of the continuum hypothesis.

In Chapter 11 we give a way to represent constructable sets in the intuitionistic models, and establish when such representatives exist.

Chapter 12 establishes the independence of the axion of constructability.

Chapter 13 is a collection of various results. We establish a connection between the sequences of intuitionistic models and the classical M_α sequence. We give some conditions under which the axiom of choice and the generalized continuum hypothesis will be valid in the intuitionistic class models (thus completing chapters 10 and 12). Finally we present Vopěnka's method for producing classical non-standard set theory models from the intuitionistic class models without countability requirements [26].

The set theory work to this point is self-contained, given a knowledge of the Gödel consistency proof ([4], in more detail [3]).

In chapter 14 we present Scott and Solovay's notion of boolean valued models for set theory [19]. We define an intuitionistic (or forcing) generalization of the R_α sequence (sets with rank) analogous to the Cohen generalization of the M_α sequence, and we establish some connections between intuitionistic and boolean valued models for set theory.

CHAPTER 1

PROPOSITIONAL INTUITIONISTIC LOGIC

SEMANTICS

§ 1. Formulas

We begin with a denumberable set of propositional variables A, B, $C, \ldots$, three binary connectives $\wedge$, $\vee$, $\supset$, and one unary connective $\sim$, together with left and right parentheses (,). We shall informally use square and curly brackets [,], {, } for parentheses, to make reading simpler. The notion of *well formed formula*, or simply *formula*, is given recursively by the following rules:

F0. If A is a propositional variable, A is a formula.

F1. If X is a formula, so is $\sim X$.

F2, 3, 4. If X and Y are formulas, so are $(X \wedge Y)$, $(X \vee Y)$, $(X \supset Y)$.

Remark 1.1: A propositional variable will sometimes be called an *atomic formula.*

It can be shown that the formation of a formula is unique. That is, for any given formula X, one and only one of the following can hold:

(1). X is A for some propositional variable A.

(2). There is a unique formula Y such that X is $\sim Y$.

(3). There is a unique pair of formulas Y and Z and a unique binary connective b ($\wedge$, $\vee$ or $\supset$) such that X is $(Y \mathrm{b} Z)$.

We make use of this uniqueness of decomposition but do not prove it here.

We shall omit writing outer parentheses in a formula when no confusion can result. Until otherwise stated, we shall use A, B and C for propositional variables, and X, Y and Z to represent any formula.

The notion of *immediate subformula* is given by the following rules:

I0. A has no immediate subformula.

I1. $\sim X$ has exactly one immediate subformula: X.

I2, 3, 4. $(X \wedge Y)$, $(X \vee Y)$, $(X \supset Y)$ each has exactly two immediate subformulas: X and Y.

The notion of *subformula* is defined as follows:

S0. X is a subformula of X.

S1. If X is an immediate subformula of Y, then X is a subformula of Y.

S2. If X is a subformula of Y and Y is a subformula of Z, then X is a subformula of Z.

By the *degree* of a formula is meant the number of occurrences of logical connectives ($\sim$, $\wedge$, $\vee$, $\supset$) in the formula.

§ 2. Models and validity

By a (*propositional intuitionistic*) *model* we mean an ordered triple $\langle \mathscr{G}, \mathscr{R}, \Vdash \rangle$, where $\mathscr{G}$ is a non-empty set, $\mathscr{R}$ is a transitive, reflexive relation on $\mathscr{G}$, and $\Vdash$ (conveniently read "forces") is a relation between elements of $\mathscr{G}$ and formulas, satisfying the following conditions:

For any $\Gamma \in \mathscr{G}$

P0. if $\Gamma \Vdash A$ and $\Gamma \mathscr{R} \Delta$ then $\Delta \Vdash A$ (recall A is atomic).

P1. $\Gamma \Vdash (X \wedge Y)$ iff $\Gamma \Vdash X$ and $\Gamma \Vdash Y$.

P2. $\Gamma \Vdash (X \vee Y)$ iff $\Gamma \Vdash X$ or $\Gamma \Vdash Y$.

P3. $\Gamma \Vdash \sim X$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, $\Delta \nVdash X$.

P4. $\Gamma \Vdash (X \supset Y)$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, if $\Delta \Vdash X$, then $\Delta \Vdash Y$.

Remark 2.1: For $\Gamma \in \mathscr{G}$, by Γ^* we shall mean any $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$. Thus "for all Γ^*, $\varphi(\Gamma^*)$" shall mean "for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, $\varphi(\Delta)$"; and "there is a Γ^* such that $\varphi(\Gamma^*)$" shall mean "there is a $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$ and $\varphi(\Delta)$". Thus P3 and P4 can be written more simply as:

P3. $\Gamma \Vdash \sim X$ iff for all Γ^*, $\Gamma^* \nVdash X$

P4. $\Gamma \Vdash (X \supset Y)$ iff for all Γ^*, if $\Gamma^* \Vdash X$, then $\Gamma^* \Vdash Y$.

A particular formula X is called *valid in the model* $\langle \mathscr{G}, \mathscr{R}, \Vdash \rangle$ if for all $\Gamma \in \mathscr{G}$, $\Gamma \Vdash X$. X is called *valid* if X is valid in all models. We will show later that the collection of all valid formulas coincides with the usual collection of propositional intuitionistic logic theorems.

When it is necessary to distinguish between validity in this sense and the more usual notion, we shall refer to the validity defined above as intuitionistic validity, and the usual notion an classical validity. This notion of an intuitionistic model is due to Saul Kripke, and is presented, in different notation, in [13]. See also [18]. Examples of models will be found in section 5, chapter 2.

§ 3. Motivation

Let $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ be a model. $\mathscr{G}$ is intended to be a collection of possible universes, or more properly, states of knowledge. Thus a particular Γ in $\mathscr{G}$ may be considered as a collection of (physical) facts known at a particular time. The relation $\mathscr{R}$ represents (possible) time succession. That is, given two states of knowledge Γ and Δ of $\mathscr{G}$, to say $\Gamma \mathscr{R} \Delta$ is to say: if we now know Γ, it is possible that later we will know Δ. Finally, to say $\Gamma \models X$ is to say: knowing Γ, we know X, or: from the collection of facts Γ, we may deduce the truth of X.

Under this interpretation condition P3 of the last section, for example, may be interpreted as follows: from the facts Γ we may conclude $\sim X$ if and only if from no possible additional facts can we conclude X.

We might remark that under this interpretation it would seem reasonable that if $\Gamma \models X$ and $\Gamma \mathscr{R} \Delta$ then $\Delta \models X$, that is, if from a certain amount of information we can deduce X, given additional information, we still can deduce X, or if at some time we know X is true, at any later time we still know X is true. We have required that this holds only for the case that X is atomic, but the other cases follow.

For other interpretations of this modeling, see the original paper [13]. For a different but closely related model theory in terms of forcing see [5].

§ 4. Some properties of models

***Lemma* 4.1:** Let $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ and $\langle \mathscr{G}, \mathscr{R}, \models' \rangle$ be two models such that for any atomic formula A and any $\Gamma \in \mathscr{G}$, $\Gamma \models A$ iff $\Gamma \models' A$. Then $\models$ and $\models'$ are identical.

Proof: We must show that for any formula X,

$$\Gamma \models X \Leftrightarrow \Gamma \models' X.$$

This is done by induction on the degree of X and is straightforward. We present one case as an example.

Suppose X is $\sim Y$ and the result is known for all formulas of degree less than that of X (in particular for Y). We show it for X:

$$\begin{aligned} \Gamma \models X &\Leftrightarrow \Gamma \models \sim Y \text{ (by definition)} \\ &\Leftrightarrow (\forall \Gamma^*)(\Gamma^* \not\models Y) \text{ (by hypothesis)} \\ &\Leftrightarrow (\forall \Gamma^*)(\Gamma^* \not\models' Y) \text{ (by definition)} \\ &\Leftrightarrow \Gamma \models' \sim Y \\ &\Leftrightarrow \Gamma \models' X. \end{aligned}$$

***Lemma* 4.2:** Let $\mathscr{G}$ be a non-empty set and $\mathscr{R}$ be a transitive, reflexive relation on $\mathscr{G}$. Suppose $\models$ is a relation between elements of $\mathscr{G}$ and *atomic* formulas. Then $\models$ can be extended to a relation $\models'$ between elements of $\mathscr{G}$ and *all* formulas in such a way that $\langle \mathscr{G}, \mathscr{R}, \models' \rangle$ is a model.

Proof: We define $\models'$ as follows:

(0). if $\Gamma \models A$ then $\Gamma^* \models' A$,

(1). $\Gamma \models' (X \wedge Y)$ if $\Gamma \models' X$ and $\Gamma \models' Y$,

(2). $\Gamma \models' (X \vee Y)$ if $\Gamma \models' X$ or $\Gamma \models' Y$,

(3). $\Gamma \models' \sim X$ if for all Γ^*, $\Gamma^* \not\models' X$,

(4). $\Gamma \models' (X \supset Y)$ if for all Γ^*, if $\Gamma^* \models' X$, then $\Gamma^* \models' Y$.

This is an inductive definition, the induction being on the degree of the formula. It is straightforward to show that $\langle \mathscr{G}, \mathscr{R}, \models' \rangle$ is a model.

From lemmas 4.1 and 4.2 we immediately have

***Theorem* 4.3:** Let $\mathscr{G}$ be a non-empty set and $\mathscr{R}$ be a transitive, reflexive relation on $\mathscr{G}$. Suppose $\models$ is a relation between elements of $\mathscr{G}$ and atomic formulas. Then $\models$ can be extended in one and only one way to a relation, also denoted by $\models$, between elements of $\mathscr{G}$ and formulas, such that $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ is a model.

***Theorem* 4.4:** Let $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ be a model, X a formula and $\Gamma, \Delta \in \mathscr{G}$. If $\Gamma \models X$ and $\Gamma \mathscr{R} \Delta$, then $\Delta \models X$.

Proof: A straightforward induction on the degree of X (it is known already for X atomic). For example, suppose the result is known for X, and $\Gamma \models \sim X$. By definition, for all Γ^*, $\Gamma^* \not\models X$. But $\Gamma \mathscr{R} \Delta$ and $\mathscr{R}$ is transitive so any $\mathscr{R}$-successor of Δ is an $\mathscr{R}$-successor of Γ. Hence for all Δ^*, $\Delta^* \not\models X$, so $\Delta \models \sim X$. The other cases are similar.

§ 5. Algebraic models

In addition to the Kripke intuitionistic semantics presented above, there is an older algebraic semantics: that of pseudo-boolean algebras. In this section we state the algebraic semantics, and in the next we prove its equivalence with Kripke's semantics. A thorough treatment of pseudo-boolean algebras may be found in [16].

Definition 5.1: A *pseudo-boolean algebra* (PBA) is a pair $\langle \mathscr{B}, \leqslant \rangle$ where $\mathscr{B}$ is a non-empty set and $\leqslant$ is a partial ordering relation on $\mathscr{B}$ such that for any two elements a and b of $\mathscr{B}$:

(1). the least upper bound $(a \cup b)$ exists.
(2). the greatest lower bound $(a \cap b)$ exists.
(3). the pseudo complement of a relative to b $(a \Rightarrow b)$, defined to be the largest $x \in \mathscr{B}$ such that $a \cap x \leqslant b$, exists.
(4). a least element $\wedge$ exists.

Remark 5.2: In the context $\Rightarrow$ is a mathematical symbol, not a meta-mathematical one.

Let $-a$ be $a \Rightarrow \wedge$ and $\vee$ be $- \wedge$.

Definition 5.3: h is called a *homomorphism* (from the set W of formulas to the PBA $\langle \mathscr{B}, \leqslant \rangle$) if $h: W \to \mathscr{B}$ and

(1). $h(X \wedge Y) = h(X) \cap h(Y)$,
(2). $h(X \vee Y) = h(X) \cup h(Y)$,
(3). $h(\sim X) \ = -h(X)$,
(4). $h(X \supset Y) = h(X) \Rightarrow h(Y)$.

If $\langle \mathscr{B}, \leqslant \rangle$ is a PBA and h is a homomorphism, the triple $\langle \mathscr{B}, \leqslant, h \rangle$ is called an *(algebraic) model* for the set of formulas W. If X is a formula, X is called *(algebraically) valid in the model* $\langle \mathscr{B}, \leqslant, h \rangle$ if $h(X) = \vee$. X is called *(algebraically) valid* if X is valid in every model.

A proof may be found in [16] that the collection of all algebraically valid formulas coincides with the usual collection of intuitionistic theorems.

§ 6. Equivalence of algebraic and Kripke validity

First let us suppose we have a Kripke model $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ (we will not use the name "Kripke model" beyond this section). We will define an algebraic model $\langle \mathscr{B}, \leqslant, h \rangle$ such that for any formula X

$$h(X) = \mathsf{v} \quad \text{iff} \quad \text{for all} \quad \Gamma \in \mathscr{G}, \;\; \Gamma \Vdash X.$$

Remark 6.1: The following proof is based on exercise LXXXVI of [2].

If $b \subseteq \mathscr{G}$, we call b *$\mathscr{R}$-closed* if whenever $\Gamma \in b$ and $\Gamma \mathscr{R} \Delta$, then $\Delta \in b$.

We take for $\mathscr{B}$ the collection of all $\mathscr{R}$-closed subsets of $\mathscr{G}$. For the ordering relation $\leqslant$ we take set inclusion $\subseteq$. Finally we define h by

$$h(X) = \{\Gamma \in \mathscr{G} \mid \Gamma \Vdash X\}.$$

It is fairly straightforward to show that $\langle \mathscr{B}, \leqslant \rangle$ is a PBA. Of the four required properties, the first two are left to the reader. We now show:

If $a, b \in \mathscr{B}$, there is a largest $x \in \mathscr{B}$ such that $a \cap x \leqslant b$.

We first note that the operations $\cup$ and $\cap$ are just the ordinary union and intersection. Now let p be the largest $\mathscr{R}$-closed subset of $(\mathscr{G} \dot{-} a) \cup b$ (where by $\dot{-}$ we mean ordinary set complementation). We will show that for all $x \in \mathscr{B}$

$$x \leqslant p \quad \text{iff} \quad a \cap x \leqslant b,$$

which suffices.

Suppose $x \leqslant p$. Then

$$\begin{aligned} x &\subseteq (\mathscr{G} \dot{-} a) \cup b, \\ a \cap x &\subseteq a \cap [(\mathscr{G} \dot{-} a) \cup b], \\ a \cap x &\subseteq a \cap b, \\ a \cap x &\subseteq b, \\ a \cap x &\leqslant b. \end{aligned}$$

Conversely suppose $a \cap x \leqslant b$. Then

$$\begin{aligned} (a \cap x) \cup (x \dot{-} a) &\subseteq b \cup (x \dot{-} a), \\ x &\subseteq b \cup (x \dot{-} a), \\ x &\subseteq b \cup (\mathscr{G} \dot{-} a), \end{aligned}$$

but $x \in \mathscr{B}$, so x is $\mathscr{R}$-closed. Hence

$$\begin{aligned} x &\subseteq p, \\ x &\leqslant p. \end{aligned}$$

The reader may verify that $\emptyset \in \mathscr{B}$ and is a least element.

Next we remark that h is a homomorphism. We demonstrate only one of the four cases, case (4). Thus we must show that $h(X \supset Y)$ is the largest

$x \in \mathscr{B}$ such that
$$h(X) \cap x \leqslant h(Y).$$
First we show
$$h(X) \cap h(X \supset Y) \leqslant h(Y),$$
that is
$$\{\Gamma \mid \Gamma \models X\} \cap \{\Gamma \mid \Gamma \models X \supset Y\} \subseteq \{\Gamma \mid \Gamma \models Y\}.$$
But it is clear from the definition that
$$\text{if } \Gamma \models X \text{ and } \Gamma \models X \supset Y, \text{ then } \Gamma \models Y.$$
Next suppose there is some $b \in \mathscr{B}$ such that $h(X) \cap b \leqslant h(Y)$ but $h(X \supset Y) < b$. Then there must be some $\Gamma \in \mathscr{G}$ such that $\Gamma \in b$ but $\Gamma \notin h(X \supset Y)$, i.e. $\Gamma \not\models X \supset Y$. Since $\Gamma \not\models X \supset Y$, there must be some Γ^* such that $\Gamma^* \models X$ but $\Gamma^* \not\models Y$. Since b is $\mathscr{R}$-closed, $\Gamma^* \in b$. But also $\Gamma^* \in h(X)$, so $\Gamma^* \in h(X) \cap b$, and so by assumption $\Gamma^* \in h(Y)$, that is $\Gamma^* \models Y$, a contradiction. Thus $h(X \supset Y)$ is largest.

Thus $\langle \mathscr{B}, \leqslant, h \rangle$ is an algebraic model. We leave it to the reader to verify that the unit element $\vee$ of $\mathscr{B}$ is $\mathscr{G}$ itself. Hence
$$h(X) = \vee \quad \text{iff} \quad \text{for all } \Gamma \in \mathscr{G}, \ \Gamma \models X.$$
Conversely, suppose we have an algebraic model $\langle \mathscr{B}, \leqslant, h \rangle$. We will define a Kripke model $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ so that for any formula X
$$h(X) = \vee \quad \text{iff} \quad \text{for all } \Gamma \in \mathscr{G}, \ \Gamma \models X.$$

***Lemma* 6.2:** Let $\mathscr{F}$ be a filter in $\mathscr{B}$ and suppose $(a \Rightarrow b) \notin \mathscr{F}$. Then the filter generated by $\mathscr{F}$ and a does not contain b.

Proof: If the filter generated by $\mathscr{F}$ and a contained b, then ([16] p. 46, 8.2) for some $c \in \mathscr{F}$, $c \cap a \leqslant b$. So $c \leqslant (a \Rightarrow b)$ and hence $(a \Rightarrow b) \in \mathscr{F}$ by [16], p. 46, 8.2 again.

***Lemma* 6.3:** Let $\mathscr{F}$ be a proper filter in $\mathscr{B}$ and suppose $-a \notin \mathscr{F}$. Then the filter generated by $\mathscr{F}$ and a is also proper.

Proof: By lemma 6.2, since $-a = (a \Rightarrow \wedge)$.

***Lemma* 6.4:** Let $\mathscr{F}$ be a filter in $\mathscr{B}$ and suppose $a \notin \mathscr{F}$. Then $\mathscr{F}$ can be extended to a prime filter $\mathscr{P}$ such that $a \notin \mathscr{P}$.

Proof: (This is a slight modification of [16], p. 49, 9.2, included for completeness.) Let $\boldsymbol{O}$ be the collection of all filters in $\mathscr{B}$ not containing a. $\boldsymbol{O}$ is partially ordered by $\subseteq$. $\boldsymbol{O}$ is non-empty since $\mathscr{F} \in \boldsymbol{O}$.

Any chain in $\boldsymbol{O}$ has an upper bound since the union of any chain of filters is a filter. So by Zorn's lemma $\boldsymbol{O}$ contains a maximal element $\mathscr{P}$. Of course $a \notin \mathscr{P}$. We need only show $\mathscr{P}$ is prime.

Suppose $\mathscr{P}$ is not prime. Then for some $a_1, a_2 \in \mathscr{B}$

$$a_1 \cup a_2 \in \mathscr{P}, \quad a_1 \notin \mathscr{P}, \quad a_2 \notin \mathscr{P}.$$

Let $\mathscr{S}_1$ be the filter generated by $\mathscr{P}$ and a_1, and $\mathscr{S}_2$ be the filter generated by $\mathscr{P}$ and a_2.

Suppose $a \in \mathscr{S}_1$ and $a \in \mathscr{S}_2$. Then [16, p. 46, 8.2] for some $c_1, c_2 \in \mathscr{P}$, $a_1 \cap c_1 \leqslant a$ and $a_2 \cap c_2 \leqslant a$. So for $c = c_1 \cap c_2$, $a_1 \cap c \leqslant a$ and $a_2 \cap c \leqslant a$, hence $(a_1 \cup a_2) \cap c \leqslant a$. But $c \in \mathscr{P}$ and $(a_1 \cup a_2) \in \mathscr{P}$, so $a \in \mathscr{P}$. But $a \notin \mathscr{P}$, so either $a \notin \mathscr{S}_1$ or $a \notin \mathscr{S}_2$.

Suppose $a \notin \mathscr{S}_1$. By definition $\mathscr{S}_1 \in \boldsymbol{O}$. But $\mathscr{S}_1$ is the filter generated by $\mathscr{P}$ and a_1, hence $\mathscr{P} \subseteq \mathscr{S}_1$. So $\mathscr{P}$ is not maximal, a contradiction. Similarly if $a \notin \mathscr{S}_2$. Thus $\mathscr{P}$ is prime.

Now we proceed with the main result. Recall that we have $\langle \mathscr{B}, \leqslant, h \rangle$. Let $\mathscr{G}$ be the collection of all proper prime filters in $\mathscr{B}$. Let $\mathscr{R}$ be set inclusion $\subseteq$. For any $\Gamma \in \mathscr{G}$ and any formula X, let $\Gamma \models X$ if $h(X) \in \Gamma$.

To show the resulting structure $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ is a model, we note property P0 is immediate. To show P1:

$$\begin{aligned} \Gamma \models (X \wedge Y) \quad & \text{iff} \quad h(X \wedge Y) \in \Gamma \\ & \text{iff} \quad h(X) \cap h(Y) \in \Gamma \\ & \text{iff} \quad h(X) \in \Gamma \text{ and } h(Y) \in \Gamma \\ & \text{iff} \quad \Gamma \models X \text{ and } \Gamma \models Y \end{aligned}$$

(using the facts that h is a homomorphism and Γ is a filter). Similarly we show P2 using the fact that Γ is prime. To show P3:

Suppose $\Gamma \models \sim X$. Then $h(\sim X) \in \Gamma$, so

$$\begin{aligned} & (\forall \Delta \in \mathscr{G})(\Gamma \subseteq \Delta \text{ implies } h(\sim X) \in \Delta), \\ & (\forall \Delta \in \mathscr{G})(\Gamma \subseteq \Delta \text{ implies } h(X) \notin \Delta), \\ & (\forall \Delta \in \mathscr{G})(\Gamma \mathscr{R} \Delta \text{ implies } \Delta \not\models X), \end{aligned}$$

i.e. for all Γ^*, $\Gamma^* \not\models X$ (using the fact that $h(\sim X) \in \Delta$ and $h(X) \in \Delta$ imply $-h(X) \cap h(X) \in \Delta$, so $\wedge \in \Delta$ and Δ is not proper).

Suppose $\Gamma \not\models \sim X$. Then $h(\sim X) \notin \Gamma$, or $-h(X) \notin \Gamma$. By lemma 6.3 the filter generated by Γ and $h(X)$ is proper. By lemma 6.4 this filter can be extended to a proper prime filter Δ. Then $\Gamma \subseteq \Delta$ and $h(X) \in \Delta$. So $(\exists \Delta \in \mathscr{G})$ $(\Gamma \mathscr{R} \Delta$ and $\Delta \models X)$, i.e. for some Γ^*, $\Gamma^* \models X$.

P4 is shown in the same way, but using lemma 6.2 instead of lemma 6.3. Thus $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ is a model.

Finally, to establish the desired equivalence, suppose first $h(X) = \vee$. Since $\vee$ is an element of every filter, for all $\Gamma \in \mathscr{G}$, $\Gamma \models X$. Conversely suppose $h(X) \neq \vee$. But $\{\vee\}$ is a filter and $h(X) \notin \{\vee\}$. By lemma 6.4 we can extend $\{\vee\}$ to a proper prime filter Γ such that $h(X) \notin \Gamma$. Thus $\Gamma \in \mathscr{G}$ and $\Gamma \not\models X$.

Thus we have shown

***Theorem* 6.5:** X is Kripke valid if and only if X is algebraically valid.

CHAPTER 2

PROPOSITIONAL INTUITIONISTIC LOGIC

PROOF THEORY

§ 1. Beth tableaus

In this section we present a modified version of a proof system due originally to Beth. It is based on [2, § 145], but at the suggestion of R. Smullyan, we have introduced signed formulas and single trees in place of the unsigned formulas and dual trees of Beth.

By a *signed formula* we mean TX or FX where X is a formula. If S is a set of signed formulas and H is a single signed formula, we will write $S \cup \{H\}$ simply as $\{S, H\}$ or sometimes S, H.

First we state the *reduction rules*, then we describe their use; S is any set (possibly empty) of signed formulas, and X and Y are any formulas:

$$T\wedge \quad \frac{S, T(X \wedge Y)}{S, TX, TY} \qquad\qquad F\wedge \quad \frac{S, F(X \wedge Y)}{S, FX \mid S, FY}$$

$$T\vee \quad \frac{S, T(X \vee Y)}{S, TX \mid S, TY} \qquad\qquad F\vee \quad \frac{S, F(X \vee Y)}{S, FX, FY}$$

$$T\sim \quad \frac{S, T(\sim X)}{S, FX} \qquad\qquad F\sim \quad \frac{S, F(\sim X)}{S_T, TX}$$

$$T\supset \quad \frac{S, T(X \supset Y)}{S, FX \mid S, TY} \qquad\qquad F\supset \quad \frac{S, F(X \supset Y)}{S_T, TX, FY}$$

In rules $F\sim$ and $F\supset$ above, S_T means $\{TX \mid TX \in S\}$.

Remark 1.1: S is a set, and hence $\{S, TX\}$ is the same as $\{S, TX, TX\}$. Thus duplication and elimination rules are not necessary.

If U is a set of signed formulas, we say one of the above rules, call it rule R, *applies to* U if by appropriate choice of S, X and Y the collection of signed formulas above the line in rule R becomes U.

By an *application of rule* R *to the set* U we mean the replacement of U by U_1 (or by U_1 and U_2 if R is $F\wedge$, $T\vee$ or $T\supset$) where U is the set of formulas above the line in rule R (after suitable substitution for S, X and Y) and U_1 (or U_1, U_2) is the set of formulas below. This assumes R applies to U. Otherwise the result is again U. For example, by applying rule $F\supset$ to the set $\{TX, FY, F(Z\supset W)\}$ we may get the set $\{TX, TZ, FW\}$. By applying rule $T\vee$ to the set $\{TX, FY, T(Z\vee W)\}$ we may get the two sets $\{TX, FY, TZ\}$ and $\{TX, FY, TW\}$.

By a *configuration* we mean a finite collection $\{S_1, S_2,\ldots, S_n\}$ of sets of signed formulas.

By an *application of the rule* R *to the configuration* $\{S_1, S_2,\ldots, S_n\}$ we mean the replacement of this configuration with a new one which is like the first except for containing instead of some S_i the result (or results) of applying rule R to S_i.

By a *tableau* we mean a finite sequence of configurations $\mathscr{C}_1, \mathscr{C}_2,\ldots, \mathscr{C}_n$ in which each configuration except the first is the result of applying one of the above rules to the preceding configuration.

A set S of signed formulas is *closed* if it contains both TX and FX for some formula X. A configuration $\{S_1, S_2,\ldots, S_n\}$ is closed if each S_i in it is closed. A tableau $\mathscr{C}_1, \mathscr{C}_2,\ldots, \mathscr{C}_n$ is closed if some $\mathscr{C}_i$ in it is closed.

By a *tableau for a set* S *of signed formulas* we mean a tableau $\mathscr{C}_1, \mathscr{C}_2,\ldots, \mathscr{C}_n$ in which $\mathscr{C}_1$ is $\{S\}$. A finite set of signed formulas S is *inconsistent* if some tableau for S is closed. Otherwise S is *consistent*. X is a *theorem* if $\{FX\}$ is inconsistent, and a closed tableau for $\{FX\}$ is called a *proof of* X. If X is a theorem we write $\vdash_I X$.

We will show in the next few sections the correctness and completeness of the above system relative to the semantics of ch. 1.

Examples of proofs in this system may be found in § 5.

The corresponding classical tableau system is like the above, but in rules $F\sim$ and $F\supset$, S_T is replaced by S (see [20]). The interpretations of the classical and intuitionistic systems are different.

In the classical system TX and FX mean X is true and X is false respectively. The rules may be read: if the situation above the line is the case, the situation below the line is also (or one of them is, if the rule is disjunctive: $F\wedge$, $T\vee$, $T\supset$). Thus TX means the same as X, and FX means $\sim X$. Classically the signs T and F are dispensable. Proof is a refutation procedure. Suppose X is not true (begin a tableau with FX). Conclude that some formula must be both true and not true (a closed configuration is reached). Since this can not happen, X is true.

In the intuitionistic case TX is to mean X is known to be true (X is proven). FX is to mean X is not known to be true (X has not been proved). The rules are to be read: if the situation above the line is the case, then the situation below the line is possible, i.e. compatible with our present knowledge (if the rule is disjunctive, one of the situations below the line must be possible). For example consider rule $F\supset$. If we have not proved $X\supset Y$, it is possible to prove X without proving Y, for if this were not possible, a proof of Y would be 'inherent' in a proof of X, and this fact would constitute a proof of $X\supset Y$.
the line in this rule and not S because in proving X we might inadvertently verify some additional previously unproven formula (some $FZ\in S$ might become TZ). Similarly for $F\sim$. The proof procedure is again by refutation. Suppose X is not proven (begin a tableau with FX). Conclude that it is possible that some formula is both proven and not proven. Since this is impossible, X is proven.

We have presented this system in a very formal fashion because it makes talking about it easier. In practice there are many simplifications which will become obvious in any attempt to use the method. Also, proofs may be written in a tree form. We find the resulting simplified system the easiest to use of all the intuitionistic proof systems, except in some cases, the system resulting by the same simplifications from the closely related one presented in ch. 6 § 4. A full treatment of the corresponding classical tableau system, with practical simplifications, may be found in [20].

§ 2. Correctness of Beth tableaus

Definition 2.1: We call a set of signed formulas

$$\{TX_1, \ldots, TX_n, FY_1, \ldots, FY_m\}$$

realizable if there is some model $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ and some $\Gamma \in \mathscr{G}$ such that $\Gamma \models X_1, \ldots, \Gamma \models X_n, \; \Gamma \not\models Y_1, \ldots, \Gamma \not\models Y_m$. We say that Γ *realizes* the set.

If $\{S_1, S_2, \ldots, S_n\}$ is a configuration, we call it realizable if some S_i in it is realizable.

***Theorem* 2.2:** Let $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$ be a tableau. If $\mathscr{C}_i$ is realizable, so is $\mathscr{C}_{i+1}$.

Proof: We have eight cases, depending on the rule whose application produced $\mathscr{C}_{i+1}$ from $\mathscr{C}_i$.

Case (1): $\mathscr{C}_i$ is $\{\ldots, \{S, T(X \vee Y)\}, \ldots\}$ and $\mathscr{C}_{i+1}$ is $\{\ldots, \{S, TX\}, \{S, TY\}, \ldots\}$. Since $\mathscr{C}_i$ is realizable, some element of it is realizable. If that element is not $\{S, T(X \vee Y)\}$, the same element of $\mathscr{C}_{i+1}$ is realizable. If that element is $\{S, T(X \vee Y)\}$, then for some model $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ and some $\Gamma \in \mathscr{G}$, Γ realizes $\{S, T(X \vee Y)\}$. That is, Γ realizes S and $\Gamma \models (X \vee Y)$. Then $\Gamma \models X$ or $\Gamma \models Y$, so either Γ realizes $\{S, TX\}$ or $\{S, TY\}$. In either case $\mathscr{C}_{i+1}$ is realizable.

Case (2): C_i is $\{\ldots, \{S, F(\sim X)\}, \ldots\}$ and $\mathscr{C}_{i+1}$ is $\{\ldots, \{S_T, TX\}, \ldots\}$. $\mathscr{C}_i$ is realizable, and it suffices to consider the case that $\{S, F(\sim X)\}$ is the realizable element. Then there is a model $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ and a $\Gamma \in \mathscr{G}$ such that Γ realizes S and $\Gamma \not\models \sim X$. Since $\Gamma \not\models \sim X$, for some $\Gamma^* \in \mathscr{G}$, $\Gamma^* \models X$. But clearly, if Γ realizes S, Γ^* realizes S_T (by theorem 1.4.4). Hence Γ^* realizes $\{S_T, TX\}$ and $\mathscr{C}_{i+1}$ is realizable.

The other six cases are similar.

***Corollary* 2.3:** The system of Beth tableaus is correct, that is, if $\vdash_I X$, X is valid.

Proof: We show the contrapositive. Suppose X is not valid. Then there is a model $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ and a $\Gamma \in \mathscr{G}$ such that $\Gamma \not\models X$. In other words $\{FX\}$ is realizable. But a proof of X would be a closed tableau $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$ in which $\mathscr{C}_1$ is $\{\{FX\}\}$. But $\mathscr{C}_1$ is realizable, hence each $\mathscr{C}_i$ is realizable. But obviously a realizable configuration cannot be closed. Hence $\not\vdash_I X$.

§ 3. Hintikka collections

In classical logic a set S of signed formulas is sometimes called downward saturated, or a Hintikka set, if

$$TX \wedge Y \in S \Rightarrow TX \in S \text{ and } TY \in S,$$
$$FX \vee Y \in S \Rightarrow FX \in S \text{ and } FY \in S,$$

$$
\begin{array}{lll}
TX \vee Y \in S & \Rightarrow & TX \in S \quad \text{or} \quad TY \in S, \\
FX \wedge Y \in S & \Rightarrow & FX \in S \quad \text{or} \quad FY \in S, \\
T \sim X \in S & \Rightarrow & FX \in S, \\
TX \supset Y \in S & \Rightarrow & FX \in S \quad \text{or} \quad TY \in S, \\
F \sim X \in S & \Rightarrow & TX \in S, \\
FX \supset Y \in S & \Rightarrow & TX \in S \quad \text{and} \quad FY \in S.
\end{array}
$$

Remark 3.1: The names Hintikka set and downward saturated set were given by Smullyan [20]. Hintikka, their originator, called them model sets.

Hintikka showed that any consistent downward saturated set could be included in a set for which the above properties hold with $\Rightarrow$ replaced by $\Leftrightarrow$. From this follows the completeness of certain classical tableau systems. This approach is thoroughly developed by Smullyan in [20].

We now introduce a corresponding notion in intuitionistic logic, which we call a Hintikka collection. While its intuitive appeal may not be as immediate as in the classical case, its usefulness is as great.

Definition 3.2: Let $\mathscr{G}$ be a collection of consistent sets of signed formulas. We call $\mathscr{G}$ a *Hintikka collection* if for any $\Gamma \in \mathscr{G}$

$$
\begin{array}{lll}
TX \wedge Y \in \Gamma & \Rightarrow & TX \in \Gamma \quad \text{and} \quad TY \in \Gamma, \\
FX \vee Y \in \Gamma & \Rightarrow & FX \in \Gamma \quad \text{and} \quad FY \in \Gamma, \\
TX \vee Y \in \Gamma & \Rightarrow & TX \in \Gamma \quad \text{or} \quad TY \in \Gamma, \\
FX \wedge Y \in \Gamma & \Rightarrow & FX \in \Gamma \quad \text{or} \quad FY \in \Gamma, \\
T \sim X \in \Gamma & \Rightarrow & FX \in \Gamma, \\
TX \supset Y \in \Gamma & \Rightarrow & FX \in \Gamma \quad \text{or} \quad TY \in \Gamma, \\
F \sim X \in \Gamma & \Rightarrow & \text{for some } \Delta \in \mathscr{G}, \Gamma_T \subseteq \Delta \quad \text{and} \quad TX \in \Delta, \\
FX \supset Y \in \Gamma & \Rightarrow & \text{for some } \Delta \in \mathscr{G}, \Gamma_T \subseteq \Delta, TX \in \Delta, FY \in \Delta.
\end{array}
$$

Definition 3.3: Let $\mathscr{G}$ be a Hintikka collection. We call $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ a *model for* $\mathscr{G}$ if

(1). $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ is a model,
(2). $\Gamma_T \subseteq \Delta \Rightarrow \Gamma \mathscr{R} \Delta$,
(3). $TX \in \Gamma \Rightarrow \Gamma \models X$,
$FX \in \Gamma \Rightarrow \Gamma \not\models X$.

***Theorem* 3.4:** There is a model for any Hintikka collection.

Proof: Let $\mathscr{G}$ be a Hintikka collection. Define $\mathscr{R}$ by: $\Gamma \mathscr{R} \Delta$ if $\Gamma_T \subseteq \Delta$.

If A is atomic, let $\Gamma \Vdash A$ if $TA \in \Gamma$, and extend $\Vdash$ to produce a model $\langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$. To show property (3) is a straightforward induction on the degree of X. We give one case as illustration. Suppose X is $\sim Y$ and the result is known for Y. Then

$$\begin{aligned} T \sim Y \in \Gamma &\Rightarrow (\forall \Delta \in \mathcal{G})(\Gamma_T \subseteq \Delta \Rightarrow T \sim Y \in \Delta) \\ &\Rightarrow (\forall \Delta \in \mathcal{G})(\Gamma_T \subseteq \Delta \Rightarrow FY \in \Delta) \\ &\Rightarrow (\forall \Delta \in \mathcal{G})(\Gamma \mathcal{R} \Delta \Rightarrow \Delta \nVdash Y) \\ &\Rightarrow \Gamma \Vdash \sim Y, \end{aligned}$$

and

$$\begin{aligned} F \sim Y \in \Gamma &\Rightarrow (\exists \Delta \in \mathcal{G})(\Gamma_T \subseteq \Delta \text{ and } TY \in \Delta) \\ &\Rightarrow (\exists \Delta \in \mathcal{G})(\Gamma \mathcal{R} \Delta \quad \text{and } \Delta \Vdash Y) \\ &\Rightarrow \Gamma \nVdash \sim Y. \end{aligned}$$

It follows from this theorem that to show the completeness of Beth tableaus we need only show the following: If $\nVdash_I X$, then there is a Hintikka collection $\mathcal{G}$ such that for some $\Gamma \in \mathcal{G}$, $FX \in \Gamma$.

§ 4. Completeness of Beth tableaus

Let S be a set of signed formulas. By $\mathcal{S}(S)$ we mean the collection of all signed subformulas of formulas in S. If S is finite, $\mathcal{S}(S)$ is finite.

Let S be a finite, consistent set of signed formulas. We define a *reduced set for* S (there may be many) as follows:

Let S_0 be S. Having defined S_n, a finite consistent set of signed formulas, suppose one of the following Beth reduction rules applies to S_n: $T\wedge$, $F\wedge$, $T\vee$, $F\vee$, $T\sim$ or $T\supset$. Choose one which applies, say $F\wedge$. Then S_n is $\{U, FX \wedge Y\}$. This is consistent, so clearly either $\{U, FX \wedge Y, FX\}$ or $\{U, FX \wedge Y, FY\}$ is consistent. Let S_{n+1} be $\{U, FX \wedge Y, FX\}$ if consistent, otherwise let S_{n+1} be $\{U, FX \wedge Y, FY\}$. Similarly if $T\wedge$ applies and was chosen, then S_n is $\{U, TX \wedge Y\}$. Since this is consistent, $\{U, TX \wedge Y, TX, TY\}$ is consistent. Let this be S_{n+1}. In this way we define a sequence $S_0, S_1, S_2, \ldots$. This sequence has the property $S_n \subseteq S_{n+1}$. Further, each S_n is finite and consistent. Since each $S_n \subseteq \mathcal{S}(S)$, there are only a finite number of different possible S_n. Consequently there must be a member of the sequence, say S_n, such that the application of any one of the rules (except $F\sim$ or $F\supset$) produces S_n again. Call such an S_n a *reduced* set of S, and denote it by S'. Clearly any finite, consistent set of signed formulas has a finite, consistent reduced set. Moreover, if S' is a reduced set, it has the following suggestive properties:

$$\begin{array}{lllll}
TX \wedge Y \in S' & \Rightarrow & TX \in S' & \text{and} & TY \in S', \\
FX \vee Y \in S' & \Rightarrow & FX \in S' & \text{and} & FY \in S', \\
TX \vee Y \in S' & \Rightarrow & TX \in S' & \text{or} & TY \in S', \\
FX \wedge Y \in S' & \Rightarrow & FX \in S' & \text{or} & FY \in S', \\
T \sim X \in S' & \Rightarrow & FX \in S', & & \\
TX \supset Y \in S' & \Rightarrow & FX \in S' & \text{or} & TY \in S', \\
S' \text{ is consistent.} & & & &
\end{array}$$

Now, given any finite, consistent set of signed formulas S, we form the collection of *associated sets* as follows:

If $F \sim X \in S$, $\{S_T, TX\}$ is an associated set.

If $FX \supset Y \in S$, $\{S_T, TX, FY\}$ is an associated set.

Let $\mathscr{A}(S)$ be the collection of all associated sets of S. $\mathscr{A}(S)$ is finite, since $U \in \mathscr{A}(S)$ implies $U \subseteq \mathscr{S}(S)$ and $\mathscr{S}(S)$ is finite. $\mathscr{A}(S)$ has the following properties: if S is consistent, any associated set is consistent and

$$\begin{array}{lll}
F \sim X \in S & \Rightarrow & \text{for some } U \in \mathscr{A}(S)\ S_T \subseteq U,\ TX \in U, \\
FX \supset Y \in S & \Rightarrow & \text{for some } U \in \mathscr{A}(S)\ S_T \subseteq U,\ TX \in U,\ FY \in U.
\end{array}$$

Now we proceed with the proof of completeness.

Suppose $\nvdash_I X$. Then $\{FX\}$ is consistent. Extend it to its reduced set S_0. Form $\mathscr{A}(S_0)$. Let the elements of $\mathscr{A}(S_0)$ be $U_1, U_2, \ldots, U_n$. Let S_1 be the reduced set of $U_1, \ldots, S_n$ be the reduced set of U_n. Thus, we have the sequence $S_0, S_1, S_2, \ldots, S_n$.

Next form $\mathscr{A}(S_1)$. Call its elements $U_{n+1}, U_{n+2}, \ldots, U_m$. Let S_{n+1} be the reduced set of U_{n+1} and so on. Thus, we have the sequence $S_0, S_1, \ldots, S_n, S_{n+1}, \ldots, S_m$. Now we repeat the process with S_2, and so on.

In this way we form a sequence $S_0, S_1, S_2, \ldots$. Since each $S_i \subseteq \mathscr{S}(S)$, there are only finitely many possible different S_i. Thus we must reach a point S_k of the sequence such that any continuation repeats on earlier member.

Let $\mathscr{G}$ be the collection $\{S_0, S_1, \ldots, S_k\}$. It is easy to see that $\mathscr{G}$ is a Hintikka collection. But $FX \in S_0 \in \mathscr{G}$. Thus we have shown:

***Theorem* 4.1:** Beth tableaus are complete.

Remark 4.2: This proof also establishes that propositional intuitionistic logic is decidable. For, if we follow the above procedure beginning with FX, after a finite number of steps we will have either a closed tableau for $\{FX\}$ or a counter-model for X. Moreover, the number of steps may be bounded in terms of the degree of X.

The completeness proof presented here is in essence the original proof of Kripke [13]. For a different tableau completeness proof see ch. 5 § 6, where it is given for first order logic. For a completeness proof of an axiom system see ch. 5 § 10, where it also is given for a first order system. The work in ch. 1 § 6 provides an algebraic completeness proof, since the Lindenbaum algebra of intuitionistic logic is easily shown to be a pseudo-boolean algebra. See [16].

§ 5. Examples

In this section, so that the reader may gain familiarity with the foregoing, we present a few theorems and non-theorems of intuitionistic propositional logic, together with their proofs or counter-models.

We show

(1). $\nvdash_{\mathrm{I}} A \vee \sim A$,

(2). $\vdash_{\mathrm{I}} \sim\sim(A \vee \sim A)$,

(3). $\nvdash_{\mathrm{I}} \sim\sim A \supset A$,

(4). $\vdash_{\mathrm{I}} (A \vee B) \supset \sim(\sim A \wedge \sim B)$,

(5). $\nvdash_{\mathrm{I}} \sim\sim(A \vee B) \supset (\sim\sim A \vee \sim\sim B)$.

For the general principle connecting (1) and (2) see ch. 4 § 8.

(1). $\nvdash_{\mathrm{I}} A \vee \sim A$.

A counter example for this is the following:

$$\mathscr{G} = \{\Gamma, \Delta\}$$
$$\Gamma \mathscr{R} \Gamma, \quad \Gamma \mathscr{R} \Delta, \quad \Delta \mathscr{R} \Delta.$$

$\Delta \models A$ is the $\models$ relation for atomic formulas, and $\models$ is extended to all formulas as usual. We may schematically represent this model by

$$\begin{array}{c} \Gamma \\ | \\ \Delta \models A \end{array}$$

We claim $\Gamma \nvDash A \vee \sim A$. Suppose not. If $\Gamma \vDash A \vee \sim A$, either $\Gamma \vDash A$ or $\Gamma \vDash \sim A$. But $\Gamma \nvDash A$. If $\Gamma \vDash \sim A$ then since $\Gamma \mathscr{R} \Delta$, $\Delta \nvDash A$. But $\Delta \vDash A$, hence $\Gamma \nvDash A \vee \sim A$.

(2). $\vdash_{\mathrm{I}} \sim\sim(A \vee \sim A)$.

A tableau proof for this is the following, where the reasons for the steps are obvious:

$$
\begin{aligned}
&\{\{F \sim\sim (A \vee \sim A)\}\},\\
&\{\{T \sim (A \vee \sim A)\}\},\\
&\{\{T \sim (A \vee \sim A), F(A \vee \sim A)\}\},\\
&\{\{T \sim (A \vee \sim A), FA, F \sim A\}\},\\
&\{\{T \sim (A \vee \sim A), TA\}\},\\
&\{\{F(A \vee \sim A), TA\}\},\\
&\{\{FA, F \sim A, TA\}\}.
\end{aligned}
$$

(3). $\nvdash_{\mathrm{I}} \sim\sim A \supset A$.

The model of example (1) has the property that $\Gamma \vDash \sim\sim A$ but $\Gamma \nvDash A$.

(4). $\vdash_{\mathrm{I}} (A \vee B) \supset \sim(\sim A \wedge \sim B)$.

The following is a proof:

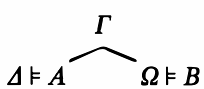
$$
\begin{aligned}
&\{\{F((A \vee B) \supset \sim(\sim A \wedge \sim B))\}\},\\
&\{\{T(A \vee B), F \sim(\sim A \wedge \sim B)\}\},\\
&\{\{T(A \vee B), T(\sim A \wedge \sim B)\}\},\\
&\{\{T(A \vee B), T \sim A, T \sim B\}\},\\
&\{\{T(A \vee B), FA, T \sim B\}\},\\
&\{\{T(A \vee B), FA, FB\}\},\\
&\{\{TA, FA, FB\}, \{TB, FA, FB\}\}.
\end{aligned}
$$

(5). $\nvdash_{\mathrm{I}} \sim\sim(A \vee B) \supset (\sim\sim A \vee \sim\sim B)$.

A counter example is the following:

$$
\begin{aligned}
&\mathscr{G} = \{\Gamma, \Delta, \Omega\},\\
&\Gamma \mathscr{R} \Gamma, \quad \Delta \mathscr{R} \Delta, \quad \Omega \mathscr{R} \Omega,\\
&\Gamma \mathscr{R} \Delta, \quad \Gamma \mathscr{R} \Omega
\end{aligned}
$$

$\Delta \vDash A$, $\Omega \vDash B$ is the $\vDash$ relation for atomic formulas, and $\vDash$ is extended as usual. We may schematically represent this model by



Now $\Delta \models A$, so $\Delta \models A \vee B$. Likewise $\Omega \models A \vee B$. It follows that $\Gamma \models \sim\sim(A \vee B)$ But if $\Gamma \models \sim\sim A \vee \sim\sim B$, either $\Gamma \models \sim\sim A$ or $\Gamma \models \sim\sim B$. If $\Gamma \models \sim\sim A$, it would follow that $\Omega \models A$. If $\Gamma \models \sim\sim B$, it would follow that $\Delta \models B$. Thus $\Gamma \not\models \sim\sim A \vee \sim\sim B$.

CHAPTER 3

RELATED SYSTEMS OF LOGIC

§ 1. f-primitive intuitionistic logic, semantics

This is an alternative formulation of intuitionistic logic in which a symbol f is taken as primitive, instead of $\sim$, which is then re-introduced as a formal abbreviation, $\sim X$ for $X \supset f$. For presentations of this type, see [15] or [17].

Specifically, we change the definition of formula by adding f to our list of propositional variables and removing $\sim$ from the set of connectives. $\sim$ is re-introduced as a metamathematical symbol as above. Our definition of subformula is also changed accordingly. The definition of model is changed as follows: replace P3 (ch. 1 § 2) by P3′: $\Gamma \nVdash f$. This leads to a new definition of validity, which we may call f-validity.

***Theorem* 1.1:** Let X be a formula (in the usual sense) and let X' be the corresponding formula with $\sim$ written in terms of f. Then X is valid if and only if X' is f-valid.

Proof: We show that in any model $\langle \mathscr{G}, \mathscr{R}, \Vdash \rangle$

$$\Gamma \Vdash X \quad \text{iff} \quad \Gamma \Vdash X'$$

(where we use two different senses of $\Vdash$). The proof is by induction on the degree of X (which is the same as the degree of X'). Actually all cases are easy except that of $\sim$ itself. So suppose the result is known for all formulas of degree less than that of X,

$$\begin{aligned} \Gamma \models X &\Leftrightarrow \Gamma \models \sim Y \\ &\Leftrightarrow \forall \Gamma^* \quad \Gamma^* \not\models Y \\ &\Leftrightarrow \forall \Gamma^* \quad \Gamma^* \not\models Y', \end{aligned}$$

but clearly this is equivalent to $\Gamma \models Y' \supset f$ since $\Gamma^* \not\models f$. Hence equivalently $\Gamma \models X'$.

§ 2. f-primitive intuitionistic logic, proof theory

In this section we still retain the altered definition of formula in the last section with f primitive. We give a tableau system for this. The new system is the same as that of ch. 2 § 1 in all but two respects. First the rules $T\sim$ and $F\sim$ are removed. Second a set S of signed formulas is called closed if it contains TX and FX for some formula X, tains Tf.

This leads to a new definition of theorem, which we may call f-theorem.

***Theorem* 2.1:** Let X be a formula (in the usual sense) and let X' be the corresponding formula with $\sim$ written in terms of f. Then X is a theorem if and only if X' is an f-theorem.

This follows immediately from the following:

***Lemma* 2.2:** Let S be a set of signed formulas (in the usual sense) and let S' be the corresponding set of signed formulas with $\sim$ replaced in terms of f. Then S is inconsistent if and only if S' is f-inconsistent.

Proof: We show this in two halves. First suppose S is inconsistent. We show the result by induction on the length of the closed tableau for S. There are only two significant cases. Suppose first that the tableau for S is $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$; $\mathscr{C}_1$ is $\{\{U, F{\sim}X\}\}$ and $\mathscr{C}_2$ is $\{\{U_T, TX\}\}$. Then by the induction hypothesis $\{U'_T, TX'\}$ is f-inconsistent. Hence so is $\{U', FX' \supset f\}$, i.e. S'. The other case is if $\mathscr{C}_1$ is $\{\{U, T{\sim}X\}\}$ and $\mathscr{C}_2$ is $\{\{U, FX\}\}$. Then by the induction hypothesis $\{U', FX'\}$ is f-inconsistent. Hence so is $\{U', TX' \supset f\}$, i.e. S'.

The converse is shown by induction on the length of the closed f-tableau for S'. If this f-tableau is of length l, either S' contains TX and

FX for some formula X, and we are done, or S' contains Tf, which is not possible since we supposed S' arose from standard set S.

The induction steps are similar to those above.

The results of this and the last sections, together with our earlier results give: X' is f-valid if and only if X' is an f-theorem. This is not the complete generality one would like since it holds only for those formulas X' which correspond to standard formulas X. The more complete result is however true, as the reader may show by methods similar to those of the last chapter.

§ 3. Minimal logic

Minimal logic is a sublogic of intuitionistic logic in which a false statement need not imply everything. The original paper on minimal logic is Johannson's [9]. Prawitz establishes several results concerning it in [15], and it is treated algebraically by Rasiowa and Sikorski [16].

Semantically, we use the f-models defined in § 1, with the change that we no longer require P3′, that is, that $\Gamma \nvDash f$. Proof theoretically, we use the f-tableaus defined in § 2, with the change that we no longer have closure of a set because it contains Tf. We leave it to the reader to show that X is provable in this tableau system if and only if X is valid in this model sense, using the methods of ch. 2.

Certainly every minimal logic theorem is an intuitionistic logic theorem, but the converse is not true. For example $(A \wedge \sim A) \supset B$ is a theorem of intuitionistic logic, but the following is a minimal counter-model for it, or rather for $(A \wedge (A \supset f)) \supset B$:

$$\mathscr{G} = \{\Gamma\},$$
$$\Gamma \mathscr{R} \Gamma,$$
$$\Gamma \vDash A, \quad \Gamma \vDash f,$$

and $\vDash$ is extended as usual. It is easily seen that $\Gamma \vDash A \wedge (A \supset f)$, but $\Gamma \nvDash B$.

§ 4. Classical logic

Beginning with this section, we return to the usual notions of formula, tableau and model, that is, with $\sim$ and not f as primitive.

Some authors call a set $\mathscr{S}$ of unsigned formulas a (classical) truth set if

$$\begin{array}{lcll} X \wedge Y \in \mathscr{S} & \Leftrightarrow & X \in \mathscr{S} \text{ and } & Y \in \mathscr{S}, \\ X \vee Y \in \mathscr{S} & \Leftrightarrow & X \in \mathscr{S} \text{ or } & Y \in \mathscr{S}, \\ \sim X \in \mathscr{S} & \Leftrightarrow & X \notin \mathscr{S}, & \\ X \supset Y \in \mathscr{S} & \Leftrightarrow & X \notin \mathscr{S} \text{ or } & Y \in \mathscr{S}. \end{array}$$

It is a standard result of classical logic that X is a classical theorem if and only if X is in every truth set. There is a proof of this in [20].

***Theorem* 4.1:** Any intuitionistic theorem is a classical theorem.

Proof: Suppose X is not a classical theorem. Then there is a truth set $\mathscr{S}$ such that $X \notin \mathscr{S}$. We define a very simple intuitionistic counter-model for X, $\langle \mathscr{G}, \mathscr{R}, \vDash \rangle$, as follows:

$$\mathscr{G} = \{\mathscr{S}\},$$
$$\mathscr{S} \mathscr{R} \mathscr{S},$$
$$\mathscr{S} \vDash A \Leftrightarrow A \in \mathscr{S},$$

for A atomic, and $\vDash$ is extended as usual. It is easily shown by induction on the degree of Y that

$$\mathscr{S} \vDash Y \Leftrightarrow Y \in \mathscr{S}.$$

Hence $\mathscr{S} \nvDash X$, and X is not an intuitionistic theorem.

That the converse is not true follows since we showed in ch. 2 § 5 that $\nvdash_I A \vee \sim A$. Thus we have: minimal logic is a proper sub-logic of intuitionistic logic which is a proper sub-logic of classical logic.

§ 5. Modal logic, S4; semantics

In this section we define the set of (propositional) S4 theorems semantically using a model due to Kripke [12] (see also [18]). S4 was originated by Lewis [14], and an algebraic treatment may be found in [16]. A natural deduction treatment is in [15].

The definition of formula is changed by adding $\Box$ to the set of unary connectives. Thus for example $\sim \Box \sim (A \vee \Box \sim A)$ is a formula. $\Box$ is read "necessarily". $\Diamond$ is sometimes taken as an abbreviation for $\sim \Box \sim$ and is read "possibly". (In [14] $\Diamond$ was primitive.)

The S4 model is defined as follows: It is an ordered triple $\langle \mathscr{G}, \mathscr{R}, \vDash \rangle$ where $\mathscr{G}$ is a non-empty set, $\mathscr{R}$ is a transitive, reflexive relation on $\mathscr{G}$, and $\models$ is a relation between elements of $\mathscr{G}$ and formulas, satisfying the following conditions:

M1. $\Gamma \models X \wedge Y$ iff $\Gamma \models X$ and $\Gamma \models Y$,
M2. $\Gamma \models X \vee Y$ iff $\Gamma \models X$ or $\Gamma \models Y$,
M3. $\Gamma \models \sim X$ iff $\Gamma \not\models X$,
M4. $\Gamma \models X \supset Y$ iff $\Gamma \not\models X$ or $\Gamma \models Y$,
M5. $\Gamma \models \Box X$ iff for all Γ^*, $\Gamma^* \models X$.

X is S4 valid in $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ if for all $\Gamma \in \mathscr{G}$, $\Gamma \models X$. X is S4 valid if X is S4 valid in all S4 models.

The intuitive idea behind this modeling is the following: $\mathscr{G}$ is the collection of all possible worlds. $\Gamma \mathscr{R} \Delta$ means Δ is a world possible relative to Γ. $\Gamma \models X$ means X is true in the world Γ. Thus M5 may be interpreted: X is necessarily true in Γ if and only if X is true in any world possible relative to Γ. This interpretation is given in [12].

