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Dedicated to

PAUL BERNAYS

and

KURT GÖDEL

PREFACE

The Logic Colloquium '78 took place in Mons, Belgium, from August 24 through September 1, and was the summer meeting of the European branch of the Association for Symbolic Logic. The conference, which attracted over 150 participants, was organized around three main themes: Constructive Mathematics, Model Theory and Set Theory. There were 16 invited talks and 57 contributed papers; in addition, three survey courses of 4 lectures each were given by G. Cherlin, S. Feferman and J.E. Fenstad. Informal evening sessions were organized by the participants; at one of these the problem of financing Logic meetings was discussed and the consensus was to continue to work with civil sources of funds.

The conference was dedicated to the memories of Paul Bernays (1888-1977) and Kurt Gödel (1906-1978), two scholars who will be remembered forever for their decisive role in the delicate process of establishing Logic as a modern scientific discipline.

The Organizing Committee consisted on M. Boffa, D. van Dalen (co-chairmen), H.P. Barendregt, P. Henrard and K. McAloon. Invaluable assistance was given by Y. Vermeulen, the administrative secretary of the meeting, and L. Bouchez, the secretary of the Mathematics Department of the Université de l'Etat à Mons.

Thanks are due to Michelle Boffa, Hendrik Chaltin, Noëlle Henrard, Françoise Point and Gilbert van den Bossche who generously volunteered their help during the meeting. We are indebted to the Faculté des Sciences de l'Université de l'Etat à Mons and the Faculté Polytechnique de Mons for their hospitality. The Mathematics Department of the Université Catholique de Louvain in Louvain-la-Neuve gave administrative assistance during the preparation of the conference.

Finally, the conference would not have been possible but for the generous financial support by l'Université de l'Etat à Mons, le Fonds National de la Recherche Scientifique, the North-Holland Publishing Company, the Netherlands Wiskundig Genootschap, the International Union for History and Philosophy of Science and les Accords Culturels Belgo-Néerlandais. We gladly express our gratitude towards the sponsors of the conference and all those who contributed to its success.

MAURICE BOFFA
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LOGIC COLLOQUIUM 78
M. Boffa, D. van Dalen, K. McAloon (eds.)

CONTINUITY IN INTUITIONISTIC SET THEORIES

Michael Beeson

I dedicate this paper to the memory of Karel de Leeuw. His real memorial lies in his influence on the lives of his students and friends. Let us not mourn but carry on his work.

The work presented here represents part of a continuing study of the connections between the two basic concepts continuity and constructivity. Both concepts have their roots deep in the soil of mathematical practice, and for as long a time as they have been considered, it has been felt that there is some deep relationship between the two. The histories of each of these concepts show striking parallels, in that (i) tremendous energies were devoted to the task of making an intuitive concept mathematically precise, and (ii) there was considerable controversy concerning the meaning and value of the resulting new mathematics. (Specifically, the ε-δ definition of continuity, and the Brouwer-Bishop-Heyting development of constructive mathematics.) Evidence that a connection between constructivity and continuity has long been perceived may be found, for example, in Hadamard's criteria for a "well-posed problem" in differential equations. One of these criteria is that the solution should depend continuously on the parameters of the problem. The rationale for this criterion was that if the problem corresponds to a physical situation, one is supposed to be able to compute the solution from (measured) approximations to the data. Further evidence lies in the central place accorded to continuity principles in the work of Brouwer. We may even go so far as to say that it was Brouwer's efforts to connect continuity and constructivity that led him in his development of intuitionism (specifically, the theory of free choice sequences).

What we believe we have now accomplished is this: we have done for the connection between continuity and constructivity what ε and δ did for continuity. More precisely, one can formulate informally the "Principle of Continuity":

> If a problem is constructively solved, then the solution depends continuously on its parameters.

Our claim is to have made this principle precise, in its most general form, by specifying

(i) exactly which "problems" it applies to.
(ii) what "constructively solved" means.
(iii) exactly what "depends continuously on parameters" means.

We consider first point (ii). We find ourselves in a unique historical position, in that for the first time formal systems are available in which the bulk of constructive mathematics can be readily formalized. Thus one way of making the Principle of Continuity precise is as a derived rule of inference: if the problem can be proved in a constructive formal system T to have a solution, then the solution depends continuously on the parameters of the problem. And, we may add, provably so in the theory T. (For those unfamiliar with the present "historical position": in 1967, Bishop published his book, demonstrating that the scope of constructive mathematics is vastly wider than was previously suspected, and also demonstrating vividly the clarity and power of the constructive approach. This work stimulated the development of formal systems by Feferman, Friedman, Myhill, and Martin-Löf, which are intended to be suitable for formalizing Bishop's work.)

Next we discuss (i). It seems that the "correct" answer here is that the principle applies to problems of the form, given a in X, find b in Y such that $\langle a,b \rangle$ is in P, where X and Y are complete separable metric spaces, and P is any subset of their Cartesian product such that for each a in X, $\{b \in Y : \langle a,b \rangle \in P\}$ is closed in Y. (There are some variants on this form and much more discussion in [B1]. It may be that the condition on X may permit some generalization, so we should thus qualify our claim to the "most general form".)

Finally, the answer to (iii) is a bit subtle. One cannot require that b be found by a continuous function of a, defined on X, as the example $\forall a \in R \, \exists b \in N \; a \leq b$ shows, where N is the integers. This example might seem to "sink the whole ship" of the Principle of Continuity, until one sees that what should really be formulated is a Principle of <u>Local</u> Continuity: We should require that for each a in X, we can find a <u>stable</u> solution b in Y, where b is called <u>stable</u> if $\forall \varepsilon > 0 \, \exists \delta > 0$ such that to any c within δ of a, there corresponds a solution d within ε of b. As a matter of fact, this definition of stability is a common one in mathematical practice (for example, see [T]). To require that b should be given by a continuous function defined on some neighborhood of a is too strong in the case of problems without a unique solution. For example, every complex number has a square root, but there is no continuous square-root function defined in a neighborhood of zero. (Thanks to M. Hyland for showing me this example.)

The proper formulation of the Principle of Local Continuity opens up two distinct lines of research:

(1) A <u>mathematical program</u>, in which one wants to systematize and clarify various stability and continuity results in mathematics, and discover new ones, by the light of the Principle of Continuity, and by making use of the body of already-developed constructive mathematics.

(2) A metamathematical program, in which one wants to clarify the nature of constructive formal systems by considering their properties in the light of the Principle of Continuity.

At the present time, the mathematical program has been carried forward mostly in unpublished work. We may mention, as illustrative examples, the following known theorems which come within the scope of the Principle of Continuity together with constructively proved existence theorems:

(a) The continuous dependence on initial or boundary conditions of the solutions of any differential equation which can be solved by a method of successive approximation (contraction mappings); for example the well-known equation $y' = f(x,y)$, where f is Lipschitz in y, and the initial value of y is the parameter.

(b) The continuous dependence on the domain D in the plane of the eigenvalues of the vibrating-membrane equation $\Delta\phi + \lambda\phi = 0$ in D, $\phi=0$ on the boundary of D.

(c) The continuous dependence on the rectifiable Jordan curve C in R^3 of the infimum of areas of surfaces bounded by C.

In [B2] a new theorem is proved, whose (ordinary mathematical) proof was first discovered by means of the Principle of Continuity. See also [B5] for another example of mathematical work inspired by the Principle of Continuity.

The metamathematical program, on the other hand, is at present nearly complete. Our aim has been to show that various formal systems have various pleasing metamathematical properties related to continuity. These properties fall roughly into two categories: derived rules of inference, and consistency/independence results. It has turned out that the Principle of Local Continuity has surprising and sweeping power to systematize and organize the various continuity properties which have been considered in the past. We may draw evidence for two conclusions from the success of this program:

(1) We have in fact found the right connection between constructivity and continuity.

(2) The formal systems in question are in fact good ones, in some sense, for formalizing constructive mathematics.

(Of course some moderation is called for, especially in relation to (2), since there may be objections to a given system having nothing to do with continuity.)

The metamathematical program outlined above was begun in [B1]. In that paper two things are accomplished: (1) General conditions on a theory T are given, such that if they hold then T is closed under the various derived rules related to continuity, which we shall describe in more detail below. (2) These conditions are verified, and a number of related consistency/independence results are obtained, for the particular formal systems introduced by Feferman [Fe] for constructive mathematics.

In this paper, our purpose is to treat the intuitionistic set theories developed by Friedman and Myhill after the same fashion as we previously treated Feferman's theories. After the work mentioned above, we do not need to consider the continuity rules directly, but only to establish that the necessary metamathematical closure properties (explicit definability etc.) are satisfied. In practice, what this entails is the development of suitable realizability and forcing interpretations for these theories; these interpretations also enable us to establish the related consistency and independence results. Before turning to a discussion of these various theories, we first wish to summarize the metamathematical conclusions of the work, by stating exactly some of the derived rules which are under discussion. A more complete list and exhaustive discussion can be found in [B1]. In this list, X and Y are complete separable metric spaces, and C(X,Y) is the set of continuous functions from X to Y which are uniformly continuous on each compact subset.

(1) (Principle of Continuity). Provably well-defined functions from X to Y can be proved to be in C(X,Y).

(2) (Continuous Choice). If $\forall a \in X\, \exists! b \in Y\ (\langle a,b\rangle \in P)$ is provable, then so is $\exists f \in C(X,Y) \forall a \in X\ (\langle a,f(a)\rangle \in P)$.

(3) (Heine-Borel's rule). If a sequence of neighborhoods I_n can be proved to cover a compact space, then for some k, the union of the first k neighborhoods can be proved to cover the space.

(4) (Principle of Local Continuity). Suppose $\forall a \in X \exists b \in Y(\langle a,b\rangle \in P)$ is provable, and the hypothesis on P mentioned above is also provable. Then $\forall a \in X \exists b \in Y(\langle a,b\rangle \in P\ \&\ b \text{ is stable})$ is provable.

There is also a Principle of Local Uniform Continuity, of which we shall say more in Section 7 below. In [B1] is is shown how the above rules all flow from the Principle of Local Continuity, with Uniform Continuity being used for Heine-Borel's rule.

We now turn to a discussion of the various formal systems to which these results apply, namely, the systems of Feferman and those of Friedman. The systems of

Feferman and Friedman are quite different. Feferman's systems are based on the idea that every object is a construction, and constructions may apply to other constructions, so we have a sort of λ-calculus of constructions; in addition, we have "classifications" (similar to sets) and an ε-relation. We do not, however, have extensionality, as there is no reason to assert it for Feferman's underlying conception. Friedman's systems, on the other hand, are modifications of classical set theory, which do contain extensionality, but are made "constructive" in some sense by weakening the axiom of choice and using intuitionistic logic (for instance, they are consistent with Church's thesis). There has been considerable discussion (in fact "controversy" is not too strong a word) over the relative merits of the two types of systems, and over the question whether they are in accord with constructive mathematics from a foundational point of view. The present contribution to this discussion is that all the systems (except perhaps the weakest) share the same closure properties under rules related to continuity, the corresponding principles of continuity are consistent with very strong intuitionistic set theories.

In this paper, besides developing forcing and realizability for these set theories, we spend considerable effort analyzing the role of the axiom of extensionality. We prove that this axiom can be eliminated from the proofs of theorems mentioning only objects of low type, such as reals or natural numbers. This seems to be necessary from a technical standpoint (or at least the most convenient way to proceed) in order to obtain the explicit definability results we need. However, it is also interesting in its own right, principally because nearly every theorem of mathematical practice can be expressed at low types (since complete separable metric spaces can be regarded as subsets of N^N). Thus extensionality is essentially irrelevant to mathematical practice. (This is not to say that it is irrelevant from the philosophical, foundational viewpoint.) Another interesting thing about this theorem is that it has applications; see [B3] and [B4].

It is a pleasure to have this opportunity to thank those who have contributed to this work, by their interest, by their criticisms, by encouraging me to prove these theorems, and by inviting me to speak at the Colloquium in Mons: H. Barendregt, D. van Dalen, S. Feferman, H. Friedman, and D. Scott. I also would like to mention that the dedication to Karel de Leeuw is especially appropriate, since this paper was written in his house, while his companionship brightened my days.

§1. Description of Some Intuitionistic Set Theories

In this section we describe the principal intuitionistic set theories, which have been invented and studied by Friedman and Myhill. First we describe Friedman's systems. (Precise statements of the axioms will be given below.)

Let ZF^- be Zermelo-Fraenkel set theory, with intuitionistic logic, and with the foundation axiom expressed as (transfinite) induction on ε, instead of the usual way. (The usual foundation axiom implies the law of the excluded middle, see [M1].) We cannot add the axiom of choice AC without getting the law of the excluded middle, but we can add (some forms of) dependent choice. The strongest set theory we consider is thus ZF^- + RDC (relativized dependent choice). (Introduced in [Fr1].)

Friedman and Myhill have directed their attention to finding subsystems of ZF^- + RDC which are formally weak and practically strong: that is, which are strong enough to formalize known constructive mathematics (e.g. Bishop's book [Bi] and yet are proof-theoretically weak. There are two principal ideas here: one is to replace the power set axiom by the axiom of exponentiation, which says that A^B exists if A and B are sets. (This was introduced by Myhill in [M1].) The other is to restrict induction to sets instead of formulae, i.e. to consider $0 \in X \ \& \ \forall n(n \in X \to n+1 \in X) \to \forall n(n \in X)$ instead of $A(0) \ \& \ \forall n(A(n) \to A(n+1)) \to \forall n A(n)$. (Note that classical second-order arithmetic with restricted induction and arithmetic comprehension is a conservative extension of arithmetic.) The use of restricted induction is the germinal idea of Friedman's work. If we use exponentiation instead of power set, and restrict induction, and restrict separation to Δ_0 formulae (no unbounded quantifiers), and add a restricted form of dependent choices, we get Friedman's theory T_1, which he showed has the same strength as arithmetic.

Friedman also studied a variant of T_1 called $\underset{\sim}{B}$, which differs from T_1 in that $\underset{\sim}{B}$ has no foundation axiom, and collection is replaced by Δ_0-abstraction, which says $\{\{u \in x: A(y,u)\} : y \in x\}$ exists, where A is a Δ_0 formula. The point of this is that $\underset{\sim}{B}$ has a model in sets of rank $< \omega + \omega$, and so is easier to justify by some constructive philosophy (see [Fr2, PartI]). $\underset{\sim}{B}$ also has the same strength as arithmetic.

In between $\underset{\sim}{B}$ and ZF^- + RDC, Friedman considers several intermediate theories, which all have the full induction schema, and have additional axioms as follows:

T_2 : T_1 + induction + RDC   Z : Zermelo set theory
T_3 : T_2 + transfinite induction
T_4 : T_3 + full separation.

Thus ZF^- + RDC is just T_4 + power set.

Myhill's first published intuitionistic set theory CST [M1] is closely related to T_2, as discussed in [Fr 2]. We do not consider CST explicitly.

One feature of all these set theories worth remarking is that they include extensionality. This is one feature which distinguishes them from other formal systems which have been shown adequate for formalizing Bishop's book, such as Feferman's systems. We shall return to this point in §3. We find it necessary (as well as interesting) to consider set theories without extensionality (even if we want results only for extensional theories). We adopt the notation T-ext for the set theory T minus the axiom of extensionality; the proper formulation of these theories requires a little care, and we give a more complete description below. One difference between the extensional and non-extensional theories is that the syntax of the extensional theories is much simpler--we need only the one binary relation of membership. We do not include equality in the extensional theories. On the other hand, we must include equality in the intensional case, as well as some constants and function symbols to be described below.

We use $\langle x,y\rangle$ for the ordered pair, defined in the usual way from unordered pairs. The integers can be developed in set theory in the usual (von Neumann) way. Each formula of arithmetic has a natural translation into set theory.

We now list the axioms we will be considering; we give them first in the form suitable when extensionality is present, i.e. in the form used by Friedman. Afterwards we shall indicate the modifications which are necessary when extensionality is dropped.

A. (extensionality) $x=y \leftrightarrow \forall a(x\in a \leftrightarrow y\in a)$

B. (pairing) $\exists x\forall y(y\in x \leftrightarrow y=a \vee y=b))$

C. (infinity) $\exists x(0\in x \;\&\; \forall y(y\in x \to y\cup\{y\}\in x) \;\&$

$$\forall z(0\in z \;\&\; \forall y(y\in z \to y\cup\{y\}\in z) \to x\subseteq z))$$

D. (union) $\exists x\forall y(y\in x\leftrightarrow \exists z(y\in z \;\&\; z\in a))$

E. Δ_0-separation) $\exists x\forall y(y\in x\leftrightarrow(y\in a \;\&\; \phi))$ where ϕ is Δ_0 and x is not free in ϕ

F. (strong collection). $\forall x \in a \exists y\ \phi(x,y) \to \exists z(\forall x \in a \exists y \in z\ \phi(x,y)$ $\& \forall y \in z \exists x \in a\ \phi(x,y))$ (Ordinary collection doesn't have the second clause on z.)

G. (foundation). $(\forall a,b\ ((a \in b \ \&\ b \in x) \to a \in x)$ $\& \forall y(y \in x \ \&\ y \subseteq z) \to y \in z)) \to x \subseteq z$, (in other words, transfinite induction on $\in$ with respect to sets only, not formulae.)

H. (exponentiation). $\exists x \forall y(y \in x \leftrightarrow Fcn(y) \ \&\ Dom(y)=a \ \&\ Rng(y) \subseteq b)$

I. (bounded dependent choice). $\forall x \in a \exists y \in a\ Q(x,y) \to$ $\forall x \in a \exists z(Fcn(z) \ \&\ Dom(z) = \omega \ \&\ z(0)=x \ \&\ \forall n \in \omega\ Q(z(n),z(n+1))$ $\&\ Rng(z) \subseteq a)$, for Q a Δ_0 formula.

This much constitutes T_1; note that restricted induction follows from foundation together with our basic axioms.

J. (induction). $A(0) \ \&\ \forall n(A(n) \to A(n')) \to \forall n A(n)$, for all formulae A.

K. (relativized dependent choice RDC). Like axiom I. except that the set a is replaced by an arbitrary formula A, and Q is not required to be Δ_0. This much constitutes T_2.

L. (transfinite induction). $\forall x,y(y \in x \to P(y)) \to P(x)) \to \forall x P(x)$, where y does not appear in $P(x)$. This much constitutes T_3.

M. (separation). $\exists x \forall y(y \in x \leftrightarrow (y \in a \ \&\ Q)$, where x is not free in Q. This much constitutes T_4. To obtain ZF^- + RDC, we add:

N. (power set). $\exists x \forall y(y \subseteq a \to y \in x)$.

The theory $\underset{\sim}{B}$ consists of A-E, G, H and the axiom of "abstraction":

O. (abstraction). $\{\{u \in x: A(u,y)\}: y \in x\}$ exists.

Note that abstraction follows from collection. (If abstraction is formulated as in [Fr2], we need extensionality to deduce it from collection; we shall discuss the non-extensional theories further below.) Thus $\underset{\sim}{B}$ differs from T_1 in that collection is dropped, and a weaker consequence is added back in. It is worth noting that abstraction is restricted to Δ_0-formulae, while collection is not.

Intuitionistic Zermelo set theory Z consists of $\underset{\sim}{B}$ with full separation (so abstraction is unnecessary); dependent choice, induction, and power set. That is, Z differs from ZF^- + RDC in that it does not have foundation or transfinite induction and does not have collection. To list its axioms: A,B,C,D,J,K,M,N.

Myhill has given still another theory in [M2]; this theory has variables for both functions and sets. We do not deal with this theory here, but we expect that it can be handled (as well as $\underset{\sim}{B}$ can be) by using the interpretation in $\underset{\sim}{B}$ given by Myhill.

Now we discuss carefully the formulation of our non-extensional set theories. If T is one of the set theories including the axiom of extensionality, then T-ext is not just T with axiom A deleted. We must also (i) include a system of terms such as $\{x \in a: P(x)\}$ and (ii) modify the exponentiation axiom and abstraction axiom, choosing a form which is equivalent to the above when extensionality is present, but which is non-extensionally correct. Let $Fcn(f)$ be $\forall x,y,z(\langle x,z\rangle \in f \& \langle x,y\rangle \in f \rightarrow \forall u(u\in y \leftrightarrow u\in z)) \& \forall a\in f \exists b,c(a=\langle b,c\rangle)$. Let $Dom(f)=a$ & $Rng(f)\subseteq b$ abbreviate $\forall x\in a\ \exists y\in b(\langle x,y\rangle \in f)$ & $\forall \langle x,y\rangle \in f(x\in a \ \&\ y\in b)$. Then the exponentiation axiom says $\exists X\ \forall g(Fcn(g)\ \&\ Dom(g) = A\ \&\ Rng(g)\subseteq B \rightarrow \exists f\in X\ \forall x\in A(f(x) = g(x))$. Similarly, in the abstraction axiom, we assert $\exists X(\forall y\in a\ \exists w\in X(x\in w \leftrightarrow (x\in a\ \&\ \phi(x,y)))\ \&$ $\forall w\in X\ \exists y\in A(x\in w \leftrightarrow (x\in a\ \&\ \phi(x,y))))$.

We now specify the exact system of terms to be included in our non-extensional set theories; these terms are built up from the following constants and function symbols. We also give the defining axioms for these symbols.

(i) a constant symbol $\emptyset$, and the axiom $\forall x(x \notin \emptyset)$.

(ii) a function symbol $\{\ \}$, and the axiom $z \in \{x,y\} \leftrightarrow z = x \vee z = y$.

(iii) a function symbol for union, and the axiom

$$y \in \bigcup_{z\in a} z \leftrightarrow \exists z \in a(y \in z)$$

Then $a \cup b$ abbreviates $\bigcup_{z\in\{a,b\}} z$.

(iv) a constant symbol ω and axiom

$$\emptyset \in \omega\ \&\ \forall z(z \in \omega \rightarrow z \cup \{z\} \in \omega)\ \&$$

$$\forall X(\emptyset \in X\ \&\ \forall z(z \in X \rightarrow z \cup \{z\} \in X) \rightarrow \omega \subseteq X)$$

(v) for each formula P for which separation is allowed, a function symbol $\{x \in a: P(x,a)\}$ and the obvious axiom.

(vi) Symbols for dependent choice: if P is a formula for which dependent choice is allowed, we have a function symbol i_p with the axiom $\forall x \in \omega \exists! y \in \omega P(x,y)\ \&\ x_o \in \omega \rightarrow i_p(x_o) \in \omega^\omega\ \&\ i_p(x_o)(0) = x_o\ \&$ $\forall n \in \omega\ P(i_p(x_o)(n),\ i_p(x_o)(n+1))$.

Note that we include choice symbols only for functions from ω to ω. Thus at least we have terms for all the primitive recursive functions. so that our non-extensional theories contain arithmetic in a natural sense. Note that there are no terms corresponding to the collection axiom. Generally speaking, it seems that we get several theories of different strength by including or not including constants and functions symbols corresponding to the various axioms. We certainly need to include separation terms in order to achieve the technical results we want; the rest seem to be optional.

The above description requires a little elaboration, since the formula P in a term $\{x \in a: P(x)\}$ may itself contain other terms. One way to make our definitions completely precise is as follows: Start by adding a list f_n of function symbols to the language (for the separation terms) and similar lists for the other types of terms required. Then Gödel number all the formulae of the language, and then write $\{x \in a: P(x,y)\}$ for $f_n(a,y)$, where P has Gödel number n. Of course, we now have <u>more</u> terms than we want, since we only want such terms for certain formulae P . One can either delete the extra terms from the language, or leave them in, but add no axioms about them. To specify which formulae P are allowed, for example in the case of Δ_0-separation, we add a clause to the definition of a Δ_0-formula specifying that if terms $\{x \in b: Q(x)\}$ occur in the component formulae, then Q is already Δ_0 , and similarly for abstraction and choice terms occurring in the component formulae. Note that generally when we add more symbols to the language, there are more Δ_0-formulae.

§2. Complete Metric Spaces and some Auxiliary Theories

Since the derived rules which we wish to establish mention metric spaces, we have to discuss the formalization in intuitionistic set theories of the mathematics of complete separable spaces. This is quite straightforward and offers no difficulties. (X,ρ) is a complete separable metric space if it is a metric space, and is has a dense subset, which is the range of a function whose domain is ω, and every Cauchy sequence converges. Using the axioms of $\underset{\sim}{B}$ only, we see that from every Cauchy sequence we can extract a subsequence x_n satisfying $\rho(x_n,x_m) < 1/n + 1/m$. Letting σ be the metric on the integers induced by "pulling back" ρ from the countable dense subset of S , we see that (X,ρ) is isometric to the space of all functions $y \in N^N$ satisfying $\sigma(y_n,y_m) < 1/n + 1/m$ (using the convention $y_n = y(n-1)$ to avoid the problem of subscripts beginning at 1 and functions beginning at 0). Here we follow Bishop in not passing to equivalence classes of such functions, but allowing instead a broader equality relation in the space X ; equality of elements of X will not necessarily be set-theoretic equality. It is worth noting that having the axiom of extensionality does not force us to use equivalence classes. Thus any complete separable space can be brought to "standard form" as a set of sequences of integers. The metric on such a space will be $\rho(x,y) = \lim_{n\to\infty} \sigma(x_n, y_n)$.

The "standard form" considered above is not the most useful form for a compact space. For instance, 2^N is most naturally thought of as the space of all y in N^N with $y_n = 0$ or 1. A compact space is one such that for each n, we can find a finite $1/n$-approximation to the space, i.e. a finite set $y_1, \ldots, y_k$ such that each point of the space is within $1/n$ of some y_k. Using bounded dependent choice, we can select a countable base consisting of such points y_j for the various values of n, and associate to each point of the space a sequence of the y_i's such that $\rho(y,y_i) < 1/i$. It follows that every compact space can be brought to standard form as follows: for some non-decreasing sequence of integers M_i, X consists of all y in N^N with $y_n \le M_n$, and the metric has the same form as in the standard form for complete spaces above.

Let T be any one of the theories considered in this paper, and let X be some (provably) complete metric space, in standard form. (To be precise, this means there is a formula Q such that T proves $\exists! X\, Q(X)$ and $Q(X) \to X$ is a complete separable space in standard form.) We shall use X both for the space defined informally by the formula Q, and also to abbreviate formal expressions; thus $y \in X$ means $\forall X(Q(X)) \to y \in X)$. Let b be an (arbitrary but fixed) element of X; we shall have occasion to consider the auxiliary theory Tb, which is formed from T by adding a constant symbol $\underline{b}$, the axiom $\underline{b} \in X$, and axioms $\underline{b}(\bar{n}) = \bar{m}$ where $m = b(n)$, and $\bar{n}$ is the numeral for n.

In case X is a compact space in standard form, we shall have occasion to consider another auxiliary theory, Ta. This theory is formed by adding a constant symbol $\underline{a}$ to T, and the axiom $\underline{a} \in X$, but no other axioms.

It will sometimes be convenient to assume that the metric on the countable base of a space in one of the two standard forms is actually a recursive function. In case X is provably a complete separable space (or a compact space) this can be done without loss of generality, since for the theories we shall consider, a function provably in N^N is (provably) recursive. (See §5.)

If P is a subset of a metric space X in standard form, we say "P is extensional" if $\rho(x,y) = 0\ \&\ P(x) \to P(y)$. Note that this concept has nothing to do with whether the axiom of extensionality is assumed or not; for instance, as long as we take the reals to be defined by Cauchy sequences instead of equivalence classes, there will be non-extensional sets, in this sense.

§3. The role of extensionality

The intuitionistic set theories propounded by Friedman and Myhill contain the axiom of extensionality, while the theories of Feferman do not. The actual practice of constructive mathematics can be done straightforwardly without any underlying notion of extensionality. Of course, in the practice of mathematics we define various notions of extensional equality; in fact, Bishop takes the view that each set should come equipped with an equivalence relation to be used as an equality relation. These equivalence relations can be used quite straightforwardly without assuming that equivalent objects are equal; for instance, many different Cauchy sequences of rationals determine the same real number. In [Fr2], Friedman goes into some detail as to exactly how to formalize Bishop's book in intuitionistic set theory. Extensionality is hardly made use of; and where it is, it is easily eliminated. Why then include extensionality at all? The answer to this is that Friedman wished to make constructive mathematics formalized in his system look as much like classical mathematics as possible, in order to make it easier for the classical mathematicians to appreciate constructive mathematics.

Be that as it may, in this paper we are trying to obtain derived rules related to continuity for intuitionistic set theories, both with and without extensionality. These results rest on the following theorem.

Theorem 3.1

Let T be any of the theories (with extensionality) discussed in this paper (so $T = \underset{\sim}{B}$, T_1, T_2, T_3, T_4 or ZF^- + RDC); or let T be one of the auxiliary theories T*a or T*b, where T* is one of the theories considered in this paper. Then

(i) T can be interpreted in T-ext (without extensionality); that is, we can assign to each formula A an interpretation A* such that $T \vdash A$ implies T-ext $\vdash A^*$. Furthermore, we have $T \vdash (A \leftrightarrow A^*)$, for A a Δ_o-formula.

(ii) T is conservative over T-ext for arithmetical sentences, in fact for sentences with quantifiers over a fixed (definable) subset of N^N allowed.

(iii) Both (i) and (ii) are provable in arithmetic.

Proof: We interpret T in T-ext, assigning to each formula A a formula A* in which $\in$ is replaced by a formula ε, and sets are relativized to a formula M(a). We shall show that $T \vdash A$ implies T-ext $\vdash A^*$. In other words, we shall explain how to give a definable model of T in T-ext. In order to make the model intelligible, we first give a false attempt. The most natural thing to do is to define $x \sim y$ if

$\forall a \in x \exists b \in y(a \sim b)$ & $\forall a \in y \exists b \in x(a \sim b)$; and then to set $x \varepsilon y$ if $\exists z \in y(x \sim z)$. The first problem with this is that $\sim$ is inductively defined, instead of being given by a formula. There are ways to overcome this, and we shall explain them. The simplest way to think of what we are doing is to think we are defining a model of T (given by a class, not a set) assuming only that the axioms of T-ext are true in the universe. This can be recase in official language as an interpretation, as above.

First of all, let us discuss the case $T = ZF^- + RDC$, where we have both power set and collection. Then we can make the above inductive definition of $\sim$ explicit in the most straightforward manner, so that if X is any transitive set, then $\sim_X$ ($\sim$ restricted to X) is a set. (It is the intersection of all binary relations R in the power set of X^2 which satisfy the appropriate inductive condition.) We need collection in order to prove that every set has a transitive closure $TC(x)$. Then we can define the model (interpretation), using for $x \sim y$ the formula

$$\forall R \in \wp(TC(\{x,y\}))(I(R,TC(\{x,y\}) \rightarrow \langle x,y\rangle \in R),$$

where $I(R,z)$ says that R satisfies the inductive conditions for $\sim$ on the transitive set z. It is straightforward to verify that the axioms of $ZF^- + RDC$ are valid on this interpretation, using the axioms of $ZF^- + RDC$.

Several of the theories we have to consider, however, are not strong enough to prove the existence of $TC(x)$ for each x. Our model for such theories will therefore have to be somewhat more complicated; we take the "sets" to be pairs $\langle x,y\rangle$, where y is the transitive closure of x. To be precise, we write $Trans(y)$ for $\forall a \in y \forall b \in a(b \in y)$ and we write $y = TC(x)$ for
$Trans(y)$ & $x \in y$ & $\forall b \in y(b=x \vee \exists n \in \omega \exists a_1,a_2,\ldots,a_n\ (b \in a_1,\ldots,a_n \in x))$.
Note that $y = TC(x)$ is a Δ_o-formula (in predicates definable in $\underset{\sim}{B}$, although it is not strictly Δ_o.)

To deal with the case T = Zermelo set theory, which lacks collection but has power set, we can proceed with these sets as we did above for $ZF^- + RDC$, using power set to make an inductive definition explicit. However, to deal with weaker set theories, which lack power set, more work is required. Let $Q(R,r)$ express that y is transitive and R is $\sim_y$; to be precise, $Q(R,y)$ is

$$Trans(y)\ \&\ \forall a,b \in y(\langle a,b\rangle \in R \leftrightarrow (\forall p \in a \exists q \in b \langle p,q\rangle \in R\ \&\ \forall q \in b \exists p \in a \langle p,q\rangle \in R))$$

Note that Q' is a Δ_ρ-formula. Note that with the aid of the foundation axiom, we can prove that for each transitive y, if $Q(R,y)$ and $Q(S,y)$, then R and S are extensionally equal relations, i.e. $\langle a,b\rangle \in R \leftrightarrow \langle a,b\rangle \in S$. (We don't need transfinite induction to prove this.)

Suppose W is a fixed transitive set; then for $x \in W$, $TC(x)$ exists. What axioms does it take to prove $\forall x \in W \exists R\, Q(R,TC(x))$?
It seems to require collection and transfinite induction, as well as union. If we know this to be provable, we could go ahead and define the model; setting $x \sim y$ iff $R(Q(R,TC(\{x,y\}))$ & $\langle x,y\rangle \in R)$, or more precisely, $\langle x,u\rangle \sim \langle y,v\rangle$ iff the same condition holds, where the TC's occuring are to be extracted from u,v. This model allows us to handle T_4, which has collection, transfinite induction, and full separation. In order to handle the weaker theories, we must refine the model more.

The first thing that occurs to one is to restrict attention to those "sets" $(x,TC(x))$ such that there is an R such that $Q(R,TC(x))$. This is not enough. Let us call pairs $\langle x,y\rangle$ such that $y = TC(x)$, M_1-sets. Consider an M_1-set $\langle x,y\rangle$ such that for all M_1-sets $\langle a,b\rangle$, there is an R such that $Q(R,TC(\{x,a\}))$. Such sets $\langle x,y\rangle$ we call M_2-sets. These are the "sets" of our next model. We define $\sim$ as above, $x \sim y$ iff (more precisely) $\langle x,TC(x)\rangle \sim \langle y,TC(y)\rangle$ iff $\exists R(Q(R,TC(\{x,y\})$ & $\langle x,y\rangle \in R)$ iff $\forall R(Q(R,TC(\{x,y\}) \to \langle x,y\rangle \in R)$. This model will work for T_1,T_2,T_3, and incidentally for T_4 as well. For simplicity we write x instead of $\langle x,TC(x)\rangle$ for M_1-sets. Suppose x and y have the same ε-members, that is, every element of x is equivalent to a member of y and vice-versa. We have to show that x and y are equivalent (hence are ε-members of the same set). Since x and y are M_2-sets, there is an R such that $Q(R,TC(\{x,y\}))$. If $z \in TC(\{x,y\})$, and $Q(R^*,TC(z))$, then R^* is a subset of R; hence every member of x is equivalent to a member of y and vice-versa, using R for the equivalence; hence $\langle x,y\rangle \in R$; this verifies extensionality.

We now check Δ_o-separation. If a is a fixed M_2-set, and $\mathbf{A}$ is a Δ_o-formula, we have to find an M_2-set x such that for all M_2-sets z, $z \varepsilon x$ iff $z \varepsilon a$ & $A^*(z)$. (Here A^* is the interpretation of A.) Since ε is not given by a Δ_o-formula, and neither is M_2, the range of the quantified variables, it is not obvious how to produce x, using only Δ_o-separation. We proceed by induction on the complexity of $\mathbf{A}$. We first have to handle the case of atomic $\mathbf{A}$; here there are two possibilities, either A is $z \in b$ or A is $b \in z$. First suppose A is $z \in b$. Take x to be $\{z: z \in a \;\&\; \exists y \in b(y \sim z)\}$. This set can be formed using Δ_o-separation, since we have $\{\langle z,y\rangle: z \in a \;\&\; y \in b \;\&\; z \sim y\}$ available to use as a parameter, since a and b are M_2-sets.
Next suppose $\mathbf{A}$ is $b \in z$. This time take x to be $\{z \in a: \exists b' \in z(b' \sim b)\}$. This set can be formed, using as a parameter some relation R such that $Q(R,TC(a))$. It is easy to check that these two sets x are actually M_2-sets. This takes care of the case A atomic. Similar arguments take care of the induction steps in which A is of the form $B \;\&\; C$, $B \vee C$, or $B \to C$.

Next, observe that an easy induction on the complexity of A shows that $x \sim y \,\&\, A^*(x) \to A^*(y)$ is provable, for each fixed A. Now consider the case in which A is $\forall w \in y\, B(z,w,y)$. Since every ε-member of y is equivalent to some ε-member, the remark just made shows that it suffices to quantify over ε-members; that is the key to the verification in this case, which we now give in more detail. By the induction hypothesis, we have that $\forall w \in y \exists P(P$ is an M_2-set & $(\langle w,z\rangle \varepsilon P \to z \in a \,\&\, B^*(z,w,y)))$. By strong collection, there is a set T such that $\forall w \in y \exists P \in T(\ldots) \,\&\, \forall P \in T \exists w \in y(\ldots)$. As a matter of fact, T is not only a set but an M_2-set, if y is an M_2-set. To check this, first note that the transitive closure of T can be obtained by taking the union of the transitive closures of the elements of T and $\{T\}$; thus T is an M_1-set; if T' is some M_1-set, we have to get $\{\langle a,b\rangle : a \sim b \,\&\, a \in TC(T) \,\&\, b \in TC(T')\}$ to exist. This relation R is just the union of the corresponding relations for the members of T (which can be formed using collection), union the set $\{\langle T,b\rangle : b \in TC(T') \,\&\, T \sim b\}$, union $\{\langle a,T'\rangle : a \in TC(T) \,\&\, a \sim T'\}$, both of which are easily defined. Hence T is an M_2-set. Now form the set $S = \{z \in a : \forall w \in y \exists P \in T \;\; \langle w,z\rangle \in P\}$, which can be formed using Δ_o-separation. We have

$$z \varepsilon S \to z \varepsilon a \;\&\; w \varepsilon y\, B^*(z,w,y) \quad .$$

This completes the verification of the case in which A is formed by bounded universal quantification. The case of bounded existential quantification is much easier, and we leave it to the reader. This completes the verification of Δ_o-separation.

We turn our attention to the axiom of infinity. Here the only difficult part of the proof is to prove that $\langle\omega,\omega\rangle$ is an M_2-set; we first have to prove that if n is an integer, then $\langle n,n\rangle$ is an M_2-set. (Each integer is its own transitive closure). This is, we have to prove that

$$\forall n \in \omega \forall \text{transitive } x \exists R(\langle p,q\rangle \in R \to p \in n \,\&\, q \in x \,\&\, \langle p,p\rangle \sim \langle a,TC(q)\rangle)$$

The obvious way to prove this is by induction on n; however, in some of the weak theories we do not have full induction available, and it seems to require at least Δ_1-induction. Fortunately, we can prove it without induction. Fix an integer n and a transitive set x ; then R can be taken to be R_n , where

$$R_o = \emptyset \text{ and } R_{k+1} = \{\langle p,q\rangle : p \in n \,\&\, q \in x \,\&\, \forall a \in p \exists b \in q \;\; \langle a,b\rangle \in R_k \,\&$$
$$\forall b \in q \exists a \in p \;\; \langle a,b\rangle \in R_k\} \quad .$$

Now the desired property of R can be proved by a bounded induction.

We next undertake to verify the sound interpretation of the axiom of exponentiation. In our non-extensional set theories, this axiom takes the following form:

$$(*)\ \forall A,B\,\exists X\,\forall q(\mathrm{Fcn}(q)\ \&\ \mathrm{Dom}(q) = A\ \&\ \mathrm{Rng}(q) \subseteq B \to$$
$$\exists f \in X(\mathrm{Fcn}(f)\ \&\ \mathrm{Dom}(f) = A\ \&\ \mathrm{Rng}(f) \subseteq B\ \&\ \forall x \in A(f(x) = q(x)))).$$

Note it is not necessary to put in that X is a set of functions from A to B, because Δ_0-separation can alway be applied to get $\{f \in X : \mathrm{Fcn}(f)\ \&\ \mathrm{Dom}(f) = A\ \&\ \mathrm{Rng}(f) \subseteq B\}$.

Now we have to use this axiom to verify that the ordinary exponentiation axiom is satisfied in the model. The first point to make is that if A and B are sets of the model, then so is B^A (and indeed any X as in $(*)$). This is because any descending $\in$-chain has one of the forms

$$x_1 \in x_2 \ldots \in a \in \{a\} \in \langle a,b\rangle \in f \in X$$
$$x_1 \in x_2 \ldots \in a \in \{a,b\} \in \langle a,b\rangle \in f \in X$$
$$x_1 \in x_2 \ldots \in b \in \{a,b\} \in \langle a,b\rangle \in f \in X .$$

Hence the transitive closure $TC(X)$ can be defined as the union over all members of such sequences, using the fact that $TC(A)$ and $TC(B)$ are sets.

Now to verify $(*)$. Fix A and B. Let X be given by axiom $(*)$. (This X is <u>not</u> however the X^* we choose to verify $(*)$ - we give this X^* later). Suppose q satisfies the hypothesis of $(*)$ in the model; that is

(i) $\forall a' \in A\,\exists b' \in B\ \langle a',b'\rangle \in q$

(ii) $\langle a',b'\rangle \in q \to a' \in A\ \&\ b' \in B$

(iii) $\langle a',b'\rangle \in q\ \&\ \langle a',c'\rangle \in q \to b' \sim c'$

Let $f = \{\langle a,b\rangle : a \in A\ \&\ b \in B\ \&\ \exists a'\exists b'\ \langle a',b'\rangle \in q\ \&\ a \sim a'\ \&\ b \sim b'\}$.

Now f can be defined in $\underset{\sim}{B}$-ext, because : we can fix a transitive set W containing both A,B,and q, and then as we have shown above, $\in$ and $\sim$ restricted to W are sets and the quantifiers $\exists a'\exists b'$ can be relativized to W.

We claim f is satisfied to be a function from A to B. Suppose $\langle a',b'\rangle \in f$; then $a' \sim a\ \&\ b' \sim b$ for some $\langle a,b\rangle \in f$, so $a \in A\ \&\ b \in B$, so $a' \in A\ \&\ b' \in B$. Next suppose $\langle a,b\rangle \in f\ \&\ \langle a,c\rangle \in f$. Then $a \sim a''$, $b \sim b'$, $a \sim a'$, $c \sim c'$ with $\langle a',b'\rangle \in q\ \&\ \langle a'',c'\rangle \in q$. Hence, by (iii) $b' \sim c'$; hence $b \sim c$. Hence f is satisfied to be a function from A to B.

Moreover, f satisfies $\forall x \in A\ (f(x) = q(x))$, i.e.
$\forall a \in A \forall b,c \in B(\langle a,b\rangle \in f\ \&\ \langle a,c\rangle \in q \rightarrow b \sim c)$, since if $\langle a,b\rangle \in f$, then $a \sim a'\ \&\ b \sim b'$ where $\langle a',b'\rangle \in f$, so $a \sim a''\ \&\ b' \sim b''$ where $\langle a'',b''\rangle \in g$, and by (iii), $a \sim a''$ implies $c \sim b''$; since we have $b \sim b' \sim b'' \sim c$, we get $b \sim c$.

After these preliminaries we can give the set X^* which is to work for X in (*) in the model. Note that if f is satisfied to be a function from A to B, f may not actually be a function; but it induces a function from A to the set $\{[b] : b \in B\}$, where $[b]$ is the equivalence class of b in the equivalence relation $\sim$ on B.

Note that $\{[b] : b \in B\}$ can be formed in $\underset{\sim}{B}$-ext by the abstraction axiom; hence by exponentiation we can form the set S of all functions from A to $\{[b] : b \in B\}$
Now if $H \in S$, let $H^{\#}$ be defined by

$$\langle a,b\rangle \in H^{\#} \leftrightarrow [b] \in H(a)$$

Take $X^* = \{H^{\#} : H \in S\}$. X^* can be produced using the abstraction axiom of $\underset{\sim}{B}$-ext. To see this explicitly, we write

$$X^* = \{\{\langle a,b\rangle : \exists z \in \{[b] : b \in B\}\ (\langle a,z\rangle \in H\ \&\ b \in z)\} : H \in S\} .$$

Note also that $TC(X^*)$ exists, so X^* is a "set" of the model.
Now suppose the model satisfies

$$Fcn(q)\ \&\ Dom(q) = A\ \&\ Rng(q) \subseteq B .$$

As above, we can produce f such that f is satisfied to be a function from A to B and $\forall x(f(x) = q(x))$ holds in the model. It remains to show $f \in X^*$. Let $f^* = \{\langle a,b\rangle : a \in A\ \&\ b \in B\ \&\ \langle a,b\rangle \in f\}$. We claim $f^* \in X^*$, and $f \sim f^*$. To see $f \sim f^*$, note that $f^* \subseteq f$, and if $\langle a',b'\rangle \in f$, then $a' \in A\ \&\ b' \in B$, so $a' \sim a \in A\ \&\ b' \sim b \in B\ \&\ \langle a'',b''\rangle \in g$ for some a'' and b'' with $a' \sim a''\ \&$ $b' \sim b''$. Hence $\langle a,b\rangle \in f$ also; thus every member $\langle a',b'\rangle$ of f is equivalent to a member $\langle a,b\rangle$ of f^*. Hence $f \sim f^*$.

Finally we prove $f^* \in X^*$. Define H by $H(a) = [f(a)]$, the equivalence class of $f(a)$ in B. Technically, f is not a function (though it is satisfied to be a function) so we really must define $H(a) = \{b \in B : \exists b' \in B\ \ \langle a,b'\rangle \in f\ \&\ b' \sim b\}$.
Actually, looking at the definition of f it is enough to take
$H(a) = \{b \in B : \langle a,b\rangle \in f\}$. In any case H is a function from A to $\{[b] : b \in B\}$, and $f^* \sim H$, in fact f^* and H have exactly the same members.
But $H^{\#} \in X^*$ by definition of X^*; hence $f^* \in X^*$.

Since we have proved $f^* \varepsilon X^* \& f^* \sim f$, it follows that $f \varepsilon X^*$. Hence the conclusion of the exponentiation axiom is verified.

This leaves pairing, union, collection, foundation, induction, dependent choice and transfinite induction still to check. Because of limitations of space, we omit the details of these verifications. Thus we take it as proved that our interpretation is sound for T_1, T_2, T_3 and T_4 . We have already done ZF^- + RDC and Z , which leaves only $\underset{\sim}{B}$ to consider.

The above model uses collection quite heavily; however, in the case of $\underset{\sim}{B}$, we should be able to describe the model consisting of sets of rank less than $\omega + \omega$ quite explicitly (incidentally giving another interpretation that works for Zermelo). Define a set x to be of rank less than $\omega + n$ if every descending $\in$-chain from x terminates in an integer in $\leq n$ steps. (This is not exactly the usual definition, but it is convenient). Write $S(x)$ if for some n, x is of rank less than $\omega + n$.

Again we shall consider the "sets" of the model to be pairs $(x, TC(x))$. Note that $S(x)$ is Δ_0 in x and $TC(x)$ (we need $TC(x)$ to be able to quantify over descending $\in$-chains). Let W be a fixed transitive set of rank $\leq \omega + n$. We prove that $\sim_W$ is a set, that is, $\exists R\, Q(R,W)$, where Q is as above.

Namely, R is R_n , where $R_o = \{\langle n,n\rangle : n \in \omega\}$ and
$R_{j+1} = \{\langle x,y\rangle \in W^2 : \forall a \in x \exists b \in y \; \langle a,b\rangle \in R_j \; \& \forall b \in y \exists a \in x \; \langle a,b\rangle \in R_j\}$

As we discussed near the beginning of this proof, this is all we need to make the interpretation work. We interpret $x \in y$ as $\exists z \in y (z \sim x)$, where $z \sim x$ is

$$\exists R(Q(R, TC(\{x,y\})) \; \& \; \langle z,x\rangle \in R) .$$

We interpret sets, as mentioned, as pairs $\langle x,y\rangle$ with $S(x)$. We leave the reader to verify that the interpretation is sound for $\underset{\sim}{B}$.

Next we turn to the case of the auxiliary theories Ta and Tb, where T is ZF^-+ RDC , Z , T_1 , T_2 , T_3 or T_4 . We use the same interpretation for Ta and Tb as for T ; to finish the proof, we have to verify that the extra of Ta and Tb are soundly interpreted.

We have to tell how the constant a (or b) is to be interpreted; of course it is as $\langle a, TC(a)\rangle$ (in the case of the interpretation that works for the T_i) and just a in the case of the interpretation that works for ZF^-+ RDC and Z . This depends on the fact that $TC(a)$ is definable from a . One easily verifies that every member of N^N (remember a is a member of a subset of N^N) has a transitive closure definable from a ; note that each integer is its own transitive closure. Next we have to verify that each member of N^N is actually (with its transitive closure) an M_2-set; that is, if x is any transitive set, we can form $\{<p,q>: p \in TC(a) \ \& \ q \in x \ \& \ p \sim q\}$. (More precisely, p and q should be paired with their transitive closure). This is easy, once we know that each integer is an M_2-set, which we have already discussed. Finally, we have to verify that the axiom $a \in X$ and the axioms $b(\bar{n}) = \bar{m}$ where $m = f(n)$, for some fixed f in X , are soundly interpreted. Recall from §2 that membership in X is given by a purely universal condition on the values of a. Below we give a proof that the interpretation preserves arithmetic sentences; the same proof applies to show that it preserves the axioms in question .

We have now given the interpretation A^* for each of the set theories discussed in this paper, and proven the soundness of the interpretation. Next we prove that $T \vdash A \leftrightarrow A^*$, for Δ_o-formulae A . This is established by induction on the complexity of A ; to be quite precise, $A \leftrightarrow A^*$ is only for closed formulae ; for formulae with free variables, say x , we should say

$$S(<x,y>) \rightarrow (A^*(<x,y>) \leftrightarrow A(x)).$$

(Here we are considering the interpretation that works for T_1,T_2,T_3,T_4). The basis is $S(<x,y>) \ \& \ S(<a,b>) \rightarrow (<x,y> \varepsilon <a,b> \rightarrow x \in a)$, which can be established using the foundation axiom; extensionality is used here. The induction step proceeds smoothly, using the fact that members of M_2-sets determine M_2-sets; we leave the details to the reader. Note that we cannot seem to get $A^* \leftrightarrow A$ for all formulae A but only for Δ_o-formulae. (For $ZF^- + RDC$ we can get it for all formulae, because the interpretation does not require that transitive closures be tacked on.) The same argument works for the interpretation of $\underset{\sim}{B}$ using sets of rank less than $\omega + \omega$; here the induction step over a bounded quantifier uses the fact that the members of sets of rank less than $\omega + \omega$ are themselves sets of rank less than $\omega + \omega$. A similar induction works for the interpretation used for Zermelo set theory. This completes the proof of (i) of the theorem.

We next consider the question of which sentences are preserved by the interpretations; it is for these sentences that we get a conservative extension result. First arithmetic sentences are preserved. This is shown by induction on the complexity of an arithmetic formula; actually, as above we have to prove

$$<m,x> \in \omega \ \& \ n \in \omega \ \& \ <m,x> \sim <n,TC(n)>$$
$$\rightarrow (A\ (<m,x>) \leftrightarrow A(n)) \quad \text{(this time without extensionality)}$$

Note that every integer has a transitive closure, namely itself; in fact, every integer is (part of) an M_2-set.
Here A is a formula of set theory translating a formula of arithmetic, which we also call A ; the induction is on the complexity of the arithmetic formula. The details are easy but cumbersome; we leave them to the reader. Next note that every f in N^N has a transitive closure; this allows us to extend the above induction to formulae involving quantifiers over such objects. Actually, we must verify that each such f is (part of) an M_2-set; to do this, we must be able to form for each transitive set x , the set $\{\langle a,b\rangle : a \in TC(f)\ \&\ b \in x\ \&\ a \sim b\}$. This boils down to the fact that we can form the corresponding set with an integer m in place of f , in other words that each integer is an M_2-set, a fact alluded to above.

This completes the proof of part (ii) of the theorem. Part (iii) of the theorem, which says that the first two parts are provable in arithmetic, is proved by examining the above proof, and noticing that only arithmetic is needed. In other words, we proved by induction on (Gödel numbers of) proofs in T that the interpretation of the last formula of the proof is provable in T-ext . This completes the proof of theorem 3.1.

§4. Realizability for Set Theories

In this section, we give a variant of q-realizability adapted to set theories. This type of realizability has been used before for arithmetic and the theory of species to obtain explicit definability theorems [Tr]. Here we extend this program to set theories. The extension to set theories without extensionality is relatively straightforward, but there seems to be no simple way to handle set theories with extensionality. (Myhill gave [M1] a complicated realizability for his extensional set theory; but it cannot be made to work for our purposes.) For this reason, even if we want to obtain derived rules only for extensional theories, we have to consider the non-extensional ones and use the results of the previous section.

The plan of the present is to give the realizability interpretation we need and prove its soundness both for the basic set theories T-ext and for Ta-ext and Tb-ext . Our definition of realizability will proceed by associating to each formula A another formula e r A ("e realizes A"). We will then prove soundness theorems of the form, if $T \vdash A$, then for some e , $T \vdash \bar{e}\ r\ a$. Here e is an integer; all our realizing objects are integers, not arbitrary sets. (We use e , n , m etc. to indicate variables whose range is restricted to ω .)

We begin by assigning to each set variable x another variable x^*, in the manner discussed in [B1]. The free variables of $e\ r\ A$ are e, x and x^*, where x are the free variables of A.
(Our convention is that a single letter can denote a finite list of variables.) We now give the clauses defining $e\ r\ A$, for the notion of realizability that works for theories without extensionality:

$e\ r\ x = y$	is	$x^* = y^*$
$e\ r\ x \in y$	is	$\langle e,x,x^*\rangle \in y^*$
$e\ r(A \& B)$	is	$(e)_0\ r\ A\ \&\ (e_1)\ r\ B$
$e\ r(A \vee B)$	is	$((e)_0 = 0 \to (e)_1\ r\ A\ \&\ A)$ $\&\ ((e)_0 \neq 0 \to (e)_1\ r\ B\ \&\ B)$
$e\ r(A \to B)$	is	$a\ r\ A\ \&\ A \to \{e\}(a)\ r\ B$; or more precisely $\forall a(a\ r\ A\ \&\ A \to \exists k(T(e,a,k)\ \&\ U(k)\ r\ B)$
$e\ r \forall x\ A$	is	$\forall x,x^*\ e\ r\ A$
$e\ r \exists x\ A$	is	$\exists x,x^*\ (A\ \&\ e\ r\ A)$

In order to complete the definition, we have to define $e\ r\ A$ for atomic A involving terms t of the non-extensional set theories. This can be done by the same clauses as above, as soon as we associate to each term t another term t^* which intuitively defines $\{\langle e,x,x^*\rangle : e\ r\ x \in t\}$. These terms t^* will be given in the course of the soundness proof below; they could be listed here, but would be unintelligible. For instance we define $\omega^* = \{\langle n,\langle n,n\rangle\rangle : n \in \omega\}$. As another example, if $t = \{y \in a : B(y)\}$, then $t^* = \{\langle e,y,y^*\rangle : \langle (e)_0,y,y^*\rangle \in a^*\ \&\ (e)_1\ r\ B(y)\}$. We give this example in order to clarify the following point: There is no vicious circle in the fact that $(e)_1\ r\ B(y)$ appears in the definition of t^*, which must precede the definition of $e\ r\ x \in t$; for, as discussed above, the definitions of Δ_0-formulae and terms proceed by simultaneous induction, so that $B(y)$ contains only less-complicated terms than t. To make this completely precise, we could assign a measure of complexity to both terms and formulae, say $C(t)$, giving atomic formulae without compound terms complexity zero, and atomic formulae $t=s$ or $t \in s$ the complexity $\max(C(t),C(s))$; let propositional connectives and quantifiers increase the complexity by 1, and let separation terms $\{x \in a: B(x)\}$ have complexity $1+\max(C(a),C(B))$; similarly for union, pair, and choice terms. Then our definition of $e\ r\ A$ proceeds by induction on the complexity of A.

Remarks:

(1) If we were doing 1945-realizability (see [Tr]), we would not need the extra variables with stars, but could avoid them by defining $e \; r \; x \in a$ to be $\langle e,x \rangle \in y$. Trying to do something similar for q-realizability is more trouble than it is worth.

(2) One cannot define $e \; r \; x \in y$ to be $\langle e,x \rangle \in y^*$, though this may seem tempting. In this case, all the axioms except dependent choice will be realized (including extensionality), but one will not be able to get anything realized to be a function. Consider Fcn(f) which says

$$\forall x,y,w \; (\langle x,y \rangle \in f \;\&\; \langle x,w \rangle \in f \rightarrow y = w).$$

In order to get $y = w$ realized, there will have to be some relation between y^* and w^*, which we cannot get from having the antecedent realized, with this definition of realizability. This is somewhat interesting because it points up the absolute necessity of the axiom of choice in proving the existence of functions.

(3) The motivation behind the definition of $e \; r \; x \in y$ is that y^* is thought of as the set of $\langle e,x \rangle$ such that e proves, or verifies, or realizes, that $x \in y$. Remember that Kleene's original motivation for realizability was that realizing numbers were thought of as like proofs. It is no wonder that extensionality gives trouble, because one can have x and y extensionally equal, without any relationship at all between x^* and y^*; yet if $\forall a (x \in a \leftrightarrow y \in a)$ is to be realized, there has to be some relationship between x^* and y^*.

Theorem 4.1. (soundness of q-realizability).

Let T be any of the set theories considered in this paper, without extensionality. Then for the notion of realizability just given, if $T \vdash A$, then for some number e, we have $T \vdash \bar{e} \; r \; A$.

Proof: As usual for realizability soundness theorems, we proceed by induction on the length of the proof of A, proving that the universal closures of all statements in the proof are realized. Thus we have to verify that the universal closures of all statements in the proof are realized. Thus we have to verify that the universal closures of all the axioms are realized, and the rules of inference preserve realizability. The logical axioms and rules of inference are handled in the usual way (see [Tr]). We have to check the non-logical axioms.

B. (Pairing). $\langle p,q\rangle \; r \; \exists x \forall y (y \in x \to (y=a \vee y=b)$

Take $x^* = \{\langle e,y,y^*\rangle : e \in \omega \;\&\; y \in \{a,b\} \;\&\; y^* \in \{a^*,b^*\} \;\&\; e \; r (y=a \vee y=b)\}$.

This set can be formed in $\underset{\sim}{B}$ without extensionality. Take $x = \{a,b\}$.

C. (Infinity).

$$\emptyset \in \omega \;\&\; \forall y \in \omega \; (y \cup \{y\} \in \omega) \;\&\; \forall z (\emptyset \in z \;\&\; \forall y \in z (y \cup \{y\} \in z) \to \omega \subseteq z)$$

Take $\omega^* = \{\langle n, \langle n,n\rangle\rangle : n \in \omega\}$

Then $\emptyset \in \omega \;\&\; \forall y \in \omega \; (y \cup \{y\} \in \omega)$ is realized and true.

We have to show that

$\forall z (\emptyset \in z \;\&\; \forall y \in z (y \cup \{y\} \in z) \to \omega \subseteq z)$ is realized and true. Suppose z and z^* are given, so that $\emptyset \in z \;\&\; \forall y \in z (y \cup \{y\} \in z)$ is true and realized, say by $\langle a,b\rangle$.

Then, first of all, $\omega \subseteq z$ is true; in order to get $\omega \subseteq z$ realized, we introduce a recursive function $\{p\}$ by the recursion theorem to satisfy the equation

$$\{p\} \; (o) = a$$
$$\{p\} \; (y + 1) = \{b\} \; (\{p\} \; (y))$$

Then we prove by induction that $\{p\} \; (y) \; r \; y \in z$; that is, $\langle \{p\}(y), y, y^*\rangle \in z^*$. (What we are proving by induction has a free variable y^*.) Note that only the restricted induction axiom is needed.

D. (Union). $y \in \bigcup_{z \in a} z \leftrightarrow \exists z (y \in z \;\&\; z \in a)$. If t is the term $\bigcup_{z \in a} z$, we take t^* to be $\{\langle e,y,y^*\rangle : e \; r \exists z (y \in z \;\&\; z \in a)\}$; this can be formed in $\underset{\sim}{B}$.

E. (Separation). Let t be the function symbol such that the following is an axiom: $\forall y (y \in t(a) \leftrightarrow (y \in a \;\&\; B(y))$. To get this realized, we define a function t^* by

$$t^*(a) = \{\langle e,y,y^*\rangle : \langle (e)_o, y, y^*\rangle \in a^* \;\&\; (e)_1 \; r \; B(y)\}$$

t^* can be proved to exist in $\underset{\sim}{B}$-ext, since $u \; r \; B$ is a Δ_o-formula if B is.

Then $t^*(a) = \{\langle e,y,y^*\rangle : e \; r \; (y \in a \; \& \; B(y))\}$; this finishes the verification. Note that Δ_0-separation suffices to interpret Δ_0-separation, and full separation for full separation.

F. (strong collection).

$$\forall a(\forall x \in a \exists y \; A \rightarrow \exists z(\forall x \in a \exists y \in z \; A \; \& \forall y \in z \; \exists x \in a \; A)).$$

Suppose a and a^* are given, and suppose $p \; r \forall x \in a \exists y \; A$, and $\forall x \in a \exists y \; A$. Let $Q_{x,x^*} = \{c: c \; r \; x \in a\} = \{c: \langle c,x,x^*\rangle \in a^*\}$. Then $\forall x \in a \forall x^* \in \mathrm{Rng} \; \mathrm{Rng}(a^*) \forall c \in Q_{x,x^*} \exists y(A \; \& \; \{p\}(c) \; r \; A)$; applying collection, we get the existence of some z_0 such that

$$\forall x \in a \forall x^* \in \mathrm{Rng} \; \mathrm{Rng}(a^*) \; \forall c \in Q_{x,x^*} \exists y \in z_0 \ldots$$

Also, applying collection to $\forall x \in a \exists y \; A$, we get some z_1 such that $\forall x \in a \exists y \in z_1 \; A \; \& \forall y \in z_1 \; \exists x \in a \; A$. Take $z = z_0 \cup z_1$. Then $\forall x \in a \exists y \in z \; A \; \& \forall y \in z \; \exists x \in a \; A$, i.e. the conclusion of axiom F is true. Note that this works because we have strong collection, not just plain collection; the extra conclusion indicated by ... in the choice of z_0 is needed. We need to show that this conclusion is not only true but realized.

Define

$z^* = \{\langle\langle e,c,x,x^*\rangle,y\rangle : c \; r \; x \in a \; \& \; x \in a \; \& \; y \in z \; \& \; x^* \in \mathrm{Rng} \; \mathrm{Rng}(a^*) \; \& \; e \; r \; A(x,y)\}$

First we show that $\forall x \in a \exists y \in z \; A$ is realized (by a number depending recursively on p). Suppose $c \; r \; x \in a$ and $x \in a$. Then for some y in $z_0, A(x,y)$ and $\{p\}(c) \; r \; A(x,y)$. Hence $\langle\langle\{p\}(c),c,x,x^*\rangle,y\rangle \in z^*$, so $\forall x \in a \exists y \in z \; A$ is realized.

Similarly, we have to show $\forall y \in z \exists x \in a \; A$ is realized. Suppose $b \; r \; y \in z$; then b has the form $\langle\langle e,c,x,x^*\rangle,y\rangle$ where $e \; rA(x,y)$ and $c \; r \; x \in a$ and $x \in a$. Hence $\forall y \in z \exists x \in a \; A$ is realized, by a simple combination of unpairing functions.

