LANGUAGES WITH EXPRESSIONS OF INFINITE LENGTH

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PREFACE

My interest in infinitary logic dates back to a February day in 1956 when I remarked to my thesis supervisor, Professor Leon Henkin, that a particularly vexing problem would be so simple if only I could write a formula that would say $x = 0$ or $x = 1$ or $x = 2$ etc. To my surprise he replied, "Well, go ahead". The problem is now long-forgotten, but that reply has led to this monograph.

Techniques for proving completeness theorems in logic and representation theorems for Boolean algebras combined to yield a completeness theorem: Valid formulas of denumerable length in which only finitely many variables can be quantified at a time are provable in a system very much like the ordinary first-order predicate calculus. It was clear that this system was not adequate for deductions from assumptions and that no formal system with denumerable proofs could be. It was known from representation theory that additional propositional axiom schemes were required to deal with non-denumerable formulas and I suspected that new quantificational rules were needed as well. The more powerful systems that I formulated in 1957 proved to be complete for many of the infinitary languages; I did not, however, have independence proofs for the new quantificational schemes till later.

The development of infinitary languages was encouraged by Professor Alfred Tarski who, with Professor Henkin, organized a seminar on this topic at Berkeley in the fall of 1956. It gave me an invaluable opportunity to report on my work while it was still at an early stage. Professor Tarski's interest in the area led to a series of new developments in set theory that grew out of William Hanf's work on models of infinitary languages, reported in 1960. Dana Scott's incompleteness theorem, appearing here in print for the first time, was announced at about the same time.

My doctoral dissertation, submitted to the University of Southern

California in 1958, contained most of the material of Chapters 2–11. However, the results of Tarski, Hanf, and Scott in 1960 gave it a focus that it did not have before. The central problem could now be formulated as, "For which cardinals α, β do there exist definable complete formal systems for formulas of length less than α in which fewer than β variables can be quantified at a time?", a question that is almost completely answered in this monograph.

The present version of the material also owes much to the referee (not known to me) of an article submitted to the Journal of Symbolic Logic in 1962. He suggested a way of using the Boolean algebraic representation theorems to make the completeness proofs clearer; I had only mentioned that such proofs could be given. As a result, the completeness proofs are done syntactically only in the propositional case, the Boolean representation theorems are derived from them as they were in the dissertation, and from that point on, algebraic methods are used freely.

This monograph owes its very existence to Professor Leon Henkin at whose suggestion work was begun on it in 1960. I am grateful to Professor Beth for his cooperation during the unexpectedly long preparation period. The work was partially supported by National Science Foundation grant G-11294.

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FOREWARD ON SET THEORY

The various infinitary languages and formal systems under discussion here, are developed informally within a standard set theory with the axiom of choice, but without the continuum hypothesis. For convenience, let it be assumed that the membership relation is primitive, that the empty set $\varnothing$ is the only set having no elements, and that the theory is developed along lines of Godel's monograph [6] or Suppes' book [41]. The ordered pair (x, y) is $\{\{x\}\{x, y\}\}$. A relation is a set of ordered pairs. If R is a relation, $\mathrm{Dom}(R)$ is its domain, $\mathrm{Rng}(R)$ its range. A function is a relation not containing any pairs (x, y), (x, z) with $y \neq z$. The value of function f at x is $f(x)$. If f is a one-one function, f^{-1} is its inverse, If f is a function, S any set, $f|S$ is the restriction of f to $\mathrm{Dom}(f) \cap S$. If f, g are functions, $f \bigcirc g$ is their composition, that function having values $f(g(x))$ for $x \in \mathrm{Dom}(g)$ with $g(x) \in \mathrm{Dom}(f)$. If S, T are sets, $S \times T$ is their Cartesian product, S^T is the set of all functions having domain T, range included in S. The power set of S is $\mathscr{P}S = \{x : x \subseteq S\}$. The union of S is $\bigcup S = \{x:$ there is $y \in S$ such that $x \in y\}$. The intersection of a non-empty set S is $\bigcap S = \{x : x \in y$ whenever $y \in S\}$. The union $\bigcup\{S_i : i \in I\}$ of an indexed family of sets may be written

$$\bigcup_{i \in I} S_i.$$

Similarly for intersection. The cardinal product $\Pi\{S_i : i \in I\}$ is the set of all functions on I such that $f(i) \in S_i$ for all $i \in I$.

An ordinal is a set S with the property that $x \in S$ implies $x \subseteq S$ and $x, y \in S$ implies $x \in y$ or $x = y$ or $y \in x$. The ordinals are well-ordered by the membership relation. If δ, ε are ordinals, we may write "$\delta < \varepsilon$" for $\delta \in \varepsilon$. The successor of δ is $s(\delta) = \delta \cup \{\delta\}$. Every ordinal δ is equal either to $\bigcup\delta$ or to $s(\bigcup\delta)$. In the first case we say that δ is a limit ordinal and write "$\delta \in \mathrm{Lim}$". In the second case δ is a non-limit ordinal. The natural numbers are identified with the finite ordinals. Zero is the empty set, $n + 1 = \{0, \ldots, n\}$. The set of all finite ordinals is ω, the first infinite ordinal. The operations of addition and multiplication for ordinals are defined recursively as follows:

$$\begin{aligned}
&\delta + 0 = \delta \\
&\delta + s(\varepsilon) = s(\delta + \varepsilon) \\
&\delta + \varepsilon = \bigcup\{\delta + \xi : \xi < \varepsilon\} \text{ if } \varepsilon \in \mathrm{Lim}. \\
&\delta \cdot 0 = 0 \\
&\delta \cdot s(\varepsilon) = (\delta \cdot \varepsilon) + \delta \\
&\delta \cdot \varepsilon = \bigcup\{\delta \cdot \xi : \xi < \varepsilon\} \text{ if } \varepsilon \in \mathrm{Lim}.
\end{aligned}$$

A sequence is any function whose domain is an ordinal. The domain of a sequence may also be spoken of as its length. A sequence of length δ may be referred to as a δ-tuple. Thus S^δ is the set of all δ-tuples with terms in S. Special brackets "$\langle$" and "$\rangle$" are reserved for sequences. The sequence $\langle x_\xi : \xi < \varepsilon\rangle$ is that function whose domain is ε and whose value for $\xi < \varepsilon$ is x_ξ. A positional notation is sometimes convenient for sequences. Thus $\langle x\rangle$ is that sequence whose length is 1 and whose value at 0 is x; that is, $\langle x\rangle = \{(0, x)\}$. Similarly, $\langle x\ y\rangle = \{(0, x)\ (1, y)\}$ and so on.

A δ-place relation is a set of δ-tuples, a δ-place function is a function whose domain is a set of δ-tuples. Note that though the two notions are closely connected, a function is not the same thing as a one-place function. The one-place functions are used only where it is convenient to treat finitary and infinitary functions uniformly. A δ-place function on a set S is a function whose domain is S^δ. An $\nearrow \delta$-place function on S is a function whose domain is $\bigcup\{S^\xi : \xi < \delta\}$. Therefore a δ-place function on S has as its domain all δ-tuples with terms in S, while an $\nearrow \delta$-place function has as its domain all ξ-tuples with terms in S and $\xi < \delta$.

The cardinal number or power of a set S is the smallest ordinal number cardinally equivalent to S. The smallest infinite cardinal number is $\omega = \omega_0$. If θ is an ordinal number not zero, then ω_θ is the smallest cardinal number greater than all ω_ξ for $\xi < \theta$. If α is a cardinal number, α^+ is the first cardinal greater than α. Therefore $\omega_\theta{}^+ = \omega_{\theta+1}$. A limit cardinal is one of the form ω_θ, $\theta \in \mathrm{Lim}$. The other cardinals are non-limit cardinals. Note that every infinite cardinal is a non-limit ordinal. The operation of cardinal exponentiation will be written "exp". Thus $\alpha \exp \beta = \mathrm{card}(\alpha^\beta)$. It can be shown for infinite sets S that if $\alpha \leqslant \text{card}(S)$, then $(\text{card}(S)) \exp \alpha$ is the cardinal of the set of subsets of S having power at most α. It is also the cardinal of the set of subsets of S having power α. We assume that the reader is familiar with König's Theorem which says that if $\text{card}(S_i) < \text{card}(T_i)$ for all $i \in I$, then $\text{card} \bigcup\{S_i : i \in I\} < < \text{card} \Pi\{T_i : i \in I\}$.

A cardinal number a is regular if $\text{card}(I) < \alpha$ and $\text{card}(S_i) < \alpha$ for all $i \in I$ implies $\text{card} \bigcup\{S_i : i \in I\} < \alpha$. Any infinite non-limit cardinal is regular. A regular limit cardinal is called inaccessible. A regular cardinal α with the property that $\gamma < \alpha$ implies $2 \exp \gamma < \alpha$ is called strongly inaccessible. Any strongly inaccessible cardinal is inaccessible. The smallest strongly inaccessible cardinals are 0, ω. Familiar systems of set theory remain consistent when it is assumed that these are the only ones. However there is no reason to believe that assuming the existence of non-denumerable strongly inaccessible cardinals renders them inconsistent. In order to avoid having to make such an assumption, we will be careful to state theorems on strong inaccessibles in such a way that they remain valid when it is assumed that there are no such non-denumerable cardinal numbers. A singular cardinal is one that is not regular.

A set S is hereditarily of power less than α if $\text{card}(S) < \alpha$ and $\text{card}(T) < \alpha$ whenever T is a set linked to S by a finite membership chain. Thus elements of S have power less than α, elements of elements of S have power less than α and so on. Note that if S is hereditarily of power less than α, then so is any element of S.

Suppose α regular, infinite. Let $T_0 = \{\phi\}$, and for $\xi < \alpha$ let $T_\xi = \{S : S \subseteq \bigcup\{T_\nu : \nu < \xi\} \text{ and } \text{card}(S) < \alpha\}$. Let $T_\alpha = \bigcup\{T_\xi : \xi < \alpha\}$. An easy transfinite induction shows that every set in T_α is hereditarily of power less than α. Moreover, by regularity, $S \subseteq T_\alpha$ and $\text{card}(S) < \alpha$ implies $S \in T_\alpha$. It follows that T_α is the collection of all sets hereditarily of power less than α, for the existence of such a set not in T_α would lead to an infinite descending membership chain. Note that $\text{card}(T_{\gamma+}) = 2 \exp \gamma$.

FOREWARD ON ALGEBRA

Throughout this book, an *algebra* will be a system $\langle D \ \mathsf{O} \ \mathsf{C}\rangle$ where D is a non-empty set (the *underlying set* or *domain*), and O, C are functions on indexing sets such that $\mathsf{C}(j)$ is an element of D for each $j \in \mathrm{Dom}(\mathsf{C})$ and $\mathsf{O}(i)$ is an operation on D with a given number of places for each $i \in \mathrm{Dom}(\mathsf{O})$. The operations may be δ-place or $\nearrow \delta$-place operations for any ordinal δ. A *relational system* or *structure* will be a system $\langle D \ \mathsf{O} \ \mathsf{C} \ \mathsf{R}\rangle$ with an additional function R such that $\mathsf{R}(k)$ is a relation on D with a given number of places for each $k \in \mathrm{Dom}(\mathsf{R})$. It is convenient also to allow $\mathsf{R}(k)$ to be a function on D to truth values $\mathbf{0}$, $\mathbf{1}$. The relation corresponding to $\mathsf{R}(k)$ is, of course, the set of all sequences $\langle x_0 \ldots x_\xi \ldots\rangle$ such that $\mathsf{R}(k)(\langle x_0 \ldots x_\xi \ldots\rangle) = \mathbf{1}$.

Two relational systems $\langle D \ \mathsf{O} \ \mathsf{C} \ \mathsf{R}\rangle$ and $\langle D' \ \mathsf{O}' \ \mathsf{C}' \ \mathsf{R}'\rangle$ have the same similarity type if $\mathrm{Dom}(\mathsf{O}) = \mathrm{Dom}(\mathsf{O}')$, $\mathrm{Dom}(\mathsf{C}) = \mathrm{Dom}(\mathsf{C}')$ and $\mathrm{Dom}(\mathsf{R}) = \mathrm{Dom}(\mathsf{R}')$ and for each $i \in \mathrm{Dom}(\mathsf{O})$, $k \in \mathrm{Dom}(\mathsf{R})$, $\mathsf{O}(i)$, $\mathsf{O}'(i)$ have the same number of places, and so have $\mathsf{R}(k)$, $\mathsf{R}'(k)$. A function h on D into D' *preserves operations* O if $h(\mathsf{O}(i)(\langle x_0 \ldots x_\xi \ldots\rangle)) = \mathsf{O}'(i)(\langle h(x_0) \ldots h(x_\xi) \ldots\rangle)$ for all $i \in \mathrm{Dom}(\mathsf{O})$ and sequences with the number of places of $\mathsf{O}(i)$. Similarly, h *preserves relations* R if $\langle x_0 \ldots x_\xi \ldots\rangle \in \mathsf{R}(k)$ if and only if $\langle h(x_0) \ldots h(x_\xi) \ldots\rangle \in \mathsf{R}'(k)$. A function h is a homomorphism on $\langle D \ \mathsf{O} \ \mathsf{C} \ \mathsf{R}\rangle$ to $\langle D' \ \mathsf{O}' \ \mathsf{C}' \mathsf{R}'\rangle$ if h maps D onto D', is relation and operation-preserving, and maps each $\mathsf{C}(j)$ to $\mathsf{C}'(j)$. A one-one homomorphism is an isomorphism. Other standard algebraic concepts can be adapted in a similar way. When it is convenient to do so, the various individual constants, operations and relations of a structure will simply be listed. Which ones are constants, which are operations, which are relations, and the associated numbers of places, will have to be determined from context. Thus we will speak of the system "$\langle \omega \ 0 \ s\rangle$ of natural numbers under the successor operation" and of the system "$\langle S \in\rangle$ where $\in$ is the membership relation restricted to S".

We will deal at some length with Boolean algebras. For background, we refer the reader to Sikorski's book [38]. Many equivalent formulations have been given, but we use the one given there; namely, a Boolean algebra is an algebra $\mathfrak{D} = \langle B \neg \wedge \vee\rangle$ where $\neg$ is a one-place operation on B, $\wedge$ and $\vee$ are two-place operations on B, satisfying the following equations:

Commutative Laws: $a \vee b = b \vee a$, $a \wedge b = b \wedge a$,

Associative Laws: $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

Laws of Absorption: $(a \wedge b) \vee b = b$, $a \vee (a \wedge b) = a$.

Distributive Laws: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Complementation Properties: $(a \wedge \neg a) \vee b = b$, $(a \vee \neg a) \wedge b = b$.

The *complement* of a is $\neg a$, the *join* of a, b is $a \vee b$, the *meet* is $a \wedge b$. The unique element $a \wedge \neg a$ is the zero $\mathbf{0}$; the element $a \vee \neg a$ is the unit $\mathbf{1}$. The algebra is called *degenerate* if $\mathbf{0} = \mathbf{1}$. The smallest non-degenerate Boolean algebra is $\mathfrak{B}_0$, the algebra of truth values, with underlying set $\{\mathbf{0}, \mathbf{1}\}$.

A *field of sets* is an algebra $\langle S \sim \cap \cup\rangle$ consisting of a family of subsets of a set U closed under two-place union and intersection and complementation with respect to U. Every field of sets is a Boolean algebra. If a topology is given for U, the *algebra of regular open sets* is the family of all subsets S of U such that $S = \text{in cl } S$, under operations $\neg S = \text{in cl } (\sim S)$, $S \wedge T = \text{in cl } (S \cap T)$, $S \vee T = \text{in cl } (S \cup T)$. These are also Boolean algebras.

The relation $a \leqslant b$ if and only if $a = a \wedge b$, partially orders the elements of a Boolean algebra. The greatest lower bound of a and b is $a \wedge b$, the least upper bound $a \vee b$. More generally, if C is a subset of B and the greatest lower bound of C exists in B, then that element is called the *meet* of C and written "$\bigwedge C$". Similarly, the least upper bound, if it exists in B, is called the *join* and written "$\bigvee C$". If α is a cardinal number and $\bigwedge C$ exists in B for all subsets C having power at most α, then $\mathfrak{B}$ is an *α-complete Boolean algebra.* The α-joins also exist in α-complete Boolean algebras since $\bigvee C = \neg \bigwedge \{\neg c : c \in C\}$. If $\mathfrak{B}$ is γ-complete for all $\gamma < \alpha$, then $\mathfrak{B}$ is an *$\nearrow \alpha$-complete Boolean algebra*. If $\mathfrak{B}$ is γ-complete for all γ, then $\mathfrak{B}$ is a *complete Boolean algebra.* Algebras of regular open sets are always complete. If $\mathfrak{S} = \langle S \sim \cap \cup\rangle$ is a field of sets and T a subset of S such that $\bigcap T \in S$, then $\bigcap T = \bigwedge T$ in the algebraic sense; however, the algebraic meet of T can exist in S when $\bigcap T \notin S$. In case $\bigcap T$ is in S for all subsets T having power at most α, then $\mathfrak{S}$ is an *α-field of sets*. Similarly if $\mathfrak{S}$ is a γ-field of sets for all $\gamma < \alpha$, then $\mathfrak{S}$ is an *$\nearrow \alpha$-field of sets*, if $\mathfrak{S}$ is a γ-field of sets for all cardinals γ, then $\mathfrak{S}$ is a *complete field of sets*.

A *subalgebra* of a Boolean algebra $\mathfrak{B}$ is an algebra whose underlying set is a subset $B' \subseteq B$ and whose operations are the restrictions of the operations of $\mathfrak{B}$ to B'. Such an algebra is again a Boolean algebra. A subalgebra $\mathfrak{B}'$ is an *α-regular subalgebra* of $\mathfrak{B}$ if $C \subseteq B'$, card$(C) \leqslant \alpha$, and $\bigwedge C$ exists in $\mathfrak{B}$ implies $\bigwedge C \in B'$. It is then also the meet of C in $\mathfrak{B}'$. A subalgebra $\mathfrak{B}'$ is an *$\nearrow \alpha$-regular subalgebra* of $\mathfrak{B}$ if it is a γ-subalgebra for all $\gamma < \alpha$, a *regular subalgebra* if it is γ-regular for all cardinals γ. The reader is assumed to be familiar with one of the constructions for representing a given Boolean algebra as a complete subalgebra of a complete Boolean algebra. A set $C \subseteq B$ *α-generates* $\mathfrak{B}$ if the only α-complete α-subalgebra of $\mathfrak{B}$ containing C is $\mathfrak{B}$ itself.

A homomorphism h from $\mathfrak{B}$ onto $\mathfrak{B}'$ is an *α-homomorphism* if $h(\bigwedge C)$ is the meet of $\{h(c): c \in C\}$ in $\mathfrak{B}'$ whenever $C \subseteq B$, card$(C) \leqslant \alpha$ and $\bigwedge C$ exists in $\mathfrak{B}$. A homomorphism is an *$\nearrow \alpha$-homomorphism* if it is a γ-homomorphism for all $\gamma < \alpha$, an *∞-homomorphism* if it is a γ-homomorphism for all cardinals γ. There is the expected correspondence between homomorphisms and ideals, subsets closed under finite joins and containing with b all $a \leqslant b$, and this correspondence carries over to one between α-homomorphisms on α-complete Boolean algebras and α-complete ideals, ideals closed under α-joins. Similarly between $\nearrow \alpha$-complete homomorphisms on $\nearrow \alpha$-complete Boolean algebras and $\nearrow \alpha$-complete ideals. Maximal ideals correspond to homomorphisms to $\mathfrak{B}_0$. The reader is assumed to be familiar with Stone's Representation Theorem which says that every Boolean algebra is isomorphic to a field of sets. The proof may be split into two parts, the first being a proof that for any $a \neq \mathbf{0}$ there is a maximal ideal of $\mathfrak{B}$ containing $\neg a$. More generally, for any subset C such that no finite join of its elements is $\mathbf{1}$, there is a maximal ideal containing C. The second part is the proof that the mapping taking a to the set of all maximal ideals containing $\neg a$ is an isomorphism on $\mathfrak{B}$ to a field of subsets of maximal ideals. The second part goes through unchanged for arbitrary infinite cardinals α. Therefore an $\nearrow\alpha$-complete Boolean algebra is isomorphic to an $\nearrow\alpha$-field of sets if and only if for every $a \neq \mathbf{0}$ there is a maximal $\nearrow\alpha$-complete ideal containing $\neg a$. Equivalent, of course, is the condition that for any $a \neq \mathbf{0}$ there is an $\nearrow\alpha$-homomorphism of $\mathfrak{B}$ to $\mathfrak{B}_0$ mapping a to $\mathbf{1}$. The first part of the proof fails for $\alpha > \omega$.

Another condition on existence of homomorphisms to $\mathfrak{B}_0$ is characteristic of $\nearrow\alpha$-complete Boolean algebras representable as $\nearrow\alpha$-homomorphic images of $\nearrow\alpha$-fields of sets, the so-called $\nearrow\alpha$*-representable* algebras. See Chang [1] and Scott [35]. We will only use the necessity of this condition; the sufficiency is an immediate consequence of Theorem 6.4.4 of Chapter 6.

Theorem. If $\mathfrak{B}$ is an $\nearrow\alpha$-homomorphic image of an $\nearrow\alpha$-field of sets, then given $a \neq \mathbf{0}$ and a collection of fewer than α meets of subsets of B having power less than α, there exists a homomorphism of $\mathfrak{B}$ to $\mathfrak{B}_0$ mapping a to $\mathbf{1}$ and preserving the given meets.

PROOF: Suppose $\mathfrak{S} = \langle S \sim \cap \cup \rangle$ is an $\nearrow\alpha$-field of subsets of U, h an $\nearrow\alpha$-homomorphism on $\mathfrak{S}$ to $\mathfrak{B} = \langle B \neg \wedge \vee \rangle$. Partition B into two subsets B', B'' in such a way that B' contains no complementary pairs and every element of B'' is the complement of an element of B'. This can be done by listing the elements of B in a transfinite sequence $\langle b_\xi : \xi < \mathrm{card}(B) \rangle$, putting b_0 into B' and thereafter putting b_ξ into B' if and only if $\neg b_\xi$ has not already been placed in B'. Let k be an arbitrary inverse of h on B' and let $k(b) = \sim k(\neg b)$ for $b \in B''$. Then k is an inverse of h on all of B and k preserves complements.

Suppose the given meets are $\bigwedge C_i$, $i \in I$, where $\mathrm{card}(I) < \alpha$ and $\mathrm{card}(C_i) < \alpha$ for all $i \in I$. Let $C = \{a, \neg a, \mathbf{0}, \mathbf{1}\} \cup \bigcup_i C_i \cup \{\bigwedge C_i : i \in I\} \cup \{\neg c : c \in \bigcup_i C_i\} \cup \{\neg \bigwedge C_i : i \in I\}$. Note that we may assume α regular, for if α singular every $\nearrow\alpha$-field of sets is an α-field of sets and every $\nearrow\alpha$-homomorphism is an α-homomorphism. The regular cardinal α^+ might just as well have been under consideration in this case. For regular α, $\mathrm{card}(C) < \alpha$. Let $\mathscr{K}$ be the following collection of fewer than α subsets of B: $\{\neg a\}$, $\{\mathbf{0}\}$, $\{\neg \bigwedge C_i\} \cup C_i$ for $i \in I$, $\{\neg b_0, \ldots, \neg b_n, (b_0 \vee \ldots \vee b_n)\}$ for each finite subset $\{b_0, \ldots, b_n\}$ of C. Since every one of these sets has meet $\mathbf{0}$ except $\{\neg a\}$, $\bigvee \{\bigwedge D : D \in \mathscr{K}\} = \neg a$. Let $\mathscr{L}$ be the collection of all sets $\{k(b) : b \in D\}$ for $D \in \mathscr{K}$. Then $\bigcup \{\bigcap E : E \in \mathscr{L}\} \neq U$ since the $\nearrow\alpha$-homomorphism h maps this set to $\neg a \neq \mathbf{1}$. By the distributive law in $\mathfrak{S}$, there is $T \in S$, $T \neq \phi$, such that for every $D \in \mathscr{K}$ there is

$b \in D$ such that $T \subseteq \sim k(b)$. Let $C' = C \cap \{b \colon T \subseteq \sim k(b)\}$. Then C' contains an element from each $D \in \mathscr{K}$ and contains no complementary pairs since $T \subseteq \sim k(b)$ and $T \subseteq \sim k(\neg b) = k(b)$ would imply $T = \phi$. These are the only properties of C' that we need.

Note that $\neg a \in C'$, $\mathbf{0} \in C'$, and that C' contains either $\neg \wedge C_i$ or an element of C_i for each $i \in I$. No finite join of elements of C' is $\mathbf{1}$ since C' contains an element of each set of the form $\{\neg b_0, \ldots, \neg b_n, (b_0 \vee \ldots \vee b_n)\}$ and cannot contain $\mathbf{1}$. The homomorphism to $\mathfrak{B}_0$ given by a maximal ideal including C' has the desired properties.

When $\alpha = \omega_1$ the representability condition on $\mathfrak{B}$ and the restriction on the cardinals of the C_i may be dropped. The resulting theorem appears in work of Rasiowa and Sikorski as a lemma for an algebraic proof of the strong completeness of the first-order predicate calculus. See [30]. We will see in Chapter 11 that it also implies the completeness of the infinitary calculus $\mathfrak{P}_{\omega_1\omega}$.

Theorem. Let $\mathfrak{B}$ be any Boolean algebra. Then given $a \neq \mathbf{0}$ and a denumerable collection of meets $\wedge C_i$, $i = 1, 2, \ldots$, in $\mathfrak{B}$, there is a homomorphism of $\mathfrak{B}$ to $\mathfrak{B}_0$ mapping a to $\mathbf{1}$ and preserving the given meets.

PROOF: Let $C'_0 = \{\neg a\}$. Suppose that C'_n is finite, $\vee C'_n \neq \mathbf{1}$, and contains $\neg a$ and either $\neg \wedge C_i$ or an element of C_i for each $i \leqslant n$. If $\vee C'_n \vee \neg \wedge C_{n+1} \neq \mathbf{1}$ let $C'_{n+1} = C'_n \cup \{\neg \wedge C_{n+1}\}$. On the other hand, if $\vee C'_n \vee \neg \wedge C_{n+1} = \mathbf{1}$ there must be an element c of C_{n+1} such that $\vee C'_n \vee c \neq \mathbf{1}$. For $\vee C'_n \vee c = \mathbf{1}$ for all $c \in C_{n+1}$ implies $\vee C'_n \vee \wedge C_{n+1} = \mathbf{1}$. But since $\vee C'_n \vee \neg \wedge C_{n+1} = \mathbf{1}$, it then follows that $\vee C'_n = \mathbf{1}$, a contradiction. In this case, let $C'_{n+1} = C'_n \cup \{c\}$. Then $\vee C'_{n+1} \neq \mathbf{1}$ and contains $\neg a$ and either $\neg \wedge C_i$ or an element of C_i for each $i \leqslant n + 1$. Let $C' = \bigcup \{C'_n \colon n < \omega\}$. Since no finite join of elements of C' is $\mathbf{1}$, there is a maximal ideal of $\mathfrak{B}$ including C'. The corresponding homomorphism to $\mathfrak{B}_0$ has the desired properties.

A Boolean algebra $\mathfrak{B}$ is *α-distributive* if

$$\bigwedge_{i \in I} \bigvee_{j \in J} b_{ij} = \bigvee_{f \in F} \bigwedge_{i \in I} b_{if(i)}, \text{ where } F = J^I,$$

whenever I, J have power at most α and joins $\bigvee_{j \in J} b_{ij}$ exist in $\mathfrak{B}$ for all $i \in I$ and meets $\bigwedge_{i \in I} \bigvee_{j \in J} b_{ij}$ and $\bigwedge_{i \in I} b_{if(i)}$ exist in $\mathfrak{B}$ for $f \in F$. For special background in distributivity see Smith-Tarski [40] and Pierce's papers [25] and [26] as well as Sikorski's book [38].

CHAPTER 1

INTRODUCTION

Let α be a regular infinite cardinal, β a cardinal either 0 or $\omega \leqslant \beta \leqslant \alpha$. The first-order predicate languages $\mathbf{L}_{\alpha\beta}$ that we will study have the same kinds of symbols that ordinary first-order predicate languages have; the difference lies in the rules of formation of formulas.

1.1 An Informal Look at Infinitary Predicate Languages

A language $\mathbf{L}_{\alpha\beta}$ has a supply of α individual variables to be interpreted as ranging over a non-empty domain D. It may also have individual constants to be interpreted as fixed elements of D. It has a supply of n-place predicate symbols, $0 < n < \omega$, to be interpreted as fixed n-place relations over D; one of these is $\bar{=}$, to be interpreted as the two-place equality relation over D. The language may also have n-place operation symbols, $0 < n < \omega$, to be interpreted as fixed n-place operations over D. Other symbols are the brackets [and], symbols $\neg$ and $\rightarrow$ for negation and material implication, the symbol $\bigwedge$ of conjunction, and $\forall$ of universal quantification. Such other symbols as $\leftrightarrow$, $\wedge$, $\vee$, of logical equivalence, ordinary conjunction and disjunction, as well as the sign $\bigvee$ of infinitary disjunction and $\exists$ of existential quantification, are introduced by abbreviation. The atomic formulas of $\mathbf{L}_{\alpha\beta}$ are the same as the atomic formulas of an ordinary first-order predicate language with the same symbols. Formulas of $\mathbf{L}_{\alpha\beta}$ are built up from atomic formulas using the ordinary rules for $\neg$, $\rightarrow$, $\leftrightarrow$, $\wedge$, $\vee$, but extraordinary rules for $\bigwedge$, $\bigvee$, $\forall$, $\exists$, as follows:

If $0 < \delta < \alpha$ and $\langle A_\xi : \xi < \delta \rangle$ is a sequence of formulas, then $[\bigwedge A_0 \ldots A_\xi \ldots]$ and $[\bigvee A_0 \ldots A_\xi \ldots]$ are formulas.

If $0 < \varepsilon < \beta$, A is a formula, and $\langle x_\xi : \xi < \varepsilon \rangle$ is a sequence of individual variables, then $[\forall x_0 \ldots x_\xi \ldots A]$ and $[\exists x_0 \ldots x_\xi \ldots A]$ are formulas.

The regularity of α guarantees that each formula has length less than α. When it is necessary to convey to the reader information about the lengths of the sequences of formulas and variables, a subscript "$\xi < \delta$" or "$\xi < \varepsilon$" will be placed under the appropriate symbol. It is to be understood, of course, that such subscripts, as well as the dots, are not part of the language $\mathbf{L}_{\alpha\beta}$ itself; they are in the informal metalanguage we use to describe $\mathbf{L}_{\alpha\beta}$.

A model $\mathfrak{M}$ for a language $\mathbf{L}_{\alpha\beta}$ consists of a non-empty domain D of individuals, together with fixed elements of D and relations and operations over D, with the appropriate number of places, to serve as denotations for the non-logical constants. The reader will have no difficulty in extending Tarski's definition of the truth value of an ordinary formula for an assignment of variables to D, to apply to these infinitely long formulas. The existence and uniqueness of such a valuation function is a consequence of the recursion principle in Chapter 8. If formula A is a sentence, that is, if it has no free variables, then the truth value of A in $\mathfrak{M}$ does not depend on the assignment to variables. A formula *holds in* $\mathfrak{M}$ if its value is truth for all assignments of its variables to D. We also say in this case that $\mathfrak{M}$ *is a model of* A. A formula that holds in all models is *valid*.

1.1.1 *Example.* The system $\langle N\ 0\ s\rangle$ of natural numbers under the successor operation is characterized up to isomorphism by the sentence $A = [A_1 \wedge A_2 \wedge A_3]$ of language $\mathbf{L}_{\omega_1\omega}$ with individual constant $\bar{0}$ and one-place operation symbol $\bar{s}$, where

$$A_1 = [\forall x[\neg \bar{0} \mathrel{\bar{=}} \bar{s}x]]$$

$$A_2 = [\forall xy[\bar{s}x \mathrel{\bar{=}} \bar{s}y \rightarrow x \mathrel{\bar{=}} y]]$$

$$A_3 = [\forall x[\bigvee_{0<n<\omega} [x \mathrel{\bar{=}} \bar{0}][x \mathrel{\bar{=}} \bar{s}\bar{0}]\, C_1 \ldots C_n \ldots]],$$

where for $0 < n < \omega$,

$$C_n = [\exists y_1 \ldots y_n [\bigwedge_{i<n} y_1 \mathrel{\bar{=}} \bar{s}\bar{0} \ldots y_{i+1} \mathrel{\bar{=}} \bar{s}y_i \ldots x \mathrel{\bar{=}} \bar{s}y_n]].$$

We mean, of course, that A holds in exactly those models isomorphic to $\langle N\ 0\ s\rangle$.

1.1.2 *Remark.* We know that the system of natural numbers cannot be characterized up to isomorphism by a set of ordinary first-order sentences. Other examples of classes of algebraic systems that can be characterized by single (ω_1, ω)-sentences but not by sets of ordinary first-order sentences, are the class of non-Archimedean fields and the class of torsion-free groups.

1.1.3 *Example.* The sentence $A = [A_1 \wedge A_2]$ of language $\mathbf{L}_{\omega_1\omega_1}$ with two-place predicate symbol $\bar{<}$ characterizes the class of well-ordered systems, where

$$A_1 = [\forall xy[x \bar{<} y \vee y \bar{<} x \vee x \bar{=} y]]$$

$$A_2 = [\neg \underset{n<\omega}{\exists} x_0 \ldots x_n \ldots [\underset{n<\omega}{\wedge} x_1 \bar{<} x_0 \ldots x_{n+1} \bar{<} x_n \ldots]].$$

This class also cannot be characterized by a set of first-order sentences, but it differs from the other examples in that an infinite quantification is required for its definition. For no regular infinite cardinal α is there a sentence of an (α, ω)-language in predicates $\bar{=}$, $\bar{<}$ whose models are precisely the well-ordered systems. [1]

The next example, taken from Hanf [7], is essentially the same as Theorem 3 in Mostowski [24].

1.1.4 *Example.* The sentence $[A_1 \wedge A_2]$ of an (ω_1, ω_1)-language with two-place predicate symbol $\bar{\in}$ characterizes the class of all systems isomorphic to a transitive family of sets under the membership relation (A collection S of sets is transitive if $x \in S$ implies $x \subseteq S$):

$$A_1 = [\forall xy[[\forall z[z \bar{\in} x \leftrightarrow z \bar{\in} y]] \rightarrow x \bar{=} y]$$

$$A_2 = \neg \underset{n<\omega}{\exists} x_0 \ldots x_n \ldots [\underset{n<\omega}{\wedge} x_1 \bar{\in} x_0 \ldots x_{n+1} \bar{\in} x_n \ldots].$$

Hereafter A_1 will be referred to as the *axiom of extensionality*, A_2 as the *axiom of well-foundedness.*

PROOF: It is clear that A_1 and A_2 hold in any transitive family of sets. Suppose A_1 and A_2 hold in model $\langle D \in' \rangle$. It is required to show that there is a transitive family S of sets such that $\langle S \in \rangle$ is isomorphic to $\langle D \in' \rangle$. The axiom of well-foundedness guarantees the existence of an element d_0 of D such that $x \in' d_0$ is false for all x in D. The axiom of extensionality guarantees the uniqueness of this element. Call it $0'$.

An induction principle holds in $\langle D \in' \rangle$: Suppose $E \subseteq D$ satisfies conditions (i) $0' \in E$, (ii) $x \in D$ and $y \in E$ for all $y \in' x$ implies $x \in E$. Then $E = D$.

[1] Note added in proof, July 9, 1964: This result is in an article "Finite Quantifier Equivalence" to appear in the Proceedings of the Symposium on Model Theory, Berkeley, California, 1963.

Notice that if $d \in D \sim E$, then $d \neq 0'$ by (i). By (ii), there is $d_1 \in' d$ such that $d_1 \in D \sim E$. Continuing by ordinary mathematical induction, we obtain an infinite descending $\in'$-sequence, thus contradicting the axiom of well-foundedness.

Let T be the family of all sets hereditarily of power at most card(D). Call a relation $R \subseteq D \times T$ *acceptable* if

(a) $(0', \phi) \in R$.

(b) If $x \in D$ and for each $y \in' x$, pair $(y, S_y) \in R$, then $(x, \{S_y : y \in' x\}) \in R$.

Since $D \times T$ is an acceptable relation, we can form $R_0 = \bigcap \{R : R \text{ is acceptable}\}$ and R_0 will also be an acceptable relation. Let $E = \{x : \text{there is a unique } S \in T \text{ such that } (x, S) \in R_0\}$. Then E satisfies condition (i) of the induction principle since $(0', \phi) \in R_0$ and if $S \neq \phi$, $R_0 \sim \{(0', S)\}$ is acceptable. Similarly E satisfies condition (ii). Hence $E = D$ and R_0 is a function on D into T. Note that R_0 satisfies the condition $R_0(x) = \{R_0(y) : y \in' x\}$. Hence the range of R_0 is a transitive field of sets. Let $E' = \{x : x \in D$ and there is no $y \in D$, $y \neq x$, such that $R_0(y) = R_0(x)\}$. If $y \neq 0'$, then there is $z \in D$ such that $z \in' y$, from which we infer $R_0(z) \in R_0(y)$. Hence $R_0(y) \neq \phi$ and $0' \in E'$. If $x \in D$ and $y \in E'$ whenever $y \in' x$, then we can show $x \in E'$. For if $x \notin E'$, there is $u \neq x$ such that $R_0(u) = R_0(x)$. By the axiom of extensionality, either there is $y \in' x$ such that not $y \in' u$, or there is $y' \in' u$ such that not $y' \in' x$. Either alternative contradicts the assumption that the elements of x with respect to $\in'$ are in E'. Hence E' satisfies both conditions (i) and (ii) and therefore is equal to D. Hence R_0 is one-one. Clearly also $x \in' y$ if $R_0(x) \in R_0(y)$. This completes the proof.

It can also be shown that an infinite quantification is required to characterize transitive families of sets.

1.1.5 *Example.* (Hanf, Scott). Consider a language $\mathbf{L}_{\gamma^+\gamma^+}$ with two-place predicates $\bar{=}$, $\bar{\in}$, γ any infinite cardinal. Let A_1, A_2 be the sentences of Example 1.1.4,

$$A_3 = \underset{\xi<\gamma}{\forall}\ x_0 \ldots x_\xi \ldots \exists y \forall z [z \bar{\in} y \leftrightarrow [\underset{\xi<\gamma}{\bigvee}\ z \bar{=} x_0 \ldots z \bar{=} x_\xi \ldots]]$$

$$A_4 = \forall y [C \rightarrow [\exists x_0 \ldots x_\xi \ldots \forall z [z \bar{\in} y \leftrightarrow [\underset{\xi<\gamma}{\bigvee}\ z \bar{=} x_0 \ldots z \bar{=} x_\xi \ldots]]]$$

where $C = [\exists z\, z \bar{\in} y]$.

Then $[\bigwedge A_1\, A_2\, A_3\, A_4]$ characterizes up to isomorphism the family T_{γ^+} of sets hereditarily of power at most γ.

PROOF: Clearly sentences A_1, A_2, A_3, A_4 hold in $\langle T_{\gamma^+} \in \rangle$. Conversely, if $\langle S \in \rangle$ is a transitive family of sets in which A_3 and A_4 hold, then elements of S are subsets of S and have power at most γ by A_4. They are therefore hereditarily of power at most γ. An easy induction on $\xi < \gamma^+$ shows that the sets T_ξ described in the Foreward on Set Theory are all subsets of S. Hence $S = T_{\gamma^+}$. This completes the proof.

At this point it should be mentioned that the theory of models has had to be declared outside the scope of this book. Instead, we shall focus our attention on formal systems for the infinitary languages. Literature on models of infinitary languages is scarce. An early contribution is M. Krasner's article [19] of 1938. The first appearance of a model theory for infinitary languages along the lines of Tarski seems to be in P. Jordan's article [14], 1949. The underlying languages are, however, not explicitly formulated. See also Frayne's review [4]. Infinite conjunctions and disjunctions are treated in A. Robinson's book [34], 1951. Infinitary predicate languages are formulated in A. Tarski's article [46], 1958, and his model-theoretic concepts for ordinary predicate languages are extended to apply to the infinite case. W. Hanf's important work in this area is reported in a summary [7], 1960. Some of his results are included here because they bear on the problem of the existence of complete formal systems.

1.2 An Informal Look at Formal Systems Based on Infinitary Predicate Languages

This brings us to the main topic of this book. For which of the languages can we provide a notion of provability with the property that exactly the valid formulas are provable? Further, for which of the languages can we provide a notion of provability with the property that exactly those formulas which hold in all models of a set Γ of sentences are the ones provable when Γ is adjoined to the set of axioms? But to make these questions reasonable, we have to say what we are willing to admit as a notion of provability.

Only Hilbert-style systems will be considered here. A set of formulas of $\mathbf{L}_{\alpha\beta}$ will be singled out to serve as axioms, closure conditions will be specified to serve as rules of inference. A formal proof will then consist of a sequence of fewer than α formulas, each of which is either an axiom, or is the result of applying a rule of inference to formulas appearing earlier in the sequence. A formula is *provable* (written "$\vdash$ A") if it is the last formula of a formal proof. Infinitary systems with tree-proofs are of considerable interest. Such systems, complete for α strongly inaccessible, are formulated and studied in a paper of S. Maehara and G. Takeuti, [21], 1961. Related systems have been treated by E. Engeler, [3], 1961.

As a rule, ordinary formal systems are required to meet a condition of effectiveness. There must be a procedure for deciding whether or not a given sequence of formulas is a formal proof in a finite number of steps. The only reason why a condition of α-effectiveness, that is, of decidability in fewer than α steps, is not imposed on (α, β)-formal systems, is that a completely satisfactory definition of α-effectiveness is not yet available. Work has been done along this line by M. Machover, announced in [20], 1961. He gives a definition of α-recursiveness of $\nearrow \alpha$-place functions and predicates on the set α. However, in order to obtain the expected properties in such a context, it is necessary to have a reversible α-recursive one-one mapping on $\bigcup \{\alpha^\delta \colon \delta < \alpha\}$ to α. To obtain such a mapping. Machover assumes Godel's axiom of constructibility. In order to escape having to make such an assumption, it might very well be necessary to consider functions and predicates on a set already closed under formation of sequences of length less than α. [1])

The condition on (α, β)-formal systems imposed here is one of definability, a condition apparently much less stringent than one of α-effectiveness would be. As a matter of fact, the (γ^+, β)-formal systems with Chang's distributive laws of level γ, 5.1.3, probably would not meet a γ^+-effectiveness requirement. The (α, β)-systems with α strongly inaccessible probably would be admitted.

1.2.1 *Criterion of Definability.* In the text, we will deal with (α, β)-predicate languages with infinite atomic formulas, but for the

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[1] Note added in proof, April 2, 1964: The axiom of constructibility has been shown to be eliminable in Machover's work. In his study of transfinite effectivity [The Theory of Transfinite Effectivity, Technical Report No. 12, July, 1963, U.S. Office of Naval Research, Information Systems Branch], Azriel Lévy develops an equivalent theory using the notion of relative constructibility instead of constructibility. It should also be mentioned that an independent treatment of transfinite recursiveness was given by Takeuti and Kino in articles in the Journal of the Mathematical Society of Japan [Takeuti, On the Recursive Functions of Ordinal Numbers, vol. 12 (1960), pp. 119–128, and Takeuti-Kino, On Hierarchies of Predicates of Ordinal Numbers, vol. 14 (1962), pp. 199–232]. Machover's approach is through the Herbrand-Gödel definition of computability, Takeuti-Kino's through primitive recursive functionals. Lévy develops a a theory of transfinite computability by Turing machine and proves that the three notions are equivalent.

moment, consider only languages having predicate and operation symbols with fixed finite numbers of places. Then the *α-metalanguage* $\mathbf{ML}^\alpha$ is an ordinary predicate language, having only formulas of finite length, and the following non-logical constants:

Individual constants: $\bar{\delta}$ for each $\delta < \alpha$, $\overline{eq}$, $\overline{lb}$, $\overline{rb}$, $\bar{n}$, $\bar{\imath}$, $\bar{c}$, $\bar{q}$.
One-place predicate symbols: IC, IV.
Two-place predicate symbols: $\overline{=}$, $\bar{\in}$, OP, PR.

Let $\mathbf{L}_{\alpha\beta}$ be a given (α, β)-predicate language. One-one functions on the symbols of $\mathbf{L}_{\alpha\beta}$ into α are called *Gödel-numberings*. If E is any sequence of fewer than α symbols of $\mathbf{L}_{\alpha\beta}$, the *Gödel-sequence* of E is the sequence of Gödel-numbers of its symbols. Relative to a Gödel-numbering g, consider the following model $\mathfrak{M}(\mathbf{L}_{\alpha\beta}, g)$ of $\mathbf{ML}^\alpha$:

Domain: T_α, the family of sets hereditarily of power less than α.
Denotation for non-logical constants:

$\bar{0}$	0	$\overline{rb}$	$g(])$
$\vdots$	$\vdots$	$\bar{n}$	$g(\neg)$
$\bar{\delta}$	δ	$\bar{\imath}$	$g(\rightarrow)$
$\vdots$	$\vdots$	$\bar{c}$	$g(\wedge)$
$\overline{eq}$	$g(\overline{=})$	$\bar{q}$	$g(\forall)$
$\overline{lb}$	$g([)$		
IC	$\{\langle g(a)\rangle : a$ is an individual constant of $\mathbf{L}_{\alpha\beta}\}$		
IV	$\{\langle g(x)\rangle : x$ is an individual variable of $\mathbf{L}_{\alpha\beta}\}$		
OP	$\{\langle n\ g(\varphi^n)\rangle : \varphi^n$ is an n-place operation symbol of $\mathbf{L}_{\alpha\beta}\}$		
PR	$\{\langle n\ g(Q^n)\rangle : Q^n$ is an n-place predicate symbol of $\mathbf{L}_{\alpha\beta}\}$		
$\bar{\in}$	The two-place membership relation.		

In Chapter 13 we will show that there is a formula Form(x) of $\mathbf{ML}^\alpha$ with one free variable such that for all Gödel-numberings g, a set $S \in T_\alpha$ satisfies Form(x) in $\mathfrak{M}(\mathbf{L}_{\alpha\beta}, g)$ if and only if S is the Gödel-sequence (with respect to g) of a formula. The *criterion of definability* is this:

> In order for $\mathfrak{P}$ to qualify as a formal system for $\mathbf{L}_{\alpha\beta}$ there must exist a formula $PRV(x)$ of $\mathbf{ML}^\alpha$ with one free variable, such that for all Gödel-numberings g, a set $S \in T_\alpha$ satisfies $PRV(x)$ in $\mathfrak{M}(\mathbf{L}_{\alpha\beta}, g)$ if and only if S is the Gödel-sequence (with respect to g) of a formula provable in $\mathfrak{P}$.

We will show that all the (α, β)-formal systems introduced in this book satisfy this condition.

1.2.2 *Summary.* A formal system for $\mathbf{L}_{\alpha\beta}$ is *complete* if exactly the valid formulas are provable, and is *strongly complete* if exactly the formulas that hold in all models of a set Γ of formulas are provable when Γ is adjoined to the set of axioms.

The study of infinitary formal systems will begin with the extension of ordinary first-order predicate logic to apply to the (α, β)-predicate languages. These are the basic formal systems. Only in cases $\alpha = \omega$ or ω_1 and $\beta \leqslant \omega$ are such systems necessarily complete. Axioms and rules are added to provide complete formal systems for all languages $\mathbf{L}_{\alpha\beta}$ with α strongly inaccessible, also for $\alpha = \gamma^+$ such that $\gamma \exp \varepsilon = \gamma$ for all $\varepsilon < \beta$ and for all other regular α such that for $\gamma < \alpha$ there is $\kappa < \alpha$ such that $\gamma \leqslant \kappa$ and $\kappa \exp \varepsilon = \kappa$ for all $\varepsilon < \beta$. For cases $\alpha = \beta = \gamma^+$, γ infinite, we are able to show that a complete definable formal system does not exist, even if the underlying language has only one two-place predicate symbol in addition to $\overline{\equiv}$. The argument is due to Dana Scott who outlined it informally at the International Congress for Logic, Methodology and Philosophy of Science in 1960. Even if the metalanguages $\mathbf{ML}^{\gamma^+}$ were broadened to admit γ-conjunctions and quantifications, a definable complete (γ^+, γ^+)-formal system still would not exist. The reader familiar with the undefinability-of-truth argument of Tarski may already see the possibility of such a proof from Example 1.1.5. In the remaining cases, the problem of the existence of complete definable formal systems is open.

It is well-known that the basic formal systems are strongly complete in case $\alpha = \omega$. However, we are able to show that there do not exist strongly complete formal systems for languages $\mathbf{L}_{\alpha\beta}$ whenever α is an infinite non-limit cardinal, or one of a large class of non-denumerable inaccessible cardinals. An easy argument shows that if $\mathbf{L}_{\alpha\beta}$ has a set Γ of sentences such that every subset of power less than α has a model, but Γ itself has no model, then there is no strongly complete formal system for $\mathbf{L}_{\alpha\beta}$. This follows from the assumption that formal proofs have length less than α. Of course, there are no such sets Γ when $\alpha = \omega$. It is easy to construct such sets for infinite non-limit α. The question of the existence of such sets for α non-denumerable and inaccessible, is a highly non-trivial matter. Examples for certain such cardinals were announced by W. Hanf in 1960 and the one for the first non-denumerable strongly inaccessible number (assuming that there is such a number) is given in Sect. 10.2. For discussion and applications to set theory, see Tarski [47].

CHAPTER 2

INFINITARY CONCATENATION

In our treatment of infinitary languages, terms and formulas are certain sequences of symbols; that is to say, they are functions on ordinal numbers to the set of symbols of the language. Since their definition depends on the notion of concatenation, it might be well to take the time to point out some of the properties of this operation.

2.1 Properties of Infinitary Ordinal Addition

Among the familiar properties of ordinary ordinal addition, we find the following:

Associative Law: $\gamma + (\delta + \varepsilon) = (\gamma + \delta) + \varepsilon$.
Law of Monotony: If $\gamma \leqslant \delta$ then $\gamma + \varepsilon \leqslant \delta + \varepsilon$.
Left Cancellation Laws: $\gamma + \delta = \gamma + \varepsilon$ if and only if $\delta = \varepsilon$,
$\gamma + \delta < \gamma + \varepsilon$ if and only if $\delta < \varepsilon$.

Finite Right Cancellation Law: If γ finite, then $\delta + \gamma = \varepsilon + \gamma$ if and only if $\delta = \varepsilon$.
Existence of Ordinal Difference: If $\gamma \leqslant \delta$, then there exists a unique ordinal ε such that $\delta = \gamma + \varepsilon$. This ordinal ε is referred to as $\delta - \gamma$.

The operation of infinitary ordinal addition assigns an ordinal to a sequence of ordinals. A recursive definition is

$$\Sigma\langle\gamma_\xi : \xi < 0\rangle = 0$$

$$\Sigma\langle\gamma_\xi : \xi < \delta\rangle = (\Sigma\langle\gamma_\xi : \xi < \delta - 1\rangle) + \gamma_{\delta-1} \text{ if } \delta \notin \mathrm{Lim},$$

$$\Sigma\langle\gamma_\xi : \xi < \delta\rangle = \bigcup_{\varepsilon<\delta} \Sigma\langle\gamma_\xi : \xi < \varepsilon\rangle \text{ if } \delta \in \mathrm{Lim}.$$

2.1.1 *Theorem.* If α is a regular cardinal and if $\delta < \alpha$ and $\gamma_\xi < \alpha$ for all $\xi < \delta$, then $\Sigma\langle\gamma_\xi \colon \xi < \delta\rangle < \alpha$.

PROOF: The statement is trivial for the finite regular cardinals 0, 1, 2. Assuming α infinite, $\gamma < \alpha$ and $\gamma' < \alpha$ implies $\gamma + \gamma' < \alpha$. The theorem then follows by transfinite induction on δ.

2.1.2 $\Sigma\langle\gamma_\xi \colon \xi < \delta + \varepsilon\rangle = \Sigma\langle\gamma_\xi \colon \xi < \delta\rangle + \Sigma\langle\gamma_{\delta+\xi} \colon \xi < \varepsilon\rangle$.

2.1.3 $\Sigma\langle\gamma_\xi \colon \xi < \bigcup_{\nu<\varepsilon} \delta_\nu\rangle = \bigcup_{\nu<\varepsilon} \Sigma\langle\gamma_\xi \colon \xi < \delta_\nu\rangle$.

Hence also

2.1.4 $\Sigma\langle\gamma_\xi \colon \xi < \Sigma\langle\delta_\nu \colon \nu < \varepsilon\rangle\rangle = \bigcup_{\lambda<\varepsilon} \Sigma\langle\gamma_\xi \colon \xi < \Sigma\langle\delta_\nu \colon \nu < \lambda\rangle\rangle$

provided $\varepsilon \in \mathrm{Lim}$.

It is well-known that if $\theta < \gamma_0 + \gamma_1$, then either $\theta < \gamma_0$ or there is a unique $\theta' < \gamma_1$ such that $\theta = \gamma_0 + \theta'$. More generally,

2.1.5 If $\theta < \Sigma\langle\gamma_\xi \colon \xi < \delta\rangle$ then there are unique ordinals $\delta' < \delta$ and $\theta' < \gamma_{\delta'}$ such that $\theta = \Sigma\langle\gamma_\xi \colon \xi < \delta'\rangle + \theta'$.

PROOF: If $\Sigma\langle\gamma_\xi \colon \xi < \delta'\rangle + \theta' = \Sigma\langle\gamma_\xi \colon \xi < \delta''\rangle + \theta''$ for ordinals $\delta' < \delta'' < \delta$, $\theta' < \gamma_{\delta'}$, $\theta'' < \gamma_{\delta''}$, then after expressing $\Sigma\langle\gamma_\xi \colon \xi < \delta''\rangle$ in the form $\Sigma\langle\gamma_\xi \colon \xi < \delta'\rangle + \gamma_{\delta'} + \varepsilon$, a form we know exists by 2.1.2, we can cancel on the left of the assumed equality to obtain $\gamma_{\delta'} \leqslant \theta' < \gamma_{\delta'}$, a contradiction. Since we would arrive at a similar contradiction by assuming $\delta'' < \delta'$, we must have $\delta' = \delta''$. Another left cancellation yields $\theta' = \theta''$. Hence ordinals $\delta' < \delta$, $\theta' < \gamma_{\delta'}$, if they exist, are unique.

The existence of δ' and θ' can be proved by transfinite induction on δ. If $\delta = 0$, there are no numbers θ to be concerned with, and if $\delta = 1$ the statement is trivial. If $\theta < \Sigma\langle\gamma_\xi \colon \xi < \delta\rangle$ and $\delta \notin \mathrm{Lim}$, then either $\theta < \Sigma\langle\gamma_\xi \colon \xi < \delta - 1\rangle$ or θ has form $\Sigma\langle\gamma_\xi \colon \xi < \delta - 1\rangle + \theta'$ where $\theta' < \gamma_{\delta-1}$. In the first case the δ' and θ' assumed to exist for $\delta - 1$ can be used, while in the second, the representation is given. If $\theta < \Sigma\langle\gamma_\xi \colon \xi < \delta\rangle$ and $\delta \in \mathrm{Lim}$, then there is $\varepsilon < \delta$ such that $\theta < \Sigma\langle\gamma_\xi \colon \xi < \varepsilon\rangle$. The δ' and θ' for ε can be used.

2.2 Properties of Infinitary Concatenation of Sequences

If E_ξ is a sequence for each $\xi < \delta$, we define $\frown\langle E_\xi \colon \xi < \delta\rangle$ to be that sequence having domain $\Sigma\langle\mathrm{Dom}(E_\xi) \colon \xi < \delta\rangle$ whose value for θ is $E_{\delta'}(\theta')$, where δ' and θ' are the unique ordinals $\delta' < \delta$, $\theta' < \mathrm{Dom}(E_{\delta'})$ such that $\theta = \Sigma\langle\mathrm{Dom}(E_\xi) \colon \xi < \delta'\rangle + \theta'$. Furthermore, we let $E_0{\frown}E_1 = \frown\langle E_\xi \colon \xi < 2\rangle$.

2.2.1 $E{\frown}\phi = \phi{\frown}E = E$ and $\frown\langle E_\xi \colon \xi < 0\rangle = \phi$.

2.2.2 $\frown\langle E_\xi : \xi < 1\rangle = E_0$ and $\frown\langle E_\xi : \xi < \delta + 1\rangle = (\frown\langle E_\xi : \xi < \delta\rangle)$ $\frown E_\delta$.

Any sequence is the concatenation of the sequence of one-termed sequences consisting of its symbols standing alone. That is,

2.2.3 $E = \frown\langle\langle E(\xi)\rangle : \xi < \mathrm{Dom}(E)\rangle$.

The infinitary concatenation operation satisfies associative laws like those for ordinal addition:

2.2.4 $\frown\langle E_\xi : \xi < \delta_0 + \delta_1\rangle = (\frown\langle E_\xi : \xi < \delta_0\rangle)\frown(\frown\langle E_{\delta_0+\xi} : \xi < \delta_1\rangle)$.
The ordinary associative law follows.

2.2.5 $(E_0\frown E_1)\frown E_2 = E_0\frown(E_1\frown E_2)$.
A sequence is a set of ordered pairs and is uniquely determined by its domain and its values for elements of its domain. Hence

2.2.6 If $\delta_0 \leqslant \delta_1$, then $\frown\langle E_\xi : \xi < \delta_0\rangle \subseteq \frown\langle E_\xi : \xi < \delta_1\rangle$.

2.2.7 $\frown\langle E_\xi : \xi < \bigcup_{\nu<\varepsilon} \delta_\nu\rangle = \bigcup_{\nu<\varepsilon} \frown\langle E_\xi : \xi < \delta_\nu\rangle$.
As a corollary we have

2.2.8 If $\varepsilon \in \mathrm{Lim}$, then

$$\frown\langle E_\xi : \xi < \Sigma\langle\delta_\nu : \nu < \varepsilon\rangle\rangle = \bigcup_{\lambda<\varepsilon} \frown\langle E_\xi : \xi < \Sigma\langle\delta_\nu : \nu < \lambda\rangle\rangle.$$

In view of 2.2.1, 2.2.2 and 2.2.7, we see that sequential concatenation satisfies recursion equations similar to those for ordinal addition:

$$\begin{aligned}
&\frown\langle E_\xi : \xi < 0\rangle = \phi\\
&\frown\langle E_\xi : \xi < \delta\rangle = \frown\langle E_\xi : \xi < \delta - 1\rangle\frown E_{\delta-1} \text{ if } \delta \notin \mathrm{Lim},\\
&\frown\langle E_\xi : \xi < \delta\rangle = \bigcup_{\varepsilon<\delta} \frown\langle E_\xi : \xi < \varepsilon\rangle \text{ if } \delta \in \mathrm{Lim}.
\end{aligned}$$

2.2.9 *Generalized Associative Law.* $\frown\langle E_\xi : \xi < \Sigma\langle\delta_\nu : \nu < \varepsilon\rangle\rangle = \frown\langle\frown\langle E_{\gamma_\nu+\xi} : \xi < \delta_\nu\rangle : \nu < \varepsilon\rangle$, where $\gamma_\nu = \Sigma\langle\delta_\mu : \mu < \nu\rangle$.

PROOF: Let $G_\varepsilon = \frown\langle E_\xi : \xi < \Sigma\langle\delta_\nu : \nu < \varepsilon\rangle\rangle$, and for $\nu < \varepsilon$, let $H_\nu = \frown\langle E_{\gamma_\nu+\xi} : \xi < \delta_\nu\rangle$. We show by transfinite induction that $G_\varepsilon = \frown\langle H_\nu : \nu < \varepsilon\rangle$.

If $\varepsilon = 0$, then $G_\varepsilon = \phi$ and $\frown\langle H_\nu : \nu < \varepsilon\rangle = \phi$. If $\varepsilon \notin \mathrm{Lim}$ and $G_{\varepsilon-1} = \frown\langle H_\nu : \nu < \varepsilon - 1\rangle$, then $G_\varepsilon = \frown\langle E_\xi : \xi < (\Sigma\langle\delta_\nu : \nu < \varepsilon - 1\rangle) + \delta_{\varepsilon-1}\rangle = G_{\varepsilon-1}\frown H_{\varepsilon-1} = \frown\langle H_\nu : \nu < \varepsilon\rangle$ by 2.2.2 and 2.2.4.

If $\varepsilon \in \mathrm{Lim}$ and $G_\lambda = \frown\langle H_\nu : \nu < \lambda\rangle$ for all $\lambda < \varepsilon$, then $G_\varepsilon = \bigcup_{\lambda<\varepsilon} G_\lambda = \bigcup_{\lambda<\varepsilon} \frown\langle H_\nu : \nu < \lambda\rangle = \frown\langle H_\nu : \nu < \varepsilon\rangle$ by 2.2.7 and 2.2.8.

We say that E' is a *consecutive part* of E if there are sequences E_0, E_1 such that $E = E_0\frown E'\frown E_1$. In case $E = E'\frown E_1$, we say that

E' is an *initial part* of E and write $E' \preccurlyeq E$. If $E = E_0{}^\frown E'$, then we say that E' is a *terminal part* of E.

2.2.10 The relation $\preccurlyeq$ is a partial ordering relation; that is, $E \preccurlyeq E$, $E_0 \preccurlyeq E_1$ and $E_1 \preccurlyeq E_0$ implies $E_0 = E_1$, and $E_0 \preccurlyeq E_1$ and $E_1 \preccurlyeq E_2$ implies $E_0 \preccurlyeq E_2$.

The anti-symmetry follows from the fact that $E_0 \preccurlyeq E_1$ implies $E_0 \subseteq E_1$. The relation of being a terminal part is clearly not anti-symmetric.

2.2.11 If $\delta_0 \leqslant \delta_1$, then ${}^\frown\langle E_\xi \colon \xi < \delta_0\rangle \preccurlyeq {}^\frown\langle E_\xi \colon \xi < \delta_1\rangle$.

Using the definition of function restriction in the Foreward, if $\delta \leqslant \mathrm{Dom}(E)$, then $E|\delta \preccurlyeq E$ since $E = {}^\frown\langle\langle E(\xi)\rangle \colon \xi < \mathrm{Dom}(E)\rangle = {}^\frown\langle\langle E(\xi)\rangle \colon \xi < \delta\rangle{}^\frown({}^\frown\langle\langle E(\delta + \xi)\rangle \colon \xi < \mathrm{Dom}(E) - \delta\rangle)$ and $E|\delta = {}^\frown\langle\langle E(\xi)\rangle \colon \xi < \delta\rangle$ by 2.2.3 and 2.2.4. Conversely, if $E' \preccurlyeq E$, then $E' = E|\mathrm{Dom}(E')$. Therefore the initial parts of E are precisely the restrictions of E to ordinals less than $\mathrm{Dom}(E)$.

2.2.12 If $\delta_0 \leqslant \delta_1 \leqslant \mathrm{Dom}(E)$, then $E|\delta_0 \preccurlyeq E|\delta_1$.

2.2.13 If $\bigcup_{\xi<\varepsilon} \delta_\xi \leqslant \mathrm{Dom}(E)$, then $E|\bigcup_{\xi<\varepsilon} \delta_\xi = \bigcup_{\xi<\varepsilon} E|\delta_\xi$.

2.2.14 If $E_0 \preccurlyeq E$ and $E_1 \preccurlyeq E$ then $E_0 \preccurlyeq E_1$ or $E_1 \preccurlyeq E_0$. Moreover, if $\mathrm{Dom}(E_0) = \mathrm{Dom}(E_1)$ then $E_0 = E_1$.

If $E' \preccurlyeq E$ and $E' \neq E$, then we say E' is a proper initial part of E and write $E' \prec E$.

2.2.15 $E' \prec E$ if and only if there is a sequence $E_1 \neq \phi$ such that $E = E'{}^\frown E_1$.

2.2.16 *Left Cancellation Laws.* (1) $E_0{}^\frown E_1 = E_0{}^\frown E_2$ if and only if $E_1 = E_2$. (2) $E_0{}^\frown E_1 \prec E_0{}^\frown E_2$ if and only if $E_1 \prec E_2$.

The general right cancellation law does not hold, just as it does not hold for ordinal addition. However,

2.2.17 *Finite Right Cancellation Law.* If E_0 finite, then $E_1{}^\frown E_0 = E_2{}^\frown E_0$ if and only if $E_1 = E_2$.

Note that we must have either $E_1 \preccurlyeq E_2$ or $E_2 \preccurlyeq E_1$ by 2.2.14. The equality must hold by virtue of the finite right cancellation property of ordinal addition.

2.2.18 *Existence of Sequential Difference.* If $E_0 \preccurlyeq E_1$ then there is a unique sequence E such that $E_1 = E_0{}^\frown E$.

The sequence E in 2.2.18 will be referred to as $E_1 - E_0$. Then $E_1 = E_0{}^\frown(E_1 - E_0)$ if $E_0 \preccurlyeq E_1$.

2.2.19 If $E \prec {}^\frown\langle E_\xi \colon \xi < \delta\rangle$, then there exist unique ordinals $\delta' < \delta$ and $\theta' < \mathrm{Dom}(E_{\delta'})$ such that $E = {}^\frown\langle E_\xi \colon \xi < \delta'\rangle{}^\frown E_{\delta'}|\theta'$.

The ordinals δ' and θ' of 2.2.19 are easily seen to be the ordinals of 2.1.5 for $\theta = \mathrm{Dom}(E)$ and $\gamma_\xi = \mathrm{Dom}(E_\xi)$.

When we deal with terms and formulas of a formal language, we often encounter a situation where symbols standing alone must be deleted and replaced by other sequences. The following lemma is needed to tell us exactly how a sequence is decomposed by such a deletion process.

2.2.20 *Lemma.* Let E be a sequence and $\langle \iota_\nu : \nu < \sigma \rangle$ a strictly increasing sequence of ordinals less than $\mathrm{Dom}(E)$. Let $\iota_\sigma = \mathrm{Dom}(E)$ and

$$E_\nu = E|\iota_\nu - (E|\iota_{\nu-1}{}^\frown\langle E(\iota_{\nu-1})\rangle) \text{ if } \nu \notin \mathrm{Lim},$$
$$E_\nu = E|\iota_\nu - E|\bigcup_{\theta<\nu} \iota_\theta \text{ if } \nu \in \mathrm{Lim},$$

for $\nu \leqslant \sigma$. Then $E = {}^\frown\langle E_\nu{}^\frown\langle E(\iota_\nu)\rangle : \nu < \sigma\rangle{}^\frown E_\sigma$.

PROOF: We show by transfinite induction on ordinals $\lambda \leqslant \sigma$ that $E|\iota_\lambda = {}^\frown\langle E_\nu{}^\frown\langle E(\iota_\nu)\rangle : \nu < \lambda\rangle{}^\frown E_\lambda$. Clearly $E|\iota_0 = E_0$, so the equation holds for $\lambda = 0$. If $\lambda \notin \mathrm{Lim}$ and $E|\iota_{\lambda-1} = {}^\frown\langle E_\nu{}^\frown\langle E(\iota_\nu)\rangle : \nu < \lambda - 1\rangle{}^\frown E_{\lambda-1}$, then using the definition of E_λ and the associative law, $E|\iota_\lambda = (E|\iota_{\lambda-1}{}^\frown\langle E(\iota_{\lambda-1})\rangle){}^\frown E_\lambda = ({}^\frown\langle E_\nu{}^\frown\langle E(\iota_\nu)\rangle : \nu<\lambda-1\rangle){}^\frown(E_{\lambda-1}{}^\frown\langle E(\iota_{\lambda-1})\rangle){}^\frown E_\lambda$. The equation holds for λ by 2.2.2.

If $\lambda \in \mathrm{Lim}$ and the equation holds for all $\theta < \lambda$, then $E|\iota_\lambda = (E|\bigcup_{\theta<\lambda} \iota_\theta){}^\frown E_\lambda = (\bigcup_{\theta<\lambda} E|\iota_\theta){}^\frown E_\lambda$ by definition of E_λ and 2.2.13. The induction hypothesis tells us that $\bigcup_{\theta<\lambda} E|\iota_\theta = \bigcup_{\theta<\lambda} ({}^\frown\langle E_\nu{}^\frown\langle E(\iota_\nu)\rangle : \nu < \theta\rangle{}^\frown E_\theta)$, but this sequence is the same as $\bigcup_{\theta<\lambda} ({}^\frown\langle E_\nu{}^\frown\langle E(\iota_\nu)\rangle : \nu < \theta\rangle)$ since ${}^\frown\langle E_\nu{}^\frown\langle E(\iota_\nu)\rangle : \nu < \theta\rangle{}^\frown E_\theta \leqslant {}^\frown\langle E_\nu{}^\frown\langle E(\iota_\nu)\rangle : \nu < \theta + 1\rangle$. One application of 2.2.7 yields the equation for λ.

This completes the proof. Note that this particular sequence $\langle E_\nu : \nu \leqslant \sigma\rangle$ need not be the only solution to the equation $E = ({}^\frown\langle E_\nu{}^\frown\langle E(\iota_\nu)\rangle : \nu < \sigma\rangle){}^\frown E_\sigma$. For example, if E is a constant sequence of length ω and $\sigma = 1$, $\iota_0 = 1$, the lemma yields the solution $E_0 = \langle c\rangle$, $E_1 = E$. Any finite initial part of E could serve as E_0.

2.3 Lemma for Replacement of Non-overlapping Parts within Sequences

This lemma is designed to deal with the replacement of terms within terms, of formulas within formulas, of terms within formulas. In each of these cases we have classes T_1, T_2 of sequences satisfying the following condition:

$$(C) \quad E_0{}^\frown E_1 \in \mathsf{T}_2 \text{ and } E_0 \neq \phi \text{ and } \phi \prec E_1 \preccurlyeq A \text{ implies } A \notin \mathsf{T}_1.$$

That is, non-empty proper terminal parts of sequences in T_2 are not initial parts of sequences in T_1. Suppose we are given sequences $A_\xi \in \mathsf{T}_1$ for each $\xi < \delta$, $C_\nu \in \mathsf{T}_2$ for each $\nu < \sigma$, and are given the decomposition

$$(*) \quad {}^\frown\langle A_\xi \colon \xi < \delta\rangle = {}^\frown\langle E_\nu{}^\frown C_\nu \colon \nu < \sigma\rangle{}^\frown E_\sigma.$$

We show that each A_ξ is either a consecutive part of one of the E_ν, or else A_ξ has form $E'_\lambda{}^\frown C_\lambda{}^\frown({}^\frown\langle E_{\lambda+1+\nu}{}^\frown C_{\lambda+1+\nu} \colon \nu < \tau - (\lambda + 1)\rangle){}^\frown E''_\tau$ where E'_λ is a terminal part of E_λ, E''_τ is an initial part of E_τ and $\lambda < \tau$. Thus each A_ξ either misses a given C_ν entirely, or has C_ν as a consecutive part.

2.3.1 *Lemma.* Suppose T_1 and T_2 are classes of sequences satisfying (C) and that lengths of sequences in T_2 are non-limit ordinals. Then there do not exist sequences $A_\xi \in \mathsf{T}_1$ for $\xi < \delta$, E_0, E_1 such that $E_0{}^\frown E_1 \in \mathsf{T}_2$, $E_0 \neq \phi$ and $\phi \prec E_1 \preccurlyeq {}^\frown\langle A_\xi \colon \xi < \delta\rangle$.

Proof: The proof is by transfinite induction on ordinals δ. The statement is trivial for $\delta = 0$. Suppose $\delta \notin \mathrm{Lim}$ and that $\delta - 1$ has the desired property. Suppose there did exist sequences $A_\xi \in \mathsf{T}_1$ for $\xi < \delta$, E_0, E_1 such that $E_0{}^\frown E_1 \in \mathsf{T}_2$, $E_0 \neq \phi$, $\phi \prec E_1 \preccurlyeq {}^\frown\langle A_\xi \colon \xi < \delta\rangle$. Since we cannot then have $E_1 \preccurlyeq {}^\frown\langle A_\xi \colon \xi < \delta - 1\rangle$, E_1 must have form ${}^\frown\langle A_\xi \colon \xi < \delta - 1\rangle{}^\frown E'_1$ and $\phi \prec E'_1 \preccurlyeq A_{\delta-1}$. Since $(E_0{}^\frown({}^\frown\langle A_\xi \colon \xi < \delta - 1\rangle){}^\frown E'_1$ is in T_2, (C) would then imply $A_{\delta-1} \notin \mathsf{T}_1$, a contradiction.

Suppose $\delta \in \mathrm{Lim}$ and that ordinals smaller than δ have the desired property. Again suppose that there were sequences $A_\xi \in \mathsf{T}_1$ for $\xi < \delta$, E_0, E_1 such that $E_0{}^\frown E_1 \in \mathsf{T}_2$, $E_0 \neq \phi$, and $\phi \prec E_1 \preccurlyeq {}^\frown\langle A_\xi \colon \xi < \delta\rangle$. Then E_1 would be a proper initial part of ${}^\frown\langle A_\xi \colon \xi < \delta\rangle$ since $\mathrm{Dom}(E_1) \notin \mathrm{Lim}$, while $\mathrm{Dom}{}^\frown\langle A_\xi \colon \xi < \delta\rangle \in \mathrm{Lim}$. It follows by 2.2.19 that there would be $\delta' < \delta$ such that $E_1 \preccurlyeq {}^\frown\langle A_\xi \colon \xi < \delta' + 1\rangle$. But then $\delta' + 1$ would not have the desired property.

2.3.2 *Lemma.* Suppose T_1 and T_2 are classes of sequences satisfying condition (C) and that the lengths of sequences in T_2 are non-limit ordinals. Suppose $A_\xi \in \mathsf{T}_1$ for all $\xi < \delta$ and that $C_\nu \in \mathsf{T}_2$ for all $\nu < \sigma$ and

$$(*) \quad {}^\frown\langle A_\xi \colon \xi < \delta\rangle = ({}^\frown\langle E_\nu{}^\frown C_\nu \colon \nu < \sigma\rangle){}^\frown E_\sigma.$$

Then for each ordinal $\lambda \leqslant \delta$ there is a unique ordinal $g(\lambda)$ and a unique initial part $E_{g(\lambda)\lambda}$ of $E_{g(\lambda)}$ such that $\frown\langle A_\xi : \xi < \lambda\rangle = (\frown\langle E_\nu{}^\frown C_\nu : \nu < g(\lambda)\rangle)^\frown E_{g(\lambda)\lambda}$.

PROOF: The uniqueness is by 2.2.19. From decomposition (*) we see that either $\frown\langle A_\xi : \xi < \lambda\rangle \prec \frown\langle E_\nu{}^\frown C_\nu : \nu < \sigma\rangle$ or $\frown\langle A_\xi : \xi < \lambda\rangle$ has form $\frown\langle E_\nu{}^\frown C_\nu : \nu < \sigma\rangle^\frown E'$, $\phi \leqslant E' \leqslant E_\sigma$. In the second case we take $g(\lambda) = \sigma$, $E_{g(\lambda)\lambda} = E'$. In the first case, we know that $\lambda < \delta$ and there is an ordinal $\sigma' < \sigma$ and sequence $E'' \prec E_{\sigma'}{}^\frown C_{\sigma'}$ such that

$$(**) \quad \frown\langle A_\xi : \xi < \lambda\rangle = \frown\langle E_\nu{}^\frown C_\nu : \nu < \sigma'\rangle^\frown E''.$$

Take $g(\lambda) = \sigma'$, $E_{g(\lambda)\lambda} = E''$. It remains only to show $E'' \leqslant E_{\sigma'}$. If this were not the case, we would have $E'' = E_{\sigma'}{}^\frown E''_1$, $\phi \prec E''_1 \prec C_{\sigma'}$. Using (*), (**) and the properties of concatenation listed in 2.2, we could then cancel $\frown\langle A_\xi : \xi < \lambda\rangle$ on the left in equation (*) to obtain $\phi \prec C_{\sigma'} - E''_1 \leqslant \frown\langle A_{\lambda+\xi} : \xi < \delta - \lambda\rangle$, thus contradicting 2.3.1. Therefore we must have $E_{g(\lambda)\lambda} \leqslant E_{g(\lambda)}$.

Under conditions of 2.3.2, $\frown\langle A_\xi : \xi < \lambda\rangle \prec \frown\langle E_\nu{}^\frown C_\nu : \nu < g(\lambda) + 1\rangle$ since $C_{g(\lambda)} \neq \phi$. Therefore, if $g(\lambda) < g(\lambda')$, we have $\frown\langle A_\xi : \xi < \lambda\rangle \prec \frown\langle A_\xi : \xi < \lambda'\rangle$, which in turn implies $\lambda < \lambda'$. It follows that the function g is increasing. Clearly $g(0) = 0$, $g(\delta) = \sigma$, $E_{g(0)0} = \phi$. $E_{g(\delta)\delta} = E_\sigma$. Using the left cancellation and other properties of concatenation listed in 2.2, the desired representation of the A_λ is seen to be the following:

2.3.3 *Corollary.* Under conditions of 2.3.2, if $g(\lambda) = g(\lambda + 1)$, then $A_\lambda = E_{g(\lambda)\lambda+1} - E_{g(\lambda)\lambda}$. If $g(\lambda) < g(\lambda + 1)$, then $A_\lambda = (E_{g(\lambda)} - E_{g(\lambda)\lambda})^\frown C_{g(\lambda)}{}^\frown E^\frown E_{g(\lambda+1)\lambda+1}$, where

$$E = \frown\langle E_{g(\lambda)+1+\nu}{}^\frown C_{g(\lambda)+1+\nu} : \nu < g(\lambda + 1) - (g(\lambda) + 1)\rangle.$$

We are now in a position to see that the replacement of sequences C'_ν for the parts C_ν within $\frown\langle A_\xi : \xi < \delta\rangle$ can be accomplished by replacing the C'_ν for the C_ν within the A_ξ and then forming the concatenation.

2.3.4 *Lemma for Replacement.* Assume conditions of 2.3.2. Let C'_ν be a sequence for each $\nu < \delta$, let $A'_\lambda = A_\lambda$ when $g(\lambda) = g(\lambda + 1)$, let $A'_\lambda = (E_{g(\lambda)} - E_{g(\lambda)\lambda})^\frown C'_{g(\lambda)}{}^\frown E'^\frown E_{g(\lambda+1)\lambda+1}$ when $g(\lambda) < g(\lambda + 1)$, where $E' = \frown\langle E_{g(\lambda)+1+\nu}{}^\frown C'_{g(\lambda)+1+\nu} : \nu < g(\lambda + 1) - (g(\lambda) + 1)\rangle$ Then $\frown\langle A'_\xi : \xi < \delta\rangle = \frown\langle E_\nu{}^\frown C'_\nu : \nu < \sigma\rangle^\frown E_\sigma$.

PROOF: We show by induction that $\frown\langle A'_\xi : \xi < \lambda\rangle = \frown\langle E_\nu{}^\frown C'_\nu : \nu < g(\lambda)\rangle^\frown E_{g(\lambda)\lambda}$ for ordinals $\lambda \leqslant \delta$. The equation holds trivially for

$\lambda = 0$. Assume that holds it for ordinals less than λ and consider four cases:

Case 1. $\lambda \in \mathrm{Lim}$ and for each $\theta < \lambda$ there is $\theta < \theta' < \lambda$ such that $g(\theta) < g(\theta')$. Then $\frown\langle A'_\xi : \xi < \lambda\rangle = \bigcup_{\theta<\lambda} \frown\langle A'_\xi : \xi < \theta\rangle = \bigcup_{\theta<\lambda} \frown\langle E_\nu \frown C'_\nu : \nu < g(\theta)\rangle \frown E_{g(\theta)\theta} = \bigcup_{\theta'<\lambda} \frown\langle E_\nu \frown C'_\nu : \nu < g(\theta')\rangle = \frown\langle E_\nu \frown C'_\nu : \nu < \bigcup_{\theta<\lambda} g(\theta)\rangle$. Similarly $\frown\langle A_\xi : \xi < \lambda\rangle = \frown\langle E_\nu \frown C_\nu : \nu < \bigcup_{\theta<\lambda} g(\theta)\rangle$. Hence $g(\lambda) = \bigcup_{\theta<\lambda} g(\theta)$ and $E_{\lambda g(\lambda)} = \phi$. Hence the desired equation holds for λ.