§ 6. Modal logic, S4; proof theory

We define a tableau system for S4 as follows: Everything in the definition of Beth tableaus in ch. 2 § 1 remains the same except the reduction rules themselves. These are replaced by

$MT\wedge$ $\quad \dfrac{S, TX \wedge Y}{S, TX, TY}$ $\qquad$ $MF\wedge$ $\quad \dfrac{S, FX \wedge Y}{S, FX \mid S, FY}$

$MT\vee$ $\quad \dfrac{S, TX \vee Y}{S, TX \mid S, TY}$ $\qquad$ $MF\vee$ $\quad \dfrac{S, FX \vee Y}{S, FX, FY}$

$MT\sim$ $\quad \dfrac{S, T\sim X}{S, FX}$ $\qquad$ $MF\sim$ $\quad \dfrac{S, F\sim X}{S, TX}$

$MT\supset$ $\quad \dfrac{S, TX \supset Y}{S, FX \mid S, TY}$ $\qquad$ $MF\supset$ $\quad \dfrac{S, FX \supset Y}{S, TX, FY}$

$MT\Box$ $\quad \dfrac{S, T\Box X}{S, TX}$ $\qquad$ $MF\Box$ $\quad \dfrac{S, F\Box X}{S_\Box, FX}$

where in rule $MF\Box$ $S_\Box$ is $\{T\Box X \mid T\Box X \in S\}$. Again the methods of ch. 2 can be adapted to S4 to establish the identity of the set of S4 theorems and the set of S4 valid formulas. This is left to the reader. The original proof is in [12]. We are more interested in the relation between S4 and intuitionistic logic.

§ 7. S4 and intuitionistic logic

A map from the set of intuitionistic formulas to the set of S4 formulas is defined by (see [18])

$$\begin{aligned}
\boldsymbol{M}(A) \quad &= \Box A \quad \text{for } A \text{ atomic},\\
\boldsymbol{M}(X \vee Y) &= \boldsymbol{M}(X) \vee \boldsymbol{M}(Y),\\
\boldsymbol{M}(X \wedge Y) &= \boldsymbol{M}(X) \wedge \boldsymbol{M}(Y),\\
\boldsymbol{M}(\sim X) \quad &= \Box \sim \boldsymbol{M}(X),\\
\boldsymbol{M}(X \supset Y) &= \Box(\boldsymbol{M}(X) \supset \boldsymbol{M}(Y)).
\end{aligned}$$

We wish to show

***Theorem* 7.1:** If X is an intuitionistic formula, X is intuitionistically valid if and only if $\boldsymbol{M}(X)$ is S4-valid.

This follows from the next three lemmas.

***Lemma* 7.2:** Let $\langle \mathscr{G}, \mathscr{R}, \models_{\mathrm{I}} \rangle$ be an intuitionistic model and $\langle \mathscr{G}, \mathscr{R}, \models_{\mathrm{S4}} \rangle$ be an S4 model, such that for any $\Gamma \in \mathscr{G}$ and any atomic A

$$\Gamma \models_{\mathrm{I}} A \Leftrightarrow \Gamma \models_{\mathrm{S4}} \boldsymbol{M}(A).$$

Then for any formula X

$$\Gamma \models_{\mathrm{I}} X \Leftrightarrow \Gamma \models_{\mathrm{S4}} \boldsymbol{M}(X).$$

Proof: A straightforward induction on the degree of X.

***Lemma* 7.3:** Given an intuitionistic counter-model for X, there is an S4 counter-model for $\boldsymbol{M}(X)$.

Proof: We have $\langle \mathscr{G}, \mathscr{R}, \models_{\mathrm{I}} \rangle$, an intuitionistic model such that for some $\Gamma \in \mathscr{G}$ $\Gamma \not\models_{\mathrm{I}} X$. We take for our S4 model $\langle \mathscr{G}, \mathscr{R}, \models_{\mathrm{S4}} \rangle$ where $\models_{\mathrm{S4}}$ is defined by

$$\Delta \models_{\mathrm{S4}} A \quad \text{if} \quad \Delta \models_{\mathrm{I}} A$$

for A atomic and any Δ in $\mathscr{G}$, and $\models_{\mathrm{S4}}$ is extended to all formulas. If A is atomic

$$\begin{aligned}
\Delta \models_{\mathrm{S4}} \boldsymbol{M}(A) &\Leftrightarrow \Delta \models_{\mathrm{S4}} \Box A\\
&\Leftrightarrow (\forall \Delta^*)\ \Delta^* \models_{\mathrm{S4}} A\\
&\Leftrightarrow (\forall \Delta^*)\ \Delta^* \models_{\mathrm{I}} A\\
&\Leftrightarrow \Delta \models_{\mathrm{I}} A
\end{aligned}$$

and the result follows by lemma 7.2.

***Lemma* 7.4:** Given an S4 counter-model for $M(X)$, there is an intuitionistic counter-model for X.

Proof: We have $\langle \mathscr{G}, \mathscr{R}, \models_{S4} \rangle$, an S4 model such that for some $\Gamma \in \mathscr{G}$ $\Gamma \not\models_{S4} M(X)$. We take for our intuitionistic model $\langle \mathscr{G}, \mathscr{R}, \models_{I} \rangle$ where $\models_I$ is defined by

$$\Delta \models_I A \quad \text{if} \quad \Delta \models_{S4} M(A)$$

for A atomic and any Δ in $\mathscr{G}$, and $\models_I$ is extended to all formulas. Now the result follows by lemma 7.2.

CHAPTER 4

FIRST ORDER INTUITIONISTIC LOGIC

SEMANTICS

§ 1. Formulas

We begin with the following:

(1). denumerably many individual variables $x, y, z, w, \ldots$

(2). denumerably many individual parameters $a, b, c, d, \ldots$

(3). for each positive integer n, a denumerable list of n-ary predicates $A^n, B^n, C^n, D^n, \ldots$

(4). connectives, quantifiers, parentheses, $\wedge$, $\vee$, $\supset$, $\sim$, $\exists$, $\forall$, (,).

An *atomic formula* is an n-ary predicate symbol A^n followed by an n-tuple of individual symbols (variables or parameters), thus $A^n(\alpha_1, \ldots, \alpha_n)$. A *formula* is anything resulting from the following recursive rules:

F0. Any atomic formula is a formula.

F1. If X is a formula, so is $\sim X$.

F2, 3, 4. If X and Y are formulas, so are $(X \wedge Y)$, $(X \vee Y)$, $(X \supset Y)$.

F5, 6. If X is a formula and x is a variable, $(\forall x)X$ and $(\exists x)X$ are formulas.

Subformulas and the *degree* of a formula are defined as usual. The property of uniqueness of composition of a formula still holds. We note the usual properties of substitution, and we use the following notation: If X is a formula and α and β are individual symbols, by $X(^{\alpha}_{\beta})$ we mean the result of substituting β for every occurrence of α in X (every free occurrence in case α is a variable). We usually denote this informally as follows: we write X as $X(\alpha)$ and $X(^{\alpha}_{\beta})$ as $X(\beta)$. It will be clear from the context what is meant. We again use parentheses in an informal manner and we omit superscripts on predicates.

Although the definition of formula as stated allows unbound occurrences of variables in formulas, we shall assume, unless otherwise stated, that all variables in a formula are bound. Notation like $X(x)$ however, indicates that x may have free occurrences in X.

§ 2. Models and validity

In this section we define the notion of a first order intuitionistic model, and first order intuitionistic validity, referred to respectively as model and validity. This modeling structure is due to Kripke and may be found, in different notation, in [13] (see also [18]). The notions of ch. 1, if needed, will be referred to as propositional notions to distinguish them.

If $\mathscr{P}$ is a map from $\mathscr{G}$ to sets of parameters, by $\hat{\mathscr{P}}(\Gamma)$ we mean the set of all formulas which may be constructed using only parameters of $\mathscr{P}(\Gamma)$. By a *(first order intuitionistic) model* we mean an ordered quadruple $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{P}\rangle$, where $\mathscr{G}$ is a non-empty set, $\mathscr{R}$ is a transitive, reflexive relation on $\mathscr{G}$, $\models$ is a relation between elements of $\mathscr{G}$ and formulas, and $\mathscr{P}$ is a map from $\mathscr{G}$ to non-empty sets of parameters, satisfying the following conditions:

for any $\Gamma \in \mathscr{G}$

Q0. $\mathscr{P}(\Gamma) \subseteq \mathscr{P}(\Gamma^*)$,

Q1. $\Gamma \models A \Rightarrow A \in \hat{\mathscr{P}}(\Gamma)$ for A atomic,

Q2. $\Gamma \models A \Rightarrow \Gamma^* \models A$ for A atomic,

Q3. $\Gamma \models (X \wedge Y) \Leftrightarrow \Gamma \models X$ and $\Gamma \models Y$,

Q4. $\Gamma \models (X \vee Y) \Leftrightarrow (X \vee Y) \in \hat{\mathscr{P}}(\Gamma)$ and $\Gamma \models X$ or $\Gamma \models Y$,

Q5. $\Gamma \models \sim X \Leftrightarrow \sim X \in \hat{\mathscr{P}}(\Gamma)$ and for all Γ^* $\Gamma^* \not\models X$,

Q6. $\Gamma \models (X \supset Y) \Leftrightarrow (X \supset Y) \in \hat{\mathscr{P}}(\Gamma)$ and for all Γ^*, if $\Gamma^* \models X$, $\Gamma^* \models Y$,

Q7. $\Gamma \models (\exists x) X(x) \Leftrightarrow$ for some $a \in \mathscr{P}(\Gamma)$ $\Gamma \models X(a)$,

Q8. $\Gamma \models (\forall x) X(x) \Leftrightarrow$ for every Γ^* and for every $a \in \mathscr{P}(\Gamma^*)$ $\Gamma^* \models X(a)$.

We call a particular formula X *valid in the model* $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{P}\rangle$ if for all $\Gamma \in \mathscr{G}$ such that $X \in \hat{\mathscr{P}}(\Gamma)$ $\Gamma \models X$. X is called *valid* if X is valid in all models.

§ 3. Motivation

The intuitive interpretation given in ch. 1 § 3 for the propositional case may be extended to this first order situation.

In one's usual mathematical work, parameters may be introduced as one proceeds, but having introduced a parameter, of course it remains introduced. This is what the map $\mathscr{P}$ is intended to represent. That is, for $\Gamma \in \mathscr{G}$ Γ is a state of knowledge, and $\mathscr{P}(\Gamma)$ is the set of all parameters introduced to reach Γ. (Or in a stricter intuitive sense, $\mathscr{P}(\Gamma)$ is the set of all mathematical entities constructed by time Γ.) Since parameters, once introduced, do not disappear, we have Q0. Q2–6 are as in the propositional case. Q7 should be obvious. Q8 may be explained: to know $(\forall x)\, X(x)$ at Γ, it is not enough merely to know $X(a)$ for every parameter a introduced so far (i.e. for all $a \in \mathscr{P}(\Gamma)$). Rather one must know $X(a)$ for all parameters which can ever be introduced (i.e. for all $a \in \mathscr{P}(\Gamma^*)$ $\Gamma^* \models X(a)$).

The restrictions Q1, and in Q4, Q5 and Q6 are simply to the effect that it makes no sense to say we know the truth of a formula X if X uses parameters we have not yet introduced. It would of course make sense to add corresponding restrictions to Q3, Q7 and Q8, but it is not necessary. The original explanation of Kripke may be found in [13]. For a different but related model theory in terms of forcing see [5].

§ 4. Some properties of models

***Theorem* 4.1:** In any model $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$, for any $\Gamma \in \mathscr{G}$, if $\Gamma \models X$, $X \in \hat{\mathscr{P}}(\Gamma)$.

Proof: A straightforward induction on the degree of X.

***Theorem* 4.2:** In any model $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$, for any formula X, if $\Gamma \models X$, $\Gamma^* \models X$.

Proof: Also a straightforward induction on the degree of X.

***Theorem* 4.3:** Let $\mathscr{G}$ be a non-empty set, $\mathscr{R}$ be a transitive reflexive relation on $\mathscr{G}$, and $\mathscr{P}$ be a map from $\mathscr{G}$ to non-empty sets of parameters such that $\mathscr{P}(\Gamma) \subseteq \mathscr{P}(\Gamma^*)$ for all $\Gamma \in \mathscr{G}$. Suppose $\models$ is a relation between elements of $\mathscr{G}$ and atomic formulas such that $\Gamma \models A \Rightarrow A \in \hat{\mathscr{P}}(\Gamma)$. Then $\models$ can be extended in one and only one way to a relation, also denoted by $\models$, between $\mathscr{G}$ and formulas, such that $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ is a model.

Proof: A straightforward extension of the corresponding propositional proof.

Definition 4.4: Let $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ be a model and suppose a is some parameter such that $a \notin \bigcup_{\Gamma \in \mathscr{G}} \mathscr{P}(\Gamma)$. By $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle \binom{b}{a}$ we mean the model $\langle \mathscr{G}, \mathscr{R}, \models', \mathscr{P}' \rangle$ defined as follows:

$\mathscr{P}'(\Gamma)$ is the same as $\mathscr{P}(\Gamma)$ except for containing a in place of b if $\mathscr{P}(\Gamma)$ contains b.

For A atomic $\Gamma \models A \Rightarrow \Gamma \models' A\binom{b}{a}$, and $\models'$ is extended to all formulas.

***Lemma* 4.5:** Let $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ be a model, $a \notin \bigcup_{\Gamma \in \mathscr{G}} \mathscr{P}(\Gamma)$, $\langle \mathscr{G}, \mathscr{R}, \models', \mathscr{P}' \rangle$ be $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle \binom{b}{a}$. Then for any formula X not containing a

$$\Gamma \models X \Leftrightarrow \Gamma \models' X\binom{b}{a}.$$

Proof: By an easy induction on the degree of X.

Definition 4.6: Let $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ be a model and suppose a is some parameter such that $a \notin \bigcup_{\Gamma \in \mathscr{G}} \mathscr{P}(\Gamma)$. By $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle_{b=a}$ we mean the model $\langle \mathscr{G}, \mathscr{R}, \models', \mathscr{P}' \rangle$ defined as follows:

$\mathscr{P}'(\Gamma)$ is the same as $\mathscr{P}(\Gamma)$ except for containing a as well as b whenever $\mathscr{P}(\Gamma)$ contains b.

For A atomic $\Gamma \models A \Rightarrow \Gamma \models' A'$, where A' is like A except for containing a at zero or more places where A contains b, and $\models'$ is extended to all formulas.

***Lemma* 4.7:** Let $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ be a model $a \notin \bigcup_{\Gamma \in \mathscr{G}} \mathscr{P}(\Gamma)$, and let $\langle \mathscr{G}, \mathscr{R}, \models', \mathscr{P}' \rangle$ be $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle_{b=a}$. Then if X is any formula not containing a, and if X' is like X except for containing a at zero or more places where X contains b

$$\Gamma \models X \Leftrightarrow \Gamma \models' X'.$$

Proof: Again an easy induction on the degree of X.

§ 5. Examples

We show that two theorems of classical logic are not intuitionistically valid:

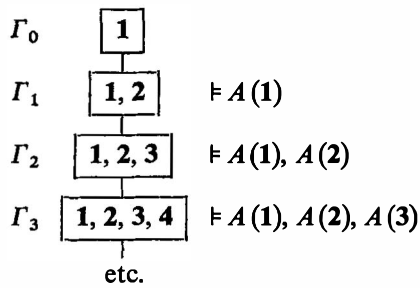
(1). $\vdash_c \sim\sim(\forall x)(A(x) \vee \sim A(x))$,

but the following is an intuitionistic counter-model for it. We take the natural numbers as parameters.

Let

$$\mathscr{G} = \{\Gamma_i \mid i = 0, 1, 2, \ldots\},$$
$$\Gamma_i \mathscr{R} \Gamma_j \quad \text{iff} \quad i \leqslant j$$
$$\mathscr{P}(\Gamma_i) = \{\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{i}, \boldsymbol{i+1}\}$$

$\Gamma_n \models A(i)$ iff $i \leqslant n$ and $\models$ is extended to all formulas. We may give this model schematically by



We claim no $\Gamma_i \models \sim\sim(\forall x)\,(A(x) \vee \sim A(x))$. Suppose instead that

$$\Gamma_i \models \sim\sim(\forall x)\,(A(x) \vee \sim A(x)).$$

Then for some $j \geqslant \mathrm{i}$

$$\Gamma_j \models (\forall x)\,(A(x) \vee \sim A(x)).$$

But $j+1 \in \mathscr{P}(\Gamma_j)$, so

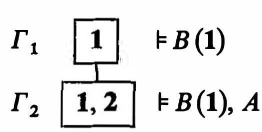
$$\Gamma_j \models A(j+1) \vee \sim A(j+1).$$

But $\Gamma_j \not\models A(j+1)$ since $j+1>j$, and if $\Gamma_j \models \sim A(j+1)$, then since $\Gamma_j \mathscr{R} \Gamma_{j+1}$, $\Gamma_{j+1} \not\models A(j+1)$, a contradiction.

(2). $\vdash_c (\forall x)(A \vee B(x)) \supset (A \vee (\forall x)B(x))$,
but an intuitionistic counter-model is the following, where parameters are again integers:

$$\mathscr{G} = \{\Gamma_1, \Gamma_2\},$$
$$\Gamma_1 \mathscr{R} \Gamma_2,\ \Gamma_1 \mathscr{R} \Gamma_1,\ \Gamma_2 \mathscr{R} \Gamma_2,$$
$$\mathscr{P}(\Gamma_1) = \{1\},\ \mathscr{P}(\Gamma_2) = \{1, 2\},$$
$$\Gamma_1 \models B(1),\ \Gamma_2 \models B(1),\ \Gamma_2 \models A,$$

and $\models$ is extended to all formulas. Schematically, this is



To show this is a counter-model, first we claim

$$\Gamma_1 \models (\forall x)\,(A \vee B(x)).$$

This follows because $\Gamma_1 \Vdash B(\mathbf{1})$. Hence

$$\Gamma_1 \Vdash A \vee B(\mathbf{1})$$

and $\Gamma_2 \Vdash A$, so

$$\Gamma_2 \Vdash A \vee B(\mathbf{1}) \quad \text{and} \quad \Gamma_2 \Vdash A \vee B(\mathbf{2}).$$

But $\Gamma_1 \nVdash A$ and moreover $\Gamma_1 \nVdash (\forall x)\, B(x)$ since $\Gamma_2 \nVdash B(\mathbf{2})$. Thus $\Gamma_1 \nVdash A \vee (\forall x)\, B(x)$.

§ 6. Truth and almost-truth sets

In classical first order logic, a set $\mathscr{S}$ of formulas is sometimes called a *truth set* if

(1). $X \wedge Y \in \mathscr{S} \Leftrightarrow X \in \mathscr{S}$ and $Y \in \mathscr{S}$,
(2). $X \vee Y \in \mathscr{S} \Leftrightarrow X \in \mathscr{S}$ or $Y \in \mathscr{S}$,
(3). $\sim X \in \mathscr{S} \Leftrightarrow X \notin \mathscr{S}$,
(4). $X \supset Y \in \mathscr{S} \Leftrightarrow X \notin \mathscr{S}$ or $Y \in \mathscr{S}$,
(5). $(\exists x)\, X(x) \in \mathscr{S} \Leftrightarrow X(a) \in \mathscr{S}$ for some parameter a,
(6). $(\forall x)\, X(x) \in \mathscr{S} \Leftrightarrow X(a) \in \mathscr{S}$ for every parameter a,

where there is some fixed set of parameters, X and Y are formulas involving only these parameters, and (5) and (6) refer to this set of parameters.

We now call $\mathscr{S}$ an *almost-truth* set if it satisfies (1)–(5) above and

(6a). $(\forall x)\, X(x) \in \mathscr{S} \Rightarrow X(a) \in \mathscr{S}$ for every parameter a.

It is one form of the classical completeness theorem that for any pure (i.e. with no parameters) formula X, X is a classical theorem if and only if X is in every truth set.

We leave the reader to show

***Theorem* 6.1:** If X is pure and contains no occurrence of the universal quantifier, X is in every truth set if and only if X is in every almost-truth set.

§ 7. Complete sequences

The method used in this section was adapted from forcing techniques, and is due to Cohen [3].

Definition 7.1: In the model $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$, we call $\mathscr{C} \subseteq \mathscr{G}$ an *$\mathscr{R}$-chain* if

$$\Gamma, \Delta \in \mathscr{C} \Rightarrow \Gamma \mathscr{R} \Delta \quad \text{or} \quad \Delta \mathscr{R} \Gamma .$$

If $\mathscr{C}$ is an $\mathscr{R}$-chain, by $\mathscr{C}'$ we mean $\{X \mid \text{for some } \Gamma \in \mathscr{C}, \ \Gamma \models X\}$.

If $\mathscr{C}$ is an $\mathscr{R}$-chain, $\mathscr{C}$ is called *complete* if for every formula X with parameters used in $\mathscr{C}'$ $X \vee \sim X \in \mathscr{C}'$.

***Lemma* 7.2:** Let $\mathscr{C}$ be a complete $\mathscr{R}$-chain in the model $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$. Then $\mathscr{C}'$ is an almost-truth set.

Proof: This is a straightforward verification of the cases. We give case (4) as an illustration.

Suppose $(X \supset Y) \in \mathscr{C}'$. Then for some $\Gamma \in \mathscr{C}$ $\Gamma \models X \supset Y$. Now either $X \notin \mathscr{C}'$ or $X \in \mathscr{C}'$. If $X \in \mathscr{C}'$, then for some $\Delta \in \mathscr{C}$ $\Delta \models X$. Let Ω be the $\mathscr{R}$-last of Γ and Δ. Then $\Omega \models X$ and $\Omega \models X \supset Y$, so $\Omega \models Y$ and $Y \in \mathscr{C}'$. Thus $X \notin \mathscr{C}'$ or $Y \in \mathscr{C}'$.

Conversely suppose $(X \supset Y) \notin \mathscr{C}'$. Then $\sim X \notin \mathscr{C}'$, since $\mathscr{C}'$ is closed under modus ponens and contains $\sim X \supset (X \supset Y)$ as is easily shown. But $X \vee \sim X \in \mathscr{C}'$, hence $X \in \mathscr{C}'$. Further $Y \notin \mathscr{C}'$, since again $Y \supset (X \supset Y) \in \mathscr{C}'$.

***Lemma* 7.3:** Let $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ be a model, $\Gamma \in \mathscr{G}$ and $X \in \hat{\mathscr{P}}(\Gamma)$. There is some $\Gamma^* \in \mathscr{G}$ such that $\Gamma^* \models X \vee \sim X$.

Proof: Either some $\Gamma^* \models X$ and we are done, or no $\Gamma^* \models X$ in which case $\Gamma \models \sim X$ and we are done.

***Theorem* 7.4:** Let $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ be a model and $\Gamma \in \mathscr{G}$. Then Γ can be included in some complete $\mathscr{R}$-chain $\mathscr{C}$ such that $\mathscr{C}'$ is an almost-truth set.

Proof: There are only countably many formulas, $X_1, X_2, X_3, \ldots$. We define a countable $\mathscr{R}$-chain $\{\Gamma_0, \Gamma_1, \Gamma_2, \ldots\}$ as follows:

Let Γ_0 be Γ.

Having defined Γ_n, if $X_{n+1} \notin \hat{\mathscr{P}}(\Gamma_n^*)$ for any Γ_n^*, let Γ_{n+1} be Γ_n. If $X_{n+1} \in \hat{\mathscr{P}}(\Gamma_n^*)$ for some Γ_n^*, then Γ_n^*, by lemma 7.3, has an $\mathscr{R}$-successor Γ_n^{**} such that $\Gamma_n^{**} \models X_{n+1} \vee \sim X_{n+1}$. Let Γ_{n+1} be this Γ_n^{**}.

Let $\mathscr{C}$ be $\{\Gamma_0, \Gamma_1, \Gamma_2, \ldots\}$. Clearly $\mathscr{C}$ is complete, and by lemma 7.2 $\mathscr{C}'$ is an almost-truth set.

§ 8. A connection with classical logic

The first theorem of this section is essentially theorem 59(b) of [10 p. 492], but there it is proved prooftheoretically and here semantically.

***Theorem* 8.1:** Let X be a pure formula. If X is in every classical almost-truth set, $\sim\sim X$ is intuitionistically valid.

Proof: Suppose $\sim\sim X$ is not valid. Then there is a model $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ and a $\Gamma \in \mathscr{G}$ such that $\Gamma \not\models \sim\sim X$. Then for some $\Gamma^* \in \mathscr{G}$ $\Gamma^* \models \sim X$. Now Γ^* can, by theorem 7.4, be included in an $\mathscr{R}$-chain $\mathscr{C}$ such that $\mathscr{C}'$ is an almost-truth set. But $\sim X \in \mathscr{C}'$, so that $X \notin \mathscr{C}'$.

***Theorem* 8.2:** If X is intuitionistically valid, then X is classically valid (for X pure).

Proof: As before, if X is not classically valid, there is a truth set $\mathscr{S}$ not containing X. But it is easily shown that if $\mathscr{G} = \{\mathscr{S}\}$, $\mathscr{S}\mathscr{R}\mathscr{S}$, $\mathscr{S} \models Y$ iff $Y \in \mathscr{S}$, and $\mathscr{P}(\mathscr{S})$ is the set of all parameters occurring in $\mathscr{S}$, the resulting $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ is a model in which X is not valid.

***Theorem* 8.3:** If X is a pure formula with no occurrence of the universal quantifier, then X is classically valid if and only if $\sim\sim X$ is intuitionistically valid.

Proof:

$$\begin{aligned} \sim\sim X \text{ intuitionistically valid} &\Rightarrow \sim\sim X \text{ classically valid} \\ &\Rightarrow X \text{ classically valid.} \end{aligned}$$

Conversely

$$\begin{aligned} X \text{ classically valid} &\Rightarrow X \text{ is in every truth set} \\ &\Rightarrow X \text{ is in every almost-truth set} \\ &\Rightarrow \sim\sim X \text{ is intuitionistically valid.} \end{aligned}$$

Remark 8.4: This result will be of fundamental importance in part II.

***Corollary* 8.5:** First order intuitionist logic is undecidable.

Proof: Classical first order logic is undecidable, and every classical formula is classically equivalent to a formula with no universal quantifiers.

Remark 8.6: That theorem 8.3 cannot be extended to all formulas is shown by example (1) in § 5.

CHAPTER 5

FIRST ORDER INTUITIONISTIC LOGIC

PROOF THEORY

§ 1. Beth tableaus

The following is an extension of the system of ch. 2 § 1 to the first order case (see [2]). Everything is as it was there, except that four reduction rules are added to the list. These are

$$T\exists \quad \frac{S,\, T(\exists x)\, X(x)}{S,\, TX(a)} \quad \text{provided } a \text{ is new}$$

$$F\exists \quad \frac{S,\, F(\exists x)\, X(x)}{S,\, FX(a)}$$

$$T\forall \quad \frac{S,\, T(\forall x)\, X(x)}{S,\, TX(a)}$$

$$F\forall \quad \frac{S,\, F(\forall x)\, X(x)}{S_T,\, FX(a)} \quad \text{provided } a \text{ is new}$$

(Note the S_T in rule $F\forall$.) In rules $F\exists$ and $T\forall$, a may be any parameter whatsoever. In rules $T\exists$ and $F\forall$, the parameter a introduced must not occur in any formula of S, or in the formula $X(x)$.

The corresponding classical tableau system is like the above, but in rule $F\forall$ S_T is replaced by S. As in ch. 2 § 1 interpretations differ. Classically the interpretation is as it was in the propositional case. The restrictions on parameters in $T\exists$ and $F\forall$ are for obvious reasons. In the intuitionistic system the difference between $T\exists$ and $F\forall$ may be explained as follows. Suppose we have proved $(\exists x)\, X(x)$. Since (intuitionistically) the only existence proofs are constructive, there must already be an instance $X(a)$ which we have proved. Thus rule $T\exists$. But suppose we have not proved $(\forall x)\, X(x)$. We might have proved all instances so far encountered, but it must be possible (i.e. compatible with our present knowledge) that we will at some time encounter an instance for which we will have no proof. However, this might happen at some time in the future, by which time we may have proved some things we do not now (some $FZ \in S$ might become TZ). Hence the restriction to S_T in rule $F\forall$.

As in the propositional case, we proceed to show correctness and completeness (in two ways) of this system.

The following two examples illustrate proofs in the system:

(1). $\vdash_I (\forall x)\, X(x) \supset \sim(\exists x) \sim X(x)$.

The proof is

$$\begin{aligned}
&\{\{F(\forall x)\, X(x) \supset \sim(\exists x) \sim X(x)\}\},\\
&\{\{T(\forall x)\, X(x),\, F \sim (\exists x) \sim X(x)\}\},\\
&\{\{T(\forall x)\, X(x),\, T(\exists x) \sim X(x)\}\},\\
&\{\{T(\forall x)\, X(x),\, T \sim X(a)\}\},\\
&\{\{TX(a),\, T \sim X(a)\}\},\\
&\{\{TX(a),\, FX(a)\}\}.
\end{aligned}$$

(2). $\vdash_I \sim(\exists x) \sim [X(x) \supset Y(x)] \supset (\forall x)[\sim Y(x) \supset \sim X(x)]$.

The proof is

$$\begin{aligned}
&\{\{F \sim (\exists x) \sim [X(x) \supset Y(x)] \supset (\forall x)\, [\sim Y(x) \supset \sim X(x)]\}\},\\
&\{\{T \sim (\exists x) \sim [X(x) \supset Y(x)],\, F(\forall x)\, [\sim Y(x) \supset \sim X(x)]\}\},\\
&\{\{T \sim (\exists x) \sim [X(x) \supset Y(x)],\, F\, [\sim Y(a) \supset \sim X(a)]\}\},\\
&\{\{T \sim (\exists x) \sim [X(x) \supset Y(x)],\, T \sim Y(a),\, F \sim X(a)\}\},\\
&\{\{T \sim (\exists x) \sim [X(x) \supset Y(x)],\, T \sim Y(a),\, TX(a)\}\},\\
&\{\{F(\exists x) \sim [X(x) \supset Y(x)],\, T \sim Y(a),\, TX(a)\}\},\\
&\{\{F \sim [X(a) \supset Y(a)],\, T \sim Y(a),\, TX(a)\}\},\\
&\{\{T[X(a) \supset Y(a)],\, T \sim Y(a),\, TX(a)\}\},\\
&\{\{FX(a),\, T \sim Y(a),\, TX(a)\},\, \{TY(a),\, T \sim Y(a),\, TX(a)\}\},\\
&\{\{FX(a),\, T \sim Y(a),\, TX(a)\},\, \{TY(a),\, FY(a),\, TX(a)\}\}.
\end{aligned}$$

§ 2. Correctness of Beth tableaus

Definition 2.1: Let $S = \{TX_1, \ldots, TX_n, FY_1, \ldots, FY_m\}$ be a set of signed formulas, $\langle \mathscr{G}, \mathscr{R}, \vDash, \mathscr{P} \rangle$ a model, and $\Gamma \in \mathscr{G}$. We say Γ *realizes* S if $X_i \in \hat{\mathscr{P}}(\Gamma)$, $Y_j \in \hat{\mathscr{P}}(\Gamma)$, and $\Gamma \vDash X_i$, $\Gamma \nvDash Y_j$ $(i=1 \ldots n, j=1 \ldots m)$.

A set S is *realizable* if something realizes it.

A configuration $\mathscr{C}$ is *realizable* if one of its elements is realizable.

***Lemma* 2.2:** Let Q stand for either the sign T or the sign F. If $S, QX(b)$ is realizable and if a is a parameter which does not occur in S or in X (so $a \neq b$) then $S, QX(a)$ is realizable.

Proof: Suppose in the model $\langle \mathscr{G}, \mathscr{R}, \vDash, \mathscr{P} \rangle$ Γ realizes $S, QX(b)$. Choose a new parameter $c \notin \bigcup_{\Gamma \in \mathscr{G}} \mathscr{P}(\Gamma)$ (we can always construct a new parameter). Let $\langle \mathscr{G}, \mathscr{R}, \vDash', \mathscr{P}' \rangle$ be $\langle \mathscr{G}, \mathscr{R}, \vDash, \mathscr{P} \rangle \binom{a}{c}$ (see ch. 4 § 4). Since a does not occur in S or X, by lemma 4.4.5, in this new model Γ realizes $S, QX(b)$. But now $a \notin \bigcup_{\Gamma \in \mathscr{G}} \mathscr{P}'(\Gamma)$, so we may define a third model $\langle \mathscr{G}, \mathscr{R}, \vDash'', \mathscr{P}'' \rangle$ as $\langle \mathscr{G}, \mathscr{R}, \vDash', \mathscr{P}' \rangle_{b=a}$. By lemma 4.4.7 in this third model Γ realizes $S, QX(a)$.

***Lemma* 2.3:** If $S, T(\exists x) X(x)$ is realizable, and if a does not occur in S or $X(x)$, then $S, TX(a)$ is realizable.

Proof: Suppose in the model $\langle \mathscr{G}, \mathscr{R}, \vDash, \mathscr{P} \rangle$ Γ realizes $S, T(\exists x) X(x)$. Then $\Gamma \vDash (\exists x) X(x)$, so for some $b \in \mathscr{P}(\Gamma)$ $\Gamma \vDash X(b)$. Thus Γ realizes $S, TX(b)$. If $a=b$ we are done. If not, by lemma 2.2 we are done.

***Lemma* 2.4:** If $S, F(\exists x) X(x)$ is realizable and if a is any parameter, $S, FX(a)$ is realizable.

Proof: Suppose in the model $\langle \mathscr{G}, \mathscr{R}, \vDash, \mathscr{P} \rangle$ Γ realizes $S, F(\exists x) X(x)$. Then $\Gamma \nvDash (\exists x) X(x)$. If $a \in \mathscr{P}(\Gamma)$, $\Gamma \nvDash X(a)$ and we are done. If $a \notin \mathscr{P}(\Gamma)$, a cannot occur in S or X by the definition of realizability. But $\mathscr{P}(\Gamma) \neq \emptyset$ so there is a $b \in \mathscr{P}(\Gamma)$ with $b \neq a$ and $\Gamma \nvDash X(b)$. Thus $S, FX(b)$ is realizable Now use lemma 2.2.

***Lemma* 2.5:** If $S, T(\forall x) X(x)$ is realizable and if a is any parameter, $S, TX(a)$ is realizable.

Proof: Similar to that of lemma 2.4.

***Lemma* 2.6:** If $S, F(\forall x) X(x)$ is realizable and if a is any parameter which does not occur in S or $X(x)$, then $S_T, FX(a)$ is realizable.

Proof: Suppose in the model $\langle \mathscr{G}, \mathscr{R}, \vDash, \mathscr{P} \rangle$ Γ realizes $S, F(\forall x) X(x)$. Then $\Gamma \nvDash (\forall x) X(x)$. But $X(x) \in \hat{\mathscr{P}}(\Gamma)$, so there is a Γ^* such that $\Gamma^* \nvDash X(b)$ for some $b \in \mathscr{P}(\Gamma^*)$. Of course Γ^* realizes S_T. If $b=a$ we are done. If not, since $S_T, X(b)$ is realizable, by lemma 2.2 we are done.

***Theorem* 2.7:** Let $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$ be a tableau. If $\mathscr{C}_i$ is realizable, so is $\mathscr{C}_{i+1}$.

Proof: We pass from $\mathscr{C}_i$ to $\mathscr{C}_{i+1}$ by the application of some reduction rule. All the propositional rules were dealt with in ch. 2. The four new (first order) rules are handled by lemmas 2.3–2.6.

***Corollary* 2.8:** If X is provable, X is valid.

Proof: Exactly as in the propositional situation.

§ 3. Hintikka collections

This section generalizes the definitions of ch. 2 § 3 to the first order setting. Recall that a finite set of signed formulas is consistent if no tableau for it is closed. We say an infinite set is consistent if every finite subset is.

Let $\mathscr{G}$ be a collection of sets of signed formulas. If $\Gamma \in \mathscr{G}$, by $\mathscr{P}(\Gamma)$ we mean the collection of all parameters occurring in formulas in Γ. If $\Gamma, \Delta \in \mathscr{G}$, by $\Gamma \mathscr{R} \Delta$ we mean $\mathscr{P}(\Gamma) \subseteq \mathscr{P}(\Delta)$ and $\Gamma_T \subseteq \Delta$.

Definition 3.1: We call $\mathscr{G}$ a *(first order) Hintikka collection* if, for any $\Gamma \in \mathscr{G}$, Γ is consistent and

$$
\begin{array}{lll}
TX \wedge Y \in \Gamma & \Rightarrow TX \in \Gamma \text{ and } TY \in \Gamma, \\
FX \vee Y \in \Gamma & \Rightarrow FX \in \Gamma \text{ and } FY \in \Gamma, \\
TX \vee Y \in \Gamma & \Rightarrow TX \in \Gamma \text{ or } TY \in \Gamma, \\
FX \wedge Y \in \Gamma & \Rightarrow FX \in \Gamma \text{ or } FY \in \Gamma, \\
T \sim X \in \Gamma & \Rightarrow FX \in \Gamma, \\
TX \supset Y \in \Gamma & \Rightarrow FX \in \Gamma \text{ or } TY \in \Gamma, \\
F \sim X \in \Gamma & \Rightarrow \text{for some } \Delta \in \mathscr{G}\ \Gamma \mathscr{R} \Delta \text{ and } TX \in \Delta, \\
FX \supset Y \in \Gamma & \Rightarrow \text{for some } \Delta \in \mathscr{G},\ \Gamma \mathscr{R} \Delta \text{ and } TX \in \Delta,\ FY \in \Delta, \\
T(\forall x) X(x) \in \Gamma & \Rightarrow TX(a) \in \Gamma \text{ for all } a \in \mathscr{P}(\Gamma), \\
F(\exists x) X(x) \in \Gamma & \Rightarrow FX(a) \in \Gamma \text{ for all } a \in \mathscr{P}(\Gamma), \\
T(\exists x) X(x) \in \Gamma & \Rightarrow TX(a) \in \Gamma \text{ for some } a \in \mathscr{P}(\Gamma), \\
F(\forall x) X(x) \in \Gamma & \Rightarrow \text{for some } \Delta \in \mathscr{G}\ \Gamma \mathscr{R} \Delta \text{ and} \\
& \quad \text{for some } a \in \mathscr{P}(\Delta)\ TX(a) \in \Delta.
\end{array}
$$

Definition 3.2: If $\mathscr{G}$ is a Hintikka collection, we call $\langle \mathscr{G}, \mathscr{R}, \models,$ *model for* $\mathscr{G}$ if

(1). $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ is a model,

(2). $\mathscr{P}$ and $\mathscr{R}$ are as above,
(3). for all $\Gamma \in \mathscr{G}$ $TX \in \Gamma \Rightarrow \Gamma \vDash X$ and $FX \in \Gamma \Rightarrow \Gamma \nvDash X$.

***Theorem* 3.3:** There is a model for any Hintikka collection.

Proof: Suppose we have a Hintikka collection $\mathscr{G}$. $\mathscr{P}$ and $\mathscr{R}$ are as defined above. If A is atomic, let $\Gamma \vDash A$ if $TA \in \Gamma$, and extend $\vDash$ to all formulas. The result $\langle \mathscr{G}, \mathscr{R}, \vDash, \mathscr{P} \rangle$ is a model. We claim it is a model for $\mathscr{G}$. We show property (3) by induction on the degree of X.

The propositional cases were done in ch. 2 § 3. Of the four new cases we only do two as an illustration.

Suppose the result known for all subformulas of the formula in question. Then

$$\begin{aligned} T(\forall x)\, X(x) \in \Gamma &\Rightarrow (\forall \Delta \in \mathscr{G})\,(\Gamma \mathscr{R} \Delta \Rightarrow T(\forall x)\, X(x) \in \Delta) \\ &\quad \text{(since } \Gamma_T \subseteq \Delta \text{ if } \Gamma \mathscr{R} \Delta) \\ &\Rightarrow (\forall \Delta \in \mathscr{G})\,(\Gamma \mathscr{R} \Delta \Rightarrow ((\forall a \in \mathscr{P}(\Delta))\, TX(a) \in \Delta)) \\ &\Rightarrow (\forall \Delta \in \mathscr{G})\,(\Gamma \mathscr{R} \Delta \Rightarrow ((\forall a \in \mathscr{P}(\Delta))\, \Delta \vDash X(a))) \\ &\Rightarrow \Gamma \vDash (\forall x)\, X(x) \end{aligned}$$

Conversely

$$\begin{aligned} F(\forall x)\, X(x) \in \Gamma &\Rightarrow (\exists \Delta \in \mathscr{G})\,(\Gamma \mathscr{R} \Delta \text{ and } (\exists a \in \mathscr{P}(\Delta))\,(FX(a) \in \Delta)) \\ &\Rightarrow (\exists \Delta \in \mathscr{G})\,(\Gamma \mathscr{R} \Delta \text{ and } (\exists a \in \mathscr{P}(\Delta))\,(\Delta \nvDash X(a))) \\ &\Rightarrow \Gamma \nvDash (\forall x)\, X(x). \end{aligned}$$

Thus, as in the propositional case, to establish the completeness of Beth tableaus we need only show that if X is not provable, there is a Hintikka collection $\mathscr{G}$ and a $\Gamma \in \mathscr{G}$ such that $FX \in \Gamma$.

§ 4. Hintikka elements

Definition 4.1: Let Γ be a set of signed formulas and $\boldsymbol{P}$ a set of parameters. We call Γ a *Hintikka element with respect to* $\boldsymbol{P}$ if Γ is consistent and

$$\begin{array}{lllll} TX \wedge Y \in \Gamma & \Rightarrow TX \in \Gamma & \text{and} & TY \in \Gamma, \\ FX \vee Y \in \Gamma & \Rightarrow FX \in \Gamma & \text{and} & FY \in \Gamma, \\ TX \vee Y \in \Gamma & \Rightarrow TX \in \Gamma & \text{or} & TY \in \Gamma, \\ FX \wedge Y \in \Gamma & \Rightarrow FX \in \Gamma & \text{or} & FY \in \Gamma, \\ T \sim X \in \Gamma & \Rightarrow FX \in \Gamma, & & \\ TX \supset Y \in \Gamma & \Rightarrow FX \in \Gamma & \text{or} & TY \in \Gamma, \end{array}$$

$$T(\forall x)\,X(x) \in \Gamma \Rightarrow TX(a) \in \Gamma \quad \text{for each} \quad a \in \boldsymbol{P},$$
$$F(\exists x)\,X(x) \in \Gamma \Rightarrow FX(a) \in \Gamma \quad \text{for each} \quad a \in \boldsymbol{P},$$
$$T(\exists x)\,X(x) \in \Gamma \Rightarrow TX(a) \in \Gamma \quad \text{for some} \quad a \in \boldsymbol{P}.$$

***Theorem* 4.2:** Let Γ be an at most countable, consistent set of signed formulas. Let $\boldsymbol{S}$ be the set of all parameters occurring in formulas in Γ. Let $a_1, a_2, a_3, \ldots$ be a countable list of parameters not in $\boldsymbol{S}$. Let $\boldsymbol{P} = \boldsymbol{S} \cup \{a_1, a_2, a_3, \ldots\}$. Then Γ can be extended to a Hintikka element with respect to $\boldsymbol{P}$.

Proof: Order the (countable) set of all subformulas of formulas in Γ, using only parameters of $\boldsymbol{P}$: $X_1, X_2, X_3, \ldots$. We define a (double) sequence of sets of signed formulas:

Let $\Gamma_0 = \Gamma$. Suppose we have defined Γ_n which is a consistent extension of Γ_0, using only finitely many of $a_1, a_2, a_3, \ldots$. Let $\Delta_n^1 = \Gamma_n$. We define $\Delta_n^2, \ldots, \Delta_n^{n+1}$ and let $\Gamma_{n+1} = \Delta_n^{n+1}$. We do this as follows:

Suppose we have defined Δ_n^k for some k $(1 \leqslant k \leqslant n)$. Consider the formula X_k. At most one of TX_k, FX_k can be in Δ_n^k (since it is consistent). If neither is, let $\Delta_n^{k+1} = \Delta_n^k$. If one is in Δ_n^k, we have several cases.

Case (1a). X_k is $Y \vee Z$ and $TX_k \in \Delta_n^k$. Then one of Δ_n^k, TY or Δ_n^k, TZ is consistent. Let Δ_n^{k+1} be Δ_n^k, TY if consistent, and Δ_n^k, TZ otherwise.

Case (1b). X_k is $Y \vee Z$ and $FX_k \in \Delta_n^k$. Then Δ_n^k, FY, FZ is consistent. Let this be Δ_n^{k+1}.

The cases

(2a). $TX \wedge Y$,

(2b). $FX \wedge Y$,

(3). $T{\sim}X$,

(4). $TX \supset Y$,

are all treated in a similar manner.

Case (5a). X_k is $(\exists x)\,X(x)$ and $TX_k \in \Delta_n^k$. Since Δ_n^k uses only finitely many of $a_1, a_2, a_3, \ldots$, let a_i be the first one unused. Let Δ_n^{k+1} be $\Delta_n^k, TX(a_i)$. Since a_i is new, this must also be consistent.

Case (5b). X_k is $(\exists x)\,X(x)$ and $FX_k \in \Delta_n^k$. Let Δ_n^{k+1} be Δ_n^k together with $FX(\alpha)$ for each $\alpha \in \boldsymbol{S}$, and each $\alpha = a_i$ which has been used so far. Then Δ_n^{k+1} is again consistent.

Case (6). $T(\forall x)\,X(x)$, is treated as we did case (5b).

Case (7). If the signed formula does not come under one of the above cases let $\Delta_n^{k+1} = \Delta_n^k$.

Thus we have defined a sequence $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$. Let $\Pi = \bigcup \Gamma_n$. We claim Π is a Hintikka collection with respect to $\boldsymbol{P}$. The verification of the properties is straightforward.

§ 5. Completeness of Beth tableaus

Supposing X to be not provable, we give a procedure for constructing a sequence of Hintikka elements.

First we order our countable collection of parameters as follows:

$$\begin{array}{llll} \boldsymbol{S}_1: & a_1^1, & a_2^1, & a_3^1, \ \ldots \\ \boldsymbol{S}_2: & a_1^2, & a_2^2, & a_3^2, \ \ldots \\ \boldsymbol{S}_3: & a_1^3, & a_2^3, & a_3^3, \ \ldots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

where we have placed all the parameters of X in $\boldsymbol{S}_1$, and let $\boldsymbol{P}_n = \boldsymbol{S}_1 \cup \boldsymbol{S}_2 \cup \ldots \cup \boldsymbol{S}_n$.

For this section only, by an F-formula we mean a signed formula of the form $F{\sim}X$, $FX \supset Y$ or $F(\forall x)\, X$. We many assume once and for all an ordering of all formulas. Now we proceed:

Step (0). X is not provable, so $\{FX\}$ is consistent. Extend it to a Hintikka element with respect to $\boldsymbol{P}_1$. Call the result Γ_1.

Step (1). Take the first F-formula of Γ_1. If this is $F{\sim}X$, consider Γ_{1T}, TX. This is consistent. Extend is to a Hintikka element with respect to $\boldsymbol{P}_2$, call it Γ_2. If the first F-formula is $FX \supset Y$, extend Γ_{1T}, TX, FY to a Hintikka element with respect to $\boldsymbol{P}_2, \Gamma_2$. If the first F-formula is $F(\forall x)\, X(x)$, extend $\Gamma_{1T}, FX(a_1^2)$ to a Hintikka element with respect to $\boldsymbol{P}_2, \Gamma_2$. In any event Γ_2 is a consistent Hintikka element with respect to $\boldsymbol{P}_2$. Now call the first F-element of Γ_1 "used". The result of step (1) is $\{\Gamma_1, \Gamma_2\}$.

Suppose at the end of step (n) we have the sequence $\{\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_{2^n}\}$ where each Γ_i is a Hintikka element with respect to $\boldsymbol{P}_i$.

Step ($n+1$). Take the first "unused" F-formula of Γ_1, proceed as in step (1) depending on whether the formula is $F{\sim}X$, $FX \supset Y$ or $F(\forall x)\, X$. Produce from Γ_{1T}, TX or Γ_{1T}, TX, FY or $\Gamma_{1T}, FX(a_1^{2^n+1})$ a Hintikka element with respect to $\boldsymbol{P}_{2^n+1}$, call it Γ_{2^n+1}, and call the formula in question "used". Repeat the same procedure with the first "unused" F-formula of Γ_2, producing a Hintikka element with respect to $\boldsymbol{P}_{2^n+2}$, call it Γ_{2^n+2}. Continue to Γ_{2^n}, producing a Hintikka element with respect to $\boldsymbol{P}_{2^n+1}$, call it Γ_{2^n+1}. The result of the $n+1$st step is thus

$$\{\Gamma_1, \Gamma_2, \ldots, \Gamma_{2^{n+1}}\}.$$

Let $\mathscr{G}$ be the collection of all Γ_n generated in the above process. We claim $\mathscr{G}$ is a Hintikka collection.

Each $\Gamma_n \in \mathscr{G}$ is a Hintikka element with respect to $\boldsymbol{P}_n$, so $\mathscr{P}(\Gamma_n)$ is $\boldsymbol{P}_n$. Since Γ_n is a Hintikka element with respect to $\mathscr{P}(\Gamma_n)$, to show $\mathscr{G}$ is a Hintikka collection we have only three properties to show.

Suppose for some $\Gamma_n \in \mathscr{G}$, $F(\forall x)\, X(x) \in \Gamma_n$. By the above construction there must be some $\Gamma_k \in \mathscr{G}$ such that $\Gamma_{nT} \subseteq \Gamma_k$, $\mathscr{P}(\Gamma_n) \subseteq \mathscr{P}(\Gamma_k)$ and $FX(a) \in \Gamma_k$ for some parameter a. Thus $(\exists \Gamma_k \in \mathscr{G})\Gamma_n \mathscr{R} \Gamma_k$ and $FX(a) \in \Gamma_k$ for some $a \in \mathscr{P}(\Gamma_k)$.

The cases $F\sim$ and $F\supset$ are similar.

Thus $\mathscr{G}$ is a Hintikka collection and $FX \in \Gamma_1 \in \mathscr{G}$, so our completeness theorem is established. We note that in the Hintikka collection $\mathscr{G}$ resulting, every formula is a subformula of X. We remark also that the construction of § 4 and of this section could be combined into a single sequence of steps.

This proof is a modification of the original proof of Kripke [13].

§ 6. Second completeness proof for Beth tableaus

The following is a Henkin type proof and serves as a transition to the completeness of the axiom system presented in the next few sections. A proof along the same lines but using unsigned formulas was discovered independently by Thomason [21] and by Aczel [1]. The similarity to the algebraic work of ch. 1 § 6 is also noted.

Recall that a finite set of signed formulas Γ is consistent if no tableau for it is closed. An infinite set is consistent if every finite subset is.

Definition 6.1.: Let $\boldsymbol{P}$ be a set of parameters and Γ a set of signed formulas. We call Γ *maximal consistent with respect to* $\boldsymbol{P}$ if

(1). every signed formula in Γ uses only parameters of $\boldsymbol{P}$,

(2). Γ is consistent,

(3). for every formula X with all its parameters from $\boldsymbol{P}$, either $TX \in \Gamma$ or $FX \in \Gamma$ or both Γ, TX and Γ, FX are inconsistent.

***Lemma* 6.2:** Let Γ be a consistent set of signed formulas, and $\boldsymbol{P}$ be a non-empty set of parameters containing at least every parameter used in Γ. Then Γ can be extended to a set Δ which is maximal consistent with respect to $\boldsymbol{P}$.

Proof: $\boldsymbol{P}$ is countable, so we may enumerate all formulas with parameters from $\boldsymbol{P}$: $X_1, X_2, X_3, \ldots$

Let $\Delta_0 = \Gamma$. Having defined Δ_n, consider X_{n+1}. If Δ_n, TX_{n+1} is consistent, let it be Δ_{n+1}. If not, but if Δ_n, FX_{n+1} is consistent, let it be Δ_{n+1}. If neither holds, let Δ_{n+1} be Δ_n.

Let $\Delta = \bigcup \Delta_n$. The conclusion of the lemma is now obvious.

Definition 6.3: Let Γ be a set of signed formulas and $\boldsymbol{P}$ a set of parameters. We call Γ *good with respect to* $\boldsymbol{P}$ if

(1). Γ is maximal consistent with respect to $\boldsymbol{P}$,

(2). $T(\exists x)\, X(x) \in \Gamma \Rightarrow TX(a) \in \Gamma$ for some $a \in \boldsymbol{P}$.

***Lemma* 6.4:** Let Γ be a consistent set of signed formulas, and $\boldsymbol{S}$ be the set of parameters occurring in Γ. Let $\{a_1, a_2, a_3, \ldots\}$ be a countable set of distinct parameters not in $\boldsymbol{S}$, and let $\boldsymbol{P} = \boldsymbol{S} \cup \{a_1, a_2, a_3, \ldots\}$. Then Γ can be extended to a set Δ which is good with respect to $\boldsymbol{P}$.

Proof: $\boldsymbol{P}$ is countable, order the set of formulas with parameters from $\boldsymbol{P}$: $X_1, X_2, X_3, \cdots$. We proceed as follows:

(1). Let $\Delta_0 = \Gamma$.

(2). Extend Δ_0 to a set Δ_1 maximal consistent with respect to $\boldsymbol{S}$.

(3). Take the first X_i (in the above ordering) of the form $T(\exists x)\, X(x)$ such that $T(\exists x)\, X(x) \in \Delta_1$ but for no $\alpha \in \boldsymbol{S}$ is $TX(\alpha) \in \Delta_1$. Let $\Delta_2 = \Delta_1, TX(a_1)$. Since a_1 is "new", Δ_2 is consistent.

(4). Extend Δ_2 to a set Δ_3 maximal consistent with respect to $\boldsymbol{S} \cup \{a_1\}$.

(5). Take the first X_i of the form $T(\exists x)\, X(x)$ such that $T(\exists x)\, X(x) \in \Delta_3$ but for no $\alpha \in \boldsymbol{S} \cup \{a_1\}$ is $TX(\alpha) \in \Delta_3$. Let $\Delta_4 = \Delta_3, TX(a_2)$. Again Δ_4 is consistent.

(6). Extend Δ_4 to a set Δ_5 maximal consistent with respect to $\boldsymbol{S} \cup \{a_1, a_2\}$

And so on.

Let $\Delta = \bigcup \Delta_n$. We claim Δ is good with respect to $\boldsymbol{P}$.

First Δ is consistent since each Δ_n is consistent.

If X has all its parameters in $\boldsymbol{P}$, then for some n all the parameters of X are in $\boldsymbol{S} \cup \{a_1, a_2, \ldots, a_n\}$. But in step $(2n)$ we extend Δ_{2n} to Δ_{2n+1}, a set maximal consistent with respect to $\boldsymbol{S} \cup \{a_1, a_2, \ldots, a_n\}$. Thus TX or

FX is in Δ_{2n+1} and hence in Δ, or neither can be added consistently. Thus Δ is maximal consistent with respect to $\boldsymbol{P}$.

Finally suppose $T(\exists x)\, X(x) \in \Delta$. We note that the formula dealt with in step (5) is different from the one dealt with in step (3), and the one dealt with in step (7) is different again. Thus we must eventually reach $T(\exists x)\, X(x)$, and so for some $\alpha \in \boldsymbol{P}$ $TX(\alpha) \in \Delta$. Hence Δ is good with respect to $\boldsymbol{P}$.

Now let us order our countably many parameters as follows:

$$\begin{array}{llll} \boldsymbol{S}_1: & a_1^1, & a_2^1, & a_3^1, \ \dots \\ \boldsymbol{S}_2: & a_1^2, & a_2^2, & a_3^2, \ \dots \\ \boldsymbol{S}_3: & a_1^3, & a_2^3, & a_3^3, \ \dots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

and let $\boldsymbol{P}_n = \boldsymbol{S}_1 \cup \boldsymbol{S}_2 \cup \dots \cup \boldsymbol{S}_n$. Let $\mathscr{G}$ be the collection of all sets of signed formulas which are good with respect to some $\boldsymbol{P}_n$. We claim $\mathscr{G}$ is a Hintikka collection.

Suppose $\Gamma \in \mathscr{G}$. Then Γ is good with respect to some $\boldsymbol{P}_i$, say $\boldsymbol{P}_n$. Then $\mathscr{P}(\Gamma)$ (the collection of all parameters of Γ) is $\boldsymbol{P}_n$.

Suppose $TX \wedge Y \in \Gamma$ but $TX \notin \Gamma$. If $\Gamma, TX \wedge Y$ is consistent, so is $\Gamma, TX \wedge Y, TX$, and so Γ is not maximal. Thus $TX \in \Gamma$. Similarly $TY \in \Gamma$. Hence

$$TX \wedge Y \in \Gamma \Rightarrow TX \in \Gamma \quad \text{and} \quad TY \in \Gamma .$$

Similarly we may show

$$\begin{array}{lll} FX \vee Y \in \Gamma & \Rightarrow & FX \in \Gamma \ \text{ and } \ FY \in \Gamma, \\ TX \vee Y \in \Gamma & \Rightarrow & TX \in \Gamma \ \text{ or } \ TY \in \Gamma, \\ FX \wedge Y \in \Gamma & \Rightarrow & FX \in \Gamma \ \text{ or } \ FY \in \Gamma, \\ T \sim X \in \Gamma & \Rightarrow & FX \in \Gamma, \\ TX \supset Y \in \Gamma & \Rightarrow & FX \in \Gamma \ \text{ or } \ TY \in \Gamma, \\ T(\forall x)\, X(x) \in \Gamma & \Rightarrow & TX(a) \in \Gamma \ \text{ for every } \ a \in \mathscr{P}(\Gamma), \\ F(\exists x)\, X(x) \in \Gamma & \Rightarrow & FX(a) \in \Gamma \ \text{ for every } \ a \in \mathscr{P}(\Gamma). \end{array}$$

Moreover

$$T(\exists x)\, X(x) \in \Gamma \Rightarrow TX(a) \in \Gamma \ \text{ for some } \ a \in \mathscr{P}(\Gamma),$$

since Γ is good with respect to $\boldsymbol{P}_n$.

Suppose $F{\sim}X\in\Gamma$. Since Γ is consistent, Γ_T, TX is consistent. Extend it to a set Δ which is good with respect to $\boldsymbol{P}_{n+1}$. Then $\mathscr{P}(\Gamma)\subseteq\mathscr{P}(\Delta)$ and $\Gamma_T\subseteq\Delta$, so $\Gamma\mathscr{R}\Delta$ and $TX\in\Delta$.