G. (foundation). $\forall a,b(a \in b \; \& \; b \in x \rightarrow a \in x) \; \& \forall y \in x(y \subseteq z \rightarrow y \in z) \rightarrow x \subseteq z$. Suppose z, z^* are given, and $p \; r \forall a,b(a \in b \; \& \; b \in x \rightarrow a \in x)$, and $q \; r \forall y \in x \; (y \subseteq z \rightarrow y \in z)$, and the formulae realized by p and q are true. Introduce a recursive function $\{f\}$ by the recursion theorem so that

$$\{f\}(e) = \{\{p\}(e)\}(\Lambda u\{f\}(\{p\}(\langle u,e\rangle))) \quad .$$

(Here $\Lambda u\ h$ is an index of $\lambda\ u\ h$, so $\{\Lambda\ u\ h(u,v)\}\ (u) = h(u,v)$; this is an old and useful notation of Kleene.) We claim $f\ r\ x \subseteq z$, the conclusion of the foundation axiom. (Which will finish the proof, since the conclusion of the axiom is true, because we have assumed the hypothesis.) We must show $f\ r\ \forall y(y \in x \rightarrow y \in z)$; that is, whenever $y \in x$, and $e\ r\ y \in x$, we have $\{f\}(e)\ r\ y \in z$. We prove this by transfinite induction on $\langle y,y^*\rangle$ (then later show how to get by with only the foundation axiom). Our induction hypothesis is that for all $a \in y$ and $a^* \in \mathrm{Rng\ Rng}\ (y^*)$, $(e\ r\ a \in x) \rightarrow \{f\}(e)\ r\ a \in z$. Suppose $e\ r\ y \in x$; we must show $\{f\}(e)\ r\ y \in z$. Note that if $u\ r\ a \in y$, then $\{p\}(\langle u,e\rangle)\ r\ a \in x$ by our hypothesis on p . Applying our induction hypothesis (substituting $\{p\}(\langle u,e\rangle)$ for e) , we see that if $a \in y$ and $u\ r\ a \in y$, we have $\{f\}(\{p\}(\langle u,e\rangle))\ r\ a \in z$. That is, $\Lambda\ u\{f\}(\{p\}(\langle u,e\rangle))\ r\ y \in z$. Now, applying the hypothesis on q, and the definition of f , we reach the desired conclusion, that $\{f\}(e)\ r\ y \in z$. This completes our proof by transfinite induction; now we have to show how to get by with only foundation. The foundation axiom amounts to proof by transfinite induction, where what is proved is membership in some set. Here the set in question is

$$\{\langle y,y^*\rangle \in x \times \mathrm{Rng\ Rng}\ (x^*)\ ;\ \forall e \in \omega\ (\langle e,y,y^*\rangle \in x^* \rightarrow \langle \{f\}(e),y,y^*\rangle \in z^*)\}$$

(Recasting the induction on the pair $\langle y,y\rangle$ as an induction on a single variable is left to the reader.) This completes the verification of foundation.

H. (Exponentiation). $\exists x \forall y(y \in x \leftrightarrow \mathrm{Fcn}(y)\ \&\ \mathrm{Dom}(y) = a\ \&\ \mathrm{Rng}(y) \subseteq b)$. (Even the strong version of the exponentiation axiom is realized.) Suppose a,b,a^* , and b^* are given; we have to produce x and x^* . Let x be b^a ; the problem is to produce x^* . Suppose for the moment that we had a set Q such that if $\mathrm{Fcn}(y)\ \&\ \mathrm{Dom}(y) = a\ \&\ \mathrm{Rng}(y) \subseteq b$ is realized, then $y^* \in Q$. (An "a priori bound" on the complexity of y^*.) Then we would like to take x^* to be

$$\{\langle e,y,a^*\rangle \in \omega \times x \times Q\text{: } e\ r\ (\mathrm{Fcn}(y)\ \&\ \mathrm{Dom}(y) = a\ \&\ \mathrm{Rng}(y) \subseteq b).\}$$

Since $\mathrm{Fcn}(y)$ involves an unbounded quantifier, it is not immediate that this set can be formed using the axioms of $\underset{\sim}{B}$. However, we can instead take

$$x^* = \{\langle e,y,y^*\rangle\ \ \omega \times x \times Q\text{: } e\ r\ (\forall s \in a \forall t,t' \in b(\langle s,t\rangle \in y\ \&\ \langle s,t'\rangle \in y \rightarrow t = t')$$
$$\&\ \mathrm{Dom}(y) = a\ \&\ \mathrm{Rng}(y) \subseteq b)\}\ \text{(here } = \text{ is extensional equality).}$$

From a member of this x^* , we can recursively pass to a realizer of $\mathrm{Fcn}(y)\ \&\ \mathrm{Dom}(y) = a\ \&\ \mathrm{Rng}(y) \subseteq b$ and vice versa, so that the exponentiation axiom will be realized. The difficult part, namely producing the set Q , is still ahead of us. At first, this seems to be a serious problem. A typical element of y^* will be $\langle e,\langle s,t\rangle,\langle s^*,t^*\rangle\rangle$ with $t=y(s)$: but t^* is not uniquely determined (even extensionally) so we cannot seem to use the exponentiation axiom to get a set in which y^* must lie, and power set is not available. However, the following argument gets us around the difficulty. First, set $x^* \sim y^*$ iff $\forall z \in b(z \in x \leftrightarrow z \in y)$ is realized. Thus $\{\langle x^*,y^*\rangle \in \mathrm{RngRng}(b^*)^2 : x^* \sim y^*\}$ can be formed in $\underset{\sim}{B}$. Now, we can form in $\underset{\sim}{B}$ the equivalence classes $[x^*]$ under the relation $\sim$ and the set $B_0 = \{[x^*] : x^* \in \mathrm{RngRng}(b^*)\}$ (using the abstraction axiom). Next, observe that if $\langle x,z\rangle \in y$ and $\langle x,w\rangle \in y$ are realized, and $\mathrm{Fcn}(y)$ is realized then $x^* \sim w^*$. Hence, although t^* is not uniquely determined, the equivalence class $[t^*]$ is uniquely determined (extensionally). Suppose

$$p \ r \ s \in a \to \exists t \langle s,t\rangle \in y; \quad \text{that is,}$$

$$\langle e,s,s^*\rangle \in a^* \to \exists t,t^*(\langle \{p\}(e),\langle s,t\rangle,\langle s^*, t^*\rangle\rangle \in y^*) .$$

Then if y^* is such that $\mathrm{Fcn}(y) \ \& \ \mathrm{Dom}(y) = a \ \& \ \mathrm{Rng}(y) \subseteq b$ is realized, then y^* has the form

$$\{\langle d,\langle s,y(s)\rangle,\langle s^*,t^*\rangle\rangle : t^* \in f(s,s^*) \ \& \ d \in p\}$$

for some function f from $a \times \mathrm{RngRng}(a^*)$ to B_0 and some set $P \subseteq \omega$, where P has the form $\{\{q\}(\{p\}(e)) : \langle e,s,s^*\rangle \in a^*\}$, for some q in ω. We do not need the power set axiom to form the set of all such P; quantification over integers q is enough. Using exponentiation, we can form the set of all such functions f; thus the set Ω of all such y^* can be formed; this completes the verification of the exponentiation axiom.

I. (Bounded dependent choice). $\forall x \in a \exists y \in a \ P \to \forall x \in a \exists f(\mathrm{Fcn}(f) \ \& \ \mathrm{Rng}(f) \subseteq a$

$$\& \ \mathrm{Dom}(f) = \omega \ \& \ f(0) = x \ \& \ \forall n \in \omega \ P(f(n),f(n+1)))\text{, with } P \ \Delta_0 .$$

Suppose $p \ r \ \forall x \in a \exists y \in a \ P$ and $s \ r \ x \in a$, and these formulae are true, as well as realized. Define a recursive function h by

$$h(0) = \langle s,0\rangle$$

$$h(n+1) = \{p\}((h(n))_0)$$

and introduce e by

$$\{e\}(0) = \langle s,(\{p\}(s))_1\rangle$$

$$\{e\}(n+1) = h(n)$$

Now, we will prove the existence of a (set-theoretic) function f such that $\{e\}(n) \ r \ (f(n) \in a \ \& \ P(f(n),f(n+1)))$. First note that the principle of countable independent choice $\forall n \in \omega \exists y \in a \ldots$ can be derived from dependent choice.

Then prove by induction on n that

$$(*)\ \forall n \in \omega \exists y_o,\ldots,y_n \in a\ \exists\ y_o^*,\ldots,y_n^* \in \mathrm{RngRng}(a^*)\ \forall j \le n$$

$$(\{e\}(j)\ r\ (y_j \in a\ \&\ P(y_j,y_{j+1})))\ \&\ y_o = x\ \&\ P(y_j,y_{j+1})\ \&\ y_o^* = x^*)$$

This can be done in $\underset{\sim}{B}$ since the formula being proved by induction is a Δ_o-formula. Apply countable choice to get two functions $\overline{f}$ and $\overline{h}$ such that $\overline{f}(n) = \langle y_o,\ldots,y_{n-1}\rangle$ and $\overline{h}(n) = \langle y_o^*,\ldots,y_{n-1}^*\rangle$ as in $(*)$. Then the functions f and h themselves can be defined. Let f^* be defined by

$$f^* = \{\langle u,\langle n,y\rangle,\langle n,y^*\rangle\rangle \in \omega \times (\omega < a) \times (\omega \times \mathrm{RngRng}(a^*)) :$$

$$y = f(n)\ \&\ y^* = h(n)\}$$

Then we will show that, with f and f^* substituted in, the conclusion of the dependent choices axiom is true and realized; that is,

$$(\mathrm{Fcn}(f)\ \&\ \mathrm{Dom}(f) = \omega\ \&\ f(O) = x\ \&\ \forall n \in \omega\ \ P(f(n),f(n+1)\ \&\ \mathrm{Rng}(f) \subseteq a)$$

is true and realized.

First we show $\forall n \in \omega\ P(f(n),f(n+1))$ is true and realized. The truth follows from the last clause in the formula $(*)$ used to define f. For the realizability, we have to use the definition of ω^*, which tells us that a realizer of $n \in \omega$ contains n, specifically, if $s\ r\ n \in \omega$ then $(s)_o = n$.

Now, $P(f(n),f(n+1))$ is actually an abbreviation for

$$(\langle n,y\rangle \in f\ \&\ \langle n+1,w\rangle \in f \rightarrow P(y,w)).\ \text{Take}$$

$$q = \Lambda s\ \ \Lambda p\ \ (\{e\}(\ (s)_o\ ))_1\ ;\ \text{then}$$

$$q\ r\ \forall n \in \omega\ P(f(n),f(n+1))\ .$$

Next we show $\mathrm{Fcn}(f)$ is true and realized; $\mathrm{Fcn}(f)$ is $\langle n,y\rangle \in f\ \&\ \langle n,w\rangle \in f \rightarrow y = w$. Suppose p realizes the left side of this; then p is a pair $\langle k,j\rangle$ with $\langle k,\langle n,y\rangle,\langle n,y^*\rangle\rangle \in f^*$ and $\langle j,\langle n,w\rangle,\langle n,w^*\rangle\rangle \in f^*$; hence $y = w = f(n)$ and $y^* = w^* = h(n)$; here equality is extensional. But since $w = y$ and $w^* = y^*$, we have $y = w$ realized. Hence $\mathrm{Fcn}(f)$ is realized. $\mathrm{Dom}(f) = \omega$ is realized since if $s\ r\ n \in \omega$, then $n = (s)_o$ and

$$\langle O,\langle n,f(n)\rangle,\langle n,h(n)\rangle\rangle \in f^*,\ \text{so}\ \exists y \in a\ y = f(n)$$

is realized. Finally, $f(0) = x$ is realized, because $h(0) = x^*$ and $f(0) = x$. This completes the verification of the axiom of bounded dependent choices.

J. (induction). The verification of this axiom is standard.

K. (relativized dependent choice RDC). Like bounded dependent choice, except that we use full induction and separation instead of bounded induction and separation.

L. (transfinite induction). Like foundation, except that transfinite induction must be used to make the verification, instead of foundation.

M. (full separation). Like Δ_0-separation.

N. (power set). $\exists x \forall y(y \subseteq a \rightarrow y \in x)$. Take $x = \wp(a)$ and

$$x^* = \{\langle e,y,y^*\rangle \in \omega \times \wp(a) \times \wp(\omega \times \mathrm{Rng}(a^*)) :$$

$$y \subseteq a \ \& \ e \ r \ y \subseteq a\}$$

Suppose y and y^* are given and $y \subseteq a$ is realized, say by e, and $y \subseteq a$. Then we have to check that y^* is a subset of $\omega \times \mathrm{Rng}(a^*)$. Suppose $\langle p,z,z^*\rangle \in y^*$ Since $y \subseteq a$ is realized, for some p_0, we have $\langle p_0,z,z^*\rangle \in a^*$; hence $\langle z,z^*\rangle \in \mathrm{Rng}(a^*)$. This completes the verification of power set.

O. (abstraction). For A a Δ_0-formula,

$$\forall x \ \exists z \forall w(w \in z \leftrightarrow \exists y(y \in x \ \& \forall u(u \in w \leftrightarrow A(u,y) \ \& \ u \in x))).$$

Take $z = \{\{u \in x: A(u,y)\} : y \in x\}$, formed by abstraction. We want to prove the existence of z^* such that

$$\langle e,w,w^*\rangle \in z^* \leftrightarrow \exists y,y^*(e \ r \ (y \in x \ \& \forall u(u \in w \leftrightarrow A(u,y) \ \& \ u \in x))).$$

To prove that z^* exists, we again need to use the equivalence relation, $w^* \sim v^*$ iff w extensionally $= v$ is realized iff

$$\exists p,q \in \omega \forall e,u,u^*((\langle e,u,u^*\rangle \in w^* \rightarrow \langle\{p\}(e),u,u^*\rangle \in v^*) \ \&$$
$$(\langle e,u,u^*\rangle \in v^* \rightarrow \langle\{q\}(e),u,u^*\rangle \in w^*) .$$

Note that $\sim$ is given by a Δ_0-formula. Now, by abstraction, we can form the set

$$P = \{\{\langle q,u,u^*\rangle : q \ r(A(u,y) \ \& \ u \in x)\} ; \langle c,y,y\rangle \in \omega \times \mathrm{Rng}(x^*)\} .$$

Now define $z^* = \{\langle e,w,w^*\rangle ; \exists v^* \in P(v^* \sim w^* \ \&$

$$\exists \langle y,y^*\rangle \in Rng(x^*) \ (e \ r \ (y \in x \ \& \forall u(u \in w \to A(u,y) \ \& \ u \in x))))\}$$

In order to show that the z^* so defined is the z^* we are seeking, we have to prove that if $\exists y,y^*(e \ r(y \in x \ \& \forall u(u \in w \leftrightarrow A(u,y) \ \& \ u \in x)))$, then for some v^* in P, $v^* \sim w^*$. Let $v^* = \{\langle q,u,u^*\rangle \in \omega \times Rng(x^*) : q \ r(A(u,y) \ \& \ u \in x)\}$; then $w^* \sim v^*$. Since $\langle (e)_0,y,y^*\rangle \in x^*$, we have $v^* \in P$. This completes the verification of abstraction, and with it, the proof of the soundness theorem 4.1.

We turn now to the auxiliary theories Ta and Tb, and discuss the notion of realizability appropriate for them. These theories, as defined in §2, depend on a particular definable metric space X, which is a subset of N^N with a metric in "standard form" as discussed in §2, the new constant $\underline{a}$ or $\underline{b}$ stands for an element of X, that is, an element of N^N satisfying an additional condition.

Instead of recursive functions, we use functions recursive in $\underline{a}$ (or $\underline{b}$, as the case may be) to realize the theory Ta (or Tb). The theory of functions recursive in $\underline{b}$ can be formalized in Tb, and the verification that all the set-theoretical axioms are realized proceeds exactly as in theorem 4.1, using $\{e\}^{\underline{b}}$ in place of $\{e\}$ throughout. This leads to

Theorem 4.2

If q-realizability is defined using functions recursive in $\underline{b}$, then the interpretation is sound for Tb; similarly for Ta. (Here T is any non-extensional set theory considered in this paper.)

Proof:

Actually, we first have to give a complete description of the interpretation. We have to explain what $e \ r \ \underline{b} \in x$ and $e \ r \ x \in \underline{b}$ are. We shall take $e \ r \ \underline{b} \in x$ to be $\langle e,\underline{b},\underline{b}^*\rangle \in x^*$ and $e \ r \ x \in \underline{b}$ to be $\langle e,x,x^*\rangle \in \underline{b}^*$. Here $\underline{b}^*$ is a particular set (more precisely, a particular term of our non-extensional set theory Tb); to be explicit, $\underline{b}^* = \{\langle k,\langle n,m\rangle,\langle n,m\rangle\rangle : \underline{b}(n) = m\}$. Thus $(e \ r \ \underline{b}(n) = m) \leftrightarrow b(n) = m$. All the logical and set-theoretical axioms can now be verified exactly as before. It remains only to check the extra axioms involving $\underline{a}$ or $\underline{b}$. First, consider the axiom $\underline{a} \in X$. This has the form
$Fcn(\underline{a}) \ \& \ \forall n \in \omega \ \exists m \in \omega(\langle n,m\rangle \in a) \ \& \forall n,m \in \omega(\rho(a_n,a_m) < 1/m + 1/n)$; here ρ may be taken to be some recursive function, as discussed in §2. The second of these three clauses is realized essentially by $\underline{a}$ itself; of course $\underline{a}$ is extensionally equal to $\{e\}^{\underline{a}}$ for a certain number e.

Next, consider the axioms $\underline{b}(\bar{n})=\bar{m}$ for $b(n)=m$. (Remember that Tb is based on a particular function b, while Ta is not based on any particular a.) We have $(e\ r\ \underline{b}(\bar{n})=\bar{m}) \to \underline{b}(\bar{n})=\bar{m}$, so that these axioms also are realized in Tb. This completes the proof of Theorem 4.2.

§5. Explicit Definability

In this section we consider the old metamathematical property, if $T \vdash \exists n \in \omega\ P(n)$, then for some n, $T \vdash P(\bar{n})$. We call this the "numerical explicit definability" property. Our goal is to derive various formalized versions of this property for set theories T and the auxiliary theories Ta and Tb, which will suffice to get the desired continuity rules. A few general remarks are in order. The numerical explicit definability property should be compared and contrasted with the set explicit definability property, if $t \vdash \exists x\ P(x)$ then for some explicitly defined $\hat{x}$, $T \vdash P(\hat{x})$. (One might give different meanings to the words "explicitly defined" here; but for example, any set given by a term of our non-extensional set theories is explicitly defined.) These explicit definability properties are already known for certain intuitionistic set theories (See [Fr 3], [M1], and [M3].) These theories have replacement instead of collection. (However, the double-negation interpretation has not been made to work for ZF with replacement, but has been made to work for ZF with collection; see [Fr1].) Friedman and Myhill use a variant of Kleene's "slash", which becomes quite complicated because extensionality is dealt with directly. This realizability is not enough for the needs of the present paper, because it is not recursive, and it is not easily formalized.

Numerical explicit definability results for the auxiliary theories Ta and Tb provide information generalizing what is usually known as "Church's rule", which says that if $\forall n \exists m\ A(n,m)$ is provable, then for some e, $\forall n\ A(n,\{\bar{e}\}(n))$ is provable. If we take the complete separable metric space X to be the integers N, then to say $\forall n \exists m\ A(n,m)$ is provable (the hypothesis of Church's rule) is to say that Ta proves $\exists m\ A(\underline{a},m)$ (the hypothesis of explicit definability for Ta). In the case $X=N^N$ or $X=2^N$ (not to mention the reals or certain function spaces) we get other interesting information. The exact form of these results will be given below. We begin with the most straightforward explicit definability theorem.

Theorem 5.1 Let T be one of the non-extensional set theories considered in this paper. If $T \vdash \exists x \in \omega P(x)$ then $T \vdash P(\bar{n})$ for some numeral $\bar{n}$.

Proof: Suppose $T \vdash \exists x \in \omega\ P(x)$. Then, by the soundness of q-realizability for T, there is some e such that $T \vdash \bar{e}\ r\ \exists x \in \omega\ P(x)$. That is, T proves $\exists x,x^*(\bar{e}_o\ r\ x \in \omega\ \&\ \bar{e}_1\ r\ P(x)\ \&\ P(x))$, where $e = \langle e_o,e_1\rangle$. According to the definition of ω^*, T proves $(s\ r\ x \in \omega \to x=(s)_o)$. Hence T proves $P((\bar{e}_o)_o)$. Now since

T contains arithmetic, T proves $(\bar{e})_o=\bar{n}$, where $n=(e)_o$. Hence T proves $P(\bar{n})$, which completes the proof.

Now consider explicit definability for extensional theories. Our methods yield numerical explicit definability, not for all formulae P, but only for P of the form $x \in Q$, where Q is a specific definable set. We take "definable set" to mean "set given by one of the terms of the non-extensional set theory T". What we would ideally want is a larger system of terms, adequate to prove the set explicit definability theorem. In trying to get such a system of terms, there is a problem in that the choice and collection axioms assert the existence of a set, without there being any obvious definable one. This is why the set explicit definablity property is known only for theories with replacement, and not for theories with collection. Although this is an interesting phenomenon, we regard it as a side issue, since our focus here is on continuity rules. We therefore restrict our attention to sets defined by terms. Actually, we could include exponentiation terms as well; if this is done, the definable sets seem to encompass most sets needed for mathematical practice.

Lemma 5.1 Let T be one of the extensional set theories considered in this paper. Let A^* be the interpretation of A in the non-extensional set theory T-ext, given in Theorem 3.1. Let Q be a definable set in T. Then $T \vdash (x \in Q)^* \leftrightarrow x \in Q$.

Proof: First we must explain precisely what is meant by $(x \in Q)^*$. Here Q is a term, which belongs to T-ext, but not to T; so $x \in Q$ must be interpreted as the formula of T obtained by writing out the definitions of the terms composing Q. Secondly, if A has a free variable x, then A^* has two free variables, x and y, where y is supposed to "witness" that x is a set. (Technically x is interpreted as the pair $\langle x,y \rangle$.) Thus $(x \in Q)^* \leftrightarrow (x \in Q)$ really means $((x \in Q)^*(x,y) \rightarrow x \in Q)$ & $(x \in Q \rightarrow \exists y (x \in Q)^*(x,y))$. Now, the proof proceeds by induction on the complexity of the term Q. For instance, if Q is $\{x \in a: P(x)\}$, where a is a term and P is Δ_o, we have $(x \in Q)^*(x,y) \leftrightarrow S(\langle x,y \rangle)$ & $(x \in a)^*$ & $P^*(x,y)$, where S is the formula defining the sets of the model of Section 3. According to Theorem 3.1(i), we have $S(\langle x,y \rangle) \rightarrow (P(x,y) \leftrightarrow P^*(x,y)$; so $(x \in Q)^*(x,y) \leftrightarrow S(\langle x,y \rangle)$ & $(x \in a)^*$ & $P(x,y)$. By induction hypothesis $(x \in a)^* \leftrightarrow x \in a$, since we can prove by induction on terms that the transitive closure of a definable set is definable, so that $S(\langle a,b \rangle)$ for some term b. Thus $(x \in Q)^*(x,y) \leftrightarrow S(\langle x,y \rangle)$ & $x \in a$ & $P(x,y)$, i.e. $\leftrightarrow S(\langle x,y \rangle)$ & $x \in Q$. But $x \in Q$ implies $\exists y\, S(\langle x,y \rangle)$, since TC(x) can be defined if x is known to belong to some transive set, and as we have just mentioned, the transitive closure of Q is definable. Thus $x \in Q \rightarrow \exists y (x \in Q)^*(x,y)$ and $(x \in Q)^*(x,y) \rightarrow x \in Q$. For reasons of space limitation we omit the other cases in the induction on Q.

Theorem 5.2 Let T be one of the set theories discussed in this paper, including extensionality. Suppose $T \vdash \exists x \in \omega(x \in Q)$, where Q is a definable set in T. Then for some numeral $\bar{n}$, $T \vdash \bar{n} \in Q$.

Proof: Suppose $T \vdash \exists x \in \omega(x \in Q)$. Then, by Theorem 3.1, T-ext $\vdash (\exists x \in \omega(x \in Q))^*$. Hence T-ext $\vdash \exists x \in \omega(x \in Q)^*$, since $x \in \omega$ is equivalent to its *-interpretation, by Theorem 3.1 (ii). By Theorem 5.1, T-ext $\vdash (\bar{n} \in Q)^*$, for some numeral $\bar{n}$. Hence, by Lemma 5.1, $T \vdash \bar{n} \in Q$. This completes the proof.

Next we turn to explicit definability results for the auxiliary theories Ta and Tb.

Theorem 5.3 (i) Suppose $Tb \vdash \exists x \in \omega P(x)$, where T is without extensionality. Then $Tb \vdash P(\bar{n})$, for some numeral $\bar{n}$. If T has extensionality, then the same result holds for P of the form $x \in Q$, where Q is a definable set in T.
(ii) Suppose $Ta \vdash \exists x \in \omega\ P(x)$, where T is without extensionality. Then for some numeral $\bar{e}$, $Ta \vdash \{\bar{e}\}^{\underline{a}}(0) \in \omega$ & $P(\{\bar{e}\}^{\underline{a}}(0))$. If T has extensionality, the same result holds for P of the form $x \in Q$, where Q is a definable set in T.

Proof: Exactly like Theorems 5.1 and 5.2, appealing to the realizability used in Theorem 4.2 instead of 4.1. For (i), we also have to observe that in Tb, if $\{\bar{e}\}^{\underline{b}}(0) \in \omega$ is provable, then for some numeral $\bar{n}$, $\{\bar{e}\}^{\underline{b}}(0)=\bar{n}$ is provable. This is proved just like the corresponding result for T; it consists in observing that the axioms of Tb suffice to formalize the computations of a Turing machine; when a value $b(\bar{n})$ is called for in the course of a computation, one of the axioms of Tb is there to formalize the step in which the "oracle" answers. Of course, this cannot be carried out in Ta, which is why the theorem takes the form it does. This completes the proof of the theorem.

Formalized Explicit Definability

We have to discuss the formalization of the preceding results on explicit definability. They cannot be formalized as they stand (see the general discussion in [B1]), but instead we have to show that there is a sequence of subsystems T_n of each set theory T, such that the explicit definability theorems for T_n can be proved in T, for each fixed integer n. This may not be possible for systems T which have only restricted induction. Here we carry it out for the other theories considered in this paper, which have full induction.

The complexity of a formula of set theory is an integer defined by induction so that prime formulae have complexity zero, and the complexity increases by one at each logical connective and quantifier. We can, for each fixed n, introduce a truth-definition Tr_n (a formula of two free variables; one of which is a number variable, i.e. a variable bound to ω, and prove $T \vdash Tr_n('A',x) \leftrightarrow A(x)$. To be technically precise, we have to worry about the fact that A can have more than one free variable (x can be a list of variables), and code these variables into the single variable on the left, so that we should actually say

$$T \vdash Tr_n(\text{'A)}, x) \leftrightarrow A((x)_1, \ldots (x)_m).$$

We neglect this distinction where it is safe to do so. The construction of Tr_n is standard; $Tr_n(\text{'A'}, x)$ is a disjunction, according to the finitely many possible forms of A.

If T is one of our set theories, let T_n be T with all proofs restricted to contain formulae of complexity $\leq n$; and the axioms of T_n are those axioms of T which are of complexity $\leq n$ and occur among the first n axioms of T in some natural enumeration. Thus T_n has finitely many axioms. Note that T_n is not a formal system in the usual sense, since a formula of complexity $\leq n$ might be provable from axioms of complexity $\leq n$, but only through intermediate steps of greater complexity. Nevertheless, T_n is useful for our purposes (chiefly because it saves us from having to use formalized cut-elimination theorems, which in some cases are not even proved yet). By the reflection principle for S, we mean

$$Pr_S(\text{'A'}) \to A \quad \text{, for all formulae A.}$$

Lemma 5.2 Let T be one of the set theories considered in this paper with full induction (i.e. T_2, T_3, T_4, Z, ZF^- + RDC). Then T proves the reflection principle for T_n, for each fixed n.

Proof: We have $Tr_n(\text{'A'}) \leftrightarrow A$; and we prove $Prf_n(j, \text{'P'}) \to Tr_n(\text{'A'})$ by induction on j. It seems that bounded induction will not suffice.

By the 1-consistency of a theory S, (terminology due to Kreisel and Levy) we mean, for A recursive with one free variable,

$$Pr_S(\exists n \in \omega\ A(n)) \to \exists n \in \omega\ Pr_S(\text{'}A(\bar{n})\text{'}).$$

For such A, we have $A(n) \to Pr_S(\text{'}A(\bar{n})\text{'})$, if S contains a modicum of arithmetic, and indeed this fact itself is provable in any theory which proves that S contains a modicum of arithmetic. Hence 1-consistency follows from the reflection principle for S, for all the S we have reason to consider.

Next we discuss the formalization of the soundness theorems for q-realizability. Let us first discuss what goes wrong with a straightforward attempt to formalize the theorem, $Pr(\text{'A'}) \to \exists e\ Pr(\text{'}\bar{e}\ r\ A\text{'})$. One would try to prove this by induction on the length of the proof of A; the induction step involves proving

$$Pr(\text{'}\bar{a}\ r\ (A \to B)\text{'})\ \&\ Pr(\text{'}\bar{b}\ r\ A\text{'}) \to \exists m\ Pr(\text{'}m\ r\ B\text{'}).$$

Now from Pr('a r (A → B)') & Pr('b r A'), we easily get Pr('∃n(Tabn & U(n) r B)'). To pass from that to the desired conclusion of the induction step requires the 1-consistency of the theory. However, this is the only obstacle to the straightforward formalization of the proof. In other words, if we have the 1-consistency of a theory S, and we can prove the axioms of S are realized, then we can prove the soundness theorem for S, using nothing more complicated than bounded induction. Thus we obtain

Lemma 5.3 (Formalized realizability). Let T be one of the non-extensional set theories discussed in this paper, and let T_n be as in Lemma 5.2. Then for each fixed n, there is some n* such that

$$T \vdash (Pr_n('A') \to \exists e\ Pr_{n*}('\bar{e}\ r\ A')),$$

where Pr_n is the provability predicate of T_n.

Remark: By writing Pr('$\bar{e}$ r A'), to be perfectly explicit, we mean Pr_n(Sub(Num(e),'x r A'), where Num is a primitive recursive function producing from e a Gödel number of the numeral $\bar{e}$, and Sub is a function producing a Gödel number of P(t) from Gödel numbers of a term t and a formula P.

Proof: As sketched above, we go by induction on the length of the proof of A, using 1-consistency in the induction step for modus ponens; it is provided by Lemma 5.2. The use of n* on the right in place of n is necessary because the complexity of e r A is usually greater than that of A. We also have to check that for each axiom A of T_n, T proves $\exists e\ Pr_{n*}(\bar{e}\ r\ A)$. If n* is chosen large enough, this will be a true Σ^0_1 sentence, by Theorem 4.1; therefore provable in arithmetic, hence in T. (Here we use that T_n has only finitely many axioms.) This completes the proof of the lemma.

Lemma 5.4 (Formalized explicit definability). Let T be one of the (non-extensional) set theories discussed in this paper. Let Ta_n and Tb_n be as in Lemma 5.2. For each fixed n, there is an n* such that

(i) $T \vdash (Pr_n('A') \to \exists e\ Pr_{n*}('\bar{e}\ r\ A'))$, where Pr_n is the provability predicate of T_b.

(ii) $T \vdash (Pr_n('A') \to \exists e\ Pr_{n*}('\{\bar{e}\}^{\underline{a}}(0)\ r\ A'))$, where Pr_n is the provability predicate of Ta_n.

Proof: Like Lemma 5.3, appealing to Theorem 4.2 instead of Theorem 4.1.

Theorem 5.4 (Formalized explicit definability).

Let T be any of the set theories discussed in this paper. Then T proves numerical explicit definability for T_n, Tb_n, and Ta_n, for each fixed n. To be precise, if Theorems 5.1, 5.2, and 5.3 are altered by changing T, Tb, and Ta to T_n, Tb_n, and Ta_n in the hypothesis and to T_{n^*}, Tb_{n^*}, and Ta_{n^*} in the conclusion, then the resulting statements are provable in T, for some n* depending on n.

Proof: We first choose n* so large that T_{n^*} will prove $s \; r \; x \in \omega \rightarrow (s)_o = x$. Now the proof of Theorem 5.1 can be formalized directly, appealing to Lemma 5.3 where the soundness of q-realizability is used. Next, note that the proof of Lemma 5.1 can be formalized in T (with T replaced by T_n in the conclusion), since the statement proved by induction there is Δ_o. The appeal to Theorem 3.1 is all right because, as noted in Theorem 3.1 (iii), the relevant part of Theorem 3.1 can be proved in arithmetic; so if n* is chosen large enough, Theorem 3.1 will be provable in T_{n^*}. Now the proof of Theorem 5.2 can be directly formalized, appealing to Lemma 5.4 where the soundness of q-realizability for Ta and Tb is needed. This completes the proof of Theorem 5.4.

§6. Uniform Continuity and Forcing.

The results of the previous sections are sufficient to establish the derived rules concerning local continuity, but not those concerning local uniform continuity. It is worth reviewing the reasons why the preceding results are not sufficient. What we need to establish is condition (iii) of [B1], which says roughly that each provably recursive function from a compact metric space X to the integers N is provably uniformly continuous (hence provably bounded). Now, as discussed in [B1], we cannot hope to prove all functions from X to N are uniformly continuous in any theory consistent with Church's thesis, because there is a recursive functional defined on all recursive members of 2^N, but not uniformly continuous there. (To compute this functional at an argument y, examine the values y(0), y(1)...until you come to y(n) such that in n steps of computation, you can verify that y cannot be a separation of two fixed recursively inseparable r.e. sets; then set the output equal to y(n+1).) This is our first observation. Our second observation is that any provably recursive functional can be proved to be continuous, by the derived rules which follow from the results already proved; hence, classically, it is uniformly continuous, since X is compact. However, this is not enough; we want to know that it is provably uniformly continuous.

Our solution to this problem lies in using forcing to add a generic real to the universe; any function which is defined on all members of a compact space, including generic ones, will have to be uniformly continuous. We used forcing in [B1] to establish these rules for Feferman's theories; here we apply a similar technique to Friedman's theories. It turns out to be rather complicated to give a suitable definition of forcing that works for the exponentiation axiom, although for theories containing power set it is straightforward.

Suppose the compact space X, whose members are the members x of N^N with $x(n) \le M_n$, for some fixed recursive sequence M_n, is fixed once and for all. Let C be the set of finite sequences of integers $p = \langle p_0, p_1, \dots p_n \rangle$ such that $p_i \le M_i$. We use the usual notations $(p)_i$ for p_i, lh(p) for n+1; and we use the notation (borrowed from forcing) $p \le q$ to mean that q is an initial segment of p (so p gives more information than q). No harm will result from using $\emptyset$ to denote the empty sequence. We use p, q, and r for members of C; thus $\forall p$ means $\forall p \in C$. We are going to assign to each formula A of a set theory $T\dot{a}$ (with an extra constant $\underline{a}$ for a member of X), a formula $p \Vdash A$ of T (without $\underline{a}$), which is read "p forces A". The free variables of $p \Vdash A$ are p together with x', where x are the free variables of A. (The use of x' here is purely for intelligibility; we may technically assume x' is the same variable as x.) We write $p, n \Vdash A$ to abbreviate $\forall q \le p(\mathrm{lh}(q) \le n + \mathrm{lh}(p) \rightarrow q \Vdash A)$. We are now ready to give the clauses defining the forcing interpretation that works for theories containing the power set axiom (below we shall discuss the modifications needed to treat theories with the exponentiation axiom).

$p\Vdash A\vee B$ is $p\Vdash A \vee p\Vdash B$

$p\Vdash A\&B$ is $p\Vdash A \;\&\; p\Vdash B$

$p\Vdash \exists xA$ is $\exists x' p\Vdash A$

$p\Vdash \forall xA$ is $\forall x' \exists n(p,n\Vdash A)$

$p\Vdash (A\rightarrow B)$ is $\forall q\leq p(q\Vdash A \rightarrow \exists n(q,n\Vdash B))$

$p\Vdash x\in y$ is $<p,x'>\in y'$

$p\Vdash x=y$ is $\forall w(w\in x' \leftrightarrow w\in y')$; $p \Vdash \bot$ is $\bot$

These last clauses will also serve to define what it means for p to force an atomic formula containing terms of the non-extensional set theories, once we associate to each such term t another term t' to use in these clauses. As in the soundness proof for realizability, the choice of t' will be apparent in the course of the soundness proof for forcing, and we postpone the definitions of the terms t' until then. We do, however, now give the term $\underline{a}'$ which is necessary in order that the above clauses should determine what it means to force an atomic formula involving $\underline{a}$. Namely, $\underline{a}' = \{<p,<n,m>>: n<lh(p) \;\&\; m=(p)_n\}$. Note that $\underline{a}'$ does not involve $\underline{a}$, so that generally $p\Vdash A$ is a formula without $\underline{a}$.

Remark: We have logic with no negation symbol, and instead a falsum symbol in terms of which negation can be defined. The above definition shows that $p\Vdash\neg A$ iff $\forall q\leq p\neg q\Vdash A$, which is the usual clause. Since $\forall$ and $\rightarrow$ are classically superfluous, if we use classical logic our definition reduces to the usual notion of forcing.

Our next goal is to give the modification of the above interpretation that will suffice for theories with the exponentiation axiom. We introduce some

Notation: $C_p = \{q\in C: q\leq p\}$

$q\leq_n p$ means $q\leq p \;\&\; lh(q) = n+lh(p)$

$x'/p = \{<q,u'>\in x': q\leq p\}$; x'/p is read "x' restricted to p".

Our forcing interpretation will be defined in the following way: we shall first associate to every formula A with free variables x another formula $R_A(p,x')$; we then write $p\Vdash A$ to abbreviate $R_A(p,x'/p)$. In what follows, we make the convention that the variables x' are restricted to so-called "good sets"; that is, $\forall x'$ means $\forall x'(G(x')\rightarrow$, and $\exists x'$ means $\exists x'(G(x')\;\&$, where $G(x')$ is a Δ_0 formula defining the good sets, as follows. With $\sim$ for extensional equality, $G(y')$ is $\forall p\forall q\leq p\forall w(<p,w>\in y' \rightarrow \exists v(<q,v>\in y' \;\&\; v\sim w/q))$. It is not obvious that any good sets exist; we shall encounter our first ones in Lemma 6.3 below. Now here are the clauses defining the formulae $R_A(p,x')$:

$R_{A\&B}$ is $R_A\&R_B$; $R_{A\vee B}$ is $R_A \vee R_B$; $R_{\exists yA}$ is $\exists y'A$

$R_{x\in y}(p,x',y')$ is $\langle p,x'\rangle \in y'$

$R_{x=y}(p,x',y')$ is $x'=y'$; $R_{\perp}$ (p) is $\perp$

$R_{\forall y}(p,x')$ is $\forall y' \exists n \forall q \leq_n p\ R_A(q,x'/q)$

$R_{A\to B}(p,x')$ is $\forall q \leq p (R_A(q,x'/q) \to \exists n \forall r \leq_n q\ R_B(r,x'/r))$

Now, using the abbreviation $p \Vdash A$ for $R_A(p,x'/p)$, the clauses for implication and universal quantification can be rewritten in exactly the form we gave for the simpler version of forcing!

Lemma 6.1 If $p \Vdash A$ and $q \leq p$ then $q \Vdash A$.

Proof: A straightforward induction on the complexity of A, using crucially that the primed variables are restricted to good sets, which in fact is built into the definition just to make the atomic case of this lemma work.

Lemma 6.2 If $p,j \Vdash A$ and $p,m \Vdash (A\to B)$, then for some k, we have $p,k \Vdash B$.

Proof: Let $n = \max(k,j)$; by Lemma 6.1, $p,n \Vdash A$ and $p,n \Vdash (A\to B)$. So for each $q \leq_n p$, $q \Vdash (A\to B)$. Hence, for each $q \leq_n p$, $\exists i(q,i \Vdash B)$. Now there are only finitely many $q \leq_n p$. Let k_o be larger than any of the values of i which work for these finitely many q. Then $q \leq p \to q,k_o \Vdash B$. Set $k = n+k_o$. Then $p,k \Vdash B$.

Lemma 6.3 $p \Vdash n \in \omega$ iff $n \in \omega$; more precisely, there is for each n a term n' such that $p \Vdash n \in \omega$ implies $n \in \omega$, and $n \in \omega$ implies $p \Vdash n \in \omega$ with n' substituted for the corresponding free variable of the formula $p \Vdash n$.

Remark: This is the analogue of saying $n \in \omega$ is "self-realizing". We might call a formula with this property "self-forcing".

Proof: We first define a function n' of n for use in the lemma, by induction: 0' is $\emptyset$, and (n+1)' is $\{\langle p,u'/p\rangle : u \leq n\}$. By restricted induction on n, one proves that for all integers n, n' is a good set. We now define ω' (which we promised to do in order to complete the definition of forcing) as $\{\langle p,n'/p\rangle : n \in \omega\}$. The assertions of the lemma may now be proved by a straightforward induction on n.

Lemma 6.4 Let A be an arithmetical predicate. Then for $x \in \omega$, $p \Vdash A(x)$ iff $A(x)$.

Proof: By induction on the complexity of A. The basis case consists of the relations $x=y+z$, $y=x\cdot z$, and successor. These relations have their set-theoretical definitions, so matters are technically complicated. Consider how to prove ($p \Vdash x=y+1$ iff $y=y+1$). This is done by induction on x, first proving $p \Vdash 0 \in z$ by induction on z. Then we proceed to + and $\cdot$, just as in the set-theoretical development of arithmetic. (This can all be done in $\underset{\sim}{B}$).

Next we do the induction step of the lemma, in which A is, for instance, $\forall z \in \omega B(x,z)$. Suppose $p \Vdash A(x)$; then for some n, we have $p,n \Vdash (z \in \omega \to B(x,z)$. Let $z \in \omega$ be given; using z' produced in Lemma 6.3, we have $p \Vdash z \in \omega$; hence for some j, we have $p,j \Vdash B(x,z)$. Hence $B(x,z)$, by the induction hypothesis. Since z was arbitrary, we have $A(x)$. Conversely, suppose $A(x)$. We will show $\emptyset \Vdash A(x)$; that is, $\emptyset \Vdash \forall z \in \omega\ B(x,z)$. We claim $q \Vdash z \in \omega$ implies $q \Vdash B(x,z)$. Indeed, if $q \Vdash z \in \omega$ then $z \in \omega$, so $B(x,z)$; hence, by induction hypothesis, $\emptyset \Vdash B(x,z)$.

Lemma 6.5 If y' is substituted for x' in the formula $p \Vdash A(x)$, the result is logically equivalent to $p \Vdash A(y)$. In other words $R_{A(x)}(p,x'/p)$ is equivalent to $R_{A(y)}(p,y'/p)$.

Proof: By induction on the complexity of A.

We are now ready to state the soundness theorem for forcing. Let Ta be the auxiliary theory described in §2, with a constant $\underline{a}$ for an element of the compact space X.

Theorem 6.1 Let T be any of the non-extensional set theories discussed in this paper, except B-ext. Thus T can be (non-extensional) T_1, T_2, T_3, T_4, Z, or ZF^-+RDC. Then $Ta \vdash A$ implies $T \vdash \exists n(\emptyset,n \Vdash A)$.

Proof: By induction on the length of the proof of A. We have to check the logical axioms and rules, then the set-theoretical axioms. We begin with modus ponens. Suppose $\emptyset,n \Vdash A$ and $\emptyset,m \Vdash (A\to B)$; then by Lemma 6.2, $\emptyset,k \Vdash B$ for some k. (Note that Lemma 6.2 was proved within $\underset{\sim}{B}$.) We leave the reader to check the other propositional axioms and rules (using e.g. the list on page 3 of [Tr]). We turn to the quantifier axioms and rules. Consider the axiom ($\forall x Ax \to At$), for some term t. Consider first the case when t is a variable. Suppose $q \Vdash \forall x Ax$; that is, $\forall x' \exists n \forall p \leq_n q\ R_A(p,x'/p)$. Substitute t' for x'; then for some n we have $q,n \Vdash A(t)$, using Lemma 6.5. The case in which t is a term other than a variable is handled the same way, provided we have at hand a corresponding term t' with the property of Lemma 6.5. We shall give, in the course of verifying the set-theoretical axioms, such a term t' for each term t. The other quantifier axioms and rules can be treated similarly.

We now turn to the non-logical axioms, beginning with the axiom $\underline{a} \in X$, which has the following form when written out:

$$\forall n \in \omega \exists m \in \omega(\langle n,m\rangle \in \underline{a}\ \&\ m \leq M_n)\ \&\ \forall x \in a \exists n,m \in \omega(x=\langle n,m\rangle)\ \&$$

$$\forall n,m(\langle n,m\rangle \in a\ \&\ \langle n,r\rangle \in a \to m=r)\ \&\ \forall n,m(\rho(\underline{a}(n),\underline{a}(m)) \leq 1/(n+1)+1/(m+1))$$

where ρ is some recursive function and M_n is some recursive sequence (see §2). First we show $\emptyset \Vdash \forall n \in \omega \exists m \in \omega(\langle n,m\rangle \in \underline{a}\ \&\ m \leq M_n)$. Let n and p be given; we claim $p \Vdash n \in \omega \to \exists m \in \omega(\langle n,m\rangle \in \underline{a}\ \&\ m \leq M_n)$. Suppose $q \leq p$ has $q \Vdash n \in \omega$; then $n \in \omega$, by Lemma 6.3; we must show $q,j \Vdash \exists m \in \omega(\langle n,m\rangle \in \underline{a}\ \&\ m \leq M_n)$ for some j. Take $j=M_n+1+n$. Then $r \Vdash \langle n,m\rangle \in \underline{a}$. Also by Lemma 6.4, $r \Vdash m \leq M_n$, since by definition of C, $m \leq M_n$ is true.

Next consider the conjunct $x \in \underline{a} \to \exists n,m \in \omega(x=\langle n,m\rangle)$. Suppose $p \Vdash x \in \underline{a}$; then $x=\langle n,m\rangle$ where $n<lh(p)$ & $m=(p)_n$. Thus $p \Vdash n \in \omega$ & $m \in \omega$ & $(x=\langle n,m\rangle)$ by Lemmas 6.3 and 6.4. Hence $\emptyset \Vdash (x \in \underline{a} \to \exists n,m \in \omega(x=\langle n,m\rangle))$. The last two conjuncts can be verified similarly; this completes the verification of the axiom $\underline{a} \in X$.

We now turn to the set-theoretical axioms. Consider an axiom of the form $\exists x \forall y(y \in x \leftrightarrow A(y))$. Pairing, union, separation, exponentiation, and power set are all of this form. If we can form $x' = \{\langle p,y'/p\rangle : p \Vdash A(y)\}$, then this axiom will be forced by $\emptyset$, as is easily checked. (If we had not been so careful in our definition, we would have to form $\{\langle p,z'\rangle : p \Vdash A(y)\}$; which cannot be done for the exponentiation axiom.) We now check the axioms of this form one by one.

Separation: Here we have to form $x' = \{\langle p,y'/p\rangle : p \Vdash (B(y)$ & $y \in a)\}$; that is, $\{\langle p,y'/p\rangle \in a' : p \Vdash (B(y)$ & $y \in a)\}$. This can be formed using separation and abstraction. To check Δ_0-separation, we have to prove that $p \Vdash B(y)$ is a Δ_0 formula if B is; this is a simple induction on the complexity of B.

The definition of x' just given also determines the term t' corresponding to the term t associated with this separation axiom. It has to be checked that x' as just defined is a good set. Generally if x' is defined as $\{\langle p,y'/p\rangle : p \Vdash C\}$ then x' is good, since if $q \le p$ and p is in x', then q also forces C, so $\langle q,y'/q\rangle$ is in x', by definition of x'; but $\langle q,y'/q\rangle$ is exactly $\langle q,(y'/p)/q\rangle$, which is what we must prove is in x' in order to show x' is good. All the terms t' which we shall exhibit in verifying the set-theoretical axioms have this form, so we need not repeat the argument in each case.

Union: Here we have to form $x' = \{\langle p,y'/p\rangle : p \Vdash \exists z(y \in z$ & $z \in a)\}$; that is, we want x' to consist of all $\langle p,y'/p\rangle$ such that for some z', we have $\langle p,y'/p\rangle \in z'$ and $\langle p,z'/p\rangle \in a'$. Now $\langle p,y'/p\rangle \in z'$ is equivalent to $\langle p,y'/p\rangle \in z'/p$. Hence we want x' to consist of all $\langle p,y'/p\rangle$ belonging to some u with $\langle p,u\rangle \in a'$. This set can be formed in B, using the union axiom to take a union over Rng(a).

Pairing: Take $x' = \bigcup_{q \in C} \{a'/q,b'/q\}$. Thus $\langle q,y'/q\rangle \in x'$ iff $q \Vdash (y=a \vee y=b)$.

Exponentiation: This is the most difficult axiom to verify. Here we have to show how to form $x'=\{\langle p,y'/p\rangle : p \Vdash (Fcn(y)$ & $Dom(y)=a$ & $Rng(y) \subseteq b\}$. The problem is to give in advance a set to which y'/p must belong, so that x' can be formed by separation. Suppose $\emptyset \Vdash (Fcn(y)$ & $Dom(y)=a$ & $Rng(y) \subseteq b)$; then where must y' lie? (Remember, we do not have power set available.)

Introduce an equivalence relation $\approx$ on Rng(b') by defining $z' \approx w'$ iff $\emptyset,n \Vdash z=w$ for some n. Then let $[z']$ be the equivalence class of z' under this relation; $[z']$ can be formed using Δ_0-separation. Let S_2 be the set of all $[z']$

for z' in $\mathrm{Rng}(b')$ (which exists by abstraction) and let S_o be the set of all finite subsets of S_2 (using exponentiation, union, and abstraction). Let S_1 be the set of all functions from a' to S_o. If f is in S_1, let $F(f)$ be $\{\langle q,\langle w',z'\rangle/q\rangle : \langle q,w'/q\rangle \in a' \ \& \ \forall z' \forall x'([z'] \in f(q,w'/q) \ \& \ [x'] \in f(q,w'/q) \to z' \approx x')\}$. (We can form this set by the Δ_o-separation axiom, since the equivalence relation $\approx$ is defined by a Δ_o formula.)

It is tempting to think that if $\emptyset \Vdash \mathrm{Fcn}(y) \ \& \ \mathrm{Dom}(y)=a \ \& \ \mathrm{Rng}(y) \subseteq b$, as we have assumed, then y' must be (extensionally) $F(f)$ for some f in S_1. Now, we can check that $y' \subseteq F(f)$, where

$$f(\langle q,w'/q\rangle) = \{ [z'] : z' \in \mathrm{Rng}(b') \ \& \ \exists r \leq q \ r \Vdash \langle w,z\rangle \in y\}.$$

(This set is finite, because $\emptyset, n \Vdash \exists ! z \in b(\langle w,z\rangle \in y)$ for some n; and it can be formed using abstraction and separation. We note that "finite" means to be the range of some function defined on some integer, in the intuitionistic context; i.e. to be of bounded size.) However, possibly some $\langle q,\langle w',z'\rangle/q\rangle \in F(f)$ may not have $q \Vdash \langle w,z\rangle \in y$, although z is unique such that $q,n \Vdash \langle w,z\rangle \in y$. To solve this problem, let us say $y'_o \sim_{n,w} y'_1$ iff

$$\forall z' \in \mathrm{Rng}(b') \forall q \leq_n \emptyset (q \Vdash \langle w,z\rangle \in y_o \leftrightarrow q \Vdash \langle w,z\rangle \in y_1).$$

So $\forall w' \in \mathrm{Rng}(a') \exists n (y' \sim_{n,w} F(f))$. Now, if $y'_o \sim_{n,w} y'_1$, then

$$y'_o = \{\langle q,\langle w',z'\rangle/q\rangle : q \leq_n \emptyset \ \& \ \langle q,\langle w',z'\rangle/q\rangle \in y'_1 \vee q \in S\}$$

for some finite subset S of C. Since the set of all finite subsets of C exists, we can form (by Δ_o-separation) $\{y'_o : y'_o \sim_{n,w} y'_1\}$ for each n,w',y'_1. Thus

$$\forall w' \in a' (y' \in \bigcup_{n \in \omega} \{y' : y' \sim_{n,w} F(f)\}), \text{ that is,}$$

$$y' \in \bigcap_{w' \in a'} \ \bigcup_{n \in \omega} \{y' : y' \sim_{n,w} F(f)\} \quad \text{(using abstraction and union).}$$

Thus, if we form by the collection axiom the set

$$S_3 = \{ \bigcap_{w' \in a'} \ \bigcup_{n \in \omega} \{y' : y' \sim_{n,w} F(f)\} : f \in S_1\}, \text{ then}$$

$\emptyset \Vdash (\mathrm{Fcn}(y) \ \& \ \mathrm{Dom}(y)=a \ \& \ \mathrm{Rng}(y) \subseteq b)$ implies $y' \in S_3$. Similarly, we can construct a set S^p_3 such that $p \Vdash (\mathrm{Fcn}(y) \ \& \ \mathrm{Dom}(y)=a \ \& \ \mathrm{Rng}(y) \subseteq b)$ implies $y'/p \in S^p_3$. Using collection again, set $S = \bigcup_{p \in C} S^p_3$, and set

$$x' = \{\langle p,y'/p\rangle \in C \times S : p \Vdash (\mathrm{Fcn}(y) \ \& \ \mathrm{Dom}(y)=a \ \& \ \mathrm{Rng}(y) \subseteq b)\}$$

This completes the verification of the exponentiation axiom. There seems to be no hope of eliminating the need for collection in forming S_3; abstraction is designed for collecting sets formed by separation, but here we have to collect sets formed by union. It is worth remarking that replacement would suffice in place of collection.

Infinity. We have already given the constant ω its associated term ω', and we have defined a function $n \mapsto n'$, in the proof of Lemma 6.3., with whose aid the formula $n \in \omega$ was proved to be "self-forcing". Let $P(z)$ be the property, $0 \in z \,\&\, \forall x(x \in z \to x+1 \in z)$, where $x+1$ is the set-theoretic successor function. Using n' and the definition of ω', we easily see that $\emptyset \Vdash P(\omega)$. Moreover, $P(z) \to \omega \subseteq z$ is also forced, for if $q \Vdash P(z)$ then one proves by induction that for every integer n, some extension of q forces $n \in z$. We note that only restricted induction is required.

Foundation. We first note some facts about well-founded relations. We say $(R,\prec)$ is well-founded if TI holds on $(R,\prec)$ for sets (not formulae): thus the foundation axiom says that $(W,\in)$ is well-founded, for each transitive set W. Suppose (W,Q) is a well-founded relation, and $(R,\prec)$ is a relation such that for some function $F{:}R \to W$, we have $a \prec b \to Q(F(a),F(b))$. Then $(R,\prec)$ is well-founded. Next, let (W,Q) be a well-founded relation, and define $R_0=Q$, R_{n+1} by $a\, R_{n+1}\, b$ iff $\exists x \in W(a\, R_n\, x \,\&\, x\, R_n b)$. Then each R_n (as well as their union) is a well-founded relation on W. A special case of this is when Q is $\in$; then, for example, if $\langle x,y\rangle \in z$ $y\, R_3\, z$.

Now consider the relation R on any subset of $C \times A$, defined by $\langle r,v\rangle R \langle q,u\rangle$ iff $\langle r,v/r\rangle \in u/q$. Define $F(\langle r,v\rangle) = v/r$. Then $\langle r,v\rangle R\langle q,u\rangle \to \langle r,F(\langle r,v\rangle)\rangle \in F(\langle q,u\rangle)$. Hence $\langle r,v\rangle R \langle q,u\rangle \to F(\langle r,v\rangle)\; R_3\; F(\langle q,u\rangle)$. Hence, by the general facts discussed above, R is a well-founded relation.

We now prove that the foundation axiom is forced. Suppose $p \Vdash \mathrm{Trans}\,(W)\;\&$ $\forall y(y \in W \,\&\, y \subseteq z \to y \in z)$. We must prove $p \Vdash \forall u(u \in W \to u \in z)$. Define for each set z', the set $z'_0 = \{\langle p,y\rangle\colon \exists n\, \forall q \leq_n p \langle q,y/q\rangle \in z'\}$. Then $p \Vdash y \subseteq z$ can be written $\forall u\, \forall q \leq p(\langle p,u/q\rangle \in y' \to \langle q,u/q\rangle \in z'_0)$, which is equivalent to $\forall u\, \forall q \leq p(\langle q,u/q\rangle \in y'_0 \to \langle q,u/q\rangle \in z'_0)$, which is equivalent to $\forall \langle u,q\rangle R\langle y'_0,p\rangle(\langle u,q\rangle R\langle z'_0,p\rangle)$. Thus $p \Vdash \forall y(y \in W \,\&\, y \subseteq z \to y \in z)$ is equivalent to $\langle y',p\rangle \in W'_0 \,\&\, \forall\langle u,q\rangle\, R\, \langle y'_0,p\rangle(\langle u,q\rangle R\langle z'_0,p\rangle) \to \langle y',p\rangle R\langle z'_0,p\rangle$. Since R is a well-founded relation, and since R is defined by a Δ_0 formula, we conclude $\forall\langle y',p\rangle \in W'_0(\langle y',p\rangle R\langle z'_0,p\rangle)$. That is equivalent to $p \Vdash \forall y(y \in W \to y \in z)$, which was what we had to prove. This completes the verification of the foundation axiom.

Strong Collection. Suppose $p \Vdash \forall x \in a \exists y\, A(x,y)$. That is,

$\forall \langle q,x\rangle \in a'/p \exists y', n(q,n \Vdash A(x,y))$. Applying collection, we get some W containing a y' such that $\exists n(q,n \Vdash A(x,y))$ for each $\langle q,x'\rangle \in a'/p$, and such that each y' in W arises this way. Put $z' = \{\langle q,y'\rangle : y' \in W \ \&\ \exists x' \in Rng(y')(\langle q,x'\rangle \in a \ \&\ q,n \Vdash A(x,y))\}$. Then $p \Vdash (\forall x \in a \exists y \in z\, A(x,y) \ \&\ \forall y \in z \exists x \in a\, A(s,y))$, so $\emptyset$ forces the strong collection axiom.

Bounded Dependent Choice. Suppose $p \Vdash \forall x \in a \exists y \in a\, Q(x,y) \ \&\ x \in a$. We will produce z' such that $p \Vdash (Fcn(y) \ \&\ Dom(z)=\omega \ \&\ z(0)=x \ \&\ \forall n \in \omega\, Q(z(n),z(n+1)))$.
From the hypothesis about p, we get

$\forall \langle q,x'\rangle \in a' \exists y' \in Rng(y') \exists r \leq_n q(r \Vdash y \in a \ \&\ Q(x,y))$; that is

$\forall \langle q,x'\rangle \in a' \exists \langle r,y'\rangle \in a'\ r \Vdash Q(x,y)$. Applying dependent choice, we get a sequence $\langle q_n, x'_n\rangle \in a'$, with $q_o = p$ and $x'_o = x'$, and $q_{n+1} \Vdash Q(x_n, x_{n+1})$. Now we define

$z' = \{\langle p, \langle n, x_n\rangle'/p\rangle : p \leq q_n\}$. Here n' is as defined in Lemma 6.3, and $\langle n,x_n\rangle'$ is a term built from n' and x'_n as discussed in the verification of the pairing axiom. The rest of the argument is routine.

Now only four axioms remain to be checked: (numerical) induction, power set, relativized dependent choices, and transfinite induction. We omit these verifications, since the proofs for RDC and TI are exactly like the proofs for DC and foundation, and since the proof for numerical induction is completely straightforward. As regards power set: when considering theories with power set, there is no need to use the complicated forcing interpretation given here; instead, one should return to the simpler definition first given. With that definition, the verification of power set is also completely straightforward. This completes the proof of Theorem 6.1.

Lemma 6.6 Let A be arithmetic in $\underline{a}$, but not containing $\rightarrow$ (in its formulation as a formula of second order arithmetic). Then $p \Vdash A(\underline{a})$ if and only if $\forall f \in X(\bar{f}(lh(p))=p \rightarrow A(f))$. For each fixed A this is provable in B.
Proof: By induction on the complexity of A, like Lemma 6.4.

Theorem 6.2 Let T be any of the set theories discussed in this paper except B; T may be either extensional or non-extensional. Suppose $Ta \vdash \{\bar{e}\}^{\underline{a}}(0) \in \omega$. Then, for some m_o and k, $Ta \vdash (\{\bar{e}\}^{\underline{a}}(0) \leq \bar{k} \ \&\ \underline{\bar{a}}(\bar{m}_o)$ determines $\{e\}^{\underline{a}}(0))$.

Remark: The phrase in the last line of the theorem means that the Turing machine computing $\{e\}^{\underline{a}}(0)$ halts using only the first m_o values of a, and yields a value $\leq k$. This can be expressed in arithmetic using the T-predicate, without mentioning the constant $\underline{a}$, which seems to appear in the formula. Note that, in this case,

m will be a modulus of uniform continuity for $\{\bar{e}\}^{a}(0)$ regarded as a function of a.
Proof: Suppose Ta $\vdash \{\bar{e}\}^{\underline{a}}(0) \in \omega$. By the results of Section 3, if T is extensional we can replace T by the non-extensional version, and still Ta will prove $\{\bar{e}\}^{\underline{a}}(0) \in \omega$. Hence, by Theorem 6.1, arguing in Ta, for some n, we have $\emptyset, n \Vdash \exists i, m \in \omega\,(i = \{\bar{e}\}^{\underline{a}}(0)$ is determined by $\bar{\underline{a}}(m))$. Since this statement does not involve $\underline{a}$, as discussed above, it is provable in T, not just Ta. Now argue in T: Let $p \le_n \emptyset$. Then (using Lemma 6.3), for some i and m, we have for some j,

$p, j \Vdash (i = \{\bar{e}\}^{\underline{a}}(0)$ is determined by $\bar{\underline{a}}(m))$. If we choose j large enough, the same j will work for all $p \le_n \emptyset$, so that by increasing n, we may assume that for $p \le_n \emptyset$ we have some i and m, depending on p, such that $p \Vdash (i = \{\bar{e}\}^{\underline{a}}(0)$ is determined by $\bar{\underline{a}}(m))$. Let m_0 be the largest of these values of m, over all $p \le_n \emptyset$, and let k be the largest of the values of i. Then if $p \le_n \emptyset$, we have

$p, j \Vdash (\{\bar{e}\}^{\underline{a}}(0)$ is determined by $\bar{\underline{a}}(m)$ & $\{\bar{e}\}^{\underline{a}}(0) \le k)$, for some value of j depending on p, since the formula in the last line is a consequence of the one forced by p. Taking the maximum of these values of j, we have

$\emptyset, n+j \Vdash (\{\bar{e}\}^{\underline{a}}(0)$ is determined by $\bar{\underline{a}}(m_0)$ & $\{\bar{e}\}^{\underline{a}}(0) \le k)$.