Case 2. $\lambda \in \mathrm{Lim}$ and there is $\theta' < \lambda$ such that $g(\theta) = g(\theta')$ for $\theta' < \theta < \lambda$. Then $\frown\langle A'_\xi : \xi < \lambda\rangle = \bigcup_{\theta<\lambda} \frown\langle A'_\xi : \xi < \theta\rangle = \bigcup_{\theta<\lambda} \frown\langle E_\nu \frown C'_\nu : \nu < g(\theta)\rangle \frown E_{g(\theta)\theta} = \bigcup_{\theta'<\theta<\lambda} \frown\langle E_\nu \frown C'_\nu : \nu < g(\theta')\rangle \frown E_{g(\theta)\theta}$. For the θ such that $\theta' < \theta < \lambda$, the $E_{g(\theta)\theta}$ are increasing initial parts of $E_{g(\theta')}$. Hence $\bigcup_{\theta'<\theta<\lambda} E_{g(\theta)\theta} \leqslant E_{g(\theta')}$ and $\frown\langle A'_\xi : \xi < \lambda\rangle = \frown\langle E_\nu \frown C'_\nu : \nu < g(\theta')\rangle \frown \bigcup_{\theta'<\theta<\lambda} E_{g(\theta)\theta}$. The analogous equation holds for $\frown\langle A_\xi : \xi < \lambda\rangle$ without the primes. By the uniqueness of $g(\lambda)$ and $E_{g(\lambda)\lambda}$ we see that $g(\lambda) = g(\theta')$ and $E_{g(\lambda)\lambda} = \bigcup_{\theta'<\theta<\lambda} E_{g(\theta)\theta}$. Hence the desired equation holds for λ.

Case 3. $\lambda \notin \mathrm{Lim}$ and $g(\lambda - 1) < g(\lambda)$. In this case, $\frown\langle A'_\xi : \xi < \lambda\rangle = \frown\langle E_\nu \frown C'_\nu : \nu < g(\lambda - 1)\rangle \frown E_{g(\lambda-1)\lambda-1} \frown A'_{\lambda-1}$. The desired equation follows immediately from the associative law and the definition of $A'_{\lambda-1}$.

Case 4. $\lambda \notin \mathrm{Lim}$ and $g(\lambda - 1) = g(\lambda)$. In this case $\frown\langle A'_\xi : \xi < \lambda\rangle = \frown\langle E_\nu \frown C'_\nu : \nu < g(\lambda - 1)\rangle \frown E_{g(\lambda-1)\lambda-1} \frown A_{\lambda-1}$. The desired equation follows immediately from the associative law and expression for $A_{\lambda-1}$ in 2.3.3.

Note that the proof just given contains a proof of the following:

2.3.5 *Corollary.* If $\lambda \in \mathrm{Lim}$, then $g(\lambda) = \bigcup_{\theta<\lambda} g(\theta)$.

CHAPTER 3

ALGEBRAS OF TERMS OF INFINITE LENGTH

At this time, we consider only the underlying systems of terms of infinitary predicate languages. However, the notion of term considered here is broader than that discussed informally in Chapter 1, for these terms may be infinitely long.

Attached to an infinitary predicate language is a regular infinite cardinal o to serve as a bound on the lengths of terms. The language has a supply of primitive symbols among which are brackets [,], and symbols classified as follows:

Individual symbols
Special two-place operation symbols
ζ-place operation symbols, $0 < \zeta < o$
Infinitary operation symbols.

The other kinds of symbols need not concern us now. It is assumed that there is at least one individual symbol, that the symbols are all distinct and are not themselves sequences. An *o-expression* is a sequence of symbols having length less than o.

As a rule, the sequential brackets will be dropped when writing expressions. Thus if φ is a symbol, the expression written merely "φ" is the sequence $\langle\varphi\rangle$. Similarly, the signs for concatenation will be dropped when writing expressions. If T_ξ is an expression for each $\xi < \zeta$, the expression written "$T_0 \ldots T_\xi \ldots$" is $\frown\langle T_\xi : \xi < \zeta\rangle$. When there is danger of misinterpretation, the notation of Chapter 2 will be used in full.

3.1 The o-algebras of Terms

An *atomic term* of an infinitary predicate language is an expression consisting of an individual symbol standing alone. The set of *terms* is the intersection of all sets T of expressions containing the atomic terms and closed under rules of o-term-formation:

(1) If $T_0, T_1 \in \mathsf{T}$ and ψ is a special two-place operation symbol, then $[T_0 \psi T_1] \in \mathsf{T}$.

(2) If $\langle T_\xi : \xi < \zeta \rangle$ is a ζ-tuple of expressions in T and φ^ζ is a ζ-place operation symbol, then $[\varphi^\zeta T_0 \ldots T_\xi \ldots] \in \mathsf{T}$.

(3) If $\langle T_\xi : \xi < \zeta \rangle$ is a ζ-tuple of expressions in T, $0 < \zeta < o$, and φ is an infinitary operation symbol, then $[\varphi T_0 \ldots T_\xi \ldots] \in \mathsf{T}$.

Of course, "$[\varphi T_0 \ldots T_\xi \ldots]$" is short-hand for $\langle [\varphi \rangle ^\frown (^\frown \langle T_\xi : \xi < \zeta \rangle) {}^\frown \langle] \rangle$. The special two-place symbols do not add anything to the expressive power of the language. We have included them only because it seems more natural to write something like "$[T_0 + T_1]$" than "$[+ T_0 T_1]$".

By virtue of 2.1.1, every term has length less than o. The algebra $\mathfrak{F}_o$ of terms has the set F_o of terms as its underlying set, and has one ζ-place operation for each ζ-place operation symbol, one $\nearrow o$-place operation for each infinitary operation symbol, with values given by (1), (2), (3). For example, the value of the operation assigned to ψ is $[T_0 \psi T_1]$ for argument $\langle T_0 T_1 \rangle$.

3.1.1 *Induction Principle for Terms.* If T is a set of expressions containing the atomic terms and closed under rules (1), (2), (3) of o-term-formation, then T contains all terms.

3.2 Algebras for the Interpretation of Terms

Let $\langle D\ \mathsf{O} \rangle$ be an algebraic system with non-empty underlying set D, having the same similarity type as $\mathfrak{F}_o$. That is, O assigns a ζ-place operation over D to each ζ-place operation symbol and an $\nearrow o$-place operation over D to each infinitary operation symbol. Such an algebra will be called an *algebra for the interpretation of terms.*

3.2.1 *Recursion Principle for Terms.* Let $\langle D\ \mathsf{O} \rangle$ be an algebra for the interpretation of terms. Then if s is any function on the individual symbols of the language to D, there is a unique homomorphism s^* on the algebra of terms to $\langle D\ \mathsf{O} \rangle$ such that $s^*(\langle x \rangle) = s(x)$ for all individual symbols x.

This means of course, that s^* satisfies the following equations for all terms T, special two-place operation symbols ψ, ζ-place operation symbols φ^ζ, infinitary operation symbols φ:

(1) $s^*([T_0 \psi T_1]) = \mathsf{O}(\psi)(\langle s^*(T_0)\, s^*(T_1) \rangle)$

(2) $s^*([\varphi^\zeta T_0 \ldots T_\xi \ldots]) = \mathsf{O}(\varphi^\zeta)(\langle s^*(T_0) \ldots s^*(T_\xi) \ldots \rangle)$

(3) $s^*([\varphi T_0 \ldots T_\xi \ldots]) = \mathsf{O}(\varphi)(\langle s^*(T_0) \ldots s^*(T_\xi) \ldots \rangle)$.

The rather lengthy proof is deferred to the next section. For the moment, let us look at some examples.

3.2.2 *Example.* Consider a language with regular infinite cardinal α as bound on lengths of terms, having one-place operation symbol $\neg$, special two-place operation symbols $\wedge$, $\vee$, infinitary operation symbols $\bigwedge$, $\bigvee$. The intended algebra for the interpretation of terms is an algebra $\langle B\ \mathsf{O}\rangle$, where B is the underlying set of an $\nearrow\alpha$-complete Boolean algebra, $\mathsf{O}(\neg)$ is the one-place complementation operation, $\mathsf{O}(\wedge)$, $\mathsf{O}(\vee)$ are the two-place meet and join, $\mathsf{O}(\bigwedge)$, $\mathsf{O}(\bigvee)$ are the $\nearrow\alpha$-place meet and join, respectively. Then $s^*(T)$ is the result of substituting $s(x)$ for individual symbols x in T and calculating the value in B.

3.2.3 *Example.* Consider a language with regular infinite cardinal α as bound on length of terms, having one-place operation symbol $\boldsymbol{\neg}$, special two-place operation symbol $\boldsymbol{\rightarrow}$, infinitary operation symbol $\boldsymbol{\bigwedge}$. The terms of the language can be thought of as α-propositional formulas. The intended algebra for interpretation is $\langle B_0\ \mathsf{O}\rangle$, where B_0 is the set of truth values $\mathbf{0}$, $\mathbf{1}$, $\mathsf{O}(\boldsymbol{\neg})$ is the one-place complementation operation, $\mathsf{O}(\boldsymbol{\bigwedge})$ is the infinitary meet, $\mathsf{O}(\boldsymbol{\rightarrow})$ has value $\neg x \vee y$ for $x, y \in B_0$. If s is any function assigning truth values $\mathbf{0}$, $\mathbf{1}$ to the individual symbols (called *propositional symbols* in this context), $s^*(T)$ is the calculated truth value of T.

3.2.4 *Example.* Let F_α be the set of α-propositional formulas of Example 3.2.3. Consider a second language with the symbols of F_α together with additional two-place operation symbols $\boldsymbol{\leftrightarrow}$, $\boldsymbol{\wedge}$, $\boldsymbol{\vee}$, and additional infinitary operation symbol $\boldsymbol{\bigvee}$. This time, take $\langle F_\alpha\ \mathsf{O}\rangle$ as algebra for interpretation of the second language, where for $T_0, \ldots, T_\xi, \ldots$ in F_α,

$$\begin{aligned}
&\mathsf{O}(\boldsymbol{\neg})(\langle T\rangle) = [\boldsymbol{\neg} T]\\
&\mathsf{O}(\boldsymbol{\rightarrow})(\langle T_0 T_1\rangle) = [T_0 \boldsymbol{\rightarrow} T_1]\\
&\mathsf{O}(\boldsymbol{\leftrightarrow})(\langle T_0 T_1\rangle) = [\boldsymbol{\bigwedge}\ [T_0 \boldsymbol{\rightarrow} T_1][T_1 \boldsymbol{\rightarrow} T_0]]\\
&\mathsf{O}(\boldsymbol{\wedge})(\langle T_0 T_1\rangle) = [\boldsymbol{\bigwedge}\ T_0 T_1]\\
&\mathsf{O}(\boldsymbol{\vee})(\langle T_0 T_1\rangle) = [\boldsymbol{\neg}\ [\boldsymbol{\bigwedge}\ [\boldsymbol{\neg} T_0][\boldsymbol{\neg} T_1]]]\\
&\mathsf{O}(\boldsymbol{\bigwedge})(\langle T_0 \ldots T_\xi \ldots\rangle) = [\boldsymbol{\bigwedge}\ T_0 \ldots T_\xi \ldots]\\
&\mathsf{O}(\boldsymbol{\bigvee})(\langle T_0 \ldots T_\xi \ldots\rangle) = [\boldsymbol{\neg}\ [\boldsymbol{\bigwedge}\ [\boldsymbol{\neg} T_0] \ldots [\boldsymbol{\neg} T_\xi] \ldots]].
\end{aligned}$$

Then if s is the function assigning $\langle x\rangle$ to propositional symbols x, and T is a formula of the second language, $s^*(T)$ is the formula of the first language that results from eliminating symbols $\boldsymbol{\leftrightarrow}$, $\boldsymbol{\wedge}$, $\boldsymbol{\vee}$, $\boldsymbol{\bigvee}$, by the usual rules of abbreviation.

3.3 Recursion Principle

As soon as we know that no proper initial part of a term is a term, we shall be well on our way to a proof of the existence and uniqueness of the functions s^*.

3.3.1 *Lemma.* If T is a term that is not atomic, then exactly one of conditions (i), (ii), (iii) holds for T:

(i) There exist terms T_0, T_1 and a special two-place operation symbol ψ such that $T = [T_0 \psi T_1]$.

(ii) There is a ζ-place operation symbol φ^ζ and a ζ-tuple of terms such that $T = [\varphi^\zeta T_0 \ldots T_\xi \ldots]$.

(iii) There is an ordinal ζ, $0 < \zeta < o$, and an infinitary operation symbol φ such that $T = [\varphi T_0 \ldots T_\xi \ldots]$.

PROOF: The set T of terms containing the atomic terms and satisfying at least one of the above conditions is obviously closed under the rules of o-term-formation. By the induction principle, it follows that T contains all terms. A comparison of symbols $T(1)$ shows that two of the above conditions could not hold simultaneously. For the operation symbols are distinct from one another and a trivial induction shows that no term begins with an operation symbol.

3.3.2 *Lemma.* If T is a term and $E \prec T$, then E is not a term.

PROOF: It is customary to introduce a bracket-counter for proofs like this. Though we could do that here as well, it seems to be faster to proceed by induction.

Let $\mathsf{T} = \{T\colon T$ is a term and $E \prec T$ implies E is not a term, and $T \prec E'$ implies E' is not a term$\}$. If T atomic and $E \prec T$, then $E = \phi$ and therefore is not a term. If $T \prec E'$, then E' begins with an individual symbol and has length greater than 1. A trivial induction shows there is no such term. Hence atomic terms are in T.

Closure condition 3.1 (1): Suppose T_0, $T_1 \in \mathsf{T}$ and ψ a special two-place operation symbol. Let $T = [T_0 \psi T_1]$. Use 2.2.19 to list the proper initial parts of T that could possibly be terms, after noting that terms are not ϕ, and if a term begins with [, it ends with]. We obtain

Case 1. $E = [E_0, \phi \prec E_0 \leqslant T_0$
Case 2. $E = [T_0 \psi E_1, \phi \prec E_1 \leqslant T_1$.

Such an expression E could not be an atomic term, nor could it have one of the forms 3.3.1 (ii) or (iii) since $T_0(0)$ not an operation symbol. It follows that there are terms T'_0, T'_1 and a special two-place operation symbol such that $E = [T'_0\psi' T'_1]$ if E is a term at all. In Case 1, after cancellation of [, $T'_0 \prec E_0 \preccurlyeq T_0$. This contradicts $T_0 \in \mathsf{T}$. In Case 2, after cancellation of [, we see that $T_0 \prec T'_0$ or $T_0 = T'_0$ or $T'_0 \prec T_0$ by 2.2.14. The equality must hold since $T_0 \in \mathsf{T}$. After left cancellation of $[T_0$, we see that $\psi = \psi'$. Cancelling that as well, we obtain $T'_1 \prec E_1 \preccurlyeq T_1$, thus contradicting $T_1 \in \mathsf{T}$. Thus E is not a term.

Next, suppose $[T_0\psi T_1] \preccurlyeq E'$, where E' is a term. Again E' has form $[T'_0\psi' T'_1]$ according to 3.3.1. The argument above shows $T_0 = T'_0$, $\psi = \psi'$, $T_1 = T'_1$. Hence $E' = [T_0\psi T_1] = T$. Therefore $T \in \mathsf{T}$.

Closure conditions 3.1 (2) and 3.1 (3): Suppose φ is a ζ-place or infinitary operation symbol and $\langle T_\xi : \xi < \zeta\rangle$ is a ζ-tuple of terms in T. Let $T = [\varphi T_0 \ldots T_\xi \ldots]$. We must show $T \in \mathsf{T}$. Use 2.2.19 to list the proper initial parts of T that could possibly be terms, remembering that terms are non-empty, and that a term beginning in [must end in]. The cases reduce to

(a) $E = \langle[\varphi\rangle^\frown(^\frown\langle T_\xi : \xi < \varepsilon\rangle)^\frown E_\varepsilon$, where $\varepsilon \leqslant \zeta$, $\phi \preccurlyeq E_\varepsilon \preccurlyeq T_\varepsilon$.

If E is a term, it is not atomic, nor does it have form 3.3.1 (i). Clearly, there must be a ζ'-tuple of terms such that

(b) $E = \langle[\varphi\rangle^\frown(^\frown\langle T'_\xi : \xi < \zeta'\rangle)^\frown\langle]\rangle$.

After left cancellation on (a) and (b), we obtain

(c) $^\frown\langle T_\xi : \xi < \varepsilon\rangle^\frown E_\varepsilon = {}^\frown\langle T'_\xi : \xi < \zeta'\rangle^\frown\langle]\rangle$.

Let $\lambda = \varepsilon \cap \zeta'$. We show by transfinite induction that $T_\xi = T'_\xi$ for all $\xi < \lambda$. Suppose $\theta < \lambda$ and $T_\xi = T'_\xi$ for all $\xi < \theta$. Cancelling $^\frown\langle T_\xi : \xi < \theta\rangle$ on the left in (c), we obtain $^\frown\langle T_{\theta+\xi} : \xi < \varepsilon - \theta\rangle^\frown E_\varepsilon = {}^\frown\langle T'_{\theta+\xi} : \xi < \zeta' - \theta\rangle^\frown\langle]\rangle$. By 2.2.14, either $T_\theta \prec T'_\theta$ or $T_\theta = T'_\theta$ or $T'_\theta \prec T_\theta$. Since $T_\theta \in \mathsf{T}$, the equality must hold. This completes the induction.

If $\varepsilon < \zeta'$, we have after left cancellation on (c), $E_\varepsilon = {}^\frown\langle T'_{\varepsilon+\xi} : \xi < \zeta' - \varepsilon\rangle^\frown\langle]\rangle$. This implies $T'_\varepsilon \prec E_\varepsilon \preccurlyeq T_\varepsilon$, contradicting $T_\varepsilon \in \mathsf{T}$. If $\zeta' < \varepsilon$, we have after left cancellation on (c), $\langle]\rangle = {}^\frown\langle T_{\zeta'+\xi} : \xi < \varepsilon - \zeta'\rangle^\frown E_\varepsilon$, clearly impossible. If $\varepsilon = \zeta'$, $E_\varepsilon = \langle]\rangle$, contradicting (a) since no term begins with]. Hence E is not a term.

Finally, suppose $[\varphi T_0 \ldots T_\xi \ldots] \preccurlyeq E'$, E' a term. The argument above with T_ξ and T'_ξ interchanged, yields $T = [\varphi T_0 \ldots T_\xi \ldots] = E'$. Hence $T \in \mathsf{T}$. This completes the proof.

Note that a proof of the following theorem was included in the one just given.

3.3.3 *Corollary*. Suppose $\langle T_\xi : \xi < \delta\rangle$ and $\langle T'_\xi : \xi < \delta'\rangle$ are arbitrary sequences of terms and that σ and σ' are arbitrary symbols. Then

(1) If $T_0\sigma T'_0 = T_1\sigma' T'_1$, then $T_0 = T'_0$, $\sigma = \sigma'$, $T_1 = T'_1$.

(2) If $\frown\langle T_\xi : \xi < \delta\rangle = \frown\langle T'_\xi : \xi < \delta'\rangle$, then $\delta = \delta'$ and $T_\xi = T'_\zeta$ for all $\xi < \delta$.

3.3.4 *Recursion Principle*. Let $\langle D\ \mathsf{O}\rangle$ be an algebra for interpretation of infinitary terms, sect. 3.2. Then given any function s on individual symbols to D, there is a unique homomorphism s^* of the algebra $\mathfrak{F}_o$ of terms to $\langle D\ \mathsf{O}\rangle$ such that $s^*(\langle x\rangle) = s(x)$ for individual symbols x.

PROOF: The uniqueness of s^* follows immediately from the induction principle 3.1.1. We proceed to show the existence. Call a relation $R \subseteq F_o \times D$ *acceptable* if and only if it satisfies the following conditions:

(0′) $\langle x\rangle R s(x)$ whenever x is an individual symbol.

(1′) $T_0 R d_0$ and $T_1 R d_1$ implies $[T_0\psi T_1] R \mathsf{O}(\psi)(\langle d_0 d_1\rangle)$ whenever T_0, T_1 are terms, ψ a special two-place operation symbol.

(2′) $T_\xi R d_\xi$ for all $\xi < \zeta$ implies $[\varphi^\varsigma T_0 \ldots T_\xi \ldots] R \mathsf{O}(\varphi^\varsigma)(\langle d_0 \ldots d_\xi \ldots\rangle)$ whenever $\langle T_\xi : \xi < \zeta\rangle$ is a ζ-tuple of terms, φ^ς is a ζ-place operation symbol.

(3′) $T_\xi R d_\xi$ for all $\xi < \zeta$ implies $[\varphi T_0 \ldots T_\xi \ldots] R \mathsf{O}(\varphi)(\langle d_0 \ldots d_\xi \ldots\rangle)$ whenever $0 < \zeta < o$, $\langle T_\xi : \xi < \zeta\rangle$ is a ζ-tuple of terms, φ an infinitary operation symbol.

Then $F_o \times D$ is an acceptable relation. Let $R_0 = \bigcap \{R : R \text{ is acceptable}\}$. Then R_0 is itself acceptable. We claim that R_0 is a function with domain F_o. The proof will then be complete since this function obviously satisfies the conditions 3.2.1 (1), (2), (3).

Let $\mathsf{T} = \{T : T$ is a term and there exists a unique $d \in D$ such that $T R_0 d\}$. If x is an individual symbol, then $\langle x\rangle R_0 s(x)$. If $d \neq s(x)$, then $R_0 \sim \{(\langle x\rangle, d)\}$ is still acceptable. Therefore $(\langle x\rangle, d) \notin R_0$. Hence $\langle x\rangle \in \mathsf{T}$.

Suppose $T_0, T_1 \in \mathsf{T}$ and ψ a special two-place operation symbol. Let d_0, d_1 be the unique elements of D such that $T_0 R_0 d_0$, $T_1 R_0 d_1$. Then $[T_0\psi T_1] R_0 \mathsf{O}(\psi)(\langle d_0 d_1\rangle)$ by (1′). If $d \neq \mathsf{O}(\psi)(\langle d_0 d_1\rangle)$, then

$R_0 \sim \{([T_0\psi T_1], d)\}$ still satisfies (0′), (2′), (3′) since $[T_0\psi T_1]$ is not an atomic term, and since $T_0(0)$ is not an operation symbol. This relation also satisfies (1′), for suppose (T'_0, d'_0) and (T'_1, d'_1) are in $R_0 \sim \{([T_0\psi T_1], d)\}$, but not the pair $([T'_0\psi' T'_1], \mathsf{O}(\psi')(\langle d'_0 d'_1\rangle)$. Since this pair is in R_0, it must be the case that

$$([T'_0\psi' T'_1], \mathsf{O}(\psi')(\langle d'_0 d'_1\rangle)) = ([T_0\psi T_1], d).$$

By 3.3.3, $T_0 = T'_0$, $\psi = \psi'$, $T_1 = T'_1$. Since they are in T, $d'_0 = d_0$ and $d'_1 = d_1$. But then $d = \mathsf{O}(\psi)(\langle d_0 d_1\rangle)$, contradicting the assumption on d. Hence $R_0 \sim \{([T_0\psi T_1], d)\}$ is acceptable and therefore, $([T_0\psi T_1], d) \notin R_0$. Therefore $[T_0\psi T_1] \in \mathsf{T}$.

The proof that T is closed under 3.1 (2) and (3) is entirely similar to the proof of closure under 3.1 (1), using 3.3.3 once more.

3.4 Replacement of Equivalent Parts

This section deals with congruence relations on the algebra $\mathfrak{F}_o$ of terms of an infinitary language. Using the ordinary algebraic definition, a congruence relation on F_o is an equivalence relation on terms such that for all terms T_ξ, T'_ξ, two-place operation symbols ψ, ζ-place operation symbols φ^ζ, infinitary operation symbols φ,

(1) $T_0 \equiv T'_0$ and $T_1 \equiv T'_1$ implies $[T_0\psi T_1] \equiv [T'_0\psi T'_1]$
(2) $T_\xi \equiv T'_\xi$ for all $\xi < \zeta$ implies
$[\varphi^\zeta T_0 \ldots T_\xi \ldots] \equiv [\varphi^\zeta T'_0 \ldots T'_\xi \ldots]$
(3) $T_\xi \equiv T'_\xi$ for all $\xi < \zeta$ and $0 < \zeta < o$ implies
$[\varphi T_0 \ldots T_\xi \ldots] \equiv [\varphi T'_0 \ldots T'_\xi \ldots]$.

The algebras of congruence classes of terms are homomorphic images of $\mathfrak{F}_o$, and are therefore also algebras for the interpretation of terms.

3.4.1 *Example.* Let $\langle D\ \mathsf{O}\rangle$ be a particular algebra for the interpretation of terms, s an assignment of individual symbols to D. Then the relation $T \equiv T'$ if and only if $s^*(T) = s^*(T')$ is a congruence relation on the algebra of terms. The algebra of congruence classes is isomorphic to the subalgebra of $\langle D\ \mathsf{O}\rangle$ generated by $\mathrm{Rng}(s)$. Similarly, if S is a set of functions on individual symbols to D, the relation $T \equiv T'$ if and only if $s^*(T) = s^*(T')$ for all $s \in S$, is a congruence relation on terms. One could consider sets of assignments to sets of interpretive algebras as well.

3.4.2 *Example.* Consider the α-propositional formulas of Example

3.2.3. The relation $T \equiv T'$ if and only if $s^*(T) = s^*(T')$ for all functions s on individual symbols to truth values, is a congruence relation on the algebra of formulas. The notions of provability for these languages, to be introduced in Chapter 5, will have the property that the relation $T \equiv T'$ if and only if $\vdash T \rightarrow T'$ and $\vdash T' \rightarrow T$, will be a congruence relation on the algebra of formulas.

The Replacement Principle tells us that when consecutive parts T_ν of a term T, the T_ν being themselves terms, are replaced by terms T'_ν congruent to T_ν, the result is a term congruent to T. This follows easily from the result of Sect. 2.3 once we have established condition (C) for terms.

3.4.3 *Lemma.* $E_0{}^\frown E_1$ a term, $E_0 \neq \phi$ and $\phi \prec E_1 \preccurlyeq T$ implies T is not a term.

PROOF: Let $\mathsf{T} = \{T: T$ is a term and $T = E_0{}^\frown E_1$ and $\phi \prec E_1 \preccurlyeq T'$, where T' is a term, implies $E_0 = \phi\}$. It suffices to show that all terms are in T. Clearly atomic terms are in T.

Closure condition 3.1 (1): Suppose T_0, $T_1 \in \mathsf{T}$ and ψ is a special two-place operation symbol. Suppose $T = [T_0 \psi T_1] = E_0{}^\frown E_1$ and $\phi \prec E_1 \preccurlyeq T'$, where T' is a term. Then E_1 cannot begin with ψ or]. Therefore if $E_0 \neq \phi$, one of the following four cases arises:

Case 1. $E_0 = \langle[\rangle$. Then after left cancellation, $T_0 \prec E_1 \preccurlyeq T'$ contradicting 3.3.2.

Case 2. $E_0 = \langle[\rangle^\frown T'_0$, $\phi \prec T'_0 \prec T_0$. Then after left cancellation of E_0, $E_1 = (T_0 - T'_0)^\frown\langle\psi\rangle^\frown T_1{}^\frown\langle]\rangle$. Since $T_0 = T'_0{}^\frown(T_0 - T'_0)$, and $\phi \prec (T_0 - T'_0) \prec E_1 \preccurlyeq T'$, this contradicts $T_0 \in \mathsf{T}$.

Case 3. $E_0 = \langle[\rangle^\frown T_0{}^\frown\langle\psi\rangle$. After left cancellation, $T_1 \prec E_1 \preccurlyeq T'$, contradicting 3.3.2.

Case 4. $E_0 = \langle[\rangle^\frown T_0{}^\frown\langle\psi\rangle^\frown T'_1$, $\phi \prec T'_1 \prec T_1$. Then after left cancellation of E_0, $E_1 = (T_1 - T'_1)^\frown\langle]\rangle$. Since $T_1 = T'_1{}^\frown(T_1 - T'_1)$ and $\phi \prec (T_1 - T'_1) \prec E_1 \preccurlyeq T'$, this contradicts $T_1 \in \mathsf{T}$.

Thus the only possibility is $E_0 = \phi$.

Closure conditions 3.1 (2), (3): Suppose $\langle T_\xi: \xi < \zeta\rangle$ is a ζ-tuple of terms in T and $T = [\varphi T_0 \ldots T_\xi \ldots]$, where φ is a ζ-place or infinitary operation symbol. Again assume $T = E_0{}^\frown E_1$, $\phi \prec E_1 \preccurlyeq T'$, T' a term. Then E_1 cannot begin with φ or with]. Therefore if the initial part E_0 of T is non-empty, it must have one of the following forms:

Case 1. $E_0 = \langle[\varphi\rangle$. Cancelling E_0 on the left, $T_0 \prec E_1 \preccurlyeq T'$, contradicting 3.3.2.

Case 2. $E_0 = \langle[\varphi\rangle^\frown(^\frown\langle T_\xi\colon \xi < \zeta'\rangle)^\frown T'_{\zeta'}, \phi \prec T'_{\zeta'} \preccurlyeq T_{\zeta'}, \zeta' < \zeta$. Cancelling E_0 on the left, $(T_{\zeta'} - T'_{\zeta'}) \prec E_1 \preccurlyeq T'$. Since $T_{\zeta'} = T'_{\zeta'}{}^\frown(T_{\zeta'} - T'_{\zeta'})$, this contradicts $T_{\zeta'} \in \mathsf{T}$.

Case 3. $E_0 = \langle[\varphi\rangle^\frown(^\frown\langle T_\xi\colon \xi < \zeta'\rangle)$, $\zeta' < \zeta$. Cancelling E_0 on the left, $T_{\zeta'} \prec E_1 \preccurlyeq T'$, contradicting 3.3.2.

The only possibility is $E_0 = \phi$. This completes the proof.

The lemmata of Sect. 2.3 are now available for terms.

3.4.4 *Definition.* A *subterm* of a term T is a consecutive part of T which is itself a term. A *proper subterm* of T is a subterm T_0 such that T has form $E_0{}^\frown T_0{}^\frown E_1$, $E_0 \neq \phi$, $E_1 \neq \phi$.

Note that in the definition of proper subterm, it would have been sufficient to insist that one of E_0, E_1 be non-empty. For if one is empty, the other must be also in view of 3.3.2 and 3.4.3.

3.4.5 *Theorem.* If term T has form $[T_0\psi T_1]$, ψ a special two-place operation symbol, T_0, T_1 terms, then a proper subterm of T is a subterm of T_0 or of T_1. If T has form $[\varphi T_0 \ldots T_\xi \ldots]$, where $\langle T_\xi\colon \xi < \zeta\rangle$ is a ζ-tuple of terms and φ a ζ-place or infinitary operation symbol, then a proper subterm of T is a subterm of some T_ξ.

PROOF: If C_0 is a proper subterm of $[T_0\psi T_1]$, then there are expressions E_0, E_1 such that $T_0{}^\frown\langle\psi\rangle^\frown T_1 = (E_0{}^\frown C_0)^\frown E_1$. Let T_1 consist of the set of all terms together with the expression $\langle\psi\rangle$, let T_2 be the set of all terms. Lemma 3.4.3 implies that condition (C) of Sect. 2.3 is satisfied for T_1, T_2. Since the additional condition on Lemma 2.3.2 is also satisfied, it follows that there is an increasing function g having domain 4, satisfying 2.3.2, 2.3.3 with $A_0 = T_0$, $A_1 = \langle\psi\rangle$, $A_2 = T_1$. Moreover, $g(0) = 0$, $g(3) = 1$, and since $\langle\psi\rangle$ cannot contain C_0 as a part, $g(1) = g(2)$. Therefore, either $g(0) < g(1)$ or $g(2) < g(3)$. In the first case, C_0 is a consecutive part of T_0, in the second, of T_1.

Similarly, if C_0 is a proper subterm of $[\varphi T_0 \ldots T_\xi \ldots]$, then there are expressions E_0, E_1 such that $^\frown\langle T_\xi\colon \xi < \zeta\rangle = (E_0{}^\frown C_0)^\frown E_1$. This time take T_1 and T_2 both to be the set of terms. Again there is an increasing function g satisfying 2.3.2 and 2.3.3. This time, $\mathrm{dom}(g) = \zeta + 1$, $g(0) = 0$, $g(\zeta) = 1$. Since $g(\xi) = \bigcup_{\theta<\xi} g(\theta)$ for $\xi \in \mathrm{Lim}$ by 2.3.5, there must exist ξ such that $g(\xi) < g(\xi + 1)$. This T_ξ contains C_0 as a consecutive part.

3.4.6 *Definition.* The *principal subterms* of a term having form $[T_0\psi T_1]$ are T_0, T_1. The principal subterms of a term $[\varphi T_0 \ldots T_\xi \ldots]$

where φ is a ζ-place or infinitary operation symbol, are $T_0, \ldots, T_\xi, \ldots$.

Note that 3.3.3 implies that principal subterms are unique. The theorem just proved says that a proper subterm of a term T is a subterm of a principal subterm of T.

3.4.7 *Replacement Principle.* Let $\equiv$ be a congruence relation on the algebra of terms. If term $T = {}^\frown\langle E_\nu{}^\frown C_\nu : \nu < \sigma\rangle{}^\frown E_\sigma$, where the C_ν are terms, and if $C_\nu \equiv C'_\nu$ for all $\nu < \sigma$, then $T' = {}^\frown\langle E_\nu{}^\frown C'_\nu : \nu < \sigma\rangle{}^\frown E_\sigma$ is a term and $T \equiv T'$.

PROOF: If $\sigma = 0$, the result is trivial. If $\sigma \neq 0$ and $E_0 = \phi$, then $T = C_0$, $\sigma = 1$, $E_\sigma = \phi$ by 3.3.2. If $\sigma \neq 0$ and $E_\sigma = \phi$, then $\sigma \notin \mathrm{Lim}$ since $\mathrm{dom}(T) \notin \mathrm{Lim}$. Then $T = C_{\sigma-1}$, $\sigma = 1$, $E_0 = \phi$ by 3.4.3. The proof for the non-trivial cases depends on 2.3.3 and the lemma for replacement, 2.3.4. We proceed by induction. For the case $T = [T_0\psi T_1] = {}^\frown\langle E_\nu{}^\frown C_\nu : \nu < \sigma\rangle{}^\frown E_\sigma$ with $E_0 \neq \phi$, $E_\sigma \neq \phi$, E_0 has form $\langle[\rangle{}^\frown E'_0$, E_σ has form $E'_\sigma{}^\frown\langle]\rangle$. Let $E'_\nu = E_\nu$ for $0 < \nu < \sigma$. Then $T_0{}^\frown\langle\psi\rangle{}^\frown T_1 = {}^\frown\langle E'_\nu{}^\frown C_\nu : \nu < \sigma\rangle{}^\frown E'_\sigma$. Take T_1 to be the set of terms together with expression $\langle\psi\rangle$, T_2 to be just the set of terms. Then condition (C) and the additional hypothesis of 2.3.3 and 2.3.4 are satisfied. We know that there is an increasing function g such that $g(0) = 0$, $g(1) = g(2)$ since $\langle\psi\rangle$ does not contain a term, $g(3) = \sigma$, and T_0, T_1 have decompositions of 2.3.3. Replacing C_ν by C'_ν in these forms, the resulting terms T'_0, T'_1 are equivalent to T_0, T_1 by the induction hypothesis. Since $\equiv$ is a congruence relation, $T \equiv [T'_0\psi T'_1]$. But $T' = [T'_0\psi T'_1]$ by 2.3.4. Hence T' is a term equivalent to T.

The proof for the case $T = [\varphi T_0 \ldots T_\xi \ldots]$ is similar.

3.5 Substitution

If T is a term, X a set of individual symbols, f a function on X to expressions of some language, then $S_f^X T$ is the result of replacing parts $\langle x\rangle$, $x \in X$, by $f(x)$. In case f maps X to expressions of length 1, $S_f^X T$ is that expression having the same length as T, and values $fT(\theta)(0)$ for those θ such that $T(\theta) \in X$, $T(\theta)$ otherwise. However it will be necessary to talk about substitutions of more general expressions for $x \in X$. By an *occurrence* of x in T we mean an ordinal $\theta < \mathrm{Dom}(T)$ such that $T(\theta) = x$. If we were to use the procedure for ordinary predicate languages, we would form the sequence $\langle\iota_\nu : \nu < \sigma\rangle$ of all occurrences in T of symbols in X, in increasing order, then write T in the form

$$T = {}^\frown\langle E_\nu {}^\frown \langle T(\iota_\nu)\rangle : \nu < \sigma\rangle {}^\frown E_\sigma,$$

and let $S_f^X T = {}^\frown\langle E_\nu {}^\frown f(T(\iota_\nu)) : \nu < \sigma\rangle {}^\frown E_\sigma$. The only difficulty with this is that in the infinitary case the expressions E_ν are not uniquely determined by the above equation. This difficulty can be avoided by adding the stipulation that the E_ν contain no occurrences of $x \in X$. In fact, the correct E_ν have already been calculated in Lemma 2.2.20.

3.5.1 *Lemma.* Let $\langle \iota_\nu : \nu < \sigma\rangle$ be the sequence of all occurrences of $x \in X$ in T in increasing order. Let $\iota_\sigma = \mathrm{Dom}(T)$ and

$$\begin{aligned} E_\nu &= T|\iota_\nu - (T|\iota_{\nu-1}{}^\frown\langle T(\iota_\nu)\rangle) \text{ if } \nu \notin \mathrm{Lim}, \\ E_\nu &= T|\iota_\nu - T|\bigcup_{\theta<\nu} \iota_\theta \text{ if } \nu \in \mathrm{Lim}. \end{aligned}$$

Then $T = {}^\frown\langle E_\nu {}^\frown \langle T(\iota_\nu)\rangle : \nu < \sigma\rangle {}^\frown E_\sigma$ and whenever

$$T = {}^\frown\langle E'_\nu {}^\frown \langle T(\iota_\nu)\rangle : \nu < \sigma\rangle {}^\frown E'_\sigma$$

and the E'_ν contain no occurrences of $x \in X$, $E_\nu = E'_\nu$ for all $\nu \leqslant \sigma$.

PROOF: Lemma 2.2.20 implies that $T = {}^\frown\langle E_\nu {}^\frown \langle T(\iota_\nu)\rangle : \nu < \sigma\rangle {}^\frown E_\sigma$. If $\nu \notin \mathrm{Lim}$, an occurrence θ of $x \in X$ in E_ν would satisfy $\iota_{\nu-1} + 1 + \theta < \iota_\nu$, impossible since the sequence $\langle \iota_\nu : \nu < \sigma\rangle$ contains all occurrences of $x \in X$ in T. Therefore E_ν contains no occurrence of any $x \in X$. Similarly, if $\nu \in \mathrm{Lim}$ it can be shown that E_ν contains no $x \in X$.

Conversely, if $T = {}^\frown\langle E'_\nu {}^\frown \langle T(\iota_\nu)\rangle : \nu < \sigma\rangle {}^\frown E'_\sigma$ and the E'_ν contain no occurrence of $x \in X$, then we can show $E_\nu = E'_\nu$ by transfinite induction. Suppose $E_\nu = E'_\nu$ for all $\nu < \lambda$ and $\lambda \leqslant \sigma$. Cancelling on the left in the two representations of T, ${}^\frown\langle E_{\lambda+\nu} {}^\frown \langle T(\iota_{\lambda+\nu})\rangle : \nu < \sigma - \lambda\rangle {}^\frown E_\sigma = {}^\frown\langle E'_{\lambda+\nu} {}^\frown \langle T(\iota_{\lambda+\nu})\rangle : \nu < \sigma - \lambda\rangle {}^\frown E'_\sigma$. If $\lambda = \sigma$ the equation yields $E_\lambda = E'_\lambda$ immediately. If $\lambda < \sigma$, then since neither E_λ nor E'_λ contain occurrences of $x \in X$, $\mathrm{Dom}(E_\lambda)$ and $\mathrm{Dom}(E'_\lambda)$ are both the first occurrence of a symbol of X in the expression. Hence $\mathrm{Dom}(E_\lambda) = \mathrm{Dom}(E'_\lambda)$. Since E_λ and E'_λ are initial parts of the same expression, it follows that $E_\lambda = E'_\lambda$. This completes the induction.

3.5.2 *Definition.* Let $\langle \iota_\nu : \nu < \sigma\rangle$ be the sequence of all occurrences of $x \in X$ in T in increasing order, let $\langle E_\nu : \nu \leqslant \sigma\rangle$ be the unique sequence of expressions containing no occurrence of $x \in X$ such that

$$T = {}^\frown\langle E_\nu{}^\frown\langle T(\iota_\nu)\rangle : \nu < \sigma\rangle^\frown E_\sigma.$$

Then $S_f^X T = {}^\frown\langle E_\nu{}^\frown f T(\iota_\nu) : \nu < \sigma\rangle^\frown E_\sigma$.

3.5.3 *Theorem.* (i) $S_f^X\langle y\rangle = \langle y\rangle$ if $y \notin X$, $S_f^X\langle y\rangle = f(y)$ if $y \in X$.

(ii) $S_f^X[T_0\psi T_1] = [S_f^X T_0 \psi S_f^X T_1]$ for special two-place operation symbols ψ.

(iii) $S_f^X[\varphi T_0 \ldots T_\xi \ldots] = [\varphi S_f^X T_0 \ldots S_f^X T_\xi \ldots]$ for ζ-place or infinitary operation symbols φ.

PROOF: (i) is obvious. Since the proofs of (ii) and (iii) are similar, we prove only (iii). Suppose

$$T = [\varphi T_0 \ldots T_\xi \ldots] = {}^\frown\langle E_\nu{}^\frown\langle T(\iota_\nu)\rangle : \nu < \sigma\rangle^\frown E_\sigma$$

where the E_ν contain no occurrences of $x \in X$, $\langle \iota_\nu : \nu < \sigma\rangle$ is the sequence of all occurrences of $x \in X$ in T. Since X was assumed to be a set of individual symbols, it follows that if $E_0 = \phi$ or $E_\sigma = \phi$ then $\sigma = 0$ and $T = E_0 = \phi$, clearly impossible. If $\sigma = 0$ then T contains no occurrences of $x \in X$, a trivial case. We therefore assume $E_0 \neq \phi$, $E_\sigma \neq \phi$ and $\sigma \neq 0$. Let $E_0 = [\varphi E_0'$, $E_\sigma = E_\sigma']$, $E_\nu = E_\nu'$ for $0 < \nu < \sigma$. Then

$$^\frown\langle T_\xi : \xi < \zeta\rangle = {}^\frown\langle E_\nu'{}^\frown\langle T(\iota_\nu)\rangle : \nu < \sigma\rangle^\frown E_\sigma'.$$

Condition (C) of Sect. 2.3 is trivially satisfied by T_1 the set of all terms, T_2 the set of atomic terms $\langle x\rangle$, $x \in X$. Therefore each T_ξ has the representation of 2.3.3:

$T_\xi = E'_{g(\xi)\xi+1} - E'_{g(\xi)\xi}$ if T_ξ has no occurrence of $x \in X$,
$T_\xi = (E'_{g(\xi)} - E'_{g(\xi)\xi})^\frown\langle T(\iota_{g(\xi)})\rangle^\frown E'^\frown E'_{g(\xi+1)\xi+1}$ if T_ξ has an occurrence of $x \in X$, where E' is the concatenation of

$$E'_{g(\xi)+1+\nu}{}^\frown\langle T(\iota_{g(\xi)+1+\nu})\rangle \text{ for } \nu < g(\xi+1) - (g(\xi)+1).$$

These are the representations of Definition 3.5.2 for the T_ξ since the E_ν' contain no occurrence of $x \in X$. Therefore by Lemma 2.3.4,

$$^\frown\langle S_f^X T_\xi : \xi < \zeta\rangle = {}^\frown\langle E_\nu'{}^\frown f T(\iota_\nu) : \nu < \sigma\rangle^\frown E_\sigma'.$$

Hence $[\varphi S_f^X T_0 \ldots S_f^X T_\xi \ldots] = S_f^X T$.

It follows from Theorem 3.5.3 that if f maps X to terms of the given language, then S_f^X is an endomorphism of the algebra of terms.

Let $\langle D\ \mathsf{O}\rangle$ be an algebra for the interpretation of terms, and let s be an assignment of individual symbols to D. Then the value of s^* at $S_f^X T$ is the same as the value at T for the assignment which is like s except that it assigns values $s^*f(x)$ to $x \in X$. This sort of replacement of values in assignments occurs so often when we deal with formal languages, that we introduce a special notation for it.

3.5.4 *Definition.* If s is a function and t a function whose domain includes X, then $\mathrm{Repl}_t^X s$ is that function whose domain is $X \cup \mathrm{Dom}(s)$, assigning values t on X, s on $\mathrm{Dom}(s) \sim X$. In other words, $\mathrm{Repl}_t^X s = (s|\mathrm{Dom}(s) \sim X) \cup t|X$.

3.5.5 *Theorem.* For any assignment s to any algebra $\langle D\ \mathrm{O}\rangle$ for the interpretation of terms, and any term T,

(i) $s^*(T) = s'^*(T)$ if s, s' agree on individual symbols in T.

(ii) $s^*(S_f^X T) = (\mathrm{Repl}_{s^* \circ f}^X s)^*(T)$.

PROOF: Both are trivial for atomic terms T. Since all the operations s^*, s'^*, S_f^X, $(\mathrm{Repl}_{s^* \circ f}^X s)^*$, are homomorphisms on the algebra of terms, the extension to all terms by induction is trivial.

CHAPTER 4

INFINITARY PROPOSITIONAL LANGUAGES

4.1 The α-Propositional Languages

Let α be a regular infinite cardinal. The underlying system of formulas of an α-propositional language $\mathbf{L}_\alpha$ is a system F_α of terms having one-place operation symbol $\neg$, special two-place operation symbol $\rightarrow$, infinitary operation symbol $\bigwedge$. In this context, the individual symbols are called *propositional symbols*, and the terms are called *propositional formulas*. When the induction principle for terms is rewritten to apply to formulas, we have the following:

4.1.1 *Induction Principle for Propositional Formulas.* Let Δ be any set of formulas containing the atomic formulas $\langle p \rangle$ and satisfying

(1) If $A \in \Delta$, then $[\neg A] \in \Delta$,
(2) If $A_0, A_1 \in \Delta$. then $[A_0 \rightarrow A_1] \in \Delta$,
(3) If $0 < \delta < \alpha$ and $A_\xi \in \Delta$ for all $\xi < \delta$, then $[\bigwedge A_0 \ldots A_\xi \ldots] \in \Delta$.

Then Δ is the set of all propositional formulas.

To complete the passage from F_α to language $\mathbf{L}_\alpha$ we must tell how the formulas are to be interpreted. The intended algebra for the interpretation of formulas is $\langle B_0\ \mathrm{O}' \rangle$ where B_0 is the set of truth values $\mathbf{0}$, $\mathbf{1}$, where $\mathrm{O}'(\neg)$, $\mathrm{O}'(\bigwedge)$ are the one-place complementation and $\nearrow \alpha$-place meet operations of $\mathfrak{B}_0$, respectively, and $\mathrm{O}'(\rightarrow)$ is the two-place operation whose value for $\langle a\ b \rangle$ is $\neg a \vee b$. According to the recursion principle for terms, given any assignment s of truth values to the propositional symbols, there is a unique function s^* on the set of all formulas such that $s^*(\langle p \rangle) = s(p)$ for propositional symbols p and

(1) $s^*[\neg A] = \mathbf{1}$ if and only if $s^*(A) = \mathbf{0}$
(2) $s^*[A_0 \rightarrow A_1] = \mathbf{1}$ if and only if $s^*(A_0) = \mathbf{0}$ or $s^*(A_1) = \mathbf{1}$

(3) $s^*[\bigwedge A_0 \ldots A_\xi \ldots] = \mathbf{1}$ if and only if $s^*(A_\xi) = \mathbf{1}$ for all $\xi < \delta$ for all formulas $A, A_0, \ldots, A_\xi, \ldots$.

The additional special two-place operation symbols $\leftrightarrow$, $\wedge$, $\vee$, and the infinitary operation symbol $\bigvee$ are introduced by the customary abbreviations:

$$\begin{aligned}
&[A_0 \leftrightarrow A_1] \text{ for } [\bigwedge [A_0 \rightarrow A_1][A_1 \rightarrow A_0]] \\
&[A_0 \wedge A_1] \text{ for } [\bigwedge A_0 A_1] \\
&[A_0 \vee A_1] \text{ for } [\neg[\bigwedge [\neg A_0][\neg A_1]]] \\
&[\bigvee A_0 \ldots A_\xi \ldots] \text{ for } [\neg[\bigwedge [\neg A_0] \ldots [\neg A_\xi] \ldots]].
\end{aligned}$$

A function s'^* for eliminating these symbols was defined recursively in Chapter 3, Example 3.2.4. When we speak of "the formula $[A_0 \leftrightarrow A_1]$ of $\mathbf{L}_\alpha$, let it be understood that the formula $s'^*([A_0 \leftrightarrow A_1])$ is intended. If s is an assignment of truth values to the propositional symbols, the function s^* extends over formulas with these defined symbols as follows:

(4) $s^*[A_0 \leftrightarrow A_1] = \mathbf{1}$ if and only if $s^*(A_0) = s^*(A_1)$
(5) $s^*[A_0 \wedge A_1] = \mathbf{1}$ if and only if $s^*(A_0) = s^*(A_1) = \mathbf{1}$
(6) $s^*[A_0 \vee A_1] = \mathbf{1}$ if and only if $s^*(A_0) = \mathbf{1}$ or $s^*(A_1) = \mathbf{1}$
(7) $s^*[\bigvee A_0 \ldots A_\xi \ldots] = \mathbf{1}$ if and only if there is $\xi < \delta$ such that $s^*(A_\xi) = \mathbf{1}$.

In this context, s^*A is called the *truth value* of A for assignment s. A formula A is *valid* (written "$\Vdash A$") if and only if its truth value is $\mathbf{1}$ for all assignments. A set Γ of formulas is *satisfiable* if and only if there exists an assignment s to truth values such that $s^*(A) = \mathbf{1}$ for all $A \in \Gamma$. A formula A is a *semantic consequence* of Δ (written "$\Vdash_\Delta A$") if $s^*(A) = \mathbf{1}$ for all assignments s satisfying Δ. A set Γ is *semantically consistent* if and only if every subset of Γ having power less than α is satisfiable. The set Γ has A as a *strict semantic consequence* (written "$\Gamma \Vdash A$") if either $\Vdash A$ or there is a subset $\{A_\xi : \xi < \delta\}$ of Γ having power less than α such that $\Vdash [\bigwedge A_0 \ldots A_\xi \ldots] \rightarrow A$. Clearly a strict semantic consequence of Γ is a semantic consequence of Γ.

It is well-known that a set of ordinary finite propositional formulas is satisfiable if and only if every finite subset of is satisfiable. That is to say, if $\alpha = \omega$, a set of formulas is satisfiable if and only if it is semantically consistent. It is not difficult to see that this is false for infinite non-limit cardinals α.

4.1.2 *Example.* Let γ be an infinite cardinal. Consider a propositional language $\mathbf{L}_{\gamma^+}$ with a doubly-indexed set $p_{\mu\nu}$, $\mu < \gamma^+$, $\nu < \gamma$, of propositional symbols. It is easy to see that the set

$$\Gamma_\gamma = \{[\bigvee_{\nu<\gamma} p_{\mu 0} \ldots p_{\mu\nu} \ldots] : \mu < \gamma^+\} \cup \{[\neg[p_{\mu\nu} \wedge p_{\mu'\nu}] : \mu \neq \mu', \mu, \mu' < \gamma^+, \nu < \gamma\}$$

is semantically consistent but not satisfiable. This amounts to saying that any set having power at most γ can be mapped one-one into γ, while a set having power γ^+ cannot be. Note that card $(\Gamma_\gamma) = \gamma^+$.

In the preceding example, an effort was made to keep the cardinal of Γ down to γ^+. If we allow card(Γ) to be 2 exp γ, then we need only γ propositional symbols.

4.1.3 *Example.* Let γ be an infinite cardinal. Consider a propositional language $\mathbf{L}_{\gamma^+}$ with a doubly-indexed set $p_{\mu\nu}$, $\mu < \gamma$, $\nu < 2$, of propositional symbols. Then the set

$$\Gamma = \{[\bigwedge_{\mu<\gamma} [p_{00} \vee p_{01}] \ldots [p_{\mu 0} \vee p_{\mu 1}] \ldots]\} \cup \{[\neg[\bigwedge_{\mu<\gamma} p_{0f(0)} \ldots p_{\mu f(\mu)} \ldots]] : f \in 2^\gamma\}$$

is semantically consistent but not satisfiable.

Examples of semantically consistent, not satisfiable, sets of formulas of a full (α, α)-predicate language have been given by Hanf for a wide class of strongly inaccessible cardinals greater than ω. These sets can be reduced to sets of α-propositional formulas with this property, but they are much too complicated to describe here. Hanf's example is given in Chapter 10, Sect. 10.2, for the first strong inaccessible greater than ω. The reduction is given in Sect. 10.4. For those inaccessibles for which those methods fail, the problem remains open.

The useful lemma that follows, is a generalization of the propositional form of a lemma in Henkin's completeness paper [9].

4.1.4 *Criterion for Satisfiability.* Let $\mathbf{L}_\alpha$ be an α-propositional language. Then a set Γ of formulas is satisfiable if and only if there is a set $\bar{\Gamma} \supseteq \Gamma$ such that if Δ is the set of all sub-formulas of formulas in Γ,

(1) If $A \in \Delta$, $A \in \bar{\Gamma}$ iff $[\neg A] \notin \bar{\Gamma}$,
(2) If $[A_0 \rightarrow A_1] \in \Delta$, then $[A_0 \rightarrow A_1] \in \bar{\Gamma}$ iff $A_0 \notin \bar{\Gamma}$ or $A_1 \in \bar{\Gamma}$,
(3) If $[\bigwedge A_0 \ldots A_\xi \ldots] \in \Delta$, then $[\bigwedge A_0 \ldots A_\xi \ldots] \in \bar{\Gamma}$ iff $A_\xi \in \bar{\Gamma}$ for all $\xi < \delta$.

PROOF: If assignment s to truth values satisfies Γ, then obviously the set $\bar{\Gamma} = \{A : s^*(A) = \mathbf{1}\}$ has properties (1), (2), (3) and includes Γ. Conversely, if $\bar{\Gamma} \supseteq \Gamma$ has properties (1), (2), (3), let s be the assignment giving p value $\mathbf{1}$ if $\langle p \rangle \in \bar{\Gamma}$, $\mathbf{0}$ if not. Let $\Delta' = \{A$: If $A \in \Delta$, then $A \in \bar{\Gamma}$ iff $s^*(A) = \mathbf{1}\}$. Clearly Δ' contains the atomic formulas $\langle p \rangle$ and is closed under the rules of formation of α-propositional formulas. By the induction principle, 4.1.1, Δ' contains all formulas. Since $\bar{\Gamma} \supseteq \Gamma$, s satisfies Γ.

4.2 Algebras of Equivalence Classes of Formulas Modulo a Semantically Consistent Set

Let Γ be a semantically consistent set of formulas of an α-propositional language $\mathbf{L}_\alpha$. Then the relation

$$A \equiv A' \text{ iff } \Gamma \Vdash A \leftrightarrow A'$$

is easily seen to be a congruence relation on the algebra of formulas. Therefore if we let $|A|_\Gamma = \{A' : A \equiv A'\}$, and let $B(\mathbf{L}_\alpha; \Gamma)$ be the set of all these equivalence classes, we obtain a homomorphic image of the algebra of formulas having operations

$$\neg|A|_\Gamma = |\neg A|_\Gamma,\ |A_0|_\Gamma \to |A_1|_\Gamma = |[A_0 \to A_1]|_\Gamma$$
$$\bigwedge (\langle |A_0|_\Gamma \ldots |A_\xi|_\Gamma \ldots \rangle) = |[\bigwedge A_0 \ldots A_\xi \ldots]|_\Gamma,$$

the latter being an $\nearrow \alpha$-place operation. The defined propositional operation symbols yield the additional operations $|A_0|_\Gamma \wedge |A_1|_\Gamma = |[A_0 \wedge A_1]|_\Gamma$, $|A_0|_\Gamma \vee |A_1|_\Gamma = |[A_0 \vee A_1]|_\Gamma$. Let $\mathfrak{B}(\mathbf{L}_\alpha; \Gamma)$ be the algebra $\langle B(\mathbf{L}_\alpha; \Gamma) \neg \wedge \vee \rangle$. The semantic consistency of Γ implies that the algebra has at least two elements.

4.2.1 *Theorem.* (i) If Γ semantically consistent, then $\mathfrak{B}(\mathbf{L}_\alpha; \Gamma)$ is a non-degenerate $\nearrow \alpha$-representable Boolean algebra.

(ii) If Γ satisfiable and $\bar{\Gamma} = \{A : \Vdash_\Gamma A\}$, then $\mathfrak{B}(\mathbf{L}_\alpha; \bar{\Gamma})$ is isomorphic to a non-degenerate $\nearrow \alpha$-field of sets.

PROOF: It is convenient to prove (ii) first. Suppose Γ is satisfiable. Let S be the set of those assignments satisfying Γ, and for each formula A, let $h(A) = \{s: s \in S$ and s satisfies $A\}$. Then $h([\neg A]) = S \sim h(A)$, $h([A_0 \wedge A_1]) = h(A_0) \cap h(A_1)$, $h([A_0 \vee A_1]) = h(A_0) \cup h(A_1)$, and $h([\bigwedge A_0 \ldots A_\xi \ldots]) = \bigcap \{h(A_\xi) : \xi < \delta\}$. Thus $\langle \mathrm{Rng}(h) \sim \cap \cup \rangle$ is an $\nearrow \alpha$-field of sets. The set $\bar{\Gamma}$ was chosen so that $\bar{\Gamma} \Vdash A \leftrightarrow A'$ iff $h(A) = h(A')$. Theretore the mapping g such that $g(|A|_{\bar{\Gamma}}) = h(A)$ is a one-one function on $B(\mathbf{L}_\alpha; \bar{\Gamma})$ onto $\mathrm{Rng}(h)$. It is obviously an isomorphism.

To prove (i), take $\Gamma = \phi$ in (ii). Then the set $\bar{\Gamma}$ of (ii) is the set of valid formulas. The mapping $g(|A|_{\bar{\Gamma}}) = g(|A|_{\phi}) = |A|_{\Gamma}$ is an $\nearrow\alpha$-homomorphism from $\mathfrak{B}(\mathbf{L}_\alpha; \bar{\Gamma})$ onto $\mathfrak{B}(\mathbf{L}_\alpha; \Gamma)$.

The order in $\mathfrak{B}(\mathbf{L}_\alpha; \Gamma)$ is $|A|_\Gamma \leqslant |A'|_\Gamma$ iff $\Gamma \Vdash [A \rightarrow A']$, the zero is $|[p \wedge [\neg p]]|_\Gamma$, the unit $|[p \vee [\neg p]]|_\Gamma$, the $\nearrow\alpha$-place meet and join are

$$\wedge \{|A_\xi|_\Gamma : \xi < \delta\} = |[\wedge A_0 \ldots A_\xi \ldots]|_\Gamma, \vee \{|A_\xi|_\Gamma : \xi < \delta\} = |[\vee A_0 \ldots A_\xi \ldots]|_\Gamma.$$

If Γ is a semantically consistent, non-satisfiable set of formulas of $\mathbf{L}_\alpha$, then $\mathfrak{B}(\mathbf{L}_\alpha; \Gamma)$ is an example of an $\nearrow\alpha$-representable Boolean algebra having no $\nearrow\alpha$-homomorphism to $\mathfrak{B}_0$. For if it had such a homomorphism h, we could form the set $\bar{\Gamma} = \{A : h|A|_\Gamma = \mathbf{1}\}$. This set would contain Γ since $|A|_\Gamma = \mathbf{1}$ for $A \in \Gamma$, and it would have properties (1), (2), (3) of the criterion for satisfiability, 4.1.4. Γ would therefore be satisfiable. Thus we have

4.2.2 *Theorem.* If Γ is a semantically consistent, non-satisfiable set of formulas, then $\mathfrak{B}(\mathbf{L}_\alpha; \Gamma)$ has no maximal $\nearrow\alpha$-complete ideal. (It is therefore not isomorphic to an $\nearrow\alpha$-field of sets.)

In the next section, we show how to construct semantically consistent non-satisfiable sets of formulas from $\nearrow\alpha$-representable Boolean algebras that are not isomorphic to $\nearrow\alpha$-fields of sets. The correspondence therefore is one-one. Thus we see clearly that the theorem that says that every semantically consistent set of ordinary ω-propositional formulas is satisfiable, is equivalent to the theorem that every homomorphic image of a field of sets is isomorphic to a field of sets. This theorem is, of course, weaker than the Stone Representation Theorem that says that every Boolean algebra is isomorphic to a field of sets. The ω-propositional equivalent to Stone's Theorem does not enter the picture until the propositional calculi are introduced.

Let us look at the algebras of equivalence classes generated by the semantically consistent, non-satisfiable sets of formulas in Examples 4.1.2 and 4.1.3.

4.2.3 *Example.* Let $b_{\mu\nu} = |p_{\mu\nu}|_{\Gamma_\gamma}$ for $\mu < \gamma^+, \nu < \gamma$, in the algebra $\mathfrak{B}(\mathbf{L}_{\gamma^+}; \Gamma_\gamma)$ for the language and set Γ_γ of Example 4.1.2. Then $\bigvee_{\nu<\gamma} b_{\mu\nu} = \mathbf{1}$ for all $\mu < \gamma^+$ and $b_{\mu\nu} \wedge b_{\mu'\nu} = \mathbf{0}$ for all $\mu \neq \mu'$, μ, $\mu' < \gamma^+$, $\nu < \gamma$. It is apparent that this γ-representable Boolean algebra cannot be isomorphic to a γ-field of sets.

In particular, consider the case $\gamma = \omega$, and represent $\mathfrak{B}(\mathbf{L}_{\omega_1}; \Gamma_\omega)$ as a regular subalgebra of a complete algebra $\mathfrak{B}^*$. Then $\mathfrak{B}^*$ does not satisfy the inequality

$$\bigwedge_{\mu<\omega_1} \bigvee_{\nu<\omega} x_{\mu\nu} \leqslant \bigvee_{\mu\neq\mu'<\omega_1} \bigvee_{\nu<\omega} x_{\mu\nu} \wedge x_{\mu'\nu}.$$

Since this inequality would have to hold in an ω_1-representable Boolean algebra, it is clear that $\mathfrak{B}^*$ is an example of a complete Boolean algebra that is not ω_1-representable. This is the first such example given without the aid of the continuum hypothesis. The extension to arbitrary regular cardinals α is given in the author's note [16] and in Chapter 7.

4.2.4 *Example.* Let $b_{\mu\nu} = |p_{\mu\nu}|_\Gamma$ for $\mu < \gamma$, $\nu < 2$, in the algebra $\mathfrak{B}(\mathbf{L}_{\gamma^+}; \Gamma)$ for the language and set Γ of Example 4.1.3. Then $\bigwedge_{\mu<\gamma} (b_{\mu0} \vee b_{\mu1}) = \mathbf{1}$ and $\bigwedge_{\mu<\gamma} b_{\mu f(\mu)} = \mathbf{0}$ for all $f \in 2^\gamma$. These algebras are therefore γ-representable, but not $(\gamma, 2)$-distributive. Such examples are familiar from the literature. It is more difficult to give such examples that are complete Boolean algebras. For regular γ, such examples appear in Smith [39], Scott [36]. The algebras in Chapter 7 are also examples. Whether or not there exists a complete γ-representable, not $(\gamma, 2)$-distributive Boolean algebra seems to be an open question for singular infinite cardinals γ.

4.3 Interpreting α-Propositional Languages in $\nearrow\alpha$-complete Boolean Algebras

The idea of interpreting ordinary propositional formulas in algebras other than the algebra of truth values, goes back to the early matrix methods for proving non-deducibility in formal systems. The modal and intuitionistic propositional calculi are treated algebraically in McKinsey-Tarski [22], 1948, and Rieger [32], 1949.

Let $\mathfrak{B} = \langle B \neg \wedge \vee\rangle$ be an $\nearrow\alpha$-complete Boolean algebra. When formulas of an α-propositional language $\mathbf{L}_\alpha$ are interpreted in $\mathfrak{B}$ we understand that the interpretive algebra is $\langle B\, \mathsf{O}'\rangle$, where $\mathsf{O}'(\neg)$, $\mathsf{O}'(\bigwedge)$ are the one-place complementation and $\nearrow\alpha$-place meet operations of $\mathfrak{B}$, respectively, and $\mathsf{O}'(\rightarrow)$ is the two-place operation assigning $\neg a \vee b$ to $\langle a\, b\rangle$. According to the recursion principle for terms, given any assignment s of propositional symbols to B, there is a unique function s^* on the set of all formulas to B such that $s^*(\langle p\rangle) = s(p)$ for propositional symbols p, and the following conditions hold for all formulas $A, A_0, \ldots, A_\xi, \ldots$:

(1) $s^*[\neg A] = \neg s^* A$
(2) $s^*[A_0 \rightarrow A_1] = \neg s^* A_0 \vee s^* A_1$
(3) $s^*[\bigwedge A_0 \ldots A_\xi \ldots] = \bigwedge \{s^* A_\xi : \xi < \delta\}$.

Making use of simple properties of Boolean algebras, it follows that s^* also satisfies

(4) $s^*[A_0 \leftrightarrow A_1] = (\neg s^* A_0 \vee s^* A_1) \wedge (\neg s^* A_1 \vee s^* A_0)$
(5) $s^*[A_0 \wedge A_1] = s^* A_0 \wedge s^* A_1$.
(6) $s^*[A_0 \vee A_1] = s^* A_0 \vee s^* A_1$
(7) $s^*[\bigvee A_0 \ldots A_\xi \ldots] = \bigvee \{s^* A_\xi : \xi < \delta\}$.

We say that s *satisfies* Γ *in* $\mathfrak{B}$ if $s^* A = \mathbf{1}$ for all $A \in \Gamma$. Formula A *holds in* $\mathfrak{B}$ if $s^* A = \mathbf{1}$ for all assignments s to $\mathfrak{B}$.

Note that the intended interpretation of $\mathbf{L}_\alpha$ is the same as the interpretation (in the algebraic sense) in $\mathfrak{B}_0$. If s is an assignment to $\mathfrak{B}$, then clearly $\langle \mathrm{Rng}(s^*) \neg \wedge \vee \rangle$ is the $\nearrow \alpha$-complete $\nearrow \alpha$-subalgebra of $\mathfrak{B}$ generated by $\mathrm{Rng}(s)$.

A trivial induction on formulas yields

4.3.1 *Lemma.* If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are $\nearrow \alpha$-complete Boolean algebras and h is an $\nearrow \alpha$-homomorphism from $\mathfrak{B}_1$ onto $\mathfrak{B}_2$, then $h(s^* A) = (h \bigcirc s)^* A$ for all assignments s to $\mathfrak{B}_1$, formulas A.

4.3.2 *Lemma.* Let Γ be a semantically consistent set of formulas of $\mathbf{L}_\alpha$. Then the assignment $s(p) = |\langle p \rangle|_\Gamma$ to $\mathfrak{B}(\mathbf{L}_\alpha; \Gamma)$ yields the value $s^* A = |A|_\Gamma$ for each formula A.

It follows that every semantically consistent set of formulas is satisfiable in an $\nearrow \alpha$-representable Boolean algebra. Therefore, according to Examples 4.1.2 and 4.1.3, for every infinite non-limit cardinal α, and for certain strongly inaccessible cardinals α as well, there exist sets of formulas which are satisfiable in an $\nearrow \alpha$-representable Boolean algebra, but not in the algebra of truth values. It turns out that satisfiability in $\nearrow \alpha$-representable Boolean algebras is equivalent to the intended semantic satisfiability for single formulas. For sets of formulas, semantic satisfiability is equivalent to satisfiability in $\nearrow \alpha$-fields of sets.

4.3.3 *Theorem.* Let Γ be a set of formulas of an α-propositional language $\mathbf{L}_\alpha$, let $\mathfrak{S}$ be a non-degenerate $\nearrow \alpha$-field of sets. Then the following conditions are equivalent:

(i) Γ is semantically satisfiable (i.e., in $\mathfrak{B}_0$)
(ii) Γ is satisfiable in $\mathfrak{S}$

(iii) There is an assignment s to $\mathfrak{S}$ such that $\{s^*A : A \in \Gamma\}$ has a non-empty lower bound.

PROOF: Taking implications in the order (ii) to (iii) to (i) to (ii), we see that the only non-trivial one is (iii) implies (i). Suppose $\phi \neq b \leqslant s^*A$ for all $A \in \Gamma$. Then we know that there is an $\nearrow\alpha$-homomorphism on $\mathfrak{S}$ to $\mathfrak{B}_0$ such that $h(b) = \mathbf{1}$ (see Foreward on Algebra). Then $h(s^*A) = (h \bigcirc s)^*A = \mathbf{1}$ for all $A \in \Gamma$. Hence (i).

4.3.4 *Theorem.* Let A be a formula of an α-propositional language $\mathbf{L}_\alpha$, let $\mathfrak{B}$ be a non-degenerate $\nearrow\alpha$-representable Boolean algebra. Then the following conditions are equivalent:

(i) A is semantically satisfiable.

(ii) A is satisfiable in $\mathfrak{B}$.

(iii) There is an assignment s to $\mathfrak{B}$ such that $s^*A \neq \mathbf{0}$.

PROOF: Again, the only non-trivial implication is (iii) implies (i). To each subformula of A of the form $[\bigwedge A_0 \ldots A_\xi \ldots]$ associate the meet $\bigwedge \{s^*A_\xi : \xi < \delta\}$ of $\mathfrak{B}$. Since A has fewer than α subformulas, there are fewer than α such meets, each having fewer than α terms. According to the theorem proved in the Foreward on Algebra, there is a homomorphism h on $\mathfrak{B}$ to $\mathfrak{B}_0$ mapping s^*A to $\mathbf{1}$ and preserving all these meets. Then $h(s^*[\bigwedge A_0 \ldots A_\xi \ldots]) = (h \bigcirc s)^*[\bigwedge A_0 \ldots A_\xi \ldots]$ whenever $[\bigwedge A_0 \ldots A_\xi \ldots]$ is a subformula of A. Since h is a homomorphism, we also have $h(s^*[\neg A]) = (h \bigcirc s)^*[\neg A]$ and $h(s^*[A_0 \rightarrow A_1]) = (h \bigcirc s)^*[A_0 \rightarrow A_1]$ for all formulas A, A_0, A_1. Clearly $\bar{\Gamma} = \{C : h(s^*C) = \mathbf{1}\}$ contains A and satisfies conditions (1), (2), (3) of the criterion for satisfiability, 4.1.4. Hence A is satisfiable.

As an application of the algebraic method, we show how to construct semantically consistent, non-satisfiable sets of formulas from $\nearrow\alpha$-representable Boolean algebras not isomorphic to $\nearrow\alpha$-fields of sets.

4.3.5 *Theorem.* Let α be a regular infinite cardinal, γ an arbitrary infinite cardinal. Then the following conditions are equivalent:

(i) There is a semantically consistent, non-satisfiable set of formulas of an α-propositional language with γ propositional symbols.

(ii) There is an $\nearrow\alpha$-representable Boolean algebra, $\nearrow\alpha$-generated by a subset of power γ, not isomorphic to an $\nearrow\alpha$-field of sets.

PROOF: Assume (i). Then (ii) follows by Theorem 4.2.2 since $\mathfrak{B}(\mathbf{L}_\alpha; \Gamma)$ is $\nearrow\alpha$-generated by elements $|\langle p \rangle|_\Gamma$. If the number of classes $|\langle p \rangle|_\Gamma$ reduces to fewer than γ, adjoin γ new symbols p'_ν,

$\nu < \gamma$. For them we will have $|\langle p'_\nu \rangle| \neq |\langle p'_\nu \rangle|$ for $\nu \neq \nu'$. Hence (ii).

Suppose $\mathfrak{B} = \langle B \neg \wedge \vee \rangle$ is $\nearrow \alpha$-representable, not isomorphic to an $\nearrow \alpha$-field of sets, and has an $\nearrow \alpha$-generating set C of power γ. Form the α-propositional language $\mathbf{L}_\alpha$ with one propositional symbol p_c for each $c \in C$. Interpret $\mathbf{L}_\alpha$ in $\mathfrak{B}$ by assignment $t(p_c) = c$. Then $\mathrm{Rng}(t^*) = B$. Since $\mathfrak{B}$ is not isomorphic to an $\nearrow \alpha$-field of sets, B has an element $b_0 \neq \mathbf{0}$ such that there is no homomorphism h on $\mathfrak{B}$ to $\mathfrak{B}_0$ mapping b_0 to $\mathbf{1}$. Let $\Gamma = \{A : A$ is a formula of $\mathbf{L}_\alpha$ and $t^*A \geqslant b_0\}$. Then Γ is semantically consistent by Theorem 4.3.4, for if $A_0, \ldots, A_\xi, \ldots$ are in Γ, $\xi < \delta$, $0 < \delta < \alpha$, $t^*[\bigwedge A_0 \ldots A_\xi \ldots] \geqslant b_0 \neq \mathbf{0}$. Thus every subset having power less than α is satisfiable. If, contrariwise, there were an assignment s to $\mathfrak{B}_0$ satisfying Γ, then the mapping $h(t^*A) = s^*A$ would define an $\nearrow \alpha$-homomorphism to $\mathfrak{B}_0$ taking b_0 to $\mathbf{1}$. Note that h would be a function since $t^*A = t^*A'$ implies $[A \leftrightarrow A'] \in \Gamma$, hence $s^*A = s^*A'$.

4.3.6 *Theorem*. Let α be a regular infinite cardinal. Then the following conditions are equivalent:

(i) There is a semantically consistent, non-satisfiable set of α-propositional formulas.

(ii) There is an $\nearrow \alpha$-representable Boolean algebra having no $\nearrow \alpha$-complete maximal ideal.

(iii) There is an $\nearrow \alpha$-representable Boolean algebra not isomorphic to an $\nearrow \alpha$-field of sets.

PROOF: By 4.3.5 and 4.2.2.

Theorem 10.4.3 says that the existence of semantically consistent, non-satisfiable sets of formulas of the full (α, α)-predicate languages is also equivalent to these conditions.

INFINITARY PROPOSITIONAL LOGIC

It has already been explained (Chapter 1, Sect. 1.2) that we are only going to consider Hilbert-style formal systems. Therefore, an α-propositional formal system is to consist of an α-propositional language, a set of its formulas singled out to serve as axioms, a collection of closure conditions, each with fewer than α premisses, to serve as rules of inference. A formula A is provable (written "$\vdash A$") if and only if there is a sequence of fewer than α formulas (called a *formal proof*), the last of which is A, such that each formula in the sequence is either an axiom, or is the result of applying a rule of inference to formulas appearing earlier. A formula A is *provable from Δ* (written "$\vdash_\Delta A$") if and only if there is a formal proof of A when formulas of Δ are added to the set of axioms. A system is *complete* if $\vdash A$ if and only if $\Vdash A$ for all formulas A. A system is *strongly complete* if $\vdash_\Delta A$ if and only if $\Vdash_\Delta A$ for all formulas A and sets of formulas Δ.

We have already encountered α-propositional languages for which no strongly complete formal system exists. Let $\mathbf{L}_\alpha$ be any α-propositional language having a set Δ of formulas which is semantically consistent but not satisfiable. Then $\Vdash_\Delta \neg[p \to p]$ since the set of assignments to $\mathfrak{P}_0$ satisfying Δ is empty. If there were a strongly complete formal system for $\mathbf{L}_\alpha$, $\vdash_\Delta \neg[p \to p]$ in this system. Since fewer than α formulas would appear in the formal proof, Δ would have a subset Δ' of power less than α such that $\vdash_{\Delta'} \neg[p \to p]$. But then $\Vdash_{\Delta'} \neg[p \to p]$, contradicting the satisfiability of Δ'. It follows that there is no strongly complete formal system for languages $\mathbf{L}_{\gamma^+}$, γ infinite, having γ or more propositional symbols. For γ propositional symbols could be doubly-indexed to yield the semantically consistent, non-satisfiable set of Example 4.1.3. If, however, $\mathbf{L}_{\gamma^+}$ has $\gamma' < \gamma$ propositional symbols, and if $2 \exp \gamma' \leqslant \gamma$, then there is a strongly complete formal system for $\mathbf{L}_{\gamma^+}$. See Theorem 5.5.4. Ordinary classical propositional systems are known to be strongly complete for languages $\mathbf{L}_\omega$, but no strongly complete formal system exists for languages $\mathbf{L}_\alpha$, α strongly inaccessible and greater than ω, having α propositional symbols, for which examples like Hanf's in Chapter 10, Sect. 10.2, can be found.

5.1 Description of the Formal Systems for α-Propositional Languages

Let α be a regular infinite cardinal. The systems will be described in terms of a special α-propositional language called the *α-propositional scheme language*, which will be used again to describe the propositional part of the full formal systems for (α, β)-predicate languages. The scheme language is an ordinary α-propositional language with propositional symbols A_ξ, $\xi < \alpha$, $A_{\mu\nu}$, $\mu < \alpha$, $\nu < \alpha$. A formula of this language is called an *α-propositional scheme.* Note that symbols "A_ξ", "$A_{\mu\nu}$" are now used in two ways: First, they are still being used informally to stand for arbitrary formulas of a specific language; secondly, they are used as primitive symbols of the scheme language. An *instance* of scheme $\mathscr{A}$ in $\mathbf{L}_\alpha$ is a substitution $S_f^V\mathscr{A}$, where V is a set containing all the propositional symbols appearing in $\mathscr{A}$ and f maps V to formulas of $\mathbf{L}_\alpha$. In view of the substitution properties, "A is an instance of scheme $[A_0 \to [A_1 \to A_0]]$" has the same meaning as "A has form $[A_0 \to [A_1 \to A_0]]$". In the first case, the A_0, A_1 are used as symbols of the scheme-language; in the second case, they are used informally to stand for formulas of $\mathbf{L}_\alpha$.

5.1.1 *Basic α-propositional Calculus* $\mathfrak{P}_\alpha$. The *axiom schemes* of $\mathfrak{P}_\alpha$ are the following:

$$\mathscr{I}_1 = [A_0 \to [A_1 \to A_0]]$$
$$\mathscr{I}_2 = [[A_0 \to [A_1 \to A_2]] \to [[A_0 \to A_1] \to [A_0 \to A_2]]]$$
$$\mathscr{N} = [[[\neg A_0] \to [\neg A_1]] \to [A_1 \to A_0]]$$
$$\mathscr{C}_{1,\delta} = [[\bigwedge [A_\delta \to A_0] \ldots [A_\delta \to A_\xi] \ldots] \to [A_\delta \to [\bigwedge A_0 \ldots A_\xi \ldots]]]$$

where sequence $\langle A_0 \ldots A_\xi \ldots \rangle$ has length δ, $0 < \delta < \alpha$.

$\mathscr{C}_{2,\delta,\nu} = [[\bigwedge A_0 \ldots A_\xi \ldots] \to A_\nu]$, where sequence $\langle A_0 \ldots A_\xi \ldots \rangle$ has length δ, $0 < \delta < \alpha$, $\nu < \delta$.

The *rules of inference* are:

Modus Ponens: From A_0, $[A_0 \to A_1]$ infer A_1.

Conjunction: From A_ξ for all $\xi < \delta$ infer $[\bigwedge A_0 \dots A_\xi \dots]$, where $0 < \delta < \alpha$.