Similarly, if $FX\supset Y\in\Gamma$, there is a $\Delta\in\mathscr{G}$ such that $\Gamma\mathscr{R}\Delta$, $TX\in\Delta$ and $FY\in\Delta$.

Finally, if $F(\forall x)\,X(x)\in\Gamma$, since a_1^{n+1} does not occur in Γ, $\Gamma_T, FX(a_1^{n+1})$ is consistent. Extend it to a set Δ which is good with respect to $\boldsymbol{P}_{n+1}$. Again $\Gamma\mathscr{R}\Delta$ and $FX(a_1^{n+1})\in\Delta$ for $a_1^{n+1}\in\mathscr{P}(\Delta)$.

Thus $\mathscr{G}$ is a Hintikka collection.

To complete the proof, suppose X is not provable. Then $\{FX\}$ is consistent. Since it has only finitely many parameters, they must all lie in some $\boldsymbol{P}_n$. Extend $\{FX\}$ to a set Γ good with respect to $\boldsymbol{P}_n$. Then $\Gamma\in\mathscr{G}$ and $FX\in\Gamma$. This establishes completeness.

Remark 6.5: The model resulting from this Hintikka collection is a "universal" model in that it is a counter-model for every non-theorem. This is not the case for the model of § 5.

We will show later that, in a sense, this Hintikka collection is the analog of a classical truth set.

§ 7. An axiom system, $\mathscr{A}_1$

The following system was chosen to give a fairly quick completeness proof. It is very close to the system of [10] p. 82.

Axiom schemas:

1. $X\supset(Y\supset X)$,
2. $(X\supset Y)\supset((X\supset(Y\supset Z))\supset(X\supset Z))$,
3. $((X\supset Z)\wedge(Y\supset Z))\supset((X\vee Y)\supset Z)$,
4. $(X\wedge Y)\supset X$,
5. $(X\wedge Y)\supset Y$,
6. $X\supset(Y\supset(X\wedge Y))$,
7. $X\supset(X\vee Y)$,
8. $Y\supset(X\vee Y)$,
9. $(X\wedge{\sim}X)\supset Y$,
10. $(X\supset{\sim}X)\supset{\sim}X$,
11. $X(a)\supset(\exists x)\,X(x)$,
12. $(\forall x)\,X(x)\supset X(a)$.

Rules:

13. $\dfrac{X(a) \supset Y}{(\exists x)\, X(x) \supset Y}$,

14. $\dfrac{Y \supset X(a)}{Y \supset (\forall x)\, X(x)}$,

15. $\dfrac{X, \ X \supset Y}{Y}$.

In rules 13 and 14 the parameter a must not occur in Y. In a deduction from premises the parameter a must not occur in the premises either. We use the usual notation, if X can be deduced from a finite subset of S, we write $S \vdash X$. We use $\vdash X$ for $\emptyset \vdash X$.

In the next three sections we establish the correctness and completeness of $\mathscr{A}_1$. We introduce a second system $\mathscr{A}_2$, equivalent to $\mathscr{A}_1$, to aid in showing correctness. For use in showing completeness we need the following three lemmas:

***Lemma* 7.1:** The deduction theorem holds for $\mathscr{A}_1$.

Proof: The standard one (e.g. [10] §§ 21, 22).

***Lemma* 7.2:** $\dfrac{\vdash (W \wedge Y) \supset X, \quad \vdash (W \wedge Z) \supset X, \quad \vdash W \supset (Y \vee Z)}{\vdash W \supset X}$

Proof:

(1).	$(W \wedge Y) \supset X$	by hypothesis, theorem,
(2).	$(W \wedge Z) \supset X$	by hypothesis, theorem,
(3).	$W \supset (Y \vee Z)$	by hypothesis, theorem,
(4).	W	premise,
(5).	$Y \vee Z$	by (3), (4), rule 15,
(6).	$W \supset (Y \supset (W \wedge Y))$	axiom 6,
(7).	$Y \supset (W \wedge Y)$	by (4), (6), rule 15,
(8).	$W \supset (Z \supset (W \wedge Z))$	axiom 6,
(9).	$Z \supset (W \wedge Z)$	by (4), (8), rule 15,
(10).	$Y \supset X$	via (1), (7),
(11).	$Z \supset X$	via (2), (9),
(12).	$(Y \vee Z) \supset X$	via (10, (11), axiom 3,
(13).	X	by (5), (12), rule 15,
(14).	$W \supset X$	deduction theorem cancelling premise (4).

***Lemma* 7.3:** If a does not occur in W, $Y(x)$ or X,

$$\frac{\vdash (W \wedge Y(a)) \supset X, \quad \vdash W \supset (\exists x)\, Y(x)}{\vdash W \supset X}$$

Proof:

(1).	$(W \wedge Y(a)) \supset X$	by hypothesis, theorems,
(2).	$W \supset (\exists x)\, Y(x)$	by hypothesis, theorems,
(3).	W	premise,
(4).	$(\exists x)\, Y(x)$	by (2), (3), rule 15,
(5).	$W \supset (Y(a) \supset (W \wedge Y(a)))$	axiom 6,
(6).	$Y(a) \supset (W \wedge Y(a))$	by (3), (5), rule 15,
(7).	$Y(a) \supset X$	via (1), (6),
(8).	$(\exists x)\, Y(x) \supset X$	by (7), rule 13,
(9).	X	by (4), (8), rule 15,
(10).	$W \supset X$	deduction theorem cancelling premise (3).

§ 8. A second axiom system, $\mathscr{A}_2$

We introduce a second, very similar, axiom system, and prove equivalence.

$\mathscr{A}_2$ has the same axioms as $\mathscr{A}_1$, as well as rules 13 and 14. It does not have rule 15. Instead it has rules

14a. $$\frac{X(a)}{(\forall x)\, X(x)}$$

15a. $$\frac{(\forall x_1) \dots (\forall x_n)\, X, \quad (\exists x_1) \dots (\exists x_n)\, X \supset Y}{Y}$$

provided all parameters of $(\forall x_1) \dots (\forall x_n)\, X$ are also in Y (n may be 0).

To show the two systems are equivalent, it suffices to show 14a and 15a are derived rules of $\mathscr{A}_1$, and 15 is a derived rule of $\mathscr{A}_2$.

To show 14a is a derived rule of $\mathscr{A}_1$, suppose in $\mathscr{A}_1$ we have $X(a)$. Let T be any theorem of $\mathscr{A}_1$ with no parameters. By axiom 1, $X(a) \supset (T \supset X(a))$, so by rule 15, $T \supset X(a)$. Since a is not in T, by rule 14, $T \supset (\forall x)\, X(x)$. But also T, so by rule 15, $(\forall x)\, X(x)$.

To show 15a is a derived rule of $\mathscr{A}_1$, suppose in $\mathscr{A}_1$ we have $(\forall x_1) \dots (\forall x_n)\, X(x_1, \dots, x_n)$ and $(\exists x_1) \dots (\exists x_n)\, X(x_1, \dots, x_n) \supset Y$, and all parameters of $(\forall x_1)\ldots(\forall x_n)\,X(x_1,\ldots,x_n)$ are in Y. From $(\forall x_1)\ldots(\forall x_n)$ $X(x_1,\ldots,x_n)$, by axiom 12, $X(a_1,\ldots,a_n)$. From axiom 11, $X(a_1,\ldots a_n)$ $\supset(\exists x_1)\ldots(\exists x_n)\,X(x_1,\ldots,x_n)$, so by rule 15, $(\exists x_1)\ldots(\exists x_n)\,X(x_1,\ldots,x_n)$ and by rule 15 again, Y.

Finally to show rule 15 is a derived rule of $\mathscr{A}_2$, suppose we have X and $X\supset Y$ in $\mathscr{A}_2$. Let $a_1, a_2\ldots, a_n$ be those parameters of X not in Y. Since we have $X(a_1,\ldots,a_n)$, by rule 14a, $(\forall x_1)\ldots(\forall x_n)\,X(x_1,\ldots,x_n)$. Similarly, since $X(a_1,\ldots,a_n)\supset Y$ and $a_1,\ldots,a_n$ do not occur in Y, by rule 13, $(\exists x_1)\ldots(\exists x_n)\,X(x_1,\ldots,x_n)\supset Y$. Now by rule 16a, Y.

Thus $\mathscr{A}_1$ and $\mathscr{A}_2$ are equivalent. For use in the next section we state the straightforward

***Lemma* 8.1:** If in $\mathscr{A}_2$ we can prove $X(a)$, there is a proof of the same length of $X(b)$ for any parameter b. (note: a does not occur in $X(b)=X(a)\binom{a}{b}$).

§ 9. Correctness of the system $\mathscr{A}_2$

***Theorem* 9.1:** If X is provable in $\mathscr{A}_2$, X is valid.

Proof: By induction on the length of the proof for X. If the proof is of length 1, X is an axiom and we leave the reader to show validity of the axioms.

Suppose the result is known for all formulas with proofs of length less than n steps, and X is provable in n steps. We investigate the steps involved in the proof of X. Axioms have been treated.

Suppose $X(a)\supset Y$ in rule 13 is provable in less than n steps where a is not in Y. Then $X(a)\supset Y$ is valid. Then $(\exists x)\,X(x)\supset Y$ is provable. We wish to show it is valid. Take any model $\langle\mathscr{G},\mathscr{R},\models,\mathscr{P}\rangle$ and any $\Gamma\in\mathscr{G}$ and suppose $((\exists x)\,X(x)\supset Y)\in\hat{\mathscr{P}}(\Gamma)$. Suppose $\Gamma^*\models(\exists x)\,X(x)$. Then $\Gamma^*\models X(b)$ for some b. But $X(a)\supset Y$ is provable, so by lemma 8.1 $(X(a)\supset Y)\binom{a}{b}$ is provable with a proof of the same length, hence by hypothesis, valid. Since a is not in Y, this is $X(b)\supset Y$. By validity, $\Gamma^*\models X(b)\supset Y$, hence $\Gamma^*\models Y$. Thus $\Gamma\models(\exists x)\,X(x)\supset Y$.

Rules 14 and 14a are similar.

Rule 15a: Suppose $(\forall x_1)\ldots(\forall x_n)\,X$ and $(\exists x_1)\ldots(\exists x_n)\,X\supset Y$ are both provable and valid. Then Y is provable. We wish to show Y is valid. Let $\langle\mathscr{G},\mathscr{R},\models,\mathscr{P}\rangle$ be any model and $\Gamma\in\mathscr{G}$. Suppose $Y\in\hat{\mathscr{P}}(\Gamma)$. Then

$(\forall x_1)\ldots(\forall x_n)\, X$ and $(\exists x_1)\ldots(\exists x_n)\, X \supset Y$ are both in $\hat{\mathscr{P}}(\Gamma)$, and since they are valid, $\Gamma \models (\forall x_1)\ldots(\forall x_n)\, X$ and $\Gamma \models (\exists x_1)\ldots(\exists x_n)\, X \supset Y$. By the latter, either $\Gamma \not\models (\exists x_1)\ldots(\exists x_n)\, X$ or $\Gamma \models Y$. If $\Gamma \not\models (\exists x_1)\ldots(\exists x_n)\, X$, for any $a_1, \ldots, a_n \in \mathscr{P}(\Gamma)$, $\Gamma \not\models X(a_1, \ldots, a_n)$, contradicting $\Gamma \models (\forall x_1)\ldots(\forall x_n)\, X$. Hence $\Gamma \models Y$.

§ 10. Completeness of the system $\mathscr{A}_1$

The following Henkin type proof was discovered independently by Thomason [21], Aczel [1], and the author.

We work in the system $\mathscr{A}_1$. Let Γ be a set of *unsigned* formulas and $\boldsymbol{P}$ a collection of parameters. Suppose all the parameters of Γ are among those in $\boldsymbol{P}$.

Definition 10.1: By the *deductive completion* of Γ with respect to $\boldsymbol{P}$ we mean the smallest set of formulas Δ involving only parameters of $\boldsymbol{P}$, such that for any X over $\boldsymbol{P}$

$$\Gamma \vdash X \Rightarrow X \in \Delta .$$

We call Γ *deductively complete with respect to* $\boldsymbol{P}$ if it is its own deductive completion with respect to $\boldsymbol{P}$.

We say Γ has the *Or-property* if

$$X \vee Y \in \Gamma \Rightarrow X \in \Gamma \quad \text{or} \quad Y \in \Gamma .$$

We say Γ has the $\exists$*-property* if, for some parameter a,

$$(\exists x)\, X(x) \in \Gamma \Rightarrow X(a) \in \Gamma .$$

We call Γ *nice with respect to* $\boldsymbol{P}$ if

(1). Γ is deductively complete with respect to $\boldsymbol{P}$,

(2). Γ has the *Or*-property,

(3). Γ has the $\exists$-property,

(4). Γ is consistent.

Remark 10.2: Consistency here has its usual meaning.

***Lemma* 10.3:** Let Γ be a set of formulas and X a single formula. Let $\boldsymbol{P}$ be the set of all parameters of Γ or X. Let $\{a_1, a_2, a_3, \ldots\}$ be a countable collection of distinct parameters not in $\boldsymbol{P}$, and let $\boldsymbol{Q}=\boldsymbol{P}\cup\{a_1, a_2, a_3 \dots\}$. If $\Gamma \nvdash X$, then Γ can be extended to a set Δ which is nice with respect to $\boldsymbol{Q}$ such that $X \notin \Delta$.

Proof: Let $Z_1, Z_2, Z_3 \dots$ be an enumeration of all formulas with parameters from $\boldsymbol{Q}$ of the form $Y \vee Z$ or $(\exists x)\, Y(x)$.

Since $\Gamma \nvdash X$, Γ is consistent. We define a sequence $\{\Gamma_n\}$ as follows:

Let Γ_0 be the deductive completion of Γ with respect to $\boldsymbol{P}$. Then Γ_0 is consistent and $\Gamma_0 \nvdash X$. Suppose we have defined Γ_n so that Γ_n is deductively complete with respect to $\boldsymbol{P}\cup\{a_1, a_2, \dots, a_n\}$ and $\Gamma_n \nvdash X$. Let $\Delta_n^0 = \Gamma_n$.

Suppose we have defined Δ_n^j $(j<n)$ so that it is consistent and $\Delta_n^j \nvdash X$. Let $\Delta_n^{j+1} = \Delta_n^j$ if (1) $Z_j \notin \Delta_n^j$, or (2a) $Z_j \in \Delta_n^j$, $Z_j = Y \vee Z$ and $Y \in \Delta_n^j$ or $Z \in \Delta_n^j$, or (2b) $Z_j \in \Delta_n^j$, $Z_j = (\exists x) Y(x)$ and $Y(a) \in \Delta_n^j$ for some a.

This leaves the two key cases:

(3). Suppose $Z_j \in \Delta_n^j$ and Z_j is $Y \vee Z$ but $Y \notin \Delta_n^j$, $Z \notin \Delta_n^j$. We claim we can add one of Y or Z to Δ_n^j so that the result still does not yield X. For otherwise

$$\Delta_n^j, Y \vdash X$$
$$\Delta_n^j, Z \vdash X$$
$$\Delta_n^j \vdash Y \vee Z$$

(since $Y \vee Z \in \Delta_n^j$). But then by lemma 7.2 $\Delta_n^j \vdash X$, a contradiction. So add to Δ_n^j one of Y or Z so that the result does not yield X. Call the result Δ_n^{j+1}.

(4). Suppose $Z_j \in \Delta_n^j$ and Z_j is $(\exists x)\, Y(x)$, but $Y(a) \notin \Delta_n^j$ for any a. Take the first unused a_i of $\{a_1, a_2, \dots\}$. We claim we can add $Y(a_i)$ to Δ_n^j and the result will not yield X. This is as above but by lemma 7.3. Thus $\Delta_n^j, Y(a_i) \nvdash X$. Let Δ_n^{j+1} be $\Delta_n^j, Y(a_i)$.

Thus in any case Δ_n^{j+1} is consistent and $X \notin \Delta_n^{j+1}$. Let Γ_{n+1} be the deductive completion of Δ_n^n with respect to $\boldsymbol{P}\cup\{a_1, a_2, \dots, a_k\}$ where a_k is the last parameter used in Δ_n^n. Let $\Delta = \bigcup \Gamma_n$, then Δ has the following properties:

Δ uses exactly the parameters of $\boldsymbol{Q}$.

$X \notin \Delta$ since $X \notin \Gamma_n$ for any n.

Δ is deductively complete with respect to $\boldsymbol{Q}$.

Δ has the *Or*-property. For if $Y \vee Z \in \Delta$, say $Y \vee Z = Z_n$, then $Y \vee Z \in \Delta_m$ for some m. We can take $m>n$. Then $Y \vee Z = Z_n \in \Delta_m^n$, so either Y or Z is in $\Delta_m^{n+1} \subseteq \Delta$.

Similarly, Δ has the $\exists$-property.

***Lemma* 10.4:** If Γ is nice with respect to $\boldsymbol{P}$:

(1). $X \wedge Y \in \Gamma \quad \Leftrightarrow X \in \Gamma$ and $Y \in \Gamma$,
(2). $X \vee Y \in \Gamma \quad \Leftrightarrow X \in \Gamma$ or $Y \in \Gamma$,
(3). $\sim X \in \Gamma \quad \Rightarrow X \notin \Gamma$,
(4). $X \supset Y \in \Gamma \quad \Rightarrow X \notin \Gamma$ or $Y \in \Gamma$,
(5). $(\exists x)\, X(x) \in \Gamma \Leftrightarrow X(a) \in \Gamma$ for some $a \in \boldsymbol{P}$,
(6). $(\forall x)\, X(x) \in \Gamma \Rightarrow X(a) \in \Gamma$ for every $a \in \boldsymbol{P}$.

Proof: (1). By axioms 4, 5 and 6, since Γ is deductively complete with respect to $\boldsymbol{P}$.

(2). $X \vee Y \in \Gamma \Rightarrow X \in \Gamma$ or $Y \in \Gamma$, since Γ has the *Or*-property. The converse holds by axioms 7 and 8.

(3). If $\sim X \in \Gamma$, $X \notin \Gamma$ since Γ is consistent (using axiom 9).

(4). If $X \supset Y \in \Gamma$, either $X \notin \Gamma$ or $Y \in \Gamma$ since Γ is deductively complete with respect to $\boldsymbol{P}$.

(5). If $(\exists x)\, X(x) \in \Gamma$, $X(a) \in \Gamma$ for some $a \in \boldsymbol{P}$ since Γ has the $\exists$-property. The converse is by axiom 11.

(6). By axiom 12.

***Lemma* 10.5:** Suppose Γ is nice with respect to $\boldsymbol{P}$, and $\{a_1, a_2, a_3 \ldots\}$ is a set of distinct parameters not in $\boldsymbol{P}$. Let $\boldsymbol{Q} = \boldsymbol{P} \cup \{a_1, a_2, a_3 \ldots\}$. Then

(1). If X has all its parameters in $\boldsymbol{P}$ but $\sim X \notin \Gamma$, Γ can be extended to a set Δ nice with respect to $\boldsymbol{Q}$ such that $X \in \Delta$.

(2). If $X \supset Y$ has all its parameters in $\boldsymbol{P}$ but $X \supset Y \notin \Gamma$, Γ can be extended to a set Δ nice with respect to $\boldsymbol{Q}$ such that $X \in \Delta$ and $Y \notin \Delta$.

(3). If $X(x)$ has all its parameters in $\boldsymbol{P}$ but $(\forall x)\, X(x) \notin \Gamma$, Γ can be extended to a set Δ nice with respect to $\boldsymbol{Q}$ such that for some $a \in \boldsymbol{Q}$, $X(a) \notin \Delta$.

Proof:

(1). Since $\sim X \notin \Gamma$, $\{\Gamma, X\}$ is consistent, for otherwise $\Gamma, X \vdash \sim X$. So by the deduction theorem $\Gamma \vdash X \supset \sim X$, and by axiom 10 $\Gamma \vdash \sim X$, so $\sim X \in \Gamma$. Since $\{\Gamma, X\}$ is consistent, there is some Y such that $\Gamma, X \nvdash Y$. Now use lemma 10.3.

(2). $\Gamma, X \nvdash Y$ for otherwise, by the deduction theorem $\Gamma \vdash X \supset Y$, so $X \supset Y \in \Gamma$. Since $\Gamma, X \nvdash Y$, use lemma 10.3.

(3). $a_1 \notin \boldsymbol{P}$. We claim $\Gamma \nvdash X(a_1)$. Suppose $\Gamma \vdash X(a_1)$. For the conjunction, call it W, of some finite subset of Γ, $\vdash W \supset X(a_1)$. But a_1 does not occur in W. By rule 14 $\vdash W \supset (\forall x)\, X(x)$, so $\Gamma \vdash (\forall x)\, X(x)$, $(\forall x)\, X(x) \in \Gamma$. Since $\Gamma \nvdash X(a_1)$, use lemma 10.3.

Now we proceed to show completeness. We arrange the parameters as follows:

$$\begin{array}{llll} \boldsymbol{S}_1: & a_1^1, & a_2^1, & a_3^1, \;\dots \\ \boldsymbol{S}_2: & a_1^2, & a_2^2, & a_3^2, \;\dots \\ \boldsymbol{S}_3: & a_1^3, & a_2^3, & a_3^3, \;\dots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

and let $\boldsymbol{P}_n = \boldsymbol{S}_1 \cup \boldsymbol{S}_2 \cup \cdots \cup \boldsymbol{S}_n$. Let $\mathscr{G}$ be the collection of all nice sets with respect to any $\boldsymbol{P}_i$. If $\Gamma \in \mathscr{G}$, Γ is nice with respect to, say, $\boldsymbol{P}_n$. Let $\mathscr{P}(\Gamma) = \boldsymbol{P}_n$. Let $\Gamma \mathscr{R} \Delta$ if $\mathscr{P}(\Gamma) \subseteq \mathscr{P}(\Delta)$ and $\Gamma \subseteq \Delta$. For any X, let $\Gamma \models X$ iff $X \in \Gamma$. By lemmas 10.4 and 10.5 $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ is a model.

Finally, suppose $\nvdash X$. All the parameters are in, say, $\boldsymbol{P}_n$. Since $\emptyset \nvdash X$, by lemma 10.3 we can extend $\emptyset$ to a set Γ, nice with respect to $\boldsymbol{P}_n$ such that $X \notin \Gamma$. Thus $\Gamma \in \mathscr{G}$, $X \in \hat{\mathscr{P}}(\Gamma)$ and $\Gamma \not\models X$.

Remark 10.6: This is a "universal" model in the sense of § 6.

In ch. 6 § 4 we will show that the set of all theorems using only parameters of $\boldsymbol{P}_n$ is itself a nice set with respect to $\boldsymbol{P}_n$. This would make the final use of lemma 10.3 above unnecessary.

CHAPTER 6

ADDITIONAL FIRST ORDER RESULTS

§ 1. Compactness

We call an infinite set $\boldsymbol{S}$ of signed formulas realizable if there is a model $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{P}\rangle$ and a $\Gamma \in \mathscr{G}$ such that for any formula X

$$TX \in \boldsymbol{S} \Rightarrow X \in \hat{\mathscr{P}}(\Gamma) \quad \text{and} \quad \Gamma \models X,$$
$$FX \in \boldsymbol{S} \Rightarrow X \in \hat{\mathscr{P}}(\Gamma) \quad \text{and} \quad \Gamma \not\models X.$$

There is a similar concept for sets of unsigned formulas $\boldsymbol{U}$. We say $\boldsymbol{U}$ is satisfiable if there is a model $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{P}\rangle$ and a $\Gamma \in \mathscr{G}$ such that for any formula X

$$X \in \boldsymbol{U} \Rightarrow X \in \hat{\mathscr{P}}(\Gamma) \quad \text{and} \quad \Gamma \models X.$$

***Lemma* 1.1:** Let $\boldsymbol{U}$ be a set of unsigned formulas and define a set $\boldsymbol{S}$ of signed formulas to be $\{TX \mid X \in \boldsymbol{U}\}$. Then

(1). $\boldsymbol{U}$ is satisfiable if and only if $\boldsymbol{S}$ is realizable

(2). $\boldsymbol{U}$ is consistent if and only if $\boldsymbol{S}$ is consistent.

Proof: Part (1) is obvious.

To show part (2), suppose $\boldsymbol{U}$ is not consistent. Then some finite subset $\{u_1, \ldots, u_n\}$ is not consistent, so from it we can deduce any formula. Let A be an atomic formula having no predicate symbols or parameters in common with $\{u_1, \ldots, u_n\}$. Then

$$\vdash_{\mathrm{I}} (u_1 \wedge \cdots \wedge u_n) \supset A.$$

Hence there is a closed tableau for

$$\{F(u_1 \wedge \cdots \wedge u_n) \supset A\},$$

so there is a closed tableau for

$$\{T(u_1 \wedge \cdots \wedge u_n), FA\}.$$

By the way we have chosen A, there must be a closed tableau for $\{T(u_1 \wedge \cdots \wedge u_n)\}$ and hence for $\{Tu_1, \ldots, Tu_n\}$. Thus S is not consistent.

The converse is trivial.

Because we have this lemma, we will only discuss realizability and consistency of sets of signed formulas.

***Lemma* 1.2:** Let S be a set of signed formulas. If S is realizable, S is consistent.

Proof: If S is not consistent, some finite subset Q is not consistent. That is, there is a closed tableau $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$ in which $\mathscr{C}_1$ is $\{Q\}$. If Q were realizable, by theorem 5.2.7 every $\mathscr{C}_i$ would be, but a closed configuration is not realizable.

***Lemma* 1.3:** Let S be a *finite* set of signed formulas. If S is consistent, S is realizable.

Proof: Let S be $\{TX_1, \ldots, TX_n, FY_1, \ldots, FY_m\}$.
S is consistent if and only if

$$\{F(X_1 \wedge \cdots \wedge X_n) \supset (Y_1 \vee \cdots \vee Y_m)\}$$

is consistent. If this is consistent, $(X_1 \wedge \cdots \wedge X_n) \supset (Y_1 \vee \cdots \vee Y_m)$ is a non-theorem, so by the completeness theorem, there is a model $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ and a $\Gamma \in \mathscr{G}$ such that $X_i \in \hat{\mathscr{P}}(\Gamma)$, $Y_j \in \hat{\mathscr{P}}(\Gamma)$ and

$$\Gamma \not\models (X_1 \wedge \cdots \wedge X_n) \supset (Y_1 \vee \cdots \vee Y_m).$$

But then for some Γ^*

$$\Gamma^* \models X_1 \wedge \cdots \wedge X_n, \quad \Gamma^* \not\models Y_1 \vee \cdots \vee Y_m,$$

so Γ^* realizes S.

This method does not work if S is infinite, but the lemma remains true, at least for sets with no parameters. The result can be extended to sets with some parameters, but we will not do so.

***Lemma* 1.4:** Let S be an infinite set of signed formulas with no parameters. If S is consistent, S is realizable.

Proof: The proof can be based on either of the two tableau completeness proofs.

If we use the first proof, that of ch. 5 § 5, change step 0 to: "S is consistent. Extend it to a Hintikka element with respect to P_1. Call the result Γ_1". Continue the proof as written. The lemma is then obvious.

If we use the proof of ch. 5 § 6 the result is even easier. S is consistent, so by lemma 5.6.4, we can extend S to a set Γ which is good with respect to P_1. The result follows immediately.

***Theorem* 1.5:** If S is any set of signed formulas with no parameters, S is consistent if and only if S is realizable.

***Corollary* 1.6:** If every finite subset of S is realizable, so is S.

***Corollary* 1.7:** If U is any set of unsigned formulas with no parameters, U is consistent if and only if U is satisfiable.

Remark 1.8: The last corollary could have been established directly by adapting the completeness proof of ch. 5 § 10.

Definition 1.9: For a set of formulas U, by $\Gamma \models U$ we mean $\Gamma \models X$ for all $X \in U$.

***Corollary* 1.10** (strong completeness): Let U be any set of unsigned formulas with no parameters. Then $U \vdash_I X$ if and only if in any model $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$, for any $\Gamma \in \mathscr{G}$, if $\Gamma \models U$, $\Gamma \models X$.

Proof: $U \vdash_I X$ if and only if $\{TY \mid Y \in U\} \cup \{FX\}$ is inconsistent.

***Corollary* 1.11:** (cut elimination, Gentzen's Hauptsatz): If S is a set of signed formulas with no constants and $\{S, TX\}$ and $\{S, FX\}$ are inconsistent, so is $\{S\}$.

Remark 1.12: This may be extended to sets S with some parameters. To be precise, to any set S which leaves unused a countable collection of parameters. It follows that in the completeness proof of ch. 5 § 6 a set Δ maximal consistent with respect to P actually contains TX or FX for each X with parameters from P.

§ 2. Concerning the excluded middle law

If S is a set of unsigned formulas, by $S \vdash_c X$ and $S \vdash_I X$ we mean classical and intuitionistic derivability respectively.

Let $X(\alpha_1, \ldots, \alpha_n)$ be a formula having exactly the parameters $\alpha_1, \ldots, \alpha_n$. By the closure of X we mean the formula

$$(\forall x_{i_1}) \ldots (\forall x_{i_n})\, X(x_{i_1}, \ldots, x_{i_n})$$

(where x_{i_j} does not occur in $X(\alpha_1, \ldots, \alpha_n)$).

Let M be the collection of the closures of all formulas of the form $X \vee \sim X$. We wish to show:

***Theorem* 2.1:** If X has no parameters,

$$\vdash_c X \Leftrightarrow M \vdash_I X.$$

We first show:

***Lemma* 2.2:** Let $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ be a model, $\Gamma \in \mathscr{G}$, and suppose $Y \in M \Rightarrow \Gamma \models Y$. Then Γ can be included in a complete $\mathscr{R}$-chain $\mathscr{C}$ such that $\mathscr{C}'$ is a *truth* set (see ch. 4 § 6).

Proof: Enumerate all formulas beginning with a universal quantifier: $X_1, X_2, X_3, \ldots$.

Let $\Gamma_0 = \Gamma$. Having defined Γ_n, consider X_{n+1}. If $X_{n+1} \notin \mathscr{P}(\Gamma_n^*)$ for any Γ_n^*, let $\Gamma_{n+1} = \Gamma_n$. Otherwise there is some Γ_n^* such that $X_{n+1} \in \mathscr{P}(\Gamma_n^*)$. Say X_{n+1} is $(\forall x) X(x)$. We have two cases: (1). If $\Gamma_n^* \models (\forall x) X(x)$, let $\Gamma_{n+1} = \Gamma_n^*$. (2). If $\Gamma_n^* \not\models (\forall x) X(x)$, there is a Γ_n^{**} and an $\alpha \in \mathscr{P}(\Gamma_n^{**})$ such that $\Gamma_n^{**} \not\models X(\alpha)$. Let Γ_{n+1} be this Γ_n^{**}.

Let the $\mathscr{R}$-chain $\mathscr{C}$ be $\{\Gamma_0, \Gamma_1, \Gamma_2, \ldots\}$. Since $Y \in M \Rightarrow \Gamma \models Y$ and $\Gamma = \Gamma_0$, $\mathscr{C}$ is a complete $\mathscr{R}$-chain by the definition of M, and so $\mathscr{C}'$ is an almost-truth set. Thus we have only one more fact to show:

$$Y(\alpha) \in \mathscr{C}' \quad \text{for every parameter } \alpha \text{ of } \quad \mathscr{C}' \Rightarrow (\forall x)\, Y(x) \in \mathscr{C}'.$$

Suppose $(\forall x)\, Y(x, \alpha_1, \ldots, \alpha_n) \notin \mathscr{C}'$ (where $\alpha_1, \ldots, \alpha_n$ are all the parameters of Y). If some α_i is not a parameter of $\mathscr{C}'$, we are done. So suppose each α_i occurs in $\mathscr{C}'$. Then for some $\Gamma_n \in \mathscr{C}$, all $\alpha_i \in \mathscr{P}(\Gamma_n)$ and $\Gamma_n \not\models (\forall x)\, Y(x, \alpha_1, \ldots, \alpha_n)$. But by the construction of $\mathscr{C}$, there is a Γ_m $(m \geqslant n)$ such that $\Gamma_m \not\models Y(b, \alpha_1, \ldots, \alpha_n)$ for some $b \in \mathscr{P}(\Gamma_m)$. But

$$\Gamma \models (\forall x_1) \ldots (\forall x_n)(\forall x)\, [Y(x, x_1, \ldots, x_n) \vee \sim Y(x, x_1, \ldots, x_n)]$$

and $\Gamma \mathcal{R} \Gamma_m$, so

$$\Gamma_m \models Y(b, \alpha_1, \ldots, \alpha_n) \vee \sim Y(b, \alpha_1, \ldots, \alpha_n),$$

thus $\Gamma_m \models \sim Y(b, \alpha_1, \ldots, \alpha_n)$. $\sim Y(b, \alpha_1, \ldots, \alpha_n) \in \mathscr{C}'$, so $Y(b, \alpha, \ldots, \alpha_n) \notin \mathscr{C}'$ for a parameter b of $\mathscr{C}'$.

Now to prove the theorem itself:

If $M \vdash_{\mathrm{I}} X$ then for some finite subset $\{m_1, \ldots, m_n\}$ of M

$$\vdash_{\mathrm{I}} (m_1 \wedge \cdots \wedge m_n) \supset X.$$

By theorem 4.8.2 (and the completeness theorems)

$$\vdash_{\mathrm{c}} (m_1 \wedge \cdots \wedge m_n) \supset X.$$

But $\vdash_{\mathrm{c}} m_1 \wedge \cdots \wedge m_n$, hence $\vdash_{\mathrm{c}} X$.

Conversely, if $M \nvdash_{\mathrm{I}} X$, let S be the set of signed formulas

$$\{FX\} \cup \{TY \mid Y \in M\}.$$

Since $M \nvdash_{\mathrm{I}} X$, S is consistent. Then by the results of the last section, S is realizable. Thus there is a model $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ and a $\Gamma \in \mathscr{G}$ such that $Y \in M \Rightarrow \Gamma \models Y$, $X \in \hat{\mathscr{P}}(\Gamma)$ and $\Gamma \nvDash X$. But X has no parameters, so $X \vee \sim X \in M$. Thus $\Gamma \models X \vee \sim X$, so $\Gamma \models \sim X$. Now by lemma 2.2 there is a truth set containing $\sim X$. Hence $\nvdash_{\mathrm{c}} X$.

§ 3. Skolem–Löwenheim

By the domain of a model $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ we mean $\bigcup_{\Gamma \in \mathscr{G}} \mathscr{P}(\Gamma)$. So far we have only considered models in which the domain was at most countable. Suppose now we have an uncountable number of parameters and we change the definitions of formula, model and validity accordingly, but not the definition of proof.

***Theorem* 3.1:** X is valid in all models if and only if X is valid in all models with countable domains.

Proof: One half is trivial.

Suppose there is a model $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ with an uncountable domain in which X is not valid. The correctness proof of ch. 5 §§ 2 or 9 is still applicable. Thus X is not provable. Since X is not provable, if we reduce the collection of parameters to a countable number (including those of X), X still will not be provable. Then any of the completeness proofs will furnish a counter-model for X with a countable domain.

This method may be combined with that of § 1 to show

***Theorem* 3.2:** If S is any countable set of signed formulas with no parameters, S is consistent if and only if S is realizable in a model with a countable domain.

***Theorem* 3.3:** If U is any countable set of unsigned formulas with no parameters, U is consistent if and only if U is satisfiable in a model with a countable domain.

Remark 3.4: In part II, we will be using models with domains of arbitrarily high cardinality.

§ 4. Kleene tableaus

The system of this section is based on the intuitionistic system G3 of [10]. The modifications are due to Smullyan. The resulting system is like that of Beth except that sets of signed formulas never contain more than one F-signed formula. Explicitely, everything is as it was in ch. 2 § 1 and ch. 5 § 1 except that the reduction rules are replaced by the following, where S is a set of signed formulas with at most one F-signed formula.

$$\mathrm{K}T\vee \quad \frac{S, TX \vee Y}{S, TX \mid S, TY} \qquad \mathrm{K}F\vee \quad \frac{S_T, FX \vee Y}{S_T, FX} \quad \frac{S_T, FX \vee Y}{S_T, FY}$$

$$\mathrm{K}T\wedge \quad \frac{S, TX \wedge Y}{S, TX, TY} \qquad \mathrm{K}F\wedge \quad \frac{S_T, FX \wedge Y}{S_T, FX \mid S_T, FY}$$

$$\mathrm{K}T\sim \quad \frac{S, T\sim X}{S_T, FX} \qquad \mathrm{K}F\sim \quad \frac{S_T, F\sim X}{S_T, TX}$$

$$\mathrm{K}T\supset \quad \frac{S, TX \supset Y}{S_T, FX \mid S, TY} \qquad \mathrm{K}F\supset \quad \frac{S_T, FX \supset Y}{S_T, TX, FY}$$

$$\mathrm{K}T\exists \quad \frac{S, T(\exists x)X(x)}{S, TX(a)} \qquad \mathrm{K}F\exists \quad \frac{S_T, F(\exists x)X(x)}{S_T, FX(a)}$$

$$\mathrm{K}T\forall \quad \frac{S, T(\forall x)X(x)}{S, TX(a)} \qquad \mathrm{K}F\forall \quad \frac{S_T, F(\forall x)X(x)}{S_T, FX(a)}$$

where in $\mathrm{K}T\exists$ and $\mathrm{K}F\forall$ the parameter a does not occur in S or $X(x)$.

There are several ways of showing this is actually a proof system for intuitionistic logic. We choose to show it is directly equivalent to the Beth tableau system, that is, we give a proof translation procedure.

We leave it to the reader to show the almost obvious fact that anything provable by Kleene tableaus is provable by Beth tableaus. To show the converse, we need

***Lemma* 4.1:** If a Beth tableau for $\{TX_1, \ldots, TX_n, FY_1, \ldots, FY_m\}$ closes, then there is a closed Kleene tableau for

$$\{TX_1, \ldots, TX_n, F(Y_1 \vee \cdots \vee Y_m)\}.$$

Proof: The proof is by induction on the length of the closed Beth tableau. If the tableau is of length 1, the result is obvious. Now suppose we know the result for all closed Beth tableaus of length less than n, and a closed tableau for the set in question is of length n. We have several cases depending on the first step of the tableau.

If the first step is an application of rule $F\wedge$, the Beth tableau begins

$$\begin{array}{l}\{\{S_T, FX_1, \ldots, FX_n, FY \wedge Z\}\}, \\ \{\{S_T, FX_1, \ldots, FX_n, FY\}, \quad \{S_T, FX_1, \ldots, FX_n, FZ\}\},\end{array}$$

and proceeds to closure. Now by the induction hypothesis there are closed Kleene tableaus for $\{S_T, F(X_1 \vee \cdots \vee X_n \vee Y)\}$ and $\{S_T, F(X_1 \vee \cdots \vee X_n \vee Z)\}$. We have two possibilities:

(1). If Y is not "used" in the first tableau, or if Z is not "used" in the second tableau, a Kleene tableau beginning

$$\begin{array}{l}\{\{S_T, F(X_1 \vee \cdots \vee X_n \vee (Y \wedge Z))\}\}, \\ \{\{S_T, F(X_1 \vee \cdots \vee X_n)\}\},\end{array}$$

must close.

(2). If both Y and Z are "used", a Kleene tableau beginning

$$\begin{array}{l}\{\{S_T, F(X_1 \vee \cdots \vee X_n \vee (Y \wedge Z))\}\}, \\ \quad\vdots \\ \{\{S_T, F(Y \wedge Z)\}\}, \\ \{\{S_T, FY\}, \{S_T, FZ\}\},\end{array}$$

must close.

The other cases are similar and are left to the reader.

Thus the two tableau systems are equivalent. Now we verify a remark made at the end of ch. 5 § 10.

***Lemma* 4.2:** (Gödel, McKinsey and Tarski): $\vdash_I X \vee Y$ iff $\vdash_I X$ or $\vdash_I Y$.
Proof: Immediate from the Kleene tableau formulation.

***Lemma* 4.3:** (Rasiowa and Sikorski): If $\vdash_I (\exists x)\, X(x, a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ are all the parameters of X, then $\vdash_I X(b, a_1, \ldots, a_n)$ where b is one of the a_i. If X has no parameters, b is arbitrary and $\vdash_I (\forall x)\, X(x)$.
Proof: A Kleene tableau proof of $(\exists x)\, X(x, a_1, \ldots, a_n)$ begins

$$\{\{F(\exists x)\, X(x, a_1, \ldots, a_n)\}\},$$
$$\{\{FX(b, a_1, \ldots, a_n)\}\},$$

and proceeds to closure. If b is some a_i, we are done. If not, we actually have a proof, except for a different first line, of

$$(\forall x)\, X(x, a_1, \ldots, a_n).$$

§ 5. Craig interpolation lemma

***Theorem* 5.1:** If $\vdash_I X \supset Y$ and X and Y have a predicate symbol in common, then there is a formula Z involving only predicates and parameters common to X and Y such that $\vdash_I X \supset Z$ and $\vdash_I Z \supset Y$; if X and Y have no common predicates, either $\vdash_I \sim X$ or $\vdash_I Y$.

The classical version of this theorem was first proved by Craig, hence the name. The intuitionistic version is due to Schütte [17]. Essentially the same proof was given for a natural deduction system by Prawitz [15]. We give basically the same proof in the Kleene tableau system. For another proof in this system see [11].

We find it convenient to temporarily introduce two symbols t and f into our collection of logical symbols, letting them be atomic formulas, and letting them combine according to the following rules.

$$X \vee t = t \vee X = t,$$
$$X \vee f = f \vee X = X,$$
$$X \wedge t = t \wedge X = X,$$
$$X \wedge f = f \wedge X = f,$$
$$\sim t = f, \quad \sim f = t,$$

$$X \supset t = f \supset X = t,$$
$$t \supset X = X \quad X \supset f = \sim X,$$
$$(\exists x)\,t = (\forall x)\,t = t,$$
$$(\exists x)\,f = (\forall x)\,f = f.$$

By a *block* we mean a finite set of signed formulas containing at most one F-signed formula. When we call a block inconsistent, we mean there is a closed Kleene tableau for it. By an *initial part* of a block we mean any subset of the T-signed formulas. We make the convention that if S is the finite set of unsigned formulas $\{X_1, \ldots, X_n\}$ then TS is the set $\{TX_1, \ldots, TX_n\}$. We further make the convention that for a set S of formulas, S_1 and S_2 represent subsets such that $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = S$. By $[S]$ we mean the set of predicates and parameters of formulas of S, together with t and f.

Now we define an interpolation formula X for the block $\{TS, FY\}$ (where S is a set of unsigned formulas and Y is a formula) with respect to the initial part TS_1, which we denote by $\{TS, FY\}/\{TS_1\}$, as follows (X may be t or f, but we assume t and f are not part of S or Y): X is an $\{TS, FY\}/\{TS_1\}$ if

(1). $[X] \subseteq [S_1] \cap [S_2, Y]$,

(2). $\{TS_1, FX\}$ is inconsistent,

(3). $\{TX, TS_2, FY\}$ is inconsistent

(we have temporarily added to the closure rules: closure of a set of signed formulas if it contains Tf or Ft).

***Lemma* 5.2:** An inconsistent block has an interpolation formula with respect to every initial part.

Proof: We show this by induction on the length of the closed tableau for the block. If this is of length 1, the block must be of the form

$$\{TS, TX, FX\}.$$

We have two cases:

Case (1). The initial part is $\{TS_1, TX\}$. Then X is an interpolation formula.

Case (2). The initial part is $\{TS_1\}$. Then $\{TS_2, TX, FX\}$ is inconsistent and t is an interpolation formula.

Now suppose we have an inconsistent block, and the result is known for all inconsistent blocks with shorter closed tableaus. We have several cases depending on the first reduction rule used.

K$T\vee$: The block is $\{TS, TX\vee Y, FZ\}$, and $\{TS, TX, FZ\}$ and $\{TS, TY, FZ\}$ are both inconsistent.

Case (1). The initial part is $\{TS_1, TX\vee Y\}$. Then by the induction hypothesis there are formulas U_1 and U_2 such that

$$U_1 \text{ is an } \{TS, TX, FZ\}/\{TS_1, TX\},$$
$$U_2 \text{ is an } \{TS, TY, FZ\}/\{TS_1, TY\}.$$

Then $U_1\vee U_2$ is an $\{TS, TX\vee Y, FZ\}/\{TS_1, TX\vee Y\}$.

Case (2). The initial part is $\{TS_1\}$. Again, by hypothesis, there are U_1, U_2 such that

$$U_1 \text{ is an } \{TS, TX, FZ\}/\{TS_1\},$$
$$U_2 \text{ is an } \{TS, TY, FZ\}/\{TS_1\}.$$

Then $U_1\wedge U_2$ is an $\{TS, TX\vee Y, FZ\}/\{TS_1\}$.

K$F\vee$: The block is $\{TS, FX\vee Y\}$, and $\{TS, FX\}$ or $\{TS, FY\}$ is inconsistent. Suppose the first. Let the initial part be $\{TS_1\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS, FX\}/\{TS_1\}.$$

Then U is an $\{TS, FX\vee Y\}/\{TS_1\}$.

K$T\wedge$: The block is $\{TS, TX\wedge Y, FZ\}$, and $\{TS, TX, TY, FZ\}$ is inconsistent.

Case (1). The initial part is $\{TS_1, TX\wedge Y\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS, TX, TY, FZ\}/\{TS_1, TX, TY\}.$$

Then U is an $\{TS, TX\wedge Y, FZ\}/\{TS_1, TX\wedge Y\}$.

Case (2). The initial part is $\{TS_1\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS, TX, TY, FZ\}/\{TS_1\}.$$

Then U is an $\{TS, TX\wedge Y, FZ\}/\{TS_1\}$.

K$F\wedge$: The block is $\{TS, FX\wedge Y\}$, and $\{TS, FX\}$ and $\{TS, FY\}$ are both inconsistent. Suppose the initial part is $\{TS_1\}$. By hypothesis there are U_1, U_2 such that

$$U_1 \text{ is an } \{TS, FX\}/\{TS_1\},$$
$$U_2 \text{ is an } \{TS, FY\}/\{TS_1\}.$$

Then $U_1\wedge U_2$ is an $\{TS, FX\wedge Y\}/\{TS_1\}$.

$\mathrm{K}F\sim$: The block is $\{TS, F\sim X\}$, and $\{TS, TX\}$ is inconsistent. Suppose the initial part is $\{TS_1\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS, TX\}/\{TS_1\}.$$

Then U is an $\{TS, F\sim X\}/\{TS_1\}$.

$\mathrm{K}T\sim$: The block is $\{TS, T\sim X, FY\}$, and $\{TS, FX\}$ is inconsistent.

Case (1). The initial part is $\{TS_1\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS, FX\}/\{TS_1\}.$$

Then U is an $\{TS, T\sim X, FY\}/\{TS_1\}$.

Case (2). The initial part is $\{TS_1, T\sim X\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS, FX\}/\{TS_2\}.$$

We claim that

$$\sim U \text{ is an } \{TS, T\sim X, FY\}/\{TS_1\}.$$

First we verify its predicates and parameters are correct. By hypothesis $[U]\subseteq[S_2]\cap[S_1, X]$, so immediately $[\sim U]\subseteq[S_1, \sim X]\cap[S_2, Y]$. We have the following two blocks are inconsistent:

$$\{TS_2, FU\},$$
$$\{TS_1, TU, FX\}.$$

It follows that the following two blocks are also inconsistent:

$$\{TS_1, T\sim X, F\sim U\},$$
$$\{TS_2, T\sim U, FY\},$$

and we are done.

$\mathrm{K}F\supset$: The block is $\{TS, FX\supset Y\}$, and $\{TS, TX, FY\}$ is inconsistent. Suppose the initial part is $\{TS_1\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS, TX, FY\}/\{TS_1\}.$$

Then U is an $\{TS, FX\supset Y\}/\{TS_1\}$.

$\mathrm{K}T\supset$: The block is $\{TS, TX\supset Y, FZ\}$, and $\{TS, FX\}$ and $\{TS, TY, FZ\}$ are both inconsistent.

Case (1). The initial part is $\{TS_1\}$. By hypothesis there are U_1, U_2 such that

$$U_1 \text{ is an } \{TS, FX\}/\{TS_1\},$$
$$U_2 \text{ is an } \{TS, TY, FZ\}/\{TS_1\},$$

Then $U_1 \wedge U_2$ is an $\{TS, TX \supset Y, FZ\}/\{TS_1\}$.

Case (2). The initial part is $\{TS_1, TX \supset Y\}$. By hypothesis there are U_1, U_2 such that

$$U_1 \text{ is an } \{TS, FX\}/\{TS_2\},$$
$$U_2 \text{ is an } \{TS, TY, FZ\}/\{TS_1, TY\}.$$

We claim $U_1 \supset U_2$ is an $\{TS, TX \supset Y, FZ\}/\{TS_1, TX \supset Y\}$.

By hypothesis

$$[U_1] \subseteq [S_2] \cap [S_1, X],$$
$$[U_2] \subseteq [S_1, Y] \cap [S_2, Z],$$

so

$$[U_1 \supset U_2] \subseteq [S_1, X \supset Y] \cap [S_2, Z].$$

We have that the following four blocks are inconsistent:

(1). $\{TS_2, FU_1\}$,

(2). $\{TU_1, TS_1, FX\}$,

(3). $\{TS_1, TY, FU_2\}$,

(4). $\{TU_2, TS_2, FZ\}$,

and we must show the following two blocks are inconsistent:

$$\{TS_1, TX \supset Y, FU_1 \supset U_2\},$$
$$\{TU_1 \supset U_2, TS_2, FZ\}.$$

The first follows from (2) and (3), and the second from (1) and (4).

K$F\exists$: The block is $\{TS, F(\exists x)\, X(x)\}$, and $\{TS, FX(a)\}$ is inconsistent. Suppose the initial part is $\{TS_1\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS, FX(a)\}/\{TS_1\}.$$

Then $[U] \subseteq [S_1] \cap [S_2, X(a)]$.

Case (1). $a \notin [U]$.

Then U is an $\{TS, F(\exists x)\, X(x)\}/\{TS_1\}$

Case (2). $a \in [U]$, $a \in [S_2]$

Again U is an $\{TS, F(\exists x)\, X(x)\}/\{TS_1\}$

Case (3). $a \in [U]$, $a \notin [S_2]$.

Then $(\exists x)\, U(^a_x)$ is an

$$\{TS, F(\exists x)\, X(x)\}/\{TS_1\}.$$

K$T\exists$: The block is $\{TS, T(\exists x)\, X(x), FZ\}$, and $\{TS, TX(a), FZ\}$ is inconsistent, where $a \notin [S, X(x), Z]$.

Case (1). The initial part is $\{TS_1, T(\exists x)\, X(x)\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS,\, TX(a),\, FZ\}/\{TS_1,\, TX(a)\}.$$

Then U is an $\{TS, T(\exists x)\, X(x), FZ\}/\{TS_1, T(\exists x)\, X(x)\}$.

Case (2). The initial part is $\{TS_1\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS,\, TX(a),\, FZ\}/\{TS_1\}.$$

Then U is an $\{TS, T(\exists x)\, X(x), FZ\}/\{TS_1\}$.

K$F\forall$: The block is $\{TS, F(\forall x)\, X(x)\}$, and $\{TS, FX(a)\}$ is inconsistent, where $a \notin [S, X(x)]$. Suppose the initial part is $\{TS_1\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS,\, FX(a)\}/\{TS_1\}.$$

Then U is an $\{TS, F(\forall x)\, X(x)\}/\{TS_1\}$.

K$T\forall$: The block is $\{TS, T(\forall x)\, X(x), FZ\}$, and $\{TS, TX(a), FZ\}$ is inconsistent.

Case (1). The initial part is $\{TS_1, T(\forall x)\, X(x)\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS,\, TX(a),\, FZ\}/\{TS_1,\, TX(a)\}.$$

Case (1a). $a \notin [U]$.
Then U is an

$$\{TS,\, T(\forall x)\, X(x),\, FZ\}/\{TS_1,\, T(\forall x)\, X(x)\}.$$

Case (1b). $a \in [U]$, $a \in [S_1, X(x)]$.
Again

$$U \text{ is an } \{TS,\, T(\forall x)\, X(x),\, FZ\}/\{TS_1,\, T(\forall x)\, X(x)\}.$$

Case (1c). $a \in [U]$, $a \notin [S_1, X(x)]$.
Then $(\forall x)\, U\binom{a}{x}$ is an $\{TS, T(\forall x)\, X(x), FZ\}/\{TS_1, T(\forall x)\, X(x)\}$.

Case (2). The initial part is $\{TS_1\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS,\, TX(a),\, FZ\}/\{TS_1\}.$$

Case (2a). $a \notin [U]$.
Then U is an $\{TS, T(\forall x)\, X(x), FZ\}/\{TS_1\}$.

Case (2b). $a \in [U]$, $a \in [S_2, X(x), Z]$.
Again U is an $\{TS, T(\forall x)\, X(x), FZ\}/\{TS_1\}$.

Case (2c). $a \in [U]$, $a \notin [S_2, X(x), Z]$.

Then $(\exists x)\, U(\substack{a \\ x})$ is an $\{TS, T(\forall x)\, X(x), FZ\}/\{TS_1\}$.

Now to prove the original theorem 5.1:

Suppose $\vdash_{\mathrm{I}} X \supset Y$. Then $\{TX, FY\}$ is inconsistent. By the lemma, there is a U such that U is an $\{TX, FY\}/\{TX\}$. We have three cases:

(1). $U = t$.

Then since $\{Tt, FY\}$ is inconsistent, $\vdash_{\mathrm{I}} Y$.

(2). $U = f$.

Then since $\{TX, Ff\}$ is inconsistent, $\{F \sim X\}$ is also inconsistent (f is not in X). Thus $\vdash_{\mathrm{I}} \sim X$.

(3). $U \neq t,\ \ U \neq f$.

Then U is a formula not involving t or f, all the parameters and predicates of U are in X and Y, and since $\{TX, FU\}$ and $\{TU, FY\}$ are both inconsistent, $\vdash_{\mathrm{I}} X \supset U$ and $\vdash_{\mathrm{I}} U \supset Y$.

§ 6. Models with constant $\mathscr{P}$ function

In part II we will be concerned with finding countermodels for formulas with no universal quantifiers, and we will confine ourselves to models with a constant $\mathscr{P}$ function. To justify this restriction, we show in this section

***Theorem* 6.1:** If X is a formula with no universal quantifiers and $\nvdash_{\mathrm{I}} X$, then there is a counter-model $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ for X in which $\mathscr{P}$ is a constant function.

Definition 6.2: For this section only, let $a_1, a_2, a_3, \ldots$ be an enumeration of all parameters. We call a set Γ of signed formulas a Hintikka element if Γ is a Hintikka element with respect to some initial segment of $a_1, a_2, a_3, \ldots$ (see ch. 5 § 4).

***Lemma* 6.3:** If S is a finite, consistent set of signed formulas with no universal quantifiers, S can be extended to a *finite* Hintikka element.

Proof: Suppose S is the set $\{X_1, X_2, \ldots, X_n\}$ where each X_i is a *signed* formula. We define the two sequences $\{P_k\}$, $\{Q_k\}$ as follows:

Let
$$P_0 = \emptyset, \qquad Q_0 = X_1, \ldots, X_n.$$

Suppose we have defined P_k and Q_k where

$$P_k = Y_1, \ldots, Y_r, \qquad Q_k = W_1, \ldots, W_s,$$

and $\boldsymbol{P}_k \cup \boldsymbol{Q}_k$ (considered as a set) is consistent. To define $\boldsymbol{P}_{k+1}$ and $\boldsymbol{Q}_{k+1}$ we have several cases depending on W_1:

Case atomic: If W_1 is a signed atomic formula, let

$$\boldsymbol{P}_{k+1} = Y_1, \ldots, Y_r, W_1, \qquad \boldsymbol{Q}_{k+1} = W_2, \ldots, W_s.$$

Case $T\vee$: If W_1 is $TX \vee Y$, either TX or TY is consistent with $\boldsymbol{P}_k \cup \boldsymbol{Q}_k$, say TX. Let

$$\boldsymbol{P}_{k+1} = Y_1, \ldots, Y_r, TX \vee Y, \qquad \boldsymbol{Q}_{k+1} = W_2, \ldots, W_s, TX.$$

Case $F\vee$: If W_1 is $FX \vee Y$ then FX, FY is consistent with $\boldsymbol{P}_k \cup \boldsymbol{Q}_k$. Let

$$\boldsymbol{P}_{k+1} = Y_1, \ldots, Y_r, FX \vee Y, \qquad \boldsymbol{Q}_{k+1} = W_2, \ldots, W_s, FX, FY.$$

Cases $T\wedge$, $F\wedge$, $T\sim$, $T\supset$ are similar.