By Lemma 6.6, this last formula is true, since it is forced by every condition of length n+j. (Note that it can be expressed without using implication.) Remembering that we have been arguing in T, we have just proved that $T \vdash \exists k \exists m_0 (\{\bar{e}\}^{\underline{a}}(0)$ is determined by $\bar{\underline{a}}(m_0)$ & $\{\bar{e}\}(0) \le k)$. Applying explicit definability for T, we get $T \vdash (\{\bar{e}\}^{\underline{a}}(0)$ is determined by $\bar{\underline{a}}(\bar{m}_0)$ & $\{\bar{e}\}^{\underline{a}}(0) \le \bar{k})$ for some numerals $\bar{m}_0$ and $\bar{k}$. This completes the proof of the theorem.

§7. The Main Theorems about Continuity

In the introduction, we have discussed the various derived rules related to continuity and local continuity, which form the focus of our work. In this section, we plan to establish results for intuitionistic set theories analogous to those obtained for Feferman's theories in [B1]. Those results are of two kinds: derived rules, and consistency or independence results. In the preceding sections, we have done all the necessary work to establish the general metamathematical properties which were shown in [B1] to be sufficient for the derived rules of continuity to hold. For the consistency results, we have something yet to prove in this section.

Before we give our main results, we shall fulfill the promise made in the introduction to explain further the Principle of Local Uniform Continuity. In formulating such a principle, we wish to state something like the Principle of Local Continuity, except that we want δ to depend only on ε. The most curious thing about the Principle of Local Uniform Continuity is that we cannot express exactly what we mean in the usual predicate calculus. What we really mean is

$$\left.\begin{matrix}\forall a \in X\, \exists b \in Y \\ \forall \varepsilon > 0\, \exists \delta > 0\end{matrix}\right\} \forall a_o \text{ within } \delta \text{ of } a\ \exists b_o \text{ within } \varepsilon \text{ of } a \ldots$$

That is, b depends only on a, and δ only on ε . Of course, one can say something in the usual predicate calculus which is equivalent to this under some axiom of choice, but that is beside the point. We have not pursued this matter further, but instead have formulated a weaker version of Local Uniform Continuity, which nevertheless has all the interesting consequences mentioned in the introduction. When we refer to Local Uniform Continuity in the rest of this paper, what we mean is this version, called LUC(X,Y) in [B1]. We now state our main theorem on derived rules.

Theorem 7.1 Let T be T_2 (similar to Myhill's CST), T_3,T_4, Z, or ZF^- + RDC, or the non-extensional version of any of these theories. Then T is closed under the rules of local continuity, continuous choice, local uniform continuity, Heine-Borel's rule, and all the rules discussed in [B1].

Corollary If the hypothesis of one of these rules is provable in $\underset{\sim}{B}$, the conclusion is provable in T_2, i.e. requires at most some instances of induction and collection beyond $\underset{\sim}{B}$.

Proof: The necessary conditions laid out in [B1] have been derived for these set theories in Theorem 5.4, Lemma 5.2, and Theorem 6.2.

Remark: In case T is a non-extensional theory, we can allow an arbitrary formula in place of a definable set in the rule of local continuity, local uniform continuity, and continuous choice.

We now shall obtain some consistency results complementing these results on derived rules. These results concern the principles corresponding to the derived rules we have already studied; their statements are obtained from the rules in the obvious way, namely: if a rule says, from A infer B, then the corresponding principle is $A \to B$. Foremost among the principles we study are the Principle of Local Continuity, the Principle of Local Uniform Continuity, and the Principle of Continuous Choice. Note that the Principle of Local Continuity implies the simpler Principle of Continuity, that every function from a complete separable metric space to a separable metric space is continuous; and the Principle of Local Uniform Continuity implies that every function from a compact metric space to a separable metric space is uniformly continuous. Thus our results include the conjectures of [Fr 2] concerning functions from N^N to N and from 2^N to N. The Principle of Local Uniform Continuity also implies Heine-Borel's theorem.

At the end of [M1], there is a "postscript" announcing certain theorems of Friedman, which include special cases of some of the consistency results in this section; see also [B3,§3] for related results. The theorem announced for Friedman in the last line of [M1], about the axiom of choice for the reals with the prefix $\forall x \exists y\ A(x,y)$ instead of $\forall x \exists ! y\ A(x,y)$, is false; the axiom is refutable, as discussed in [B1].

We are going to prove our consistency theorems using realizability. The key to these proofs lies in the construction in [B1] of a so-called "weak BRFT" in which all operations on N^N are continuous. To explain what this means: Let S be a set, and suppose C is a class of partial functions from S to S, of several variables, including a pairing function and a binary "universal function" $\emptyset(e,x)$ such that each unary function f in C is $\lambda x \emptyset(e,x)$ for some e. If (S,C) satisfies some other, less important conditions spelled out in [B1], then it is called a "weak BRFT". The use of such structures is that the functions in C can be used in place of recursive functions for realizability. As mentioned, in [B1] a specific weak BRFT is constructed in which all operations from N^N to N are continuous. (Each weak BRFT contains a copy of the integers, calling some element O and using $\underline{p}(x,O)$ for successor, where $\underline{p}$ is the pairing function. Thus it makes sense to speak of operations from N^N to N.) As a matter of fact, two specific weak BRFT's are constructed with this property: in one of them, all operations on 2^N to N are uniformly continuous, and in the other, there is a continuous, but not uniformly continuous function on 2^N to N. Call these weak BRFT's S_o and

S_1, respectively.

Theorem 7.2 ZF^- + RDC is consistent with Church's thesis CT plus "All functions from a complete separable space X to a separable space Y are continuous".

Remark: Continuity cannot be improved to uniform continuity without dropping Church's thesis.

Theorem 7.3 ZF^- + RDC is consistent with the Principle of Local Continuity plus "There is a non-uniformly continuous, continuous function from 2^N to N."

Theorem 7.4 ZF^- + RDC is consistent with the Principle of Local Uniform Continuity.

Corollary: ZF^- + RDC is consistent with "All functions from a compact metric space to a separable metric space are uniformly continuous."

Proof: The idea of all the proofs is to use realizability to prove the theorem for ZF^- + RDC - ext, and then use the ideas of Theorem 3.1 to prove it for ZF^- + RDC. We first show how to use realizability to prove the theorems for T-ext, where for simplicity we write T for ZF^- + RDC.

Generally speaking, realizability interpretations can be either formal or informal; that is, e r A can be either a formula of the formal system, or an informal predicate. For instance, Kleene's original interpretation for arithmetic can be taken either way. The q-realizability given earlier in this paper for set theory is necessarily formal, however, since it is not clear how to interpret x* informally. Of course we can also do (formal) "1945-realizability" for set theory, which is analogous to Kleene's original "1945-realizability" (as it has come to be called) for arithmetic. Here are the clauses defining this interpretation:

$e \ r \ x \in y$	is	$\langle e,x\rangle \in y$
$e \ r \ (A \ \& \ B)$	is	$(e)_0 \ r \ A \ \& \ (e)_1 \ r \ B$
$e \ r \ (A \vee B)$	is	$((e)_0 = 0 \to (e)_1 \ r \ A) \ \&$ $(e)_0 \neq 0 \to (e)_1 \ r \ B))$
$e \ r \ (A \to B)$	is	$\forall a(a \ r \ A \to \{e\}(a) \ r \ B)$
$e \ r \ \exists x \ A$	is	$\exists x \ e \ r \ A$
$e \ r \ \forall x \ A$	is	$\forall x \ e \ r \ A$

We then have the soundness theorem, T-ext $\vdash$A implies T-ext $\vdash \bar{e}$ r A for some numeral $\bar{e}$. The proof of the soundness theorem is so similar to the soundness theorem for q-realizability that we do not write it out here.

Now Kreisel-Lacombe-Schoenfield's theorem asserts that every effective operation from N^N to N is continuous; and the same theorem is true for complete separable metric spaces in place of N^N and separable spaces in place of N. (Kreisel-Lacombe-Schoenfield's theorem is proved, for instance, in Rogers [R, p. 362]; the reader will have no difficulty making the extension mentioned.) Moreover, if X and Y are complete separable and separable metric spaces, respectively, with X in standard form, then KLS(X,Y) (in obvious notation) is 1945-realized, as is proved in [B2]. Now, it is easy to see that (1) Church's thesis is realized using 1945-realizability, and (2) with Church's thesis, KLS(X,Y) is equivalent to the assertion that all functions from X to Y are continuous. It follows that "all functions from X to Y are continuous" is realized, and hence consistent with T-ext, by the soundness theorem for 1945-realizability. Thus Theorem 7.2 is proved with T-ext in place of T.

Now we turn to the proofs of Theorems 7.3 and 7.4. In the definition of realizability given above, there is nothing particularly sacred about the recursive functions. If we have any weak BRFT which can be defined and proved to be a weak BRFT in T-ext, we can use it for (formal) 1945-realizability. That is, instead of using $\{e\}(a)=y$ we use $\emptyset(e,a)=y$, where $\emptyset$ is the universal function of the weak BRFT. To be precise, instead of using $\exists n(T(e,a,n)\ \&\ U(n)=y)$ we use the formula defining $\emptyset(e,a)=y$ in T-ext. A priori, it is possible that we might have a weak BRFT which could be proved to be a weak BRFT without having a definable universal function, but that possibility doesn't occur here. Also, one should add to the definition of e r $(A \vee B)$ a proviso that $(e)_o$ is an integer of the BRFT (each weak BRFT contains a copy of the integers).

One can verify by reading the construction of S_o and S_1 in [B1] that their universal functions are definable, and that they can be proved in a very weak non-extensional set theory to be weak BRFT's. To verify that S_o has the property that all functions from 2^N to N are uniformly continuous requires something like König's lemma, which goes beyond intuitionistic systems, but that doesn't affect the usefulness of S_o for formal realizability, which only requires that we be able to prove that it is a weak BRFT.

It follows from the above discussion, and from a soundness proof inessentially different from the one give for q-realizability, that we can assign to each formula A another formula e r_i A, for e realizes A in S_i, and prove that T-ext $\vdash$ A implies T-ext $\vdash \exists e \in S_i (e\ r_i\ A)$.

If A is a sentence, then "$\exists e(e\ r_o\ A)$ is true" can be regarded as an informal realizability interpretation of A. This interpretation can play the same role that "$\mathcal{M} \models e(e\ r\ A)$" does for Feferman's theories in [B1]. If one reads the arguments of §3.2 of [B1], making the substitution just mentioned, one finds that Theorem 7.3 is proved, with T-ext in place of T. Similarly, if one reads the arguments of §3.3 of B1, substituting "$\exists e(e\ r_1\ A)$" for "$\mathcal{N} \models e(e\ r\ A)$", one finds that Theorem 7.4 is proved, with T-ext in place of T. That is, the proofs that Local Continuity and related properties are or are not realized are not dependent on the particular theory in detail, but only on the existence of a sound realizability interpretation in some weak BRFT with the properties of S_o or S_1, respectively.

Thus Theorems 7.2, 7.3, and 7.4 are proved for T-ext. Now we discuss how to improve this result to T instead of T-ext. We have to consider the interpretation * from T to T-ext given in Theorem 3.1. Suppose for simplicity that we are working with a two-sorted version of T, with variables for integers and variables for sets.[1] Recall that two sets are $\sim$ if each member of one is $\sim$ some member of the other, and $a \varepsilon b$ if $a \sim$ some $c \in b$. The interpretation for two-sorted T leaves numbers alone. It is then easy to check that two functions from N to N (as sets of ordered pairs) are $\sim$ if and only if they have the same values. Thus if X is any "extensional" subset of N^N, i.e. whenever $a \in X$ and b has the same values as a then $b \in X$, we have $a \varepsilon X$ iff $a \in X$. In particular, any complete separable metric space in standard form (see §2) is such an extensional subset of N^N. Similarly, if X and Y are complete separable spaces and P is an extensional subset of $X \times Y$ in the sense of §2, then $x \varepsilon P$ iff $x \in P$. We shall now prove the following: Let A be an instance of the Principle of Local Continuity. Then T-ext $\vdash (A \leftrightarrow A^*)$. We consider the conjuncts of the hypothesis of Local Continuity one-by-one.

Making use of the "standard form" of X, we may suppose that all references to X in the Principle of Local Continuity are implicit: that is, "$a \in X$" actually is "$a \in N^N$ & Conv(a)", where Conv is a formula expressing the convergence conditions on a, i.e. $\text{Conv}(a) \leftrightarrow \forall n,m \in N(\sigma(a_n, a_m) < (1/m)+(1/n))$, where σ is a certain recursive function. Hence, the hypothesis "X is a complete separable metric space" no longer needs to actually occur. Similarly for Y. Consider the hypothesis "P is extensional", which says $d(a,a')=0$ & $d'(b,b')=0$ & $P(a,b) \to P(a',b')$, where d and d' are the metrics of X and Y respectively. We have seen already that P is equivalent to P*; by a similar argument it follows that $(d(a,a')=0)$ is equivalent to $(d(a,a')=0)^*$; hence, "P is extensional" is equivalent to its * interpretation.

Consider the hypothesis, "$\forall a \in X$ {b: P(a,b) & $b \in Y$} is closed". We do not have to check this one, since the special case Y=N implies the general case, provably in a very weak theory plus a simple axiom of choice AC_N which is realized, as is shown in [B1]. However, the reader who wishes can verify directly that this hypothesis, too, is equivalent to its * interpretation.

Finally, consider $\forall a \in X \exists b \in Y$ P(a,b). The interpretation of this is $\forall a \varepsilon X \exists b \varepsilon Y$ P*(a,b); which we have seen is just the original formula again. The conclusion of Local Continuity can be dealt with similarly. Hence, each instance A of the Principle of Local Continuity is provably equivalent to A*.

Now we prove the consistency of T + LC(X,Y). If it is not consistent, then some conjunction of instances of LC(X,Y), say B, implies 0=1 in T. Then, by Theorem 3.1, B* implies 0=1 in T-ext. But B* is provably equivalent to B, in T-ext. Hence LC(X,Y) is inconsistent with T-ext, which possibility we have already ruled out by realizability.

The proofs of Theorems 7.2, 7.3, and 7.4 can be completed by checking that the other statements involved are also equivalent to their * interpretations. The basic reason why this works seems to be that none of these statements mention objects of type higher than functions from N^N to N. The use of standard form for complete separable spaces reduces everything to low types. We check, for instance, that "All functions from N^N to N are continuous" is equivalent to its * interpretation. Now "F: $N^N \to N$" is Fcn(F) & $\forall a,b(a \in N^N$ & $\langle a,b\rangle \in F \to b \in N)$. Now Fcn(F)* says that if $\langle a,b\rangle \in F$ and $\langle a,c\rangle \in F$ then $b \sim c$. But $\langle a,b\rangle \in F$ says a $\sim$some a' and b=F(a'). But then $a \in N^N$ and so b=F(a). Hence Fcn(F)* is equivalent to Fcn(F). The argument shows also that $\langle a,b\rangle \varepsilon F$ iff $\langle a,b\rangle \in F$. Together with what we have already proved, this suffices to complete the proof that "F: $N^N \to N$" is equivalent to its interpretation. Next, "m is a modulus for F at y" is $\forall y \in N^N(\forall i \leq m(z(i)=y(i)) \to F(z)=F(y)$, which is equivalent to its own interpretation in view of the fact that $\langle a,b\rangle \varepsilon F$ iff $\langle a,b\rangle \in F$, as proved above. Hence, "All functions from N^N to N are continuous" is equivalent to its own interpretation, as claimed. The rest of the statements in Theorems 7.2, 7.3, and 7.4 can be treated similarly. This completes the proof.

Footnote 1: We have formulated our set theories T with a constant ω for the von Neumann integers. Alternately one may use a two-sorted theory T^2 with one sort of variables for numbers and one sort for sets (or equivalently, one can use two unary predicates.) At first glance it may seem that T and T^2 are trivially equivalent, but the problem is more subtle. T^2 can be easily interpreted in T. But the converse is more difficult, since T^2 does not necessarily prove the existence of the von Neumann integers. However, if T contains collection, then T^2 does prove the existence of the von Neumann integers, and T can easily be interpreted in T^2. In particular, the application we make of T^2 in the consistency proofs of §7 fall under this remark.

Errata: (1) In Lemma 0.2 of [B1], page 260, the hypothesis should state that for $i, j \leq k$, we have $\rho(a_i, a_j) < (1/4i) + (1/4j)$. When the lemma is applied on page 298, we may assume that b satisfies this hypothesis, by replacing b_n by b_{4n}.
(2) Theorem 2.4 of [B1], p. 303, is correct, but something must be added to the proof at line 26, for as it stands, the proof applies only if X' is provably compact, which we could only assure in general if X is locally compact. (This is related to a defect of Bishop's definition of continuity pointed out by Hayashi: continuity on compact sets does not guarantee pointwise continuity unless the space is locally compact). To correct the proof, we appeal at line 26 to the rule of local uniform continuity with a parameter X' for a compact subspace of X. Note that a compact subset X' of X can be coded as a function from N to N, since it is given by a function assigning to each rational $\varepsilon > 0$ a finite ε-approximation to X', and since X is in standard form, each member of X is a sequence of integers. Thus, in the notation of Section 2.8 of [B1], $Q(e) \leftrightarrow$ e codes a compact subset of X is an allowable choice of a set of parameters. In Section 2.8, the rule of local continuity with parameters is derived; the rule of local uniform continuity with parameter may be similarly treated.

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STABLE ALGEBRAIC THEORIES [1]

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Abstract: We survey the classification of stable theories of modules, rings, and groups.[2]

I. Introduction

In § 1 we give an elementary introduction to stability theory.

§ 1. The stability spectrum theorem.

1.1 Stable theories.

Let T be a first order theory, let A be a model of T , and set:

Def(A) = the Boolean algebra of definable subsets of A .

Here we take "definable" to mean: first order definable using parameters from A .

Stability theory starts with the question: how can we measure the complexity of the Boolean algebra Def(A) ? To study Def(A) it is convenient to pass to its Stone space:

SA = the Stone space of Def(A) .

Thus SA is a topological space whose points are the maximal filters in Def(A) , or to put the matter more explicitly: a point p of SA is a collection of definable subsets of A possessing the finite intersection property p and such that p is maximal with respect to this property.

A coarse measure of the complexity of Def(A) is given by the cardinality of its Stone space SA - and this suffices for stability theory. Clearly card(SA) $\geq$ card (A) (just assign to each $a \in A$ the corresponding principal filter

p_a = all definable sets which contain a .)

Call a model A of T *tame* iff card(SA) = card(A) . Then our basic notion is the following:

Definition: For any cardinal λ we say that the theory T is λ-*stable* if all models A of T which have cardinality λ are tame. We also say that λ is a *stability cardinal* for T .

1.2 Examples.

There are essentially only four relevant examples:

Example 1. Let T be simply the theory of infinite sets (equipped with the equality relation and no further structure). Then for any

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2 Bibliographical and historical notes will be found mingled at the end of the article. Both sorts are thoroughly incomplete.

model X of T, $Def(X)$ contains just the finite sets and their complements, the cofinite sets. The Stone space SX has in addition to the principal filters just one more filter, the filter of all cofinite sets. Thus T is λ-stable for all λ.

<u>Example 2.</u> Let T be the theory of sets P which are equipped with $\aleph_0$ independent subsets P_n (represented in the language of T by unary predicates). The independence condition means:

(ind) Given two disjoint finite sets of indices I,J, then

$$\bigcap_{i \in I} P_i \cap \bigcap_{j \in J} (P-P_j) \quad \text{is nonempty.}$$

Then the definable sets in a model P of T will be generated (as a Boolean algebra) by the sets P_n and the finite sets. The independence condition (ind) is just the finite intersection of the sets P_i and the complements of the rest of the sets P_i. This immediately gives us $2^{\aleph_0}$ extra elements of the Stone space SP, regardless of the cardinality of P. It is then easy to see that T is λ-stable for $\lambda \geq 2^{\aleph_0}$, and λ-unstable for $\lambda < 2^{\aleph_0}$.

<u>Example 3.</u> Let T be the theory of sets E equipped with $\aleph_0$ equivalence relations such that:

1. E_1 has infinitely many equivalence classes.
2. E_{n+1} subdivides each equivalence class of E_n into infinitely many equivalence classes.

For E a model of T the Boolean algebra $Def(E)$ is generated by the equivalence classes of the various E_n together with the finite sets. The main way to construct elements of the Stone space SE is to choose a sequence of sets $C_1 \supseteq C_2 \supseteq C_3 \ldots$ so that each C_i is an E_i-equivalence class, and then to extend $\{C_i\}$ to a maximal nonprincipal filter. Now if E has cardinality λ then the cardinality of SE will depend somewhat on the structure of E, but in the worst case it will be possible to choose each of $C_1, C_2, C_3, \ldots$ in λ different ways, producing $\lambda^{\aleph_0}$ different elements of SE. It is then easy to show that the stability cardinals for T are exactly those cardinals λ satisfying:

$$\lambda^{\aleph_0} = \lambda \quad .$$

<u>Example 4.</u> Let T be the theory of dense linear orderings Q. Then for any model Q of T the Boolean algebra $Def(Q)$ is generated by the rays:

$(-\infty, a)$ and (a, ∞) where $a \in Q$ and $-\infty < Q < \infty$.

The elements of SQ are basically Dedekind cuts, and there are many of them. In fact T is λ-unstable for all λ, which translates into a classical fact:

For all λ there is a dense linear ordering Q of cardinality λ sitting as a dense subset of a linear ordering of cardinality greater than λ .

It remains to be seen in what respect these examples are typical.

1.3 <u>The stability spectrum theorem.</u>

For a first order theory T in a countable language the set of stability cardinals for T must be as in one of the four examples above:

1. all cardinals.
2. all cardinals from $2^{\aleph_0}$ on .
3. just cardinals of the form $\lambda = \lambda^{\aleph_0}$.
4. no cardinals.

This is the content of the stability spectrum theorem, which may be reformulated to apply to uncountable languages as follows. Call a theory T *stable* if it is λ-stable for some λ , and *unstable* otherwise. For T stable, let $\delta(T)$ be the smallest stability cardinal for T .

<u>Theorem 1 (Stability Spectrum Theorem).</u> If T is stable then there is a cardinal $\kappa(T)$ such that the stability cardinals λ for T are characterized by:

1. $\lambda \geq \delta(T)$
2. $\lambda^{<\kappa(T)} = \lambda$

<u>Theorem 2 (Supplement).</u> $\delta(T) \leq 2^{\mathrm{card}(T)}$ and $\kappa(T) \leq \mathrm{card}(T)^+$. This corresponds nicely to the situation in Examples 2,3 above.

<u>Terminology:</u> $\kappa(T)$ is the *stability exponent* for T . T is said to be *superstable* if T is stable for all cardinals above $\delta(T)$, which is equivalent to the condition that T have countable stability exponent.

Tradition dictates that we write "ω-stable" for "$\aleph_0$-stable" . It is a basic fact that ω-stability implies stability in all cardinals; this can be read off from Theorem 1 .

Unfortunately a sketch of the proof of this theorem would lead us too far from our main topic. The basic idea of the proof may be summarized as follows.

Obviously the first step is to get a manageable definition of the invariant $\kappa(T)$. Once this has been done, one is obliged to prove two theorems:

The Instability Cardinal Theorem. Given $\kappa < \kappa(T)$ and λ such that:

$$\lambda^{\kappa} > \lambda$$

then T is λ-unstable.

The Stability Cardinal Theorem. Given $\lambda > \delta(T)$ such that $\lambda^{<\kappa(T)} = \lambda$, then T is λ-stable.

Roughly speaking, $\kappa(T)$ is defined as the largest cardinal for which it is easy to prove the Instability Cardinal Theorem, after which one discovers that with considerable effort the Stability Cardinal Theorem can be proved.

This concludes our introduction to stability theory. It would be interesting to discuss applications of stability theory to specific problems, e.g. the structure of differentially closed fields, but we will omit this topic, referring the reader to [2,20] . The work of Garavaglia [16] is interesting in this connection.

§ 2. Stable algebraic theories.

We consider the following question:

Let A be a module, a ring, or a group. If the theory of A is stable, then what can be said about the structure of A ?

In the present section we summarize many of the known results. The following convention is useful: if a structure A has λ-stable theory, we say more briefly that A is λ-stable.

2.1 Modules.

The basic fact is:

Theorem 1. All modules are stable.

This does not really trivialize the basic question, since one also wants a characterization of superstable and ω-stable modules. Such characterizations have been found; they involve descending chain conditions, and will be presented under that heading at the end of the next section.

2.2 Rings.

Theorem 2. The Jacobson radical of a stable ring is nilpotent.

Theorem 3. A semisimple stable ring is a finite direct sum of matrix rings $M_n(D)$ over stable division rings D .

Stable division rings have not been classified, so we must settle for less:

Theorem 4. A superstable division ring is finite dimensional over its center (which is a superstable field).

Theorem 5. An ω-stable field is algebraically closed or finite.

The day before this talk was delivered Shelah completed the proof that Theorem 5 applies also to superstable fields, so we can combine Theorems 3,4 , and the strengthened Theorem 5 to get:

Corollary. A semisimple superstable ring is the direct sum of a finite ring and finitely many matrix rings $M_n(F)$ over algebraically closed fields F .

We will prove Theorem 5 in § 5 using techniques developed in the study of ω-stable groups; this is the argument that generalizes to the superstable case.

The classification of stable fields and division rings is very much an open problem. A few nontrivial stable fields are known, namely the separably closed fields. It is certainly possible that stability alone implies separable closure.

The largest gap in the results above lies in Theorem 2 .

Problem. What can be said about a nilpotent ring if it is known to be stable (or superstable, or ω-stable)?

A simple construction shows that this problem is at least difficult. Let A_1, A_2, A_3 be abelian groups and let

$$B : A_1 \times A_2 \to A_3$$

be a bilinear map. We associate to B a ring R defined by:

$$R = A_1 \oplus A_2 \oplus A_3$$

as an additive group, with the following multiplication:

$$(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = (0, 0, B(a_1, b_2) + B(b_1, a_2)) .$$

This ring is nilpotent: $xyz = 0$ for any x, y, z in R . R is even commutative. Furthermore if the structure consisting of A_1, A_2, A_3 equipped with the map B is a λ-stable structure then R will inherit this λ-stability.

To apply this construction, take any λ-stable ring A and take for the map B the multiplication map on R :

$$B : A \times A \to A , \ B(r,s) = rs .$$

Then the above construction produces a new ring R whose underlying set is $A \oplus A \oplus A$. This suggests that the classification of stable nilpotent rings of exponent 3 is already very difficult.

2.3 Groups.

The methods available for the classification of stable groups will be described in detail below. The following special results have been obtained, among others:

Theorem 6. If G is an $\aleph_0$-categorical ω-stable group then G is abelian-by-finite.

(A group is abelian-by-finite if it has an abelian subgroup of finite index.)

Theorem 7. If G is an $\aleph_0$-categorical stable group then it is nilpotent-by-finite.

Theorem 8. If G is a stable locally nilpotent group then G is solvable.

The techniques discussed below were developed in the course of proving these three results.

Some further results on groups of Morley rank at most three will be discussed in § 6 (we introduce Morley rank in § 3.1).

What are the prospects for obtaining some general results about stable (superstable, ω-stable) groups? This discussion in § 6 will indicate one direction in which such results might lie. On the negative side, just as in the case of rings, it is a fact of life that there are many examples of stable groups, and hence general results cannot be trivial.

The main source of stable groups is the following. Let R be a stable commutative ring with identity. Then any of the usual matrix groups taken with coefficients in R will inherit the stability properties of R. One example, exploited by Mal'cev, is obtained by using upper triangular unipotent 3×3 matrices, i.e.:

$$G(R) = \{ \begin{vmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{vmatrix} : a,b,c \in R\} \quad .$$

For this particular example it is not necessary to have R commutative, or even associative.

II. ω-stable groups.

We will describe the specific model-theoretic machinery applicable to the structural analysis of ω-stable groups. In III. applications of these techniques are discussed.

§ 3. Chain conditions for stable groups.

3.1 The ω-stable DCC.

Definition. The group G satisfies the *ω-stable DCC* (= descending chain condition) iff every descending chain of definable subgroups of G is finite.

Theorem 1. If G is ω-stable then G satisfies the ω-stable DCC

The proof of this theorem rests on model-theoretic machinery:

Morley rank:

If A is any ω-stable structure then it is possible to assign to each definable subset S of A an ordinal:

$$\alpha = \text{rank}(S)$$

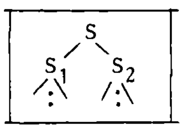
called the (Morley) rank of S in such a way that the following axioms are satisfied:

(mor) If S contains infinitely many mutually disjoint definable subsets $S_1, S_2, \ldots$ then for some i $\text{rank}(S_i) < \text{rank}(S)$.

(ee) If S is defined in A by the formula $\phi(x)$, if A' is an elementary extension of A , and if S' is the subset of A' defined by the same formula, then $\text{rank}(S) = \text{rank}(S')$.

Indeed, if we try to assign to S the least ordinal compatible with (mor) and (ee) then an inductive definition of rank emerges. (In particular the rank 0 sets will be just the finite sets.) The only problem is to show that every set S eventually does get assigned a rank, and this is proved using the ω-stability of A and the following "splitting argument" :

If we suppose that $S \subseteq A$ remains unranked, then it is easy to show that S splits as the disjoint union of two unranked definable subsets S_1, S_2 . If we split S_1, S_2 in the same way and iterate this construction indefinitely, we will obtain a binary branching tree of unranked definable sets:



where each set in the tree is the disjoint union of the pair below. If A_0 is a countable elementary substructure of A such that every set in this tree is definable over A_0 , then just as in Example 3,§1 the paths through this tree will produce $2^{\aleph_0}$ distinct elements of the Stone space SA_0 , contradicting the ω-stability of A .

<u>Supplement: Morley degree.</u>

if A is ω-stable and $S \subseteq A$ is definable and of rank α , then it is easy to show that there is a largest integer d such that in some elementary extension of A :

(deg) S can be decomposed into the disjoint union of d definable subsets of rank α .

This maximal integer d is called the Morley degree of S . The following is straightforward:

(sum) If S of rank α is the disjoint union of S_1 and S_2 , definable subsets of rank α , then :

$$\text{degree}(S) = \text{degree}(S_1) + \text{degree}(S_2) .$$

Lemma 2. Let $G_1 \subseteq G_2$ be definable subgroups of the ω-stable structure A. If $G_1 \subsetneq G_2$ then:

either $\mathrm{rank}(G_1) < \mathrm{rank}(G_2)$ or else the ranks are equal and then $\mathrm{degree}(G_1) < \mathrm{degree}(G_2)$

Proof:

The right cosets of G_1 in G_2 all have the same rank and degree (they can be identified by means of definable 1-1 correspondences). The lemma therefore follows from (sum) above and the fact that there are at least two such cosets.

Proof of Theorem 1 :

Immediate from Lemma 2 .

3.2 The superstable DCC .

Definition. A group G satisfies the *superstable DCC* iff there is no infinite descending chain $G_1 > G_2 > \ldots$ of definable subgroups of G such that the indices:

$$[G_i : G_{i+1}]$$

are all infinite.

Theorem 3. If G is superstable then G satisfies the superstable DCC.

This depends on the following model-theoretic machinery:

Shelah degree:

If A is any superstable structure then it is possible to assign to each definable subset S of A and ordinal:

$$\alpha = \mathrm{Deg}(S)$$

called the (Shelah) degree of S in such a way that the following axioms are satisfied (jargon to be explicated below):

(she) If S contains very many n-mutually disjoint uniformly definable subsets $S_1, S_2, \ldots$ then for some i

$$\mathrm{Deg}(S_i) < (\mathrm{Deg}(S) .$$

(ee) as for Morley rank .

Terminology:

1. n-mutually disjoint: the intersection of any n of them is empty.
2. uniformly definable: there is a single formula $\phi(x,\bar{y})$ such that each of the sets S_i is definable by a formula of the form $\phi(x,\bar{a})$ for a suitable choice of the elements $\bar{a}$ in A .

As in § 3.1 a splitting argument shows that every definable set can be assigned a Shelah degree. Then Theorem 3 follows directly from the following fact, whose proof the reader should supply:

Lemma 4. If $G_1 \subseteq G_2$ are definable subgroups of the superstable structure A and $[G_1 : G_2]$ is infinite then the Shelah degree of G_2 is smaller than that of G_1.

3.3 The stable chain conditions.

Definition. 1. G satisfies the *obvious stable CC* iff any chain of uniformly definable subgroups of G is finite.

2. G satisfies the (full) *stable CC* iff any chain of intersections of uniformly definable subgroups of G is finite.

Example. The centralizers $C(g)$ of single elements g in G are uniformly definable, and their intersections are the centralizers of arbitrary subsets of G. Thus the **stable** CC includes a chain condition on centralizers.

Another example occurs in § 5.

The following result is nontrivial even for superstable groups.

Theorem 5. If G is stable then G satisfies the stable CC.

We will sketch a proof of this theorem.

Remark. Let A be a stable structure, n an integer, and R a definable binary relation between n-tuples in A. Then R cannot linearly order any infinite subset of A^n.

Proof: (sketch):

An elementary argument shows that A^n inherits the stability of A, and hence we may take $n = 1$ without loss of generality. Then if R linearly orders an infinite subset X of A it is easy to find an elementary extension A' of A in which R linearly orders a dense linear ordering of any desired form. Then the Dedekind cuts in this ordering may be used to construct elements of the Stone space SA', contradicting the stability of A (cf. Example 4, § 1).

At this point we see that the obvious stable CC really is obvious for stable groups. Indeed, an infinite collection of uniformly definable groups linearly ordered under inclusion induces a definable linear ordering of the parameters used to define the groups, contradicting the above remark.

The point in the proof of the full stable CC is that it can be reduced to the obvious stable CC, namely:

Lemma 6. Let F be a family of uniformly definable subgroups of a stable group G. Then there is an integer n such that any intersection of elements of F is in fact an intersection of n elements of F.

Corollary 7. With F,G as above let F' be the family of arbitrary intersections of groups belonging to F. Then the groups in

F' are uniformly definable.

Clearly Lemma 6 gives the corollary, and the corollary reduces the stable CC to the obvious CC. The point then is to prove Lemma 6.

This depends on the following, which can be proved directly on the basis of $2^\lambda > \lambda$. (Proof omitted).

Remark. If A is a stable structure and F is a family of independent uniformly definable subsets of A then F is finite. (Independence was defined in Example 2, § 1.2 .)

Proof of Lemma 6 (sketch):

If for each n we can find n independent groups in F, then in an elementary extension of A we can find an infinite set of independent uniformly definable groups, contradicting the preceding remark. Thus for some n we cannot find $n+1$ independent groups.

We claim every intersection of elements of F can be reduced to an intersection of n elements of F. Assuming the contrary, we obtain an intersection of $n+1$ groups $G_1,\ldots,G_{n+1}$ in F which is not an intersection of any n of them. It suffices to show that this implies that the groups G_i are independent. If one thinks of this as a variant of the Chinese Remainder Theorem with the G_i playing the role of maximal ideals then the proof is evident.

3.4 Classification of modules.

We will give the classification of superstable and ω-stable modules without proof. Consider modules over a fixed ring R. For any system E of linear equations involving parameters from R, an unspecified parameter x from M, and unknowns $x_1,\ldots,x_k$ let $M(E)$ denote the set of parameters m in M for which the system E becomes solvable in M upon setting $x = m$. Then $M(E)$ is a subgroup of M. Let $\Phi = \Phi_M$ denote the family of all such subgroups of M.

Theorem 8. The following are equivalent:

1. M is ω-stable.
2. M satisfies the descending chain condition relative to groups in Φ.

The analog of Theorem 8 for superstability is also correct.

§ 4. Connected groups.

4.1 The identity component.

Definition. The group G is *connected* iff it contains no definable proper subgroup of finite index.

<u>Definition.</u> If G contains a connected definable subgroup H of finite index in G, then it is easy to see that H is unique, so we may set:

$$H = G^{o} .$$

We emphasize that G^{o} may not exist. When G^{o} exists we say that G is *connected-by-finite*. Note that G^{o} is normal in G when it exists.

<u>Theorem 1.</u> If G is ω-stable then G is connected-by-finite.

<u>Proof:</u>

This is an application of the ω-stable DCC. If G is not itself connected then it has a disconnecting subgroup H (i.e. a definable proper subgroup of finite index). If H is not connected it has a disconnecting subgroup H'. Continuing in this manner we construct a chain of definable subgroups of G. Where the chain stops we will find the identity component of G.

(In a similar vein the stable CC can be used to prove:

<u>Theorem 2.</u> If G is stable and $\aleph_0$-categorical then G is connected-by-finite.

This is a useful first step in the proof of Theorem 7 of § 2.3)

There is another variation on this theme which plays an important role in § 5 :

<u>Theorem 3.</u> Let F be an infinite stable field. Then the additive group of F is connected.

<u>Proof:</u>

Suppose A is a definable additive subgroup of finite index in F. We will show that $A = F$.

For any nonzero element a of F the scalar multiple aA is again an additive subgroup of finite index in F. Let A_o be the intersection of all groups in Φ. By the stable CC A_o is a finite intersection of such groups, and as such is of finite index in F. However by construction the additive group A_o is closed under multiplication by elements of F, i.e. is an ideal of F. Thus A_o is F or (0), and since (0) is not of finite index in F we have $A_o = F$, hence $A = F$, as desired.

4.2 <u>Consequences of connectedness.</u>

We have seen that connected groups arise in cases of interst. We will now establish some of their properties.

<u>Theorem 4 (Surjectivity Theorem).</u> Let G be a connected superstable group and let

$$h : G \to G$$

be a definable endomorphism of G with finite kernel. Then h is surjective.

Proof:

Let G have Shelah degree α and let H be the image of h. Viewed somewhat abstractly, H is the result of collapsing G by a definable equivalence relation whose equivalence classes are of fixed finite size. Such a collapse preserves Shelah degree, so H also has Shelah degree α (details omitted). Looking at the way G breaks up into cosets modulo H, we see that the index $[G:H]$ must be finite (by (she), §3). But G is connected, so $H = G$, as claimed.

The following result illustrates a certain proof technique.

Theorem 5. Let G be a connected group and let N be a finite normal subgroup of G. Then N is contained in the center of G.

Proof:

Let the elements of G act on the elements of N via conjugation, so that each element $g \in G$ induces a permutation σ_g of N. In this way we get a homomorphism:

$$\sigma : G \to \text{Permutations of } N$$

whose kernel K is the centralizer of N.

Since G/K is isomorphic with a group of permutations of N, the index $[G:K]$ is finite. Since K is definable and G is connected we get $K = G$, so G centralizes N, as claimed.

The main result on connected groups is the following:

Theorem 6 (Indecomposability Theorem). Let G be an ω-stable group. Then the following are equivalent:

1. G is connected.
2. G has Morley degree 1.

The proof closely resembles the proof of Theorem 5, except for technical difficulties which will be glossed over.

Proof (so-called):

$2 \to 1$: entirely trivial.

$1 \to 2$:

Let G have Morley rank α and Morley degree d. We know that there is a decomposition:

(dec) $$G = S_1 \dot\cup S_2 \cup \ldots \cup S_d$$

into d definable sets of rank α and degree 1. Furthermore it is not hard to show that this decomposition is essentially unique in the sense that given a second such decomposition:

$$G = T_1 \dot\cup T_2 \cup \ldots \cup T_d$$

there will be a unique permutation σ of the indices such that

$\operatorname{rank}(S_i \Delta T_{i\sigma}) < \alpha$ (Δ = symmetric difference) .

It is convenient to abbreviate this condition by:

$$S_i = T_{i\sigma}$$

We can use the decomposition (dec) to define an action of G as a group of permutations on $1,\ldots,d$ as follows. Right translation by an element g of G converts the decomposition (dec) into a similar decomposition:

$$G = S_1 g \dot{\cup} S_2 g \dot{\cup} \ldots \dot{\cup} S_d g \quad ,$$

which must essentially coincide with (dec) as explained above after a permutation σ_g of the indices $1,\ldots,d$.

More explicitly: $S_i g \equiv S_{i\sigma_g}$. Thus we have a permutation representation σ of G , and we may consider its kernel K . Arguing as in the proof of Theorem 5 we conclude that $K = G$ (however it is no longer obvious that K is definable, and this presents the main technical complication in the argument). Making this explicit, we have that for all g in G :

(fix) $\qquad S_i g \equiv S_i$ for $i = 1,\ldots,d$.

Now with a considerable amount of "hand-waving" we are in sight of a contradiction if $d > 1$. For fixed g in S_1 and for most s_2 in S_2 the left-handed version of (fix) yields:

$$g s_2 \in S_2 \ .$$

Hence if we can find an element s in S_2 which is in some sense "generic" relative to the elements of S_2 we conclude:

$$S_1 s \subseteq S_2 \quad ,$$

which contradicts (fix). This genericity argument can easily be made rigorous by going to an elementry extension G' of G .

III. Classification theorems.

We present the classification of ω-stable fields in § 5 and we describe results on ω-stable groups of low Morley rank in § 6 .

§ 5. ω-stable fields.

We will prove Theorem 5 of § 2.2 : an ω-stable field is either algebraically closed or finite. The proof involves the following specific algebraic information:

Fact A. Let K be a Galois extension of prime degree q over a field F such that x^q-1 splits in F . Let F have characteristic p (possible $p = 0$) . Then either:

1. $q \neq p$ and K is a ***Kummer extension***, i.e. $K = F(\alpha)$ with $\alpha^q \in F$, or

2. $q = p$ and K is an ***Artin-Schreier extension***, i.e. $K = F(\alpha)$ where $\alpha^p - \alpha \in F$.

This will be combined with the following result:

Lemma 1. Let F be an infinite ω-stable field. Let h be one of the following maps:

1. $h(a) = a^n$
2. $h(a) = a^p - a$ if F has characteristic $p > 0$.

Then h is surjective.

Let us first see how to complete the proof of Theorem 5 of § 2.2 using Lemma 1.

Proof of Theorem 2.2.5:

Suppose toward a contradiction that F is an infinite ω-stable field which is not algebraically closed. By Lemma 1 with $n = p$ (if F has characteristic $p > 0$) F is perfect, hence has a Galois extension K of finite degree. Among all pairs (F,K) of fields satisfying:

i. F is infinite and ω-stable.

ii. K is a Galois extension of F of finite degree q,

choose a pair for which the degree q is minimal. It is then easy to verify that we have arrived at the situation described by Fact A, namely that q is prime and $x^q - 1$ splits in F. Hence we have a Kummer extension or an Artin-Schreier extension. However it is an immediate consequence of Lemma 1 that F has no such extensions, and we have the desired contradiction.

Proof of Lemma 1:

Let G_1, G_2 be respectively the multiplicative group of nonzero elements of F and the additive group of F. If these two groups are connected, then the desired surjectivity results follow immediately from the Surjectivity Theorem (§ 4.2). Theorem 3 of § 4.1 tells us that G_2 is connected. It remains to be seen that G_1 is connected.

By the Indecomposability Theorem (Theorem 6 of § 4.2) the following are equivalent:

1. $(F,+)$ is a connected group
2. F has Morley degree 1
3. $(F^{\bullet},\cdot)$ is a connected group.

Hence the connectivity of G_2 implies the connectivity of G_1, and we are finished.

§ 6. Groups of small Morley rank.

6.1 Results.

Using the Indecomposability Theorem of § 4.2 it is possible to give a rather thorough analysis of ω-stable groups of ranks one and two, as well as - to a more limited extent - rank three. Since an arbitrary ω-stable group is connected-by-finite (§ 4.1) we may confine ourselves to the connected case.

Theorem 1. Let G be a connected ω-stable group of Morley rank n.

1. If $n = 1$ then G is abelian.
2. If $n = 2$ then G is solvable.
3. If $n = 3$ and if G contains a subgroup of rank 2 then either G is solvable or else G is isomorphic to a group of the form $SL(2,F)$ or $PSL(2,F)$ for some algebraically closed field F.

(Notation: $GL(2,F)$ is the group of invertible 2×2 matrices over F, $SL(2,F)$ is the subgroup of matrices of determinant 1, and $PSL(2,F)$ is the quotient of $SL(2,F)$ by its center. $PSL(2,F)$ is simple.)

6.2 Groups of Morley rank one.

We begin with a simple group-theoretic lemma:

Lemma 2. Let H be a group in which all elements different from 1 are conjugate and of finite order. Then H has at most two elements.

Proof:

Assume H is nontrivial and fix a in H of prime order p. Then H is of exponent p, and if $p = 2$ a standard exercise yields that H is commutative, hence equal to Z_2.

If $p > 2$ we obtain a contradiction as follows. Let $g \in G$ conjugate a to a^{-1}. Then since p is odd we get:

$$a = a^{g^p} = a^{(-1)^p} = a^{-1} ,$$

contradicting $p > 2$.

We can combine this lemma with the Indecomposability Theorem of § 4.2 to obtain:

Theorem 3. If G is an infinite ω-stable group then G contains an infinite abelian definable subgroup.

Proof:

Suppose that G is a counterexample. We may take G to have least possible Morley rank and then least possible Morley degree. All proper definable subgroups H of G are finite, since otherwise the theorem would apply to H and hence to G. In particular

G is connected, and hence by the Indecomposability Theorem has Morley degree 1 .

Let the rank of G be α and let the center of G be Z . We will show that in G/Z all nonidentity elements are conjugate and of finite order, which in conjunction with the above lemma yields a contradiction.

For a in $G-Z$ the centralizer $C(a)$ of a in G is a proper definable subgroup of G , hence finite. In particular a is of finite order, since a is in $C(a)$, and it remains to consider the conjugacy class of a , which we denote a^G .

There is a natural identification of A^G with cosets of G modulo $C(a)$, and thus as in the proof of the Surjectivity Theorem of § 4.2 it follows that a^G has rank α . Thus since G has Morley degree 1 it follows that there is room in G for only one such equivalence class, i.e. all noncentral elements of G are conjugate, and hence all nontrivial elements of G/Z are conjugate. Thus Lemma 2 applies to complete the argument.

Part 1 of Theorem 1 is now an immediate consequence of Theorem 3.

6.3 Algebraic groups.

It is appropriate at this point to summarize some basic facts from the theory of algebraic groups which motivate the analysis used to establish the second and third parts of Theorem 1 , even though I will not give any of that analysis in detail.

Consider an algebraic matrix group G over a field F . G is a subgroup of $GL(n,F)$ defined as the set of matrices with coefficients in F whose entries satisfy certain polynomial equations:

$p_1(a_{ij}) = 0,\ldots,p_k(a_{ij}) = 0$.

Typical groups defined in this manner are:

$SL(n,F)$, defined by $\det(a_{ij})-1 = 0$

$T(n,F)$ (the upper triangular group), defined by the equations $a_{ij} = 0$ for $j < i$.

We take the base field F to be algebraically closed. Then all such groups are ω-stable.

If a topology is placed on G by taking the zero-sets of arbitrary systems of polynomials to be closed then one proves that the connected component of the identity in G is a subgroup, denoted G^o , and that the index of G^o in G is finite.

The aspect of the theory that is of interest at this point is the structure of simple algebraic matrix groups, and specifically the

Bruhat decomposition, which we will briefly describe. As a preliminary we need to discuss *Borel subgroups* and *Weyl groups*.

1. Borel subgroups.

Let G be a connected algebraic matrix group. A maximal solvable connected subgroup B of G is called a *Borel subgroup*. In this context a good deal of information is obtainable without further hypotheses, such as:

Fact B. With the above hypotheses and notation:

1. B is its own normalizer in G.
2. G is the union of the conjugates of B.

We can consider Borel subgroups of ω-stable groups using the same definition: maximal solvable connected definable subgroups. Of course the general results concerning Borel subgroups of algebraic matrix groups may not transfer to this extended context, and indeed it is not known whether any of them remain valid.

The Bruhat decomposition of G is simply its decomposition into double cosets modulo B. In the context of algebraic groups this is tied up with the Weyl group, which we briefly describe next.

2. The Weyl group.

Example: Let $G = GL(n,F)$. The symmetric group S_n on n letters sits inside G as the set of permutation matrices. Furthermore the action of S_n as a group of permutations can be seen in G by looking at the action of the permutation matrices on the group D of diagonal matrices (conjugation by a permutation matrix induces a permutation of the diagonal entries).

Now if we take as a Borel subgroup of G the group of upper triangular matrices, $B = T(n,F)$, then the double coset decomposition of G relative to B can be put in the form:

$$G = \bigcup_{\sigma} B\sigma B \ ; \quad \sigma \text{ varies over permutation matrices,}$$

or in other words the double cosets are naturally parametrized by the group S_n, which is the Weyl group of $GL(n,F)$.

The notion of a Weyl group does not seem to generalize much beyond the class of matrix groups. First one needs the notion of a *torus*, which may be defined as a connected *diagonalizable* subgroup D of G. If one has this notion, then one can take a maximal torus within a Borel subgroup B, and let W (the Weyl group) be the group of automorphisms of D which are induced by inner automorphisms of G, i.e. letting C be the centralizer of D and N the normalizer of D, set $W = N/C$.

Then one gets:

Fact C . 1. W is a finite group.

2. If G is e.g. simple and W is a system of representatives for W in G then $G = BWB$.

(This is the Bruhat decomposition.)

All of this becomes rather trivial when we focus our attention on PSL(2,F). Here the Weyl group is cyclic of order 2 (as in GL(2,F)) and the generator of W is represented by the matrix:

$$w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{in } SL(2,F) .$$

3. Philosophy.

The relevance of this material to groups of small Morley rank lies in the following. In the analysis of groups of rank 2 or 3 one attempts first to get a "Bruhat decomposition" for G (a double coset decompostion for G relative to a Borel subgroup, made reasonably explicit). Depending on the assumptions made on G this can lead either to a contradiction or to generators and relations for G.

The content of Theorem 3 is that any infinite ω-stable group has a nontrivial Borel subgroup. This is a weak result, and hence we were forced in the third part of Theorem 1 to make an additional assumption which simply amounts to the existence of a Borel subgroup of rank 2 (taking into account the result of part 2).

6.4 Groups of rank 2 :

In proving part two of Theorem 1 , we consider a connected ω-stable group G of rank 2 which is not solvable, make a detailed structural analysis of G , and eventually obtain a contradiction. The final contradiction may be put in various forms. One such is as follows.

By a preliminary reduction one may assume that G has no nontrivial proper definable normal subgroup. Let I denote the set of involutions in G (i.e. elements of order two). We can derive the following information:

1. I is nonempty.

2. The elements of I commute with each other.

Thus $I \cup \{1\}$ is a definable commutative normal subgroup of G , and we obtain a contradiction. Facts 1,2 are obtained on the basis of the following:

1.1 $G = B \cup BwB$ for some involution w (here B is a Borel subgroup).

2.1 $N(B) = B$ for any Borel subgroup ($N(B)$ is the normalizer of B in G).

2.2 $B_1 \cap B_2 = 1$ for distinct Borel subgroups B_1, B_2.

2.3 The centralizer of any nontrivial element of G is a Borel subgroup.

Clearly 1.1 yields 1. It remains to be seen that 2.1-3 together yield 2. This goes as follows.

Let i,j be distinct involutions in G and let $b = ij$. Let B be the centralizer of b in G. If we conjugate b by i a simple calculation shows that the result is b^{-1}. It follows that B and the conjugate of B by i have a nontrivial intersection, so 2.2 and 2.3 imply that i normalizes B. Then 2.1 implies that i is in B, i.e. i centralizes ij. It follows immediately that i and j commute, as desired.

We need to bring out more clearly the role of the Bruhat decomposition in all of this. We have seen it in a minor role, as the context in which a nontrivial involution occurs. In fact the first step in the group-theoretic analysis, on which the others depend (except 2.2, which is almost trivial), is the following:

1.1.1 If $x \in G$ does not normalize the Borel subgroup B, then $G = N(B) \cup BxB$.

This is already a version of the Bruhat decomposition, and reduces to the desired decomposition when we subsequently show that $N(B) = B$.

The proof of 1.1.1 is instructive. One shows that $N(B)$ has rank 1 and BxB has rank 2 for x outside $N(B)$. Since G is connected, the Indecomposability Theorem tells us that G can have only one double coset BxB of rank 2. 1.1.1 follows.

Now from 1.1.1 we can already get our involution with the help of 2.2. Namely, 1.1.1 implies that there is an equation:

$$x^{-1} = b_1 x b_2 \quad \text{with } b_1, b_2 \text{ in } B.$$

Set $w = xb_2$ and $b = b_1^{-1} b_2$. An easy computation shows that

$$w^2 = b.$$

Hence:

$$b = b^w \in B \cap B^w = 1 \quad \text{(by 2.2)},$$

so $w^2 = 1$, as desired.

This illustrates the way in which even a partial Bruhat decomposition facilitates an analysis of the structure of G.

6.5 Open questions.

Present techniques for analyzing simple ω-stable groups will probably not be able to handle the problem of proving that Borel subgroups are reasonably large. The theory of algebraic groups suggests that it would be more reasonable to attempt an analysis of simple *linear* ω-stable groups, i.e. groups of linear transformations acting on a vector space V in such a way that the group together with its action on V constitutes an ω-stable structure. We conjecture accordingly:

Conjecture A. Any simple linear ω-stable group of finite Morley rank is an algebraic group.

A more specialized variant would be:

Conjecture B. Any locally finite simple ω-stable group of finite Morley rank is an algebraic group. A proof of this would use the techniques of finite group theory blown up in the manner of [22] .

Notes.

§ 1. The stability spectrum theorem.

This section is based on [4] , primarily Chapter III . ω-Stability is found first in [1] , as a peripheral notion. Stability becomes a central notion in Shelah's papers,e.g. [3] .

A calm introduction to the subject is found in [2] .

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§ 2. Stable algebraic theories.

For a survey of stability, including model-theoretic and algebraic applications, see [18] .

2.1 Modules.

A straightforward version of the stability of modules is given in [18] . Fisher announced an equivalent theorem in [14] , and Baur found a proof independently [7] .

The characterizations of ω-stable and superstable modules were developed by Garavaglia and Macintyre, and Garavaglia alone, respectively [16]. Garavaglia has shown that there is substantially more to the theory of ω-stable modules than the classification problem, i.e. they occur in profusion, and they have nontrivial properties [16] .

2.2 Rings.

The classification of stable semisimple rings modulo stable division rings is in [5,12] . Superstable division rings are classified modulo superstable fields in [1o] . The stimulus to this work came from Macintyre's classification of the ω-stable fields in [17] .

2.3 Groups.

The theorems on $\aleph_0$-categorical stable groups are in [9] and [13]. The solvability of stable locally nilpotent groups is in [6] .

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§ 3. Chain conditions.

The ω-stable DCC is discussed in [19] . The superstable DCC is given in [18] . The stable CC is implicit in the argument on p. 274 of [6] . The study of chain conditions in group theory has been going on for some time [2o] .

Morley rank is developed and exploited in [1]. Shelah degree is one of various modification of this notion manufactured by Shelah; see the first two chapters of [3].

References:

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2o. D.Robinson, Finiteness Conditions and Generalized Solvable Groups , Springer, New York, 1972 .

§ 4. Connected groups.

The use of a suitable notion of connectedness is suggested by [22] .

The Indecomposability Theorem for ω-stable groups is proved in [21] . A weaker version is in [23] .

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§ 6. Groups of small Morley rank.

The details are in [21] . The rank 1 case is in [25] (given the Indecomposability Theorem). Background on algebraic groups is in [24] .

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Un résultat de non contradiction relative au sujet de la conjecture de SOLOVAY

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SOLOVAY avait conjecturé : si $a \subset On$ est un ensemble qui ne construit pas $0^{\#}$, alors a est générique pour un ensemble de conditions de L. JENSEN (cf. JE) a montré que ceci pouvait être faux. Nous montrons ici, en utilisant la preuve de JENSEN, que la conjecture de SOLOVAY peut être niée par un réel a qui est singleton Π^1_2 dans $L(a)$.

<u>THEOREME</u>

Il existe une formule $\varphi(x)$ qui est Π^1_2 telle que l'on puisse démontrer dans ZF:

Cons (ZF) $\rightarrow$ Cons (ZFC + GCH + $\neg 0^{\#}$ + $\exists ! x \varphi(x)$ + $\forall \alpha$ (α cardinal dans L $\rightarrow$ α cardinal) + $\forall x(\varphi(x) \rightarrow V = L(x)$ & $x \notin L$ & x n'est pas générique sur L pour un ensemble de conditions.))

On obtient le modèle cherché comme extension générique d'un modèle obtenu par JENSEN (cf. JE) qui vérifie:

(*) 1) ZFC + GCH + $\exists A(A \subset \aleph_2$ + $V = L(A)$ + A n'est pas générique sur L pour un ensemble de conditions)

2) $\forall x\ (x \subset \aleph_1 \rightarrow x \in L)$

3) $\forall \alpha$ (α cardinal dans L $\rightarrow$ α cardinal & $cf(\alpha)^L = cf(\alpha)$)

(A n'est pas générique sur L parce que, dans L(A), pour tout $K \geq \aleph_2$ il existe $B \in P(K) - L$ tel que pour tout $\alpha < K$, $B \cap \alpha \in L$.)

Notre preuve utilise alors les arbres de Souslin et les ensembles

presque disjoints pour coder A par un réel qui est singleton Π^1_2.
(voir (JE-JO); (JE-SO); (H))

Dans la suite nous adoptons la terminologie suivante. Pour un ordre partiel P, "P a la condition de chaine K" signifie que toute K chaine décroissante dans P a un minorant dans P; "P a la condition d'antichaine < K" signifie que toute famille d'éléments de P deux à deux incompatibles est de cardinalité inférieure à K; si G est M générique sur P, $V_G(x)$ est l'interprétation par G dans M(G) des termes $x \in M$ du langage forcing pour P.
Soit donc M_0 un modèle de (*); dans la suite L désignera $L^{M_0} = L \cap M_0$.
Dans M_0 soit Q l'ensemble de conditions qui permet de coder génériquement A par une partie de $\aleph_1$:

$$Q = \{(s,u) \ / \ s \subset \aleph_1 \ ; \ u \subset A \ ; \ |s|, \ |u| \leqslant \aleph_0\}$$

avec l'ordre: $(s,u) \leqslant (s',u') \leftrightarrow s \supset s' \ \& \ u \supset u' \ \&$

$$\& \ \forall \alpha \in u' \quad s_\alpha \cap s \subset s',$$

où $(s_\alpha)_{\alpha<\aleph_2}$ est une famille, <u>dans L</u>, de parties presque disjointes de $\aleph_1$ (M_0 et L ayant les mêmes cardinaux, la notion est absolue).

<u>Lemme 1</u>

1) Q est inclus dans L et si $q, q' \in Q$, $q \leqslant q'$ ssi $L \models q \leqslant q'$.

2) Q a la condition de chaine dénombrable et la condition d'antichaine $< \aleph_2$.

Soit G un M_0 générique sur Q et $M_1 = M_0(G)$. On a : $M_1 \models \exists B \subset \aleph_1$ $V = L(B)$.

<u>Définition</u>

Soit $((T_n)_{n\in\omega}, f)$ la famille d'arbres construite, <u>dans L</u>, par (JE-JO); soit P l'ensemble de conditions associé:

$$P = \{p \ / \ \text{dom } p = n \ \& \ \forall i < n \ p_i \in T_i \ \& \ f(p_{i+1}) \geqslant p_i\}$$

avec l'ordre : $p \leqslant q \leftrightarrow \text{dom } p \supset \text{dom } q \ \& \ \forall i \in \text{dom } q \quad q_i \leqslant p_i$

$$P_n = \{p \in B \ / \ \text{dom } p = n\}.$$

<u>Lemme 2</u>

Soit N un modèle de ZF et T un arbre de Souslin dans N. Si C est un ensemble de conditions de forcing ayant la condition de chaine dénombrable et si G est N générique sur C alors N(G) satisfait : T est un arbre de Souslin.

preuve : voir (DE-JO).

<u>Lemme 3</u>

Dans M_1 P a la condition d'antichaine $< \aleph_1$.

<u>preuve</u>: $P \blacksquare \bigcup_n P_n$; il suffit donc de montrer que chaque P_n a la condition d'antichaine $< \aleph_1$. Or $P_{n+1} = P_n * T(b_n)$ où * désigne l'itération des forcings et $T(b_n)$ est l'image par f^{-1} de la branche b_n ajoutée dans T_n par le forcing P_n. $T(b_n)$ est un sous-arbre normal de T_{n+1}; il suffit donc de montrer que :

1) P_0 est un arbre de Souslin dans M_1.

2) $\forall n \; T(b_n)$ est un arbre de Souslin dans $M_1(b_n)$.

Dans (JE-JO) il est montré que : $\forall n \; T(b_n)$ est un arbre de Souslin dans $L(b_n)$;

* T_0 est Souslin dans L, donc par (* - 2) il l'est aussi dans M_0 et par les lemmes 1 et 2 il l'est aussi dans M_1.

* soit n le premier entier tel que $T(b_n)$ n'est pas Souslin dans $M_1(b_n)$. P_n a donc la condition d'antichaine $< \aleph_1$.

soit $p \in P_n$ et $\underline{X} \in M_1$ tel que M_1 satisfait $\Psi(p,\underline{X})$ où $\Psi(p,\underline{X})$ est la formule : $p \Vdash \underline{X}$ est une antichaine non dénombrable de $T(\Gamma)$ (où Γ est le nom canonique pour le générique sur P_n)

soit $q \in Q$ et $\underline{\underline{X}} \in M_0$ tel que M_0 satisfait : $q \Vdash \Psi(\check{p},\underline{\underline{X}})$

soit $H \blacksquare H_1 \times H_2$ un M_0 générique sur $P_n \times Q$ tel que $(p,q) \in H$ et $M_2 = M_0(H) \blacksquare M_0(H_1)(H_2) = M_0(H_2)(H_1)$.

$q \in H_2$ donc $M_0(H_2)$ satisfait : $p \Vdash V_{H_2}(\underline{\underline{X}})$ est une antichaine non dénombrable de $T(\Gamma)$. $p \in H_1$ donc M_2 satisfait : $X' = V_{H_1}(V_{H_2}(\underline{\underline{X}}))$ est une antichaine non dénombrable dans $T(b'_n)$. (où b'_n est la branche dans T_n associée à H_1)

Montrons (ce qui donnera la contradiction cherchée) que M_2 satisfait: $T(b'_n)$ est un arbre de Souslin.

On sait que $T = T(b'_n)$ est Souslin dans $L(b'_n)$; donc aussi dans $M_0(H_1) = M_0(b'_n)$ (car $T \subset \aleph_1$; si A est dans $M_0(b'_n)$ une antichaine de T, A a dans M_0 un nom $\underline{A}$ qui est dans L (par *-2) donc A est dans $L(b'_n)$ donc dénombrable). Pour montrer que T est Souslin dans M_2 il suffit, d'après le lemme 2 de voir que Q a, dans $M_0(H_1)$, la condition de chaine dénombrable. Or si $(q_1)_{1\in\omega}$ est dans $M_0(H_1)$ une chaine décroissante d'éléments de Q, $(q_1)_{1\in\omega}$ est en fait dans M_0 donc (lemme 1) a un minorant.

Cela résulte immédiatement du lemme bien connu suivant et d'une induction triviale.