If $\mathbf{L}_\alpha$ is an α-propositional language, the formal system $\mathfrak{P}_\alpha(\mathbf{L}_\alpha)$ has as axioms all instances of axiom schemes of $\mathfrak{P}_\alpha$, and as rules of inference, modus ponens and conjunction. If Σ is any set of α-propositional schemes, $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ is the system like $\mathfrak{P}_\alpha(\mathbf{L}_\alpha)$ except that it has all instances of schemes in Σ added to the set of axioms. The calculus $\mathfrak{P}_\alpha(\Sigma)$ is called *complete* if all systems $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ are complete.

5.1.2 *Distributive Laws.* For each cardinal γ let there be given a fixed ordering $\langle g_\xi : \xi < 2 \exp \gamma \rangle$ of the set γ^γ of all functions on γ to γ. Moreover, to improve readability, let expressions $[\bigwedge E_0 \dots E_\xi \dots]$ be written

$$[\bigwedge_{\xi<\delta} E_\xi],$$

where δ is the length of the sequence $\langle E_0 \dots E_\xi \dots \rangle$. Similarly for $\bigvee$. Then

$$\mathscr{D}_\gamma = [[\bigwedge_{\mu<\gamma} [\bigvee_{\nu<\gamma} A_{\mu\nu}]] \to [\bigvee_{\xi<2\exp\gamma} [\bigwedge_{\mu<\gamma} A_{\mu g_\xi(\mu)}]]].$$

Note that $\mathscr{D}_\gamma$ is an α-propositional scheme only for $\alpha \geqslant (2 \exp \gamma)^+$.

The set of instances of $\mathscr{D}_\gamma$ does not satisfy the criterion of definability, 1.2.1, Chapter 1, as it stands, because we have no way of defining an ordering of γ^γ. However, the set of all instances of all schemes $\mathscr{D}_\gamma$ for all possible orderings $\langle g_\xi : \xi < 2 \exp \gamma \rangle$ of γ_γ is definable, and all these schemes are provable from $\mathscr{D}_\gamma$. We have offered the single scheme for the fixed ordering only to emphasize the point that only one scheme is needed on each level γ. This is in contrast with the distributive laws of Chang, where for most γ, it is an open question whether or not the set of schemes at level γ can be reduced to one.

5.1.3 *Chang's Distributive Laws.* Let γ be an infinite cardinal number. Then Π_γ is the collection of all schemes

$$[\bigvee_{\mu<\gamma} [\bigwedge_{\nu<\gamma} \mathscr{A}_{\mu\nu}]]$$

where each $\mathscr{A}_{\mu\nu}$ is either A_ξ or $[\neg A_\xi]$ for some $\xi < \gamma$, and for every $g \in \gamma^\gamma$, the set $\{\mathscr{A}_{\mu g(\mu)} : \mu < \gamma\}$ contains a complementary pair A_ξ, $[\neg A_\xi]$ for some $\xi < \gamma$.

Note that every scheme in Π_γ is a γ^+-propositional scheme, and therefore is also an α-propositional scheme for every $\alpha \geqslant \gamma^+$.

We will show that the set of all instances of Π_γ satisfies the criterion of definability for $\gamma^+ = \alpha$, but it is doubtful that it would satisfy a criterion of effectiveness.

As an example of a scheme in Π_γ, let $\gamma = 2 \exp \gamma'$, and let $\mathscr{A}_{\mu\nu}$ be given by the following diagram for $\mu < \gamma$, $\nu < \gamma$, where $\{g_\xi \colon \xi < \gamma\} = \gamma'^{\gamma'}$, the ordering of 5.1.2:

	$\nu < \gamma'$	$\gamma' \leqslant \nu < \gamma$
$\mu < \gamma'$	$\mathscr{A}_{\mu\nu} = [\neg A_{\mu\nu}]$	$\mathscr{A}_{\mu\nu} = [\neg A_{\mu 0}]$
$\xi < \gamma$	$\mathscr{A}_{\gamma'+\xi,\nu} = A_{\nu g_\xi(\nu)}$	$\mathscr{A}_{\gamma'+\xi,\nu} = A_{0 g_\xi(0)}$

Every set $\{\mathscr{A}_{\mu g(\mu)} \colon \mu < \gamma\}$ contains a formula $[\neg A_{\mu\nu_\mu}]$ from each of the first γ' rows. It must also contain a formula $A_{\mu g_\xi(\mu)}$ from that row such that $g_\xi(\mu) = \nu_\mu$ for all $\mu < \gamma'$. Hence every such set contains a complementary pair. It requires only a little development of the basic systems to see that it follows that every instance of $\mathscr{D}_{\gamma'}$ in $\mathbf{L}_\alpha$ is provable in $\mathfrak{P}_\alpha(\Pi_\gamma)(\mathbf{L}_\alpha)$ for $\alpha \geqslant \gamma^+$.

The Chang distributive laws are a translation of the algebraic conditions of Chang in [1] for Boolean α-representability. The systems $\mathfrak{P}_\alpha(\bigcup_{\gamma<\alpha} \Pi_\gamma)(\mathbf{L}_\alpha)$ are nearly the same as the systems formulated by Scott and Tarski in [37].

5.1.4 *Lemma.* Let $\mathfrak{B}$ be an $\nearrow \alpha$-complete Boolean algebra, s any assignment of propositional symbols of $\mathbf{L}_\alpha$ to $\mathfrak{B}$. Then if $A = S^V_f \mathscr{A}$ is an instance of α-propositional scheme $\mathscr{A}$, $s^*A = (\mathrm{Repl}^V_{s^* \circ f} s)^* \mathscr{A}$.

PROOF: By 3.5.5.

5.1.5 *Theorem.* Let $\mathbf{L}_\alpha$ be any α-propositional language. Then

(i) If $\mathscr{A}$ is a valid α-propositional scheme, any instance of $\mathscr{A}$ in $\mathbf{L}_\alpha$ is valid. Hence if Σ is a set of valid α-propositional schemes, and if Δ is any set of formulas of $\mathbf{L}_\alpha$, $\vdash_\Delta A$ in $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ implies $\Vdash_\Delta A$.

(ii) The axiom schemes of $\mathfrak{P}_\alpha$, the schemes $\mathscr{D}_\gamma$, all schemes in Π_γ, are valid.

PROOF: If scheme $\mathscr{A}$ is valid, then $A = S_f^V\mathscr{A}$ is valid by 5.1.4, since validity is equivalent to validity in $\mathfrak{B}_0$. If s is any assignment to truth values, then clearly $s^*(A_0) = s^*[A_0 \rightarrow A_1] = \mathbf{1}$ implies $s^*(A_1) = \mathbf{1}$, and $s^*(A_\xi) = \mathbf{1}$ for all $\xi < \delta$ implies $s^*[\bigwedge A_0 \ldots A_\xi \ldots] = \mathbf{1}$, for all formulas $A_0, \ldots, A_\xi, \ldots$ of $\mathbf{L}_\alpha$. Therefore, if s satisfies Δ, then s satisfies all formulas of a formal proof of A from Δ in $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$. Hence $s^*(A) = \mathbf{1}$.

Axiom schemes of $\mathfrak{P}_\alpha$ are clearly valid, and, using the axiom of choice, it is also clear that schemes $\mathscr{D}_\gamma$ are valid. Suppose

$$\mathscr{A}' = [\bigvee_{\mu<\gamma} [\bigwedge_{\nu<\gamma} \mathscr{A}_{\mu\nu}]]$$

is a scheme of Π_γ. Let s be any assignment of the symbols of the $\mathscr{A}_{\mu\nu}$ to B_0. If, contrariwise, $s^*\mathscr{A}' = \mathbf{0}$, for each $\mu < \gamma$ there would be $\nu_\mu < \gamma$ such that $s^*\mathscr{A}_{\mu\nu_\mu} = \mathbf{0}$. But then the set $\{\mathscr{A}_{\mu g(\mu)} : \mu < \gamma\}$ with $g(\mu) = \nu_\mu$ could not contain a pair A_ξ, $[\neg A_\xi]$.

5.2 Development of the Formal Systems for α-Propositional Languages

If A is a valid formula of $\mathbf{L}_\alpha$ containing only propositional operation symbols $\neg$, $\rightarrow$, then A has finite length, and is known to be provable using only schemes $\mathscr{I}_1$, $\mathscr{I}_2$, $\mathscr{N}$, and the rule of modus ponens. See Church [2], page 149. Hence

5.2.1 *Theorem.* If A is a valid formula of $\mathbf{L}_\alpha$ in which only propositional operation symbols $\neg$, $\rightarrow$, appear, then $\vdash A$ in $\mathfrak{P}_\alpha(\mathbf{L}_\alpha)$.

5.2.2 *Theorem.* If $\mathbf{L}_\alpha$, $\mathbf{L}'_\alpha$ are α-propositional languages and $\vdash A$ in $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$, then $\vdash S_f^X A$ in $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}'_\alpha)$, where f is any function on the set X of propositional symbols in A to formulas of $\mathbf{L}'_\alpha$.

PROOF: An easy induction using the substitution property 3.5.3 shows that if A is any formula of $\mathbf{L}_\alpha$ whose propositional symbols are all in X, and if f is any function on X to formulas of $\mathbf{L}'_\alpha$, then $S_f^X A$ is a formula of $\mathbf{L}'_\alpha$.

Let $\langle C_\nu : \nu < \sigma + 1\rangle$ be a formal proof of A in $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$, let X' be the set of all propositional symbols appearing in the proof. Extend f to X' by mapping $X' \sim X$ to arbitrary formulas of $\mathbf{L}'_\alpha$. We claim that $\langle S_f^{X'} C_\nu : \nu < \sigma + 1\rangle$ is a formal proof of $S_f^X A$ in $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}'_\alpha)$. Its length is, of course, less than α, and its last formula is $S_f^{X'} C_\sigma = S_f^X A$. If C_ν is in the given proof of A because it is an instance $S_g^V \mathscr{A}$ of an axiom scheme of $\mathfrak{P}_\alpha(\Sigma)$, then $S_f^{X'} C_\nu = S_h^V \mathscr{A}$ by 3.5.3, where for symbols $A_\xi \in V$, $h(A_\xi) = S_f^{X'}(g(A_\xi))$. As we have just observed, h maps V to formulas of $\mathbf{L}'_\alpha$ and therefore, $S_f^{X'} C_\nu$ is an instance of $\mathscr{A}$ in $\mathbf{L}'_\alpha$. If C_ν follows from earlier formulas $C_{\nu(0)}$, $\ldots, C_{\nu(\xi)}, \ldots$ by modus ponens or conjunction, then $S_f^{X'} C_\nu$ follows from earlier formulas $S_f^{X'} C_{\nu(0)}, \ldots, S_f^{X'} C_{\nu(\xi)}, \ldots$ by the same rule, since operator $S_f^{X'}$ distributes over $\rightarrow$ and $\bigwedge$.

5.2.3 *Definition.* Let Γ be a set of formulas of $\mathbf{L}_\alpha$. Then $\Gamma \vdash_\Delta A$ iff $\vdash_\Delta A$ or there are formulas $C_\xi \in \Gamma$, $\xi < \delta$, $0 < \delta < \alpha$, such that $\vdash_\Delta [\bigwedge C_0 \ldots C_\xi \ldots] \rightarrow A$.

Note the parallel in the definitions of $\Gamma \vdash A$ and $\Gamma \Vdash A$. Obviously a formal system is complete if and only if these two notions are equivalent for all sets Γ, formulas A.

5.2.4 *Theorem.* In all systems $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$,

(i) $\Gamma \vdash_\Delta A_0$ and $\Gamma \vdash_\Delta [A_0 \rightarrow A_1]$ implies $\Gamma \vdash_\Delta A_1$.

(ii) $\Gamma \vdash_\Delta A_\xi$ for all $\xi < \delta$ implies $\Gamma \vdash_\Delta [\bigwedge A_0 \ldots A_\xi \ldots]$.

Proof: If $\Gamma = \phi$, then (i) and (ii) are rules of inference. If $\Gamma \neq \phi$ and $\Gamma \vdash_\Delta A$ then there exists $0 < \delta < \alpha$, formulas $C_\xi \in \Gamma$ for $\xi < \delta$, such that $\vdash_\Delta [\bigwedge C_0 \ldots C_\xi \ldots] \rightarrow A$. For if $\vdash_\Delta A$, we can take $\delta = 1$, any $C_0 \in \Gamma$ and $\vdash_\Delta [\bigwedge C_0] \rightarrow A$ follows by $\mathscr{I}1$ and modus ponens.

Suppose $\Gamma \neq \phi$ and that $\vdash_\Delta [C_0 \rightarrow A_0]$ and $\vdash_\Delta [C_1 \rightarrow [A_0 \rightarrow A_1]]$ where C_0 and C_1 are conjunctions of formulas in Γ. Let C be the conjunction whose terms are those of C_0 together with whose of C_1. Then $\vdash C \rightarrow C'$ whenever C' is a principal subformula of C_0 or C_1 by $\mathscr{C}_2$, and $\vdash C \rightarrow C_0$, $\vdash C \rightarrow C_1$ after use of conjunction and $\mathscr{C}_1$. We can then conclude $\vdash_\Delta C \rightarrow A_1$ by four uses of modus ponens on a substitution of a finite valid scheme in $\rightarrow$ alone.

Suppose $\Gamma \neq \phi$ and that $\vdash_\Delta C_\xi \rightarrow A_\xi$ for $\xi < \delta$, where each C_ξ is a conjunction of formulas in Γ. Since fewer than α formulas appear in the totality of these conjunctions, we can form the conjunction C of all the principal subformulas of all the C_ξ. Then $\vdash C \rightarrow C_\xi$ for all $\xi < \delta$ using schemes $\mathscr{C}$ and the rules of inference. It follows that $\vdash_\Delta C \rightarrow A_\xi$ for all $\xi < \delta$ using a finite valid scheme in $\rightarrow$ alone. Hence $\vdash_\Delta C \rightarrow [\bigwedge A_0 \ldots A_\xi \ldots]$.

5.2.5 *Deduction Theorem.* In all systems $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$,

(i) $\Gamma \vdash_\Delta A$ if and only if $\vdash_{\Delta \cup \Gamma} A$.

(ii) If $\Gamma = \{C_\xi : \xi < \delta\}$ where $0 < \delta < \alpha$, then

$$\vdash_\Delta [\bigwedge C_0 \ldots C_\xi \ldots] \to A$$

if and only if $\vdash_{\Delta \cup \Gamma} A$.

PROOF: The left-to-right implications are trivial and (ii) follows easily from (i). If $\vdash_{\Delta \cup \Gamma} A$, then there is a formal proof $\langle D_\nu : \nu < \sigma + 1\rangle$ of A from assumptions $\Delta \cup \Gamma$. It can be shown that $\Gamma \vdash_\Delta D_\nu$ for all $\nu \leqslant \sigma$ by transfinite induction. If D_ν is an axiom or a formula of Δ, then $\vdash_\Delta D_\nu$ and therefore $\Gamma \vdash_\Delta D_\nu$. If $D_\nu \in \Gamma$, then $\vdash [\bigwedge D_\nu] \to D_\nu$ by $\mathscr{C}_2$ and therefore $\Gamma \vdash_\Delta D_\nu$. Theorem 5.2.4 tells us that if D_ν follows from earlier formulas D_μ by modus ponens or conjunction, then $\Gamma \vdash_\Delta D_\nu$ follows from $\Gamma \vdash_\Delta D_\mu$ for $\mu < \nu$. Since $A = D_\sigma$, $\Gamma \vdash_\Delta A$.

5.2.6 *Equivalence Theorem.* In $\mathfrak{P}_\alpha(\mathbf{L}_\alpha)$,

(i) $\vdash [A \leftrightarrow A]$ and $\vdash [A_0 \leftrightarrow A_1] \to [A_1 \leftrightarrow A_0]$.

(ii) $\vdash [[A_0 \leftrightarrow A_1] \wedge [A_1 \leftrightarrow A_2]] \to [A_0 \leftrightarrow A_2]$.

(iii) $\vdash [A_0 \leftrightarrow A_1] \to [[\neg A_0] \leftrightarrow [\neg A_1]]$.

(iv) $\vdash [[A_0 \leftrightarrow A_0'] \wedge [A_1 \leftrightarrow A_1']] \to [[A_0 \to A_1] \leftrightarrow [A_0' \to A_1']]$.

(v) $\vdash [\bigwedge [A_0 \leftrightarrow A_0'] \ldots [A_\xi \leftrightarrow A_\xi'] \ldots] \to$
$[[\bigwedge A_0 \ldots A_\xi \ldots] \leftrightarrow [\bigwedge A_0' \ldots A_\xi' \ldots]]$.

PROOF: For example, (v). Let $\Delta = \{[A_\xi \leftrightarrow A_\xi'] : \xi < \delta\}$, let $\Gamma = \{A_\xi : \xi < \delta\}$. Then $\vdash_{\Delta \cup \Gamma} A_\xi'$ for all $\xi < \delta$ and by conjunction, $\vdash_{\Delta \cup \Gamma} [\bigwedge A_0' \ldots A_\xi' \ldots]$. Then $\vdash_\Delta [\bigwedge A_0 \ldots A_\xi \ldots] \to [\bigwedge A_0' \ldots A_\xi' \ldots]$ by the deduction theorem. Similarly we can show the implication in reverse order provable from Δ. By conjunction, $\vdash_\Delta [\bigwedge A_0 \ldots A_\xi \ldots] \leftrightarrow [\bigwedge A_0' \ldots A_\xi' \ldots]$. Another use of the deduction theorem yields (v).

5.2.7 *Replacement Principle.* If $A = E_0 C_0 \ldots E_\nu C_\nu \ldots E_\sigma$, where $\langle C_\nu : \nu < \sigma\rangle$ is a sequence of formulas, and if $\vdash_\Delta [C_\nu \leftrightarrow C_\nu']$ for all $\nu < \sigma$, then $\vdash_\Delta A \leftrightarrow E_0 C_0' \ldots E_\nu C_\nu' \ldots E_\sigma$ in $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$.

PROOF: The equivalence theorem implies that the relation $A \equiv A'$ iff $\vdash_\Delta [A \leftrightarrow A']$ in $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ is a congruence relation on the algebra of formulas. Therefore this theorem is a corollary of the replacement principle for terms, 3.4.7.

5.2.8 *Theorem.* If A is a valid formula of $\mathbf{L}_\alpha$ having finite length, then $\vdash A$ in $\mathfrak{P}_\alpha(\mathbf{L}_\alpha)$.

PROOF: It is well-known that a finite conjunction $[\bigwedge A_0 \ldots A_n]$ is semantically equivalent to a $(\neg, \to)$-formula; namely, to $C = \neg[A_0 \to [\ldots [A_{n-1} \to [\neg A_n]] \ldots]]$. Using the deduction theorem and the provability of substitutions of finite $(\neg, \to)$-formulas, it is easy to see that $[\bigwedge A_0 \ldots A_n] \leftrightarrow C$ is provable in $\mathfrak{P}_\alpha(\mathbf{L}_\alpha)$.

Let $\mathbf{L}_\omega$ be the ω-propositional language having the same symbols as $\mathbf{L}_\alpha$. The recursion principle for terms guarantees the existence of a unique function f on formulas of $\mathbf{L}_\omega$ to formulas of $\mathbf{L}_\omega$ such that $f(\langle p\rangle) = \langle p\rangle$, for propositional symbols p, and for all formulas $A_0, \ldots, A_n, \ldots$ of $\mathbf{L}_\omega$,

$$f([\neg A_0]) = [\neg f(A_0)]$$
$$f([A_0 \rightarrow A_1]) = [f(A_0) \rightarrow f(A_1)]$$
$$f([\bigwedge A_0 \ldots A_n]) = [\neg[f(A_0) \rightarrow [\ldots[f(A_{n-1}) \rightarrow [\neg f(A_n)]]\ldots]]].$$

With the aid of the replacement theorem 5.2.7, we can prove by induction on the set of formulas of $\mathbf{L}_\omega$ that $\vdash A \leftrightarrow f(A)$ in $\mathfrak{P}_\omega(\mathbf{L}_\omega)$, and that $f(A)$ is always a finite formula in $\neg, \rightarrow$. Therefore if A is a finite valid formula of $\mathbf{L}_\alpha$, then A is a valid formula of $\mathbf{L}_\omega$, and moreover, so is $f(A)$. Since $\vdash f(A)$ by 5.2.1, $\vdash A$ in $\mathfrak{P}_\omega(\mathbf{L}_\omega)$. Then also $\vdash A$ in $\mathfrak{P}_\alpha(\mathbf{L}_\alpha)$.

5.2.9 *Conjunction Theorem.* For all formulas $A, A_0, \ldots, A_\xi, \ldots$ of $\mathbf{L}_\alpha$, the following are formal theorems of $\mathfrak{P}_\alpha(\mathbf{L}_\alpha)$:

(i) Laws of Absorption.

$$\vdash[\bigwedge A\ldots A\ldots] \leftrightarrow A \quad\text{and}\quad \vdash A \leftrightarrow [\bigvee A\ldots A\ldots].$$

(ii) Generalized Associative Laws. If $\delta = \Sigma\langle\delta_\nu : \nu < \varepsilon\rangle$, and $\gamma_\nu = \Sigma\langle\delta_\mu : \mu < \nu\rangle$ for each $\nu < \varepsilon$, then

$$\vdash[\bigwedge_{\xi<\delta} A_\xi] \leftrightarrow [\bigwedge_{\nu<\varepsilon}[\bigwedge_{\xi<\delta_\nu} A_{\gamma_\nu+\xi}]]$$

and

$$\vdash[\bigvee_{\xi<\delta} A_\xi] \leftrightarrow [\bigvee_{\nu<\varepsilon}[\bigvee_{\xi<\delta_\nu} A_{\gamma_\nu+\xi}]].$$

(iii) Generalized Commutative Laws. If

$$\{A_\xi : \xi < \delta\} = \{A_{\nu(\xi)} : \xi < \delta'\},$$

then

$$\vdash[\bigwedge_{\xi<\delta} A_\xi] \leftrightarrow [\bigwedge_{\xi<\delta'} A_{\nu(\xi)}] \quad\text{and}\quad \vdash[\bigvee_{\zeta<\delta} A_\xi] \leftrightarrow [\bigvee_{\xi<\delta'} A_{\nu(\xi)}].$$

(iv) If $\delta = \delta_0 + \delta_1$, then

$$\vdash[\bigwedge_{\xi<\delta_1} A_{\delta_0+\xi}] \rightarrow [[\bigwedge_{\xi<\delta} A_\xi] \leftrightarrow [\bigwedge_{\xi<\delta_0} A_\xi]]$$

$$\vdash[\bigwedge_{\xi<\delta_1} [\neg A_{\delta_0+\xi}]] \rightarrow [[\bigvee_{\xi<\delta} A_\xi] \leftrightarrow [\bigvee_{\xi<\delta_0} A_\xi]].$$

PROOF: With the aid of the deduction theorem, the replacement theorem, and the combination of 5.2.8 and 5.2.2, saying that every substitution of a valid finite formula is provable, these proofs are routine. As an example, we give (ii).

Let $\Gamma = \{A_\xi : \xi < \delta\}$, let $C_\nu = [\bigwedge_{\xi<\delta_\nu} A_{\gamma_\nu+\xi}]$ for $\nu < \varepsilon$. Then $\vdash_\Gamma C_\nu$ for each $\nu < \varepsilon$ by conjunction. Again by conjunction, $\vdash_\Gamma [\bigwedge_{\nu<\varepsilon} C_\nu]$. Hence $\vdash [\bigwedge_{\xi<\delta} A_\xi] \to [\bigwedge_{\nu<\varepsilon} C_\nu]$ by deduction theorem.

Conversely, for each $\xi < \delta$, there is unique $\nu' < \varepsilon$, $\theta < \delta_{\nu'}$ such that $\xi = \gamma_{\nu'} + \theta$ by 2.1.5, Chapter 2. Therefore $\vdash C_{\nu'} \to A_\xi$. If $\Gamma' = \{C_\nu : \nu < \varepsilon\}$, $\vdash_{\Gamma'} A_\xi$ for each $\xi < \delta$ by modus ponens. Hence $\vdash_{\Gamma'} [\bigwedge_{\xi<\delta} A_\xi]$ by conjunction. The deduction theorem implies

$$\vdash [\bigwedge_{\nu<\varepsilon} C_\nu] \to [\bigwedge_{\xi<\delta} A_\xi].$$

The dual can be proved by beginning with a substitution in the formula just proved. By 5.2.2,

$$\vdash [\bigwedge_{\xi<\delta} [\neg A_\xi]] \leftrightarrow [\bigwedge_{\nu<\varepsilon} [\bigwedge_{\xi<\delta_\nu} [\neg A_{\gamma_\nu+\xi}]]].$$

Taking negations, 5.2.6 (iii) implies

$$\vdash [\bigvee_{\xi<\delta} A_\xi] \leftrightarrow [\neg [\bigwedge_{\nu<\varepsilon} [\bigwedge_{\xi<\delta_\nu} [\neg A_{\gamma_\nu+\xi}]]]].$$

After replacing parts $[\neg\neg A]$ by A in the formula $[\bigvee_{\nu<\varepsilon} [\bigvee_{\xi<\delta_\nu} A_{\gamma_\nu+\xi}]]$ with $\bigvee$ eliminated, one more use of the equivalence theorem gives the result.

5.3 Completeness of the Basic Formal Systems when $\alpha = \omega_1$

A set of formulas is called *contradictory* if it contains a formula with its negation. A *choice set* for a doubly-indexed system $\langle A_{\mu\nu} : \mu < \gamma, \nu < \gamma_\mu \rangle$ of formulas is a set containing at least one formula from each row.

5.3.1 *Completeness Lemma.* Let Σ be a set of valid α-propositional schemes. Suppose that whenever $\langle A_{\mu\nu} : \mu < \gamma, \nu < \gamma_\mu \rangle$ is a doubly-indexed system of formulas of $\mathbf{L}_\alpha$ with $\gamma < \alpha$, $\gamma_\mu \leqslant \gamma$ for $\mu < \gamma$, such that every choice set is contradictory, we have

$$\vdash \bigvee_{\nu<\gamma_\mu} A_{\mu\nu} \text{ for all } 0 < \mu < \gamma \text{ implies } \vdash [\neg [\bigvee_{\nu<\gamma_0} A_{0\nu}]]$$

in $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$.

Then $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ is complete.

PROOF: We have already seen that formulas provable in these systems with valid axiom schemes are valid. Conversely, suppose A is valid. If A is finite we already know A is provable. Suppose A infinite and let γ be the smallest cardinal number that is an upper bound of the cardinals of the lengths of the conjunctions in A. Then $\omega \leqslant \gamma < \alpha$. Associate sequences of formulas to the subformulas of A as follows: To a subformula of the form $[\neg A_0]$, associate $\langle[\neg A_0]A_0\rangle$; to a subformula of the form $[A_0 \rightarrow A_1]$, associate the three sequences

$\langle[A_0 \rightarrow A_1]A_0\rangle$, $\langle[A_0 \rightarrow A_1][\neg A_1]\rangle$, $\langle[\neg[A_0 \rightarrow A_1]][\neg A_0]A_1\rangle$;

and, finally, to a subformula of the form $[\bigwedge A_0 \ldots A_\xi \ldots]$, associate a sequence with elements $[\bigwedge A_0 \ldots A_\xi \ldots]$, $[\neg A_0]$, $\ldots$, $[\neg A_\xi]$, $\ldots$ arranged in length at most γ, and associate also the sequence $\langle[\neg[\bigwedge A_0 \ldots A_\xi \ldots]]A_\nu\rangle$ for each A_ν. There are γ such sequences. Index them by ordinals μ, $0 < \mu < \gamma$. Let $\langle A_{\mu\nu}: \nu < \gamma_\mu\rangle$ be the sequence with index μ, let $\gamma_0 = 1$ and $A_{00} = [\neg A]$. Consider the resulting doubly-indexed system $\langle A_{\mu\nu}: \mu < \gamma, \nu < \gamma_\mu\rangle$.

First, we notice that every choice set is contradictory. For every choice set contains $[\neg A]$, and if, moreover, one such set $\bar{\Gamma}$ were not contradictory, $\bar{\Gamma}$ would obviously satisfy conditions (1), (2), (3) of the criterion for satisfiability 4.1.4. It would follow that $[\neg A]$ is satisfiable, contradicting the validity of A.

Secondly, we notice that $\vdash[\bigvee A_{\mu 0} \ldots A_{\mu\nu} \ldots]$ for $0 < \mu < \gamma$, in $\mathfrak{P}_\alpha(\mathbf{L}_\alpha)$. In case the sequence with index μ comes from a subformula of A of the form $[\neg A_0]$, or $[A_0 \rightarrow A_1]$, then γ_μ is finite and the disjunction $[\bigvee A_{\mu 0} \ldots A_{\mu\gamma_\mu - 1}]$ is a substitution of a valid formula of finite length. We have seen such formulas to be provable, 5.2.8, 5.2.2. If the sequence comes from a subformula of form $[\bigwedge A_0 \ldots A_\xi \ldots]$, then $A'_\mu = [\bigvee A_{\mu 0} \ldots A_{\mu\nu} \ldots]$ has form

(a) $A'_\mu = [\bigvee [\bigwedge A_0 \ldots A_\xi \ldots][\neg A_{\nu(0)}] \ldots [\neg A_{\nu(\xi)}] \ldots]$

where $\{A_\xi: \xi < \delta\} = \{A_{\nu(\xi)}: \xi < \delta'\}$, or it has form

(b) $A'_\mu = [\bigvee [\neg[\bigwedge A_0 \ldots A_\xi \ldots]]A_\nu]$.

The provability of A'_μ; in case (a) is an immediate consequence of the commutative and associative laws of 5.2.9. In case (b) $\vdash[\bigwedge A_0 \ldots A_\xi \ldots] \rightarrow A_\nu$ by $\mathscr{C}_2$. Hence $\vdash A'_\mu$ by a substitution of the finite valid scheme $[A \rightarrow A'] \leftrightarrow [\bigvee [\neg A]A']$.

Finally $\vdash[\neg[\bigvee [\neg A]]]$ in $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ by the hypothesis. Hence $\vdash A$.

The completeness of $\mathfrak{P}_{\omega_1}(\mathbf{L}_{\omega_1})$ follows easily. The argument is essentially the same as Rieger's proof of the fact that free ω-

complete Boolean algebras are isomorphic to ω-fields of sets. See [33]. This completeness theorem appears also in Scott-Tarski [37].

5.3.2 *Theorem.* Systems $\mathfrak{P}_{\omega_1}(\mathbf{L}_{\omega_1})$ are complete.

PROOF: Suppose $\langle A_{n\nu}: n < \omega, \nu < \gamma_n\rangle$ is a doubly-indexed system of formulas of $\mathbf{L}_{\omega_1}$ such that $0 < \gamma_n \leqslant \omega$ for $n < \omega$, and every choice set is contradictory. Suppose, moreover, $\vdash[\mathbf{V}\, A_{n0}\ldots A_{n\nu}\ldots]$ for every $0 < n < \omega$. We must show $\vdash[\neg[\mathbf{V}\, A_{00}\ldots A_{0\nu}\ldots]]$.

Call a formula A *consistent* if not $\vdash[\neg A]$. Note that if $[A \wedge [\mathbf{V}\, A_0\ldots A_\xi\ldots]]$ is consistent, then there is ξ such that $[A \wedge A_\xi]$ is consistent. For if, contrariwise, $\vdash[\neg[A \wedge A_\xi]]$ for all ξ, then $\vdash[A \rightarrow [\neg A_\xi]]$ for all ξ, using a finite valid scheme. Then $\vdash[A \rightarrow [\mathbf{\Lambda}[\neg A_0]\ldots[\neg A_\xi]\ldots]$ by 5.2.4, and $\vdash[A \rightarrow [\neg[\mathbf{V}\, A_0\ldots A_\xi\ldots]]]$ using the replacement theorem. This contradicts the consistency of $[A \wedge [\mathbf{V}\, A_0\ldots A_\xi\ldots]]$.

Suppose, contrariwise, not $\vdash[\neg[\mathbf{V}\, A_{00}\ldots A_{0\nu}\ldots]]$. Then $[\mathbf{V}\, A_{00}\ldots A_{0\nu}\ldots]$ is consistent, and there is ν_0 such that $A_{0\nu_0}$ is consistent. Suppose that there are $\nu_i < \gamma_i$ for all $i \leqslant n$ such that $[A_{0\nu_0} \wedge \ldots \wedge A_{n\nu_n}]$ consistent. Call this conjunction A_n'. Since $\vdash A_n' \leftrightarrow [A_n' \wedge [\mathbf{V}\, A_{n+1,0}\ldots A_{n+1,\nu}\ldots]]$, this formula must also be consistent. Therefore there is $\nu_{n+1} < \gamma_{n+1}$ such that $[A_n' \wedge A_{n+1\nu_{n+1}}]$ is consistent. Let $\Gamma = \{A_{n\nu_n}: n < \omega\}$. This is a choice set that cannot be contradictory because finite conjunctions of its formulas are consistent.

5.4 Completeness of the Basic Formal Systems with Chang's Distributive Laws

The denumerability of the formulas is essential to the proof of the completeness theorem just given. In Chapter 7 we will show that additional schemes are needed to guarantee completeness when $\alpha > \omega_1$. A review of the definition of Chang's distributive laws, 5.1.3, tells us that

$$[\mathop{\mathbf{V}}_{\mu<\gamma} [\mathop{\mathbf{\Lambda}}_{\nu<\gamma} A_{\mu\nu}]]$$

is an instance of a scheme in Π_γ if and only if every choice set of the array $\langle A_{\mu\nu}: \mu < \gamma, \nu < \gamma\rangle$ is contradictory.

5.4.1 *Lemma.* If $\gamma' < \gamma < \alpha$, then every instance in $\mathbf{L}_\alpha$ of a scheme of $\Pi_{\gamma'}$ is provable in $\mathfrak{P}_\alpha(\Pi_\gamma)(\mathbf{L}_\alpha)$.

PROOF: Suppose array $\langle A_{\mu\nu}: \mu < \gamma', \nu < \gamma'\rangle$ of formulas of $\mathbf{L}_\alpha$ has the property that every choice set is contradictory. First, fill out the existing rows to length γ by repeating a formula: $A_{\mu\nu} = A_{\mu 0}$ for $\mu < \gamma'$, $\gamma' \leqslant \nu < \gamma$. By 5.2.9 (iii),

(1) $\vdash [\bigwedge_{\nu<\gamma} A_{\mu\nu}] \leftrightarrow [\bigwedge_{\nu<\gamma'} A_{\mu\nu}]$ for $\mu < \gamma'$.

The array is then filled out to length γ by repeating row 0: $A_{\mu\nu} = A_{0\nu}$ for $\gamma' \leqslant \mu < \gamma$. By 5.2.9 (iii),

(2) $\vdash [\bigvee_{\mu<\gamma} [\bigwedge_{\nu<\gamma'} A_{\mu\nu}]] \leftrightarrow [\bigvee_{\mu<\gamma'} [\bigwedge_{\nu<\gamma'} A_{\mu\nu}]]$.

The resulting array still has the property that every choice set is contradictory. Hence

(3) $\vdash [\bigvee_{\mu<\gamma} [\bigwedge_{\nu<\gamma} A_{\mu\nu}]]$ in $\mathfrak{P}_\alpha(\Pi_\gamma)(\mathbf{L}_\alpha)$.

Replacing parts in (3) by their formal equivalents according to (1), yields the left-hand formula of (2). Hence

$$\vdash [\bigvee_{\mu<\gamma'} [\bigwedge_{\nu<\gamma'} A_{\mu\nu}]] \text{ in } \mathfrak{P}_\alpha(\Pi_\gamma)(\mathbf{L}_\alpha).$$

5.4.2 *Lemma.* If $\gamma < \alpha$, $\gamma_\mu \leqslant \gamma$ for every $\mu < \gamma$, and if every choice set of array $\langle A_{\mu\nu} : \mu < \gamma, \nu < \gamma_\mu \rangle$ is contradictory, then

$$\vdash [\bigvee_{\mu<\gamma} [\bigwedge_{\nu<\gamma_\mu} A_{\mu\nu}]]$$

in $\mathfrak{P}_\alpha(\Pi_\gamma)(\mathbf{L}_\alpha)$.

PROOF: Similar to the one just given. Fill out rows to length γ by repeating a formula already present.

5.4.3 *Theorem.* Let γ be an infinite cardinal. Then systems $\mathfrak{P}_{\gamma^+}(\Pi_\gamma)(\mathbf{L}_{\gamma^+})$ are complete.

PROOF: According to the completeness lemma, 5.3.1, it suffices to show that given a doubly-indexed system $\langle A_{\mu\nu} : \mu < \gamma', \nu < \gamma_\mu \rangle$ of formulas of $\mathbf{L}_{\gamma^+}$ with $\gamma' < \gamma^+$, $\gamma_\mu \leqslant \gamma'$ for $\mu < \gamma'$, such that every choice set is contradictory,

$\vdash \bigvee_{\nu<\gamma_\mu} A_{\mu\nu}$ for all $0 < \mu < \gamma'$ implies $\vdash [\neg [\bigvee_{\nu<\gamma_0} A_{0\nu}]]$

in $\mathfrak{P}_{\gamma^+}(\Pi_\gamma)(\mathbf{L}_{\gamma^+})$.

The array $\langle [\neg A_{\mu\nu}] : \mu < \gamma', \nu < \gamma_\mu \rangle$ also has the property that every choice set is contradictory. By 5.4.2 and 5.4.1,

(1) $\vdash [\bigvee_{\mu<\gamma'} C_\mu]$, where $C_\mu = [\bigwedge_{\nu<\gamma_\mu} [\neg A_{\mu\nu}]]$.

By our assumption, $\vdash [\neg C_\mu]$ for $0 < \mu < \gamma'$. Hence

(2) $\vdash[\bigwedge_{1\leqslant\mu<\gamma'}[\neg C_\mu]]$ by conjunction.

By 5.2.9 (iv),

(3) $\vdash[\bigwedge_{1\leqslant\mu<\gamma'}[\neg C_\mu]]\to[[\bigvee_{\mu<\gamma'} C_\mu]\to C_0]$.

Two uses of modus ponens with (1), (2), yields $\vdash C_0$. Hence

$$\vdash[\neg[\bigvee_{\nu<\gamma_0} A_{0\nu}]].$$

5.4.4 *Theorem.* Let α be an inaccessible cardinal. Then systems $\mathfrak{P}_\alpha(\bigcup\{\Pi_\gamma\colon \gamma<\alpha\})(\mathbf{L}_\alpha)$ are complete.

PROOF: We already know $\mathfrak{P}_\omega(\mathbf{L}_\omega)$ is complete. If $\alpha>\omega$, an easy induction on formulas shows that every formula of $\mathbf{L}_\alpha$ is a formula of a language $\mathbf{L}_\gamma$, $\gamma<\alpha$, with the same symbols. Hence this is a corollary of Theorem 5.4.3.

Theorems 5.4.3 and 5.4.4 are based directly on work of Chang on the representability of Boolean algebras, [1]. They appear in Scott-Tarski [37] as well as my doctoral dissertation. See also Sect. 6.4.

5.5 The Basic Formal Systems with Ordinary Distributive Laws

Let γ be a cardinal number, let α be a regular infinite cardinal such that $\alpha\geqslant(2\exp\gamma)^+$. Since any instance of $\mathscr{D}_\gamma$ in $\mathbf{L}_\alpha$ is valid, it is provable in $\mathfrak{P}_\alpha(\Pi_{2\exp\gamma})(\mathbf{L}_\alpha)$. It is easy to see that any instance of a scheme in Π_γ is provable in $\mathfrak{P}_\alpha(\mathscr{D}_\gamma)(\mathbf{L}_\alpha)$. For if every choice set of $\langle A_{\mu\nu}\colon \mu<\gamma,\ \nu<\gamma\rangle$ is contradictory, then for $\xi<2\exp\gamma$,

$$\vdash[\bigvee_{\mu<\gamma} A_{\mu g_\xi(\mu)}] \text{ in } \mathfrak{P}_\alpha(\mathbf{L}_\alpha),$$

$\langle g_\xi\colon \xi<2\exp\gamma\rangle$ being the ordering of γ^γ chosen in advance according to 5.1.2. By conjunction,

(1) $\vdash[\bigwedge_{\xi<2\exp\gamma}[\bigvee_{\mu<\gamma} A_{\mu g_\xi(\mu)}]]$ in $\mathfrak{P}_\alpha(\mathbf{L}_\alpha)$.

Scheme $\mathscr{D}_\gamma$ with $\mathscr{N}$ and modus ponens yields

(2) $\vdash[\neg[\bigvee_{\xi<2\exp\gamma}\bigwedge_{\mu<\gamma}[\neg A_{\mu g_\xi(\mu)}]]]\to[\neg[\bigwedge_{\mu<\gamma}\bigvee_{\nu<\gamma}[\neg A_{\mu\nu}]]]$.

Hence $\vdash[\bigvee_{\mu<\gamma}[\bigwedge_{\nu<\gamma} A_{\mu\nu}]]$ in $\mathfrak{P}_\alpha(\mathscr{D}_\gamma)(\mathbf{L}_\alpha)$ using modus ponens on (1) and (2) after replacing parts $[\neg\neg A]$ by A.

Equally simple arguments yield the following lemma.

5.5.1 *Lemma.* If α is a regular infinite cardinal such that $\alpha \geqslant (2 \exp \gamma)^+$, then

(1) Instances of schemes of Π_γ are provable in $\mathfrak{P}_\alpha(\mathscr{D}_\gamma)(\mathbf{L}_\alpha)$.

(2) If cardinals γ_0, γ_1 are less than or equal to γ, and if $\gamma_0^{\gamma_1} = \{f_\xi \colon \xi < \gamma_0 \exp \gamma_1\}$, then

$$\vdash [\bigwedge_{\mu<\gamma_0} [\bigvee_{\nu<\gamma_1} A_{\mu\nu}]] \rightarrow [\bigvee_{\xi<\gamma_0 \exp \gamma_1} [\bigwedge_{\mu<\gamma_0} A_{\mu f_\xi(\mu)}]]$$

in $\mathfrak{P}_\alpha(\mathscr{D}_\gamma)(\mathbf{L}_\alpha)$.

5.5.2 *Theorem.* If α is strongly inaccessible, then systems $\mathfrak{P}_\alpha(\{\mathscr{D}_\gamma \colon \gamma < \alpha\})(\mathbf{L}_\alpha)$ are complete.

PROOF: By 5.4.4 and 5.5.1.

5.5.3 *Remark.* There is yet another case where we have been able to obtain a complete α-propositional calculus using $\mathscr{D}_\gamma$ for $\gamma < \alpha$. If γ is a union of ω smaller cardinals γ_n, $n < \omega$, such that $2 \exp \gamma_n < \gamma$, then $\mathfrak{P}_{\gamma^+}([\bigwedge_{n<\omega} \mathscr{D}_{\gamma_n}])(\mathbf{L}_{\gamma^+})$ is always complete. The proof is being delayed so that we may use algebraic techniques to simplify the computations. It appears in Sect. 6.5. These cases are the only ones except $\alpha = \omega$ and $\alpha = \omega_1$ where we have been able to obtain a complete α-propositional calculus by adjoining a single scheme to the basic calculus.

We do not know whether or not there are any cardinals except these, the strong inaccessibles, and ω_1, such that $\mathfrak{P}_\alpha(\{\mathscr{D}_\gamma \colon 2 \exp \gamma < \alpha\})$ is a complete α-propositional calculus. It is doubtful that calculi $\mathfrak{P}_{(2 \exp \gamma)^+}(\mathscr{D}_\gamma)$ are complete, but we have not been able to give a proof. See Sect. 7.1 for the algebraic version of this problem. It is not difficult to see that these calculi are complete, in fact, strongly complete, for languages with at most γ propositional symbols. The proof is essentially the same as Tarski's proof that a completely distributive Boolean algebra is isomorphic to a set-of-all-subsets algebra. See [42].

5.5.4 *Theorem.* Let γ be an infinite cardinal, $\alpha \geqslant (2 \exp \gamma)^+$, and $\mathbf{L}_\alpha$ be an α-propositional language with at most γ propositional symbols. Then $\mathfrak{P}_\alpha(\mathscr{D}_\gamma)(\mathbf{L}_\alpha)$ is strongly complete.

PROOF: Let $P = \{p_\lambda \colon \lambda < \sigma\}$ be the set of all propositional symbols of the language, indexed by an ordinal $\sigma \leqslant \gamma$. For each assignment s of P to truth values, let $C_s = [\bigwedge s'(p_0) \ldots s'(p_\lambda) \ldots]$, where $s'(p_\lambda) = p_\lambda$ if $s(p_\lambda) = \mathbf{1}$, $s'(p_\lambda) = [\neg p_\lambda]$ if $s(p_\lambda) = \mathbf{0}$. Then s satisfies C_s and is the only assignment that does. Since formal theorems are valid, $\vdash [C_s \rightarrow A]$ implies that s satisfies A.

Let $\Delta = \{A : A$ is a formula and s satisfies A implies $\vdash [C_s \to A]$ and s satisfies $[\neg A]$ implies $\vdash [C_s \to [\neg A]]$ in $\mathfrak{P}_\alpha(\mathbf{L}_\alpha)\}$. Then Δ obviously contains the atomic formulas and is closed under formation of negations. The proof that Δ is also closed under formation of implications and conjunctions, is an easy exercise in basic propositional calculus. Hence s satisfies A if and only if $\vdash [C_s \to A]$ in $\mathfrak{P}_\alpha(\mathbf{L}_\alpha)$.

Suppose Γ is a set of formulas, A is a formula, and $\Vdash_\Gamma A$. We must show $\vdash_\Gamma A$. Enumerate the set of all assignments: $B_0^P = \{s_\xi : \xi < 2 \exp \sigma\}$. Split this set into two subsets

$$S_1 = \{s : s \text{ satisfies } \Gamma\},\ S_2 = \{s : s \text{ does not satisfy } \Gamma\}.$$

Then

(1) $\vdash C_s \to A$ in $\mathfrak{P}_\alpha(\mathbf{L}_\alpha)$, for all $s \in S_1$.

For $s \in S_2$, there is $C \in \Gamma$ such that $\vdash C_s \to [\neg C]$. Hence

(2) $\vdash_\Gamma [\neg C_s]$ in $\mathfrak{P}_\alpha(\mathbf{L}_\alpha)$, for all $s \in S_2$.

Clearly,

(3) $\vdash [\bigwedge_{\lambda<\sigma} [p_\lambda \vee [\neg p_\lambda]]]$.

By $\mathscr{D}_\gamma$ and 5.5.1,

(4) $\vdash [\bigvee_{\xi<2\exp\sigma} C_{s_\xi}]$.

If $S_2 = \phi$, then by (1), 5.2.7 and finite valid schemes,

$$\vdash [\bigvee_{\xi<2\exp\sigma} C_{s_\xi}] \to A.$$

Hence $\vdash A$ in this case by (4) and modus ponens.

If $S_1 = \phi$, then by (2) and finite valid schemes,

$$\vdash_\Gamma [\neg [\bigvee_{\xi<2\exp\sigma} C_{s_\xi}].$$

Hence $\vdash_\Gamma A$ in this case by (4), finite valid scheme $[\neg A_0] \to [A_0 \to A]$ and modus ponens.

If $S_1 \neq \phi$ and $S_2 \neq \phi$, let $S_1 = \{s_{p(\xi)} : \xi < \gamma'\}$, $S_2 = \{s_{q(\xi)} : \xi < \gamma''\}$. Then $\vdash [\bigvee_{\xi<\gamma'} C_{s_{p(\xi)}}] \to A]$ and $\vdash_\Gamma [\bigwedge_{\xi<\gamma''} [\neg C_{s_{q(\xi)}}]]$ by

(1) and (2). Then by (4) and 5.2.9, $\vdash_{\Gamma}[\bigvee_{\xi<\gamma'} C_{s_{p(\xi)}}]$. Hence $\vdash_{\Gamma} A$.

5.5.5 *Theorem.* If α is strongly inaccessible and $\mathbf{L}_\alpha$ has fewer than α propositional symbols, then $\mathfrak{P}_\alpha(\{\mathscr{D}_\gamma : \gamma < \alpha\})$ is strongly complete for $\mathbf{L}_\alpha$.

PROOF: For if $\mathbf{L}_\alpha$ has γ propositional symbols, $\mathfrak{P}_\alpha(\mathscr{D}_\gamma)(\mathbf{L}_\alpha)$ is already strongly complete.

CHAPTER 6

REPRESENTATION THEORY FOR BOOLEAN ALGEBRAS

The idea of relating representation theorems for Boolean algebras to completeness theorems for propositional calculi is not new. Henkin proved in [11] that Stone's representation theorem for Boolean algebras follows from the completeness of the ordinary classical propositional calculus. However, when this argument is adapted to α-propositional calculi, α regular, infinite, a different ingredient appears. For it is the strong completeness of an α-propositional calculus that is equivalent to representability by $\nearrow\alpha$-fields of sets. Ordinary completeness is equivalent to the weaker representability by $\nearrow\alpha$-homomorphic images of $\nearrow\alpha$-fields of sets.

6.1 Boolean Algebraic Equations Corresponding to α-Propositional Schemes

It is convenient to introduce officially the two-place Boolean algebraic operation $\rightarrow$ by the equation

$$a \rightarrow b = \neg a \vee b.$$

Then an α-propositional scheme becomes an $\nearrow\alpha$-complete Boolean algebraic term when symbols $\boldsymbol{\neg}$, $\boldsymbol{\rightarrow}$, $\boldsymbol{\bigwedge}$, are replaced by $\neg$, $\rightarrow$, $\bigwedge$. The intended interpretation of an $\nearrow\alpha$-complete Boolean algebraic term is, naturally, in an $\nearrow\alpha$-complete Boolean algebra $\mathfrak{B} = \langle B \neg \wedge \vee \rangle$ with $\rightarrow$ the operation just defined, $\bigwedge$ the $\nearrow\alpha$-place meet. For an assignment s of the symbols of the scheme-language to $\mathfrak{B}$, the value s^*T of such a term is its computed value using operations of $\mathfrak{B}$ (Example 3.2.2). According to the rules of Sect. 4.3 for interpreting α-propositional formulas in $\mathfrak{B}$, the value of scheme $\mathscr{A}$ in $\mathfrak{B}$ for assignment s is exactly the same as the value of the corresponding term $T_{\mathscr{A}}$ in $\mathfrak{B}$. Thus scheme $\mathscr{A}$ holds in $\mathfrak{B}$ if and only if the equation $T_{\mathscr{A}} = \mathbf{1}$ holds in $\mathfrak{B}$.

The reader can easily check for himself that all of the axiom schemes of the basic calculus $\mathfrak{P}_\alpha$ hold in every $\nearrow$ α-complete Boolean algebra. If $\alpha \geqslant (2 \exp \gamma)^+$, the scheme $\mathscr{D}_\gamma$ holds in $\mathfrak{B}$ if and only if $\mathfrak{B}$ is (γ, γ)-distributive. Note that if $\mathbf{L}_\alpha$ is an arbitrary α-propositional language, the condition "every instance of $\mathscr{A}$ holds in $\mathfrak{B}$" may not be the same as "$\mathscr{A}$ holds in $\mathfrak{B}$". These conditions are only equivalent if $\mathbf{L}_\alpha$ has at least α propositional symbols. However, if scheme $\mathscr{A}$ holds in $\mathfrak{B}$, so does every instance of $\mathscr{A}$ in $\mathbf{L}_\alpha$ by Lemma 5.1.4. Hence

6.1.1 *Theorem.* Let $\mathbf{L}_\alpha$ be an α-propositional language, Σ a set of α-propositional schemes, $\mathfrak{B}$ an $\nearrow$ α-complete Boolean algebra in which equations $T_{\mathscr{A}} = \mathbf{1}$ hold for $\mathscr{A} \in \Sigma$. Then if $\vdash_\Gamma A$ in $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ and s is an assignment to $\mathfrak{B}$ such that $\{s^*C\colon C \in \Gamma\}$ has lower bound $b \in B$, then $s^*A \geqslant b$.

PROOF: Obvious since the axioms all hold in $\mathfrak{B}$ and the property $s^*(A) \geqslant b$ is preserved by modus ponens and conjunction.

Clearly, under the conditions of the theorem, if formulas of Γ hold in $\mathfrak{B}$, so does A.

6.2 Formally Consistent Sets of Formulas

We say that a set Γ of formulas is *formally consistent* with respect to $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ if and only if there is a formula A such that not $\vdash_\Gamma A$. This is equivalent to the condition not $\vdash_\Gamma [p \wedge [\neg p]]$, where p is any propositional symbol. If $\vdash [p \wedge [\neg p]]$ in $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$, then no set is consistent with respect to the system. Such a system is *inconsistent*. According to the deduction theorem, if $\vdash_\Gamma [p \wedge [\neg p]]$, then Γ has a subset Γ' of power less than α such that $\vdash_{\Gamma'} [p \wedge [\neg p]]$. Hence Γ is formally consistent with respect to $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ if and only if every subset of Γ having power less than α is formally consistent. A formula A is formally consistent with respect to $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ if and only if $\{A\}$ is formally consistent. Using finite valid formulas, it is easy to see that A is consistent if and only if not $\vdash [\neg A]$.

Let Σ be a set of valid schemes. Then since every formal theorem of $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ is valid, Γ semantically consistent implies Γ formally consistent. Moreover, if Γ formally consistent, but not semantically consistent, then Γ has a subset $\{A_\xi\colon \xi < \delta\}$, $0 < \delta < \alpha$, such that $\Vdash [\wedge\, A_0 \ldots A_\xi \ldots] \rightarrow [p \wedge [\neg p]]$, but not $\vdash [\wedge\, A_0 \ldots A_\xi \ldots] \rightarrow [p \wedge [\neg p]]$. Hence $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ is not complete. Thus we have shown that if $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ is complete, then Γ is semantically consistent if and only if it is formally consistent. The converse is equally easy to prove. These and other properties of the notion of consistency that follow easily from the theorems of Chapter 5, Sect. 5.2, are summarized in the following lemma.

6.2.1 Let $\mathbf{L}_\alpha$ be an α-propositional language, $\mathfrak{P}_\alpha(\Sigma)$ an α-propositional calculus. Then with respect to $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$,

(i) Γ is formally consistent if and only if every subset of having power less than α is formally consistent.

(ii) If Γ formally consistent and $\vdash_\Gamma A$ for all $A \in \Gamma'$, then $\Gamma \cup \Gamma'$ is formally consistent.

(iii) $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ is complete if and only if for every formula A, A is formally consistent iff A is satisfiable.

(iv) $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ is complete if and only if for every set Γ of formulas, Γ is formally consistent iff Γ is semantically consistent.

(v) $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ is strongly complete if and only if for every set Γ of formulas, Γ is formally consistent iff Γ is satisfiable.

(vi) $\Gamma \cup \{A\}$ is formally consistent if and only if not $\vdash_\Gamma [\neg A]$.

(vii) If Γ is formally consistent, then either $\Gamma \cup \{A\}$ or $\Gamma \cup \{[\neg A]\}$ is formally consistent.

(viii) If $\Gamma \cup \{[\mathbf{V}\, A_0 \ldots A_\xi \ldots]\}$ is formally consistent, then there is $\nu < \delta$ such that $\Gamma \cup \{[\mathbf{V}\, A_0 \ldots A_\xi \ldots], A_\nu\}$ is formally consistent.

6.3 Algebras of Equivalence Classes of Formulas Modulo a Formally Consistent Set

Let Γ be a set of formulas formally consistent with respect to $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$. According to the equivalence theorem, 5.2.6, the relation

$$A \equiv A' \text{ if and only if } \vdash_\Gamma A \leftrightarrow A'$$

is a congruence relation on the algebra of formulas. The consistency of Γ guarantees that there are at least two classes $|A|_\Gamma = \{A' : A \equiv A'\}$. Moreover, the following equations define one-place, two-place, $\nearrow\alpha$-place operations on the set $B(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha);\Gamma)$ of all these equivalence classes:

$$\begin{aligned}
&\neg |A|_\Gamma = |[\neg A]|_\Gamma \\
&|A_0|_\Gamma \to |A_1|_\Gamma = |[A_0 \to A_1]|_\Gamma \\
&\wedge \langle |A_0| \ldots |A_\xi| \ldots \rangle = |[\mathbf{\Lambda}\, A_0 \ldots A_\xi \ldots]|_\Gamma.
\end{aligned}$$

In addition, two-place operations are defined by the equations

$$\begin{aligned}
|A_0|_\Gamma \wedge |A_1|_\Gamma &= |[A_0 \wedge A_1]|_\Gamma \\
|A_0|_\Gamma \vee |A_1|_\Gamma &= |[A_0 \vee A_1]|_\Gamma.
\end{aligned}$$

From the provability of substitutions of finite valid schemes, it follows that the defining equations for Boolean algebras in the Foreward on Algebra all hold in

$$\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \Gamma) = \langle B(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \Gamma) \neg \wedge \vee \rangle.$$

Therefore these algebras are non-degenerate Boolean algebras. The order is

$$|A|_\Gamma \leqslant |A'|_\Gamma \text{ if and only if } \vdash_\Gamma A \rightarrow A',$$

the zero is $|[p \wedge [\neg p]|_\Gamma$, the unit $|[p \vee [\neg p]|_\Gamma$. Since $|[\bigwedge A_0 \ldots A_\xi \ldots]|_\Gamma \leqslant |A_\nu|_\Gamma$ for all $\nu < \delta$, and $\vdash_\Gamma [C \rightarrow A_\xi]$ for all $\xi < \delta$ implies $\vdash_\Gamma [C \rightarrow [\bigwedge A_0 \ldots A_\xi \ldots]]$, the algebras are $\nearrow\alpha$-complete Boolean algebras with $|[\bigwedge A_0 \ldots A_\xi \ldots]|_\Gamma$ as the meet of $|A_0|_\Gamma, \ldots, |A_\xi|_\Gamma, \ldots$ The join is $|[\bigvee A_0 \ldots A_\xi \ldots]|_\Gamma$.

If $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ is complete, then $\vdash_\Gamma A \leftrightarrow A'$ if and only if $\Gamma \Vdash A \leftrightarrow A'$. Therefore, in this case, the algebra $\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \Gamma)$ is exactly the same as the algebra $\mathfrak{B}(\mathbf{L}_\alpha; \Gamma)$ of Sect. 4.2. Repeating Theorem 4.2.1 we have

6.3.1 *Theorem.* Suppose $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ is complete. Then

(i) If Γ formally consistent, then $\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \Gamma)$ is a non-degenerate $\nearrow\alpha$-representable Boolean algebra.

(ii) If Γ satisfiable, and $\bar{\Gamma} = \{A : \Vdash_\Gamma A\}$, then $\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \bar{\Gamma})$ is isomorphic to a non-degenerate $\nearrow\alpha$-field of sets.

It follows from the next two theorems that algebras $\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)) = \mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \phi)$ are the free $\nearrow\alpha$-complete Boolean algebras in which equations $T_{\mathscr{A}} = \mathbf{1}$ hold for all $\mathscr{A} \in \Sigma$. By this we mean that given any $\nearrow\alpha$-complete Boolean algebra $\mathfrak{B}$ such that these equations hold, there is an α-propositional language $\mathbf{L}_\alpha$ and an $\nearrow\alpha$-homomorphism from $\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha))$ onto $\mathfrak{B}$. In particular, algebras $\mathfrak{B}(\mathfrak{P}_\alpha(\mathbf{L}_\alpha))$ are the free $\nearrow\alpha$-complete Boolean algebras. In fact, the construction given here is the same as Rieger's in [33] except for terminology. Note first that if $\Gamma_0 \subseteq \Gamma_1$ and Γ_1 consistent, then the mapping $h(|A|_{\Gamma_0}) = |A|_{\Gamma_1}$ defines an $\nearrow\alpha$-homomorphism on

$$\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \Gamma_0) \text{ to } \mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \Gamma_1).$$

6.3.2 *Theorem.* If Γ consistent with respect to $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$, then equations $T_{\mathscr{A}} = \mathbf{1}$ hold in $\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \Gamma)$ for all $\mathscr{A} \in \Sigma$.

Proof: This is equivalent to saying that schemes $\mathscr{A} \in \Sigma$ hold in $\mathfrak{B} = \mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \Gamma)$. Let s be any assignment of propositional symbols of the scheme-language to $\mathfrak{B}$. We must show $s^*\mathscr{A} = \mathbf{1}$ for all $\mathscr{A} \in \Sigma$. Since s assigns an equivalence class $|A|_\Gamma$ to each symbol A_ξ or $A_{\mu\nu}$ of the scheme-language, there exists a function g on the set V of all such symbols such that $s(A_\xi) = |g(A_\xi)|_\Gamma$. Then $s^*\langle A_\xi\rangle = |S_g^V\langle A_\xi\rangle|_\Gamma$ for all atomic schemes $\langle A_\xi\rangle$. Since s^*, S_g^V and the function taking a formula A of $\mathbf{L}_\alpha$ to the element $|A|_\Gamma$ of $\mathfrak{B}$, are all $\nearrow\alpha$-homomorphisms, an easy induction shows that $s^*\mathscr{A} = |S_g^V\mathscr{A}|_\Gamma$ for all schemes $\mathscr{A}$. In particular, if $\mathscr{A} \in \Sigma$, then $S_g^V\mathscr{A}$ is an instance of $\mathscr{A}$ in $\mathbf{L}_\alpha$, and is therefore an axiom of the system. Hence $s^*\mathscr{A} = |S_g^V\mathscr{A}|_\Gamma = \mathbf{1}$.

6.3.3 *Theorem.* Suppose $\mathfrak{B}$ is a non-degenerate $\nearrow\alpha$-complete Boolean algebra in which equations $T_{\mathscr{A}} = \mathbf{1}$ hold for all $\mathscr{A} \in \Sigma$. Suppose $C \subseteq B$ $\nearrow\alpha$-generates $\mathfrak{B}$. Let $\mathbf{L}_\alpha$ be the α-propositional language having one propositional symbol p_c for each $c \in C$. Interpret $\mathbf{L}_\alpha$ in $\mathfrak{B}$ by the assignment $s(p_c) = c$ for each $c \in C$. Let $\Gamma = \{A : A$ is a formula of $\mathbf{L}_\alpha$ and $s^*A = \mathbf{1}\}$. Then Γ is consistent with respect to $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ and $\mathfrak{B} \cong \mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \Gamma)$. It then follows that $\mathfrak{B}$ is an $\nearrow\alpha$-homomorphic image of $\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha))$.

PROOF: By Theorem 6.1.1, if $\vdash_\Gamma A$, then $s^*A = \mathbf{1}$. Since $\mathbf{1} \neq \mathbf{0}$ in $\mathfrak{B}$, Γ is consistent. Since C $\nearrow\alpha$-generates $\mathfrak{B}$, $\operatorname{Rng} s^* = B$. If $s^*A = s^*A'$ then $[A \leftrightarrow A'] \in \Gamma$, hence $|A|_\Gamma = |A'|_\Gamma$. Therefore the mapping h assigning $|A|_\Gamma$ to s^*A is a function from B onto $B(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \Gamma)$. If $|A|_\Gamma = |A'|_\Gamma$ then $\vdash_\Gamma [A \leftrightarrow A']$, from which it follows by 6.1.1 that $s^*A = s^*A'$. Therefore h is one-one. Since s^* and the function taking formula A to $|A|_\Gamma$ are both $\nearrow\alpha$-homomorphisms, it is clear that h is also an $\nearrow\alpha$-homomorphism. Therefore $\mathfrak{B}$ and $\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \Gamma)$ are isomorphic. Since the mapping assigning $|A|_\Gamma$ to $|A|_\phi$ is also an $\nearrow\alpha$-homomorphism, $\mathfrak{B}$ is an $\nearrow\alpha$-homomorphic image of $\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha))$.

6.4 Metamathematical Proofs of Some Boolean Representation Theorems

When a language $\mathbf{L}_\alpha$ is interpreted in an algebra $\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \Gamma)$ the assignment $s(p) = |\langle p\rangle|_\Gamma$ will give formulas A values $s^*A = |A|_\Gamma$. Therefore if Γ is consistent with respect to $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$, Γ is satisfiable in a non-degenerate $\nearrow\alpha$-complete Boolean algebra in which equations $T_{\mathscr{A}} = \mathbf{1}$ hold for all $\mathscr{A} \in \Sigma$ by 6.3.2. Conversely, if Γ is satisfiable in such an algebra, Γ is consistent with respect to $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ by 6.1.1. Hence

6.4.1 *Theorem.* Let $\mathbf{L}_\alpha$ be an α-propositional language, Σ a set of α-propositional schemes. Then the following conditions are equivalent:

(i) Γ is formally consistent with respect to $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$.

(ii) Γ is satisfiable in a non-degenerate $\nearrow\alpha$-complete Boolean algebra in which equations $T_{\mathscr{A}} = \mathbf{1}$ hold for all $\mathscr{A} \in \Sigma$.

(iii) There is a non-degenerate $\nearrow\alpha$-complete Boolean algebra $\mathfrak{B}$ in which equations $T_{\mathscr{A}} = \mathbf{1}$ hold for all $\mathscr{A} \in \Sigma$, and an assignment s to $\mathfrak{B}$ such that $\{s^*A : A \in \Gamma\}$ has a non-zero lower bound.

6.4.2 *Theorem.* Let α be a regular infinite cardinal, γ any infinite cardinal. Then for sets Σ of valid α-propositional schemes, the following conditions are equivalent:

(i) $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ is complete whenever $\mathbf{L}_\alpha$ has γ propositional symbols.

(ii) If $\mathbf{L}_\alpha$ has γ propositional symbols, $\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha))$ is isomorphic to an $\nearrow\alpha$-field of sets.

(iii) Every $\nearrow\alpha$-complete Boolean algebra with a set of γ $\nearrow\alpha$-generators, in which equations $T_{\mathscr{A}} = \mathbf{1}$ hold for all $\mathscr{A} \in \Sigma$, is $\nearrow\alpha$-representable.

PROOF: (i) implies (ii) by 6.3.1 (ii) with $\Gamma = \phi$. (ii) implies (iii) by Theorem 6.3.3. We show (iii) implies (i). Suppose $\mathbf{L}_\alpha$ has γ propositional symbols. Then $\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha))$ is $\nearrow\alpha$-generated by $\{|\langle p\rangle| : p$ is a propositional symbol$\}$ and these elements are all distinct. Since equations $T_{\mathscr{A}} = \mathbf{1}$ hold in this algebra for all $\mathscr{A} \in \Sigma$, $\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha))$ is $\nearrow\alpha$-representable by (iii). If $\vdash A$ then $\Vdash A$ since the schemes are all valid. If $\Vdash A$, then A holds in all $\nearrow\alpha$-representable Boolean algebras by Theorem 4.3.4. In particular, for the assignment $s(p) = |\langle p\rangle|$, $s^*A = |A| = \mathbf{1}$. Hence $\vdash A$. Therefore the system is complete.

Dropping references to γ, Theorem 6.4.2 says that $\mathfrak{P}_\alpha(\Sigma)$ is a complete α-propositional calculus if and only if every $\nearrow\alpha$-complete Boolean algebra in which equations $T_{\mathscr{A}} = \mathbf{1}$ hold for all $\mathscr{A} \in \Sigma$ is $\nearrow\alpha$-representable. The completeness theorems of Sect. 5.3, Sect. 5.4 and Sect. 5.5 are then seen to be equivalent to known Boolean algebraic representation theorems.

6.4.3 *Loomis Representation Theorem.* Every ω-complete Boolean algebra is ω-representable.

PROOF: By 5.3.2 and 6.4.2.

Similarly, Theorem 5.4.3 yields the criterion of Chang in [1] for γ-representability.

6.4.4 *Theorem.* Let γ be an infinite cardinal. Then a γ-complete

Boolean algebra is γ-representable if and only if it satisfies the following conditions:

(C_γ) If every choice set of a doubly-indexed system $\langle b_{\mu\nu} : \mu < \gamma, \nu < \gamma\rangle$ of elements of B contains a complementary pair, then $\bigvee_{\mu<\gamma} \bigwedge_{\nu<\gamma} b_{\mu\nu} = \mathbf{1}$.

PROOF: If $\mathfrak{B}$ satisfies (C_γ) then schemes of Π_γ obviously hold in $\mathfrak{B}$. Then $\mathfrak{B}$ is γ-representable by 5.4.3 and 6.4.2. Conversely, if $\mathfrak{B}$ is γ-representable and $\bigvee_{\mu<\gamma} \bigwedge_{\nu<\gamma} b_{\mu\nu} \neq \mathbf{1}$, then according to the theorem in the Foreward on Algebra, there is a homomorphism h to $\mathfrak{B}_0$ sending $\bigvee_{\mu<\gamma} \bigwedge_{\nu<\gamma} b_{\mu\nu}$ to $\mathbf{0}$ and preserving all meets $\bigwedge_{\nu<\gamma} b_{\mu\nu}$ for $\mu < \gamma$ and the join $\bigvee_{\mu<\gamma} \bigwedge_{\nu<\gamma} b_{\mu\nu}$. Since $\bigwedge_{\mu<\gamma} \bigvee_{\nu<\gamma} \neg h(b_{\mu\nu}) = \mathbf{1}$ in $\mathfrak{B}_0$, there is a choice function $f \in \gamma^\gamma$ such that $\neg h(b_{\mu f(\mu)}) = \mathbf{1}$ for all $\mu < \gamma$. Then $\{b_{\mu f(\mu)} : \mu < \gamma\}$ is a choice set for $\langle b_{\mu\nu} : \mu < \gamma, \nu < \gamma\rangle$ containing no complementary pair.

Theorems 5.4.4 and 5.5.2 yield

6.4.5 *Theorem.* If α is strongly inaccessible, an $\nearrow\alpha$-complete Boolean algebra is $\nearrow\alpha$-representable if and only if it satisfies conditions (C_γ) for all $\gamma < \alpha$.

6.4.6 *Theorem.* If α is strongly inaccessible, an $\nearrow\alpha$-complete Boolean algebra is $\nearrow\alpha$-representable if and only if it is (γ, γ)-distributive for all $\gamma < \alpha$.

The analogue to 6.4.2 for strong completeness is the following:

6.4.7 *Theorem.* Let α be a regular infinite cardinal, γ any infinite cardinal. Then for sets Σ of valid α-propositional schemes the following conditions are equivalent:

(i) $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ is strongly complete whenever $\mathbf{L}_\alpha$ has γ propositional symbols.

(ii) Every $\nearrow\alpha$-complete Boolean algebra with a set of γ $\nearrow\alpha$-generators in which equations $T_{\mathscr{A}} = \mathbf{1}$ hold for all $\mathscr{A} \in \Sigma$, is isomorphic to an $\nearrow\alpha$-field of sets.

PROOF: Assume (i) and that $\mathfrak{B}$ is an $\nearrow\alpha$-complete Boolean algebra with a set of γ $\nearrow\alpha$-generators, in which equations $T_{\mathscr{A}} = \mathbf{1}$ hold for all $\mathscr{A} \in \Sigma$. Then $\mathfrak{B} \cong \mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \Gamma)$ for the language and set Γ of 6.3.3. By (i), if $\bar{\Gamma} = \{A : \Vdash_\Gamma A\}$, $\vdash_\Gamma A$ iff $\Vdash_\Gamma A$ iff $\bar{\Gamma} \Vdash A$. Hence $\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \Gamma) = \mathfrak{B}(\mathbf{L}_\alpha; \bar{\Gamma})$ which is isomorphic to an $\nearrow\alpha$-field of set by 4.2.1.

Conversely, if Γ is a satisfiable set of formulas, then Γ is consistent with respect to $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ since the schemes of Σ are valid. Assume (ii) and that Γ is consistent. We must show Γ satisfiable. But Γ is satisfiable in $\mathfrak{B}(\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha); \Gamma)$, a non-degenerate $\nearrow \alpha$-complete Boolean algebra in which equations $T_{\mathscr{A}} = \mathbf{1}$ hold for $\mathscr{A} \in \Sigma$. It is generated by elements $|\langle p\rangle|_\Gamma$, and if this set reduces to fewer than γ elements, γ new propositional symbols can be adjoined. For them, $|\langle p\rangle|_\Gamma \neq |\langle p'\rangle|_\Gamma$. Therefore the algebra has an $\nearrow \alpha$-generating set of power γ. By (ii), it is isomorphic to an $\nearrow \alpha$-field of sets, and by Theorem 4.3.4, it follows that Γ is semantically satisfiable.

This theorem with Theorem 5.5.5 yields this version of Tarski's Representation Theorem minus the property of atomicity:

6.4.8 *Theorem.* Let α be strongly inaccessible. For $\nearrow \alpha$-complete Boolean algebras with an $\nearrow \alpha$-generating set of power less than α the following conditions are equivalent:

(i) $\mathfrak{B}$ is isomorphic to an $\nearrow \alpha$-field of sets.

(ii) $\mathfrak{B}$ is (γ, γ)-distributive for all $\gamma < \alpha$.

6.5 Algebraic Proof of a Completeness Theorem

We proceed to the proof of the completeness theorem mentioned in Remark 5.5.3. Let $\gamma = \bigcup \{\gamma_n : n < \omega\}$, where $2 \exp \gamma_n < \gamma$ for all $n < \omega$. We must show that the γ^+-propositional calculus $\mathfrak{P}_{\gamma^+}([\bigwedge_{n<\omega} \mathscr{D}_{\gamma_n}])$ is complete. In view of Theorem 6.4.2, this is equivalent to showing that every γ-complete Boolean algebra that is γ_n-distributive for all $n < \omega$ is γ-representable. It is convenient to use a modification of the Chang distributive law (C_γ) of 6.4.4 for γ-representability.

6.5.1 *Lemma.* Let γ be an arbitrary infinite cardinal. Then a γ-complete Boolean algebra is γ-representable if and only if it satisfies the following condition:

(C'_γ) Given $b \neq \mathbf{0}$ and doubly-indexed system $\langle b_{\mu\nu} : \mu < \gamma, \nu < \gamma\rangle$ such that $\bigvee \langle b_{\mu\nu} : \nu < \gamma\rangle = \mathbf{1}$ for all $\mu < \gamma$, there is a choice set C such that $C \cup \{b\}$ does not contain a complementary pair.

PROOF: We show (C'_γ) equivalent to (C_γ). Assume (C_γ), let $b \neq \mathbf{0}$ and doubly-indexed system $\langle b_{\mu\nu} : \mu < \gamma, \nu < \gamma\rangle$ be given such that $\bigvee \langle b_{\mu\nu} : \nu < \gamma\rangle = \mathbf{1}$ for all $\mu < \gamma$. Make a new array with rows $\langle \neg b_{\mu\nu} : \nu < \gamma\rangle$ for $\mu < \gamma$, and a row $\langle \neg b \ldots \neg b \ldots\rangle$ of length γ. Call the new array $\langle b'_{\mu\nu} : \mu < \gamma, \nu < \gamma\rangle$. If for every choice set C of the original array, $C \cup \{b\}$ contained a complementary pair, then every choice set of the new array would contain a complementary pair. But then we would have

$$\bigvee_{\mu<\gamma} \bigwedge_{\nu<\gamma} b'_{\mu\nu} = \mathbf{1} = \neg b$$

by (C_γ). This would imply $b = \mathbf{0}$, a contradiction.

Conversely, assume (C'_γ) and let $\langle b_{\mu\nu} : \mu < \gamma, \nu < \gamma \rangle$ be a doubly-indexed system such that every choice set contains a complementary pair. Suppose, contrariwise,

$$\neg b = \bigvee_{\mu<\gamma} \bigwedge_{\nu<\gamma} b_{\mu\nu} \neq \mathbf{1}.$$

Then $b \neq \mathbf{0}$. Make a new array with the following rows, each of length γ:

$$\langle \neg \bigwedge_{\nu<\gamma} b_{\mu\nu}\, \neg b \ldots \neg b \ldots \rangle,\ \langle \bigwedge_{\nu<\gamma} b_{\mu\nu}\, \neg b_{\mu 0} \ldots \neg b_{\mu\nu} \ldots \rangle$$

for each $\mu < \gamma$. Then each of the rows has join $\mathbf{1}$. By (C'_γ), there is a choice set C' for the new array such that $C' \cup \{b\}$ does not contain a complementary pair. But this would imply that the original array had a choice set that did not contain a complementary pair.

6.5.2 *Theorem.* Let $\gamma = \bigcup \{\gamma_n : n < \omega\}$, where the γ_n are infinite cardinals less than γ. Then every γ-complete Boolean algebra that is (γ_n, γ_n)-distributive for all $n < \omega$ is γ-representable.

PROOF: Let $b \neq \mathbf{0}$ and doubly-indexed system $\langle b_{\mu\nu} : \mu < \gamma, \nu < \gamma \rangle$ be given such that $\bigvee \langle b_{\mu\nu} : \nu < \gamma \rangle = \mathbf{1}$ for all $\mu < \gamma$. We must show that there is a choice set C for the array such that $C \cup \{b\}$ does not contain a complementary pair. We may as well assume that $\gamma_n < \gamma_{n+1}$ for all $n < \omega$. Let $\Theta_n = \gamma_n \sim \bigcup_{i<n} \gamma_i$, $b'_{\mu n} = \bigvee \langle b_{\mu\nu} : \nu \in \Theta_n \rangle$ for $n < \omega$, $\mu < \gamma$. If $\Theta \subseteq \gamma$, we will call a set C a *choice set for* Θ if C contains at least one element from each row μ of the given array for $\mu \in \Theta$.

We claim

(1) Given $b' \neq \mathbf{0}$, $n < \omega$, there is f on Θ_n to ω such that
$$b' \wedge \bigwedge \langle b'_{\mu f(\mu)} : \mu \in \Theta_n \rangle \neq \mathbf{0}.$$

For $\mathbf{1} = \bigvee \langle b_{\mu\nu} : \nu < \gamma \rangle = \bigvee \langle b'_{\mu n} : n < \omega \rangle$ for all $\mu < \gamma$. Therefore, $b' = b' \wedge \bigwedge \langle \bigvee \langle b'_{\mu n} : n < \omega \rangle : \mu \in \Theta_n \rangle$. The existence of f is a consequence of the (γ_n, ω)-distributive law.

Similarly, we claim

(2) Given $b' \neq \mathbf{0}$, $n < \omega$, $m < \omega$ and $\Theta \subseteq \Theta_m$, if $b' \leqslant \wedge \langle b'_{\mu n} : \mu \in \Theta \rangle$, there is a choice set C for Θ such that

$$b' \wedge \wedge C \neq \mathbf{0}.$$

For $\mathbf{0} \neq b' = b' \wedge \wedge \langle \vee \langle b_{\mu\nu} : \nu \in \Theta_n \rangle : \mu \in \Theta \rangle$. The existence of C is a consequence of the (γ_m, γ_n)-distributive law.

By (1), there is f_0 on Θ_0 to ω such that $b \wedge \wedge \langle b'_{\mu f_0(\mu)} : \mu \in \Theta_0 \rangle \neq \mathbf{0}$. Although we cannot find a choice set for all of Θ_0 whose meet with b is not $\mathbf{0}$, we can do the following: Let $\Theta_{0m} = \{\mu : \mu \in \Theta_0$ and $f_0(\mu) = m\}$. Since $b \wedge \wedge \langle b'_{\mu f_0(\mu)} : \mu \in \Theta_0 \rangle \leqslant \wedge \langle b'_{\mu 0} : \mu \in \Theta_{00} \rangle$, there is a choice set C_0 for Θ_{00} such that $b \wedge \wedge \langle b'_{\mu f_0(\mu)} : \mu \in \Theta_0 \rangle \wedge \wedge C_0 \neq \mathbf{0}$ by (2). The choices for the other Θ_{0m} will be picked up later.

Suppose that for $k \leqslant n$ we have found functions f_k on Θ_k to ω and C_k such that

$$(P_k) \qquad b \wedge \bigwedge_{i \leqslant k} \wedge \langle b'_{\mu f_i(\mu)} : \mu \in \Theta_i \rangle \wedge \bigwedge_{i \leqslant k} \wedge C_i \neq \mathbf{0},$$

and C_k is a choice set for $\cup \{\Theta_{ij} : i + j = k\}$, where $\Theta_{ij} = \{\mu : \mu \in \Theta_i$ and $f_i(\mu) = j\}$. We must find f_{n+1}, C_{n+1} satisfying (P_{n+1}).