Case $T\exists$: If W_1 is $T(\exists x)\, X(x)$, let a be the first in the sequence $a_1, a_2, \ldots$ not occurring in $\boldsymbol{P}_k$ or $\boldsymbol{Q}_k$. Then $TX(a)$ is consistent with $\boldsymbol{P}_k \cup \boldsymbol{Q}_k$. Let

$$\boldsymbol{P}_{k+1} = Y_1, \ldots, Y_r, T(\exists x)\, X(x), \qquad \boldsymbol{Q}_{k+1} = W_2, \ldots, W_s, TX(a).$$

Case $F\exists$: If W_1 is $F(\exists x)\, X(x)$, let $\{a_{i_1}, \ldots, a_{i_t}\}$ be the set of parameters occurring in $\boldsymbol{P}_k \cup \boldsymbol{Q}_k$ such that no $FX(a_{i_j})$ occurs in $\boldsymbol{P}_k \cup \boldsymbol{Q}_k$. Then $\{FX(a_{i_1}), \ldots, FX(a_{i_t})\}$ is consistent with $\boldsymbol{P}_k \cup \boldsymbol{Q}_k$. Let

$$\boldsymbol{P}_{k+1} = \boldsymbol{P}_k, \qquad \boldsymbol{Q}_{k+1} = W_2, \ldots, W_s, FX(a_{i_1}), \ldots, FX(a_{i_t}), F(\exists x)\, X(x),$$

After finitely many steps there will be no T-signed formulas left in the $\boldsymbol{Q}$-sequence because each rule $T\vee$, $T\wedge$, $T\sim$, $T\supset$, $T\exists$ reduces degree, and no rule $F\vee$, $F\wedge$, $F\exists$ introduces new T-signed formulas. When no T-signed formulas are left in the $\boldsymbol{Q}$-sequence, no new parameters can be introduced since rule $T\exists$ no longer applies. After finitely many more steps we must reach an unusable $\boldsymbol{Q}$-sequence. The corresponding $\boldsymbol{P} \cup \boldsymbol{Q}$-sequence is finite, consistent, and clearly a Hintikka element.

Remark 6.4: The above proof also shows the following which we will need later:

Let $\boldsymbol{R}$ be a finite Hintikka element. Suppose we add (consistently) a finite set of F-signed formulas to $\boldsymbol{R}$ and extend the result to a finite Hintikka element $\boldsymbol{S}$ by the above method. Then

$$\boldsymbol{R}_T = \boldsymbol{S}_T.$$

Since $R \subseteq S$, certainly $R_T \subseteq S_T$. That $S_T \subseteq R_T$ also holds follows by an inspection of the above proof; no new T-signed formulas will be added.

Now we turn to the proof of the theorem itself. We have no universal quantifiers to consider, so we may use the definition of associated sets in ch. 2 § 4.

Suppose X is a formula with no universal quantifiers, and $\nvdash_I X$. Then $\{FX\}$ is consistent. Extend it to a finite Hintikka element S_0^0. Let $T_1, \ldots, T_n$ be the associated sets of S_0^0. Extend each to a finite Hintikka element, $S_1^0, \ldots, S_n^0$ respectively. Thus we have

$$S_0^0, S_1^0, \ldots, S_n^0 .$$

For each parameter a of some S_i^0 and each formula of the form $F(\exists x)\, X(x)$ in S_0^0, adjoin $FX(a)$ to S_0^0 and extend the result to a Hintikka element S_0^1. Do the same for $S_1^0, \ldots. S_n^0$, producing $S_1^1, \ldots, S_n^1$ respectively. Thus we have now

$$S_0^1, S_1^1, \ldots, S_n^1 .$$

Let $T_{n+1}, \ldots, T_m$ be the associated sets of $S_0^1, S_1^1, \ldots, S_n^1$. Extend each to a Hintikka element, $S_{n+1}^0, \ldots, S_m^0$ respectively. Thus we have now

$$S_0^1, S_1^1, \ldots, S_n^1, S_{n+1}^0, \ldots, S_m^0 .$$

For each parameter a used so far, and for each formula of the form $F(\exists x)\, X(x)$ in S_0^1, adjoin $FX(a)$ to S_0^1 and extend the result to a finite Hintikka element S_0^2. Do the same for each. Thus we have now

$$S_0^2, S_1^2, \ldots, S_n^2, S_{n+1}^1, \ldots, S_m^1 .$$

Again take the associated sets, and extend to finite Hintikka elements, producing now

$$S_0^2, S_1^2, \ldots, S_n^2, S_{n+1}^1, \ldots, S_m^1, S_{m+1}^0, \ldots, S_p^0 .$$

Continue in this manner. Let

$$S_0 = \bigcup_{k=0}^{\infty} S_0^k, \quad S_1 = \bigcup_{k=0}^{\infty} S_1^k, \quad \text{etc.}$$

By the remark above, for each n,

$$S_{nT} = S_{nT}^0 = S_{nT}^1 = \cdots .$$

Thus if S_n^k has as an associated set S_m^j, $S_{nT} \subseteq S_m$.

It now follows that $\{S_0, S_1, \ldots\}$ is a Hintikka collection. For example, suppose $F{\sim}Y \in S_j$. Let k be the least integer such that $F{\sim}Y \in S_j^k$. By the above construction, there is some set S_r^0 such that S_r^0 is an associated set of S_j^k and $TY \in S_r^0$. But then $S_{jT}^k \subseteq S_r^0$, so by the above $S_{jT} \subseteq S_r$, and $TY \in S_r$. The other properties are shown similarly.

Moreover, $\mathscr{P}(S_n) = \mathscr{P}(S_m)$ for all m and n, as is easily seen. (Recall that $\mathscr{P}(S)$ is the collection of all parameters used in S.) Now as in ch. 5 § 3 there is a model for this Hintikka collection, and this model will have a constant $\mathscr{P}$ map, so the theorem is shown.

CHAPTER 7

INTUITIONISTIC M_α GENERALIZATIONS

§ 1. Introduction

Here and in the rest of part II we restrict our considerations to the following language: a countable collection of bound variables $x, y, z, \ldots$, a collection of parameters (or constants) of arbitrarily high cardinality $f, g, h, \ldots$, one two-place predicate symbol $\in$ (we write $\in(x, y)$ as $(x \in y)$), and the usual connectives, quantifiers and parentheses.

In all the models $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{P} \rangle$ which we will consider in part II, the map $\mathscr{P}$ will be constant, and so we will simply write the range $\mathscr{S}$ of $\mathscr{P}$ instead of $\mathscr{P}$, thus $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$, where $\mathscr{P}(\Gamma) = \mathscr{S}$ for all $\Gamma \in \mathscr{G}$.

We call a model $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ an *intuitionistic* ZF *model* if classical equivalents of all the axioms of Zermelo-Fraenkel set theory, expressed *without the use of the universal quantifier*, are valid in it. As a special case, suppose $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is an intuitionistic ZF model and $\mathscr{G}$ has only one element Γ. Then this is (isomorphically) a classical model for ZF. If we define a truth function on all formulas over $\mathscr{S}$ by

$$v(X) = T \quad \text{if} \quad \Gamma \models X,$$
$$v(X) = F \quad \text{if} \quad \Gamma \not\models X,$$

v will be a classical truth function, and all the axioms of ZF map to T. Thus the notion of intuitionistic ZF model is a generalization of the classical notion.

Suppose $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ were an intuitionistic ZF model such that

$\sim$AC was valid in it, where AC is some classically equivalent form of the axiom of choice expressed without use of the universal quantifier. It follows that the axiom of choice is *classically* unprovable from the axioms of ZF. For otherwise

$$\mathrm{ZF} \vdash_c \mathrm{AC},$$

so for some finite subset $A_1, \ldots, A_n$ of ZF

$$A_1, \ldots, A_n \vdash_c \mathrm{AC}.$$

We may suppose $A_1, \ldots, A_n$ stated without the universal quantifier.

$$\vdash_c (A_1 \wedge \cdots \wedge A_n) \supset \mathrm{AC}.$$

So by the results of ch. 4 § 8

$$\vdash_I \sim\sim ((A_1 \wedge \cdots \wedge A_n) \supset \mathrm{AC}),$$

equivalently,

$$\vdash_I (A_1 \wedge \cdots \wedge A_n) \supset \sim\sim \mathrm{AC}.$$

But $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is an intuitionistic model in which $A_1, \ldots, A_n$, $\sim$AC are valid, a contradiction. Thus to show the classical independence of the axiom of choice it suffices to construct an intuitionistic ZF model in which $\sim$AC is valid. Similar results hold for the independence of the continuum hypothesis and of the axiom of constructability.

In this chapter we will define intuitionistic generalizations of the classical M_α sequence of Gödel [4], which provide intuitionistic generalizations of L, the class of constructable sets. We will show these generalizations are intuitionistic ZF models. In later chapters we will give specific intuitionistic generalizations of L establishing the independence of the axiom of choice, the continuum hypothesis and the axiom of constructability. The specific models constructed, and most of the general methods will be those of forcing, due to Cohen [3]. It is the point of view that is different. No classical models are constructed, complete sequences and countable ZF models are not used.

In [5], Gregorzyk noted the foundations of a connection between forcing and intuitionistic logic. In [13] Kripke discussed the relationship between forcing and his models.

Remark 1.1: For the rest of part II we shall distinguish informally between constants, bound variables, and free variables. We shall use

$x, y, z, \ldots$ for both bound and free variables. This is an *informal* distinction. Formally, free variables and constants are both parameters in the sense of part I since free variables are simply place holders for arbitrary constants.

§ 2. The classical M_α sequence

Let V be a classical ZF model. In [4] Gödel defined over V the sequence M_α of sets as follows.

$M_0 = \emptyset$.

$M_{\alpha+1}$ is the collection of all (first order) definable subsets of M_α.

$M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$ for limit ordinals, λ.

Let the class L be $\bigcup_{\alpha\in V} M_\alpha$. Gödel showed that L was a classical ZF model. As an introduction to the intuitionistic generalization, we re-state the Gödel construction using characteristic functions instead of sets. Now of course "$\in$" is to be considered as a formal predicate symbol, not as set membership.

Let M be some collection and let v be a truth function on the set of formulas with constants from M. We say a (characteristic) function f is *definable over* $\langle M, v\rangle$ if *domain* $(f) = M$, *range* $(f) \subseteq \{T, F\}$, and for some formula $X(x)$ with one free variable and all constants from M, for all $a \in M$

$$f(a) = v(X(a)).$$

Let M' be the elements of M together with all functions definable over $\langle M, v\rangle$.

We define a truth function v' on the set of formulas with constants from M' by defining it for atomic formulas. If $f, g \in M'$ we have three cases:

(1). $f, g \in M$; let $v'(f \in g) = v(f \in g)$.

(2). $f \in M$, $g \in M' - M$; let $v'(f \in g) = g(f)$.

(3). $f \in M' - M$; let $X(x)$ be the formula which defines f over $\langle M, v\rangle$. If there is an $h \in M$ such that

$$v((\forall x)(x \in h \equiv X(x))) = T,$$

and

$$v'(h \in g) = T,$$

let

$$v'(f \in g) = T.$$

Otherwise let

$$v'(f \in g) = F.$$

(Case (3) reduces the situation to case (1) or case (2).) We call the pair $\langle M', v' \rangle$ the derived model of $\langle M, v \rangle$.

Now let $M_0 = \emptyset$ and let v_0 be the obvious truth function. Thus we have $\langle M_0, v_0 \rangle$. Let $\langle M_{\alpha+1}, v_{\alpha+1} \rangle$ be the derived model of $\langle M_\alpha, v_\alpha \rangle$. If λ is a limit ordinal, let $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$. Let $v_\lambda(f \in g) = T$ if for some $\alpha < \lambda$, $v_\alpha(f \in g) = T$. Otherwise let $v_\lambda(f \in g) = F$. Thus we have $\langle M_\lambda, v_\lambda \rangle$. Let

$$L = \bigcup_{\alpha \in V} M_\alpha.$$

Let $v(f \in g) = T$ if for some $\alpha \in V$, $v_\alpha(f \in g) = T$. Otherwise let $v(f \in g) = F$. Thus we have the "class" model $\langle L, v \rangle$.

The reader may convince himself that this construction is essentially equivalent to Gödel's, so that if A is any axiom of ZF, $v(A) = T$. Thus $\langle L, v \rangle$ is a classical ZF model, though not a standard one.

For a boolean generalization of this type of sequence see ch. 14 § 7.

§ 3. The intuitionistic M_α sequence

Suppose we have a model $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ (recall that $\mathscr{S}$ is a set, the range of the $\mathscr{P}$ map, and that there is only one predicate symbol $\in$). For convenience, let $\boldsymbol{P}$ be the collection of all $\mathscr{R}$-closed subsets of $\mathscr{G}$.

We say a function f is *definable over* $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ if *domain* $(f) = \mathscr{S}$, *range* $(f) \subseteq \boldsymbol{P}$, and for some formula $X(x)$ with one free variable, all constants from $\mathscr{S}$, and no *universal quantifiers*, for any $a \in \mathscr{S}$

$$f(a) = \{\Gamma \mid \Gamma \models X(a)\}.$$

Let $\mathscr{S}'$ be the elements of $\mathscr{S}$ together with all functions definable over $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$. We define a $\models'$ relation by giving it for atomic formulas over $\mathscr{S}'$. If $f, g \in \mathscr{S}'$ we have three cases:

(1). $f, g \in \mathscr{S}$; then let $\Gamma \models' (f \in g)$ if $\Gamma \models (f \in g)$.

(2). $f \in \mathscr{S}$, $g \in \mathscr{S}' - \mathscr{S}$: let $\Gamma \models' (f \in g)$ if $\Gamma \in g(f)$.

(3). $f \in \mathscr{S}' - \mathscr{S}$; let $X(x)$ be the formula which defines f over

$\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$. Let $\Gamma \models' (f \in g)$ if there is an $h \in \mathscr{S}$ such that

$$\Gamma \models \sim (\exists x) \sim (x \in h \equiv X(x))$$

and

$$\Gamma \models' (h \in g).$$

(This reduces the situation to case (1) or case (2).) We call the model $\langle \mathscr{G}, \mathscr{R}, \models', \mathscr{S}' \rangle$ the *derived model of* $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

Now let V be a *classical* (first order) model for ZF. We define a sequence of intuitionistic models in V as follows:

Let $\langle \mathscr{G}, \mathscr{R}, \models_0, \mathscr{S}_0 \rangle$ be any intuitionistic model satisfying the following five conditions:

(1). $\langle \mathscr{G}, \mathscr{R}, \models_0, \mathscr{S}_0 \rangle \in V$

(2). $\mathscr{S}_0$ is a collection of functions such that, if $f \in \mathscr{S}_0$, *domain* $(f) \subseteq \mathscr{S}_0$ and *range* $(f) \subseteq \boldsymbol{P}$.

(3). for $f, g \in \mathscr{S}_0$, $\Gamma \models_0 (f \in g)$ iff $\Gamma \in g(f)$.

(4) (extensionality). for $f, g, h \in \mathscr{S}_0$, if $\Gamma \models_0 \sim (\exists x) \sim (x \in f \equiv x \in g)$ and $\Gamma \models_0 \sim (f \in h)$ then $\Gamma \models_0 \sim (g \in h)$.

(5) (regularity). $\mathscr{S}_0$ is well-founded with respect to the relation $x \in$ *domain* (y).

Remark 3.1: If we consider the symbols $\vee$, $\wedge$, $\sim$, $\supset$, $\forall$, $\exists$, (,), $\in$, $x_1, x_2, x_3, \ldots$ to be suitable "code" sets, formulas are sequences of sets, and hence sets. It is in this sense that (1) is meant. See also § 14.

Next let $\langle \mathscr{G}, \mathscr{R}, \models_{\alpha+1}, \mathscr{S}_{\alpha+1} \rangle$ be the derived model of $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, \mathscr{S}_\alpha \rangle$. If λ is a limit ordinal, let $\mathscr{S}_\lambda = \bigcup_{\alpha < \lambda} \mathscr{S}_\alpha$. Let $\Gamma \models_\lambda (f \in g)$ if for some $\alpha < \lambda$, $\Gamma \models_\alpha (f \in g)$. Thus we have $\langle \mathscr{G}, \mathscr{R}, \models_\lambda, \mathscr{S}_\lambda \rangle$.

Finally, let $\mathscr{S} = \bigcup_{\alpha \in V} \mathscr{S}_\alpha$. Let $\Gamma \models (f \in g)$ if for some $\alpha \in V$, $\Gamma \models_\alpha (f \in g)$. Thus we have the "class" model $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

We will spend the rest of this chapter showing

***Theorem* 3.2:** $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is an intuitionistic ZF model.

Remark 3.3: If as a special case we let $\mathscr{S}_0$ be empty and $\mathscr{G} = \{\Gamma\}$, and we identify T with $\{\Gamma\}$ and F with $\emptyset$, the result is the characteristic function version of the M_α sequence in § 2. (The truth functions become $v_\alpha(X) = \{\Gamma \mid \Gamma \models_\alpha X\}$.) Thus as a special case of the above theorem, L is a classical ZF model.

Notation: Sometimes we will write $g_X \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$ where by the subscript X we mean g is the function defined over the model $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, \mathscr{S}_\alpha \rangle$ by the formula $X(x)$. Then part (2) of the definition of $\models'$ for the derived model may be restated as:

$$\text{If} \quad f \in \mathscr{S}, \ g_X \in \mathscr{S}' - \mathscr{S}, \quad \text{then} \quad \Gamma \models' (f \in g_X) \quad \text{if} \quad \Gamma \models X(f).$$

§ 4. Dominance

Definition 4.1: Let $X(x_1, \ldots, x_n)$ be a formula with no constants and with all its free variables among $x_1, \ldots, x_n$. We call X *dominant* if for any $\Gamma \in \mathscr{G}$, any α and any $c_1, \ldots, c_n \in \mathscr{S}_\alpha$

$$\Gamma \models_\alpha X(c_1, \ldots, c_n) \Leftrightarrow \Gamma \models X(c_1, \ldots, c_n).$$

Definition 4.2: Let
(1). $(f \subseteq g)$ stand for $\sim(\exists x) \sim (x \in f \supset x \in g)$,
(2). $(f = g)$ stand for $(f \subseteq g) \wedge (g \subseteq f)$.

***Theorem* 4.3:** $(x \in y)$, $(x \subseteq y)$ and $(x = y)$ are dominant.

Proof: That $(x \in y)$ is dominant is obvious. If $(x \subseteq y)$ is dominant, so is $(x = y)$. That $(x \subseteq y)$ is dominant follows from the next three lemmas:

***Lemma* 4.4:** If $f, g \in \mathscr{S}_\alpha$ and $\Gamma \models (f \subseteq g)$, then $\Gamma \models_\alpha (f \subseteq g)$.

Proof: Suppose for some Γ^* and some $h \in \mathscr{S}_\alpha$, $\Gamma^* \models_\alpha (h \in f)$. By the dominance of $(x \in y)$, $\Gamma^* \models (h \in f)$. But $\Gamma^* \models \sim(\exists x) \sim (x \in f \supset x \in g)$, so by intuitionistic logic $\Gamma^* \models \sim\sim(h \in g)$. By dominance again $\Gamma^* \models_\alpha \sim\sim(h \in g)$. Thus $\Gamma \models_\alpha (\forall x)(x \in f \supset \sim\sim x \in g)$, which is equivalent to $\Gamma \models_\alpha \sim(\exists x) \sim (x \in f \supset x \in g)$.

Remark 4.5: The reader may show the two simple facts used above, and often later: X is dominant implies $\sim X$ is dominant and

$$\vdash_I (\forall x)(X(x) \supset \sim\sim Y(x)) \equiv \sim(\exists x) \sim (X(x) \supset Y(x)).$$

***Lemma* 4.6:** If $f, g \in \mathscr{S}_\alpha$ and $\Gamma \models_\alpha (f \subseteq g)$ then $\Gamma \models_{\alpha+1} (f \subseteq g)$.

Proof: $\Gamma \models_\alpha (f \subseteq g)$. Suppose for some Γ^* and some $h \in \mathscr{S}_{\alpha+1}$ $\Gamma^* \models_{\alpha+1} (h \in f)$. If $h \in \mathscr{S}_\alpha$, by dominance $\Gamma^* \models_\alpha (h \in f)$. But $\Gamma^* \models_\alpha (f \subseteq g)$ so as above $\Gamma^* \models_\alpha \sim\sim(h \in g)$, and by dominance $\Gamma^* \models_{\alpha+1} \sim\sim(h \in g)$.

If $h \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$, since $f \in \mathscr{S}_\alpha$ and $\Gamma^* \models_{\alpha+1} (h \in f)$, it must be the case that h is h_X for some formula X over $\mathscr{S}_\alpha$, and there is some $k \in \mathscr{S}_\alpha$ such that

$\Gamma^* \models_{\alpha+1} (k \in f)$ and $\Gamma^* \models_\alpha \sim(\exists x)\sim(x \in k \equiv X(x))$. Since both $k, f \in \mathscr{S}_\alpha$, by dominance $\Gamma^* \models_\alpha (k \in f)$. Thus $\Gamma^* \models_\alpha \sim\sim(k \in g)$, and by dominance $\Gamma^* \models_{\alpha+1} \sim\sim(k \in g)$. That is for any Γ^{**} there is some Γ^{***} such that $\Gamma^{***} \models_{\alpha+1} (k \in g)$. But also $\Gamma^{***} \models_\alpha \sim(\exists x)\sim(x \in k \equiv X(x))$, $k \in \mathscr{S}_\alpha$, so by definition $\Gamma^{***} \models_{\alpha+1} (h_X \in g)$. Thus $\Gamma^* \models_{\alpha+1} \sim\sim(h_X \in g)$.

Hence $\Gamma \models_{\alpha+1} (\forall x)(x \in f \supset \sim\sim x \in g)$, so $\Gamma \models_{\alpha+1} \sim(\exists x)\sim(x \in f \supset x \in g)$.

***Lemma* 4.7:** If $f, g \in \mathscr{S}_\alpha$ and $\Gamma \models_\alpha (f \subseteq g)$, then $\Gamma \models (f \subseteq g)$.

Proof: First, by transfinite induction, for any $\beta \geqslant \alpha$, $\Gamma \models_\beta (f \subseteq g)$. The successor ordinal step is given by lemma 4.6. Suppose λ is a limit ordinal, $\lambda > \alpha$, and the result is known for all β such that $\alpha \leqslant \beta < \lambda$. If $\Gamma^* \models_\lambda (h \in f)$, then for some $\beta < \lambda$ $\Gamma^* \models_\beta (h \in f)$. But $\Gamma^* \models_\beta (f \subseteq g)$, so $\Gamma^* \models_\beta \sim\sim(h \in g)$. By dominance $\Gamma^* \models_\lambda \sim\sim(h \in g)$. So $\Gamma \models_\lambda (f \subseteq g)$.

Finally, that $\Gamma \models (f \subseteq g)$ follows just as in the limit ordinal case.

§ 5. A little about equality

***Theorem* 5.1:** If $f \in \mathscr{S}_\alpha$ and $g_X \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$ then $\Gamma \models_\alpha \sim(\exists x)\sim(x \in f \equiv X(x))$ if and only if $\Gamma \models_{\alpha+1} (f = g_X)$.

This follows from the next two lemmas:

***Lemma* 5.2:** If $f \in \mathscr{S}_\alpha$, $g_X \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$ and $\Gamma \models_{\alpha+1} (f = g_X)$, then

$$\Gamma \models_\alpha \sim(\exists x)\sim(x \in f \equiv X(x)).$$

Proof: Suppose for some Γ^* and some $h \in \mathscr{S}_\alpha$, $\Gamma^* \models_\alpha (h \in f)$. Then $\Gamma^* \models_{\alpha+1} (h \in f)$, so $\Gamma^* \models_{\alpha+1} \sim\sim(h \in g_X)$. Thus for any Γ^{**} there is a Γ^{***} such that $\Gamma^{***} \models_{\alpha+1} (h \in g_X)$. But $h \in \mathscr{S}_\alpha$, $g_X \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$, so $\Gamma^{***} \in g_X(h)$, that is $\Gamma^{***} \models_\alpha X(h)$. Thus $\Gamma^* \models_\alpha \sim\sim X(h)$, so $\Gamma \models_\alpha (\forall x)(x \in f \supset \sim\sim X(x))$ or $\Gamma \models_\alpha \sim(\exists x)\sim(x \in f \supset X(x))$. Similarly $\Gamma \models_\alpha \sim(\exists x)\sim(X(x) \supset x \in f)$. The result follows since $\sim(\exists x)\sim X_1(x) \wedge \sim(\exists x)\sim X_2(x) \vdash_I \sim(\exists x)\sim(X_1(x) \wedge X_2(x))$.

***Lemma* 5.3:** If $f \in \mathscr{S}_\alpha$, $g_X \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$ and $\Gamma \models_\alpha \sim(\exists x)\sim(x \in f \equiv X(x))$, then $\Gamma \models_{\alpha+1} (f = g_X)$.

Proof: $\Gamma \models_\alpha \sim(\exists x)\sim(x \in f \equiv X(x))$. Suppose for some Γ^* and some $h \in \mathscr{S}_{\alpha+1}$, $\Gamma^* \models_{\alpha+1} (h \in f)$.

If $h \in \mathscr{S}_\alpha$, trivially $\Gamma^* \models_{\alpha+1} \sim\sim(h \in g_X)$.

If $h \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$, then since $f \in \mathscr{S}_\alpha$ h must be h_Y for some formula Y over $\mathscr{S}_\alpha$, and there is some $k \in \mathscr{S}_\alpha$ such that $\Gamma^* \models_{\alpha+1} (k \in f)$ and

$\Gamma^* \Vdash_\alpha \sim(\exists x)\sim(x \in k \equiv Y(x))$. By dominance $\Gamma^* \Vdash_\alpha (k \in f)$, so $\Gamma^* \Vdash_\alpha \sim\sim X(k)$. So for every Γ^{**} there is a Γ^{***} such that $\Gamma^{***} \Vdash_\alpha X(k)$. Thus $\Gamma^{***} \Vdash_{\alpha+1} (k \in g_X)$. But also $\Gamma^{***} \Vdash_\alpha \sim(\exists x)\sim(x \in k \equiv Y(x))$, so by definition $\Gamma^{***} \Vdash_{\alpha+1} (h_Y \in g_X)$. Thus $\Gamma^* \Vdash_{\alpha+1} \sim\sim(h \in g_X)$. Hence $\Gamma \Vdash_{\alpha+1} (f \subseteq g_X)$.

In a similar manner it can be shown that $\Gamma \Vdash_{\alpha+1} (g_X \subseteq f)$.

For later use we show the following most useful corollary:

***Theorem* 5.4:** If $\Gamma \Vdash_\alpha (f \in g)$, then there is an $h \in$ *domain*(g) such that $\Gamma \Vdash_\alpha (f = h) \wedge (h \in g)$.

Proof: By induction on α:

If $\alpha = 0$, and $\Gamma \Vdash_0 (f \in g)$, by definition f must be in the domain of g.

Suppose the result is known for α, and $\Gamma \Vdash_{\alpha+1} (f \in g)$. We have three cases:

(1). If $f, g \in \mathscr{S}_\alpha$, the result is by induction hypothesis.

(2). If $f \in \mathscr{S}_\alpha$, $g \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$, the result is trivial since $f \in$ *domain*(g).

(3). If $f \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$, by definition and theorem 5.1 for some $k \in \mathscr{S}_\alpha$ $\Gamma \Vdash_{\alpha+1} (k \in g) \wedge (k = f)$. Since $\Gamma \Vdash_{\alpha+1} (k \in g)$, by case (1) or case (2) there is some $h \in$ *domain*(g) such that $\Gamma \Vdash_{\alpha+1} (h \in g) \wedge (h = k)$. But trivially if $\Gamma \Vdash_{\alpha+1} (h = k) \wedge (k = f)$, then $\Gamma \Vdash_{\alpha+1} (h = f)$.

The limit ordinal step is simple.

Remark 5.6: By dominance of $(x \in g)$ and $(x = g)$, the result follows also for the class model.

§ 6. Weak substitutivity of equality

***Theorem* 6.1:** Let $X(x)$ be a formula with one free variable and no universal quantifiers. If $\Gamma \Vdash_\alpha (f = g)$ and $\Gamma \Vdash_\alpha \sim X(f)$, then $\Gamma \Vdash_\alpha \sim X(g)$. Similarly if $\Gamma \Vdash (f = g)$ and $\Gamma \Vdash \sim X(f)$, then $\Gamma \Vdash \sim X(g)$.

Proof: Suppose the result is known in the model $\langle \mathscr{G}, \mathscr{R}, \Vdash_\alpha, \mathscr{S}_\alpha \rangle$ (or in $\langle \mathscr{G}, \mathscr{R}, \Vdash, \mathscr{S} \rangle$) for all *atomic* formulas $X(x)$. It then follows for all formulas $X(x)$ by the following intuitionistic theorems:

$$
\begin{aligned}
\sim X \equiv \sim Y \vdash_{\mathrm{I}} \; & \sim(X \wedge Z) \equiv \sim(Y \wedge Z), \\
& \sim(X \vee Z) \equiv \sim(Y \vee Z), \\
& \sim(\sim X) \quad \equiv \sim(\sim Y), \\
& \sim(X \supset Z) \equiv \sim(Y \supset Z), \\
& \sim(Z \supset X) \equiv \sim(Z \supset Y), \\
(\forall x)[\sim X(x) \equiv \sim Y(x)] \vdash_{\mathrm{I}} \; & \sim(\exists x)\, X(x) \equiv \sim(\exists x)\, Y(x).
\end{aligned}
$$

Thus we must show the result for atomic formulas. Over $\langle \mathscr{G}, \mathscr{R}, \Vdash_0, \mathscr{S}_0 \rangle$ an atomic formula must be either $(a \in x)$, $(x \in a)$ or $(a \in b)$ for $a, b \in \mathscr{S}_0$. The case $(a \in b)$ is trivial. For the case $(a \in x)$, we are given: $\Gamma \Vdash_0 \sim (\exists x) \sim (x \in f \equiv x \in g)$ and $\Gamma \Vdash_0 \sim (a \in f)$. The result $\Gamma \Vdash_0 \sim (a \in g)$ follows by intuitionistic logic. For the case $(x \in a)$ the result is condition (4) on $\langle \mathscr{G}, \mathscr{R}, \Vdash_0, \mathscr{S}_0 \rangle$ in § 3.

Suppose the result is known for all formulas over $\mathscr{S}_\alpha$. We show it for atomic formulas of $\langle \mathscr{G}, \mathscr{R}, \Vdash_{\alpha+1}, \mathscr{S}_{\alpha+1} \rangle$. Again an atomic formula must be either $(a \in x)$, $(x \in a)$ or $(a \in b)$ for $a, b \in \mathscr{S}_{\alpha+1}$. As above $(x \in a)$ is the only difficult case. Thus we are given $\Gamma \Vdash_{\alpha+1} (f = g)$ and $\Gamma \Vdash_{\alpha+1} \sim (f \in a)$. We have eight subcases:

(1) $a, f, g \in \mathscr{S}_\alpha$,
(2) $a, f \in \mathscr{S}_\alpha$, $g \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$,
(3). $a, g \in \mathscr{S}_\alpha$, $f \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$,
(4). $a \in \mathscr{S}_\alpha$, $f, g \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$,
(5). $a \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$, $f, g \in \mathscr{S}_\alpha$,
(6). $a, g \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$, $f \in \mathscr{S}_\alpha$,
(7). $a, f \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$, $g \in \mathscr{S}_\alpha$,
(8). $a, f, g \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$.

We treat these cases separately.

Case (1). The result follows by dominance of $(x \in y)$ and $(x = y)$, and the induction hypothesis.

Case (2). Suppose $\Gamma \nVdash_{\alpha+1} \sim (g \in a)$. Then for some Γ^* $\Gamma^* \Vdash_{\alpha+1} (g \in a)$. By theorem 5.4 there is an $h \in \mathscr{S}_\alpha$ such that $\Gamma^* \Vdash_{\alpha+1} (g = h) \wedge (h \in a)$. But $\Gamma^* \Vdash_{\alpha+1} (f = g)$, hence $\Gamma^* \Vdash_{\alpha+1} (f = h)$. By dominance $\Gamma^* \Vdash_\alpha (f = h) \wedge (h \in a)$. By induction hypothesis $\Gamma^* \Vdash_\alpha \sim \sim (f \in a)$. By dominance $\Gamma^* \Vdash_{\alpha+1} \sim \sim (f \in a)$, so $\Gamma \nVdash_{\alpha+1} \sim (f \in a)$.

Case (3). Suppose $\Gamma \nVdash_{\alpha+1} \sim (g \in a)$. Then for some Γ^* $\Gamma^* \Vdash_{\alpha+1} (g \in a)$. But $\Gamma^* \Vdash_{\alpha+1} (f = g)$. Now by theorem 5.1 and the definitions $\Gamma^* \Vdash_{\alpha+1} (f \in a)$, so $\Gamma \nVdash_{\alpha+1} \sim (f \in a)$.

Case (4). This is an elaboration of (2) and (3).

Case (5). a is $a_X \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$. Suppose $\Gamma \nVdash_{\alpha+1} \sim (g \in a_X)$. Then for some Γ^* $\Gamma^* \Vdash_{\alpha+1} (g \in a_X)$, so $\Gamma^* \Vdash_\alpha X(g)$. But $\Gamma^* \Vdash_{\alpha+1} (f = g)$, so by dominance $\Gamma^* \Vdash_\alpha (f = g)$. By hypothesis $\Gamma^* \Vdash_\alpha \sim \sim X(f)$, so it follows that $\Gamma^* \Vdash_{\alpha+1} \sim \sim (f \in a_X)$. Hence $\Gamma \nVdash_{\alpha+1} (f \in a_X)$.

Case (6). Suppose $\Gamma \nVdash_{\alpha+1} \sim (g \in a)$. For some Γ^* $\Gamma^* \Vdash_{\alpha+1} (g \in a)$. By theorem 5.4 for some $h \in \mathscr{S}_\alpha$ $\Gamma^* \Vdash_{\alpha+1} (g = h) \wedge (h \in a)$. But $\Gamma^* \Vdash_{\alpha+1} (f = g)$

so $\Gamma^* \Vdash_{\alpha+1} (f=h)$. By dominance $\Gamma^* \Vdash_\alpha (f=h)$. Moreover a must be $a_X \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$. Since $\Gamma^* \Vdash_{\alpha+1} (h \in a)$, $\Gamma^* \Vdash_\alpha X(h)$. By hypothesis $\Gamma^* \Vdash_\alpha \sim\sim X(f)$, and so $\Gamma^* \Vdash_{\alpha+1} \sim\sim (f \in a_X)$. Thus $\Gamma \nVdash_{\alpha+1} \sim (f \in a_X)$.

Case (7). Suppose $\Gamma \nVdash_{\alpha+1} \sim (g \in a)$. Then for some Γ^* $\Gamma^* \Vdash_{\alpha+1} (g \in a)$. But $\Gamma^* \Vdash_{\alpha+1} (f=g)$, so by theorem 5.1 and the definitions $\Gamma^* \Vdash_{\alpha+1} (f \in a)$. Thus $\Gamma \nVdash_{\alpha+1} \sim (f \in a)$.

Case (8). This is an elaboration of (6) and (7).

Thus, we have the result for successor models. The result for atomic formulas in limit models and in the class model is straightforward.

§ 7. More on dominance

Definition 7.1: A formula X is called *stable* if $\vdash_I X \equiv \sim\sim X$.

Definition 7.2: A formula X (with no universal quantifiers) is said to have its quantifiers *bounded* if every subformula beginning with a quantifier is of the form

$$(\exists x)((x \in v) \wedge Y(x)),$$

where v is a variable or a constant. Moreover, if Y is stable we say X has *strongly bounded* quantifiers.

***Theorem* 7.3:** Let X be any formula with no constants, no universal quantifiers and all its quantifiers strongly bounded. Then X is dominant.

Proof: By induction on the degree of X. If X is atomic the result is just the dominance of $(x \in y)$.

Suppose X is not atomic and the result is known for all formulas of lesser degree. The four cases X is $(Y \vee Z)$, $(Y \wedge Z)$, $\sim Y$ or $(Y \supset Z)$ are simple. Suppose $X(y, z, \ldots)$ is $(\exists x)[(x \in y) \wedge Y(x, y, z, \ldots)]$ where Y is stable and by hypothesis dominant. Suppose $a, b, \ldots \in \mathscr{S}_\alpha$.

If $\Gamma \Vdash_\alpha X(a, b, \ldots)$ then $\Gamma \Vdash_\alpha (\exists x)[(x \in a) \wedge Y(x, a, b, \ldots)]$. For some $f \in \mathscr{S}_\alpha$ $\Gamma \Vdash_\alpha (f \in a) \wedge Y(f, a, b, \ldots)$. By hypothesis both of these are dominant, so $\Gamma \Vdash (f \in a) \wedge Y(f, a, b, \ldots)$. $\Gamma \Vdash (\exists x)[(x \in a) \wedge Y(x, a, b, \ldots)]$. Hence $\Gamma \Vdash X(a, b, \ldots)$.

Conversely suppose $\Gamma \Vdash X(a, b, \ldots)$. $\Gamma \Vdash (\exists x)[(x \in a) \wedge Y(x, a, b, \ldots)]$. Then for some $f \in \mathscr{S}$, $\Gamma \Vdash (f \in a) \wedge Y(f, a, b, \ldots)$. $a \in \mathscr{S}_\alpha$, so by theorem 5.4 there is a $g \in \mathscr{S}_\alpha$ such that $\Gamma \Vdash (f=g) \wedge (g \in a)$. By weak substitutivity of equality $\Gamma \Vdash \sim\sim Y(g, a, b, \ldots)$. But Y is stable so $\Gamma \Vdash Y(g, a, b, \ldots)$. Now by dominance

$$\Gamma \models_\alpha (g \in a) \wedge Y(g, a, b, \ldots),$$
$$\Gamma \models_\alpha (\exists x)\,[(x \in a) \wedge Y(x, a, b, \ldots)].$$

Hence $\Gamma \models_\alpha X(a, b, \ldots)$.

We define the following formula abbreviations:

$y = \emptyset$	for	$\sim(\exists x)\,(x \in y)$,
$\emptyset \in y$	for	$(\exists x)\,(x \in y \wedge x = \emptyset)$,
$y = x'$	for	$\sim(\exists w) \sim [w \in y \equiv (w \in x \vee w = x)]$,
$x' \in y$	for	$(\exists w)\,(w \in y \wedge w = x')$,
$\omega \subseteq y$	for	$\sim\sim(\emptyset \in y) \wedge \sim(\exists x) \sim [x \in y \supset x' \in y]$,
$x = \{y, z\}$	for	$\sim(\exists w) \sim [w \in x \equiv (w = y \vee w = z)]$,
$x = \bigcup y$	for	$\sim(\exists z) \sim [z \in x \equiv (\exists w)\,(w \in y \wedge z \in w)]$.

***Theorem* 7.4:** The above formulas are dominant.

Proof: $y = \emptyset$ and $\emptyset \in y$ are directly by theorem 7.3.

$y = x'$ is equivalent to the conjunction of the following two formulas:

$$\sim(\exists w)\,[w \in y \wedge \sim(w \in x \vee w = x)],$$
$$\sim(\exists w) \sim [(w \in x \vee w = x) \supset w \in y].$$

The dominance of the first is by the above theorem. That of the second is simple to show.

In a similar fashion the remaining formulas follow, making use of

$$\vdash_{\mathrm{I}} \sim(\exists x) \sim [X(x) \supset Y(x)] \equiv \sim(\exists x)\,[X(x) \wedge \sim Y(x)]$$

and

$$\vdash_{\mathrm{I}} \sim(\exists x) \sim [X(x) \equiv Y(x)] \equiv \sim(\exists x) \sim [X(x) \supset Y(x)] \wedge \sim(\exists x) \sim [Y(x) \supset X(x)]$$

§ 8. Axiom of extensionality

***Theorem* 8.1:** The following is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$:

$$\sim(\exists x)\,(\exists y) \sim \{\sim(\exists w) \sim [w \in x \equiv w \in y] \supset \sim(\exists z) \sim [x \in z \equiv y \in z]\}$$

In addition it is valid in every model $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, \mathscr{S}_\alpha \rangle$.

Proof: For any $\Gamma \in \mathscr{G}$ and any $f, g \in \mathscr{S}$, if $\Gamma \models (f = g)$, by weak substitutivity of equality, $\Gamma \models \sim(f \in d) \equiv \sim(g \in d)$. But this holds for every $d \in \mathscr{S}$, so $\Gamma \models (\forall z)[\sim(f \in z) \equiv \sim(g \in z)]$, and by intuitionistic logic $\Gamma \models \sim(\exists z) \sim[f \in z \equiv g \in z]$. Thus the result follows. (The same proof also works for every α.)

§ 9. Null set axiom

***Theorem* 9.1:** The following is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$:

$$(\exists x) \sim (\exists y)(y \in x).$$

In addition it is valid in any model $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, \mathscr{S}_\alpha \rangle$ for $\alpha > 0$.

Proof: Suppose we show the formula is valid in $\langle \mathscr{G}, \mathscr{R}, \models_1, \mathscr{S}_1 \rangle$. If $\Gamma \in \mathscr{G}$ $\Gamma \models_1 (\exists x) \sim (\exists y)(y \in x)$, so for some $f \in \mathscr{S}_1$ $\Gamma \models_1 \sim (\exists y)(y \in f)$, i.e. $\Gamma \models_1 f = \emptyset$. The result then follows by dominance of $x = \emptyset$.

Let $X(x)$ be the formula $\sim(x = x)$. There is an $f_X \in \mathscr{S}_1 - \mathscr{S}_0$. We claim for any $\Gamma \in \mathscr{G}$ $\Gamma \models_1 \sim (\exists y)(y \in f_X)$. Suppose otherwise $\Gamma \not\models_1 \sim (\exists y)(y \in f_X)$. Then for some Γ^* $\Gamma^* \models_1 (\exists y)(y \in f_X)$. For some $d \in \mathscr{S}_1$ $\Gamma^* \models_1 (d \in f_X)$. By theorem 5.4 there is an $e \in \mathscr{S}_0$ such that $\Gamma^* \models_1 (d = e) \wedge (e \in f_X)$. Since $\Gamma^* \models_1 (e \in f_X)$, by definition $\Gamma^* \models_0 X(e)$, i.e. $\Gamma^* \models_0 \sim\sim(\exists x) \sim (x \in e \equiv x \in e)$ which is not possible by intuitionistic logic.

§ 10. Unordered pairs axiom

***Theorem* 10.1:** The following is valid in the class model and in any limit model:

$$\sim(\exists x)(\exists y) \sim (\exists z) \sim (\exists w) \sim [w \in z \equiv (w = x \vee w = y)].$$

Proof: If we show that for any $f, g \in \mathscr{S}_\alpha$ there is an $h \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$ such that $h = \{f, g\}$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models_{\alpha+1}, \mathscr{S}_{\alpha+1} \rangle$, the result will follow by dominance of $x = \{y, z\}$.

Let $f, g \in \mathscr{S}_\alpha$. Let $X(x)$ be the formula $(x = f) \vee (x = g)$. There is an $h_X \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$. We show $h_X = \{f, g\}$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models_{\alpha+1}, \mathscr{S}_{\alpha+1} \rangle$. Let $\Gamma \in \mathscr{G}$.

Suppose $\Gamma^* \models_{\alpha+1} (a \in h_X)$. Then there is some $b \in \mathscr{S}_\alpha$ such that $\Gamma^* \models_{\alpha+1} (a = b) \wedge (b \in h_X)$. Since $\Gamma^* \models_{\alpha+1} (b \in h_X)$, $\Gamma^* \models_\alpha X(b)$. $\Gamma^* \models_\alpha (b = f) \vee (b = g)$. By dominance $\Gamma^* \models_{\alpha+1} (b = f) \vee (b = g)$. But $\Gamma^* \models_{\alpha+1} (a = b)$, so by intuitionistic logic $\Gamma^* \models_{\alpha+1} (a = f) \vee (a = g)$. Thus $\Gamma \models_{\alpha+1} (\forall x)(x \in h_X \supset (x = f \vee x = g))$.

Conversely suppose $\Gamma^* \models_{\alpha+1} (a = f) \vee (a = g)$. Then either $\Gamma^* \models_{\alpha+1} (a = f)$

or $\Gamma^* \vDash_{\alpha+1} (a=g)$. Say $\Gamma^* \vDash_{\alpha+1} (a=f)$. It is trivial to show $\Gamma^* \vDash_{\alpha+1} (f \in h_X)$ so by weak substitutivity of equality $\Gamma^* \vDash_{\alpha+1} \sim\sim (a \in h_X)$. Thus $\Gamma \vDash_{\alpha+1} (\forall x)\, ((x=f \vee x=g) \supset \sim\sim x \in h_X)$. The result follows easily.

§ 11. Union axiom

***Theorem* 11.1:** The following is valid in the class model and in any limit model:

$$\sim (\exists x) \sim (\exists y) \sim (\exists z) \sim [z \in y \equiv (\exists w)\,(z \in w \wedge w \in x)].$$

Proof: If we show that for any $f \in \mathscr{S}_\alpha$ there is a $g \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$ such that $g = \bigcup f$ is valid in $\langle \mathscr{G}, \mathscr{R}, \vDash_{\alpha+1}, \mathscr{S}_{\alpha+1} \rangle$, the result will follow by dominance of $x = \bigcup y$.

Let $f \in \mathscr{S}_\alpha$. Let $X(x)$ be the formula $(\exists w)\,(x \in w \wedge w \in f)$. There is a $g_X \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$. We claim $g_X = \bigcup f$ is valid in $\langle \mathscr{G}, \mathscr{R}, \vDash_{\alpha+1}, \mathscr{S}_{\alpha+1} \rangle$. Let $\Gamma \in \mathscr{G}$.

Suppose $\Gamma^* \vDash_{\alpha+1} (\exists w)\,(h \in w \wedge w \in f)$. Then for some $k \in \mathscr{S}_{\alpha+1}$ $\Gamma^* \vDash_{\alpha+1} (h \in k) \wedge (k \in f)$. Since $\Gamma^* \vDash_{\alpha+1} (k \in f)$, there is some $t \in \mathscr{S}_\alpha$ such that $\Gamma^* \vDash_{\alpha+1} (k=t) \wedge (t \in f)$. By weak substitutivity of equality $\Gamma^* \vDash_{\alpha+1} \sim\sim (h \in t)$. Thus for every Γ^{**} there is a Γ^{***} such that $\Gamma^{***} \vDash_{\alpha+1} (h \in t)$. But $t \in \mathscr{S}_\alpha$ so there is an $s \in \mathscr{S}_\alpha$ such that $\Gamma^{***} \vDash_{\alpha+1} (s=h) \wedge (s \in t)$. But $\Gamma^{***} \vDash_{\alpha+1} (h \in k) \wedge (k \in f)$ and $\Gamma^{***} \vDash_{\alpha+1} (s=h) \wedge (k=t)$, so $\Gamma^{***} \vDash_{\alpha+1} \sim\sim [(s \in t) \wedge (t \in f)]$. Now $s, t, f \in \mathscr{S}_\alpha$, so by dominance $\Gamma^{***} \vDash_\alpha \sim\sim [(s \in t) \wedge (t \in f)]$. $\Gamma^{***} \vDash_\alpha (\exists w) \sim\sim [(s \in w) \wedge (w \in f)]$. By intuitionistic logic, $\Gamma^{***} \vDash_\alpha \sim\sim (\exists w)\, [s \in w \wedge w \in f]$, that is $\Gamma^{***} \vDash_\alpha \sim\sim X(s)$, so $\Gamma^{***} \vDash_{\alpha+1} \sim\sim (s \in g_X)$. But $\Gamma^{***} \vDash_{\alpha+1} (s=h)$, so $\Gamma^{***} \vDash_{\alpha+1} \sim\sim (h \in g_X)$. Thus for every Γ^{**} there is a Γ^{***} such that $\Gamma^{***} \vDash_{\alpha+1} \sim\sim (h \in g_X)$. Then $\Gamma^* \vDash_{\alpha+1} \sim\sim (h \in g_X)$. So we have shown

$$\Gamma \vDash_{\alpha+1} (\forall x)\, [(\exists w)\,(x \in w \wedge w \in f) \supset \sim\sim x \in g_X].$$

Conversely suppose $\Gamma^* \vDash_{\alpha+1} (h \in g_X)$. There is some $k \in \mathscr{S}_\alpha$ such that $\Gamma^* \vDash_{\alpha+1} (h=k) \wedge (k \in g_X)$. So $\Gamma^* \vDash_\alpha X(k)$ or $\Gamma^* \vDash_\alpha (\exists w)\,(k \in w \wedge w \in f)$. For some $t \in \mathscr{S}_\alpha$ $\Gamma^* \vDash_\alpha (k \in t) \wedge (t \in f)$. By dominance $\Gamma^* \vDash_{\alpha+1} (k \in t) \wedge (t \in f)$. $\Gamma^* \vDash_{\alpha+1} (\exists w)\,(k \in w \wedge w \in f)$, hence $\Gamma^* \vDash_{\alpha+1} \sim\sim (\exists w)\,(h \in w \wedge w \in f)$.

So we have shown

$$\Gamma \vDash_{\alpha+1} (\forall x)\, [x \in g_X \supset \sim\sim (\exists w)\,(x \in w \wedge w \in f)]$$

The result follows easily.

§ 12. Axiom of infinity

***Theorem* 12.1:** The following is valid in $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ and in $\langle\mathscr{G}, \mathscr{R}, \models_\alpha, \mathscr{S}_\alpha\rangle$ for $\alpha>\omega$:

$$(\exists x)\,[\emptyset\in x \wedge \sim(\exists y)\sim(y\in x \supset y'\in x)]\,.$$

Proof: If we show there is an $f\in\mathscr{S}_{\omega+1}-\mathscr{S}_\omega$ such that $\omega\subseteq f$ is valid in $\langle\mathscr{G}, \mathscr{R}, \models_{\omega+1}, \mathscr{S}_{\omega+1}\rangle$, the result will follow by dominance of $\omega\subseteq x$.

Let $X(x)$ be the formula

$$\sim(\exists y)\sim\{[\sim(\exists z)\sim(z\in y\supset z'\in y)\wedge\emptyset\in y]\supset x\in y\}\,.$$

There is an $f_X\in\mathscr{S}_{\omega+1}-\mathscr{S}_\omega$. We claim $\omega\subseteq f_X$ is valid in $\langle\mathscr{G}, \mathscr{R}, \models_{\omega+1}, \mathscr{S}_{\omega+1}\rangle$. This follows from the next four lemmas:

***Lemma* 12.2:** If $\Gamma\models_\alpha f=\emptyset\wedge g=\emptyset$ then $\Gamma\models_\alpha f=g$.

Proof: $\Gamma\models_\alpha\sim(\exists x)\,(x\in f)\wedge\sim(\exists x)\,(x\in g)$ so by intuitionistic logic $\Gamma\models_\alpha\sim(\exists x)\sim(x\in f\equiv x\in g)$, $\Gamma\models_\alpha f=g$.

***Lemma* 12.3:** $\Gamma\models_{\omega+1}\emptyset\in f_X$.

Proof: By the results of § 9 for some $g\in\mathscr{S}_\omega$ $\Gamma\models_\omega g=\emptyset$. Suppose for some Γ^*

$$\Gamma^*\models_\omega\sim(\exists z)\sim(z\in k\supset z'\in k)\wedge\emptyset\in k\,.$$

Then $\Gamma^*\models_\omega\emptyset\in k$, that is $\Gamma^*\models_\omega(\exists w)\,(w=\emptyset\wedge w\in k)$, so for some $s\in\mathscr{S}_\omega$ $\Gamma^*\models_\omega s=\emptyset\wedge s\in k$. By lemma 12.2 $\Gamma^*\models_\omega s=g$, so $\Gamma^*\models_\omega\sim\sim(g\in k)$. We have shown

$$\Gamma\models_\omega(\forall x)\,\{[\sim(\exists z)\sim(z\in x\supset z'\in x)\wedge\emptyset\in x]\supset\sim\sim(g\in x)\}$$

or equivalently

$$\Gamma\models_\omega\sim(\exists x)\sim\{[\sim(\exists z)\sim(z\in x\supset z'\in x)\wedge\emptyset\in x]\supset g\in x\}\,,$$
$$\Gamma\models_\omega X(g)\,,$$
$$\Gamma\models_{\omega+1} g\in f_X\,.$$

But $\Gamma\models_{\omega+1}g=\emptyset$, so by definition $\Gamma\models_{\omega+1}\emptyset\in f_X$.

***Lemma* 12.4:** If $g\in\mathscr{S}_\alpha$, there is an $h\in\mathscr{S}_{\alpha+1}-\mathscr{S}_\alpha$ such that $h=g'$ is valid in $\langle\mathscr{G}, \mathscr{R}, \models_{\alpha+1}, \mathscr{S}_{\alpha+1}\rangle$.

Proof: Let $Y(x)$ be the formula $(x\in g)\vee(x=g)$. There is an $h_Y\in\mathscr{S}_{\alpha+1}-\mathscr{S}_\alpha$. We will show

$$\Gamma\models_{\alpha+1}\sim(\exists w)\sim[w\in h_Y\equiv(w\in g\vee w=g)]\,.$$

Suppose for some Γ^* $\Gamma^* \vDash_{\alpha+1} (s \in h_Y)$. Then for some $t \in \mathscr{S}_\alpha$

$$\begin{aligned}
&\Gamma^* \vDash_{\alpha+1} (s = t) \wedge (t \in h_Y),\\
&\Gamma^* \vDash_\alpha Y(t),\\
&\Gamma^* \vDash_\alpha (t \in g) \vee (t = g),\\
&\Gamma^* \vDash_{\alpha+1} (t \in g) \vee (t = g),\\
&\Gamma^* \vDash_{\alpha+1} \sim\sim ((s \in g) \vee (s = g)),
\end{aligned}$$

so

$$\Gamma \vDash_{\alpha+1} \sim (\exists w) \sim [w \in h_Y \supset (w \in g \vee w = g)].$$

Conversely suppose $\Gamma^* \vDash_{\alpha+1} (s \in g) \vee (s = g)$. We have two cases:

(1). $\Gamma^* \vDash_{\alpha+1} (s \in g)$. Since $g \in \mathscr{S}_\alpha$ there is some $t \in \mathscr{S}_\alpha$ such that

$$\begin{aligned}
&\Gamma^* \vDash_{\alpha+1} (s = t) \wedge (t \in g),\\
&\Gamma^* \vDash_\alpha (t \in g),\\
&\Gamma^* \vDash_\alpha (t \in g) \vee (t = g),\\
&\Gamma^* \vDash_\alpha Y(t),\\
&\Gamma^* \vDash_{\alpha+1} (t \in h_Y),\\
&\Gamma^* \vDash_{\alpha+1} \sim\sim (s \in h_Y).
\end{aligned}$$

(2). $\Gamma^* \vDash_{\alpha+1} (s = g)$. Since trivially $\Gamma^* \vDash_{\alpha+1} (g \in h_Y)$,

$$\Gamma^* \vDash_{\alpha+1} \sim\sim (s \in h_Y).$$

Thus we have

$$\Gamma \vDash_{\alpha+1} \sim (\exists w) \sim [(w \in g \vee w = g) \supset w \in h_Y].$$

***Lemma* 12.5:** If $\Gamma \vDash_{\omega+1} (g \in f_X)$, then $\Gamma \vDash_{\omega+1} (g' \in f_X)$.

Proof: $\Gamma \vDash_{\omega+1} (g \in f_X)$, so there is an $h \in \mathscr{S}_\omega$ such that $\Gamma \vDash_{\omega+1} (g = h) \wedge (h \in f_X)$. Since $h \in \mathscr{S}_\omega$, for some $\alpha < \omega$ $h \in \mathscr{S}_\alpha$. By lemma 12.4 there is some $k \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$ such that $\Gamma \vDash_{\alpha+1} k = h'$, so by dominance $\Gamma \vDash_\omega k = h'$. But also $\Gamma \vDash_{\omega+1} (h \in f_X)$, $\Gamma \vDash_\omega X(h)$, so

$$\Gamma \vDash_\omega \sim (\exists y) \sim \{[\sim (\exists z) \sim (z \in y \supset z' \in y) \wedge \emptyset \in y] \supset h \in y\}.$$

By intuitionistic logic it follows that

$$\Gamma \vDash_\omega \sim (\exists y) \sim \{[\sim (\exists z) \sim (z \in y \supset z' \in y) \wedge \emptyset \in y] \supset k \in y\},$$

that is $\Gamma \vDash_\omega X(k)$, $\Gamma \vDash_{\omega+1} (k \in f_X)$. But $\Gamma \vDash_{\omega+1} k = h'$, so by definition $\Gamma \vDash_{\omega+1} h' \in f_X$.

§ 13. Axiom of regularity

***Theorem* 13.1:** The following is valid in all models:

$$\sim(\exists x) \sim \{(\exists y)(y \in x) \supset (\exists y)[y \in x \wedge \sim(\exists z)(z \in x \wedge z \in y)]\}.$$

Proof: All the elements of the class $\mathscr{S}$ are functions. We have assumed $\mathscr{S}_0$ is well-founded by the relation $x \in domain(y)$. It then follows that $\mathscr{S}$ is also well-founded by $x \in domain(y)$.

The formula

$$\sim\sim\{(\exists y)(y \in x) \supset (\exists y)[y \in x \wedge \sim(\exists z)(z \in x \wedge z \in y)]\} \qquad (*)$$

is equivalent to

$$\sim\{(\exists y)(y \in x) \wedge \sim(\exists y)[y \in x \wedge \sim(\exists z)(z \in x \wedge z \in y)]\}$$

which is obviously dominant.