<u>Lemme 4</u>

Si T est un arbre de Souslin dans N, si G est N générique sur T et si $f : \omega \to N$ est dans N(G) alors f est en fait dans N.

Ceci achève la preuve du lemme 3.

<u>Définition</u>

Si $B \subset \aleph_1$ soit C(B) l'ensemble ordonné défini par :

$c \in C(B) \leftrightarrow c \blacksquare (p,u)\ p \in P\ \&\ u \subset B$ fini

$(p,u) \leqslant (p',u') \leftrightarrow p \leqslant p'\ \&\ u \supset u'\ \&\ \forall\alpha \in u'\ s_\alpha \cap \sigma(p) \subset \sigma(p')$

(où $(S_\alpha)_{\alpha<\aleph_1}$ est, <u>dans L</u>, une famille de parties presque disjointes de ω et où $\sigma(p)$ désigne l'ensemble des éléments de hauteur 0 sous $\{p_i\ /\ i \in \text{dom}\ p\}$ - pour les détails voir (H) -).

<u>Lemme 5</u>

Soit $B \subset \aleph_1$ tel que $M_1 \models V \blacksquare L(B)$. Alors dans M_1, C(B) a la condition d'antichaine $< \aleph_1$.

<u>preuve</u>: si $\sigma(p) \blacksquare \sigma(p')$, (p,u) et (p',u') sont compatibles si et seulement si p et p' le sont; si $(p_\alpha, u_\alpha)_{\alpha<\aleph_1}$ est une antichaine de C(B) on peut supposer (puisque $\sigma(p)$ est une partie finie de ω) que:

$\forall\alpha,\beta\ \sigma(p_\alpha) = \sigma(p_\beta)$ donc $(p_\alpha)_{\alpha<\aleph_1}$ est une antichaine de P, ce qui est impossible.

Fin de la preuve du théorème

Soit G un M_1 générique sur C(B). Dans $M_1(G)$ soit
$a = \{n \in \omega \ / \ \exists(p,u) \in G \quad n \in \sigma(p)\}$
alors $M_1(G)$ satisfait : $V = L(a)$ & a est singleton Π^1_2.
en effet C(B) réalise en même temps les propriétes de codage de B et rend le générique associé a définissable par une formule Π^1_2. (pour les détails voir (H)). La définition de a est indépendante de M_1 et cela achéve donc la preuve du théorème.

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REMARKS ON CONSTRUCTIVE MATHEMATICAL ANALYSIS

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The paper is devoted to some questions of constructive mathematics (CM) in the sense of the Markov school. Its aim is first a short outline of the fundamental principles of this trend in mathematics, and secondly a survey of basic results concerning constructive mathematical analysis, obtained within the Prague seminar.

§ 1. Introduction

CM, in the sense of the Russian school (headed by A.A. Markov and N.A. Šanin), has been created on the basis of criticism of foundations of mathematics, as formulated consistently by L.E.J. Brouwer and H. Weyl. The aim was to find foundations for mathematics that would be as simple and secure as possible and would be free from too far reaching idealisations, foundations in which the concept of effectivity would be the principal.

In this connection, the following facts are important:

1) intuitive natural numbers (NNs) are indispensable for mathematics,

2) from the historical point of view, the development of mathematics was substantially influenced by applications of mathematics where solutions of problems consisted, de facto, in transformation of particular information coded by words.

We should notice that words in an alphabet can be taken for material representatives of intuitive NNs. Thus, the means necessary for algorithmic processing of words are indispensable for any sufficiently rich mathematical theory. The constructive program and its realization shows that these means necessary for processing of words are also sufficient for the development of wide mathematical theories, comparable to classical mathematics as to richness of results and methods.

Let us characterize the basic features of constructive mathematics:

1) in CM are studied so called constructive objects, i.e. the words

in certain alphabets and the objects which can be coded in a finite way by words,

2) the abstraction of potential realizability and theory of algorithms are used,

3) a specific interpretation of mathematical propositions is used.

To put it shortly, the main subject of CM is the study of possibilities of algorithmical transformation of coded information about mathematical objects. This fact delimitates the space in which CM can be developed and, at the same time, it distinguishes CM from the intuitionistic program as well from other constructive trends as, e.g. Bishop's one. In this context we would like to mention that the use of theory of algorithms, mentioned above, results in some similarity of constructive mathematical analysis and recursive analysis.

The use of theory of algorithms and the way how the system of constructive concepts is built require the use of specific interpretation of mathematical propositions. As for our opinion the classical Aristotle logic is not in harmony with the needs of theory of algorithms. N.A. Šanin suggested successfully the constructive interpretation of mathematical propositions congenial to ideas of A.N. Kolmogorov and S.C. Kleene. It is based on the intuitionistic interpretation of logical connectives and quantifiers, theory of algorithms and Markov's principle. The substance of constructive interpretation is the algorithmical, i.e. the effective interpretation of the existential quantifier and the disjunction.

The basic concepts and results of constructive mathematical analysis can be acquainted with, for example, in [2], [13], [14], [16], [17]. A summary exposition and detailed bibliography up to 1971 can be found in [6].

The present paper deals with the structure of the real line in CM, with the questions of differentiability and with the theory of integral. Corresponding results of classical mathematics can be found in [1] and [2]. Note, that a list of necessary concepts and notations, used in the sequel, is introduced in short in §2.

§2. Basic concepts

The basic objects of constructive mathematical analysis studied in this paper are words in the alphabet Ξ containing, among others, the letters: $0, |, -, /, \lozenge, \square, \Delta, \nabla$ etc.. By Ξ^* we denote the set of all words in the alphabet Ξ. The signs Λ, $\overline{\underline{o}}$, $\simeq$ denote empty word, graphical equality and conditional (graphical) equality, respectively. The symbols U, V play the role of variables for the elements of Ξ^*.

Markov algorithms, which we deal with, are the algorithms over the alphabet Ξ. The applicability of an algorithm $\mathfrak{A}$ to a word P is denoted by $!\mathfrak{A}(P)$.

The term "set" is understood in the same sense as in [13]. We write $\mathfrak{M} \cup \mathfrak{N}$, $\mathfrak{M} \cap \mathfrak{N}$, $\mathfrak{M} \setminus \mathfrak{N}$ for the union, the intersection and the difference of $\mathfrak{M}$ and $\mathfrak{N}$, respectively.

By a system of words of a certain type we mean any list of words of the type. Systems of words will be denoted by $\{V_i\}_{i=0}^n$. By a finite set we mean a set for which a list of all its elements can be given, and by an infinite set we mean a set different from any finite set.

By NNs we mean the words 0, 0|, 0||,... . The set of all NNs will be denoted by N. The symbols i,j,k,m,n,p,q,s,t play the role of variables for NNs. The rational numbers (RtNs) are introduced as certain words in Ξ. The set of all RtNs will be denoted by Q. The symbols a,b,c,d play the role of variables for RtNs. As abbreviational notation for NNs and RtNs we shall employ the standard notation of the form 2, -3, $\frac{2}{5}$ etc. . Let us note that the arithmetical operations over these numbers are, in line with the requirements of CM, realized algorithmically.

We use the constructive interpretation of mathematical propositions ([13], [14]). The substance of it is the algorithmical interpretation of $\exists$ and $\vee$. For example, the formula $\forall U \exists V\, A(U,V)$ holds iff there exists a Markov algorithm $\mathfrak{A}$ such that $\forall U(!\mathfrak{A}(U)\ \&\ A(U,\mathfrak{A}(U)))$ holds. A disjunctive formula holds iff it is possible algorithmically determine the member of it which holds. Let us recall the importance of so-called normal formulas (i.e. formulas not containing $\exists$ and $\vee$). E.g., any normal formula is equivalent to its own double negation.

By a normal set we mean any set for which the relation of membership can be given by a normal formula. It should be noted that we deal only with variables for which the domains of admissible values are normal sets.

Let us fix Markov algorithms <u>wd</u> and <u>en</u> establishing one-to-one correspondence between Ξ^* and N, where <u>wd</u> maps N onto Ξ^* and <u>en</u> is algorithm inverse to <u>wd</u>. With the help of this numbering we carry over concepts introduced initially for sets of NNs to the sets of words. Also, the well-known equivalence of Markov algorithms and partial recursive functions (PRFs) can be expressed in this way. The use of either Markov algorithms of PRFs depends on practical needs of the context.

As for the relative computability we use the constructive reformulation of its characterization, given in [11]. The only difference is caused by the constructive interpretation of the existential quantifier. In fact, if B is a set of NNs and m, k and n are NNs then by $\langle m\rangle^B$ we mean the B-PRF with the index m, by $\langle m\rangle^B(k) \simeq n$ we denote $\neg\neg\exists st(\langle k,n,s,t\rangle \in W_{\rho(m)}\ \&\ D_s \subseteq B\ \&\ D_t \subseteq N\setminus B)$ and by $!\langle m\rangle^B(k)$ we denote $\neg\neg\exists p(\langle m\rangle^B(k) \simeq p)$, (for notation see [11], §9.2). For details and for the employment of relative computability in CM see [54]. We consider the relativized PRFs as predicatively defined correspondences. Let us note, that these predicates are equivalent to normal formulas.

$\emptyset$-PRFs are just the PFRs and therefore $\langle m\rangle^\emptyset$ is an indexing of PRFs. We usually write $\langle m\rangle$ instead of $\langle m\rangle^\emptyset$.

It should be noted that we are interested, owing to the natural connection between concepts of constructive mathematical analysis and arithmetical predicates, only in the computability relative to jumps of empty set. It is known from the results of E.M. Gold and P. Putnam that the $\emptyset^{(n)}$-PRFs $(1 \leqslant n)$ can be represented on the basis of recursive functions by means of non-effective limits. Without leaving constructive program concerning effective processes we improve, by the use of relative computability, our ability to handle effective procedures. The advantage of the improvement consists in both substantial simplification and clearness of formulations.

Relativized algorithms can be introduced on the basis of relativized PRFs. Let B be a set of NNs and let m be a NN. The correspondence $[\![\langle m\rangle^B]\!]$ defined by $[\![\langle m\rangle^B]\!](U) \simeq \underline{wd}(\langle m\rangle^B(\underline{en}(U)))$ is called a B-algorithm with the index m. Let us note that $\emptyset$-algorithms are just the correspondences realizable by Markov algorithms.

If B is a set of NNs and F is a B-algorithm then for any word P we denote by F_P a B-algorithm such that $\forall V(F_P(V) \simeq F(PV))$.

In such a manner as the constructive interpretation of $\exists$ and $\vee$ is expressed by means of Markov algorithms we can analogically define the relativized existential quantifier and the relativized disjunction on the basis of relativized algorithms. Let B be a set of NNs. We write $\exists^B UA_1(U)$ ("B-exists a word U ... ") for $\exists m(\,![\![\langle m\rangle^B]\!](O)\ \&\ A_1([\![\langle m\rangle^B]\!](O)))$, and we write $A_1 \overset{B}{\vee} A_2$ for

$\exists^B V((V \asymp \Lambda \supset A_1) \ \& \ (V \asymp \Lambda \supset A_2))$, where A_1, A_2 have no occurences of variables m and V. In such a manner the interpretation of the quantifier $\exists^B$ is reduced to the interpretation of the quantifier $\exists$. Of course, $\exists$ and $\exists^{\emptyset}$ are equivalent. Further, $\exists$ implies $\exists^B$ for any set B, and $\exists^B$ implies $\neg\neg\exists$.

Let B be a set of NNs and let $\mathfrak{M}$ be a normal set. A B-algorithm F is said to be

1) a B-sequence of elements of $\mathfrak{M}$ if $\forall n(!F(n) \ \& \ F(n) \in \mathfrak{M})$,
2) a B-sequence of B-algorithms of a certain type if for any NN n $F_{n\square}$ is a B-algorithm of this type.

For any set $\mathfrak{M}$ and any word P we denote by $\tilde{\mathfrak{M}}_P$ the set $\{U : PU \in \mathfrak{M}\}$. By a sequences of sets of a certain type we mean a set $\mathfrak{N}$ such that $\forall U(U \in \mathfrak{N} \supset \exists n V(U \asymp n\square V))$ holds and for any NN n $\tilde{\mathfrak{N}}_{n\square}$ is a set of this type.

In the sequel we present B-sequences (of words or B-algorithms) and sequences of sets by their "members" using notations $\{\ldots\}_n^B$ and $\{\ldots\}_n$, respectively. Of course, e.g. "$\{P_n\}_n^B$ is a B-sequence of words" means that there exists a B-algorithm F such that $\forall n(!F(n) \ \& \ F(n) \asymp P_n)$.

In what follows we deal with various $\emptyset^{(n)}$-concepts. If n = 0 we, as a rule, omit the sign $\emptyset$ in the corresponding notations. E.g., we speak of sequences instead of $\emptyset$-sequences, continuity instead of $\emptyset$-continuity etc..

The central concept of mathematical analysis is the concept of real number. There exist more constructive formulas mutually nonequivalent that characterize, from the classical point of view, the fundamentality i.e. cauchyness. Indeed, a sequence F of RtNs is said to be

i) fundamental if $\forall n \exists m \forall k(m \leqslant k \supset |F(k)-F(m)| \leqslant 2^{-n})$,
ii) pseudo-fundamental if $\forall n \neg\neg\exists m \forall k(m \leqslant k \supset |F(k)-F(m)| \leqslant 2^{-n})$.

Let us note that in accordance with constructive interpretation of propositions the fundamentally of F means the existence of an algorithmical regulator of fundamentality of F, i.e. the existence of an algorithm transforming every NN n to the corresponding m. In the case of pseudo-fundamentality of F the formula is equivalent to a normal formula and its interpretation does not differ from the classical one, specifically the existence of algorithmical regulator is not required. As known, in the case of Specker sequence such algorithmical regulator does not exist, indeed.

For sequences of RtNs we obtain the concept of $\emptyset^{(n)}$-fundamentality so that we replace in i) $\exists$ by $\exists^{\emptyset(n)}$.

It should be noted that a sequence of RtNs is pseudo-fundamental iff it is $\emptyset'$-fundamental.

Let us mention that the similar situation, we have met in the case of fundamentality, is also found in the case of constructive formulations of other concepts such as convergence, continuity, uniform continuity etc.. In quite analogical way we receive concepts of $\emptyset^{(n)}$-convergence and pseudo-continuity etc.. Let us agree that the sign $\to$ is used for denoting the convergence, i.e. $\emptyset$-convergence.

A constructive real number (CRN) is a RtN of a word of the form $m\Diamond n$, where m,n are NNs, $[\![\langle m\rangle]\!]$ is a sequence of RtNs, $[\![\langle n\rangle]\!]$ is a sequence of NNs being a regulator of fundamentality of $[\![\langle m\rangle]\!]$.

A pseudo-number (PN) is a RtN or a word of the form $m\Diamond$, where m is a NN and $[\![\langle m\rangle]\!]$ is a pseudo-fundamental sequence of RtNs. The set of all PNs is denoted by π.

The symbols x,y,z,v,w play the role of variables for CRNs, the symbols ξ,η play the role of variables for PNs.

For any RtN a we denote by $\llcorner a \lrcorner$ a sequence of RtNs such that $\forall n(\llcorner a \lrcorner(n) \asymp a)$, for any NN m we denote by $\llcorner m\Diamond \lrcorner$ the algorithm $[\![\langle m\rangle]\!]$.

Let us note that on the basis of $\emptyset^{(n)}$-fundamental $\emptyset^{(n)}$-sequences of RtNs we can obtain "arithmetical real numbers". In this connection it is worthwhile to note that PNs are, de facto, real numbers constructive relatively to $\emptyset'$ ($\emptyset'$-CRNs).

On the set of all CRNs and PNs the relations of equality and order are defined as predicates in the obvious way. Basic algebraic operations over these numbers are realized algorithmically.

As for the arithmetical complexity, the set of CRNs is Π_2-complete and the set of PNs (i.e. π) is Π_3-complete.

It is known that the set of CRNs (with euclidean metric) forms a complete separable metric space (cf. [14]).

By a segment (resp. interval) we mean any word of the form $x \Delta y$ (resp. $x \nabla y$), where x,y are CRNs and $x < y$. If H is a segment (resp. interval) then $E_\ell(H)$, $E_r(H)$ denote the left and right endpoints, respectively, $|H|$ denotes the length of H, i.e. $E_r(H) - E_\ell(H)$, and $(H)^0$ denotes the interval $E_\ell(H) \nabla E_r(H)$. A segment (resp. interval) the endpoints of which are RtNs is called rational segment (resp. rational interval). Relation of membership for CRNs and PNs to a segment (resp. interval) is defined in a natural way.

It is known (Lacombe, Cejtin, Zaslavskij) that for any RtN $\varepsilon > 0$ there exists a sequence of rational intervals which effectively covers all CRNs and such that the sum of the lengths of an arbitrary finite set of these intervals is less than ε. In the sequel we use the following type of coverings ([17]).

A sequence Φ of non-overlapping rational segments (resp. segments) contained in $0 \Delta 1$ is said to be a covering (resp. a real covering) if $E_\ell(\Phi(0)) = 0$, $E_r(\Phi(1)) = 1$ and
$\forall x(x \in 0 \Delta 1 \supset \exists pq(E_r(\Phi(p)) = E_\ell(\Phi(q)) \;\&\; E_\ell(\Phi(p)) \leq x$
$\& \; x \leq E_r(\Phi(q))))$ hold.

A covering Φ is said to be

a) regular if the series $\sum_k |\Phi(k)|$ converges to 1,

b) singular if $\neg\neg\exists w \forall p(\sum_{k \leq p} |\Phi(k)| \leq w < 1)$.

Let us note that there exists both regular and singular coverings.

The following lemma (cf. [17]) enables us to give a reasonable definition of the constructive concept of almost everywhere for CRNs.

Lemma 2.1. Let Φ be a sequence of segments, $x \Delta y$ a segment, and let the series $\sum_k |\Phi(k)|$ converges to a CRN less than $|x \Delta y|$. Then there exists a CRN w such that $w \in x \nabla y \;\&\; -\exists k(w \in \Phi(k))$.

Corollary. For any covering Φ

a) if the series $\sum_k |\Phi(k)|$ converges then Φ is regular;

b) if the series $\sum_k |\Phi(k)|$ does not converge then ϕ is singular and this series pseudo-converges to a PN not equal to any CRN.

A sequence $\{H_n\}_n$ of non-overlapping segments is termed an s_σ-set and a CRN z is termed a measure of $\{H_n\}_n$ if the series $\sum_n |H_n|$ converges to z. If P is a CRN or a PN and $\{H_n\}_n$ is an s_σ-set then we write $P \in \{H_n\}_n$ for $\neg\neg\exists n(P \in H_n)$.

A property A of CRNs is said to hold for almost every (a.e.) CRN x (resp. for a.e. CRN x from a segment H) if there exists a sequence of S_σ-sets $\{\mathcal{S}^n\}_n$ such that for any NN n the measure of $\mathcal{S}^n$ is less than 2^{-n} and for any CRN x (resp. $x \in H$),

$\neg(x \in \mathcal{S}^n) \supset A(x)$ holds.

"A property A of PNs holds for a.e. PN" can be introduced either by full relativization to $\emptyset'$ of the definition just given (recall, that PNs are, de facto, $\emptyset'$-CRNs), or without using relativized concepts (cf. [44]).

Let us consider constructive analogues of the concept of a function of a real variable.

An algorithm f is called a constructive function of a real variable (CFRV) if the following conditions are satisfied:

1) for any CRN x, if $!f(x)$, then $f(x)$ is a CRN,

2) $\forall xy(!f(x) \,\&\, x = y \supset !f(y) \,\&\, f(x) = f(y))$.

An algorithm f is called a π-operator if the following condition is satisfied $\xi\eta(!f(\xi) \,\&\, f(\xi) \in \pi \,\&\, (\xi = \eta \supset f(\xi) = f(\eta)))$.

One of the well-known results concerning CFRVs is the theorem stating the continuity of any CRFV at every CRN in its domain (proved by Kreisel, Lacombe, Shoenfield, Moschovakis, Cejtin). On the other hand, everywhere defined CFRVs (though continuous) need not be either uniformly continuous or bounded on $0 \Delta 1$. As for domains of CFRVs let us only note that they are the sets of the type G_δ (in the effective sense) (cf. Černov [4], independently in [46]), but they need not be open (Friedberg). For further results on domains see [46].

As for the arithmetical complexity, the set of (indices of) CFRVs is Π_3-complete ([64]).

For brevity, everywhere defined CFRVs, constant on both $\{x : x \leqslant 0\}$ and $\{x : 1 \leqslant x\}$, are called simply functions.

Let F,G be functions. By F*G we denote a superposition of F and G, i.e. a function such that $\forall x(F{*}G(x) = F(G(x)))$. If F is an increasing on $0 \Delta 1$ function, $F(0) = 0 \,\&\, F(1) = 1$, then we denote by F^{-1} the inverse function of F.

Let $\{H_n\}_n$ be a sequence of segments, $x_0 \Delta x_1$ a segment, and F a function. Then,

1) $\mathcal{K}(\{H_n\}_n)$ means: $\{H_n\}_n$ is a sequence of non-overlapping segments, $|H_n| \xrightarrow[n\to\infty]{} 0$, and $\neg\exists n(0 \in (H_n)^0 \vee 1 \in (H_n)^0)$,

2) if $\mathcal{K}(\{H_n\}_n)$ holds, then $[F,\{H_n\}_n]$ denotes a function, linear on any H_n, satisfying $\forall x(\neg\exists n(x \in (H_n)^0) \supset [F,\{H_n\}_n](x) = F(x))$,

3) $F^{[x_0 \Delta x_1]}$ denotes a function such that $\forall z(F^{[x_0 \Delta x_1]}(z) = F(\max(\min(z,x_1),x_0)))$.

Let us note that for any covering Φ we have $\overline{\mathcal{H}}(\Phi)$.

Further, for any function F pseudo-uniformly continuous on $0 \Delta 1$ a π-operator denoted by Op[F] can be constructed such that $\forall x\xi(x = \xi \supset F(x) = \mathrm{Op}[F](\xi))$. Let us note that a function is pseudo-uniformly continuous iff it is $\emptyset'$-uniformly continuous.

Let F be a function, $x \Delta y$ a segment, P (resp. R) a word being a CRN or a PN. Then

1) $\Delta(F, x\Delta y)$ denotes $(F(y) - F(x))$,

2) by D(R,F,P) we denote

$$\forall k \exists m \forall ab(a<p<b \;\&\; b-a<2^{-m} \supset \left|\frac{\Delta(F,a\ b)}{|a\Delta b|} - R\right| < 2^{-k}) \qquad (1)$$

("R is a derivative of F at P"),

3) by $D_{cl}(R,F,P)$ we denote (1) where $\exists$ is replaced by $\neg\neg\exists$

("R is a pseudo-derivative of F at P"),

4) by $D_{cl}(F,P)$ we denote $\forall k \neg\neg\exists m \forall abcd(a<P<b \;\&\; c<P<d \;\&\; b-a<2^{-m} \;\&\; d-c<2^{-m} \supset \left|\frac{\Delta(F,a\ b)}{|a\Delta b|} - \frac{\Delta(F,c\ d)}{|c\Delta d|}\right| < 2^{-k})$

("F is finitely pseudo-differentiable at P"),

5) we define $D_{c\ell}(+\infty,F,P)$, $D_{c\ell}(-\infty,F,P)$, and $\overline{D}_{c\ell}(+\infty,F,P)$, $\underline{D}_{c\ell}(-\infty,F,P)$ (upper and lower pseudo-derivate, respectively), in a natural way.

Let us note that if a function is finitely pseudo-differentiable at a PN then the value of the corresponding "pseudo-derivative" need not be a PN but, in general, a real number constructive relatively to $\emptyset^{(2)}$. The questions of upper and lower pseudo-derivatives are still more complicated. For details see [51].

By Theorem 5.4 of [16] we can construct for any uniformly continuous function F

1) algorithms $\langle S,F\rangle$ and $\langle I,F\rangle$ transforming any segment H to a CRN being a l.u.b. and g.l.b., respectively, of values assumed by F on H,

2) algorithms $\langle \omega,F\rangle$ and $\langle \mathcal{O},F\rangle$ such that for any segment H
$\langle \omega,F\rangle(H) \simeq \langle S,F\rangle(H) - \langle I,F\rangle(H)$ and
$\langle \mathcal{O},F\rangle(H) \simeq \langle I,F\rangle(H) \;\Delta\; \langle S,F\rangle(H)$.

<u>Lemma 2.2.</u> Let F be a uniformly continuous function. Then there exists a sequence of CRNs $\{z_k\}_k$ such that $\forall abcy(0 \leqslant b < c \leqslant 1 \;\&\; (F(a) = y \vee \langle S,F\rangle(a\Delta b) = y \vee \langle I,F\rangle(x\Delta y) = y) \supset \exists k(y = z_k))$, and, consequently, for any rational segment $a \Delta b \subseteq 0 \Delta 1$ and for

any CRN y such that $\neg\exists n(y = z_n)$ we have

1) $(\exists x(x \in a\Delta b \ \&\ F(x) = y) \vee \neg\exists x(x \in a\Delta b \ \&\ F(x) = y))$,

2) if $y \in \langle 0,F\rangle(a\Delta b)$, then CRNs x_1, x_2 can be constructed such that $a < x_1 \leqslant x_2 < b \ \&\ F(x_1) = F(x_2) = y \ \&\ \forall x(x \in a\Delta b \ \&\ F(x) = y \supset x_1 \leqslant x \leqslant x_2)$.

Let F be a function, $H \subseteq 0\Delta 1$ a segment, and z a CRN. By BVS(z,F,H) we mean that variation sums of F on H are bounded by z. By Var(z,F,H) we denote BVS(z,F,H) & $\forall n$ -BVS($z-2^{-n}$,F,H). Further, F is called to be

1) a function of bounded variation on H if $\exists v$ Var(v,F,H),

2) a function of weakly (resp. quasi-weakly) bounded variation on H if there exists (resp. cannot fail to exist) a NN m such that BVS(m,F,H).

If F is a function of bounded variation on $0\Delta 1$ then by [16] it is uniformly continuous and there exists a function V[F] such that $\forall w(0 < w \leqslant 1 \supset \mathrm{Var}(V[F](w),F,0\Delta w))$. Further, any function of quasi-weakly bounded variation on a segment is pseudo-uniformly continuous on the segment.

Obviously, the class of functions of weakly bounded variation is closed under the basic arithmetical operations but this important property does not hold in the case of functions of bounded variation.

For any CRN z we denote by h_z a function such that $\forall x(h_z(x) = z.\max\min(x,1),0))$.

We introduce the condition α important in the sequel.

A function F is said to fulfil the condition α (in symbols, $\alpha(F)$) if $\forall a \exists z$ Var(z,f - h_a,$0\Delta 1$).

<u>Example 2.1.</u> There exist non-decreasing functions F_1, F_2 fulfilling the condition α such that the function $F_1 - F_2$ is not a function of bounded variation on any segment contained in $0\Delta 1$.

<u>Lemma 2.3.</u> If F is a function increasing on $0\Delta 1$ such that $F(0) = 0 \ \&\ F(1) = 1$ then $\alpha(F)$ iff $\alpha(F^{-1})$.

A function F is said to fulfil the condition $\mathcal{A}$ (resp. $\mathcal{A}_{c\ell}$)- in symbols, $\mathcal{A}(F)$ (resp. $\mathcal{A}_{c\ell}(F)$)- if for any NN m there exists (resp. there cannot fail to exist) a NN n such that for any system of non-overlapping rational segments $\{a_i \Delta b_i\}_{i=0}^{s}$

$\sum_{i=0}^{s} |a_i \Delta b_i| < 2^{-n} \supset \sum_{i=0}^{s} | \ (F, a_i \Delta b_i)| < 2^{-m}$ holds.

Remark 2.1. Let F be a function.

1) If $\mathcal{A}(F)$ then F is a uniformly continuous function of weakly bounded variation on $0\Delta 1$ (but, in general, not being a function of bounded variation on $0\Delta 1$).

2) $\mathcal{A}(F)$ iff for any S_σ-set $\{H_n\}_n$ such that $\overline{\mathcal{H}}(\{H_n\}_n)$, the series $\sum_n |\Delta(F, H_n)|$ converges.

3) If $\mathcal{A}_{c\ell}(F)$ then F is a function of quasi-weakly bounded variation on $0\Delta 1$.

A function F is said to be absolutely continuous on a segment H, $H \subseteq 0\Delta 1$, if there exists a sequence of polygonal functions $\{G_n\}_n$ such that $\forall n\ BVS(2^{-n}, F-G_n, H)$ holds; AC(F) means that F is absolutely continuous on $0\Delta 1$.

Let us note that any absolutely continuous function (on $0\Delta 1$) is a function of bounded variation on $0\Delta 1$ fulfilling the condition $\mathcal{A}$. But a stronger result, being of interest in the sequel, holds.

Theorem 2.1. For any function F, AC(F) iff (F) & α(F) iff $\mathcal{A}_{c\ell}(F)$ & $\alpha(F)$. (Cf. [33], [34]).

Let K be a class of functions. K is called

1) $\overline{A}$-closed if for any functions F_1, F_2 from K, any CRNs v,w, any S_σ-set $\{H_n\}_n$ such that $\overline{\mathcal{H}}(\{H_n\}_n)$, any segment H contained in $0\Delta 1$ and any function φ increasing on $0\Delta 1$ such that $\varphi(0) = 0$ & $\varphi(1) = 1$ & $AC(\varphi)$ & $AC(\varphi^{-1})$ the following conditions are satisfied: $|F_1| \in K$, $(F_1+F_2) \in K$, $(v.F_1+w) \in K$, $(F_1.F_2) \in K$, $[F_1, \{H_n\}_n] \in K$, $(F_1)^{[H]} \in K$, $F_1{*} \in K$ and

$$(\exists m(|F_1| \geqslant \frac{1}{m}) \supset \frac{1}{F_1} \in K),$$

2) A-closed if it is $\overline{A}$-closed and for any function F and any increasing sequence $\{x_n\}_n$ of CRNs from $0\nabla 1$, such that $x_n \xrightarrow[n\to\infty]{} 1$, the following holds

$$\forall n (F^{[0\Delta x_n]} \in K) \supset F \in K;$$

3) V-closed if for any sequence of functions $\{F_n\}_n$ and any function F we have $\forall n\ F_n \in K$ & $BVS(2^{-n}, F_n - F, 0\Delta 1)) \supset F \in K$.

It turns out that the class of functions absolutely continuous on $0\Delta 1$ is both $\overline{A}$-closed and V-closed.

§3. Some results concerning CRNs and PNs

There is a close connection between coverings and properties of r.e. sets. Cejtin ([3]) studied pseudocuts, i.e. r.e. sets A of of RtNs such that $A = \{a : \exists b(b \in A \ \& \ a < B)\}$, and proved: a pseudocut $A(A \neq \emptyset \ \& \ A \neq Q)$ is strongly undecidable iff there exists a strong lowering algorithm for A, i.e. an algorithm transforming any CRN x to a rational interval H containing x and such that for the set B, $B = \{a : a \in H\}$, $\neg\neg(B \subseteq A \vee B \cap A = \emptyset)$ holds. Note that from [3], [7] it follows: if A is a pseudocut for which $0 \in A \ \& \ 1 \notin A$ then A is strongly undecidable iff there exists a covering Φ such that $\forall b(b > 0 \supset (b \in A \equiv \exists n(0\Delta b \subseteq \bigcup_{i \leq n} \Phi(i))))$. Further, a pseudocut A is wtt-complete $(A \neq \emptyset \ \& \ A \neq Q)$ iff there exists an algorithm transforming any index of a r.e. set W of RtNs contained in $Q \backslash A$ to a RtN $\varepsilon > 0$ less than the "distance" between W and A ([5], independently [63]).

These facts have further applications in constructive topology. We can construct a topological linear space of pairs of CRNs with an effectively separable topology which is not euclidean. It turns out that spaces of this type are complete iff the convergence in them is the euclidean one. Note that both complete and non-complete spaces of this type were constructed ([61], [62]).

It is possible by means of coverings to give examples of functions having peculiar properties. In this connection, the sorting of coverings is of interest. For example, let us take functions increasing on $0\Delta 1$ and mapping $0\Delta 1$ onto $0\Delta 1$. Any such function transforms, in a natural way, any real covering to a real covering. Two coverings are called equivalent if they can be transformed to each other in this way.

Lemma 3.1.

1) Any equivalence class of real coverings contains a regular covering.
2) There exists an equivalence class of real coverings containing only regular real coverings.

Example 3.1. There exist a function g increasing on $0\Delta 1$, $g(0) = 0 \ \& \ g(1) = 1$, and a singular covering Φ and a regular covering Ψ such that $g = [g,\Psi]$ and g transforms any $\Psi(n)$ onto $\Phi(n)$. From Theorem 2.1 and Lemma 2.3 it follows that $AC(g^{-1})$, $\alpha(g)$,

$\neg\mathcal{Q}_{c\ell}(g)$ and, hence, $\neg AC(g)$.

Definition. A real covering Φ is called hereditarily regular if any equivalent real covering is regular.

Let us note that hereditarily regular coverings are used in constructive theory of non-absolutely convergent integrals (see [48]).

Studying the pseudo-differentiability it is useful to divide PNs into two classes according to existence of a certain weak algorithmical regulator of fundamentality (see [43]).

Definition. A PN ξ is said to be

a) a PN of the first class (^{1}PN) if there exists a sequence of non-infinite recursive sets of NNs $\{C_m\}_m$ such that for any NNs m,q the Lebesque measure of the set $\{x : \neg\neg\exists n(0 \leqslant n \leqslant q \;\&\; n \notin C_m \;\&\; \min(\xi(n),\xi(n+1)) \leqslant x \leqslant \max(\xi(n), \xi(n+1)))\}$ is less than 2^{-m};

b) a PN of the second class (^{2}PN) if ξ is not a ^{1}PN;

c) the set of all ^{1}PNs (resp. ^{2}PNs) is denoted by $^1\pi$ (resp. $^2\pi$).

As for the arithmetical complexity we have the following.

Theorem 3.1. ([64]) The set of all ^{1}PNs is Π_3-complete (and, hence, recursively isomorphic to the set of all PNs).

The existence of "weak algorithmical regulator of fundamentality" for ^{1}PNs causes the existence of coverings which, moreover, "weakly cover", i.e. pseudo-cover, $^1\pi$.

Theorem 3.2. For any NN t there exists a sequence of rational segments $\{K_s^t\}_s$ such that

1) $\forall p(\sum_{s=0}^{p} |K_s^t| < 2^{-t})$,

2) $\forall x \exists s(E_\ell(K_s^t) < x < E_r(K_s^t))$,

3) for any ^{1}PN ξ there exists a non-infinite r.e. set of NN s C such that the segments K_s^t, $s \in C$, are non-overlapping and $\neg\neg\exists s(s \in C \;\&\; \xi \in K_s^t)$;

4) if $\{D_m\}_m$ is a sequence of r.e. sets (resp. non-infinite r.e. sets) of rational segments such that for any NN m the sum of the

lengths of an arbitrary finite set of segments from D_n is less than 2^{-m}, then there exist a NN m_0 and a r.e. set (resp. non-infinite r.e. set) of NNs C such that for any NN p, $p \in C$, there exists a segment H from D_{m_0} for which $K_p^t \subseteq H$ holds, and any segment H from D_{m_0} is covered by a finite number of segments K_p^t, $p \in C$.

Corollary 1. If ξ is a PN then $\xi \in {}^1\pi$ iff $\forall t \neg\neg\exists s(\xi \in K_s^t)$.

Corollary 2. Almost every PN is a ^{2}PN.

Definition. A covering Φ is called a ${}^1\pi$-covering if $\forall \xi(\xi \in {}^1\pi \ \& \ \xi \in 0\Delta 1 \supset \neg\neg\exists k(\xi \in \Phi(k)))$.
From Theorem 3.2 we obtain the following.

Theorem 3.3. For any NN t there exists a ${}^1\pi$-covering Φ^t such that $\forall m(\sum_{k=0}^{m} |\Phi^t(k)| < 2^{-t}) \ \& \ \forall s \exists k(K_s^t \cap 0\Delta 1 \subseteq \bigcup_{p=0}^{k} \Phi^t(p))$.

Lemma 3.2.

1) There is no regular ${}^1\pi$-covering.
2) For any ${}^1\pi$-covering Φ the series $\sum_k |\Phi(k)|$ pseudo-converges to a ^{2}PN and, consequently, there exists a ^{2}PN ξ_0 such that ξ_0 is a non-decreasing sequence of RtNs.

Remark 3.1. There exist singular coverings which are not ${}^1\pi$-coverings.

It turns out that any equality class of PNs containing a ^{1}PN-consists entirely of ^{1}PNs. On the other hand, ^{1}PNs are not closed under the basic arithmetical operations.

Theorem 3.4.

1) For any PN ξ, $\xi \in {}^1\pi$ iff $\neg\neg\exists\eta(\eta \in {}^1\pi \ \& \ \eta = \xi)$.
2) For any PN η there exist ^{1}PNs ξ_1, ξ_2 such that $\eta = \xi_1 + \xi_2$.

Theorem 3.5. Let ξ be a PN, $\{Q_k\}_k$ a sequence of segments such that $\neg\neg\forall m \exists p(\sum_{k=0}^{p} |Q_k| < m) \ \& \ \forall p \neg\neg\exists q(p < q \ \& \ \xi \in Q_q)$.
Then $\xi \in \pi^1$.

Corollary. Let $\{\mathcal{L}^k\}_k$ be a sequence of S_σ-sets, ξ a ^{2}PN, and for any NN k let the measure of $\mathcal{L}^k$ be less than 2^{-k}. Then $\neg\neg\exists k(\xi \notin \mathcal{L}^k)$.

Obviously, no (real) covering can pseudo-cover all PNs from $0\Delta 1$. Nevertheless, PNs can be "covered" in a more general way (see [8] and Theorem 3 from [43]).

Theorem 3 from [43] implies the existence of a sequence $\{\mathfrak{N}_n\}_n$ of non-infinite r.e. sets of non-overlapping rational segments, contained in 0 1, such that

$$\bigcup_{J\in\mathfrak{N}_{p+1}} J \subseteq \bigcup_{J\in\mathfrak{N}_p} J, \quad \sum_{J\in\mathfrak{N}_p} |J| \geq \frac{1}{2}, \quad \forall x\exists p(x \notin \bigcup_{J\in\mathfrak{N}_p} J) \text{ and, for any}$$

PN ξ, $\neg\neg\exists p(\xi \notin \bigcup_{J\in\mathfrak{N}_p} J)$.

Remark 3.2. The behaviour of functions is "reasonable" in a neighbourhood of any ^{2}PN (see §6). Therefore, the $^1\pi$-coverings are of a special interest in constructive function theory.

§4. Lebesgue measurable and integrable objects.

We introduce the sets S and L_1 as constructive analogues of the classes of Lebesgue measurable a.e. finite on [0,1] functions and Lebesgue integrable on [0,1] functions. The members of these sets are indices (codes) of sequences of "step functions" having the corresponding properties. On this basis we introduce also the concept of Lebesgue measurability of sets of CRNs.

The detailed study of the subject, including the n-dimensional case, can be found in [20] (see also [18], [21] - [23], [27], [29], [30], [32], [36] and [40]).

By rational step frames (s-frames) we mean the words of the type $a_0\gamma a_1\ldots\gamma a_n\delta b_1\gamma b_2\ldots b_n$, where $n \geq 1$, $\forall i(0 \leq i \leq n \supset a_i \in Q)$ & $0 = a_0 < a_1\ldots < a_n = 1$ & $\forall i(1 \leq i \leq n \supset b_i \in Q)$.

There exists an algorithm $\mathcal{E}$ such that for any s-frame R the algorithm $\tilde{\mathcal{E}}_R$ is a polygonal function which is an "indefinite integral of R on $0\Delta 1$". For any CRNs $x_0, x_1, x_0 \leq x_1$, by

${}^0\!\int_{x_0}^{x_1} R$ we denote $(\tilde{\mathcal{E}}_R(x_1) - \tilde{\mathcal{E}}_R(x_0))$.

The operations absolute value, addition, subtraction and multiplication (denoted by $||_o$, $\underset{o}{+}$, $\underset{o}{-}$, $\underset{o}{\cdot}$) over s-frames can be defined in a natural way. Further, operation over s-frames corresponding to the CFRV $\frac{x}{1+|x|}$ we denote by ω_o.

Let us note that for any s-frame R we have $V[\widetilde{\mathcal{E}}_R] = \widetilde{\mathcal{E}}_{|R|_o}$.

By T we denote the set of all words of the form βm, where m is a NN and $[\![\langle m\rangle]\!]$ is a sequence of s-frames.

Let R be an s-frame, $R \rightleftharpoons a_0\gamma a_1\ldots\gamma a_n\delta b_1\gamma b_2\ldots\gamma b_n$, let $\beta m \in T$ and x,y CRNs, and let U (resp. V) be a word being a CRN or a PN. Then

1) by $P_o(V,R,U)$ we denote $(\neg\exists i(0 \leqslant i \leqslant n \,\&\, U = a_i) \,\&\, ((U < 0 \vee 1 < U) \supset V = 0) \,\&\, \forall i(1 \leqslant i \leqslant n \,\&\, a_{i-1} < U < a_i \supset V = b_i))$;

2) by $P(y,\beta m,x)$ we mean that there exists a sequence of CRNs $\{y_n\}_n$ such that $\forall n P_o(y_n,[\![\langle m\rangle]\!](n),x)$ and $y_n \xrightarrow[n\to\infty]{} y$;

3) by $P_{c\ell}(V,\beta m,U)$ we mean that there exists a sequence of PNs $\{\xi_n\}_n$ such that $\forall n P_o(\xi_n,[\![\langle m\rangle]\!](n),U)$ and $\{\xi_n\}_n$ pseudo-converges to V.

By S and L_1 we denote the sets of elements of T such that for any $\beta m \in T$

$$\beta m \in S \equiv \forall n({}^o\!\!\int_0^1 \omega_o([\![\langle m\rangle]\!](n) \underset{o}{-} [\![\langle m\rangle]\!](n+1)) < 2^{-n}) \text{ and}$$

$$\beta m \in L_1 \equiv \forall n({}^o\!\!\int_0^1 |[\![\langle m\rangle]\!](n) \underset{o}{-} [\![\langle m\rangle]\!](n+1)|_o < 2^{-n}).$$

Let us note that $L_1 \subseteq S$. There exists an algorithm $\mathcal{V}$ such that $\forall v(!\mathcal{V}(v) \,\&\, \mathcal{V}(v) \in L_1 \,\&\, \forall x(x \in 0\nabla 1 \supset P(v,\mathcal{V}(v),x)))$.

Algorithmical operations absolute value, addition, subtraction and multiplication, denoted by $||$, $+$, $-$, $\cdot$, and ω over elements of T are introduced as natural extensions of the corresponding operations over s-frames to sequences of s-frames (cf. [27]). Let us note that

1) for any $\beta m \in S$, $\beta p \in S$ $|\beta m|$, $\beta m+\beta p$, $\beta m-\beta p$ (2)
and $\beta m.\beta p$ are elements of S and $\omega(\beta m) \in L_1$,

2) for any $\beta m \in L_1$, $\beta p \in L_1$ and any CRN v, (2) and $\mathcal{V}(v)\cdot\beta m$ are elements of L_1.

Let $\beta m \in T$, $\beta p \in T$ and let v be a CRN. Then

1) $\beta m = 0$ (resp. $0 \leqslant \beta m$) means that $P_{c\ell}(0,\beta m,x)$ (resp. $\exists y(P_{c\ell}(y,\beta m,x) \,\&\, 0 \leqslant y)$) holds for a.e. CRN x (from $0\Delta 1$);

2) we write $\beta m = \beta p$ (resp. $\beta m \leqslant \beta p$) for $\beta p-\beta m = 0$ (resp. $0 \leqslant \beta p-\beta m$);

3) we write $\beta m - v$ (resp. $v.\beta m$) for $\beta m - (v)$ (resp. $(v).\beta m$).

Theorem 4.1. Let $\beta m \in S$. Then there exist a sequence of $S\sigma$ -sets $\{\mathscr{E}^n\}_n$ and a sequence of uniformly continuous functions $\{\varphi_n\}_n$ such that for any NN n the measure of $\mathscr{E}^n$ is less than 2^{-n}, $\mathscr{E}^{n+1} \subseteq \mathscr{E}^n$ and $\forall x(\neg(x \in \mathscr{E}^n) \supset P(\varphi_n(x), \beta m, x)$ & $\forall \xi \ (\neg(\xi \in \mathscr{E}^n) \supset P_{c\ell}(Op[\varphi_n](\xi), \beta m, \xi))$.

Definitions. 1) A function F is said to be Lebesgue measurable (resp. Lebesgue integrable) on $0\Delta 1$, if there exists a $\beta m \in S$ (resp. a $\beta m \in L_1$) such that $P(F(x), \beta m, x)$ holds for a.e. CRN x from $0\Delta 1$.

2) An object $\beta m \in S$ is called summable if there exists a $\beta p \in L_1$ such that $\beta m = \beta p$.

Theorem 4.2. Let $\beta m \in S$ and $\beta p \in L_1$. Then

1) if $|\beta m| \leqslant \beta p$, then βm is summable and, consequently,

2) βm is summable iff $|\beta m|$ is summable.

Lemma 4.1. Any function of weakly bounded variation on $0\Delta 1$ is Lebesgue integrable on $0\Delta 1$.

There exist algorithms $\int$, $\| \ \|_{L_1}$ and ρ_S such that for any NNs m_0 and m_1 and any CRNs x_0 and x_1, $x_0 \leqslant x_1$,

1) if $\beta m_0 \in L_1$, then

a) $\int$ is applicable to the word $x_0 \square x_1 \square m_0$ and transforms it into a CRN being the limit of the fundamental sequence of CRNs

$$0\int_{x_0}^{x_1} [\![< m_0 >]\!] (n) \ {}_n \quad \text{(we write } \int_{x_0}^{x_1} m_0 \text{ for } \int(x_0 \square x_1 \square \beta m_0) \text{) ;}$$

b) $\| \beta m_0 \|_{L_1} \simeq \int_0^1 | \beta m_0 |$;

2) if $\beta m_0 \in S$ and $\beta m_1 \in S$, then

$$\rho_S(\beta m_0 \boxplus \beta m_1) \simeq \int_0^1 \omega(\beta m_0 - \beta m_1).$$

Theorem 4.3. 1) A function F is absolutely continuous on $0\Delta 1$ iff there exists a $\beta m \in L_1$ such that

$$\forall x \ (0 \leqslant x \leqslant 1 \supset F(x) - F(0) = \int_0^x \beta m \ ). \tag{3}$$

2) Let F be a function and let $\beta m \in L_1$ such that (3) holds, then

$$\forall xy \ (0 \leqslant x < y \leqslant 1 \supset \mathrm{Var}(\int_x^y |\beta m|, F, x\Delta y) \ ).$$

Theorem 4.4. ([36]) Let $\beta m \in L_1$. Then there exist a sequence of $S\sigma$-sets $\{ \ ^n\}_n$, a sequence of uniformly continuous functions $\{\varphi\}$ and an increasing sequence of NNs $\{p_k\}_k$ such that for any NN n the measure of $\ ^n$ is less than 2^{-n} and

$$\ ^{n+1} \ \ ^n \ \& \ \forall x \ (\neg(x \in \ ^n) \supset P(\varphi_n(x), \beta m, x) \ \& \ \forall kab \ (n \leqslant k \ \&$$

$$a \leqslant x \leqslant b \ \& \ 0 < b - a < 2^{-p_k} \supset \int_a^b |\beta m - \varphi_n(x)| < 2^{-k}.|a\Delta b| \ ) \ );$$

consequently, a.e. CRN from $0\Delta 1$ is a Lebesgue point.

Corollary 1. Let F be a function and $\beta m \in L_1$ such that (3). Then $\exists y \ (P(y, \beta m, x) \ \& \ D(y, F, x))$ holds for a.e. CRN x.

Corollary 2. Let F be a function and let $\beta m \in L_1$ such that (3.) Let H be a segment. Then

1) F is constant on H iff for a.e. CRN x from H holds $P(0, \beta m, x)$;
2) F is non-decreasing on H iff for a.e. CRN x from H $\exists y \ (P(y, \beta m, x) \ \& \ 0 \leqslant y)$ holds.

The following theorem can be proved on the basis of preceding results.

Theorem 4.5. ([27]) 1) $\| \ \|_{L_1}$ is a norm for the set L_1. $(L_1, \| \ \|_{L_1})$ is a complete separable normed linear space.

2) ρ_S is a metric for the set S. (S, ρ_S) is a complete separable linear metric space.

Definition. Let $\{F_n\}_n$ be a sequence of CFRVs, $\{\beta m_n\}_n$ a sequence of elements of the set T, and let $\beta m \in T$.

1) $\{F_n\}_n$ (resp. $\{\beta m_n\}_n$) is said to be almost uniformly fundamental if for any NN k there exist an S_σ-set $\mathcal{S}^k$ with measure less than 2^{-k} and a sequence of NNs $\{p_t\}_t$ such that for any CRN

x, $x \in 0\Delta 1$ & $\neg$ $(x \in \mathcal{S}^k)$, $\forall n$ $(!F_n(x))$ holds (resp. there exists a sequence of CRNs $\{z_n\}_n$ such that $\forall n$ $P(z_n, \beta m_n, x)$ holds) and $\{p_t\}_t$ is a regulator of fundamentality of the sequence of CRNs $\{F_n(x)\}_n$ (resp. $\{z_n\}_n$).

2) " $\{F_n\}_n$ (resp. $\{\beta m_n\}_n$) almost uniformly converges to βm" can be defined in an analogical way.

Theorem 4.6. ([40]) Let $\{\beta m_n\}_n$ be a sequence of elements of the set S. Then

1) if the series $\sum_n \rho_S(\beta m_n \boxplus \beta m_{n+1})$ converges, then the sequence $\{\beta m_n\}_n$ is almost uniformly fundamental,

2) if the sequence $\{\beta m_n\}_n$ is almost uniformly fundamental, then

  a) it is fundamental in the space (S, ρ_S) and

  b) for any $\beta m \in S$, $\{\beta m_n\}_n$ almost uniformly converges to βm iff $\rho_S(\beta m_n \boxplus \beta m) \xrightarrow[n\to\infty]{} 0$.

Corollary. Let $\{F_n\}_n$ be an almost uniformly fundamental sequence of uniformly continuous functions. Then there exists a $\beta m \in S$ such that $\{F_n\}_n$ almost uniformly converges to βm.

The following analogue of Levi's theorem can be proved with help of Theorems 4.5 and 4.6.

Theorem 4.7. Let $\{\beta m_n\}_n$ be a sequence of elements of the set L_1 such that the series $\sum_n \int_0^1 |\beta m_n|$ converges. Then the sequence $\{\sum_{p=0}^{a} \beta m_p\}_q$ is both fundamental in $(L_1, \| \|_{L_1})$ and almost uniformly fundamental and, consequently, there exists a $\beta m \in L_1$ such that $\int_0^1 |\sum_{p=0}^{a} \beta m_p - \beta m| \xrightarrow[q\to\infty]{} 0$ and the sequence $\{\sum_{p=0}^{a} \beta m_p\}q$ almost uniformly converges to βm.

Example 4.1. There are a function F and a π-operator $\mathfrak{A}$ such that $\forall x \xi (\xi = x \supset \mathfrak{A}(\xi) = F(x)$ & $|F(x)| \leqslant 1)$ and F is not Lebesgue measurable. Consequently, we can construct a sequence of polygonal functions $\{F_n\}_n$ such that $\forall n$ $(|F_n| \leqslant 1)$ and for any CRN x and any PN ξ the sequence of CRNs $\{F_n(x)\}_n$ converges to F(x) and the sequence of PNs $\{Op[F_n](\xi)\}_n$ ϕ'-converges to $\mathcal{O}\!\mathcal{L}(\xi)$. Thus, this type of convergence is weaker than both almost uniform convergence in (S,ρ_S).

Definitions. 1) A set of CRNs $\mathcal{N}$ will be called regular if $\forall xy\ (x\in\mathcal{N} \supset x\in 0\Delta 1\ \&\ (x = y \supset y\in\mathcal{N}))$.
2) A regular set of CRNs $\mathcal{N}$ will be termed Lebesgue measurable and a CRN z will be called the measure of this set (in symbols, $\mathcal{M}(z,\mathcal{N})$) if there exists a $\beta m\in L_1$ such that $(P(0,\beta m,x) \vee P(1,\beta m,x))\ \&\ (x\in\mathcal{N} \equiv P(1,\beta m,x))$ holds for a.e. CRN x and $z = \int_0^1 \beta m$.

Remark 4.1. 1) For any Lebesgue measurable regular sets of CRNs $\mathcal{M}_1$ and $\mathcal{M}_2$ and for any segment H the following sets

$\mathcal{M}_1 \cup \mathcal{M}_2$, $\mathcal{M}_1 \cap \mathcal{M}_2$, $\mathcal{M}_1 \setminus \mathcal{M}_2$ and $\mathcal{M}_1 \cap H$, where $\mathcal{M}_1 \cap H \rightleftharpoons \{x : x\in \mathcal{M}_1\ \&\ x\in H\}$, are Lebesgue measurable regular sets of CRNs.
2) For any Lebesgue measurable regular set of CRNs $\mathcal{N}$ there exists an absolutely continuous on $0\Delta 1$ function F such that $\forall xy\ (x < y \supset 0 \leqslant \Delta(F,x\Delta y) \leqslant |x\Delta y|\ \&\ \mathcal{M}(\Delta(F,x\Delta y), \mathcal{N} \cap x\Delta y))$ holds and $(D(0,F,x) \vee D(1,F,x))\ \&\ (D(1,F,x) \equiv x\in\mathcal{N})\ \&\ (D(0,F,x) \equiv \neg(x\in\mathcal{N}))$ holds for a.e. CRN x.
Any CRN x such that D(1.F.x) (resp.D(0,F,x)) is said to be a point of density (resp. dispersion) for the set $\mathcal{N}$.
3) For any S_σ-set $\mathcal{S}$, $\mathcal{S} \subseteq 0\Delta 1$, and any CRN z, which is the measure of $\mathcal{S}$ (in the sense of the definition from §2), the set $\{x : x\in\mathcal{S}\}$ is a Lebesgue measurable regular set of CRNs and $\mathcal{M}(z,\{x : x\in\mathcal{S}\})$ holds.

As a corollary of Levi's theorem we have :

Theorem 4.8. ([29]) Let $\{\mathcal{N}_n\}_n$ be a sequence of Lebesgue measurable regular sets of CRNs and let $\{z_n\}_n$ be a sequence of CRNs such that $\forall n\ \mathcal{M}(z_n, (\mathcal{N}_n \setminus \mathcal{N}_{n\dot{+}1}) \cup (\mathcal{N}_{n+1} \setminus \mathcal{N}_n))$ holds and the series $\sum_n z_n$ converges. Then the set $\{x : \neg\neg\exists n \forall k (n < k \supset x\in\mathcal{N}_k)\}$ is a Lebesgue measurable regular set of CRNs.

The following statement reflects peculiar properties of singular coverings.

<u>Theorem 4.9.</u> ([49]) Let Φ be a covering, let $\mathfrak{N}$ be a Lebesgue measurable regular set of CRNs, let z be the measure of $\mathfrak{N}$ and let $\forall k\ (\sum_{i=0}^{k} |\Phi(i)| < 1 - z)$ hold. Then $\forall p \exists q y\ (p < q\ \&$ $\mathfrak{M}(y, \mathfrak{N} \cap \Phi(q))\ \&\ y < 2^{-p} . |\Phi(q)|)$ holds.

<u>Example 4.2.</u> Let Φ be a singular covering. Then
a) the regular set of CRNs $\mathfrak{N}$, $\mathfrak{N} \rightleftharpoons \{x : \exists n\ (E_\ell (\Phi(n)) < x < \frac{1}{2} . (E_\ell (\Phi(n)) + E_r(\Phi(n))))\}$, is (in the effective sense) an open set which is not Lebesgue measurable ;
b) there exists a uniformly continuous function F such that $\forall x\ (0 < F(x) \equiv x \in \mathfrak{N})$ holds (cf. Theorem 4.11).

Nevertheless, we can prove the following statement.

<u>Theorem 4.10.</u> ([29]) Let $\mathfrak{N}$ be a Lebesgue measurable regular set of CRNs and let z be the measure of this set. Then for any NN m there exists a Lebesgue measurable open regular set of CRNs $\mathfrak{M}$ with measure less than $z + 2^{-m}$ such that $\mathfrak{N} \cap 0\nabla 1 \subseteq \mathfrak{M}$ (thus, the set $\mathfrak{N}$ is "equivalent" to a Lebesgue measurable set of the type G_δ).

A relation between Lebesgue measurability of objects (or functions) and that of regular sets of CRNs is described in the following statements (cf. [40]).

<u>Theorem 4.11.</u> Let $\beta m \in S$. Then there exist a sequence of CRNs $\{y_k\}_k$ and an increasing sequence of NNs $\{p_q\}_q$ such that for any CRN y $, \neg \exists k\ (y = y_k)$,
a) the set

$$\{x : x \in 0\Delta 1\ \&\ \exists z\ (P(z,\beta m,x)\ \&\ z*y)\}, \tag{4}$$

where

$$* \rightleftharpoons <\ \vee\ * \rightleftharpoons \leqslant\ \vee\ * \rightleftharpoons =\ \vee\ * \rightleftharpoons \geqslant\ \vee\ * \rightleftharpoons >, \tag{5}$$

is a Lebesgue measurable regular set of CRNs,
b) if we denote by $w^{*,y}$ (for $*$ satisfying (5)) the measure of the set (4), then $w^{=,y} = 0\ \&\ w^{>,y} = w^{\geqslant,y} = 1 - w^{\leqslant,y} = 1 - w^{<,y}$ $\&\ \forall q\ ((p_q < y \supset w^{>,y} < 2^{-q})\ \&\ (y < -p_q \supset w^{>,y} > 1 - 2^{-q}))$.

Theorem 4.12. Let $\beta m \in T$ and let $*$ be one of the signs $>$ and $\geqslant$ such that

1) $\exists z\, P(z, \beta m, x)$ holds for a.e. CRN x from $0\Delta 1$.

2) there exist an everywhere dense sequence of CRNs $\{v_k\}_k$ and a sequence of CRNs $\{w_k\}_k$ such that

a) for any NN k the set $\{x : x \in 0\Delta 1 \;\&\; \exists z\, (P(z, \beta m, x) \;\&\; z * v_k)\}$ is a Lebesgue measurable regular set of CRNs and w_k is the measure of this set,

b) $\forall p \exists q \forall k\, ((q < v_k \supset w_k < 2^{-p}) \;\&\; (v_k < -q \quad w_k > 1 - 2^{-p}))$ holds.

Then there exists a $\beta t \in S$ such that $\beta t = \beta m$.

Theorem 4.11 allows us to define for S the comcepts of both "convergence in measure" and "fundamentality in measure" in the way quite analogical to the classical one.

Theorem 4.13. Let $\{\beta m_n\}_n$ be a sequence of elements of the set S and let $\beta m \in S$. Then

1) $\{\beta m_n\}_n$ is fundamental in measure iff it is fundamental in (S, ρ_S) ;

2) $\{\beta m_n\}_n$ is convergent in measure to βm iff $\rho_S(\beta m_n \boxplus \beta m) \xrightarrow[n\to\infty]{} 0$ holds.

We can prove the following analogue of Lebesgue's theorem.

Theorem 4.14. Let $\beta q \in L_1$ and let $\{\beta m_n\}_n$ be a sequence of elements of the set S, fundamental in (S, ρ_S), such that $\forall n\, (|\beta m_n| \leqslant \beta q)$. Then there exist a sequence of elements of the set L_1 $\{\beta p_n\}_n$ and a $\beta p \in L_1$ for which

$\forall n\, (\beta m_n = \beta p_n) \;\&\; (\int_0^1 |\beta p_n - p| \xrightarrow[n\to\infty]{} 0)$ holds.

Example 4.3. There exists a sequence of polygonal functions $\{F_n\}_n$ such that $\forall n\, (0 \leqslant F_{n+1} \leqslant F_n \leqslant 1) \;\&\; \forall x \exists n\, (F_n(x) = 0)$ holds and for any NN n the Lebesgue integral of F_n over $0\Delta 1$ is greater than $\frac{1}{2}$.

Let us note that the theorem on integration by parts and the first mean value theorem follow from preceding results. Let us consider the second mean value theorem ([30]).

Definition. A $\beta m \in T$ will be termed non-decreasing, if for any NN n there exists an S_σ-set $\mathcal{S}^n$ with measure less than 2^{-n} such that for any CRNs x and y
$0 \leqslant x < y \leqslant 1$ & $-$ $(x \in \mathcal{S}^n)$ & $-$ $(y \in \mathcal{S}^n) \supset \exists$ vw $(P(v,\beta m,x)$ & $P(w,\beta m,y)$ & $v \leqslant w)$ holds.

Lemma 4.2. Let βm be a non-decreasing element of the set T. Then there exists a $\beta k \in S$ such that $\beta m = \beta k$.

Theorem 4.15. Let $\beta p \in L_1$, let βm be a non-decreasing element of the set T, and let y_0 and y_1 be CRNs such that $y_0 \leqslant \beta m \leqslant y_1$. Then there exists a $\beta q \in L_1$ such that $\beta q = \beta m . \beta p$ and

$$- - \exists x \ (0 \leqslant x \leqslant 1 \ \& \ \int_0^1 \beta q = y_0 . \int_0^1 \beta p + y_1 . \int_x^1 \beta p) . \qquad (6)$$

Let us note that the double negation in (6) cannot be omitted.

§5. Differentiability of functions at CRNs

Example 5.1. There exist functions F_1 and F_2 such that
1) F_1 is an increasing on $0\Delta 1$ function which fulfils the Lipschitz condition on $0\Delta 1$ but which is not differentiable at any CRN from $0\Delta 1$ (cf. [26]) ;
2) $\mathcal{A}(F_2)$ holds but F_2 is not pseudo-differentiable at any CRN (resp. at any ^{1}PN) from $0\Delta 1$ (cf. [44]).

The example above shows that the direct analogues of classical results concerning the differentiability of functions of bounded variation does not hold in constructive mathematical analysis.

In classical mathematics we have : any finite function differentiable almost everywhere on a segment is necessarily almost uniformly differentiable on it and the corresponding derivative is Lebesgue measurable. This fact leads us to the following definition.

Definition. Let F be a function and $\beta p \in S$.
1) By $\mathcal{D}(F)$ we denote the following : there exist a sequence of S_σ-sets $\{\mathcal{S}^n\}_n$ and a sequence of NNs $\{k_m\}_m$ such that for any NN n the measure of $\mathcal{S}^n$ is less than 2^{-n} and
$\forall mxabcd \ (n \leqslant m \ \& \ x \in 0\Delta 1 \ \& \ \neg \ (x \in \mathcal{S}^n) \ \& \ a < x < b \ \& \ c < x < d \ \&$
$\max(|a\Delta b|, |c\Delta d|) < 2^{-k_m} \supset \left| \frac{\Delta(F,a\Delta b)}{|a\Delta b|} - \frac{\Delta(F,c\Delta d)}{|c\Delta d|} \right| < 2^{-m})$, .