Letting $b_n = b \wedge \bigwedge_{i \leqslant n} \wedge \langle b'_{\mu f_i(\mu)} : \mu \in \Theta_i \rangle \wedge \bigwedge_{i \leqslant n} \wedge C_i$, use (1) to find f_{n+1} on Θ_{n+1} to ω such that $b_n \wedge \wedge \langle b'_{\mu f_{n+1}(\mu)} : \mu \in \Theta_{n+1} \rangle = b'_{n+1} \neq \mathbf{0}$. We must still find choice function C_{n+1} for $\cup \{\Theta_{ij} : i + j = n + 1\} = \Theta_{0, n+1} \cup \Theta_{1, n} \cup \ldots \cup \Theta_{i, n+1-i} \cup \ldots \cup \Theta_{n+1, 0}$. Since $b'_{n+1} \leqslant \wedge \langle b'_{\mu f_0(\mu)} : \mu \in \Theta_0 \rangle \leqslant \wedge \langle b'_{\mu n+1} : \mu \in \Theta_{0,n+1} \rangle$, (2) implies that there is a choice set C^0_{n+1} for $\Theta_{0, n+1}$ such that $b'_{n+1} \wedge \wedge C^0_{n+1} \neq \mathbf{0}$. Given $i < n + 1$ and choice sets C^i_{n+1} for $\Theta_{i, n+1-i}$ such that $b'_{n+1} \wedge \wedge C^0_{n+1} \wedge \ldots \wedge \wedge C^i_{n+1} \neq \mathbf{0}$, (2) again implies that choice set C^{i+1}_{n+1} for $\Theta_{i+1, n-i}$ can be found so that $b'_{n+1} \wedge \bigwedge_{j \leqslant i+1} \wedge C^j_{n+1} \neq \mathbf{0}$. Let $C_{n+1} = \cup \{C^i_{n+1} : i \leqslant n + 1\}$. Then C_{n+1} has the desired property (P_{n+1}).

Having completed the definition of the C_n by induction, let $C = \cup \{C_n : n < \omega\}$. Then C is a choice set for the entire array, and $C \cup \{b\}$ is non-contradictory since $b \wedge \bigwedge_{i \leqslant n} \wedge C_i \neq \mathbf{0}$ for all $n < \omega$.

6.5.3 *Theorem.* Let $\gamma = \cup \{\gamma_n : n < \omega\}$ where $2 \exp \gamma_n \leqslant \gamma$. Then $\mathfrak{P}_{\gamma^+}([\bigwedge_{n<\omega} \mathscr{D}_{\gamma_n}])$ is complete:

Proof: By 6.5.2 and 6.4.2. The conditions $2 \exp \gamma_n \leqslant \gamma$ are added only to insure that the distributive scheme is a γ^+-propositional scheme.

CHAPTER 7

NON-DEDUCIBILITY IN INFINITARY PROPOSITIONAL LOGIC

We will not take time to give independence proofs for the schemes and rules of basic propositional calculus, but turn instead to questions of independence of the various distributive laws that have been introduced.

7.1 Summary of Results and Open Problems

Let α be a regular infinite cardinal, Σ, Σ' sets of α-propositional schemes. Then we say

$$\mathfrak{P}_\alpha(\Sigma) \approx \mathfrak{P}_\alpha(\Sigma')$$

if and only if for every α-propositional language L_α, $\mathfrak{P}_\alpha(\Sigma)(\mathsf{L}_\alpha)$ and $\mathfrak{P}_\alpha(\Sigma')(\mathsf{L}_\alpha)$ have the same formal theorems. Similarly,

$$\mathfrak{P}_\alpha(\Sigma) \leqslant \mathfrak{P}_\alpha(\Sigma')$$

if and only if for every α-propositional language L_α, every formula provable in $\mathfrak{P}_\alpha(\Sigma)(\mathsf{L}_\alpha)$ is provable in $\mathfrak{P}_\alpha(\Sigma')(\mathsf{L}_\alpha)$. Finally,

$$\mathfrak{P}_\alpha(\Sigma) \prec \mathfrak{P}_\alpha(\Sigma')$$

if $\mathfrak{P}_\alpha(\Sigma) \leqslant \mathfrak{P}_\alpha(\Sigma')$ and not $\mathfrak{P}_\alpha(\Sigma) \approx \mathfrak{P}_\alpha(\Sigma')$.

7.1.1 *Summary.* Relative to the Chang distributive laws,

$$\mathfrak{P}_\alpha(\bigcup \{\Pi_\gamma : \gamma < \alpha\})$$

is complete and

(i) If $\gamma^+ < \alpha$ is infinite, then $\mathfrak{P}_\alpha(\Pi_\gamma) \prec \mathfrak{P}_\alpha(\Pi_{\gamma_+})$ if γ regular. If γ singular, $\mathfrak{P}_\alpha(\Pi_\gamma) \leqslant \mathfrak{P}_\alpha(\Pi_{\gamma_+})$ but the question whether or not the inequality is strict is open.

(ii) If $\gamma < \alpha$ is an infinite limit cardinal, then if $\gamma = \omega$, $\mathfrak{P}_\alpha(\Pi_\gamma) \approx \mathfrak{P}_\alpha$. If γ is a union of ω smaller cardinals γ_n, $n < \omega$, such that $2 \exp \gamma_n < \alpha$, then $\mathfrak{P}_\alpha(\bigcup \{\Pi_\beta : \beta < \gamma\}) \approx \mathfrak{P}_\alpha(\Pi_\gamma)$. For all other such

γ, $\mathfrak{P}_\alpha(\bigcup\{\Pi_\beta: \beta < \gamma\}) \leqslant \mathfrak{P}_\alpha(\Pi_\gamma)$ but the question whether or not the equality is strict seems to be open.

Relative to the ordinary distributive laws,

(iii) If $\gamma < \alpha$ is an infinite regular cardinal such that $2 \exp \gamma < \alpha$, then $\mathfrak{P}_\alpha(\{\mathscr{D}_\beta: \beta < \gamma\}) \prec \mathfrak{P}_\alpha(\mathscr{D}_\gamma)$.

(iv) If $\gamma < \alpha$ is an infinite singular cardinal such that $2 \exp \gamma < \alpha$, then $\mathfrak{P}_\alpha(\{\mathscr{D}_\beta: \beta < \gamma\}) \leqslant \mathfrak{P}_\alpha(\mathscr{D}_\gamma)$ but the question whether or not the inequality is strict seems to be open.

(v) If $\gamma < \alpha$ is an infinite cardinal such that $2 \exp \gamma < \alpha$, then $\mathfrak{P}_\alpha(\Pi_\gamma) \leqslant \mathfrak{P}_\alpha(\mathscr{D}_\gamma)$. The inequality is strict if γ regular. Whether or not it is strict if γ singular is open.

(vi) If $\gamma < \alpha$ is an infinite cardinal such that $2 \exp \gamma < \alpha$, then $\mathfrak{P}_\alpha(\mathscr{D}_\gamma) \leqslant \mathfrak{P}_\alpha(\Pi_{2 \exp \gamma})$. Whether or not the inequality is strict is open.

We have already remarked that the only cases $\alpha > \omega$ where we have found complete α-propositional calculi by adding a single scheme to $\mathfrak{P}_\alpha$, are the cases $\alpha = \gamma^+$ where γ is a union of ω smaller cardinals γ_n such that $2 \exp \gamma_n \leqslant \gamma$. Once we have established (i) it is clear that it is not possible to find such calculi for α inaccessible. For if $\mathscr{A}$ is an α-propositional scheme and α inaccessible, then there is a regular infinite cardinal $\gamma < \alpha$ such that $\mathscr{A}$ is a γ-propositional scheme. Hence if $\mathscr{A}$ is valid, $\mathfrak{P}_\alpha(\mathscr{A}) \leqslant \mathfrak{P}_\alpha(\Pi_\gamma) \prec \mathfrak{P}_\alpha(\Pi_{\gamma^+})$. Thus $\mathfrak{P}_\alpha(\mathscr{A})$ is not complete.

7.1.2 *Algebraic Summary*. The statements of 7.1.1 all follow from the following:

(i) Every γ^+-representable Boolean algebra is γ-representable. If γ regular, there is a complete γ-representable Boolean algebra that is not γ^+-representable. If γ singular, the question of the existence of such an algebra, even one that is only γ^+-complete, is open.

(ii) Every ω-complete Boolean algebra is ω-representable. If γ is a union of ω smaller cardinals γ_n, $n < \omega$, such that $2 \exp \gamma_n < \gamma$, then a γ-complete Boolean algebra is γ-representable if and only if it is γ_n-representable for all $n < \omega$. For all other infinite limit cardinals γ, the question of the existence of a γ-complete $\nearrow \gamma$-representable Boolean algebra which is not γ-representable, is open.

(iii) If γ regular, infinite, there exists a complete Boolean algebra which is β-distributive for all $\beta < \gamma$ but not γ-distributive.

(iv) If γ singular and infinite, the question of the existence of a $(2 \exp \gamma)$-complete Boolean algebra which is β-distributive for all $\beta < \gamma$ but not γ-distributive, is open.

(v) Every γ-complete γ-distributive Boolean algebra is γ-representable. There is a complete γ-representable, not γ-distributive Boolean algebra for each regular infinite γ. The existence of such an algebra is an open question for infinite singular γ, even if only $(2 \exp \gamma)$-completeness is required,

(vi) Every $(2 \exp \gamma)$-representable Boolean algebra is γ-distributive. The question of the existence of a $(2 \exp \gamma)$-complete γ-distributive, not $(2 \exp \gamma)$-representable Boolean algebra, is open.

The various completeness assumptions are essential. The γ-representable, not γ-distributive algebras of Example 4.2.4 do not suffice for (v). The statements of 7.1.2 all follow from the theorems of Chapter 6 and from the examples in the next section of complete Boolean algebras $\mathfrak{B}_\gamma$ for each infinite regular cardinal γ having the following properties: $\mathfrak{B}_\gamma$ is γ-representable but not (γ^+)-representable, and β-distributive for all $\beta < \gamma$ but not γ-distributive. In fact, $\mathfrak{B}_\gamma$ is (β, κ)-distributive for all $\beta < \gamma$ and all cardinals κ. These examples appear also in my note [16].

Examples of complete Boolean algebras that are γ-representable, (β, κ)-distributive for all $\beta < \gamma$ and all κ, not γ-distributive, were given for regular infinite γ by Smith in [39], Scott in [36]. These algebras are also not (γ^+)-representable if we assume the continuum hypothesis $\gamma^+ = 2 \exp \gamma$, for then the γ-distributive law is a (γ^+)-equation. However, the examples of Sect. 7.2 can be proved not (γ^+)-representable making no use of the continuum hypothesis. The algebras were suggested by the example of 4.1.2 of a non-satisfiable set of (γ^+)-propositional formulas which is semantically consistent. If the construction could be modified to show that the existence of a non-satisfiable set of γ-propositional formulas which is semantically consistent, implies the existence of a complete $\nearrow \gamma$-representable, not γ-representable Boolean algebra, then the open questions (ii) would at least be settled for those infinite regular γ for which such sets exist; namely, for the incompact cardinals of Sect. 10.2.

Smith in [39] proves the existence of a complete, γ-distributive Boolean algebra not $(2 \exp \gamma)$-representable for infinite γ, but only under an assumption that is itself in question. For example, if $\gamma = \omega$ it implies that the answer to Souslin's Problem is negative. This result serves to enforce the conjecture that examples for (vi) can be found without making such an assumption. For more information on distributivity see Pierce's papers [25], [26], [27], [28], and Smith-Tarski [40] as well as the papers already cited.

7.2 Examples of Complete γ-Representable, not γ^+-Representable Boolean Algebras

Let γ be a regular infinite cardinal. We construct complete γ-representable Boolean algebras $\mathfrak{B}_\gamma$ not satisfying the inequality

$$(1)_\gamma \qquad \bigwedge_{\nu<\gamma^+} \bigvee_{\mu<\gamma} x_{\nu\mu} \leqslant \bigvee_{\nu\neq\nu'<\gamma} \bigvee_{\mu<\gamma} (x_{\nu\mu} \wedge x_{\nu'\mu}).$$

Since this inequality holds in γ^+-fields of sets, it follows that $\mathfrak{B}_\gamma$ is not γ^+-representable.

Considering the set X of all one-one functions on γ into γ^+ as points, take as a basis for open subsets of X the empty set, together with sets $S(g) = \{f : f \in X \text{ and } f|\text{Dom } g = g\}$, where g is a one-one function on a subset of γ having power less than γ, into γ^+.

If $\{S(g_i) : i \in I\}$ is a collection of fewer than γ non-empty basic sets, then we see that $\bigcap \{S(g_i) : i \in I\} \neq \phi$ if and only if $\bigcup \{g_i : i \in I\}$ is a one-one function. Since the regularity of γ guarantees card $\bigcup \{\text{Dom}(g_i) : i \in I\} < \gamma$, $\bigcap \{S(g_i) : i \in I\}$ is either empty or is equal to $S(g)$, where $g = \bigcup \{g_i : i \in I\}$. Thus the collection of basic open sets is closed under intersections of fewer than γ elements. Moreover, since $\bigcup \{g_i : i \in I\}$ is a one-one function if and only if $g_i \cup g_{i'}$ is a one-one function for each pair i, $i' \in I$, we have the following compactness property:

7.2.1 *Lemma.* If C is a collection of fewer than γ non-empty basic open sets such that no pair has an empty intersection, then $\bigcap C$ is a non-empty basic open set.

Basic sets are open-closed, since
$X \sim S(g) = X \sim \bigcap \{S(\{(\mu,\nu)\}) : (\mu,\nu) \in g\} = \bigcup \{X \sim S(\{(\mu,\nu)\}) : (\mu,\nu) \in g\}$,
while for any pair $(\mu, \nu) \in \gamma \times \gamma^+$,

$$X \sim S(\{(\mu, \nu)\}) = \bigcup \{S(\{(\mu, \nu')\}) : \nu \neq \nu' < \gamma^+\}.$$

Let $\mathfrak{B}_\gamma$ be the algebra of regular open sets of this space. This algebra is described in the Foreward on Algebra. It is complete, as are all algebras of regular open sets in any topological space. The infinitary operations are

$$\bigvee_\xi S_\xi = \text{in cl} \bigcup_\xi S_\xi, \quad \bigwedge_\xi S_\xi = \text{in cl} \bigcap_\xi S_\xi.$$

The zero is ϕ, the unit X.

7.2.2 *Lemma.* (i) If $\beta < \gamma$ and the $S_\xi \in B_\gamma$, then $\bigcap \{S_\xi : \xi < \beta\} = \bigwedge \{S_\xi : \xi < \beta\}$ in $\mathfrak{B}_\gamma$.

(ii) If $\beta < \gamma$, then a union of β nowhere dense sets is nowhere dense.

The second statement follows by the classical Baire category argument extended to β with the aid of compactness property 7.2.1. The first follows from the fact that β-intersections of open sets are open.

7.2.3 *Theorem.* $\mathfrak{B}_\gamma$ is neither γ^+-representable nor γ-distributive.

PROOF: If $S(g)$ is any non-empty basic open set with $\nu \in \mathrm{Rng}(g)$ then $S(g) \subseteq S(\{(\mu, \nu)\})$ where $\mu = g^{-1}(\nu)$. If $S(g)$ is a non-empty basic open set with $\nu \notin \mathrm{Rng}(g)$ then we can choose $\mu \in \gamma \sim \mathrm{Dom}(g)$ since $\mathrm{Dom}(g)$ has power less than γ. For such a μ, $g \cup \{(\mu, \nu)\}$ is a one-one function and therefore $S(g) \cap S(\{(\mu, \nu)\}) \neq \phi$. Since every non-empty open set intersects $\bigcup \{S(\{(\mu, \nu)\}) : \mu < \gamma\}$, we see that $\bigwedge_{\nu<\gamma^+} \bigvee_{\mu<\gamma} S(\{(\mu, \nu)\}) = X$ in $\mathfrak{B}_\gamma$. On the other hand, $S(\{(\mu, \nu)\}) \cap S(\{(\mu, \nu')\})$ is empty for any $\nu \neq \nu' < \gamma^+$ and $\mu < \gamma$. Hence $\mathfrak{B}_\gamma$ is not γ^+-representable since $(1)_\gamma$ fails.

To show $\mathfrak{B}_\gamma$ not γ-distributive, it suffices to show that it is not (γ, γ^+)-distributive, for it is known that a $(2 \exp \gamma)$-complete γ-distributive Boolean algebra is $(\gamma, 2 \exp \gamma)$-distributive. See Smith-Tarski [40] or Pierce [26]. Let $S_{\mu\nu} = S(\{(\mu, \nu)\})$ for $\mu < \gamma$, $\nu < \gamma^+$. Then $\bigcup \{S_{\mu\nu} : \nu < \gamma^+\} = X$ for all $\mu < \gamma$. Hence all these joins are X in $\mathfrak{B}_\gamma$. On the other hand, if h maps γ to γ^+, then $\bigcap \{S_{\mu h(\mu)} : \mu < \gamma\}$ is either empty or is a single point. In either case $\bigwedge \{S_{\mu h(\mu)} : \mu < \gamma\} = \phi$. Therefore the (γ, γ^+)-distributive law fails in $\mathfrak{B}_\gamma$.

7.2.4 *Theorem.* $\mathfrak{B}_\gamma$ is γ-representable and (β, κ)-distributive for all $\beta < \gamma$, all cardinals κ.

PROOF: For the γ-representability, we check condition (C'_γ) of 6.5.1. Suppose $S \in B_\gamma$ and $\langle S_{\mu\nu} : \mu < \gamma, \nu < \gamma \rangle$ is a doubly-indexed array of elements of B_γ such that $S \neq \phi$ and $\bigvee \{S_{\mu\nu} : \nu < \gamma\} = X$ for all $\mu < \gamma$. We must show that that the array has a choice set C such that $C \cup \{S\}$ does not contain a complementary pair.

Choose a non-empty basic set $S(g) \subseteq S$. Suppose for $\mu < \gamma$, we have chosen $\nu_\lambda < \gamma$ and basic sets $S(g_\lambda)$ for all $\lambda < \mu$ such that

$$\phi \neq S(g) \cap \bigcap \{S(g_\xi) : \xi \leqslant \lambda\} \subseteq S_{\lambda\nu_\lambda}.$$

We show that such a choice ν_μ and $S(g_\mu)$ can be made for μ. By the compactness property, $S(g) \cap \bigcap \{S(g_\xi) : \xi < \mu\}$ is a non-empty basic open set. Since $\mathrm{cl} \bigcup \{S_{\mu\nu} : \nu < \gamma\} = X$, there is $\nu_\mu < \gamma$ such that $S(g) \cap \bigcap \{S(g_\xi) : \xi < \mu\} \cap S_{\mu\nu_\mu} \neq \phi$. This set is open and therefore contains a non-empty basic open set $S(g_\mu)$. Then

$$\phi \neq S(g) \cap \bigcap \{S(g_\xi) : \xi \leqslant \mu\} \subseteq S_{\mu\nu_\mu}.$$

Let $C = \{S_{\mu\nu_\mu} : \mu < \gamma\}$. Clearly the sequence $\langle \nu_\mu : \mu < \gamma \rangle$ was chosen so that $C \cup \{S\}$ cannot contain a complementary pair.

Finally, to show $\mathfrak{B}_\gamma(\beta, \kappa)$-distributive, we use an argument like that of Scott in [36]. It suffices to show that if $\langle T_{\mu\nu} : \mu < \beta, \nu < \kappa \rangle$ is a doubly-indexed array of elements of B_γ such that all the joins $\bigvee \{T_{\mu\nu} : \nu < \kappa\} = T \neq \phi$ for $\mu < \beta$, then the array has a choice set C such that $\bigwedge C \neq \phi$. If, contrariwise, all the meets $\bigwedge C = \phi$ for choice sets C, then since β-meets are β-intersections in $\mathfrak{B}_\gamma$, all intersections $\bigcap C = \phi$. By the (β, κ)-distributive law for sets, $\bigcup \{\bigcap C : C \text{ is a choice set}\} = \bigcap_{\mu<\beta} \bigcup_{\nu<\kappa} T_{\mu\nu} = \phi$. But $T \sim \bigcup_{\nu<\kappa} T_{\mu\nu}$ is nowhere dense for each $\mu < \beta$, since the joins of the rows are all T. By 7.2.2 (ii), $\bigcup_{\mu<\beta} (T \sim \bigcup_{\nu<\kappa} T_{\mu\nu}) = T \sim \bigcap_{\mu<\beta} \bigcup_{\nu<\kappa} T_{\mu\nu}$ is nowhere dense. But this set is $T \neq \phi$, $T \in B_\gamma$. This contradicts $T =$ in cl T.

CHAPTER 8

SYSTEMS OF FORMULAS OF INFINITE LENGTH

At this point, we turn our attention to the full systems of formulas of infinitary predicate languages. The notion of formula is broader than that discussed informally in Chapter 1, for these formulas may be built from infinitely long atomic formulas.

Attached to an infinitary predicate language are regular infinite cardinals α, o, to serve as bounds on the lengths of conjunctions and on the lengths of terms, respectively. Also attached to such a language are infinite cardinals β, π, to serve as bounds on the lengths of quantifications, and on the number of places of predicate symbols, respectively. The language has a supply of primitive symbols among which are the brackets [,], and other symbols classified as follows:

Individual variables
Individual constants
Special two-place operation symbols
ζ-place operation symbols, $0 < \zeta < o$.
Infinitary operation symbols
Special two-place predicate symbols
η-place predicate symbols, $0 < \eta < \pi$.
Infinitary predicate symbols
Special two-place propositional operation symbols
δ-place propositional operation symbols, $o < \delta < \alpha$
Infinitary propositional operation symbols
Quantifiers

The individual variables together with the individual constants will be referred to as *individual symbols*. It is assumed that the symbols are all distinct and not themselves sequences. *Expressions* are sequences of symbols having length less than the smallest regular cardinal that is at least as large as $\alpha \cup \beta \cup o \cup \pi$. As a rule, sequential brackets and signs for concatenation will be dropped when writing expressions. Thus "$[\Phi A_0 \ldots A_\xi \ldots]$" is one way of writing the expression $\langle[\Phi\rangle^\frown(^\frown\langle A_\xi : \xi < \delta\rangle)^\frown\langle]\rangle$.

8.1 The (α, β, o, π)-Systems of Formulas

The terms of an infinitary language have been described in Chapter 3. *Atomic formulas* are those expressions having one of the following forms:

(1′) $[T_0 P T_1]$, where T_0, T_1 are terms and P is a special two-place predicate symbol.

(2′) $[Q^\eta T_0 \ldots T_\xi \ldots]$, where $\langle T_\xi : \xi < \eta\rangle$ is an η-tuple of terms and Q^η is an η-place predicate symbol.

(3′) $[Q T_0 \ldots T_\xi \ldots]$, where $0 < \eta < \pi$, $\langle T_\xi : \xi < \eta\rangle$ is an η-tuple of terms and Q an infinitary predicate symbol.

The set of *formulas* is the intersection of all sets Δ of expressions containing the atomic formulas and closed under rules of (α, β)-formula-formation:

(1) If $A_0, A_1 \in \Delta$ and Ψ is a special two-place propositional operation symbol, then $[A_0 \Psi A_1] \in \Delta$.

(2) If $\langle A_\xi : \xi < \delta\rangle$ is a δ-tuple of expressions in Δ and Φ^δ a δ-place propositional operation symbol, then $[\Phi^\delta A_0 \ldots A_\xi \ldots] \in \Delta$.

(3) If $\langle A_\xi : \xi < \delta\rangle$ is a δ-tuple of expressions in Δ, $0 < \delta < \alpha$, and Φ is an infinitary propositional operation symbol, then $[\Phi A_0 \ldots A_\xi \ldots] \in \Delta$.

(4) If $A \in \Delta$, χ is a quantifier, v a sequence of individual variables such that $0 < \mathrm{Dom}(v) < \beta$, then $[\chi\, v\, A] \in \Delta$.

8.1.1 *Induction Principle for Formulas.* If Δ is any set of expressions containing the atomic formulas and closed under rules (1), (2), (3), (4) of (α, β)-formula-formation, then Δ contains all formulas.

Let $F_{\alpha\beta o\pi}$ be the set of all formulas of an infinitary language. The *algebra of formulas* has underlying set $F_{\alpha\beta o\pi}$, has a δ-place operation for each δ-place propositional operation symbol, an $\nearrow\alpha$-place operation for each infinitary propositional operation symbol, with values given by (1), (2), (3). For example, the operation assigned to Φ has value $[\Phi A_0 \ldots A_\xi \ldots]$ for δ-tuple $\langle A_\xi : \xi < \delta\rangle$.

Note that the structure generated by the quantifiers is not reflected in the algebra of formulas. One way of bringing the quantifiers into the picture is to associate to each pair (χ, v), v a sequence of variables having length less than β, a one-place function whose value for A is $[\chi v A]$. This is the approach that leads to the infinitary analogue of the cylindric and polyadic algebras. The quantifier structure is built into the systems for interpretation of formulas in such a way that this appears as a special case.

8.2 Systems for the Interpretation of Formulas

These systems are very broad. They comprehend not only the intended semantic interpretations of the formulas, the ones used informally in Chapter 1, but also interpretations where formulas take values other than truth values. For example, values are elements of arbitrary complete Boolean algebras in Example 8.2.4, formulas of other systems in Example 8.2.5, ordinals in Example 8.2.6. The general models of Chapter 12 are also examples.

8.2.1 *Definition.* A *system for the interpretation* of formulas is a structure $\langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ B\ \mathsf{O}'\ \mathsf{Q}\ S\rangle$ where D, B are non-empty sets and

(i) $\langle D\ \mathsf{O}\rangle$ is an algebra having the similarity type of the algebra of terms.

(ii) $\langle B\ \mathsf{O}'\rangle$ is an algebra having the similarity type of the algebra of formulas.

(iii) C assigns an element of D to each individual constant.

(iv) R is a function on predicate symbols assigning an η-place operation on D to B for each η-place predicate symbol, an $\nearrow\pi$-place operation on D to B for each infinitary predicate symbol.

(v) S is a set of functions on individual variables to D (to be called *assignments*).

(vi) Q assigns to pairs (χ, v) consisting of quantifiers and sequences v of variables having length less than β, a function on non-empty subsets of B having power at most card(S) to B.

8.2.2 *Recursion Principle for Formulas.* Let $\langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ B\ \mathsf{O}'\ \mathsf{Q}\ S\rangle$ be a system for the interpretation of formulas. Extend functions $s \in S$ over all individual symbols by letting s be C on individual constants. Let s^* be the unique homomorphism on the algebra of terms to $\langle D\ \mathsf{O}\rangle$ such that $s^*(\langle x\rangle) = s(x)$ for individual symbols x.

Then there exists a unique function V on $S \times F_{\alpha\beta o\pi}$ to B satisfying the following conditions:

(i) For all terms T, special two-place predicate symbols P, η-place predicate symbols Q^η, infinitary predicate symbols Q,

$$V(s, [T_0 P T_1]) = \mathsf{R}(P)(\langle s^*(T_0)\, s^*(T_1)\rangle)$$
$$V(s, [Q^\eta T_0 \ldots T_\xi \ldots]) = \mathsf{R}(Q^\eta)(\langle s^*(T_0) \ldots s^*(T_\xi) \ldots\rangle)$$
$$V(s, [Q T_0 \ldots T_\xi \ldots]) = \mathsf{R}(Q)(\langle s^*(T_0) \ldots s^*(T_\xi) \ldots\rangle).$$

(ii) For each $s \in S$, the function $h(A) = V(s, A)$ is a homomorphism from the algebra of formulas to $\langle B\ \mathsf{O}'\rangle$.

(iii) For formulas A, quantifiers χ, sequences v of variables having length less than β,

$$V(s, [\chi v A]) = \mathsf{Q}(\chi, v)\{V(s_{v,t}, A) : t \in D^{\mathrm{Rng}(v)} \text{ and } s_{v,t} \in S\}$$

where $s_{v,t} = \mathrm{Repl}_t^{\mathrm{Rng}(v)} s$; i.e., t on $\mathrm{Rng}(v)$, s elsewhere.

The function V is called the *valuation function*, B the set of values.

The proof of the existence and uniqueness of V is deferred to the next section in order that we may first see some examples.

8.2.3 *Example.* The underlying systems of formulas of the languages $\mathbf{L}_{\alpha\beta o\pi}$ that will be dealt with in the succeeding chapters, are systems $F_{\alpha\beta o\pi}$ with certain restrictions on the cardinal bounds, having special two-place propositional operation symbol $\rightarrow$, one-place propositional operation symbol $\neg$, infinitary symbol $\bigwedge$, quantifier $\forall$. They are to be interpreted in systems $\langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ B_0\ \mathsf{O}'\ \mathsf{Q}\ S\rangle$ where S is the set of all functions on individual variables to D, B_0 is the set of truth values $\mathbf{0}$, $\mathbf{1}$, $\mathsf{O}'(\rightarrow)$ is that two-place operation on B_0 with values $\neg x \vee y$, $\mathsf{O}'(\neg)$ is the operation of complementation, $\mathsf{O}'(\bigwedge)$ is the $\nearrow\alpha$-place meet, $\mathsf{Q}(\forall, v)$ is the meet. Such a system depends only on the domain D and assignments O, C, R to constants, and is called a *model* for the language.

The valuation function in this case is the obvious generalization of the truth-valuation function for ordinary predicate languages. It satisfies the following conditions for all formulas A_ξ and sequences v of variables having length less than β:

(1) $V(s, [A_0 \rightarrow A_1]) = \mathbf{1}$ if and only if
$$V(s, A_0) = \mathbf{0} \text{ or } V(s, A_1) = \mathbf{1}.$$

(2) $V(s, [\neg A_0]) = \mathbf{1}$ if and only if $V(s, A_0) = \mathbf{0}$.

(3) $V(s, [\bigwedge A_0 \ldots A_\xi \ldots]) = \mathbf{1}$ if and only if $V(s, A_\xi) = \mathbf{1}$ for all $\xi < \delta$.

(4) $V(s, [\forall v A_0]) = \mathbf{1}$ if and only if $V(\mathrm{Repl}_t^{\mathrm{Rng}(v)} s, A_0) = \mathbf{1}$ for all $t \in D^{\mathrm{Rng}(v)}$.

8.2.4 *Example.* Consider the system of formulas of Example 8.2.3. A closely related interpretation is in a system $\langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ B\ \mathsf{O}'\ \mathsf{Q}\ S\rangle$, where S is the set of all functions on individual variables to D, B the underlying set of a complete Boolean algebra, $\mathsf{O}'(\rightarrow)$, $\mathsf{O}'(\neg)$, $\mathsf{O}'(\bigwedge)$, $\mathsf{Q}(\forall, v)$ the operations described in 8.2.3 on B instead of on B_0. This method for extending the interpretation of propositional formulas in Boolean algebras to apply to formulas with quantifiers, is due to Mostowski, and has been studied in papers of Rasiowa, Sikorski and Henkin for ordinary predicate languages. See Sect. 12.2 for discussion.

8.2.5 *Example.* The recursion principle can be used to eliminate defined propositional operation symbols and quantifiers. Again consider the system of formulas of Example 8.2.3, the formulas of the languages $\mathbf{L}_{\alpha\beta o\pi}$. Consider also a second system of formulas with additional special two-place propositional operation symbols, $\leftrightarrow$ $\wedge$, $\vee$, additional infinitary propositional operation symbol $\bigvee$, additional quantifier $\exists$. Let F_o be the set of terms common to the two languages, $F_{\alpha\beta o\pi}$ the set of formulas of the first language, $F'_{\alpha\beta o\pi}$ the set of formulas of the second language. As a system for interpretation of formulas of the second language, consider $\langle F_o\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ F_{\alpha\beta o\pi}\ \mathsf{O}'\ \mathsf{Q}\ S\rangle$, where $\langle F_o\ \mathsf{O}\rangle$ is the algebra of terms, $\mathsf{C}(c) = \langle c\rangle$ for individual constants c, $S = \{s\}$ where $s(x) = \langle x\rangle$ for variables x, R assigns to each η-place or $\nearrow\pi$-place predicate symbol that η-place or $\nearrow\pi$-place operation on F_o having as values the atomic formulas $[T_0PT_1]$ or $[Q^\eta T_0\ldots T_\xi\ldots]$ or $[QT_0\ldots T_\xi\ldots]$, as the case may be, for arguments $\langle T_0T_1\rangle$ or $\langle T_\xi : \xi < \eta\rangle$. Finally, let

$$
\begin{aligned}
&\mathsf{O}'(\neg)(\langle A_0\rangle) = [\neg A_0]\\
&\mathsf{O}'(\rightarrow)(\langle A_0A_1\rangle) = [A_0 \rightarrow A_1]\\
&\mathsf{O}'(\leftrightarrow)(\langle A_0A_1\rangle) = [\bigwedge [A_0 \rightarrow A_1][A_1 \rightarrow A_0]]\\
&\mathsf{O}'(\wedge)(\langle A_0A_1\rangle) = [\bigwedge A_0A_1]\\
&\mathsf{O}'(\vee)(\langle A_0A_1\rangle) = [\neg[\bigwedge [\neg A_0][\neg A_1]]]\\
&\mathsf{O}'(\bigwedge)(\langle A_0\ldots A_\xi\ldots\rangle) = [\bigwedge A_0\ldots A_\xi\ldots]\\
&\mathsf{O}'(\bigvee)(\langle A_0\ldots A_\xi\ldots\rangle) = [\neg[\bigwedge [\neg A_0]\ldots[\neg A_\xi]\ldots]]\\
&\mathsf{Q}(\forall, v)\{A_0\} = [\forall v A_0]\\
&\mathsf{Q}(\exists, v)\{A_0\} = [\neg[\forall v[\neg A_0]]]
\end{aligned}
$$

for all formulas A_ξ, ε-tuples v of variables, $0 < \varepsilon < \beta$.

In this case, whenever A' is a formula in $F'_{\alpha\beta o\pi}$, $V(s, A')$ is that formula in $F_{\alpha\beta o\pi}$ that results from eliminating symbols $\leftrightarrow$, $\wedge$, $\vee$, $\bigvee$, $\exists$, by their usual definitions in terms of $\neg$, $\rightarrow$, $\bigwedge$, $\forall$.

8.2.6 *Example.* Consider an arbitrary system with set $F_{\alpha\beta o\pi}$ of formulas. We sometimes encounter situations where an inductive proof is called for, but a straight induction on formulas is not possible. It may, however, be possible to carry out a transfinite induction on ordinals that measure the number of steps required to build a formula from atomic formulas. Such a measure is the rank function ρ:

$\rho(A_0) = 0$ if A_0 atomic.

$\rho([A_0 \Psi A_1]) = (\rho(A_0) \cup \rho(A_1)) + 1$, if Ψ a special two-place propositional operation symbol.

$\rho([\Phi A_0 \ldots A_\xi \ldots]) = (\bigcup_{\xi<\delta} \rho(A_\xi)) + 1$ if Φ is a δ-place or infinitary propositional operation symbol.

$\rho([\chi v A_0]) = \rho(A_0) + 1$, if χ is a quantifier.

The recursion principle can be used to prove the existence and uniqueness of the rank function. Consider interpretive system $\langle F_o\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ \alpha\ \mathsf{O}'\ \mathsf{Q}\ S\rangle$, where O, C, S are the same as in Example 8.2.5, R assigns to predicate symbols functions identically 0, and

$\mathsf{O}'(\Psi)(\langle\rho_0\rho_1\rangle) = (\rho_0 \cup \rho_1) + 1$, if Ψ is a special two-place propositional operation symbol.

$\mathsf{O}'(\Phi)(\langle\rho_0 \ldots \rho_\xi \ldots\rangle) = \bigcup_{\xi<\delta} \rho_\xi + 1$, if Φ is a δ-place or infinitary propositional symbol.

$\mathsf{Q}(\chi, v)\{\rho\} = \rho + 1$ if χ is a quantifier.

An easy induction on formulas shows that $\rho(A) = V(s, A)$ is the desired function. Every formula has a unique rank less than α.

8.3 Recursion Principle

As was the case with terms, we are well on our way to a proof of the recursion principle once we have proved that no proper initial part of a formula is a formula.

8.3.1 *Lemma.* If A is a formula that is not atomic, then exactly one of conditions (i), (ii), (iii), (iv) holds for A:

(i) There exist formulas A_0, A_1 and a special two-place propositional operation symbol such that $A = [A_0 \Psi A_1]$.

(ii) There is a δ-place propositional operation symbol Φ^δ and a δ-tuple of formulas such that $A = [\Phi^\delta A_0 \ldots A_\xi \ldots]$.

(iii) There is an ordinal δ, $0 < \delta < \alpha$, an infinitary propositional operation symbol Φ and a δ-tuple of formulas such that $A = [\Phi A_0 \ldots A_\xi \ldots]$.

(iv) There is an ordinal ε, $0 < \varepsilon < \beta$, an ε-tuple v of individual variables, a quantifier χ, and a formula A_0 such that $A = [\chi v A_0]$.

PROOF: The set of all formulas which are either atomic or have one of the forms (i), (ii), (iii), (iv) is closed under the rules of (α, β)-formula-formation of Sect. 8.2. Hence every non-atomic formula has one of these forms. Since every formula begins with [, a comparison of $A(1)$ shows that no formula could have two of the these forms.

8.3.2 *Lemma.* If $A = E_0 {}^\frown C {}^\frown E_1$ is an atomic formula, and if C is a formula, then $E_0 = E_1 = \phi$ and $A = C$.

PROOF: Every non-atomic formula contains a symbol that could not appear in an atomic formula. Therefore, C is atomic. Suppose first that A has form $[T_0 P T_1]$. Since each atomic formula has exactly one occurrence of a predicate symbol, C must have form $[T'_0 P T'_1]$ and $\langle[\rangle {}^\frown T_0 = E_0 {}^\frown \langle[\rangle {}^\frown T'_0$. Left cancellation yields $T_1 {}^\frown \langle]\rangle = T'_1 {}^\frown \langle]\rangle {}^\frown E_1$ as well. Lemma 3.3.2 implies $T_1 = T'_1$ since neither can be a proper initial part of the other. Hence $E_1 = \phi$. If $E_0 \neq \phi$, it must have form $\langle[\rangle {}^\smile E'_0$. Cancelling [, $T_0 = E'_0 {}^\frown \langle[\rangle {}^\frown T'_0$, contradicting 3.4.3. Hence $E_0 = \phi$ and $A = C$.

Suppose A has form $[Q T_0 \ldots T_\xi \ldots]$, Q a η-place or infinitary predicate symbol. Since C has exactly one occurrence of a predicate symbol, $C = [Q T'_0 \ldots T'_\xi \ldots]$ and $E_0 = \phi$. A left cancellation yields ${}^\frown\langle T_\xi : \xi < \eta \rangle {}^\frown \langle]\rangle = {}^\frown\langle T'_\xi : \xi < \eta' \rangle {}^\frown \langle]\rangle {}^\frown E_1$. Since neither can be a proper initial part of the other, $T_0 = T'_0$. Continuing by transfinite induction after left cancellations of equal terms, we see that $T_\xi = T'_\xi$ for all $\xi < \eta \cap \eta'$. If $\eta = \eta'$, then $E_1 = \phi$ after left cancellation. Either $\eta < \eta'$ or $\eta' < \eta$ would imply the existence of a term beginning with]. Hence $A = C$.

8.3.3 *Definition.* A *subformula* of a formula A is a consecutive part of A that is itself a formula. It is *proper* if it is different from A. The *principal terms* of atomic formula $[T_0 P T_1]$ are T_0, T_1, and of atomic formulas $[Q T_0 \ldots T_\xi \ldots]$ are $T_0, \ldots, T_\xi, \ldots$.

8.3.4 *Lemma.* Every atomic formula has precisely one of the forms 8.1 (1′), (2′), (3′). Moreover, $[T_0 P T_1] = [T'_0 P' T'_1]$ implies $T_0 = T'_0$, $P = P'$, $T_1 = T'_1$. Similarly,

$$[Q T_0 \ldots T_\xi \ldots] = [Q T'_0 \ldots T'_\xi \ldots]$$

implies that the lengths of the sequences of terms are the same, and all $T_\xi = T'_\xi$. Hence the principal terms of atomic formulas are unique.

PROOF: A comparison of $A(1)$ shows no atomic formula A has two of the forms 8.1 (1′), (2′), (3′). The uniqueness statements are corollaries of 3.3.3.

8.3.5 *Lemma.* Every formula has an atomic subformula.

PROOF: Trivial by induction.

8.3.6 *Lemma.* If A is a formula and $E \leqslant \underset{k}{[\ldots[}A$, where $1 \leqslant k < \omega$, then E is not an atomic formula.

PROOF: Let Δ be the set of all formulas A such that no initial part of expression $\underset{k}{[\ldots[}A$ is an atomic formula, where $1 \leqslant k < \omega$. If A is atomic and has form $[T_0 P T_1]$, then the only way an initial part $E \leqslant \underset{k}{[\ldots[}A$ could possibly be an atomic formula would be for E to have form $[T_0' P T_1']$. Since an atomic formula only contains one occurrence of a predicate symbol, it would follow that $\langle[\rangle^\frown T_0' = \langle\underset{k+1}{[\ldots[}\rangle^\frown T_0$. Left cancellation of one left bracket produces an equation that contradicts Lemma 3.4.3. If A is an atomic formula of form $[Q T_0 \ldots T_\xi \ldots]$, then clearly $E \leqslant \underset{k}{[\ldots[}A$ is not an atomic formula for $E(1)$ would have to be Q. Hence atomic formulas A are in Δ.

If $A_0, \ldots, A_\xi, \ldots$ is a δ-tuple of formulas in Δ and $A = [\Phi A_0 \ldots A_\xi \ldots]$, then no initial part of $\underset{k}{[\ldots[}A$ could be an atomic formula since no such formula contains an occurrence of Φ. Hence $A \in \Delta$. Similarly, if $A_0 \in \Delta$ and $A = [\chi v A_0]$ then $A \in \Delta$. If $A_0, A_1 \in \Delta$ and $A = [A_0 \Psi A_1]$, then $E \leqslant \underset{k}{[\ldots[}A$ and E an atomic formula implies $E \leqslant \underset{k+1}{[\ldots[}A_0$, since E contains no occurrence of Ψ. But then $A_0 \notin \Delta$. Hence in this case as well, $A \in \Delta$. By the induction principle for formulas, 8.1.1, Δ contains all formulas.

8.3.7 *Corollary.* If A is a formula and $E \prec A$, then E is not an atomic formula.

PROOF: Trivial by 8.3.2 and 8.3.6.

8.3.8 *Lemma.* If A is a formula and $E \prec A$, then E is not a formula.

PROOF: Let $\Delta = \{A : A$ is a formula and $E \prec A$ implies E is not a formula, and $A \prec E'$ implies E' is not a formula$\}$. Atomic formulas are in Δ by 8.3.2 and 8.3.7. For the closure of Δ under rules 8.1 (1), (2), (3) of formula-formation by propositional operation symbols, we refer the reader to the proof of the corresponding lemma for terms, 3.3.2. The proofs are nearly the same. Lemma 8.3.7 can be used to rule out the case E atomic.

Finally, we show Δ closed under rule (4) of (α, β)-formula formation. Suppose $A_0 \in \Delta$, χ a quantifier, v an ε-tuple of variables, $0 < \varepsilon < \beta$, and $A = [\chi v A_0]$. Using 2.2.19 to list the proper initial parts of A that could possibly be formulas, noting that no formula could end in χ or in a variable, we obtain $E = [\chi v E_0]$, $E_0 \preccurlyeq A_0$, $E_0 \neq \phi$. By 8.3.1, if E is a formula, it must have form $E = [\chi v' A'_0]$, A'_0 a formula and v' a sequence of variables. After cancelling $\langle[\chi\rangle$ on the left on the two expressions for E, we see that one of v, v' is an initial part of the other. Since no formula begins with a variable, $v = v'$. Another left cancellation yields $A'_0 \prec E_0 \preccurlyeq A_0$, contradicting $A_0 \in \Delta$.

If $A = [\chi v A_0] \preccurlyeq E'$ and E' a formula, again E' must have form $[\chi v' A'_0]$. Again $v = v'$, and since one of A_0, A'_0 must then be an initial part of the other, and since neither can be a proper initial part of the other without contradicting condition $A_0 \in \Delta$, we must have $A_0 = A'_0$. Hence $A = E'$ and $A \in \Delta$.

8.3.9 *Definition.* The principal subformulas of a formula $[A_0 \Psi A_1]$ are A_0, A_1. The principal subformulas of a formula $[\Phi A_0 \ldots A_\xi \ldots]$, are $A_0, \ldots, A_\xi, \ldots$ The only principal subformula of $[\chi v A_0]$ is A_0.

A proof of the following theorem was included in the proof just given and in the proof of 3.3.2 adapted to formulas.

8.3.10 *Corollary.* Suppose $\langle A_\xi : \xi < \delta\rangle$ and $\langle A'_\xi : \xi < \delta'\rangle$ are arbitrary sequences of formulas, σ, σ', are symbols, and v, v' are sequences of variables. Then

(1) $A_0{}^\frown\langle\sigma\rangle{}^\frown A_1 = A'_0{}^\frown\langle\sigma'\rangle{}^\frown A'_1$ implies $A_0 = A'_0, \sigma = \sigma', A_1 = A'_1$.

(2) ${}^\frown\langle A_\xi : \xi < \delta\rangle = {}^\frown\langle A'_\xi : \xi < \delta'\rangle$ implies $\delta = \delta'$ and $A_\xi = A'_\xi$ for all $\xi < \delta$.

(3) $v^\frown A_0 = v'^\frown A'_0$ implies $v = v'$ and $A_0 = A'_0$.

It follows that principal subformulas are unique.

8.3.11 *Proof of Recursion Principle* 8.2.2. The uniqueness of V follows by an easy induction making use of 3.3.3 and 8.3.10 to infer the uniqueness of the principal terms of an atomic formula and the principal subformulas of a non-atomic formula. We turn to a proof of the existence. We are to prove the existence of a function on $S \times F_{\alpha\beta o\pi}$ to B, satisfying conditions 8.2.2 (i), (ii), (iii). Elements of V are to be pairs $((s, A), b)$, where $s \in S$, A is a formula, $b \in B$. Throughout this proof, we write "(s, A, b)" for $((s, A), b)$.

Call a relation $R \subseteq S \times F_{\alpha\beta o\pi} \times B$ *acceptable* if for all terms T_ξ, formulas A_ξ, sequences v of variables having length less than β, $s \in S$,

(i′) $(s, [T_0 P T_1], \mathsf{R}(P)(\langle s^*(T_0) s^*(T_1)\rangle)) \in R$, all special two-place predicate symbols P.

(ii′) $(s, [Q^\eta T_0 \ldots T_\xi \ldots], \mathsf{R}(Q^\eta)(\langle s^*(T_0) \ldots s^*(T_\xi) \ldots\rangle)) \in R$, for all η-place predicate symbols Q^η, η-tuples of terms.

(iii′) $(s, [Q T_0 \ldots T_\xi \ldots], \mathsf{R}(Q)(\langle s^*(T_0) \ldots s^*(T_\xi) \ldots\rangle)) \in R$, for all infinitary predicate symbols Q, sequences of terms having length less than π.

(iv′) $(s, A_0, b_0) \in R$ and $(s, A_1, b_1) \in R$ implies $(s, [A_0 \Psi A_1], \mathsf{O}'(\Psi)(\langle b_0 b_1\rangle)) \in R$, for all special two-place propositional operation symbols Ψ.

(v′) $(s, A_\xi, b_\xi) \in R$ for all $\xi < \delta$ implies $(s, [\Phi^\delta A_0 \ldots A_\xi \ldots], \mathsf{O}'(\Phi^\delta)(\langle b_0 \ldots b_\xi \ldots\rangle)) \in R$, for all δ-place propositional operation symbols Φ^δ.

(vi′) $(s, A_\xi, b_\xi) \in R$ for all $\xi < \delta$ implies $(s, [\Phi A_0 \ldots A_\xi \ldots], \mathsf{O}'(\Phi)(\langle b_0 \ldots b_\xi \ldots\rangle)) \in R$, for all $0 < \delta < \alpha$, all infinitary propositional operation symbols Φ.

(vii′) $(s_{v,t}, A_0, b_t) \in R$ for all $t \in D^{\mathrm{Rng}(v)}$ such that $s_{v,t} \in S$, implies $(s, [\chi v A_0], \mathsf{Q}(\chi, v)\{b_t : t \in D^{\mathrm{Rng}(v)}$ and $s_{v,t} \in S\}) \in R$, whenever χ is a quantifier.

The relation $S \times F_{\alpha\beta o\pi} \times B$ is itself acceptable. Therefore we can form R_0, the intersection of all acceptable relations. From the form of the conditions, it is clear that R_0 is itself acceptable. The proof will be complete once we have shown that R_0 is a function with domain $S \times F_{\alpha\beta o\pi}$. It obviously satisfies 8.2.2 (i), (ii), (iii).

Let $\Delta = \{A : A$ is a formula and for all $s \in S$ there is a unique $b \in B$ such that $(s, A, b) \in R_0\}$. If A is atomic, conditions (i′), (ii′), (iii′) insure that there is $b \in B$ such that $(s, A, b) \in R_0$. If A has form $[Q^\eta T_0 \ldots T_\xi \ldots]$ and $b \neq \mathsf{R}(Q^\eta)(\langle s^*(T_0) \ldots s^*(T_\xi) \ldots\rangle)$, consider $R_0 \sim \{(s, A, b)\}$. It is easy to see with the aid of 8.3.4 that this relation is still acceptable. Hence $(s, A, b) \notin R_0$. The other cases with A atomic are similar. The proofs of the closure of Δ

under the rules of (α, β)-formula-formation by propositional operation symbols, are so similar to the proofs in the analogous cases for terms in 3.3.4, that we omit them here. Consider the case $A = [\chi v A_0]$ with $A_0 \in \Delta$. Given s and $t \in D^{\mathrm{Rng}(v)}$ such that $s_{v,t} \in S$, let b_t be the unique element of B such that $(s_{v,t}, A_0, b_t) \in R_0$. Let $X = \{b_t \colon t \in D^{\mathrm{Rng}(v)}$ and $s_{v,t} \in S\}$. Then by (vii′), $(s, A, \mathsf{Q}(\chi, v)(X)) \in R_0$. If $b \neq \mathsf{Q}(\chi, v)(X)$, consider $R_0' = R_0 \sim \{(s, A, b)\}$. Obviously R_0' still satisfies conditions (i′)-(vi′). To see that R_0' satisfies (vii′) as well, let $s' \in S$ and suppose $(s'_{v',t'}, A_0', b'_{t'}) \in R_0'$ for all $t' \in D^{\mathrm{Rng}(v')}$ such that $s'_{v',t'} \in S$. Since $R_0' \subseteq R_0$, the triple $(s', [\chi' v' A_0'], \mathsf{Q}(\chi', v')(X'))$ is in R_0, where $X' = \{b'_{t'} \colon t' \in D^{\mathrm{Rng}(v')}$ and $s'_{v',t'} \in S\}$. If this triple were not in R_0', it would have to be equal to (s, A, b). But then $s = s'$, $\chi = \chi'$, $v = v'$, $A_0 = A_0'$, and since $A_0 \in \Delta$, $X = X'$. Hence $b = \mathsf{Q}(\chi, v)(X)$, contradicting choice of b. Therefore the triple is in R_0'. Since then R_0' is acceptable, $(s, A, b) \notin R_0$. Hence $A \in \Delta$.

This completes the proof of the recursion principle.

8.4 Replacement of Equivalent Parts

A relation $\equiv$ is a *congruence relation on formulas* if and only if it is an equivalence relation that is a congruence relation on the algebra of formulas and satisfies

$$A \equiv A' \text{ implies } [\chi v A] \equiv [\chi v A']$$

for all quantifiers χ and sequences v of variables having length less than β. According to the ordinary algebraic definition of congruence relation, such a relation also satisfies

$$A_0 \equiv A_0' \text{ and } A_1 \equiv A_1' \text{ implies } [A_0 \Psi A_1] \equiv [A_0' \Psi A_1']$$

$$A_\xi \equiv A_\xi' \text{ for all } \xi < \delta \text{ implies } [\Phi A_0 \ldots A_\xi \ldots] \equiv [\Phi A_0' \ldots A_\xi' \ldots]$$

for the propositional operation symbols Ψ, Φ.

8.4.1 *Example.* Given any interpretive system for a system of formulas, the relation

$$A \equiv A' \text{ if and only if } V(s, A) = V(s, A') \text{ for all } s \in S$$

is a congruence relation.

In particular, if ρ is the rank function of Example 8.2.6, the relation $A \equiv A'$ if and only if $\rho(A) = \rho(A')$ is a congruence relation on formulas.

8.4.2 *Example.* If K is any class of models for formulas of a predicate language $\mathbf{L}_{\alpha\beta o\pi}$ described in Example 8.2.3, the relation

$A \equiv A'$ if and only if $V(s, A) = V(s, A')$ for all assignments s to all models in K

is a congruence relation on formulas.

The formal systems of Chapter 11 have the property that the relation $A \equiv A'$ iff $\vdash A \leftrightarrow A'$ is a congruence relation on the formulas of $\mathbf{L}_{\alpha\beta o\pi}$.

The Replacement Principle for formulas says that when consecutive parts C_ν of a formula A, the C_ν being themselves formulas, are replaced by formulas C'_ν congruent to C_ν, the result is a formula congruent to A. As was the case in Sect. 3.4 for terms, the result follows easily from the theorems of Sect. 2.3 once we have established condition (C) for formulas.

8.4.3 *Lemma.* If $E_0{}^\frown E_1$ is a term and $E_1 \neq \phi$, then E_1 is not an initial part of a formula.

PROOF: Let Δ be the set of all formulas A such that whenever $\phi \prec E_1 \leqslant A$, E_1 is not a terminal part of a term. If A is an atomic formula of form $[T_0 P T_1]$, then the only possible candidates for initial part E_1 would have form $E_1 = \langle[\rangle{}^\frown E'_1$, $E'_1 \leqslant T_0$. This case is ruled out by Lemma 3.4.3. A non-empty initial part of an atomic formula of form $[Q T_0 \ldots T_\xi \ldots]$ is clearly not a terminal part of a term since Q never occurs in a term. Hence atomic formulas are in Δ.

If $A_0 \in \Delta$, $A_1 \in \Delta$, and $A = [A_0 \Psi A_1]$, then the only possible candidates for E_1 would have form $E_1 = \langle[\rangle{}^\frown E'_1$, $E'_1 \leqslant A_0$. The induction hypothesis on A_0 implies E_1 not a terminal part of a term. That Δ is closed under the other rules of (α, β)-formula-formation is trivial.

8.4.4 *Lemma.* If $E_0{}^\frown E_1$ is a formula, $E_0 \neq \phi$ and $\phi \prec E_1 \leqslant C$ then C is not a formula. (Condition (C) of Sect. 2.3 for formulas.)

PROOF: Let $\Delta = \{A\colon A$ is a formula and $A = E_0{}^\frown E_1$ and $\phi \prec E_1 \leqslant C$ where C is a formula, implies $E_0 = \phi\}$. The only possible candidates for terminal parts E_1 of atomic formulas A that could be initial parts of formulas, either begin with a terminal part of a term, or have $E_0 = \phi$. Lemmas 8.4.3 and 8.3.8 imply that atomic formulas are therefore in Δ. For the proof of the closure of Δ under formation of formulas by propositional operation symbols, we refer the reader to the proof of Lemma 3.4.3. It is almost identical. References to Lemma 3.3.2 are made to Lemma 8.3.8 instead.

Finally, suppose $A_0 \in \Delta$, $A = [\chi v A_0] = E_0{}^\frown E_1$, and $\phi \prec E_1 \preccurlyeq C$ for a formula C. Since every formula begins with [, either $E_0 = \phi$ or $E_0 = [\chi v E_0'$ and $\phi \preccurlyeq E_0' \prec A_0$. After cancelling E_0 on the left in the equality $A = E_0{}^\frown E_1$, we obtain $E_1 = (A_0 - E_0')^\frown\langle]\rangle \preccurlyeq C$. The induction hypothesis on A_0 implies $E_0' = \phi$. But then $A_0 \prec E_1 \preccurlyeq C$, contradicting 8.3.8. Hence $E_0 = \phi$, and $A \in \Delta$.

8.4.5 *Replacement Principle.* Let $\equiv$ be a congruence relation on formulas. If formula $A = {}^\frown\langle E_\nu{}^\frown C_\nu : \nu < \sigma\rangle^\frown E_\sigma$, where the C_ν are formulas, and if $C_\nu \equiv C_\nu'$ for all $\nu < \sigma$, then $A' = {}^\frown\langle E_\nu{}^\frown C_\nu' : \nu < \sigma\rangle^\frown E_\sigma$ is a formula and $A \equiv A'$.

PROOF: If $\sigma = 0$, the result is trivial. If $\sigma \neq 0$ and $E_0 = \phi$, then $C_0 = A$, $\sigma = 1$, $E_\sigma = \phi$, by 8.3.8. If $\sigma \neq 0$ and $E_\sigma = \phi$, then $\sigma \notin \mathrm{Lim}$ since $\mathrm{Dom}(A) \notin \mathrm{Lim}$. Then $A = C_{\sigma-1}$, $\sigma = 1$, $E_0 = \phi$, by 8.4.4. These cases are trivial, so only decompositions with $E_0 \neq \phi$ and $E_\sigma \neq \phi$ need be considered. If A atomic, $\sigma = 1$, $E_0 = E_\sigma = \phi$ by 8.3.2 and 8.3.5. Hence atomic formulas have the replacement property. Proceeding by induction, suppose A_0, A_1 have the replacement property and $A = [A_0 \Psi A_1]$. Then A can be shown to have the replacement property by the argument that appears in the proof of the replacement principle for terms, 3.4.7, for the results of Sect. 2.3 are also available for formulas. The cases $A = [\Phi A_0 \ldots A_\xi \ldots]$ are similar.

Finally, suppose A_0 has the replacement property and $A = [\chi v A_0] = {}^\frown\langle E_\nu{}^\frown C_\nu : \nu < \sigma\rangle^\frown E_\sigma$ with $E_0 \neq \phi$, $E_\sigma \neq \phi$. Since C_0 begins with [, E_0 has form $[\chi v E_0'$, E_σ has form $E_\sigma']$. Let $E_\nu' = E_\nu$ for $0 < \nu < \sigma$. Then after admissible cancellations, $A_0 = {}^\frown\langle E_\nu'{}^\frown C_\nu : \nu < \sigma\rangle^\frown E_\sigma'$. By induction hypothesis, $A_0 \equiv A_0' = {}^\frown\langle E_\nu'{}^\frown C_\nu' : \nu < \sigma\rangle^\frown E_\sigma'$. Since $\equiv$ is a congruence relation, $A \equiv [\chi v A_0']$. But $[\chi v A_0'] = {}^\frown\langle E_\nu{}^\frown C_\nu' : \nu < \sigma\rangle^\frown E_\sigma$. Hence A' is a formula and $A \equiv A'$.

The proof just given includes a proof of the fundamental property of principal subformulas.

8.4.6 *Theorem.* A proper subformula of a formula A is a subformula of a principal subformula of A.

CHAPTER 9

SUBSTITUTION

In this chapter we continue the study of the general (α, β, o, π)-systems of formulas introduced in Chapter 8, Sect. 8.1. The substitutions $S_t^X T$ for terms were treated in Chapter 3, Sect. 3.5, as special kinds of replacements, where atomic terms $\langle x \rangle$, $x \in X$, are replaced by arbitrary expressions $f(x)$. Clearly the definition of 3.5.2, when applied to formulas, does not yield a formula unless measures are taken to provent the substitution of something other than an individual variable in a quantifier-sequence. The two most important such substitutions, the substitutions of terms for free occurrences of variables, and of variables for bound occurrences of variables, are the topics of this chapter.

9.1 Free and Bound Occurrences of Variables in Formulas

An *occurrence* of a symbol x in expression E is an ordinal $\iota < \mathrm{Dom}(E)$ such that $E(\iota) = x$. An ordinal $\iota < \mathrm{Dom}(E)$ may be referred to as a *position in* E. A position ι is *bound by* X *in* E if and only if E has a representation $E = E_0{}^\frown A^\frown E_1$ with $\mathrm{Dom}(E_0) \leqslant \iota < \mathrm{Dom}(E_0) + \mathrm{Dom}(A)$, where A is a formula of form $[\chi v A_0]$ with $X \cap \mathrm{Rng}(v) \neq \phi$. An occurrence ι of individual symbol x in E is *bound in* E if and only if it is bound by $\{x\}$ in E. Otherwise the occurrence ι of x is *free in* E.

Clearly, an occurrence of an individual constant in E is always free in E. Similarly, if E contains no quantifiers, any occurrence of an individual symbol is free in E. Hence all occurrences of individual symbols in terms or atomic formulas are free.

9.1.1 *Lemma.* For formulas $A_0, \ldots, A_\xi, \ldots$, special two-place propositional operation symbols Ψ, δ-place or infinitary propositional operation symbols Φ, quantifiers χ, and sequences v of individual variables having length less than β, sets X of individual symbols:

(i) If $A = [A_0 \Psi A_1]$, then position ι in A_0 is bound by X in A_0 if and only if $1 + \iota$ is bound by X in A. Also position ι in A_1 is bound by X in A_1 if and only if $1 + \mathrm{Dom}(A_0) + 1 + \iota$ is bound by X in A.

(ii) If $A = [\Phi A_0 \ldots A_\xi \ldots]$, then for each $\lambda < \delta$, position ι in A_λ is bound by X in A_λ if and only if $2 + \Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda\rangle + \iota$ is bound by X in A.

(iii) If $A = [\chi v A_0]$, then position ι in A_0 is bound by X in A_0 implies $2 + \mathrm{Dom}(v) + \iota$ is bound by X in A. Moreover, $2 + \mathrm{Dom}(v) + \iota$ is bound by X in A if and only if either ι is bound by X in A_0 or $X \cap \mathrm{Rng}(v) \neq \phi$.

PROOF: A trivial computation shows that if position ι is bound by X in a principal subformula A_ξ of A, then the corresponding position in A is also bound by X. The converses require use of the results of Chapter 8. Consider, for example, A of form (ii) above, position ι in A_λ. Suppose $\iota' = 2 + \Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda\rangle + \iota$ bound by X in A. Then A has a representation of the form $A = [\Phi A_0 \ldots A_\xi \ldots] = (E_0{}^\frown A'){}^\frown E_1$ where $\mathrm{Dom}(E_0) \leqslant \iota' < \mathrm{Dom}(E_0) + \mathrm{Dom}(A')$, and A' has form $[\chi v A_0']$ with $X \cap \mathrm{Rng}(v) \neq \phi$. We saw in Lemma 8.4.4 that formulas satisfy condition (C) of Chapter 2, Sect. 2.3. Applying 2.3.3 with $\sigma = 1$, it follows that there is a unique $\lambda' < \delta$ such that $\langle [\Phi\rangle{}^\frown({}^\frown\langle A_\xi : \xi < \lambda'\rangle) = E_{0\lambda'} \preccurlyeq E_0$ and $A_{\lambda'} = (E_0 - E_{0\lambda'}){}^\frown A'{}^\frown E_{1\lambda'+1}$ where $E_{1\lambda'+1} \preccurlyeq E_1$. Then $\lambda = \lambda'$, for if $\lambda < \lambda'$, then

$$\iota' < 2 + \Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda + 1\rangle \leqslant 2 + \Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda'\rangle \leqslant \mathrm{Dom}(E_0),$$

contradicting condition on ι'. Similarly, if $\lambda' < \lambda$, then

$$\mathrm{Dom}(E_0) + \mathrm{Dom}(A') \leqslant 2 + \Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda' + 1\rangle \leqslant 2 + \Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda\rangle \leqslant \iota'$$

also contradicting condition on ι'. Moreover, subtracting $\mathrm{Dom}(E_{0\lambda'}) = 2 + \Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda\rangle$ on the left in the inequality $\mathrm{Dom}(E_0) \leqslant \iota' < \mathrm{Dom}(E_0) + \mathrm{Dom}(A')$, we obtain the required inequality for ι; namely, $\mathrm{Dom}(E_0 - E_{0\lambda'}) \leqslant \iota < \mathrm{Dom}(E_0 - E_{0\lambda'}) + \mathrm{Dom}(A')$. Hence ι is bound by X in A_λ.

The converse in (i) is similar. As for the converse in (iii), obviously every position in $A = [\chi v A_0]$ is bound by X if $X \cap \mathrm{Rng}(v) \neq \phi$. If $X \cap \mathrm{Rng}(v) = \phi$ and $2 + \mathrm{Dom}(v) + \iota$ is bound by X in A, then A has a decomposition $A = E_0{}^\frown A'{}^\frown E_1$ with $\mathrm{Dom}(E_0) \leqslant 2 + \mathrm{Dom}(v) + \iota < \mathrm{Dom}(E_0) + \mathrm{Dom}(A')$ and A' has form $[\chi' v' A_0']$ with $\mathrm{Rng}(v') \cap X \neq \phi$. Moreover, $E_0 \neq \phi$, $E_1 \neq \phi$. Since A' does not begin with [, $[\chi v \leqslant E_0$. After admissible cancellations, $A_0 = (E_0 - [\chi v)^\frown A'{}^\frown E_1'$, where $E_1 = E_1']$. Subtracting $2 + \mathrm{Dom}(v)$ on the left in the inequality for $2 + \mathrm{Dom}(v) + \iota$, we obtain the necessary inequality for ι; namely,

$$\mathrm{Dom}(E_0 - [\chi v) \leqslant \iota < \mathrm{Dom}(E_0 - [\chi v) + \mathrm{Dom}(A').$$

Hence ι is bound by X in A_0.

9.1.2 *Corollary.* (i) If $A = [A_0 \Psi A_1]$, then ι is a free occurrence of x in A_0 if and only if $1 + \iota$ is a free occurrence of x in A. Similarly, ι is a free occurrence of x in A_1 if and only if $1 + \mathrm{Dom}(A_0) + 1 + \iota$ is a free occurrence of x in A.

(ii) If $A = [\Phi A_0 \ldots A_\xi \ldots]$, then ι is a free occurrence of x in A_λ if and only if $2 + \Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda\rangle + \iota$ is a free occurrence of x in A.

(iii) If $A = [\chi v A_0]$, then $2 + \mathrm{Dom}(v) + \iota$ is a free occurrence of x in A if and only if ι is a free occurrence of x in A_0 and $x \notin \mathrm{Rng}(v)$.

If A is a formula, $FV(A)$ is the set of all individual variables having free occurrence in A. Another corollary of 9.1.1 is the following:

9.1.3 *Corollary.* (i) $FV([A_0 \Psi A_1]) = FV(A_0) \cup FV(A_1)$.

(ii) $FV([\Phi A_0 \ldots A_\xi \ldots]) = \bigcup_{\xi < \delta} FV(A_\xi)$

(iii) $FV([\chi v A_0]) = FV(A_0) \sim \mathrm{Rng}(v)$.

A *sentence* of a system $F_{\alpha\beta o\pi}$ of formulas is a formula having no free occurrences of variables.

9.1.4 *Corollary.* Any subformula of a sentence has fewer than β free variables.

Proof: Let $\Delta = \{A : A$ is a formula and if A has fewer than β free variables, then so does every subformula of $A\}$. Atomic formulas have no proper subformulas, hence are in Δ. If $A_0, \ldots, A_\xi, \ldots$ are in Δ and $A = [\Phi A_0 \ldots A_\xi \ldots]$ has fewer than β free variables, then by the theorem, every principal subformula A_ξ has fewer than β free variables. Since every subformula of A that is different from A is a subformula of some A_ξ, clearly $A \in \Delta$. Similarly $A_0 \in \Delta$ and $A_1 \in \Delta$ implies $A = [A_0 \Psi A_1] \in \Delta$. Finally, suppose $A_0 \in \Delta$ and

$A = [\chi v A_0]$. If A has fewer than β free variables, then since $FV(A_0) \subseteq FV(A) \cup \mathrm{Rng}(v)$, both sets have fewer than β variables and β infinite, A_0 has fewer than β free variables. Since every proper subformula of A is a subformula of A_0, $A \in \Delta$.

As we would expect, the value $V(s, A)$ of a formula A of an (α, β, o, π)-predicate language $\mathsf{L}_{\alpha\beta o\pi}$ (Example 8.2.3) in a model $\langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\rangle$ does not depend on the values s assigns to variables not free in A. This extends to arbitrary interpretive systems as follows:

9.1.5 *Theorem*. Let $\langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ B\ \mathsf{O}'\ \mathsf{Q}\ S\rangle$ be a system for the interpretation of an arbitrary system of formulas. Suppose, moreover, S satisfies

(*) If $s, s' \in S$ and X is a set of individual variables having power less than β, then $\mathrm{Repl}_t^X s \in S$ implies $\mathrm{Repl}_t^X s' \in S$.

Then if $s, s' \in S$ and s, s' agree on variables with free occurrence in formula A, $V(s, A) = V(s', A)$.

Proof: Let $\Delta = \{A\colon A$ is a formula and $V(s, A) = V(s', A)$ whenever s, s' agree on variables free in $A\}$. If A atomic, every variable in A is free in A. Hence

$$V(s, [T_0 P T_1]) = \mathsf{R}(P)(\langle s^*(T_0)\, s^*(T_1)\rangle) = \mathsf{R}(P)(\langle s'^*(T_0)\, s'^*(T_1)\rangle) = V(s', [T_0 P T_1])$$

by Theorem 3.5.5 (i). A similar computation holds for the other types of atomic formulas. Hence atomic formulas are in Δ. Suppose $A = [A_0 \Psi A_1]$, $A_0 \in \Delta$, $A_1 \in \Delta$. If s, s' agree on variables free in A, then they agree on variables free in A_0, A_1 by 9.1.3. Hence $V(s, A_0) = V(s', A_0)$ and the same for A_1. Then $V(s, A) = \mathsf{O}'(\Psi)(\langle V(s, A_0)\, V(s, A_1)\rangle) = V(s', A)$. Hence $A \in \Delta$. The case $A = [\Phi A_0 \ldots A_\xi \ldots]$ with Φ a δ-place or infinitary propositional operation symbol is similar. Finally, suppose $A_0 \in \Delta$, $A = [\chi v A_0]$. For $s \in S$, $t \in D^{\mathrm{Rng}(v)}$, let $s_{v,t} = \mathrm{Repl}_t^{\mathrm{Rng}(v)} s$. Then $V(s, A) = \mathsf{Q}(\chi, v)\{V(s_{v,t}, A_0) : s_{v,t} \in S\}$ and $V(s', A) = \mathsf{Q}(\chi, v)\{V(s'_{v,t}, A_0) : s'_{v,t} \in S\}$. For $t \in D^{\mathrm{Rng}(v)}$, $s_{v,t}$ and $s'_{v,t}$ agree on $FV(A_0)$ since this set of variables is included in $FV(A) \cup \mathrm{Rng}(v)$. By induction hypothesis and condition (*), $V(s, A) = V(s', A)$. Hence $A \in \Delta$.

Note that condition (*) holds in all the systems of Examples 8.2.3–8.2.6.

9.2 Substitution for Free Variables

Let X be a set of individual symbols, f a function on X to terms, E an expression. Then $S_f^X E$ is to be the result of replacing all occurrences of $x \in X$ by occurrences of $f(x)$, while $SF_f^X E$ is to be the result of replacing only the free occurrences of $x \in X$ in E by occurrences of $f(x)$.

9.2.1 *Definition.* Let $\langle \iota_\nu : \nu < \sigma \rangle$ be the sequence of all occurrences of symbols of X in E, in strictly increasing order. Represent E in the form of 2.2.20:

$$E = {}^\frown\langle E_\nu {}^\frown \langle E(\iota_\nu)\rangle : \nu < \sigma\rangle^\frown E_\sigma$$

where for $\iota_\sigma = \mathrm{Dom}(E)$ and for $\nu \leqslant \sigma$,

$$\begin{aligned} E_\nu &= E|\iota_\nu - (E|\iota_{\nu-1}{}^\frown\langle E(\iota_{\nu-1})\rangle) && \text{if } \nu \notin \mathrm{Lim} \\ E_\nu &= E|\iota_\nu - E|\bigcup_{\theta<\nu} \iota_\theta && \text{if } \nu \in \mathrm{Lim}. \end{aligned}$$

Then $S_f^X E = {}^\frown\langle E_\nu{}^\frown f(E(\iota_\nu)) : \nu < \sigma\rangle^\frown E_\sigma$.

In Theorem 3.5.3 we proved that the operator S_f^X distributes over the operation symbols in terms to yield equations 3.5.1 (i), (ii), (iii). A slight modification of that proof yields the following:

9.2.2 *Theorem.* If the T_ξ are terms, P a special two-place predicate symbol, Q an η-place or infinitary predicate symbol,

(i) $S_f^X[T_0 P T_1] = [S_f^X T_0 P S_f^X T_1]$
(ii) $S_f^X[Q T_0 \ldots T_\xi \ldots] = [Q S_f^X T_0 \ldots S_f^X T_\xi \ldots]$.

Clearly the S_f^X-operator need not send formulas into formulas. The corresponding operator for formulas is SF_f^X which replaces $f(x)$ only at free occurrences of $x \in X$.

9.2.3 *Definition.* Let $\langle \iota_\nu : \nu < \sigma\rangle$ be the sequence of all free occurrences of symbols of X in E in strictly increasing order. Represent E in the form of 2.2.20:

$$E = {}^\frown\langle E_\nu{}^\frown\langle E(\iota_\nu)\rangle : \nu < \sigma\rangle^\frown E_\sigma.$$

Then $SF_f^X E = {}^\frown\langle E_\nu{}^\frown f E(\iota_\nu) : \nu < \sigma\rangle^\frown E_\sigma$.

The above decomposition of E will be referred to as the *decomposition for the free substitution of* X. This decomposition is more difficult to deal with than the decomposition of 9.2.1 for substitution at all occurrences of X. We know that $E = {}^\frown\langle E'_\nu{}^\frown C_\nu : \nu < \sigma'\rangle^\frown E'_\sigma$ is the correct decomposition for substitution at all occurrences if and only if each C_ν has form $\langle x\rangle$, $x \in X$, and no E'_ν contains any occurrence of $x \in X$. However for $SF^X_f E$ we have the following:

9.2.4 *Lemma.* The decomposition of E for the free substitution of X is $E = {}^\frown\langle E'_\nu{}^\frown C_\nu : \nu < \sigma'\rangle^\frown E'_\sigma$ if and only if each C_ν has length one and $\langle \Sigma\langle \mathrm{Dom}(E'_\theta) + 1 : \theta < \nu\rangle + \mathrm{Dom}(E'_\nu) : \nu < \sigma'\rangle$ is the sequence of all free occurrences of $x \in X$ in E.

PROOF: Let $E = {}^\frown\langle E_\nu{}^\frown\langle E(\iota_\nu)\rangle : \nu < \sigma\rangle^\frown E_\sigma$ be the decomposition of E for the free substitution of X. Then for each $\nu < \sigma$, $E|\iota_\nu = {}^\frown\langle E_\theta{}^\frown\langle E(\iota_\theta)\rangle : \theta < \nu\rangle^\frown E_\nu$ by Lemma 2.2.20. Hence

$$\iota_\nu = \Sigma\langle \mathrm{Dom}(E_\theta) + 1 : \theta < \nu\rangle + \mathrm{Dom}(E_\nu).$$

Therefore the condition is satisfied by this decomposition.

Since the condition given completely determines the lengths of parts E'_ν, it is a simple exercise in ordinal arithemic to show that if decomposition $E = {}^\frown\langle E'_\nu{}^\frown C_\nu : \nu < \sigma'\rangle^\frown E'_\sigma$ satisfies the condition, then $E_\nu = E'_\nu$ for all $\nu < \sigma$, and $\sigma = \sigma'$.

9.2.5 *Lemma.* Let $\langle A_\xi : \xi < \delta\rangle$ be any sequence of formulas. Then $SF^X_f{}^\frown\langle A_\xi : \xi < \delta\rangle = {}^\frown\langle SF^X_f A_\xi : \xi < \delta\rangle$.

PROOF: Let $\langle \iota_\nu : \nu < \sigma\rangle$ be the sequence of all free occurrences of $x \in X$ in $A = {}^\frown\langle A_\xi : \xi < \delta\rangle$. Let $A = {}^\frown\langle E_\nu{}^\frown\langle A(\iota_\nu)\rangle : \nu < \sigma\rangle^\frown E_\sigma$ be the decomposition for the free substitution of X in A. Then each formula A_λ has the decomposition of 2.3.3 with $\mathsf{T}_2 = \{\langle x\rangle : x \in X\}$:

$$A_\lambda = E_{g(\lambda)\lambda+1} - E_{g(\lambda)\lambda} \text{ if } g(\lambda) = g(\lambda+1)$$
$$A_\lambda = (E_{g(\lambda)} - E_{g(\lambda)\lambda})^\frown\langle A(\iota_{g(\lambda)})\rangle^\frown A'^\frown E_{g(\lambda+1)\lambda+1} \text{ where}$$
$$A' = {}^\frown\langle E_{g(\lambda)+1+\nu}{}^\frown\langle A(\iota_{g(\lambda)+1+\nu})\rangle : \nu < g(\lambda+1) - (g(\lambda)+1)\rangle$$
$$\text{if } g(\lambda) < g(\lambda+1).$$

In view of 2.3.4, it suffices to show that this decomposition of A_λ is the decomposition for the free substitution of X in A_λ.

Recall that ordinals $g(\lambda)$ and initial parts $E_{g(\lambda)\lambda} \preccurlyeq E_{g(\lambda)}$ were defined uniquely by

$$(1)\quad {}^\frown\langle A_\xi : \xi < \lambda\rangle = ({}^\frown\langle E_\nu{}^\frown\langle A(\iota_\nu)\rangle : \nu < g(\lambda)\rangle)^\frown E_{g(\lambda)\lambda}.$$

Hence

$$(2)\quad \Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda\rangle = \Sigma\langle \mathrm{Dom}(E_\nu) + 1 : \nu < g(\lambda)\rangle + \mathrm{Dom}(E_{g(\lambda)\lambda}).$$

We know that $g(\delta) = \sigma$ and $E_{g(\sigma)\sigma} = E_\sigma$. But if $\lambda < \delta$, then $E_{g(\lambda)\lambda} \prec E_{g(\lambda)}$ since $E_{g(\lambda)\lambda} = E_{g(\lambda)}$ would imply that A_λ begins with an individual symbol. By 9.2.4, if sequence $\langle \iota_\nu : \nu < \sigma \rangle$ is extended by letting $\iota_\sigma = \mathrm{Dom}(A)$,

(3) $\iota_{g(\lambda)} = \Sigma\langle \mathrm{Dom}(E_\nu) + 1 : \nu < g(\lambda) \rangle + \mathrm{Dom}(E_{g(\lambda)})$, all $\lambda \leqslant \delta$.

It follows from (2) and (3) that

(4) If $\lambda < \delta$, $\Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda \rangle < \iota_{g(\lambda)}$.

Moreover, if $\nu < g(\lambda)$, then $\iota_\nu = \Sigma\langle \mathrm{Dom}(E_\theta) + 1 : \theta < \nu \rangle + \mathrm{Dom}(E_\nu) < \Sigma\langle \mathrm{Dom}(E_\theta) + 1 : \theta < g(\lambda) \rangle$, using 9.2.4. Hence,

(5) If $\nu < g(\lambda)$, $\iota_\nu < \Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda \rangle$

by (2). Combining (4) and (5) we see that if there are any free occurrences of $x \in X$ in A past $\Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda \rangle$, then the first such is $\iota_{g(\lambda)}$.

Suppose first, $g(\lambda) = g(\lambda + 1)$. Then the decomposition of 2.3.3 for A_λ is the correct one if and only if A_λ contains no free occurrences of $x \in X$. Actually, this is the case. For if there were a free occurrence of $x \in X$ in A_λ, 9.1.2 (ii) would imply the existence of a free occurrence ι_ν of $x \in X$ in A such that $\Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda \rangle \leqslant \iota_\nu < \Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda + 1 \rangle$. Then $\nu < \sigma$ and by the minimal property of $g(\lambda)$, $\iota_{g(\lambda)} \leqslant \iota_\nu$. By (5), $g(\lambda) \neq \sigma$. Hence $\lambda + 1 \neq \delta$. By (4), $\Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda + 1 \rangle < \iota_{g(\lambda)} \leqslant \iota_\nu$, a contradiction.