Suppose $f \in \mathscr{S}_\alpha$ and $\Gamma \vDash_\alpha (\exists y)(y \in f)$. Then for some $g \in \mathscr{S}_\alpha$, $\Gamma \vDash_\alpha (g \in f)$. We claim

$$\Gamma \vDash_\alpha \sim\sim(\exists y)[y \in f \wedge \sim(\exists z)(z \in f \wedge z \in y)].$$

Suppose otherwise. Then there is some Γ^* such that

$$\Gamma^* \vDash_\alpha \sim(\exists y)[y \in f \wedge \sim(\exists z)(z \in f \wedge z \in y)].$$

We define a set W to be

$$\{x \mid x \in \mathscr{S}_\alpha \text{ and for some } \Gamma^{**} \ \ \Gamma^{**} \vDash_\alpha (x \in f)\}.$$

W is not empty since $g \in W$. The relation $x \in domain(y)$ well-founds W. Let s be a "smallest" element of W. That is, $s \in W$, but for no $t \in W$ is $t \in domain(s)$. Since $s \in W$, for some Γ^{**} $\Gamma^{**} \vDash_\alpha (s \in f)$. We claim

$$\Gamma^{**} \vDash_\alpha \sim(\exists z)(z \in f \wedge z \in s).$$

Suppose not. Then for some Γ^{***}

$$\Gamma^{***} \vDash_\alpha (\exists z)(z \in f \wedge z \in s).$$

Thus for some $r \in \mathscr{S}_\alpha$

$$\Gamma^{***} \vDash_\alpha (r \in f) \wedge (r \in s).$$

Since $\Gamma^{***} \vDash_\alpha (r \in s)$, there is some $t \in domain(s)$ such that $\Gamma^{***} \vDash_\alpha (r = t) \wedge \wedge (t \in s)$. But then $\Gamma^{***} \vDash_\alpha \sim\sim(t \in f)$, so for some Γ^{****}

$$\Gamma^{****} \vDash_\alpha (t \in f),$$

so $t \in W$, a contradiction. Thus $\Gamma^{**} \Vdash_\alpha \sim(\exists z)(z \in f \wedge z \in s)$. But $\Gamma^{**} \Vdash_\alpha (s \in f)$ so

$$\Gamma^{**} \Vdash_\alpha (\exists y)[y \in f \wedge \sim(\exists z)(z \in f \wedge z \in y)],$$

and this contradicts

$$\Gamma^{*} \Vdash_\alpha \sim(\exists y)[y \in f \wedge \sim(\exists z)(z \in f \wedge z \in y)].$$

Thus

$$\Gamma \Vdash_\alpha \sim\sim(\exists y)[y \in f \wedge \sim(\exists z)(z \in f \wedge z \in y)].$$

But Γ was arbitrary. We have shown that for each $f \in \mathscr{S}_\alpha$ the following is valid in $\langle \mathscr{G}, \mathscr{R}, \Vdash_\alpha, \mathscr{S}_\alpha \rangle$:

$$(\exists y)(y \in f) \supset \sim\sim(\exists y)[y \in f \wedge \sim(\exists z)(z \in f \wedge z \in y)].$$

The theorem now follows by the dominance of (∗).

§ 14. Definability of the models

One of our initial assumptions was that $\langle \mathscr{G}, \mathscr{R}, \Vdash_0, \mathscr{S}_0 \rangle \in V$. The definition of the sequence was an inductive definition. It should be clear that the definition can be carried out in V itself. That is, not only is $\langle \mathscr{G}, \mathscr{R}, \Vdash_\alpha, \mathscr{S}_\alpha \rangle \in V$ for each $\alpha \in V$, but moreover

***Theorem* 14.1:** There is a formula $F(x, y)$ over V which defines the sequence of $\langle \mathscr{G}, \mathscr{R}, \Vdash_\alpha, \mathscr{S}_\alpha \rangle$. That is, for $x, y \in V$ $F(x, y)$ is true over V if and only if x is some ordinal α and y is $\langle \mathscr{G}, \mathscr{R}, \Vdash_\alpha, \mathscr{S}_\alpha \rangle$. (In fact $F(x, y)$ can be absolute, as should be obvious.)

Of course $\langle \mathscr{G}, \mathscr{R}, \Vdash, \mathscr{S} \rangle$ is not in V, since, in particular, $\mathscr{S}$ is not a set. But we do have

***Theorem* 14.2:** Let $X(x_1, \ldots, x_n)$ be any formula with no constants and no universal quantifiers. There is a (classical) formula $R_X(z, x_1, \ldots, x_n)$ with constants from V such that for any $\Gamma \in \mathscr{G}$ and $c_1, \ldots, c_n \in \mathscr{S}$, $\Gamma \Vdash X(c_1, \ldots, c_n)$ if and only if $R_X(\Gamma, c_1, \ldots, c_n)$ is true over V.

Proof: By induction on the degree of X. Suppose X is atomic, $(x \in y)$. Let $R_X(z, x, y)$ be the formula

$$z \in \mathscr{G} \wedge (\exists \alpha)(\mathit{ordinal}(\alpha) \wedge x \in \mathscr{S}_\alpha \wedge y \in \mathscr{S}_\alpha \wedge z \Vdash_\alpha (x \in y))$$

(where we have used the obvious abbreviations allowed by the above theorem).

Suppose X is not atomic but the result is known for all formulas of lesser degree. If $X(x_1,\ldots, x_n)$ is $Y(x_1,\ldots, x_n) \vee Z(x_1,\ldots, x_n)$, by hypothesis there are formulas $R_Y(w, x_1,\ldots, x_n)$ and $R_Z(w, x_1,\ldots, x_n)$. Let $R_X(w, x_1,\ldots, x_n)$ be the formula $R_Y(w, x_1,\ldots, x_n) \vee R_Z(w, x_1,\ldots, x_n)$.

The case X is $Y \wedge Z$ is similar.

Suppose $X(x_1,\ldots, x_n)$ is $\sim Y(x_1,\ldots, x_n)$. By hypothesis there is a formula $R_Y(z, x_1,\ldots, x_n)$. Let $R_X(z, x_1,\ldots, x_n)$ be the formula

$$\sim (\exists w)\,(w \in \mathscr{G} \wedge z\mathscr{R}w \wedge R_Y(w, x_1,\ldots, x_n)).$$

The case X is $Y \supset Z$ is similar.

Suppose $X(x_1,\ldots, x_n)$ is $(\exists y)\; Y(y, x_1,\ldots, x_n)$. By hypothesis there is a formula $R_Y(w, y, x_1,\ldots, x_n)$. Let $R_X(w, x_1,\ldots, x_n)$ be the formula

$$(\exists y)\,(\exists \alpha)\,[\mathit{ordinal}(\alpha) \wedge y \in \mathscr{S}_\alpha \wedge R_Y(w, y, x_1,\ldots, x_n)].$$

§ 15. Power set axiom

We wish to show in this section that the power set axiom is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

Let c_0 be a *fixed* element of $\mathscr{S}$. Then for some smallest ordinal α_0 $c_0 \in \mathscr{S}_{\alpha_0}$. Thus α_0 is also *fixed*. We first want to show that for a fixed $\Gamma \in \mathscr{G}$ there is a β_0 such that for any $c \in \mathscr{S}$, if $\Gamma \models (c \subseteq c_0)$, there is some $d \in \mathscr{S}_{\beta_0}$ such that $\Gamma \models (c = d)$. After showing this we will show that in fact there is one β_0 which will do for all $\Gamma \in \mathscr{G}$.

For the above fixed c_0, α_0 and Γ, for $c_1, c_2 \in \mathscr{S}$ such that $\Gamma \models (c_1 \subseteq c_0) \wedge (c_2 \subseteq c_0)$, if for all Γ^* and for all $t \in \mathscr{S}_{\alpha_0}$

$$\Gamma^* \models ((t \in c_1) \equiv (t \in c_2)),$$

then $\Gamma \models (c_1 = c_2)$. The proof is as follows:

Suppose for some Γ^* and some $h \in \mathscr{S}$ $\Gamma^* \models (h \in c_1)$. Since $\Gamma \models (c_1 \subseteq c_0)$ $\Gamma^* \models \sim\sim (h \in c_0)$. Then for any Γ^{**} there is a Γ^{***} such that $\Gamma^{***} \models (h \in c_0)$. But $c_0 \in \mathscr{S}_{\alpha_0}$, so there is some $t \in \mathscr{S}_{\alpha_0}$ such that $\Gamma^{***} \models (h = t) \wedge (t \in c_0)$. Since $\Gamma^{***} \models (h \in c_1)$, $\Gamma^{***} \models \sim\sim (t \in c_1)$. Now by hypothesis, since $t \in \mathscr{S}_{\alpha_0}$, $\Gamma^{***} \models \sim\sim (t \in c_2)$; so $\Gamma^{***} \models \sim\sim (h \in c_2)$. Thus $\Gamma^* \models \sim\sim (h \in c_2)$. We have shown $\Gamma \models (\forall x)\,(x \in c_1 \supset \sim\sim x \in c_2)$ or $\Gamma \models (c_1 \subseteq c_2)$. Similarly $\Gamma \models (c_2 \subseteq c_1)$.

Thus (speaking intuitively) to decide if two subsets of c_0 are equal at Γ we can confine ourselves to elements of $\mathscr{S}_{\alpha_0}$ provided we look at all Γ^*.

Now let $\boldsymbol{P}$ be the collection of all elements $c \in \mathscr{S}$ such that $\Gamma \models (c \subseteq c_0)$. We define (intuitively) a function $\boldsymbol{U}$ on $\boldsymbol{P}$ by

$$\boldsymbol{U}(c) = \{\langle \Gamma^*, t\rangle \mid t \in \mathscr{S}_{\alpha_0} \quad \text{and} \quad \Gamma^* \models (t \in c)\}.$$

By the above result, for $c_1, c_2 \in \boldsymbol{P}$, if $\boldsymbol{U}(c_1) = \boldsymbol{U}(c_2)$, then $\Gamma \models (c_1 = c_2)$.

Let $\boldsymbol{B}$ be the range of $\boldsymbol{U}$ on $\boldsymbol{P}$. $\boldsymbol{U}: \boldsymbol{P} \to \boldsymbol{B}$ is a function but not one-to-one. So we cut down its domain to a new domain $\boldsymbol{P}'$ on which $\boldsymbol{U}$ is one-to-one. Thus for $u \in \boldsymbol{B}$, for $\boldsymbol{U}^{-1}(u)$ choose some single element x from the class of all $y \in \boldsymbol{P}$ such that $\boldsymbol{U}(y) = u$. Let $\boldsymbol{P}' = \{\boldsymbol{U}^{-1}(u) \mid u \in \boldsymbol{B}\}$. Let $\boldsymbol{U}'$ be $\boldsymbol{U}$ restricted to $\boldsymbol{P}'$. Then $\boldsymbol{U}'$ is an isomorphism between $\boldsymbol{P}'$ and $\boldsymbol{B}$.

Suppose we could show for some $\beta_0 \in V$ $\boldsymbol{P}' \subseteq \mathscr{S}_{\beta_0}$. Then if $c \in \mathscr{S}$ and $\Gamma \models (c \subseteq c_0)$, $c \in \boldsymbol{P}$, so there is some $d \in \boldsymbol{P}'$ such that $\boldsymbol{U}(c) = \boldsymbol{U}(d)$, so $\Gamma \models (c = d)$ and $d \in \mathscr{S}_{\beta_0}$. Thus we would have the desired result. We now show $\boldsymbol{P}' \subseteq \mathscr{S}_{\beta_0}$ for some $\beta_0 \in V$.

***Lemma* 15.1:** There is a formula $F(x)$ over V such that $x \in \boldsymbol{P}$ iff $F(x)$ is true over V.

Proof: Let $R_{\subseteq}(z, x, y)$ be the formula defining $z \models (x \subseteq y)$ as given in the last section. Let $F(x)$ be $R_{\subseteq}(\Gamma, x, c_0)$.

***Lemma* 15.2:** There is a formula $G(x, y)$ over V such that $y \in \boldsymbol{U}(x)$ iff $G(x, y)$ is true over V.

Proof: Let $R_{\in}(Z, x, y)$ be the formula defining $Z \models (x \in y)$. Let $G(x, y)$ be

$$F(x) \wedge (\exists r, s)\,[y = \langle r, s\rangle \wedge r \in \mathscr{G} \wedge s \in \mathscr{S}_{\alpha 0} \wedge \Gamma \mathscr{R} r \wedge R_{\in}(r, s, x)].$$

***Lemma* 15.3:** For any $c \in \mathscr{S}$ $\boldsymbol{U}(c) \in P(\mathscr{G} \times \mathscr{S}_{\alpha 0}) \in V$. ($P(x)$ is the power set of x in V.)

Proof: $\boldsymbol{U}(c)$ is a subset of $\mathscr{G} \times \mathscr{S}_{\alpha 0} \in V$ (and is defined by $G(c, x)$).

***Lemma* 15.4:** $\boldsymbol{B} \in V$.

Proof: By lemma 15.4 $\{\boldsymbol{U}(x) \mid x \in \mathscr{S}\}$ is a subset of $P(\mathscr{G} \times \mathscr{S}_{\alpha 0}) \in V$. (It is a definable subset, defined by

$$(\exists \alpha)\,(\textit{ordinal}\,(\alpha) \wedge (\exists c)\,(c \in \mathscr{S}_{\alpha} \wedge G(c, x)))\ ).$$

***Lemma* 15.5:** There is a formula $H(x, y)$ such that $x \in y$, for y a subset of $\mathscr{S}$, if and only if $H(x, y)$ is true over V (that is, a choice function).

Proof: That $\mathscr{S}$ can be well ordered in V is straightforward.

***Theorem* 15.6:** $\boldsymbol{P}' \subseteq \mathscr{S}_{\beta_0}$ for some $\beta_0 \in V$.

Proof: The function $\boldsymbol{U}^{-1}(u)$ can be defined by: $\boldsymbol{U}^{-1}(u)$ is that x such that $H(x, y)$ where $y = \{z \in \boldsymbol{P} \mid \boldsymbol{U}(z) = \boldsymbol{U}(u)\}$, which can be formalized. Now $\boldsymbol{P}'$ is the range of $\boldsymbol{U}^{-1}(u)$ on $\boldsymbol{B}$. By the axiom of substitution in V $\boldsymbol{P}' \in V$. Hence $\boldsymbol{P}' \subseteq \mathscr{S}_{\beta_0}$ for some $\beta_0 \in V$, since $\boldsymbol{P}' \subseteq \mathscr{S}$ and $\mathscr{S}$ is a class.

Thus we have our first assertion. We have written it out fairly completely as illustration. From now we will only indicate the steps.

Above, for fixed Γ we produced an appropriate β_0. But the procedure can itself be defined over V. Since $\mathscr{G} \in V$, by the axiom of substitution again, there is a maximum $\beta_0 \in V$ which works for all $\Gamma \in \mathscr{G}$. Thus we have shown:

There is a $\beta_0 \in V$ such that for any $c \in \mathscr{S}$ and any $\Gamma \in \mathscr{G}$, if $\Gamma \models (c \subseteq c_0)$, then for some $d \in \mathscr{S}_{\beta_0}$ $\Gamma \models (c = d)$.

Now we can show the following, from which the power set axiom follows, since c_0 was arbitrary:

***Theorem* 15.7:** The following is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$:

$$(\exists y) \sim (\exists z) \sim [(z \in y) \equiv (z \subseteq c_0)].$$

Proof: Let $X(x)$ be the formula $(x \subseteq c_0)$, with $c_0 \in \mathscr{S}_{\alpha_0}$. Let β_0 be as above, and let $\gamma = \max(\alpha_0, \beta_0)$. Then $\gamma \in V$. Consider $f_X \in \mathscr{S}_{\gamma+1} - \mathscr{S}_\gamma$. We claim $\sim (\exists z) \sim [(z \in f_X) \equiv (z \subseteq c_0)]$ is valid.

Let $\Gamma \in \mathscr{G}$ and suppose $\Gamma^* \not\models \sim (h \in f_X)$. Then for some Γ^{**} $\Gamma^{**} \models (h \in f_X)$, so there is some $t \in \mathscr{S}_\gamma$ such that $\Gamma^{**} \models (t = h) \wedge (t \in f_X)$. By dominance $\Gamma^{**} \models_{\gamma+1} (t \in f_X)$, $\Gamma^{**} \models_\gamma X(t)$, so $\Gamma^{**} \models_\gamma (t \subseteq c_0)$, by permanence $\Gamma^{**} \models (t \subseteq c_0)$. Thus $\Gamma^{**} \models \sim\sim (h \subseteq c_0)$, so Γ^*W$\sim (h \subseteq c_0)$. We have shown

$$\Gamma \models (\forall x) [\sim (h \subseteq c_0) \supset \sim (h \in f_X)]$$

or equivalently,

$$\Gamma \models \sim (\exists x) \sim [(h \in f_X) \supset (h \subseteq c_0)].$$

Conversely suppose $\Gamma^* \not\models \sim (h \subseteq c_0)$. Then for some Γ^{**} $\Gamma^{**} \models (h \subseteq c_0)$. There is some $t \in \mathscr{S}_{\beta_0}$ such that $\Gamma^{**} \models (h = t)$. So $\Gamma^{**} \models ((t \subseteq c_0)$ $(x \subseteq y$ is stable.) By dominance

$$\Gamma^{**} \models_\gamma (t \subseteq c_0),$$

$$\Gamma^{**} \models_\gamma X(t),$$

$$\Gamma^{**} \models_{\gamma+1} (t \in f_X),$$
$$\Gamma^{**} \models (t \in f_X),$$
$$\Gamma^{**} \models \sim\sim (h \in f_X).$$

Thus $\Gamma^* \not\models \sim(h \in f_X)$.

We have shown

$$\Gamma \models (\forall x)[\sim(h \in f_X) \supset \sim(h \subseteq c_0)],$$

or equivalently

$$\Gamma \models \sim(\exists h) \sim [(h \subseteq c_0) \supset (h \in f_X)],$$

and the theorem follows.

Remark 15.8: Above we obtained β_0 by two applications of the axiom of substitution. These could have been combined into one step as in Cohen [3]. This proof was based on that one, which followed a suggestion of Solovay. We find this two step approach more intuitive, but the treatment in Cohen is more elegant.

§ 16. X-equivalence

Definition 16.1: Let X be a formula with no universal quantifiers and all constants in $\mathscr{S}_\alpha$. We call $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, \mathscr{S}_\alpha \rangle$ *X-equivalent* to $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ if for every Y which is an instance of a subformula of X with all constants in $\mathscr{S}_\alpha$, for any $\Gamma \in \mathscr{G}$

$$\Gamma \models_\alpha Y \Leftrightarrow \Gamma \models Y.$$

***Theorem* 16.2:** Let X be as above, with all its constants in $\mathscr{S}_\alpha$. There is an ordinal $\beta \in V$, $\alpha \leqslant \beta$, such that $\langle \mathscr{G}, \mathscr{R}, \models_\beta, \mathscr{S}_\beta \rangle$ is X-equivalent to $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

We spend the rest of the section proving this.

Definition 16.3: Let $\beta \in V$ and X be a formula with all its constants in $\mathscr{S}_\beta$. We call X (for this section only) *β-dominant* if for any $\Gamma \in \mathscr{G}$

$$\Gamma \models_\beta X \Leftrightarrow \Gamma \models X.$$

***Lemma* 16.4:** Any atomic formula over $\mathscr{S}_\beta$ is *β-dominant*. If X and Y are *β-dominant*, so are $\sim X$, $(X \vee Y)$, $(X \wedge Y)$ and $(X \supset Y)$.

Proof: Straightforward.

***Lemma* 16.5:** Suppose for every $a \in \mathscr{S}_\beta$ $X(a)$ is *β-dominant*. Then if $\Gamma \models_\beta (\exists x)\, X(x)$, $\Gamma \models (\exists x)\, X(x)$.

Proof: $\Gamma \models_\beta (\exists x)\, X(x)$ implies $\Gamma \models_\beta X(a)$ for some $a \in \mathscr{S}_\beta$. By hypothesis $\Gamma \models X(a)$, so $\Gamma \models (\exists x)\, X(x)$.

Now for the proof of the theorem. Recall X is a formula over $\mathscr{S}_\alpha$. There are only a finite number of formulas $Y_1, Y_2, \ldots, Y_n$ with free variables but no constants, such that every subformula of X is an instance of some Y_i. By theorem 14.2 there are formulas $R_{Y_1}, R_{Y_2}, \ldots, R_{Y_n}$ over V such that

$$\Gamma \models Y_i(c_1, \ldots, c_k) \Leftrightarrow R_{Y_i}(\Gamma, c_1, \ldots, c_k)$$

is true over V.

We define informally a sequence in V. Using the above R_{Y_i}, the sequence can be formally defined over V. We note again that there is a formula over V which well-orders the class $\mathscr{S}$.

Let $\boldsymbol{D}_0 = \mathscr{S}_\alpha$. Suppose we have defined $\boldsymbol{D}_m$, which is some $\mathscr{S}_\beta$ for $\beta \in V$. $\boldsymbol{D}_m$ can be well-ordered in V, so all subformulas of X with constants from $\boldsymbol{D}_m$ and of the form $(\exists x)\, Z(x)$ can be well-ordered (isomorphically) in V. If $(\exists x)\, Z(x)$ is a subformula of X and has all its constants from $\boldsymbol{D}_m$, and if there is a $\Gamma \in \mathscr{G}$ such that $\Gamma \models (\exists x)\, Z(x)$, for some $c \in \mathscr{S}$ $\Gamma \models Z(c)$. Choose the smallest c in the well-ordering of $\mathscr{S}$ such that $\Gamma \models Z(c)$. Let $\boldsymbol{K}_{m+1}$ be $\boldsymbol{D}_m$ together with all such c. $\boldsymbol{K}_{m+1}$ can be defined as the range of a function, definable over V, whose domain is the collection of ordered pairs $\langle x, y \rangle$ where $x \in \mathscr{G}$ and y is a formula of the form $(\exists x)\, Z(x)$, a subformula of X over $\boldsymbol{D}_m$. This domain is a set, hence $\boldsymbol{K}_{m+1}$ is a set. But $\boldsymbol{K}_{m+1} \subseteq \mathscr{S}$. Thus there is a least $\gamma \in V$ such that $\boldsymbol{K}_{m+1} \subseteq \mathscr{S}_\gamma$. Let $\boldsymbol{D}_{m+1} = \mathscr{S}_\gamma$.

In this way, we define the sequence $\boldsymbol{D}_0, \boldsymbol{D}_1, \boldsymbol{D}_2, \ldots$ But this sequence can be defined formally over V. Thus $\bigcup \boldsymbol{D}_n$ is an element of V. But by the definition $\bigcup \boldsymbol{D}_n$ must be some $\mathscr{S}_\beta$ for $\beta \in V$ (since $\boldsymbol{D}_k \subseteq \boldsymbol{D}_{k+1}$).

We have produced an $\mathscr{S}_\beta \in V$, $\alpha \leqslant \beta$. We claim $\langle \mathscr{G}, \mathscr{R}, \models_\beta, \mathscr{S}_\beta \rangle$ is X-equivalent to $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$. That is, for Y any subformula of X with constants from $\mathscr{S}_\beta$,

$$\Gamma \models_\beta Y \Leftrightarrow \Gamma \models Y.$$

The proof is by induction on the degree of Y. All the cases but one are immediate by the above lemmas. The only non-trivial case is the following. Suppose $(\exists x)\, Z(x)$ is a subformula of X, has all its constants in $\mathscr{S}_\beta$, and $\Gamma \models (\exists x)\, Z(x)$. All the constants of $(\exists x)\, Z(x)$ lie in $\bigcup \boldsymbol{D}_n$, but there are only finitely many, so for some integer k, all the constants of

$(\exists x)\, Z(x)$ lie in $\boldsymbol{D}_k$. By definition there is a $c \in \boldsymbol{D}_{k+1} \subseteq \mathscr{S}_\beta$ such that $\Gamma \models Z(c)$. By induction hypothesis $\Gamma \models_\beta Z(c)$, so $\Gamma \models_\beta (\exists x)\, Z(x)$.

§ 17. Axiom of substitution

As we did for the power set axiom, we wish to show the axiom of substitution is valid over $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$. The proof is essentially that of [3].

Let $X(x, y)$ be a formula with no universal quantifiers, and constants from $\mathscr{S}$, *which defines a function at* Γ, *that is, such that*

$$\Gamma \models \sim (\exists x) \sim (\exists ! y)\, X(x, y),$$

where $(\exists ! y)\, Z(y)$ abbreviates

$$(\exists y)\, [Z(y) \wedge \sim (\exists w)\, (Z(w) \wedge \sim (w = y))].$$

Let c_0 be a *fixed* element of $\mathscr{S}$. Let α_0 be the smallest ordinal such that $c_0 \in \mathscr{S}_{\alpha_0}$. We want to show there is some $f \in \mathscr{S}$ such that

$$\Gamma \models \sim (\exists x) \sim [x \in f \equiv (\exists w)\, (w \in c_0 \wedge X(w, x))].$$

That is, roughly, f is the range of X on c_0 at Γ.

By § 14 there is a formula $R_X(z, x, y)$ over V such that $\Delta \models \sim \sim X(x, y)$ iff $R_X(\Delta, x, y)$ is true over V.

Let $g(\Delta, c)$ be the smallest ordinal β such that for some $c' \in \mathscr{S}_\beta$ $\Delta \models \sim \sim X(c, c')$ if there is such, and 0 otherwise; g is definable over V (using R_X).

Since $\alpha_0 \in V$, $\mathscr{G} \times \mathscr{S}_{\alpha_0} \in V$. By the axiom of substitution in V, the range of g on $\mathscr{G} \times \mathscr{S}_{\alpha_0}$ is a set in V. Thus also $\bigcup(\text{range } g \text{ on } \mathscr{G} \times \mathscr{S}_{\alpha_0}) \in V$. Let β_0 be this union. Then β_0 is an ordinal, $\beta_0 \in V$.

***Lemma* 17.1:** Suppose $\Gamma^* \models (\exists x)\, (x \in c_0 \wedge X(x, d))$. Then there is some $c' \in \mathscr{S}_{\beta_0}$ such that $\Gamma^* \models (c' = d)$.

Proof: $\Gamma^* \models (\exists x)\, (x \in c_0 \wedge X(x, d))$ so for some $c \in \mathscr{S}$

$$\Gamma^* \models (c \in c_0) \wedge X(c, d).$$

$c_0 \in \mathscr{S}_{\alpha_0}$ so there is some $t \in \mathscr{S}_{\alpha_0}$ such that $\Gamma^* \models (t \in c_0) \wedge (t = c)$. Hence $\Gamma^* \models \sim \sim X(t, d)$. Now $\langle \Gamma^*, t \rangle \in domain(g)$, so by definition $g(\Gamma^*, t) \leqslant \beta_0$. Thus there is some $c' \in \mathscr{S}_{\beta_0}$ such that $\Gamma^* \models \sim \sim X(t, c')$. But

$$\Gamma^* \models \sim \sim X(t, d) \quad \text{and} \quad \Gamma^* \models \sim (\exists x) \sim (\exists ! y)\, X(x, y),$$

so by intuitionistic logic

$$\Gamma^* \models (c' = d)$$

($(x=y)$ is stable).

Let $\varphi(x)$ be the formula $(\exists w)\,[w \in c_0 \wedge X(w, x)]$. There are only a finite number of constants in $\varphi(x)$ (recall that X may have constants), hence all lie in some $\mathscr{S}_\gamma$ (take $\gamma \geqslant \beta_0$). By theorem 16.2 there is some $\delta \in V$, $\gamma \leqslant \delta$, such that $\langle \mathscr{G}, \mathscr{R}, \models_\delta, \mathscr{S}_\delta \rangle$ is φ-equivalent to $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

Since φ is a formula over $\mathscr{S}_\gamma$, φ is also a formula over $\mathscr{S}_\delta$. Thus it defines a function $f_\varphi \in \mathscr{S}_{\delta+1} - \mathscr{S}_\delta$. We claim

$$\Gamma \models \sim (\exists x) \sim [x \in f_\varphi \equiv (\exists w)\,(w \in c_0 \wedge X(w, x))],$$

which is what we wanted. We now proceed with the proof.

Suppose $\Gamma^* \not\models \sim (c \in f_\varphi)$. Then for some Γ^{**} $\Gamma^{**} \models (c \in f_\varphi)$. Since $f_\varphi \in \mathscr{S}_{\delta+1} - \mathscr{S}_\delta$, there is some $d \in \mathscr{S}_\delta$ such that $\Gamma^{**} \models (c = d) \wedge (d \in f_\varphi)$. By dominance $\Gamma^{**} \models_{\delta+1} (d \in f_\varphi)$. $\Gamma^{**} \models_\delta \varphi(d)$, But $\langle \mathscr{G}, \mathscr{R}, \models_\delta, \mathscr{S}_\delta \rangle$ is φ-equivalent to $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ hence

$$\begin{aligned}
&\Gamma^{**} \models \varphi(d),\\
&\Gamma^{**} \models \sim\sim \varphi(c),\\
&\Gamma^{*} \not\models \sim \varphi(c),\\
&\Gamma^{*} \not\models \sim (\exists w)\,(w \in c_0 \wedge X(w, c)).
\end{aligned}$$

Thus we have shown

$$\Gamma \models (\forall x)\,[\sim (\exists w)\,(w \in c_0 \wedge X(w, x)) \supset \sim (x \in f_\varphi)].$$

Conversely suppose $\Gamma^* \not\models \sim (\exists w)\,(w \in c_0 \wedge X(w, c))$. Then for some Γ^{**}

$$\Gamma^{**} \models (\exists w)\,(w \in c_0 \wedge X(w, c)).$$

By the above lemma, there is some $c' \in \mathscr{S}_{\beta_0}$ such that $\Gamma^{**} \models (c' = c)$. Hence $\Gamma^{**} \models \sim\sim (\exists w)\,(w \in c_0 \wedge X(w, c'))$, that is, $\Gamma^{**} \models \sim\sim \varphi(c')$. But $c' \in \mathscr{S}_{\beta_0} \subseteq \mathscr{S}_\gamma \subseteq \mathscr{S}_\delta$, and $\langle \mathscr{G}, \mathscr{R}, \models_\delta, \mathscr{S}_\delta \rangle$ is φ-equivalent to $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$, hence

$$\begin{aligned}
&\Gamma^{**} \models_\delta \sim\sim \varphi(c'),\\
&\Gamma^{**} \models_{\delta+1} \sim\sim (c' \in f_\varphi),\\
&\Gamma^{**} \models \sim\sim (c' \in f_\varphi).
\end{aligned}$$

But

$$\Gamma^{**} \models (c' = c),$$

so

$$\Gamma^{**} \models \sim\sim (c \in f_\varphi),$$
$$\Gamma^{*} \not\models \sim (c \in f_\varphi).$$

We have shown

$$\Gamma \models (\forall x)\,[\sim (x \in f_\varphi) \supset \sim (\exists w)\,(w \in c_0 \wedge X\,(w, x))].$$

The assertion now follows.

CHAPTER 8

INDEPENDENCE OF THE AXIOM OF CHOICE

§ 1. The specific model

The model given here is adapted from the one of Cohen [3]. We have changed it from showing directly that there is an infinite set with no countable subset to showing directly that there is a set with no choice function. The change was made because the notion of countability requires much more machinery in these models. See [3, p. 136] for a brief introduction to the model.

Following ch. 7 § 3, a sequence of models and a class model are defined if the 0th model is fixed. We now define a specific $\langle \mathscr{G}, \mathscr{R}, \models_0, \mathscr{S}_0 \rangle$. All the work is relative to a classical model V.

Let e be some formal symbol. By a *forcing condition* we mean a finite consistent set Γ of statements of the form (nem) and $\sim(nem)$ $(n \geqslant 0, m \geqslant 1)$. ($(nem)$ can be some ordered triple in V, say $\langle n, 0, m \rangle$. Anything convenient. Similarly $\sim(nem)$ can be some other triple, say $\langle n, 1, m \rangle$. We have written it like this for reading ease.) Let $\mathscr{G}$ be the collection of all forcing conditions, and let $\mathscr{R}$ be set inclusion $\subseteq$.

Before defining $\mathscr{S}_0$, we define the following partition of the integers:

$$\begin{aligned} I_0 &= \{1, 3, 5, 7, \ldots\}, \\ I_1 &= \{2, 6, 10, 14, \ldots\}, \\ I_2 &= \{4, 12, 20, 28, \ldots\}, \\ &\text{etc.} \end{aligned}$$

in general

$$I_n = \{2^n(2k + 1) \mid k = 0, 1, 2, \ldots\}.$$

This partition has the properties that each I_n is infinite, and if $n \in I_m$ $n > m$.

Now we define $\mathscr{S}_0$. It consists of the functions

$$\hat{0}, \hat{1}, \hat{2}, \ldots, \quad s_1, s_2, s_3, \ldots, \quad t_0, t_1, t_2, \ldots, \quad T,$$

whose definitions are the following:

For each integer n the function $\hat{n}$ has domain $\{\hat{0}, \hat{1}, \ldots, \widehat{n-1}\}$, and for $k < n$,

$$\hat{n}(\hat{k}) = \mathscr{G}.$$

Each s_n has as domain $\{\hat{0}, \hat{1}, \hat{2}, \ldots\}$ and

$$s_n(\hat{m}) = \{\Gamma \in \mathscr{G} \mid (m \in n) \in \Gamma\}.$$

Each t_n has as domain $\{s_1, s_2, s_3, \ldots\}$ and

$$t_n(s_m) = \begin{cases} \mathscr{G} & \text{if } m \in I_n, \\ \emptyset & \text{otherwise}. \end{cases}$$

T has as domain $\{t_0, t_1, t_2, \ldots\}$ and

$$T(t_n) = \mathscr{G}.$$

From this technical definition $\Vdash_0$ for atomic formulas becomes

$$\begin{aligned} &\Gamma \Vdash_0 (\hat{m} \in \hat{n}) \quad \text{iff} \quad m < n, \\ &\Gamma \Vdash_0 (\hat{m} \in s_n) \quad \text{iff} \quad (m \in n) \in \Gamma, \\ &\Gamma \Vdash_0 (s_m \in t_n) \quad \text{iff} \quad m \in I_n, \\ &\Gamma \Vdash_0 (t_n \in T). \end{aligned}$$

We now examine the five properties of ch. 7 § 3. (1), (2), (3) and (5) are trivial. (4) is satisfied in the very strong sense that for any $\Gamma \in \mathscr{G}$ and any $a, b \in \mathscr{S}_0$, if

$$\Gamma \Vdash_0 \sim (\exists x) \sim [x \in a \equiv x \in b],$$

then a and b are the same function. This is proved by examining the various possible choices for a and b. We show only the most difficult case and leave the rest to the reader.

***Theorem* 1.1:** If $m \neq n$, $\sim (s_m = s_n)$ is valid in $\langle \mathscr{G}, \mathscr{R}, \Vdash_0, \mathscr{S}_0 \rangle$.

Proof: We show for any $\Gamma \in \mathscr{G}$ $\Gamma \nVdash_0 (s_m = s_n)$. Suppose $\Gamma \Vdash_0 (s_m = s_n)$ for some $\Gamma \in \mathscr{G}$. Since Γ is a forcing condition, it is finite, so we may choose an integer k such that neither $(k \in m)$, $\sim (k \in m)$, $(k \in n)$, $\sim (k \in n)$ belong to Γ. Let Δ be $\Gamma \cup \{(kem), \sim(ken)\}$. Then $\Delta \in \mathscr{G}$ and $\Gamma \mathscr{R} \Delta$. By definition $\Delta \models_0 (\hat{k} \in s_m)$. Since $\Delta \models_0 \sim(\exists x) \sim (x \in s_m \equiv x \in s_n)$, by intuitionistic logic $\Delta \models_0 \sim \sim (\hat{k} \in s_n)$. Then for some Δ^* $\Delta^* \models_0 (\hat{k} \in s_n)$, which means $(ken) \in \Delta^*$. But $\sim(ken) \in \Delta^*$, a contradiction.

Thus all five conditions are met so the resulting class model $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is an intuitionistic ZF model.

§ 2. Symmetries

Let $\boldsymbol{G}$ be the collection of all permutations π of integers such that π permutes the elements of one I_n and is the identity on all I_m for $m \neq n$. We may extend any $\pi \in \boldsymbol{G}$ to $\mathscr{S}$ as follows:

$$\begin{aligned} \pi(\hat{n}) &= \hat{n}, \\ \pi(s_n) &= s_{\pi(n)}, \\ \pi(t_n) &= t_n, \\ \pi(T) &= T. \end{aligned}$$

Let X be the formula $X(x, c_1, \ldots, c_n)$ where π has been defined for $c_1, \ldots, c_n$. Let $\pi(X)$ be $X(x, \pi(c_1), \ldots, \pi(c_n))$. If $f_X \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$, let $\pi(f_X)$ be $f_{\pi(X)}$. Thus π is extended to $\mathscr{S}$. We also extend π to $\mathscr{G}$ by

$$\begin{aligned} (n e m) \in \Gamma &\Leftrightarrow (n e \pi(m)) \in \pi(\Gamma), \\ \sim(n e m) \in \Gamma &\Leftrightarrow \sim(n e \pi(m)) \in \pi(\Gamma). \end{aligned}$$

We note that $\Gamma \in \mathscr{G}$ implies $\pi(\Gamma) \in \mathscr{G}$.

***Theorem* 2.1:** For any formula X with all constants in $\mathscr{S}_\alpha$, with no universal quantifiers, any $\Gamma \in \mathscr{G}$, and any $\pi \in \boldsymbol{G}$

$$\Gamma \models_\alpha X \Leftrightarrow \pi(\Gamma) \models_\alpha \pi(X),$$

and

$$\Gamma \models X \Leftrightarrow \pi(\Gamma) \models \pi(X).$$

Proof: A straightforward induction on α and the degree of X.

Definition 2.2: Let N be some collection of integers. By $\boldsymbol{G}_N$ we mean the subset of $\boldsymbol{G}$ leaving N invariant.

***Lemma* 2.3:** Let $f \in \mathscr{S}$. There is a *finite* set N of integers such that if $\pi \in \boldsymbol{G}_N$, $\pi(f) = f$.

Proof: If $f \in \mathscr{S}_0$, we have two cases. If f is not some s_n, let $N=\emptyset$. If f is s_m, let $N=\{n\}$.

Suppose the result is known for all $g \in \mathscr{S}_\alpha$. Let $f \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$. Then f is f_X for some $X(x, c_1, \ldots, c_n)$, where $c_1, \ldots, c_n \in \mathscr{S}_\alpha$. By hypothesis there are finite sets $N_1, \ldots, N_n$ of integers such that if $\pi \in \boldsymbol{G}_{N_i}$, $\pi(c_i)=c_i$. Let $N=N_1 \cup \cdots \cup N_n$. Then if $\pi \in \boldsymbol{G}_N$, $\pi(f_X)=f_{\pi(X)}=f_X$.

§ 3. Functions

We introduce the following formula abbreviations:

$$
\begin{array}{rll}
x = \langle y, z\rangle & \text{for} & \sim\sim(\exists w)\,[w\in x \wedge w = \{y, z\} \wedge x = \{y, w\}], \\
\langle x, y\rangle \in z & \text{for} & (\exists w)\,[w\in z \wedge w = \langle x, y\rangle], \\
\mathit{ordpr}(x) & \text{for} & \sim(\exists y)\sim[y\in x \supset (\exists z)\,(\exists w)\,(y = \langle z, w\rangle)], \\
\mathit{relation}(x) & \text{for} & \sim(\exists y)\sim[y\in x \supset \mathit{ordpr}(y)], \\
\mathit{function}(x) & \text{for} & \mathit{relation}(x) \wedge \sim(\exists y)\,(\exists z)\,(\exists u)\,(\exists v) \\
 & & \qquad \sim[(\langle y, z\rangle\in x \wedge \langle u, v\rangle \in x \wedge y = u) \supset z = v], \\
\mathit{domain}(x) = y & \text{for} & \sim(\exists z)\,(\exists w)\sim[\langle z, w\rangle\in x \supset z\in y] \wedge \\
 & & \qquad \sim(\exists z)\sim[z\in y \supset (\exists w)\,(\langle z, w\rangle \in x)].
\end{array}
$$

***Theorem* 3.1:** All the above formulas are dominant.

§ 4. Axiom of choice

Let $\mathrm{AC}(T)$ be the formula

$$
(\exists x)\,\{\mathit{function}(x) \wedge \mathit{domain}(x) = T \wedge \\
\qquad \sim(\exists y)\sim[y\in T \supset (\exists z)\,(z\in y \wedge \langle y, z\rangle \in x)]\}.
$$

That is, $\mathrm{AC}(T)$ says that T has a choice function. In this section we show that $\sim\mathrm{AC}(T)$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$. In fact, it is valid in $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, \mathscr{S}_\alpha\rangle$ for every α; the same proof holds for each case.

We first show a preliminary

***Lemma* 4.1:** If $f \in \mathscr{S}$ and $\Gamma \models (f \in t_n)$ then for some $m \in I_n$

$$\Gamma \models_\alpha (f = s_m).$$

Proof: $\Gamma \models (f \in t_n)$ so there is some $b \in \mathit{domain}(t_n)$ such that

$$\Gamma \models (f = b) \wedge (b \in t_n).$$

Now suppose there is some $\Gamma\in\mathscr{G}$ such that $\Gamma\models\mathrm{AC}(T)$. Then for some $F\in\mathscr{S}$,

$$\Gamma\models \mathit{function}(F)\wedge \mathit{domain}(F)=T\wedge \\ \sim(\exists y)\sim[y\in T\supset(\exists z)(z\in y\wedge\langle y,z\rangle\in F)].$$

There is a finite set N of integers such that if $\pi\in G_N$, $\pi(F)=F$. Let $n=1+\max N$.

$$\Gamma\models\sim(\exists y)\sim[y\in T\supset(\exists z)(z\in y\wedge\langle y,z\rangle\in F)]$$

and $\Gamma\models(t_n\in T)$ hence

$$\Gamma\models\sim\sim(\exists z)(z\in t_n\wedge\langle t_n,z\rangle\in F).$$

Then for some Γ^*

$$\Gamma^*\models(\exists z)(z\in t_n\wedge\langle t_n,z\rangle\in F).$$

For some $\alpha\in\mathscr{S}$

$$\Gamma^*\models(\alpha\in t_n)\wedge\langle t_n,\alpha\rangle\in F.$$

By the above lemma, for some $m\in I_n$, $\Gamma^*\models(\alpha=s_m)$. Hence

$$\Gamma^*\models\sim\sim(\langle t_n,s_m\rangle\in F),$$

so for some Γ^{**}, $\Gamma^{**}\models\langle t_n,s_m\rangle\in F$.

Now $m\in I_n$, so $m>n=1+\max N$, hence $m\notin N$. Choose an integer $k>n$ such that $k\neq m$ and neither $(p\epsilon k)$ nor $\sim(p\epsilon k)$ belongs to Γ^{**} for any integer p, but $k\in I_n$. (Γ^{**} is finite but I_n is infinite, so this is possible.) Let π be the permutation $\pi(m)=k$, $\pi(k)=m$, on all other integers π is the identity. Since $m,k\notin N$, $\pi\in G_N$. Now

$$\pi(\Gamma^{**})\models\pi(\langle t_n,s_m\rangle\in F),$$
$$\pi(\Gamma^{**})\models\langle\pi(t_n),\pi(s_m)\rangle\in\pi(F),$$
$$\pi(\Gamma^{**})\models\langle t_n,s_k\rangle\in F.$$

But $\Delta=\Gamma^{**}\cup\pi(\Gamma^{**})$ is itself a forcing condition. It is finite, and since Γ^{**} and $\pi(\Gamma^{**})$ must be the same except for statements involving m and k, and m is not (a second element of any statement) in $\pi(\Gamma^{**})$ and k is not in Γ^{**}, $\pi(\Gamma^{**})$ and Γ^{**} are compatible. Thus $\Delta\in\mathscr{G}$ and $\Gamma^{**}\mathscr{R}\Delta$ and $\pi(\Gamma^{**})\mathscr{R}\Delta$. So

$$\Delta\models\text{function}\,F,$$

(since $\Gamma\models \mathit{function}F$)

$$\Delta\models\langle t_n,s_m\rangle\in F,$$
$$\Delta\models\langle t_n,s_k\rangle\in F.$$

It then follows by intuitionistic logic that

$$\Delta \models \sim \sim (s_m = s_k),$$

or since $(x = y)$ is stable,

$$\Delta \models (s_m = s_k).$$

But $m \neq k$, contradicting theorem 1.1. Thus for all $\Gamma \in \mathscr{G}$

$$\Gamma \not\models \mathrm{AC}(T),$$

so

$$\Gamma \models \sim \mathrm{AC}(T).$$

As we showed in ch. 7 § 1 the axiom of choice is now *classically* independent.

CHAPTER 9

ORDINALS AND CARDINALS

§ 1. Definitions

Continuing ch. 8 § 3 we introduce the following formula abbreviations:

$$\mathit{range}(x)=y \quad \text{for} \quad \sim(\exists z)(\exists w)\sim[\langle z, w\rangle\in x \supset w\in y] \wedge \sim(\exists w)\sim[w\in y \supset (\exists z)\langle z, w\rangle\in x],$$

$$\text{1-1}(x) \quad \text{for} \quad \sim(\exists y)(\exists z)(\exists u)(\exists v) \sim[(\langle y, z\rangle\in x \wedge \langle u, v\rangle\in x \wedge z=v)\supset y=u]$$

$$\mathit{trans}(x) \quad \text{for} \quad \sim(\exists y)(\exists z)\sim[(y\in x \wedge z\in y)\supset z\in x],$$

$$\mathit{ordered}(x) \quad \text{for} \quad \sim(\exists y)(\exists z)\sim[(y\in x \wedge z\in x)\supset(y=z \vee y\in z \vee z\in y)],$$

$$\mathit{welord}(x) \quad \text{for} \quad \mathit{ordered}(x)\wedge\sim(\exists y)\sim\{[y\subseteq x \wedge (\exists z)(z\in y)]\supset (\exists w)[w\in y \wedge \sim(\exists u)\sim(u\in y \supset (w\in u \vee w=u))]\},$$

$$\mathit{ordinal}(x) \quad \text{for} \quad \mathit{trans}(x)\wedge\mathit{welord}(x).$$

***Theorem* 1.1:** All of the above formulas are dominant.

The proof is again primarily an application of ch. 7 § 7.

§ 2. Some properties of ordinals

In this section we establish some useful analogues of classical theorems. We use a method of proof which we call a classical-intuitionistic argument. Rather than stating it generally, we illustrate its use by writing out in full the first proof below.

***Theorem* 2.1:** $\sim(\exists x)\sim(\mathit{ordered}(x) \equiv \mathit{welord}(x))$ is valid over $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ (and by dominance, over any $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, \mathscr{S}_\alpha \rangle$).

Proof: It is a standard classical result that

$$\text{ZF}, \textit{axiom of regularity} \vdash_c \sim(\exists x) \sim (\mathit{ordered}(x) \equiv \mathit{welord}(x)).$$

So for some finite subset of ZF with no universal quantifiers

$$\vdash_c (A_1 \wedge \cdots \wedge A_n \wedge \textit{axiom of regularity}) \supset \sim(\exists x) \sim (\mathit{ordered}(x) \equiv \mathit{welord}(x)).$$

By the results of ch. 4 § 8 together with

$$\vdash_I \sim\sim(X \supset Y) \equiv (X \supset \sim\sim Y)$$

and

$$\vdash_I \sim\sim\sim X \equiv \sim X$$

we have

$$\vdash_I (A_1 \wedge \cdots \wedge A_n \wedge \textit{axiom of regularity}) \supset \sim(\exists x) \sim (\mathit{ordered}(x) \equiv \mathit{welord}(x)).$$

Since $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is an intuitionistic ZF model, $\sim(\exists x)\sim(\mathit{ordered}(x) \equiv \mathit{welord}(x))$ is valid.

***Theorem* 2.2:** If $\Gamma \models \mathit{ordinal}(f)$ and $\Gamma \models g \in f$ then $\Gamma \models \mathit{ordinal}(g)$.

Proof: By a classical-intuitionistic argument we have:

$$\sim(\exists x)(\exists y) \sim [(\mathit{ordinal}(x) \wedge y \in x) \supset \mathit{ordinal}(y)]$$

is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$. The result now follows by stability of $\mathit{ordinal}(y)$.

***Theorem* 2.3:** If $\Gamma \models \mathit{ordinal}(f) \wedge \mathit{ordinal}(g)$ then

$$\Gamma \models \sim\sim(f \in g \vee f = g \vee g \in f).$$

§ 3. General ordinal representatives

We define inductively representatives for the classical ordinals. Later we discuss their existence and uniqueness.

Definition 3.1: Suppose we have defined *general representatives* in $\mathscr{S}$ for all ordinals $\beta < \alpha$. We call $f \in \mathscr{S}$ a general representative of the ordinal α if

(1). if g represents an ordinal $< \alpha$, $(g \in f)$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$, (2). if $\Gamma \models (h \in f)$, there is some Γ^*, some $\beta < \alpha$, and some $g \in \mathscr{S}$ which represents β, such that $\Gamma^* \models (g = h)$.

***Theorem* 3.2:** If $f \in \mathscr{S}$ is a general representative of some ordinal, *ordinal*(f) is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

Proof: Suppose f represents the ordinal α and the result is known for all representatives of ordinals $\beta < \alpha$. We have three facts to show.

I. *trans*(f) is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$

Suppose $\Gamma \models (a \in f) \wedge (b \in a)$. Then for any Γ^* $\Gamma^* \models (a \in f) \wedge (b \in a)$. By property (2) there is some $a' \in \mathscr{S}$ which represents $\beta < \alpha$ and some Γ^{**} such that $\Gamma^{**} \models (a = a')$. Thus $\Gamma^{**} \models \sim\sim(b \in a')$. There is some Γ^{***} such that $\Gamma^{***} \models (b \in a')$. Again by property (2) there is some $b' \in \mathscr{S}$ which represents $\gamma < \beta$ and some Γ^{****} such that $\Gamma^{****} \models (b = b')$. By property (1) $\Gamma^{****} \models (b' \in f)$, hence $\Gamma^{****} \models \sim\sim(b \in f)$. Thus for any Γ^* there is some Δ $(= \Gamma^{****})$ such that $\Gamma^* \mathscr{R} \Delta$ and $\Delta \models \sim\sim(b \in f)$. Thus $\Gamma \models \sim\sim(b \in f)$. Since Γ was arbitrary, *trans*(f) is valid.

II. *ordered*(f) is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$

Suppose $\Gamma \models (a \in f) \wedge (b \in f)$. For any Γ^*, $\Gamma^* \models (a \in f) \wedge (b \in f)$. By property (2), there is some Γ^{**} and some $a', b' \in \mathscr{S}$ such that a' represents β and b' represents γ where $\beta < \alpha$, $\gamma < \alpha$, and $\Gamma^{**} \models (a = a') \wedge (b = b')$. By hypothesis $\Gamma^{**} \models \mathit{ordinal}(a') \wedge \mathit{ordinal}(b')$. By theorem 2.3

$$\Gamma^{**} \models \sim\sim(a' \in b' \vee a' = b' \vee b' \in a').$$

So

$$\Gamma^{**} \models \sim\sim(a \in b \vee a = b \vee b \in a).$$

Thus as above

$$\Gamma \models \sim\sim(a \in b \vee a = b \vee b \in a).$$

Again Γ is arbitrary, so *ordered*(f) is valid.

III. *ordinal*(f) is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$

By the above *trans*$(f) \wedge$ *ordered*(f) is valid. Then *welord*(f) is also valid by theorem 2.1 (*welord*(x) is stable). Thus *ordinal*(f) is valid.

***Theorem* 3.3:** If $f, g \in \mathscr{S}$ are both general representatives of the same ordinal, $(f = g)$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

Proof: Suppose f and g both represent α. If $\Gamma \models (h \in f)$, for any Γ^* $\Gamma^* \models (h \in f)$. By property (2), there is some Γ^{**}, some $\beta < \alpha$, and some k representing β, such that $\Gamma^{**} \models (h = k)$. Since g represents α and k represents β, and $\beta < \alpha$, by property (1) $\Gamma^{**} \models (k \in g)$. Thus $\Gamma^{**} \models \sim\sim(h \in g)$, so $\Gamma \models \sim\sim(h \in g)$. Similarly if $\Gamma \models (h \in g)$, $\Gamma \models \sim\sim(h \in f)$. But Γ is arbitrary, so the result follows.

§ 4. Canonical ordinal representatives

Again we postpone a discussion of existence.

Definition 4.1: We call $f \in \mathscr{S}$ a *canonical representative* of the ordinal α if
(1). f is a general representative of α,
(2). for no $g \in domain(f)$ and for no $\Gamma \in \mathscr{G}$ $\Gamma \models (f = g)$,
(3). if $\Gamma \models \sim\sim(g \in f)$, $\Gamma \models (g \in f)$ for $g \in domain(f)$.

***Theorem* 4.2:** Suppose $f \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$ is a canonical representative of some ordinal. Then f is f_X where $X(x)$ is the formula $ordinal(x)$.

Proof: We must show for any $a \in \mathscr{S}_\alpha$

$$\Gamma \models_{\alpha+1} (a \in f) \quad \text{iff} \quad \Gamma \models_\alpha ordinal(a).$$

Suppose $\Gamma \models_{\alpha+1} (a \in f)$. By theorem 3.2 $\Gamma \models ordinal(f)$, so by theorem 2.2 (and dominance) $\Gamma \models_\alpha ordinal(a)$.

Suppose $\Gamma \models_\alpha ordinal(a)$. By theorem 3.2 $\Gamma \models ordinal(f)$. So by theorem 2.3 (and dominance)

$$\Gamma \models \sim\sim(a \in f \lor a = f \lor f \in a).$$

Thus, for every Γ^* there is some Γ^{**} such that

$$\Gamma^{**} \models (a \in f) \lor (a = f) \lor (f \in a).$$

If $\Gamma^{**} \models (f \in a)$, since $a \in \mathscr{S}_\alpha$, there is some $g \in \mathscr{S}_\alpha$ such that $\Gamma^{**} \models (f = g)$ contradicting part (2) of the above definition. Similarly $\Gamma^{**} \not\models f = a$. Thus $\Gamma^{**} \models (a \in f)$. So $\Gamma \models \sim\sim(a \in f)$, and by part (3) above $\Gamma \models (a \in f)$. Now by dominance $\Gamma \models_{\alpha+1} (a \in f)$.

§ 5. Ordinalized models

We give a condition on our model (actually on $\langle \mathscr{G}, \mathscr{R}, \models_0, \mathscr{S}_0 \rangle$) which will insure existence and uniqueness of canonical representatives for the ordinals.

We call $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ *ordinalized* if
(1). no ordinal has more than one canonical representative in $\mathscr{S}_0$,

(2). if $f \in \mathscr{S}_0$ and $\Gamma \models ordinal(f)$ for some $\Gamma \in \mathscr{G}$, then there is some Γ^* and some $h \in \mathscr{S}_0$ which is a canonical representative of an ordinal, such that $\Gamma^* \models (f = h)$.

Remark 5.1: By dominance, whether $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized can be decided by considering only $\langle \mathscr{G}, \mathscr{R}, \models_0, \mathscr{S}_0 \rangle$.

***Theorem* 5.2:** If $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized and $f, g \in \mathscr{S}$ are both canonical representatives for the same ordinal, f and g are identical.

Proof: Suppose first that $g \in \mathscr{S}_\alpha$ and $f \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$. By theorem 3.3 $(f = g)$ is valid, contradicting part (2) of definition 4.1. There is a similar contradiction if $f \in \mathscr{S}_\alpha$ and $g \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$. Thus, either $f, g \in \mathscr{S}_0$, or for some $\alpha, f, g \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$. If $f, g \in \mathscr{S}_0$, by part (1) of the above definition they are identical. If $f, g \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$, they are identical by theorem 4.2

Thus if an ordinal has any canonical representatives, it has only one. From now on, by representative we will mean canonical representative, and we will denote the representative of α, if it exists, by $\hat{\alpha}$.

We give the following temporary definition. We say $\beta \in V$ has the *representative property* provided: if α is the smallest ordinal not representable by an element of $\mathscr{S}_\beta$, α is representable by an element of $\mathscr{S}_{\beta+1}$. In other words, β has the representative property provided: if for all $\gamma < \alpha$, $\hat{\gamma} \in \mathscr{S}_\beta$, but $\hat{\alpha} \notin \mathscr{S}_\beta$, then $\hat{\alpha} \in \mathscr{S}_{\beta+1} - \mathscr{S}_\beta$.

***Lemma* 5.3:** If $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized and if all ordinals $< \beta$ have the representative property, so does β.

Proof: Let α be the smallest ordinal not representable in $\mathscr{S}_\beta$. We must show $\hat{\alpha} \in \mathscr{S}_{\beta+1} - \mathscr{S}_\beta$.

Let $X(x)$ be the formula $ordinal(x)$, and let $f_X \in \mathscr{S}_{\beta+1} - \mathscr{S}_\beta$. We claim f_X is $\hat{\alpha}$.

Suppose $\Gamma \models (h \in f_X)$. Then there is some $g \in \mathscr{S}_\beta$ such that $\Gamma \models (g = h) \wedge (g \in f_X)$. But then $\Gamma \models_\beta X(g)$, $\Gamma \models_\beta ordinal(g)$. We now have three cases.

Suppose $\beta = 0$. Since $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized, there is some Γ^* and some $k \in \mathscr{S}_0$ which is an ordinal representative (and by hypothesis, of an ordinal $< \alpha$) such that $\Gamma^* \models (k = g)$. Thus $\Gamma^* \models (k = h)$.

Suppose β is a successor ordinal. By hypothesis $\beta - 1$ has the representative property. Let γ be the smallest ordinal not representable in $\mathscr{S}_{\beta-1}$. Then $\hat{\gamma} \in \mathscr{S}_\beta$. Now (theorem 3.2)

$$\Gamma \models ordinal(\hat{\gamma}) \wedge ordinal(g),$$

so by theorem 2.3

$$\Gamma \models \sim\sim (g \in \hat{\gamma} \vee g = \hat{\gamma} \vee \hat{\gamma} \in g).$$

Then for some Γ^*

$$\Gamma^* \models (g \in \hat{\gamma}) \vee (g = \hat{\gamma}) \vee (\hat{\gamma} \in g).$$

If $\Gamma^* \models (g \in \hat{\gamma})$, by definition of $\hat{\gamma}$ there is some Γ^{**} and some $\delta < \gamma$ such that $\Gamma^{**} \models (\hat{\delta} = g)$ and so $\Gamma^{**} \models (\hat{\delta} = h)$.