2) By $\mathcal{D}(F,\beta p)$ we denote the following : there exist a sequence of S_σ-sets $\{\mathcal{S}^n\}_n$ and a sequence of NNs $\{k_m\}_m$ such that for any NN n the measure of $\mathcal{S}^n$ is less than 2^{-n} and

$$\forall mxab\ (n \leqslant m\ \&\ x \in 0\Delta 1\ \&\ \neg\ (x \in \mathcal{S}^n)\ \&\ a < x < b\ \&\ |a\Delta b| < 2^{-k_m} \supset$$

$$\exists v\ (P(v,\beta p,x)\ \&\ \frac{\Delta(F,a\Delta b)}{|a\Delta b|} - v < 2^{-m}))\text{ holds.}$$

Theorems 4.3 and 4.4 lead immediately to the following theorem.

Theorem 5.1. Let F be an absolutely continuous on $0\Delta 1$ function. Then $\mathcal{D}(F)$ holds and for any $\beta m \in L_1$ such that (3) we have $\mathcal{D}(F,\beta m)$.

Lemma 5.1. Let F be a function such that $\mathcal{D}(F)$. Then
1) F is Lebesgue measurable and there exists a $\beta p \in S$ such that $\mathcal{D}(F,\beta p)$,
2) for any NN n there exist an S_σ-set $\{H_p\}_p$ with measure less than 2^{-n}, a uniformly continuous function φ and a sequence of NNs $\{k_m\}_m$ such that

a) $\{H_p\}_p$ is a sequence of mutually disjoint segments,

$$\forall p\ (H_p \subseteq 0\ 1)\ \&\ E_\ell(H_0) = 0\ \&\ E_r(H_1) = 1\ \&\ \forall a(0 < a < 1 \supset$$

$$\exists p\ (a \in (H_p)^{\circ})),$$

b) for any PN ξ, for which $\xi \in 0\nabla 1\ \&\ \neg\ \exists p\ (\xi \in (H_p)^{\circ})$ holds, we have $\forall mab\ (a \leqslant \xi \leqslant b\ \&\ 0 < b-a < 2^{-k_m} \supset$

$$\left|\frac{\Delta(F,a\Delta b)}{|a\Delta b|} - 0p\,[\varphi](\xi)\right| < 2^{-m})\text{ , and, consequently,}$$

$D\ (0p\,[\varphi](\xi),\ F,\xi)$.

In particular, $\forall x\ (x \in 0\nabla 1\ \&\ \neg\exists p\ (x \in (H_p)^{\circ})\quad D\ (\varphi(x),\ F,x))$ holds, $[F,\ \{H_p\}_p]$ is an absolutely continuous function (on $0\Delta 1$) which fulfils the Lipschitz condition on $0\Delta 1$, and if further the function F is uniformly continuous, the series $\sum_p \langle\omega,F\rangle(H_p)$ converges.

Corollary. Let F be a function such that $\mathcal{D}(F)$ and let ξ be a^2PN. Then $\neg\ \neg\ \exists\eta D(\eta,F,\xi)$ holds.

Theorem 5.2. ([48]) For every uniformly continuous function F we have : $\mathcal{D}(F)$ iff for any NN n there exists an S_σ-set $\{H_p\}_p$ with measure less than 2^{-n} such that $\mathcal{H}$ ($\{H_p\}_p$) holds, $[F,\{H_p\}_p]$ is a function absolutely continuous on 0Δ1, and the series $\sum_p \langle \omega,F\rangle (H_p)$ converges.

Let us note that the class of all functions F such that $\mathcal{D}(F)$ holds is A- and V-closed.

Theorem 5.3. ([36]) Let F be a function of bounded variation on 0Δ1. Then $\mathcal{D}(F) \equiv \alpha(F)$ holds.

Corollary. Let F_1 and F_2 be functions, $\alpha(F_1)$ & $\alpha(F_2)$. Then $\alpha(F_1 + F_2)$ holds iff $(F_1 + F_2)$ is a function of bounded variation on 0Δ1.

Theorem 5.4. Let F be a function of bounded variation on 0Δ1. Then a) we have $\alpha(F) \equiv \alpha(V[F])$ and $\mathcal{A}(F) \equiv \mathcal{A}(V[F])$ and, consequently, $AC(F) \equiv AC(V[F])$,

b) for any $\beta m \in S$ such that $\mathcal{D}(F,\beta m)$ we have $\mathcal{D}(V[F],|\beta m|)$.

The following theorem gives information concerning the connection between almost uniform differentiability and pseudo-differentiability almost everywhere.

Theorem 5.5. Let F be a uniformly continuous function. Then $\mathcal{D}(F)$ holds iff for any NN n there exist an S_σ-set $\{H_p\}_p$ with measure less than 2^{-n} and a uniformly continuous function φ such that $\{H_p\}_p \subset 0\Delta 1$, $\forall x\ (x \in 0\nabla 1\ \&\ \neg\exists p\ (x \in H_p) \supset$

$D_{c\ell}(\varphi(x),[F,\{H_p\}_p],x))$ and the series $\sum_p \langle \omega,F\rangle (H_p)$ converges.

On the basis of this theorem and theorem 2.1 we have the following.

Theorem 5.6. ([32]) A function F is absolutely continuous on 0Δ1 iff $\mathcal{A}(F)$ holds and there exists a $\beta m \in S$ such that for a.e. CRN x from 0Δ1 $\exists y\ (P(y,\beta m,x)\ \&\ D_{c\ell}(y,F,x))$ holds.

Example 5.2. There exist a uniformly continuous function F, a Lebesgue measurable function G and a π-operator $\mathfrak{A}$ such that $\forall x\, D(G(x),F,x)\ \&\ \forall \xi D_{c\ell}(\mathfrak{A}(\xi),F,\xi)\ \&\ \neg\ \mathcal{D}(F)$.

Theorem 5.7. ([58]) For every $\beta m \in S$ there exists a uniformly continuous function F such that $\mathcal{D}(F,\beta m)$.

Definition. A function F will be termed singular (on $0\Delta 1$) , if $\alpha(F)$ and for a.e. CRN x from $0\Delta 1$ $D_{c\ell}(0,F,x)$ holds.

Theorem 5.8. ([35]) Let F be a function and let v be a CRN such that $Var(v,F,0\Delta 1)$. Then F is singular iff the following holds $\forall z\ Var(v + |z|, F - h_z, 0\Delta 1)$.

Theorem 5.9. ([35]) Each of the following two conditions (a) and (b) is necessary and sufficient for a function F to be expressible as the sum of an absolutely continuous function and a singular function.

(a) F is a function of bounded variation on $0\Delta 1$ and there exists a $\beta m \in L_1$ such that $\mathcal{D}(F,\beta m)$ holds.

(b) $\alpha(F)$ holds and the sequence of CRNs

$\{\Delta(V^+[F - h_n] + V^-[F + h_n], 0\Delta 1)\}_n$ converges. (V^+ and V^- denote positive and negative variation, respectively).

Remark 5.1. 1) The function g from Example 3.1 is an increasing on $0\Delta 1$ function which fulfils the condition α but which is not the sum of an absolutely continuous function and a singular function.

2) Every uniformly continuous function F is expressible as $\psi * \varphi$ where φ is an increasing on $0\Delta 1$ singular function and for function ψ $\exists m\ (\beta m \in S\ \&\ \beta m = 0\ \&\ \mathcal{D}(\psi,\beta m))$ holds.

We introduce the following concept as a constructive analogue of the concept of aproximate differentiability almost everywhere which is important for the Denjoy integral.

Definition. ([50]) Let F be a function and let $\beta m \in S$. Then $\mathcal{D}^{ap}(F)$ (resp. $\mathcal{D}^{ap}(F,\beta m)$) means : for any NN n there exists an S_σ-set $\{H_k\}_k$ with measure less than 2^{-n} , $\overline{\mathcal{H}}(\{H_k\}_k)$, and such that $\mathcal{D}(F,[\{H_k\}_k])$ (resp. $\mathcal{D}([F,\{H_k\}_k])$ and $(\neg\exists k\,(x\in H_k) \supset \exists y\,(P(y,\beta m,x)\ \&\ D_{c\ell}(y,[F,\{H_k\}_k],x)))$ holds for a.e. CRN x from $0\Delta 1$).

Theorem 5.10. 1) For any function F and for any $\beta m\in S$ we have

a) $\mathcal{D}(F,\beta m) \supset \mathcal{D}^{ap}(F,\beta m)$;

b) $\mathcal{D}^{ap}(F)$ holds iff there exists a $\beta q\in S$ such that $\mathcal{D}^{ap}(F,\beta q)$ holds.

2) If F is a function of bounded variation on $0\Delta 1$ then $\mathcal{D}(F) \equiv \mathcal{D}^{ap}(F)$ holds.

3) The class of all functions F such that $\mathcal{D}^{ap}(F)$ holds is A- and V-closed.

Example 5.3. There exist a uniformly continuous function F and an S_σ-set $\{H_n\}_n$ with measure less than ½ such that $\mathcal{H}(\{H_n\}_n)$ & $\mathcal{Q}_{c\ell}(F)$ & $\mathcal{D}^{ap}(F)$ & $AC([F,\{H_n\}_n])$ & $\neg\exists x\,(x\in 0\Delta 1\ \&\ \neg\exists n\,(x\in H_n)\ \&\ D_{c\ell}(F,x))$ holds.

Example 5.4. There exist a uniformly continuous function F, a Lebesgue measurable function f and a covering Φ such that

$\forall n\ AC(F^{[\Phi(n)]})\ \&\ \forall x\ D(f(x),F,x)\ \&\ \forall\xi D_{c\ell}(F,\xi)\ \&\ \neg\ \mathcal{D}^{ap}(F)$.

§6. Pseudo-differentiability of functions

By Example 5.1 there exist monotone functions satisfying the Lipschitz condition which are not differentiable at any CRN from $0\Delta 1$. At the same time, the importance of the concept of pseudo-differentiability is indicated by Theorems 5.6 and 5.5.

Example 6.1. ([49]) There exists a non-decreasing function F such that for any Lebesgue measurable regular set of CRNs $\mathcal{R}$ with measure less than 1 there exists a CRN x such that $x\in 0\nabla 1$ & $\neg\,(x\in\mathcal{R})$ & $D_{c\ell}(+\infty,F,x)$ holds.

On the other hand, the following propositions hold.

Lemma 6.1. ([44]) For any function F and for any ^{2}PN ξ we have $\neg\,(D_{c\ell}(-\infty,F,\xi) \vee D_{c\ell}(+\infty,F,\xi))$.

Lemma 6.2. For any non-decreasing function F and for any Lebesgue measurable regular set of CRNs $\mathfrak{N}$ of positive measure we can realize a CRN x such that $x \in 0\Delta 1$ & $x \in \mathfrak{N}$ & $- D_{c\ell}(+\infty, F, x)$ holds.

Thus, for any non-decreasing function F we have :
a) The inner measure of the set $\{x : D_{c\ell}(+\infty, F, x)\}$ is always equal to 0, but the outer measure of this set can be equal to 1 ;
b) $\neg D_{c\ell}(+\infty, F, \xi)$ holds for any ^{2}PN ξ (and, consequently, for a.e. PN ξ).

The described situation led to the following concept.

Definition. A property A of CRNs is said to hold for w-almost every CRN x from $0\Delta 1$, if there exists a non-decreasing function G such that $\forall x\ (x \in 0\nabla 1\ \&\ \neg D_{c\ell}(+\infty, G, x) \supset A(x))$ holds.

The following constructive analogue of Vitali's covering theorem holds.

Theorem 6.1. ([43] , [49]) Let V be a property of rational segments such that for any rational segment $a\Delta b \subseteq 0\Delta 1$ and for any RtN r, $0 < r < |a\Delta b|$, we have $(\exists cd\ (a < c < d < b\ \&\ r < |c\Delta d|\ \&\ V(c\Delta d))\ \vee\ \neg\exists cd\ (a < c < d < b\ \&\ r < |c\Delta d|\ \&\ V(c\Delta d)))$. Then there exist a sequence of rational segments $\{H_n\}_n$ and a non-decreasing function G such that $\bar{\mathcal{K}}(\{H_n\}_n)\ \&\ \forall n\ (H_n \subseteq 0\Delta 1 \supset H_n \subseteq 0\nabla 1\ \&\ V(H_n))\ \&\ \forall ab\ (0 < a < b < 1\ \&\ V(a\Delta b) \supset \exists n\ (\tfrac{1}{2}.|a\Delta b| \leqslant |H_n|\ \&\ a\Delta b \cap H_n \neq \phi))\ \&\ \forall \xi (\xi \in 0\nabla 1\ \&\ \neg D_{c\ell}(+\infty, G, \xi)\ \&\ \forall m\ \neg\neg\exists ab\ (a \leqslant \xi \leqslant b\ \&\ 0 < b - a < 2^{-m}\ \&\ V(a\Delta b)) \supset \neg\neg\exists n\ (\xi \in (H_n)^{o}))$.

It turned out, that many properties of rational segments, important for pseudo-differentiability of uniformly continuous functions, fulfil the presumption of this theorem. This fact and the following lemma enable us to prove Theorems 6.2, 6.4 and 6.5.

Lemma 6.3. ([44]) Let F be a function and let w and z be CRNs such that $w < z\ \&\ \forall abx\ (0 \leqslant a < b \leqslant 1\ \&\ (x = z \vee x = w) \supset \neg\ (x = \frac{\Delta(F, a\Delta b)}{|a\Delta b|}))\ \&\ F(1) - F(0) < z$ holds. Then there exists a sequence of rational segments $\{H_n\}_n$ such that

$\mathfrak{K}$ ($\{H_n\}_n$) & $\forall n$ ($H_n \subseteq 0\Delta 1 \supset w.|H_n| < \Delta(F,H_n)$) &
$\forall xy$ ($0 \leqslant x \leqslant y \leqslant 1 \supset \Delta([F,\{H_n\}_n]$, $x\Delta y) < z.|x\Delta y|$). Consequently, the function ([F, $\{H_n\}_n] - h_z$) is decreasing on $0\Delta 1$ and, thus [F, $\{H_n\}_n$] is a uniformly continuous function of weakly bounded variation on $0\Delta 1$.

Theorem 6.2. ([49]) Let F be a uniformly continuous function. Then there exists a non-decreasing function G such that for any PN $\xi, \xi \in 0\nabla 1$ & $-D_{c\ell}(+\infty,G,\xi)$, we have

$$\neg\neg(\underline{D}_{c\ell}(-\infty, F,\xi) \ \& \ \overline{D}_{c\ell}(+\infty, F,\xi) \vee D_{c\ell}(F,\xi)). \qquad (7)$$

Consequently, (7) holds for any ^{2}PN, and for w-almost every CRN x from $0\Delta 1$ we have $\neg\neg(\underline{D}_{c\ell}(-\infty,F,x) \ \& \ \overline{D}_{c\ell}(+\infty,F,x) \vee \exists\eta D_{c\ell}(\eta,F,x))$.

Corollary. Let F be a function of bounded variation on $0\Delta 1$. Then for w-almost every CTN x from 0 1 we have $\exists\eta D_{c\ell}(\eta,F,x)$.

Theorem 6.3. ([45]) Let F be a uniformly continuous function and let ξ be a PN. Then we have

$(\xi\in{}^1\pi \ \& \ Op[F](\xi)\in{}^2\pi \supset \neg\neg(\underline{D}_{c\ell}(-\infty,F,\xi) \ \& \ \overline{D}_{c\ell}(+\infty,F,\xi) \vee D_{c\ell}(-\infty,F,\xi) \vee D_{c\ell}(+\infty,F,\xi)))$

and

$((\xi\in{}^2\pi \ \& \ Op[F](\xi)\in{}^2\pi) \ \& \ \neg\neg\exists m\forall\eta(\eta\in\pi \ \& \ |\eta-\xi| < 2^{-m} \ \& \ Op[F](\xi) = Op[F](\eta) \supset \xi = \eta) \supset \neg(\underline{D}_{c\ell}(-\infty,F,\xi) \ \& \ \overline{D}_{c\ell}(+\infty,F,\xi)))$.

Corollary. Let F be a uniformly continuous function. Then for any PN ξ, $Op[F](\xi)\in{}^2\pi$, we have

$\neg\neg(\underline{D}_{c\ell}(-\infty,F,\xi) \ \& \ \overline{D}_{c\ell}(+\infty,F,\xi) \vee D_{c\ell}(-\infty,F,\xi) \vee D_{c\ell}(F,\xi) \vee D_{c\ell}(+\infty,F,\xi))$.

The assumption of uniform continuity in Theorem 6.2 cannot be omitted.

Example 6.2. There exists a function F of weakly bounded variation on $0\Delta 1$ such that

1) $\mathcal{a}_{c\ell}(F)$ & $\forall x$ ($\neg\underline{D}_{c\ell}(-\infty,F,x)$ & $\neg\overline{D}_{c\ell}(+\infty,F,x,)$) holds,

2) for any non-decreasing function G there cannot fail to exist a CRN x such that $x \in 0\nabla 1$ & $\neg D_{c\ell}(+\infty, G, x)$ & $D_{c\ell}(F, x)$.

On the other hand, we can prove the following statements.

Theorem 6.4. ([44]) For any function F of quasi-weakly bounded variation on $0\Delta 1$ and for any ^{2}PN ξ we have $D_{c\ell}(F, \xi)$.

It is useful to confront this result with Example 5.1.

Theorem 6.5. ([45]) Let F be a function. Then for a.e. PN ξ we have (7) and $(D_{c\ell}(F, \xi) \supset \exists\eta D_{c\ell}(\eta, F, \xi))$.

Example 6.3. 1) We can realize an increasing on $0\Delta 1$ function F and a ^{2}PN ξ_0 such that $\mathcal{Q}_{c\ell}(F)$ & $\forall x \exists y\, D(y, F, x)$ & $\forall \xi D_{c\ell}(F, \xi)$ & $\neg\exists\eta D_{c\ell}(\eta, F, \xi_0)$ (cf. [44]).

2) There exist a pseudo-uniformly continuous function F and ^{2}PNs ξ_1 and ξ_2 such that $\overline{D}_{c\ell}(+\infty, F, \xi_1)$ & $\neg \underline{D}_{c\ell}(-\infty, F, \xi_1)$ & $\neg \underline{D}_{c\ell}(-\infty, F, \xi_2)$ & $\neg \overline{D}_{c\ell}(+\infty, F, \xi_2)$ & $\neg D_{c\ell}(F, \xi_2)$ (cf. [45]).

Let us note, that for any pseudo-uniformly continuous functions F and G and for any ^{2}PN we have

$D_{c\ell}(F, \xi) \supset (\mathrm{Op}[F](\xi) \in {}^1\pi \equiv D_{c\ell}(0, F, \xi))$ and, consequently,

$D_{c\ell}(F, \xi)$ & $D_{c\ell}(G, \xi)$ & $(\mathrm{Op}[F](\xi) - \mathrm{OP}[G](\xi)) \in {}^1\pi \supset$

$D_{c\ell}(0, F - G, \xi)$.

We conclude this § by giving two sufficient conditions for a function to be non-decreasing.

Notation. For any function F and any CRN x we denote

$$\forall m \neg\neg\exists n \forall ab \left(a < x < b \; \& \; |a\Delta b| < 2^{-n} \supset \frac{\Delta(F, a\Delta b)}{|a\Delta b|} > -2^{-m}\right)$$

by $\underline{D}_{c\ell}[F](x) \geq 0$.

Theorem 6.6. ([51]) Let F be a function and let $\{x_n\}_n$ be a sequence of CRNs such that

1) for any CRN x such that $\neg\exists n\,(x = x_n)$ we have $\neg \underline{D}_{c\ell}(-\infty, F, x)$ (resp. $\neg\neg(D_{c\ell}(+\infty, F, x) \vee D_{c\ell}(-\infty, F, x) \vee D_{c\ell}(F, x))$),

2) $\underline{D}_{c\ell}[F](x) \geq 0$ holds for a.e. CRN x from $0\Delta 1$.

Then F is non-decreasing.

§7. Constructive analogues of the conditions (N) and (T_1)

The condition (N) , introduced by N. Lusin, is very important in the theory of the integral. As it is shown in [55] and in [56] the following condition (N)* is an appropriate constructive analogue of this condition.

Definition. Part I : A function F, $0 \leqslant F \leqslant 1$, is said to fulfil the condition (N)*, if it satisfies the following condition :
(a) for any Lebesgue measurable regular set of CRNs $\mathfrak{N}$ the set $\{y : \neg\neg\exists x\ (x \in \mathfrak{N}\ \&\ F(x) = y)\}$ is Lebesgue measurable.

With the help of Cejtin's theorem ([2]) we can prove the following statement.

Theorem 7.1. A function F, $0 \leqslant F \leqslant 1$, fulfils the condition (N)* iff F is uniformly continuous and there exists a sequence of NNs $\{k_p\}_p$ such that for any NN p and any system of rational segments $\{a_i \Delta b_i\}_{i=0}^{s}$, $\sum_{i=0}^{s} |a_i \Delta b_i| < 2^{-kp}$, the measure of $\bigcup_{i=0}^{s} \langle \mathcal{O},F \rangle (a_i \Delta b_i)$ is less than 2^{-p} .

By this theorem the definition just given can be extended as follows.

Definition. Part II : We say that a function G fulfils the condition (N)* iff G is uniformly continuous and the function F such that $F = \frac{1}{\sigma} \cdot (G - \min(\langle I,G \rangle (0\Delta 1),\ 0))$, where $\sigma = \max(\langle S,G \rangle (0\Delta 1),\ \langle \omega,G \rangle (0\Delta 1),\ 1)$, fulfils the condition (a) from the part I of the definition.

Let us give several results.

Lemma 7.1. Every function F, for which $\mathcal{A}(F)$, fulfils the condition (N)*.

Example 7.1. ([55]). We can realize a function F such that $\mathcal{A}(F)$ & $0 \leq F \leq 1$ and for any CRN y, $0 \leq y \leq 1$, there exists a sequence of mutually non-equal CRNs $\{x_n\}_n$ such that $\forall n(0 < x_n < 1 \ \& \ F(x_n) = y)$.

Lemma 7.2. Let F and G be functions fulfilling the condition $(N)^*$. Then 1) F G fulfils the conditions $(N)^*$,
2) $\forall\xi(\xi \in^1 \pi \supset Op[F](\xi) \in^1 \pi)$ & $(\exists z \ var(z,F,0\Delta 1) \supset \mathcal{A}(F))$ &
$(\neg\neg\exists m \ BVS(m,F,0\Delta 1) \equiv \mathcal{A}_{c\ell}(F))$ holds,
3) iff $\neg\neg\exists\eta(D_{c\ell}(\eta F,x)$ & $0 \leq \eta)$ holds for a.e. CRN x from $0\Delta 1$, then F is non-decreasing.

Theorem 7.2. A function F is absolutely continuous on $0\Delta 1$ iff F fulfils the condition $(N)^*$ and there exists a $\beta m \in L_1$ such that $\exists y(P(y,\beta m,x)$ & $D_{c\ell}(y,F,x))$ holds for a.e. CRN x from $0\Delta 1$.

On the basis of facts, mentioned in [55] and [42], the following conditions turned out to be proper constructive analogues of the Banach condition (T_1).

Definition. We say that a function F fulfils the condition $(T_1)^*$ (resp. $(T_1)^\Delta$) if there exist functions φ and Ψ such that Ψ is a function of bounded variation on $0\Delta 1$ (resp. $\alpha(\psi)$ holds) and $AC(\varphi)$ & $F = \varphi * \psi$ holds.

Remark 7.1. 1) The condition $(T_1)^\Delta$ implies the condition $(T_1)^*$. 2) Any function of bounded variation on $0\Delta 1$ fulfils the condition $(T_1)^*$.

Theorem 7.3. Let a function F fulfil the condition $(T_1)^*$. Then

1) F is uniformly continuous,
2) if $0 \leq F \leq 1$ holds, then there exists a $\beta m \in S$ such that
  a) for a.e. CRN y from $0\Delta 1$ there exists a NN k and an increasing system of CRNs $\{x_j\}_{j=1}^k$ for which $P(k,\beta m,y)$ & $\forall x(F(x) = y \equiv \exists j(1 \leq j \leq k \ \& \ x = x_j))$ holds,
  b) F is a function of bounded variation on $0\Delta 1$ iff βm is summable.

Theorem 7.4. A function F fulfils the condition α iff F is a function of bounded variation on $0\Delta 1$ fulfilling the condition $(T_1)^\Delta$.

From Theorem 7.4 and Theorem 5.3 it follows that the function F_1 from Example 5.1 fulfils the conditions $(T_1)^*$ and $\mathcal{Q}$, but it does not fulfil the condition $(T_1)^{\Delta}$.

Theorem 7.5. Let F be a uniformly continuous function such that $\mathcal{A}(F)$. Then 1) F fulfils the condition $(T_1)^*$ then it fulfils the condition $(T_1)^{\Delta}$, too;
2) F fulfils both the conditions $(N)^*$ and $(T_1)^{\Delta}$ iff $\forall\xi(\xi \in^1 \pi \supset Op[F](\xi) \in^1 \pi)$ holds.

Lemma 7.3. Let F be a function fulfilling both the conditions $(T_1)^*$ (resp. $(T_1)^{\Delta}$) and $(N)^*$ and let G be a function fulfilling the condition $(T_1)^*$ (resp. $(T_1)^{\Delta}$).

Let us note that any uniformly continuous function is the sum of two functions fulfilling the condition $(T_1)^{\Delta}$.

As for differentiability of functions fulfilling the condition $(N)^*$ (resp. $(T_1)^*$), in CRNs, let us remember Example 5.1. On the other hand, for any function F which fulfils the condition $(N)^*$ (resp. $(T_1)^*$) and for any ^{2}PN η there cannot fail to exist a NN p and an increasing system of PNs $\{\xi_j\}_{j=1}^{p}$ such that $\forall\xi(Op[F](\xi) = \eta \equiv \neg\neg\exists j(1 \leqslant j \leqslant p \;\&\; \xi = \xi_j))$ holds. By this fact, Theorems 6.2 and 6.3 and by Lemma 7.2 the following statement holds.

Theorem 7.6. For any uniformly continuous function F and for any PN ξ, $Op[F](\xi) \in^2 \pi$, we have:
1) if F fulfils the condition $(T_1)^*$, then $\neg\neg(D_{c\ell}(-\infty,F,\xi) \vee D_{c\ell}(F,\xi) \vee D_{c\ell}(+\infty,F,\xi))$ holds;
2) if F fulfils the condition $(N)^*$ then $D_{c\ell}(F,\xi)$ holds.

§8. Superpositions of absolutely continuous functions

Theorem 8.1. ([42]). In order that a function be a superposition of two (resp. of n, $2 \leqslant n$) absolutely continuous on $0\Delta 1$ functions, it is necessary and sufficient that the function fulfils both the conditions $(T_1)^*$ and $(N)^*$.

Corollary. A superposition of two absolutely continuous on $0\Delta 1$ functions is absolutely continuous on $0\Delta 1$ iff it is a function of bounded variation on $0\Delta 1$.

Theorem 8.2. ([38]) Let F be a uniformly continuous function such that for any rational segment $a\Delta b$, $a\Delta b \; 0\subseteq 1$, there exist an S_σ-set $\mathcal{E}$ whose measure is less than $|a\Delta b|$ and a uniformly continuous function G for which $\forall x(x\in a\Delta b \;\&\; \neg\,(x\in \mathcal{E}) \supset D(G(x),F,x))$ holds. Then F is the sum of two superpositions of absolutely continuous on $0\Delta 1$ functions.

Theorem 8.3. ([39]) Every uniformly continuous function is the sum of three superpositions of absolutely continuous on $0\Delta 1$ functions.

Example 8.1. There exists an increasing on $0\Delta 1$ function F which fulfils the Lipschitz condition on $0\Delta 1$ and which, consequently, fulfils both the conditions $(N)^*$ and $(T_1)^*$, and such that

a) $\forall x \exists y\; D(y,F,x)$ holds,

b) the function F cannot be expressed as the sum $\varphi_1 * \varphi_2 + \varphi_3 * \varphi_4 + \varphi_5$, where φ_i is absolutely continuous on $0\Delta 1$ for $1 \leqslant i \leqslant 5$.

On the other hand, we can prove the following theorem.

Theorem 8.4. ([41]) A function F fulfils both the conditions $(N)^*$ and $(T_1)^*$ iff there exist an absolutely continuous on $0\Delta 1$ function φ and a function Ψ of bounded variation on $0\Delta 1$ such that $\mathcal{A}(\Psi) \;\&\; F = \varphi * \Psi$ holds.

Example 8.2. There exists a function F such that

a) F fulfils the condition $(N)^*$,

b) for a.e. CRN y there exist a NN k and an increasing system of CRNs $\{x_j\}_{j=1}^{k}$ such that $\forall x(F(x) = y \equiv \exists j\; (1 \leqslant j \leqslant k \;\&\; x = x_j))$,

c) F is not expressible as a superposition of a finite number of functions fulfilling the condition $\mathcal{A}_{c\ell}$ (cf. [41]).

§9. Generalized absolutely continuous functions and functions of generalized bounded variation ([50])

The importance of these concepts in the theory of the Denjoy integral is well-known. As it is shown by Example 3.1, to deal with constructive analogues of these concepts it is necessary to consider not only CRNs but also PNs.

Definition. Let F be a function. Then

a) $W\alpha(F)$ means: F is a uniformly continuous function of quasi-weakly bounded variation on $0\Delta 1$ and $\mathcal{D}(F)$ holds;

b) we write WAC(F) for $W\alpha(F)$ & $\mathcal{A}_{c\ell}(F)$.

From results of §§2, 5 and 7 it follows:

Lemma 9.1.

a) For any function F we have AC(F) (resp. $\alpha(F)$) iff WAC(F) (resp. $W\alpha(F)$) holds and F is a function of bounded variation on $0\Delta 1$.

b) For any function F we have WAC(F) (resp. AC(F)) iff $W\alpha(F)$ (resp. $\alpha(F)$) and

$$\xi(\xi \in {}^1\pi \supset \mathrm{Op}[F](\xi) \in {}^1\pi). \tag{8}$$

c) The class of all functions F for which $W\alpha(F)$ (resp. WAC(F)) holds is both $\bar{A}$-closed and V-closed.

Notation. For any sequence $\{\{H^m_n\}_n\}_m$ of sequences of segments by $J(\{\{H^m_n\}_n\}_m)$ we denote $\forall m \bar{\mathcal{K}}(\{H^m_n\}_n)$ & $\forall \xi(\xi \in 0\Delta 1 \supset \neg\neg\exists m\, \neg\exists n(\xi \in H^m_n))$.

Lemma 9.2. Let $v\Delta w$ be a segment, $\{x_k \nabla y_k\}_k$ a sequence of intervals and $\{\{H^m_n\}_n\}_m$ a sequence of sequences of segments such that $v\Delta w \subseteq 0\Delta 1$ & $\neg\neg\exists\xi(\xi \in v\Delta w$ & $\neg\exists k(\xi \in x_k\nabla y_k))$ & $J(\{\{H^m_n\}_n\}_m)$ holds. Then there cannot fail to exist a rational segment $a\Delta b$, a NN m and a PN η such that $v < a < \eta < b < w$ & $\neg\exists k(\eta \in x_k\nabla y_k)$ & $\forall\xi n(\xi \in a\Delta b$ & $\xi \in (H^m_n)^o \supset \neg\neg\exists k(\xi \in x_k\nabla y_k))$ holds.

Notations. Let F be a uniformly continuous function, $\{\{H^m_n\}_n\}_m$ a sequence of sequences of segments and let Y be a word, for which

$$Y \equiv AC \lor Y \equiv \alpha \lor Y \equiv WAC \lor Y \equiv W\alpha. \tag{9}$$

Then

1) $YG(F,\{\{H^m_n\}_n\}_m)$ means: $J(\{\{H^m_n\}_n\}_m)$ & $\forall m Y(F,\{H^m_n\}_n)$;

2) $YG_*(F,\{\{H^m_n\}_n\}_m)$ (resp. $YG_o(F,\{\{H^m_n\}_n\}_m)$) means:

$YG(F,\{\{H_n^m\}_n\}_m)$, and for every NN m the series $\sum_n \langle \omega, F\rangle (H_n^m)$ converges (resp. pseudo-converges).

Definition. Let Y and Z be words such that (9) and

$$Z \doteq G \vee Z \doteq G_0 \vee Z \doteq G_*. \qquad (10)$$

A function F is said to fulfil the condition YZ (in symbols, $YZ(F)$), if F is a uniformly continuous function and if there exists a sequence of S_σ-sets $\{\{H_n^m\}_n\}_m$ such that $YZ(F,\{\{H_n^m\}_n\}_m)$.

From lemmas 9.1 and 9.2 and results of §§2 and 5 it follows:

Theorem 9.1. Let Y be a word, F a function, $x_0 \Delta y_0$ a segment and $\{K_n\}_n$ an S_σ-set such that (9) and $YG(F)$ & $x_0\Delta y_0 \subseteq 0\Delta 1$ & $\bar{\mathfrak{I}}(\{K_n\}_n)$ & $\neg\neg\exists\xi(0 < \xi < 1$ & $\neg\exists n(\xi \in (K_n)^0))$ hold. Then there cannot fail to exist rational segments $a_0\Delta b_0$ and $a_1\Delta b_1$ and a PN η such that $x_0 < a_0 < b_0 < y_0$ & $0 < a_1 < \eta < b_1 < 1$ & $\neg\exists n(\eta \in (K_n)^0)$ holds and the functions $F^{[a_0\Delta b_0]}$ and $[F,\{K_n\}_n]^{[a_1\Delta b_1]}$ fulfil the condition Y.

Theorem 9.2. Let F be a function and Y a word such that (9) holds. Then we have:

1) $(Y(F) \supset YG_*(F))$ & $(YG_*(F) \supset \quad (F)$ & $YG_0(F))$ & $(YG_0(F) \supset YG(F))$ & $(YG(F) \supset \mathfrak{D}^{ap}(F))$;
2) if $\alpha G_*(F)$ holds, then F fulfils the condition $(T_1)^\circ$;
3) if $WACG(F)$ holds then (8) and $(AC(F) \equiv \exists z\ Var(z,F,0\Delta 1))$ hold;
4) if $WACG(F)$ & $\quad (F)$ holds then F is a superposition of two absolutely continuous on $0\Delta 1$ functions;
5) if $W\alpha G(F)$ holds then F is the sum of two superpositions of absolutely continuous on $0\Delta 1$ functions.

Let us note, that for each of the following conditions ACG_*, $WACG_0$, $W\alpha G_0$, $WACG$ and $W\alpha G$ the class of all functions which fulfil the condition is A-closed, but is not V-closed. The class of all functions F such that $WACG_0(F)$ & $\mathfrak{D}(F)$ holds is both A-closed and V-closed.

Theorem 9.3. Let F be a uniformly continuous functions such that $\mathfrak{D}(F)$. Then we have:

1) a) if $\neg\exists\xi(\underline{D}_{cl}(-\infty,F,\xi)$ & $\bar{D}_{cl}(+\infty,F,\xi))$ (11)

then $\alpha G_0(F)$;

b) if (11) and F fulfils the condition $(T_1)^*$ then $\alpha G_*(F)$;

2) if $\neg\exists\xi(\underline{D}_{cl}(-\infty,F,\xi) \vee \bar{D}_{cl}(+\infty,F,\xi)$ (12)

then $ACG_*(F)$.

Theorem 9.4. Let F be a uniformly continuous functions such that $Д^{ap}(F)$ holds. Then we have:

1) if (11) then $\alpha G(F)$;
2) if (12) the $ACG(F)$.

Example 9.1. There exists a function F such that $ACG(F)$, (12) and $\neg WACG_o(F)$ hold.

Theorem 9.5. Let F be a function such that $\alpha G_*(F)$. Then there exists an increasing on $0\Delta 1$ function φ such that

$$\alpha(\varphi) \ \& \ \varphi(0) = 0 \ \& \ \varphi(1) = 1 \ \& \ AC(\varphi^{-1}) \quad (13)$$

and

$$\neg\exists\xi(\underline{D}_{cl}(-\infty,F*\varphi^{-1},\xi) \vee \bar{D}_{cl}(+\infty,F*\varphi^{-1},\xi)). \quad (14)$$

Theorem 9.6. In order that a function F fulfil the condition ACG_* it is necessary and sufficient that F is a uniformly continuous function for which $Д(F)$ holds and there exists an increasing on $0\Delta 1$ function φ such that (13), $AC(\varphi)$ and (14) hold.

Theorem 9.7. Let F be a uniformly continuous function such that $Д(F)$ holds and let φ be an increasing on $0\Delta 1$ function such that (13) and $\neg\exists\xi(\underline{D}_{cl}(-\infty,F*\varphi^{-1},\xi) \ \& \ \bar{D}_{cl}(+\infty,F*\varphi^{-1},\xi))$ hold. Then we have $W\alpha G_o(F)$.

Theorem 9.8. Let $\beta m \in S$ and let F be a uniformly continuous function such that $0 \leqslant \beta m \ \& \ Д^{ap}(F,\beta m)$ and (8) hold. Then F is a non-decreasing absolutely continuous on $0\Delta 1$ function and, consequently, $Д(F,\beta m)$ holds and βm is summable.

§10. Constructive Denjoy integrals ([52])

We base the study of the Denjoy integrals on their descriptive definition (using results of §9).

Definition. Let $\beta m \in S$.

Definitions. Let $\beta m \in S$.

1) A function F is called

a) indefinite Denjoy integral in the restricted sense ($\mathfrak{D}_*$-integral) of βm on $0\Delta 1$ if $ACG_*(F)$ & $\Pi(F,\beta m)$ holds;

b) indefinite $\mathfrak{D}'$-integral of βm on $0\Delta 1$ if $WACG_o(F)$ & $\Pi(F,\beta m)$ holds;

c) indefinite Denjoy integral in the wide sense ($\mathfrak{D}$-integral) of βm on $0\Delta 1$ if $WACG(F)$ & $\Pi^{ap}(F,\beta m)$ holds.

2) βm is called $\mathfrak{D}_*$-integrable (resp. $\mathfrak{D}'$-integrable, resp. $\mathfrak{D}$-integrable) on $0\Delta 1$ if there exists a function being an indefinite integral of the type of βm on $0\Delta 1$.

Let us note that

1) for any function F and any $\beta m \in S$ if F is an indefinite Lebesgue integral (resp. $\mathfrak{D}_*$-integral, resp. $\mathfrak{D}'$-integral) of βm on $0\Delta 1$ then F is an indefinite $\mathfrak{D}_*$-integral (resp. $\mathfrak{D}'$-integral, resp. $\mathfrak{D}$-integral) of βm on $0\Delta 1$;

2) any summable object (from S) is $\mathfrak{D}_*$-integrable and any $\mathfrak{D}_*$-integrable (resp. $\mathfrak{D}'$-integrable) object is $\mathfrak{D}'$-integrable (resp. $\mathfrak{D}$-integrable).

Examples 5.4 and 9.1 show that there exist a measurable function integrable in the sense of Newton, which is not $\mathfrak{D}$-integrable, and a $\mathfrak{D}$-integrable object not being $\mathfrak{D}'$-integrable.

Remark 10.1. Results of §9 give us immediately for any integral, introduced above, the following statements: theorems of monotonicity and of distributivity of the integral and theorem on integration by parts.

Theorem 10.1. A $\beta m \in S$ is summable iff $|\beta m|$ is $\mathfrak{D}$-integrable.

Theorem 10.2. Let φ be an increasing on $0\Delta 1$ function such that $AC(\varphi)$ & $\varphi(0) = 0$ & $\varphi(1) = 1$ & $AC(\varphi^{-1})$ and let $\beta m \in S$. Then there exists a $\beta p \in S$ such that

1) $\exists vw(P(v,\beta m,\varphi(x))$ & $D(w,\varphi,x)$ & $P(v.w,\beta p,x))$ holds for a.e. CRN x from $0\Delta 1$;

2) a function F is an indefinite Lebesgue integral (resp. $\mathfrak{D}_*$-integral, resp. $\mathfrak{D}'$-integral, resp. $\mathfrak{D}$-integral) of βm on $0\Delta 1$ iff

the function $F*\varphi$ is an indefinite integral of the type of βp on $0\Delta 1$ iff the type of βp on $0\Delta 1$;

3) βm is summable (resp. $\mathfrak{D}_*$-integrable, resp. $\mathfrak{D}'$-integrable, resp. $\mathfrak{D}$-integrable) on $0\Delta 1$ iff so is βp.

Definition. Let $\mathfrak{N}$ be a Lebesgue measurable regular set of CRNs and let $\beta p \in L_1$ such that $(P(0,\beta p,x) \vee P(1,\beta p,x))$ & $(P(1,\beta p,x) \equiv x \in \mathfrak{N})$ holds for a.e. CRN x from $0\Delta 1$. Then the object $\beta m \in S$ is said to be integrable in a given sense on the set $\mathfrak{N}$ is $\beta m.\beta p$ is integrable in the sense on $0\Delta 1$. Further, any indefinite integral of the type of $\beta m.\beta p$ on $0\Delta 1$ is said to be an indefinite integral of the type of βm on the set $\mathfrak{N}$.

Theorem 10.3. Let $\beta m \in S$, let F be a function and let $\{x_n\}_n$ be an increasing sequences of CRNs from $0\nabla 1$ which converges to 1. If for any NN n the function $F^{[0\Delta x_n]}$ is an indefinite $\mathfrak{D}_*$-integral (resp. $\mathfrak{D}'$-integral, resp. $\mathfrak{D}$-integral) of βm on $0\Delta x_n$, then F is an indefinite integral of the type of βm on $0\Delta 1$.

Theorem 10.4. Let $\beta m \in S$ and let $\{H_n\}_n$ be an S_σ-set such that $\mathcal{K}(\{H_n\}_n)$ holds. Let F be an indefinite $\mathfrak{D}_*$-integral of βm on the set $\{x : x \in 0\Delta 1 \;\&\; \neg(x \in \{H_n\}_n)\}$ and let $\{F_n\}_n$ be a sequence of functions such that

1) for any NN n

a) if $\neg(H_n \subseteq 0\Delta 1)$ then $F_n = 0$,

b) if $H_n \subseteq 0\Delta 1$ then F_n is an indefinite $\mathfrak{D}_*$-integral of βm on H_n and $F_n(0) = 0$ holds;

2) the series $\sum_n \langle \omega, F_n\rangle (H_n)$ converges.

Then the object βm is $\mathfrak{D}_*$-integrable on $0\Delta 1$ and the function $(F + \sum_{n=0}^{\infty} F_n)$ is an indefinite $\mathfrak{D}_*$-integral of βm on $0\Delta 1$.

Let us note that quite analogical theorem holds for $\mathfrak{D}'$-integral (resp. $\mathfrak{D}$-integral), too. Further, in the case of $\mathfrak{D}$-integral, if we replace the assumption 2) by

2') the series $\sum_n |\Delta(F_n,H_n)|$ converges and $\langle \omega, F_n\rangle (H_n) \xrightarrow[n\to\infty]{} 0$,

then the theorem remains valid.

§11. The constructive Perron integral ([53])

In classical mathematics the Perron integral is equivalent to the Denjoy integral in the restricted sense ($\mathfrak{D}_*$-integral). As we shall see, in constructive mathematics $\mathfrak{D}_*$-integration includes the Perron integral and Perron's process of integration includes $\mathfrak{D}'$-integral, but the Perron integral is not equivalent either to $\mathfrak{D}_*$-$\mathfrak{D}$-integral or to $\mathfrak{D}'$-integral. Let us note that this fact is connected with the existence of a function which fulfils the condition α but is not expressible as the sum of an absolutely continuous function and a singular function (cf. Remark 5.1).

Definition. Let $\beta m \in S$. A function G is termed

1) major function of βm on $0\Delta 1$ if G is uniformly continuous, $G(0) = 0$ & $\neg\exists\xi \underline{D}_{c1}(-\infty, G, \xi)$ holds and there exists a $\beta p \in S$ such that $\mathcal{D}(G, \beta p)$ & $\beta m \leqslant \beta p$;

2) minor function of βm on $0\Delta 1$ if the function $-G$ is a major function of $-\beta m$ on $0\Delta 1$.

Remark 11.1.

1) By Theorem 9.3 any major function fulfils the condition αG_0. On the other hand, there exists a major function F such that $\neg\alpha G_*(F)$.

2) By results of §§5 and 6 if G_1 and G_2 are major functions of a $\beta m \in S$ and G_3 is a minor function of βm on $0\Delta 1$ then the function $\min(G_1, G_2)$ is a major function of βm on $0\Delta 1$ and the function $(G_1 - G_3)$ is non-decreasing.

Definition. A $\beta m \in S$ is said to be integrable in the sense of Perron ($\mathcal{P}$-integrable) on $0\Delta 1$ if for any NN t there exist a major function G of βm on $0\Delta 1$ and a minor function $\bar{G}$ of βm on $0\Delta 1$ such that $\Delta(G - \bar{G}, 0\Delta 1) < 2^{-t}$ holds.

Let $\beta m \in S$ be $\mathcal{P}$-integrable on $0\Delta 1$. Then, by Remark 11.1 and §5, there exists a uniformly continuous function F_0 being both the l.u.b. of all major functions of βm on $0\Delta 1$ and the g.l.b. of all minor functions of βm on $0\Delta 1$ and, consequently, $F_0(0) = 0$ & $\mathcal{D}(F_0, \beta m)$ holds; a function F is said to be an indefinite Perron integral ($\mathcal{P}$-integral) of βm on $0\Delta 1$, if the function $(F - F_0)$ is constant on $0\Delta 1$.

Theorem 11.1. A function F is an indefinite Perron integral of $\beta m \in S$ on $0\Delta 1$ iff F is uniformly continuous, $\mathfrak{A}(F, \beta m)$ holds and there exists an increasing on $0\Delta 1$ function φ such that $\alpha(\varphi)$ & $\varphi(0) = 0$ & $\varphi(1) = 1$ & $\forall \xi(\neg D_{cl}(F,\xi) \supset D_{cl}(0, F * \varphi^{-1}, 0p[\varphi](\xi)))$.

By Theorem 11.1 and by §9 we have:

1) Any indefinite $\mathcal{P}$-integral is an indefinite $\mathfrak{D}$'-integral; any indefinite $\mathfrak{D}_*$-integral is an indefinite $\mathcal{P}$-integral. Thus, any indefinite $\mathcal{P}$-integral is the superposition of two absolutely continuous functions.

2) The class of all indefinite $\mathcal{P}$-integrals is A-closed, but it is not V-closed.

Remark 11.2. For the Perron integral (the analogues of) Remark 10.1 and Theorems 10.1 - 10.4 hold.

Theorem 11.2. Let $\beta m \in S$ and let F be a uniformly continuous function such that $\mathfrak{A}^{ap}(F, \beta m)$ & $\forall \xi(\xi \in {}^1\pi \supset 0p[F](\xi) \in {}^1\pi)$.
Then

1) F is an indefinite Perron integral of βm on $0\Delta 1$ iff there exists a major function of βm on $0\Delta 1$;

2) F is an indefinite $\mathfrak{D}_*$-integral of βm on $0\Delta 1$ iff there exists a major function of βm on $0\Delta 1$ fulfilling the condition $(N)^*$.

Example 11.1. There exists a $\beta m \in S$, $\mathfrak{D}$'-integrable on $0\Delta 1$, such that βm has no major function on $0\Delta 1$.

Example 11.2. There exists a $\beta m \in S$ which is $\mathcal{P}$-integrable on $0\Delta 1$, but which is not $\mathfrak{D}_*$-integrable on $0\Delta 1$.

Remark 11.3. Let g be function from Example 3.1. Then g is an increasing on $0\Delta 1$ function and there exists a $\beta m \in S$ such that $\mathfrak{A}(g, \beta m)$. Thus, g is a major function of βm on $0\Delta 1$ and h_0 is a minor function of βm on $0\Delta 1$. On the other hand, βm is not $\mathfrak{D}$-integrable on $0\Delta 1$ and, consequently, it is not $\mathcal{P}$-integrable on $0\Delta 1$, too.

§12. $\mathcal{T}$-integral ([48])

A generalization of the Lebesgue integral possessing many of the properties of the Perron integral can be introduced by specifically constructive means.

Definition.

1) A function F is said to fulfil the condition $\mathcal{T}$ (in symbols, $\mathcal{T}(F)$), if F is uniformly continuous and for any increasing absolutely continuous on $0\Delta 1$ function φ, $\varphi(0) = 0$ & $\varphi(1) = 1$, we have Д$(F*\varphi)$.

2) Let $\beta m \in S$. Then
   a) a function F is said to be an indefinite $\mathcal{T}$-integral of βm on $0\Delta 1$, if $\mathcal{T}(F)$ & Д$(F,\beta m)$ holds;
   b) βm is said to be $\mathcal{T}$-integrable on $0\Delta 1$ if there exists an indefinite $\mathcal{T}$-integral of βm on $0\Delta 1$.

Remark 12.1.

1) The class of all functions fulfilling the condition $\mathcal{T}$ is both A-closed and V-closed.

2) For $\mathcal{T}$-integral (the analogues of) Remark 10.1 and Theorems 10.1 - 10.3 hold and a certain analogue of Theorem 10.4 is valid.

Theorem 12.1. Let $\beta m \in S$ and let F be an indefinite $\mathcal{T}$-integral of βm on $0\Delta 1$. Then

1) if $0 \leqslant \beta m$ then F is absolutely continuous on $0\Delta 1$;

2) if βm is $\mathfrak{D}$-integrable on $0\Delta 1$ then F is an indefinite $\mathfrak{D}_*$-integral of βm on $0\Delta 1$;

3) if βm has on $0\Delta 1$ a major function then F is an indefinite $\mathfrak{D}_*$-integral of βm on $0\Delta 1$.

On the other hand, there exists a function F, $\mathcal{T}(F)$, which does not fulfil the condition $(N)^*$ and, consequently, F is not an indefinite $\mathfrak{D}$-integral. Indeed, for any uniformly continuous function G and for any hereditarily regular covering Φ we have $\mathcal{T}([G,\Phi])$. We can also realize a function $\bar{F}$, $ACG_*(\bar{F})$ & $\neg\mathcal{T}(\bar{F})$.

Let us note that the theory of $\mathcal{T}$-integral is also useful for the study of other non-absolutely convergent integrals, dealt by us.

Example 12.1. There exist a function F and a $\beta m \in S$ such that

1) $\mathcal{T}(F)$ & WAC(F) & $\mathcal{J}(F,\beta m)$ holds, and, consequently, βm is both $\mathcal{T}$-integrable and $\mathfrak{D}'$-integrable on $0\Delta 1$,
2) there is no major function of βm on $0\Delta 1$ and thus βm is not $\mathcal{P}$-integrable on $0\Delta 1$.

§13. A constructive analogue of functions of the first Baire's class ([47])

Definition.

1) An algorithm F is said to be an A^{ϕ}-operator if $\forall xy(!F(x) \ \& \ F(x) \in \pi \ \& \ (x = y \supset F(x) = F(y)) \ \& \ F(x) = F(\min(\max(x,0),1)))$ holds.
2) We say that an A^{ϕ}-operator F belongs to the first Baire's class (in symbols, $F \in B_1$), if there exists a sequence of uniformly continuous functions $\{G_n\}_n$ such that for any CRN x (from $0\Delta 1$) the sequence of CRNs $\{G_n(x)\}_n$ pseudo-converges to F(x).

Remark 13.1.

1) For any function G there exists an A^{ϕ}-operator F such that $F \in B_1 \ \& \ \forall x(F(x) = G(x))$.
2) If a function G is finitely pseudo-differentiable at any CRN from $0\nabla 1$ then there exists an A^{ϕ}-operator F such that $F \in B_1$ & $\forall x(x \in 0\nabla 1 \supset D_{cl}(F(x),G,x))$.

Theorem 13.1. Let F be an A^{ϕ}-operator, $F \in B_1$, let p be a NN and $a\Delta b$ a rational segment, $a\Delta b \subseteq 0\Delta 1$. Then there ϕ'-exists a rational segment $c\Delta d$ such that $c\Delta d \subseteq a\Delta b \ \& \ \forall xy(c \leqslant x \leqslant y \leqslant d \supset |F(x) - F(y)| < 2^{-p})$ holds.

Example 13.1.

1) We can realize a function G which fulfils the Lipschitz condition on $0\Delta 1$ and an A^{ϕ}-operator F such that $F \in B_1$ & $\forall x D_{cl}(F(x),G,x)$ holds but F is not pseudo-continuous at any CRN from $0\nabla 1$.
2) There exists an A^{ϕ}-operator F such that $\forall x \neg\neg\exists m(F(x) = m)$ & $\forall xy(0 \leqslant x < y \leqslant 1 \supset \neg(F(x) = F(y)))$ and, consequently, $\neg(F \in B_1)$.

Remark 13.2. Let F and G be A^{ϕ}-operators from B_1. Then A^{ϕ}-

operators $|F|$, $(F+G)$, $(F\cdot G)$ belong to B_1 and, further, if $\neg\exists x(G(x) = 0)$ holds then $(F/G) \in B_1$, too.

We can prove the following analogue of Lebesgue's theorem.

Theorem 13.2. An A^{ϕ}-operator F belongs to the first Baire's class iff for any RtN a the sets $\{x : F(x) \geq a\}$ and $\{x : F(x) \leq a\}$ are sets of the type G_δ (in the effective sense).

Theorem 13.3. An A^{ϕ}-operator F belongs to the first Baire's class iff there exists an A^{ϕ}-operator G such that $\forall x(F(x) = G(x))$ & $\forall xm\neg\neg\exists k\forall py(k \leq p$ & $y = x \supset |\underline{G(x)}(p) - \underline{G(y)}(p)| < 2^{-m})$ holds.

Corollary. Let F be an A^{ϕ}-operator. If there exists a sequence of functions $\{G_n\}_n$ such that for any CRN x (from $0\Delta 1$) the sequence of CRNs $\{G_n(x)\}_n$ pseudo-converges to F(x), then $F \in B_1$.

Let us consider the question of sufficient conditions for belonging an A^{ϕ}-operator to the first Baire's class.

Definition. An A^{ϕ}-operator F is termed strictly ϕ'-continuous if for any NN k there exists a ϕ'-sequence of rational intervals such that $\forall x\neg\neg\exists p(x \in \mathfrak{A}(p))$ & $\forall pxy\ x \in \mathfrak{A}(p)$ & $y \in \mathfrak{A}(p) \supset |F(x) - F(y)| < 2^{-k}$ holds.

Theorem 13.4. Each of the following five conditions is sufficient for belonging an A^{ϕ}-operator F to the first Baire's class:

(a) F is ϕ'-uniformly continuous.

(b) F is strictly ϕ'-continuous.

(c) F is non-decreasing and pseudo-continuous at any CRN from $0\Delta 1$.

(d) F is convex on $0\nabla 1$.

(e) There exists a ϕ'-sequence of CRNs $\mathfrak{R}$ such that for any CRN x, $\neg\exists m(x = \mathfrak{R}(m))$, F is continuous at x.

Example 13.2. There exist A^{ϕ}-operators F_1, F_2 and F_3 not belonging to the first Baire's class and such that

F_1 is ϕ"-uniformly continuous,

F_2 is ϕ'-continuous at any CRN and

F_3 is non-decreasing.

Example 13.3. There exist a CFRV F and an A^{ϕ}-operator G, which does not belong to the first Baire's class, such that $\forall x((!F(x) \supset 0 < x < 1) \,\&\, \neg\neg(G(x) = 0 \vee G(x) = 1) \,\&\, (G(x) = 1 \equiv !F(x))) \,\&\, \forall x \exists q \forall y(|y-x| < 2^{-q} \supset G(y) \leqslant G(x))$ and, consequently, G is upper semi-continuous at any CRN.

On the other hand, we can prove the following statement.

Theorem 13.5. We can realize a sequence of A^{ϕ}-operators $\{G_p\}_p$, belonging to the first Baire's class, such that for any CFRV F there exists a NN p for which $\forall x(0 \leqslant x \leqslant 1 \,\&\, !F(x) \supset G_p(x) = F(x))$ holds.

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THE DIOPHANTINE PROBLEM FOR POLYNOMIAL RINGS OF POSITIVE CHARACTERISTIC

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1. Introduction[1]. The main results of the present paper are Theorem A and Theorem B below.

THEOREM A. Let R be any integral domain. Let p be the characteristic of R. And let $R[T]$ be the ring of polynomials over R in one variable T. Then the diophantine problem for $R[T]$, with coefficients in $\frac{\mathbb{Z}}{p\mathbb{Z}}[T]$, is unsolvable (this terminology is defined in [8, §1]), i.e. there is no algorithm to decide whether or not a polynomial equation (in several variables) with coefficients in $\frac{\mathbb{Z}}{p\mathbb{Z}}[T]$ has a solution in $R[T]$.

We proved this in [8, § 2] for $p = 0$, by using an idea of M. Davis and H. Putnam [4]. However the method of [8] does not work for $p > 0$. In the present paper we prove Theorem A in Section 2 for $p > 2$ and in Section 3 for $p = 2$. R. M. Robinson [14] proved that the elementary theory of $R[T]$ is undecidable.

The present paper is part of a series of papers: In [6], [10] (in collaboration with Lipshitz) and [9] we proved that the diophantine problem is unsolvable for the ring of algebraic integers in a totally real number field or in a quadratic extension of a totally real number field, and we showed that every recursively enumerable relation is diophantine over such a ring of algebraic integers. (For the definition of "diophantine relation" and related terminology, see [8, §1]). In [8, § 3] we proved that the diophantine problem for the field of rational functions over a formally real field is unsolvable. And in [7] we proved that every recursively enumerable relation in $\mathbb{Z}[T]$ is diophantine over $\mathbb{Z}[T]$. Of course, all these results are based on the fact that the diophantine problem for $\mathbb{Z}$ is

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[1] We denote the set of natural numbers by $\mathbb{N}$, and the ring of integers by $\mathbb{Z}$.

unsolvable (see e.g. [3], [5]). The outstanding open problem is whether the diophantine problem for the field of rationals is unsolvable.

Let me now sketch the idea involved in the proof of Theorem A. When R has positive characteristic, it makes no sense to prove that $\mathbb{Z}$ is diophantine over R[T]. So we only can hope to define a model of $\mathbb{Z}$ in R[T]. Using Pell sequences we can define a model of $\mathbb{Z}$ in R[T], which is diophantine over R[T], and for which addition and divisibility are also diophantine over R[T] (see Section 2). However this does not prove that the diophantine problem for R[T] is unsolvable, since A. Bel'tyukov [1] and L. Lipshitz [11] proved that the existential theory of $(\mathbb{Z};+,|)$ is decidable. (The divisibility relation is denoted by $|$, thus $a|b \longleftrightarrow \exists c : b = ac$.) On the other hand, Lipshitz [12], [13] proved that the positive existential theory of $(\mathcal{O};+,|)$ is undecidable if $\mathcal{O}$ is a ring of algebraic integers (of finite degree) whose group of units is infinite and whose diophantine problem is unsolvable. (If the group of units of $\mathcal{O}$ is finite, then the existential theory of $(\mathcal{O};+,|)$ is decidable, see [11].) In Section 4 of the present paper we prove the following theorem, by using the ideas of Lipshitz [13]:

THEOREM B. Let n be a fixed integer, and $n > 1$. Denote divisibility in $\mathbb{Z}[\frac{1}{n}]$ by $|_n$, thus for all $x, y \in \mathbb{Z}$

$$x|_n y \longleftrightarrow \exists q, f \in \mathbb{Z} : y = xqn^{-f}.$$

Then the positive existential theory of $(\mathbb{Z};+,|_n)$ is undecidable, i.e. there is no algorithm to decide formulas of the form

$$\exists x_1, \ldots, x_m \in \mathbb{Z} : \bigwedge_{i=1}^{s} F_i(x_1, \ldots, x_m) |_n G_i(x_1, \ldots, x_m),$$

where F_i and G_i are polynomials over $\mathbb{Z}$ of degree one or less, and where $\bigwedge_{i=1}^{s}$ denotes a finite conjunction.

COROLLARY. Let p be a fixed prime number, $p > 1$. Define the relation $|^p$ by

$$x|^p y \longleftrightarrow \exists f \in \mathbb{N} : y = \pm xp^f .$$

Then the positive existential theory of $(\mathbb{Z};+,|,|^p)$ is undecidable.

Indeed Theorem B implies the corollary since

$$x|_p y \longleftrightarrow \exists z \in \mathbb{Z} : x|z \wedge y|^p z.$$

Thus to prove Theorem A, it is sufficient to show for some p that the interpretation of $|^p$ in our model of $\mathbb{Z}$, is diophantine over $R[T]$. It turns out that this can be done if p is the characteristic of R. The underlying idea is that for all $\ell, s \in \mathbb{N}$

$$(T+1)^s = T^\ell + 1 \longleftrightarrow \exists f \in \mathbb{N} : s = \ell = p^f,$$

when p is the characteristic of R.

2. <u>Polynomial rings of characteristic</u> $p \nmid 0, 2$.

Let R be any integral domain of characteristic $p \nmid 2$.

Let $a \in R[T]$ and $a \notin R$. Set $\alpha = \sqrt{a^2-1}$. Then α is not in the field of fractions of $R[T]$. Indeed it is sufficient to prove this when R is a field, and hence it suffices to show that $\alpha \notin R[T]$. Suppose that $\alpha \in R[T]$. Then $\alpha - a$ and $\alpha + a$ are in R, since $(\alpha-a)(\alpha+a) = -1$. Hence $2a \in R$, which contradicts $a \notin R$. This enables us to make the following definition:

DEFINITION. Let R be an integral domain of characteristic $p \nmid 2$. Let $a \in R[T]$ and $a \notin R$. Let $\alpha(a) = \sqrt{a^2-1}$. We define two sequences $X_m(a)$, $Y_m(a) \in R[T]$, $m \in \mathbb{Z}$ by

$$(1) \qquad X_m(a) + \alpha(a)Y_m(a) = (a+\alpha(a))^m = (a-\alpha(a))^{-m}.$$

LEMMA 2.1. Let R be an integral domain of characteristic $p \nmid 0, 2$. Let $a \in R[T]$, $a \notin R$. For all $m, n \in \mathbb{Z}$ we have:

1) $X_m(a)$ (resp. $Y_m(a)$) is equal to the polynomial obtained by substituting a for T in $X_m(T)$ (resp. $Y_m(T)$).

The degree of the polynomial $X_m(T)$ is m, if $m \geq 0$.
The degree of the polynomial $Y_m(T)$ is $m-1$, if $m > 0$.
$X_{-m}(a) = X_m(a)$ and $Y_{-m}(a) = -Y_m(a)$.

2) All solutions $X, Y \in R[T]$ of the Pell equation

$$X^2 - (a^2-1)Y^2 = 1 \tag{2}$$

are given by

$$X_m(a), Y_m(a) \text{ and } -X_m(a), -Y_m(a), \text{ with } m \in \mathbb{Z}.$$

3) $X_{m+n}(a) = X_m(a)X_n(a) + (a^2-1)Y_m(a)Y_n(a)$,

$Y_{m+n}(a) = X_m(a)Y_n(a) + Y_m(a)X_n(a)$,

4) $n|m \Longleftrightarrow Y_n(a)|Y_m(a)$ (At the left side $|$ denotes divisibility in $\mathbb{Z}$, but at the right side divisibility in $R[T]$.)