Finally, suppose $g(\lambda) < g(\lambda + 1)$. The decomposition for A_λ is the decomposition for the free substitution of X in A_λ if and only if the sequence of free occurrences of $x \in X$ in A_λ is $\langle \iota'_\nu : \nu < 1 + (g(\lambda + 1) - (g(\lambda) + 1)) \rangle$ where $\iota'_0 = \mathrm{Dom}(E_{g(\lambda)} - E_{g(\lambda)\lambda})$ and $\iota'_{1+\nu} = \iota'_0 + 1 + \Sigma\langle \mathrm{Dom}(E_{g(\lambda)+1+\theta}) + 1 : \theta < \nu \rangle + \mathrm{Dom}(E_{g(\lambda)+1+\nu})$, for $\nu < g(\lambda + 1) - (g(\lambda) + 1)$. Moreover, by 9.1.2, the above sequence is the sequence of free occurrences of $x \in X$ in A_λ if and only if the sequence with terms $\Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda \rangle + \iota'_\nu$ is the sequence of free occurrences of $x \in X$ in A lying within A_λ. In this case, $\iota_{g(\lambda)} < \Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda + 1 \rangle$ by (2) and (3). So $\iota_{g(\lambda)}$ lies within A_λ and is the first free occurrence of $x \in X$ in A that does. Moreover, if $\nu < g(\lambda + 1) - (g(\lambda) + 1)$, then $\iota_{g(\lambda)+1+\nu} < \iota_{g(\lambda+1)}$. These occurrences also lie within A_λ by (5). By (4), $\iota_{g(\lambda+1)}$ does not lie within A_λ. Hence the sequence of free occurrences of $x \in X$ in A lying within A_λ is $\langle \iota_{g(\lambda)}, \ldots, \iota_{g(\lambda)+1+\nu}, \ldots \rangle$ with $\nu < g(\lambda + 1) - (g(\lambda) + 1)$. Since $\Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda \rangle + \iota'_0 = \iota_{g(\lambda)}$ by (3), 9.2.3 implies that the sequence with terms $\Sigma\langle \mathrm{Dom}(A_\xi) : \xi < \lambda \rangle + \iota'_\nu$ is exactly the sequence of free occurrences of $x \in X$ in A lying within A_λ.

9.2.6 *Theorem.* If A is an atomic formula, $SF_f^X A = S_f^X A$. For all formulas A_ξ, special two-place propositional operation symbols Ψ, δ-place or infinitary propositional operation symbols Φ, quantifiers χ,

(i) $SF_f^X[A_0 \Psi A_1] = [SF_f^X A_0 \Psi SF_f^X A_1]$
(ii) $SF_f^X[\Phi A_0 \ldots A_\xi \ldots] = [\Phi SF_f^X A_0 \ldots SF_f^X A_\xi \ldots]$
(iii) $SF_f^X[\chi v A_0] = [\chi v SF_f^{X-\mathrm{Rng}(v)} A_0]$.

PROOF: The definitions of $S_f^X E$ and $SF_f^X E$ agree on atomic formulas since all occurrences are free. (ii) follows from the lemma just given. (i) can be proved by the same method, though in a case where only two formulas are involved, it is easier to begin with the decompositions for the free substitution of X in A_0, A_1. Formation of the concatenation with $\langle[\rangle$, $\langle\Psi\rangle$, $\langle]\rangle$, and a reassociation yields the decomposition for the free substitution of X in $[A_0 \Psi A_1]$. We can illustrate this method with (iii). Let $\langle \iota_\nu : \nu < \sigma\rangle$ be the sequence of all free occurrences of symbols of $X \sim \mathrm{Rng}(v)$ in A_0, and let the decomposition of A_0 for the free substitution of $X \sim \mathrm{Rng}(v)$ be $A_0 = {}^\frown\langle E_\nu {}^\frown \langle A_0(\iota_\nu)\rangle : \nu < \sigma\rangle {}^\frown E_\sigma$. By 9.1.2, the sequence of all free occurrences of $x \in X$ in $[\chi v A_0]$ is$\langle 2 + \mathrm{Dom}(v) + \iota_\nu : \nu < \sigma\rangle$. If $\sigma = 0$, the formula (iii) obviously holds. Suppose $\sigma \neq 0$. Then by the associative law, $[\chi v A_0] = {}^\frown\langle E'_\nu {}^\frown \langle A_0(\iota_\nu)\rangle : \nu < \sigma\rangle {}^\frown E'_\sigma$. where $E'_0 = \langle[\chi\rangle {}^\frown v {}^\frown E_0$, $E'_\sigma = E_\sigma {}^\frown \langle]\rangle$, $E'_\nu = E_\nu$ for $0 < \nu < \sigma$. Moreover, for $\nu < \sigma$, $\Sigma\langle \mathrm{Dom}(E'_\theta) + 1 : \theta < \nu\rangle + \mathrm{Dom}(E'_\nu) = 2 + \mathrm{Dom}(v) + \Sigma\langle \mathrm{Dom}(E_\theta) + 1 : \theta < \nu\rangle + \mathrm{Dom}(E_\nu) = 2 + \mathrm{Dom}(v) + \iota_\nu$ by 9.2.4. Hence this decomposition of $[\chi v A_0]$ is the one for the free substitution of X in $[\chi v A_0]$. By Definition 9.2.3, $SF_f^X[\chi v A_0] = {}^\frown\langle E'_\nu {}^\frown f A_0(\iota_\nu) : \nu < \sigma\rangle {}^\frown E'_\sigma = [\chi v SF_f^{X-\mathrm{Rng}(v)} A_0]$.

9.2.7 *Corollary.* If A is a formula, then $SF_f^X A$ is a formula.

PROOF: The set $\Delta = \{A : A$ is a formula and $SF_f^X A$ is a formula for all sets X of individual symbols and functions f on X to terms$\}$ contains the atomic formulas and, with the aid of the theorem, is easily shown to be closed under the rules of (α, β)-formula-formation.

Note that it is equally easy to show that SF_f^X is the unique operation satisfying conditions $SF_f^X A = S_f^X A$ for A atomic, and (i), (ii), (iii) of 9.2.6 for all formulas A_ξ. The conditions constitute a recursive definition of the operator SF_f^X on formulas.

9.2.8 *Corollary.* If A is a formula, $\rho(A) = \rho(SF_f^X A)$.

PROOF: The rank ρ of a formula was defined in Example 8.2.6.

Let Δ be the set of all formulas A such that for all sets X of individual symbols and functions f on X to terms, $\rho(A) = \rho(SF_f^X A)$. Clearly Δ contains atomic formulas, and with the aid of the theorem, is seen to be closed under rules of (α, β)-formula-formation.

For ordinary first-order predicate languages, it is well-known that the value of $SF_f^X A$ for assignment s to model $\langle D \mathsf{O} \mathsf{C} \mathsf{R}\rangle$ is the same as the value of A for the assignment like s except that it assigns $s^*f(x)$ to $x \in X$, provided that the replacement of $f(x)$ for x at free occurrences in A does not make the variables of $f(x)$ bound. For example, the value of $A = [\forall y[Qyx]$ for assignment d to x is the same as the value of the substitution $[\forall y[Qyz]]$ for the assignment d to z. However, the value of the substitution $[\forall y[Qyy]]$ for the assignment d to y is, in general, different. Theorem 9.2.9 is the extension of this valuation theorem to systems of formulas $F_{\alpha\beta o\pi}$ interpreted in the more general systems $\langle D \mathsf{O} \mathsf{C} \mathsf{R} B \mathsf{O}' \mathsf{Q} S\rangle$.

The condition required to avoid conflicts of variables in the substitution $SF_f^X A$ is this: A must have no free occurrence of x bound by $FV(f(x))$ for any $x \in X$. (Recall that $FV(E)$ is the set of all individual variables having free occurrence in E. When applied to terms, $FV(E)$ is simply the set of all variables appearing in E.) Lemma 9.2.9 is a corollary of Lemma 9.1.1.

9.2.9 *Lemma.* Suppose that f is a function on set X of individual symbols to terms and that A is a formula having no free occurrence of x bound by $FV(f(x))$ in A for any $x \in X$. Then

(i) If A has form $[A_0 \Psi A_1]$, then A_i has no free occurrence of x bound by $FV(f(x))$ in A_i for $x \in X$, $i = 0, 1$.

(ii) If A has form $[\Phi A_0 \ldots A_\xi \ldots]$ and $\xi < \delta$, then A_ξ has no free occurrence of x bound by $FV(f(x))$ in A_ξ for $x \in X$.

(iii) If A has form $[\chi v A_0]$, then A_0 has no free occurrence of x bound by $FV(f(x))$ in A_0 for $x \in X \sim \mathrm{Rng}(v)$.

9.2.10 *Theorem.* Let $\langle D \mathsf{O} \mathsf{C} \mathsf{R} B \mathsf{O}' \mathsf{Q} S\rangle$ be a system for the interpretation of formulas of $F_{\alpha\beta o\pi}$ satisfying

(*) If $s, s' \in S$ and Y is a set of variables having power less than β, then $\mathrm{Repl}_t^Y s \in S$ implies $\mathrm{Repl}_t^Y s' \in S$.

Then for any $s \in S$, set X of individual symbols, function f on X to terms, and $s' \in S$ agreeing with $\mathrm{Repl}_{s^* \circ f}^X s$ on $FV(A)$, we have $V(s, SF_f^X A) = V(s', A)$, provided A has no free occurrence of x bound by $FV(f(x))$ for $x \in X$.

PROOF: Let $\Delta = \{A : A$ is a formula and for all s, X, f chosen as above, $V(s, SF_f^X A) = V(s', A)$ whenever $s' \in S$ agrees with $\mathrm{Repl}_{s \bullet \bigcirc f}^X s$ on $FV(A)$ and A has no free occurrence of x bound by $FV(f(x))$ for $x \in X\}$. Since $SF_f^X A = S_f^X A$ when A is an atomic formula, it is an immediate consequence of Theorems 9.2.2 and 3.5.5 that such formulas are in Δ. Suppose A has form $[A_0 \Psi A_1]$ or $[\Phi A_0 \ldots A_\xi \ldots]$ and that the $A_\xi \in \Delta$. Then if X, f are chosen so that A has no free occurrence of x bound by $FV(f(x))$ for $x \in X$, the A_ξ also have this property by 9.2.9. Moreover, if $s' \in S$ agrees with $\mathrm{Repl}_{s \bullet \bigcirc f}^X s$ on $FV(A)$, then it agrees also on $FV(A_\xi)$. Hence for all ξ, $V(s, SF_f^X A_\xi) = V(s', A_\xi)$. Since the operation SF_f^X and the valuations V for s and s' are all homomorphisms on the algebra of formulas, it follows that $A \in \Delta$.

Suppose $A = [\chi v A_0]$ and $A_0 \in \Delta$. Having chosen X, f such that A has no free occurrence of x bound by $FV(f(x))$ for $x \in X$, 9.1.2 implies that

(1) For $x \in X \sim \mathrm{Rng}(v)$, if x free in A_0, $FV(f(x)) \cap \mathrm{Rng}(v) = \phi$.

Suppose $s' \in S$ agrees with $\mathrm{Repl}_{s \bullet \bigcirc f}^X s$ on $FV(A)$. Let $s_{v,t} = \mathrm{Repl}_t^{\mathrm{Rng}(v)} s$ for all $t \in D^{\mathrm{Rng}(v)}$. Then using formula 9.2.6 (iii) for distribution of SF_f^X over quantifiers,

(2) $V(s, SF_f^X A) = \mathsf{Q}(\chi, v)\{V(s_{v,t}, SF_f^{X-\mathrm{Rng}(v)} A_0) : s_{v,t} \in S\}$.

(3) $V(s', A) = \mathsf{Q}(\chi, v)\{V(s'_{v,t} A_0) : s'_{v,t} \in S\}$.

The induction hypothesis on A_0 applies to the elements of the set in equation (2). The formula A_0 has no free occurrence of x bound by $FV(f(x))$ for $x \in X \sim \mathrm{Rng}(v)$ by 9.2.9. By (1), $\mathrm{Repl}_{s \bullet \bigcirc f}^{X-\mathrm{Rng}(v)} s_{v,t}$ and $\mathrm{Repl}_{s_{v,t} \bullet \bigcirc f}^{X-\mathrm{Rng}(v)} s_{v,t}$ agree on $FV(A_0)$. Moreover, $s'_{v,t}$ and $\mathrm{Repl}_{s \bullet \bigcirc f}^{X-\mathrm{Rng}(v)} s_{v,t}$ agree on $FV(A_0)$ since $FV(A_0) \subseteq FV(A) \cup \mathrm{Rng}(v)$. By (*), $s_{v,t} \in S$ if and only if $s'_{v,t} \in S$. Hence

(4) $V(s_{v,t}, SF_f^{X-\mathrm{Rng}(v)} A_0) = V(s'_{v,t}, A_0)$ for all $s_{v,t} \in S$.

By (2), (3), (4), $V(s, SF_f^X A) = V(s', A)$. Therefore $A \in \Delta$. This completes the induction.

The theorem that follows is a corollary of the one just given for interpretive systems with $\mathrm{Repl}_{s \bullet \bigcirc f}^X s \in S$ whenever $s \in S$. However there are useful interpretive systems where this condition does not hold; for example, systems where S has only one element as in Examples 8.2.5 and 8.2.6. For them we have a weak version of 9.2.10.

9.2.11 *Theorem*. Let $\langle D \; \mathsf{O} \; \mathsf{C} \; \mathsf{R} \; B \; \mathsf{O}' \; \mathsf{Q} \; S \rangle$ be a system for the interpretation of formulas of $F_{\alpha\beta o\pi}$. Then for any $s \in S$, set X of individual symbols, functions f, f' on X to terms such that $s^* \bigcirc f$ and $s^* \bigcirc f'$ agree on $FV(A) \cap X$, $V(s, SF_f^X A) = V(s, SF_{f'}^X A)$ provided that A has no free occurrence of x bound by $FV(f(x))$ or by $FV(f'(x))$ for $x \in X$.

PROOF: Let Δ be the set of all formulas A such that for all X, f, f' such that $s^* \bigcirc f$ and $s^* \bigcirc f'$ agree on $FV(A) \cap X$, $V(s, SF_f^X A) = V(s, SF_{f'}^X A)$ provided that A has no free occurrence of x bound by $FV(f(x))$ or by $FV(f'(x))$ for $x \in X$. If A atomic, $V(s, SF_f^X A) = V(s, S_f^X A)$ and the same for f'. That $V(s, S_f^X A) = V(s, S_{f'}^X A)$ follows by Theorems 9.2.2 and 3.5.5. Suppose A has form $[A_0 \Psi A_1]$ or $[\Phi A_0 \ldots A_\xi \ldots]$ and that the $A_\xi \in \Delta$. Choose s, X, f, f' so that $s^* \bigcirc f$ and $s^* \bigcirc f'$ agree on $FV(A)$ and A has no free occurrence of x bound by $FV(f(x))$ or by $FV(f'(x))$. Then the principal subformulas also satisfy these conditions. Hence $V(s, SF_f^X A_\xi) = V(s, SF_{f'}^X A_\xi)$ for all ξ. Since the operations SF_f^X and $SF_{f'}^X$ and the valuation V for s are all homomorphisms on the algebra of formulas, it is clear that $A \in \Delta$.

Suppose $A = [\chi v A_0]$ and $A_0 \in \Delta$. Choose s, X, f, f' such that $s^* \bigcirc f$ and $s^* \bigcirc f'$ agree on $FV(A) \cap X$ and A has no free occurrence of x bound by $FV(f(x))$ or by $FV(f'(x))$ for $x \in X$. Again letting $s_{v,t}$ be $\mathrm{Repl}_t^{\mathrm{Rng}(v)} s$ for $t \in D^{\mathrm{Rng}(v)}$,

(1) $V(s, SF_f^X A) = \mathsf{Q}(\chi, v)\{V(s_{v,t}, SF_f^{X-\mathrm{Rng}(v)} A_0) : s_{v,t} \in S\}$

(2) $V(s, SF_{f'}^X A) = \mathsf{Q}(\chi, v)\{V(s_{v,t}, SF_{f'}^{X-\mathrm{Rng}(v)} A_0) : s_{v,t} \in S\}$.

We apply the induction hypothesis for A_0 to the elements of the sets in (1) and (2). The restriction on free occurrences of x in A implies $s_{v,t}{}^* \bigcirc f$ agrees with $s^* \bigcirc f$ on $X \cap FV(A)$. Similarly $s_{v,t}{}^* \bigcirc f'$ agrees with $s^* \bigcirc f'$ and hence with $s^* \bigcirc f$ on $X \cap FV(A)$. But $X \cap FV(A) = (X \sim \mathrm{Rng}(v)) \cap FV(A_0)$. Hence $s_{v,t}{}^* \bigcirc f$ and $s_{v,t}{}^* \bigcirc f'$ agree on $FV(A_0) \cap (X \sim \mathrm{Rng}(v))$. By 9.2.8 A_0 has no free occurrence of x bound by $FV(f(x))$ or by $FV(f'(x))$ for $x \in X \sim \mathrm{Rng}(v)$. Hence the sets appearing in (1) and (2) have exactly the same elements. Therefore $V(s, SF_f^X A) = V(s, SF_{f'}^X A)$ and $A \in \Delta$. This completes the induction.

9.3 Substitution for Bound Variables

Let Y be a set of variables, let g be a function on Y to variables. Then if $f(y) = \langle g(y) \rangle$ for all $y \in Y$, Definition 9.2.3 implies that

$SF_f^Y E$ is a variable-for-variable replacement; namely, $SF_f^Y E$ is that expression E' having the same length as E such that $E'(\iota) = E(\iota)$ if ι is not a free occurrence of $y \in Y$ and $E'(\iota) = gE(\iota)$ if ι is a free occurrence of $y \in Y$. It is convenient to call this expression $SF_g^Y E$ so that the passage to f need not be made.

9.3.1 *Definition.* If Y is a set of variables, g a function on Y to variables, $SF_g^Y E = SF_f^Y E$ where $f(y) = \langle g(y) \rangle$ for all $y \in Y$. The length of $SF_g^Y E$ is the same as the length of E and

$$(SF_g^Y E)(\iota) = gE(\iota) \text{ if } \iota \text{ is a free occurrence of } y \in Y \text{ in } E$$
$$(SF_g^Y E)(\iota) = E(\iota) \text{ if not.}$$

An easy induction on formulas shows that whenever variables are substituted for variables in a formula, the resulting expression is again a formula. The particular such substitution we deal with now is the substitution of variables at bound occurrences of variables.

9.3.2 *Definition.* Let Y be a set of variables, g a function on Y to variables. Then $SB_g^Y E$ is that expression having the same length as E satisfying

$$(SB_g^Y E)(\iota) = gE(\iota) \text{ if } \iota \text{ is a bound occurrence of } y \in Y \text{ in } E$$
$$(SB_g^Y E)(\iota) = E(\iota) \text{ if not.}$$

The following is a corollary of Lemma 9.1.1.

9.3.3 *Theorem.* If T is a term, $SB_g^Y T = T$. If A is an atomic formula, $SB_g^Y A = A$. For all formulas A, special two-place propositional operation symbols Ψ, δ-place or infinitary propositional operation symbols Φ,

(i) $SB_g^Y [A_0 \Psi A_1] = [SB_g^Y A_0 \Psi SB_g^Y A_1]$.

(ii) $SB_g^Y [\Phi A_0 \ldots A_\xi \ldots] = [\Phi SB_g^Y A_0 \ldots SB_g^Y A_\xi \ldots]$.

Thus the operator SB_g^Y as well as SF_g^Y, is a homomorphism on the algebra of formulas. The rule for distributing SB_g^Y over quantifiers is not simple. We only need a rule for a special case.

9.3.4 *Theorem.* If g is one-one on Y to variables not in $[\chi v A_0]$, then $SB_g^Y [\chi v A_0] = [\chi g' \bigcirc v SF_g^{Y \cap \text{Rng}(v)} SB_g^Y A_0]$, where g' is g on Y, identity elsewhere. Moreover variables of Rng(g) have no free occurrence in $SB_g^Y A_0$ and no occurrence of $y \in Y$ in $SB_g^Y A_0$ is bound by $\{g(y)\}$.

PROOF: Any occurrence ι of $g(y)$ in $SB_g^Y A_0$ must be a bound occurrence of y in A_0. Therefore ι is a position in A_0 within a subformula $[\chi' v' C']$, $y \in \text{Rng}(v')$. Then ι is a position in $SB_g^Y A_0$ within a subformula of form $[\chi' v'' C'']$, $g(y) \in \mathrm{Rng}(v'')$. The position is therefore bound.

Suppose ι is any occurrence of $y \in Y$ in $SB_g^Y A_0$. If it were bound by $\{g(y)\}$ it would follow that position ι is bound by $\{y\}$ in A_0 since $g(y)$ does not occur in A_0. Moreover ι would have to be an occurrence of y in A_0 since otherwise y would be a variable of $\mathrm{Rng}(g)$ with occurrence in A_0. Hence ι would be a bound occurrence of y in A_0. But then ι is an occurrence of $g(y)$ in $SB_g^Y A_0$ and $g(y) \neq y$. This is a contradiction. Hence ι not bound by $\{g(y)\}$ in $SB_g^Y A_0$.

Let $C = SF_g^{Y \cap \mathrm{Rng}(v)} SB_g^Y A_0$. We must show $SB_g^Y[\chi v A_0] = [\chi g' \bigcirc v C]$. It is clear that the two formulas have the same length and that their symbols are the same at all positions less than $2 + \mathrm{Dom}(v)$. We have to check symbols at positions $\iota = 2 + \mathrm{Dom}(v) + \theta$, $\theta < \mathrm{Dom}(A_0)$. Since the formulas obviously agree at occurrences of symbols other than $y \in Y$ it suffices to consider the following cases:

Case 1. θ is a free occurrence of $y \in Y$ in A_0. Then $SB_g^Y[\chi v A_0]$ at ι is $g(y)$ or y according as y is or is not in $\mathrm{Rng}(v)$. Then θ is still an occurrence of y in $SB_g^Y A_0$ and is still free since $y \notin \mathrm{Rng}(g)$. Therefore $C(\theta) = [\chi g' \bigcirc v C](\iota)$ is also $g(y)$ or y according as y is or is not in $\mathrm{Rng}(v)$.

Case 2. θ is a bound occurrence of $y \in Y$ in A_0. Then $SB_g^Y[\chi v A_0]$ at ι is $g(y)$ and $SB_g^Y A_0$ at θ is $g(y)$. Since $g(y) \notin \mathrm{Rng}(v)$, $C(\theta) = [\chi g' \bigcirc v C](\iota) = g(y)$.

The theorem that follows is a general principle for changing bound variables. It implies $V(s, A) = V(s, SB_g^Y A)$ in ordinary models for (α, β, o, π)-predicate languages. It implies the same equation for the general models of Chapter 12. It even implies $\vdash A \leftrightarrow SB_g^Y A$ in the basic formal systems for these languages.

9.3.5 *Theorem.* Let $\langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ B\ \mathsf{O}'\ \mathsf{Q}\ S\rangle$ be any system for the interpretation of formulas of $F_{\alpha\beta o\pi}$ satisfying

> (***) For all formulas A, $s \in S$, sets Y of variables, one-one functions g on Y to variables not free in A, $V(s, [\chi v A]) = V(s, [\chi g' \bigcirc v SF_g^{Y \cap \mathrm{Rng}(v)} A])$ provided that A has no free occurrence of $y \in Y \cap \mathrm{Rng}(v)$ bound by $\{g(y)\}$. (Again g' is g on Y, identity elsewhere.)

Then $V(s, A) = V(s, SB_g^Y A)$ for all $s \in S$, one-one functions g on Y to variables not in A.

PROOF: Fix Y, g and let $\Delta = \{A : V(s, A) = V(s, SB_g^Y A)$ for all

$s \in S$, provided that variables of Rng(g) do not appear in $A\}$. Atomic formulas are in Δ since $SB_g^Y A = A$ in that case. If A has form $[A_0 \Psi A_1]$ or $[\Phi A_0 \ldots A_\xi \ldots]$ with the $A_\xi \in \Delta$, then A is in Δ since the operation SB_g^Y and the valuation function at s are both homomorphisms on the algebra of formulas.

Suppose $A = [\chi v A_0]$, $A_0 \in \Delta$, and that no variable of Rng(g) appears in A. Then $SB_g^Y A = [\chi g' \bigcirc v SF_g^{Y \cap \mathrm{Rng}(v)} SB_g^Y A_0]$ by 9.3.4 and (***) applies to $SB_g^Y A_0$ to yield $V(s, SB_g^Y A) = V(s, [\chi v SB_g^Y A_0])$. Let $s_{v,t} = \mathrm{Repl}_t^{\mathrm{Rng}(v)} s$ for $t \in D^{\mathrm{Rng}(v)}$. Since $A_0 \in \Delta$, $V(s_{v,t}, A_0) = V(s_{v,t}, SB_g^Y A_0)$ for all $s_{v,t} \in S$. Hence $V(s, [\chi v SB_g^Y A_0]) = V(s, A)$ and $A \in \Delta$. This completes the induction.

For interpretetive systems like the one of Sect. 11.2, condition (***) can be checked directly. But (***) also holds in a broad class of interpretive systems where the condition is not easily checked. This class is the class of interpretive systems with the substitution property, which contains the ordinary semantic models of Example 8.2.3, the algebraic models of Example 8.2.4, the general models of Chapter 12.

9.3.6 *Definition.* A system $\langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ B\ \mathsf{O}'\ \mathsf{Q}\ S\rangle$ for the interpretation of formulas of $F_{\alpha\beta o\pi}$ is said to have the *substitution property* if $\mathsf{Q}(\chi, v) = \mathsf{Q}(\chi, v')$ for all sequences v, v' of variables having length less than β, and if S satisfies the following two conditions:

(*) If $s, s' \in S$ and Y is a set of variables having power less than β, then $\mathrm{Repl}_t^Y s \in S$ implies $\mathrm{Repl}_t^Y s' \in S$.

(**) If $s \in S$, Y is a set of variables having power less than β and $t \in \mathrm{Rng}(s^*)^Y$, then $\mathrm{Repl}_t^Y s \in S$.

If S is the set of all functions on individual variables to D, then S satisfies (*) and (**). The set S of all functions on individual variables to D such that Rng(s^*) is finite, is another example. So is the set of all functions on individual variables to D such that Rng(s^*) has power less than γ, where γ is any infinite cardinal. If D is partially ordered by a relation $\prec$, then the set S of all functions on variables to D such that Rng(s^*) contains no infinite chain with respect to $\prec$ satisfies (*) and (**).

Theorem 9.3.7 is a corollary of Theorem 9.2.10.

9.3.7 *Theorem.* Suppose $\langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ B\ \mathsf{O}'\ \mathsf{Q}\ S\rangle$ is a system for the interpretation of formulas of $F_{\alpha\beta o\pi}$ having the substitution property. Then for all $s \in S$ and sets X of variables having power less than β,

$$V(s, SF_f^X A) = V(\mathrm{Repl}_{s^* \bigcirc f}^X s, A)$$

provided formula A has no free occurrence of x bound by $FV(f(x))$ for $x \in X$.

9.3.8 *Lemma.* Suppose S satisfies conditions (*), (**) of 9.3.6. Let Y be a set of variables having power less than β, let g be a one-one function on Y to variables not in Y. Then $s \in S$ and $\mathrm{Repl}_t^Y s \in S$ implies $\mathrm{Repl}_{tg^{-1}}^{\mathrm{Rng}(g|Y)} s \in S$.

PROOF: By (**), $\mathrm{Repl}_{sg^{-1}}^{\mathrm{Rng}(g|Y)} s \in S$. Hence by (*),

$$\mathrm{Repl}_t^Y \mathrm{Repl}_{sg^{-1}}^{\mathrm{Rng}(g|Y)} s \in S.$$

Call this assignment s'. Let t' be s on Y, tg^{-1} on $\mathrm{Rng}(g|Y)$. By (**), $\mathrm{Repl}_{t'}^{Y \cup \mathrm{Rng}(g|Y)} s' = \mathrm{Repl}_{tg^{-1}}^{\mathrm{Rng}(g|Y)} s \in S$.

9.3.9 *Lemma.* Suppose S satisfies condition (*) of 9.3.6. Let X, Y be pairwise disjoint sets of variables having power less than β. Then

(i) $s \in S$ and $\mathrm{Repl}_t^{X \cup Y} s \in S$ implies $\mathrm{Repl}_t^Y s \in S$.

(ii) $s \in S$, $\mathrm{Repl}_t^X s \in S$ and $\mathrm{Repl}_{t'}^Y s \in S$ implies $\mathrm{Repl}_{t \cup t'}^{X \cup Y} s \in S$.

PROOF: In (i), $s = \mathrm{Repl}_s^X s \in S$. Then $\mathrm{Repl}_t^Y s = \mathrm{Repl}_s^X \mathrm{Repl}_t^{X \cup Y} s$ is in S by (*). (ii) is an immediate consequence of (*) since

$$\mathrm{Repl}_{t \cup t'}^{X \cup Y} s = \mathrm{Repl}_t^X \mathrm{Repl}_{t'}^Y s.$$

9.3.10 *Theorem.* Suppose $\langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ B\ \mathsf{O}'\ \mathsf{Q}\ S\rangle$ is a system for the interpretation of formulas of $F_{\alpha\beta o\pi}$ having the substitution property. Then $V(s, A) = V(s, SB_g^Y A)$ for all one-one functions g to variables that do not appear in A.

PROOF: It suffices to prove condition (***) of Theorem 9.3.5. Suppose $s \in S$, g is one-one on Y to variables not free in formula B, and that B has no free occurrence of y bound by $\{g(y)\}$ for $y \in Y \cap \mathrm{Rng}(v)$. We must show

$$V(s, [\chi v B]) = V(s, [\chi g' \circ v SF_g^{Y \cap \mathrm{Rng}(v)} B]).$$

Let $B' = SF_g^{Y \cap \mathrm{Rng}(v)} B$, $s_{v,t} = \mathrm{Repl}_t^{\mathrm{Rng}(v)} s$ for $t \in D^{\mathrm{Rng}(v)}$ and $s_{g' \circ v, t'} = \mathrm{Repl}_{t'}^{\mathrm{Rng}(g' \circ v)} s$ for $t' \in D^{\mathrm{Rng}(g' \circ v)}$. Since Q depends only on χ in the interpretive system, the definition of V yields

(1) $V(s, [\chi v B]) = \mathsf{Q}(\chi)\{V(s_{v,t}, B) : s_{v,t} \in S\}$

(2) $V(s, [\chi g' \circ v B']) = \mathsf{Q}(\chi)\{V(s_{g' \circ v, t'}, B') : s_{g' \circ v, t'} \in S\}$.

Let $X_0 = \mathrm{Rng}(v) \cap FV(B) \cap Y$, $X_1 = \mathrm{Rng}(v) \cap FV(B) \sim Y$. Let us say that t, t' are *paired* if $t'(x) = t(x)$ for $x \in X_1$, $t'g(x) = t(x)$ for $x \in X_0$. We first show that if $s_{v,t} \in S$, $s_{g' \circ v, t'} \in S$, and t, t'

are paired, then $V(s_{v,t}, B) = V(s_{g' \bigcirc v, t'}, B')$. Theorem 9.3.7 says that $V(s_{g' \bigcirc v, t'}, B') = V(s', B)$ where s' is $s_{g' \bigcirc v, t'} \bigcirc g$ on $Y \cap \mathrm{Rng}(v)$, $s_{g' \bigcirc v, t'}$ elsewhere. On X_0, $s'(x) = s_{g' \bigcirc v, t'}(g(x)) = t'g(x) = t(x) = s_{v,t}(x)$. On X_1, $s'(x) = s_{g' \bigcirc v, t'}(x) = t'(x) = t(x) = s_{v,t}(x)$. On $FV(B) \sim X_0 \sim X_1 = FV(B) \sim \mathrm{Rng}(v)$, $s'(x) = s_{g' \bigcirc v, t'}(x) = s(x) = s_{v,t}(x)$. Therefore s' and $s_{v,t}$ agree on $FV(B)$. So $V(s', B) = V(s_{v,t}, B)$ by Theorem 9.1.5.

Comparing (1) and (2) we see that paired t, t' contribute the same values to the sets on the right-hand sides. Therefore the proof will be complete once we have shown that for any $s_{v,t} \in S$ there is t' such that t, t' paired and $s_{g' \bigcirc v, t'} \in S$, and conversely. If $s_{v,t} \in S$ let t' be $t \bigcirc g^{-1}$ on $\mathrm{Rng}(g|X_0)$, t on X_1, s on $\mathrm{Rng}(g' \bigcirc v) \sim \mathrm{Rng}(g|X_0) \sim X_1$. Then t, t' are paired. By 9.3.9 (i), $\mathrm{Repl}\,{}^{X_1}_{t}\, s \in S$, $\mathrm{Repl}\,{}^{X_0}_{t}\, s \in S$. Since X_0, $\mathrm{Rng}(g|X_0)$ are disjoint, $\mathrm{Repl}^{\mathrm{Rng}(g|X_0)}_{t \bigcirc g^{-1}}\, s \in S$. Since also X_1, $\mathrm{Rng}(g|X_0)$ disjoint, $s_{g' \bigcirc v, t'} \in S$ by 9.3.9 (ii). A similar computation yields the converse. Lemma 9.3.8 can be applied with g^{-1}, $\mathrm{Rng}(g|X_0)$ for g, Y.

9.4 Function Notation for Substitution

The convenient function notation will be used for formulas in the chapters that follow, but only in a special way. First a sequence v of variables must be attached to formula A. We then write A as "$A(v)$" or sometimes "$A(v(0)\ldots v(\xi)\ldots)$". It is understood that the sequence need not contain all variables free in A and may contain variables not in A. Let f be a function mapping $\mathrm{Rng}(v)$ to terms. Then, roughly speaking, $A(f \bigcirc v) = A(fv(0)\ldots fv(\xi)\ldots)$ arises by first changing bound variables to avoid conflicts, then substituting $fv(\xi)$ for free occurrences of $v(\xi)$. It has to be assumed that there are sufficiently many variables available for the change, variables not in A nor in any of the $fv(\xi)$.

In order to give a procedure for the change of bound variables we further assume that the variables of the system are indexed by an ordinal. Let Y be the set of all variables in terms of $\mathrm{Rng}(f)$ which appear bound in A. Order the variables of Y in order of occurrence in A: $Y = \{y_\nu : \nu < \sigma\}$. Let Z be the set of all variables of the system which do not appear in A nor in terms of $\mathrm{Rng}(f)$. Let $g(y_0)$ be the variable of least index in Z, and for $\nu < \sigma$, let $g(y_\nu)$ be the variable of least index in $Z \sim \{g(y_\theta) : \theta < \nu\}$. Then g is one-one on Y, and variables of Rng(g) are not in A. Let $A(f \bigcirc v) = SF_f^{\mathrm{Rng}(v)} SB_g^Y A$. Since $SB_g^Y A$ contains no variable of $fv(\xi)$ in any of its quantifier-sequences, a free occurrence of $v(\xi)$ in $SB_g^Y A$ is not bound by $FV(fv(\xi))$. A simple argument shows that $FV(A) = FV(SB_g^Y A)$. Consequently the substitution theorems can now be restated without the provisos.

9.4.1 *Theorem.* Suppose A is a formula of a system for which $A(f \bigcirc v)$ is defined. Let $\langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ B\ \mathsf{O}'\ \mathsf{Q}\ S\rangle$ be a system for interpretation of these formulas satisfying conditions (*) of 9.2.10 and (***) of 9.3.5. Then $V(s, A(f \bigcirc v)) = V(s', A(v))$ for all $s' \in S$ agreeing with $\mathrm{Repl}_{s*\bigcirc f}^{\mathrm{Rng}(v)} s$ on $FV(A)$.

9.4.2 *Theorem.* Suppose A is a formula of a system for which $A(f \bigcirc v)$ is defined, interpreted in a system having the substitution property. Then $V(s, A(f \bigcirc v)) = V(s', A(v))$ for all $s' \in S$ agreeing with $\mathrm{Repl}_{s*\bigcirc f}^{\mathrm{Rng}(v)} s$ on $FV(A)$. Moreover, if $\mathrm{Dom}(v) < \beta$, $V(s, A(f \bigcirc v)) = V(\mathrm{Repl}_{s*\bigcirc f}^{\mathrm{Rng}(v)} s, A(v))$ since in this case we know that the replacement itself is in S.

Proofs are by 9.2.10, 9.3.5, 9.3.10.

CHAPTER 10

INFINITARY PREDICATE LANGUAGES

10.1 The (α, β, o, π)-Predicate Languages

Let α be a regular infinite cardinal, β a cardinal which is either 0 or satisfies $\omega \leqslant \beta \leqslant \alpha$. Let o be a regular infinite cardinal which is less than β if β singular, at most α if β regular. Let π be an infinite cardinal at most α. Then an (α, β, o, π)-language $\mathbf{L}_{\alpha\beta o\pi}$ has as its formulas a system $F_{\alpha\beta o\pi}$ with at least α individual variables, with equality symbol $\bar{=}$ among its special two-place predicate symbols, having one-place propositional operation symbol $\neg$, special two-place propositional operation symbol $\rightarrow$, infinitary symbol $\bigwedge$, quantifier $\forall$. A language $\mathbf{L}_{\alpha\beta}$ is an $(\alpha, \beta, \omega, \omega)$-predicate language. Such a language has finite atomic formulas. Rewriting 8.1.1 we have the following:

10.1.1 *Induction Principle for Formulas.* If Δ is any set of expressions containing the atomic formulas and closed under conditions

(1) If $A \in \Delta$, then $[\neg A] \in \Delta$,
(2) If $A_0, A_1 \in \Delta$, then $[A_0 \rightarrow A_1] \in \Delta$,
(3) If $0 < \delta < \alpha$ and $A_\xi \in \Delta$ for all $\xi < \delta$, then $[\bigwedge A_0 \ldots A_\xi \ldots] \in \Delta$,
(4) If $A \in \Delta$ and v is a sequence of individual variables having length less than β, then $[\forall v A] \in \Delta$,

then Δ contains all formulas.

To complete passage from $F_{\alpha\beta o\pi}$ to $\mathbf{L}_{\alpha\beta o\pi}$ we must tell how the formulas are to be interpreted. The intended interpretations are in models $\langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\rangle$ described in Example 8.2.3, such that R assigns the equality relation over D to $\bar{=}$. According to the recursion principle for terms, given any function s on individual variables to D, there is a unique function s^* mapping the terms to D such that $s^*\langle x\rangle = s(x)$ for individual variables x, $s^*\langle c\rangle = \mathsf{C}(c)$ for individual constants c, and such that the following conditions hold for all terms $T_0, \ldots, T_\xi, \ldots$:

(1) $s^*[T_0 \psi T_1] = \mathsf{O}(\psi)(\langle s^*T_0 s^*T_1\rangle)$ for special two-place operation symbols ψ,

(2) $s^*[\varphi T_0 \ldots T_\xi \ldots] = \mathsf{O}(\varphi)(\langle s^*T_0 \ldots s^*T_\xi \ldots\rangle)$ for $0 < \zeta < o$, ζ-place or infinitary operation symbols φ, ζ-tuples of terms.

Furthermore, according to the recursion principle for formulas, there is a unique function V on $S \times F_{\alpha\beta o\pi}$ such that for all terms $T_0, \ldots, T_\xi, \ldots$ and assignments $s \in S$,

(3) $V(s, [T_0 P T_1]) = \mathsf{R}(P)(\langle s^*T_0 s^*T_1\rangle)$ for special two-place predicate symbols P,

(4) $V(s, [Q T_0 \ldots T_\xi \ldots]) = \mathsf{R}(Q)(\langle s^*T_0 \ldots s^*T_\xi \ldots\rangle)$ for $0<\eta<\pi$, η-place or infinitary predicate symbols Q,

and for all formulas $A_0, \ldots, A_\xi, \ldots$ and sequences v of variables having length less than β,

(5) $V(s, [\neg A_0]) = \mathbf{1}$ if and only if $V(s, A_0) = \mathbf{0}$

(6) $V(s, [A_0 \rightarrow A_1]) = \mathbf{1}$ if and only if $V(s, A_0) = \mathbf{0}$ or $V(s, A_1) = \mathbf{1}$

(7) $V(s, [\bigwedge A_0 \ldots A_\xi \ldots]) = \mathbf{1}$ if and only if $V(s, A_\xi) = \mathbf{1}$ for all $\xi < \delta$,

(8) $V(s, [\forall v A_0]) = \mathbf{1}$ if and only if $V(\mathrm{Repl}_t^{\mathrm{Rng}(v)} s, A_0) = \mathbf{1}$ for all $t \in D^{\mathrm{Rng}(v)}$.

The additional special two-place propositional operation symbols $\leftrightarrow$, $\wedge$, $\vee$, infinitary symbol $\bigvee$, quantifier $\exists$, are introduced by their customary abbreviations. They are given in Chapter 8, Example 8.2.5, where also a function V' is defined for the elimination of these additional symbols from formulas. When we speak of "the formula $[A_0 \leftrightarrow A_1]$ of $\mathbf{L}_{\alpha\beta o\pi}$", let it be understood that the formula $V'([A_0 \leftrightarrow A_1])$ is meant. The valuation function extends to the defined symbols as follows:

(9) $V(s, [A_0 \leftrightarrow A_1]) = \mathbf{1}$ if and only if $V(s, A_0) = V(s, A_1)$

(10) $V(s, [A_0 \wedge A_1]) = \mathbf{1}$ if and only if $V(s, A_0) = V(s, A_1) = \mathbf{1}$

(11) $V(s, [A_0 \vee A_1]) = \mathbf{1}$ if and only if

$$V(s, A_0) = \mathbf{1} \text{ or } V(s, A_1) = \mathbf{1}$$

(12) $V(s, [\bigvee A_0 \ldots A_\xi \ldots]) = \mathbf{1}$ if and only if there is $\xi < \delta$ such that $V(s, A_\xi) = \mathbf{1}$

(13) $V(s, [\exists v A_0]) = \mathbf{1}$ if and only if there is $t \in D^{\mathrm{Rng}(v)}$ such that $V(\mathrm{Repl}_t^{\mathrm{Rng}(v)} s, A_0) = \mathbf{1}$.

10.1.2 *Remark.* Note the several restrictions on the relative magnitudes of α, β, o, π. The assumption of regularity on α and o is an inessential simplifying assumption. If, for example, α were allowed to be singular, say $\alpha = \bigcup \{\gamma_\nu : \nu < \kappa\}$ with $\kappa < \alpha$, $\gamma_\nu < \alpha$ for $\nu < \kappa$, then

$$[\bigwedge_{\xi<\alpha} A_0 \ldots A_\xi \ldots]$$

could be regarded as an abbreviation for

$$[\bigwedge_{\nu<\kappa} [\bigwedge_{\xi<\gamma_0} A_0 \ldots A_\xi \ldots] \ldots [\bigwedge_{\xi<\gamma_\nu} A_0 \ldots A_\xi \ldots] \ldots].$$

The bound might as well have been the regular cardinal α^+ in the first place.

The assumptions $\pi \leqslant \alpha$ and $o < \beta$ if β singular, $o \leqslant \alpha$ if β regular, are essential restrictions. The completeness theorems of Chapter 11 were originally obtained only for languages with finite atomic formulas. However, the proofs have to be only slightly modified to apply to these more general languages. When the restrictions on o and π are relaxed, exceptions occur with alarming frequency, and even when theorems are capable of extension (and some are), complicating features have to be introduced into the proofs. A comprehensive study of such languages has never been undertaken; however, languages with infinitary predicate symbols but finite conjunctions and quantifications, have been dealt with by Henkin in connection with his work on cylindric algebras. See [13]. H. Keisler in [17] announced the strong completeness of basic predicate calculus for languages without equality, having infinitely long atomic formulas and quantifications, but finitely long conjunctions. The absence of equality is essential.

As long as atomic formulas are restricted in length to less than α, it is no further restriction to assume $\beta \leqslant \alpha$. Corollary 9.1.3 implies that a formula could never have α or more free variables anyway. Ruling out β finite and not zero is clearly an innessential simplifying assumption. These cases are covered by letting β be ω.

Note that every term of $\mathbf{L}_{\alpha\beta o\pi}$ has length less than o, every formula has length less than α. A formula *holds* in a model $\mathfrak{M} = \langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\rangle$ if its value is $\mathbf{1}$ for all assignments to $\mathfrak{M}$. We say that $\mathfrak{M}$ is a *model of* Γ if all formulas of Γ hold in $\mathfrak{M}$. If formula A holds in all models, then A is *valid* (written "$\Vdash A$"). A set Γ of formulas is *satisfiable* if there is a model and assignment s to that model such that $V(s, A) = \mathbf{1}$ for all $A \in \Gamma$. Formula A is a *semantic consequence* of Δ (written "$\Vdash_\Delta A$") if A holds in all models of Δ. Formula A is a *strict semantic consequence of* Γ (written "$\Gamma \Vdash A$") if either $\Vdash A$ or there is a subset $\{A_\xi : \xi < \delta\}$ of Γ having power less than α such that $\Vdash [\bigwedge A_0 \ldots A_\xi \ldots] \rightarrow A$. Clearly a strict semantic consequence of Γ is a semantic consequence of Γ. Set Γ is *semantically consistent* if every subset of Γ having power less than α is satisfiable. Satisfiability implies semantic consistency and the two notions coincide for sets of power less than α.

The intended interpretation in models has the substitution property 9.3.6. As corollaries of Theorems 9.1.5, 9.2.10, 9.3.10, 9.4.2, we have the following:

10.1.3 *Theorem.* If s, s' are assignments agreeing on variables free in A, then $V(s, A) = V(s', A)$.

10.1.4 *Theorem.* If A has no free occurrence of x bound by $FV(f(x))$ for $x \in X$, then $V(s, SF_f^X A) = V(\mathrm{Repl}^X_{s*\bigcirc f} s, A)$.

10.1.5 *Theorem.* $V(s, A) = V(s, SB_g^Y A)$ for all one-one functions g to variables that do not appear in A.

10.1.6 *Theorem.* $V(s, A(f \bigcirc v)) = V(\mathrm{Repl}^{\mathrm{Rng}(v)}_{s*\bigcirc f} s, A(v))$.

The function notation for substitution can be used because we have assumed that there are α variables in the language, while fewer than α variables appear in A and in the $fv(\xi)$ for $v(\xi)$ free in A. Therefore conflicting bound variables can be changed.

By 10.1.3, to say that a sentence holds in model $\mathfrak{M}$ is equivalent to saying that it is satisfiable in $\mathfrak{M}$.

10.2 Semantic Consistency, Hanf's Example

It is a fundamental theorem for ordinary (ω, ω)-predicate languages that every semantically consistent set of formulas is satisfiable. It is clear from the examples of Chapter 4 that this compactness property fails for infinite non-limit cardinals α.

10.2.1 *Example.* Let γ be an infinite cardinal, $\mathbf{L}$ any $(\gamma^+, \beta, o, \pi)$-predicate language. Then the following set of γ^+ formulas is semantically consistent but not satisfiable:

$$\Gamma = \{[\bigvee_{\nu<\gamma} [x_\mu \overline{=} y_0]\ldots[x_\mu \overline{=} y_\nu]\ldots] : \mu < \gamma^+\} \cup$$
$$\{\neg[x_\mu \overline{=} x_{\mu'}] : \mu \neq \mu', \mu, \mu' < \gamma^+\}.$$

For a wide class of strongly inaccessible cardinals α, W. Hanf has given examples of semantically consistent, non-satisfiable sets of α sentences of an (α, α)-predicate language. See his abstract [7]. These examples are discussed in Tarski's article [47], where far-reaching applications to set theory are given.

Let us look at Hanf's example for θ_1, the first strongly inaccessible cardinal greater than ω. A cardinal is *accessible* if it is not strongly inaccessible. Referring to the definition in the Foreward, an accessible cardinal is one that is either singular, or for which there is a cardinal γ such that $\gamma < \alpha \leqslant 2 \exp \gamma$. Consider the (θ_1, θ_1)-predicate language with special two-place predicate symbol $\bar{\in}$, individual variables $x_0, \ldots, x_\xi, \ldots, \xi < \theta_1$.

10.2.2 *Example.* Let A_0 be the conjunction of the axiom of extensionality and the axiom of well-foundedness in Example 1.1.4 of Chapter 1. Then any model of A_0 is isomorphic to a transitive family of sets under the membership relation. For each cardinal $0 < \gamma < \theta_1$ let

$$C_\gamma = [\underset{\xi<\gamma}{\forall} x_0 \ldots x_\xi \ldots [\exists x_\gamma[\forall x_{\gamma+1}[x_{\gamma+1} \bar{\in} x_\gamma \leftrightarrow [\underset{\xi<\gamma}{\vee} x_{\gamma+1} \bar{=} x_\xi]]]].$$

Then any model of C_γ is closed under formation of sets of power at most γ. Next write the usual defining formulas for the following fundamental notions of set theory:

Ord Pr(x_0, x_1, x_2)	expressing	x_2 is the ordered pair of x_0, x_1,
One-one Fcn (x_0)	expressing	x_0 is a one-one function,
Union(x_0, x_1)	expressing	x_1 is the union of x_0,
$\bar{\subseteq}(x_0, x_1)$	expressing	x_0 is a subset of x_1,
Ord(x_0)	expressing	x_0 is an ordinal.

The reader can easily write these formulas for himself using the definitions in the Foreward on Set Theory. The last three formulas have their intended meanings in any transitive field of sets; that is, for them, $\langle S_0 \ldots S_n \rangle$ satisfies $A(x_0, \ldots, x_n)$ in a transitive family of sets if and only if $\langle S_0 \ldots S_n \rangle$ satisfies the intended condition. The first two formulas have their intended meanings in those transitive families of sets in which C_2 holds. Then in any transitive family of sets,

$$A_1 = [\exists x_0[\text{Ord}(x_0) \wedge \neg\exists x_1[\text{Ord}(x_1) \wedge x_0 \bar{\in} x_1]]$$

says that there is a largest ordinal in the family. Finally, we have to write a sentence that says that every ordinal is either denumerable or accessible; that is, is denumerable, or is a union of a smaller number of smaller ordinals, or is less than or equal to the cardinal of the power set of an element. The first alternative is easy to write:

$$B_0(x_0) = [\neg\exists x_1[x_1 \mathrel{\bar{\in}} x_0]] \vee [\underset{n<\omega}{\exists}\ x_1 \ldots x_n \ldots [\forall x_\omega[x_\omega \mathrel{\bar{\in}} x_0 \leftrightarrow [\underset{n<\omega}{\bigvee}\ x_\omega \mathrel{\bar{\equiv}} x_n]]]].$$

In any transitive family of sets, S_0 satisfies $B_0(x_0)$ if and only if S_0 is denumerable. To express the second alternative, we can write a formula $B_1(x_0, x_1)$ such that $\langle S_0 S_1\rangle$ satisfies $B_1(x_0, x_1)$ in a transitive family $\langle S \in\rangle$ of sets satisfying C_2 if and only if $S_0 = \bigcup S_1$ and $S_1 \subsetneq S_0$ and there is a one-one function in S mapping S_1 into an element of S_0:

$$B_1(x_0, x_1) = [\mathrm{Union}(x_1, x_0) \wedge \bar{\subsetneq}\ (x_1, x_0) \wedge [\exists x_2 x_3 B'(x_0, x_1, x_2, x_3)]],$$

where

$$B'(x_0, x_1, x_2, x_3) = [\text{One-one Fcn}(x_2) \wedge [x_3 \mathrel{\bar{\in}} x_0] \wedge [\forall x_4[x_4 \mathrel{\bar{\in}} x_1 \rightarrow [\exists x_5 x_6[x_5 \mathrel{\bar{\in}} x_3 \wedge x_6 \mathrel{\bar{\in}} x_2 \wedge \text{Ord Pr}(x_4, x_5, x_6)]]]]].$$

Similarly we can write a formula $B_2(x_0, x_1)$ such that $\langle S_0 S_1\rangle$ satisfies $B_2(x_0, x_1)$ in a transitive family $\langle S \in\rangle$ of sets satisfying C_2 if and only if there is a one-one function in S mapping S_0 into $\mathscr{P}S_1$:

$$B_2(x_0, x_1) = \exists x_2[\text{One-one Fcn}(x_2) \wedge [\forall x_3[x_3 \mathrel{\bar{\in}} x_0 \rightarrow \exists x_4 x_5[\text{Ord Pr}(x_3, x_4, x_5) \wedge x_5 \mathrel{\bar{\in}} x_2 \wedge \bar{\subseteq}\ (x_4, x_1)]]]].$$

Let $A_2 = [\forall x_0[\mathrm{Ord}(x_0) \rightarrow [B_0(x_0) \vee [\exists x_1 B_1(x_0, x_1)] \vee [\exists x_1[x_1 \mathrel{\bar{\in}} x_0 \wedge B_2(x_0, x_1)]]]]]$. Finally, let $\Gamma = \{A_0, A_1, A_2\} \cup \{C_\gamma\colon \gamma$ is a cardinal less than $\theta_1\}$.

Note first that Γ is not satisfiable. For if it were, it would have a transitive family of sets as model by A_0. All ordinals less than θ_1 would be in the family by the C_γ. Hence the largest ordinal that A_1 says is in the family would have to be at least as large as θ_1. Since the family is transitive, θ_1 is an element. But θ_1 does not satisfy any of the alternatives of A_2.

Consider any subset Γ_0 of Γ having power less than θ_1. We must show Γ_0 satisfiable. Let κ be an infinite cardinal less than θ_1 such that $C_\gamma \in \Gamma_0$ implies $\gamma \leqslant \kappa$. Let $T'_0 = \{\phi, \kappa^+, f\}$ where f is a one-one function mapping κ^+ into subsets of κ. For $\xi < \kappa^+$ let $T'_\xi = \{S: S \subseteq \bigcup\{T'_\nu : \nu < \xi\}$ and $\operatorname{card}(S) \leqslant \kappa\}$. Let $T' = \bigcup\{T'_\xi : \xi < \kappa^+\}$. Then T' includes the family T_{κ^+} of all sets hereditarily of power at most κ since it includes all the sets T_ξ, $\xi < \kappa^+$, described in the Foreward on Set Theory. The C_γ in Γ_0 obviously hold in $\langle T' \in\rangle$. Elements of κ^+ and of f are in T_{κ^+}, hence also in T'. Elements of T'_ξ, $0 < \xi < \kappa^+$, are obviously subsets of T'. Therefore $\langle T' \in\rangle$ is a transitive family of sets. The only elements of T' having power κ^+ or more are κ^+ and f. Therefore κ^+ is the largest ordinal in T' and A_1 is satisfied. To see that this is also a model of A_2, consider first an ordinal $\xi < \kappa^+$ that is not a cardinal. Then there is $\nu < \xi$ such that ν and ξ are cardinally equivalent. There is a one-one function in T_{κ^+} mapping ξ to one-element subsets of ν. Hence the third alternative of A_2 holds in this case. If $\gamma \leqslant \kappa^+$ is a cardinal then it is either denumerable or accessible. If it is denumerable then it satisfies $B_0(x_0)$. If it is singular, having form $\gamma = \bigcup\{\gamma_\nu : \nu < \lambda\}$ with $\lambda \in \gamma$ and the $\gamma_\nu \in \gamma$, then $\gamma < \kappa^+$, $\{\gamma_\nu : \nu < \lambda\} \in T_{\kappa^+}$, and there is a one-one function in T_{κ^+} mapping $\{\gamma_\nu : \nu < \lambda\}$ into λ. Hence the second alternative of A_2 holds in this case. Finally if $\gamma \leqslant 2 \exp \gamma'$, $\gamma' \in \gamma$, then if $\gamma < \kappa^+$ there is a one-one function in T_{κ^+} mapping γ to subsets of γ'. If $\gamma = \kappa^+$, such a function was placed in T'. Therefore in this case γ satisfies the third alternative of A_2. This completes the example for θ_1.

10.2.3 *Remarks.* Note that one can find similar examples for any regular cardinal $\alpha > \omega$ such that it is possible to write a sentence like A_2 in $\mathbf{L}_{\alpha\alpha}$ describing the property of being an ordinal less than α. Such examples are available not only for many higher strong inaccessibles, but for many classes of regular accessibles as well. See Tarski's article [47] for more information on this point. He calls an infinite cardinal α *incompact* if it is singular, or it is regular and there is a semantically consistent, non-satisfiable set of sentences of an (α, α)-predicate language with no individual constants. He calls such a cardinal *strongly incompact* if there is such a set of sentences having power α. Then any cardinal for which an example like Hanf's can be found is strongly incompact. For non-limit α the simpler examples of 4.1.2 are available to show strong incompactness. The cardinal ω is, of course, compact. The problem of the existence of a compact cardinal $\alpha > \omega$ remains open, but Tarski points out that we do not know any example of a cardinal $\alpha > \omega$ which would have a "constructive characterization" and for which we could not prove strong incompactness.

As an application of our basic lemma on satisfiability, we will show that the incompactness of a regular cardinal α is equivalent to the existence of an $\nearrow \alpha$-representable Boolean algebra not isomorphic to an $\nearrow \alpha$-field of sets. This and other set-theoretic and Boolean algebraic properties of incompact cardinals are discussed in Tarski's paper [47].

10.3 A Criterion for Satisfiability

This criterion is a straightforward generalization of Henkin's method for constructing models in [9]. The suggestion of Hasenjaeger in [8] is also used. It is, however, considerably complicated by the necessity for keeping track of cardinalities when $\alpha > \omega$. The set Δ of 10.3.4 must have power less than α in order for the completeness theorems to go through.

10.3.1 *Definition.*

Let γ be an infinite cardinal, β a cardinal at most γ^+. Then $\kappa(\gamma^+, \beta)$ is the smallest cardinal κ such that

(1) $\gamma \leqslant \kappa$

(2) $\kappa \exp \varepsilon = \kappa$ for all cardinals $\varepsilon < \beta$.

10.3.2 Properties of $\kappa(\gamma^+, \beta)$.

(i) $\gamma \leqslant \kappa(\gamma^+, \beta) \leqslant \gamma \exp \beta$.

(ii) $\beta \leqslant \kappa(\gamma^+, \beta)$.

(iii) If β regular, $\kappa(\gamma^+, \beta) = \bigcup \{\gamma \exp \varepsilon : \varepsilon < \beta\}$.

(iv) If $\mathrm{card}(I) < \kappa(\gamma^+, \beta)$ and $\mathrm{card}(S_i) < \beta$ for $i \in I$, then
$$\mathrm{card} \bigcup \{S_i : i \in I\} < \kappa(\gamma^+, \beta).$$

PROOF: (i) and (ii) are immediate consequences of the definition. Turn to (iii) and let $\kappa = \bigcup \{\gamma \exp \varepsilon : \varepsilon < \beta\}$. Then $\kappa = \mathrm{card}(K)$, where K is the set of all sequences of ordinals $\leqslant \gamma$ having length less than β, and κ is also $\mathrm{card}(K')$, where K' is the set of all sequences of ordinals less than γ having length less than β. It suffices to show that if $0 < \delta < \beta$, then there is a one-one mapping from K'^{δ} into K. Such a mapping is g, where for $f = \langle f_0 \ldots f_\xi \ldots \rangle \in K'^{\delta}$, $g(f) = {}^\frown\langle\langle\gamma\rangle{}^\frown f_\xi : \xi < \delta\rangle$. The sequence $g(f)$ is in K by regularity of β, and g can be seen to be one-one by using the properties of concatenation in Chapter 2.

The proof of (iv) splits into two cases. Case 1. $\beta \leqslant \text{card}(I)$. Then $\text{card} \bigcup \{S_i\colon i \in I\} \leqslant \text{card}(I \times I) = \text{card}(I) < \kappa(\gamma^+, \beta)$. Case 2. $\text{card}(I) < \beta$. Using König's Theorem, $\text{card} \bigcup \{S_i\colon i \in I\} < \text{card}(\beta^I)$. By (ii), $\text{card}(\beta^I) \leqslant \kappa(\gamma^+, \beta)$.

The cardinal $\kappa(\gamma^+, \beta)$ now entering the picture is the smallest cardinal κ such that every satisfiable (γ^+, β)-sentence has a model of power at most κ. For the satisfiability lemma, 10.3.5, implies that every satisfiable (γ^+, β)-sentence has a model of power at most $\kappa(\gamma^+, \beta)$, while the examples below show that no smaller cardinal has that property.

10.3.3 *Example.* If $\beta = \omega$ then $\kappa(\gamma^+, \beta) = \gamma$. Consider a (γ^+, ω)-language having individual constants u_ξ, $\xi < \gamma$, and the equality symbol as the only constants. Let A be the conjunction of formulas $[\neg u_\xi \mathrel{\bar{=}} u_\nu]$ for $\xi, \nu < \gamma$, $\xi \neq \nu$, and the formula

$$[\forall x_0[\bigvee_{\xi<\gamma} [x_0 \mathrel{\bar{=}} u_0]\ldots[x_0 \mathrel{\bar{=}} u_\xi]\ldots]].$$

There is up to isomorphism only one model of A and it has power $\kappa(\gamma^+, \omega) = \gamma$.

If $\beta = \gamma^+$ then $\kappa(\gamma^+, \beta) = 2 \exp \gamma$. Consider the (γ^+, γ^+)-language having predicate symbols $\bar{=}$, $\bar{\in}$, as the only constants. We saw in Example 1.1.5 of Chapter 1 that the family T_{γ^+} of sets hereditarily of power at most γ can be characterized up to isomorphism by a sentence of this language. This family has power $\kappa(\gamma^+, \gamma^+)$.

If $\omega < \beta < \gamma^+$ consider a (γ^+, β)-language having individual constants u_ξ, $\xi < \gamma$, and predicate constants $\bar{=}$, $\bar{\in}$, as only constants. Let A be the conjunction of sentences $[\neg u_\xi \mathrel{\bar{=}} u_\nu]$ for $\xi, \nu < \gamma$, $\xi \neq \nu$, with the following:

$$C_1 = \forall x_0 x_1[[[\bigwedge_{\xi<\gamma} [\neg x_0 \mathrel{\bar{=}} u_\xi]] \wedge [\bigwedge_{\xi<\gamma} [\neg x_1 \mathrel{\bar{=}} u_\xi]]] \rightarrow$$
$$[\forall x_2[[x_2 \mathrel{\bar{\in}} x_0 \leftrightarrow x_2 \mathrel{\bar{\in}} x_1] \rightarrow x_0 \mathrel{\bar{=}} x_1]]].$$

$$C_2 = \neg \mathop{\exists}_{n<\omega} x_0\ldots x_n\ldots[\bigwedge_{n<\omega} x_{n+1} \mathrel{\bar{\in}} x_n]$$

$$C_3 = \bigwedge_{\xi<\gamma} [\forall x_0[\neg x_0 \mathrel{\bar{\in}} u_\xi]].$$

$$C_4 = [\bigwedge_{0<\varepsilon<\beta} [\mathop{\forall}_{\xi<\varepsilon} x_0\ldots x_\xi\ldots[\exists x_\beta \forall x_{\beta+1} C']] \text{ where}$$

$$C' = [x_{\beta+1} \mathrel{\bar{\in}} x_\beta \leftrightarrow [\bigvee_{\xi<\varepsilon} x_{\beta+1} \mathrel{\bar{=}} x_\xi]].$$

$$C_5 = \forall x_\beta[[\bigwedge_{\xi<\gamma} \neg x_\beta \mathrel{\bar{=}} u_\xi] \rightarrow [\bigvee_{0<\varepsilon<\beta} [\mathop{\exists}_{\xi<\varepsilon} x_0\ldots x_\xi\ldots \forall x_{\beta+1} C']]].$$

This formula A characterizes up to isomorphism model $\langle S, c_0 \ldots c_\xi \ldots \in/S\rangle$, where the c_ξ are any given sets all different and not connected by a $\in$-chain, and S is the intersection of all families of sets containing the c_ξ and closed under formation of non-empty sets of power less than β. The proof is so similar to the one in Example 1.1.4, that it will not be repeated here. This family has power $\kappa(\gamma^+, \beta)$.

The first step in the satisfiability lemma is to obtain the set Δ like the one in the analogous lemma for propositional formulas, 4.1.4. If A is a (γ^+)-propositional formula, then we can take Δ to be the set of all subformulas of A and the criterion for satisfiability is this: A is satisfiable if and only if there is a set Γ of formulas containing A such that $A_0 \in \Delta$ implies $A_0 \in \Gamma$ iff $[\neg A_0] \notin \Gamma$, $[A_0 \rightarrow A_1] \in \Delta$ implies $[A_0 \rightarrow A_1] \in \Gamma$ iff $A_0 \notin \Gamma$ or $A_1 \in \Gamma$, $[\bigwedge A_0 \ldots A_\xi \ldots] \in \Delta$ implies $[\bigwedge A_0 \ldots A_\xi \ldots] \in \Gamma$ iff all $A_\xi \in \Gamma$. The fact that Δ has power at most γ figured in several of the theorems for (γ^+)-propositional languages, notably, Theorems 4.3.4 and 5.3.1.

If A is a $(\gamma^+, \beta, o, \pi)$-sentence containing no variables, then the set of subformulas of A may again serve as Δ; however, if A has variables, then it is necessary to put into Δ all substitutions of subformulas of A to a suitable set T of terms. This set T must have properties:

(1) Principal terms of atomic formulas in Δ are in T and subterms of terms in T are in T.

(2) Every formula of Δ and term of T has fewer than β free variables.

(3) $\mathrm{Card}(\mathsf{T}) = \kappa(\gamma^+, \beta)$.

10.3.4 *Lemma.* Let A be a sentence of a language $\mathbf{L}_{\gamma+\beta o\pi}$ and X be a set of $\kappa(\gamma^+, \beta)$ individual variables not in A. Then there exists a set T of terms such that if $\Delta = \{SF_f^{FV(C)}C : C$ is a subformula of A and $f \in \mathsf{T}^{FV(C)}\} \cup \{[T \overline{=} T'] : T, T' \in \mathsf{T}\}$ then conditions (1), (2), (3) hold. Moreover Δ and T have properties

(4) The free variables of formulas of Δ and all variables in terms of T are in X.

(5) $\mathrm{Card}(\Delta) = \kappa(\gamma^+, \beta)$.

Proof: Let $\kappa = \kappa(\gamma^+, \beta)$ and let T_0 be the set of all atomic terms $\langle x\rangle$ with $x \in X$ and $\langle c\rangle$ with c an individual constant of A. In case A contains no operation symbols T_0 already satisfies the required conditions. Corollary 9.1.4 is used to verify (2) and property $\kappa \exp \varepsilon = \kappa$ for all $\varepsilon < \beta$ is used to verify (5).

In case A contains operation symbols we distinguish two cases. Suppose first $o < \beta$. Then let T be the closure of T_0 under formation of terms by operation symbols in A. From the regularity of o it follows that this set is $\mathsf{T} = \bigcup \{\mathsf{T}_\nu : \nu < o\}$ where for $\nu > 0$, T_ν is the set of all terms of the form $[T_0 \psi T_1]$ or $[\varphi T_0 \ldots T_\xi \ldots]$ with ψ, φ operation symbols in A and the $T_\xi \in \bigcup \{\mathsf{T}_\lambda : \lambda < \nu\}$. Note that $\mathrm{card}(\mathsf{T}_0) = \kappa$ and that if $\nu > 0$ and $\mathrm{card}(\mathsf{T}_\lambda) = \kappa$ for $\lambda < \nu$, then $\mathrm{card}(\mathsf{T}_\nu) = \kappa$ since $\kappa \exp o = \kappa$. Hence $\mathrm{card}(\mathsf{T}) = \kappa$ and T has property (3). Property (5) follows and (1) and (4) are obvious. As for (2), terms have length less than o and in this case $o < \beta$. Since subformulas of A have fewer than β free variables, if $f \in \mathsf{T}^{FV(C)}$, $\mathrm{card}(FV(SF_f^{FV(C)}C)) \leqslant \mathrm{card}(FV(C)) \cdot o < \beta$. Hence (2).

Finally suppose $o \geqslant \beta$. Then β is regular and $o \leqslant \gamma^+$. Let $\Delta_0 = \{SF_f^{FV(C)}C : C \text{ is a subformula of } A \text{ and } f \in \mathsf{T}_0^{FV(C)}\}$. Given T_λ, Δ_λ for $\lambda < \nu$, let T_ν be the set of terms in atomic formulas of $\bigcup \{\Delta_\xi : \xi < \nu\}$ and let $\Delta_\nu = \{SF_f^{FV(C)}C : C \text{ is a subformula of } A \text{ and } f \in \mathsf{T}_\nu^{FV(C)}\}$. Let $\mathsf{T} = \bigcup \{\mathsf{T}_\nu : \nu < \beta\}$. If C is a subformula of A then C has fewer than β free variables. Therefore since β is regular, for any $f \in \mathsf{T}^{FV(C)}$ there is $\nu < \beta$ such that $f \in \mathsf{T}_\nu^{FV(C)}$. Hence $\Delta = \bigcup \{\Delta_\nu : \nu < \beta\} \cup \{[T \mathrel{\bar{=}} T'] : T, T' \in \mathsf{T}\}$. (1) follows. Properties (2), (3), (4) can be proved by induction on $\nu < \beta$. For T_0 and Δ_0 have these properties. Moreover if T_λ and Δ_λ have properties (2), (3), (4) for all $\lambda < \nu$, and if $\nu < \beta$, consider T_ν, Δ_ν. A term of T_ν is a subterm of a formula of some Δ_λ and therefore has fewer than β variables, all in X. If C is a subformula of A and $f \in \mathsf{T}_\nu^{FV(C)}$, then all variables of $SF_f^{FV(C)}C$ are in $\bigcup \{FV(f(y)) : y \in FV(C)\}$, a subset of X having power less than β by regularity. Therefore T_ν, Δ_ν have properties (2) and (4). From $\mathrm{card}(\mathsf{T}_\lambda) = \kappa$ for $\lambda < \nu$, it follows $\mathrm{card}(\Delta_\lambda) = \kappa$ for $\lambda < \nu$. Hence $\mathrm{card}(\bigcup_{\lambda<\nu} \Delta_\lambda) = \kappa$ and since each formula of this set contributes at most γ terms to T_ν, $\mathrm{card}(\mathsf{T}_\nu) = \kappa$. Hence (3).

10.3.5 *Criterion for Satisfiability.* Let A be a sentence of a language $\mathsf{L}_{\gamma^+\beta o\pi}$ having a set X of $\kappa(\gamma^+, \beta)$ individual variables not in A. Form sets T, Δ, of 10.3.4. Then the following conditions are equivalent:

(i) A is satisfiable.

(ii) A has a model of power at most $\kappa(\gamma^+, \beta)$.

(iii) There is a set Γ of formulas containing A and all formulas $[T \mathrel{\bar{=}} T]$ with $T \in \mathsf{T}$, and satisfying the following conditions:

(1) If terms $[T_0\psi T_1]$, $[T_0'\psi T_1']$ are in T, then $[T_0 \overline{=} T_0'] \in \Gamma$ and $[T_1 \overline{=} T_1'] \in \Gamma$ implies $[[T_0\psi T_1] \overline{=} [T_0'\psi T_1']] \in \Gamma$.

(2) If terms $[\varphi T_0 \ldots T_\xi \ldots]$, $[\varphi T_0' \ldots T_\xi' \ldots]$ are in T, then $[T_\xi \overline{=} T_\xi'] \in \Gamma$ for all $\xi < \zeta$ implies

$$[[\varphi T_0 \ldots T_\xi \ldots] \overline{=} [\varphi T_0' \ldots T_\xi' \ldots]] \in \Gamma.$$

(3) If atomic formulas $[T_0 P T_1]$, $[T_0' P T_1']$ are in Δ, then $[T_0 \overline{=} T_0']$, $[T_1 \overline{=} T_1']$ and $[T_0 P T_1] \in \Gamma$ implies $[T_0' P T_1'] \in \Gamma$.

(4) If atomic formulas $[Q T_0 \ldots T_\xi \ldots]$, $[Q T_0' \ldots T_\xi' \ldots]$ are in Δ, then $[T_\xi \overline{=} T_\xi'] \in \Gamma$ for all $\xi < \eta$ and $[Q T_0 \ldots T_\xi \ldots] \in \Gamma$ implies $[Q T_0' \ldots T \ldots_\xi'] \in \Gamma$.

(5) If $A_0 \in \Delta$, then $A_0 \in \Gamma$ iff $[\neg A_0] \notin \Gamma$.

(6) If $[A_0 \rightarrow A_1] \in \Delta$, then $[A_0 \rightarrow A_1] \in \Gamma$ iff $A_0 \notin \Gamma$ or $A_1 \in \Gamma$.

(7) If $[\bigwedge A_0 \ldots A_\xi \ldots] \in \Delta$, then $[\bigwedge A_0 \ldots A_\xi \ldots] \in \Gamma$ iff all the $A_\xi \in \Gamma$.

(8) If $[\forall v A_0] \in \Delta$, then $[\forall v A_0] \in \Gamma$ iff all substitutions

$$SF_f^{\mathrm{Rng}(v)} A_0 \in \Gamma \text{ for } f \in \mathsf{T}^{\mathrm{Rng}(v)}.$$

PROOF: It suffices to show that (i) implies (iii) and (iii) implies (ii). Assume (i). Suppose A holds in model $\mathfrak{M} = \langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\rangle$. We wish to find an assignment s to $\mathfrak{M}$ such that $\{C\colon V(s, C) = \mathbf{1}\}$ satisfies conditions (1)–(8) for Γ. Clearly the only problem is satisfying condition (8); the other conditions will be satisfied no matter what s is. We use the familiar procedure for selecting witnesses. Make a list having length κ of all the quantifications in Δ:

$$[\forall v_0 C_0(v_0)] \ldots [\forall v_\nu C_\nu(v_\nu)] \ldots \qquad \nu < \kappa.$$

By a sequence of witnessing formulas we mean a sequence

$$C_0(g_0 \bigcirc v_0) \ldots C_\nu(g_\nu \bigcirc v_\nu) \ldots \qquad \nu < \kappa$$

such that each g_ν is a one-one function on $\mathrm{Rng}(v_\nu)$ to X and variables in its range are not in $\mathrm{Rng}(g_\xi)$ for $\xi \neq \nu$, nor do they appear in $[\forall v_\xi C_\xi(v_\xi)]$ for $\xi \leqslant \nu$. Such a sequence of functions can be found. For suppose we are given g_ξ for $\xi < \nu$. Then 10.3.2 (iv) implies that there are κ variables not in $\mathrm{Rng}(g_\xi)$ for $\xi < \nu$ and not appearing in $[\forall v_\xi C_\xi(v_\xi)]$ for $\xi \leqslant \nu$, since there are fewer than β variables of X in each such set and each such formula. Thus a one-one function g_ν on $\mathrm{Rng}(v_\nu)$ to such variables can be found.

Let $W_\nu = [C_\nu(g_\nu \bigcirc v_\nu) \rightarrow [\forall v_\nu C_\nu(v_\nu)]]$. Given an arbitrary assignment s, we define by transfinite induction functions t_ν on $\mathrm{Rng}(g_\nu)$ to D such that if s_ν is that assignment which is t_ξ on $\mathrm{Rng}(g_\xi)$ for $\xi \leqslant \nu$, and s elsewhere, $V(s_\nu, W_\nu) = \mathbf{1}$. Suppose we have such t_ξ for $\xi < \nu$. Let s'_ν be t_ξ on $\mathrm{Rng}(g_\xi)$ for $\xi < \nu$, s elsewhere. If $V(s'_\nu, [\forall v_\nu C_\nu(v_\nu)]) = \mathbf{1}$, let t_ν be s on $\mathrm{Rng}(g_\nu)$. Then $V(s_\nu, W_\nu) = \mathbf{1}$ in this case. If $V(s'_\nu, [\forall v_\nu C_\nu(v_\nu)]) = \mathbf{0}$, pick t on $\mathrm{Rng}(v_\nu)$ to D such that $V(\mathrm{Repl}_t^{\mathrm{Rng}(v_\nu)} s'_\nu, C_\nu(v_\nu)) = \mathbf{0}$. Choose t_ν so that $t_\nu \bigcirc g_\nu$ agrees with t on $\mathrm{Rng}(v_\nu)$. Then $V(s_\nu, C_\nu(g_\nu \bigcirc v_\nu)) = V(\mathrm{Repl}_t^{\mathrm{Rng}(v_\nu)} s_\nu, C_\nu(v_\nu)) = \mathbf{0}$ by 10.1.6 and 10.1.3 since s_ν and s'_ν agree on $FV(C_\nu)$. Hence $V(s_\nu, W_\nu) = \mathbf{1}$ in this case as well. Having completed the induction, let $\Gamma = \{C : V(s_\kappa, C) = \mathbf{1}\}$. Since s_κ and s_ν agree on $FV(W_\nu)$, $V(s_\kappa, W_\nu) = \mathbf{1}$ for all $\nu < \kappa$. Therefore if $V(s_\kappa, [\forall v_\nu C_\nu(v_\nu)]) = \mathbf{0}$, then $V(s_\kappa, C_\nu(g_\nu \bigcirc v_\nu)) = \mathbf{0}$. But since no variable of X is bound in a formula of Δ, $C_\nu(g_\nu \bigcirc v_\nu) = SF_{g_\nu}^{\mathrm{Rng}(v_\nu)} C_\nu$. It follows that Γ has property (8). The others are immediate.

Assume (iii). We must construct a model of A having power at most $\kappa(\gamma^+, \beta)$. Given Γ satisfying (1)–(8) and containing A and formulas $[T \mathrel{\bar{=}} T]$ for $T \in \mathsf{T}$, note that the relation

$$T \equiv T' \text{ if and only if } [T \mathrel{\bar{=}} T'] \in \Gamma$$

is an equivalence relation on terms of T. Conditions (3) for $\bar{=}$ can be used to demonstrate symmetry and transitivity. As the domain of the model we take the set of all these equivalence classes. Then $\mathrm{card}(D) \leqslant \kappa(\gamma^+, \beta)$. As assignments O, C, R to constants we take the following, where x_0 is a fixed variable in X:

For individual constants c, $\mathsf{C}(c) = |\langle c\rangle|$ if $\langle c\rangle \in \mathsf{T}$, $|\langle x_0\rangle|$ if not.

For special two-place operation symbols ψ,

$$\mathsf{O}(\psi)(\langle d_0 d_1\rangle) = |[T_0 \psi T_1]|$$

if there are $T_0 \in d_0$, $T_1 \in d_1$ such that $[T_0 \psi T_1] \in \mathsf{T}$, $|\langle x_0\rangle|$ if there are none.

For ζ-place or infinitary operation symbols φ,

$$\mathsf{O}(\varphi)(\langle d_0 \ldots d_\xi \ldots\rangle) = |[\varphi T_0 \ldots T_\xi \ldots]|$$

if there are $T_\xi \in d_\xi$ such that $[\varphi T_0 \ldots T_\xi \ldots] \in \mathsf{T}$, $|\langle x_0\rangle|$ if there are none.

Properties (1) and (2) of Γ guarantee that these conditions define functions. Finally for special two-place predicate symbols P, η-place or infinitary predicate symbols Q,

$\mathsf{R}(P)(\langle d_0 d_1\rangle) = \mathbf{1}$ iff there are $T_0 \in d_0$, $T_1 \in d_1$ such that

$$[T_0 P T_1] \in \Delta \cap \Gamma.$$

$\mathsf{R}(Q)(\langle d_0 \ldots d_\xi \ldots\rangle) = \mathbf{1}$ iff there are $T_\xi \in d_\xi$ for $\xi < \eta$ such that

$$[Q T_0 \ldots T_\xi \ldots] \in \Delta \cap \Gamma.$$

Note that $\mathsf{R}(\overline{=})$ is the equality relation on D.