If $\Gamma^* \models (g = \hat{\gamma})$ then $\Gamma^* \models (h = \hat{\gamma})$.

Finally, we can not have $\Gamma^* \models (\hat{\gamma} \in g)$ for, since $g \in \mathscr{S}_\beta$ there is some $k \in \mathscr{S}_{\beta-1}$ such that $\Gamma^* \models (\hat{\gamma} = k) \wedge (k \in g)$. But $\hat{\gamma} \in \mathscr{S}_\beta - \mathscr{S}_{\beta-1}$ and this contradicts part (2) of definition 4.1.

Suppose β is a limit ordinal. Since $g \in \mathscr{S}_\beta$, for some $\eta < \beta$, $g \in \mathscr{S}_{\eta+1} - \mathscr{S}_\eta$. Let γ be the smallest ordinal not representable in $\mathscr{S}_\eta$. Then $\hat{\gamma} \in \mathscr{S}_{\eta+1} - \mathscr{S}_\eta$. Now proceed as above.

Thus in any case there is an ordinal $<\alpha$, a representative t of it, and a Δ such that $\Gamma \mathscr{R} \Delta$ and $\Delta \models (h = t)$. Thus f_X is a general representative of α.

Next suppose for some $g \in \mathscr{S}_\alpha$ $\Gamma \models (g = f_X)$. Since f_X is a general representative of α, by theorem 3.2 $\Gamma \models \mathit{ordinal}(f_X)$. Thus $\Gamma \models \mathit{ordinal}(g)$, so by dominance $\Gamma \models_\alpha \mathit{ordinal}(g)$, $\Gamma \models_\alpha X(g)$. Thus $\Gamma \models_{\alpha+1} (g \in f_X)$. Hence $\Gamma \models_{\alpha+1} \sim\sim (g \in g)$, $\Gamma \models \sim\sim (g \in g)$, contradicting the validity of the axiom of regularity.

Finally if $\Gamma \models \sim\sim (g \in f_X)$ for some $g \in \mathscr{S}_\alpha$, then $\Gamma \models_{\alpha+1} \sim\sim (g \in f_X)$. For every Γ^* there is some Γ^{**} such that $\Gamma^{**} \models_{\alpha+1} (g \in f_X)$. Or $\Gamma^{**} \models_\alpha X(g)$. $\Gamma^{**} \models_\alpha \mathit{ordinal}(g)$, thus $\Gamma \models_\alpha \sim\sim \mathit{ordinal}(g)$. But $\mathit{ordinal}(x)$ is stable so $\Gamma \models_\alpha \mathit{ordinal}(g)$, $\Gamma \models_\alpha X(g)$, $\Gamma \models_{\alpha+1} (g \in f_X)$.

Thus f_X is a canonical representative of α.

***Theorem* 5.4:** Suppose $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized. Then every ordinal in V is uniquely representable by an element of $\mathscr{S}$.

Proof: Immediate by lemma 5.3.

Remark 5.5: Although it seems unlikely, it is conceivable that some ordinal not in V might be representable by an element of $\mathscr{S}$. In fact this can not happen. Suppose for some $\gamma \notin V$ $\hat{\gamma} \in \mathscr{S}$. For some $\alpha \in V$ $\hat{\gamma} \in \mathscr{S}_\alpha$. The class of elements of $\mathscr{S}$ which are ordinal representatives is definable over V. The intersection of this class with $\mathscr{S}_\alpha$ is a set, i.e. an element of V. But the relation $\Gamma \models_\alpha (x \in y)$ well-orders this set, the relation is in V, and the order type must be γ (or greater). Hence $\gamma \in V$.

Thus exactly the ordinals of V are representable in ordinalized $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$.

§ 6. Properties of ordinal representatives

***Theorem* 6.1:** If $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ is ordinalized and $\alpha, \beta \in V$, then if for some $\Gamma \in \mathscr{G}$ $\Gamma \models (\hat{\alpha} = \hat{\beta})$, $\alpha = \beta$; and if $\alpha = \beta$, $(\hat{\alpha} = \hat{\beta})$ is valid.

Proof: If $\alpha < \beta$, by part (1) of definition 3.1, $\Gamma \models \hat{\alpha} \in \hat{\beta}$, but if $\Gamma \models (\hat{\alpha} = \hat{\beta})$ $\Gamma \models \sim\sim(\hat{\alpha} \in \hat{\alpha})$, contradicting the axiom of regularity. Similarly if $\beta < \alpha$. Thus if $\Gamma \models (\hat{\alpha} = \hat{\beta})$, $\alpha = \beta$. The second half is by uniqueness of representatives.

***Theorem* 6.2:** If $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ is ordinalized and $\alpha, \beta \in V$, then if for some $\Gamma \in \mathscr{G}$ $\Gamma \models (\hat{\alpha} \in \hat{\beta})$, $\alpha \in \beta$; and if $\alpha \in \beta$, $(\hat{\alpha} \in \hat{\beta})$ is valid.

Proof: If $\Gamma \models (\hat{\alpha} \in \hat{\beta})$, by part (2) of definition 3.1 for some Γ^* and some $\gamma < \beta$, $\Gamma^* \models (\hat{\alpha} = \hat{\gamma})$. By theorem 6.1 $\alpha = \gamma$, and $\gamma \in \beta$. If $\alpha \in \beta$, by part (1) of definition 3.1 $(\hat{\alpha} \in \hat{\beta})$ is valid.

***Theorem* 6.3:** Suppose $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ is ordinalized, and for some $\Gamma \in \mathscr{G}$, $\Gamma \models \mathit{ordinal}(f)$. Then there is some Γ^* and some ordinal $\alpha \in V$ such that $\Gamma^* \models f = \hat{\alpha}$.

Proof: $f \in \mathscr{S}$ so for some β $f \in \mathscr{S}_\beta$. Let γ be the smallest ordinal not representable in $\mathscr{S}_\beta$ ($\mathscr{S}_\beta \in V$ so there must be one). Then $\hat{\gamma} \in \mathscr{S}_{\beta+1} - \mathscr{S}_\beta$ (see § 5). But $\Gamma \models \mathit{ordinal}(\hat{\gamma})$. Hence $\Gamma \models \sim\sim(f \in \hat{\gamma} \vee f = \hat{\gamma} \vee \hat{\gamma} \in f)$. For some Γ^*, $\Gamma^* \models (f \in \hat{\gamma}) \vee (f = \hat{\gamma}) \vee (\hat{\gamma} \in f)$. If $\Gamma^* \models f \in \hat{\gamma}$, we are done by part (2) of definition 3.1. $\Gamma^* \not\models (f = \hat{\gamma})$ by part 2 of definition 4.1. Finally, $\Gamma^* \models \hat{\gamma} \in f$ is not possible, for otherwise, since $f \in \mathscr{S}_\beta$, there is some $g \in \mathscr{S}_\beta$ such that $\Gamma^* \models (\hat{\gamma} = g)$. But $\hat{\gamma} \in \mathscr{S}_{\beta+1} - \mathscr{S}_\beta$ and this contradicts part (2) of definition 4.1

§ 7. Types of ordinals

We introduce the following formula abbreviations:

successor ordinal(x)	for	*ordinal*$(x) \wedge (\exists y)(y \in x \wedge x = y')$,
limit ordinal(x)	for	*ordinal*$(x) \wedge \sim(\exists y) \sim (y \in x \supset y' \in x)$,
integer(x)	for	*ordinal*$(x) \wedge \sim$ *limit ordinal*$(x) \wedge \sim(\exists y)(y \in x \wedge$ *limit ordinal*$(y))$,
x *is* ω	for	*limit ordinal*$(x) \wedge \sim(\exists y)(y \in x \wedge$ *limit ordinal*$(y))$.

***Theorem* 7.1:** The above formulas are dominant.

***Theorem* 7.2:** If $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ is ordinalized, $\widehat{\alpha+1}=\hat{\alpha}'$ is valid.

Proof: We must show for all $\Gamma\in\mathscr{G}$

$$\Gamma\models\sim(\exists x)\sim[x\in\widehat{\alpha+1}\equiv(x\in\hat{\alpha}\vee x=\hat{\alpha})].$$

Suppose $\Gamma\models f\in\widehat{\alpha+1}$. Then for every Γ^* $\Gamma^*\models f\in\widehat{\alpha+1}$. There is some Γ^{**} and some $\beta<\alpha+1$, $\Gamma^{**}\models f=\hat{\beta}$. But $\beta\leqslant\alpha$, so $\Gamma^{**}\models(\hat{\beta}\in\hat{\alpha})\vee(\hat{\beta}=\hat{\alpha})$, $\Gamma^{**}\models\sim\sim(f\in\hat{\alpha}\vee f=\hat{\alpha})$. Thus $\Gamma\models\sim\sim(f\in\hat{\alpha}\vee f=\hat{\alpha})$. Similarly, if $\Gamma\models(f\in\hat{\alpha}\vee f=\hat{\alpha})$, then $\Gamma\models\sim\sim(f\in\widehat{\alpha+1})$. The result follows.

***Corollary* 7.3:** If $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ is ordinalized, *successor ordinal*$(\widehat{\alpha+1})$ is valid.

***Theorem* 7.4:** If $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ is ordinalized, and for some $f\in\mathscr{S}$ and some $\Gamma\in\mathscr{G}$ $\Gamma\models$*successor ordinal* f, then for some Γ^* and some $\alpha+1$ $\Gamma^*\models(f=\widehat{\alpha+1})$.

Proof: $\Gamma\models$*successor ordinal* f, so for some $g\in\mathscr{S}$ $\Gamma\models$*ordinal* $g\wedge g\in f\wedge f=g'$. Since $\Gamma\models$*ordinal* g, there is a Γ^* and an ordinal α such that $\Gamma^*\models g=\hat{\alpha}$. Then $\Gamma^*\models f=\hat{\alpha}'$, $\Gamma^*\models f=\widehat{\alpha+1}$.

In a similar manner we may show

***Theorem* 7.5:** Suppose $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ is ordinalized. Then

(1). If λ is a limit ordinal, *limit ordinal*$(\hat{\lambda})$ is valid.

(2). If $\Gamma\models$*limit ordinal*(f), then for some Γ^* and some limit ordinal λ $\Gamma^*\models(f=\hat{\lambda})$.

(3). If n is an integer, *integer*$(\hat{n})$ is valid.

(4). If $\Gamma\models$*integer*(f), then for some Γ^* and some integer n $\Gamma^*\models(f=\hat{n})$.

(5). $\hat{\omega}$ *is* ω is valid.

(6). If $\Gamma\models f$ *is* ω, then for some Γ^* $\Gamma^*\models(f=\hat{\omega})$.

§ 8. Cardinalized models

Let *cardinal*(x) be an abbreviation for

ordinal$(x)\wedge$
$\sim(\exists y)(\exists z)[y\in x\wedge$*function*$(z)\wedge 1\text{-}1(z)\wedge$*domain*$(z)=y\wedge$*range*$(z)=x]$

We remark that *cardinal*(x) is not dominant (probably) but it is stable.

Suppose $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized. We call it *cardinalized* if for every $\alpha \in V$, if α is a cardinal of V, *cardinal*$(\hat{\alpha})$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

By the classical-intuitionistic technique of § 2

$$\sim (\exists x) \sim [\mathit{integer}(x) \supset \mathit{cardinal}(x)]$$

and

$$\sim (\exists x) \sim [x \mathit{ is } \omega \supset \mathit{cardinal}(x)]$$

are both valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$. But then by § 7 for any integer n *cardinal*$(\hat{n})$ is valid. Also *cardinal*$(\hat{\omega})$ is valid.

Thus the troublesome cardinals of V are the uncountable ones. In the next section we give a condition due to Cohen which will take care of such cardinals.

Remark 8.1: To say $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is cardinalized is to say the cardinals of V are among those of $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$. In fact, we will show in ch. 13 that the cardinals of $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ are the same as the cardinals of L, the class of constructable sets of V.

§ 9. Countably incompatible $\mathscr{G}$

The following argument is from [3]:

Definition 9.1: Two elements $\Gamma, \Delta \in \mathscr{G}$ are called *compatible* if they have a common $\mathscr{R}$-extension, that is, if some Γ^* is some Δ^*. Otherwise Γ and Δ are incompatible.

$\mathscr{G} \in V$ is called *countably incompatible* if any subset of $\mathscr{G}$ of mutually incompatible Γ is at most countable in V.

***Lemma* 9.2:** Suppose $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized, $\mathscr{G}$ is countably incompatible, $\hat{\alpha}, \hat{\beta} \in \mathscr{S}$, $\mathit{card}(\alpha) < \mathit{card}(\beta)$ and $\aleph_0 < \mathit{card}(\beta)$ in V. Then

$$\sim (\exists f)\, [\mathit{function}(f) \wedge 1\text{-}1\,(f) \wedge \mathit{domain}(f) = \hat{\alpha} \wedge \mathit{range}(f) = \hat{\beta}]$$

is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

Proof: Let f be some fixed element of $\mathscr{S}$. We remarked earlier that the class of ordinal representatives was definable over V. Let $F(x)$ be the formula defining it. Let $A(x, y, z)$ be the formula

$$[x \in \mathscr{S} \wedge y \in \mathscr{S} \wedge F(y) \wedge z \in \mathscr{G} \wedge z \models (\mathit{function}(f) \\ \wedge 1\text{-}1\,(f) \wedge \mathit{domain}(f) = \hat{\alpha} \wedge \mathit{range}(f) = \hat{\beta} \wedge \langle x, y \rangle \in f)].$$

Suppose for $\hat{\gamma}$, $\hat{\delta}$, Δ, Δ', c that $A(c, \hat{\gamma}, \Delta)$ and $A(c, \hat{\delta}, \Delta')$ are both true over V. If Δ and Δ' are compatible, some

$$\Delta^* \Vdash \langle c, \hat{\gamma}\rangle \in f \wedge \langle c, \hat{\delta}\rangle \in f .$$

Hence $\Delta^* \Vdash \hat{\gamma} = \hat{\delta}$ so $\gamma = \delta$. Thus if $\gamma \neq \delta$, Δ and Δ' are incompatible. Thus for any fixed $c \in \mathscr{S}$ and any $\Delta \in \mathscr{G}$ there are only countably many ordinals γ such that $A(c, \hat{\gamma}, \Delta)$ is true over V, by the countable incompatibility hypothesis.

Let $B(x, y)$ be the formula

$$(\exists \Delta)(\Delta \in \mathscr{G} \wedge A(x, y, \Delta)).$$

Then for fixed $c \in \mathscr{S}$, the set defined by $B(c, y)$ is at most countable. For any ordinal α, let α^0 be $\{\hat{\gamma} \mid \gamma < \alpha\}$; $\alpha^0 \in V$ for $\alpha \in V$. Finally let $C(x)$ be the formula

$$(\exists c)(c \in \alpha^0 \wedge B(c, x)).$$

Let C' be the class in V defined by $C(x)$, and let C be $\{\gamma \mid \hat{\gamma} \in C'\}$. Since C' is a definable subset of β^0, $C \in V$. For a bound on the cardinality of C we note that for any $c \in \alpha^0$, there are at most $\aleph_0$ x such that $B(c, x)$. Thus card $C \leqslant \aleph_0$; $card(\alpha) < card(\beta)$ so $card(C) < card(\beta)$.

Next we show that, if for some $\Delta \in \mathscr{G}$

$$\Delta \Vdash (function(f) \wedge 1\text{-}1 f \wedge domain(f) = \hat{\alpha} \wedge range(f) = \hat{\beta} \wedge \langle c, d\rangle \in f),$$

then there is some Δ^* and some $\delta \in C$ such that $\Delta^* \Vdash (d = \hat{\delta})$. For since $\Delta \Vdash \langle c, d\rangle \in f$, there must be some Δ^* such that $\Delta^* \Vdash (d \in \hat{\beta})$ and hence a Δ^{**} and a $\delta \in \beta$ such that $\Delta^{**} \Vdash (d = \hat{\delta})$. Thus

$$\Delta^{**} \Vdash (function(f) \wedge 1\text{-}1(f) \wedge domain(f) = \hat{\alpha} \\ \wedge range(f) = \hat{\beta} \wedge \langle c, \hat{\delta}\rangle \in f).$$

So

$$\begin{array}{ll} A(c, \hat{\delta}, \Delta^{**}) & \text{is true over } V, \\ B(c, \hat{\delta}) & \text{is true over } V, \\ C(\hat{\delta}) & \text{is true over } V, \\ \delta \in C. & \end{array}$$

Now suppose there were some $\Gamma \in \mathscr{G}$ such that

$$\Gamma \Vdash (function(f) \wedge 1\text{-}1(f) \wedge domain(f) = \hat{\alpha} \wedge range(f) = \hat{\beta}).$$

Since $card(C) < card(\beta)$, but $C \subseteq \beta$, there is some $\gamma \in \beta$, $\gamma \notin C$. Since $\gamma \in \beta$, $\Gamma \models (\hat{\gamma} \in \hat{\beta})$. Then since $\Gamma \models (range(f) = \hat{\beta})$, for some Γ^*

$$\Gamma^* \models (\exists c)(c \in \hat{\alpha} \wedge \langle c, \hat{\gamma} \rangle \in f),$$

so for some $c \in \mathscr{S}$,

$$\Gamma^* \models \langle c, \hat{\gamma} \rangle \in f .$$

That is

$$\Gamma^* \models (function(f) \wedge 1\text{-}1(f) \wedge domain(f) = \hat{\alpha} \wedge range(f) = \hat{\beta} \wedge \langle c, \hat{\gamma} \rangle \in f)$$

By the above there is some Γ^{**} and some $\delta \in C$ such that $\Gamma^{**} \models (\hat{\gamma} = \hat{\delta})$, but then $\gamma = \delta$, so $\gamma \in C$, a contradiction.

Since f is arbitrary, the result follows.

***Theorem* 9.3:** Suppose $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized, $\mathscr{G}$ is countably incompatible, and β is a cardinal of V. Then $cardinal(\hat{\beta})$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

Proof: By the last section we need only consider $\beta > \aleph_0 = \omega$. Suppose $\Gamma \not\models cardinal(\hat{\beta})$. Then for some α, f, Γ^*,

$$\Gamma^* \models (\hat{\alpha} \in \hat{\beta} \wedge function(f) \wedge domain(f) = \hat{\alpha} \wedge range(f) = \hat{\beta}).$$

Since $\Gamma^* \models (\hat{\alpha} \in \hat{\beta})$, $\alpha \in \beta$, so $card(\alpha) < card(\beta)$ (β is a cardinal). Now, by lemma 9.2 we are done.

Remark 9.4: A simple corollary of this theorem (which should be obvious anyway) is the following. If L is the class of constructable sets of V, not only is L a classical ZF model, but, if α is a cardinal of V, α is a cardinal of L. This follows by noting that in the intuitionistic formulation of the classical M_α sequence (remark 7.3.3) $\mathscr{G}$ is trivially countably incompatible, since $\mathscr{G}$ is finite, and since $M_0 = \emptyset$, the model is ordinalized.

CHAPTER 10

INDEPENDENCE OF THE CONTINUUM HYPOTHESIS

§ 1. The specific model

Again the model is adapted from Cohen [3], with practically no change. We define a particular $\langle \mathscr{G}, \mathscr{R}, \models_0, \mathscr{S}_0 \rangle$.

Recall V was some classical ZF model. Let $\delta \in V$ be that ordinal which is $\aleph_2$ in V. δ remains fixed for the rest of this chapter.

As in ch. 8, let e be some formal symbol. By a forcing condition we mean a finite, consistent set of statements of the form $(n e \alpha)$ or $\sim(n e \alpha)$ where n is any integer and α is any ordinal $<\delta$.

Let $\mathscr{G}$ be the collection of all forcing conditions, and let $\mathscr{R}$ be set inclusion $\subseteq$.

$\mathscr{S}_0$ consists of functions which we write as $\hat{\alpha}$, a_α, $\{\hat{\alpha}\}$, $\{\hat{\alpha}, a_\alpha\}$ and $\langle \hat{\alpha}, a_\alpha \rangle$ for each $\alpha<\delta$, and W. The definitions are the following:

For each $\alpha<\delta$ the domain of $\hat{\alpha}$ is $\{\hat{\beta} \mid \beta<\alpha\}$ and for $\beta<\alpha$,

$$\hat{\alpha}(\hat{\beta}) = \mathscr{G}.$$

a_α has domain $\{\hat{0}, \hat{1}, \hat{2}, \ldots\}$ and

$$a_\alpha(\hat{n}) = \{\Gamma \in \mathscr{G} \mid (m\, e\, n) \in \Gamma\}.$$

$\{\hat{\alpha}\}$ has only $\hat{\alpha}$ in its domain, and

$$\{\hat{\alpha}\}(\hat{\alpha}) = \mathscr{G}.$$

$\{\hat{\alpha}, a_\alpha\}$ has only $\hat{\alpha}$ and a_α in its domain, and

$$\{\hat{\alpha}, a_\alpha\}(\hat{\alpha}) = \mathscr{G},$$
$$\{\hat{\alpha}, a_\alpha\}(a_\alpha) = \mathscr{G}.$$

$\langle\hat{\alpha}, a_\alpha\rangle$ has only $\{\hat{\alpha}\}$ and $\{\hat{\alpha}, a_\alpha\}$ in its domain and

$$\langle\hat{\alpha}, a_\alpha\rangle(\{\hat{\alpha}\}) = \mathscr{G},$$
$$\langle\hat{\alpha}. a_\alpha\rangle(\{\hat{\alpha}, a_\alpha\}) = \mathscr{G}.$$

Finally W has as domain all $\langle\hat{\alpha}, a_\alpha\rangle$ for $a<\delta$, and

$$W(\langle\hat{\alpha}, a_\alpha\rangle) = \mathscr{G}.$$

From this $\models_0$ for atomic formulas becomes

$$\begin{aligned}
&\Gamma \models_0 (\hat{\alpha} \in \hat{\beta}), \quad \text{if} \quad \alpha \in \beta, \\
&\Gamma \models_0 (\hat{n} \in a_\alpha), \quad \text{if} \quad (n\,e\,\alpha) \in \Gamma, \\
&\Gamma \models_0 (\hat{\alpha} \in \{\hat{\alpha}\}), \\
&\Gamma \models_0 (\hat{\alpha} \in \{\hat{\alpha}, a_\alpha\}), \\
&\Gamma \models_0 (a_\alpha \in \{\hat{\alpha}, a_\alpha\}), \\
&\Gamma \models_0 (\{\hat{\alpha}\} \in \langle\hat{\alpha}, a_\alpha\rangle), \\
&\Gamma \models_0 (\{\hat{\alpha}, a_\alpha\} \in \langle\hat{\alpha}, a_\alpha\rangle), \\
&\Gamma \models_0 (\langle\hat{\alpha}, a_\alpha\rangle \in W).
\end{aligned}$$

Thus $\langle\mathscr{G}, \mathscr{R}, \models_0, \mathscr{S}_0\rangle$ is determined. We examine the five properties of ch. 7 § 3. (1), (2), (3) and (5) are trivial; (4) is satisfied in the same sense as in the model of ch. 8, that is, if $\Gamma \models_0 (a=b)$, a and b are identical. The proof is the same as in ch. 8. Thus $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ is an intuitionistic ZF model.

That $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ is ordinalized is straightforward. For $\alpha<\delta$ $\hat{\alpha}\in\mathscr{S}_0$ is the representative of α, and if for some $a\in\mathscr{S}_0$ $\Gamma \models_0$ *ordinal* a, a must be $\hat{\alpha}$ for some $\alpha<\delta$.

Finally, in the next section we show $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ is cardinalized.

§ 2. Countable incompatibility of $\mathscr{G}$

***Theorem* 2.1** (Cohen): $\mathscr{G}$ is countably incompatible. (and hence $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ is cardinalized).

Proof: We give the argument informally, but $\mathscr{G}\in V$ and $\mathscr{R}\in V$ so the argument can be formalized. We note that for this model to say $\Gamma, \Delta \in \mathscr{G}$ are compatible is to say $\Gamma \cup \Delta \in \mathscr{G}$.

Let $\mathscr{H} \subseteq \mathscr{G}$ ($\mathscr{H} \in V$), and suppose any two elements of $\mathscr{H}$ are incompatible. We show $\mathscr{H}$ is countable.

Suppose $\mathscr{H}$ is not countable. For each $n>0$, let $\mathscr{H}_n$ be $\{\Gamma \in \mathscr{H} \mid \Gamma$ contains $<n$ statements$\}$. Since $\mathscr{H} = \bigcup \mathscr{H}_n$, some $\mathscr{H}_n$ must be uncountable. Thus let $\mathscr{H}_n$ be uncountable.

Let k be the largest integer such that for some $\Gamma \in \mathscr{H}_n$ Γ has k statements and uncountably many $\Delta \in \mathscr{H}_n$ are such that $\Gamma \subseteq \Delta$. (k must exist since $\emptyset \in \mathscr{H}_n$ and there are uncountably many $\Delta \in \mathscr{H}_n$ such that $\emptyset \subseteq \Delta$, and every $\Gamma \in \mathscr{H}_n$ has $<n$ statements, so there is a largest k.)

Pick some particular $\Gamma \in \mathscr{H}_n$ such that Γ has k statements and Γ is a subset of uncountably many elements of $\mathscr{H}_n$. Let $\mathscr{K}$ be $\{\Delta \in \mathscr{H}_n \mid \Gamma \subseteq \Delta\}$. We have the following facts:

(1). Any two elements of $\mathscr{K}$ are incompatible.

(2). $\mathscr{K}$ is uncountable.

(3). $\Delta \in \mathscr{K}$ implies $\Gamma \subseteq \Delta$.

(4). Γ has k elements.

(5). For any $\Omega \in \mathscr{K}$ with more than k elements, there are only countably many $\Delta \in \mathscr{K}$ such that $\Omega \subseteq \Delta$.

Now choose some $\Delta \in \mathscr{K}$, $\Delta \neq \Gamma$. Then $\Delta - \Gamma = \{X_1, \ldots, X_m\}$. Since Δ is incompatible with all other elements of $\mathscr{K}$, by (3) there must be uncountably many elements of $\mathscr{K}$ containing $\bar{X}_i$ for some X_i. ($\bar{X}_i$ is $\sim(ne\alpha)$ if X_i is $(ne\alpha)$, and $\bar{X}_i$ is $(ne\alpha)$ if X_i is $\sim(ne\alpha)$.)

Let $\Omega = \Gamma \cup \{\bar{X}_i\}$. Then $\Omega \in \mathscr{H}_n$ since $X_i \notin \Gamma$. But there are uncountably many $\Delta \in \mathscr{H}_n$ such that $\Omega \subseteq \Delta$ and Ω has $k+1$ statements, a contradiction.

§ 3. Cardinals and W

We now have that $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is a cardinalized model. We introduce the following abbreviations:

x ***is at least*** $\aleph_1$ for *cardinal* $x \wedge (\exists y)(y \in x \wedge y$ *is* $\omega)$
x ***is at least*** $\aleph_2$ for *cardinal* $x \wedge (\exists y)(y \in x \wedge y$ *is at least* $\aleph_1)$

Recall that in V, δ was $\aleph_2$. We wish to show ($\hat{\delta}$ *is at least* $\aleph_2$) is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$. We showed in ch. 9, that ($\hat{\omega}$ *is* ω) is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

Let γ be the ordinal of V which is $\aleph_1$. Since γ is a cardinal, (*cardinal* $\hat{\gamma}$)

is valid, and since $\omega \in \gamma$, $(\hat{\omega} \in \hat{\gamma})$ is valid. Thus ($\hat{\gamma}$ *is at least* $\aleph_1$) is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$. Finally δ is a cardinal of V, so (*cardinal* $\hat{\delta}$) is valid, and $\gamma \in \delta$, so $(\hat{\gamma} \in \hat{\delta})$ is valid. Thus ($\hat{\delta}$ *is at least* $\aleph_2$) is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

Now we list a few properties of W. The proofs are straightforward.

***Lemma* 3.1:** $\langle \hat{\alpha}, a_\alpha \rangle = \langle \hat{\alpha}, a_\alpha \rangle$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ (where the first of these expressions is the function in $\mathscr{S}_0$, and the second is the expression of ch. 8 § 3).

***Theorem* 3.2:** $(\textit{function}\, W \wedge \text{1-1}\, W \wedge \textit{domain}\, W = \hat{\delta})$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

***Theorem* 3.3:** $\sim (\exists x) \sim [x \in \textit{range}(W) \supset \sim (\exists y) \sim (y \in x \supset \textit{integer}\; y)]$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

§ 4. Continuum hypothesis

Let $(\textit{card}\, \mathscr{P}(\omega) \geqslant \aleph_2)$ be an abbreviation for

$$(\exists x)\,\{x \textit{ is at least } \aleph_2 \wedge \\ (\exists W)\,[\textit{function}(W) \wedge \text{1-1}(W) \wedge \textit{domain}(W) = x \wedge \\ \sim (\exists y) \sim (y \in \textit{range}(W) \supset \sim (\exists z) \sim (z \in y \supset \textit{integer}(z)))]\}.$$

By the results of § 3 $(\textit{card}\, \mathscr{P}(\omega) \geqslant \aleph_2)$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$. Hence $\sim$(*continuum hypothesis*) is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

Now, as we showed in ch. 7 § 1 the continuum hypothesis is classically independent of the axioms of ZF. Of course we would also like that it is independent of ZF together with the axiom of choice. That the axiom of choice is valid in this model will be shown in ch. 13.

DEFINABILITY AND CONSTRUCTABILITY

§ 1. Definitions

We introduce the following formula abbreviations:

$\mathit{partfun}(f)$ for

$$\mathit{function}(f) \wedge (\exists n)\,[\mathit{integer}(n) \wedge \mathit{domain}(f) \subseteq n],$$

$\mathit{partrel}(R)$ for

$$\sim (\exists x)\,(\exists y) \sim [(x \in R \wedge y \in R) \supset (\mathit{partfun}(x) \wedge \mathit{partfun}(y) \wedge \mathit{domain}(x) = \mathit{domain}(y))],$$

$n \in \mathit{Domain}(R)$ for

$$\sim (\exists x) \sim [(\mathit{partfun}(x) \wedge x \in R) \supset n \in \mathit{domain}(x)],$$

R *is atomic*(1) *over* X for

$$(\exists m)\,(\exists n)\,\{\mathit{integer}(m) \wedge \mathit{integer}(n) \wedge \sim (\exists f) \sim [f \in R \equiv (\mathit{partfun}(f) \wedge \mathit{domain}(f) = \{m, n\} \wedge f(m) \in X \wedge f(n) \in X \wedge f(m) \in f(n))]\},$$

R *is atomic*(2) *over* X for

$$(\exists n)\,(\exists a)\,\{\mathit{integer}(n) \wedge \sim\sim a \in X \wedge \sim (\exists f) \sim [f \in R \equiv (\mathit{partfun}(f) \wedge \mathit{domain}(f) = \{n\} \wedge f(n) \in X \wedge f(n) \in a)]\},$$

R *is atomic*(3) *over* X for

$$(\exists n)\,(\exists a)\,\{\mathit{integer}(n) \wedge \sim\sim a \in X \wedge \sim (\exists f) \sim [f \in R \equiv (\mathit{partfun}(f) \wedge \mathit{domain}(f) = \{n\} \wedge f(n) \in X \wedge a \in f(n))]\},$$

R *is atomic*(4) *over* X for

$$(\exists a)\,(\exists b)\,\{\sim\sim a \in X \wedge \sim\sim b \in X \wedge \sim (\exists f) \sim [f \in R \equiv (\mathit{partfun}(f) \wedge \mathit{domain}(f) = \emptyset \wedge a \in b)]\},$$

R is atomic over X for

$(R \textit{ is atomic}(1) \textit{ over } X) \vee (R \textit{ is atomic}(2) \textit{ over } X) \vee$
$(R \textit{ is atomic}(3) \textit{ over } X) \vee (R \textit{ is atomic}(4) \textit{ over } X)$,

R is not-S for

$\textit{partrel}(S) \wedge \sim(\exists x) \sim [x \in \textit{Domain}(R) \equiv x \in \textit{Domain}(S)] \wedge$
$\sim(\exists f) \sim [f \in R \equiv \sim f \in S]$,

$(f \restriction \textit{Domain}(S)) \in S$ for

$(\exists g)\,[g \in S \wedge \sim(\exists x) \sim [x \in \textit{Domain}(S) \supset f(x) = g(x)]]$,

R is S-and-T for

$\textit{partrel}(S) \wedge \textit{partrel}(T) \wedge$
$\sim(\exists x) \sim [x \in \textit{Domain}(R) \equiv (x \in \textit{Domain}(S) \vee x \in \textit{Domain}(T)] \wedge$
$\sim(\exists f) \sim [f \in R \equiv ((f \restriction \textit{Domain}(S)) \in S \wedge (f \restriction \textit{Domain}(T)) \in T)]$,

R is S-or-T for

$\textit{partrel}(S) \wedge \textit{partrel}(T) \wedge$
$\sim(\exists x) \sim [x \in \textit{Domain}(R) \equiv (x \in \textit{Domain}(S) \vee x \in \textit{Domain}(T))] \wedge$
$\sim(\exists f) \sim [f \in R \equiv ((f \restriction \textit{Domain}(S)) \in S \vee (f \restriction \textit{Domain}(T)) \in T)]$,

R is S-implies-T for

$\textit{partrel}(S) \wedge \textit{partrel}(T) \wedge$
$\sim(\exists x) \sim [x \in \textit{Domain}(R) \equiv (x \in \textit{Domain}(S) \vee x \in \textit{Domain}(T))] \wedge$
$\sim(\exists f) \sim [f \in R \equiv ((f \restriction \textit{Domain}(S)) \in S \supset (f \restriction \textit{Domain}(T)) \in T)]$,

$f = g \restriction \textit{Domain}(R)$ for

$\textit{domain}(f) = \textit{Domain}(R) \wedge$
$\sim(\exists x) \sim [x \in \textit{Domain}(R) \supset f(x) = g(x)]$,

R is $(\exists n) S$ *over X* for

$\textit{partrel}(S) \wedge \textit{integer}(n) \wedge$
$\sim(\exists x) \sim [x \in \textit{Domain}(R) \equiv (x \in \textit{Domain}(S) \wedge \sim x = n)] \wedge$
$\sim(\exists f) \sim [f \in R \equiv (\exists g)\,(g \in S \wedge f = g \restriction \textit{Domain}(R) \wedge g(n) \in X)]$,

R is a definable relation over X for

$(\exists F)\,(\exists n)\,\{\textit{function}(F) \wedge \textit{integer}(n) \wedge \textit{domain}(F) = n \wedge$
$\sim(\exists x) \sim [x \in n \supset F(x) \textit{ is atomic over } X \vee$
$(\exists y)\,(y \in x \wedge F(x) \textit{ is not-}F(y)) \vee$
$(\exists y)\,(\exists z)\,(y \in x \wedge z \in x \wedge F(x) \textit{ is } F(y)\textit{-and-}F(z)) \vee$
$(\exists y)\,(\exists z)\,(y \in x \wedge z \in x \wedge F(x) \textit{ is } F(y)\textit{-or-}F(z)) \vee$
$(\exists y)\,(\exists z)\,(y \in x \wedge z \in x \wedge F(x) \textit{ is } F(y)\textit{-implies-}F(z)) \vee$
$(\exists y)\,(\exists k)\,(y \in x \wedge \textit{integer}(k) \wedge F(x) \textit{ is } (\exists k) F(y) \textit{ over } X)] \wedge$
$(\exists m)\,(m \in n \wedge F(m) = R)\}$,

X is definable over Y for

$$(\exists R)(\exists n)\{partrel(R) \wedge integer(n) \wedge R\ is\ a\ definable\ relation\ over\ Y \wedge \\ \sim(\exists x)\sim[x \in Domain(R) \equiv x = n] \wedge \\ \sim(\exists x)\sim[x \in X \equiv (x \in Y \wedge (\exists f)(f \in R \wedge f(n) = x))]\}.$$

Remark 1.1: In the above we have used a few additional minor but obvious abbreviations.

This approach to first order definability using partial relaitons is due to Smullyan. Intuitively, if we have the formula $X(x_2, x_4, x_5)$ which is true over the set Y for an instance $x_2 = a$, $x_4 = b$, $x_5 = c$, we can consider *instead of the instance the partial function f with domain $\{2, 4, 5\}$ such* that $f(2) = a$, $f(4) = b$, $f(5) = c$. Instead of the formula X itself, we can consider the collection of all partial functions with domain $\{2, 4, 5\}$ which represent true instances of X as above. This collection is called a partial relation.

We leave to the reader the verification of the fact that classically (*X is definable over Y*) does indeed represent first order definability. In the next sections we consider to what extent it represents it in our intuitionistic models. We also leave to the reader such elementary facts as

$ZF \vdash_I R\ is\ atomic\ over\ X \supset partrel(R)$
$ZF \vdash_I partrel(S) \wedge R\ is\ not\text{-}S \supset partrel(R)$
$ZF \vdash_I partrel(S) \wedge partrel(T) \wedge R\ is\ S\text{-}and\text{-}T \supset partrel(R)$
etc.

§ 2. Adequacy of the definability formula

In this section we state two theorems of considerable use, whose classical analogues are reasonably intuitive. For the intuitionistic case the theorems are less obvious. The proofs are tedious and we relegate them to an appendix.

***Theorem* 2.1:** Let $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ be ordinalized and suppose for some $\Gamma \in \mathscr{G}$ and some $g, f \in \mathscr{S}$ $\Gamma \models f$ *is definable over* g. Then there is some Γ^* and some dominant formula $X(x)$ with no universal quantifiers such that

(1). every quantifier of X is bound to g,
(2). if a is a constant of X other than a quantifier bound, $\Gamma^* \models (a \in g)$,
(3). $\Gamma^* \models \sim(\exists x)\sim[x \in f \equiv (x \in g \wedge X(x))]$.

***Theorem* 2.2:** Let $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ be ordinalized and $f, g\in\mathscr{S}$. Suppose $X(x)$ is a formula with no universal quantifiers such that for some $\Gamma\in\mathscr{G}$

(1). every quantifier of X is bound to g,

(2). if a is a constant of X other than a quantifier bound $\Gamma\models\sim\sim(a\in g)$,

(3). $\Gamma\sim(\exists x)\sim[x\in f\equiv(x\in g\wedge X(x))]$.

Then $\Gamma\models\sim\sim$ *(f is definable over g)*.

***Corollary* 2.3** (to theorem 2.1): Let $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ be ordinalized, $g\in\mathscr{S}_\alpha$, and $\Gamma\models f$ *is definable over* g. Then for some $k\in\mathscr{S}_{\alpha+1}-\mathscr{S}_\alpha$ and some Γ^* $\Gamma^*\models(f=k)$.

Proof: $\Gamma\models f$ *is definable over* g, so there is a dominant formula $X(x)$ and a Γ^* as in theorem 2.1 above.

Suppose the constants of $X(x)$ other than g are $a_1, a_2, \ldots, a_n$. $\Gamma^*\models(a_1\in g)$ so there is an $h_1\in\mathscr{S}_\alpha$ such that $\Gamma^*\models(a_1=h_1)$. Similarly we find $h_2,\ldots, h_n\in\mathscr{S}_\alpha$ for $a_2,\ldots, a_n$. Let X' be

$$X\begin{pmatrix}a_1\ldots a_n\\ h_1\ldots h_n\end{pmatrix}.$$

By weak substitutivity of equality

$$\Gamma^*\models\sim(\exists x)\sim[X(x)\equiv X'(x)].$$

Let $Y(x)$ be $X'(x)\wedge x\in g$. Then all constants of Y are in $\mathscr{S}_\alpha$. Let $k_Y\in\mathscr{S}_{\alpha+1}-\mathscr{S}_\alpha$. We claim $\Gamma^*\models(k_Y=f)$. We leave the verification of this to the reader, after noting that by a classical-intuitionistic argument we have $\Gamma\models f\subseteq g$ and $g\in\mathscr{S}_\alpha$.

§ 3. ω-dominance

This definition of *ω-dominance* is not to be confused with that of ch. 7 § 16 which was used only that section.

We consider only ordinalized models. We call a formula $X(x_1,\ldots, x_n)$ with no constants *ω-dominant*, if for any $\alpha\in V$ such that $\hat{\omega}\in\mathscr{S}_\alpha$, and for any constants $c_1,\ldots, c_n\in\mathscr{S}_\alpha$

$$\Gamma\models_\alpha X(c_1,\ldots,c_n)\quad\text{iff}\quad\Gamma\models X(c_1,\ldots,c_n).$$

We wish to show all the formulas of § 1 are *ω-dominant*.

***Lemma* 3.1:** If $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized, $\sim(\exists x)\sim[x \in \hat{\omega} \equiv integer(x)]$ is valid.

Proof: Suppose $\Gamma \models (a \in \hat{\omega})$. Then for any Γ^* $\Gamma^* \models (a \in \hat{\omega})$. But $\Gamma^* \models$ *ordinal* a so there is some Γ^{**} and some ordinal α such that $\Gamma^{**} \models (a = \hat{\alpha})$. Then $\Gamma^{**} \models \sim\sim(\hat{\alpha} \in \hat{\omega})$. Then it must be that $\alpha \in \omega$, hence α is some integer n. Thus $\Gamma^{**} \models (a = \hat{n})$. But $\Gamma^{**} \models integer(\hat{n})$, so $\Gamma^{**} \models \sim\sim integer(a)$. Thus $\Gamma \models \sim\sim integer(a)$.

Conversely, if $\Gamma \models integer(a)$, for any Γ^* $\Gamma^* \models integer(a)$. Then there is some Γ^{**} and some integer n such that $\Gamma^{**} \models (a = \hat{n})$. But $n \in \omega$ so $\Gamma^{**} \models (\hat{n} \in \hat{\omega})$. Thus $\Gamma^{**} \models \sim\sim(a \in \hat{\omega})$, $\Gamma \models \sim\sim(a \in \hat{\omega})$. Since Γ is arbitrary, the result follows.

Now replace in all the formulas of § 1 $integer(x)$ by $x \in \hat{\omega}$. By the above lemma, the resulting formulas are weakly equivalent to the originals (i.e. their negations are equivalent) which is sufficient for our purposes.

We call a formula with constants dominant if the corresponding formula with free variables replacing the constants is dominant.

We leave it to the reader to show the formulas produced above are dominant. For example, $partfun(f)$ is

$$function(f) \wedge (\exists n)\,(integer(n) \wedge domain(f) \subseteq n).$$

This becomes

$$function(f) \wedge (\exists n)\,(n \in \hat{\omega} \wedge domain(f) \subseteq n),$$

and the corresponding formula with no constants is

$$function(y) \wedge (\exists n)\,(n \in x \wedge domain(y) \subseteq n),$$

which is dominant.

It then follows that the formulas of § 1 are *ω-dominant.*

§ 4. The M_α sequence

Let $(f \; is \; M(\alpha))$ be an abbreviation for

$$\begin{aligned} &ordinal(\alpha) \wedge \sim\sim(\exists F)\,\{function(F) \wedge domain(F) = \alpha' \wedge \\ &\quad \sim(\exists x)\sim[x \in \alpha' \supset [(x = \emptyset \wedge F(x) = \emptyset) \vee (\exists y)\,(x = y' \wedge \\ &\quad\quad \sim(\exists z)\sim[z \in F(x) \equiv z \text{ is definable over } F(y)]) \vee \\ &\quad (limit\ ordinal(x) \wedge \sim(\exists z)\sim[z \in F(x) \equiv (\exists w)\,(w \in x \wedge \\ &\quad\quad z \in F(w))])]] \wedge F(\alpha) = f\}. \end{aligned}$$

Remark 4.1: By a classical-intuitionistic argument we have

$$\text{ZF} \vdash_{\text{I}} \sim (\exists x)(\exists y)(\exists z) \sim \{[x \textit{ is } M(z) \wedge \sim (\exists w) \sim (w \in y \equiv w \textit{ is definable over } x)] \supset y \textit{ is } M(z')\}.$$

***Lemma* 4.2:** Suppose $\langle \mathscr{G}, \mathscr{R}, \Vdash, \mathscr{S} \rangle$ is ordinalized, $\hat{\omega}, \hat{\alpha}, f \in \mathscr{S}_\beta$, and $(f \textit{ is } M(\hat{\alpha}))$ is valid. Then there is some $g \in \mathscr{S}_{\beta+2} - \mathscr{S}_{\beta+1}$ such that $(g \textit{ is } M(\widehat{\alpha+1}))$ is valid.

Proof: Let $X(x)$ be the formula $(x \textit{ is definable over } f)$, and let $g_X \in \mathscr{S}_{\beta+2} - \mathscr{S}_{\beta+1}$. We claim $(g_X \textit{ is } M(\widehat{\alpha+1}))$ is valid. Since $(x \textit{ is } M(y))$ is stable, we may show $\sim\sim(g_X \textit{ is } M(\widehat{\alpha+1}))$ is valid. Using the above remark, it suffices to show $\sim(\exists w)\sim[w \in g_X \equiv w \textit{ is definable over } f]$ is valid.

Suppose $\Gamma \Vdash (c \in g_X)$. Since $g_X \in \mathscr{S}_{\beta+2} - \mathscr{S}_{\beta+1}$, $\Gamma \Vdash (c=d) \wedge (d \in g_X)$ for some $d \in \mathscr{S}_{\beta+1}$. So

$$\Gamma \Vdash_{\beta+2} (d \in g_X),$$
$$\Gamma \Vdash_{\beta+1} X(d),$$
$$\Gamma \Vdash_{\beta+1} (d \textit{ is definable over } f).$$

So by *ω-dominance*

$$\Gamma \Vdash (d \textit{ is definable over } f),$$
$$\Gamma \Vdash \sim\sim (c \textit{ is definable over } f).$$

Conversely, if $\Gamma \Vdash (c \textit{ is definable over } f)$, by corollary 2.3 for some $d \in \mathscr{S}_{\beta+1} - \mathscr{S}_\beta$, $\Gamma \Vdash (c=d)$. So

$$\Gamma \Vdash \sim\sim (d \textit{ is definable over } f),$$

and by *ω-dominance*

$$\Gamma \Vdash_{\beta+1} \sim\sim (d \textit{ is definable over } f),$$
$$\Gamma \Vdash_{\beta+1} \sim\sim X(d),$$
$$\Gamma \Vdash_{\beta+2} \sim\sim (d \in g_X),$$
$$\Gamma \Vdash \sim\sim (d \in g_X),$$
$$\Gamma \Vdash \sim\sim (c \in g_X).$$

Since Γ is arbitrary, the result follows.

***Lemma* 4.3:** Suppose $\langle \mathscr{G}, \mathscr{R}, \Vdash, \mathscr{S} \rangle$ is ordinalized. Let $\alpha \in V$, and let δ be the largest non-successor ordinal $\leqslant \alpha$. Then $\alpha = \delta + n$ for some integer $n \geqslant 0$. There is an $f \in \mathscr{S}_{\delta+\omega+2n+1}$ such that $(f \textit{ is } M(\hat{\alpha}))$ is valid in $\langle \mathscr{G}, \mathscr{R}, \Vdash, \mathscr{S} \rangle$.

Proof: By induction on α.

If $\alpha=0$, the result becomes: there is an $f\in\mathscr{S}_{\omega+1}$ such that $(f\ is\ M(\hat{0}))$ is valid. But by a classical-intuitionistic argument

$$\sim(\exists x)\sim[\sim(\exists y)(y\in x)\supset x\ \ is\ \ M(x)]$$

is valid, and since $\hat{0}\in\mathscr{S}_1$, we have $\sim\sim(\hat{0}\ is\ M(\hat{0}))$ is valid, or by stability $(\hat{0}\ is\ M(\hat{0}))$.

Next suppose the result is known for α. The result for $\alpha+1$ follows by lemma 4.2. Finally suppose α is a limit ordinal and the result is known for all ordinals $<\alpha$ (here $\alpha=\delta$). We must show for some $f\in\mathscr{S}_{\alpha+\omega+1}$ $f\ is\ M(\hat{\alpha})$ is valid. But it follows from the methods of ch. 9 that $\hat{\alpha}\in\mathscr{S}_{\alpha+1}$, so $\hat{\alpha}\in\mathscr{S}_{\alpha+\omega}$. Let $X(x)$ be the formula

$$(\exists y)(y\in\hat{\alpha}\wedge(\exists z)(z\ \ is\ \ M(y)\wedge x\in z))$$

and let $f_X\in\mathscr{S}_{\alpha+\omega+1}-\mathscr{S}_{\alpha+\omega}$. We claim $(f_X\ is\ M(\hat{\alpha}))$ is valid.

Since (*limit ordinal* $(\hat{\alpha})$) is valid, we must show

$$\sim(\exists x)\sim[x\in f_X\equiv(\exists y)(y\in\hat{\alpha}\wedge(\exists z)(z\ \ is\ \ M(y)\wedge x\in z))]$$

is valid. But this is *ω-dominant*, so we must show it is valid in $\langle\mathscr{G},\mathscr{R},\models_{\alpha+\omega+1},\mathscr{S}_{\alpha+\omega+1}\rangle$, but this follows from the validity of

$$\sim(\exists x)\sim[X(x)\equiv(\exists y)(y\in\hat{\alpha}\wedge(\exists z)(z\ \ is\ \ M(y)\wedge x\in z))]$$

in $\langle\mathscr{G},\mathscr{R},\models_{\alpha+\omega},\mathscr{S}_{\alpha+\omega}\rangle$ (this is valid trivially because it is an identity).

***Theorem* 4.4:** Suppose $\langle\mathscr{G},\mathscr{R},\models,\mathscr{S}\rangle$ is ordinalized and $\alpha\in V$. There is some $f\in\mathscr{S}$ such that $(f\ is\ M(\hat{\alpha}))$ is valid in $\langle\mathscr{G},\mathscr{R},\models,\mathscr{S}\rangle$.

§ 5. Representatives of constructable sets

Somewhat as we did with ordinals in ch. 9 § 3 we associate with constructable sets elements of $\mathscr{S}$ which will represent them. We find it sufficient to work with general representatives, and do not single out canonical ones.

We make the following preliminary definitions. We call a formula with no universal quantifiers *E-stable* if every subformula beginning with a quantifier is of the form $(\exists x)\ Y(x)$ where $Y(x)$ is stable. Classically any formula is equivalent to many *E-stable* formulas. For a formula X by X^y we mean the formula X with all quantifiers bound to y. That is, if a subformula of X is of the form $(\exists x)\ Y(x)$, the corresponding subformula of X^y has the form $(\exists x)\,[x \in y \wedge Y^y(x)]$. Clearly if X is *E-stable*, X^y has strongly bounded quantifiers and so by ch. 7 § 7 X^y is dominant.

Now suppose $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized. Suppose we have defined representatives in $\mathscr{S}$ for all the elements of M_α. Let $C \in M_{\alpha+1} - M_\alpha$. Then C is a classically definable subset of M_α. Let $X(x)$ be any *E-stable* formula which defines C over M_α. Suppose the constants of X are $C_1, \ldots, C_n$. These are all in M_α. Let $\hat{C}_1, \ldots, \hat{C}_n$ be any representatives in $\mathscr{S}$ of $C_1, \ldots, C_n$ respectively, and let $\hat{X}$ be

$$X\begin{pmatrix} C_1 \ldots C_n \\ \hat{C}_1 \ldots \hat{C}_n \end{pmatrix}.$$

By theorem 4.4 there is an $f \in \mathscr{S}$ such that $(f \textit{ is } M(\hat{\alpha}))$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$. Choose one such f. Le $Y(x)$ be the formula $[x \in f \wedge \hat{X}^f(x)]$. There are only finitely many constants in $Y(x)$. Let $\mathscr{S}_\beta$ contain them all. Consider $g_Y \in \mathscr{S}_{\beta+1} - \mathscr{S}_\beta$. We call g_Y a representative of the constructable set C. In this way we may associate representatives in $\mathscr{S}$ to every element of L, the class of constructable sets in V.

Representatives as defined are of course non-unique. They depend on the particular formula X chosen, on which f, on which representatives for the constants of X, and on which β. However, we will show later that if f and g both represent the same constructable set, $(f = g)$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

We shall use the ambiguous notation that $\hat{C}$ is any one of the representatives of the constructable set C. Since an ordinal α is also a constructable set, $\hat{\alpha}$ is doubly ambiguous, but it will be clear from context whether we mean the ordinal or the constructable set representative. Moreover, as we show later, these two notions are closely connected.

§ 6. Properties of constructable set representatives

Let $(x \textit{ is constructable})$ be an abbreviation for the formula

$$(\exists z)\,(\exists y)\,(\textit{ordinal}(z) \wedge y \textit{ is } M(z) \wedge x \in y).$$

In this section we show:

***Theorem* 6.1:** Let $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ be ordinalized and suppose for some $\Gamma \in \mathscr{G}$ $\Gamma \models (\exists y)\,(y \textit{ is } M(\hat{\alpha}) \wedge f \in y)$. Then there is some Γ^*, some $C \in M_\alpha$, and some $\hat{C}$ representing C such that $\Gamma^* \models (f = \hat{C})$.

***Corollary* 6.2:** If $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized and $\Gamma \models (f \textit{ is constructable})$, then for some Γ^*, some constructable set C, and some representative $\hat{C}$ of C $\Gamma^* \models (f = \hat{C})$.

***Theorem* 6.3:** If $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized, $C \in M_\alpha$, and $\hat{C}$ is any representative of C, then $\sim\sim(\exists y)\,(y \textit{ is } M(\hat{\alpha}) \wedge \hat{C} \in y)$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

***Corollary* 6.4:** If $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized, C is a constructable set, and $\hat{C}$ is any representative of C, $\sim\sim(\hat{C} \textit{ is constructable})$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

Proof of theorem 6.1: By induction on α.

If $\alpha = 0$, since $M_0 = \emptyset$, it follows that $\sim(\exists y)\,(y \textit{ is } M(\hat{0}) \wedge f \in y)$ is valid, so the result is trivial.

Suppose the result is known for α and $\Gamma \models (\exists y)\,(y \textit{ is } M(\widehat{\alpha+1}) \wedge f \in y)$. By a classical-intuitionistic argument

$$\mathrm{ZF} \vdash_{\mathrm{I}} \sim(\exists f)\,(\exists \alpha)\,(\exists y) \sim [(\textit{successor ordinal}(\alpha) \wedge y \textit{ is } M(\alpha) \wedge f \in y) \supset (\exists z)\,(\exists \beta)\,(\textit{ordinal}(\beta) \wedge \alpha = \beta' \wedge z \textit{ is } M(\beta) \wedge f \textit{ is definable over } z)].$$

Moreover, $(\textit{successor ordinal } (\widehat{\alpha+1}))$ is valid, so

$$\Gamma \models \sim\sim (\exists z)\,(\exists \beta)\,(\textit{ordinal}(\beta) \wedge \widehat{\alpha+1} = \beta' \wedge z \textit{ is } M(\beta) \wedge f \textit{ is definable over } z).$$

It then follows that for some $g \in \mathscr{S}$ and some Γ^* that $\Gamma^* \models g \textit{ is } M(\hat{\alpha}) \wedge f \textit{ is definable over } g$. But we have shown there is an $h \in \mathscr{S}$ such that $(h \textit{ is } M(\hat{\alpha}))$ is valid. Thus $\Gamma^* \models h \textit{ is } M(\hat{\alpha})$ and by a classical-intuitionistic argument $\Gamma^* \models (g = h)$. Thus $\Gamma^* \models \sim\sim(f \textit{ is definable over } h)$. There is some Γ^{**} such that $\Gamma^{**} \models (f \textit{ is definable over } h)$. Now by theorem 2.1 there is some dominant formula $X(x)$ with only existential quantifiers, with all quantifiers bound to h, and some Γ^{***} such that if a is a constant of $X(x)$ other than a quantifier bound, then $\Gamma^{***} \models (a \in h)$ and $\Gamma^{***} \models \sim(\exists x) \sim [x \in f \equiv (x \in h \wedge X(x))]$.

There are only a finite number of constants $a_1, \ldots, a_n$ in X. Consider a_1. $\Gamma^{***} \models (a_1 \in h) \wedge (h \textit{ is } M(\hat{\alpha}))$. By induction hypothesis there is some Γ^{****} and a $C \in M_\alpha$ such that $\Gamma^{****} \models (a_1 = \hat{C}_1)$. Consider a_2 similarly, starting with Γ^{****}, and so on to a_n. Thus we get some $\Gamma^{***\cdots*} = \Delta$ and some $C_1, \ldots, C_n \in M_\alpha$ such that $\Delta \models (a_1 = \hat{C}_1) \wedge \cdots \wedge (a_n = \hat{C}_n)$.

Now let X' be

$$X\begin{pmatrix} a_1 \dots a_n \\ \hat{C}_1 \dots \hat{C}_n \end{pmatrix}.$$

Then by weak substitutivity of equality

$$\Delta \models \sim (\exists x) \sim [x \in f \equiv (x \in h \wedge X'(x))].$$

Let $Y(x)$ be the formula $x \in h \wedge X'(x)$. Let $\mathscr{S}_\beta$ contain all the constants of $Y(x)$ and f, and consider $g_Y \in \mathscr{S}_{\beta+1} - \mathscr{S}_\beta$. By definition, for some $C \in M_{\alpha+1}$ g_Y represents C. We claim $\Delta \models (f = g_Y)$.

By dominance, we must show $\Delta \models_{\beta+1} (f = g_Y)$, or equivalently

$$\Delta \models_\beta \sim (\exists x) \sim [x \in f \equiv Y(x)]$$

or

$$\Delta \models_\beta \sim (\exists x) \sim [x \in f \equiv (x \in h \wedge X'(x))].$$

But this is dominant so we must show

$$\Delta \models \sim (\exists x) \sim [x \in f \equiv (x \in h \wedge X'(x))]$$

which we have. If α is a limit ordinal, the result is trivial.