5) $X_{mp^n}(a) = (X_m(a))^{p^n}$ and $X_{p^n}(a) = a^{p^n}$, if $n \geq 0$.

6) $X_m(a+1) = X_m(a) + 1 \Longleftrightarrow m = \pm p^n$ for some $n \in \mathbb{N}$.

7) $X_m(a) \equiv 1 \bmod(a-1)$. (Congruence in $R[T]$.)

PROOF. 1) From (1) follows for $m > 0$ that

$$X_m(a) = \sum_{i \text{ even}}^{m} \binom{m}{i} a^{m-i}(a^2-1)^{i/2} \tag{3}$$

$$= \Big(\sum_{i \text{ even}}^{m} \binom{m}{i}\Big) a^m + \text{terms of lower degree in } a$$

$$= 2^{m-1}a^m + (\ldots)a^{m-1} + \ldots$$

The same can be done for $Y_m(a)$. This proves the assertion about the degree.

Applying the automorphism $\alpha(a) \mapsto -\alpha(a)$ to (1), we get

$$X_m(a) - \alpha(a)Y_m(a) = (a-\alpha(a))^m$$

$$= (a+\alpha(a))^{-m}$$

$$= X_{-m}(a) + \alpha(a)Y_{-m}(a)$$

Hence $X_{-m}(a) = X_m(a)$ and $Y_{-m}(a) = -Y_m(a)$.

2) We may suppose that R is an algebraically closed field. To simplify the notation we write α instead of $\alpha(a)$. Let

$$G = \{X + \alpha Y : X, Y \in R[T] \text{ satisfy (2)}\}.$$

Notice that G is a group with respect to multiplication. We consider the field $K = R(T)(\alpha)$, it is a field of algebraic functions in one variable. We use the terminology of [2, Chapter I]. Fix $X + \alpha Y \in G$, with $X, Y \in R[T]$. Let S be the set of places of K which are poles of T. Since K has degree two over $R(T)$, S consists of at most two places (see e.g. [2, Chapter IV thm. 1 p. 52]). Let P be a place of K not in S. Obviously P is not a pole of X, Y or α. Thus P is not a pole of $X + \alpha Y$ or $X - \alpha Y$. Since $(X+\alpha Y)(X-\alpha Y) = 1$, we see that P is not a zero of $X + \alpha Y$. Thus all the zeroes and poles of $X + \alpha Y$ are contained in S. Now every element of K which is not in R has at least one zero and one pole. Thus S consists of at least two places, since $a + \alpha \in G$ and $a + \alpha \notin R$. Hence $S = \{L_1, L_2\}$, where L_1 and L_2 are two different places of K. Thus the divisor of $X + \alpha Y$ equals $qL_1 - qL_2$, for some $q \in \mathbb{Z}$. This gives us a group homomorphism $G \to \mathbb{Z} : X + \alpha Y \mapsto q$. Its kernel is $\{\pm 1\}$. Indeed if $X + \alpha Y$ is in the kernel, then $X + \alpha Y \in R$, hence $Y = 0$ and $X = \pm 1$. Thus $G/\{\pm 1\}$ is an infinite cyclic group.

Let $X + \alpha Y$, with $X, Y \in R[T]$, be a generator for $G/\{\pm 1\}$. We have $a + \alpha = \pm (X+\alpha Y)^e$, for some $e \in \mathbb{Z}$. For a good choice of the generator, we can suppose that $e \in \mathbb{N}$. Hence $a + \alpha = \pm (X+\alpha Y)^e = \pm \sum_{i=0}^{e} \binom{e}{i} X^{e-i} \alpha^i Y^i = (\ldots) + (\ldots)Y\alpha$. Thus Y divides 1 in the ring $R[T]$, hence $Y \in R$. From (2) follows

$$(X-aY)(X+aY) = 1 - Y^2 \in R.$$

If $Y \neq \pm 1$, then $X - aY \in R$ and $X + aY \in R$. Hence $2aY \in R$, and $Y = 0$ since $a \notin R$. But this is in contradiction with $Y|1$. Hence $Y = \pm 1$ and $X = \pm a$. This means that $\pm a \pm \alpha$ is a generator for $G/\{\pm 1\}$. Thus $a + \alpha$ is also a generator for $G/\{\pm 1\}$. This proves 2).

3) From (1) follows

$$X_{m+n}(a) + \alpha(a)Y_{m+n}(a) = (X_m(a)+\alpha(a)Y_m(a)) \cdot (X_n(a)+\alpha(a)Y_n(a)),$$

which proves 3).

4) It is sufficient to prove this for $n, m \geq 0$. First, suppose that $m = nq$, with $q \in \mathbb{N}$. From (1) follows

$$X_{nq}(a) + \alpha(a)Y_{nq}(a) = (X_n(a)+\alpha(a)Y_n(a))^q.$$

Hence

$$Y_{nq}(a) = \sum_{i \text{ odd}}^{q} \binom{q}{i}(X_n(a))^{q-i}(a^2-1)^{\frac{i-1}{2}} (Y_n(a))^i$$

Thus $Y_n(a) | Y_{nq}(a)$.

Conversely, suppose $Y_n(a) | Y_m(a)$. If $n = 0$, then $m = 0$, thus suppose $n > 0$. Write $m = nq + r$, with $q, r \in \mathbb{N}$ and $0 \leq r < n$. From 3) follows

$$Y_m(a) = X_{nq}(a)Y_r(a) + Y_{nq}(a)X_r(a)$$

Since $Y_n(a) | Y_m(a)$ and $Y_n(a) | Y_{nq}(a)$, we obtain

$$Y_n(a) | X_{nq}(a)Y_r(a)$$

$$Y_n(a) | (X_{nq}(a))^2 Y_r(a)$$

$$Y_n(a) | (1+(a^2-1)(Y_{nq}(a))^2)Y_r(a) \qquad \text{(by (2))}$$

$$Y_n(a) | Y_r(a)$$

Suppose $r \neq 0$, then $Y_r(a) \neq 0$. Hence the degree of $Y_n(a)$ is less or equal than the degree of $Y_r(a)$. From 1) follows then that $n \leq r$, which contradicts $r < n$. Thus $r = 0$, which proves 4).

5) From (1) follows

$$X_{mp^n}(a) + \alpha(a)Y_{mp^n}(a) = (X_m(a)+\alpha(a)Y_m(a))^{p^n}$$

$$= (X_m(a))^{p^n} + \alpha(a)(Y_m(a))^{p^n}(a^2-1)^{\frac{p^n-1}{2}}$$

This proves 5).

6) It is sufficient to prove this for $m \geq 0$. First, suppose $m = p^n$. Then 5) implies the left side of 6).

Conversely suppose $X_m(a+1) = X_m(a) + 1$. Hence $m \neq 0$. Write $m = qp^n$, with $q, n \in \mathbb{N}$ and $q \not\equiv 0 \bmod p$. From 5) follows

$$(X_q(a+1))^{p^n} = (X_q(a))^{p^n} + 1 = (X_q(a)+1)^{p^n} .$$

Hence

$$X_q(a+1) = X_q(a) + 1$$

From 1) follows

$$X_q(a) = \gamma a^q + \beta a^{q-1} + \text{terms of lower degree in } a$$

$$\begin{aligned} X_q(a+1) &= \gamma(a+1)^q + \beta(a+1)^{q-1} + \dots \\ &= \gamma a^q + (\beta+\gamma q)a^{q-1} + \dots \end{aligned}$$

where $\gamma, \beta \in R$ and $\gamma \neq 0$. Thus if $q \geq 2$, then $\beta + \gamma q = \beta$, which contradicts $q \not\equiv 0 \bmod p$. Thus $q = 1$ and $m = p^n$.

7) From (3) (in the proof of 1)) follows

$$X_m(a) \equiv a^m \equiv 1 \bmod(a-1), \quad \text{for } m > 0.$$

Moreover $X_{-m}(a) = X_m(a)$. Q.E.D.

From Lemma 2.1 2) and 7) follows for all $X, Y, a \in R[T]$ with $a \notin R$ that

(4) $$(\exists m \in \mathbb{Z} : X = X_m(a) \wedge Y = Y_m(a)) \longrightarrow$$

(5) $$X^2 - (a^2-1)Y^2 = 1 \wedge X \equiv 1 \bmod(a-1).$$

Thus the relation (4) in the variables X, Y, a is diophantine over $R[T]$. Notice that the relation (4) is only defined for $a \notin R$, but we can extend it for $a \in R$ by taking (5) as its definition (e.g. if $a = 1$ then the relation (4) is defined to hold iff $X = 1$).

From Lemma 2.1 1) follows

$$X_m(T) = X_n(T) \wedge Y_m(T) = Y_n(T) \Longrightarrow m = n,$$

thus we can consider the set

$$\{(X_m(T), Y_m(T)) : m \in \mathbb{Z}\}$$

as a model of $\mathbb{Z}$ in $R[T]$. This model is diophantine over $R[T]$ with coefficients in $\frac{\mathbb{Z}}{p\mathbb{Z}}[T]$.

Moreover, by Lemma 2.1 3) and 4), addition and divisibility in that model are diophantine over $R[T]$ with coefficients in $\frac{\mathbb{Z}}{p\mathbb{Z}}[T]$.

Thus by the corollary to Theorem B, we see that Theorem A, in the case of characteristic $p \nmid 0, 2$, follows from Lemma 2.2 below. Indeed the existential quantifier $\exists \ell, s \in \mathbb{Z}$ in Lemma 2.2 can be replaced using (4)$\longleftrightarrow$(5).

LEMMA 2.2. Let R be an integral domain of characteristic $p \nmid 0, 2$. For all $m, q \in \mathbb{Z}$ we have

$$m|^p q \longleftrightarrow (X_m(T)=X_q(T)=1) \vee \exists a \in R[T] \exists \ell, s \in \mathbb{Z} :$$

(6) $$a = X_m(T) \wedge$$

(7) $$X_q(T) = X_\ell(a) \wedge$$

(8) $$X_s(a+1) = X_\ell(a) + 1.$$

PROOF. It is sufficient to prove the Lemma for $m, q, \ell, s \in \mathbb{N}$. First suppose $m|^p q$, thus $q = mp^n$ for some $n \in \mathbb{N}$. If $m = 0$, then $q = 0$ and $X_m(T) = X_q(T) = 1$. Thus suppose $m \nmid 0$. Set $a = X_m(T) \notin R$. Set $s = \ell = p^n$. Lemma 2.1 6) implies (8). From Lemma 2.1 5) follows

$$X_q(T) = (X_m(T))^{p^n} = a^{p^n} = X_\ell(a).$$

Thus also (7) is satisfied.

Conversely, suppose (6), (7) and (8). If $m = 0$, then $a = 1$. Hence $X_\ell(a) = 1$, by (5), and $X_q(T) = 1$ by (7). Thus $q = 0$. So, suppose $m > 0$, then $a \notin R$. From Lemma 2.1 1) follows $s = \ell$, since $X_s(a+1)$ and $X_\ell(a)$

have the same degree by (8). Hence (8) and Lemma 2.1 6) imply $\ell = p^n$, for some $n \in \mathbb{N}$. Thus by Lemma 2.1 5) and (6) we have

$$X_\ell(a) = a^{p^n} = (X_m(T))^{p^n} = X_{mp^n}(T).$$

So, (7) gives $X_q(T) = X_{mp^n}(T)$. Hence $q = mp^n$. Q.E.D.

3. Polynomial rings of characteristic $p = 2$.

Let R be an integral domain of characteristic $p = 2$. Let $a \in R[T]$ and $a \notin R$. We have $\sqrt{a^2-1} = a + 1 \in R[T]$, and the formulas of Section 2 are no more valid. However the method of Section 2 can be adapted by defining other sequences $X_m(a)$, $Y_m(a)$. Notice that the equation $x^2 + ax + 1 = 0$ has no root in the fraction field of $R[T]$. Indeed if $x \in R[T]$ satisfies this equation, then x divides 1 in $R[T]$, hence $x \in R$ and $a \in R$, which contradicts $a \notin R$. This enables us to make the following definition:

DEFINITION. Let R be an integral domain of characteristic $p = 2$. Let $a \in R[T]$ and $a \notin R$. Let $\alpha(a)$ be a root of the equation $x^2 + ax + 1 = 0$. We define two sequences $X_m(a)$, $Y_m(a) \in R[T]$, $m \in \mathbb{Z}$ by

$$X_m(a) + \alpha(a)Y_m(a) = (\alpha(a))^m = (a+\alpha(a))^{-m}. \tag{1}$$

LEMMA 3.1. Let R be an integral domain of characteristic $p = 2$. Let $a \in R[T]$, $a \notin R$. For all $m, n \in \mathbb{Z}$ we have:

1) $X_m(a)$ (resp. $Y_m(a)$) is equal to the polynomial obtained by substituting a for T in $X_m(T)$ (resp. $Y_m(T)$).

The degree of the polynomial $X_m(T)$ is $m-2$, if $m \geq 2$.

The degree of the polynomial $Y_m(T)$ is $m-1$, if $m \geq 2$.

$$X_{-m}(a) = X_m(a) + aY_m(a)$$

$$Y_{-m}(a) = Y_m(a)$$

2) All solutions $X, Y \in R[T]$ of the equation

$$X^2 + aXY + Y^2 = 1 \tag{2}$$

are given by

$$X_m(a),\ Y_m(a)\ , \qquad \text{with } m \in \mathbb{Z}.$$

3) $X_{m+n}(a) = X_m(a)X_n(a) + Y_m(a)Y_n(a),$

$Y_{m+n}(a) = X_m(a)Y_n(a) + Y_m(a)X_n(a) + aY_m(a)Y_n(a).$

4) $n \mid m \Longleftrightarrow Y_n(a) \mid Y_m(a)$.

5) $Y_{m2^n}(a) = \frac{a^{2^n}}{a}(Y_m(a))^{2^n}$ and $Y_{2^n}(a) = \frac{a^{2^n}}{a}$, if $n \geq 0$.

6) $(a+1)Y_m(a+1) = aY_m(a) + 1 \Longleftrightarrow m = \pm 2^n$ for some $n \in \mathbb{N}$.

REMARK. Many of the above formulas remain true for characteristic $p \nmid 2$, if one replaces $+$ by $-$ at some places. But 5) and 6) do not generalise to characteristic $p \nmid 2$. Thus the method of this section only works in characteristic two.

PROOF. Notice that $\alpha(a)$ and $(\alpha(a))^{-1}$ are the roots of the equation $x^2 + ax + 1 = 0$, and that $(\alpha(a))^{-1} = a + \alpha(a)$. Taking this into account, the proof of Lemma 3.1 is almost the same as the proof of Lemma 2.1; except for 5) and the assertion about the degree in 1) which are proved by induction on n and m respectively. Q.E.D.

From Lemma 3.1 2), 3) and 4) follows that

$$\{(X_m(T),\ Y_m(T)) : m \in \mathbb{Z}\}$$

can be used as a model for $\mathbb{Z}$ in $R[T]$, which is diophantine over $R[T]$, and for which addition and divisibility is also diophantine over $R[T]$. Thus as in Section 2 we see that Theorem A, in the case of characteristic $p = 2$, follows from the corollary to Theorem B and Lemma 3.2 below.

LEMMA 3.2. Let R be an integral domain of characteristic $p = 2$. For all $m, q \in \mathbb{Z}$ we have:

$$m \mid^2 q \longleftrightarrow (Y_m(T) = Y_q(T) = 0) \vee \exists a \in R[T] \exists \ell,\ s \in \mathbb{Z}:$$

$$a = TY_m(T) \ \wedge$$

$$TY_q(T) = aY_\ell(a) \;\wedge$$

$$(a+1)Y_s(a+1) = aY_\ell(a) + 1.$$

PROOF. The proof is almost the same as the proof of Lemma 2.2, now using Lemma 3.1 instead of Lemma 2.1. Q.E.D.

4. The diophantine problem for addition and localised divisibility.

In this section we prove Theorem B. From now on, all variables run over the integers. We use the method of L. Lipshitz [13]. It is sufficient to show that the relation $u = z^2$ can be defined by

(1) $$u = z^2 \longleftrightarrow \exists x_1, \ldots, x_k \in \mathbb{Z} :$$
$$\bigwedge_{i=1}^{r} K_i(u, z, x_1, \ldots, x_k) \,|_n\, L_i(u, z, x_1, \ldots, x_k),$$

where K_i and L_i are polynomials of degree one or less. Indeed, this implies that also the relation $w = xy$ can be defined in this way, since

(2)
$$w = xy \longleftrightarrow 4w = (x+y)^2 - (x-y)^2$$
$$c = a + b \longleftrightarrow 0 \,|_n\, (a+b-c).$$

LEMMA 4.1. Let $n > 1$, and suppose $x \,|_n\, 1$ and $y \,|_n\, 1$. Then $y = x^2$ if and only if the following conditions (3), (4) and (5) hold:

(3) $$2nx + 1 \,|_n\, 4n^2y - 1$$

(4) $$2nx - 1 \,|_n\, 4n^2y - 1$$

(5) $$ny - kx \,|_n\, nx - k, \text{ for all } k \text{ satisfying } |k| < n.$$

PROOF. Obviously, if $y = x^2$ then (3), (4) and (5) hold.

Conversely, suppose (3), (4) and (5) are satisfied. Since n, $2nx + 1$ and $2nx - 1$ are relatively prime to one another, (3) and (4) imply

$$(2nx+1)(2nx-1) \mid 4n^2y - 1.$$

Moreover $4n^2y - 1 \neq 0$, hence the above divisibility yields

$$|(2nx+1)(2nx-1)| \leqq |4n^2y-1|, \quad \text{whence}$$

$$x^2 \leqq |y| + \frac{1}{2n^2} .$$

But x and y are integers, thus

$$x^2 \leqq |y|. \tag{6}$$

Now we shall use (5) to obtain an inequality in the other direction. For every prime number p, we define $h(p)$ (depending on p, n, x and y) by

$h(p) = 0$ if ny and x are divisible by the same powers of p,
$h(p) = 1$ otherwise.

If $h \equiv h(p) \mod p$, then

$$p^i | ny - hx \rightarrow p^i | x, \quad \text{for all positive } i. \tag{7}$$

By the Chinese Remainder Theorem there exists an h (mod n) such that for every prime p dividing n we have $h \equiv h(p) \mod p$. Moreover, we can choose h such that $|h| < n$ and $hx \geqq 0$. Thus from (7) and (5) follows

$$ny - hx | x(nx-h). \tag{8}$$

From $|h| < n$ and $x \neq 0$ follows $x(nx-h) > 0$, hence (8) yields

$$n|y| - |hx| \leqq nx^2 - hx.$$

Since $hx \geqq 0$, we obtain $|y| \leqq x^2$; and in view of (6) we conclude $y = \pm x^2$. Suppose $y = -x^2$, then (3) yields

$$2nx + 1 | -4n^2x^2 - 1.$$

Hence $2nx + 1 | -2$, and one easily derives a contradiction. Thus $y = x^2$.

Q. E. D.

LEMMA 4.2. Let $n > 1$. Suppose the following conditions (9), (10), (11), and

(12) hold:

(9) $$nz + nx - 1 \,|_n\, n^2u - (nx-1)^2$$

(10) $$2nz + 1 \,|_n\, nx - 1$$

(11) $$2nz - 1 \,|_n\, nx - 1$$

(12) $$2n^2u + 1 \,|_n\, nx - 1.$$

Then $u = z^2$.

PROOF. Since n and $nz + nx - 1$ are relatively prime we can replace $|_n$ in (9) by $|$, and we easily obtain

$$nz + nx - 1 \mid n^2u - n^2z^2.$$

Suppose $u \neq z^2$, then the above divisibility implies

(13) $$|nx-1| - n|z| \leqq n^2|u| + n^2z^2.$$

Since n, $2nz + 1$ and $2nz - 1$ are relatively prime to one another, (10) and (11) imply

$$(2nz+1)(2nz-1) \mid nx - 1.$$

But $nx - 1 \neq 0$, hence

(14) $$4n^2z^2 - 1 \leqq |nx-1|.$$

Analogously (12) yields

(15) $$2n^2|u| - 1 \leqq |nx-1|.$$

From (13), (14) and (15) we obtain

$$(n|z|)^2 - n|z| - 1 \leqq 0.$$

If $z \neq 0$, then $n|z| \geqq 2$ and the above unequality cannot hold. Thus $u = z^2$ or $z = 0$. But when $z = 0$ one easily deduces from (13) and (15) that $u = 0$. Q.E.D.

LEMMA 4.3. For any nonzero integer d there exists an integer x satisfying $x \mid_n 1$ and $d \mid_n nx - 1$.

PROOF. Write d as $d = d_0 d_1$ where $d_0 \mid_n 1$ and d_1 is relatively prime to n. Set $x = n^{\varphi(d_1)-1}$, where φ denotes the Euler Function, then by the Fermat-Euler Theorem $d_1 \mid nx - 1$. Hence $d \mid_n nx - 1$. Q.E.D.

PROOF OF THEOREM B. We claim that $u = z^2$ if and only if there exist integers x and y satisfying $x \mid_n 1$, $y \mid_n 1$, (3), (4), (5), (10), (11), (12) and

$$nz + nx - 1 \mid_n n^2u - n^2y + 2nx - 1. \tag{16}$$

Indeed suppose $u = z^2$. By Lemma 4.3 (with $d = (2nz+1)(2nz-1)(2n^2u+1)$) there is an integer x satisfying (10), (11), (12) and $x \mid_n 1$. Set $y = x^2$, then all the conditions hold.

Conversely, suppose there are x and y satisfying the above conditions. From Lemma 4.1 follows $y = x^2$. Substituting this in (16) yields (9). Hence $u = z^2$, by Lemma 4.2, and the claim is proved. Thus we have shown that the relation $u = z^2$ can be defined by a formula of the form (1). This completes the proof of Theorem B. Q.E.D.

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Algorithms and Bounds for Polynomial Rings

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§0. Introduction

Let in the following X be a sequence of variables $(X_1,..,X_n)$. Consider polynomials $f,f_1,..,f_k \in K[X]$, all of degree $\leq d$, where K is a field. If $f \in \sqrt{(f_1,..,f_k)}$, i.e. $f^e = \Sigma h_i f_i$ for some $e \in \mathbb{N}$ and certain $h_i \in K[X]$, then we can choose the exponent e and the degrees of the h_i's all below a number $H = H(n,d) \in \mathbb{N}$, which depends only on n,d (not on the polynomials or the field).

Let me recall A. Robinson's compactness trick to prove this: Let c be the vector of coefficients of the polynomials, the number of coordinates of c being determined by (n,d,k). Then we have:

$$K \models \phi(c) \quad \text{iff} \quad K \models \vee\{\phi_r(c) \mid r \in \mathbb{N}\},$$

where $\phi(c)$ expresses that f vanishes on all common zeros of $f_1,..,f_k$ in the algebraic closure $\tilde{K}$ of K, and $\phi_r(c)$ expresses that $f^e = \Sigma h_i f_i$ for some $e \leq r$ and certain $h_i \in K[X]$ of degree $\leq r$. (Note that such an open formula $\phi(y)$ exists by the quantifier elimination for algebraically closed fields, and that the equivalence above simply reformulates Hilbert's Nullstellensatz.)

So, with FL the elementary theory of fields, we have:

$$FL \models \phi(y) \leftrightarrow \vee\{\phi_r(y) \mid r \in \mathbb{N}\}.$$

By the compactness theorem there is $H = H(n,d,k) \in \mathbb{N}$ such that:

$$FL \vdash \phi(y) \leftrightarrow \vee\{\phi_r(y) \mid r < H\}.$$

We can eliminate the dependence of H on k, because we need only consider $k \leq \binom{d+n}{n} = \dim_K\{g \in K[X] \mid \deg(g) \leq d\}$.
Note that by Gödel's Completeness Theorem we can even compute such an H from (n,d).

It may be interesting that we can apply the same trick to a weak version of Hilbert's Nullstellensatz

- if $f_1,..,f_k$ have a common zero in some extension field of K, they have a common zero in an algebraic extension -

to give the following:

if $f_1,..,f_k$ have a common zero in some extension field of K, they have a common zero in a finite extension L with $[L:K] \leq H'$, where H' can be computed from (n,d).

In fact, some older proofs of Hilbert's Nullstellensatz, cf. [20, §74-75], provide such bounds H and H', but in the (very elegant) approach which is fashionable today, cf. [9, p. 255-256], there is not the least indication for their existence. Therefore it seems to me that we may welcome such compactness arguments, which enable us to recover very easily a lot of information we seem to lose by an inconstructive approach. A few new results were actually proved in this way, related to Hilbert's 17^{th} problem, [13, p. 223], and desingularization, [6], respectively.

Now there remained a lot of effective bounds in the theory of polynomial rings over fields obtained by very complicated constructive arguments, cf. [7], [19], [17], which A. Robinson was always interested in to "explain" by modeltheory, as is clear from [13], [14, p. 503], [16]. But he made only a beginning with this in [15].

One reason for such an approach is of course the very easy proofs it usually provides, another motive is that it might help in solving open problems in this area, for instance Ritt's problem in differential algebra mentioned in [14, p. 504].

In [5, Chapter 4] I proved by model theory the existence of effective bounds in a number of cases where Robinson's trick doesn't work. (At the moment I can handle in this way all the positive results in Seidenberg's [17].)

To be precise I used the following compactness result:

<u>Basic Lemma</u>

Let T be an L-theory and $(\phi_i)_{i\in I}, (\psi_j)_{j\in J}$ families of L-sentences such that

$$T \models \wedge\{\phi_i \mid i\in I\} \leftrightarrow \vee\{\psi_j \mid j\in J\}.$$

Then there are <u>finite</u> subsets I_0 of I, J_0 of J with

$$T \models \wedge\{\phi_i \mid i\in I_0\} \leftrightarrow \vee\{\psi_j \mid j\in J_0\}.$$

Moreover, for such I_0 and J_0 one has:

$$T \models \wedge\{\phi_i | i\in I\} \leftrightarrow \wedge\{\phi_i | i\in I_0\} \text{ and}$$

$$T \models \vee\{\psi_j | j\in J\} \leftrightarrow \vee\{\psi_j | j\in J_0\}.$$

If, in addition, L is recursive, T is r.e., $I = J = \omega$, and the sequences $(\phi_i)_{i\in\omega}, (\psi_j)_{j\in\omega}$ are r.e., then such sets I_0, J_0 can be computed effectively from given indices for T,I,J. (Robinson's compactness argument corresponds to the case that the set I is a singleton, i.e. there is only one ϕ_i.)

Proof

By hypothesis $T \cup \{\phi_i\} \cup \{\neg\psi_j\}$ is inconsistent. Pick finite sets $I_0 \subset J$, $J_0 \subset J$ such that $T \cup \{\phi_i | i\in I_0\} \cup \{\neg\psi_j | j\in J_0\}$ is inconsistent. Then $T \models \bigwedge_{i\in I_0} \phi_i \to \bigvee_{j\in J_0} \psi_j$.
The converse implication is obvious. The computability of I_0, J_0 follows from Gödel's completeness theorem. ⊠

In section 1 I will indicate how this lemma nicely combines with the wellknown techniques and results of commutative algebra: local-global principles ([8, p. 14-15]), Krull's intersection theorem, the primitive element theorem, etc., to obtain many of the bounds we are looking for. Also I mention some problems which seem to be open and should be investigated from this new point of view.

In section 2 we will follow a quite different path, namely a non-standard approach, again initiated by A. Robinson, cf. [15]. It often favourably compares with the methods of §1, for instance in treating associated primes and primary decomposition. Its main attraction however is that the results on bounds in their non-standard formulation express very simple relations between two rings, a polynomial ring K[X] and a certain extension K[X]*, for instance: K[X]* is a faithfully flat K[X]-algebra, an ideal I of K[X] is prime, primary, radical, respectively, iff I·K[X]* has the same property as an ideal of K[X]*.

Section 3 concludes with remarks on still other aspects to this whole subject.
My intention with this paper is not to give full and precise proofs,

but to show the underlying ideas and some (hopefully) suggestive examples.

§1. Compactness arguments

We will keep the notations and conventions of the introduction. Let us also write $(f_1,..,f_k)$ for the ideal generated by $f_1,..,f_k$ in $K[X]$.

Then clearly:

(1.1) $f \in (f_1,..,f_k) \Leftrightarrow K \models \vee\{\psi_r(c) \mid r \in \mathbb{N}\}$,

where $\psi_r(c)$ expresses in K that $f = \Sigma h_i f_i$ for certain $h_i \in K[X]$ of degree at most r.

Krull's intersection theorem, cf. [9, p. 155], combined with a local-global principle, implies:

$(f_1,..,f_k) = \cap \{\underline{m}^m + (f_1,..,f_k) \mid m \in \mathbb{N},\ \underline{m}$ a maximal ideal of $K[X]\}$.

But $K[X]/(\underline{m}^m + (f_1,..,f_k))$ is a finite-dimensional K-algebra $K[x]$, $x = X \bmod(\underline{m}^m + (f_1,..,f_k))$, so we get:

(1.2) $f \in (f_1,..,f_k) \Leftrightarrow K \models \wedge\{\phi_r(c) \mid r \in \mathbb{N}\}$,

$\phi_r(c)$ expressing in K that $f(x) = 0$ for each ring $K[x] = K[x_1,..,x_n]$ such that $f_1(x) = \ldots = f_k(x) = 0$ and $\dim_K K[x] \leqslant r$.

Combining (1.1) and (1.2) we see, that

$$FL \models \wedge\{\phi_r(y) \mid r \in \mathbb{N} \leftrightarrow \vee\{\psi_r(y) \mid r \in \mathbb{N}\}.$$

(1.3) By the Basic Lemma of §0 this implies that one can compute a number A from (n,d) such that, if $f \in (f_1,..,f_k)$, then $f = \Sigma h_i f_i$ for certain $h_i \in K[X]$ of degree at most A.

If we apply this compactness argument to submodules of the free module $K[X]^p$ (instead of to ideals of $K[X]$) we obtain the following generalization:

(1.4) Given $n,d,p \in \mathbb{N}$, we can compute $A = A(n,d,p) \in \mathbb{N}$, such that for each field K and each system of linear equations

$$\begin{matrix} f_{11} Y_1 + \ldots + f_{1q} Y_q = g_1 \\ \vdots \qquad\qquad \vdots \\ f_{p1} Y_1 + \ldots + f_{pq} Y_q = g_p \end{matrix} \qquad \text{(all } f_{ij}, g_i \in K[X] \text{ of degree at most d)}$$

there is a solution $(h_1,..,h_q) \in K[X]^q$ with all h_j of degree at most A, if there is a solution in $K[X]^q$ at all, cf. [17, p. 277].

Note that this result implies that the decision whether such a system has a solution can be reduced to the decision whether a certain systems of linear equations with coefficients in K has a solution.

(1.5) In differential algebra a similar problem is still open, as fas as I know (cf. [12, p. 177]): Let $f,g \in K\{T\} = K[T,dT,d^2T,...]$ be differential polynomials in one variable T over the differential field K. How to decide whether $g \in [f] \overset{def}{=} (f,df,d^2f,...)$?
(1.2) suggests to consider first the following type of question: Does $g \in [f]$ hold, if $g(t) = 0$ for each differential ring extension S of K and each $t \in S$, such that $f(t) = 0$ and S is (an an ordinary ring) a localization of a finitely generated K-algebra?

(1.6) <u>Problem</u>
Can we find a formula Max(y) (independent of K and the vector of coefficients c of $f_1,..,f_k$) such that:

$$K \models \mathrm{Max}(c) \Leftrightarrow (f_1,..,f_k) \text{ is a maximal ideal?}$$

By (1.3) we construct for each $r \in \mathbb{N}$ a formula $\phi_r(y)$ s.t.

(1.7) $K \models \phi_r(c) \Leftrightarrow 1 \notin (f_1,..,f_k)$, and $1 \in (g,f_1,..,f_k)$ for each $g \in K[X]\setminus(f_1,..,f_k)$ of degree at most r.
Suppose now that $(f_1,..,f_k)$ is maximal (so $K[X]/(f_1,..,f_k)$ is a field extension of finite degree over K) and assume for simplicity that char(K) = 0. Then by [9, p. 185, Th. 14]:

(1.8) $K[X]/(f_1,..,f_k) \simeq_K K[T]/p(T)$, for some irreducible $p \in K[T]$.

(1.9) Hence $K \models \vee\{\psi_r(c) \mid r \in \mathbb{N}\}$, where $\psi_r(c)$ expresses that for some irreducible p(T) of degree at most r an isomorphism as in in (1.8) holds under which the residue class of T corresponds to the residue class of a polynomial in K[X] of degree at most r.
(1.7) and (1.9) imply:

$$FL \models \wedge\{\phi_r(y) \mid r \in \mathbb{N}\} \leftrightarrow \vee\{\psi_r(y) \mid r \in \mathbb{N}\}.$$

Hence the basic lemma provides a formula Max(y) as required, and moreover we can compute a bound M = M(n,d) such that

(1.10) $(f_1,..,f_k)$ is maximal iff $1 \notin (f_1,..,f_k)$ and $1 \in (g,f_1,..,f_k)$ for each $g \in K[X]\setminus(f_1,..,f_r)$ of degree at most M.

Along these lines I showed in [5, Ch. 4, §3].

(1.11) Given $n,d \in \mathbb{N}$ one can compute $P = P(n,d) \in \mathbb{N}$, such that for each field K and each ideal I of K[X] generated by polynomials of degree at most d one has:
I is prime $\Leftrightarrow$ for all $f,g \in K[X]$ of degree at most P,
if $fg \in I$, then $f \in I$ or $g \in I$.

(1.12) Moreover, the proof also shows that, if the perfect field K is computable (in Rabin's sense, cf. [10]) and there is an algorithm for factoring polynomials in one variable over K, then there is an algorithm such that, given $f_1,..,f_k \in K[X]$, the algorithm determines the number of minimal primes of $(f_1,..,f_k)$, and a finite set of generators for each of them.

(1.13) Modeltheoretically, Ritt's problem, cf. [12, p. 178], which has origins in work of Lagrange and Laplace, can be formulated as follows:
Is there an elementary, i.e. 1^{st} order, condition on the coefficients of an irreducible differential polynomial $f(T) \in K\{T\}$ with $f(0) = 0$ and K a differential field of characteristic 0 (and where a bound on the order and degree of f is given), which expresses that $g(0) = 0$ for each $g \in K\{T\}$ vanishing on the generic zero of f over K? (These g form the unique prime differential ideal containing f but not its separant.)

It is not difficult to express this by an infinite conjunction of formulas in the language of differential fields.
So we have 'only' to find an equivalent infinite disjunction. Useful hints might be provided by [3], where a necessary and sufficient condition is given in terms of the existence of rank 1 valuations with certain properties.
Can this "existential" test be transformed in an infinite disjunction?

(1.14) If $n > 2$, $d > 1$, $k > 2$ it does not seem to be known whether there are formulas $\sigma_{n,d,k}(y)$ and $\tau_{n,d,k}(y)$ (independent of c and K) s.t.:

$$K \models \sigma_{n,d,k}(c) \Leftrightarrow (f_1,..,f_k) = (g_1,..,g_{k-1}) \text{ for certain } g_1,..,g_{k-1} \in K[X],$$

$$K \models \tau_{n,d,k}(c) \Leftrightarrow \sqrt{(f_1,..,f_k)} = \sqrt{(g_1,..,g_{k-1})} \text{ for certain } g_1,..,g_{k-1} \in K[X].$$

In both cases there are of course infinite disjunctions available. The existence of such formulas $\tau_{3,d,k}$ would have as a consequence the positive solution of the long standing problem whether every algebraic curve in $\mathbb{C}^3$ is the intersection of 2 algebraic surfaces, cf. [8, p. 214]. (This follows by modeltheory, because Cowsik & Nori, [4], have positively answered the analogue of the above set-theoretic complete intersection problem for each prime characteristic.)

(1.15) If $n > m > 1$, $d > 1$ it does not seem to be known whether there are formulas $\pi_{n,m,d,k}(y), \rho_{n,m,d,k}(y)$ such that:

$$K \models \pi_{n,m,d,k}(c) \Leftrightarrow K[X]/(f_1,..,f_k) \simeq K[Y_1,..,Y_m]$$

as K-algebras,

$$K \models \rho_{n,m,d,k}(c) \Leftrightarrow (f_1,..,f_k) \text{ is prime and the fraction}$$

field of $K[X]/(f_1,..,f_k)$ is K-isomorphic with $K(Y_1,..,Y_m)$.

For $m = 1$ and algebraically closed K, the right hand side of the second equivalence expresses that $(f_1,..,f_k)$ is the ideal defined by a curve of genus 0 in affine n-space. So, with K restricted to be algebraically closed, formulas $\rho_{n,1,d,k}$ exist.

§2. A non-standard approach.

We keep the convention that $X = (X_1,..,X_n)$

(2.1) Given $n,d,k \in \mathbb{N}$ there is a bound $B = B(n,d,k) \in \mathbb{N}$ such that for each field K and each homogeneous linear equation

$$(\alpha) \quad f_1Y_1 + .. + f_kY_k = 0 \quad (f_1,..,f_k \in K[X], \text{ all of degree at most } d)$$

the solution subset of $K[X]^k$ is generated as $K[X]$-module by solutions $g = (g_1,..,g_k)$ with $\deg(g) \leq B$ (i.e. $\deg(g_i) \leq B$ for

$i = 1,..,k)$.

(2.2) <u>Proof</u>

Let n,d,k be given and suppose such a bound B doesn't exist. So for each $m \in \mathbb{N}$ there is a field K_m and a system of type (α) over K_m with a solution in $(K_m[X])^k$ which is not generated by solutions of degree $\leq m$. Consider a structure containing all fields K_m, polynomial rings $K_m[X]$, $\mathbb{N}$, etc., and take an enlargement of this structure. By the saturatedness properties of enlargements there is an internal field K in this enlargement and an equation $f_1Y_1 + .. + f_kY_k = 0$ with $f_1,..,f_k \in K[X]^* \overset{def}{=}$ the ring of internal polynomials in X over K, each f_i of degree $\leq d$, such that at least one solution $(g_1,..,g_k) \in (K[X]^*)^k$ of the equation is not generated by solutions in $K[X]^k$, where $K[X]$ is naturally identified with the ring of finite degree polynomials in $K[X]^*$.

Note that $f_1,..,f_k \in K[X]$. But this implies that $K[X]^*$ is <u>not</u> a flat $K[X]$-module [8, p.5].

Hence (2.1) follows by contradiction from:

(2.3) $K[X]^*$ is a faithfully flat $K[X]$-module (where $K[X]^*$ is the ring of internal polynomials in X over any internal field K in an enlargement).

An easy proof of (2.3) by induction on n is given in 5, p. 128 . It is fairly obvious that (2.3) is in fact <u>equivalent</u> with the conjunction of (2.1) (flatness) and (1.3) ("faithfully").

(2.4) In most cases (and certainly in the more difficult ones), where bounds can be extracted from the proofs in [17], they can be obtained much easier using the above approach. Of course we lose some information, even in comparison with §1, namely the effectiveness of the bounds. Another difference with §1 is that the non-trivial proofs use induction on n.

Let me mention some typical results, using the notation of (2.3).

Given an ideal I of $K[X]$ the following hold:

(2.5) $K(X)^*$ ($\overset{def}{=}$ the fraction field of $K[X]^*$) is a regular extension of $K(X)$.

(2.6) I is prime iff $I.K[X]^*$ is prime in $K[X]^*$.

(2.7) $\sqrt{I}.K[X]^* = \{f \in K[X]^* \mid f^{\omega} \in I.K[X]$ for some $\omega \in^* \mathbb{N}\} = \sqrt{I.K[X]^*}$.

(2.8) If M is a K[X]-module, then

$$\mathrm{Ass}_{K[X]^*}(M \otimes_{K[X]} K[X]^*) = \{p.K[X]^* \mid p \in \mathrm{Ass}_{K[X]} M\}.$$

(2.9) If $I = I_1 \cap .. \cap I_m$ is a reduced primary decomposition of I, I_k being p_k-primary (k = 1,..,m), then

$I.K[X]^* = I_1.K[X]^* \cap .. \cap I_m.K[X]^*$ is a reduced primary

decomposition of

$I.K[X]^*, I_k.K[X]^*$ being $p_k.K[X]^*$-primary (k = 1,..,m).

(2.10) Comment

(2.5) is an important lemma in proving (2.6). The reader will check easily that (2.6) is the non-standard version of (1.11). (2.7) is an easy consequence of (2.3) and (2.6), and it implies for example that the exponent e mentioned in the first lines of the introduction can be bounded by a function of n and the degrees of $f_1,..,f_k$ only (and does not depend on the degree of f). (2.8) is an immediate consequence of (2.3), (2.6) and [2, Th. 2, p. 154].
If one takes M = K[X]/I it means that the associated primes of an ideal I of K[X] which is generated by polynomials of degree at most d, are themselves generated by polynomials of degree at most C = C(n,d), and that there are at most D = D(n,d) of them (K any field).
(2.9) follows easily from (2.3), (2.6), (2.7) and (2.8). Its standard interpretation is that an ideal I of K[X] which is generated by polynomials of degree at most d has a primary decomposition $I = I_1 \cap .. \cap I_m$ with $m \leq E(n,d)$ and where each I_k is generated by polynomials of degree at most F(n,d) (K any field).

(2.11) Perhaps one should consider the open problems mentioned in §1 also in the context of this section. For instance, the first problem of (1.14) simply becomes: if $I.K[X]^*$ is generated by m elements, is then I also generated by m elements?

§3. Concluding Remarks

(3.1) In the preceding sections not any attempt was made to obtain concrete, say exponential, expressions for the bounds. This is not possible with model theory alone. Still it seems interesting to give a model theoretic explanation why the bounds given in [17] tend to be (super) exponential in n and polynomial in d.

(3.2) Except that the results of [17] are more precise - and less easy to obtain - than in this paper, another contrast is the point of view of [17] and [19] which is thoroughly constructivistic. This is also the case in [11], [18], where some basic construction problems are solved for more general polynomial rings.

(3.3) Still another approach to some of the material treated here, is discussed in [1], where algorithms for reducing words to canonical forms are the leading theme.

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LOGIC COLLOQUIUM 78
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CONSTRUCTIVE THEORIES OF FUNCTIONS AND CLASSES[1)]

SOLOMON FEFERMAN

Dedicated to the memory of my friend and colleague, Karel de Leeuw

Introduction and contents. These lectures were designed to acquaint a general logical audience with basic features of Bishop's approach to constructive mathematics (BCM) and with work on a certain formal system T_0 in which that can be represented. Several competing and rather different systems have been proposed for the same purpose. Thus, in addition to the intrinsic interest of the subject BCM provides an excellent case study for the process of formalization.

The contents are divided into five parts, only the last of which assumes some prior background; in outline they are as follows.

I. Background and aims. Part I gives an informal introduction to BCM which contrasts it both with everyday non-constructive mathematics as well as with the schools of constructivity previously established by Brouwer and Markov. Towards the end of this part we discuss general criteria of formalization, involving questions of adequacy and accord with the informal body of mathematics being represented.

II. The theory T_0. In part II we present the language and axioms of T_0 and some natural subsystems and extensions. The adequacy of T_0 to BCM is sketched and the question of its accord is discussed. Alternative formal systems proposed by Martin-Löf and Myhill are briefly compared in this connection.

III. Models. A variety of models (in the classical sense of the word) are presented for T_0 and related theories. One main purpose which these serve is to show how developments in BCM, when formalized in T_0, generalize corresponding parts of classical mathematics and certain recursion-theoretic analogues. They are also used to obtain consistency and independence results for some statements of mathematical interest.

IV. Realizability interpretations. In contrast to models, the method of realizability (originating with Kleene) is distinctively associated with interpretations

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of constructive theories. It is here adapted to the formalism of T_0 so as to obtain more delicate consistency (and conservation) results, in particular as concern axioms of choice and continuity principles.

V. Relations with subsvstems of analysis. In this part one combines both proof-theoretical and model-theoretical methods to obtain equivalence (in strength) of various subsystems of the classical system $S = (\Sigma_2^1\text{-AC}) + (BI)$ with subsystems of T_0. For the full system T_0 one has an interpretation in S, but it is an open question whether S is equivalent to T_0.

We concentrate throughout on explanation and statement of results. Proofs are not given but some proof-ideas are indicated. The basic source is Feferman 1975; this has been enriched considerably by the work of Beeson 1977. The latter gives both models and realizability interpretations which are used particularly for continuity principles; his work is described within Parts III and IV. Important contributions to Part V have been made by Aczel, Buchholz, Friedman, Pohlers, and Sieg; detailed references are given in the text. Otherwise we draw principally on the unpublished notes Feferman 1976a, 1976b, and 1976c, which are now largely incorporated in the following.

For the reader seeking a general introduction to the subject of constructivity and its formalizations (especially stemming from the schools of Brouwer and Markov) I would suggest the excellent survey article Troelstra 1977a; this contains an extensive bibliography.

I. Background and aims

1. Ad hoc (local) vs. systematic (global) constructive mathematics. At the local level one deals with particular questions of construction without regard to general principles or methods. Frequently one knows an existential result guaranteeing the existence of a solution to a specific mathematical problem without knowing how it may be calculated, represented, or constructed. One then seeks to produce an explicit solution to the problem. For examples familiar to logicians we have: (i) decidability of p-adic fields (first existence by Ax and Kochen, followed by a concrete decision procedure by Cohen) and (ii) representability of positive definite real polynomials as sums of squares of rational functions (existence by Artin, followed by recursive representations by Robinson and primitive recursive representations by Kreisel).

At the global, systematic level one reconstructs whole portions of mathematics using entirely constructive notions and methods. One of the main reasons advanced for doing this is philosophical; it is based on a conception of mathematics which is opposed to the current underlying platonistic conception and has its source in human thought and constructions. Such systematic redevelopment according to constructive principles was initiated by Brouwer and carried on by Heyting and his students. Subsequently another school of constructivity was developed in Russia by Markov and Shanin (cf. Troelstra 1977a for references on these two schools). Finally the approach (here labeled BCM) was initiated in Bishop 1967 and continued by him and his students. The main features of the first two schools will be described briefly below and those of BCM will be described at length.

2. Constructivity in principle and constructivity in practice (feasibility). No matter how a constructive result is obtained (locally or globally) there is a question of its actual computation or execution. In this respect, even constructive existence results have a non-concrete character. A classical example is provided by Gauss' characterization of the regular polygons which are constructible by ruler and compass; the general theory had to be refined in order to give a feasible construction even of the 17-sided regular polygon. For a (negative) example familiar to logicians, we may mention Tarski's primitive recursive decision procedure for the theory of reals. It has been shown by Fischer and Rabin that any decision procedure for the reals requires exponential time and so is unfeasible by present computational methods.

3. General features of the platonistic conception. We describe these for a point-by-point comparison with the constructivist conception in §4.

3.1. Mathematical entities. These are conceived to be external to us and independent of our thoughts and constructions. In its modern form, the most general mathematical entities are sets and functions (which are interchangeable, cf. 3.3 below). Thus the platonist conception is also called the Cantorian set-theoretical conception of mathematics.

3.2. Mathematical statements are true or false. Hence the logic employed is the classical predicate calculus based on 2-valued semantics. The law of excluded middle $\phi \vee \neg \phi$ leads us to conclude $\exists x\, \psi(x) \vee \forall x \neg \psi(x)$. Thus to prove $\exists x\, \psi(x)$ it is sufficient to prove $\neg \forall x \neg \psi(x)$. This is the basis of the use of the indirect method to obtain existential results: assume $\forall x \neg \psi(x)$ and draw a contradiction. Evidently there is no explicit solution provided by such arguments.

3.3. Interchangeability of sets and functions. Of course the former are reduced to the latter via characteristic functions. Conversely, functions are regarded as many-one relations, which in turn are certain sets of ordered pairs. But the latter are definable as sets, so functions are reduced to sets.

3.4. Extensionality. The principle $\forall x\, (x \in A \leftrightarrow x \in B) \rightarrow A = B$ for sets A,B is justified by the consideration that sets exist independently of us and of any means of definition. Sets, then, may only be distinguished by their members.

3.5. Power set. Since arbitrary subsets of a set are supposed to exist independently and permanently, we may speak of their totality $\mathcal{P}(A)$. This operation may be iterated, leading to the finite-type hierarchy. For transfinite iteration the operation $A \mapsto A \cup \mathcal{P}(A)$ gives a more convenient theory (the cumulative hierarchy).

3.6. Subset formation. Any property $\phi(x)$ of elements of A determines a subset $B = \{x \in A | \phi(x)\}$ (separation or comprehension principle). ϕ may contain quantified variables ranging over other sets, in particular over $\mathcal{P}(A)$. Such comprehension principles are impredicative: B is defined in terms of the totality $\mathcal{P}(A)$, which contains B as an element.

3.7. The axiom of choice is usually agreed to be correct on the Cantorian view, since there is no question as to how the choices are to be effected. Then one has the well-ordering theorem and the theory of finite and transfinite cardinals. In consequence, such statements as the continuum hypothesis are taken to have a definite truth-value, though undecided by all set-theoretical principles so far recognized to be correct (or even having some plausibility).

4. General features of the constructivist conception.[2)]

4.1. Mathematical entities are only those which are understood directly by humans or obtained from such by successive human constructions (e.g., by combination into pairs or sequences). The natural numbers $0,1,2,\ldots$ (denoted as a whole by $\mathbb{N}$) form basic entities which are generated by repeated adjunction of a single unit. Both the processes of construction of mathematical entities and of recognition of their properties are mental activities. Such recognition is the result either of direct intuition or of proofs based on principles inherent in the specific nature of the constructions used. For example, the principle of induction for $\mathbb{N}$ directly follows the manner of its generation.

4.2. Mathematical statements do not communicate questions of truth or falsity; they can only be assertions which communicate results of completed proofs. The use of the logical particles is explained in terms of constructions and proofs, roughly as follows:
(i) a proof of $(\phi \wedge \psi)$ is given by a pair (p,q) of proofs of ϕ and ψ , resp.; (ii) a proof of $(\phi \vee \psi)$ consists of a proof of ϕ or a proof of ψ (together with the information as to which of these is proved); (iii) a proof of

2) For further information and references on §4-§8, cf. Troelstra 1977a .

of $(\phi \to \psi)$ is a constructive operation for which we recognize that it will convert any proof q of ϕ into a proof (pq) of ψ. (iv) a proof of $\exists x\, \psi(x)$ consists of a pair (p,c) where p is a proof of $\psi(c)$; (v) a proof p of $\forall x\, \psi(x)$ is a constructive operation for which we recognize that it will convert any object c (in the intended range of the variable 'x') into a proof (pc) of $\psi(c)$.

Taking $\bot$ to be an identically false statement (e.g. $0 = 1$) which has no proof, negation is defined by $(\neg \phi) = (\phi \to \bot)$; thus proof of a negation of a statement (or of its absurdity) amounts to constructive recognition of the impossibility of proof of that statement. A proof of $\phi \vee \neg \phi$ is only given when one has a proof of ϕ or a proof of its absurdity.

There is a system of intuitionistic logic which is recognized to be correct for this interpretation of the logical operations, but which does not yield such (apparently) unacceptable principles as the law of excluded middle (LEM) or its consequence $\neg \forall x \neg \phi(x) \to \exists x\, \phi(x)$. Heyting has formulated this logic in such a way that classical logic is obtainable from it simply by adjunction of LEM. No further general logical principles have been recognized as constructively evident. (However, there is no generally recognized completeness result for intuitionistic logic.)

4.3. Functions are supposed to be constructive operations, the idea of which was already contained in 4.2 (iii),(v). These are supposed to be given by algorithmic rules of construction which can be effected by finite mechanical steps of computation. For relations with the recursion-theoretic concept of computable function cf. 4.8 below.

4.4. Sets are only given by defining properties , for which we are supposed to know and understand their condition for membership. For example, the condition for $x \in \mathbb{N}$ is that x is generated from 0 by a finite number of applications of the successor operation. If A,B are sets then $A \times B$, which consists of all ordered pairs (x,y) with $x \in A \wedge y \in B$, is a set. So also is B^A , where $x \in B^A$ iff $x : A \to B$, which means that x is a constructive operation such that for each $y \in A$, $x(y)$ (is defined and) belongs to B. Finally, if A is any set and $\phi(x)$ is a well-understood property of members of A then $B = \{x \in A \mid \phi(x)\}$ is a set, with $x \in B \leftrightarrow x \in A \wedge \phi(x)$.

4.5. Non-extensionality. Two rules may have the same values at all arguments (even provably so), but they are not identified unless the rules are recognized to be the same, as rules. (This allows for minor syntactic variations in the presentation of rules.) Two sets may have the same members, but they are not identified unless they are seen to be given by the same properties. (For the notion of intensional identity implicit here, cf. 4.11 below.)

4.6. Non-interchangeability of sets and functions. If B is a subset of A and $f: A \to \{0,1\}$ is such that $\forall x \in A[x \in B \leftrightarrow f(x) = 0]$ then we say that f is a characteristic function of B (rel. to A). Not every (sub)set (of a given set) has a characteristic function. Those which do are called decidable, otherwise undecidable. E.g. the set of exponents n for which Fermat's last theorem is true is (presently) undecidable. If every constructive function on $\mathbb{N}$ is recursive then every subset of $\mathbb{N}$ which is recursively undecidable is undecidable in the constructive sense. In any case, sets are not reducible to functions.

If $f: A \to B$ then the graph of f is a set, namely $R = \{(x,y) \in A \times B \mid f(x)=y\}$. R has the property $\forall x \in A\, \exists!\, y\ (x,y) \in R$. The question whether conversely, any such $R \subseteq A \times B$ determines a function $f: A \to B$ is a special case of the following. The conclusion is that functions are not reducible to sets.

4.7. The axiom of choice is considered here in the schematic form

$$\text{(AC)} \qquad \forall x \in A\, \exists y\, \phi(x,y) \to \exists f\, \forall x \in A\, \phi(x,f(x)).$$

This looks like it ought to be admitted using the interpretation of the connectives in 4.2. However we have to be careful : a proof p of the hypothesis, written out as $\forall x[x \in A \to \exists y\, \phi(x,y)]$ gives for each x and each proof q that x belongs to A (i.e. that x has the property which defines A) a proof p^* of $\exists y\, \phi(x,y)$ which is a pair $p^* = (p_1^*, y)$ where p_1^* proves $\phi(x,y)$. But p^* depends on both x and q, i.e. $p^* = p(q,x)$, so that y also is a function of x and q, not of x only as would be required for (AC). Writing $x \in_q A$ for 'q is a proof that x has the property determining A', this informal argument does justify accepting the following modified principle:

$$\text{(AC)}' \qquad \forall x \in A\, \exists y\, \phi(x,y) \to \exists f\, \forall q\, \forall x \in_q A\, \phi(x,f(x,q)).$$

We can derive (AC) only for those sets A for which we have a canonical choice of q, i.e. a function c such that $\forall x[x \in A \to x \in_{c(x)} A]$. $\mathbb{N}$ is an example of such a set; for $n \in \mathbb{N}$, the build-up of $\mathbb{N}$ gives itself the verification that we have a natural number. It may be noted that the principle

$$\text{(AC!)} \qquad \forall x \in A\, \exists!\, y\, \phi(x,y) \to \exists f\, \forall x \in A\, \phi(x,f(x))$$

is, for the same reasons, no more assured in general than AC. (These principles are dealt with formally in the framework of T_0 in Part IV below.)

4.8. Church's thesis. Let $e, n, m, \ldots$ range over $\mathbb{N}$, and take the usual notation $\{e\}(n)$ for partial recursive function application. The thesis that every (total) constructive function on $\mathbb{N}$ is recursive is referred to as

Church's Thesis in the literature on intuitionism (though it is open to argument whether Church himself had this in mind). Formally we can express it (in a 2nd order language) by

(CT) $$\forall f \in \mathbb{N}^{\mathbb{N}}\, \exists e\, \forall n[\{e\}(n)\downarrow \wedge f(n) = \{e\}(n)].$$

Note that the converse to Church's thesis is that

$$\forall n\, \exists m\, \{e\}(n) \simeq m \rightarrow \exists f \in \mathbb{N}^{\mathbb{N}} \forall n[f(n) = \{e\}(n)].$$

This follows from $(AC)_{\mathbb{N}}$ - which is acceptable by 4.7. There are some schemes related to (CT) which are expressible in 1st order form and follow from (CT) and $(AC)_{\mathbb{N}}$, in particular:

(CT_o) $$\forall n\, \exists m\, \phi(n,m) \rightarrow \exists e\, \forall n[\{e\}(n)\downarrow \wedge \phi(n,\{e\}(n))].$$

It is of logical interest that almost every known theory T which is informally constructively acceptable is consistent with (CT_o). However, the acceptability of this or of (CT) itself is a matter of dispute. As an example of the kind of argument which can be made against it, consider the following. Let J be a mathematician who works on deep problems of set theory and whose mental behavior is not duplicable by a machine. Then the function f defined by

$$f(n_o,n_1) = \begin{cases} 1 & \text{if on the } n_o\text{th day from now, J proves the } n_1\text{th theorem of ZF} \\ 0 & \text{otherwise} \end{cases}$$

is constructive but not recursive. Perhaps a more convincing argument against (CT) is that under the constructive interpretation of the logical operations if it held we would have to be able to pass constructively from any (proof of) $f \in \mathbb{N}^{\mathbb{N}}$ to a Turing machine e which calculates f. Thus even if human mental behavior is believed to be mechanical in principle, there is no constructive method of duplicating it by Turing machines. An argument for (CT) on the other hand, goes back to what is meant by constructive operation; at least in the form explained in 4.3, this would seem to be justified by Church's thesis in the usual sense that every finite algorithmic procedure can be carried out by a Turing machine.

4.9. Function sets and power sets. If Church's thesis is accepted, the meaning of $\mathbb{N}^{\mathbb{N}}$ is perfectly clear: it consists simply of the total recursive functions. An argument can be made without CT that we understand B^A for any sets A,B (whose condition for membership was given in 4.4) because our conception of constructive operation is supposed to be basic; this does not mean that "we know the totality of all constructive operations from A to B". The question of whether for each set A we have a set $\mathcal{P}(A)$ of all subsets of A seems to be different: even if we accept CT it is not clear that we have an understanding of what constitutes an arbitrary property of elements of $\mathbb{N}$, let alone of any A. The constructive status of $\mathcal{P}(A)$ is not settled; it is of mathematical and logical interest to investigate the effect of assuming its existence.

4.10. Comprehension principles. If a set A is given (understood and accepted) then quantification over A, i.e. the logical operations $\forall x \in A(\dots)$ and $\exists x \in A(\dots)$, are understood. Hence any property built using such operations determines a subset of any given set. If the existence of power sets is assumed then this leads us to impredicative comprehension principles, i.e. existence of $\{x \in A \mid \phi(x)\}$ where in ϕ we can quantify over $\mathcal{P}(A)$. Again, the constructive character of such principles is not settled, while their role and effect is of interest.

4.11. Literal, intensional and extensional identity. In 4.6 we spoke of functions given by the same rule or sets given by the same property. If we concentrate on syntactic representation of rules, properties, etc., then the most obvious notion of sameness to consider is that of literal identity, i.e. identity of syntactic configurations, symbol by symbol. A less definite but common idea is that rules, properties, etc. are mental objects which may have a variety of syntactic representations. For example, and most trivially, this may be by a renaming of bound variables or other symbols. More generally, we may have representations in different, but intertranslatable languages (so that the structure of the formal configurations may actually change). When two syntactic objects represent the same mental object they are said to be in the relation of intensional identity.

Most frequently in mathematics we are concerned with various kinds of defined relations of "equality" $=_A$ on a set A, which are simply equivalence relations. For example, when defining the integers $\mathbb{Z}$ as $\mathbb{N} \times \mathbb{N}$, we take $(n_1, m_1) =_{\mathbb{Z}} (n_2, m_2) \leftrightarrow n_1 + m_2 = n_2 + m_1$. When defining $\mathbb{Z}_p$ we take $x =_{\mathbb{Z}_p} y \leftrightarrow p \mid (x-y)$ for $x, y \in \mathbb{Z}$. The set $F = B^A$ has defined on it the relation $f =_F g \leftrightarrow \forall x \in A\ \forall y \in A\,[x =_A y \rightarrow f(x) =_B f(y)]$. All such equality relations are sometimes lumped together (perhaps misleadingly) under the heading of extensional identity relations. In Cantorian mathematics it is common to pass from $(A, =_A)$ to $(A/ =_A)$ so as to replace all equality relations by literal identity using the axiom of extensionality for sets. This practice is neither possible (without extensionality) nor necessary constructively; one simply makes clear for each set A considered what equality relation is being used in a given context. Note. It is common practice to drop the subscript 'A' from $=_A$ once that is fixed in any given discussion.

5. Constructive theory of real numbers. We sketch here how the preceding principles are used to set up a theory of real numbers. First of all, recursion on $\mathbb{N}$ is justified directly by its manner of generation, so we can define successively $+$, $\cdot$ and all further primitive recursive operations. $\mathbb{Z}$ and $=_{\mathbb{Z}}$ are defined as explained in 4.11, and $+, \cdot, <$ are extended in the standard way to $\mathbb{Z}$. Then $\mathbb{Q}$ is taken to consist of all (x,y) with $x, y \in \mathbb{Z}$ and $y \neq 0$ and $+, \cdot, <$ are extended to it. Next $\mathbb{Q}^{\mathbb{N}}$ consists of all sequences $\langle r_n \rangle_n$ of rational numbers. Cauchy sequences of rationals are those for which the Cauchy condition is constructively satisfied, i.e. for which we have a rate-of-convergence function $\mu : \mathbb{N} \to \mathbb{N}$ such that

$$\text{(C)} \qquad \forall k > 0 \ \forall n, m \geq \mu(k)[\,|r_n - r_m| < \tfrac{1}{k}\,] .$$

By the set $\mathbb{R}$ of real numbers is meant the set of all pairs $x = (\langle r_n \rangle, \mu)$ with $\langle r_n \rangle \in \mathbb{Q}^{\mathbb{N}}$ satisfying (C). Then we put $x =_{\mathbb{R}} y$ for $y = (\langle s_n \rangle, \nu)$ if $(r_n - s_n) \to 0$. Real functions (of k arguments) are of course those operations $f: \mathbb{R}^k \to \mathbb{R}$ which preserve $=_{\mathbb{R}}$. In particular $+$ and $\cdot$ may be defined as real functions. For example, we may take $(\langle r_n \rangle, \mu_1) + (\langle s_n \rangle, \mu_2) = (\langle r_n + s_n \rangle, \nu)$ with $\nu(k) = \max(\mu_1(2k), \mu_2(2k))$.

The first essential difference is met with inverse and order. Given $x = (\langle r_n \rangle, \mu)$ we seek $x^{-1} = (\langle r_n^{-1} \rangle, \nu)$ but there is no obvious choice of ν unless we know a bound of x away from 0. Define $x > 0(m, k)$ if $\forall n \geq m \ (r_n \geq \frac{1}{k})$, and $x > 0$ if $\exists m,k(x > 0(m,k))$. Then define $x > y$ (or $y < x$) if $(x-y) > 0$ and finally $x \# y$ if $(x > y) \vee (x < y)$. We cannot establish constructively that $x \# y \vee x = y$. Inverse is defined for all $(x, (m, k))$ such that $|x| > 0\ (m,k)$. This is not strictly speaking a subset of $\mathbb{R}$, but only a subset by imbedding; such sets are dealt with systematically in BCM as will be described in § 14 below.