Let s be any assignment to this model such that $s(x) = |\langle x\rangle|$ for $x \in X$. Then $s^*\langle u\rangle = |\langle u\rangle|$ for all atomic terms in T. Continuing by induction on terms, if term $[T_0 \psi T_1] \in \mathsf{T}$, then T_0 and T_1 are in T, and from $s^*T_0 = |T_0|$ and $s^*T_1 = |T_1|$, we conclude $s^*[T_0 \psi T_1] = \mathsf{O}(\psi)(|T_0|\,|T_1|) = |[T_0 \psi T_1]|$. The cases

$$[\varphi T_0 \ldots T_\xi \ldots] \in \mathsf{T}$$

are similar. Thus we see that $s^*T = |T|$ for all $T \in \mathsf{T}$.

Finally we show that $V(s, C) = \mathbf{1}$ iff $C \in \Gamma$ for all $C \in \Delta$. Since $A \in \Gamma$, this will complete the proof. This induction is done on the ranks of formulas. A formula of rank 0 is atomic. Consider an atomic formula $[Q T_0 \ldots T_\xi \ldots] = C \in \Delta$. Since the terms are in T, $V(s, C) = \mathsf{R}(Q)(|T_0| \ldots |T_\xi| \ldots)$. The definition of $\mathsf{R}(Q)$ together with property (4) of Γ implies $V(s, C) = \mathbf{1}$ iff $C \in \Gamma$. The case $C = [T_0 P T_1]$ is similar. Property (3) of Γ is used.

Suppose formulas having rank less than that of C have the desired property. Cases where C is a negation, implication or conjunction are similar. Suppose $C \in \Delta$, $C = [\bigwedge C_0 \ldots C_\xi \ldots]$. Then $C = SF_f^{FV(C')} C'$ for a subformula C' of A and function f to T. Then C' has form $[\bigwedge C_0' \ldots C_\xi' \ldots]$ and 9.2.6 implies that $C = [\bigwedge SF_f^{FV(C')} C_0' \ldots SF_f^{FV(C')} C_\xi' \ldots]$. The uniqueness theorem for principal subformulas tells us that $C_\xi = SF_f^{FV(C')} C_\xi'$ for all ξ and that the two sequences of formulas have the same length. Therefore the $C_\xi \in \Delta$, and since their rank is smaller, $C_\xi \in \Gamma$ iff $V(s, C_\xi) = \mathbf{1}$. With property (7) of Γ, it follows $C \in \Gamma$ iff $V(s, C) = \mathbf{1}$.

Finally suppose $C = [\forall v C_0] \in \Delta$. Then $C = SF_{f'}^{FV(C')} C'$ for a subformula C' of A and function f' to T. By 9.2.6, C' has form $[\forall v C_0']$ and $C = [\forall v SF_{f'}^{FV(C')} C_0']$. Therefore, whenever f maps $\mathrm{Rng}(v)$ to T, $SF_f^{\mathrm{Rng}(v)} C_0 = SF_{f \cup f'}^{FV(C_0')} C_0' \in \Delta$. Since these substitutions have rank less than C by 9.2.8, $V(s, SF_f^{\mathrm{Rng}(v)} C_0) = \mathbf{1}$ if and only if $SF_f^{\mathrm{Rng}(v)} C_0 \in \Gamma$ by induction hypothesis. Property (8) then implies

(a) $C \in \Gamma$ iff $V(s, SF_f^{\mathrm{Rng}(v)} C_0) = \mathbf{1}$ for all $f \in \mathsf{T}^{\mathrm{Rng}(v)}$.

Since variables of X are not bound in C, $V(s, C) = \mathbf{1}$ implies all $V(s, SF_f^{\mathrm{Rng}(v)}C_0) = \mathbf{1}$ by 10.1.4. Conversely, if $V(s, C) = \mathbf{0}$, there is t on $\mathrm{Rng}(v)$ to D such that $V(\mathrm{Repl}_t^{\mathrm{Rng}(v)}s, C_0) = \mathbf{0}$. Choosing f to T such that $f(y) \in t(y)$ for $y \in \mathrm{Rng}(v)$, we see that $s^* \bigcirc f = t$ because $s^*T = |T|$ on T. Therefore $V(s, SF_f^{\mathrm{Rng}(v)}C_0) = V(\mathrm{Repl}_t^{\mathrm{Rng}(v)}s, C_0) = \mathbf{0}$ by 10.1.4. Hence

(b) $V(s, C) = \mathbf{1}$ iff $V(s, SF_f^{\mathrm{Rng}(v)}C_0) = \mathbf{1}$ for all $f \in \mathsf{T}^{\mathrm{Rng}(v)}$.

Combining (a) and (b) we have our result.

The proof that (i) implies (iii) contained a proof of the following:

10.3.6 *Lemma.* Consider any ordered set Γ of formulas $[C_\nu(g_\nu \bigcirc v_\nu) \rightarrow [\forall v_\nu C_\nu(v_\nu)]]$ such that functions g_ν are one-one on $\mathrm{Rng}(v_\nu)$ and variables of $\mathrm{Rng}(g_\nu)$ are not in $\mathrm{Rng}(g_\xi)$ for $\xi \neq \nu$, and do not appear in $[\forall v_\xi C_\xi(v_\xi)]$ for $\xi \leqslant \nu$. Then given any assignment s to model $\mathfrak{M}$, there is an assignment s' satisfying Γ that agrees with s except on $\bigcup\{\mathrm{Rng}(g_\nu) : \nu < \sigma\}$.

Another form of the same lemma is this:

10.3.7 *Lemma.* Consider any ordered set Γ of formulas $A_0(v_0)$, $\ldots, A_\nu(v_0 \ldots v_\nu), \ldots$ such that the $\mathrm{Rng}(v_\nu)$ are pairwise disjoint and no variable of $\mathrm{Rng}(v_\nu)$ appears free in A_ξ for $\xi < \nu$. Let $Y = \bigcup\{\mathrm{Rng}(v_\nu) : \nu < \sigma\}$. Then if s is an assignment to model $\mathfrak{M}$ such that formulas $[\exists v_\nu A_\nu(v_0 \ldots v_\nu)]$ all take value $\mathbf{1}$ for all assignments $\mathrm{Repl}_t^Y s$, then there exists such an assignment satisfying Γ.

10.3.8 (*Skolem-Löwenheim Theorem for Sets of Infinitary Sentences*). Let Γ be a set of $(\gamma^+, \beta, o, \pi)$-sentences with $\gamma^+ \leqslant \mathrm{card}(\Gamma)$ and $\mathrm{card}(\Gamma) \exp \varepsilon = \mathrm{card}(\Gamma)$ for all $\varepsilon < \beta$. Then Γ is satisfiable if and only if Γ is satisfiable in a model of power at most $\mathrm{card}(\Gamma)$.

PROOF: Let A be the conjunction of all the sentences in Γ. Then A is a $(\mathrm{card}(\Gamma)^+, \beta, o, \pi)$-sentence and $\kappa(\mathrm{card}(\Gamma)^+, \beta) = \mathrm{card}(\Gamma)$. Hence this follows from 10.3.5.

The following theorem was formulated by Hanf.

10.3.9 Suppose Γ is a set of (α, β, o, π)-sentences having a model with domain D. Then if γ is any cardinal such that $\gamma \exp \varepsilon = \gamma$ for all $\varepsilon < \beta$ and $\alpha \cup \mathrm{card}(\Gamma) \leqslant \gamma < \mathrm{card}(D)$, then Γ has a model of power γ.

PROOF: Introduce new constants c_ξ, $\xi < \gamma$. Let A be the conjunction of all the formulas in Γ together with formulas $[\neg c_\xi \mathrel{\overline{=}} c_{\xi'}]$ for $\xi \neq \xi'$. Then A is a $(\gamma^+, \beta, o, \pi)$-sentence and $\kappa(\gamma^+, \beta) = \gamma$. By 10.3.5, A has a model of power at most γ. Since every model has power at least γ, we have our result.

Note that the restriction to sentences in 10.3.5 and the two preceding theorems is not really serious. Free individual variables and individual constants are practically indistinguishable from the point of view of satisfiability.

10.4 Algebras of Equivalence Classes of Formulas Modulo a Semantically Consistent Set

Let Γ be a semantically consistent set of formulas of an (α, β, o, π)-predicate language $\mathbf{L}$. Then the relation

$$A \equiv A' \text{ iff } \Gamma \Vdash A \leftrightarrow A'$$

is a congruence relation on the algebra of formulas. Recall that the algebra of formulas has only the operations of negation, implication and conjunction. This relation is not a congruence relation with respect to the cylindric operations taking A to $[\forall v A]$ unless Γ is a set of sentences. Let $B(\mathbf{L}; \Gamma)$ be the set of all equivalence classes $|A|_\Gamma = \{A' : A \equiv A'\}$. The following are operations on $B(\mathbf{L}; \Gamma)$:

$$\neg|A|_\Gamma = |[\neg A]|_\Gamma,\ |A_0|_\Gamma \to |A_1|_\Gamma = |[A_0 \to A_1]|_\Gamma$$
$$\wedge\,(\langle|A_0|_\Gamma \ldots |A_\xi|_\Gamma \ldots\rangle) = |[\bigwedge A_0 \ldots A_\xi \ldots]|_\Gamma,$$

the latter being $\nearrow \alpha$-place. The defined propositional operation symbols yield additional operations

$$|A_0|_\Gamma \wedge |A_1|_\Gamma = |[A_0 \wedge A_1]|_\Gamma,\ |A_0|_\Gamma \vee |A_1|_\Gamma = |[A_0 \vee A_1]|_\Gamma.$$

Let $\mathfrak{B}(\mathbf{L}; \Gamma)$ be the algebra $\langle B(\mathbf{L}; \Gamma) \neg \wedge \vee\rangle$. The semantic consistency of Γ tells us that the algebra has at least two elements.

10.4.1 *Theorem.* (i) If Γ is semantically consistent, then $\mathfrak{B}(\mathbf{L}; \Gamma)$ is a non-degenerate $\nearrow \alpha$-representable Boolean algebra.

(ii) If Γ satisfiable and $\bar{\Gamma} = \{C$: for all models $\mathfrak{M}$, s satisfies Γ in $\mathfrak{M}$ implies s satisfies C in $\mathfrak{M}\}$. Then $\mathfrak{B}(\mathbf{L}; \bar{\Gamma})$ is isomorphic to a non-degenerate $\nearrow \alpha$-field of sets.

PROOF: We only have to make minor changes in the proof of the analogous theorem for propositional languages 4.2.1. We prove (ii) first. Let $K = \{(\mathfrak{M}, s) : s$ satisfies Γ in model $\mathfrak{M}\}$ and for formulas A let $h(A) = \{(\mathfrak{M}, s) : s$ satisfies $\Gamma \cup \{A\}$ in $\mathfrak{M}\}$. Then $h([\neg A]) = K \sim h(A)$, $h([A_0 \wedge A_1]) = h(A_0) \cap h(A_1)$, $h([A_0 \vee A_1]) = h(A_0) \cup h(A_1)$, and $h([\bigwedge A_0 \ldots A \ldots]) = \bigcap \{h(A_\xi) : \xi < \delta\}$. Thus $\langle \mathrm{Rng}(h) \sim \cap \cup\rangle$

is an $\nearrow\alpha$-field of sets. $\bar{\Gamma}$ was defined in such a way that $\bar{\Gamma} \Vdash A \leftrightarrow A'$ iff $h(A) = h(A')$. Therefore the mapping $g(|A|_\Gamma) = h(A)$ is an isomorphism on $\mathfrak{B}(\mathbf{L}; \bar{\Gamma})$ onto $\langle \mathrm{Rng}(h) \sim \cap \cup \rangle$.

To prove (i), take $\Gamma = \phi$ in (ii). Then the set $\bar{\Gamma}$ of (ii) is the set of valid formulas. The mapping $g(|A|_{\bar{\Gamma}}) = |A|_\Gamma$ clearly is an $\nearrow\alpha$-homomorphism on $\mathfrak{B}(\mathbf{L}; \bar{\Gamma}) = \mathfrak{B}(\mathbf{L}; \phi)$ onto $\mathfrak{B}(\mathbf{L}; \Gamma)$. Hence (i).

The order in $\mathfrak{B}(\mathbf{L}; \Gamma)$ is $|A|_\Gamma \leqslant |A'|_\Gamma$ iff $\Gamma \Vdash [A \rightarrow A']$, the zero is $|[\neg x = x]|_\Gamma$, unit $|[x = x]|_\Gamma$, the $\nearrow\alpha$-meet and join are $\wedge \{|A_\xi|_\Gamma : \xi < \delta\} = |[\boldsymbol{\wedge}\ A_0 \ldots A_\xi \ldots]|_\Gamma$ and $\vee \{|A_\xi|_\Gamma : \xi < \delta\} = |[\boldsymbol{\vee}\ A_0 \ldots A_\xi \ldots]|_\Gamma$.

We are now in a position to show how to construct an $\nearrow\alpha$-representable Boolean algebra having no $\nearrow\alpha$-homomorphism to $\mathfrak{B}_0$, from a semantically consistent non-satisfiable set of (α, α)-sentences.

10.4.2 *Theorem.* Let α be a regular infinite cardinal, $\mathbf{L}$ any (α, α, o, π)-predicate language. Then if $\mathbf{L}$ has a set Γ of sentences that is semantically consistent but not satisfiable, there is an (α, α, o, π)-language $\mathbf{L}'$ with the same constants, and semantically consistent set $\Gamma' \supseteq \Gamma$ such that $\mathfrak{B}(\mathbf{L}'; \Gamma')$ has no $\nearrow\alpha$-complete maximal ideal. (It is therefore not isomorphic to an $\nearrow\alpha$-field of sets. It is $\nearrow\alpha$-representable by 10.4.1.)

Proof: Let $\gamma = \bigcup \{\mathrm{card}(\Gamma) \exp \varepsilon : \varepsilon < \alpha\}$. Since α is regular, $\kappa(\gamma^+, \alpha) = \bigcup \{\gamma \exp \varepsilon : \varepsilon < \alpha\} = \gamma$ by 10.3.2 (iii). As $\mathbf{L}'$ take the (α, α, o, π)-language with the symbols of $\mathbf{L}$ plus a set X of γ new variables. Let $\mathbf{L}''$ be a $(\gamma^+, \alpha, o, \pi)$-language with all the symbols of $\mathbf{L}'$. Then A, the conjunction of all the sentences in Γ, is a sentence of $\mathbf{L}''$. Form sets T, Δ of 10.3.4 for A, X with respect to $\mathbf{L}''$. It is important to note that all the terms of T are terms of $\mathbf{L}'$ and all the formulas of Δ except A itself, are formulas of $\mathbf{L}'$. Make a list $\langle [\forall v_\nu C_\nu(v_\nu)] : \nu < \gamma \rangle$ of all the quantifications in Δ and choose for them a sequence of witnessing formulas $\langle C_\nu(g_\nu \bigcirc v_\nu) : \nu < \sigma \rangle$ satisfying conditions of 10.3.6. Then if $W_\nu = [C_\nu(g_\nu \bigcirc v_\nu) \rightarrow [\forall v_\nu C_\nu(v_\nu)]]$ for $\nu < \gamma$, $\{W_\nu : \nu < \gamma\}$ is satisfiable in any model of $\mathbf{L}'$. Let $\Gamma' = \Gamma \cup \{W_\nu : \nu < \gamma\}$. Then Γ' is also consistent.

Suppose, contrariwise, $\mathfrak{B}(\mathbf{L}', \Gamma')$ had an $\nearrow\alpha$-complete maximal ideal, or, in other words, had an $\nearrow\alpha$-homomorphism h to $\mathfrak{B}_0$. Then let $\bar{\Gamma} = \{C : C \text{ is a formula of } \mathbf{L}' \text{ and } h|C|_{\Gamma'} = \mathbf{1}\} \cup \{A\}$. We claim that $\bar{\Gamma}$ would then satisfy condition (iii) of 10.3.5 relative to $\mathbf{L}''$ and A. This would contradict the non-satisfiability of A. $\bar{\Gamma}$ obviously contains A and formulas $[T \,\overline{=}\, T]$ for $T \in \mathsf{T}$. Checking conditions (1)–(8) is a routine matter. As examples, look at (2), (7), (8). If $[T_\xi \,\overline{=}\, T'_\xi] \in \bar{\Gamma}$ for all $\xi < \zeta$, then $h|[T_\xi \,\overline{=}\, T'_\xi]|_{\Gamma'} = \mathbf{1}$ for all $\xi < \zeta$. Since $\zeta < o \leqslant \alpha$,

$$C = [\bigwedge_{\xi<\zeta} [T_\xi \,\overline{=}\, T'_\xi]] \rightarrow [\varphi T_0 \ldots T_\xi \ldots] \,\overline{=}\, [\varphi T'_0 \ldots T'_\xi \ldots]$$

is a valid formula of $\mathbf{L}'$. Hence $|C|_{\Gamma'} = \mathbf{1}$. Passing to operations of $\mathfrak{B}(\mathbf{L}', \Gamma')$, it then follows that $h|[\varphi T_0 \ldots T_\xi \ldots] \,\overline{=}\, [\varphi T'_0 \ldots T'_\xi \ldots]|_{\Gamma'} = \mathbf{1}$. Hence (2). In (7), the conjunction A must be considered separately from the other conjunctions. $A \in \bar{\Gamma}$ and the principal subformulas of A are in Γ. They therefore also belong to $\bar{\Gamma}$. Then (7) follows by a computation much like the one just given. As for (8), $|[\forall v A_0]|_{\Gamma'} \leqslant |SF_f^{\mathrm{Rng}(v)} A_0|_{\Gamma'}$ for all $f \in \mathsf{T}^{\mathrm{Rng}(v)}$. Therefore $[\forall v A_0] \in \bar{\Gamma}$ implies that the substitutions to T are also in $\bar{\Gamma}$. If $[\forall v A_0] \notin \bar{\Gamma}$, then $h|[\forall v A_0]|_{\Gamma'} = \mathbf{0}$. But there is a witnessing formula $A_0(g_\nu \bigcirc v)$ such that $W_\nu = [A_0(g_\nu \bigcirc v) \rightarrow [\forall v A_0]] \in \Gamma'$. Passing to operations of $\mathfrak{B}(\mathbf{L}'; \Gamma')$, $h|W_\nu| = \neg h|A_0(g_\nu \bigcirc v)|_{\Gamma'} = \mathbf{1}$. Hence $A_0(g_\nu \bigcirc v) \notin \bar{\Gamma}$. Thus we have (8).

Combining this theorem with 4.3.6 we obtain

10.4.3 *Theorem.* Let α be a regular infinite cardinal. Then the following conditions are equivalent:

(i) α is incompact.

(ii) There is a semantically consistent, non-satisfiable set of α-propositional formulas.

(iii) There is an $\nearrow\alpha$-representable Boolean algebra having no $\nearrow\alpha$-complete maximal ideal.

(iv) There is an $\nearrow\alpha$-representable Boolean algebra not isomorphic to an $\nearrow\alpha$-field of sets.

The completeness of the propositional calculus $\mathfrak{P}_\alpha(\Pi_{\nearrow\alpha})$ together with Theorems 6.4.2 and 6.4.7 yields a fifth equivalent condition:

(v) $\mathfrak{P}_\alpha(\Pi_{\nearrow\alpha})$ is not strongly complete.

CHAPTER 11

INFINITARY PREDICATE LOGIC

11.1 Description of the Formal Systems for (α, β, o, π)-Predicate Languages

The restrictions on α, β, o, π are assumed to be in force: α regular infinite, $\beta = 0$ or $\omega \leqslant \beta \leqslant \alpha$, $\pi \leqslant \alpha$, o regular infinite and $o < \beta$ if β singular, $o \leqslant \alpha$ if β regular.

For an (α, β, o, π)-language $\mathbf{L}$, the *basic formal system* $\mathfrak{P}_{\alpha\beta}(\mathbf{L})$ has as axioms all substitutions to $\mathbf{L}$ of the axiom schemes of $\mathfrak{P}_{\alpha}$, 5.1.1, together with all formulas of $\mathbf{L}$ of the form

$\mathcal{Q}1$. $[\forall v[A_0 \rightarrow A_1]] \rightarrow [A_0 \rightarrow [\forall v A_1]]$, provided that no variable of $\mathrm{Rng}(v)$ is free in A_0.

$\mathcal{Q}2$. $[[\forall v A_0] \rightarrow SF_f^{\mathrm{Rng}(v)} A_0]$, provided that A_0 has no free occurrence of $x \in \mathrm{Rng}(v)$ bound by $FV(f(x))$.

The function f in $\mathcal{Q}2$ may be any function on $\mathrm{Rng}(v)$ to terms of $\mathbf{L}$. In addition, $\mathfrak{P}_{\alpha\beta}(\mathbf{L})$ has as axioms all formulas of the following forms where the T_ξ, T'_ξ are terms, ψ is a special two-place operation symbol, φ a ζ-place or infinitary operation symbol, P a special two-place predicate symbol, Q an η-place or infinitary predicate symbol:

$\mathcal{E}\mathcal{Q}1$. $[T \mathrel{\overline{=}} T]$.

$\mathcal{E}\mathcal{Q}2$. $[[T_0 \mathrel{\overline{=}} T'_0] \wedge [T_1 \mathrel{\overline{=}} T'_1]] \rightarrow [[T_0 \psi T_1] \mathrel{\overline{=}} [T'_0 \psi T'_1]]$.

$\mathcal{E}\mathcal{Q}3$. $[\bigwedge [T_0 \mathrel{\overline{=}} T'_0] \ldots [T_\xi \mathrel{\overline{=}} T'_\xi] \ldots] \rightarrow$
$$[[\varphi T_0 \ldots T_\xi \ldots] \mathrel{\overline{=}} [\varphi T'_0 \ldots T'_\xi \ldots]].$$

$\mathcal{E}\mathcal{Q}4$. $[[T_0 \mathrel{\overline{=}} T'_0] \wedge [T_1 \mathrel{\overline{=}} T'_1]] \rightarrow [[T_0 P T_1] \rightarrow [T'_0 P T'_1]]$.

$\mathcal{E}\mathcal{Q}5$. $[\bigwedge [T_0 \mathrel{\overline{=}} T'_0] \ldots [T_\xi \mathrel{\overline{=}} T'_\xi] \ldots] \rightarrow$
$$[[Q T_0 \ldots T_\xi \ldots] \rightarrow [Q T'_0 \ldots T'_\xi \ldots]].$$

The rules of inference are modus ponens, conjunction, and

Generalization: From A_0 infer $[\forall v A_0]$.

If Σ is any set of α-propositional schemes, $\mathfrak{P}_{\alpha\beta}(\Sigma)(\mathbf{L})$ is that system like $\mathfrak{P}_{\alpha\beta}(\mathbf{L})$ except that instances of Σ in $\mathbf{L}$ are added to the set of axioms.

11.1.1 *Laws of Independent and Dependent Choices.* Let γ be an infinite cardinal. Then a formula A of $\mathbf{L}$ is an instance of the *law of γ independent choices* if and only if A has form

$$\mathscr{C}\mathscr{H}_\gamma.\ [\bigwedge_{\xi<\gamma} [\exists v_0 A_0]\ldots[\exists v_\xi A_\xi]\ldots] \rightarrow [\exists_{\xi<\gamma} v_0\ldots v_\xi\ldots[\bigwedge_{\xi<\gamma} A_0\ldots A_\xi\ldots]],$$

provided sequences v_ξ have pairwise disjoint ranges and no variable of $\mathrm{Rng}(v_\xi)$ is free in A_ν for $\nu \neq \xi$.

Similarly, a formula A of $\mathbf{L}$ is an instance of the *law of γ dependent choices* if and only if A has form

$$\mathscr{D}\mathscr{C}\mathscr{H}_\gamma.\ [\bigwedge_{\xi<\gamma} [\exists v_0 A_0]\ldots[\forall_{\nu<\xi} v_0\ldots v_\nu\ldots[\exists v_\xi A_\xi]]\ldots] \rightarrow [\exists_{\xi<\gamma} v_0\ldots v_\xi\ldots[\bigwedge_{\xi<\gamma} A_0\ldots A_\xi\ldots]],$$

provided sequences v_ξ have pairwise disjoint ranges and no variable of $\mathrm{Rng}(v_\xi)$ is free in A_ν for $\nu < \xi$.

Note that $\mathbf{L}$ only has instances of $\mathscr{C}\mathscr{H}_\gamma$ or $\mathscr{D}\mathscr{C}\mathscr{H}_\gamma$ if $\gamma < \alpha$ and $\gamma < \beta$. It must be possible to quantify sequence $\frown\langle v_\xi \colon \xi < \gamma\rangle$ of variables. If Σ is any set of α-propositional schemes or schemes $\mathscr{C}\mathscr{H}_\gamma$ or $\mathscr{D}\mathscr{C}\mathscr{H}_\gamma$ with $\gamma < \alpha \cap \beta$, $\mathfrak{P}_{\alpha\beta}(\Sigma)(\mathbf{L})$ is that system like $\mathfrak{P}_{\alpha\beta}(\mathbf{L})$ except that formulas of $\mathbf{L}$ having forms of Σ are added to the set of axioms.

11.1.2 *Rules of Independent and Dependent Choices.* Let γ be an infinite cardinal. When β is small relative to α, the laws $\mathscr{C}\mathscr{H}_\gamma$ and $\mathscr{D}\mathscr{C}\mathscr{H}_\gamma$ are not very useful because of the restriction $\gamma < \beta$. They can however be changed in form to rules of inference which will apply to all the languages with $\gamma < \alpha$.

Rule of γ Independent Choices: From $[\bigvee_{\xi<\gamma} A_0\ldots A_\xi\ldots]$ infer $[\bigvee_{\xi<\gamma} [\forall v_0 A_0]\ldots[\forall v_\xi A_\xi]\ldots]$ under proviso of $\mathscr{C}\mathscr{H}_\gamma$.

Rule of γ Dependent Choices: From $[\bigvee_{\xi<\gamma} A_0\ldots A_\xi\ldots]$ infer

$[\bigvee_{\xi<\gamma} [\forall v_0 A_0] \ldots [\exists w_\xi [\forall v_\xi A_\xi]] \ldots]$ under proviso of $\mathscr{D}\mathscr{C}\mathscr{H}_\gamma$, provided also that A_ξ has fewer than β free variables in

$$\bigcup \{\mathrm{Rng}(v_\nu) : \nu < \xi\}$$

and that

$$\mathrm{Rng}(w_\xi) \supseteq FV(A_\xi) \cap \bigcup \{\mathrm{Rng}(v_\nu) : \nu < \xi\}.$$

If Σ is any set of α-propositional schemes or of schemes $\mathscr{C}\mathscr{H}_\gamma$ or $\mathscr{D}\mathscr{C}\mathscr{H}_\gamma$ with $\gamma < \alpha \cap \beta$ and if Ω is any set of rules of γ independent or dependent choices with $\gamma < \alpha$, then $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$ is that system like $\mathfrak{P}_{\alpha\beta}(\mathbf{L})$ except that formulas of $\mathbf{L}$ having forms of Σ are added to the set of axioms and rules of Ω are added to the rules of inference.

11.1.3 *Theorem.* Let $\mathbf{L}$ be an (α, β, o, π)-predicate language. Then instances in $\mathbf{L}$ of valid α-propositional schemes and of schemes $\mathscr{Q}$, $\mathscr{E}\mathscr{Q}$, $\mathscr{C}\mathscr{H}_\gamma$, $\mathscr{D}\mathscr{C}\mathscr{H}_\gamma$ are all valid. The rules of modus ponens, conjunction, generalization, and of γ independent and γ dependent choices not only preserve validity, but preserve the property of holding in a given model.

Proof: Suppose formula A is an instance of valid α-propositional scheme $\mathscr{A}$. Then if W is the set of all propositional symbols in $\mathscr{A}$, A has form $S_f^W \mathscr{A}$ where f maps W to formulas of $\mathbf{L}$. Let s be any assignment to any model $\mathfrak{M}$ of $\mathbf{L}$. For symbols $A_\xi \in W$ let $s'(A_\xi) = V(s, f(A_\xi))$. Then s' is an assignment of W to truth values $\mathbf{0}$, $\mathbf{1}$. Therefore $s'^* \mathscr{A} = \mathbf{1}$. Since V at s is a homomorphism on the algebra of formulas of $\mathbf{L}$ and s'^* and S_f^W are homomorphisms on the algebra of schemes, it follows that $V(s, A) = s'^*(\mathscr{A}) = \mathbf{1}$. Hence A is valid.

Formulas of forms $\mathscr{Q}1$, $\mathscr{Q}2$ are valid by 10.1.3, 10.1.4. Formulas of forms $\mathscr{D}\mathscr{C}\mathscr{H}_\gamma$ and $\mathscr{C}\mathscr{H}_\gamma$ are valid by 10.3.7 and 10.1.3.

If $\mathfrak{M}$ is a model and s an assignment making a formula of the form of the conclusion of the rule of γ dependent choices false, then formulas $[\exists v_\xi [\neg A_\xi]]$ all take value $\mathbf{1}$ for assignments $\mathrm{Repl}_t^Y s$, $Y = \bigcup \{\mathrm{Rng}(v_\xi) : \xi < \gamma\}$, by 10.1.3. By 10.3.7, $[\bigvee A_0 \ldots A_\xi \ldots]$ does not hold in $\mathfrak{M}$. Hence this rule preserves the property of holding in $\mathfrak{M}$. A similar argument shows that the rules of γ independent choices also preserve this property. This is obvious for the other rules.

A formal proof in one of the systems $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$ is a sequence of formulas having length less than α, where every formula is either an axiom or follows from formulas earlier in the list by a rule of inference. If A is the last formula of such a sequence, A is provable and we write "$\vdash A$". If A is provable when formulas of Δ are added to the axioms, we write "$\vdash_\Delta A$". A formal system $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$ is *complete* if exactly the valid formulas are provable. A *calculus* $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)$ consists of the basic schemes and rules plus those of Σ and Ω. It is *complete* if and only if $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$ is complete for all (α, β, o, π)-languages $\mathbf{L}$. There is no point in discussing strong completeness for these systems, since the situation is the same as for the propositional systems. The argument in the introduction to Chapter 5 shows that no formal system for $\mathbf{L}$ can be strongly complete if $\mathbf{L}$ has a semantically consistent, non-satisfiable set of formulas. Thus no strongly complete calculus exists for α non-limit or for any of the incompact cardinals of 10.2.3. As a corollary to 11.1.3 we have

11.1.4 *Corollary.* Let Σ be any set of valid α-propositional schemes or schemes $\mathscr{C}\mathscr{H}_\gamma$ or $\mathscr{D}\mathscr{C}\mathscr{H}_\gamma$ for $\gamma < \alpha \cap \beta$. Let Ω be any collection of rules of independent or dependent choices with $\gamma < \alpha$. Then provable formulas of an (α, β, o, π)-language $\mathbf{L}$ are valid. Moreover, if $\vdash_\Delta A$ then A holds in all models of Δ.

11.2 Development of the Formal Systems for (α, β, o, π)-Predicate Languages

When we treated the α-propositional formal systems, we proved the completeness theorems syntactically and then deduced the Boolean algebraic representation theorems from them. However, the completeness theorems for predicate logic will be proved algebraically, making full use of the representation theorems for $\nearrow\alpha$-complete Boolean algebras. Such basic theorems as the substitution rule for equality and the rule of change of bound variables will also be proved algebraically, making use of the model-theoretic theorems of Chapter 9.

11.2.1 *Theorem.* If $\mathbf{L}_\alpha$ is an α-propositional language, $\mathbf{L}$ an (α, β, o, π)-predicate language, and Σ a set of α-propositional schemes, then $\vdash A$ in $\mathfrak{P}_\alpha(\Sigma)(\mathbf{L}_\alpha)$ implies $\vdash S^X_f A$ in $\mathfrak{P}_{\alpha\beta}(\Sigma)(\mathbf{L})$, where f is any function on the set X of propositional symbols in A to formulas of $\mathbf{L}$.

PROOF: An easy induction using substitution properties 3.5.3

shows that if A is any formula of $\mathbf{L}_\alpha$ whose propositional symbols are all in X, and if f is any function on X to formulas of $\mathbf{L}$, then $S_f^X A$ is a formula of $\mathbf{L}$. The theorem follows by the argument of Theorem 5.2.2.

11.2.2 *Definition.* Let Γ be a set of formulas of $\mathbf{L}$. Then $\Gamma \vdash_\Delta A$ iff $\vdash_\Delta A$ or there are formulas $C_\xi \in \Gamma$, $\xi < \delta$, $0 < \delta < \alpha$ such that $\vdash_\Delta [\bigwedge C_0 \ldots C_\xi \ldots] \rightarrow A$.

Note the parallel in the definitions of $\Gamma \vdash A$ and $\Gamma \Vdash A$. Obviously a formal system for $\mathbf{L}$ is complete if and only if the two notions are equivalent for all formulas A, sets Γ.

11.2.3 *Theorem.* Let Γ be a set of formulas of $\mathbf{L}$. Then in any of the systems $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$ of Sect. 11.1,

(i) $\Gamma \vdash_\Delta A_0$ and $\Gamma \vdash_\Delta [A_0 \rightarrow A_1]$ implies $\Gamma \vdash_\Delta A_1$.

(ii) $\Gamma \vdash_\Delta A_\xi$ for all $\xi < \delta$ implies $\Gamma \vdash_\Delta [\bigwedge A_0 \ldots A_\xi \ldots]$.

(iii) $\Gamma \vdash_\Delta A$ and variables of Rng(v) not free in Γ implies $\Gamma \vdash_\Delta [\forall v A]$.

(iv) If the rule of γ independent choices is in Ω, then under proviso of that rule, $\Gamma \vdash_\Delta [\bigvee_{\xi<\gamma} A_0 \ldots A_\xi \ldots]$ implies

$$\Gamma \vdash_\Delta [\bigvee_{\xi<\gamma} [\forall v_0 A_0] \ldots [\forall v_\xi A_\xi] \ldots]$$

if variables of $\bigcup \{\text{Rng}(v_\xi) : \xi < \gamma\}$ are not free in Γ.

(v) If the rule of γ dependent choices is in Ω, then under the proviso of that rule, $\Gamma \vdash_\Delta [\bigvee_{\xi<\gamma} A_0 \ldots A_\xi \ldots]$ implies

$$\Gamma \vdash_\Delta [\bigvee_{\xi<\gamma} [\forall v_0 A_0] \ldots [\exists w_\xi [\forall v_\xi A_\xi]] \ldots]$$

if variables of $\bigcup \{\text{Rng}(v_\xi) : \xi < \gamma\}$ are not free in Γ.

PROOF: The proof of (i) and (ii) is like the proof of 5.2.4. In case $\Gamma = \phi$, all of (i)–(v) are ordinary rules of inference. In case $\Gamma \neq \phi$, the argument in the proof of Theorem 5.2.4 shows that if $\Gamma \vdash_\Delta A$ then there is a conjunction C of formulas in Γ such that $\vdash_\Delta C \rightarrow A$. Therefore, if $\Gamma \vdash_\Delta A$ and variables of Rng(v) are not free in Γ, $\vdash_\Delta C \rightarrow A$ for a conjunction C of formulas of Γ and variables of Rng(v) are not free in C. Hence $\vdash_\Delta C \rightarrow [\forall v A]$ by $\mathscr{Q}1$ and modus ponens. Hence (iii).

Suppose $\mathscr{CH}_\gamma$ is a rule of Ω and $\Gamma \vdash_\Delta [\bigvee_{\xi<\gamma} A_0 \ldots A_\xi \ldots]$, where attached sequences $v_0, \ldots, v_\xi, \ldots$ of variables satisfy the proviso of the rule and are not free in Γ. Then $\vdash_\Delta C \rightarrow [\bigvee_{\xi<\gamma} A_0 \ldots A_\xi \ldots]$ for a conjunction of formulas in Γ. Then $\vdash_\Delta [\bigvee_{\zeta<\gamma} A_0[\neg C]\ldots A_\xi\ldots]$ using a substitution of an α-propositional formula provable in $\mathfrak{P}_\alpha$, 11.2.1 and modus ponens. To the principal subformulas of the new disjunction attach sequences $v_0, \langle x\rangle, v_1, \ldots, v_\xi, \ldots$ of variables where x is a variable not in C or in any of the A_ξ or v_ξ. This sequence again has length γ since γ was an infinite cardinal, and this sequence again satisfies the proviso of $\mathscr{CH}_\gamma$. Hence

$$\vdash_\Delta [\bigvee_{\xi<\gamma} [\forall v_0 A_0][\forall x[\neg C]]\ldots[\forall v_\xi A_\xi]\ldots].$$

Again using an α-propositional scheme provable in $\mathfrak{P}_\alpha$,

$$\vdash_\Delta [\exists x C] \rightarrow [\bigvee_{\xi<\gamma} [\forall v_0 A_0]\ldots[\forall v_\xi A_\xi]\ldots].$$

Since $\vdash [\forall x[\neg C]] \rightarrow [\neg C]$ by $\mathscr{Q}2$, $\vdash C \rightarrow [\exists x C]$ by ordinary propositional calculus. Using a valid scheme in $\rightarrow$ alone and modus ponens,

$$\vdash_\Delta C \rightarrow [\bigvee_{\xi<\gamma} [\forall v_0 A_0]\ldots[\forall v_\xi A_\xi]\ldots].$$

Hence (iv). (v) is similar.

11.2.4 *Deduction Theorem.* In any of the systems $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$ of Sect. 11.1, $\Gamma \vdash_\Delta A$ if and only if $\vdash_{\Delta\cup\Gamma} A$ for sets Γ of sentences of $\mathbf{L}$.

PROOF: Immediate from 11.2.3 by induction on the length of any given proof of A from assumptions $\Delta \cup \Gamma$.

11.2.5 *Properties of Quantification.* In $\mathfrak{P}_{\alpha\beta}(\mathbf{L})$,

(i) $\vdash [\forall v[A_0 \rightarrow A_1]] \rightarrow [[\forall v A_0] \rightarrow [\forall v A_1]]$.

(ii) $\vdash [\forall v[A_0 \wedge A_1]] \leftrightarrow [[\forall v A_0] \wedge A_1]$ if variables of $\mathrm{Rng}(v)$ not free in A_1.

(iii) $\vdash [\forall v A] \leftrightarrow [\forall v_0[\forall v_1 A]]$ if $\mathrm{Rng}(v) = \mathrm{Rng}(v_0) \cup \mathrm{Rng}(v_1)$.

(iv) $\vdash [\forall v A] \leftrightarrow [\forall g \bigcirc v SF_g^{\mathrm{Rng}(v)} A]$ if g one-one to variables not free in A, and A has no free occurrence of $v(\xi)$ bound by $g(v(\xi))$ for $\xi < \mathrm{Dom}(v)$.

PROOF: These proofs are the same as proofs in ordinary predicate calculus. Perhaps, however, for (iv) we should point out that $[[\forall g \bigcirc v SF_g^{\mathrm{Rng}(v)} A] \rightarrow A]$ is an instance of $\mathscr{Q}2$. For this, it suffices to show that $A = SF_{g^{-1}}^{\mathrm{Rng}(g \bigcirc v)} SF_g^{\mathrm{Rng}(v)} A$ and that no free occurrence of $g(v(\xi))$ in $SF_g^{\mathrm{Rng}(v)} A$ is bound by $v(\xi)$ for $\xi \in \mathrm{Dom}(v)$. Let $A' = SF_g^{\mathrm{Rng}(v)} A$, $A'' = SF_{g^{-1}}^{\mathrm{Rng}(g \bigcirc v)} A'$. If ι is an occurrence of a variable $y \notin \mathrm{Rng}(v)$ in A, then ι is still an occurrence of y in A' and since the SF-operator does not affect quantifications, ι is free in A if and only if it is free in A'. Since ι is then not a free occurrence of a variable of $\mathrm{Rng}(g \bigcirc v)$ in A', $A''(\iota) = y = A(\iota)$. If ι is a free occurrence of $v(\xi)$ in A, then ι is an occurrence of $g(v(\xi))$ in A', free by proviso. Hence $A''(\iota) = A(\iota) = v(\xi)$. Finally, if ι is a bound occurrence of $v(\xi)$ in A, it is still a bound occurrence of $v(\xi)$ in A'. Again $A''(\iota) = A(\iota) = v(\xi)$. Hence $A = A''$. A free occurrence of $g(v(\xi))$ in A' must have been a free occurrence of $v(\xi)$ in A. If ι had been bound by $v(\xi)$ in A', it would also have been bound by $v(\xi)$ in A, making the occurrence ι bound. Hence proviso on $\mathscr{Q}2$ is satisfied.

Consider the relation

$$A \equiv A' \text{ iff } \vdash_\Delta [A \leftrightarrow A'] \text{ in } \mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$$

on formulas of an (α, β, o, π)-predicate language $\mathbf{L}$. By the Equivalence Theorem 5.2.6 and 11.2.1, $\equiv$ is an equivalence relation which is also a congruence relation with respect to negation, implication, and conjunction. By 11.2.5 (i), it is also a congruence relation with respect to the cylindric operations taking A into $[\forall v A]$; that is,

$$A \equiv A' \text{ implies } [\forall v A] \equiv [\forall v A'].$$

It is therefore a congruence relation in the sense of Sect. 8.4. As a corollary to 8.4.5, we have

11.2.6 *Replacement Principle.* If $A = E_0 C_0 \ldots E_\nu C_\nu \ldots E_\sigma$, where $\langle C_\nu : \nu < \sigma \rangle$ is a sequence of formulas of $\mathbf{L}$, and if $\vdash_\Delta [C_\nu \leftrightarrow C'_\nu]$ for all $\nu < \sigma$ in $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$, then $\vdash_\Delta A \leftrightarrow [E_0 C'_0 \ldots E_\nu C'_\nu \ldots E_\sigma]$.

Consider also the relation

$$T \equiv T' \text{ iff } \vdash_\Delta [T \mathrel{\overline{=}} T'] \text{ in } \mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$$

on terms of $\mathbf{L}$. It is reflexive by $\mathscr{E}\mathscr{Q}1$, symmetric and transitive by $\mathscr{E}\mathscr{Q}4$ with $P = \overline{=}$. It is a congruence relation on the algebra of terms by $\mathscr{E}\mathscr{Q}2$ and $\mathscr{E}\mathscr{Q}3$. Moreover, $\mathscr{E}\mathscr{Q}4$ and $\mathscr{E}\mathscr{Q}5$ tell us that this congruence relation on the algebra of terms and the above congruence relation on formulas are linked as follows:

$$T_0 \equiv T'_0 \text{ and } T_1 \equiv T'_1 \text{ implies } [T_0 P T_1] \equiv [T'_0 P T'_1]$$
$$T_\xi \equiv T'_\xi \text{ for all } \xi < \eta \text{ implies } [Q T_0 \ldots T_\xi \ldots] \equiv [Q T'_0 \ldots T'_\xi \ldots]$$

for all special two-place predicate symbols P, all infinitary or η-place predicate symbols Q. Therefore the following describes a system $\langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ B\ \mathsf{O}'\ \mathsf{Q}\ S \rangle$ for the interpretation of formulas of $\mathbf{L}$, a system we will call $\mathfrak{S}(\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L}), \Delta)$:

For terms T, let $|T| = \{T' : T \equiv T'\}$, for formulas A, let $|A| = \{A' : A \equiv A'\}$. Let

$D = \{|T| : T$ is a term of $\mathbf{L}\}$.
$B = \{|A| : A$ is a formula of $\mathbf{L}\}$.
$S = \{s\}$ where $s(x) = |\langle x\rangle|$ for all variables x.
$C(c) = |\langle c\rangle|$ if c is an individual constant.
$\mathsf{O}(\psi)(\langle |T_0|\,|T_1|\rangle) = |[T_0\psi T_1]|$ if ψ is a special two-place operation symbol.

$\mathsf{O}(\varphi)(\langle |T_0|\ldots|T_\xi|\ldots\rangle) = |[\varphi T_0\ldots T_\xi\ldots]|$ if φ is a ζ-place or infinitary operation symbol, $0 < \zeta < o$, and ζ is the length of the sequence $\langle T_0\ldots T_\xi\ldots\rangle$.
$\mathsf{R}(P)(\langle |T_0|\,|T_1|\rangle) = |[T_0 P T_1]|$ if P is a special two-place predicate symbol.
$\mathsf{R}(Q)(\langle |T_0|\ldots|T_\xi|\ldots\rangle) = |[Q T_0\ldots T_\xi\ldots]|$ if Q is an η-place or infinitary predicate symbol, $0 < \eta < \pi$, and η is the length of the sequence $\langle T_0\ldots T_\xi\ldots\rangle$.

$$\begin{aligned}
&\mathsf{O}'(\neg)(|A|) = |\neg A|\\
&\mathsf{O}'(\rightarrow)(\langle |A_0|\,|A_1|\rangle) = |[A_0 \rightarrow A_1]|\\
&\mathsf{O}'(\bigwedge)(\langle |A_0|\ldots|A_\xi|\ldots\rangle) = |[\bigwedge A_0\ldots A_\xi\ldots]|\\
&\mathsf{Q}(\forall, v)(\{|A|\}) = |[\forall v A]|.
\end{aligned}$$

An induction on terms shows that the unique homomorphism s^* on terms such that $s^*(|\langle x\rangle|) = s(x)$ for variables x is the following:

$$s^*T = |T| \text{ for all terms } T.$$

An equally easy induction on formulas shows that the valuation function V is the following:

$$V(s, A) = |A| \text{ for all formulas } A.$$

11.2.7 *Substitution Rule for Equality.* In any of the systems $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$ of Sect. 11.1, if $\vdash_\Delta f(x) \mathrel{\overline{=}} f'(x)$ for all $x \in X$ then $\vdash_\Delta SF_f^X A \leftrightarrow SF_{f'}^X A$, where f, f' map variables X to terms, provided that A has no free occurrence of $x \in X$ bound by $FV(f(x))$ or by $FV(f'(x))$.

PROOF: Form $\mathfrak{S}(\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L}), \Delta)$ and let s be the unique assignment in S. Then $s^* \bigcirc f$ and $s^* \bigcirc f'$ agree on X since the values are $|f(x)| = |f'(x)|$. By Theorem 9.2.11, $V(s, SF_f^X A) = V(s, SF_{f'}^X A)$. This is exactly what we wished to prove.

11.2.8 *Rule for Change of Bound Variable.* $\vdash A \leftrightarrow SB_g^Y A$ in $\mathfrak{P}_{\alpha\beta}(\mathbf{L})$ where g is a one-one function on Y to variables not in A.

PROOF: Form $\mathfrak{S}(\mathfrak{P}_{\alpha\beta}(\mathbf{L}), \phi)$. According to Theorem 9.3.5, the result will follow once we know

$$\vdash [\forall v A] \leftrightarrow [\forall g' \circ v SF_g^{Y \cap \mathrm{Rng}(v)} A]$$

for all one-one functions g to variables not free in A such that A has no free occurrence of $y \in Y \cap \mathrm{Rng}(v)$ bound by $g(y)$, where g' is g on Y, identity elsewhere. This follows by 11.2.5. Let v_0, v_1 be chosen so that $\mathrm{Rng}(v_0) = Y \cap \mathrm{Rng}(v)$, $\mathrm{Rng}(v_1) = \mathrm{Rng}(v) \sim Y$. Apply 11.2.5 (iv) for v_0, then use (i), (iii) and replacement.

11.2.9 *Corollary.* $\vdash [\forall v A] \to A(f \circ v)$ in $\mathfrak{P}_{\alpha\beta}(\mathbf{L})$.

11.3 Completeness of the Basic Formal Systems with Chang's Distributive Laws and the Rule of Dependent Choices for Certain α, β

The completeness theorems make use of the criterion for satisfiability, 10.3.5. Since that theorem was formulated only for sentences, we must first verify that our systems are complete if only the valid sentences are provable. In case $\beta = \alpha$ there is no problem about this for if A is a valid formula and sequence v contains all variables free in A, then $[\forall v A]$ is a valid sentence. From $\vdash [\forall v A]$ we conclude $\vdash A$. In case $\beta < \alpha$ it may not be possible to form the closure $[\forall v A]$, but the following lemma says that the familiar device of replacing the variables by individual constants still applies.

11.3.1 *Lemma.* Let $A(x_0, \ldots, x_\xi, \ldots)$ be a formula of an (α, β, o, π)-language $\mathbf{L}$ and suppose sequence $\langle x_0 \ldots x_\xi \ldots \rangle$ contains all variables free in A. Let $\mathbf{L}'$ be the language like $\mathbf{L}$ except that it has one new constant c_ξ for each x_ξ. Then

(i) $\Vdash A(x_0, \ldots, x_\xi, \ldots)$ if and only if $\Vdash A(c_0, \ldots, c_\xi, \ldots)$.

(ii) For any of the formal systems of Sect. 11.1, $\vdash A(c_0, \ldots, c_\xi, \ldots)$ in $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L}')$ implies $\vdash A(x_0, \ldots, x_\xi, \ldots)$ in $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$.

PROOF: Since no bound variables are changed in passage from $A(x_0, \ldots, x_\xi, \ldots)$ to $A(c_0, \ldots, c_\xi, \ldots)$, $A(c_0, \ldots, c_\xi, \ldots)$ is the result of changing free occurrences of x_ξ to c_ξ. Therefore in any model $\mathfrak{M}'$ of $\mathbf{L}'$, $V(s, A(c_0, \ldots, c_\xi, \ldots)) = V(s, A(x_0, \ldots, x_\xi, \ldots))$ for assignment $s(x_\xi) = \mathbf{C}(c_\xi)$, by Theorem 9.2.10. Thus $A(c_0, \ldots, c_\xi, \ldots)$

is valid if and only if $A(x_0, \ldots, x_\xi, \ldots)$ holds in all models $\mathfrak{M}'$ of $\mathbf{L}'$. But clearly this is the same as saying $A(x_0, \ldots, x_\xi, \ldots)$ is valid. Hence (i).

Suppose that $\langle C_\nu : \nu < \sigma + 1 \rangle$ is a formal proof of $A(c_0, \ldots, c_\xi, \ldots)$ in $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L}')$. Let Y be the set of all variables appearing in the proof and let g be a one-one function on Y to variables of $\mathbf{L}$ not in $A(x_0, \ldots, x_\xi, \ldots)$. Let C'_ν arise from C_ν by changing all occurrences of variables y by $g(y)$. Since we have only relabeled the variables, $\langle C'_\nu : \nu < \sigma + 1 \rangle$ is still a formal proof. Let C''_ν arise from C'_ν by changing all occurrences of the c_ξ by x_ξ. Since no variable x_ξ appears in C'_ν, all occurrences of x_ξ in C''_ν are free. It is therefore not difficult to see that $\langle C''_\nu : \nu < \sigma + 1 \rangle$ is still a formal proof, and since its formulas are all in $\mathbf{L}$, it is a formal proof in $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$. We claim that $C''_\sigma = SB^Y_g A(x_0, \ldots, x_\xi, \ldots)$. For if ι is a free occurrence of a variable in A, it is a free occurrence of some x_ξ. Hence ι is an occurrence of c_ξ in $A(c_0, \ldots, c_\xi, \ldots) = C_\sigma$ and also in C'_σ. Then ι is an occurrence of x_ξ in C''_σ. In this case $C''_\sigma(\iota) = (SB^Y_g A)(\iota)$. If ι is a bound occurrence of a variable y in A, then it is a bound occurrence of y in C_σ. Hence $y \in Y$ and ι is an occurrence of $g(y)$ in C'_σ and in C''_σ. In this case as well, $C''_\sigma(\iota) = (SB^Y_g A)(\iota)$. Hence $\vdash SB^Y_g A(x_0, \ldots, x_\xi, \ldots)$ in $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$. Then $\vdash A(x_0, \ldots, x_\xi, \ldots)$ by the rule for changing bound variables.

Let $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$ be any one of the systems of Sect. 11.1. We have already seen that

$$A \equiv A' \text{ if and only if } \vdash A \leftrightarrow A'$$

is a congruence relation on the algebra of formulas. Let $B(\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L}))$ be the set of all these equivalence classes. Operations on these classes induced by the propositional operations are

$$\neg|A| = |[\neg A]|, \ |A_0| \to |A_1| = |[A_0 \to A_1]|,$$
$$\bigwedge \langle |A_0| \ldots |A_\xi| \ldots \rangle = |[\mathbf{\bigwedge} A_0 \ldots A_\xi \ldots]|.$$

The defined propositional operations yield additional operations on $B(\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L}))$:

$$|A_0| \wedge |A_1| = |[A_0 \wedge A_1]|, \ |A_0| \vee |A_1| = |[A_0 \vee A_1]|,$$
$$\bigvee \langle |A_0| \ldots |A_\xi| \ldots \rangle = |[\mathbf{\bigvee} A_0 \ldots A_\xi \ldots]|. \text{ Let } \mathbf{1} = |[A \vee [\neg A]]|.$$

Then the defining equations for Boolean algebras all hold, for Theorem 11.2.1 implies this and more:

11.3.2 *Lemma.* Let L_α be the α-propositional scheme language and Σ' the set of α-propositional schemes in Σ. Interpret L_α in $\mathfrak{B} = \mathfrak{B}(\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathsf{L}))$ letting $\mathsf{O}'(\neg) = \neg$, $\mathsf{O}'(\rightarrow) = \rightarrow$, $\mathsf{O}'(\wedge) = \wedge$. Then if $\vdash \mathscr{A}$ in $\mathfrak{P}_\alpha(\Sigma')$, then $s^*\mathscr{A} = \mathbf{1}$ for all assignments s to $\mathfrak{B}$.

PROOF: Let W be the set of all propositional symbols in $\mathscr{A}$. For $A_\xi \in W$ let $f(A_\xi)$ be a formula of L in the equivalence class $s(A_\xi)$. Then $s^*\langle A_\xi\rangle = s(A_\xi) = |f(A_\xi)|$. An easy induction on formulas of L_α shows that $s^*\mathscr{B} = |S_f^W\mathscr{B}|$ for all schemes $\mathscr{B}$ with symbols in W. Hence if $\vdash \mathscr{A}$ in $\mathfrak{P}_\alpha(\Sigma')$, $s^*\mathscr{A} = |S_f^W\mathscr{A}| = \mathbf{1}$ by 11.2.1.

It is a consequence of the lemma that $\mathfrak{B} = \mathfrak{B}(\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathsf{L})) = \langle B \neg \wedge \vee\rangle$ is an $\nearrow \alpha$-complete Boolean algebra. Moreover if α-propositional scheme $\mathscr{A} \in \Sigma$ then $\mathscr{A}$ holds in $\mathfrak{B}$ and, what is the same thing, the equation $T_{\mathscr{A}} = \mathbf{1}$ holds in $\mathfrak{B}$. Therefore it is a consequence of the completeness of $\mathfrak{P}_\alpha(\Pi_{\nearrow\alpha})$ and 6.4.2 that if the Chang distributive laws (5.1.3) of levels less than α are in Σ then $\mathfrak{B}$ is $\nearrow \alpha$-representable. We use this fact in the completeness theorems.

The order in $\mathfrak{B}(\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathsf{L}))$ is

$$|A_0| \leqslant |A_1| \text{ if and only if } \vdash[A_0 \rightarrow A_1].$$

It follows by the rule of change of bound variable and $\mathscr{Q}2$ that $|[\forall v A(v)]| \leqslant |A(f \bigcirc v)|$ for all substitutions of $\mathrm{Rng}(v)$ to terms. It is not difficult to see that $|[\forall v A(v)]|$ is in fact the greatest lower bound of all the substitutions $|A(f \bigcirc v)|$.

11.3.3 *Theorem.* If $\gamma \exp \varepsilon = \gamma$ for all $\varepsilon < \beta$, the calculus $\mathfrak{P}_{\gamma^+\beta}(\Pi_\gamma; \Omega_\gamma)$ where Ω_γ contains only the rule of γ dependent choices, is comp'ete.

PROOF: We have seen that provable formulas are valid (11.1.4). We have seen also (11.3.1) that to show this calculus complete it suffices to show that every valid sentence A of a $(\gamma^+, \beta, o, \pi)$-language L is provable in $\mathfrak{P}_{\gamma^+\beta}(\Pi_\gamma; \Omega_\gamma)(\mathsf{L})$. To do this, form the algebra $\mathfrak{B} = \mathfrak{B}(\mathfrak{P}_{\alpha\beta}(\Pi_\gamma; \Omega_\gamma)(\mathsf{L}))$ and choose a set X of $\kappa(\gamma^+, \beta) = \gamma$ variables not in A and form sets T of terms, Δ of formulas in accordance with 10.3.4. Both have power γ. If, contrariwise, not $\vdash A$ then this algebra has a homomorphism to $\mathfrak{B}_0$ sending $|[\neg A]|$ to $\mathbf{1}$ and preserving any given collection of γ meets and joins, each with at most γ terms. This follows from the γ-representability of $\mathfrak{B}$ and the theorem in the Foreward on Algebra. We intend to choose this collection in such a way that for such a homomorphism h, $\Gamma = \{C : h|C| = \mathbf{1}\}$ satisfies 10.3.5 (1)–(8). Since $[\neg A]$ will be in Γ

it will follow that $[\neg A]$ is satisfiable, contradicting the validity of A.

It is necessary to first go through the procedure of 10.3.5 for selecting witnesses. This step can be omitted if A contains no quantifications. Make a list of length γ of all quantifications in Δ:

$$[\forall v_0 C_0(v_0)], \ldots, [\forall v_\nu C_\nu(v_\nu)], \ldots$$

Select witnessing formulas

$$C_0(g_0 \bigcirc v_0), \ldots, C_\nu(g_\nu \bigcirc v_\nu), \ldots$$

so that each g_ν is one-one on $\mathrm{Rng}(v_\nu)$ to X and variables of $\mathrm{Rng}(g_\nu)$ are not in $\mathrm{Rng}(g_\xi)$ for $\xi \neq \nu$ and do not appear in $[\forall v_\xi C_\xi(v_\xi)]$ for $\xi \leqslant \nu$. Let $W_\nu = [C_\nu(g_\nu \bigcirc v_\nu) \to [\forall v_\nu C_\nu(v_\nu)]]$. We claim that $|[\neg A] \wedge [\bigwedge W_0 \ldots W_\nu \ldots]| \neq \mathbf{0}$ in $\mathfrak{B}$. To see this, suppose that this element were $\mathbf{0}$. Then

$$\vdash \neg[[\neg A] \wedge [\bigwedge W_0 \ldots W_\nu \ldots]].$$

Since the underlying propositional logic is complete, substitutions of valid γ^+-schemes are all provable. Using such a scheme,

$$\neg A \vdash [\bigvee [\neg W_0] \ldots [\neg W_\nu] \ldots]$$

To each W_ν attach sequence $g_\nu \bigcirc v_\nu$ for an application of the rule of dependent choices. They have pairwise disjoint ranges and variables of $\mathrm{Rng}(g_\nu)$ are not free in W_ξ for $\xi < \nu$. Moreover each formula of Δ has fewer than β free variables. It is therefore possible to associate to each W_ν a sequence w_ν of length less than β containing all variables of $\bigcup \{\mathrm{Rng}(g_\xi) : \xi < \nu\}$ free in W_ν. Then

(a) $\neg A \vdash [\bigvee [\forall g_0 \bigcirc v_0 [\neg W_0]] \ldots [\exists w_\nu [\forall g_\nu \bigcirc v_\nu [\neg W_\nu]]] \ldots]$

Since $\vdash [\neg W_\nu] \leftrightarrow [C_\nu(g_\nu \bigcirc v_\nu) \wedge [\neg \forall v_\nu C_\nu(v_\nu)]]$, and variables of $\mathrm{Rng}(g_\nu)$ do not appear in v_ν or C_ν,

$$\vdash [\forall g_\nu \bigcirc v_\nu [\neg W_\nu]] \leftrightarrow [[\forall g_\nu \bigcirc v_\nu C_\nu(g_\nu \bigcirc v_\nu)] \wedge [\neg \forall v_\nu C_\nu(v_\nu)]]$$

for each $\nu < \gamma$ by 11.2.5 (ii). By 11.2.5 (iv) and ordinary propositional calculus and generalization, it is apparent that the negation of each of the principal subformulas of the disjunction (a) is provable. By the deduction theorem, $\vdash A$. But we were assuming not $\vdash A$. Therefore we must have $|[\neg A] \wedge [\bigwedge W_0 \ldots W_\nu \ldots]| \neq \mathbf{0}$.

We choose the collection of meets and joins to be preserved by a homomorphism h to $\mathfrak{B}_0$ taking $|[\neg A] \wedge [\bigwedge W_0 \ldots W_\nu \ldots]|$ to $\mathbf{1}$ as follows:

The joins

$\vee\{\neg|[T_\xi \overline{=} T'_\xi]|: \xi < \zeta\} \vee |[\varphi T_0 \ldots T_\xi \ldots] \overline{=} [\varphi T'_0 \ldots T'_\xi \ldots]|$

for all terms of form $[\varphi T_0 \ldots T_\xi \ldots]$, $[\varphi T'_0 \ldots T'_\xi \ldots]$ in **T**. These joins are **1** by $\mathscr{E}\mathscr{Q}3$.

The joins

$\vee\{\neg|[T_\xi \overline{=} T'_\xi]|: \xi < \eta\} \vee (\neg|[QT_0 \ldots T_\xi \ldots]|) \vee |[QT'_0 \ldots T'_\xi \ldots]|$

for all atomic formulas of the form $[QT_0 \ldots T_\xi \ldots]$, $[QT'_0 \ldots T'_\xi \ldots]$ in Δ. These joins are **1** by $\mathscr{E}\mathscr{Q}5$.

The meets $\wedge\{|A_\xi|: \xi < \delta\} = |[\mathbf{\wedge}\, A_0 \ldots A_\xi \ldots]|$ for all conjunctions in Δ.

There are at most γ such joins and meets each with at most γ terms. Let $\Gamma = \{C: h|C| = \mathbf{1}\}$, h being be the preserving homomorphism. Then $[\neg A] \in \Gamma$ and Γ satisfies 10.3.5 (1)–(8). Formulas $[T \overline{=} T] \in \Gamma$ by $\mathscr{E}\mathscr{Q}1$. If $A_0 = [T_0 \overline{=} T'_0]$ and $A_1 = [T_1 \overline{=} T'_1]$ are in Γ, then $A_2 = [[T_0 \psi T_1] \overline{=} [T'_0 \psi T'_1]] \in \Gamma$ because $|A_0| \wedge |A_1| \leqslant |A_2|$ by $\mathscr{E}\mathscr{Q}2$ and because h preserves finite operations and order. Hence 10.3.5 (1). The verification of the other conditions is similar. The preservation of the special joins and meets is needed for (2), (4), (7). Checking (8) for the quantifiers, note that $|[\forall v A_0]| \leqslant |SF_f^{\mathrm{Rng}(v)} A_0|$ for all f on $\mathrm{Rng}(v)$ to **T** by $\mathscr{Q}2$. The proviso is satisfied because terms in **T** have variables in X and such variables are never in quantifier-sequences of formulas of Δ. Since h preserves order, $[\forall v A_0] \in \Gamma$ implies that all substitutions $SF_f^{\mathrm{Rng}(v)} A_0 \in \Gamma$. Conversely, if $[\forall v A_0] \notin \Gamma$, then since there is ν such that $W_\nu = [A_0(g_\nu \bigcirc v) \rightarrow [\forall v A_0]]$ and since $h|W_\nu| = \mathbf{1}$ and $h|[\forall v A_0]| = \mathbf{0}$, it follows that $h|A_0(g_\nu \bigcirc v)| = \mathbf{0}$. Hence $A_0(g_\nu \bigcirc v) \notin \Gamma$. But in this case $A_0(g_\nu \bigcirc v) = SF_{g_\nu}^{\mathrm{Rng}(v)} A_0$. Therefore condition (8) holds. We then conclude $[\neg A]$ satisfiable, thus contradicting the validity of A. Therefore A must be provable.

11.3.4 *Theorem.* If α is strongly inaccessible then $\mathfrak{P}_{\alpha\alpha}(\mathscr{D}_{\nearrow\alpha}, \mathscr{D}\mathscr{C}\mathscr{H}_{\nearrow\alpha})$ is complete. ($\mathscr{D}_{\nearrow\alpha}$ is the set of all γ-distributive laws for $\gamma < \alpha$, $\mathscr{D}\mathscr{C}\mathscr{H}_{\nearrow\alpha}$ the set of laws of dependent choices for all $\gamma < \alpha$.)

PROOF: Let **L** be any (α, α, o, π)-language, A any valid fromula of **L**. Let γ be the least cardinal upper bound of the length of A and of $\{o, \pi\} \sim \{\alpha\}$. Then $\gamma < \alpha$ and $o \leqslant \gamma$ if $o < \alpha$ and $\pi \leqslant \gamma$ if $\pi < \alpha$. Let **L**$'$ be the $((2 \exp \gamma)^+, \gamma^+, o', \pi')$-language with the constants of A where o' is o if $o < \alpha$, $o' = \gamma^+$ if $o = \alpha$, and π' is π if $\pi < \alpha$, $\pi' = \gamma^+$ if $\pi = \alpha$. Then A is a formula of **L**$'$ and every formula of $\mathbf{L}'$ is a formula of $\mathbf{L}$. By 11.3.3 A is provable in $\mathfrak{P}_{(2\exp\gamma)^+,\gamma^+}(\Pi_{2\exp\gamma};\Omega)$ where Ω consists of the rule of $2\exp\gamma$ dependent choices. But $2\exp\gamma<\alpha$ and instances of schemes of $\Pi_{2\exp\gamma}$ are provable from the distributive law of level $2\exp\gamma$ and the rule of $2\exp\gamma$ dependent choices is provable from $\mathscr{DCH}_{2\exp\gamma}$. It follows that A is provable in the given system.

11.3.5 *Theorem.* If α is inaccessible and $\kappa(\gamma^+,\beta)<\alpha$ for all $\gamma<\alpha$ then $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha};\Omega)$ is complete, where Ω consists of all rules of γ dependent choices for $\gamma<\alpha$. Moreover if β is regular the assumption $\gamma\exp\varepsilon<\alpha$ for all $\gamma<\alpha$ and $\varepsilon<\beta$ suffices. If α strongly inaccessible $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha};\Omega)$ is complete for all $\beta\leqslant\alpha$.

PROOF: If $\beta=\alpha$ the hypothesis on α, β, reduces to the strong inaccessibility of α. In this case the completeness follows by 11.3.4. For β less than α and regular, $\kappa(\gamma^+,\beta)=\bigcup\{\gamma\exp\varepsilon:\varepsilon<\beta\}<\alpha$ for all $\gamma<\alpha$ if and only if $\gamma\exp\varepsilon<\alpha$ for all $\gamma<\alpha$, $\varepsilon<\beta$. Hence the reduction in this case. Moreover, if α strongly inaccessible and $\beta<\alpha$ then $\kappa(\gamma^+,\beta)\leqslant\gamma\exp\beta<\alpha$ for all $\gamma<\alpha$ by 10.3.2. Therefore it remains only to show that $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha};\Omega)$ is complete for $\beta<\alpha$, provided that $\kappa(\gamma^+,\beta)<\alpha$ for all $\gamma<\alpha$.

Suppose A a valid formula of an (α,β,o,π)-language $\mathbf{L}$. Let γ be the least cardinal upper bound of the length of A and of $\{o,\pi\}\sim\{\alpha\}$. Then $\gamma<\alpha$ and $o\leqslant\gamma$ if $o<\alpha$ and $\pi\leqslant\gamma$ if $\pi<\alpha$. Let $\mathbf{L}'$ be the (κ^+,β,o',π')-language with the constants of A where $\kappa=\kappa(\gamma^+,\beta)$, $o'=o$ if $o<\alpha$, $o'=\gamma^+$ if $o=\alpha$, and $\pi'=\pi$ if $\pi<\alpha$, $\pi'=\gamma^+$ if $\pi=\alpha$. Then A is a formula of $\mathbf{L}'$ and every formula of $\mathbf{L}'$ is a formula of $\mathbf{L}$. Since $\kappa\exp\varepsilon=\kappa$ for all $\varepsilon<\beta$, A is provable in $\mathfrak{P}_{\kappa^+,\beta}(\Pi_\kappa;\Omega_\kappa)(\mathbf{L}')$ by 11.3.3. This proof will also be a proof in $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha};\Omega)(\mathbf{L})$.

11.3.6 *Theorem.* If α is inaccessible, $\mathfrak{P}_{\alpha 0}(\Pi_{\nearrow\alpha})$ is complete. For any infinite cardinal γ, $\mathfrak{P}_{\gamma^+0}(\Pi_\gamma)$ is complete. Also $\mathfrak{P}_{\omega_1 0}$ is complete.

PROOF: Quantificational schemes were not needed to carry out the steps of the proof of 11.3.3 for quantifier-free formulas. Schemes of Π_ω are known to be provable in basic propositional calculus $\mathfrak{P}_{\omega_1}$. Therefore instances of such schemes are provable in basic predicate calculus $\mathfrak{P}_{\omega_1 0}$ by 11.2.1. Hence $\mathfrak{P}_{\omega_1 0}$ is complete.

11.4 Completeness of the Basic Formal Systems when $\alpha = \omega_1$, $\beta = \omega$

For $\alpha = \omega_1$, the completeness theorem 11.3.3 applies only for $\beta = 0$ and $\beta = \omega$. We have seen that the completeness of $\mathfrak{P}_{\omega_1 0}$ is a corollary. Moreover since instances of schemes of Π_ω are provable in basic calculus, 11.3.3 tells us that $\mathfrak{P}_{\omega_1\omega}(\phi; \Omega_\omega)$ is complete, where Ω_ω is the rule of ω dependent choices. But we can say more; $\mathfrak{P}_{\omega_1\omega}$ itself is complete.

We will give a proof that depends only on the existence of a homomorphism on a given Boolean algebra to $\mathfrak{B}_0$ carrying a given non-zero element to **1** and preserving a given denumerable collection of joins and meets. See the Foreward on Algebra for a proof of this theorem. The collection of joins and meets can be chosen so that each has denumerably many terms as well. This suggests that $\mathfrak{P}_{\gamma^+,\omega}(\Pi_\gamma)$ may also be complete for $\gamma > \omega$, for the algebras of equivalence classes have the analogous property for collections of γ joins and meets. However the argument we now give does not generalize, for the homomorphisms are taken not on the algebras of equivalence classes $\mathfrak{B}(\mathfrak{P}_{\omega_1\omega}(\mathsf{L}))$ but on subalgebras of them. The question of the completeness of calculi $\mathfrak{P}_{\gamma^+,\omega}(\Pi_\gamma)$ for $\gamma > \omega$ remains open.

11.4.1 *Theorem.* $\mathfrak{P}_{\omega_1\omega}$ is complete.

PROOF: Let L be any $(\omega_1, \omega, o, \pi)$-predicate language, A any valid sentence of L. We must show $\vdash A$. Choose a set X of $\kappa(\omega_1, \omega) = \omega$ variables not in A and form sets T of terms, Δ of formulas, in accordance with 10.3.4. Both have power ω.

Form the algebra $\mathfrak{B}(\mathfrak{P}_{\omega_1\omega}(\mathsf{L}))$ and let $\mathfrak{B}'$ be the subalgebra with underlying set $B' = \{|C|: FV(C) \subseteq \text{a finite subset of } X\}$. Since formulas of Δ have only finitely many free variables, and all are in X, $|C| \in B'$ if $C \in \Delta$. The algebra $\mathfrak{B}'$ is clearly not ω-complete, nor is it an ω-regular subalgebra of $\mathfrak{B}(\mathfrak{P}_{\omega_1\omega}(\mathsf{L}))$. For in $\mathfrak{B}'$, if $[\forall v C] \in \Delta$,

(a) $|[\forall v C]| = \bigwedge \{|C(f \bigcirc v)|: f \in X^{\mathrm{Rng}(v)}\}$, with respect to $\mathfrak{B}'$.

We have $|[\forall v C]| \leqslant |C(f \bigcirc v)|$ for all $f \in X^{\mathrm{Rng}(v)}$ in $\mathfrak{B}(\mathfrak{P}_{\omega_1\omega}(\mathsf{L}))$, but in that algebra, the meet on the right side of (a) is a conjunction, not a quantification. To see that (a) holds in $\mathfrak{B}'$, note that if $|C'| \in B'$ and $|C'| \leqslant |C(f \bigcirc v)|$ for all $f \in X^{\mathrm{Rng}(v)}$, then such an f can be chosen one-one to variables not in C'. Then from $\vdash C' \rightarrow$

$C(f \bigcirc v)$ it follows $\vdash C' \to [\forall f \bigcirc v C(f \bigcirc v)]$. By 11.2.5 (iv) and replacement, $\vdash C' \to [\forall v C]$. Hence (a).

Suppose, contrariwise, not $\vdash A$. Then $|A| \neq \mathbf{1}$ and $|\neg A| \neq \mathbf{0}$. Let h be a homomorphism on $\mathfrak{B}'$ to $\mathfrak{B}_0$ taking $|\neg A|$ to $\mathbf{1}$ and preserving the following collection of meets and joins in $\mathfrak{B}'$:

The meets of form (a) for quantifications in Δ.

The meets $\wedge \{|A_\xi|: \xi < \delta\} = |[\mathbf{\wedge}\, A_0 \ldots A_\xi \ldots]|$ for conjunctions in Δ. These meets are the same is $\mathfrak{B}'$ as in $\mathfrak{B}$.

The joins arising from infinitary terms in T, infinitary atomic formulas in Δ, as in the proof of Theorem 11.3.3. These joins are still $\mathbf{1}$ in $\mathfrak{B}'$.

Let $\Gamma = \{C: |C| \in B' \text{ and } h|C| = \mathbf{1}\}$. Then $[\neg A] \in \Gamma$ and Γ has properties (1)–(8) of 10.3.5. Hence $[\neg A]$ is satisfiable, contradicting the validity of A. Therefore A must be provable.

11.5 The Reduction to the Rule of Independent Choices when $\beta = \omega$

The question of the completeness of $\mathfrak{P}_{\alpha\omega}(\Pi_{\nearrow\alpha})$ is open for $\alpha > \omega_1$, but we can show that there are cases where the rules of dependent choices in the completeness theorems can be replaced by rules of independent choices when $\beta = \omega$. It is a theorem of Chapter 12 that such reductions are not possible when $\beta > \omega$.

To simplify notation, let

$\Omega_{I,\gamma}$ contain only the rule of γ independent choices
$\Omega_{I,\nearrow\gamma} = \bigcup \{\Omega_{I,\kappa}: \kappa < \gamma\}$
$\Omega_{D,\gamma}$ contain only the rule of γ dependent choices
$\Omega_{D,\nearrow\gamma} = \bigcup \{\Omega_{D,\kappa}: \kappa < \gamma\}$.

Then we can prove

11.5.1 *Theorem.* If α is strongly inaccessible, then $\mathfrak{P}_{\alpha\omega}(\Pi_{\nearrow\alpha};\ \Omega_{I,\nearrow\alpha})$ is complete.

The proof combines techniques of the completeness theorems already given and needs only to be outlined at this point. Consider a valid sentence A in a formal system

$$\mathfrak{Q}_{\alpha\omega}(\mathbf{L}) = \mathfrak{P}_{\alpha\omega}(\Pi_{2\exp\gamma};\ \Omega_{I,\gamma})(\mathbf{L}),$$

A itself being a (γ^+, ω)-sentence, where $2 \exp \gamma < \alpha$. Passage from γ^+ to α needs to be made so that the γ-distributive law will be available. Choose a set X of $\kappa(\gamma^+, \omega) = \gamma$ variables not in A and form sets T of terms, Δ of formulas in accordance with 10.3.4. Split X into ω pairwise disjoint sets $X_0, \ldots, X_n, \ldots$, each of power γ. Form the algebra $\mathfrak{B} = \mathfrak{B}(\mathfrak{Q}_{\alpha\omega}(\mathbf{L}))$ of equivalence classes and let $\mathfrak{B}'_n$ be the the subalgebra with underlying set $B'_n = \{|C|: FV(C) \subseteq \bigcup \{X_i: i \leqslant n\}\}$. Let $\mathfrak{B}' = \bigcup \{\mathfrak{B}'_n: n < \omega\}$. Remember that formulas C in Δ have finitely many free variables, so such elements $|C|$ are in $\mathfrak{B}'$.

$\mathfrak{B}$ is $\nearrow \alpha$-complete and γ-distributive as are the $\mathfrak{B}'_n$. If $\delta < \alpha$ and $FV([\bigwedge C_0 \ldots C_\xi \ldots]) \subseteq \bigcup \{X_i: i \leqslant n\}$ then $|[\bigwedge C_0 \ldots C_\xi \ldots]| = \bigwedge \{|C_\xi|: \xi < \delta\}$ in $\mathfrak{B}'_n$, in $\mathfrak{B}'$ and in $\mathfrak{B}$. The $\mathfrak{B}'_n$ are therefore $\nearrow \alpha$-subalgebras of $\mathfrak{B}'$ and of $\mathfrak{B}$. However $\mathfrak{B}'$ is not even ω-complete and not even an ω-regular subalgebra of $\mathfrak{B}$. It is easy to see that for $|[\forall v C(v)]| \in B'$

(a) $|[\forall v C(v)]| = \bigwedge \{|C(f \bigcirc v)|: f \in X^{\mathbf{Rng}(v)}\}$ in $\mathfrak{B}'$.

The meet on the right is a conjunction in $\mathfrak{B}$.

The rule $\Omega_{I,\gamma}$ imposes additional structure on the meets (a). Given any formula D with free variables in $\bigcup \{X_i: i \leqslant n\}$, $|D| \neq 0$, and collection $|[\forall v_\nu C_\nu(v_\nu)]|$, $\nu < \gamma$, of elements of B'_n, the rule of γ independent choices tells us that for one-one functions g_ν on $\mathrm{Rng}(v_\nu)$ to X_{n+1} with pairwise disjoint ranges we have

$$|D| \wedge \bigwedge_{\nu<\gamma} |C_\nu(g_\nu \bigcirc v_\nu) \rightarrow [\forall v_\nu C_\nu(v_\nu)]| \neq \mathbf{0}$$

in $\mathfrak{B}'_{n+1}$, hence also in $\mathfrak{B}'$ and in $\mathfrak{B}$. The argument is similar to the one given in the proof of 11.3.3.

Suppose, contrariwise, the formula A not provable in $\mathfrak{Q}_{\alpha\omega}(\mathbf{L})$. Then $|[\neg A]| \neq \mathbf{0}$. We wish to show that there is a homomorphism h on $\mathfrak{B}'$ to the two-element Boolean algebra $\mathfrak{B}_0$ sending $|[\neg A]|$ to $\mathbf{1}$ and preserving the following collection of γ joins and meets in $\mathfrak{B}'$:

a) The meets (a) arising from quantifiers in Δ.

b) The joins $\mathbf{1} = |[\bigwedge A_0 \ldots A_\xi \ldots]| \vee \bigvee \{|\neg A_\xi|: \xi < \delta\}$ for conjunctions in Δ.

c) The joins arising from infinitary terms and atomic formulas of T and Δ as in the proof of 11.3.3. They are all $\mathbf{1}$ in $\mathfrak{B}'$.

Note that each one of these joins and meets has at most γ terms. With such a homomorphism we can let $\Gamma = \{C: h|C| = \mathbf{1}\}$ as before, and Γ will be seen to contain $[\neg A]$, formulas $[T \,\overline{=}\, T]$ for $T \in \mathsf{T}$ and to satisfy conditions 10.3.5 (1)–(8). We can then conclude that

$[\neg A]$ is satisfiable, contradicting the validity of A. Therefore A must have been provable.