***Lemma* 6.5** (for theorem 6.3): Suppose $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized. Suppose that for any $C \in M_\alpha$ and for any representative $\hat{C}$ of C $\sim\sim(\exists y)$ (y *is* $M(\hat{\alpha}) \wedge \hat{C} \in y$) is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$. Then for any $C \in M_{\alpha+1}$ and for any representative $\hat{C}$ of C $\sim\sim(\exists y)$ (y *is* $M(\widehat{\alpha+1}) \wedge \hat{C} \in y$) is valid.

Proof: Let $C \in M_{\alpha+1}$, and let $\hat{C}$ represent C. Since $\hat{C}$ represents C, $\hat{C}$ is $f_Y \in \mathscr{S}_{\gamma+1} - \mathscr{S}_\gamma$, where $Y(x)$ is $(x \in h \wedge \hat{X}^h(x))$, where $X(x)$ is *E-stable*, $X(x)$ defines C classically over M_α, and (h *is* $M(\hat{\alpha})$) is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

But $\sim(\exists x)\sim[x \in \hat{C} \equiv (x \in h \wedge \hat{X}^h(x))]$ is valid (remember that $\hat{X}^h(x)$ is dominant, and $h \in \mathscr{S}_\gamma$). Moreover, suppose a is some constant of $\hat{X}^h(x)$ other than a quantifier bound. By definition a must represent some element of M_α, so by hypothesis $\sim\sim(\exists y)$ (y *is* $M(\hat{\alpha}) \wedge a \in y$) is valid. But again ($h$ *is* $M(\hat{\alpha})$) is valid, so by a classical-intuitionistic argument $\sim\sim(a \in h)$ is valid. Now by theorem 2.2 $\sim\sim$($\hat{C}$ *is definable over* h) is valid and $\widehat{\alpha+1} = \alpha'$ is valid, so by another classical-intuitionistic argument $\sim\sim(\exists y)$ (y *is* $M(\widehat{\alpha+1}) \wedge \hat{C} \in y$) is valid.

Now theorem 6.3 follows by a straightforward induction on α.

§ 7. The principal result

This section is devoted to showing the following:

***Theorem* 7.1:** Let $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ be ordinalized. Then

(1). If $C, D \in L$, and $\hat{C}$, $\hat{D}$ are representatives of C, D respectively, then $C \in D$ iff $\sim\sim(\hat{C} \in \hat{D})$ is valid, and $C \notin D$ iff $\sim(\hat{C} \in \hat{D})$ is valid.

(2). If f and g both represent the same constructable set, $(f = g)$ is valid.

(3). If f represents the ordinal α in an ordinal sense and g represents α in a constructable set sense, $(f = g)$ is valid.

We first show

***Lemma* 7.2:** Let $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ be ordinalized. Let X be an *E-stable* formula with no universal quantifiers, with all quantifiers bound to M_α, and with all constants other than quantifier bound elements of M_α. By X' we mean (in this lemma) any formula which is like X except for having some representative $\hat{C}$ in place of C, for every non-quantifier-bounding constant of X, and having all its quantifiers bound to h instead of M_α, where $h \in \mathscr{S}$ is such that $(h \textit{ is } M(\hat{\alpha}))$ is valid. Then for the following to hold for all such formulas X, it is sufficient that they hold for atomic X:

$$X \text{ is true over } M_\alpha \Rightarrow \sim\sim X' \text{ is valid in } \langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle,$$
$$X \text{ is false over } M_\alpha \Rightarrow \sim X' \text{ is valid in } \langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle.$$

Proof: By induction on the degree of X. Suppose the result is known for all formulas of degree less than that of X. We have five cases:

(1)–(4). Since

$$\begin{aligned}
(Y \wedge Z)' &= Y' \wedge Z', \\
(Y \vee Z)' &= Y' \vee Z', \\
(\sim Y)' &= \sim Y', \\
(Y \supset Z)' &= Y' \supset Z',
\end{aligned}$$

the four propositional cases follow easily.

(5). Suppose X is $(\exists x)(x \in M_\alpha \wedge Y(x))$ (where $Y(x)$ is stable) and the result is known for Y. X' is $(\exists x)(x \in h \wedge Y'(x))$.

$$X \text{ is true over } M_\alpha \Rightarrow \text{for some } C \in M_\alpha \;\; Y(C) \text{ is true}.$$

But then by induction hypothesis $\sim\sim Y'(\hat{C})$ is valid (for any representative $\hat{C}$). Since $C \in M_\alpha$, by theorem 6.3 $\sim\sim(\exists y)(y \textit{ is } M(\hat{\alpha}) \wedge \hat{C} \in y)$ is valid. It follows that $\sim\sim(\hat{C}\in h)$ is valid. Thus $\sim\sim(\hat{C}\in h)\wedge\sim\sim Y'(\hat{C})$ is valid, which implies $\sim\sim(\exists x)\,(x\in h\wedge Y'(x))$ is valid, i.e. $\sim\sim X'$.

Conversely

$$X \text{ is false over } M_\alpha \Rightarrow \text{for every } C\in M_\alpha \quad Y(C) \text{ is false over } M_\alpha.$$

Suppose for some Γ $\Gamma\nvDash\sim X'$. Then for some Γ^* $\Gamma^*\vDash X'$ and

$$\Gamma^*\vDash(\exists x)\,(x\in h\wedge Y'(x)).$$

For some $a\in\mathscr{S}$ $\Gamma^*\vDash(a\in h\wedge Y'(a))$. But $\Gamma^*\vDash h\ is\ M(\hat{\alpha})$, so by theorem 6.1 for some $C\in M_\alpha$ and some Γ^{**}

$$\Gamma^{**}\vDash(a=\hat{C}),$$
$$\Gamma^{**}\vDash\sim\sim Y'(\hat{C}).$$

But by hypothesis $\sim Y'(\hat{C})$ is valid. Thus $\sim X'$ is valid.

Now we show part (1) of theorem 7.1. The proof is by induction on the order of D (D is of order α if $D\in M_{\alpha+1}-M_\alpha$).

Suppose D is of order α and the result is known for all constructable sets of lower order. $D\in M_{\alpha+1}-M_\alpha$ so D is a definable subset of M_α. Let $\hat{D}$ be some corresponding element $f_Y\in\mathscr{S}_{\beta+1}-\mathscr{S}_\beta$, where $Y(x)$ is the formula $(x\in h\wedge\hat{X}^h(x))$, where $(h\ is\ M(\hat{\alpha}))$ is valid and X defines D over M_α.

$C\in D$ iff $X(C)$ is true over M_α. By induction hypothesis, the conclusion of the above lemma is known for all atomic formulas over M_α, and hence for all formulas. Thus

$$C\in D\Rightarrow X(C) \text{ is true over } M_\alpha$$
$$\Rightarrow\sim\sim X'(\hat{C}) \text{ is valid}.$$

But $C\in M_\alpha$ and $(h\ is\ M(\hat{\alpha}))$ is valid so $\sim\sim(\hat{C}\in h)$ is valid. Thus $\sim\sim[\hat{C}\in h\wedge\hat{X}^h(\hat{C})]$ is valid. By dominance $\sim\sim[\hat{C}\in h\wedge\hat{X}^h(\hat{C})]$ is valid in $\langle\mathscr{G},\mathscr{R},\vDash_\beta,\mathscr{S}_\beta\rangle$, that is $\sim\sim Y(\hat{C})$. Then $\sim\sim(\hat{C}\in\hat{D})$ is valid in $\langle\mathscr{G},\mathscr{R},\vDash_{\beta+1},\mathscr{S}_{\beta+1}\rangle$ and hence in $\langle\mathscr{G},\mathscr{R},\vDash,\mathscr{S}\rangle$.

The second half is similar, and the result follows.

Next we show part (2). Suppose f and g both represent the same constructable set $D\in M_{\alpha+1}-M_\alpha$. Suppose $\Gamma\vDash(a\in f)$. Since $D\in M_{\alpha+1}$, by theorem 6.3 $\Gamma\vDash\sim\sim(\exists y)\,(y\ is\ M(\widehat{\alpha+1})\wedge f\in y)$. By a classical-intuitionistic argument $\Gamma\vDash\sim\sim(\exists y)\,(y\ is\ M(\hat{\alpha})\wedge a\in y)$. Then for any Γ^* $\Gamma^*\vDash\sim\sim(\exists y)\,(y\ is\ M(\hat{\alpha})\wedge a\in y)$. Now by theorem 6.1 there is some $C\in M_\alpha$ and some

Γ^{**} such that $\Gamma^{**} \models (a = \hat{C})$. But then $\Gamma^{**} \models \sim\sim(\hat{C} \in f)$, so by part (1) of the theorem, $C \in D$ is true (since f represents D). But since g also represents D, $\Gamma^{**} \models \sim\sim(\hat{C} \in g)$. So $\Gamma^{**} \models \sim\sim(a \in g)$, $\Gamma \models \sim\sim(a \in g)$. Since Γ is arbitrary and the argument with f and g is symmetric, part (2) holds.

Finally, to show part (3) we proceed by induction on the ordinal α. Suppose the result is known for all $\beta < \alpha$. Let $O(\alpha)$ be some ordinal representative of α, and $C(\alpha)$ be some constructable set representative.

If $\Gamma \models a \in O(\alpha)$, for any Γ^* $\Gamma^* \models a \in O(\alpha)$. But $\Gamma^* \models$ *ordinal* $O(\alpha)$, so $\Gamma^* \models$ *ordinal* a. Now by the results of ch. 9, there is an ordinal β and a Γ^{**} such that $\Gamma^{**} \models a = O(\beta)$. Thus $\Gamma^{**} \models O(\beta) \in O(\alpha)$ so it must be the case that $\beta \in \alpha$. But then by part 1 above $\Gamma^{**} \models C(\beta) \in C(\alpha)$, and by induction hypothesis $\Gamma^{**} \models O(\beta) = C(\beta)$. Thus $\Gamma^{**} \models \sim\sim(O(\beta) \in C(\alpha))$, $\Gamma^{**} \models \sim\sim(a \in C(\alpha))$, so $\Gamma \models \sim\sim(a \in C(\alpha))$. Since Γ is arbitrary, $O(\alpha) \subseteq C(\alpha)$ is valid. The converse inclusion is similar.

CHAPTER 12

INDEPENDENCE OF THE AXIOM OF CONSTRUCTABILITY

§ 1. The specific model

Once again the model presented is adapted from Cohen [3]. Let e and a be formal symbols. By a forcing condition we mean any finite consistent set of statements of the form (nea) or $\sim(nea)$ for any integer n.

Let $\mathscr{G}$ be the collection of all forcing conditions, and let $\mathscr{R}$ be set inclusion $\subseteq$. $\mathscr{S}_0$ consists of the functions $\hat{0}, \hat{1}, \hat{2}, \ldots$, and a. The definitions are as follows:

For each integer n, $\hat{n}$ has as domain $\{\hat{0}, \hat{1}, \ldots, \widehat{n-1}\}$, and if $m<n$,

$$\hat{n}(\hat{m}) = \mathscr{G}.$$

a has as domain $\{\hat{0}, \hat{1}, \hat{2}, \ldots\}$, and

$$a(\hat{n}) = \{\Gamma \mid (n\,e\,a) \in \Gamma\}.$$

Then $\models_0$ for atomic formulas is simply

$$\begin{aligned} &\Gamma \models_0 (\hat{m} \in \hat{n}) \quad \text{if} \quad m \in n, \\ &\Gamma \models_0 (\hat{n} \in a) \quad \text{if} \quad (n e a) \in \Gamma. \end{aligned}$$

We leave to the reader the verification that $\langle \mathscr{G}, \mathscr{R}, \models_0, \mathscr{S}_0 \rangle$ satisfies the five properties of ch. 7 § 3. Property (4) is shown just as in chs. 8 or 10. Thus $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is an intuitionistic ZF model. We also leave to the reader the straightforward verification that $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized.

§ 2. Axiom of constructability

***Theorem* 2.1:** $(\exists x) \sim [x \textit{ is constructable}]$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

Proof: We show in particular that $\sim (a \textit{ is constructable})$ is valid.

Suppose for some $\Gamma \in \mathscr{G}$ $\Gamma \models (a \textit{ is constructable})$. By corollary 11.6.2 for some constructable set $C \in V$ and some Γ^* $\Gamma^* \models (a = \hat{C})$. We will show this is not possible.

Let Γ^{*+} be $\{n \mid (n \varepsilon a) \in \Gamma^*\}$. We have two cases:

Case (1): every integer of C is in Γ^{*+}. Choose some integer n such that $(n \varepsilon a)$ is not in Γ^* (recall that Γ^* is finite). Let Γ^{**} be $\Gamma^* \cup \{(n \varepsilon a)\}$. Then $\Gamma^{**} \in \mathscr{G}$ and $\Gamma^* \mathscr{R} \Gamma^{**}$. But $n \notin C$, so $\Gamma^{**} \models \sim (\hat{n} \in \hat{C})$. Since $(n \varepsilon a) \in \Gamma^{**}$, $\Gamma^{**} \models (\hat{n} \in a)$, which is not possible.

Case (2): some integer of C is not in Γ^{*+}. Let n be such an integer. Let Γ^{**} be $\Gamma^* \cup \{\sim (n \varepsilon a)\}$. Again $\Gamma^{**} \in \mathscr{G}$ and $\Gamma^* \mathscr{R} \Gamma^{**}$. But $n \in C$, so $\Gamma^{**} \models \sim \sim (\hat{n} \in \hat{C})$. Since $\sim (n \varepsilon a) \in \Gamma^{**}$, it follows easily that $\Gamma^{**} \models \sim (\hat{n} \in a)$, which is again impossible.

Hence $\Gamma \not\models (a \textit{ is constructable})$, and since Γ is arbitrary, the theorem follows.

Now we have classical independence by the results of ch. 7 § 1. In ch. 13 we will show that the axiom of choice and the generalized continuum hypothesis are both valid in this model, so the full independence is established.

Remark 2.2: This proof actually shows the stronger fact that *every set of integers is constructable* is independent, since $a \subseteq \omega$ is valid in the above model.

CHAPTER 13

ADDITIONAL RESULTS

§ 1. $\mathscr{S}_\alpha$ representatives

Definition 1.1: We say $s \in \mathscr{S}$ *represents* $\mathscr{S}_\alpha$ if

(1). $g \in \mathscr{S}_\alpha$ implies $\sim\sim(g \in s)$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$,

(2). if $\Gamma \models (g \in s)$, then for some Γ^* and some $h \in \mathscr{S}_\alpha$ $\Gamma^* \models (g = h)$.

***Lemma* 1.2**: Suppose $X(x_1, \ldots, x_n)$ is a formula with no universal quantifiers, and with all constants from $\mathscr{S}_\alpha$. Then for any $c_1, \ldots, c_n \in \mathscr{S}_\alpha$ and any $\Gamma \in \mathscr{G}$

$$\Gamma \models_\alpha \sim X(c_1, \ldots, c_n) \quad \text{iff} \quad \Gamma \models \sim X^s(c_1, \ldots, c_n)$$

(X^s is X relativized to s).

Proof: A straightforward induction on the degree of X.

***Lemma* 1.3**: Suppose s represents $\mathscr{S}_\alpha$. Then for any $f \in \mathscr{S}$:

(1). If $f \in \mathscr{S}_{\alpha+1}$, $\sim\sim(f$ *is definable over* $s)$ is valid.

(2). If $\Gamma \models (f$ *is definable over* $s)$ then for some Γ^* and some $h \in \mathscr{S}_{\alpha+1}$ $\Gamma^* \models (f = h)$.

Proof: Suppose $f \in \mathscr{S}_{\alpha+1}$. If $f \in \mathscr{S}_\alpha$, the result is simple. If $f \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$, then f is f_X for some formula X over $\mathscr{S}_\alpha$. We claim $\sim(\exists x)\sim[x \in f_X \equiv (x \in s \wedge X^s(x))]$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$. We leave this to the reader, using the above lemma. It then follows by theorem 11.2.2 that $\sim\sim(f$ *is definable over* $s)$ is valid.

Suppose conversely that $\Gamma \models (f$ *is definable over* $s)$. By theorem 11.2.1

there is some Γ^* and a dominant formula $X(x)$ with no universal quantifiers, bound to s, with every non-quantifier-bounding constant a such that $\Gamma^* \models (a \in s)$, such that

$$\Gamma^* \models \sim (\exists x) \sim [x \in f \equiv (x \in s \wedge X(x))].$$

Now for any a of $X(x)$ $\Gamma^* \models (a \in s)$, so for some $a' \in \mathscr{S}_\alpha$ and some Γ^{**} $\Gamma^{**} \models (a = a')$. Similarly with all constants of $X(x)$ (other than s). Thus we have $\Delta = \Gamma^{**\cdots*}$ such that if b is any constant of $X(x)$ other than s, there is some $b' \in \mathscr{S}_\alpha$ such that $\Delta \models (b = b')$. Now let X' be like X except for containing $a' \in \mathscr{S}_\alpha$ for each a of X. Then it follows that

$$\Delta \models \sim (\exists x) \sim [x \in f \equiv (x \in s \wedge X'(x))].$$

Let X'' be like X' except for having unbounded quantifiers. Then X'' is a formula over $\mathscr{S}_\alpha$. Let $h_{X''} \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$. We claim $\Delta \models (f = h_{X''})$. This follows immediately by lemma 1.2.

***Lemma* 1.4:** If s represents $\mathscr{S}_\alpha$ and t represents $\mathscr{S}_{\alpha+1}$, then

$$\sim (\exists x) \sim [x \in t \equiv x \textit{ is definable over } s]$$

is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

Proof: By lemma 1.3 and definition 1.1.

***Lemma* 1.5:** If s represents $\mathscr{S}_\alpha$ and $\sim (\exists x) \sim [x \in t \equiv x \textit{ is definable over } s]$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$, then t represents $\mathscr{S}_{\alpha+1}$.

Proof: Again straightforward.

Remark 1.6: Every $\mathscr{S}_\alpha$ is, of course, representable. Let $X(x)$ be the formula $x = x$, and let $f_X \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$. Then f_X represents $\mathscr{S}_\alpha$.

§ 2. Definition functions

Let (F *is a* β *length* s *function*) be an abbreviation for

$$\begin{aligned} &\textit{function}\ (F) \wedge \textit{ordinal}\ (\beta) \wedge \textit{domain}\ (F) = \beta \wedge \\ &\quad \sim (\exists \gamma) \sim \{\gamma \in \beta \supset [(\gamma = \emptyset \wedge F(\gamma) = s) \vee \\ &\qquad\qquad (\exists \delta)[\delta \in \gamma \wedge \gamma = \delta' \wedge \sim (\exists x) \sim (x \in F(\gamma) \equiv \\ &\qquad\qquad\qquad x \textit{ is definable over } F(\delta))] \vee \\ &\quad [\textit{limit ordinal}\,(\gamma) \wedge \sim (\exists x) \sim (x \in F(\gamma) \equiv (\exists \delta)(\delta \in \gamma \wedge x \in F(\delta)))]]\}. \end{aligned}$$

The following is left to the reader.

***Lemma* 2.1:** If

$\Gamma \models [(\beta \in \gamma) \wedge F \textit{ is a } \beta \textit{ length s function} \wedge G \textit{ is a } \gamma \textit{ length s function}]$ then $\Gamma \models (F \subseteq G)$.

For the rest of this section we assume our models are ordinalized.

***Lemma* 2.2:** Let $s \in \mathscr{S}_1 - \mathscr{S}_0$ represent $\mathscr{S}_0$. Then for any $\beta \geqslant 0$ there is an $F \in \mathscr{S}_{\beta+3} - \mathscr{S}_{\beta+2}$ such that $[F \textit{ is a } \widehat{\beta+1} \textit{ length s function}]$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$, and for any $\gamma < \beta$, if $\Gamma \models (h = F(\hat{\gamma}))$, then h represents $\mathscr{S}_\gamma$.

Proof: By induction on β.
If $\beta = 0$, let $X(x)$ be the formula $x = \langle \hat{0}, s \rangle$ and consider $F_X \in \mathscr{S}_3 - \mathscr{S}_2$. Suppose the result is known for β. Then there is an $F \in \mathscr{S}_{\beta+3} - \mathscr{S}_{\beta+2}$ satisfying the lemma. Let $f \in \mathscr{S}_{\beta+2} - \mathscr{S}_{\beta+1}$ represent $\mathscr{S}_{\beta+1}$. Let $X(x)$ be the formula $x \in F \vee x = \langle \widehat{\beta+1}, f \rangle$ and let $G_X \in \mathscr{S}_{\beta+4} - \mathscr{S}_{\beta+3}$. If β is a limit ordinal and the result is known for all lesser ordinals, let $X(x)$ be the formula

$$(\exists \gamma)(\exists F)(\gamma \in \hat{\beta} \wedge F \textit{ is a } \gamma \textit{ length s function} \wedge x \in F)$$

and let $G_X \in \mathscr{S}_{\beta+3} - \mathscr{S}_{\beta+2}$.

We leave verifications to the reader.

***Theorem* 2.3:** Let $s \in \mathscr{S}_1 - \mathscr{S}_0$ represent $\mathscr{S}_0$. Then

$$\sim (\exists x) \sim (\exists \beta)(\exists F)[F \textit{ is a } \beta' \textit{ length s function} \wedge x \in F(\beta)]$$

is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

§ 3. Restriction on ordinals representable

We devote this section to a brief sketch of the proof of

***Theorem* 3.1:** Suppose $\langle \mathscr{G}, \mathscr{R}, \models_\Omega, \mathscr{S}_\Omega \rangle$ is itself an ordinalized intuitionistic ZF model, where $\Omega > 0$. Then exactly the ordinals $< \Omega$ are representable in $\mathscr{S}_\Omega$.

Proof: Trivially Ω must be a limit ordinal, so by the work of ch. 9 at least the ordinals $< \Omega$ are representable in $\mathscr{S}_\Omega$. We show now that $\hat{\Omega} \notin \mathscr{S}_\Omega$.

Since $\Omega > 0$ there is an $s \in \mathscr{S}_1 - \mathscr{S}_0$ (and hence $s \in \mathscr{S}_\Omega$) such that s represents $\mathscr{S}_0$ (see § 1). By the work in § 2 the following is valid in $\langle \mathscr{G}, \mathscr{R}, \models_\Omega, \mathscr{S}_\Omega \rangle$:

$$\sim (\exists x) \sim (\exists \beta)(\exists F)\,[F \textit{ is a } \beta' \textit{ length } s \textit{ function} \wedge x \in F(\beta)].$$

Suppose $\hat{\Omega} \in \mathscr{S}_\Omega$. It then follows that

$$\left.\begin{array}{l} \sim (\exists x) \sim (\exists \beta \in \hat{\Omega})(\exists F)[F \textit{ is a } \beta' \textit{ length } s \textit{ function} \wedge x \in F(\beta)] \\ \textit{is valid in } \langle \mathscr{G}, \mathscr{R}, \models_\Omega, \mathscr{S}_\Omega \rangle. \end{array}\right] \quad (1)$$

Moreover β length s functions form a chain, that is the following is valid in $\langle \mathscr{G}, \mathscr{R}, \models_\Omega, \mathscr{S}_\Omega \rangle$:

$$\begin{array}{l} \sim (\exists \alpha \in \hat{\Omega})(\exists \beta \in \hat{\Omega})(\exists F)(\exists G) \\ \sim [(\alpha \in \beta \wedge F \textit{ is an } \alpha \textit{ length } s \textit{ function} \wedge G \textit{ is a } \beta \textit{ length } s \textit{ function}) \supset \\ \hfill F \subseteq G] \end{array}$$

(see § 2).

It then follows that the following is valid in $\langle \mathscr{G}, \mathscr{R}, \models_\Omega, \mathscr{S}_\Omega \rangle$ (using obvious abbreviations)

$$\sim \sim (\exists y)(y = \bigcup \{F \mid F \textit{ is a } \beta' \textit{ length } s \textit{ function}, \textit{for } \beta \in \hat{\Omega}\}). \quad (2)$$

From (1) and (2) the validity of $\sim\sim(\exists z)\sim(\exists x)\sim(x \in z)$ follows, which is not possible.

§ 4. A classical connection

The result of ch. 11 § 7 may be extended to

***Theorem* 4.1:** Suppose $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized. Let X be any formula with no universal quantifiers, no free variables, and all constants from L. Let X' be like X except for having constants $\hat{C}$ where X has C, and having all its quantifiers bound to the formula (x *is constructable*). Then

$$\begin{array}{lll} X \text{ is true over } L & \text{iff} & \sim\sim X' \text{ is valid in } \langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle. \\ X \text{ is false over } L & \text{iff} & \sim X' \text{ is valid in } \langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle. \end{array}$$

Proof: By induction on the degree of X. If X is atomic, the result is theorem 11.7.1.

Suppose the result is known for all formulas of degree less than that of X. The four cases X is $Y \supset Z$, $\sim Y$, $Y \vee Z$ or $Y \wedge Z$ are simple.

Suppose X is $(\exists x)\, Y(x)$. Then X' is $(\exists x)\,(x$ *is constructable* $\wedge\, Y'(x))$. If X is true over L, for some $C \in L$ $Y(C)$ is true over L. By induction hypothesis $\sim\sim Y'(\hat{C})$ is valid. But by corollary 11.6.4 $\sim\sim(\hat{C}$ *is constructable*) is also valid. Hence $(\exists x)(\sim\sim x$ *is constructable* $\wedge \sim\sim Y'(x))$ is valid. But this implies $\sim\sim(\exists x)(x$ *is constructable* $\wedge\ Y'(x))$ is valid, i.e. $\sim\sim X'$.

Conversely suppose X is false over L. Then $Y(C)$ is false over L for every $C \in L$. By induction hypothesis $\sim Y'(\hat{C})$ is valid for every $C \in L$. Now suppose for some $\Gamma \in G$ $\Gamma \nVdash \sim X'$. Then for some Γ^* $\Gamma^* \Vdash X'$ or $\Gamma^* \Vdash (\exists x)(x$ *is constructable* $\wedge\, Y'(x))$. For some $a \in \mathscr{S}$ $\Gamma^* \Vdash (a$ *is constructable* $\wedge\ Y'(a))$. By corollary 11.6.2 for some Γ^{**} and some $C \in L$ $\Gamma^{**} \Vdash (a = \hat{C})$, so $\Gamma^{**} \Vdash \sim\sim Y'(\hat{C})$, a contradiction.

Remark 4.2: Suppose $\langle \mathscr{G}, \mathscr{R}, \Vdash_\Omega, \mathscr{S}_\Omega \rangle$ were itself an ordinalized intuitionistic ZF model. We showed in § 3 that exactly the ordinals $< \Omega$ are representable in $\mathscr{S}_\Omega$. It then follows that for any $C \in M_\Omega$ $\hat{C} \in \mathscr{S}_\Omega$, and conversely. This may be shown by adapting the methods of ch. 11. Now the above theorem may be restricted to

***Theorem* 4.3:** Suppose $\langle \mathscr{G}, \mathscr{R}, \Vdash_\Omega, \mathscr{S}_\Omega \rangle$ is an ordinalized intuitionistic ZF model. Let X and X' be as above, save that X has constants only from M_Ω. Then

$$\begin{aligned} &X \text{ is true over } M_\Omega \quad \text{iff} \quad \sim\sim X' \text{ is valid in } \langle \mathscr{G}, \mathscr{R}, \Vdash_\Omega, \mathscr{S}_\Omega \rangle, \\ &X \text{ is false over } M_\Omega \quad \text{iff} \quad \sim X' \text{ is valid in } \langle \mathscr{G}, \mathscr{R}, \Vdash_\Omega, \mathscr{S}_\Omega \rangle. \end{aligned}$$

Proof: This may be shown exactly as theorem 4.1 was shown. It is simple to establish that the theorem 7.1 relativizes to $\langle G, \mathscr{R}, \Vdash_\Omega, \mathscr{S}_\Omega \rangle$ in the obvious manner.

§ 5. Sets which are models

Classically certain of the M_α themselves may be ZF models. For example M_Ω, where Ω is the first inaccessible cardinal, is such a model. We now examine the intuitionistic counterpart.

***Theorem* 5.1:** Suppose M_α is a classical ZF model, and $\langle \mathscr{G}, \mathscr{R}, \Vdash_0, \mathscr{S}_0 \rangle \in M_\alpha$. Then $\langle \mathscr{G}, \mathscr{R}, \Vdash_\alpha, \mathscr{S}_\alpha \rangle$ is an intuitionistic ZF model.

Proof: In the proofs of ch. 7 V was any arbitrary classical ZF model.

If we take V to be M_α, all the results still hold. But now the class model $\langle G, \mathscr{R}, \models, \mathscr{S}\rangle$ with respect to M_α is actually $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, \mathscr{S}_\alpha\rangle$.

***Theorem* 5.2:** Suppose $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, \mathscr{S}_\alpha\rangle$ is an ordinalized intuitionistic ZF model. Then M_α is a classical ZF model.

Proof: Let X be any ZF axiom stated with no universal quantifiers. Since X has no constants, X' as in theorem 4.3 is simply X relativized to the constructable sets. It is shown in the course of the Gödel consistency proofs that $\mathrm{ZF}\vdash_c X'$ (for example see [3]). Hence as usual $\mathrm{ZF}\vdash_I \sim\sim X'$. Thus $\sim\sim X'$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, \mathscr{S}_\alpha\rangle$. Now if X were not true over M_α, by theorem 4.3 $\sim X'$ would be valid in $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, \mathscr{S}_\alpha\rangle$. Hence X is true over M_α.

§ 6. Restriction on cardinals representable

In ch. 9 § 8 we called $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ cardinalized if all the cardinals of V were cardinals of $\mathscr{S}$. We now want to verify the remark made there that the cardinals of $\mathscr{S}$ were the same as the cardinals of L. More precisely:

***Theorem* 6.1:** Suppose $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ is ordinalized, and for some $\alpha\in V$ and some $\Gamma\in\mathscr{G}$ $\Gamma\models(\textit{cardinal}\ (\hat{\alpha}))$. Then α is a cardinal of L, the class of constructable sets of V.

Proof: Suppose α is not a cardinal of L. Then for some $\beta\in\alpha$ and some $F\in L$ the following formula is true over L:

$$[\textit{function}\ (F)\wedge 1\text{-}1(F)\ \wedge\ \textit{domain}(F)=\beta\wedge\textit{range}(F)=\alpha].$$

But $\beta\in\alpha$, so $\sim\sim(\hat{\beta}\in\hat{\alpha})$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$. By theorem 4.1

$$\sim\sim[\textit{function}(F)\wedge 1\text{-}1(F)\wedge\textit{domain}(F)=\beta\wedge\textit{range}(F)=\alpha]'$$

is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$. But this is

$$\sim\sim[\textit{function}^L(\hat{F})\wedge 1\text{-}1^L(\hat{F})\wedge\textit{domain}^L(\hat{F})=\hat{\beta}\wedge\textit{range}^L(\hat{F})=\hat{\alpha}]$$

where the superscript L means the formula has been relativized to (x *is constructable*). But classically

$$\mathrm{ZF}\vdash_c\sim(\exists x)\sim[(x\ \textit{is constructable}\ \wedge\ \textit{function}^L(x))\supset\textit{function}(x)]$$

and similarly for 1-1, *domain* and *range*. By corollary 11.6.4

$$\sim\sim(\hat{F}\ \textit{is constructable})\wedge$$
$$\sim\sim(\hat{\alpha}\ \textit{is constructable})\wedge\sim\sim(\hat{\beta}\ \textit{is constructable})$$

is valid. Hence

$$\sim \sim [\mathit{function}(\hat{F}) \wedge 1\text{-}1(\hat{F}) \wedge \mathit{domain}(\hat{F}) = \hat{\beta} \wedge \mathit{range}(\hat{F}) = \hat{\alpha}]$$

is valid. This contradicts $\Gamma \models (\mathit{cardinal}(\hat{\alpha}))$.

Remark 6.2: In the above it does not matter whether $\hat{\alpha}$ and $\hat{\beta}$ are ordinal or constructable set representatives. See theorem 11.7.1.

§ 7. Axiom of choice

By $\mathscr{F}(X)$ we mean the collection of all classically definable subsets of the set X. Suppose we can define classically a sequence of sets as follows:

$$\begin{aligned} \mathscr{S}_0 &= X, \\ \mathscr{S}_{\alpha+1} &= \mathscr{F}(\mathscr{S}_\alpha), \\ \mathscr{S}_\lambda &= \bigcup_{\alpha<\lambda} \mathscr{S}_\alpha \quad \text{(for limit ordinals } \lambda), \end{aligned}$$

and let the class $\mathscr{S} = \bigcup \mathscr{S}_\alpha$. If X can be well ordered by some relation R, then it is easy to show there is a class which well orders S, or, any set in S can be well ordered. Formally we have

$$\begin{aligned} \mathrm{ZF} \vdash_c \sim \sim (\exists X) \sim (\exists x) \sim (\exists \beta)(\exists F)[(F \textit{ is a } \beta' \textit{ length } X \textit{ function} \wedge \\ x \in F(\beta)) \wedge (\exists R)(R \textit{ well orders } X)] \supset \\ \sim (\exists y) \sim (\exists t)(t \textit{ well orders } y). \end{aligned}$$

Now by a classical-intuitionistic argument we have

***Theorem* 7.1:** Let $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ be ordinalized. Suppose $s \in \mathscr{S}_1 - \mathscr{S}_0$ represents $\mathscr{S}_0$. The if $\Gamma \models (\exists R)(R \textit{ well orders } s)$ then $\Gamma \models \textit{axiom of choice}$.

Now we consider the specific models constructed earlier.

In the model of ch. 12, if $X(x)$ is the formula $x = x$ and $s_X \in \mathscr{S}_1 - \mathscr{S}_0$, s_X represents $\mathscr{S}_0$. We wish to show $(\exists R)(R \textit{ well orders } s_X)$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

Let $Y(x)$ be the formula

$$\begin{aligned} (\exists y)(\exists z)\{[\mathit{integer}\ (y) \wedge \mathit{integer}\ (z) \wedge y \in z \wedge x = \langle y, z \rangle] \vee \\ [\mathit{integer}\ (y) \wedge z = a \wedge x = \langle y, z \rangle]\}, \end{aligned}$$

and let $R_Y \in \mathscr{S}_{\omega+3} - \mathscr{S}_{\omega+2}$. Then $(R_Y \textit{ well orders } s_X)$ is valid. Thus the axiom of choice is valid in the model of ch. 12.

In the model of ch. 10, as above s_X represents $\mathscr{S}_0$. A reasonable well-ordering of $\mathscr{S}_0$ would be (schematically) $\hat{0}, \hat{1}, \hat{2},\ldots, a_0, a_1, a_2,\ldots, \{\hat{0}\}, \{\hat{1}\}, \{\hat{2}\},\ldots, \{\hat{0}, a_0\}, \{\hat{1}, a_1\}, \{\hat{2}, a_2\},\ldots, \langle\hat{0}, a_0\rangle, \langle\hat{1}, a_1\rangle, \langle\hat{2}, a_2\rangle, \ldots, W$.

We leave it to the reader to show that this well-ordering can be expressed in the model. The only nontrivial part of the well-ordering is $a_0, a_1, a_2,\ldots,$ since the subscripts are not part of the model. But W itself provides this ordering.

Thus the axiom of choice is valid in the model of ch. 10.

§ 8. Continuum hypothesis

In this section we show that the generalized continuum hypothesis is valid in the model of ch. 12. More generally we show the following:

***Theorem* 8.1:** Suppose $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ is ordinalized, $\langle\mathscr{G}, \mathscr{R}, \models_0, \mathscr{S}_0\rangle\in L$, and $\mathscr{G}$ and $\mathscr{S}_0$ are countable in L. Then the generalized continuum hypothesis is valid in $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$.

We devote the rest of this section to the proof.

We remarked in ch. 7 § 14 that the definition of the sequence of intuitionistic models is absolute. If L is the class of constructable sets of V, since $\langle\mathscr{G}, \mathscr{R}, \models_0, \mathscr{S}_0\rangle\in L$, the construction of the sequence is the same over V or over L. Thus in this case we may assume in all the preceding work V was L. (We use the continuum hypothesis in L.)

Trivially $card(\mathscr{S}_{\alpha+1})=\aleph_0\cdot card(\mathscr{S}_\alpha)$ in L. Since $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ is ordinalized and $\mathscr{S}_0$ is countable in L, it follows by the work of ch. 9 that for any ordinal α of L, if $\alpha\geqslant\omega$ and if β is the least ordinal such that $\hat{\alpha}\in\mathscr{S}_\beta$, then $card(\alpha)=card(\mathscr{S}_\beta)$ in L.

We use $\boldsymbol{P}(x)$ to denote the power set operation both in L and in $\langle\mathscr{G}, \mathscr{R}, \models, \mathscr{S}\rangle$ in an obvious way.

***Lemma* 8.2:** Under the conditions of the theorem, if $\alpha,\beta\in L$ and $card(\alpha)\geqslant\aleph_0$ in L, and if for some $\Gamma\in\mathscr{G}$

$$\Gamma\models(card(\boldsymbol{P}(\hat{\alpha}))=card(\hat{\beta})),$$

then $card(\boldsymbol{P}(\alpha))\geqslant card(\hat{\gamma})$ *in* L.

Proof: As we showed in ch. 7 § 15, for fixed α there is some $\gamma\in L$ such that if $\Gamma \models (f \subseteq \hat{\alpha})$, there is some $g \in \mathscr{S}_\gamma$ such that $\Gamma \models (f = g)$. Assume Γ is fixed.

$\mathscr{S}_\gamma \in L$. We have the axiom of choice in L so we can define a set $P \in L$ such that $P \subseteq \mathscr{S}_\gamma$ and if $\Gamma \models (f \subseteq \hat{\alpha})$, there is some $g \in P$ such that $\Gamma \models (f = g)$, and if $f, g \in P$ and $f \neq g$, $\Gamma \not\models (f = g)$.

Now as in ch. 7 § 15 the following is definable (as a class) over L: the function $\boldsymbol{U}$ such that for $u \in P$

$$\boldsymbol{U}(u) = \{\langle \Gamma^*, t\rangle \mid t \in \mathscr{S}_{\alpha_0} \wedge \Gamma^* \models (t \in u)\},$$

where α_0 is the least ordinal such that $\hat{\alpha} \in \mathscr{S}_{\alpha_0}$. In this case since $P \in L$, $\boldsymbol{U}$ is a set in L, i.e. $\boldsymbol{U} \in L$.

As we showed in ch. 7, for $u, v \in P$, if $\boldsymbol{U}(u) = \boldsymbol{U}(v)$, then $\Gamma \models (u = v)$ and hence $u = v$ here. Thus $u = v$ if and only if $\boldsymbol{U}(u) = \boldsymbol{U}(v)$ for $u, v \in P$. Thus if R is the range of $\boldsymbol{U}$ on P, since $\boldsymbol{U}$ is 1-1, $card(P) = card(R)$ in L. But $R \subseteq \boldsymbol{P}(\mathscr{G} \times \mathscr{S}_{\alpha_0})$ so $card\,(R) \leqslant card(\boldsymbol{P}(\mathscr{G} \times \mathscr{S}_{\alpha_0}))$.
Since

$$\begin{aligned} card(\mathscr{G} \times \mathscr{S}_{\alpha_0}) &= card(\mathscr{G}) \cdot card(\mathscr{S}_{\alpha_0}) \\ &= \aleph_0 \cdot card(\alpha) \\ &= card(\alpha), \end{aligned}$$

then

$$\begin{aligned} card(R) &\leqslant card(\boldsymbol{P}(\alpha)), \\ card(P) &\leqslant card(\boldsymbol{P}(\alpha)). \end{aligned}$$

We have $\Gamma \models (card(\boldsymbol{P}(\hat{\alpha})) = card(\hat{\beta}))$, so for some $F \in \mathscr{S}$,

$$\Gamma \models [\mathit{function}(F) \wedge \text{1-1}(F) \wedge \mathit{domain}(F) = \hat{\beta} \wedge \mathit{range}(F) = \boldsymbol{P}(\hat{\alpha})].$$

We can thus define a function $\boldsymbol{G} \in L$ to satisfy domain $(\boldsymbol{G}) = \beta$ and for $\delta < \beta$ $\boldsymbol{G}(\delta)$ is that element e of P such that $\Gamma \models (F(\hat{\delta}) = e)$ (there is only one such element e for each δ). $\boldsymbol{G}$ is a function in L, range $\boldsymbol{G} \subseteq P$, and it is easy to see $\boldsymbol{G}$ is 1-1. Thus $card(\beta) \leqslant card(P)$ in L. So $card(\beta) \leqslant card(\boldsymbol{P}(\alpha))$ in L.

Now we show the theorem itself.

Suppose for some $\Gamma \in \mathscr{G}$, $\Gamma \not\models$ *generalized continuum hypothesis*. Then for some $\alpha, \beta, \gamma \in L$ and some Γ^*

$$\begin{aligned} \Gamma^* \models\ & \mathit{cardinal}(\hat{\alpha}) \wedge \mathit{cardinal}(\hat{\beta}) \wedge \mathit{cardinal}(\hat{\gamma}) \wedge \\ & \hat{\alpha} \in \hat{\beta} \wedge \hat{\beta} \in \hat{\gamma} \wedge (\hat{\omega} \in \hat{\alpha} \vee \hat{\omega} = \hat{\alpha}) \wedge card(\boldsymbol{P}(\hat{\alpha})) = card(\hat{\gamma}). \end{aligned}$$

Then by § 3 α, β and γ are cardinals of L. Moreover $\alpha \in \beta$, $\beta \in \gamma$, $\omega \in \alpha$ or $\omega = \alpha$, so $card(\alpha) \geqslant \aleph_0$ in L.

By the above lemma

$$card(\boldsymbol{P}(\alpha)) \geqslant card(\gamma) \text{ in } L.$$

Thus β is a cardinal in L between α and $\boldsymbol{P}(\alpha)$ contradicting the continuum hypothesis in L.

§ 9. Classical counter models

In the foregoing we have obtained independence result in set theory without constructing any classical models. In more traditional treatments of forcing, classical models are constructed by a method due to Cohen; for example see [3], but countable classical ZF models are used. Essentially this method was used in ch. 4 § 7 to prove the theorem there. It is possible, using an ultralimit construction, to construct suitable non-standard classical models without countability requirements. The following method is from Vopěnka [22] and is simply translated from the topological intuitionistic models used there to the Kripke semantic models we use. It can be applied in more general settings but we only give it in a form which applies directly to intuitionistic ZF models.

Let $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ be a class model over the classical model V and suppose the axiom of choice is true over V. As we showed in ch. 1 § 6, if $\mathscr{P}$ is the collection $\mathscr{R}$-closed subsets of $\mathscr{G}$, $\langle \mathscr{P}, \subseteq \rangle$ is a pseudo-boolean algebra. Let $\boldsymbol{F}$ be any maximal filter in $\mathscr{P}$. See [16] pp. 44, 66.

Define the class $\overline{\boldsymbol{S}}$ to be the collection of all functions f such that $domain(f) \in \boldsymbol{F}$, $range(f) \subseteq \mathscr{S}$. Define $\in \subseteq \overline{\boldsymbol{S}} \times \overline{\boldsymbol{S}}$ by:

$f \in g$ is true if and only if

$$\{\Gamma \in \mathscr{G} \mid \Gamma \in dom(f),\ \ \Gamma \in dom(g),\ \ \Gamma \models (f(\Gamma) \in g(\Gamma))\} \in \boldsymbol{F}.$$

We claim that for any formula $X(x_1, \ldots, x_n)$ with no universal quantifiers $X(f_1, \ldots, f_n)$ is true over $\overline{\boldsymbol{S}}$ if and only if

$$\{\Gamma \in \mathscr{G} \mid \Gamma \in dom(f_1) \cap \cdots \cap dom(f_n),\ \ \Gamma \models X(f_1(\Gamma), \ldots, f_n(\Gamma))\} \in \boldsymbol{F}.$$

The proof is by induction on the degree of X. We have the result for atomic formulas by definition. The propositional cases are straightforward, using the various properties of maximal filters. We show the existential quantifier case. Suppose X is $(\exists x)\, Y(x, f_1, \ldots, f_n)$ and the result is known for formulas of lesser degree.

Suppose $(\exists x)\, Y(x, f_1, \ldots, f_n)$ is true over $\boldsymbol{\mathfrak{S}}$. Then for some $g \in \boldsymbol{\mathfrak{S}}$ $Y(g, f_1, \ldots, f_n)$ is true over $\boldsymbol{\mathfrak{S}}$. By inductive hypothesis

$$\{\Gamma \mid \Gamma \in dom(g) \cap dom(f_1) \cap \cdots \cap dom(f_n),\\ \Gamma \models Y(g(\Gamma), f_1(\Gamma), \ldots, f_n(\Gamma))\} \in \boldsymbol{F}.$$

But this set is contained in

$$\{\Gamma \mid \Gamma \in dom(f_1) \cap \cdots \cap dom(f_n),\ \ \Gamma \models (\exists x)\, Y(x, f_1(\Gamma), \ldots, f_n(\Gamma))\}$$

so this is an element of $\boldsymbol{F}$.

Conversely suppose

$$\{\Gamma \mid \Gamma \in dom(f_1) \cap \cdots \cap dom(f_n),\ \Gamma \models (\exists x)\, Y(x, f_1(\Gamma), \ldots, f_n(\Gamma))\} \in \boldsymbol{F}.$$

Let this set be A. We define a function g on $A \in \boldsymbol{F}$ as follows. Suppose $\Gamma \in A$, then

$$\Gamma \models (\exists x)\, Y(x, f_1(\Gamma), \ldots, f_n(\Gamma)).$$

So for some $a \in \mathscr{S}$

$$\Gamma \models Y(a, f_1(\Gamma), \ldots, f_n(\Gamma)).$$

choose one such a, and let $g(\Gamma) = a$. Thus, by definition, for $\Gamma \in A$

$$\Gamma \models (\exists x)\, Y(x, f_1(\Gamma), \ldots, f_n(\Gamma)) \quad \text{iff} \quad \Gamma \models Y(g(\Gamma), f_1(\Gamma), \ldots, f_n(\Gamma)).$$

Thus

$$A = \{\Gamma \mid \Gamma \in dom(f_1) \cap \cdots \cap dom(f_n) \cap dom(g),\\ \Gamma \models Y(g(\Gamma), f_1(\Gamma), \ldots, f_n(\Gamma))\} \in \boldsymbol{F}.$$

So by hypothesis $Y(g, f_1, \ldots, f_n)$ is true over $\boldsymbol{\mathfrak{S}}$, so $(\exists x)\, Y(x, f_1, \ldots, f_n)$ is true over $\boldsymbol{\mathfrak{S}}$.

As a special case we have: If X has no universal quantifiers and no constants, X is true over $\boldsymbol{\mathfrak{S}}$ iff $\{\Gamma \mid \Gamma \models X\} \in \boldsymbol{F}$.

Since the unit element of $\langle \mathscr{P}, \subseteq \rangle$ is $\mathscr{G}$, we have $\mathscr{G} \in \boldsymbol{F}$. Thus if X has no universal quantifiers and no constants, and X is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$, X is true over $\boldsymbol{\mathfrak{S}}$.

CHAPTER 14

ADDITIONAL CLASSICAL MODEL GENERALIZATIONS

§ 1. Introduction

All of the preceding work in part II has been with intuitionistic M_α generalizations, but other kinds of generalizations are possible. In this chapter we briefly examine some of them.

Classically two particular models have proved of great use: the model of constructable sets, and the model of sets with rank. We have discussed an intuitionistic generalization of the first. In a similar fashion an intuitionistic generalization of the R_α sequence is possible.

Scott and Solovay have developed what they call boolean valued models for set theory [19]. These are really boolean valued generalizations of the classical R_α sequence, in a sense to be given later. A similar boolean valued generalization of the M_α sequence is possible.

§ 2. Boolean valued logics

This section is intended as a preliminary to boolean valued models for set theory. The subject is treated completely in [16]. Also see ch. 1 § 5.

In a pseudo boolean algebra, if $-a$, the pseudo-compliment of a, has the property $a \cup -a = \vee$, then $-a$ is called the compliment of a. A pseudo boolean algebra in which every element has a complement is called a boolean algebra.

Let $\mathscr{B}$ be a boolean algebra and let v be a map from W, the set of formulas, to $\mathscr{B}$. v is called a (propositional) *homomorphism* if

$$\begin{aligned} v(X \wedge Y) &= v(X) \cap v(Y), \\ v(X \vee Y) &= v(X) \cup v(Y), \\ v(\sim X) &= -v(X), \\ v(X \supset Y) &= v(X) \Rightarrow v(Y) = \\ &= -v(X) \cup v(Y). \end{aligned}$$

In addition v is called a (Q)-*homomorphism* if

$$v((\exists x)\, X(x)) = \bigcup_{a \in T} v(X(a)),$$

$$v((\forall x)\, X(x)) = \bigcap_{a \in T} v(X(a)),$$

where T is the collection of all parameters. The infinite sups and infs corresponding to quantifiers are assumed to exist.

It can be shown that for X a formula with no parameters, X is a theorem of classical logic if and only if $v(X) = \vee$ for any Q-homomorphism into any boolean algebra.

One way of generating a theory (a collection of formulas called true, closed under modus ponens, and containing all valid formulas) is to give a boolean algebra $\mathscr{B}$ and a Q-homomorphism v, and to call a formula X true in the theory being described if $v(X) = \vee$.

§ 3. Boolean valued R_α generalizations

This generalization is from [19], though the particular formulation of it is different.

As usual V is a classical ZF model. Let $\mathscr{B}$ be a complete boolean algebra such that $\mathscr{B} \in V$. ($\mathscr{B}$ is complete if all sups and infs exist. Any boolean algebra can be imbedded in a complete one. See [16].) We define a transfinite sequence $R_\alpha^{\mathscr{B}}$, and simultaneously a sequence of homomorphisms v_α from $W_\alpha^{\mathscr{B}}$ to $\mathscr{B}$ where $W_\alpha^{\mathscr{B}}$ is the collection of all formulas with constants from $R_\alpha^{\mathscr{B}}$. (Note that to define a homomorphism it is sufficient to define it for atomic formulas.) Let $R_0^{\mathscr{B}} = \emptyset$; v_0 is trivially defined. Having defined $R_\alpha^{\mathscr{B}}$, if f is a function from $R_\alpha^{\mathscr{B}}$ to $\mathscr{B}$, call f *extensional* if for each $g, h \in R_\alpha^{\mathscr{B}}$

$$f(g) \cap v_\alpha((\forall x)(x \in g \equiv x \in h)) \leq fh)).$$

Let $R^{\mathscr{B}}_{\alpha+1}$ be the elements of $R^{\mathscr{B}}_\alpha$ together with all extensional functions from $R^{\mathscr{B}}_\alpha$ to $\mathscr{B}$. Suppose $f, g \in R^{\mathscr{B}}_{\alpha+1}$:

(1). if $f, g \in R^{\mathscr{B}}_\alpha$ let

$$v_{\alpha+1}(f \in g) = v_\alpha(f \in g)$$

(2). if $f \in R^{\mathscr{B}}_\alpha$ and $g \in R^{\mathscr{B}}_{\alpha+1} - R^{\mathscr{B}}_\alpha$ let

$$v_{\alpha+1}(f \in g) = g(f)$$

(3). if $f \in R^{\mathscr{B}}_{\alpha+1} - R^{\mathscr{B}}_\alpha$ let

$$v_{\alpha+1}(f \in g) = \bigcup_{h \in dom(g)} \{g(h) \cap \bigcap_{x \in R_\alpha^{\mathscr{B}}} (f(x) \Leftrightarrow v_\alpha(x \in h))\}.$$

Remark 3.1: If an equality symbol is defined in the usual way, condition (3) is the same as

$$v_{\alpha+1}(f \in g) = \bigcup_{h \in dom(g)} \{g(h) \cap v_{\alpha+1}(f = g)\}.$$

If λ is a limit ordinal, let $R^{\mathscr{B}}_\lambda = \bigcup_{\alpha<\lambda} R^{\mathscr{B}}_\alpha$. If $f, g \in R^{\mathscr{B}}_\lambda$, for some $\alpha < \lambda$, $f, g \in R^{\mathscr{B}}_\alpha$; let $v_\lambda(f \in g) = v_\alpha(f \in g)$. Finally, let $R^{\mathscr{B}} = \bigcup_{\alpha \in V} R^{\mathscr{B}}_\alpha$.

If $f, g \in R^{\mathscr{B}}$, for some $\alpha \in V$, $f, g \in R^{\mathscr{B}}_\alpha$, let

$$v(f \in g) = v_\alpha(f \in g).$$

Thus we have a class $R^{\mathscr{B}}$ and a Q-homomorphism v from $W^{\mathscr{B}}$ to $\mathscr{B}$. As we remarked in the last section, all the classically valid formulas map to $\mathbf{v}$. In [19] moreover, it is shown that all the axioms of ZF (as well as the axiom of choice, if true in V) map to $\mathbf{v}$. Thus $R^{\mathscr{B}}$ is called a boolean valued model for ZF.

Finally in [19] a specific model of this kind is produced in which the continuum hypothesis does not map to $\mathbf{v}$, which establishes independence. Similarly for the axiom of constructability.

§ 4. Intuitionistic R_α generalizations

Let V be a classical ZF model. We define a (class of) transfinite sequence of intuitionistic models $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, R^{\mathscr{G}}_\alpha \rangle$ and a class model $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{R}^{\mathscr{G}} \rangle$ as follows:

Let $\mathscr{G}$ be some non-empty element of V, and let $\mathscr{R}$ be some arbitrary reflexive, transitive relation on $\mathscr{G}$, also a member of V. Let $\mathscr{P}$ be the collection of all $\mathscr{R}$-closed subsets of $\mathscr{G}$. As we showed in ch. 1 § 6 $\mathscr{P}$ under the ordering $\subseteq$ is a pseudo-boolean algebra. An element $a \in \mathscr{P}$ is called regular if $--a=a$. We call a function with range $\mathscr{P}$ regular if every member of the range is regular. We define a sequence of intuitionistic models $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, R_\alpha^{\mathscr{G}} \rangle$ as follows. Let

$$R_0^{\mathscr{G}} = \emptyset,$$

so $\langle \mathscr{G}, \mathscr{R}, \models_0, R_0^{\mathscr{G}} \rangle$ is trivial. Suppose $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, R_\alpha^{\mathscr{G}} \rangle$ has been defined. If f is a regular function from $R_\alpha^{\mathscr{G}}$ to $\mathscr{P}$, call f *extensional* if, for each $g, h \in R_\alpha^{\mathscr{G}}$

$$f(g) \cap \{\Gamma \mid \Gamma \models_\alpha (g = h)\} \subseteq f(h).$$

Let $R_{\alpha+1}^{\mathscr{G}}$ be the elements of $R_\alpha^{\mathscr{G}}$ together with all extensional functions from $R_\alpha^{\mathscr{G}}$ to $\mathscr{P}$. If $\Gamma \in \mathscr{G}$ and $f, g \in R_{\alpha+1}^{\mathscr{G}}$, let $\Gamma \models_{\alpha+1} (f \in g)$ if

(1). $f, g \in R_\alpha^{\mathscr{G}}$ and $\Gamma \models_\alpha (f \in g)$,
(2). $f \in R_\alpha^{\mathscr{G}}$, $g \in R_{\alpha+1}^{\mathscr{G}} - R_\alpha^{\mathscr{G}}$ and $\Gamma \in g(f)$,
(3). $f \in R_{\alpha+1}^{\mathscr{G}} - R_\alpha^{\mathscr{G}}$ and for some $h \in domain(g)$

$$\Gamma \in g(h) \quad \text{and} \quad \Gamma \in (f(x) \Leftrightarrow \{\Delta \mid \Delta \models_\alpha \sim\sim (x \in h)\})$$

for every $x \in R_\alpha^{\mathscr{G}}$

Remark 4.2: The expression in part (3) is an element of the pseudo-boolean algebra $\mathscr{P}$, $\Leftrightarrow$ is the operation of $\mathscr{P}$. The definition could have been stated without such a use of $\mathscr{P}$, but less concisely.

Thus we have $\langle \mathscr{G}, \mathscr{R}, \models_{\alpha+1}, R_{\alpha+1}^{\mathscr{G}} \rangle$. If λ is a limit ordinal let

$$R_\lambda^{\mathscr{G}} = \bigcup_{\alpha < \lambda} R_\alpha^{\mathscr{G}}.$$

For $f, g \in R_\lambda^{\mathscr{G}}$ let $\Gamma \models_\lambda (f \in g)$ if for some $\alpha < \lambda$ $\Gamma \models_\alpha (f \in g)$. Thus we have $\langle \mathscr{G}, \mathscr{R}, \models_\lambda, R_\lambda^{\mathscr{G}} \rangle$. Finally let

$$R^{\mathscr{G}} = \bigcup_{\alpha \in V} R_{\mathscr{G}}^{\alpha}$$

and for $f, g \in R^{\mathscr{G}}$ let $\Gamma \models (f \in g)$ if for some $\alpha \in V$ $\Gamma \models_\alpha (f \in g)$.

Thus we have a sequence of models $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, R_\alpha^{\mathscr{G}} \rangle$ and a class model $\langle \mathscr{G}, \mathscr{R}, \models, R^{\mathscr{G}} \rangle$, determined by specifying $\mathscr{G}$ and $\mathscr{R}$. In the next section we show, by translation to a boolean valued $\mathscr{R}_\alpha$ sequence, that $\langle \mathscr{G}, \mathscr{R}, \models, R^{\mathscr{G}} \rangle$ is an intuitionistic ZF model.