We could of course define $x \geq 0$ by $x > 0 \vee x = 0$, but it is more useful to take $x \geq 0$ to be $\forall k > 0 \ \exists m \ \forall n \geq m\,(r_n \geq -\frac{1}{p})$; these are not constructively equivalent definitions, nor is $x \geq 0 \vee x < 0$ constructively justified. Classically, the expression

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

defines a function on $\mathbb{R}$ which is discontinuous, but this does not make constructive sense as a definition. Indeed there is no evident way to obtain a discontinuous function; theoretical reasons for this will be produced later.

6. Brouwer's intuitionism. [3] Brouwer both explored general constructive concepts (e.g. constructive operations, sets or "species", ordinals, etc.) and carried out particular mathematical developments, especially in analysis. He thought it should be possible to prove constructively that every (total) real function is continuous and that every real function on a closed bounded interval $F:[a,b]\to\mathbb{R}$ is uniformly continuous. For this purpose he introduced a new concept of free choice sequence (f.c.s.) $\langle r_n\rangle$ of which we know only a finite amount of information $(r_0,\ldots,r_k)$ at any given time, though we can proceed as far out as needed to make a calculation. The sequence may be produced randomly, e.g. by rolls of a die or observations of some random physical phenomena, rather than by some mechanical law. [4] It makes sense to operate constructively on such sequences to obtain values in $\mathbb{N}$ or $\mathbb{Q}$ or new f.c.s. themselves. For example, the operations + and . are easily defined for f.c.s. Now if $f(\langle r_n\rangle)=\langle s_n\rangle$ and a value s_m has been established, it can only have used a finite amount of information about $\langle r_n\rangle$; from this principle follows the statement of continuity of real functions where the reals are understood in the extended sense to include all those given by f.c.s. By some further (less immediately evident) principles Brouwer also derived the statement concerning uniform continuity.

Choice sequences need not be completely 'free'. They can be considered with or without restrictions on their values. For example, we can consider sequences $\langle r_n\rangle$ restricted by $|r_n|\le M_n$ where $\langle M_n\rangle$ is given in advance, or is itself produced by some rule depending on earlier values of $\langle r_n\rangle$. Lawless sequences are those which are given without any restriction whatever. At the opposite end, lawlike sequences are those which are completely determined in advance by rules. The theory of reals sketched in § 5 may be interpreted as applying to the latter kinds of Cauchy sequences; for this reason it is sometimes called lawlike analysis.

Brouwer's analysis based on f.c.s. has been studied in various logical formalisms by Kleene, Vesley, Kreisel, Troelstra, van Dalen and others (cf. Troelstra 1977a 1977b for references). Various parts of this have taken settled and coherent form (and have, incidentally, been shown consistent). But efforts to treat the most general concept of f.c.s. have not yet had a convincing outcome. For mathematicians, Brouwer's theory has remained a curiosity;

3) A perusal of Brouwer 1975 (collected works, vol.I) is rewarding here.

4) Following a remark of Troelstra, H. Jervell has traced back the idea of f.c.s. to papers of E. Borel in 1912 which grew out of earlier discussions by the French mathematicians on the axiom of choice. At first Brouwer rejected the idea but later (1917) accepted it and expanded it into a theory.

it has largely been of interest to logicians. Moreover, the concepts are rather special to analysis and topology and seem to have little to do with other parts of mathematics. Historically, the actual development of intuitionistic mathematics got hung up around analysis because of the need to clarify Brouwer's ideas there.

It should be remarked that the intuitionistic theory of f.c.s. is inconsistent with classical mathematics, for we can prove $\neg \forall (r_n)[\exists m(r_m=0) \vee \forall m(r_m \neq 0)]$, as is intuitively evident from the 'finite-information' principle. Relatedly, one can disprove $\forall x \in \mathbb{R}(x \geq 0 \vee x < 0)$, etc. This is in contrast to lawlike analysis, which is a part of classical mathematics (if one does not assume (CT_0)).

7. The (Russian) school of Markov and Shanin. Here one accepts the scheme (CT_0) and the laws of intuitionistic logic, but also the following non-intuitionistic law, called Markov's principle:

(MP) $$\forall n[\phi(n) \vee \neg \phi(n)] \wedge \neg \forall n \neg \phi(n) \rightarrow \exists n \phi(n)$$

(where 'n' ranges over $\mathbb{N}$). The intuitive idea for (MP) is that under the hypothesis we can constructively find a solution n of the conclusion simply by performing a search through $\mathbb{N}$. It may be shown that $(CT_0)+(MP)$ is consistent over number theory though (CT_0) is inconsistent with full classical logic there. (The consistency proof can be given by Kleene's recursive realizability, which will be described in IV.) Various parts of analysis can be carried out under these assumptions, continuing the line sketched in § 5. For example, if f is continuous on $[a,b]$ then $\inf_{a \leq x \leq b} f(x)$ and $\sup_{a \leq x \leq b} f(x)$ exist. However, it cannot be proved that f takes on its minimum (resp. maximum) in $[a,b]$. The reason is provided by a well-known example due to Specker of a recursively continuous function on $[0,1]$ which has no recursive point at which f takes on its minimum. Various other basic results of classical analysis may also be contradicted by suitable recursion-theoretic examples, e.g. that if f is continuous on $[a,b]$ and $f(a) < 0$ then $\exists x(a < x < b \wedge f(x) = 0)$. (In the Russian school it is admitted that there are some 'peculiarities' to their approach.)

8. Recursive analogues to classical mathematics. We have in mind here a series of studies concerning analogues to classical notions where one uses recursive functions (or functionals or sets) in place of arbitrary objects of the same type. To be mentioned in particular is the work of Dekker and Myhill for set theory, Crossley for order theory, Malcev and Rabin for algebra, and Specker and Lacombe for analysis and topology (cf. Feferman 1975 for references). These have been

carried out informally, with no restriction on the logic or methods employed. In effect, though, at least (CT) is assumed (though not (CT_0), which is classically inconsistent), and indeed a corresponding stronger principle identifying partial functions on $\mathbb{N}$ to $\mathbb{N}$ with partial recursive functions. For example, in the Dekker-Myhill theory of recursive equivalence types one defines

$$(A \sim B) \leftrightarrow \exists f, g[f, g \text{ partial recursive} \wedge f \upharpoonright A : A \to B \wedge g \upharpoonright B : B \to A \wedge (gf) \upharpoonright A = 1_A \wedge (fg) \upharpoonright B = 1_B] ,$$

where A, B may be arbitrary subsets of $\mathbb{N}$. One positive result which is proved for this is a form of the Cantor-Bernstein Theorem:

$$A \sim (B \dotplus C) \wedge B \sim (A \dotplus D) \to A \sim B.$$

In analysis one considers recursive real numbers (i.e. $x = (\langle r_n \rangle, \mu)$ with both $\langle r_n \rangle, \mu$ recursive) and recursive functions of reals (defined in an appropriate way via recursive functionals on $\mathbb{N}^{\mathbb{N}}$). As with the Russian school, a number of 'peculiarities' are met in this version of analysis.

While these pursuits are not constructive they can be relevant to constructive approaches in the following ways. Where a recursion-theoretic analogue gives a positive result, i.e. where a classical theorem carries over, one can often prove the same theorem constructively. On the other hand, when a negative result is obtained by suitable counter example, it is usually possible to use such to get underivability of the classical theorem in a constructive system. However, neither of these is automatic. For example, the least number principle

$$\exists n\, \phi(n) \to \exists n[\phi(n) \wedge \forall m < n \neg \phi(m)]$$

which is frequently applied in recursion-theoretic arguments is not constructively derivable except for decidable ϕ.

9. Bishop's approach. In 1967 Bishop published his Foundations of constructive analysis in which he carried out an informal development of constructive analysis which looked much more like modern analysis than anything done previously by constructivists and which went substantially further mathematically. Bishop works with general notions of function and set regarded in informal constructive terms. He rejects the notion of f.c.s. as being obscure and unnecessary. Instead of trying to prove that all functions of reals are continuous, his view is: there is not much of interest we can say about arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$ or from $[a,b]$ to $\mathbb{R}$. Define $C([a,b], \mathbb{R})$ to be the uniformly continuous functions on $[a,b]$ to $\mathbb{R}$ and $C(\mathbb{R}, \mathbb{R})$ to be the functions which

are uniformly continuous on each compact interval. (These definitions will be explained in more detail below.) These are classes of central mathematical interest. In a sense, Bishop is working in lawlike analysis and the notions and principles he uses are contained either directly or implicitly in Brouwer's intuitionism, but simply without f.c.s. What is novel about Bishop's work is its spirit and execution, which is much more like everyday modern mathematics than anything previously done in a systematic constructive way. Indeed, a (philosophically unprepared) mathematician could pick up Bishop 1967 and read it as a straight piece of classical Cantorian mathematics. What would be puzzling to him is the more involved choice of certain notions and proofs, unless he also saw in what sense these were dictated by constructive requirements. It is this which is least successfully explained by Bishop. One of the main aims of the logical study of BCM is to elicit its underlying principles and to show how they may be interpreted constructively, as well as classically. One is led to consider constructive theories of functions, sets and classes which relate to BCM as theories like Zermelo - Fraenkel relate to Cantorian mathematics. Such systems could have been developed years ago, before Bishop, but it must be acknowledged that the work itself provided both the stimulus and a test for the adequacy of proposed theories. [5)]

10. Note on personal viewpoints. Bishop is a confirmed constructivist, as was Brouwer. Just as with Brouwer, he places the doing of constructive mathematics ahead of its logical study, regarding the latter as inessential. I am not a constructivist (nor a Platonist - it is harder to say what I am.) My main interests are logical and as a logician I am particularly interested in various forms of explicit mathematics (constructive, recursive, predicative, hyperarithmetic, inductive, Borelian, etc.) Of course this kind of position lends itself to greater objectivity, but there is also the possibility of insensitivity to, or neglect of, what are considered by a given school to be essential points.

11. Criteria of formalization. How well does a formal theory T represent an informal body of mathematics M? We judge this in terms of its adequacy and accordance.

(i) T is an adequate formalization of M if every concept, argument and result of M may be represented by a (basic or defined) concept, proof and theorem, resp. of T.

(ii) T is in accordance with (or faithful to) M if every basic concept of T corresponds to a basic concept of M and every axiom and rule of T corresponds to or is implicit in the assumptions and reasoning followed in M (i.e. T does not go beyond M conceptually or in principle).

5) Actually an informal constructive theory of functions and sets was outlined about the same time as Bishop's work in Tait 1968.

Remark. Formalisms always go syntactically beyond what is of ordinary interest, e.g. in practice we never assert $\phi \wedge \phi \wedge \phi$ or $\phi \rightarrow \psi$ where ϕ, ψ are unrelated.

We may refine (i),(ii) by considering whether the representation is direct or indirect. The idea of being (i)' directly adequate, resp. (ii)' directly in accordance with M seems clear. We would say that

(i)" T is indirectly adequate to M if there is a theory T^* directly adequate to M which can be translated into T (or otherwise reduced to T in an elementary way).

(ii)" T is indirectly in accordance with M if T can be translated or reduced to a theory T^+ which is directly in accordance with M.

A good formalization of M is one which is both directly adequate to and in accordance with M.

12. Illustrations of these criteria.

12.1. M = elementary number theory (non-analytic and non-algebraic).

$Z = Z^1$ = Peano's arithmetic with all primitive recursive function symbols.

$Z^1(+,\cdot)$ = Peano's arithmetic with just $+,\cdot$.

Z^2 = 2nd order arithmetic with full comprehension.

Z^1 is directly adequate to and directly in accordance with M.

$Z^1(+,\cdot)$ is directly in accordance with M but only indirectly adequate to it (by translation of Z^1).

Z^2 is directly adequate to M but not in accordance with M since the concept of $\mathcal{P}(\mathbb{N})$ as a completed totality is implicitly assumed in the comprehension scheme of Z^2.

12.2. M = classical analysis.

Z^ω = arithmetic in all finite types.

Z^ω is directly adequate to M.

Z^2 is indirectly adequate to M by reduction of the concepts that actually occur in practice to second order terms.

Z^ω is not in accordance with M; for in classical analysis we assume the totality $\mathbb{R}$, but not $\mathbb{R}^{\mathbb{R}}$ as a totality. [$C(\mathbb{R}, \mathbb{R})$ is assumed as a totality in the calculus of variations. Functionals in $C(\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$ are treated in modern analysis, but not higher type objects in any essential way.]

12.3. M = Cantorian set theory.

ZF = Zermelo - Fraenkel set theory.

ZF is directly adequate to M; is it in accordance with M? The question is raised since the idea of the cumulative hierarchy does not seem essential to M.

It is evident from these examples that the application of the criteria of formalization are reasonably objective, though there are cases of uncertainty.

13. Formal systems which have been proposed for BCM.

13.1. Bishop 1970, Goodman-Myhill 1972, both considered formalization in HA^{ω} + AC where HA^{ω} is intuitionistic arithmetic extended to finite types (HA = Heyting's intuitionistic arithmetic). HA^{ω} is directly in accordance with BCM. The question of Bishop's views on, and use of, AC is more delicate and will be taken up below. HA^{ω} + AC is inadequate to Bishop's theory of sets.

13.2. Martin-Löf 1975 (transfinite type theory). This is directly in accordance with BCM and adequate to everything but Bishop's theory of inductively defined classes (ordinals, Borel sets, etc.); it may also be naturally supplemented for the latter. It thus constitutes a good formalization of BCM. However, it is syntactically complicated, and not as flexible to work with as other theories to be discussed. This will be explained in more detail later. It should be added that Scott 1970 anticipated Martin-Löf 1975 in various ways.

13.3. Myhill 1975 CST (Constructive set theory), Friedman 1977. CST is a subtheory of IZFC/ZFC, intuitionistic ZFC, which like ZFC assumes extensionality and identifies functions with many one relations. Thus it is not directly in accord with constructive views, let alone BCM. It is indirectly adequate to BCM, as will be explained later. Friedman 1977 has considered a number of such theories and characterized their strength; he has also sketched interpretation into constructively justified theories, thus indirectly in accord with BCM.

13.4. Feferman 1975(T_0). This will be described in detail in Part II below. It is directly adequate to all of BCM. Accordance however is a matter of dispute; I shall argue that it is in accord, at least indirectly. T_0 is a type-free theory which is very amenable to metamathematical study and applications.

14. Some general features of BCM. As already said these incorporate the general features of constructivist mathematics outlined in §4, §5: the logic is intuitionistic, functions are given by rules, sets by defining properties; these are not interchangeable, and extensionality is not assumed. We detail in the following slight variants from §4-§5 above; novel points come with the treatment of subsets in 14.6.

14.1. Mathematical entities. The only objects which appear to be considered by Bishop are natural numbers, operations and sets, and such as are generated from these by pairing. Each such is considered to be presented by a finite symbolic expression.

14.2. Identity and equality. Two symbolic expressions are identical if they are presented in the same way - as in 4.11. We may take this to be literal identity or intensional identity. Each set considered has attached to it one or more relations of 'equality'. Notation: Bishop writes $\equiv$ for literal or intensional identity, $=$ for an equality relation on a set. We shall write instead $=$ for the first and $=_A$ for the second (but when there is no ambiguity we drop the subscript).

14.3. Operations and functions: Given sets $(A, =_A)$, $(B, =_B)$, f is an operation from A to B if it is a rule which assigns to each a in A an element $f(a)$ in B. f is a function from A to B if $a_1 =_A a_2 \to f(a_1) =_B f(a_2)$.

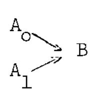
14.4. Function sets. Bishop says that for each A,B there is the set of all functions $F(A,B)$. Of course, if we consider A,B as sets endowed with the literal identity relation this implies that there is the set $O(A,B)$ of all operations from A to B. Since we can form these sets we can consider operations applied to operations, etc. Iterating F (or O) starting with N allows us to obtain the finite type hierarchy over N. We write B^A for $O(A,B)$.

14.5. Integers, Rationals, Reals. This follows §5 for $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$. However, Bishop defines $\mathbb{R}$ to consist of the following special class of Cauchy sequences: those $\langle x_n \rangle_{n \geq 1}$ such that $|x_n - x_m| \leq \frac{1}{n} + \frac{1}{m}$ for all n, m, (so the limit x satisfies $|x_n - x| \leq \frac{1}{n}$). $\mathbb{R}^+$ consists of pairs (x,k) where $x = \langle x_n \rangle$ is a real and $x_k > \frac{1}{k}$; for $\epsilon = (x_k - \frac{1}{k})$ this means $x \geq \epsilon > 0$. $(x,k) =_{\mathbb{R}^+} (y,\ell)$ is defined to mean $x =_{\mathbb{R}} y$.

14.6. Subsets. $\mathbb{R}^+$ is not a subset of $\mathbb{R}$ in the usual sense, but is one in the following sense of Bishop. By a subset (A,i) of B is meant a set A and operation $i : A \to B$ such that $a_1 =_A a_2 \leftrightarrow i(a_1) =_B i(a_2)$ (i.e. i is an injection of A into B). For x in B we say $x \in A$ if $x = i(y)$ with $y \in A$. Thus if we take $i(x,k) = x$ we have $(\mathbb{R}^+, i)$ a subset of $\mathbb{R}$. Similarly, intervals (a,b), $[a,b]$ etc. in $\mathbb{R}$ are given as subsets in this generalized sense. But we can also consider subsets in the usual sense where $i =$ identity operation on A.

14.7. Separation. Bishop recognizes that each property P applicable to elements of a set S determines the subset $A = \{x | x \in S \wedge P(x)\}$. (Implicitly, this is a subset in the usual sense; equality is defined to be $=_P$ restricted to A.)

14.8. Operations on subsets of a set. Using the more general concept of subset from 14.6, the operations of union and intersection take on more general (categorical) forms. Suppose given subsets (A_o, i_o) and (A_1, i_1) of B:



$(A_0 \cup A_1, i)$ is defined by Bishop to consist of all pairs (k,a) where $k=0$ and $a \in A_0$ or $k=1$ and $a \in A_1$; further, $i(k,a) = i_k(a)$ for $k=0,1$. Note this is essentially a form of disjoint union. $(A_0 \cap A_1, j)$ is further defined to consist of all (a_0, a_1) with $a_0 \in A_0$ and $a_1 \in A_1$ and $i_0(a_0) = i_1(a_1)$; then one takes $(j(a_0,a_1) = i_0(a_0)$. Note that $A_0 \cap A_1$ in this sense is contained in $A_0 \times A_1$.

14.9. Families of subsets of a set. By a family of subsets of a set B, indexed by T, is meant an operation f which associates with each $t \in T$ a subset $f(t) = (A_t, i_t)$ of B, in such a way that equal sets are associated with equal indices. The family is indicated by $\langle (A_t, i_t) \rangle_{t \in T}$ or $\langle A_t \rangle_{t \in T}$ or just $\langle A_t \rangle$. For simplicity, we shall only consider in the following the cases that the index set T is simply supplied with literal identity as its equality relation.

14.10. Operations on families. As an extension of the operations in 14.8 one defines $(\bigcup_{t \in T} A_t, i)$ and $(\bigcap_{t \in T} A_t, j)$ as subsets of B by:

$$\bigcup_{t \in T} A_t = \{(x,t) \mid x \in A_t\} \quad \text{and} \quad i(x,t) = i_t(x);$$

$$\bigcap_{t \in T} A_t = \{g \mid \forall t \in T(g(t) \in A_t) \wedge \forall t_1, t_2 \in T(i_{t_1}(g(t_1)) = i_{t_2}(g(t_2)))\} \quad \text{and}$$

$$j(g) = i_{t_0}(g(t_0)) \quad \text{for} \quad t_0 \in T.$$

The union is again a form of disjoint sum that we call the join of $\langle A_t \rangle$; it will be denoted below by $\sum_{t \in T} A_t$. In effect, the intersection is formed by separation from the cartesian product $\prod_{t \in T} A_t = \{g \mid \forall t \in T(g(t) \in A_t)\}$, on which equality $g_1 = g_2$ is defined by $\forall t \in T(g_1(t) =_{A_t} g_2(t))$, i.e. $\forall t \in T(i_t(g_1(t)) =_B i_t(g_2(t)))$.

14.11. Pre-joined families. An alternative definition of family which Bishop says could be considered is a subset A of $B \times T$. Certainly, given any such A we can define $f(t) = A_t = \{x \mid (x,t) \in A\}$. Then $A = \bigcup_{t \in T} A_t$ (extensionally). Thus we call such a family pre-joined, i.e. its prescription already guarantees existence of its join. In general though we need a join axiom which tells us that if f is a family $\langle A_t \rangle_{t \in T}$ then the join $J(T,f)$ exists.

14.12. Borel sets. These are inductively generated in a topological space from certain basic sets by the operations of union and intersection applied to countable families.

Abstractly this has the form:

(i) $B_0 \subseteq B$

(ii) $(f: \mathbb{N} \to B)$ implies $(J(\mathbb{N}, f) \in B \wedge I(\mathbb{N}, f) \in B)$.

(iii) if a property holds of all elements of B_0 and holds of $J(\mathbb{N}, f)$ and $I(\mathbb{N}, f)$ whenever it holds of $f(n)$ for each n, then it holds of all elements of B.

14.13 Principles in the general theory of integration. The Borel sets in a measure space are used in the development of integration theory in Bishop 1967, Ch.7. The theory of measure and integration was redeveloped by Bishop-Cheng 1972 without the use of Borel sets. This is an abstract theory, i.e. one starts with an arbitrary 'integration space' X and associates with it a certain completion L(X). It was pointed out by Friedman that the basic definitions in the latter approach make prima-facie use of the power-set operation which, as we have seen, is constructively problematic. However, this is only necessary if one wants L(X) again to be a set. The notion of being a member of L(X) does not require the power-set axiom and in that sense one can carry out abstract integration theory without this principle or the generalized inductive principles behind Borel sets. For more detailed examination of the issue here cf. Feferman 1978 §4. In any case, the potential (albeit marginal) mathematical utility of both generalized inductive and power-set principles in BCM makes them of interest for logical study. The former are incorporated directly in T_0, since they have constructive character.

15. General features of BCM, contd: Existential definitions and witnessing information. Notions which are defined classically using existential information are frequently replaced in BCM (as well as in other schools of constructive mathematics) by corresponding notions in which witnessing information is explicitly shown. This is required to carry out constructive operations on the objects satisfying the given notion.

15.1. Examples.

(i) $\mathbb{R}^+$. We have already explained its definition in §5 (as needed to make the operation of inverse constructive).

(ii) Limits of sequences of reals. By a convergent sequence of reals is meant a triple $(\langle x_n\rangle, x_0, m)$ where x_0 and each $x_n (n \geq 1)$ belongs to $\mathbb{R}$ and m is a function of positive integers such that (for all $k \geq 1$) $|x_n - x_0| \leq \frac{1}{k}$ for all $n \geq m(k)$; m is called a modulus-of-convergence function for the sequence.

(iii) Continuous functions. By a (uniformly) continuous function f on a compact interval $I = [a,b]$ is meant a pair (f,ω) with $f \in F(I,\mathbb{R})$ (i.e. $f: I \to \mathbb{R}$ preserving $=_{\mathbb{R}}$) and $\omega: \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(x)-f(y)| \leq \epsilon$ whenever $|x-y| \leq \omega(\epsilon)$; ω is called a modulus-of-uniform-continuity function for f. The set of all continuous functions from I to $\mathbb{R}$ is denoted by $C(I, \mathbb{R})$; it is a subset of $F(I,\mathbb{R})$ by $i\langle f,\omega\rangle = f$. (This function ω is needed for example when performing the operation of integration of f over I.)

15.2. The logical pattern. In each case we are spelling out a property $P(x)$ involving existential quantifiers. In the above examples (i)-(iii) $P(x)$ is, respectively:

(i) $\exists k \geq 1 (x_k > \frac{1}{k})$

(ii) $\exists x_o \forall k \geq 1 \exists m \forall n \geq m (|x_n - x_o| \leq \frac{1}{k})$

(iii) $\forall \epsilon > 0 \exists \delta > 0 \forall x \forall y (|x-y| < \delta \rightarrow |f(x)-f(y)| < \epsilon)$.

With each such property is associated another $P^*(x,w)$ where w is some witnessing information that realizes or verifies $P(x)$. These properties are related by

(1) $\exists w P^*(x,w) \rightarrow P(x)$ and

(2) if (AC) is assumed, $P(x) \rightarrow \exists w P^*(x,w)$.

We shall also call P^* a spelled-out form of P. For example, in case (ii), w consists of x_o together with m as a function of k.

15.3. Having your cake and eating it too. Often Bishop defines a set A in what appears to be the form $\{x|P(x)\}$ (the unofficial definition) but then says that he is really defining the set $A^* = \{(x,w)|P(x,w)\}$ (the official definition).[6] He speaks of x being a member of A^* when it is really only x coupled with some side information w which can be considered to be a member. Then one treats operations on A^* as if they were operations on A. For example, the set A of (uniformly) continuous functions on I to $\mathbb{R}$ is officially defined as the set A^* of all (f,ω) satisfying the condition of 15.1(iii). But ω is not explicitly revealed in the operation $\int: C(I,\mathbb{R}) \rightarrow \mathbb{R}$ when written in the form $\int_a^b f(x)dx$.

There is a certain casualness in Bishop 1967 about mentioning the witnessing information as one goes along. Constructivity in theory requires that it be mentioned, but one is looser in practice in order to keep that from getting too heavy. Practice then looks very much like everyday analysis and it is hard to see what the difference is unless one takes the official definitions seriously.

15.4. A concrete illustration. The preceding is illustrated by a relatively simple example of a proof from Bishop 1967, but in which several spelling-out steps are already implicitly involved. This is for the theorem that every continuous function on a compact interval has a l.u.b. The reader should compare the following with the original as indicated by the page references.

Definition (p.34). c is called a l.u.b. of A (for $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$) if (i) $x \leq c$ for all x in A and (ii) for each $\epsilon > 0$ there exists x in A with $c - x < \epsilon$.

Spelled out, (ii) requires that we provide a function g such that $\forall \epsilon > 0$ ($g(\epsilon)$ in A and $c - g(\epsilon) < \epsilon$).

6) With reference to 15.1(i)-(iii) the reader should compare Bishop 1967 pp. 18, 19, pp. 26, 27 and p.34, resp.

Theorem (p.34). Suppose A is a subset of $\mathbb{R}$ such that for each $\epsilon > 0$ there exist $y_1,\ldots,y_n$ in A such that for each x in A at least one of the numbers $|x-y_1|,\ldots,|x-y_n|$ is $\leq \epsilon$. (Such a set is called totally bounded). Then l.u.b. A exists.

Spelled out, the definition of being totally bounded (contained in the statement of this theorem) requires that we have two functions h and j such that for each $\epsilon > 0$, $h(\epsilon)$ is a finite sequence $(n,\langle y_1,\ldots,y_n\rangle)$ with each y_i in A (so $n = \ell h(h(\epsilon))$) and for each x in A and $\epsilon > 0$,

$$j(x,\epsilon) \leq \ell h(h(\epsilon)) \quad \text{and} \quad |x-y_{j(x,\epsilon)}| \leq \epsilon .$$

With this understanding, the proof of the above theorem proceeds as follows. Given any $k \geq 1$, choose $y_1,\ldots,y_n$ in A such that for each x in A, some $|x-y_j| \leq \epsilon$. The choice of $\langle y_1,\ldots,y_n\rangle$ is given by $h(1/k)$ and of j by $j(x, 1/k)$. Next it is shown that

(1) for some m with $1 \leq m \leq n$ we have $y_m \geq \max\{y_1,\ldots,y_n\} - k^{-1}$.

For, each y_i is given as a Cauchy sequence of rationals $y_i = \langle y_p^i\rangle_{p\geq 1}$, i.e. each y_p^i is in Q and $|y_p^i - y_q^i| \leq \frac{1}{p}+\frac{1}{q}$ (from p.15). Take $p = 4k$ and find m such that $y_p^m \geq y_p^i$ for $i=1,\ldots,n$. Then for $q \geq p$

$$(y_q^m - y_q^i) + \frac{1}{k} = (y_q^m - y_p^m) + (y_p^m - y_p^i) + (y_p^i - y_q^i) + \frac{1}{k} \geq (y_q^m - y_p^m) + (y_p^i - y_q^i) + \frac{1}{k} \geq 0$$

since $|y_q^i - y_p^i| \leq \frac{2}{p} = \frac{1}{2k}$. It follows that $y_q^m \geq (y_q^i - \frac{1}{k})$ for $q \geq p$ and then that $y_m \geq y_i - \frac{1}{k}$ as required for (1). Note that m in(1) is found as a function of k, say $m = m(k)$. Let $x_k = y_{m(k)}$. It is easily shown that $\langle x_k\rangle$ is a Cauchy sequence of reals. Using a result from p.27, it is shown that $\langle x_k\rangle$ converges and we can find its limit x_o. We claim that

(2) x_o is a l.u.b. for A.

For, given any x in A, choose $y_1,\ldots,y_n$ as above; then $x-x_k = (x-y_j)+(y_j-y_{m(k)}) < \frac{2}{k}$ when $j = j(x, 1/k)$. Hence $(x-x_o) = \lim_{k\to\infty}(x-x_k) \leq \lim_{k\to\infty}(2/k) = 0$; thus $x \leq x_o$ for all x in A. The function g required by the definition of l.u.b. is provided by $g(\epsilon) = x_k$ where k is chosen so that $\frac{2}{k} < \epsilon$, such k being calculable from the information which presents ϵ as a member of $\mathbb{R}$.

Corollary (p.35). If $f:[a,b]\to\mathbb{R}$ is continuous then $\{f(x)|x\epsilon[a,b]\}$ has a l.u.b.

Proof. f is implicitly provided with a modulus-of-continuity function ω. Given any $\epsilon > 0$ choose $a = a_o < a_1 < \ldots < a_n = b$ such that $(a_{i+1} - a_i) < \omega(\epsilon)$ for each

$i = 0,\dots,n-1$. Let $a \leq x \leq b$. It is claimed that we can find j such that $|x-a_j| < \omega(\varepsilon)$. To do this we consider $x = \langle x_p \rangle$ and $a_i = \langle a_p^i \rangle$ each presented as a Cauchy sequence of rationals. Every x_p can be compared with each of $a_p^0,\dots,a_p^i,\dots,a_p^n$. For p sufficiently large (determined by $\omega(\varepsilon)$), the required j is found from this comparison. This shows that $A = \{f(x) \mid x \in [a,b]\}$ is totally bounded (as required by the official definition), and we can apply the preceding theorem.

The reader may wish to reconstitute one or two other proofs from Bishop 1967 in the same manner. (Another instructive example is provided by the proof of the Baire category theorem, p.87).

15.5 A theoretical setting for 15.3. In Part IV we shall present a theory T_0^* extending T_0 in which the basic membership relation is refined to a 3-placed relation $x \varepsilon_w A$ and in which $x \varepsilon A$ is defined as $\exists w(x \varepsilon_w A)$. With reference to the logical pattern of 15.2, 15.3 one has $A^* = \{(x,w) \mid x \varepsilon_w A\}$, so that the 'unofficially' defined A actually determines the 'officially' defined A^*. T_0^* can be reduced to T_0 so that in this theory we can have our (constructive) cake and eat it too.

15.6 Remark on witnessing data in classical mathematics. The practice of suppressing official parts of the defining data is also frequent in classical mathematics, e.g. algebraic or topological structures are simply referred to by their underlying sets. However the practice is more wholesale in BCM.

II. The theory T_0.

As presented here this theory is a minor modification of that introduced in Feferman 1975; the differences are explained below. For the reader's convenience a good deal of the material from secs. 2-3 loc. cit. is incorporated in the following. There are also some novel points.

1. The language of T_0; syntactical notions.

1.1 Variables and constants. The language $\mathcal{L}(T_0)$ is two-sorted.

Individual (or general) variables	$a,b,c,\dots,x,y,z$
Class variables	$A,B,C,\dots,X,Y,Z$
Individual constants	$k,s,p,p_1,p_2,d,0,s_{\mathbb{N}},p_{\mathbb{N}},c_n,i,j$
Class constants	$\mathbb{N}$

We use $t,t_1,t_2,\dots$ to range over variables or constants of either sort.

1.2 Atomic formulas are all those of the form $t_1 = t_2$, $App(t_1,t_2,t_3)$ and $t_1 \varepsilon t_2$. In addition there is an atomic formula $\bot$ with no free variables, interpreted as falsity.

1.3 Logical operations: $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$

1.4 Formulas are generated from atomic formulas by applying $\wedge, \vee, \rightarrow$, $\exists x, \forall x, \exists X, \forall X$.

Notation: ϕ, ψ, θ range over arbitrary formulas. $(\neg\phi) =_{def} (\phi \rightarrow \bot)$. $(\phi \leftrightarrow \psi)$, $\exists! x\phi$, $(\forall x \in A)\phi$, $(\exists x \in A)\phi$ are defined as usual. $\underline{x}, \underline{X}, \underline{t}$ represent sequences of variables or terms, $\phi(\underline{x}, \underline{X})$ is written for a formula all of whose free variables are among $\underline{x}, \underline{X}$. In such expressions as $\exists y \psi(\underline{x}, y, \underline{X})$ it is assumed that y is not in the list $\underline{x}$ and similarly for $\exists Y \psi(\underline{x}, \underline{X}, Y)$.

2. Informal interpretation of the language. The individual variables range over the full universe of discourse of T_0, hence are also called general variables. These are to be thought of as mental objects (including rules and properties) or as symbolic representations of such objects. Then $=$ is interpreted as intensional identity or, in the latter view as literal identity of syntactic objects. The relation $App(t_1, t_2, t_3)$ is understood to hold when t_1 is a (rule or) operation which has value t_3 at the argument t_2. Since it is not assumed that every operation is total we shall write $t_1 t_2 \simeq t_3$ for $App(t_1, t_2, t_3)$. We write $t_1 t_2 \downarrow$ for $\exists z\, App(t_1, t_2, z)$. (This notation is expanded below.)

The class variables range over one-placed properties. $t \in X$ is interpreted as: t has the property (given by) X. (We may also think of 'class' as short for 'classification'; classes are not conceived extensionally.) Class variables only range over a part of the universe of discourse. To express that an individual t happens to be a class we simply use the formula

$$C\ell(t) =_{def} \exists X(t = X).$$

The constants k, s are certain basic combinatory operations which permit one to form the constant operations and carry out the process of substitution, resp.; p, p_1, p_2 are operations of pairing and projection. d is an operation which gives definition-by-cases on $\mathbb{N}$, where $\mathbb{N}$ denotes the class of natural numbers, with least element 0 and operators of successor $s_{\mathbb{N}}$ and predecessor $p_{\mathbb{N}}$. The constants c_n, i, j represent certain class-formation operations, corresponding respectively to comprehension, inductive generation and join.

3. Application terms. The language $\mathcal{L}(T_0)$ is formally extended by a binary operator $\alpha(-,-)$ which is interpreted as the operation of application. We use $\tau, \tau_1, \tau_2, \ldots$ to range over the terms of this extended language, which are called application terms (a.t.'s). We write $\tau_1\tau_2$ for $\alpha(\tau_1 \tau_2)$. Thus the a.t.'s are generated from the variables and constants of $\mathcal{L}(T_0)$ by closure under α. Since a.t.'s may not have defined values, relations between these are explained as formulas in $\mathcal{L}(T_0)$ in the following way:

$$(\tau \simeq x) =_{\text{def}} \begin{cases} (t=x) & \text{when } \tau \text{ is a term } t \text{ of } \mathcal{L}(T_o) \\ \exists y_1 \exists y_2 (\tau_1 \simeq y_1 \wedge \tau_2 \simeq y_2 \wedge y_1 y_2 \simeq x) & \text{when } \tau \text{ is } \tau_1\tau_2 \end{cases}$$

$(\tau_1 \simeq \tau_2) =_{\text{def}} \forall x [\tau_1 \simeq x \leftrightarrow \tau_2 \simeq x]$

$(\tau \downarrow) =_{\text{def}} \exists x (\tau \simeq x)$.

We write $(\tau_1 = \tau_2)$ for $(\tau_1 \simeq \tau_2)$ when $(\tau_1\downarrow) \wedge (\tau_2\downarrow)$ is known or assumed.

$\phi(\tau) =_{\text{def}} \exists x(\tau \simeq x \wedge \phi(x))$.

In particular, $(\tau \in X)$ is $\exists x(\tau \simeq x \wedge x \in X)$.

Parentheses in $\tau_1\tau_2 \ldots \tau_n$ are supposed to be associated to the left as $(\ldots(\tau_1\tau_2)\ldots)\tau_n$. We write (τ) for τ, (τ_1,τ_2) for $p\tau_1\tau_2$ and $(\tau_1,\tau_2,\ldots,\tau_n)$ for $(\tau_1,(\tau_2,\ldots,\tau_n))$. This explains the notation $z(\tau_1,\ldots,\tau_n)$ or $z(\underline{\tau})$ for any $n \geq 1$. Finally, $\tau' =_{\text{df}} s_{\mathbb{N}}\tau$.

4. Further syntactical notions .

4.1 Stratified formulas are those (in $\mathcal{L}(T_o)$) which contain class variables or constants only on the right-hand side of $\in$ atomic formulas, i.e. only in contexts of the form $t \in X$ or $t \in \mathbb{N}$, where t is an individual variable or constant. Thus all the other atomic formulas of a stratified formula are relations of equality and applications between individual terms. Formally, stratified formulas may be thought of as 2nd order formulas with the sort of individuals specifying the 1st order level.

4.2 Elementary formulas are those stratified formulas which contain no bound class variables. These are also sometimes called predicative formulas, the others being impredicative. In an elementary formula $\phi(\underline{x}, \underline{X})$ classes are not referred to in any general way; we only require understanding membership in the given X_i . Elementary formulas may also be considered as the 1st order (stratified) formulas.

4.3 Comprehension notation. Let n be the Gödel number $\ulcorner\phi(x, \underline{y}, \underline{Z})\urcorner$ of ϕ with a specified inclusive list of variables $x, \underline{y}, \underline{Z}$. We put

$$\{x \mid \phi(x,\underline{y}, \underline{Z})\} =_{\text{def}} c_n(\underline{y}, \underline{Z}).$$

This shows the process of class formation by comprehension as a uniform function of the parameters $\underline{y}, \underline{Z}$ of the defining formula ϕ. The instances of comprehension are not all equally evident; the mc ⊦ evident are those corresponding to elementary ϕ, and only those are immediately accepted in T_o .

5. Axioms and logic of T_0.

5.1 The logic of T_0 is that of the intuitionistic two-sorted predicate calculus with equality. There is in addition a basic ontological axiom relating the sorts, namely $\forall X \exists x(X = x)$. Note that this justifies the formalism in 4.3 above where one applies operations to classes.

5.2 Non-logical axioms. These fall into several groups.

APP (Applicative axioms)

(o) (unicity) $x\,y \simeq z \wedge x\,y \simeq w \to z = w$

(i) (constants) $kxy\downarrow \wedge kxy = x$

(ii) (substitution) $sxy\downarrow \wedge sxyz \simeq xz(yz)$

(iii) (Pairing, projections) $pxy\downarrow \wedge p_1z\downarrow \wedge p_2z\downarrow \wedge p_1(pxy) = x \wedge p_2(pxy) = y$

(iv) (definition by cases on $\mathbb{N}$) $x,y \in \mathbb{N} \to (x = y \to dxyab = a) \wedge (x \neq y \to dxyab = b)$.

(v) (zero, successor, predecessor) $x,y \in \mathbb{N} \to x'\downarrow \wedge p_{\mathbb{N}}y\downarrow \wedge p_{\mathbb{N}}(x') = x \wedge x' \neq 0 \wedge (x' = y' \to x = y)$.

The remaining axioms are class existence axioms. Note from I(iii) that $c_n(y_1,\ldots,y_m, Z_1,\ldots,Z_p)\downarrow$ for all $\underline{y}, \underline{Z}$.

ECA (Elementary comprehension). For each elementary $\phi(x,\underline{y},\underline{z})$ we take:

$$\exists X\,[\{x|\phi(x,\underline{y},\underline{Z})\} = X \wedge \forall x[x \in X \leftrightarrow \phi(x,\underline{y},\underline{Z})]]$$

$\mathbb{N}$ (Natural numbers)

(i) (closure) $0 \in \mathbb{N} \wedge \forall x(x \in \mathbb{N} \to x' \in \mathbb{N})$

(ii) (induction) $\phi(0) \wedge \forall x(\phi(x) \to \phi(x')) \to \forall x \in \mathbb{N}\,\phi(x)$

where $\phi(x) = \phi(x,\ldots)$ is an arbitrary formula of $\mathcal{L}(T_0)$.

J (Join)

$\forall x \in A \exists Y(fx \simeq Y) \to \exists X[j(A,f) = X \wedge \forall z[z \in X \leftrightarrow \exists x \exists y(z = (x,y) \wedge x \in A \wedge y \in fx)]]$.

IG (Inductive generation)

(i) (closure) $\exists I[i(A,R) = I \wedge \forall y((y,x) \in R \to y \in I) \to x \in I]]$

(ii) (induction) $\forall x \in A[\forall y((y,x) \in R \to \phi(y)) \to \phi(x)) \to (\forall x \in i(A,R))\phi(x)$

where $\phi(x) = \phi(x,\ldots)$ is an arbitrary formula of T_0. This completes the list of axioms: $T_0 = APP + ECA + \mathbb{N} + J + IG$.

6. Relation of T_0 as given here with that of Feferman 1975.

(a) Previously we took a one-sorted language with an additional predicate $C\ell(x)$; the variables $A,B,C,\ldots,X,Y,Z$ were introduced by convention to range over $C\ell$.

(b) We defined $x' = (x,0)$ and $p_{\mathbb{N}} = p_1$ and took the axiom $(x,y) \neq 0$; then the axiom APP(v) as given here was derived.

(c) The constant $\mathbb{N}$ was defined as $i(A,R)$ for

$$A = \{x|x = 0 \vee x \simeq (p_{\mathbb{N}}x)'\} \text{ and}$$

$$R = \{z|\exists x,y\,(z = (y,x) \wedge x = y')\} \qquad \text{(predecessor relation).}$$

The reason for listing $\mathbb{N}$ as a special axiom here is so as to consider the strength of T_0 - IG while still including $\mathbb{N}$.

(d) Previously we used definition by cases on the universe instead of just on $\mathbb{N}$, as will be explained next.

7. Some variants of the axioms which will be considered.

D_V (Definition by cases on the universe). This is the following in place of APP(v):

$(x = y \vee x \neq y) \wedge dxyab\downarrow \wedge (x = y \rightarrow dxyab = a) \wedge (x \neq y \rightarrow dxyab = b)$.

Then $T_0 + D_V$ is equivalent to the system T_0 of Feferman 1975.

Note. The clause $(x,y \in \mathbb{N} \rightarrow x = y \vee x \neq y)$ was not needed in APP(v) since it is derivable from the other axioms.

CA_1 (First order comprehension). This is another denotation of ECA.

CA_2 (Second order comprehension). The same scheme as for CA_1 except that ϕ may now be any stratified formula in $\{x|\phi(x,\underline{y},\underline{Z})\}$.

SEP_1 (First order separation). This is CA_1 restricted to formulas of the form $x \in A \wedge \psi(x,\underline{y},\underline{Z})$ (with parameters $\underline{y},\underline{Z},A$). $\{x \in A|\psi(x,\underline{y},\underline{Z})\}$ is written for $\{x|x \in A \wedge \psi(x,\underline{y},\underline{Z})\}$.

SEP_2 (Second order separation). The same as SEP_1 with any stratified ψ.

$\mathbb{N}\restriction$ (Restricted induction on $\mathbb{N}$). Here one replaces the induction schema $\mathbb{N}$(ii) by the specific instance

$$0 \in X \wedge \forall x\,(x \in X \rightarrow x' \in X) \rightarrow \mathbb{N} \subseteq X$$

(where $(X \subseteq Y) =_{def} \forall x(x \in X \rightarrow x \in Y)$). Note that $\mathbb{N}\restriction$ can be used to derive any instance

$$\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(x')) \rightarrow \forall x \in \mathbb{N}\,\phi(x)$$

for which $\{x|\phi(x,\ldots)\}$ is known to exist as a class.

IG$\restriction$ (Restricted induction for IG). Analogously replaces IG(ii) by

$$\forall x \in A\,\{\forall y((y,x) \in R \rightarrow y \in X) \rightarrow x \in X\} \rightarrow i(A,R) \subseteq X\,.$$

Remark. We could formulate a generalized inductive definition axiom GID expressing for any elementary $\psi(x,X)(= \psi(x,X,\ldots))$ in which X occurs only positively, the existence of a class I which is the least predicate satisfying $\forall x[\psi(x,X) \rightarrow x \in X]$. This stronger axiom is not evidently constructive (GID$\restriction$ can be derived if one accepts the impredicative comprehension principle CA_2).

Note. IG itself becomes still more evident if one writes $y <_R x$ for $(y,x) \in R$. Then $i(A,R)$ is the class of $<_R$ - accessible elements of A, which may be pictured as the elements of A sitting atop well-founded trees branching by the $<_R$ relation.

MIG$\restriction$ (Restricted monotone inductive definition)

$$\forall X \exists Y[fX \simeq Y] \wedge \forall X_1,X_2[X_1 \subseteq X_2 \rightarrow fX_1 \subseteq fX_2] \rightarrow \exists I\{fI \subseteq I \wedge \forall X[fX \subseteq X \rightarrow I \subseteq X]\}\,.$$

By adjunction of a suitable constant, we could also express I uniformly as a function of f. Here f represents a monotone operation on classes to classes. This is stronger than GID$\upharpoonright$ (in the presence of ECA), since positivity is weaker than monotonicity.

8. Product and power class axioms.

P (Product axiom) $\forall x \in A\, \exists Y(fx \simeq Y) \rightarrow \exists X\, \forall z[z \in X \leftrightarrow \forall x \in A(zx \in fx)]$.

We shall prove (11.2) that P follows from T_0; however, it does not if CA_1 is replaced by SEP_1.

POW^+ (Strong power class axiom). $\forall A\, \exists B\, \forall x[x \in B \leftrightarrow \exists X(X \subseteq A \wedge x = X)]$.

POW (Weak power class axiom).

$$\forall A\, \exists B\, \forall x[x \in B \rightarrow \exists X(X \subseteq A \wedge x = X) \wedge \forall X(X \subseteq A \rightarrow \exists Y(X \equiv Y \wedge Y \in B))]$$

where $X \equiv Y$ is the relation of extensional equality between classes,

$$(X \equiv Y) =_{def} (X \subseteq Y) \wedge (Y \subseteq X).$$

As remarked previously, the constructive status of POW^+ (or even POW) is unclear.

Note. Each of these can be expressed uniformly by adjunction of suitable constants.

9. Theories related to T_0 which are to be considered here.

9.1 $EM_0 = APP + ECA + \mathbb{N}$

$EM_0\upharpoonright = APP + ECA + \mathbb{N}\upharpoonright$.

Remark. The notation 'EM' comes from 'Explicit mathematics'. $EM_0\upharpoonright$ is a minimal constructive theory in the present framework. We shall consider the effect of adding, variously, D_V, J, IG$\upharpoonright$ and IG to these theories.

9.2 A theory of sets S_0.

$$S_0 = APP + SEP_1 + \mathbb{N} + J + P + IG.$$

The class variables here can be interpreted as sets ("small" classes). (The product axiom P is added here according to the remark made in § 8.)

9.3 A theory of sets and classes $T_0(S)$. This may be obtained from T_0 by adjoining a constant S for the class of all sets, for which suitable closure conditions are expressed by additional axioms. A theory of this character was presented in Feferman 1978, but of greater generality, since sets there are taken to be pairs (A,E) where E is an equality relation on A.

10. Consequences of $EM_0\upharpoonright$. Throughout the following all statements are to be considered semi-formally. They are all provable in $EM_0\upharpoonright$.

Notation. We shall often use letters like 'f', 'g', 'h' for individual variables being treated in operator situations. The letters 'k','n','m','p' are now reserved to range over $\mathbb{N}$.

10.1 Abstraction. For each application term τ and variable x we can find a term τ^* with variables $\subseteq$ var $(\tau) - \{x\}$ such that

$$\tau^* \downarrow \wedge \forall x [\tau^* x \simeq \tau] .$$

τ^* is denoted $\lambda x(\tau)$ or $\lambda x(\tau[x])$. (Intuitively, $\tau^*\downarrow$ because it always denotes a rule, whether or not $\tau[x]\downarrow$ at any x.) The proof is carried out by induction on τ (just as in total combinatory theories):

(i) If τ is x, we take skk for $\lambda x(x)$ since $skkx \simeq kx(kx) = x$.
(ii) If τ is an individual term $t \not\equiv x$, we take kt for $\lambda x(t)$ since $ktx = t$.
(iii) If $\tau = \tau_1\tau_2$ we want $\tau^* x \simeq \tau_1[x]\tau_2[x]$. But $\tau_1[x]\tau_2[x] \simeq (\tau_1^* x)(\tau_2^* x) \simeq s\tau_1^* \tau_2^* x$, so we can take $\tau^* = s\tau_1^* \tau_2^*$ in this case. (Only APP(i),(ii) are applied here).

10.2 The recursion theorem. We can find a fixed r such that for all f:

$$rf\downarrow \text{ and for } g = rf,\ \forall x(gx \simeq fgx).$$

For the proof take $h = \lambda y\, \lambda x f(yy)x$, so $hy\downarrow$ for all y. In particular $hh\downarrow$ and $hh = \lambda x(f(hh)x)$. Thus $g = hh$ serves as rf. We can take

$$r = \lambda f((\lambda y \lambda x f(yy)x)(\lambda y \lambda x f(yy)x)).$$

10.3 Recursion on $\mathbb{N}$. For any a,f we can find g satisfying:

$$gx \simeq \begin{cases} a & \text{if } x = 0 \\ f(x, g(p_{\mathbb{N}} x)) & \text{if } x \in \mathbb{N} \text{ and } x \neq 0 \end{cases} .$$

Namely, g is found by the recursion theorem so as to make

$$gx \simeq dx\, 0\, a(f(x,g(p_{\mathbb{N}} x))).$$

It follows that

(i) $g0 = a$

(ii) $gn' \simeq f(n,gn)$ for any n.

Then for any a,A,f we have:

(iii) $a \in A \wedge \forall n \forall y [y \in A \rightarrow f(n,y) \in A] \rightarrow \forall n(gn \in A)$,

with the conclusion being proved from the hypothesis by restricted induction, since $\{x | gx \in A\}$ is a class.

10.4 Arithmetic in $EM_0\restriction$. It follows from the preceding that all primitive recursive functions can be defined in $EM_0\restriction$. Furthermore, every arithmetical formula is equivalent in this theory to an elementary formula, hence defines a class. Thus the scheme of induction for arithmetical formulas holds in $EM_0\restriction$. Hence the intuionistic system HA of (Heyting's) arithmetic is contained in $EM_0\restriction$.

10.5 Bounded and unbounded minima. Using recursion on $\mathbb{N}$ we can define $\prod_{m \leq n} fm$ so that $\forall m \leq n(fm \in N) \leftrightarrow \prod_{m \leq n} fm \in N$ and $\prod_{m \leq n} fm = 0 \leftrightarrow \exists m \leq n(fm \leq 0)$.

Then further we can obtain $(\mu m \leq n)(fm \simeq 0)$ which is defined under the same conditions. By the recursion theorem one finds g such that

$$g(f,n) \simeq \begin{cases} (\mu m \leq n)\ (fm \simeq 0) & \text{if } \exists m \leq n(fm \simeq 0) \\ g(f,n') & \text{otherwise.} \end{cases}$$

Let $\mu f = g(f,0)$. It is seen that μf is defined and equal to $\mu n(fn \simeq 0)$ just in case $\exists n(fn = 0 \wedge \forall m < n(fm \in N))$.

10.6 Partial recursive functions; forms of Church's Thesis. The enumeration of partial recursive functions $\{k\}$ for $k \in \mathbb{N}$ can now be defined as usual in $EM_0 \upharpoonright$ with $\{k\}(n) \simeq U(\mu n T_1(k,n,m))$. We now introduce the function-mapping notation:

$$(f : A \to B) =_{def} \forall x \in A\,(fx \in B\,).$$

The three forms of Church's thesis described in I.4.8 can be formulated in $EM_0 \upharpoonright$ as follows.

CT_0 is the scheme $\forall n \exists m \phi(n,m) \to \exists k\ \forall n[\{k\}(n) \downarrow \wedge \phi(n, \{k\}(n))]$

CT_1 is $\forall f[(f:\mathbb{N} \to \mathbb{N}) \to \exists k\, \forall n(fn = \{k\}(n))]$

CT_2 is $\forall f[\forall n(fn \downarrow \to fn \in \mathbb{N}) \to \exists k\, \forall n\,(fn \simeq \{k\}(n))]$.

(CT_2 was suggested by Beeson; it expresses that every partial function on $\mathbb{N}$ coincides with a partial recursive function there.) Obviously $CT_2 \to CT_1$, and as we remarked in 4.8, $CT_1 + AC_{\mathbb{N}} \to CT_0$. The consistency of these with T_0 will be taken up in Parts III and IV.

10.7 Elementary operations on classes. We now turn to uses of ECA. The following give operations having class values as functions of the individual and class parameters shown.

$V = \{x | x = x\}$

$\Lambda = \{x | \bot\}$

$\{a,b\} = \{x | x = a \vee x = b\}$

$-A = \{x | x \notin A\}$

$A \cup B = \{x | x \in A \vee x \in B\}$, $A \cap B = \{x | x \in A \wedge x \in B\}$

$A \times B = \{z | \exists x \in A\, \exists y \in B\ \ z = (x,y)\}$

$B^A = \{f | f : A \to B\}$ (also denoted $A \to B$)

$Df = \{x | fx \downarrow\}$

$f[A] = \{y | \exists x \in A(fx \simeq y)\}$.

Evidently all these have the form $\{x | \phi(x,\ldots)\}$ with ϕ elementary.

10.8 The finite type hierarchy and HA^{ω}. The finite type symbols (f.t.s.) are generated by the following elementary inductive definition: $\dot{0}$ is a f.t.s. and if ρ, σ are f.t.s. then so also are $\rho \dot{\times} \sigma$ and $\rho \dot{\to} \sigma$, where $\dot{0} = (0,0)$, $u \dot{\times} v = (1,u,v)$ and $(u \dot{\to} v) = (2,u,v)$. The f.t.s. are enumerated by a function on

$\mathbb{N}$ whose range thus forms a class that we denote by FTS. Using recursion on $\mathbb{N}$ we can define a function g satisfying:

$$g\dot{0} = \mathbb{N}\,,\quad g(u \dot{\times} v) = gu \times gv \quad \text{and} \quad g(u \dot{\rightarrow} v) = (gu \rightarrow gv)$$

where $\times$, $\rightarrow$ are the operations defined in 10.7. For each $\sigma \in$ FTS we denote by N_σ the value $g\sigma$. Thus

$$N_{\dot{0}} = \mathbb{N}\,,\quad N_{\rho \dot{\times} \sigma} = N_\rho \times N_\sigma \quad \text{and} \quad N_{\rho \dot{\rightarrow} \sigma} = (N_\rho \rightarrow N_\sigma) = N_\sigma^{N_\rho} \quad .$$

For each particular σ in FTS we can prove in $EM_0\restriction$ that

$$C\ell(N_\sigma).$$

However, the statement

$$\forall \sigma \in \text{FTS}\,[C\ell\,(N_\sigma)]$$

requires a proof by induction using the impredicative property $\exists X(g\sigma \simeq X)$. This can be carried out in EM_0 but not $EM_0\restriction$.

Gödel's notion of primitive recursive functional of finite type (Gödel 1958) can be interpreted in $EM_0\restriction$ simply by using recursion on $\mathbb{N}$. The basic scheme is to pass from $f_0 \in N_{\rho \dot{\rightarrow} \sigma}$ and $f_1 \in N_{(\rho \dot{\times} 0 \dot{\times} \sigma \dot{\rightarrow} \sigma)}$ to a g satisfying

$$g(x,0) \simeq f_0(x) \quad \text{and} \quad g(x,n') \simeq f_1(x,n,g(x,n)),$$

which is obtained using 10.3 uniformly in x. Now we can prove by restricted induction on $\mathbb{N}$ that $x \in N_\rho \rightarrow g(x,n) \in N_\sigma$, hence $g \in N_{\rho \dot{\times} 0 \dot{\rightarrow} \sigma}$ as required.

The theory HA^ω of intuitionistic arithmetic in all finite types has variables of each type $\sigma \in$ FTS and constants for the primitive recursive functionals.[7)] The axioms are those of HA together with the defining schemata for all these functionals and, finally, the induction scheme for all formulae of the language. Interpreting the variables of type σ to range over N_σ, each formula of HA^ω is equivalent to an elementary formula with finitely many constants $N_{\sigma_1}, \ldots, N_{\sigma_m}$. Hence it defines a class under ECA. It follows that

$$HA^\omega \subseteq EM_0\restriction \quad .$$

Remark. An intensional form of HA^ω, denoted $I\text{-}HA^\omega$ is obtained by adjoining a functional at each type level which decides equality between objects of that type. It is easily seen that

$$I\text{-}HA^\omega \subseteq EM_0\restriction + D_V \quad .$$

10.9 The extensional finite type hierarchy. The system $E\text{-}HA^\omega$ obtained from HA^ω by adjoining extensionality axioms in all finite types can also be interpreted in $EM_0\restriction$. However, here we interpret the variables of type σ to

7) Cf. Troelstra 1973, Part I §6 for a precise description of HA^ω and the systems $I\text{-}HA^\omega$, $E\text{-}HA^\omega$ below.

range over M_σ, defined together with an equality relation $=_\sigma$ by induction as follows:

$$M_{\dot 0} = \mathbb{N} \quad \text{and} \quad n =_0 m \leftrightarrow n = m$$

$$M_{\rho\dot\times\sigma} = M_\rho \times M_\sigma \quad \text{and} \quad x =_{\rho\dot\times\sigma} y \leftrightarrow p_1 x =_\rho p_1 y \wedge p_2 x =_\sigma p_2 y$$

$$M_{\rho \dot\rightarrow \sigma} = \{f \mid f \in M_\rho \rightarrow M_\sigma \wedge \forall x, y \in M_\rho (x =_\rho y \rightarrow fx =_\sigma fy)\} \quad \text{and}$$

$$f =_{(\rho \dot\rightarrow \sigma)} g \leftrightarrow \forall x \in M_\rho (fx =_\sigma gx) .$$

Equality between objects of type σ is interpreted as $=_\sigma$ so as to obtain $\text{E-HA}^\omega \subseteq EM_0 \upharpoonright$.

10.10 Classes with equality relations. These are simply pairs (A,E) where $E \subseteq A \times A$ and E is an equivalence relation on A. Then we write $x =_A y$ for $(x,y) \in E$ (though this notation is ambiguous, since E is not uniquely associated with A). While we can operate on classes-with-equality in T_0 (or its subtheories) we proceed more generally than in Bishop 1967 and work with classes per se.

10.11 Integers, rationals and reals in $EM_0 \upharpoonright$. Our definitions here follow I, 14.5 (i.e. essentially Bishop 1967 Ch.2)

$\mathbb{Z} = \mathbb{N} \times \mathbb{N}$; $(n,m) =_{\mathbb{Z}} (p,q) \leftrightarrow n + q = m + p$.

$+$ is defined on $\mathbb{Z}$ by $(n,m) +_{\mathbb{Z}} (p,q) = (n+p, m+q)$, and so on for $\cdot$, $<$ on $\mathbb{Z}$. $\mathbb{N}$ is embedded in $\mathbb{Z}$, and subscripts are dropped.

$\mathbb{Q} = \{(x,y) \mid x \in \mathbb{Z} \wedge y \in \mathbb{Z} \wedge y \neq 0\}$; $(x,y) =_{\mathbb{Q}} (u,v) \leftrightarrow x \cdot v = y \cdot u$

$(x,y) +_{\mathbb{Q}} (u,v) = (xv + yu, uv)$, and so on for $\cdot$, $<$ on $\mathbb{Q}$.

$\mathbb{Z}$ is embedded in $\mathbb{Q}$, and subscripts are dropped.

$\mathbb{Z}^+ = \{n \mid n \in \mathbb{Z} \wedge n > 0\}$. We write x_n for xn when $x \in \mathbb{Z}^+ \rightarrow A$

$\mathbb{R} = \{x \mid x \in \mathbb{Z}^+ \rightarrow \mathbb{Q} \wedge \forall n, m \in \mathbb{Z}^+ (|x_n - x_m| \leq \frac{1}{n} + \frac{1}{m})\}$

$x =_{\mathbb{R}} y \leftrightarrow \forall k \in \mathbb{Z}^+ \exists m \in \mathbb{Z}^+ \forall n \geq m (|x_n - y_n| \leq \frac{1}{k})$.

$x +_{\mathbb{R}} y = \lambda u (x_{2u} + y_{2u})$,

$x \cdot_{\mathbb{R}} y = \lambda u (x_{2ku} \cdot y_{2ku})$ where $k = \max(k_x, k_y)$ and for each x, k_x is the least integer greater than $|x_1| + 2$.

$\mathbb{R}^+ = \{(x,n) \mid x \in \mathbb{R} \wedge x_n > \frac{1}{n}\}$ and $\mathbb{R}^{0+} = \{x \mid x \in \mathbb{R} \wedge \forall n \in \mathbb{Z}^+ (x_n \geq -\frac{1}{n})\}$.

$\mathbb{Q}$ is embedded in $\mathbb{R}$, and the subscripts are dropped.

The elementary properties of these number systems can be developed in $EM_0 \upharpoonright$ directly following Bishop 1967. Then the complex numbers $\mathbb{C}$ can be introduced as usual and their properties derived in the same way.

10.12 Continuous functions and classical analysis. Given any two classes A, B with equality relations $=_A$, $=_B$ respectively, tne function class F(A,B) is defined by

$$F(A,B) = \{f \mid f : A \to B \wedge \forall x,y \in A(x =_A y \to f(x) =_B f(y))\}.$$

This is a subclass of $(A \to B)$. In particular in analysis one is interested in $F(\mathbb{R}, \mathbb{R})$ and $F([a,b], \mathbb{R})$ where $a,b \in \mathbb{R}$ and $[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$. Next for the (uniformly) continuous functions on [a,b], following I.15.1, one takes

$$C([a,b],\mathbb{R}) = \{(f,w) \mid f \in F([a,b], \mathbb{R}) \wedge w \in F(\mathbb{R}^+, \mathbb{R}^+) \wedge \\ \forall e \in \mathbb{R}^+ \forall x,y \in [a,b](|x-y| \leq w(e) \to |f(x) - f(y)| < e)\},$$

and one takes

$$C(\mathbb{R}, \mathbb{R}) = \{(f,u) \mid f \in F(\mathbb{R}, \mathbb{R}) \wedge \forall a, b \in \mathbb{R}(a < b \to (f,u(a,b)) \in C([a,b], \mathbb{R}))\}$$

i.e. the functions continuous on each compact interval [a,b]. Starting with this as basis, classical real analysis is pursued in $EM_0\restriction$ just as in Bishop 1967, Ch. 2. The only point which requires careful checking is that only restricted induction on $\mathbb{N}$ is applied throughout. We shall return to this observation in §14 below.