To see that there is such a homomorphism, begin by listing all joins of type b) and c) arising from formulas with free variables in X_0. By the γ-distributive law there is a formula C_0' with free variables in X_0 such that $\mathbf{0} \neq |C_0'| \leq |\neg A|$ and for each join $\bigvee_\nu |D_\nu| = \mathbf{1}$ in the collection there is ν such that $|C_0'| \leqslant |D_\nu|$. Then list meets of type a) arising from quantifier-formulas $[\forall v_\nu C_\nu(v_\nu)]$ with free variables in X_0 and attach to them one-one functions g_ν on $\mathrm{Rng}(v_\nu)$ to X_1 with pairwise-disjoint ranges. Let $C_0'' = [C_0' \wedge [\bigwedge_{\nu<\gamma} [C_\nu(g_\nu \bigcirc v_\nu) \rightarrow [\forall v_\nu C_\nu(v_\nu)]]]]$. Then $|C_0''| \neq \mathbf{0}$. Then list all joins of type b) and c) arising from formulas with free variables in $X_0 \cup X_1$. Use the γ-distributive law to find a formula C_1' with free variables in $X_0 \cup X_1$ such that $|C_1'| \neq \mathbf{0}$ and for each join $\bigvee_\nu |D_\nu| = \mathbf{1}$ in the collection there is ν such that $|C_1'| \leqslant |D_\nu| \wedge |C_0''|$. Then form C_1'' for quantifications with free variables in $X_0 \cup X_1$ choosing witnessing formulas by substitutions to X_2. Continue by ordinary induction. We will have a sequence $|C_0''| \geqslant |C_1''| \geqslant \ldots |C_n''| \geqslant \ldots$ of non-zero elements of B' such that for each join $\bigvee_\nu |D_\nu| = \mathbf{1}$ of type b) or c) there are n, ν such that $|C_n''| \leqslant |D_\nu|$ and for each meet of type a) there is n and there is g on $\mathrm{Rng}(v)$ to X such that $|C_n''| \leqslant \neg\, |C(g \bigcirc v)| \vee |[\forall v C(v)]|$. The homomorphism h determined by a maximal ideal containing $\{\neg\, |C_n''| : n < \omega\}$ preserves all joins and meets a), b), c).

CHAPTER 12

NON-DEDUCIBILITY IN INFINITARY PREDICATE LOGIC

The restrictions on α, β, o, π for formation of infinitary languages are assumed to be still in force: α regular, infinite, $\beta = 0$ or $\omega \leqslant \beta \leqslant \alpha$, $\pi \leqslant \alpha$, o regular, infinite and $o < \beta$ if β singular, $o \leqslant \alpha$ if β regular. The problem of the independence of the various α-propositional schemes was the topic of Chapter 7. We have nothing to add to the discussion at this point and consider only (α, β)-systems with the Chang distributive laws, so that the underlying propositional calculi are complete. We take up the problem of the independence of the various choice schemes and rules that had to be adjoined to carry out the completeness theorems. It is not hard to see that $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha})$ is incomplete if $\beta > \omega$. Let $\mathbf{L}$ be an (α, β, o, π)-language with one-place predicate symbols Q_n, $n < \omega$, and let

$$CH_\omega = [\bigwedge_{n<\omega} [\exists x Q_n x]] \rightarrow [\bigexists_{n<\omega} x_0 \ldots x_n \ldots [\bigwedge_{n<\omega} Q_n x_n]].$$

Then CH_ω is an example of a valid formula not provable in $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha})(\mathbf{L})$. More generally,

Theorem A. Suppose $\omega \leqslant \gamma < \beta$, γ regular. Let $\mathbf{L}$ be the (α, β)-language having one-place predicate symbols Q_ξ, $\xi < \gamma$. Let

$$CH_\gamma = [\bigwedge_{\xi<\gamma} [\exists x Q_\xi x]] \rightarrow [\bigexists_{\xi<\gamma} x_0 \ldots x_\xi \ldots [\bigwedge_{\xi<\gamma} Q_\xi x_\xi]].$$

Then CH_ω is valid and not provable in $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha})(\mathbf{L})$ and CH_γ is valid and not provable in $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha} \cup \mathscr{DCH}_{\nearrow\gamma})(\mathbf{L})$.

A language with a single two-place predicate symbol Q' would serve just as well for Theorem A. Replace parts $[Q_\xi x_\xi]$ by $[Q' x_\xi y_\xi]$. Note that CH_γ is an instance of $\mathscr{CH}_\gamma$.

It is more difficult to see that $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha}; \Omega_{I,\nearrow\alpha})$ is incomplete if $\beta > \omega$. Let $\mathbf{L}$ be an (α, β, o, π)-language with two-place predicate symbol Q and let

$$DCH_\omega = [\forall x \exists y Qxy] \rightarrow [\exists x_0 \ldots x_n \ldots [\bigwedge_{n<\omega} Qx_n x_{n+1}]].$$

This is an example of a valid unprovable formula. More generally,

Theorem B. Let $\mathbf{L}$ be an (α, β)-predicate language with $\beta > \omega$ having two-place predicate symbol Q. Then DCH_ω is valid and not provable in $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha}; \Omega_{I,\nearrow\alpha})(\mathbf{L})$. For $\omega \leqslant \gamma < \beta$, γ regular, let

$$DCH_\gamma = [\forall x_0 \exists x_1 Q x_0 x_1] \wedge [\bigwedge_{1<\xi<\gamma} [\bigvee_{\nu<\xi} x_0 \ldots x_\nu \ldots \exists x_\xi C_\xi]] \rightarrow$$
$$[\exists_{\xi<\gamma} x_0 \ldots x_\xi \ldots [\bigwedge_{0<\xi<\gamma} \bigwedge_{\nu<\xi} Qx_\nu x_\xi]],$$

where for $\xi \geqslant 2$,

$$C_\xi = [[\bigwedge_{0<\theta<\xi} \bigwedge_{\nu<\theta} Qx_\nu x_\theta] \rightarrow [\bigwedge_{0<\theta\leqslant\xi} \bigwedge_{\nu<\theta} Qx_\nu x_\theta]].$$

Then DCH_γ is valid and not provable in $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha} \cup \mathscr{DCH}_{\nearrow\gamma}; \Omega_{I,\nearrow\alpha})(\mathbf{L})$.

Note that DCH_γ follows from scheme $\mathscr{DCH}_\gamma$ by basic calculus. Let $v_\xi = \langle x_\xi \rangle$, $A_0 = [x_0 \equiv x_0]$, $A_1 = [Qx_0x_1]$, $A_\xi = C_\xi$ for $\xi \geqslant 2$. It is necessary to check

$$\vdash [\bigwedge_{\xi<\gamma} A_0 \ldots A_\xi \ldots] \rightarrow [\bigwedge_{0<\xi<\gamma} \bigwedge_{\nu<\xi} Qx_\nu x_\xi] \text{ in } \mathfrak{P}_{\alpha\beta}(\mathbf{L}).$$

Let $A = [\bigwedge A_0 \ldots A_\xi \ldots]$ and note that if there are $0 < \xi < \gamma$, $\nu < \xi$ such that not $\vdash A \rightarrow Qx_\nu x_\xi$, then there must be a least ordinal ξ_0 for which such a ν_0 exists, $\nu_0 < \xi_0$. Then $\xi_0 \neq 0, \neq 1$, and $\vdash A \rightarrow [\bigwedge_{0<\xi<\xi_0} \bigwedge_{\nu<\xi} Qx_\nu x_\xi]$. Using C_{ξ_0}, $\vdash A \rightarrow [\bigwedge_{0<\xi\leqslant\xi_0} \bigwedge_{\nu<\xi} Qx_\nu x_\xi]$. Therefore $\vdash A \rightarrow Qx_{\nu_0} x_{\xi_0}$, contradicting choice of ξ_0, ν_0.

Theorems A and B appear in my paper [15] for the case $\alpha = \omega_1$.

12.1 Summary of Results and Open Problems

Using the symbols $\approx$ and $\prec$ for comparing calculi as in Chapter 7, we have the following:

(i) Cases $\beta = \omega$. $\mathfrak{P}_{\omega_1\omega}$ is complete. For $\alpha > \omega_1$ the question of the completeness of $\mathfrak{P}_{\alpha\omega}(\Pi_{\nearrow\alpha})$ is open. It goes without saying that the question of the independence of the quantificational rules is also open.

(ii) If $\omega < \beta$, $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha}) \prec \mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha} \cup \{\mathscr{CH}_\omega\})$. More generally, if $\omega \leqslant \gamma < \beta$ and γ regular, $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha} \cup \mathscr{CH}_{\nearrow\gamma}) \prec \mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha} \cup \{\mathscr{CH}_\gamma\})$. If γ singular, $\mathfrak{P}_{\alpha\beta}(\mathscr{CH}_{\nearrow\gamma}) \approx \mathfrak{P}_{\alpha\beta}(\mathscr{CH}_\gamma)$.

(iii) Same as (ii) with "$\mathscr{CH}$" replaced by "$\mathscr{DCH}$".

(iv) If $\omega \leqslant \gamma < \beta$ and γ regular,

$$\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha} \cup \mathscr{DCH}_{\nearrow\gamma}) \prec \mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha} \cup \mathscr{DCH}_{\nearrow\gamma} \cup \{\mathscr{CH}_\gamma\}).$$

(v) If $\omega < \beta$, $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha}; \Omega_{I,\nearrow\alpha}) \prec \mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha} \cup \{\mathscr{DCH}_\omega\}; \Omega_{I,\nearrow\alpha})$. More generally, if $\omega \leqslant \gamma < \beta$ and γ regular,

$$\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha} \cup \mathscr{DCH}_{\nearrow\gamma}; \Omega_{I,\nearrow\alpha}) \prec \mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha} \cup \{\mathscr{DCH}_\gamma\}; \Omega_{I,\nearrow\alpha}).$$

The proofs of the equivalences in (ii) and (iii) are entirely routine, depending only on the grouping of terms in the relevant conjunctions and disjunctions. They are omitted. The other statements follow from Theorems A and B.

For $\beta = \alpha$ inaccessible it follows that calculi

$$\mathfrak{P}_{\alpha\alpha}(\Pi_{\nearrow\alpha} \cup \mathscr{DCH}_\gamma \cup \mathscr{CH}_{\nearrow\alpha})$$

are incomplete whenever $\gamma < \alpha$. The calculus $\mathfrak{P}_{\alpha\alpha}(\Pi_{\nearrow\alpha} \cup \mathscr{DCH}_{\nearrow\alpha})$ is complete when α strongly inaccessible, but the question of the completeness is open for α inaccessible but not strongly inaccessible. For α an infinite non-limit cardinal not only is the calculus $\mathfrak{P}_{\alpha\alpha}(\Pi_{\nearrow\alpha} \cup \mathscr{DCH}_{\nearrow\alpha})$ incomplete but we will see in Chapter 14 that there is no definable calculus in this case.

12.2 Boolean Algebraic Methods

We have obtained our non-deducibility results by means of a semantic interpretation of the languages with a restricted set S of assignments. It may very well be that this method will become obsolete as we learn more about the independence of Boolean algebraic equations in complete Boolean algebras. As an example we show that the unprovability of CH_ω in basic calculus follows from the independence of the ω-distributive law in complete Boolean algebras. If we knew an example of a complete $\nearrow\alpha$-representable, not (ω, α)-distributive Boolean algebra for $\alpha > \omega_1$, then we could show CH_ω not provable in $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha})(\mathbf{L}_{\alpha\beta})$ in exactly the same way. As it is, we must use the method of general models to prove this.

12.2.1 *Interpreting* (α, β, o, π)-*predicate Languages in Complete*

Boolean Algebras. When an (α, β, o, π)-predicate language in interpreted in a complete Boolean algebra $\mathfrak{B} = \langle B \neg \wedge \vee \rangle$, we understand that the system for interpretation is one of form $\langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ B\ \mathsf{O}'\ \mathsf{Q}\ S \rangle$ where B is the underlying set of the algebra, S is the set of all functions on variables to D, $\mathsf{O}'(\boldsymbol{\neg}) = \neg$, $\mathsf{O}'(\boldsymbol{\rightarrow}) = \rightarrow$, $\mathsf{O}'(\boldsymbol{\wedge}) = \wedge$, the operations being taken in $\mathfrak{B}$. Moreover, operations $\mathsf{Q}(\boldsymbol{\forall}, v)$ are all meets in $\mathfrak{B}$ and it is assumed that the assignments O, C, R are made in such a way that instances of schemes $\mathscr{E}\mathscr{Q}$ in $\mathbf{L}$ take value $\mathbf{1}$ for all assignments of variables to D. One way to guarantee the equality property is to let $\mathsf{R}(\overline{=})(d_0, d_1) = \mathbf{1}$ if $d_0 = d_1$, $\mathbf{0}$ if not. But there are other possible assignments to $\overline{=}$ as well. The valuation rules are given in 8.2.2. The interpretive system depends only on the algebra $\mathfrak{B}$, domain D and the assignments O, C, R to constants. We will refer to the resulting system as an *algebraic model* $\langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ \mathfrak{B} \rangle$. Formula A *holds* in such a model if $V(s, A) = \mathbf{1}$ for all assignments s. A set Γ is satisfiable in such a model if there is an assignment s such that $V(s, C) = \mathbf{1}$ for all $C \in \Gamma$. According to 9.1.5 and 9.2.10,

$V(s, A) = V(s', A)$ if s, s' agree on $FV(A)$.

$V(s, SF_f^X A) = V(\mathrm{Repl}_{s^* \circ f}^X s, A)$ provided A has no free occurrence of x bound by $FV(f(x))$ for $x \in X$.

The following theorem is immediate.

12.2.2 *Theorem.* Let $\mathfrak{M} = \langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ \mathfrak{B} \rangle$ be an algebraic model for (α, β, o, π)-language $\mathbf{L}$. Then

(i) If A is an instance of an axiom scheme of $\mathfrak{P}_{\alpha\beta}$ then A holds in $\mathfrak{M}$.

(ii) Modus ponens, conjunction and generalization preserve the property of holding in $\mathfrak{M}$.

(iii) If Σ is a set of α-propositional schemes and if equations $T_{\mathscr{A}} = \mathbf{1}$ hold in $\mathfrak{B}$ for all $\mathscr{A} \in \Sigma$, then any instance of a scheme of Σ in $\mathbf{L}$ holds in $\mathfrak{M}$.

12.2.3 *Theorem.* Let α be a regular infinite cardinal, suppose $\omega < \beta \leqslant \alpha$. Let $\mathbf{L}_{\alpha\beta}$ be the (α, β)-predicate language with one-place predicate symbols Q_n, $n < \omega$. Then CH_ω is not provable in $\mathfrak{P}_{\alpha\beta}(\mathbf{L}_{\alpha\beta})$.

PROOF: Let $\mathfrak{B}$ be a complete Boolean algebra that is not (ω, ω_1)-distributive. The examples of Sect. 7.2 will suffice. It has elements

$b_{n\nu}$, $n < \omega$, $\nu < \omega_1$, such that $c \nleq d$, where $c = \bigwedge_{n<\omega} \bigvee_{\nu<\omega_1} b_{n\nu}$ and $d = \bigvee \{ \bigwedge_{n<\omega} b_{nf(n)} : f \in \omega_1^\omega\}$. As model for $\mathbf{L}_{\alpha\beta}$ take $\langle \omega_1 \ \mathsf{O} \ \mathsf{C} \ \mathsf{R} \ \mathfrak{B}\rangle$ where $\mathsf{R}(Q_n)(\nu) = b_{n\nu}$. Computing the value of CH_ω, we obtain for any assignment s to variables,

$V(s, [\bigwedge_{n<\omega} [\exists x Q_n x]]) = \bigwedge_{n<\omega} \bigvee \{V(\mathrm{Repl}_\nu^x s, Q_n x) : \nu < \omega_1\} = c$, while

if $X = \{x_n : n < w\}$ then $V(s, [\exists x_0 \ldots x_n \ldots [\bigwedge_{n<\omega} Q_n x_n]]) =$

$\bigvee \{V(\mathrm{Repl}_t^X s, [\bigwedge_{n<\omega} Q_n x_n]) : t \in \omega_1^X\} = d.$

Hence the value of CH_ω is $c \to d = \neg c \vee d \neq \mathbf{1}$.

According to 12.2.2, all formal theorems of $\mathfrak{P}_{\alpha\beta}(\mathbf{L}_{\alpha\beta})$ hold in every algebraic model. Hence CH_ω is not provable.

The idea of interpreting the quantifiers algebraically is due to Mostowski who used it to prove non-deducibility in non-classical predicate logics in [23]. These interpretations have been explored further in work of Rasiowa, Sikorski, Henkin. See [29], [31], [10]. Their methods apply also to infinitary predicate logic. We can, for example, prove that a set of formulas is consistent with respect to $\mathfrak{P}_{\alpha\beta}(\mathbf{L}_{\alpha\beta})$ if and only if it is satisfiable in a Boolean algebraic model, but only if we extend the interpretation to apply to $\nearrow \alpha$-complete Boolean algebras that are not necessarily complete. A system $\langle D \ \mathsf{O} \ \mathsf{C} \ \mathsf{R} \ \mathfrak{B}\rangle$ must be admitted as an algebraic model whenever $\mathfrak{B}$ is an $\nearrow \alpha$-complete Boolean algebra and a valuation function exists. That is to say that all the meets

$$V(s, [\forall v A]) = \bigwedge \{V(\mathrm{Repl}_t^{\mathrm{Rng}(v)} s, A) : t \in D^{\mathrm{Rng}(v)}\}$$

must exist in $\mathfrak{B}$. The theorem is a routine extension of the corresponding theorem for ordinary predicate languages that appears in Rasiowa-Sikorski [31], Henkin [10], and its proof is not included here.

12.3 General Models

Let $\mathbf{L}$ be an (α, β, o, π)-predicate language. A *general model* for $\mathbf{L}$ is a system $\langle D \ \mathsf{O} \ \mathsf{C} \ \mathsf{R} \ B_0 \ \mathsf{O}' \ \mathsf{Q} \ S\rangle$ for interpretation of formulas of $\mathbf{L}$ where B_0 is the set of truth values $\mathbf{0}$, $\mathbf{1}$, $\mathsf{O}'(\neg) = \neg$, $\mathsf{O}'(\to) = \to$,

$\mathsf{O}'(\boldsymbol{\wedge}) = \wedge$, the $\mathsf{Q}(\boldsymbol{\forall}, v)$ are meets, just as for ordinary models, but where S is allowed to be any set of functions on variables to D having substitution properties:

(*) If s, $s' \in S$ and Y is a set of variables having power less than β, then $\mathrm{Repl}_t^Y s \in S$ implies $\mathrm{Repl}_t^Y s' \in S$.

(**) If $s \in S$ and Y is a set of variables having power less than β, and if $t \in (\mathrm{Rng}\ s^*)^Y$, then $\mathrm{Repl}_t^Y s \in S$.

The set S is called the set of assignments. We assume moreover that $\mathsf{R}(\overline{=})$ is the equality relation.

Since general models depend only on D, O, C, R, S, they will be referred to as systems $\mathfrak{M} = \langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\ S\rangle$. The valuation function for $\mathfrak{M}$ is described in 8.2.2. Its behavior is the same as that of the valuation function for an ordinary model except that for quantifiers we now have the following:

$V(s, [\boldsymbol{\forall} vA]) = \mathbf{1}$ if and only if $V(\mathrm{Repl}_t^{\mathrm{Rng}(v)} s, A) = \mathbf{1}$ for all $t \in D^{\mathrm{Rng}(v)}$ such that $\mathrm{Repl}_t^{\mathrm{Rng}(v)} s \in S$.

A formula A is *satisfiable in* $\mathfrak{M}$ if there exists $s \in S$ such that $V(s, A) = \mathbf{1}$. It *holds in* $\mathfrak{M}$ if $V(s, A) = \mathbf{1}$ for all $s \in S$.

12.3.1 *Example.* Formula

$$CH_\omega = [\bigwedge_{n<\omega} [\boldsymbol{\exists} x Q_n x]] \rightarrow [\boldsymbol{\exists}_{n<\omega} x_0 \ldots x_n \ldots [\bigwedge_{n<\omega} Q_n x_n]]$$

holds in all ordinary models, but not in all general models. For example, consider $\mathfrak{M} = \langle \omega\ \mathsf{R}\ S\rangle$ where $\mathsf{R}(Q_n)(i) = \mathbf{1}$ if $n = i$, $\mathbf{0}$ if not, and S is the set of all functions on variables to ω having finite range. Then $V(s, [\boldsymbol{\exists} x Q_n x]) = \mathbf{1}$ for all $n < \omega$, but

$$V(s, [\boldsymbol{\exists}_{n<\omega} x_0 \ldots x_n \ldots [\bigwedge_{n<\omega} Q_n x_n]]) = \mathbf{0}$$

since no assignment in S gives value n to each x_n, $n < \omega$. Hence CH_ω is false in $\mathfrak{M}$ for all assignments s.

Since general models have the substitution property of 9.3.6, the basic valuation theorems of Chapter 9 hold for them.

12.3.2 *Lemma.* If s, $s' \in S$ agree on $FV(A)$, then

$$V(s, A) = V(s', A).$$

12.3.3 *Lemma.* If $s \in S$ and $\mathrm{card}(X) < \beta$, then $V(s, SF_f^X A) = V(\mathrm{Repl}_{s^* \circ f}^X s, A)$ provided that A has no free occurrence of x bound by $FV(f(x))$, $x \in X$. Moreover, if X is any set of individual symbols and $s' \in S$ agrees with $\mathrm{Repl}^{X}_{s^* \bigcirc f} s$ on $FV(A)$, then $V(s, SF^X_f A) = V(s', A)$ under the same proviso.

12.3.4 *Lemma.* If $s \in S$ and sequence v has length less than β, then $V(s, A(f \bigcirc v)) = V(\mathrm{Repl}^{\mathrm{Rng}(v)}_{s^* \bigcirc f} s, A)$. Moreover, if v is any sequence of individual symbols and $s' \in S$ agrees with $\mathrm{Repl}^{\mathrm{Rng}(v)}_{s^* \bigcirc f} s$ on $FV(A)$, then $V(s, A(f \bigcirc v)) = V(s', A)$.

12.3.5 *Theorem.* Let $\mathbf{L}$ be an (α, β, o, π)-predicate language. Then instances in $\mathbf{L}$ of valid α-propositional schemes and formulas of forms $\mathscr{Q}$, $\mathscr{E}\mathscr{Q}$, all hold in all general models. The rules of modus ponens, conjunction, generalization preserve the property of holding in a given general model.

PROOF: If $\mathscr{A}$ is an α-propositional scheme, $\mathfrak{M}$ a general model, $s \in S$, then for any instance $S^W_f \mathscr{A}$ of $\mathscr{A}$ in $\mathbf{L}$, $s'(A_\xi) = V(s, f(A_\xi))$ is an assignment of truth values to the symbols A_ξ of $\mathscr{A}$. An easy induction shows $s'^* \mathscr{A} = V(s, S^W_f \mathscr{A})$. Hence if $\mathscr{A}$ is valid, $S^W_f \mathscr{A}$ holds in $\mathfrak{M}$.

Formulas of form $\mathscr{Q}1$ hold in $\mathfrak{M}$ by 12.3.2, formulas of form $\mathscr{Q}2$ hold in $\mathfrak{M}$ by 12.3.3 and condition (**) for S. There is no problem in showing formulas of form $\mathscr{E}\mathscr{Q}$ hold in $\mathfrak{M}$ and that the basic rules of inference preserve the property of holding in $\mathfrak{M}$.

We now have the first part of Theorem A.

12.3.6 *Corollary.* Let $\mathbf{L}_{\alpha\beta}$ be the (α, β)-predicate language with one-place symbols $Q_n, n < \omega$. Then CH_ω is not provable in $\mathfrak{P}_{\alpha\beta}(\Pi_{\not\gamma}\,\alpha)(\mathbf{L}_{\alpha\beta})$.

PROOF: By 12.3.5, 12.3.1.

The more general statement of Theorem A is almost as easy to prove.

12.3.7 *Lemma.* Let $\mathfrak{M}$ be a general model for (α, β, o, π)-predicate language $\mathbf{L}$ satisfying

(*)$_\gamma$ If $s \in S$, and $\mathrm{Repl}^{X_\xi}_{t_\xi} s \in S$ for all $\xi < \gamma$, where the X_ξ are pairwise disjoint and have power less than β, then $\mathrm{Repl}^{X}_{t} s \in S$, where $X = \bigcup \{X_\xi : \xi < \gamma\}$ and $t = \bigcup \{t_\xi : \xi < \gamma\}$.

Then instances of $\mathscr{D}\mathscr{C}\mathscr{H}_\gamma$ in $\mathbf{L}$ hold in $\mathfrak{M}$.

PROOF: The proof is nearly the same as that of Lemma 10.3.7. Let

$$A = [\bigwedge_{\xi<\gamma} [\exists v_0 A_0] \ldots [\bigvee_{\nu<\xi} v_0 \ldots v_\nu \ldots [\exists v_\xi A_\xi]] \ldots] \rightarrow [\bigexists_{\xi<\gamma} v_0 \ldots v_\xi \ldots [\bigwedge_{\xi<\gamma} A_0 \ldots A_\xi \ldots]]$$

where the $\mathrm{Rng}(v_\xi)$ have pairwise disjoint ranges and no variable of

$\mathrm{Rng}(v_\xi)$ is free in A_ν for $\nu < \xi$. Suppose $s \in S$ satisfies the antecedent. We define by transfinite induction functions t_ξ on $\mathrm{Rng}(v_\xi)$ to D such that $\mathrm{Repl}_{t_\xi}^{\mathrm{Rng}(v_\xi)} s \in S$, and if s_ξ is that assignment, in S by $(*)_\gamma$, that is t_ν on $\mathrm{Rng}(v_\nu)$ for $\nu \leqslant \xi$ and s elsewhere, then $V(s_\xi, A_\xi) = \mathbf{1}$. Supposing that we already have such t_ν for all $\nu < \xi$, let s'_ξ be t_ν on $\mathrm{Rng}(v_\nu)$ for $\nu < \xi$ and s elsewhere. Then $s'_\xi \in S$ by $(*)_\gamma$. If $V(s'_\xi, A_\xi) = \mathbf{1}$ let t_ξ be s on $\mathrm{Rng}(v_\xi)$. Then $V(s_\xi, A_\xi) = \mathbf{1}$. If $V(s'_\xi, A_\xi) = \mathbf{0}$, then since $V(s, [\forall v_0 \ldots v_\nu \ldots [\exists v_\xi A_\xi]]) = \mathbf{1}$, $V(s'_\xi, [\exists v_\xi A_\xi]) = \mathbf{1}$. We can choose t_ξ on $\mathrm{Rng}(v_\xi)$ to D such that $V(s_\xi, A_\xi) = \mathbf{1}$, where $s_\xi = \mathrm{Repl}_{t_\xi}^{\mathrm{Rng}(v_\xi)} s'_\xi$.

We also have $\mathrm{Repl}_{t_\xi}^{\mathrm{Rng}(v_\xi)} s \in S$ by substitution property $(*)$.

Finally, consider s_γ, that assignment, in S by $(*)_\gamma$, which is t_ξ on $\mathrm{Rng}(v_\xi)$ for $\xi < \gamma$ and s elsewhere. Since s_γ and s_ξ agree on $FV(A_\xi)$ whenever $\xi < \gamma$, $V(s_\gamma, A_\xi) = \mathbf{1}$ by 12.3.2.

12.3.8 *Theorem* A. Suppose γ regular, $\omega \leqslant \gamma < \beta$, and that language $\mathbf{L}_{\alpha\beta}$ has one-place predicate symbols Q_ξ, $\xi < \gamma$. Then CH_γ is not provable in $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha} \cup \{\mathscr{DCH}_{\nearrow\gamma}\})(\mathbf{L}_{\alpha\beta})$.

PROOF: Consider general model $\mathfrak{M} = \langle \gamma\ \mathsf{R}\ S\rangle$ where $\mathsf{R}(Q_\xi)(\nu) = \mathbf{1}$ if $\xi = \nu$, $\mathbf{0}$ if not, and S is the set of all functions on variables to γ whose range has power less than γ. The regularity of γ implies that this is a general model satisfying condition $(*)_\kappa$ of 12.3.7 for all $\kappa < \gamma$. By 12.3.7 and 12.3.5, all formal theorems of

$$\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha} \cup \{\mathscr{DCH}_{\nearrow\gamma}\})(\mathbf{L}_{\alpha\beta})$$

hold in $\mathfrak{M}$. But CH_γ is obviously false in $\mathfrak{M}$.

12.4 Non-deducibility of the Axiom of Dependent Choices in Systems with Distributive Laws and Rules of Independent Choices

It is convenient to prove the non-deducibility of DCH_ω in $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha}; \Omega_{I,\nearrow\alpha})$ first. The construction can be generalized to prove Theorem B for regular $\gamma < \beta$.

For a subset $\Theta \subseteq \alpha$ let $\mathscr{E}(\Theta) = \bigcup \{\Theta^n : n < \omega\}$, the set of all finite sequences of elements of Θ. Let $\leqslant$ be the partial ordering discussed in Chapter 2: $E' \leqslant E$ if and only if E' is an initial part of E. For $E \in \mathscr{E}(\alpha)$ let $I(E) = \{E' : E' \leqslant E\}$. Then $I(E)$ is finite and totally ordered by $\leqslant$. For $\mathscr{D} \subseteq \mathscr{E}(\alpha)$ let $I(\mathscr{D}) = \bigcup \{I(E) : E \in \mathscr{D}\}$. Then $I(\bigcup_\xi \mathscr{D}_\xi) = \bigcup_\xi I(\mathscr{D}_\xi)$.

Let $\mathbf{L}_{\alpha\beta}$ be any (α, β)-language with $\beta > \omega$ having only two-place predicate symbol Q as non-logical constant. As a general model for $\mathbf{L}_{\alpha\beta}$ consider $\mathfrak{M} = \langle \mathscr{E}(\alpha) \prec S \rangle$ where the ordering $\prec$ is assigned to Q and S is the set of all functions s on variables to $\mathscr{E}(\alpha)$ such that $I(\mathrm{Rng}(s))$ contains no infinite chains [1]. If $s, s' \in S$ and $\mathrm{Repl}_t^Y s \in S$ then $I(\mathrm{Rng}(\mathrm{Repl}_t^Y s')) \subseteq I(\mathrm{Rng}(t|Y) \cup \mathrm{Rng}(s')) = I(\mathrm{Rng}(t|Y)) \cup I(\mathrm{Rng}(s'))$ contains no infinite chains since $I(\mathrm{Rng}(t|Y))$ and $I(\mathrm{Rng}(s'))$ do not. Hence condition (*) holds for S. If $s \in S$ and $t \in \mathrm{Rng}(s)^Y$ then $I(\mathrm{Rng}(\mathrm{Repl}_t^Y s)) \subseteq I(\mathrm{Rng}(s))$ contains no infinite chains. Hence also (**) holds for S and $\mathfrak{M}$ is a general model.

12.4.1 *Lemma*. The formula

$$DCH_\omega = [\forall x \exists y Qxy] \rightarrow [\underset{n<\omega}{\exists} x_0 \ldots x_n \ldots [\underset{n<\omega}{\bigwedge} Qx_n x_{n+1}]]$$

is false in $\mathfrak{M} = \langle \mathscr{E}(\alpha) \prec S \rangle$.

PROOF: For $s \in S$ and $\mathrm{Repl}_E^x s \in S$, $\mathrm{Repl}_{E'}^y (\mathrm{Repl}_E^x s) \in S$ and makes $[Qxy]$ true, where $E' = E^\frown\langle 0 \rangle$. Therefore the antecedent is true. But no assignment in S makes $[\underset{n<\omega}{\bigwedge} Qx_n x_{n+1}]$ true. So DCH_ω is false.

The basic axioms and instances of schemes in $\Pi_{\nearrow\alpha}$ all hold in $\mathfrak{M}$ by 12.3.5 and the rules of basic calculus preserve this property. It remains only to show that rules of δ independent choices also preserve the property of holding in $\mathfrak{M}$ whenever $\delta < \alpha$. The proof goes as follows: We call an isomorphism h on a subsystem of $\mathscr{E}(\alpha)$ to $\mathscr{E}(\alpha)$ *distance-preserving* if $\mathrm{Dom}(E) = \mathrm{Dom}(h(E))$ for all $E \in \mathrm{Dom}(h)$. We show that the value of a formula A for assignment s is invariant under distance-preserving isomorphisms on $\mathscr{E}(\alpha)$. Then if a formula of form

$$[\underset{\xi<\delta}{\bigvee} [\forall v_0 A_0] \ldots [\forall v_\xi A_\xi] \ldots],$$

$$\text{with } \mathrm{Rng}(v_\xi) \cap FV(A_\nu) = \mathrm{Rng}(v_\xi) \cap \mathrm{Rng}(v_\nu) = \phi \text{ for } \xi \neq \nu,$$

takes value $\mathbf{0}$ for $s \in S$, functions t_ξ on $\mathrm{Rng}(v_\xi)$ such that $\mathrm{Repl}_{t_\xi}^{\mathrm{Rng}(v_\xi)} s$ are in S and give A_ξ value $\mathbf{0}$ can be transformed by distance-preserving isomorphisms to functions t'_ξ in such a way that necessary elements of $\mathrm{Rng}(s)$ remain fixed and $I(\bigcup \{\mathrm{Rng}(t'_\xi) : \xi < \delta\})$ contains no infinite chains. Then $\mathrm{Repl}_{t'}^{\mathrm{Rng}(v)} s \in S$ and gives $[\underset{\xi<\delta}{\bigvee} A_0 \ldots A_\xi \ldots]$ value $\mathbf{0}$, where $v = {}^\frown \langle v_\xi : \xi < \delta \rangle$ and $t' = \bigcup \{t'_\xi : \xi < \delta\}$.

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[1] In my paper [15] it was only stipulated that $\mathrm{Rng}(s)$ contain no infinite chains. This was an error. The proof of Lemma 12.4.6 was incorrect.

12.4.2 *Lemma.* If h is a distance-preserving isomorphism on a subsystem of $\langle \mathscr{E}(\alpha) \leqslant \rangle$ such that $I(E) \subseteq \mathrm{Dom}(h)$, then $h(E|m) = h(E)|m$ for all $m \leqslant \mathrm{Dom}(E)$. Therefore if $\mathrm{Dom}(h) = \mathscr{E}(\alpha)$, $E \in \mathrm{Rng}(h)$ implies $I(E) \subseteq \mathrm{Rng}(h)$. Also $I(\mathscr{D}) \subseteq \mathrm{Dom}(h)$ implies

$$I(\mathrm{Rng}(h|\mathscr{D})) = \mathrm{Rng}(h|I(\mathscr{D})).$$

PROOF: Since $h(E)$ has only one initial part of length m and $h(E|m) \leqslant h(E)$ and $m = \mathrm{Dom}(h(E|m))$, it follows that $h(E|m) = h(E)|m$.

12.4.3 *Lemma.* If h is a distance-preserving isomorphism on $\langle \mathscr{E}(\alpha) \leqslant \rangle$, then $s \in S$ implies $h \bigcirc s \in S$.

PROOF: By 12.4.2,

$$I(\mathrm{Rng}(h \bigcirc s)) = I(\mathrm{Rng}(h|\mathrm{Rng}(s))) = \mathrm{Rng}(h|I(\mathrm{Rng}(s)))$$

contains no infinite chains because $I(\mathrm{Rng}(s))$ does not.

12.4.4 *Lemma.* Suppose $\mathscr{D} \subseteq \mathscr{E}(\alpha)$ and that h is a distance-preserving isomorphism on a subsystem of $\langle \mathscr{E}(\alpha) \leqslant \rangle$ such that $I(\mathscr{D}) \subseteq \mathrm{Dom}(h)$ and there is a set $\Theta \subseteq \alpha$ of power α containing no coordinates of elements of $\mathrm{Rng}(h|\mathscr{D})$. For each $E \in \mathscr{E}(\alpha)$ let $r(E) = \bigcup (I(E) \cap I(\mathscr{D}))$ (that is, the largest initial part of E in $I(\mathscr{D})$). Then for any one-one function f on α to Θ the function h_f with values

$$h_f(E) = h(r(E))^\frown f \bigcirc (E - r(E))$$

is a distance-preserving isomorphism on $\mathscr{E}(\alpha)$ such that $h_f|I(\mathscr{D}) = h|I(\mathscr{D})$. Moreover, any sequence in $\mathrm{Rng}(h_f)$ has a unique representation in the form $H^\frown F$ where $H \in \mathrm{Rng}(h|I(\mathscr{D}))$ and $F \in \mathscr{E}(\Theta)$. For two such sequences $H^\frown F$ and $H'^\frown F'$ in $\mathrm{Rng}(h_f)$, $H^\frown F \leqslant H'^\frown F'$ implies $H \prec H'$ and $F = \phi$ or $H = H'$ and $F \leqslant F'$.

PROOF: If H, $H' \in \mathrm{Rng}(h|I(\mathscr{D}))$ and F, $F' \in \mathscr{E}(\Theta)$ and $H^\frown F \leqslant H'^\frown F'$ then if $H = H'$, $F \leqslant F'$ after left-cancellation. If $H \prec H'$ then $F \neq \phi$ is excluded since $F(0)$ would be a coordinate of H'. Hence $H \prec H'$ implies $F = \phi$. If $H' \prec H$ then $F'(0)$ would be a coordinate of H, also excluded. Since one of H, H' is an initial part of the other, these cases are the only possible ones. It follows that representations of elements of $\mathrm{Rng}(h_f)$ in form $H^\frown F$ are unique.

$\mathrm{Dom}(h_f(E)) = \mathrm{Dom}(h(r(E))) + \mathrm{Dom}(f \bigcirc (E - r(E))) = \mathrm{Dom}(r(E)) + \mathrm{Dom}(E - r(E)) = \mathrm{Dom}(E)$ and therefore h_f is distance-preserving. Suppose $E_0 \leqslant E_1$. Then if $E_0 \in I(\mathscr{D})$, $E_0 \leqslant r(E_1)$, so

$h_f(E_0) = h(E_0) \leqslant hr(E_1) \leqslant h_f(E_1)$. If $E_0 \notin I(\mathscr{D})$, then $r(E_0) = r(E_1)$ and $E_0 - r(E_0) \leqslant E_1 - r(E_1)$ by left cancellation. Therefore $h_f(E_0) = hr(E_0)^\frown f \bigcirc (E_0 - r(E_0)) \leqslant hr(E_1)^\frown f \bigcirc (E_1 - r(E_1)) = h_f(E_1)$. Conversely, suppose $h_f(E_0) \leqslant h_f(E_1)$. Then as we have just seen, it follows that $hr(E_0) = hr(E_1)$ and $f \bigcirc (E_0 - r(E_0)) \leqslant f \bigcirc (E_1 - r(E_1))$ or $hr(E_0) \prec hr(E_1)$ and $f \bigcirc (E_0 - r(E_0)) = \phi$. In the first case, $r(E_0) = r(E_1)$ and $E_0 - r(E_0) \leqslant E_1 - r(E_1)$, so that $E_0 \leqslant E_1$. In the second case, $r(E_0) \prec r(E_1)$ and $E_0 = r(E_0)$. In this case too, $E_0 \leqslant E_1$. Therefore h_f is an isomorphism.

12.4.5 *Lemma.* If h is a distance-preserving isomorphism on $\mathscr{E}(\alpha)$ then $V(s, A) = V(h \bigcirc s, A)$ for all $s \in S$, formulas A of $\mathbf{L}_{\alpha\beta}$.

Proof: Let $\Delta = \{A : V(s, A) = V(h \bigcirc s, A)$ for all $s \in S$ and distance-preserving isomorphisms $h\}$. Atomic formulas have form $x \equiv y$ or Qxy and are obviously in Δ. It is clear that Δ is closed under formation of negations, implications and conjunctions. Suppose $A = [\boldsymbol{\forall} v A_0]$, $A_0 \in \Delta$. If $V(s, A) = \mathbf{0}$ there is a function t on $\mathrm{Rng}(v)$ to $\mathscr{E}(\alpha)$ such that $s_{v,t} = \mathrm{Repl}_t^{\mathrm{Rng}(v)} s$ is in S and $V(s_{v,t}, A_0) = \mathbf{0}$. Then $V(h \bigcirc s_{v,t}, A_0) = \mathbf{0}$ by induction hypothesis. Since $h \bigcirc s_{v,t} = \mathrm{Repl}_{h \bigcirc t}^{\mathrm{Rng}(v)} h \bigcirc s$, $V(h \bigcirc s, [\boldsymbol{\forall} v A_0]) = \mathbf{0}$.

Conversely, suppose $V(h \bigcirc s, [\boldsymbol{\forall} v A_0]) = \mathbf{0}$. Then there is t on $\mathrm{Rng}(v)$ to $\mathscr{E}(\alpha)$ such that $\mathrm{Repl}_t^{\mathrm{Rng}(v)}(h \bigcirc s) \in S$ and $V(\mathrm{Repl}_t^{\mathrm{Rng}(v)}(h \bigcirc s), A_0) = \mathbf{0}$. Let $\mathscr{D} = \mathrm{Rng}((h \bigcirc s)|FV(A))$. By 12.4.2, $I(\mathscr{D}) \subseteq \mathrm{Rng}(h) = \mathrm{Dom}(h^{-1})$. Since $\mathrm{Rng}(h^{-1}|\mathscr{D}) = \mathrm{Rng}(s|FV(A))$ has power less than α, there is a subset of α having power α containing no coordinates of elements of $\mathrm{Rng}(h^{-1}|\mathscr{D})$. Therefore we can apply 12.4.4 with h^{-1} as the given isomorphism. There is then a distance-preserving isomorphism k on $\mathscr{E}(\alpha)$ agreeing with h^{-1} on $I(\mathscr{D})$. By induction hypothesis, $V(k \bigcirc \mathrm{Repl}_t^{\mathrm{Rng}(v)}(h \bigcirc s), A_0) = \mathbf{0}$. But $\mathrm{Repl}_{k \bigcirc t}^{\mathrm{Rng}(v)} s$ is in S and agrees with this assignment on $FV(A_0)$. By 12.3.2,

$$V(\mathrm{Repl}_{k \bigcirc t}^{\mathrm{Rng}(v)} s, A_0) = \mathbf{0}$$

and therefore $V(s, [\boldsymbol{\forall} v A_0]) = \mathbf{0}$. This completes the proof.

12.4.6 *Lemma.* The rules of δ independent choices, $\delta < \alpha$, preserve the property of holding in $\mathfrak{M} = \langle \mathscr{E}(\alpha) \prec S \rangle$.

Proof: Suppose $s \in S$ and $V(s, A) = \mathbf{0}$, where A has form $[\bigvee_{\xi<\delta} [\boldsymbol{\forall} v_0 A_0] \ldots [\boldsymbol{\forall} v_\xi A_\xi] \ldots]$, where $\mathrm{Rng}(v_\xi) \cap \mathrm{Rng}(v_\nu) = \phi$ and $\mathrm{Rng}(v_\xi) \cap FV(A_\nu) = \phi$ for $\xi \neq \nu$. We must show that there is $s' \in S$ such that $V(s', [\bigvee_{\xi<\delta} A_0 \ldots A_\xi \ldots]) = \mathbf{0}$. For each $\xi < \delta$ let t_ξ be a function on $\mathrm{Rng}(v_\xi)$ to $\mathscr{E}(\alpha)$ such that $s_{v_\xi,t_\xi} = \mathrm{Repl}_{t_\xi}^{\mathrm{Rng}(v_\xi)} s$ is in S and $V(s_{v_\xi,t_\xi}, A_\xi) = \mathbf{0}$. Let $\mathscr{D} = \mathrm{Rng}(s|FV(A))$. Let Θ be a subset of α having power α not containing any coordinates of elements of $I(\mathscr{D})$. Partition Θ into sets Θ_ξ, $\xi < \delta$, each of power α, pairwise disjoint. Use 12.4.4 with the identity isomorphism as the given isomorphism to find distance-preserving isomorphisms h_ξ, $\xi < \delta$, such that $h_\xi|I(\mathscr{D})$ is identity on $I(\mathscr{D})$ and every element of $\mathrm{Rng}(h_\xi)$ has a unique representation of the form $H_\xi{}^\frown F_\xi$, $H_\xi \in I(\mathscr{D})$, $F_\xi \in \mathscr{E}(\Theta_\xi)$. Let $s' = \mathrm{Repl}_{t'}^{\mathrm{Rng}(v)} s$, where $v = {}^\frown <v_\xi: \xi < \delta>$ and $t' = \bigcup \{h_\xi \bigcirc t_\xi : \xi < \delta\}$. We claim that $s' \in S$. For

$$I(\mathrm{Rng}(s')) \subseteq \bigcup \{I(\mathrm{Rng}(h_\xi \bigcirc t_\xi) : \xi < \delta\} \cup I(\mathrm{Rng}(s)).$$

Since $s \in S$, $I(\mathrm{Rng}(s))$ contains no infinite chain and since the $s_{v_\xi,t_\xi} \in S$, the $I(\mathrm{Rng}(h_\xi \bigcirc t_\xi))$ contain no infinite chains. Therefore if $I(\mathrm{Rng}(s'))$ did contain an infinite chain there would be distinct $\xi_0, \ldots, \xi_n, \ldots$ and sequences $H_{\xi_n}{}^\frown F_{\xi_n}$ such that

$$H_{\xi_0}{}^\frown F_{\xi_0} \prec \ldots \prec H_{\xi_n}{}^\frown F_{\xi_n} \prec \ldots$$

with the $H_{\xi_n} \in I(\mathscr{D})$, $F_{\xi_n} \in \mathscr{E}(\Theta_{\xi_n})$. But the Θ_{ξ_n} are pairwise disjoint and contain no coordinates of elements of $I(\mathscr{D})$. The argument in 12.4.4 shows that all the $F_{\xi_n} = \phi$. But then $I(\mathscr{D})$ would contain an infinite chain, contradicting condition $s \in S$. Hence $s' \in S$. Since $V(h_\xi \bigcirc s_{v_\xi,t_\xi}, A) = V(s_{v_\xi,t_\xi}, A) = \mathbf{0}$ by 12.4.5 and $h_\xi \bigcirc s_{v_\xi,t_\xi}$ agrees with s' on $FV(A_\xi)$, $V(s', A_\xi) = \mathbf{0}$ for all $\xi < \delta$. Hence

$$V(s', [\bigvee_{\xi<\delta} A_0 \ldots A_\xi \ldots]) = \mathbf{0}.$$

12.4.7 *Theorem.* The formula DCH_ω is not provable in systems $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha}; \Omega_{I,\nearrow\alpha})(\mathbf{L}_{\alpha\beta})$ with $\beta > \omega$ and two-place predicate symbol Q.

PROOF: By 12.4.1, 12.3.5, 12.4.6.

To prove the more general statement of Theorem B, we must show that if γ regular, $\omega \leqslant \gamma < \beta$, then DCH_γ is not provable in $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha} \cup \mathscr{DCH}_{\nearrow\gamma}; \Omega_{I,\nearrow\alpha})(\mathbf{L}_{\alpha\beta})$ where

$$DCH_\gamma = [\forall x_0 \exists x_1 Q x_0 x_1] \wedge [\bigwedge_{1<\xi<\gamma} [\forall_{\nu<\xi} x_0 \ldots x_\nu \ldots \exists x_\xi C_\xi]] \rightarrow$$

$$[\exists_{\xi<\gamma} x_0 \ldots x_\xi \ldots [\bigwedge_{0<\xi<\gamma} \bigwedge_{\nu<\xi} Q x_\nu x_\xi]],$$

and

$$C_\xi = [[\bigwedge_{0<\theta<\xi} \bigwedge_{\nu<\theta} Q x_\nu x_\theta] \rightarrow [\bigwedge_{0<\theta\leqslant\xi} \bigwedge_{\nu<\theta} Q x_\nu x_\theta]].$$

This can be done by modifying the construction just given.

For $\Theta \subseteq \alpha$ let $\mathscr{E}_\gamma(\Theta) = \bigcup \{\Theta^\eta : \eta < \gamma\}$. Then for $E \in \mathscr{E}_\gamma(\alpha)$, $I(E)$ has power less than γ and is well-ordered by $\leqslant$. If $\mathscr{D} \subseteq \mathscr{E}(\alpha)$ is totally ordered by $\leqslant$ and has power less than γ, then $\bigcup \mathscr{D}$ is an element of $\mathscr{E}_\gamma(\alpha)$. Its domain is $\bigcup \{\mathrm{Dom}(E) : E \in \mathscr{D}\}$, less than γ by regularity. As a general model for $\mathbf{L}_{\alpha\beta}$ consider $\mathfrak{M}_\gamma = \langle \mathscr{E}_\gamma(\alpha) \prec S_\gamma \rangle$ where S_γ is the set of all functions s on variables to $\mathscr{E}_\gamma(\alpha)$ such that $I(\mathrm{Rng}(s))$ contains no chains of length γ. An argument similar to the case where $\gamma = \omega$ shows that this is a general model.

12.4.8 *Lemma.* Instances of schemes $\mathscr{DCH}_\kappa$ hold in $\mathfrak{M}_\gamma$ whenever $\kappa < \gamma$. The formula DCH_γ is false in $\mathfrak{M}_\gamma$.

PROOF: We may use 12.3.7 to show $\mathscr{DCH}_\kappa$ holds in $\mathfrak{M}_\gamma$. Suppose $s \in S_\gamma$, and $\mathrm{Repl}_{t_\xi}^{X_\xi} s \in S_\gamma$ for all $\xi < \kappa$, the X_ξ being pairwise disjoint. Then for each ξ, $I(\mathrm{Rng}(\mathrm{Repl}_{t_\xi}^{X_\xi} s)) \subseteq I(\mathrm{Rng}(t_\xi|X_\xi)) \cup I(\mathrm{Rng}(s))$ contains no chain of length γ. The regularity of γ implies that $I(\mathrm{Rng}(\mathrm{Repl}_t^X s)) \subseteq \bigcup \{I(\mathrm{Rng}(t_\xi|X_\xi)) : \xi < \kappa\} \cup I(\mathrm{Rng}(s))$ contains no chain of length γ. Therefore condition $(*)_\kappa$ holds for $\mathfrak{M}_\gamma$.

$[\forall x_0 \exists x_1 Q x_0 x_1]$ obviously holds in $\mathfrak{M}_\gamma$. If $1 < \xi < \gamma$ and $s \in S_\gamma$, $X_\xi = \{x_\nu : \nu < \xi\}$ and $\mathrm{Repl}_t^{X_\xi} s \in S_\gamma$, then if $\mathrm{Repl}_t^{X_\xi} s$ satisfies the antecedent of C_ξ, $t(x_\nu) \prec t(x_\theta)$ for every $\nu < \theta$, $0 < \theta < \xi$. Let $E = \bigcup \{t(x_\theta) : \theta < \xi\}$. Then $E \in \mathscr{E}_\gamma(\alpha)$ and $\mathrm{Repl}_{E'}^{x_\xi} \mathrm{Repl}_t^{X_\xi} s \in S_\gamma$, where $E' = E^\frown\langle 0\rangle$. This assignment satisfies $[\bigwedge_{0<\theta\leqslant\xi} \bigwedge_{\nu<\theta} Q x_\nu x_\theta]$. Therefore the C_ξ hold in $\mathfrak{M}_\gamma$. But $[\bigwedge_{0<\xi<\gamma} \bigwedge_{\nu<\xi} Q x_\nu x_\xi]$ is not satisfied by any assignment in S_γ since the elements assigned to the x_ξ would form a chain of length γ. Hence DCH_γ is false in $\mathfrak{M}_\gamma$.

In view of 12.3.5 to show DCH_γ unprovable in

$$\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha} \cup \mathscr{DCH}_{\nearrow\gamma};\, \Omega_{I,\nearrow\alpha})(\mathbf{L}_{\alpha\beta})$$

it suffices to show that the rules of δ independent choices preserve the property of holding in $\mathfrak{M}_\gamma$. We proceed as before.

12.4.9 *Lemma.* Lemma 12.4.2 and Lemma 12.4.3 hold for $\mathfrak{M}_\gamma$.

PROOF: The proofs go through unchanged.

Lemma 12.4.4 must be changed in passage from $\gamma = \omega$ to arbitrary regular $\gamma < \beta$. The difficulty is that $\bigcup (I(E) \cap I(\mathscr{D}))$ need no longer be an element of $I(\mathscr{D})$. The role of $I(\mathscr{D})$ is now played by the set of unions of chains of fewer than γ elements of $I(\mathscr{D})$.

For $\mathscr{D} \subseteq \mathscr{E}_\gamma(\alpha)$ let $Un_\gamma(\mathscr{D})$ be the set of all unions of chains of fewer than γ elements of $\mathscr{D}$. We have already observed that the regularity of γ implies $Un_\gamma(\mathscr{D}) \subseteq \mathscr{E}_\gamma(\alpha)$. Since $\mathrm{Dom}(\bigcup \{E_\xi : \xi < \gamma'\}) = \bigcup \{\mathrm{Dom}(E_\xi) : \xi < \gamma'\}$, it follows that $E \prec \bigcup \{E_\xi : \xi < \gamma'\}$ implies $E \prec E_\xi$ for some $\xi < \gamma'$. Hence $I(Un_\gamma(\mathscr{D})) \subseteq Un_\gamma(I(\mathscr{D}))$. Note that every coordinate of $\bigcup \{E_\xi : \xi < \gamma'\}$ is a coordinate of some E_ξ.

12.4.10 *Lemma.* If h is a distance-preserving isomorphism on a subsystem of $\mathscr{E}_\gamma(\alpha)$ and $\mathscr{D}$ is a chain in $\mathrm{Dom}(h)$ of length less than γ such that $\bigcup \mathscr{D} \in \mathrm{Dom}(h)$, then $h(\bigcup \mathscr{D}) = \bigcup \{h(E) : E \in \mathscr{D}\}$. Therefore if $\mathrm{Dom}(h) = \mathscr{E}_\gamma(\alpha)$ then $\mathscr{D} \subseteq \mathrm{Rng}(h)$ implies $Un_\gamma(\mathscr{D}) \subseteq \mathrm{Rng}(h)$.

PROOF: Since h preserves order, $h(E) \leqslant h(\bigcup \mathscr{D})$ for all $E \in \mathscr{D}$. Therefore $\bigcup \{h(E) : E \in \mathscr{D}\} \leqslant h(\bigcup \mathscr{D})$ and since $\mathrm{Dom}(h(\bigcup \mathscr{D})) = \mathrm{Dom}(\bigcup \mathscr{D}) = \bigcup \{\mathrm{Dom}(E) : E \in \mathscr{D}\} = \bigcup \{\mathrm{Dom}\, h(E) : E \in \mathscr{D}\} = \mathrm{Dom} \bigcup \{h(E) : E \in \mathscr{D}\}$, the two sequences must be equal.

12.4.11 *Lemma.* Suppose h is a distance-preserving isomorphism on a subsystem of $\langle \mathscr{E}_\gamma(\alpha) \leqslant \rangle$ such that $Un_\gamma I(\mathscr{D}) \subseteq \mathrm{Dom}(h)$ and there is a set $\Theta \subseteq \alpha$ of power α containing no coordinates of elements of $\mathrm{Rng}(h|\mathscr{D})$. For each $E \in \mathscr{E}_\gamma(\alpha)$ let $r_\gamma(E) = \bigcup (I(E) \cap I(\mathscr{D}))$. Then for any one-one function f on α to Θ the function $h_{\gamma, f}$ with values

$$h_{\gamma,f}(E) = h(r_\gamma(E))^\frown f \bigcirc (E - r_\gamma(E))$$

is a distance-preserving isomorphism on $\mathscr{E}_\gamma(\alpha)$ such that

$$h_{\gamma,f}|Un_\gamma I(\mathscr{D}) = h|Un_\gamma I(\mathscr{D}).$$

Moreover any sequence in $\mathrm{Rng}(h_{\gamma,f})$ has a unique representation in the form $H^\frown F$ where $H \in \mathrm{Rng}(h|Un_\gamma I(\mathscr{D}))$ and $F \in \mathscr{E}_\gamma(\Theta)$. For such sequences, $H^\frown F \leqslant H'^\frown F'$ implies $H \prec H'$ and $F = \phi$ or $H = H'$ and $F \leqslant F'$.

PROOF: Every coordinate of an element of $\mathrm{Rng}(h|Un_\gamma I(\mathscr{D}))$ is a coordinate of an element of $\mathrm{Rng}(h|\mathscr{D})$ by 12.4.10. Therefore the first part of the proof of 12.4.4 goes through unchanged. For the second part it is necessary to check that if $E_0 \leqslant E_1$ then $E_0 \in Un_\gamma I(\mathscr{D})$ implies $E_0 \leqslant r_\gamma(E_1)$ and $E_0 \notin Un_\gamma I(\mathscr{D})$ implies $r_\gamma(E_0) = r_\gamma(E_1)$. In the first case E_0 is a union of a subset of $I(E_1) \cap I(\mathscr{D})$ which implies $E_0 \leqslant r_\gamma(E_1)$. In the second case we still have $r_\gamma(E_0) \leqslant r_\gamma(E_1)$, while $r_\gamma(E_0) \prec r_\gamma(E_1)$ would imply $r_\gamma(E_0) \prec E'$ for some $E' \in I(E_1) \cap I(\mathscr{D})$. Then one of E_0, E' is an initial part of the other, but $E_0 \leqslant E'$ would put E_0 into $Un_\gamma I(\mathscr{D})$ while $E' \prec E_0$ would make $E' \leqslant r_\gamma(E_0)$. Therefore we must have $r_\gamma(E_0) = r_\gamma(E_1)$. The rest of the proof of 12.4.4 applies.

12.4.12 *Lemma.* If h is a distance-preserving isomorphism on $\mathscr{E}_\gamma(\alpha)$ then $V(s, A) = V(h \bigcirc s, A)$ for all $s \in S_\gamma$, formulas A of $\mathbf{L}_{\alpha\beta}$.

PROOF: The proof of 12.4.5 still applies using 12.4.11 instead of 12.4.4. Lemma 12.4.10 is used to guarantee $Un_\gamma I(\mathscr{D}) \subseteq \mathrm{Rng}(h) = \mathrm{Dom}(h^{-1})$.

12.4.13 *Lemma.* The rules of δ independent choices, $\delta < \alpha$, preserve the property of holding in $\mathfrak{M}_\gamma$.

PROOF: The proof of Lemma 12.4.6 will apply once we have shown that $Un_\gamma I(\mathscr{D})$ contains no chain of length γ if $I(\mathscr{D})$ does not.

Suppose there were a chain $\langle H_\xi : \xi < \gamma \rangle$ of length γ, $H_\nu \prec H_{\nu'}$ for $\nu < \nu' < \gamma$, the $H_\xi \in Un_\gamma I(\mathscr{D})$. Then each H_ξ has a representation $H_\xi = \bigcup \mathscr{D}_\xi$ as a union of a non-empty totally-ordered subset of fewer than γ elements of $I(\mathscr{D})$. We wish to show that there is then a totally-ordered subset of $I(\mathscr{D})$ having power γ. This can be done by using transfinite induction to attach to each non-limit ordinal $\xi < \gamma$ an element E_ξ of $\mathscr{D}_\xi$ in such a way that $E_\nu \prec E_{\nu'}$ for $\nu < \nu' \leqslant \xi$. Let E_1 be any element of $\mathscr{D}_1$. Having selected $E_\mu \in \mathscr{D}_\mu$ for all non-limit $\mu < \xi$ in such a way that $E_\nu \prec E_{\nu'}$ for $\nu < \nu' \leqslant \mu$, consider H_ξ, ξ non-limit. If $\xi - 1$ is also a non-limit ordinal, then $E_{\xi-1} \leqslant H_{\xi-1} \prec H_\xi = \bigcup \mathscr{D}_\xi$ implies that there is an element of $\mathscr{D}_\xi$ having $E_{\xi-1}$ as a proper initial part. Such an element can be chosen for E_ξ. If $\xi - 1$ is a limit ordinal, then $H_\mu \prec H_{\xi-1}$ for all non-limit $\mu < \xi$. So $E_\mu \prec H_{\xi-1}$ for all non-limit $\mu < \xi$. Then $H = \bigcup \{E_\xi : \mu < \xi\} \leqslant H_{\xi-1} \prec H_\xi$. Therefore $\mathscr{D}_\xi$ has an element with H as a proper initial part. Such an element can be chosen for E_ξ. This complete the induction.

12.4.14 *Theorem* B. Let $\mathbf{L}$ be an (α, β)-predicate language with $\beta > \omega$ having two-place predicate symbol Q. Then DCH_ω is valid and not provable in $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha}; \Omega_{I,\nearrow\alpha})(\mathbf{L})$. For $\omega < \gamma < \beta$, γ regular, DCH_γ is valid and not provable in $\mathfrak{P}_{\alpha\beta}(\Pi_{\nearrow\alpha} \cup \mathscr{DCH}_{\nearrow\gamma}; \Omega_{I,\nearrow\alpha})(\mathbf{L})$.

PROOF: By 12.4.7, 12.3.5, 12.4.8, 12.4.13.

CHAPTER 13

THE DEFINABILITY OF THE INFINITARY FORMAL SYSTEMS

13.1 The Metalanguage for (α, β, o, π)-Formal Systems

The usual restrictions on α, β, o, π are assumed to be in force: α regular, infinite, $\beta = 0$ or $\omega \leqslant \beta \leqslant \alpha$, o regular, infinite, and less than β if β singular, at most α if β regular, $\omega \leqslant \pi \leqslant \alpha$. The infinitary languages $\mathbf{L}_{\alpha\beta o\pi}$ and their calculi can be described by formulas of an ordinary (ω, ω)-predicate language interpreted in a set-theoretic model. The individual variables of the meta-language $\mathbf{ML}^{\alpha}$ for (α, β, o, π)-predicate languages are $x_0, \ldots, x_n, \ldots, n < \omega$. The individual constants are $\overline{lb}$, $\overline{rb}$, $\bar{n}$, , $\bar{c}$, $\bar{q}$, $\overline{eq}$, along with a constant $\bar{\delta}$ for each $\delta < \alpha$. One-place predicate symbols are IV, IC, BOP, IOP, BPR, IPR, special two-place predicate symbols are $\bar{=}$, $\bar{\in}$, two-place predicate symbols are OP, PR. The intended models are $\mathfrak{M}(\mathbf{L}, g) = \langle T_\alpha, \mathbf{C}, \mathbf{R}\rangle$, one for each (α, β, o, π)-language $\mathbf{L}$ and one-one function g (called a *Gödel-numbering*) on the symbols of $\mathbf{L}$ into α, where T_α is the family of sets hereditarily of power less than α and $\mathbf{C}$ and $\mathbf{R}$ take values as follows:

$\bar{0}$	0	$\overline{lb}$	$g(\mathbf{[})$
·	·	$\overline{rb}$	$g(\mathbf{]})$
·	·	$\bar{n}$	$g(\neg)$
$\bar{\delta}$	δ	$\bar{i}$	$g(\rightarrow)$
·	·	$\bar{c}$	$g(\bigwedge)$
·	·	$\bar{q}$	$g(\forall)$
·	·	$\overline{eq}$	$g(\bar{=})$
IV	$\{\langle g(x)\rangle$: x is an individual variable of $\mathbf{L}\}$		
IC	$\{\langle g(a)\rangle$: a is an individual constant of $\mathbf{L}\}$		
BOP	$\{\langle g(\psi)\rangle$: ψ is a special two-place operation symbol of $\mathbf{L}\}$		

IOP	$\{\langle g(\varphi)\rangle$: φ is an infinitary operation symbol of $\mathbf{L}\}$
BPR	$\{\langle g(P)\rangle$: P is a special two-place predicate symbol of $\mathbf{L}\}$
IPR	$\{\langle g(Q)\rangle$: Q is an infinitary predicate symbol of $\mathbf{L}\}$
$\bar{\in}$	The membership relation on T_α
OP	$\{\langle \zeta g(\varphi^\zeta)\rangle$: φ^ζ is a ζ-place operation symbol of $\mathbf{L}\}$
PR	$\{\langle \eta g(Q^\eta)\rangle$: Q^η is an η-place predicate symbol of $\mathbf{L}\}$.

Note that the formula $[BPR\overline{eq}]$ holds in any model $\mathfrak{M}(\mathbf{L}, g)$. If $\mathbf{L}$ is an (α, β, o, π)-language with $o < \alpha$ then $[\forall x_0[\neg OP\bar{\zeta} x_0]]$ holds in $\mathfrak{M}(\mathbf{L}, g)$ for all $o \leqslant \zeta < \alpha$. Also if $\pi < \alpha$, $[\forall x_0[\neg PR\bar{\eta} x_0]]$ holds in $\mathfrak{M}(\mathbf{L}, g)$ for all $\pi \leqslant \eta < \alpha$.

A Gödel-numbering g of symbols of $\mathbf{L}$ is extended over expressions of $\mathbf{L}$ in the natural way: The Gödel-sequence of an expression is the sequence of Gödel-numbers of its symbols. In symbols, $g(E) = g \bigcirc E$. Similarly we can speak of the Gödel-sequence of a sequence of expressions, of a sequence of such sequences and so on. Since α is regular the Gödel-sequence of an expression is an element of T_α and the Gödel-sequence of a sequence of fewer than α expressions is again an element of T_α. In this chapter we show that for any one of the systems $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$ of Sect. 11.1 there is a formula $PRV(x_0)$ such that S satisfies $PRV(x_0)$ in $\mathfrak{M}(\mathbf{L}, g)$ if and only if S is the Gödel sequence (with respect to g) of a formula provable in $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$.

In order to describe these formulas $PRV(x_0)$ is is necessary to introduce new constants into $\mathbf{ML}^\alpha$ by definition. Since we are dealing with ordinary predicate languages an elimination procedure for defined constants is already available. See e.g., Kleene's book [18]. The elimination procedure of Chapter 14 for a particular infinitary language can be modified to apply to all (α, β, o, π)-languages, but would be unnecessarily complicated in the finitary case, for in this case it is enough to know how to eliminate a single occurrence of a single defined symbol at a time. This by no means suffices in the infinitary case.

13.1.1 *Defined constants.* Let $\mathbf{L}_{\omega\omega}$ be an (ω, ω)-language, $\mathfrak{M} = \langle D\ \mathsf{O}\ \mathsf{C}\ \mathsf{R}\rangle$ a model of $\mathbf{L}_{\omega\omega}$. Then the elimination procedure for defined constants has the following properties:

(i) A new n-place predicate symbol Q^n can be introduced by a definition of the form

$$[\forall x_0 \ldots x_{n-1}[Q^n x_0 \ldots x_{n-1} \leftrightarrow C(x_0, \ldots, x_{n-1})]]$$

where $C(x_0, \ldots, x_{n-1})$ is a formula of $\mathbf{L}_{\omega\omega}$ and $x_0, \ldots, x_{n-1}$ are its only free variables. If $\mathbf{L}'_{\omega\omega}$ is the language like $\mathbf{L}_{\omega\omega}$ except that it has the additional constant Q^n, if $R_C = \{\langle d_0 \ldots d_{n-1}\rangle : \langle d_0 \ldots d_{n-1}\rangle$ satisfies $C(x_0, \ldots, x_{n-1})$ in $\mathfrak{M}\}$ and if $\mathfrak{M}' = \langle D, \mathsf{O}, \mathsf{C}, \mathsf{R} \cup \{\langle Q^n R_C\rangle\}\rangle$, then $V_{\mathfrak{M}'}(s, A) = V_{\mathfrak{M}}(s, A_e)$ for all assignments s to D, and formulas A of $\mathbf{L}'_{\omega\omega}$, where A_e is the result of eliminating Q^n. The formula $C(x_0, \ldots, x_{n-1})$ is said to define R_C relative to $\mathfrak{M}$.

(ii) A new individual constant $\bar{c}$ can be introduced by a definition of the form

$$[\forall x_0[x_0 \bar{=} \bar{c} \leftrightarrow C(x_0)]]$$

provided that $[\exists x_0 C(x_0)]$ and $[\forall x_0 \forall x_1[[C(x_0) \wedge C(x_1)] \rightarrow x_0 \bar{=} x_1]]$ hold in $\mathfrak{M}$ and only x_0 is free in $C(x_0)$. If $\mathbf{L}'_{\omega\omega}$ is that language like $\mathbf{L}_{\omega\omega}$ except that it has the additional constant $\bar{c}$, if c is the unique element of D satisfying $C(x_0)$ in $\mathfrak{M}$, and if $\mathfrak{M}' = \langle D, \mathsf{O}, \mathsf{C} \cup \{\langle \bar{c} c\rangle\}, \mathsf{R}\rangle$ then $V_{\mathfrak{M}'}(s, A) = V_{\mathfrak{M}}(s, A_e)$ for all formulas A of $\mathbf{L}'_{\omega\omega}$, and assignments s to D, where A_e is the result of eliminating $\bar{c}$ from A. The formula $C(x_0)$ is said to define c relative to $\mathfrak{M}$.

(iii) A new n-place operation symbol φ^n can be introduced by a definition of the form

$$[\forall x_0 \ldots x_{n-1} y[y \bar{=} [\varphi^n x_0 \ldots x_{n-1}] \leftrightarrow C(x_0, \ldots, x_{n-1}, y)]]$$

provided that $[\forall x_0 \ldots x_{n-1}[\exists y C(x_0, \ldots, x_{n-1}, y)]]$ and $[\forall x_0 \ldots x_{n-1} yz[[C(x_0, \ldots, x_{n-1}, y) \wedge C(x_0, \ldots, x_{n-1}, z)] \rightarrow y \bar{=} z]]$ hold in $\mathfrak{M}$ and only $x_0, \ldots, x_{n-1}, y$ are free in $C(x_0, \ldots, x_{n-1}, y)$. If $\mathbf{L}'_{\omega\omega}$ is that language like $\mathbf{L}_{\omega\omega}$ except that it has the additional constant φ^n, if O_C assigns to n-tuple $\langle d_0 \ldots d_{n-1}\rangle$ of elements of D the unique e such that $\langle d_0 \ldots d_{n-1} e\rangle$ satisfies $C(x_0, \ldots, x_{n-1}, y)$, and if $\mathfrak{M}' = \langle D, \mathsf{O} \cup \{\langle \varphi^n O_C\rangle\}, \mathsf{C}, \mathsf{R}\rangle$, then $V_{\mathfrak{M}'}(s, A) = V_{\mathfrak{M}}(s, A_e)$ for all formulas A of $\mathbf{L}'_{\omega\omega}$, assignments s to D, where A_e is the result of eliminating φ^n from A. The formula $C(x_0, \ldots, x_{n-1}, y)$ is said to define the operation O_C relative to $\mathfrak{M}$.

13.2 The Definability of Fundamental Notions of Set Theory

In this section $\mathbf{L}_{\omega\omega}$ is an (ω, ω)-predicate language having only the equality and membership symbols $\bar{=}$, $\bar{\in}$, α is a regular infinite cardinal, $\mathfrak{T}_\alpha = \langle T_\alpha \in\rangle$ the model with domain the set T_α of all sets hereditarily of power less than α, assigning the membership relation on T_α to $\bar{\in}$.

Defining formulas for such notions as set-theoretic inclusion, ordered pair, relation, function, and others, are so easily obtained from their definitions in the Foreward on Set Theory that they are omitted. We will be content merely to introduce the symbols for these notions with their informal definitions using the variables both formally and informally. Note that we must be careful. The power-set function, for example, may not be definable in T_α for if $\alpha = \gamma^+$, then $\gamma \in T_\alpha$ but $\mathscr{P}\gamma \notin T_\alpha$. The provisos of 13.1.1 must be carefully checked.

13.2.1 *Definition.* $y \mathrel{\bar{=}} \overline{\mathrm{Pr}}\, x_0 x_1$ if and only if $y = (x_0, x_1)$.

13.2.2 *Definition.* $y \mathrel{\bar{=}} [x_0 \mathbin{\bar{\times}} x_1]$ if and only if $y = x_0 \times x_1$.

13.2.3 *Definition.* $y \mathrel{\bar{=}} [x_0 \mathbin{\bar{|}} x_1]$ if and only if y is the set of all ordered pairs in x_0 with first coordinates in x_1.

13.2.4 *Definition.* If n is a non-zero natural number, $y \mathrel{\bar{=}} [\overline{\mathrm{Set}}_n\, x_0 \ldots x_{n-1}]$ if and only if $y = \{x_0, \ldots, x_{n-1}\}$.

13.2.5 *Definition.* $y \mathrel{\bar{=}} \overline{\mathrm{Dom}}\, x_0$ if and only if y is the set of all first coordinates of ordered pairs in x_0.

13.2.6 *Definition.* $y \mathrel{\bar{=}} \bar{0}$ if and only if $y = \phi$.

13.2.7 *Definition.* $y \mathrel{\bar{=}} \overline{\mathrm{Rng}}\, x_0$ if and only if y is the set of all second coordinates of ordered pairs in x_0.

13.2.8 *Definition.* $\overline{\mathrm{Rel}}\, x_0$ if and only if x_0 is a relation.

13.2.9 *Definition.* $\overline{\mathrm{Fcn}}\, x_0$ if and only if x_0 is a function.

13.2.10 *Definition.* $\overline{\text{One-one}}\ \overline{\mathrm{Fcn}}\, x_0$ if and only if x_0 is a one-one function.

13.2.11 *Definition.* $y \mathrel{\bar{=}} [x_0 \bar{o}\!f x_1]$ if and only if $y = x_0(x_1)$ in case x_0 is a function and $x_1 \in \mathrm{Dom}(x_0)$, ϕ if not.

13.2.12 *Definition.* $x_0 \mathrel{\bar{\subseteq}} x_1$ if and only if $x_0 \subseteq x_1$.

13.2.13 *Definition.* $\overline{\mathrm{Ord}}\, x_0$ if and only if x_0 is an ordinal.

13.2.14 *Definition.* $\overline{\mathrm{Seq}}\, x_0$ if and only if x_0 is a sequence.

13.2.15 *Definition.* For $1 \leqslant n < \omega$ individual constants $\bar{n}$ are defined recursively by

$$x_0 \mathrel{\bar{=}} \bar{n} \leftrightarrow x_0 \mathrel{\bar{=}} [\overline{\mathrm{Set}}_n\, \bar{0} \ldots \overline{n-1}].$$

13.2.16 *Definition.* For $0 < n < \omega$ the n-place operation symbols $\overline{\mathrm{Seq}}_n$ are defined by

$$y \mathrel{\bar{=}} [\overline{\mathrm{Seq}}_n\, x_0 \ldots x_{n-1}] \leftrightarrow y \mathrel{\bar{=}} \overline{\mathrm{Set}}_n\, [\overline{\mathrm{Pr}}\, \bar{0}\, x_0] \ldots [\overline{\mathrm{Pr}}\, \overline{n-1}\, x_{n-1}]$$

Note that $[\overline{\mathrm{Ord}}\ \bar{n}]$ and $\overline{\mathrm{Dom}}[\overline{\mathrm{Seq}}_n\, x_0 \ldots x_{n-1}] \mathrel{\bar{=}} \bar{n}$ hold in $\mathfrak{T}_\alpha$.

13.2.17 *Definition.* $y \mathrel{\bar{=}} \bar{\bigcup}\, x_0$ if and only if $y = \bigcup x_0$.

The regularity of α is required for the definability of the union operation.

13.2.18 *Definition.* $y \mathrel{\bar{=}} x_0 \mathbin{\bar{\cup}} x_1 \leftrightarrow y \mathrel{\bar{=}} \bar{\bigcup}\, [\overline{\mathrm{Set}}_2\, x_0 x_1]$.

13.2.19 *Definition.* $y \mathrel{\bar{=}} \bar{\bigcap}\, x_0$ if and only if $y = \bigcap x_0$ in case $x_0 \neq \phi$, ϕ otherwise.

13.2.20 *Definition.* $y \mathrel{\bar{=}} x_0 \mathbin{\bar{\cap}} x_1 \leftrightarrow y \mathrel{\bar{=}} \bar{\bigcap}\, [\overline{\mathrm{Set}}_2\, x_0 x_1]$.

Note that the formula $[\overline{\mathrm{Ord}}\ x_0 \rightarrow [[x_0 \mathrel{\bar{=}} \bar{\bigcup}\, x_0] \vee [x_0 \mathrel{\bar{=}} \bar{\bigcup}\, x_0 \mathbin{\bar{\cup}} \overline{\mathrm{Set}}_1\, \bar{\bigcup}\, x_0]]]$ holds in $\mathfrak{T}_\alpha$.

13.2.21 *Definition.* $y \mathrel{\bar{=}} \bar{s}\, x_0 \leftrightarrow y \mathrel{\bar{=}} x_0 \mathbin{\bar{\cup}} \overline{\mathrm{Set}}_1\, x_0$.

Recursive definitions are not difficult to deal with. Consider ordinal addition defined recursively by conditions:

$$\begin{aligned} &\delta + 0 = \delta \\ &\delta + s(\varepsilon) = s(\delta + \varepsilon) \\ &\delta + \varepsilon = \bigcup \{\delta + \xi : \xi < \varepsilon\} \text{ if } \varepsilon \in \mathrm{Lim}. \end{aligned}$$

Then $y = \delta + \varepsilon$ if and only if there is a sequence z such that $\varepsilon \in \mathrm{Dom}(z)$, $z(\varepsilon) = y$, $z(0) = \delta$ and

(i) For all ξ such that $s(\xi) \in \mathrm{Dom}(z)$, $z(s(\xi)) = s(z(\xi))$

(ii) For all $\xi \neq 0$ such that $\xi = \bigcup \xi \in \mathrm{Dom}(z)$, $z(\xi) = \bigcup \mathrm{Rng}(z|\xi)$.

This condition is clearly definable by a formula of $\mathbf{L}_{\omega\omega}$.

13.2.22 *Definition.* $y \mathrel{\bar{=}} x_0 \mathbin{\bar{+}} x_1$ if and only if $y = x_0 + x_1$ in case x_0, x_1 are ordinals, ϕ if not.

The formal definition of $\bar{+}$ is $y \mathrel{\bar{=}} x_0 \mathbin{\bar{+}} x_1 \leftrightarrow [\neg[\overline{\mathrm{Ord}}\, x_0 \wedge \overline{\mathrm{Ord}}\, x_1] \rightarrow y \mathrel{\bar{=}} \bar{0}] \wedge [[\overline{\mathrm{Ord}}\ x_0 \wedge \overline{\mathrm{Ord}}\ x_1] \rightarrow [\exists z[\overline{\mathrm{Seq}}\ z \wedge x_1 \mathrel{\bar{\in}} \overline{\mathrm{Dom}}\ z \wedge [z \mathbin{\overline{of}} x_1] \mathrel{\bar{=}} y \wedge [z \mathbin{\overline{of}} \bar{0}] \mathrel{\bar{=}} x_0 \wedge A_1(z) \wedge A_2(z)]]]$ where $A_1(z)$ and $A_2(z)$ express conditions (i), (ii) above.

The operations of ordinal multiplication and infinitary ordinal addition can be shown definable by a formula of $\mathbf{L}_{\omega\omega}$ in exactly the same way. The formal definitions are omitted.

13.2.23 *Definition.* $y \mathrel{\bar{=}} x_0 \mathbin{\bar{\cdot}} x_1$ if and only if $y = x_0 \cdot x_1$ in case x_0, x_1 ordinals, ϕ if not.

13.2.24 *Definition.* $y \mathrel{\bar{=}} \bar{\Sigma} x_0$ if and only if $y = \Sigma x_0$ in case x_0 is a sequence of ordinals, ϕ if not.

The regularity of α is required to show the definability of infinitary ordinal addition. The recursive definition of infinitary concatenation following 2.2.8 is easier to handle than the explicit definition.

13.2.25 *Definition.* $y \mathrel{\overline{=}} x_0 \overline{{}^\frown} x_1 \leftrightarrow [\neg C(x_0,x_1) \rightarrow y \mathrel{\overline{=}} \bar{0}] \wedge [C(x_0,x_1) \rightarrow \overline{\mathrm{Seq}}\, y \wedge [\overline{\mathrm{Dom}}\, y \mathrel{\overline{=}} \overline{\mathrm{Dom}}\, x_0 \mathbin{\overline{+}} \overline{\mathrm{Dom}}\, x_1] \wedge [\forall t A(t, y, x_0, x_1)]]$ where $C(x_0, x_1) = [\overline{\mathrm{Seq}}\, x_0 \wedge \overline{\mathrm{Seq}}\, x_1]$ and $A(t, y, x_0, x_1) = [t \mathrel{\overline{\in}} \overline{\mathrm{Dom}}\, x_0 \rightarrow [y \overline{of} t] \mathrel{\overline{=}} [x_0 \overline{of} t]] \wedge [t \mathrel{\overline{\in}} \overline{\mathrm{Dom}}\, x_1 \rightarrow [y \overline{of} [\overline{\mathrm{Dom}}\, x_0 \mathbin{\overline{+}} t]] \mathrel{\overline{=}} [x_1 \overline{of} t]]$.