§ 5. $\langle \mathscr{G}, \mathscr{R}, \models, R^{\mathscr{G}} \rangle$ is an intuitionistic ZF model

As we remarked in the last section, $\mathscr{P}$, the collection of all $\mathscr{R}$-closed subsets of $\mathscr{G}$, is a pseudo boolean algebra. Moreover it is complete, i.e. all sups and infs exist. This follows since in this case a sup is an infinite union, and the union of $\mathscr{R}$-closed subsets is an $\mathscr{R}$-closed subset, and similarly for infs.

The results of ch. 1 § 6 concerning the relationship of $\mathscr{P}$ and $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, R_\alpha^{\mathscr{G}} \rangle$ may be stated as: for any formulas X and Y $\{\Gamma \mid \Gamma \models_\alpha X\} \in \mathscr{P}$ and

$$\begin{aligned}
\{\Gamma \mid \Gamma \models_\alpha X\} \cup \{\Gamma \mid \Gamma \models_\alpha Y\} &= \{\Gamma \mid \Gamma \models_\alpha X \vee Y\}, \\
\{\Gamma \mid \Gamma \models_\alpha X\} \cap \{\Gamma \mid \Gamma \models_\alpha Y\} &= \{\Gamma \mid \Gamma \models_\alpha X \wedge Y\}, \\
\{\Gamma \mid \Gamma \models_\alpha X\} \Rightarrow \{\Gamma \mid \Gamma \models_\alpha Y\} &= \{\Gamma \mid \Gamma \models_\alpha X \supset Y\}, \\
- \{\Gamma \mid \Gamma \models_\alpha X\} &= \{\Gamma \mid \Gamma \models_\alpha \sim X\}.
\end{aligned}$$

In this case the relationship extends to

$$\begin{aligned}
\bigcup_{f \in R_\alpha^{\mathscr{G}}} \{\Gamma \mid \Gamma \models_\alpha X(f)\} &= \{\Gamma \mid \Gamma \models_\alpha (\exists x)\, X(x)\}, \\
\bigcap_{f \in R_\alpha^{\mathscr{G}}} \{\Gamma \mid \Gamma \models_\alpha X(f)\} &= \{\Gamma \mid \Gamma \models_\alpha (\forall x)\, X(x)\}.
\end{aligned}$$

Similar results hold between the class models.

Now we construct a boolean valued R_α sequence as in § 2.

An element $a \in \mathscr{P}$ is called dense if $-a = \wedge$ or equivalently, if $- - a = \vee$. Let F be the collection of all dense elements of $\mathscr{P}$. F is a filter and (see [16] p. 132-5.8) $\mathscr{P}/F = \mathscr{B}$ is a boolean algebra. Moreover $\mathscr{B} \in V$. ($\mathscr{P}/F$ is the collection of all equivalence classes of $\mathscr{P}$ where a and b are equivalent if $(a \Rightarrow b) \in F$ and $(b \Rightarrow a) \in F$.) In fact, denoting the equivalence class of $a \in \mathscr{P}$ by $|a| \in \mathscr{B}$, we have

$$\begin{aligned}
|a| \cup |b| &= |a \cup b|, \\
|a| \cap |b| &= |a \cap b|, \\
|a| \Rightarrow |b| &= |a \Rightarrow b|, \\
- |a| &= |- a|,
\end{aligned}$$

and the unit of $\mathscr{B}$ is $|\vee| = |\mathscr{G}|$. Furthermore $\mathscr{B}$ is complete and for any index set T

$$\bigcup_{x \in T} |a_x| = \Big|\bigcup_{x \in T} a_x\Big|.$$

Remark 5.1: This relation does not extend generally to $\bigcap$, but since in a boolean algebra $\bigcap$ is equivalent to $-\bigcup-$, the above is sufficient for completeness.

We include the proof of this last statement as it is so useful.

***Lemma* 5.2:** For $a, b \in \mathscr{P}$

$$--(a \Rightarrow b) = (a \Rightarrow --b).$$

Proof: By [16] p. 62, -37]

$$--(a \Rightarrow b) \leqslant (a \Rightarrow --b).$$

Conversely

$$--(--c \Rightarrow c) = \mathsf{V}$$

[16] p. 132, -5.7, and

$$a \cap --b \leqslant --b.$$

so

$$--[(a \cap --b) \Rightarrow b] = \mathsf{V},$$

([16] p. 60-14)

$$--[(a \cap (a \Rightarrow --b)) \Rightarrow b] = \mathsf{V},$$

([16] p. 60-18)

$$--[(a \Rightarrow --b) \Rightarrow (a \Rightarrow b)] = \mathsf{V},$$

([16] p. 60-37)

$$(a \Rightarrow --b) \Rightarrow --(a \Rightarrow b) = \mathsf{V},$$

$$(a \Rightarrow --b) \leqslant --(a \Rightarrow b).$$

***Lemma* 5.3:** In $\mathscr{P}$ for any index set T

$$\bigcap_{x \in T} --(a_x \Rightarrow b) = --\bigcap_{x \in T} (a_x \Rightarrow b).$$

Proof:

$$
\begin{aligned}
&--\bigcap_{x \in T} (a_x \Rightarrow b) = && \text{([16] p. 136, -7)}\\
&= --\left(\bigcup_{x \in T} a_x \Rightarrow b\right) = && \text{(lemma 5.2)}\\
&= \bigcup_{x \in T} a_x \Rightarrow --b \quad = && \text{([16] p. 136, -7)}\\
&= \bigcap_{x \in T} (a_x \Rightarrow --b) = && \text{(lemma 5.2)}\\
&= \bigcap_{x \in T} --(a_x \Rightarrow b).
\end{aligned}
$$

***Theorem* 5.4:** $\bigcup_{x \in T} |a_x| = |\bigcup_{x \in T} a_x|$.

Proof: In $\mathscr{P}$ for any $x \in T$

$$a_x \leqslant \bigcup_{x \in T} a_x ,$$

So

$$--(a_x \Rightarrow \bigcup_{x \in T} a_x) = \mathbf{v} ,$$

$$(a_x \Rightarrow \bigcup_{x \in T} a_x) \in \boldsymbol{F} .$$

So for all $x \in T$

$$|a_x| \leqslant |\bigcup_{x \in T} a_x| .$$

Conversely suppose for some $b \in \mathscr{P}$, for all $x \in T$

$$|a_x| \leqslant |b| .$$

Then for all $X \in T$

$$--(a_x \Rightarrow b) = \mathbf{v} .$$

and since $\mathscr{P}$ is complete,

$$\bigcap_{x \in T} --(a_x \Rightarrow b) = \mathbf{v} ,$$

$$-- \bigcap_{x \in T} (a_x \Rightarrow b) = \mathbf{v} ,$$

([16] p. 136-7)

$$--(\bigcup_{x \in T} a_x \Rightarrow b) = \mathbf{v} ,$$

so

$$|\bigcup_{x \in T} a_x| \leqslant |b| .$$

Thus $\mathscr{B} = \mathscr{P}/\boldsymbol{F}$ is a complete boolean algebra. As shown in § 2, this determines the sequence $R_\alpha^{\mathscr{B}}$, the homomorphisms v_α, and the class model $R^{\mathscr{B}}$ and v. We now wish to investigate the relationship between this and the intuitionistic model from which it arose.

First we claim there is an isomorphism between $R_\alpha^{\mathscr{G}}$ and $R_\alpha^{\mathscr{B}}$ (and between $R^{\mathscr{G}}$ and $R^{\mathscr{B}}$) of a rather substantial kind. We show this by induction on α. $R_0^{\mathscr{G}}$ and $R_0^{\mathscr{B}}$ are identical.

Suppose we have a mapping between $R_\alpha^{\mathscr{G}}$ and $R_\alpha^{\mathscr{B}}$ (pairing $f \in R_\alpha^{\mathscr{G}}$ with $f' \in R_\alpha^{\mathscr{B}}$). If $g \in \mathscr{P}^{R_\alpha^{\mathscr{G}}}$, let $g' \in \mathscr{B}^{R_\alpha^{\mathscr{B}}}$ be the function whose value at $f' \in R_\alpha^{\mathscr{B}}$ is

$$g'(f') = |g(f)| .$$

It follows from the proof of theorem 5.5 below that g is extensional if and only if g' is extensional. We will assume this now.

This map from $R^{\mathscr{G}}_{\alpha+1}$ to $R^{\mathscr{B}}_{\alpha+1}$ is one to one. For suppose $g,h\in R^{\mathscr{G}}_{\alpha+1}-R^{\mathscr{G}}_{\alpha}$ are distinct functions. If g and h are different, there must be some $f\in R^{\mathscr{G}}_{\alpha}$ such that $g(f)\neq h(f)$. If $|g(f)|=|h(f)|$ then by definition

$$g(f) \Rightarrow h(f)\in \boldsymbol{F},$$

or

$$--(g(f) \Rightarrow h(f)) = \mathsf{v}\,,$$

or by lemma 5.2

$$(g(f) \Rightarrow --h(f)) = \mathsf{v}\,.$$

But h is a regular function, so

$$(g(f) \Rightarrow h(f)) = \mathsf{v}\,,$$
$$g(f) \leqslant h(f)\,.$$

Similarly

$$h(f) \leqslant g(f)\,,$$

so

$$g(f) = h(f)\,.$$

Secondly, this map from $R^{\mathscr{G}}_{\alpha+1}$ to $R^{\mathscr{B}}_{\alpha+1}$ is onto. For let $h\in R^{\mathscr{B}}_{\alpha+1}-R^{\mathscr{B}}_{\alpha}$. Let s be any function from $R^{\mathscr{G}}_{\alpha}$ to $\mathscr{P}$ defined by:

for $f\in R^{\mathscr{G}}_{\alpha}$, $s(f)$ is some particular element of $h(f')$.

Let g be the function defined by $g(x)=--s(x)$. Then g is regular, with domain $R^{\mathscr{G}}_{\alpha}$, so $g\in R^{\mathscr{G}}_{\alpha+1}-R^{\mathscr{G}}_{\alpha}$. Moreover, for $f\in R^{\mathscr{G}}_{\alpha}$

$$g'(f') = |g(f)| = |--s(f)| = --|s(f)| = |s(f)| = h(f')\,,$$

and so h is g' for $g\in R^{\mathscr{G}}_{\alpha+1}-R^{\mathscr{G}}_{\alpha}$.

Next we establish the essential identity of the two models.

***Theorem* 5.5:** Let X be a formula over $R^{\mathscr{G}}_{\alpha}$ with no universal quantifiers. Then $X=X(f_1,\dots,f_n)$ for $f_1,\dots,f_n\in R^{\mathscr{G}}_{\alpha}$. Let $X'=X(f_1',\dots,f_n')$ where $f_i'\in R^{\mathscr{B}}_{\alpha}$ is the image of f_i as above. Then

$$v_\alpha(X') = |\{\Gamma \mid \Gamma \models_\alpha X\}|\,.$$

(Similarly for the class models)

Corollary 5.6: If X is any formula with no universal quantifiers and no constants, X is valid in the boolean model $R^{\mathscr{B}}_{\alpha}$ (that is $v_\alpha(X)=\mathsf{v}$) if and only if $\sim\sim X$ is valid in $\langle \mathcal{G}, \mathcal{R}, \models_\alpha, R_\alpha^{\mathcal{G}} \rangle$. (And similarly for the class models.)

Proof: The unit element of $\mathcal{B}$ is $|\mathcal{G}|$ so

$$
\begin{aligned}
v_\alpha(X) = \mathsf{v} \quad &\text{iff} \quad v_\alpha(X) = |\mathcal{G}| \\
&\text{iff} \quad |\{\Gamma \mid \Gamma \models_\alpha X\}| = |\mathcal{G}| \\
&\text{iff} \quad --\{\Gamma \mid \Gamma \models_\alpha X\} = --\mathcal{G} \\
&\text{iff} \quad \{\Gamma \mid \Gamma \models_\alpha \sim\sim X\} = \mathcal{G}.
\end{aligned}
$$

***Corollary* 5.7:** $\langle \mathcal{G}, \mathcal{R}, \models, R^{\mathcal{G}} \rangle$ is an intuitionistic ZF model (and the axiom of choice is valid if it is true over V).

Proof: By corollary 5.6 and the results reported in § 2.

We now turn to the proof of theorem 5.5.

Suppose the result is known for atomic formulas over $R_\alpha^{\mathcal{G}}$. It then follows for all formulas over $R_\alpha^{\mathcal{G}}$ by induction on the degree. For example suppose X is $\sim Y$ and the result is known for Y. Then

$$
\begin{aligned}
v_\alpha(X') &= v_\alpha(\sim Y') \\
&= -\, v_\alpha(Y') \\
&= -\, |\{\Gamma \mid \Gamma \models_\alpha Y\}| \\
&= |-\{\Gamma \mid \Gamma \models_\alpha Y\}| \\
&= |\{\Gamma \mid \Gamma \models_\alpha \sim Y\}| \\
&= |\{\Gamma \mid \Gamma \models_\alpha X\}|.
\end{aligned}
$$

Also suppose the result is known for all formulas $Y(f)$, and X is $(\exists x)\, Y(x)$. Then

$$
\begin{aligned}
v_\alpha(X') &= v_\alpha((\exists x)\, Y'(x)) \\
&= \bigcup_{f' \in R_\alpha^{\mathcal{B}}} v_\alpha(Y'(f')) \\
&= \bigcup_{f' \in R_\alpha^{\mathcal{B}}} |\{\Gamma \mid \Gamma \models_\alpha Y(f)\}| \\
&= \bigcup_{f \in R_\alpha^{\mathcal{G}}} |\{\Gamma \mid \Gamma \models_\alpha Y(f)\}| \\
&= \Big|\bigcup_{f \in R_\alpha^{\mathcal{G}}} \{\Gamma \mid \Gamma \models_\alpha Y(f)\}\Big| \\
&= |\{\Gamma \mid \Gamma \models_\alpha (\exists x)\, Y(x)\}| \\
&= |\{\Gamma \mid \Gamma \models_\alpha X\}|.
\end{aligned}
$$

The other cases are similar. Thus we must show the result holds for atomic formulas. Suppose the result holds for all formulas over $R_\alpha^{\mathscr{G}}$. Let $f, g \in R_{\alpha+1}^{\mathscr{G}}$. We have three cases:

Case (1): $f, g \in R_\alpha^{\mathscr{G}}$

The result is then trivial.

Case (2): $f \in R_\alpha^{\mathscr{G}}$, $g \in R_{\alpha+1}^{\mathscr{G}} - R_\alpha^{\mathscr{G}}$

Then

$$\begin{aligned} v_{\alpha+1}(f' \in g') &= g'(f') \\ &= |g(f)| \\ &= |\{\Gamma \mid \Gamma \Vdash_{\alpha+1} f \in g\}|. \end{aligned}$$

Case (3): $f \in R_{\alpha+1}^{\mathscr{G}} - R_\alpha^{\mathscr{G}}$

We first note that the following holds in any complete pseudo boolean algebra:

$$\bigcap_{x \in T} (-a_x \Leftrightarrow -b_x) = -\bigcup_{x \in T} -(a_x \Leftrightarrow b_x).$$

Now for any $h \in domain(g)$ let

$$\mathscr{P}_h = \{\Gamma \mid \Gamma \in g(h)\}$$

and

$$\Gamma \in \bigcap_{x \in R_\alpha \mathscr{G}} (f(x) \Leftrightarrow \{\Delta \mid \Delta \Vdash_\alpha \sim\sim x \in h\})\}.$$

Then

$$\bigcup_{h \in dom(g)} \mathscr{P}_h = \{\Gamma \mid \Gamma \Vdash_{\alpha+1} f \in g\}.$$

But also

$$\mathscr{P}_h = g(h) \cap \bigcap_{x \in R_\alpha \mathscr{G}} (f(x) \Leftrightarrow -- \{\Delta \mid \Delta \Vdash_\alpha x \in h\}),$$

so, since f is regular,

$$\mathscr{P}_h = g(h) \cap -\bigcup_{x \in R_\alpha \mathscr{G}} -(f(x) \Leftrightarrow \{\Delta \mid \Delta \Vdash_\alpha x \in h\}).$$

Thus

$$\begin{aligned} |\mathscr{P}_h| &= |g(h)| \cap -\bigcup_{x \in R_\alpha \mathscr{G}} -(|f(x)| \Leftrightarrow |\{\Delta \mid \Vdash_\alpha x \in h\}|) \\ &= g'(h') \cap \bigcap_{x' \in R_\alpha \mathscr{B}} (f'(x') \Leftrightarrow v_\alpha(x' \in h')). \end{aligned}$$

And so

$$v_{\alpha+1}(f' \in g') = \bigcup_{h' \in dom(g')} |\mathscr{P}_h|$$

$$= \Big|\bigcup_{h \in dom(g)} \mathscr{P}_h\Big| = \big|\{\Gamma \mid \Gamma \vDash_{\alpha+1} f \in g\}\big|.$$

The case of limit ordinals, and of the class models, is straightforward.

§ 6. Equivalence of the R_α generalizations

In the last section we showed that for any intuitionistic R_α generalization there is a corresponding equivalent boolean valued R_α generalization. In this section we show, under restricted conditions, a converse.

Let $\mathscr{B}$ be a complete boolean algebra. A maximal (=prime) filter $\boldsymbol{F}$ is called a Q-*filter* if, whenever $\bigcup_{X \in T} a_X \in \boldsymbol{F}$, $a_t \in \boldsymbol{F}$ for some $t \in T$, for any index set T. We say $\mathscr{B}$ has property (1) if every non-zero element of $\mathscr{B}$ belongs to some Q-filter ([16] pp. 86–88).

Suppose we have a boolean valued R_α sequence as in § 3, and suppose the algebra $\mathscr{B}$ has property (1). Let $\mathscr{G}$ be the collection of all Q-filters of $\mathscr{B}$, and let $\mathscr{R}$ be $\subseteq$ (which is actually equality, since all Q-filters are maximal). As we showed in § 3, this determines an intuitionistic R_α sequence. We now proceed to show these two models are equivalent.

Let s be the function from $\mathscr{B}$ to ($\mathscr{R}$-closed) subsets of $\mathscr{G}$ defined by: $s(a)$ is the collection of all Q-filters with a as an element. Since $\mathscr{B}$ has property (1), s is an isomorphism between $\mathscr{B}$ and the power set of $\mathscr{G}$ (any subset is $\mathscr{R}$-closed), where the boolean operations in $\mathscr{G}$ are the ordinary set-theoretic ones ([16] p. 87).

We define a reasonable isomorphism between $R_\alpha^{\mathscr{B}}$ and $R_\alpha^{\mathscr{G}}$ as follows: $R_0^{\mathscr{B}}$ and $R_0^{\mathscr{G}}$ are identical.

Suppose an isomorphism has been defined between $R_\alpha^{\mathscr{B}}$ and $R_\alpha^{\mathscr{G}}$ (pairing $f \in R_\alpha^{\mathscr{B}}$ with $f' \in R_\alpha^{\mathscr{G}}$). If $g \in \mathscr{B}^{R_\alpha^{\mathscr{B}}}$ let g' be that element of $\mathscr{P}^{R_\alpha^{\mathscr{G}}}$ defined by

$$g'(f') = s(g(f)).$$

It follows from the proof of theorem 6.1 below that g is extensional if and only if g' is extensional, so we have an isomorphism between $R_{\alpha+1}^{\mathscr{B}}$ and $R_{\alpha+1}^{\mathscr{G}}$.

Now we give the key theorem.

***Theorem* 6.1:** Let X be a formula over $R_\alpha^{\mathscr{B}}$. Then $X = X(f_1,\dots,f_n)$ for $f_1,\dots,f_n \in R_\alpha^{\mathscr{B}}$. Let $X' = X(f_1' \dots, ,f_n')$ where $f_i' \in R_\alpha^{\mathscr{G}}$ is the image of f_i as above. Then

$$\{\Gamma \mid \Gamma \models_\alpha X'\} = s(v_\alpha(x)).$$

(Similarly for the class models.)

Proof: Suppose the result is known for all atomic formulas over $R_\alpha^{\mathscr{B}}$. It then follows for all formulas X by induction on the degree of X. Suppose the result is known for all formulas of degree less than that of X.

If X is $\sim Y$,

$$\{\Gamma \mid \Gamma \models_\alpha X'\} = \{\Gamma \mid \Gamma \models_\alpha \sim Y'\} = -\{\Gamma \mid \Gamma \models_\alpha Y\}$$

(where this is the complement in the boolean algebra of all subsets of $\mathscr{G}$. Since $\Gamma \mathscr{R} \Delta$ implies $\Gamma = \Delta$, it follows that either $\Gamma \models_\alpha Y'$ or $\Gamma \models_\alpha \sim Y'$, so this follows.)

$$= -s(v_\alpha(Y)) = s(-v_\alpha(Y)) = s(v_\alpha(\sim Y)) = s(v_\alpha(X)).$$

Similarly, if X is $(\exists x)\, Y(x)$,

$$\begin{aligned}
\{\Gamma \mid \Gamma \models_\alpha X'\} &= \{\Gamma \mid \Gamma \models_\alpha (\exists x)\, Y'(x)\} \\
&= \bigcup_{f' \in R_\alpha^{\mathscr{G}}} \{\Gamma \mid \Gamma \models_\alpha Y'(f')\} \\
&= \bigcup_{f \in R_\alpha^{\mathscr{B}}} s(v_\alpha(Y(f))) \\
&= s\Big(\bigcup_{f \in R_\alpha^{\mathscr{B}}} v_\alpha(Y(f))\Big) \\
&= s(v_\alpha((\exists x)\, Y(x))) \\
&= s(v_\alpha(x)).
\end{aligned}$$

The other cases are similar.

Thus, we must show the result for atomic formulas. Suppose the result holds for all formulas over $R_\alpha^{\mathscr{B}}$. Let $f,g \in R_{\alpha+1}^{\mathscr{B}}$. We have three cases:

Case (1): $f,g \in R_\alpha^{\mathscr{B}}$

Then the result is trivial.

Case (2): $f \in R_\alpha^{\mathscr{B}}$, $g \in R_{\alpha+1}^{\mathscr{B}} - R_\alpha^{\mathscr{B}}$

Then

$$\begin{aligned}\{\Gamma \mid \Gamma \vDash_{\alpha+1} f' \in g'\} &= g'(f') \\ &= s(g(f)) \\ &= s(v_{\alpha+1}(f \in g)).\end{aligned}$$

Case (3): $f \in R^{\mathscr{B}}_{\alpha+1} - R^{\mathscr{B}}_{\alpha}$.

Then

$$\begin{aligned} s(v_{\alpha+1}(f \in g)) &= s\Big(\bigcup_{h \in \operatorname{dom} g} \big(g(h) \cap \bigcap_{x \in R_\alpha{}^{\mathscr{B}}} (f(x) \Leftrightarrow v_\alpha(x \in h))\big)\Big) \\ &= \bigcup_{h \in \operatorname{dom} g} \big(s(g(h)) \cap \bigcap_{x \in R_\alpha{}^{\mathscr{B}}} (s(f(x)) \Leftrightarrow s(v_\alpha(x \in h)))\big) \\ &= \bigcup_{h' \in \operatorname{dom} g'} \big(g'(h') \cap \bigcap_{x' \in R_\alpha{}^{\mathscr{G}}} (f'(x') \Leftrightarrow \{\Gamma \mid \Gamma \vDash_\alpha x' \in h'\})\big) \\ &= \{\Gamma \mid \Gamma \vDash_{\alpha+1} f' \in g'\}. \end{aligned}$$

The limit ordinal and class cases are straight-forward.

From this theorem, the essential equivalence of the two models follows.

As a special case, suppose V, the underlying classical ZF model, is countable. Then ([16] p. 87-9.3) if $\mathscr{B} \in V$ is a complete boolean algebra, $\mathscr{B}$ also has property (1). Thus if we assume there is a countable ZF model, the two R_α generalizations are equal in power.

The following results would be interesting, but are as yet undone:

(1). A direct proof that $\langle \mathscr{G}, \mathscr{R}, \vDash, R^{\mathscr{G}} \rangle$ is an intuitionistic ZF model.

(2). A more general set of circumstances under which a boolean valued R_α sequence has a corresponding equivalent intuitionistic R_α sequence.

(3). A direct proof that there are intuitionistic R_α generalizations providing counter models for the continuum hypothesis, or the axiom of constructability (preferably not using countability of V).

§ 7. Boolean valued M_α generalizations

Let V be a classical ZF model, and let $\mathscr{B} \in V$ be a complete boolean algebra. We define simultaneously a sequence $M^{\mathscr{B}}_\alpha$ of boolean valued functions, and a sequence v_α of homomorphisms from $M^{\mathscr{B}}_\alpha$ to $\mathscr{B}$. This is a direct generalization of the sequence of ch. 7 § 2.

Let $M^{\mathscr{B}}_0$ be some arbitrary collection of functions with domains subsets of $M^{\mathscr{B}}_0$ and ranges subsets of $\mathscr{B}$. We assume $M^{\mathscr{B}}_0$ is well-founded with respect to the relation $x \in \text{domain}\, y$. We assume $M_0^{\mathscr{B}} \in V$. v_0 is defined by the condition: for $f,g \in M_0^{\mathscr{B}}$

$$v_0(f \in g) = g(f).$$

We require that $M_0^{\mathscr{B}}$ and v_0 satisfy the equality condition

$$v_0((\forall x)(x \in f \equiv x \in g)) \cap v_0(f \in h) \leqslant v_0(g \in h),$$

for any f, g, $h \in M_0^{\mathscr{B}}$.

Suppose we have defined $M_\alpha^{\mathscr{B}}$ and v_α. If $X(x)$ is any formula over $M_\alpha^{\mathscr{B}}$ with one free variable, by f_X we mean the function whose domain is $M_\alpha^{\mathscr{B}}$, whose range is $\mathscr{B}$, and which is defined by

$$f_X(x) = v_\alpha(X(x)),$$

for all $x \in M_\alpha^{\mathscr{B}}$.

Let $M_{\alpha+1}^{\mathscr{B}}$ be $M_\alpha^{\mathscr{B}}$ together with all f_X for all formulas $X(x)$ over $M_\alpha^{\mathscr{B}}$. We define $v_{\alpha+1}$ for atomic formulas as follows. If $f,g \in M_{\alpha+1}^{\mathscr{B}}$,

(1). if $f,g \in M_\alpha^{\mathscr{B}}$, let

$$v_{\alpha+1}(f \in g) = v_\alpha(f \in g);$$

(2). if $f \in M_\alpha^{\mathscr{B}}$, $g \in M_{\alpha+1}^{\mathscr{B}} - M_\alpha^{\mathscr{B}}$ let

$$v_{\alpha+1}(f \in g) = g(f);$$

(3). if $f_X \in M_{\alpha+1}^{\mathscr{B}} - M_\alpha^{\mathscr{B}}$, let

$$v_{\alpha+1}(f \in g) = \bigcup_{h \in M_\alpha^{\mathscr{B}}} \{v_{\alpha+1}(h \in g) \cap \bigcap_{x \in M_\alpha^{\mathscr{B}}} (f(x) \Leftrightarrow v_\alpha(x \in h))\}$$

(where $v_{\alpha+1}(h \in g)$ has been defined in case (1) or case (2)).

If λ is a limit ordinal, let

$$M_\lambda^{\mathscr{B}} = \bigcup_{\alpha < \lambda} M_\alpha^{\mathscr{B}}.$$

If $f,g \in M_\lambda^{\mathscr{B}}$, then for some $\alpha < \lambda$ $f,g \in M_\alpha^{\mathscr{B}}$. Let

$$v_\lambda(f \in g) = v_\alpha(f \in g).$$

Finally let

$$M^{\mathscr{B}} = \bigcup_{\alpha \in V} M_\alpha^{\mathscr{B}}.$$

If $f,g \in M^{\mathscr{B}}$, for some $\alpha \in V$ $f,g \in M_\alpha$. Let

$$v(f \in g) = v_\alpha(f \in g).$$

Thus we have a boolean valued generalization of the M_α sequence, and of L.

§ 8. Equivalence of the M_α generalizations

Let $\langle \mathscr{G}, \mathscr{R}, \models_\alpha, \mathscr{S}_\alpha \rangle$ be any intuitionistic M_α generalization, satisfying the conditions of ch. 1. We proceed almost as we did in § 5.

If $f, g \in \mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$ call f and g equivalent if $(f = g)$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models_{\alpha+1}, \mathscr{S}_{\alpha+1} \rangle$. Let $\mathscr{S}'_\alpha$ be some subset of $\mathscr{S}_\alpha$ containing only one from each collection of equivalent elements.

$\mathscr{P}$ is the collection of all $\mathscr{R}$-closed subsets of $\mathscr{G}$. $\mathscr{P}$ under $\subseteq$ is a pseudo boolean algebra. If $\boldsymbol{F}$ is the filter of all dense elements of $\mathscr{P}$, $\mathscr{B} = \mathscr{P}/\boldsymbol{F}$ is a boolean algebra. Define $M_0^{\mathscr{B}}$ from $\mathscr{S}_0$ by induction on the well-founded relation $x \in domain\,(y)$, so that for $f, g \in \mathscr{S}_0$ the corresponding elements $f', g' \in M_0^{\mathscr{B}}$ satisfy

$$g'(f') = |g(f)|\,.$$

Under this definition $M_0^{\mathscr{B}}$ and $\mathscr{S}'_0$ are isomorphic by induction on the well founded relation $x \in domain\,(y)$. For if $g' = h'$, then for all $f' \in dom\,(g') = dom(h')$, $g'(f') = h'(f')$, so $|g(f)| = |h(f)|$. It follows that for al $\Gamma \in \mathscr{G}$ $\Gamma \models_0 \sim\sim(f \in g) \equiv \sim\sim(f \in h)$, and so $\Gamma \models_0 \sim(\exists x)\sim(x \in g \equiv x \in h)$, so $\Gamma \models_0 g = h$. Then if g, h are in $\mathscr{S}'_0$, g is h. Next we may show $\mathscr{S}'_\alpha$ and $M_\alpha^{\mathscr{B}}$ are isomorphic, and the mapping still satisfies $g'(f') = |g(f)|$. Then following the procedure of § 5, we may show

***Theorem* 8.1:** If X is any formula with no universal quantifiers and no constants, X is valid in the boolean valued model $M^{\mathscr{B}}$ if and only if $\sim\sim X$ is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

Similarly, following the procedure of § 6, we may show

***Theorem* 8.2:** Let $\mathscr{B}$ be a complete boolean algebra satisfying property (1), and let $M_0^{\mathscr{B}}$ and v_0 satisfy the conditions in § 6. Then there is an intuitionistic sequence such that if X is any formula with no constants, X is valid in $M^{\mathscr{B}}$ if and only if X is valid in $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$.

Again the following results would be interesting:

(1). A direct proof that $M^{\mathscr{B}}$ is a boolean valued ZF model.

(2). A more general set of circumstances under which a boolean valued M_α sequence has a corresponding equivalent intuitionistic M_α sequence.

(3). A direct proof that there are boolean valued M_α sequences which establish the various set theory independence results.

APPENDIX

(to ch. 11 § 2)

§ 1. Corresponding formulas

Definition 1.1: Suppose $\Gamma \models partrel(R)$. We say *R corresponds to the formula X over g with respect to* Γ if there is a Γ^* and a finite set of integers $\{i_1, \ldots i_n\}$ such that X is $X(x_{i_1}, \ldots, x_{i_n})$ and

(1). X is dominant,

(2). all the quantifiers (existential only) are bound to g,

(3). for any constant a of X not a quantifier bound, $\Gamma^* \models (a \in g)$,

(4). $\Gamma^* \models \sim (\exists x) \sim [x \in Domain(R) \equiv (x = \hat{\imath}_1 \vee \cdots \vee x = \hat{\imath}_n)]$,

(5). $\Gamma^* \models \sim (\exists x_{i_1}) \ldots (\exists x_{i_n}) \sim [X(x_{i_1}, \ldots, X_{i_n}) \equiv$
$$(\exists f)(f \in R \wedge f(\hat{\imath}_1) = x_{i_1} \wedge \cdots \wedge f(\hat{\imath}_n) = x_{i_n})].$$

***Lemma* 1.2:** Suppose $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized. If $\Gamma \models (R$ *is atomic over* $g)$ then R corresponds to an atomic formula over g, with respect to Γ.

Proof: There are four cases, all treated similarly. We show only one. Thus suppose $\Gamma \models (R$ *is atomic*(2) *over* $g)$. Then for some $a, b \in \mathscr{S}$

$$\Gamma \models [integer\,(b) \wedge \sim \sim (a \in g) \wedge \\ \sim (\exists f) \sim (f \in R \equiv (partfun(f) \wedge domain(f) = \{b\} \wedge f(b) \in a))].$$

Since $\Gamma \models integer(b)$, there is some Γ^* and some integer n such that $\Gamma^* \models (b = \hat{n})$. Since $\Gamma^* \models \sim \sim (a \in g)$, there is some Γ^{**} such that $\Gamma^{**} \models (a \in g)$. Let $\Delta = \Gamma^{**}$. Then

$$\Delta \models [integer(\hat{n}) \wedge a \in g \wedge \\ \sim (\exists f) \sim (f \in R \equiv (partfun(f) \wedge domain(f) = \{\hat{n}\} \wedge f(\hat{n}) \in a))].$$

Now we claim R corresponds to the formula $(x_n \in a)$ over g. If we take the set of integers to be $\{n\}$, properties (1)–(4) are immediate. Property (5) becomes

$$\Delta \models \sim (\exists x_n) \sim [x_n \in a \equiv (\exists f)(f \in R \wedge f(\hat{n}) = x_n)].$$

We show this in two parts:

Suppose $\Delta^* \models (\exists f)(f \in R \wedge f(\hat{n}) = b)$. Then for some $f \in \mathscr{S}$ $\Delta^* \models (f \in R \wedge f(\hat{n}) = b)$. Since $\Delta^* \models (f \in R)$, by the above $\Delta^* \models \sim\sim f(\hat{n}) \in a$. But also $\Delta^* \models f(\hat{n}) = b \wedge \mathit{function}(f)$, so $\Delta^* \models \sim\sim (b \in a)$. Thus

$$\Delta \models \sim (\exists x) \sim [(\exists f)(f \in R \wedge f(\hat{n}) = x) \supset x \in a].$$

Conversely suppose $\Delta^* \models (b \in a)$. Let $Z(x)$ be the formula $x = \langle \hat{n}, b \rangle$, and let w_z be in some suitable $\mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$. The reader may verify

$$\Delta^* \models [\mathit{partfun}(w_z) \wedge \mathit{domain}(w_z) = \hat{n} \wedge w_z(\hat{n}) = b].$$

But $\Delta^* \models b \in a$, so $\Delta^* \models \sim\sim (w_z \in R)$. Thus

$$\begin{aligned} &\Delta^* \models (\exists f)(\sim\sim f \in R \wedge f(\hat{n}) = b), \\ &\Delta^* \models \sim\sim (\exists f)(f \in R \wedge f(\hat{n}) = b), \\ &\Delta \models \sim (\exists x) \sim [x \in a \supset (\exists f)(f \in R \wedge f(\hat{n}) = x)]. \end{aligned}$$

***Lemma* 1.3:** Suppose $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized. If S corresponds to a formula X over g with respect to Γ, and $\Gamma \models (R \text{ is not-} S)$ then R corresponds to the formula $\sim X$ over g with respect to Γ.

Proof: Suppose without loss of generality that the finite set of integers for S is $\{1, 2, \ldots, n\}$. We keep the same set for R. By hypothesis X is dominant, hence so is $\sim X$, thus property (1). Properties (2), (3) and (4) are immediate.

Property (5) becomes

$$\Gamma^* \models \sim (\exists x_1) \ldots (\exists x_n) \sim [\sim X(x_1, \ldots, x_n) \equiv (\exists f)(f \in R \wedge f(\hat{1}) = x_1 \wedge \cdots \wedge f(\hat{n}) = x_n)].$$

But we are given

$$\Gamma^* \models \sim (\exists x_1) \ldots (\exists x_n) \sim [X(x_1, \ldots, x_n) \equiv (\exists f)(f \in S \wedge f(\hat{1}) = x_1 \wedge \cdots \wedge f(\hat{n}) = x_n)],$$

and $\Gamma \models (R \text{ is not-} S)$. We show property (5) in two parts. Suppose $\Gamma^* \mathscr{R} \Delta$. If

$$\Delta \models (\exists f)(f \in R \wedge f(\hat{1}) = c_1 \wedge \cdots \wedge f(\hat{n}) = c_n),$$

then for some $f \in S$

$$\Delta \models (f \in R \wedge f(\hat{1}) = c_1 \wedge \cdots \wedge f(\hat{n}) = c_n).$$

But

$$\Gamma \models \sim (\exists f) \sim [f \in R \equiv \sim f \in S],$$

so $\Delta \models \sim (f \in S)$. We claim that from this follows

$$\Delta \models \sim X(c_1, \dots, c_n).$$

For otherwise for some Δ^* $\Delta^* \models X(c_1, \dots, c_n)$. Then

$$\Delta^* \models \sim \sim (\exists f)(f \in S \wedge f(\hat{1}) = c_1 \wedge \cdots \wedge f(\hat{n}) = c_n),$$

so for some $g \in S$

$$\Delta^* \models \sim \sim (g \in S) \wedge g(\hat{1}) = c_1 \wedge \cdots \wedge g(\hat{n}) = c_n.$$

But

$$\Delta^* \models \sim \sim (g \in S) \wedge (f \in R)$$

and

$$\Delta^* \models \sim (\exists x) \sim [x \in \mathit{Domain}(R) \equiv x \in \mathit{Domain}(S)],$$

so it follows that

$$\Delta^* \models \mathit{domain}(f) = \mathit{domain}(g),$$
$$\Delta^* \models \mathit{domain}(f) = \{\hat{1}, \dots, \hat{n}\}.$$

And

$$\Delta^* \models f(\hat{1}) = g(\hat{1}) \wedge \cdots \wedge f(\hat{n}) = g(\hat{n}),$$

thus

$$\Delta^* \models f = g.$$

But $\Delta^* \models \sim (f \in S) \wedge \sim \sim (g \in S)$, a contradiction. Hence $\Delta \models \sim X(c_1, \dots, c_n)$. Thus

$$\Gamma^* \models \sim (\exists x_1) \dots (\exists x_n) \sim [(\exists f)(f \in R \wedge f(\hat{1}) = x_1 \wedge \cdots \wedge f(\hat{n}) = x_n) \supset \sim X(x_1, \dots, x_n)].$$

Suppose conversely $\Delta \models \sim X(c_1, \dots, c_n)$. Then

$$\Delta \models \sim (\exists f)(f \in S \wedge f(\hat{1}) = c_1 \wedge \cdots \wedge f(\hat{n}) = c_n).$$

Let $Y(x)$ be the formula

$$x = \langle \hat{1}, c_1 \rangle \vee \cdots \vee x = \langle \hat{n}, c_n \rangle,$$

and consider g_Y in some suitable $\mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$. The reader may verify that $\Delta \models [partfun(g_Y) \wedge domain(g_Y) = \{\hat{1}, \ldots, \hat{n}\} \wedge g_Y(\hat{1}) = c_1 \wedge \cdots \wedge g_Y(\hat{n}) = c_n]$. It follows that $\Delta \models \sim (g_Y \in \mathscr{S})$. Hence $\Delta \models \sim \sim (g_Y \in R)$. That is

$$\Delta \models \sim \sim (g_Y \in R) \wedge g_Y(\hat{1}) = c_1 \wedge \cdots \wedge g_Y(\hat{n}) = c_n .$$

$$\Delta \models \sim \sim (\exists f)[f \in R \wedge f(\hat{1}) = c_1 \wedge \cdots \wedge f(\hat{n}) = c_n],$$

so

$$\Gamma \models \sim (\exists x_1) \ldots (\exists x_n) \sim [\sim X(x_1, \ldots, x_n) \supset (\exists f)(f \in R \wedge f(\hat{1}) = x_1 \wedge \cdots \wedge f(\hat{n}) = x_n)].$$

We may in a similar fashion show

***Lemma* 1.4:** Suppose $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized. Suppose S corresponds to a formula X over g and T corresponds to a formula Y over g with respect to Γ. Then

(1). if $\Gamma \models R$ *is S-and-T*, R corresponds to $X \wedge Y$ over g,
(2). if $\Gamma \models R$ *is S-or-T*, R corresponds to $X \vee Y$ over g,
(3). if $\Gamma \models R$ *is S-implies-T*, R corresponds to $X \supset Y$ over g.

Finally we show

***Lemma* 1.5:** Suppose $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized. Suppose S corresponds to a formula $X(x_1, \ldots, x_n)$ over g with respect to Γ, and $\Gamma \models R$ *is* $(\exists j)$ *S over g*. Then R corresponds to the formula

$$(\exists x_j)[(x_j \in g) \wedge \sim \sim X(x_1, \cdots, x_n)]$$

over g with respect to Γ.

Proof: The finite set of integers for S is $\{1, \ldots, n\}$. We may take j to be 1. Then let the set of integers for R be $\{2, \ldots, n\}$. Now property (1) follows by theorem 7.7.3. Properties (2) and (3) are immediate, and (4) is straightforward.

Property (5) becomes

$$\Gamma^* \models \sim (\exists x_2) \ldots (\exists x_n) \sim [(\exists x_1)(x_1 \in g \wedge \sim \sim X(x_1, \ldots, x_n)) \equiv (\exists f)(f \in R \wedge f(\hat{2}) = x_2 \wedge \cdots \wedge f(\hat{n}) = x_n)].$$

We are given

$$\Gamma^* \models \sim (\exists x_1) \ldots (\exists x_n) \sim [X(x_1, \ldots, x_n) \equiv (\exists f)(f \in S \wedge f(\hat{1}) = x_1 \wedge \cdots \wedge f(\hat{n}) = x_n)].$$

We show property (5) in two parts: Let $\Gamma^* \mathscr{R} \Delta$.

Suppose

$$\Delta \models (\exists f)(f \in R \wedge f(\hat{2}) = c_2 \wedge \cdots \wedge f(\hat{n}) = c_n).$$

Then for some $f \in \mathscr{S}$

$$\Delta \models f \in R \wedge f(\hat{2}) = c_2 \wedge \cdots \wedge f(\hat{n}) = c_n.$$

But

$$\Delta \models R \ \textit{is} \ (\exists 1)S \ \text{over} \ g,$$

so

$$\Delta \models \sim \sim (\exists h)(h \in \mathscr{S} \wedge f = h \restriction \textit{Domain}(R) \wedge h(\hat{1}) \in g).$$

Then for any Δ^* there is a Δ^{**} such that

$$\Delta^{**} \models h \in S \wedge f = h \restriction \textit{Domain}(R) \wedge h(\hat{1}) \in g).$$

For some $a \in \mathscr{S}$

$$\Delta^{**} \models h(\hat{1}) = a \wedge a \in g.$$

It now follows that

$$\Delta^{**} \models h(\hat{1}) = a \wedge h(\hat{2}) = c_2 \wedge \cdots \wedge h(\hat{n}) = c_n.$$

So

$$\begin{aligned}
&\Delta^{**} \models \sim \sim X(a, c_2, \ldots, c_n), \\
&\Delta^{**} \models (\exists x_1)\,[\sim \sim X(x_1, c_2, \ldots, c_n) \wedge x_1 \in g], \\
&\Delta^{**} \models \sim \sim (\exists x_1)\,[X(x_1, c_2, \ldots, c_n) \wedge x_1 \in g], \\
&\Delta \models \sim \sim (\exists x_1)\,[X(x_1, c_2, \ldots, c_n) \wedge x_1 \in g],
\end{aligned}$$

This establishes one half.

Conversely suppose

$$\Delta \models (\exists x_1)\,[x_1 \in g \wedge \sim \sim X(x_1, c_2, \ldots, c_n)],$$

then for some $a \in \mathscr{S}$

$$\Delta \models a \in g \wedge \sim \sim X(a, c_2, \ldots, c_n).$$

Thus

$$\Delta \models \sim \sim (\exists f)(f \in S \wedge f(\hat{1}) = a \wedge f(\hat{2}) = c_2 \wedge \cdots \wedge f(\hat{n}) = c_n).$$

So for any Δ^* there is a Δ^{**} such that

$$\begin{aligned}
&\Delta^{**} \models (\exists f)(f \in S \wedge f(\hat{1}) = a \wedge f(\hat{2}) = c_2 \wedge \cdots \wedge f(\hat{n}) = c_n) \\
&\Delta^{**} \models f \in S \wedge f(\hat{1}) = a \wedge f(\hat{2}) = c_2 \wedge \cdots \wedge f(\hat{n}) = c_n.
\end{aligned}$$

Let $Y(x)$ be the formula

$$x = \langle \hat{2}, c_2 \rangle \vee \cdots \vee x = \langle \hat{n}, c_n \rangle ,$$

and let h_Y be in some $\mathscr{S}_{\alpha+1} - \mathscr{S}_\alpha$. The reader may show

$$\Delta^{**} \models partfun(h_Y) \wedge h_Y = f \upharpoonright \hat{Domain} R \wedge f(\hat{1}) \in g .$$

So

$$\begin{aligned}
&\Delta^{**} \models h_Y \in R , \\
&\Delta^{**} \models (h_Y \in R \wedge h_Y(\hat{2}) = c_2 \wedge \cdots \wedge h_Y(\hat{n}) = c_n), \\
&\Delta^{**} \models (\exists h)(h \in R \wedge h(\hat{2}) = c_2 \wedge \cdots \wedge h(\hat{n}) = c_n), \\
&\Delta \models \sim \sim (\exists h)(h \in R \wedge h(\hat{2}) = c_2 \wedge \cdots \wedge h(\hat{n}) = c_n) .
\end{aligned}$$

This establishes the second half.

***Theorem* 1.6:** Suppose $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized and

$$\Gamma \models (R \textit{ is a definable relation over } g).$$

Then R corresponds to a dominant formula X over g with respect to Γ.

Proof: $\Gamma \models (R$ *is a definable relation over* $g)$ so, for some $F \in \mathscr{S}$, some integer n and some Γ^*

$$\begin{aligned}
\Gamma^* \models & \textit{function}(F) \wedge \textit{integer}(\hat{n}) \wedge \textit{domain}(F) = \hat{n} \wedge \\
& \sim (\exists x) \sim [x \in \hat{n} \supset F(x) \textit{ is atomic over } g \vee \\
& \qquad (\exists y)(y \in x \wedge F(x) \textit{ is not-}F(y)) \vee \cdots \vee \\
& \qquad (\exists y)(\exists k)(y \in x \wedge \textit{integer}(k) \wedge F(x) \textit{ is} \\
& \qquad (\exists k)\, F(y) \textit{ over } X)] \wedge \\
& (\exists m)(m \in \hat{n} \wedge F(m) = R).
\end{aligned}$$

Now n is some particular integer. We examine $0, 1, \ldots, n-1$. That is $\Gamma^* \models \hat{0} \in \hat{n}$, so

$$\Gamma^* \models \sim \sim [F(\hat{0}) \textit{ is atomic over } g \vee (\exists y)(y \in \hat{0} \wedge F(\hat{0}) \textit{ is not } F(y)) \vee \cdots].$$

So for some Γ^{**}

$$\Gamma^{**} \models F(\hat{0}) \textit{ is atomic over } g \vee \cdots$$

In fact, since $\Gamma^{**} \models \sim (\exists y)(y \in \hat{0})$,

$$\Gamma^{**} \models F(\hat{0}) \textit{ is atomic over } g .$$

Next $\Gamma^{**} \models \hat{1} \in \hat{n}$, so similarly there is a Γ^{***} such that

$$\Gamma^{***} \models F(\hat{1}) \textit{ is atomic over } g \vee (\exists y)(y \in \hat{1} \wedge F(\hat{1}) \textit{ is not-}F(y)) \vee \cdots$$

and also

$$\Gamma^{***} \models F(\hat{0}) \textit{ is atomic over } g\,.$$

We proceed similarly for each m<n. Thus we have some $\Delta = \Gamma^{**\cdots *}$ such that for each m<n

$$\Delta \models F(\hat{m}) \textit{ is atomic over } g \vee (\exists y)(y \in \hat{m} \wedge F(\hat{m}) \textit{ is not-}F(y)) \vee \cdots.$$

Now by the above lemmas $F(\hat{0})$ corresponds to a dominant formula over g with respect to Δ (hence to Γ). So $F(\hat{1})$ corresponds to a dominant formula over g with respect to $\Delta(\Gamma)$, and so on to $F(\widehat{n-1})$. Finally

$$\Delta \models (\exists m)\,(m \in \hat{n} \wedge F(m) = R),$$

so in some Δ^*

$$\Delta^* \models \hat{m} \in \hat{n} \wedge F(\hat{m}) = R).$$

§ 2. Completeness of the definability formula

***Theorem* 2.1:** Suppose $\langle \mathcal{G}, \mathcal{R}, \models, \mathcal{S} \rangle$ is ordinalized and for some $\Gamma \in \mathcal{G}$ $f, g \in \mathcal{S}$ and

$$\Gamma \models (f \textit{ is definable over } g).$$

Then there is some Γ^* and some dominant formula $X(x)$ with one free variable, no universal quantifier, all quantifiers bound to g, such that if a is a constant of $X(x)$ not a quantifier bound, $\Gamma^* \models (a \in g)$ and

$$\Gamma^* \models \sim (\exists x) \sim [x \in f \equiv (x \in g \wedge X(x))].$$

Proof: $\Gamma \models (f \textit{ is definable over } g)$ so for some Γ^* $R \in \mathcal{S}$, *integer* (n),

$$\begin{aligned} \Gamma^* \models \textit{partrel}(R) \wedge \textit{integer}(\hat{n}) \wedge R \textit{ is a definable relation over } g \wedge \\ \sim (\exists x) \sim [x \in \textit{Domain}(R) \equiv x = \hat{n}] \wedge \\ \sim (\exists x) \sim [x \in f \equiv (x \in g \wedge (\exists h)(h \in R \wedge h(\hat{n}) = x))]. \end{aligned}$$

By theorem 1.6, R corresponds to a permanent formula X over g with respect to Γ. X must be one-placed, $X = X(x_n)$. Moreover, X is dominant, has no universal quantifiers, and has all quantifiers bound to g. There is some Γ^{**} such that for any a of X not a quantifier bound

$\Gamma^{**} \models a \in g$, and

$$\Gamma^{**} \models \sim (\exists x_n) \sim [X(x_n) \equiv (\exists f)(f \in R \wedge f(\hat{n}) = x_n)].$$

Now if $\Gamma^{**} \mathscr{R} \Delta$ and $\Delta \models c \in f$, then

$$\Delta \models \sim \sim (c \in g \wedge (\exists h)(h \in R \wedge h(\hat{n}) = c)),$$

so

$$\Delta \models \sim \sim (c \in g \wedge X(c)).$$

Conversely, if $\Delta \models c \in g \wedge X(c)$, then

$$\Delta \models c \in g \wedge \sim \sim (\exists f)(f \in R \wedge f(\hat{n}) = c).$$

$$\Delta \models \sim \sim [c \in g \wedge (\exists f)(f \in R \wedge f(\hat{n}) = c)],$$

so $\Delta \models \sim \sim c \in f$. Thus

$$\Gamma^{**} \models \sim (\exists x) \sim [x \in f \equiv (x \in g \wedge X(x))].$$

Thus we have established theorem 11.2.1.

§ 3. Adequacy of the definability formula

The proof of theorem 11.2.2 is rather like that of theorem 11.2.1, so we only sketch the general steps.

Definition 3.1: Suppose $X(x_{i_1}, \ldots, x_{i_n})$ is a formula with no universal quantifiers, with all quantifiers bound to $g \in \mathscr{S}$, and such that if a is a constant of X other than a quantifier bound, $\Gamma \models \sim \sim (a \in g)$. We say X corresponds to the partial relation R with respect to Γ if

(1). $\Gamma \models \sim (\exists x) \sim [x \in Domain(R) \equiv (x = \hat{1}_1 \vee \cdots \vee x = \hat{1}_n)]$,

(2). $\Gamma \models \sim (\exists x_{i_1}) \ldots (\exists x_{i_n}) \sim [X(x_{i_1}, \ldots, x_{i_n}) \equiv$
$(\exists f)(f \in R \wedge f(\hat{1}_1) = x_i \wedge \cdots \wedge f(\hat{1}_n) = x_{i_n})]$,

(3). $\Gamma \models \sim \sim (R \textit{ is a definable relation over } g)$.

We wish to show

***Theorem* 3.2:** Suppose $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized and X is a formula with no universal quantifiers, with all quantifiers bound to $g \in \mathscr{S}$, and such that for $\Gamma \in \mathscr{G}$, for any constant a of X other than a quantifier bound $\Gamma \models \sim \sim (a \in g)$. Then X corresponds to some partial relation R with respect to Γ.

To show this we must show a sequence of lemmas similar to those of § 1. For example:

***Lemma* 3.3:** If $\langle \mathscr{G}, \mathscr{R}, \models, \mathscr{S} \rangle$ is ordinalized, $g, a \in \mathscr{S}$, and $\Gamma \models \sim \sim (a \in g)$. Then the formula $x_n \in a$ corresponds to a partial relation R with respect to Γ such that

$$\Gamma \models R \ \ \textit{is atomic}\ (2)\ \textit{over}\ \ g\,.$$

Proof: Let $Y(x)$ be the formula

$$\textit{partfun}\,(x) \wedge \textit{domain}\,(x) = \{\hat{n}\} \wedge x\,(\hat{n}) \in a\,.$$

Let $R_Y \in \mathscr{S}_{\alpha+1} - \mathscr{S}_{\alpha}$ (where $a, \hat{n} \in \mathscr{S}_{\alpha}$). Then

$$\Gamma \models R_Y \ \ \textit{is atomic}\ (2)\ \textit{over}\ \ g\,,$$

and $x_n \in a$ corresponds to R_Y.

Similarly we may show the analogues of the other lemmas of § 1.

Finally, to show theorem 3.2, in a sense we reverse the procedure of the proof in § 1. We proceed through subformulas of X, using the lemmas referred to above, concluding with X.

Given theorem 3.2, theorem 11.2.2 is straightforward.

BIBLIOGRAPHY

[1] Aczel, P. H. G., Some results on intuitionistic predicate logic, abstract in *J. Symb. Logic.*, **32** (1967) 556.

[2] Beth, E. W., *The foundations of mathematics*, North Holland, Amsterdam (1959); also, Harper and Row, New York (1966).

[3] Cohen, P., *Set theory and the continuum hypothesis*, W. A. Benjamin, New York (1966).

[4] Gödel, K., Consistency proof for the generalized continuum hypothesis, *Proc. Nation. Acad. Sci. U.S.A.* **25** (1939) 220–224.

[5] Gregorzyk, A., A philosophically plausible formal interpretation of intuitionistic logic, *Indag. Math.* **26** (1964) 596–601.

[6] Hajek, P. and P. Vopěnka, Some permutation submodels of the model ∇, *Bull. Acad. Polon. Sci.* **14** (1966) 1–7.

[7] Jech, T. and A. Sochor, On Θ model of the set theory, *Bull. Acad. Polon. Sci.* **14** (1966) 297–303.

[8] Jech, T. and A. Sochor, Applications of the Θ model, *Bull. Acad. Polon. Sci.* **14** (1966) 351–355.

[9] Johansson, I., Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus, *Comp. Math.* **4**, (1937) 119–136.

[10] Kleene, S. C., *Introduction to metamathematics*, Van Nostrand, New York (1952).

[11] Kleene, S. C., *Mathematical logic*, Wiley, New York (1967).

[12] Kripke, S., Semantical analysis of modal logic I, *Z. math. Logik u. Grundl. Math.* **9** (1963) 67–96.

[13] Kripke, S., Semantical analysis of intuitionistic logic I, in *Formal systems and recursive functions*, North Holland, Amsterdam (1965), 92–130.

[14] Lewis, C. I., and C. H. Cooper Langford, *Symbolic logic*, 2nd ed., Dover, New York (1959).

[15] Prawitz, D., Natural deduction, a proof-theoretical study, *Acta Universitatis Stockholmiensis*, Stockholm Studies in Philosophy No. 3, Almqvist and Wiksell, Stockholm (1965).

[16] Rasiowa, H. and R. Sikorski, *The mathematics of metamathematics*, Panstwowe Wydawnictwo Naukowe, Warszawa (1963).

[17] Schütte, K. Der Interpolationssatz der intuitionistischen Prädikatenlogik, *Math. Ann.* **148** (1962) 192–200.
[18] Schütte, K., *Vollständige Systeme modaler und intuitionistischer Logik*, Springer, Berlin (1968).
[19] Scott, D. and R. Solovay, Boolean-valued models for set theory, Summer Inst. Axiomatic set theory, Univ. of Cal., Los Angeles, 1967; to appear in *Proc. Am. Math. Soc.*
[20] Smullyan, R. *First order logic*, Springer, New York (1968).
[21] Thomason, R. H., On the strong semantical completeness of the intuitionistic predicate calculus, *J. Symb. Logic* **33** (1968) 1–7.
[22] Vopěnka, P., The limits of sheaves and applications on constructions of models, *Bull. Acad. Polon. Sci.* **13** (1965) 189–192.
[23] Vopěnka, P., On ∇ model of set theory, *Bull. Acad. Polon. Sci.* **13** (1965) 267–272.
[24] Vopěnka, P., Properties of ∇ model, *Bull. Acad. Polon. Sci.* **13** (1965) 441–444.
[25] Vopěnka, P., ∇ models in which the generalized continuum hypothesis does not hold, *Bull. Acad. Polon. Sci.* **14** (1966) 95–99.
[26] Vopěnka, P., The limits of sheaves over extremally disconnected compact hausdorff spaces, *Bull. Acad. Polon. Sci.* **15** (1967) 1–4.
[27] Vopěnka, P. and P. Hájek, Permutation submodels of the model ∇, *Bull. Acad. Polon. Sci.* **13** (1965) 611–614.
[28] Vopěnka, P. and P. Hájek, Concerning the ∇ models of set theory, *Bull. Acad. Polon. Sci.* **15** (1967) 113–117.

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