If x_0 is a sequence of sequences then $y = {}^\frown x_0$ if and only if there is a sequence z such that $\mathrm{Dom}(x_0) \in \mathrm{Dom}(z)$, $y = z(\mathrm{Dom}(x_0))$, $z(0) = \phi$ and

(i) For all t such that $s(t) \in \mathrm{Dom}(z)$, $z(s(t)) = z(t)^\frown x_0(t)$.

(ii) For all t such that $t = \bigcup t \in \mathrm{Dom}(z)$, $z(t) = \bigcup \mathrm{Rng}(z|t)$. Conditions (i) and (ii) can be directly translated to formulas of $\mathbf{L}_{\omega\omega}$ to yield a formal definition of infinitary concatenation.

13.2.26 *Definition.* $y \mathrel{\overline{=}} \overline{{}^\frown} x_0$ if and only if $y = {}^\frown x_0$ in case x_0 is a sequence of sequences, ϕ if not.

13.3 The Definability of the Formal Systems

In this section $\mathbf{ML}^\alpha$ is the metalanguage for (α, β, o, π)-languages, $\mathbf{L}$ is a particular such language and g a Gödel-numbering for the symbols of $\mathbf{L}$. When we say that a set Δ of expressions of $\mathbf{L}$ is definable let it be understood that we mean that the set of Gödel-sequences of expressions in Δ is definable by a formula of $\mathbf{ML}^\alpha$ with respect to $\mathfrak{M}(\mathbf{L}, g)$.

Atomic terms have form $\langle x \rangle$ or $\langle c \rangle$ where x is an individual variable, c an individual constant. Hence the following formula defines the set of atomic terms.

13.3.1 *Definition.* $\overline{\mathrm{AtTerm}}\, x_0 \leftrightarrow [\exists x_1[[ICx_1 \vee IVx_1] \wedge x_0 \mathrel{\overline{=}} \overline{\mathrm{Seq}}_1\, x_1]]$.

13.3.2 *Theorem.* The set of terms of $\mathbf{L}$ is definable.

PROOF: Going back to the definition of terms in Chapter 3, Sect. 3.1, it is clear that an expression E is a term of $\mathbf{L}$ if and only if there is a sequence x_1 such that $E \in \mathrm{Rng}(x_1)$ and for all $x_2 \in \mathrm{Dom}(x_1)$ either $x_1(x_2)$ is an atomic term or one of the following conditions holds:

(i) $x_1(x_2) = \langle [\rangle^\frown x_1(x_3)^\frown \langle \psi \rangle^\frown x_1(x_4)^\frown \langle] \rangle$ where ψ is a special two-place operation symbol and $x_3, x_4 \in x_2$.

(ii) $x_1(x_2) = \langle [\varphi \rangle^\frown ({}^\frown x_3)^\frown \langle] \rangle$ where φ is a ζ-place operation symbol,

$0 < \zeta < o$, x_3 is a sequence of length ζ with range included in $\mathrm{Rng}(x_1|x_2)$.

(iii) $x_1(x_2) = \langle[\varphi\rangle^\frown(^\frown x_3)^\frown\langle]\rangle$ where φ is an infinitary operation symbol, x_3 a non-empty sequence of length less than o with range included in $\mathrm{Rng}(x_1|x_2)$.

Conditions (i), (ii), (iii) are translated by formulas

$$A_1(x_1, x_2) = [\exists x_3x_4x_5[x_3 \mathrel{\bar\in} x_2 \wedge x_4 \mathrel{\bar\in} x_2 \wedge BOPx_5 \wedge [x_1\overline{of}\,x_2] \mathrel{\bar\equiv} \overline{\mathrm{Seq}}_1\,\overline{lb}^\frown[x_1\overline{of}\,x_3]^\frown\overline{\mathrm{Seq}}_1\,x_5{}^\frown[x_1\overline{of}\,x_4]^\frown\overline{\mathrm{Seq}}_1\,\overline{rb}]].$$

$$A_2(x_1, x_2) = [\exists x_3x_4[\overline{\mathrm{Seq}}\,x_3 \wedge \overline{\mathrm{Rng}}\,x_3 \mathrel{\bar\subseteq} \overline{\mathrm{Rng}}[x_1\bar|x_2] \wedge [OP[\overline{\mathrm{Dom}}\,x_3]x_4] \wedge [x_1\overline{of}\,x_2] = [\overline{\mathrm{Seq}}_2\,\overline{lb}\,x_4]^\frown[^\frown x_3]^\frown[\overline{\mathrm{Seq}}_1\,\overline{rb}]]].$$

$$A_3(x_1, x_2) = [\exists x_3x_4[\overline{\mathrm{Seq}}\,x_3 \wedge \overline{\mathrm{Rng}}\,x_3 \mathrel{\bar\subseteq} \overline{\mathrm{Rng}}[x_1\bar|x_2] \wedge IOPx_4 \wedge \neg x_3 \mathrel{\bar\equiv} \bar 0 \wedge [\overline{\mathrm{Dom}}\,x_3] \mathrel{\bar\in} \bar o \wedge [x_1\overline{of}\,x_2] \mathrel{\bar\equiv} [\overline{\mathrm{Seq}}_2\,\overline{lb}\,x_4]^\frown[^\frown x_3]^\frown[\overline{\mathrm{Seq}}_1\,\overline{rb}]]],$$

where clause $[\overline{\mathrm{Dom}}\,x_3] \mathrel{\bar\in} \bar o$ is omitted in case $o = \alpha$.

Note that if $o = \alpha$ sequence x_3 will automatically have length less than o since no sequence of length α can be in T_α.

Therefore the following formula defines the set of terms:

$$A(x_0) = \exists x_1[\overline{\mathrm{Seq}}\,x_1 \wedge x_0 \mathrel{\bar\in} \overline{\mathrm{Rng}}\,x_1 \wedge [\forall x_2[x_2 \mathrel{\bar\in} \overline{\mathrm{Dom}}\,x_1 \to [\overline{\mathrm{AtTerm}}[x_1\overline{of}\,x_2] \vee A_1(x_1, x_2) \vee A_2(x_1, x_2) \vee A_3(x_1, x_2)]]]].$$

Since the set of terms is definable we may introduce the new predicate symbol $\overline{\mathrm{Term}}$ by definition.

13.3.3 *Definition.* $\overline{\mathrm{Term}}\,x_0$ if and only if x_0 is the Gödel-sequence of a term of **L**.

The classes of atomic formulas and formulas may be shown definable by a method so similar to that used for terms that the proofs are omitted. Their predicate symbols are introduced informally.

13.3.4 *Definition.* $\overline{\mathrm{AtForm}}\,x_0$ if and only if x_0 is the Gödel-sequence of an atomic formula.

13.3.5 *Definition.* $\overline{\mathrm{Form}}\,x_0$ if and only if x_0 is the Gödel-sequence of a formula.

13.3.6 *Theorem.* The sets of instances in **L** of the basic propositional axiom schemes $\mathscr{I}$, $\mathscr{N}$, $\mathscr{C}$, are definable.

PROOF: Consider, for example, the set of formulas of **L** having form

$$\mathscr{C}_{1,\delta} = [[\bigwedge [A_\delta \to A_0]\ldots[A_\delta \to A_\xi]\ldots] \to [A_\delta \to [\bigwedge A_0\ldots A_\xi\ldots]]], \quad 0 < \delta < \alpha.$$

Then A has form $\mathscr{C}_1$ if and only if there is a non-empty sequence x_1 of formulas and a formula x_2 such that $A = \langle[[\,\boldsymbol{\wedge}\rangle{}^\frown({}^\frown x_3){}^\frown\langle] \to [\rangle{}^\frown x_2{}^\frown\langle \to [\boldsymbol{\wedge}\rangle{}^\frown({}^\frown x_1){}^\frown\langle]]]\rangle$, where x_3 is that sequence having the same length as x_1 such that for $x_4 \in \mathrm{Dom}(x_3)$, $x_3(x_4) = \langle[\rangle{}^\frown x_2{}^\frown\langle\to\rangle{}^\frown x_1(x_4){}^\frown\langle]\rangle$. Translating this into $\mathbf{ML}^\alpha$ we have

$\mathrm{Inst}\,\mathscr{C}_1(x_0) \leftrightarrow \exists x_1 x_2[\overline{\mathrm{Seq}}\, x_1 \wedge [\neg x_1 \mathrel{\overline{\equiv}} \bar{0}] \wedge \overline{\mathrm{Form}}\, x_2 \wedge$
$[\forall x_4[x_4 \mathrel{\overline{\in}} \overline{\mathrm{Dom}}\, x_1 \to \overline{\mathrm{Form}}[x_1 \overline{of}\, x_4]]] \wedge [\forall x_3[\overline{\mathrm{Dom}}\, x_3 \mathrel{\overline{\equiv}} \overline{\mathrm{Dom}}\, x_1 \wedge$
$A(x_1, x_2, x_3)] \to x_0 \mathrel{\overline{\equiv}} \overline{\mathrm{Seq}}_3\, \overline{lb}\,\overline{lb}\,\bar{c}\,\overline{\frown}[\overline{\frown} x_3]\overline{\frown}\,\overline{\mathrm{Seq}}_3\, \overline{rb}\,\bar{i}\,\overline{lb}\,\overline{\frown} x_2 \frown$
$\overline{\mathrm{Seq}}_3\, \bar{i}\,\overline{lb}\,\bar{c}\,\overline{\frown}[\overline{\frown} x_1]\overline{\frown}\,\overline{\mathrm{Seq}}_3\, \overline{rb}\,\overline{rb}\,\overline{rb}]]$, where $A(x_1, x_2, x_3) =$
$\forall x_4[x_4 \mathrel{\overline{\in}} \overline{\mathrm{Dom}}\, x_1 \to [x_3 \overline{of}\, x_4] \mathrel{\overline{\equiv}} \overline{\mathrm{Seq}}_1\, \overline{lb}\,\overline{\frown} x_2 \overline{\frown}\,\overline{\mathrm{Seq}}_1\, \bar{i}\,\overline{\frown}[x_1 \overline{of}\, x_4]\overline{\frown}\,\overline{\mathrm{Seq}}_1\, \overline{rb}]$.
Hence the set of instances of schemes $\mathscr{C}_{1,\delta}$ for $0 < \delta < \alpha$ is definable. The sets for $\mathscr{I}, \mathscr{N}, \mathscr{C}_2$ can be shown definable in the same way.

13.3.7 *Theorem.* a) For any cardinal $0 < \gamma < \alpha$ the set of all instances in $\mathbf{L}$ of the Chang distributive laws Π_γ is definable. b) The set of all formulas of $\mathbf{L}$ of the form $[\bigvee_{\mu<\delta}[\bigwedge_{\nu<\delta} A_{\mu\nu}]]$, $0 < \delta < \alpha$, such that every choice set of $\langle A_{\mu\nu}: \mu < \delta, \nu < \delta\rangle$ contains a formula and its negation, is definable. That is, the set of instances of $\Pi_{\nearrow\alpha}$ is definable.

PROOF: Let γ be a cardinal, $0 < \gamma < \alpha$. Then the set of instances of Π_γ in $\mathbf{L}$ is the set of all formulas of the form

$$[\bigvee_{\mu<\gamma}[\bigwedge_{\nu<\gamma} A_{\mu\nu}]] = [\neg[\boldsymbol{\wedge} \ldots[\neg[\boldsymbol{\wedge} \ldots A_{\mu\nu}\ldots]]\ldots]]$$

where every choice set of $\langle A_{\mu\nu}: \mu < \gamma, \nu < \gamma\rangle$ contains a formula with its negation. A formula A has this form if and only if there is a sequence x_1 of length γ such that $x_1(x_2)$ is a sequence of formulas having length γ for every $x_2 \in \gamma$ and

(i) If for every $x_2 \in \gamma$ there is $x_3 \in \gamma$ such that $x_1(x_2)(x_3) \in x_4$, then x_4 contains a formula and its negation.

(ii) $A = \langle[\,\neg\,[\,\boldsymbol{\wedge}\rangle{}^\frown({}^\frown x_3){}^\frown\langle]]\rangle$ where $\mathrm{Dom}(x_3) = \mathrm{Dom}(x_1) = \gamma$ and for every $x_2 \in \gamma$, $x_3(x_2) = \langle[\,\neg\,[\,\boldsymbol{\wedge}\rangle{}^\frown({}^\frown x_1(x_2)){}^\frown\langle]]\rangle$.
This condition can readily be translated into $\mathbf{ML}^\alpha$.

Part b) is similar. Begin by asserting the existence of a non-empty sequence x_1 and thereafter replace γ by $\mathrm{Dom}(x_1)$.

13.3.8 *Theorem.* For every cardinal $\gamma \neq 0$ such that $2\exp\gamma < \alpha$ the set of all formulas of the form

$$[\bigwedge_{\mu<\gamma}[\bigvee_{\nu<\gamma} A_{\mu\nu}]] \to [\bigvee_{\xi<2\exp\gamma}[\bigwedge_{\mu<\gamma} A_{\mu g_\xi(\mu)}]],$$

where $\gamma^\gamma = \{g_\xi: \xi < 2\exp\gamma\}$, is definable.

PROOF: Eliminating abbreviated symbols and restoring missing brackets, the formulas in question have form

$$[[\bigwedge \ldots [\neg[\bigwedge \ldots [\neg A_{\mu\nu}] \ldots]] \ldots] \rightarrow [\neg[\bigwedge \ldots [\neg[\bigwedge \ldots A_{\mu g_{\xi}(\mu)} \ldots]] \ldots]]].$$

Then A has this form if and only if there is a sequence x_1 of length γ such that $x_1(x_2)$ is a sequence of formulas having length γ for every $x_2 \in \gamma$, and a one-one sequence x_3 of functions on γ into γ such that

(i) Every function on γ into γ is in $\mathrm{Rng}(x_3)$.

(ii) Let x_4 be that sequence of length γ such that for $x_2 \in \gamma$. $x_4(x_2) = \langle[\ \neg\ [\ \bigwedge\rangle^\frown(^\frown x_5)^\frown\langle]]\rangle$, where x_5 is that sequence of length γ such that $x_5(x_6) = \langle[\ \neg\rangle^\frown x_1(x_2)(x_6)^\frown\langle]\rangle$, $x_6 \in \gamma$. Further let x_7 be that sequence of length $\mathrm{Dom}(x_3)$ such that for $x_8 \in \mathrm{Dom}(x_3)$, $x_7(x_8) = \langle[\ \neg\ [\ \bigwedge\rangle^\frown(^\frown x_9)^\frown\langle]]\rangle$, where x_9 is that sequence of length γ such that $x_9(x_2) = (x_1(x_2))(x_3(x_8)(x_2))$ for $x_2 \in \gamma$. Then $A = \langle[\ [\ \bigwedge\rangle^\frown(^\frown x_4)^\frown\langle] \rightarrow [\ \neg\ [\ \bigwedge\rangle^\frown(^\frown x_7)^\frown\langle]]]\rangle$. This condition can be translated into $\mathbf{ML}^\alpha$.

13.3.9 *Theorem.* The set of equality axioms of $\mathbf{L}$ is definable.

PROOF: Consider, for example, the set of formulas of $\mathbf{L}$ of form $\mathscr{E}\mathscr{Q}5$:

$$[[\bigwedge [T_0 \equiv T_0'] \ldots [T_\xi \equiv T_\xi'] \ldots] \rightarrow [[Q T_0 \ldots T_\xi \ldots] \rightarrow [Q T_0' \ldots T_\xi' \ldots]]]$$

where Q is an η-place or infinitary predicate symbol, $0 < \eta < \pi$, and $\langle T_0 \ldots T_\xi \ldots\rangle$ and $\langle T_0' \ldots T_\xi' \ldots\rangle$ are sequences of terms having length η.

Suppose first $\pi < \alpha$. Then a formula A has this form if and only if there are non-empty sequences x_1, x_2 of terms such that $\mathrm{Dom}(x_1) = \mathrm{Dom}(x_2) \in \pi$ and $A = \langle[\ [\ \bigwedge\rangle^\frown(^\frown x_3)^\frown\langle] \rightarrow [\ [\ Q\rangle^\frown(^\frown x_1)^\frown\langle] \rightarrow [\ Q\rangle^\frown(^\frown x_2)^\frown\langle]]]\rangle$, where Q is a $\mathrm{Dom}(x_1)$-place or infinitary predicate symbol and x_3 is that sequence of length $\mathrm{Dom}(x_1)$ such that for all $x_4 \in \mathrm{Dom}(x_1)$, $x_3(x_4) = \langle[\rangle^\frown x_1(x_4)^\frown\langle\overline{=}\rangle^\frown x_2(x_4)^\frown\langle]\rangle$. This can easily be translated into $\mathbf{ML}^\alpha$. In case $\pi = \alpha$ references to π are simply dropped.

The quantificational schemes $\mathscr{Q}$, $\mathscr{CH}_\gamma$, $\mathscr{DCH}_\gamma$ all involve the notion of free and bound occurrences of variables.

13.3.10 *Lemma.* Let three-place predicate symbol $\overline{\mathrm{Bound}}$ be defined as follows:

$$\overline{\text{Bound}}\, x_0 x_1 x_2 \leftrightarrow \exists x_3 x_4 x_5 x_6 [x_0 \mathrel{\bar{\equiv}} x_3 \mathbin{\bar{\frown}} \overline{\text{Seq}}_2\, \overline{lb}\, \bar{q} \mathbin{\bar{\frown}} x_4 \mathbin{\bar{\frown}} x_5 \mathbin{\bar{\frown}} \overline{\text{Seq}}_1\, \overline{rb} \mathbin{\bar{\frown}} x_6] \wedge \overline{\text{Seq}}\, x_4 \wedge \neg x_4 \mathrel{\bar{\equiv}} \bar{0} \wedge \overline{\text{Dom}}\, x_4 \mathrel{\bar{\in}} \bar{\beta} \wedge \forall x_7 [x_7 \mathrel{\bar{\in}} \overline{\text{Dom}}\, x_4 \rightarrow \overline{IV}[x_4 \overline{of} x_7]] \wedge \neg [x_1 \mathbin{\bar{\cap}} \overline{\text{Rng}}\, x_4 \mathrel{\bar{\equiv}} \bar{0}] \wedge \overline{\text{Form}}\, x_5 \wedge \overline{\text{Seq}}\, x_3 \wedge \overline{\text{Seq}}\, x_6 \wedge [x_2 \mathrel{\bar{\equiv}} \overline{\text{Dom}}\, x_3 \vee [\overline{\text{Dom}}\, x_3 \mathrel{\bar{\in}} x_2 \wedge x_2 \mathrel{\bar{\in}} \overline{\text{Dom}}\, x_3 \mathbin{\bar{+}} \bar{2} \mathbin{\bar{+}} \overline{\text{Dom}}\, x_4 \mathbin{\bar{+}} \overline{\text{Dom}}\, x_5]].$$

Then if x_0 is the Gödel-sequence of a formula A, if x_1 is the set of Gödel-numbers of a set X of variables and if $x_2 \in \text{Dom}(A)$, then $\langle x_0 x_1 x_2 \rangle$ satisfies $\overline{\text{Bound}}\, x_0 x_1 x_2$ if and only if x_2 is bound by X in A. If $\beta = \alpha$ the clause $[\overline{\text{Dom}}\, x_4 \mathrel{\bar{\in}} \bar{\beta}]$ is dropped.

PROOF: The formal definition will be recognized as a translation into $\mathbf{ML}^{\alpha}$ of the definition of bound position in Chapter 9, Sect. 9.1.

13.3.11 *Lemma.* Let the two-place operation symbol $\overline{\text{FreeOcc}}$ be defined as follows:

$$x_2 \mathrel{\bar{\equiv}} \overline{\text{FreeOcc}}\, x_0 x_1 \leftrightarrow [\neg C(x_0, x_1) \rightarrow x_2 \mathrel{\bar{\equiv}} \bar{0}] \wedge [C(x_0, x_1) \rightarrow \forall x_3 [x_3 \mathrel{\bar{\in}} x_2 \leftrightarrow x_3 \mathrel{\bar{\in}} \overline{\text{Dom}}\, x_0 \wedge [x_0 \overline{of} x_3] \mathrel{\bar{\in}} x_1 \wedge \neg [\overline{\text{Bound}}\, x_0\, \overline{\text{Set}}_1\, [x_0 \overline{of} x_3] x_3]]]$$

where $C(x_0, x_1) = [\overline{\text{Form}}\, x_0 \wedge \forall x_3 [x_3 \mathrel{\bar{\in}} x_1 \rightarrow \overline{IV}\, x_3]]$.

Then $\langle x_0 x_1 x_2 \rangle$ satisfies $x_2 = \overline{\text{FreeOcc}}\, x_0 x_1$ if and only if $x_2 = \{\iota\colon \iota$ is a free occurrence of a variable with Gödel-number in x_1 in the formula with Gödel-sequence $x_0\}$ if x_0 is the Gödel-sequence of a formula and x_1 a set of Gödel-numbers of variables, ϕ otherwise.

13.3.12 *Definition.* $x_1 = \overline{FV}\, x_0 \leftrightarrow [\neg\, \overline{\text{Form}}\, x_0 \rightarrow x_1 \mathrel{\bar{\equiv}} \bar{0}] \wedge [\overline{\text{Form}}\, x_0 \rightarrow \forall x_2 [x_2 \mathrel{\bar{\in}} x_1 \leftrightarrow \neg [\overline{\text{FreeOcc}}\, x_0 [\overline{\text{Set}}_1\, x_2] \mathrel{\bar{\equiv}} \bar{0}]]]$.

Note that if x_0 is the Gödel-sequence of a formula A then $\langle x_0 x_1 \rangle$ satisfies $x_1 \mathrel{\bar{\equiv}} \overline{FV}\, x_0$ if and only if x_1 is the set of Gödel-numbers of $FV(A)$.

We are now in a position to prove the definability of all the quantificational schemes except $\mathscr{Q}2$, the substitution scheme. Recall that $SF_f^X A = {}^\frown\langle E_\nu {}^\frown f A(\iota_\nu)\colon \nu < \sigma \rangle {}^\frown E_\sigma$ where $\langle \iota_\nu\colon \nu < \sigma \rangle$ is the strictly increasing sequence of all free occurrences of variables of X in A and

$$E_{s(\nu)} = A|\iota_{s(\nu)} - (A|\iota_\nu {}^\frown \langle A(\iota_\nu)\rangle) \text{ if } s(\nu) \leqslant \sigma,$$
$$E_\nu = A|\iota_\nu - A|\bigcup_{\theta<\nu} \iota_\theta \text{ if } \nu = \bigcup \nu,\ \nu \leqslant \sigma.$$

It is convenient to introduce special notation for the sequence $\langle E_\nu : \nu < \sigma + 1\rangle$.

13.3.13 *Definition.* $x_2 \equiv x_0 \bar{-} x_1 \leftrightarrow [\neg C(x_0, x_1) \rightarrow x_2 \equiv \bar{0}] \wedge [C(x_0, x_1) \rightarrow x_0 \equiv x_1 \bar{\frown} x_2]$ where $C(x_0, x_1) = \overline{\text{Seq}}\, x_1 \wedge \exists x_3[\overline{\text{Seq}}\, x_3 \wedge x_0 \equiv x_1 \bar{\frown} x_3]$.

13.3.14 *Lemma.* If x_0 is a sequence, x_4 a strictly increasing sequence of ordinals in $\text{Dom}(x_0)$, let $\text{Rem Seq}(x_0, x_4)$ be that sequence x_5 of length $\text{Dom}(x_4) + 1$ such that

$$x_5(s(\nu)) = x_0|x_4(s(\nu)) - (x_0|x_4(\nu)\frown\langle x_0(x_4(\nu))\rangle) \text{ if } s(\nu) \in \text{Dom}(x_4) + 1,$$
$$x_5(\nu) = x_0|x_4(\nu) - x_0|\bigcup \text{Rng}(x_4|\nu) \text{ if } \nu = \bigcup \nu \in \text{Dom}(x_4) + 1.$$

Otherwise, let $\text{Rem Seq}(x_0, x_4) = \phi$. Then Rem Seq is definable.

PROOF: The definability is obvious from the form of the informal definition. The operations involved are all definable.

13.3.15 *Definition.* $x_5 \equiv \overline{\text{Rem}}\,\overline{\text{Seq}}\, x_0 x_4$ if and only if $x_5 = \text{Rem Seq}(x_0, x_4)$.

13.3.16 *Lemma.* If x_0 is the Gödel-sequence of a formula A, x_1 a set of Gödel-numbers of variables, x_2 a function on x_1 to Gödel sequences of terms, let $SF(x_0, x_1, x_2)$ be the Gödel-sequence of $SF_f^X A$, where $X = \{x : g(x) \in x_1\}$ and $f = g^{-1} \bigcirc x_2 \bigcirc g$ on X. Otherwise let $SF(x_0, x_1, x_2)$ be ϕ. Then SF is definable.

PROOF: Using the new terminology,

$$g \bigcirc SF_f^X A = {}^\frown\langle x_5(\nu)\frown x_2(x_0(x_4(\nu))) : \nu < \sigma\rangle\frown x_5(\sigma)$$

where x_4 is the increasing sequence with range $\text{Free Occ}(x_0, x_1)$, $\sigma = \text{Dom}(x_4)$, and $x_5 = \text{Rem Seq}(x_0, x_4)$. The definability is obvious.

13.3.17 *Definition.* $x_3 \equiv \overline{SF}\, x_0 x_1 x_2$ if and only if

$$x_3 = SF(x_0, x_1, x_2).$$

13.3.18 *Theorem.* The sets of instances in $\mathbf{L}$ of each of the quantificational schemes $\mathcal{Q}1$, $\mathcal{Q}2$, $\mathcal{CH}_\gamma$, $\mathcal{DCH}_\gamma$, $\omega \leqslant \gamma < \alpha$, are definable. The sets of formulas of form $\mathcal{DCH}_{\nearrow\alpha}$ and of form $\mathcal{CH}_{\nearrow\alpha}$ are definable.

PROOF: As examples consider $\mathcal{Q}2$, $\mathcal{DCH}_\gamma$ and $\mathcal{DCH}_{\nearrow\alpha}$. An instance of $\mathcal{Q}2$ has form $A = [[\forall v A_0] \rightarrow SF_f^{\text{Rng}(v)} A_0]$, where A_0 has no free occurrence of a variable $x \in \text{Rng}(v)$ bound by $FVf(x)$. If T is a term, $x \in FV(T)$ if and only if x is a variable and has an occurrence in T. The set of Gödel-numbers of $FV(T)$ is defined by

$$x_1 \,\overline{\equiv}\, \overline{FVT}x_0 \leftrightarrow [\neg\overline{\mathrm{Term}}\, x_0 \rightarrow x_1 \,\overline{\equiv}\, \bar{0}] \wedge [\overline{\mathrm{Term}}\, x_0 \rightarrow \forall x_2[x_2 \,\overline{\in}\, x_1 \leftrightarrow IV x_2 \wedge \exists x_3[x_3 \,\overline{\in}\, \overline{\mathrm{Dom}}\, x_0 \wedge [x_0\overline{of}x_3] \,\overline{\equiv}\, x_2]]].$$

Therefore a defining formula for the set of Gödel-numbers of instances of $\mathscr{Q}2$ is

$$\mathrm{Inst}\,\mathscr{Q}2(x_0) \leftrightarrow \exists x_1x_2x_3[x_0 \,\overline{\equiv}\, \overline{\mathrm{Seq}}_3\overline{lb}\,\overline{lb}\,\bar{q}^\frown x_1{}^\frown x_2{}^\frown\overline{\mathrm{Seq}}_2\overline{rb}\,\bar{i}^\frown [\overline{SF}x_2[\overline{\mathrm{Rng}}\, x_1]x_3]^\frown\overline{\mathrm{Seq}}_1\overline{rb} \wedge \overline{\mathrm{Seq}}\, x_1 \wedge \neg x_1 \,\overline{\equiv}\, \bar{0} \wedge \overline{\mathrm{Dom}}\, x_1 \,\overline{\in}\, \bar{\beta} \wedge \forall x_4[x_4 \,\overline{\in}\, \overline{\mathrm{Dom}}\, x_1 \rightarrow IV[x_1\overline{of}x_4]] \wedge \overline{\mathrm{Form}}\, x_2 \wedge \overline{\mathrm{Fcn}}\, x_3 \wedge \overline{\mathrm{Rng}}\, x_1 \,\overline{\subseteq}\, \overline{\mathrm{Dom}}\, x_3 \wedge \forall x_4[x_4 \,\overline{\in}\, \overline{\mathrm{Dom}}\, x_3 \rightarrow \overline{\mathrm{Term}}[x_3\overline{of}x_4]] \wedge \forall x_5[x_5 \,\overline{\in}\, \overline{\mathrm{FreeOcc}}\, x_2[\overline{\mathrm{Rng}}\, x_1] \rightarrow \neg\overline{\mathrm{Bound}}\, x_2\overline{FVT}[x_3\overline{of}[x_2\overline{of}x_5]]x_5]].$$

In case $\beta = \alpha$ the clause $\overline{\mathrm{Dom}}\, x_1 \in \bar{\beta}$ is dropped.

An instance of $\mathscr{DCH}_\gamma$ has form

$$A = [\bigwedge_{\xi<\gamma} [\exists v_0 A_0]\ldots[\bigforall_{\nu<\xi} v_0\ldots v_\nu\ldots[\exists v_\xi A_\xi]]\ldots] \rightarrow [\bigexists_{\xi<\gamma} v_0\ldots v_\xi\ldots[\bigwedge_{\xi<\gamma} A_0\ldots A_\xi\ldots]],$$

provided $\mathrm{Rng}(v_\xi) \cap \mathrm{Rng}(v_\nu) = \phi$ for $\xi \neq \nu$ and $\mathrm{Rng}(v_\xi) \cap FV(A_\nu) = \phi$ for $\nu < \xi$.

Dropping abbreviated symbols and restoring brackets,

$$A = [[\bigwedge [\neg[\forall v_0[\neg A_0]]]\ldots[\forall v_0\ldots v_\nu\ldots[\neg[\forall v_\xi[\neg A_\xi]]]]\ldots] \rightarrow [\neg[\forall v_0\ldots v_\xi\ldots[\neg[\bigwedge A_0\ldots A_\xi\ldots]]]]].$$

Therefore a formula is an instance of $\mathscr{DCH}_\gamma$ if and only if there is a sequence x_1 of formulas having length γ and a sequence x_2 of sequences of variables having length γ, each sequence $x_2(\xi)$ having length less than β, such that $\mathrm{Rng}\, x_2(\xi) \cap \mathrm{Rng}\, x_2(\nu) = \phi$ for $\xi \neq \nu$, $\mathrm{Rng}\, x_2(\xi) \cap FV x_1(\nu) = \phi$ for $\nu < \xi$, and $A = \langle[[\bigwedge\rangle^\frown(^\frown x_3)^\frown\langle]\rightarrow[\neg[\forall\rangle^\frown(^\frown x_2)^\frown\langle[\neg[\bigwedge\rangle^\frown(^\frown x_1)^\frown\langle]]]]]\rangle$ where x_3 is that sequence of length γ such that $x_3(0) = \langle[\neg[\forall\rangle^\frown x_2(0)^\frown\langle[\neg\rangle^\frown x_1(0)^\frown\langle]]]\rangle$ and for $\xi > 0$, $x_3(\xi) = \langle[\forall\rangle^\frown(^\frown x_2|\xi)^\frown\langle[\neg[\forall\rangle^\frown x_2(\xi)^\frown\langle[\neg\rangle^\frown x_1(\xi)^\frown\langle]]]]\rangle$. This can readily be translated into $\mathbf{ML}^\alpha$. If $\beta = \alpha$ the condition on the lengths of the $x_2(\xi)$ is dropped.

The case $\mathscr{DCH}_{\nearrow\alpha}$ is similar. Assert the existence of a non-empty sequence x_1 of formulas and thereafter replace γ by $\mathrm{Dom}(x_1)$.

13.3.19 *Theorem.* Let Ω be any one of the collections $\Omega_{I,\gamma}$ or $\Omega_{D,\gamma}$ for $\omega \leqslant \gamma < \alpha$, or $\Omega_{I,\nearrow\gamma}$ or $\Omega_{D,\nearrow\gamma}$ for $\omega \leqslant \gamma \leqslant \alpha$. Then there is a formula Infer $\Omega(x_0, x_1)$ such that $\langle x_0 x_1\rangle$ satisfies Infer $\Omega(x_0, x_1)$ if and only if x_0, x_1 are Gödel-sequences of formulas C_0, C_1 such that C_1 follows from C_0 by a rule in Ω.

Proof: Consider for example, $\Omega = \Omega_{D,\gamma}$. Then C_1 follows from C_0 by a rule of Ω if and only if C_0 has form $[\bigvee_{\xi<\gamma} A_0 \ldots A_\xi \ldots]$ and C_1 has form $[\bigvee_{\xi<\gamma} [\forall v_0 A_0] \ldots [\exists w_\xi [\forall v_\xi A_\xi]] \ldots]$ where $\mathrm{Rng}(v_\xi) \cap \mathrm{Rng}(v_\nu) = \phi$ for $\xi \neq \nu$, $\mathrm{Rng}(v_\xi) \cap FV(A_\nu) = \phi$ for $\nu < \xi$, and $\mathrm{Rng}(w_\xi) \supseteq FV(A_\xi) \cap \bigcup_{\nu<\xi} \mathrm{Rng}(v_\nu)$. Dropping abbreviated symbols and restoring brackets, $C_0 = [\neg[\bigwedge [\neg A_0] \ldots [\neg A_\xi] \ldots]]$ and $C_1 = [\neg[\bigwedge [\neg[\forall v_0 A_0]] \ldots [\neg[\neg[\forall w_\xi[\neg[\forall v_\xi A_\xi]]]]] \ldots]]$. Therefore C_1 follows from C_0 by Ω if and only if there is a sequence x_2 of formulas having length γ and sequences x_3, x_4 of sequences of variables having length γ, each sequence $x_3(\xi)$, $x_4(\xi)$ having length less than β, such that $\mathrm{Rng}\, x_3(\xi) \cap \mathrm{Rng}\, x_3(\nu) = \phi$ for $\xi \neq \nu$, $\mathrm{Rng}\, x_3(\xi) \cap FV x_2(\nu) = \phi$ for $\nu < \xi$, $\mathrm{Rng}\, x_4(\xi) \supseteq FV(x_2(\xi)) \cap \bigcup \mathrm{Rng}(x_3|\xi)$ and $C_0 = \langle[\neg[\bigwedge\rangle^\frown(^\frown x_5)^\frown\langle]]\rangle$ where x_5 is that sequence of length γ such that $x_5(\xi) = \langle[\neg\rangle^\frown x_2(\xi)^\frown\langle]\rangle$, while $C_1 = \langle[\neg[\bigwedge\rangle^\frown(^\frown x_6)^\frown\langle]]\rangle$ where x_6 is that sequence of length γ such that $x_6(0) = \langle[\neg[\forall\rangle^\frown x_3(0)^\frown x_2(0)^\frown\langle]]\rangle$ and for $\xi > 0$, $x_6(\xi) = \langle[\neg[\neg[\forall\rangle^\frown x_4(\xi)^\frown\langle[\neg[\forall\rangle^\frown x_3(\xi)^\frown x_2(\xi)^\frown\langle]]]]]\rangle$. This can readily be translated into $\mathbf{ML}^\alpha$. In case $\beta = \alpha$ conditions on the lengths of sequences $x_2(\xi)$, $x_3(\xi)$, $x_4(\xi)$, are dropped.

The other cases are similar.

13.3.20 *Theorem.* Let $\mathfrak{P}_{\alpha\beta}(\Sigma; \Omega)(\mathbf{L})$ be any of the systems of Sect. 11.1 for $\mathbf{L}$. Then there is a formula $PRV(x_0)$ of $\mathbf{ML}^\alpha$ such that x_0 satisfies $PRV(x_0)$ in $\mathfrak{M}(\mathbf{L}, g)$ if and only if x_0 is the Gödel-sequence of a provable formula.

Proof: From Theorems 13.3.6, 13.3.7, 13.3.8, 13.3.9, 13.3.18, 13.3.19 it follows that there is a formula $AX(x_0)$ such that x_0 satisfies $AX(x_0)$ if and only if x_0 is the Gödel-sequence of an axiom of the system, and there is a formula Infer $\Omega(x_0, x_1)$ such that $\langle x_0 x_1\rangle$ satisfies Infer $\Omega(x_0, x_1)$ if and only if x_0, x_1 are Gödel-sequences of formulas C_0, C_1 such that C_1 follows from C_0 by a rule in Ω. Therefore, the following formula has the desired property: $PRV(x_0) = \exists x_1[\overline{\mathrm{Seq}}\, x_1 \wedge \forall x_2[x_2 \mathrel{\overline{\in}} \overline{\mathrm{Rng}}\, x_1 \rightarrow \overline{\mathrm{Form}}\, x_2] \wedge x_0 \mathrel{\overline{\in}} \overline{\mathrm{Rng}}\, x_1 \wedge \forall x_3[x_3 \mathrel{\overline{\in}} \overline{\mathrm{Dom}}\, x_1 \rightarrow [AX([x_1 \overline{of} x_3]) \vee \exists x_4[x_4 \mathrel{\overline{\in}} x_3 \wedge \mathrm{Infer}\Omega([x_1 \overline{of} x_4], [x_1 \overline{of} x_3])] \vee A_1(x_1, x_3) \vee A_2(x_1, x_3) \vee A_3(x_1, x_3)]]]$, where

$A_1(x_1, x_3) = \exists x_4 x_5[x_4 \bar{\in} x_3 \wedge x_5 \bar{\in} x_3 \wedge [x_1 \overline{of} x_5] \bar{\equiv} \overline{\mathrm{Seq}_1}\, \overline{lb}^\frown$
$[x_1 \overline{of} x_4]^\frown \overline{\mathrm{Seq}_1}\, \bar{i}^\frown [x_1 \overline{of} x_3]^\frown \overline{\mathrm{Seq}_1}\, \overline{rb}]$. (Modus ponens)
$A_2(x_1, x_3) = \exists x_4[\overline{\mathrm{Seq}}\, x_4 \wedge \overline{\mathrm{Rng}}\, x_4 \bar{\subseteq} \overline{\mathrm{Rng}}[x_1 \bar{|} x_3] \wedge$
$[x_1 \overline{of} x_3] \bar{\equiv} \overline{\mathrm{Seq}_2}\, \overline{lb}\, \bar{c}^\frown [\bar{^\frown} x_4]^\frown \overline{\mathrm{Seq}_1}\, \overline{rb}]$. (Conjunction)
$A_3(x_1, x_3) = \exists x_4 x_5[x_4 \bar{\in} x_3 \wedge \overline{\mathrm{Seq}}\, x_5 \wedge \overline{\mathrm{Dom}}\, x_5 \bar{\in} \bar{\beta} \wedge$
$\forall x_6[x_6 \bar{\in} \overline{\mathrm{Rng}}\, x_5 \rightarrow IVx_6] \wedge [x_1 \overline{of} x_3] \bar{\equiv} \overline{\mathrm{Seq}_2}\, \overline{lb}\, \bar{q}^\frown x_5^\frown$
$[x_1 \overline{of} x_4]^\frown \overline{\mathrm{Seq}_1}\, \overline{rb}]$. (Generalization)

Clause $\overline{\mathrm{Dom}}\, x_5 \bar{\in} \bar{\beta}$ is dropped in case $\beta = \alpha$.

INCOMPLETENESS IN INFINITARY PREDICATE LOGIC

At the International Congress of Logic, Methodology and the Philosophy of Science at Stanford University in 1960, Dana Scott circulated an outline of a proof of the impossibility of a definable complete formal system for (γ^+, γ^+)-languages with a single two-place predicate symbol in addition to the equality symbol. This proof, an adaptation of the classical arguments of Gödel in [5] and Tarski [44], is based on his ideas.

Throughout the chapter α is a regular infinite cardinal, $\mathbf{L}^0$ is the (α, α)-language with special two-place predicate symbol $\bar{\in}$, $\mathbf{L}^\alpha$ is the $(\alpha, \alpha, \alpha, \omega)$-language with additional individual constants $\bar{\delta}$, one for each ordinal $\delta < \alpha$, and the $\nearrow$ α-place operation symbol $\overline{\mathrm{Sq}}$. It is convenient to assume that both languages have as individual variables x_ξ, $\xi < \alpha$. The Gödel-numbering g of the symbols of $\mathbf{L}^0$ and $\mathbf{L}^\alpha$ is fixed throughout the chapter as follows:

$$\begin{array}{ccccccccc}
\bar{=}, & \bar{\in}, & \overline{\mathrm{Sq}}, & [, &], & \neg, & \rightarrow, & \bigwedge, & \forall, \\
| & | & | & | & | & | & | & | & | \\
0 & 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24
\end{array}$$

$$\begin{array}{ccccc}
\bar{0}, & \bar{1}, & \bar{2}, & \ldots\ldots\ldots\ldots\ldots & \bar{\delta}\ldots \\
| & | & | & & | \\
1 & 4 & 7 & & \delta\cdot 3 + 1
\end{array}$$

$$\begin{array}{ccccc}
x_0, & x_1, & x_2, & \ldots\ldots\ldots\ldots\ldots & x_\xi\ldots \\
| & | & | & & | \\
2 & 5 & 8 & & \xi\cdot 3 + 2
\end{array}$$

Both languages are interpreted set-theoretically. For $\mathbf{L}^0$ the intended set-theoretic model is the model $\mathfrak{T}_\alpha = \langle T_\alpha \in\rangle$ of sets hereditarily of power less than α, for $\mathbf{L}^\alpha$ the intended model is $\mathfrak{M}_\alpha =$

$\langle T_\alpha, \mathrm{Sq}, 0, \ldots, \delta, \ldots, \in\rangle$ where Sq is the $\nearrow\alpha$-place identity function. Note that $\mathrm{Sq}(\langle S_0 \ldots S_\xi \ldots\rangle) = \langle S_0 \ldots S_\xi \ldots\rangle$, so that assignment s satisfies $[x_\theta \mathrel{\bar{\equiv}} [\overline{\mathrm{Sq}}\, x_0 \ldots x_\xi \ldots]]$ if and only if $s(x_\theta) = \langle s(x_0) \ldots s(x_\xi) \ldots\rangle$.

Finite formulas of $\mathbf{L}^\alpha$ serve equally well as a metalanguage for both $\mathbf{L}^0$ and $\mathbf{L}^\alpha$. Individual symbols of $\mathbf{ML}^\alpha$, the meta-language of Chapter 13, are translated to individual symbols of $\mathbf{L}^\alpha$ by the mapping e with values $e(x_n) = x_n$ for $n < \omega$, $e(\delta) = \bar{\delta}$ for $\delta < \alpha$, $e(lb) = \bar{9}$, $e(rb) = \overline{12}$, $e(\bar{n}) = \overline{51}$, $e(\bar{\imath}) = \overline{81}$, $e(\bar{c}) = \overline{21}$, $e(\bar{q}) = \overline{24}$, $e(\overline{eq}) = \bar{0}$. Then if y is any individual symbol of $\mathbf{ML}^\alpha$ and s any assignment of variables to T_α, $s^*\langle y\rangle$ in $\mathfrak{M}(\mathbf{L}^0, g)$ is $s^*\langle y\rangle$ in $\mathfrak{M}(\mathbf{L}^\alpha, g)$ and is $s^*\langle e(y)\rangle$ in $\mathfrak{M}_\alpha$. Atomic formulas A of $\mathbf{ML}^\alpha$ are translated to finite formulas $E_0(A)$, $E_\alpha(A)$, one for interpretation in $\mathfrak{M}(\mathbf{L}^0, g)$, the other for interpretation in $\mathfrak{M}(\mathbf{L}^\alpha, g)$:

A	$E_\alpha(A)$	$E_0(A)$
$[y_0 \mathrel{\bar{\equiv}} y_1]$	$[e(y_0) \mathrel{\bar{\equiv}} e(y_1)]$	Same
$[y_0 \mathrel{\bar{\in}} y_1]$	$[e(y_0) \mathrel{\bar{\in}} e(y_1)]$	Same
$[IVy]$	$[\exists x_0[\overline{\mathrm{Ord}}\, x_0 \wedge [e(y) \mathrel{\bar{\equiv}} x_0 \bar{\cdot} \bar{3} \mathbin{\bar{+}} \bar{2}]]$	Same
$[ICy]$	$[\exists x_0[\overline{\mathrm{Ord}}\, x_0 \wedge [e(y) \mathrel{\bar{\equiv}} x_0 \bar{\cdot} \bar{3} \mathbin{\bar{+}} \bar{1}]]$	$\neg[e(y) \mathrel{\bar{\equiv}} e(y)]$
$[BOPy]$	$\neg[e(y) \mathrel{\bar{\equiv}} e(y)]$	Same
$[IOPy]$	$[e(y) \mathrel{\bar{\equiv}} \bar{6}]$	$\neg[e(y) \mathrel{\bar{\equiv}} e(y)]$
$[BPRy]$	$[e(y) \mathrel{\bar{\equiv}} \bar{0} \vee e(y) \mathrel{\bar{\equiv}} \bar{3}]$	Same
$[IPRy]$	$\neg[e(y) \mathrel{\bar{\equiv}} e(y)]$	Same
$[OPy_0y_1]$	$\neg[e(y_0) \mathrel{\bar{\equiv}} e(y_0) \wedge e(y_1) \mathrel{\bar{\equiv}} e(y_1)]$	Same
$[PRy_0y_1]$	$\neg[e(y_0) \mathrel{\bar{\equiv}} e(y_0) \wedge e(y_1) \mathrel{\bar{\equiv}} e(y_1)]$	Same

Symbols $\overline{\mathrm{Ord}}$, $\bar{\cdot}$, $\bar{+}$, are to be eliminated using definitions of Sect. 13.2. Then if s is any assignment of variables to T_α, A any atomic formula of $\mathbf{ML}^\alpha$, $V(s, A)$ in $\mathfrak{M}(\mathbf{L}^0, g)$ is $V(s, E_0(A))$ in $\mathfrak{M}_\alpha$ and $V(s, A)$ in $\mathfrak{M}(\mathbf{L}^\alpha, g)$ is $V(s, E_\alpha(A))$ in $\mathfrak{M}_\alpha$. Functions E_0, E_α are extended to all formulas of $\mathbf{ML}^\alpha$ by recursion:

$$\begin{aligned}
&E_i([\neg A]) = [\neg E_i(A)]\\
&E_i([A_0 \rightarrow A_1]) = [E_i(A_0) \rightarrow E_i(A_1)]\\
&E_i([\bigwedge A_0 \ldots A_n]) = [\bigwedge E_i(A_0) \ldots E_i(A_n)]\\
&E_i([\forall v A]) = [\forall v E_i(A)],\ i = 0, \alpha.
\end{aligned}$$

The recursion principle guarantees the existence and uniqueness of functions E_0, E_α. Note that the free variables of A are the same as the free variables of $E_0(A)$ and $E_\alpha(A)$. Moreover an easy induction on formulas shows that for all formulas of $\mathbf{ML}^\alpha$, $V(s, A)$ in $\mathfrak{M}(\mathbf{L}^0, g)$ is $V(s, E_0(A))$ in $\mathfrak{M}_\alpha$ and $V(s, A)$ in $\mathfrak{M}(\mathbf{L}^\alpha, g)$ is $V(s, E_\alpha(A))$ in $\mathfrak{M}_\alpha$.

To say that there is a definable complete formal system for $\mathbf{L}^0$ is to say that there is a formula $PRV(x_0)$ of $\mathbf{ML}^\alpha$ such that for all $S \in T_\alpha$, S satisfies $PRV(x_0)$ in $\mathfrak{M}(\mathbf{L}^0, g)$ if and only if S is the Gödelsequence of a valid formula of $\mathbf{L}^0$. But if there is such a formula $PRV(x_0)$, then $E_0(PRV(x_0))$ is a formula of $\mathbf{L}^\alpha$ satisfied in $\mathfrak{M}_\alpha$ by precisely the Gödel-sequences of valid formulas of $\mathbf{L}^0$. Therefore if there is a definable complete formal system for $\mathbf{L}^0$ then the set of valid formulas of $\mathbf{L}^0$ is definable in $\mathfrak{M}_\alpha$ by a formula of $\mathbf{L}^\alpha$. We go on to show that this is not possible for infinite non-limit cardinals α.

14.1 The Basic Undefinability Argument

Let Δ be the set of formulas of $\mathbf{L}^\alpha$ that hold in $\mathfrak{M}_\alpha$. The classical undefinability-of-truth argument of Tarski can be used to show that Δ is not definable by a formula of $\mathbf{L}^\alpha$ relative to $\mathfrak{M}_\alpha$. The elegant formulation in [45] is especially useful.

Let m be that function on T_α such that $m(S)$ is the Gödel-sequence of the term $[\overline{\mathrm{Sq}}\,\overline{S(0)}\ldots\overline{S(\xi)}\ldots]$ if S is a non-empty sequence of ordinals, ϕ if not. Note that the desired Gödel-sequence is $\langle 9\ 6\ (S(0)\cdot 3)+1\ldots(S(\xi)\cdot 3)+1\ldots 12\rangle$. Using definitions of Sect. 13.2 to eliminate defined symbols, we see that m is definable relative to $\mathfrak{M}_\alpha$ by a finite formula in $\overline{\equiv}$, $\overline{\in}$ alone:

$$M(x_0, x_2) = [\neg C(x_0) \rightarrow x_2 \overline{\equiv} \bar{0}] \wedge [C(x_0) \rightarrow \overline{\mathrm{Seq}}\, x_2 \wedge \overline{\mathrm{Dom}}\, x_2 \overline{\equiv} \bar{2} \mathbin{\overline{+}} \overline{\mathrm{Dom}}\, x_0 \mathbin{\overline{+}} \bar{1} \wedge [x_2 \mathbin{\overline{of}} \bar{0}] \overline{\equiv} \bar{9} \wedge [x_2 \mathbin{\overline{of}} \bar{1}] \overline{\equiv} \bar{6} \wedge [x_2 \mathbin{\overline{of}} [2 \mathbin{\overline{+}} \overline{\mathrm{Dom}}\, x_0]] \overline{\equiv} \overline{12} \wedge \forall x_3[x_3 \overline{\in} \mathrm{Dom}\, x_0 \rightarrow [x_2 \mathbin{\overline{of}} \bar{2} \mathbin{\overline{+}} x_3] \overline{\equiv} [x_0 \mathbin{\overline{of}} x_3] \mathbin{\overline{\cdot}} \bar{3} \mathbin{\overline{+}} 1]] \text{ where } C(x_0) \overline{\equiv} \overline{\mathrm{Seq}}\, x_0 \wedge \neg x_0 \overline{\equiv} \bar{0} \wedge \forall x_3[x_3 \overline{\in} \overline{\mathrm{Dom}}\, x_0 \rightarrow \overline{\mathrm{Ord}}[x_0 \mathbin{\overline{of}} x_3]].$$

Let SF be the function described in Lemma 13.3.16. It is definable relative to $\mathfrak{M}(\mathbf{L}^\alpha, g)$ by a formula of $\mathbf{ML}^\alpha$. Applying the operator E_α to the defining formula, we obtain a finite formula of $\mathbf{L}^\alpha$ with free variables x_0, x_1, x_2, x_3 which is satisfied in $\mathfrak{M}_\alpha$ by exactly those four-tuples $\langle S_0 S_1 S_2 S_3\rangle$ such that $S_3 = SF(S_0, S_1, S_2)$. Call this formula $\mathrm{Sub}(x_0, x_1, x_2, x_3)$. Then if S_0 is the Gödel-sequence of a formula, S_1 a set of Gödel-numbers of variables, S_2 a function on S_1 to Gödel-sequences of terms of $\mathbf{L}^\alpha$, then $\langle S_0 S_1 S_2 S_3 \rangle$ satisfies $\mathrm{Sub}(x_0, x_1, x_2, x_3)$ if and only if $S_3 = g \bigcirc SF_f^X A$ where $A = g^{-1} \bigcirc S_0$, $X = \{x : g(x) \in S_1\}$ and $f = g^{-1} \bigcirc S_2 \bigcirc g$ on X.

14.1.1 *Theorem.* The set Δ is not definable by a formula of $\mathbf{L}^\alpha$ relative to $\mathfrak{M}_\alpha$.

PROOF: Let

$N(x_0, x_1) = [\exists x_2[\mathrm{M}(x_0, x_2) \wedge \mathrm{Sub}(x_0, [\overline{\mathrm{Set}}_1\, \bar{2}], [\overline{\mathrm{Set}}_1[\overline{\mathrm{Pr}}\, \bar{2}\, x_2]], x_1)]]$. This is again a finite formula of $\mathbf{L}^\alpha$. If S_0 is the Gödel-sequence of a formula C of $\mathbf{L}^\alpha$, then $\langle S_0 S_1 \rangle$ satisfies $N(x_0, x_1)$ in $\mathfrak{M}_\alpha$ if and only if $S_1 = g \bigcirc SF_f^{\{x_0\}} C$, where $f(x_0) = [\overline{\mathrm{Sq}}\, \overline{gC(0)} \ldots \overline{gC(\xi)} \ldots]$.

Suppose, contrariwise, there is a formula $D(x_1)$ of $\mathbf{L}^\alpha$ defining Δ relative to $\mathfrak{M}_\alpha$. Then let $A = [\forall x_1[N(x_0, x_1) \rightarrow \neg D(x_1)]]$. Let $B = SF_f^{\{x_0\}} A$ where $f(x_0) = [\overline{\mathrm{Sq}}\, \overline{gA(0)} \ldots \overline{gA(\xi)} \ldots]$. Then B holds in $\mathfrak{M}_\alpha$ if and only if $g \bigcirc A$ satisfies A. But then B holds in $\mathfrak{M}_\alpha$ if and only if $SF_f^{\{x_0\}} A$ is not in Δ. That is to say, B holds in $\mathfrak{M}_\alpha$ if and only if B does not hold in $\mathfrak{M}_\alpha$. We can only conclude that there was no such formula $D(x_1)$.

14.2 The Incompleteness of Definable Formal Systems when α Non-limit, $\beta = \alpha$

Let Δ be the set of formulas of $\mathbf{L}^\alpha$ that hold in $\mathfrak{M}_\alpha$ as before, let Γ be the set of formulas of $\mathbf{L}^0$ that hold in $\mathfrak{T}_\alpha$. We first show that if Γ is definable relative to $\mathfrak{M}_\alpha$ by a formula of $\mathbf{L}^\alpha$ then Δ is also so definable. The proof rests on the fact that ordinals $\delta < \alpha$ and the $\nearrow \alpha$-place function Sq can be defined by formulas of $\mathbf{L}^0$. Therefore the additional symbols $\bar{\delta}$ and $\overline{\mathrm{Sq}}$ of $\mathbf{L}^\alpha$ are really superfluous. They were only introduced because it seems to be very difficult to write a Tarski-sentence without constant terms to stand for the Gödel-sequences.

Let $EQ_0(x_0) = [\forall x_1[\neg x_1 \mathrel{\bar{\in}} x_0]]$ and for $0 < \delta < \alpha$ let $EQ_\delta(x_\delta) =$

$$\underset{\nu<\delta}{\exists} x_0 \ldots x_\nu \ldots [[\forall x_{\delta+1}[\neg x_{\delta+1} \mathrel{\bar{\in}} x_0]] \wedge [\underset{1 \leqslant \nu < \delta+1}{\bigwedge} [\forall x_{\delta+1}[x_{\delta+1} \mathrel{\bar{\in}} x_\nu \leftrightarrow [\underset{\eta<\nu}{\bigvee} x_{\delta+1} \mathrel{\bar{\equiv}} x_\eta]]]]]].$$

It is easy to see that S satisfies $EQ_\delta(x_\delta)$ in $\mathfrak{T}_\alpha$ if and only if $S = \delta$.

Similarly for each $\delta < \alpha$, $\delta \neq 0$, let $SEQ_\delta(x_0, \ldots, x_\xi, \ldots, x_\delta) = \forall x_{\delta+1}[x_{\delta+1} \bar{\in} x_\delta \leftrightarrow [\bigvee_{\xi<\delta} [\forall x_{\delta+2} x_{\delta+3}[[\forall x_{\delta+4}[x_{\delta+4} \bar{\in} x_{\delta+2} \leftrightarrow EQ_\xi(x_{\delta+4})]] \wedge [\forall x_{\delta+4}[x_{\delta+4} \bar{\in} x_{\delta+3} \leftrightarrow EQ_\xi(x_{\delta+4}) \vee x_{\delta+4} \bar{\equiv} x_\xi]]] \rightarrow \forall x_{\delta+4}[x_{\delta+4} \bar{\in} x_{\delta+1} \leftrightarrow x_{\delta+4} \bar{\equiv} x_{\delta+2} \vee x_{\delta+4} \bar{\equiv} x_{\delta+3}]]]]$. Then $\langle S_0 \ldots S_\xi \ldots S_\delta \rangle$ satisfies $SEQ_\delta(x_0, \ldots, x_\xi, \ldots, x_\delta)$ in $\mathfrak{T}_\alpha$ if and only if elements of S_δ are $\{\{\xi\}, \{\xi, S_\xi\}\}$ for $\xi < \delta$, which is to say, if and only if $S_\delta = \langle S_0 \ldots S_\xi \ldots \rangle$. Let Γ' be the set of all formulas $[\forall x_\delta[x_\delta \bar{\equiv} \bar{\delta} \leftrightarrow EQ_\delta(x_\delta)]]$ or

$$[\forall x_0 \ldots x_\xi \ldots x_\delta[x_\delta = \overline{\mathrm{Sq}}\, x_0 \ldots x_\xi \ldots \leftrightarrow SEQ_\delta(x_0, \ldots, x_\xi, \ldots, x_\delta)]]$$

for some $\delta < \alpha$. Formulas of Γ' hold in $\mathfrak{M}_\alpha$.

14.2.1 *Lemma*. A formula A of $\mathbf{L}^\alpha$ holds in $\mathfrak{M}_\alpha$ if and only if $\vdash_{\Gamma \cup \Gamma'} A$ in $\mathfrak{P}_{\alpha\alpha}(\mathbf{L}^\alpha)$.

PROOF: Terms of $\mathbf{L}^\alpha$ are built from atomic terms $\langle x_\xi \rangle$ and $\langle \bar{\delta} \rangle$ by means of the $\nearrow \alpha$-place operation symbol $\overline{\mathrm{Sq}}$. The recursion principle for terms guarantees the existence of a unique function F on terms of $\mathbf{L}^\alpha$ to formulas of $\mathbf{L}^0$ such that

$$F(\langle x_\xi \rangle)(x_0) = [x_0 \bar{\equiv} x_\xi], \quad \xi < \alpha$$
$$F(\langle \bar{\delta} \rangle)(x_0) = EQ_\delta(x_0), \quad \delta < \alpha$$
$$F([\overline{\mathrm{Sq}}\; T_0 \ldots T_\xi \ldots])(x_0) = [\bigvee_{\xi<\delta} x_{\mu_0} \ldots x_{\mu_\xi} \ldots [[\bigwedge_{\xi<\delta} F(T_\xi)(x_{\mu_\xi})] \rightarrow$$

$SEQ_\delta(x_{\mu_0} \ldots x_{\mu_\xi} \ldots x_0)]]$, for all δ-tuples of terms $\langle T_\xi \colon \xi < \delta \rangle$ where $\langle x_{\mu_\xi} \colon \xi < \delta \rangle$ is the sequence of variables of least index not in any of the T_ξ and different from x_0.

Let $\mathsf{T} = \{T \colon T$ is a term of $\mathbf{L}^\alpha$ and $\vdash_{\Gamma'} \forall x_0[F(T)(x_0) \leftrightarrow x_0 \bar{\equiv} T]$ in $\mathfrak{P}_{\alpha\alpha}(\mathbf{L}^\alpha)\}$. We wish to show that all terms of $\mathbf{L}^\alpha$ are in T. In case T is a variable standing alone, $F(T)(x_0) = [x_0 \bar{\equiv} T]$. Such T are obviously in T. In case $T = \langle \bar{\delta} \rangle$, $[F(T)(x_0) \leftrightarrow x_0 \bar{\equiv} T]$ is a substitution of a quantification in Γ'. Such a term is in T by 11.2.9 and generalization. Suppose $T = [\overline{\mathrm{Sq}}\; T_0 \ldots T_\xi \ldots]$ where $T_\xi \in \mathsf{T}$ for all $\xi < \delta$. Since this formula is a substitution of a quantification in Γ',

(1) $\vdash_{\Gamma'} x_0 \bar{\equiv} T \leftrightarrow SEQ_\delta(T_0, \ldots, T_\xi, \ldots, x_0)$.

By another substitution

(2) $\vdash F(T)(x_0) \rightarrow [[\bigwedge_{\xi<\delta} F(T_\xi)(T_\xi)] \rightarrow SEQ_\delta(T_0, \ldots, T_\xi, \ldots, x_0)]$.

By induction hypothesis,

(3) $\vdash_{\Gamma'} F(T_\xi)(T_\xi) \leftrightarrow T_\xi \equiv T_\xi$ for all $\xi < \delta$.

Since $\vdash_{\Gamma'} [\bigwedge_{\xi<\delta} F(T_\xi)(T_\xi)]$ by (3) and the equality axioms,

(4) $\vdash_{\Gamma'} F(T)(x_0) \rightarrow x_0 \equiv T$

by propositional calculus and replacement of (1) in (2).

Conversely, the equality axioms yield

(5) $\vdash x_0 \equiv T \rightarrow [[\bigwedge_{\xi<\delta} x_{\mu_\xi} \equiv T_\xi] \rightarrow x_0 \equiv \overline{\mathrm{Sq}}\, x_{\mu_0} \ldots x_{\mu_\xi} \ldots]$.

By the induction hypothesis, 11.2.9, and the Equivalence Theorem,

(6) $\vdash_{\Gamma'} [\bigwedge_{\xi<\delta} F(T_\xi)(x_{\mu_\xi}) \leftrightarrow [\bigwedge_{\xi<\delta} x_{\mu_\xi} \equiv T_\xi]$.

Another substitution of a quantification in Γ' yields

(7) $\vdash_{\Gamma'} x_0 \equiv \overline{\mathrm{Sq}}\, x_{\mu_0} \ldots x_{\mu_\xi} \ldots \leftrightarrow SEQ_\delta(x_{\mu_0}, \ldots, x_{\mu_\xi}, \ldots, x_0)$.

After replacement of (6), (7) in (5) and generalization on $x_{\mu_0}, \ldots, x_{\mu_\xi}, \ldots,$

(8) $\vdash_{\Gamma'} x_0 \equiv T \rightarrow F(T)(x_0)$.

Hence with (4) we have $T \in \mathbf{T}$.

Suppose formula A of $\mathbf{L}^\alpha$ holds in $\mathfrak{M}_\alpha$. Write A in the form $E_0C_0 \ldots E_\nu C_\nu \ldots E_\sigma$ where $\langle C_\nu : \nu < \sigma \rangle$ is a list containing all the atomic subformulas with symbols $\bar{\delta}$, $\delta < \alpha$, and $\overline{\mathrm{Sq}}$. Then each C_ν has form $[T_0 \equiv T_1]$ or $[T_0 \mathrel{\bar{\in}} T_1]$. Let $C'_\nu = [\exists x_0 x_1 [F(T_0)(x_0) \wedge F(T_1)(x_1) \wedge x_0 \equiv x_1]$ in the first case and $[\exists x_0 x_1 [F(T_0)(x_0) \wedge F(T_1)(x_1) \wedge x_0 \mathrel{\bar{\in}} x_1]$ in the second. From $\vdash_{\Gamma'} \forall x_0 [F(T_i)(x_0) \leftrightarrow x_0 \equiv T_i]$, $i = 0, 1$, it follows that $\vdash_{\Gamma'} C_\nu \leftrightarrow C'_\nu$ for all $\nu < \sigma$. According to the Replacement Principle, $\vdash_{\Gamma'} A \leftrightarrow A'$ where $A' = E_0 C'_0 \ldots E_\nu C'_\nu \ldots E_\sigma$. But A' is a formula of $\mathbf{L}^0$ since it contains no occurrences of constants $\bar{\delta}$ or $\overline{\mathrm{Sq}}$ and since formulas provable from Γ' hold in $\mathfrak{M}_\alpha$, A' holds in $\mathfrak{M}_\alpha$. But for formulas of $\mathbf{L}^0$, this is the same as saying that the formula holds in $\mathfrak{T}_\alpha$. Hence $A' \in \Gamma$. Therefore $\vdash_{\Gamma \cup \Gamma'} A$.

14.2.2 *Lemma.* The set Γ' of definitions of constants $\bar{\delta}$, $\overline{\mathrm{Sq}}$ is definable relative to $\mathfrak{M}_\alpha$ by a formula of $\mathbf{L}^\alpha$.

PROOF: Though we make no use of this, Γ' is in fact definable by a finite formula of $\mathbf{L}^0$. Since the bound variables of $EQ_\delta(x_\delta)$ are

$\{x_\nu : \nu < \delta\} \cup \{x_{\delta+1}\}$, $EQ_\delta(x_\kappa)$ is the result of substituting all occurrences of x_δ by x_κ in $EQ_\delta(x_\delta)$ whenever $\kappa \geqslant \delta$ and $\kappa \neq \delta + 1$. No bound variables need to be changed. Eliminating defined quantifiers and propositional symbols and restoring missing brackets,

$$EQ_0(x_\kappa) = [\forall x_1[\neg[x_1 \bar{\in} x_\kappa]]], \quad \kappa \neq 1.$$
$$EQ_\delta(x_\kappa) = [\neg[\bigvee_{\nu<\delta} x_0 \ldots x_\nu \ldots [\neg[\bigwedge [\forall x_{\delta+1}[\neg[x_{\delta+1} \bar{\in} x_0]]] C_1 \ldots C_\nu \ldots]]]]$$

for $\delta \neq 0$, $\kappa \geqslant \delta$, $\kappa \neq \delta + 1$, where

$$C_\nu = [\forall x_{\delta+1}[\bigwedge [[x_{\delta+1} \bar{\in} x_\nu] \to C'_\nu][C'_\nu \to [x_{\delta+1} \bar{\in} x_\nu]]]], \quad 0 < \nu < \delta,$$
$$C_\nu = [\forall x_{\delta+1}[\bigwedge [[x_{\delta+1} \bar{\in} x_\kappa] \to C'_\nu][C'_\nu \to [x_{\delta+1} \bar{\in} x_\kappa]]]], \quad \nu = \delta,$$

and

$$C'_\nu = [\neg[\bigwedge_{\eta<\nu} \ldots [\neg[x_{\delta+1} \bar{\equiv} x_\eta]] \ldots]], \quad 0 < \nu \leqslant \delta.$$

Let $p(S)$ be the Gödel-sequence of the expression $\langle x_\nu : \nu < S\rangle$ if S is an ordinal, ϕ if not. Then $p(S) = \langle \nu \cdot 3 + 2 : \nu < S\rangle$ or ϕ, obviously a definable function. Let $P(x_0, x_1)$ be its defining formula. Let $q'(S_0, S_1)$ be the Gödel-sequence of C'_{S_1} for $\delta = S_0$ if S_0, S_1 are ordinals and $0 \in S_1 \in S_0 + 1$, ϕ if not. This function is defined by the following formula:

$$Q'(x_0, x_1, x_2) = [\neg D \to x_2 \bar{\equiv} \bar{0}] \wedge [D \to \forall x_3[\overline{\mathrm{Seq}}\, x_3 \wedge [\overline{\mathrm{Dom}}\, x_3] \bar{\equiv} x_1 \wedge$$
$$\forall x_4[x_4 \bar{\in} x_1 \to [x_3 \,\overline{of}\, x_4] \bar{\equiv}$$
$$\overline{\mathrm{Seq}}_8\, \bar{9}\, \overline{15}\, \bar{9}\, [[x_0 \bar{+} \bar{1}] \bar{\cdot} \bar{3} \bar{+} \bar{2}]\, \bar{0}\, [x_4 \bar{\cdot} \bar{3} \bar{+} \bar{2}]\, \overline{12}\, \overline{12}] \to$$
$$x_2 \bar{\equiv} [\overline{\mathrm{Seq}}_4\, \bar{9}\, \overline{15}\, \bar{9}\, \overline{21}] \bar{\frown} [\bar{\frown} x_3] \bar{\frown} [\overline{\mathrm{Seq}}_2\, \overline{12}\, \overline{12}]]]],$$

where

$$D = [\overline{\mathrm{Ord}}\, x_0 \wedge \overline{\mathrm{Ord}}\, x_1 \wedge \bar{0} \bar{\in} x_1 \wedge x_1 \bar{\in} x_0 \bar{+} \bar{1}].$$

The defined symbols are to be eliminated using their definitions in Sect. 13.2. Let $q(S_0, S_1, S_2)$ be the Gödel-sequence of C_{S_1} for $\delta = S_0$, $\kappa = S_2$ if S_0, S_1, S_2 are ordinals and $0 \in S_1 \in S_0 + 1$, $S_0 \in S_2 + 1$, $S_2 \neq S_0 + 1$, ϕ if not. Then q is defined by

$$Q(x_0, x_1, x_2, x_3) = [\neg D \to x_3 \bar{\equiv} \bar{0}] \wedge [D \to \forall x_4[Q'(x_0, x_1, x_4) \to$$
$$[[x_1 \bar{\in} x_0 \to x_3 \bar{\equiv} [\overline{\mathrm{Seq}}_{12}\, \bar{9}\, \overline{24}\, [[x_0 \bar{+} \bar{1}] \bar{\cdot} \bar{3} \bar{+} \bar{2}]\, \bar{9}\, \overline{21}\, \bar{9}\, \bar{9}\, [[x_0 \bar{+} \bar{1}] \bar{\cdot} \bar{3} \bar{+} \bar{2}]$$
$$\bar{3}\, [x_1 \bar{\cdot} \bar{3} \bar{+} \bar{2}]\, \overline{12}\, \overline{18}] \bar{\frown} x_4 \bar{\frown} \overline{\mathrm{Seq}}_2\, \overline{12}\, \bar{9} \bar{\frown} x_4 \bar{\frown} [\overline{\mathrm{Seq}}_9\, \overline{18}\, \bar{9}\, [[x_0 \bar{+} \bar{1}] \bar{\cdot} \bar{3} \bar{+} \bar{2}]$$
$$\bar{3}\, [x_1 \bar{\cdot} \bar{3} \bar{+} \bar{2}]\, \overline{12}\, \overline{12}\, \overline{12}\, \overline{12}]] \wedge [x_1 \bar{\equiv} x_0 \to x_3 \bar{\equiv} T]]]],$$

where T is the term like the one written out except that x_1 is replaced by x_2, and

$$D = [\overline{\mathrm{Ord}}\, x_0 \wedge \overline{\mathrm{Ord}}\, x_1 \wedge \overline{\mathrm{Ord}}\, x_2 \wedge \bar{0} \mathrel{\overline{\in}} x_1 \wedge x_1 \mathrel{\overline{\in}} x_0 \mathbin{\overline{+}} \bar{1} \wedge \\ x_0 \mathrel{\overline{\in}} x_2 \mathbin{\overline{+}} \bar{1} \wedge \neg x_2 \mathrel{\overline{\equiv}} x_0 \mathbin{\overline{+}} \bar{1}].$$

Finally, let $r(S_0, S_2)$ be the Gödel-sequence of $EQ_{S_0}(x_{S_2})$ if S_0, S_2 are ordinals and $S_0 \in S_2 + 1$ and $S_2 \neq S_0 + 1$, ϕ otherwise. A defining formula for r is

$$R(x_0, x_2, x_3) = [\neg D \rightarrow x_3 \mathrel{\overline{\equiv}} \bar{0}] \wedge [D \rightarrow [[x_0 \mathrel{\overline{\equiv}} \bar{0} \wedge x_3 \boxminus \overline{\mathrm{Seq}}_{12}\, \bar{9}\, \overline{24}\, \bar{5}\, \bar{9} \\ \overline{15}\, \bar{9}\, \bar{5}\, \bar{3}\, [x_2 \mathbin{\bar{\cdot}} \bar{3} \mathbin{\overline{+}} \bar{2}]\, \overline{12}\, \overline{12}\, \overline{12}] \vee [\neg x_0 \mathrel{\overline{\equiv}} \bar{0} \wedge \forall x_4 x_5 [[P(x_0, x_4) \wedge \\ \overline{\mathrm{Seq}}\, x_5 \wedge \overline{\mathrm{Dom}}\, x_5 \mathrel{\overline{\equiv}} [x_0 \mathbin{\overline{+}} \bar{1}] \wedge [x_5 \mathbin{\overline{of}} \bar{0}] \mathrel{\overline{\equiv}} \bar{0} \wedge \forall x_1 [\bar{0} \mathrel{\overline{\in}} x_1 \wedge \\ x_1 \mathrel{\overline{\in}} [x_0 \mathbin{\overline{+}} \bar{1}] \rightarrow Q(x_0, x_1, x_2, [x_5 \mathbin{\overline{of}} x_1])]] \rightarrow x_3 \mathrel{\overline{\equiv}} [\overline{\mathrm{Seq}}_4\, \bar{9}\, \overline{15}\, \bar{9}\, \overline{24}] \mathbin{\overline{\frown}} x_4 \mathbin{\overline{\frown}} \\ \overline{\mathrm{Seq}}_{16}\, \bar{9}\, \overline{15}\, \bar{9}\, \overline{21}\, \bar{9}\, \overline{24}\, [[x_0 \mathbin{\overline{+}} \bar{1}] \mathbin{\bar{\cdot}} \bar{3} \mathbin{\overline{+}} \bar{2}]\, \bar{9}\, \overline{15}\, \bar{9}\, [[x_0 \mathbin{\overline{+}} \bar{1}] \mathbin{\bar{\cdot}} \bar{3} \mathbin{\overline{+}} \bar{2}] \\ \bar{3}\, \bar{2}\, \overline{12}\, \overline{12}\, \overline{12}] \mathbin{\overline{\frown}} [\overline{\frown} x_5] \mathbin{\overline{\frown}} \overline{\mathrm{Seq}}_4\, \overline{12}\, \overline{12}\, \overline{12}\, \overline{12}]]]],$$

where

$$D = [\overline{\mathrm{Ord}}\, x_0 \wedge \overline{\mathrm{Ord}}\, x_2 \wedge x_0 \mathrel{\overline{\in}} x_2 \mathbin{\overline{+}} \bar{1} \wedge \neg x_2 \mathrel{\overline{\equiv}} x_0 \mathbin{\overline{+}} \bar{1}].$$

By now it should be clear that the function assigning the Gödel-sequence of $SEQ_\delta(x_0, \ldots, x_\xi, \ldots, x_\delta)$ to ordinals δ is also definable. The function r was defined as a function of two variables so that there would be no difficulty bringing the formulas $EQ_\xi(x_{\delta+4})$ into this definition. With formulas to provide passage from ordinals δ to formulas $EQ_\delta(x_\delta)$ and $SEQ_\delta(x_0, \ldots, x_\xi, \ldots, x_\delta)$, there is no difficulty in writing a defining formula for Γ'.

14.2.3 *Lemma.* The set Γ of formulas of $\mathbf{L}^0$ holding in $\mathfrak{T}_\alpha$ is not definable relative to $\mathfrak{M}_\alpha$ be a formula of $\mathbf{L}^\alpha$.

PROOF: If Γ were definable, then it would follow by 14.2.2 that $\Gamma \cup \Gamma'$ is also definable. It is easy to see that then the set of formulas of $\mathbf{L}^\alpha$ provable from $\Gamma \cup \Gamma'$ in $\mathfrak{P}_{\alpha\alpha}(\mathbf{L}^\alpha)$ would also be definable.

To see this, adjoin to the metalanguage $\mathbf{ML}^\alpha$ a new one-place predicate symbol $\bar{G}$ to be interpreted in $\mathfrak{M}(\mathbf{L}^\alpha, g)$ as the set G of Gödel-sequences of formulas in $\Gamma \cup \Gamma'$. Replace the clause $\mathrm{AX}([x_1 \mathbin{\overline{of}} x_3])$ in the formula $PRV(x_0)$ of Theorem 13.3.20 by $B([x_1 \mathbin{\overline{of}} x_3]) \vee \bar{G}[x_1 \mathbin{\overline{of}} x_3]$ where $B(x_0)$ is a formula defining the logical axioms of $\mathfrak{P}_{\alpha\alpha}(\mathbf{L}^\alpha)$ relative to $\mathfrak{M}(\mathbf{L}^\alpha, g)$. Then in the enlarged model $PRV(x_0)$ is satisfied by exactly the Gödel-sequences of formulas provable from $\Gamma \cup \Gamma'$ in $\mathfrak{P}_{\alpha\alpha}(\mathbf{L}^\alpha)$. Extending the mapping E_α of the introduction to this chapter so that $E_\alpha([\bar{G}y]) = C(e(y))$, $C(x_0)$ being the defining formula for G assumed to exist, the formula $E_\alpha(PRV(x_0))$ will be satisfied in $\mathfrak{M}_\alpha$ by exactly the Gödel-sequences of formulas provable from $\Gamma \cup \Gamma'$ in $\mathfrak{P}_{\alpha\alpha}(\mathbf{L}^\alpha)$. But Lemma 14.2.1 tells us that this set of formulas is Δ, the set of formulas of $\mathbf{L}^\alpha$ that hold in $\mathfrak{M}_\alpha$, while Theorem 14.1.1 says that this very set is not definable. We can only conclude that there was no such formula $C(x_0)$ and that therefore Γ was not definable.

14.2.4 *Lemma.* If there is a single sentence of $\mathbf{L}^0$ characterizing the model $\mathfrak{T}_\alpha$ up to isomorphism, then there is no definable complete formal system for $\mathbf{L}^0$.

PROOF: If sentence B characterizes $\mathfrak{T}_\alpha$ and $PRV(x_0)$ is a formula of $\mathbf{ML}^\alpha$ defining the set of valid formulas of $\mathbf{L}^0$ relative to $\mathfrak{M}(\mathbf{L}^0, g)$, then $\mathrm{Val}(x_0) = E_0(PRV(x_0))$ is a defining formula for the set of valid formulas of $\mathbf{L}^0$ relative to $\mathfrak{M}_\alpha$. But since a formula A of $\mathbf{L}^0$ holds in $\mathfrak{T}_\alpha$ if and only if $\Vdash [B \to A]$, Γ can now be defined relative to $\mathfrak{M}_\alpha$ as follows:

$$C(x_0) = \forall x_1[x_1, \bar{\bar{=}}\, [[\overline{\mathrm{Sq}}\bar{9}\overline{gB(0)} \ldots \overline{gB(\xi)} \ldots \bar{1}\underline{\bar{8}}]^\frown x_0{}^\frown \overline{\mathrm{Sq}}\,\overline{12}] \to \mathrm{Val}(x_1)].$$

After eliminating the concatenation symbol $^\frown$ with the aid of 13.2.25, we are left with a formula of $\mathbf{L}^\alpha$ defining Γ. Since we have just seen that Γ is not so definable, either there is no such formula B or no such formula $PRV(x_0)$.

The main conclusion is that since we have already exhibited a sentence characterizing $\mathfrak{T}_\alpha$ up to isomorphism when α is a non-limit cardinal (Example 1.1.5, Chapter 1), there can be no definable complete formal system for $\mathbf{L}^0$ in this case.

14.2.5 *Theorem.* (Scott). For infinite non-limit cardinals α, there is no definable complete formal system for the (α, α)-language $\mathbf{L}^0$. More generally, there is no complete formal system for $\mathbf{L}^0$ definable relative to $\mathfrak{M}_\alpha$ by a formula of $\mathbf{L}^\alpha$.

PROOF: By 14.2.4, Example 1.1.5.

It follows incidentally that for no regular infinite α can there exist a definable (relative to $\mathfrak{M}_\alpha$) set Γ' of sentences of $\mathbf{L}^0$ and a definable formal system for $\mathbf{L}^0$ such that for all formulas A of $\mathbf{L}^0$, $\vdash_{\Gamma'} A$ if and only if A holds in $\mathfrak{T}_\alpha$.

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INDEX OF SYMBOLS

SET-THEORETIC

ALGEBRAIC

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