11. Consequences of the join axiom. Here, unless otherwise specified, we work within $EM_0\restriction + J$.

11.1 Families and joins. By a family of classes $\langle B_x \rangle_{x \in A}$ is meant an operation f such that $\forall x \in A\ \exists Y(fx \simeq Y)$ and where we write B_x for fx. The join axiom guarantees the existence of a class $\sum_{x \in A} B_x$ with the defining property

$$z \in \sum_{x \in A} B_x \leftrightarrow \exists x \in A\, \exists y[z = (x,y) \wedge y \in B_x].$$

From this we can define the union operation on classes-with-equality as explained by Bishop (I.14.10) above. By a pre-joined family on A is meant a class $B \subseteq A \times V$. Associated with such is a family in the preceding sense by $fx (= B_x) = \{y \mid (x,y) \in B\}$; extensionally this makes $B \equiv \sum_{x \in A} B_x$.

11.2 Products. Suppose given a family $f = \langle B_x \rangle_{x \in A}$. Let $J = \sum_{x \in A} B_x$. Then we can define

$$\prod_{x \in A} B_x = \{g \mid \forall x \in A((x,gx) \in J)\}$$

since $(x,gx) \in J \leftrightarrow gx \in B_x$ for $x \in A$. From this we can treat intersection of a family of classes-with-equality (I.14.10).

Remark. There is no evident way to derive the product axiom P from the remaining axioms in the theory S_0 of sets in 9.2.

11.3 The passage to transfinite types. Using the operations of 10.7 we know that

$$C\ell(a) \wedge C\ell(b) \rightarrow C\ell(a \times b) \wedge C\ell(a \rightarrow b) .$$

Then by enumerability of FTS in 10.8 we can carry out an induction to prove

$$\forall \sigma \in \mathrm{FTS}\, [C\ell(N_\sigma)] .$$

However this requires unrestricted induction on $\mathbb{N}$ in the theory EM_0. Using J we can then form $\sum_{\sigma \in FTS} N_\sigma$ and $\prod_{\sigma \in FTS} N_\sigma$, which is the first move to transfinite types, at type level ω. Then by successively applying the operations $\times$ and $\rightarrow$ again we can move up to level $\omega \cdot 2$, then $\omega \cdot 3, \ldots, \omega^2$, etc. A general pursuit of this would be based on a theory of ordinals, which are treated in terms of well-founded trees ("tree ordinals") in constructive mathematics and based on IG in the framework of T_0. This is taken up next. In any case we see that the passage to lower transfinite types can be effected in $EM_0 + J$. Precise limits for this are provided by the proof theory of $EM_0 + J$ (Part V).

12. Consequences of the inductive generation axiom. Here we move to full T_0.

12.1 Tree ordinals. Define $\underline{0} = (0,0)$, $x^+ = (1,x)$, $\sup_a f = (2,a,f)$. These are distinct. Note $\sup_{\mathbb{N}} f = (2,\mathbb{N},f)$. $\mathcal{O}_1$ is inductively generated as the least class such that

$$\underline{0} \in \mathcal{O}_1 , \; x \in \mathcal{O}_1 \rightarrow x^+ \in \mathcal{O}_1 , \; \text{and} \; (f: \mathbb{N} \rightarrow \mathcal{O}_1) \rightarrow \sup_{\mathbb{N}} f \in \mathcal{O}_1 .$$

Then $\mathcal{O}_1 = i(A,R)$ for suitable A, R. Classically the members of $\mathcal{O}_1$ represent countable ordinals with

$$|\underline{0}| = 0, \; |x^+| = |x| + 1, \; |\sup_{\mathbb{N}} f| = \sup_{n \in \mathbb{N}} |f(n)| + 1 .$$

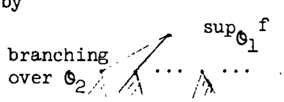
(Note that x^+ can be dropped in favor of $\sup_{\mathbb{N}} \lambda n(x)$.) We can picture members of $\mathcal{O}_1$ as well-founded trees:

Definition by recursion on $\mathcal{O}_1$ is a consequence of the recursion theorem. $\mathcal{O}_2$ is inductively generated as the least class such that

$$\underline{0} \in \mathcal{O}_2 , \; x \in \mathcal{O}_2 \rightarrow x^+ \in \mathcal{O}_2 , \; (f: \mathbb{N} \rightarrow \mathcal{O}_2) \rightarrow \sup_{\mathbb{N}} f \in \mathcal{O}_2$$

$$\text{and} \; (f: \mathcal{O}_1 \rightarrow \mathcal{O}_2) \rightarrow \sup_{\mathcal{O}_1} f \in \mathcal{O}_2 .$$

The last is pictured by



Then we can obtain analogously the existence of a class $\mathcal{O}_n$ for each $n \in \mathbb{N}$. Using join we can carry this on to define transfinite tree classes: $\mathcal{O}_a$ for $a \in \mathcal{O}_1$, and more generally for $a \in \mathcal{O}_b$ with any given b. (For more details on this material cf. Feferman 1975 pp.99-100, also Feferman 1978 § 5.)

12.2 IG and Borel sets. For simplicity we indicate the treatment in Baire space $\mathbb{N}^{\mathbb{N}}$. Let $s_n (n \in \mathbb{N})$ be an enumeration of all finite sequences in $\mathbb{N}$ and $G_n = \{g \in \mathbb{N}^{\mathbb{N}} \mid \bar{g}(\ell h(s_n)) = s_n\}$; these are the basic clopen sets. We take $\mathfrak{B}$ to be the smallest class such that each $G_n \in \mathfrak{B}$ and if $f : \mathbb{N} \to \mathfrak{B}$ then $\bigcap_{n \in \mathbb{N}} fn \in \mathfrak{B}$ and $\bigcup_{n \in \mathbb{N}} fn \in \mathfrak{B}$. In more general spaces we can follow Bishop's treatment via complemented sets. In any case IG suffices for this.

It would be appropriate at this point to take up the questions of adequacy and accord of T_0 and its subtheories with BCM. However we first complete our discussion of the consequences of various axioms with a look at D_V and POW (which are only marginally related to actual BCM).

13. Non-extensionality as a consequence of D_V.

13.1 Non-extensionality of operations. The following is proved in Feferman 1975 §3.4 as a consequence of $APP + D_V$:

$$\neg \forall f,g[\forall x(fx \simeq gx) \to f = g].$$

The idea of the proof is first (using D_V) to associate with each f an f^* such that $Df = Df^*$ and $f^*x \simeq 0$ whenever $x \in Df$. Then, if extensionality is assumed we have f total (i.e. $\forall x[fx\downarrow]$) iff $f^* = \lambda x(0) = 0^*$. Again using full definition by cases D_V we obtain a total operation e such that $Tot(f) \leftrightarrow ef = 0^*$. Diagonalization produces a contradiction.

13.2 Consistency of extensionality of operations in T_0. Denote by EXT_{Op} the statement $\forall f,g[\forall x(fx \simeq gx) \to f = g]$. Using extensional term models for APP (due to Barendregt) it will be shown in Part III that $T_0 + EXT_{Op}$ is consistent. Hence the use of D_V in 13.1 is essential. (It will also be shown that $T_0 + D_V$ is consistent.)

13.3 Non-extensionality of classes. Denote by $EXT_{C\ell}$ the statement $\forall A,B [\forall x(x \in A \to x \in B) \to A = B]$. It is also proved in Feferman 1975 §3.4 that $\neg EXT_{C\ell}$ holds under the assumptions $APP + ECA + D_V$. The idea is to associate with each f the class $cf = \{x \mid fx\downarrow\}$; c itself is total. Then if $EXT_{C\ell}$ held we would have $Tot(f) \leftrightarrow cf = V$, from which one can proceed as in 13.1. With reference to 13.2 we have the following.

Question: Is $T_0 + EXT_{C\ell}$ consistent?
We can of course ask similar questions for the addition of $EXT_{C\ell}$ to sub-theories like EM_0, S_0 etc., for all of which the answer is not known.

13.4 Discussion. D_V is a perfectly reasonable axiom if we regard the entities of our universe as being syntactic objects and = as literal identity. It is less evident if the entities are viewed as mental objects and = is interpreted as intensional identity; however, it appears from writings of Kreisel and of Troelstra (cf. Troelstra 1975) that here also D_V is to be accepted. Then, far from being disturbing, the results of 13.1 and 13.3 add support to the basic non-extensional viewpoint of constructive mathematics as presented in I.4.5, 4.11.

14. Status of the power-class axioms.

14.1 Inconsistency of POW with Join. To be more precise it is shown that APP + ECA + J proves $\neg$POW, the weak power-class axiom of §8 above. Indeed, suppose that there is a weak power-class C of V, so $\forall x[x \in C \to C\ell(x)] \wedge \forall X \exists Y(Y \in C \wedge X \equiv Y)$. Let $B = \sum_{x \in C} x$ and $A = \{x \mid x \in C \wedge (x,x) \notin B\}$. Then $A \equiv a$ for some a in C, and $a \in a \leftrightarrow a \in A \leftrightarrow a \in C \wedge (a,a) \notin B \leftrightarrow a \in C \wedge (a \notin a) \leftrightarrow a \notin a$, which gives a contradiction.

14.2 Consistency of POW with EM_0. This will be proved in Part III. It may also be shown that S_0 + POW is consistent where S_0 is the theory of sets described in 9.2. Even though S_0 contains J we cannot derive a contradiction as in 14.1, since we don't have a universal class V in S_0.

The consistency of further axioms introduced in §7 above (such as 2nd order comprehension) will also be taken up in Part III.

15. Adequacy of (subtheories of) T_0 to BCM.

15.1 Adequacy of T_0. The development outlined in §§10-12 provides a basis for the formalization of BCM in T_0. Moreover, this is accomplished by following the informal mathematics as explained in I.14 and I.15. The official intended definitions come to the forefront in the process of formalization and must always be kept in mind. When informal concepts and proofs are spelled out accordingly, one is in a position from which formalization in T_0 can proceed directly. (This was illustrated in I.15.4). One may thus conclude that T_0 is directly adequate to BCM (as exemplified in Bishop 1967). It is of logical interest to see next how much of BCM can be carried out in theories weaker than T_0.

15.2 The role of IG. Obviously IG is used only for the theory of Borel sets in Bishop 1967, which in turn figured in the theory of measure and integration. As was explained in I.14.13, this was superseded by a treatment without Borel sets in Bishop-Cheng 1972. The latter makes prima facie use of the axiom POW, but just to form (a complete integration space) L(X) as a class from any integration space X; however integration theory only requires the notion of f being a member of L(X), which is definable without the assumption of POW.[8] The conclusion is that IG is unnecessary for the development of abstract integration theory in this sense.

8) The role of axioms like POW in abstract constructive integration theory is studied in Feferman 1978 §4.3. A modified form of POW for this purpose can be derived in the theory of sets and classes $T_0(S)$ (cf.9.3 above); this yields the class of subsets of any given class.

15.3 Dispensability of the join axiom. We have seen in 11.3 that the axiom J is needed to effect the passage to transfinite types (e.g. to $\sum_{\sigma \in FTS} N_\sigma$ and $\prod_{\sigma \in FTS} N_\sigma$). However in actual analysis one never considers families of varying type but only families of subsets of a given set. In these cases one can try to eliminate J by replacing the notion of family by that of pre-joined family (11.1). It may be verified that, except for the theory of Borel sets, this replacement does indeed leave the treatment of analysis in BCM unaffected.

15.4 Adequacy of restricted induction on $\mathbb{N}$. An example where unrestricted induction on $\mathbb{N}$ was used in an essential way was given in 11.3, namely to prove that for all $\sigma \in FTS$, N_σ is a class. Similarly, the principle of unrestricted induction in IG is used only to show that the objects in the Borel hierarchy actually are classes. But for BCM without transfinite types and without Borel theory it appears that only restricted induction on $\mathbb{N}$ is needed. This has been verified in detail by Friedman (unpublished, but cf. 18.2 below).

15.5 Adequacy of $EM_0\restriction$. Putting 15.1 - 15.4 together we conclude that $EM_0\restriction$ is adequate to all of Bishop 1967 except for that part involving the theory of Borel sets, and to all of Bishop-Cheng 1972 except for treating L(-) as an operation from classes to classes. This is of logical (and epistemological) interest because, as will be shown in Part V, $EM_0\restriction$ is conservative over HA.

16. The question of accordance of T_0 (or its subtheories) with BCM.

16.1 Sets vs. classes. Bishop does not speak of classes and it is questionable whether he would countenance a universal class V. In this respect, T_0 is not explicitly in accordance with BCM. The theory of sets S_0 (9.2) is here in greater direct accord. Incidentally, S_0 is adequate to the same part of practice as T_0.

16.2 The question of operations with unbounded domains. There is no explicit discussion by Bishop of operations with unbounded domains like k, s and the resulting $d = \lambda x(x)$, $e = \lambda x \lambda y(xy)$, etc. However, the idea of such does seem to be implicit in his view of operations simply as rules; it is further implicit in his use of operations such as Cartesian product and power on sets, since no class of all sets is assumed as an object. It is my conclusion from these arguments that the use of operations with unbounded domains is implicitly in accordance with BCM. However, this is clearly subject to debate, especially since it leads us to talk about combinations like (xx) which appear foreign to practice.

Remark. There is a simple formal device which permits us to replace unbounded combinatory operations by corresponding bounded ones and still achieve much the same mathematical effects. Namely, one introduces formal "external" operation symbols on (variable) classes (A,B,...) e.g. $k_{A,B}$, id_A, $e_{A,B}$ etc. (writing the arguments as subscripts) with axioms like:

$k_{A,B} \in (A \to (B \to A))$, $\forall x \in A \forall y \in B\ (k_{A,B}xy = x)$,

$id_A \in (A \to A)$, $\forall x \in A(id_A(x) = x)$,

$e_{A,B} \in (A \to ((A \to B) \to B))$, etc.

What is lost here is the possibility of reducing recursion (on $\mathbb{N}$, or any $i(A,R)$) to the combinators, since those reductions make essential use of the possibility of self-application (xx). Thus in such a step one must supplement the $\mathbb{N}$, resp. IG axioms, by suitable axioms for recursion operators.

16.3 The other principles. These are comprehension, natural numbers, join and inductive generation. If we are to judge the axioms for these separately from the issues in 16.1, 16.2, we must naturally consider them in weaker forms that apply as well to sets and are given by external rather than internal operations. In particular CA_1 is to be replaced by SEP_1. With such modifications in mind, it should be clear from I.14, 15 and 10-12 above that these principles are called for in BCM.

Remark. Beeson has also raised a question (in conversation) about the constructivity of the join axoim, as formulated uniformly using j. His point is that the result should depend not only on A and f but also on a proof of $\forall x \in A[C\ell(fx)]$.

16.4 Conclusion. The issues in dispute are those in 16.1 and 16.2. I believe a case can be made - based on Bishop's views of operations given by rules and sets by properties - that the use of both operations with unbounded domains and of classes (as well as sets) makes T_0 implicitly in accordance with BCM. However, there is little support for explicit, direct accordance.

Remark. Since $EM_0\restriction$ is conservative over HA and the latter is certainly in direct accordance with BCM, the former is consequently in indirect accordance with it.

17. Comparison with Martin-Löf 1975.

17.1 Character of Martin-Löf's system. This is a kind of logic-free transfinite type theory which is denoted TT.[9] There are terms for objects a,b,c,..., and terms for types A,B,C,... . The informal idea is that each object is of a unique type. The basic propositions are of the form

$$a \in A \quad \text{and} \quad a = b ,$$

where $a \in A$ is read: a is of type A. TT is based on a natural deduction system (cf. Prawitz 1971) for deriving such propositions from hypotheses of the form $x_i \in A_i$. For example, suppose one has inferred $b[x] \in B[x]$ from $x \in A$.[10]

9) As stated by Martin-Löf, a significant earlier attempt to formulate such a theory was made in Scott 1970.

10) We simplify here the form of assumptions actually given by Martin-Löf for TT.

Then we have terms for application, abstraction and Cartesian product related by the rules

$$\lambda x\, b[x] \in (\Pi x \in A)\, B[x], \quad \frac{a \in A}{(\lambda x\, b[x])(a) = b[a]}, \text{ and } \frac{a \in A,\ c \in (\Pi x \in A)B[x]}{c(a) \in B[a]}.$$

Similarly there are rules for pairing, projection and join $(\Sigma x \in A)\, B[x]$. Special cases of product and join are $(A \rightarrow B)$ and $A \times B$. There are rules for the natural numbers N using 0, s (successor) and recursion on N. (Finite initial segments N_k of N are provided for too.) Further, with each A is associated the identity relation I on A as a function of $(x,y) \in A \times A$. Finally, there is a V_0 which is supposed to be the type of all small types, and is closed under the introduction rules for types; moreover, there is for each V_n a corresponding V_{n+1}. To prove that A is a type in the system one proves $A \in V_n$ for some n. The syntax of the predicate calculus is represented in TT via the (Curry-Howard) correspondence between formulas and types. When a type A is thought of as a proposition then $(a \in A)$ is thought of as 'a is a proof of the proposition A'. From this, the intuitionistic predicate calculus is derived using the (Brouwer-Heyting) explanation of the logical operators in terms of proofs (I.4.2 above)

Remark. Logic is assumed informally in the explanation of the rules.

17.2 Comparison of the system with T_0. TT does not provide for inductively generated types in general, but rules for them can be adjoined along the same lines, following Martin-Löf 1971. With or without such rules, the system can be interpreted in T_0 (each V_n is inductively generated by certain closure conditions involving $V_0,\dots,V_{n-1}$). Furthermore, the system with no universes contains $EM_0\!\upharpoonright$.[11)]

17.3 Adequacy of TT to BCM. By the preceding, TT (as given by Martin-Löf) is adequate to the same portion of BCM as $EM_0\!\upharpoonright$; when supplemented by inductively generated types as suggested in 17.2 it is also adequate to the same portion of BCM as all of T_0 actually serves to formalize.

17.4 Accordance with BCM. The types of TT can be interpreted as sets in Bishop's sense. Following I.14 - I.15 above it should be granted that TT is in direct accordance with BCM, at least insofar as concerns basic concepts and principles. The one reservation has to do with its heavily syntactic formulation for the conditions to introduce and use the various kinds of terms. This is in turn necessitated by the requirement that each object is assigned a type. Thus we cannot have an 'internal' function f of which it is proved $\forall x \in A$[fx is a type] (as done in T_0 by $\forall x \in A[C\ell(fx)]$) but must use 'external' objects B[x] of which $B[x] \in V_n$ is proved (for some n) under the hypothesis $x \in A$. There are no indications in Bishop's writings that would lead one necessarily to take such a formal approach. In this respect, the looseness which T_0 enjoys owing to its type-free character seems more in accord with BCM.

11) The exact relationships are not known to me.

Remark. The syntax of TT is evidently somewhat more complicated than that of T_0. Some simplification could presumably be made by assuming all of intuitionistic logic at the outset. In any case, it is much easier to form a variety of models and interpretations of T_0, as we shall see in Parts III, IV.

18. Comparison with Myhill's and Friedman's extensional systems.

18.1 The character of these systems. The system CST introduced in Myhill 1975 is a subsystem of IZF(N) + DC by which is meant Zermelo-Fraenkel set theory over the natural numbers (as urelements) with the logic restricted to be intuitionistic and with the axiom scheme of dependent choices added. The notions of pair and of function are both defined here just as usual in ZF : in other words functions are identified with graphs of many-one relations. One takes (over the usual axioms of N) the axioms of extensionality, unordered pair, union, Δ_0-separation, domain and ranges of functions, the set $(A \to B)$ of all functions $f:A \to B$ for given sets A, B (which is taken in place of the power set axiom) and the replacement scheme

$$(\forall x \in A)\exists! y\, \phi(x,y) \to \exists z[\mathrm{Fun}(z) \wedge \mathrm{Dom}(z) = A \wedge \forall x \in A\, \phi(x, z(x))].$$

Finally, the principle DC

$$(\forall x \in A\, \exists y \in A\, \phi(x,y) \to \forall x \in A\, \exists z[z:N \to A \wedge z(0) = x \wedge \forall n \in N\, \phi(z(n), z(n'))]$$

is taken, but not AC, since that is shown to contradict Church's thesis in the system.[12]

Friedman 1977 considers a number of subsystems and extensions of CST (all contained in IZF/N). The weakest of these is denoted $\underset{\sim}{B}$. In $\underset{\sim}{B}$, induction on N is restricted, replacement is taken only to form $\{\{x \in A|\, \phi(x,y)\}|y \in A\}$ for Δ_0 formulas ϕ, and DC is also taken only for such ϕ. The other systems considered are denoted T_1, T_2, T_3 and T_4. We shall not describe them here;[13] however CST is equivalent to T_2.

Remarks. (i) Axioms of inductive generation are not taken in CST. They are derivable in T_4.

(ii) Friedman 1977 shows that $\underset{\sim}{B}$ itself is reducible to HA and is conservative for Π^0_2 sentences. (Beeson 1979 shows that it is conservative for all sentences.) By contrast, $\underset{\sim}{B}$ + classical logic is equivalent to Zermelo set theory.

18.2 Adequacy of these systems to BCM. The system $\underset{\sim}{B}$ is adequate to the same portion of BCM as the system $EM_0\!\restriction$ (cf. the discussion in Friedman 1977 p.7). Though formally stronger, the system CST does not seem to have any further power for the actual mathematics involved. The adequacy in both cases is indirect. The definitions of concepts do not follow Bishop's official spelled-out definitions, but rather the corresponding classical ones which use extensionality. For example,

12) The reason why AC but not DC is problematic in the framework of T_0 will be explained in Part IV below.

13) The reader may find it useful to read my review of Friedman 1977 which appeared in Math. Reviews 55(1978) No.7748.

the real numbers $\mathbb{R}$ are taken to be equivalence classes of Cauchy sequences. The positive real numbers $\mathbb{R}^+$ are those $x \in \mathbb{R}$ such that $(\exists n \in \mathbb{N}^+)(x > \frac{1}{n})$. Thus all the distinctions and use of witnessing data required by Bishop in order to carry out constructive operations are essentially ignored.

18.3 The question of accordance. These systems differ in two essential respects from the constructive point of view which is basic to BCM, at least as described in I.4. Namely, extensionality is accepted, in violation of I.4.5, and functions are defined in terms of sets (of ordered pairs), in violation of I.4.6. It is plain then that any set theory which contains the extensionality axiom and defines the notion of function in this way - and in particular CST and $\underset{\sim}{B}$ - is not in direct accordance with BCM.

By the reduction of $\underset{\sim}{B}$ to HA due to Friedman (and Beeson) referred to above, $\underset{\sim}{B}$ is certainly indirectly in accordance with BCM.[14] As to CST, Myhill 1975 gives a constructive reduction via a realizability interpretation. More sharply, Friedman 1977 obtains reduction of the equivalent T_2 to intuitionistic ramified analysis in levels $< \varepsilon_0$ (which in turn is interpretable in our T_0 minus IG, cf.Part V below). The system T_3 is also reduced loc. cit. to an intuitionistic theory of one inductively defined set, which is certainly justified by BCM and is contained in our T_0. Finally, the theory T_4 is reducible to the full 2nd order theory of species which is contained in $EM_0 + CA_2$; but the accordance of the latter with BCM is open to dispute.

It should be mentioned that in Beeson's contribution to this volume he shows for a number of intuitionistic extensional theories of sets how to interpret them in their subtheories without extensionality. This is followed in Beeson 1979 by certain realizability interpretations to reduce the latter theories to sub-theories of T_0, in particular of $\underset{\sim}{B}$ to HA (conservatively).

III. Models

Throughout this part models will be understood in the usual set-theoretical sense and thus will satisfy classical logic. This does not hold for the interpretations to be dealt with in Part IV.

1. A model of T_0 over any model of APP (presented in Feferman 1975 sec. 4.1.) Let

$$\mathfrak{A} = \langle V, App, k, s, d, p, p_1, p_2, 0, s_N, p_N \rangle$$

be any model of the axioms APP of T_0 (in II.5). Here $x \in \mathbb{N}$ is interpreted as $x \in \omega$ where ω is the least subset of V containing 0 and closed under $x \to x' = s_N x$. (The identification of $\mathbb{N}$ as a member of V will be explained in a moment.) Abbreviations for application terms, pairing, comprehension are

14) Friedman also gives an informal argument for the constructive justification of $\underset{\sim}{B}$ (and stronger theories) by interpretation in a theory of species of finite type.

taken just as in II.3,4. Now we take codes in V for the class constants and operations, e.g. as follows:

$$\mathbb{N} = (0,0),\ c_n z = (1,n,z),\ j(a,f) = (2,a,f) \text{ and } i(a,r) = (3,a,r).$$

Next $C\ell_\alpha$ and ε_α are defined by transfinite recursion on α; at stage α one has a structure $(\mathfrak{A}, C\ell_\alpha, \varepsilon_\alpha)$ in which the formulas of $\mathfrak{L}(T_0)$ are interpreted by taking ε_α for 'ε' and letting the class variables range over $C\ell_\alpha$. In this definition we shall also use 'ε' in its ordinary set-theoretic extensional sense; the context serves to avoid ambiguity.[15)]

(1) $C\ell_0 = \{\mathbb{N}\}$ and $x \,\varepsilon_0 \mathbb{N} \rightarrow x \in \omega$.

(2) (i) $C\ell_\alpha \subseteq C\ell_{\alpha+1}$ and $x\,\varepsilon_{\alpha+1} a \rightarrow x\,\varepsilon_\alpha a$, for $a \in C\ell_\alpha$;

(ii) for each elementary $\phi(x,y_1,\dots,y_m, Z_1,\dots,Z_p)$ and $n = \ulcorner\phi(x,\underline{y},\underline{Z})\urcorner$, and for any $y_1,\dots,y_m \in V$ and $a_1,\dots,a_p \in C\ell_\alpha$ we have $c = c_n(y_1,\dots,y_m, a_1,\dots,a_p) \in C\ell_{\alpha+1}$ and

$$x\,\varepsilon_{\alpha+1} c \leftrightarrow (\mathfrak{A}, C\ell_\alpha, \varepsilon_\alpha) \models \phi(x,y_1,\dots,y_m, a_1,\dots,a_p);$$

(iii) for each $a \in C\ell_\alpha$ and $f \in V$ such that $\forall x(x\,\varepsilon_\alpha a \rightarrow fx \in C\ell_\alpha)$, we have $c = j(a,f) \in C\ell_{\alpha+1}$ and $z\,\varepsilon_{\alpha+1} c \rightarrow \exists x,y(x\,\varepsilon_\alpha a \wedge y\,\varepsilon_\alpha fx)$;

(iv) for each $a \in C\ell_\alpha$ and $r \in C\ell_\alpha$ we have $c = i(a,r) \in C\ell_{\alpha+1}$ and $x\,\varepsilon_{\alpha+1} c \rightarrow \forall I \subseteq V\{\forall u[u\,\varepsilon_\alpha a \wedge \forall w((w,u)\varepsilon_\alpha r \rightarrow w \in I) \rightarrow u \in I] \rightarrow x \in I\}$;

(v) $C\ell_{\alpha+1}$ has only those elements obtained by (i)-(iv).

(3) For limit λ, $C\ell_\lambda = \bigcup_{\alpha<\lambda} C\ell_\alpha$ and $\varepsilon_\lambda = \bigcup_{\alpha<\lambda} \varepsilon_\alpha$.

For the final model of T_0 we take

(4) $C\ell = \bigcup_\alpha C\ell_\alpha$ and $\varepsilon = \bigcup_\alpha \varepsilon_\alpha$, so that for $a \in C\ell_\alpha$ and any x, $x\,\varepsilon\, a \leftrightarrow x\,\varepsilon_\alpha a$.

The axioms for T_0 are verified to hold in $\mathfrak{B} = (\mathfrak{A}, C\ell, \varepsilon)$ in a straightforward way. In particular, by (1) we have full induction on $\mathbb{N}$ (with respect to any properties) and by (2)(iv) we have full induction on $i(A,R)$ for any A,R. It is further to be noted that in checking elementary comprehension for $c = \{x \mid \phi(x,y_1,\dots,y_m, a_1,\dots,a_p)\}$ we need only know the meaning of $x\,\varepsilon\, a_i (1 \le i \le p)$. This is where the predicative character of ϕ is used in an essential way. A new idea is needed if one wishes to satisfy stronger comprehension schemes; that is explained in the next section.

Remarks. (i) V, defined by $\{x \mid x=x\}$, is interpreted as a certain code c_n in the domain V of $\mathfrak{A}$; then $x\,\varepsilon\, V$ has the same meaning in the model as extensionally.

15) In Feferman 1975 we used the symbol 'η' in place of 'ε' in $\mathfrak{L}(T_0)$ in order to distinguish the two uses.

(ii) If the constant d of $\mathfrak{A}$ satisfies $\forall x,y,a,b[dxyab\downarrow]$ and $(x=y \to dxyab = a) \wedge (x \neq y \to dxyab = b)$ then the full definition-by-cases axiom D_V is of course satisfied so we have a model of $T_0 + D_V$ in this case.

(iii) There is an obvious modification of the construction above to get a model of T_0 starting with any $C\ell_0$ and ϵ_0 such that $\mathbb{N} \in C\ell_0$ and $x \epsilon_0 \mathbb{N} \leftrightarrow x \in \omega$, as long as card $(C\ell_0) \leq$ card (V) (so that there is room for all the codes).

2. Modification to obtain a model of $T_0 + CA_2$. (Feferman 1975, Addendum). The argument here will be much less constructive. Starting with any model $\mathfrak{A}$ of APP as in § 1, let $\mathfrak{M} = (\mathfrak{A}, \mathcal{P}(V), \epsilon)$ where $\mathcal{P}(V)$ is the set of all subsets of V and ϵ is the standard membership relation. To each stratified formula $\psi(x,\underline{y},\underline{Z},X)$ is assigned a Skolem function $F_\psi(x,\underline{y},\underline{Z}) = X$ which has the property:

$$\mathfrak{M} \models \{\exists X\, \psi(x,\underline{y},\underline{Z},X) \to \psi(x,\underline{y},\underline{Z},F_\psi(x,\underline{y},\underline{Z}))\} .$$

Given any stratified $\phi(x,\underline{y},\underline{Z})$ take $\psi(x,\underline{y},\underline{Z},X) = \forall x[x \in X \leftrightarrow \phi(x,\underline{y},\underline{Z})]$, and $G_\phi = F_\psi$, so $G_\phi(\underline{y},\underline{Z}) = \{x \in V \mid \mathfrak{M} \models \phi(x,\underline{y},\underline{Z})\}$. We choose codes f_n for the F_ψ for each stratified ψ and in place of (2)(ii) in § 1 take, for $n = \ulcorner\psi(x,\underline{y},\underline{Z},X)\urcorner$:

(2)(ii)' for each $x, y_1,\ldots,y_m \in V$ and $a_1,\ldots,a_p \in C\ell_\alpha$ we have
$c = f_n(x,y_1,\ldots,y_m,a_1,\ldots,a_p) \in C\ell_{\alpha+1}$ and
$x \epsilon_{\alpha+1} c \leftrightarrow x \in F_\psi(x,y_1,\ldots,y_m,A_1,\ldots,A_p)$, where $A_i = \{x \mid x \epsilon_\alpha a_i\}$ for each $i=1,\ldots,p$.

With the resulting $(\mathfrak{A},C\ell,\epsilon)$ then defined as in §1(4), let a^* be the extension $\{x \mid x \epsilon a\} \subseteq V$ for each a in $C\ell$, and let $C\ell^* = \{a^* \mid a \in C\ell\}$. Then $C\ell^*$ is closed under the F_ψ and so $\mathfrak{M}^* = (\mathfrak{A},C\ell^*,\epsilon)$ is an elementary substructure of $\mathfrak{M}$. It follows that for each stratified ϕ and any sets $A_1,\ldots,A_p$ we have $\{x \mid \mathfrak{M}^* \models \phi(x,y_1,\ldots,y_m,A_1,\ldots,A_p\}$ $(= G_\phi(\underline{y},A))$ in $C\ell^*$. Finally it is proved by induction on stratified ϕ that

$$(\mathfrak{A},C\ell,\epsilon) \models \phi(x,\underline{y},\underline{a}) \leftrightarrow \mathfrak{M}^* \models \phi(x,\underline{y},\underline{a}^*).$$

From this it follows that $(\mathfrak{A}, C\ell, \epsilon)$ is a model of $T_0 + CA_2$.

Remarks. The axioms for $\mathbb{N}$ and IG could be subsumed under CA_2 since their 2nd order definitions in $\mathfrak{M}^*$ are absolute.

3. The recursion-theoretic model. Take $V = \omega$ and $App(x,y,z) \leftrightarrow \{x\}(y) \simeq z$ in the sense of ordinary recursion theory. Taking 0 and $x \mapsto x'$ to be standard, we can easily choose constants k,s,d,p,p_1,p_2,S_N,P_N so as to obtain $\mathfrak{A}$ satisfying the axioms APP. Note that D_V is automatically satisfied. In addition then to T_0 (or $T_0 + CA_2$ if we follow § 2) we also have Church's thesis for partial functions

(CT_2) $\qquad \forall f\, \exists e\, \forall n(fn \simeq \{e\}(n))$

true in $(\mathfrak{A}, C\ell, \epsilon)$ simply by taking $e = f$. $\mathbb{N}^{\mathbb{N}}$ is just the class of recursive functions. Hence CT_1 is also true. On the other hand CT_0, being classically false, is not satisfied.

With reference to II 10.8-10.9, it may be seen that $\langle N_\sigma \rangle_{\sigma \in FTS}$ is interpreted in this model as the hierarchy HRO of <u>hereditarily recursive operations</u> and $\langle M_\sigma \rangle_{\sigma \in FTS}$ as the hierarchy HEO of <u>hereditarily effective operations</u> (cf. Troelstra 1973, 124-127). Going on to 10.11 - 10.12 one sees that the reals $\mathbb{R}$ are interpreted as in <u>recursive analysis</u>, and so on for $C([a,b], \mathbb{R})$, etc. Finally, with reference to 12.1, it is seen that $\mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_a, \ldots$ are interpreted as forms of the Church-Kleene <u>constructive ordinal notation classes</u>.

<u>Remark</u>. Any "enumerative" generalization of recursion theory gives rise to a model of APP which, when extended to a model of T_0 as in § 1 yields other interesting interpretations of its concepts. For further examples of such and §§5-6 below cf. Feferman 1978, 3.2 - 3.4.

4. <u>Independence results from BCM</u>. Since all of BCM can be safely formalized in T_0, any model $\mathfrak{B} = (\mathfrak{A}, C\ell, \varepsilon)$ of T_0 automatically provides independence results for ϕ which are classically true but for which $\mathfrak{B} \not\models \phi$. For example, if we take the recursion-theoretic model $\mathfrak{A}$ of § 3 to begin with then the example due to Specker of a recursively (uniformly) continuous function on [0,1] which does not take on a recursive minimum shows that <u>the theorem of the minimum is underivable in BCM</u>. Indeed, to be more precise and even stronger, by §2 <u>it is not derivable in</u> $T_0 + D_V + CA_2$ <u>with classical logic</u>. Similarly for the other examples giving 'peculiarities' of recursive analysis and of the Russian school of constructive analysis (cf. I.7-8).

<u>Remark</u>. The obverse of the point here is that if ϕ is a mathematical statement for which (classical) $T_0 + D_V + CA_2 \vdash \phi$ then ϕ has a recursion-theoretic interpretation or 'analogue' .

5. <u>Generating models of</u> $APP + D_V$. Given any infinite set V, we can generate a model $\mathfrak{A}$ of APP from the following information: (i) a pairing operation $P : V^2 \xrightarrow{1-1} V$ and projections $P_i : V \to V$ for which $P_i(P(x_1, x_2)) = x_i$, (ii) an embedding of $(\omega, 0, ')$ in V, and (iii) any collection $\mathfrak{F}$ of partial $F : V \xrightarrow{\sim} V$ for which card $(\mathfrak{F}) \leq$ card (V). Then we can define constants for $\mathfrak{A}$ so that $APP + D_V$ is satisfied and $pxy = P(x,y)$, $p_i x = P_i(x)$, $s_N n = n'$, $(p_N n') = n$ and such that for each $F \in \mathfrak{F}$ there exists $f \in V$ which represents F, i.e. $fx \simeq F(x)$ for all x. (By non-extensionality, each F will have many representations.) To obtain $\mathfrak{A}$ we simply use pairing to build codes for the constants k, s, ... of $\mathfrak{A}$ as well as for each $F \in \mathfrak{F}$. Then we regard the axioms of APP as inductive closure conditions on the relation App(x,y,z). In particular (seeing to it that sx and sxy are always defined in a simple way) one wants

$$xz \simeq u \wedge yz \simeq w \wedge uw \simeq v \rightarrow (sxy)z \simeq v.$$

6. Full set-theoretic models of $APP + D_V$. In particular, let $V = R_\lambda$ (the set of sets in the cumulative hierarchy of rank $< \lambda$) for some limit $\lambda > \omega$. Define 0,' and predecessor on ω and pairing and projections as usual. Let $\mathfrak{F}$ be the class of all functions which (as sets) are members of R_λ. By § 5 we obtain a model $\mathfrak{A}$ of $APP + D_V$ in which every set-theoretic function is represented. Proceed to build a model $\mathfrak{B} = (\mathfrak{A}, C\ell, \epsilon)$ of $T_0 + D_V + CA_2$ over $\mathfrak{A}$. In $\mathfrak{B}$, $\mathbb{N}$ has the same elements as ω and $(\mathbb{N} \to \mathbb{N})$ consists of representatives of all functions from ω to ω. For the type symbols $1 = (0 \dot{\to} 0)$, $2 = (1 \dot{\to} 0)$, etc. we have $(M_1 / =_1) \cong (\omega \to \omega)$ and $(M_2 / =_2) \cong ((\omega \to \omega) \to \omega)$, etc. Further $(\mathbb{R} / =_{\mathbb{R}})$ is isomorphic to the reals in the set-theoretical sense, and the class of all functions from $\mathbb{R}$ to $\mathbb{R}$ which preserve $=_{\mathbb{R}}$ is isomorphic (modulo the defined equality between such functions) with the set of all real functions in the set-theoretic sense. Now $C([a,b], \mathbb{R})$ consists of representatives of all uniformly continuous functions.

Suppose λ is inaccessible. Each element a of $\mathfrak{O}_1$ has a naturally associated ordinal $|a| < \omega_1$ and $\omega_1 = \{|a| : a \in \mathfrak{O}_1\}$. More generally for any $\mathfrak{O}_a$, we have $\omega_{|a|} = \{|b| : b \in \mathfrak{O}_a\}$. The Borel hierarchy in $\mathbb{N}^{\mathbb{N}}$ as explained in II.12.2 consists of representatives of the full Borel hierarchy in Baire space in the set-theoretic sense.

7. Generalizing classical, recursive and constructive mathematics.

7.1 It follows from § 3 and § 6 that any mathematical theorem ϕ of $T_0 + D_V + CA_2$ with classical logic automatically generalizes a theorem of recursive mathematics and of classical set-theoretic mathematics.

7.2 It also follows that for any sub-theory T of $T_0 + D_V + CA_2$ which is recognized as being constructively valid (so, the logic may be restricted) any mathematical theorem ϕ of T generalizes one from classical, recursive and constructive mathematics. In particular, this applies to $T = T_0$ (if II.15.4 is accepted).

Remarks. (i) In a certain sense Martin-Löf's TT can also be considered to have both set-theoretic and recursion-theoretic models, so 7.2 would also apply to it. (ii) Myhill's CST (and related theories) has immediate set-theoretic models, but no direct recursion-theoretic model and, as we have seen in II.17, its constructive interpretation is in dispute.

8. Term models. In the framework of T_0 these have been given by Beeson 1977 (1.3) which is followed here; however, the ideas are familiar from combinatory calculi (cf. Barendregt 1971, 1977). As will be explained in 8.2, the method works to give a model $\mathfrak{A}$ of APP but not of $APP + D_V$.

8.1 Reduction of terms. Let $\tau, \tau_1, \tau_2, \ldots$ range over application terms as explained in II.1.3. A reduction relation $\tau_1 \geq \tau_2$ is defined inductively by the following clauses, where we write $\bar{n}$ for $0\underbrace{''\cdots'}_{n}$:

(i) $\tau \geq \tau$

(ii) $\tau_1 \geq \tau_2 \wedge \tau_2 \geq \tau_3 \to \tau_1 \geq \tau_3$

(iii) $\tau_1 \geq \tau_1^* \wedge \tau_2 \geq \tau_2^* \to \tau_1\tau_2 \geq \tau_1^*\tau_2^*$

(iv) $k\tau_1\tau_2 \geq \tau_1$

(v) $s\tau_1\tau_2\tau_3 \geq \tau_1\tau_3(\tau_2\tau_3)$

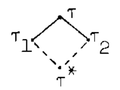
(vi) $p_1(p\tau_1\tau_2) \geq \tau_1$, $p_2(p\tau_1\tau_2) \geq \tau_3$

(vii) $p_N(S_N\tau) \geq \tau$

(viii) $d\bar{n}\bar{n}\tau_1\tau_2 \geq \tau_1$, $n \neq m \to d\bar{n}\bar{m}\tau_1\tau_2 \geq \tau_2$

(ix) $\tau_1 \geq \tau_2$ only as required by (i) - (viii).

We shall use $\tau_1 \geq \tau_2$ for literal identity of terms. τ_1 is said to be in normal form (or irreducible) if whenever $\tau_1 \geq \tau_2$ we have $\tau_1 = \tau_2$. The set of terms in normal form is denoted by NF. Note every term reduces to a term in NF. The Church-Rosser theorem (or $\diamond$ property) for $\geq$ is proved by standard methods:

CR. If $\tau \geq \tau_1$ and $\tau \geq \tau_2$ then for some τ^*, $\tau_1 \geq \tau^*$ and $\tau_2 \geq \tau^*$. 

As a corollary one has unicity of normal form in the sense that

$$\tau_1, \tau_2 \in NF \wedge \tau \geq \tau_1 \wedge \tau \geq \tau_2 \to \tau_1 = \tau_2 .$$

8.2 The model of normal terms. The domain V of the model $\mathfrak{A}$ taken by Beeson 1977 is NF (which is also used here to denote $\mathfrak{A}$) . The application relation is:

$$A_{pp}(\tau_1, \tau_2, \tau_3) \leftrightarrow \tau_1, \tau_2, \tau_3 \in NF \wedge (\tau_1\tau_2 \geq \tau_3) .$$

The constants, all of which are in NF, denote themselves in this model. NF is a model of APP because all $\bar{n}$ are in NF and $\bar{n} \neq \bar{m}$ for $n \neq m$. The following easy lemmas are proved by Beeson, where $APP_{\mathbb{N}}$ denotes APP plus the $\mathbb{N}$-closure axiom $0 \in \mathbb{N} \wedge \forall x (x \in \mathbb{N} \to x' \in \mathbb{N})$.

(a) If $\tau \in NF$ then $APP_{\mathbb{N}} \vdash (\tau\downarrow)$.

(b) If τ_1, τ_2 are closed terms and $\tau_1 \geq \tau_2$ then $APP_{\mathbb{N}} \vdash (\tau_1\downarrow \to \tau_2\downarrow \wedge \tau_1 = \tau_2)$.

(c) If τ is closed and $APP_{\mathbb{N}} \vdash (\tau\downarrow)$ then $\exists\tau^* \in NF(\tau \geq \tau^*)$.

(d) If T is any sub-theory of $T_0 + CA_2$ with $APP_{\mathbb{N}} \subseteq T$ and if $T \vdash (\tau \in \mathbb{N})$ then for some $n \in \omega$, $T \vdash (\tau = \bar{n})$.

When coupled with realizability methods in Part IV the lemma (d) allows one to obtain the numerical instantiation property and the disjunction property for such T.

Remark. An essential difference of D_V from the other applicative axioms appears here. In the proof of Lemma (a) we use that if $n \neq m$ then $APP_{\mathbb{N}} \vdash \bar{n} \neq \bar{m}$. To try to obtain a corresponding result for D_V we would want that if $\tau_1, \tau_2 \in NF$ and $\tau_1 \neq \tau_2$ then $APP_{\mathbb{N}} + D_V \vdash (\tau_1 \neq \tau_2)$. But that isn't so - for example, $k, 0 \in NF$ and $k \neq 0$ but $(k \neq 0)$ is not provable (since we can construct an applicative model $\mathfrak{A}$ in which $k = 0$). For the same reason we can't prove the disjunction property for $T_0 + D_V$.[16)]

8.3 An extensional term model (Barendregt 1971). Instead of taking V as in 8.2, one takes V to be the set of all equivalence classes $[\tau]$ of terms for the least equivalence relation $\equiv$ such that $\tau_1 \geq \tau_2 \to \tau_1 \equiv \tau_2$. Then we take $App([\tau_1],[\tau_2],[\tau_3]) \leftrightarrow \tau_1\tau_2 \equiv \tau_3$. By the CR property if an equivalence class contains some $\tau \in NF$, that τ is unique. Then one sees that $[\bar{n}] \neq [\bar{m}]$ whenever $n \neq m$ so that we obtain a model of APP. Since every application term τ denotes $[\tau]$ in the model, it satisfies $(\tau\downarrow)$, i. e. every operation here is total. In addition the model may be shown to satisfy the axiom of extensionality for terms. This shows D_V to be essentially required for the non-extensionality result of II.13.1.

9. Continuous function models. There are again models due to Beeson 1977 (1.2), once more without D_V.

9.1 Continuous partial function application. The idea here is to form a model of APP which is a kind of untyped version of the class of countable functionals of finite type (which are hereditarily continuous in a certain sense) due to Kleene 1959 (and Kreisel, same volume). One takes V to be the class of all partial functions f from ω to ω. A relation $App(f,g,h)$ (or $fg \simeq h$) is defined for members of V as follows. For each n, the value of h at n is supposed to depend on only a finite amount of information about g. Let $(f)_n = \lambda x.f(x,n)$ with (x,y) a primitive recursive pairing function. More precisely, $h(n)$ is obtained, when defined, by $(f)_n$ acting continuously on $(g)_n$ so that if $(f)_n(\overline{(g_n)}(m)) = 0$ no information is given by the initial segment $\overline{(g)_n}(m)$, and if $(f)_n(\overline{(g_n)}(m)) = k+1$ then k is unique and we put $h(n) = k$. It is shown by Beeson that the natural numbers can be embedded in V and the constants interpreted in such a way as to form a model $\mathfrak{A}$ of APP. (We can't do the same for $APP + D_V$ because definition by cases on V is not continuous.) Now form a model $\mathfrak{B}$ of (classical) $T_0 + CA_2$ from APP by § 2.

9.2 Consistency of continuity properties. It can be shown that the model $\mathfrak{B}$ satisfies the following statements of interest:

(i) Any operation $f : \mathbb{N} \to \mathbb{N}$ is continuous (in the product topology).

(ii) Any function from a complete separable metric space X to a separable metric space Y is continuous.

16) Another explanation of the difficulty is due to Klop 1977, who has shown that there is no Church-Rosser theorem for the calculus with the $\geq$ relation augmented as follows to correspond to the D_V axiom: $d\tau\tau\tau_3\tau_4 \geq \tau_3$ and $\tau_1, \tau_2 \in NF \wedge \tau_1 \neq \tau_2 \to d\tau_1\tau_2\tau_3\tau_4 \geq \tau_4$.

It follows that these continuity properties are consistent with $T_0 + CA_2$, even allowing classical logic. Moreover, by modifying the model so as to take V to be the class of all partial recursive functions we can also satisfy Church's thesis CT_2 for partial functions. Hence, to the extent that constructive mathematics is contained in $T_0 + CA_2 + CT_2$, we cannot prove constructively the existence of discontinuous functions on the spaces of interest to us in ordinary analysis. The main results of Beeson 1977 are in certain respects stronger positive results for a variety of intuitionistic theories T, to the effect that if a term can be proved in T to define a function (between suitable spaces) then it can proved to be continuous. This will be explained more precisely in Part IV (cf. also Beeson's corresponding results for intuitionistic theories of sets in this volume).

10. Topological models. (The material of this section and its application in 11.4 was developed in collaboration with my student Jan Stone.) Let S be a topological space and $\mathfrak{F}$ a family of partial continuous functions from S to S with card $(\mathfrak{F}) \leq$ card (S). ω is assumed disjoint from S and is considered with its discrete topology. We use $+$, Σ for the operations of disjoint sum of topological spaces. Let $J = \{0\}^*$ be the closure of $\{0\}$ under pairing. Then define S_a for $a \in J$ by

$$S_0 = \omega + S \quad \text{and} \quad S_{(a,b)} = S_a \times S_b \,.$$

Finally, take $V = \sum_{a \in J} S_a$. Thus pairing and projection make sense on V and we have $\omega \subseteq V$. A model $\mathfrak{A} = (V, \simeq, k, s, p, p_1, p_2, d, 0, s_N, p_N)$ of APP is generated as indicated in §5, only now defining $dxyuv$ just for $x, y \in \omega$. The choice of codes can be arranged in such a way that

for each f, the partial function $\lambda x(fx)$ is continuous on V.

We illustrate the argument for k, s where we take $k=1$, $kx = (1,x)$, $kxy = x$, $s=2$, $sx = (2,x)$, $sxy = (2,x,y)$ and $sxyz \simeq xz(yz)$. It is proved by induction on the generation of the relation App that if $f^* \to f$ and $x^* \to x$ then $f^*x^* \to fx$. For example, if f is $(2,x,y) = (2,(x,y))$ and $f^* \to f$ and $z^* \to z$ then in a suitable neighborhood of f, we have $f^* = (2, x^*, y^*)$ where $x^* \to x$, $y^* \to y$. Hence by induction it follows that $x^*z^*(y^*z^*) \to xz(yz)$. Again the model does not work to give the axiom V because full definition by cases is not continuous.

11. Applications to independence of Cantor-Bernstein statements.

11.1 Cardinality relations. These relations between classes are defined in the language of T_0 as follows:

(i) $(X \sim Y) \leftrightarrow \exists f, g[f: X \to Y \wedge g: Y \to X \wedge \forall x \in X(g(fx) = x) \wedge \forall y \in Y(f(gy) = y)]$

(ii) $(X \preccurlyeq_1 Y) \leftrightarrow \exists f[f: X \to Y \wedge \forall x_1, x_2 \in X(fx_1 = fx_2 \to x_1 = x_2)]$

(iii) $(X \prec_2 Y) \leftrightarrow \exists g[g: Y \to X \wedge \forall x \in X \, \exists y \in Y(gy = x)]$

(iv) $(X \prec_3 Y) \leftrightarrow \exists Z(Y \sim X + Z)$.

(The operation $X_o + X_1$ is $\sum_{i \in \{0,1\}} X_i$.) The statement of Cantor-Bernstein can be given in one of three forms corresponding to (ii)-(iv):

$(CB)_i$ $$X \preccurlyeq_i Y \wedge Y \preccurlyeq_i X \rightarrow X \sim Y .$$

The converse in each case is trivial. It will be shown that each of these statements is constructively unprovable, by suitable independence arguments. The first such results were obtained by van Dalen 1968 in the informal framework of Brouwer's theory of free choice sequences where maps between suitable topological spaces are necessarily continuous. We give different arguments here for the framework of T_o .

11.2 Independence of CB_1 from $T_o + D_V + CA_2$. This is by failure of the recursion-theoretic analogue of CB_1 . Let $\mathfrak{B} = (\mathfrak{A}, C\ell, \epsilon)$ be a model of $T_o + D_V + CA_2$ built from the Structure $\mathfrak{A}$ of ordinary recursion theory in § 3 . Let $X = \mathbb{N}$ and $Y \subseteq \mathbb{N}$ any member of $C\ell$ with $\mathbb{N} \preccurlyeq_1 Y$ but Y not recursively enumerable (e.g. such Y can be chosen co-r.e.). There is no map in the model from $\mathbb{N}$ onto Y , otherwise Y would be r.e.; thus $\mathbb{N} \not\sim Y$.

11.3 Independence of CB_2 from $T_o + CA_2$. Here we use an example from van Dalen 1968 but apply §10 instead to get the independence result. Let X be $2^{\mathbb{N}}$ considered as a topological space, and $Y = X + E$ where E consists of a single point, thus isolated in Y. There are continuous maps $F : X \underset{\text{onto}}{\rightarrow} Y$ and $G: Y \underset{\text{onto}}{\rightarrow} X$. Let $S = X + Y$ and $\mathfrak{F} = \{F,G\}$. By §11 we form a model $\mathfrak{B} = (\mathfrak{A}, C\ell, \epsilon)$ of T_o+CA_2 with F,G represented in $\mathfrak{A}$, and every $\lambda x(fx)$ in $\mathfrak{A}$ being partial continuous on $V = \Sigma_a S_a$. Thus if $X \sim Y$ in this model we would have X homeomorphic to $X + E$, which is false.

Question: Is CB_2 independent from $T_o + D_V + CA_2$? That would of course follow if the recursion-theoretic analogue of CB_2 is false.

11.4 Independence of CB_3 from $T_o + CA_2$. Here we use an example due to Hanf (cf. Halmos 1963) of a pair of topological spaces X, Z with $X \sim X + Z + Z$ but $X \not\sim X + Z$, where $\sim$ is the relation of being homeomorphic. It follows for $Y = X + Z$ that $X \sim Y + Z$ so CB_3 fails for this topological interpretation. Now we form a model $\mathfrak{B}$ of $T_o + CA_2$ over $S = X + Z + Z$ by §10, in which the maps giving the homeomorphism $X \sim X + Z + Z$ are included and every map is continuous on V. By Hanf's result, $X \not\sim Y$ in the sense of cardinal equivalence in this model. (Another example of van Dalen 1968 can also be adapted to this purpose.)

Remark. The recursion-theoretic analogue of CB_3 is true by Dekker-Myhill 1960. In their argument the fact that the universe V is $\mathbb{N}$ is used in an essential way. This is an example of a positive recursive analogue of a classical set-theoretic result which is not subsumed under a theorem of T_o or even $T_o + CA_2$.

Question. Is CB_3 independent from $T_o + D_V + CA_2$?

12. A model of the weak power-class axiom. In the preceding we mostly chose different models of APP to get various consistency and independence results; the only exception was in §2. Here we modify the construction of $C\ell$ and ε so as to get a model of POW. This will also satisfy $EM_0 + D_V + IG$ but not the join axiom J. (Recall inconsistency of POW with J from II.14.1).

Let $\mathfrak{A}$ be the recursion-theoretic model of $APP + D_V$. We introduce a new code $\mathbb{C}$ for the "class of all classes". Now instead of defining $C\ell_\alpha$, ε_α simultaneously we first define $C\ell$ and then ε. (This procedure would not be possible if closure under join were required.) Take $C\ell_0 = \{\mathbb{N}, \mathbb{C}\}$ and

$$C\ell_{n+1} = C\ell_n \cup \{c_k(\underline{y},\underline{a}) \mid k = \ulcorner\phi(x,\underline{y},\underline{Z})\urcorner \text{ with } \phi \text{ elementary and } a_1,\ldots,a_m \in C\ell_n\}$$

$$\cup\ \{i(a,r) \mid a,r \in C\ell_n\}.$$

Put $C\ell = \bigcup_{n<\omega} C\ell_n$. For $a \in C\ell_n$ we define $x \,\varepsilon_n a$ as follows:

$$x \,\varepsilon_0\, \mathbb{N} \leftrightarrow x \in \omega\ , \quad x\,\varepsilon_0\,\mathbb{C} \leftrightarrow x \in C\ell\ .$$

For $c \in C\ell_n$, $x\,\varepsilon_{n+1} c \leftrightarrow x\,\varepsilon_n c$. For $c = c_k(\underline{y},\underline{a})$, $k = \ulcorner\phi(x,\underline{y},\underline{Z})\urcorner$, with the $a_i \in C\ell_n$, and $c \notin C\ell_n$ we put

$$x\,\varepsilon_{n+1} c \leftrightarrow (\mathfrak{A}, C\ell, \varepsilon_n) \models \phi(x,\underline{y},\underline{a})\ ,$$

and for $a,r \in C\ell_n$ and $c = i(a,r)$ with $c \notin C\ell_n$ we put

$$x\,\varepsilon_{n+1} c \leftrightarrow \forall I \subseteq \omega\{[\forall u(u\varepsilon_n a \wedge \forall w((w,u)\,\varepsilon_n r \to w \in I) \to u \in I] \to x \in I\}.$$

It may be seen that the resulting model satisfies $EM_0 + D_V + IG$ (plus CT_2 as in §3). Furthermore it satisfies

$$\forall x[x \,\varepsilon\, \mathbb{C} \leftrightarrow \exists X(x = X)].$$

Given any A we can form a weak power class $P(A)$ of A by taking $P(A) = \{fAb \mid b \,\varepsilon\, \mathbb{C}\}$ where $fAB = A \cap B$.

Remark. One can also arrange to satisfy CA_2 by using the method of §2. CA_2 can't be derived from $CA_1 + POW$ without join.

IV Realizability interpretations

1. Background. The distinctive effect of restriction of the logic to be intuitionistic is of course not shown by standard models of the kind considered in the previous part III. The following are some special properties which are typically enjoyed to some extent or other by various intuitionistic theories T: (i) The disjunction property (DP), i.e. if $T \vdash (\phi \vee \psi)$ then $T \vdash \phi$ or $T \vdash \psi$; (ii) the existential definability property (ED), i.e. if $T \vdash \exists x\,\phi(x)$ then for some term τ, we have $T \vdash \phi(\tau)$; and for T containing arithmetic: (iii) the property $(ED)_{\mathbb{N}}$ holds, i.e. if $T \vdash \exists n\,\phi(n)$ then for some (specific) n, $T \vdash \phi(\bar{n})$; (iv) T is consistent with the schematic form of Church's thesis CT_0; (v) T is closed under Church's Rule CR_0, i.e. if $T \vdash \forall n\, \exists m\, \phi(n,m)$ then for some (specific)

e, $T \vdash \forall n\, \phi(n, \{\bar{e}\}(n))$; and finally for T containing function variables: (vi) T is consistent with various forms of the <u>axiom of choice</u> AC, and is closed under corresponding choice rules. While, as remarked by Kreisel and Troelstra, these properties are neither necessary nor sufficient for T to be constructive, much of the metamathematics of constructive theories revolves around their verification.

The basic methods to obtain such results are by <u>realizability interpretations</u>.[17] These were introduced by Kleene in 1945 with his notion of <u>recursive realizability</u>. Many extensions and variants have since been applied, due to Kreisel, Troelstra, de Jongh, J.R. Moschovakis , Friedman, Beeson and others. A rather complete survey can be found in Troelstra 1973 Ch.III or Troelstra 1977a §4; it may be helpful for the reader to look at these references in connection with this part.

It is useful to distinguish <u>formal</u> or <u>internal realizability</u> from <u>informal</u> or <u>external</u> realizability interpretations, though very often these are coupled. In the former one associates with each formula ϕ of $\mathcal{L}(T)$ a new formula ϕ_r with one additional free variable f, written $f\,r\,\phi$. In the latter one defines a <u>relation</u> between mathematical objects f of some sort and formulas ϕ. (Kleene's recursive realizability was of this type: he defined a relation between numbers $f \in \omega$ and formulas of arithmetic.) External realizability interpretations can often be regarded as the reading of a formal $f\,r\,\phi$ in a specific model M; that is the approach we shall take here. In any case the idea of $f\,r\,\phi$ is that f packages the constructive information (witnesses, proofs) which verifies ϕ; the definitions are thus closely related to the informal interpretation of the logical connectives in I.4.2.

By a <u>realizability interpretation of</u> $\mathcal{L}(T)$ <u>in</u> $\mathcal{L}(T')$ is meant an association $\phi \mapsto f\,r\,\phi$ with each formula ϕ (of the language of T) of a formula $f\,r\,\phi$ (of the language of T') having at most one additional free variable f. (Thus every sort of variable of $\mathcal{L}(T)$ must also be included among those of $\mathcal{L}(T')$.) This interpretation is said to be <u>sound for</u> T <u>in</u> T' if for each <u>theorem</u> ϕ of T we have a term τ such that $T' \vdash (\tau\, r\, \phi)$.

2. <u>Formal realizability of</u> $\mathcal{L}(T_0)$ <u>in itself</u>. This was introduced in Feferman 1975;[18] variants from Feferman 1976b and Beeson 1977 will be explained below. When ϕ is written $\phi(\underline{x}, \underline{X})$ we write $f\,r\,\phi(\underline{x},\underline{X})$ for $f\,r\,\phi$; when concentrating on a distinguished variable as in $\exists x\, \phi$ we may write $fr(\exists x\, \phi(x))$. The interpretation is defined inductively as follows:

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17) Another method to obtain some of these properties is due to Kripke; cf. Smorynski's chapter on Kripke models in Troelstra 1973. These models will be applied at one point in Part V below.

18) It was pointed out by Beeson that the clause there for disjunction needed correction, as given in (iii) below.

(i) $[f\, r\, \phi] = \phi$ for ϕ atomic

(ii) $[f\, r(\phi \wedge \psi)] = [(p_1 f) r\, \phi \wedge (p_2 f) r \psi]$

(iii) $[f\, r(\phi \vee \psi)] = [p_1 f \in \mathbb{N} \wedge (p_1 f = 0 \rightarrow (p_2 f) r\, \phi) \wedge (p_1 f \neq 0 \rightarrow (p_2 f) r\, \psi)]$

(iv) $[f\, r(\phi \rightarrow \psi)] = \forall z[z\, r\, \phi \rightarrow (fz) r\, \psi]$

(v) $[f\, r(\forall x\, \phi)] = \forall x[(fx) r\, \phi]$

(vi) $[f\, r(\exists x\, \phi(x))] = [(p_1 f) r\, \phi(p_2 f)]$

(vii) $[f\, r(\forall X\, \phi)] = \forall X[(fX) r\, \phi]$

(viii) $[f\, r\, \exists X\, \phi(X)] = [C\ell(p_2 f) \wedge (p_1 f) r\, \phi\, (p_2 f)]$. [19)]

When it is necessary to distinguish this from other realizability interpretations to be defined later, we shall subscript this r as r_1. Note that $fr(\neg \phi)$ is equivalent to $\forall z \neg (zr\, \phi)$.

3. Essentially $(\vee, \exists)$-free formulas. This class of formulas are such as can be realized in a canonical way (if at all) and for that reason play a distinguished role. We call ϕ essentially $(\vee, \exists)$-free if it is built up from formulas of the form $(\tau\downarrow)$, $C\ell(\tau)$, $(\tau \in X)$ and $(\tau_1 \simeq \tau_2)$ by $\wedge$, $\rightarrow$ and $\forall$ applied to either sort of variable. Note that the existential information in the first three formulas, written as $\exists x(\tau \simeq x)$, $\exists X(\tau \simeq X)$ and $\exists x(\tau \simeq x \wedge x \in X)$ can be represented by the application term τ itself. The following lemmas are easily established for $r = r_1$.

(1) For each ϕ, the formula $(fr\phi)$ is essentially $(\vee, \exists)$-free.

(2) With each essentially $(\vee, \exists)$-free ϕ is associated a term τ_ϕ with free variables contained in those of ϕ such that $APP_{\mathbb{N}} \vdash [\phi \rightarrow (\tau_\phi\, r\, \phi)]$.

(3) If ϕ is essentially $(\vee, \exists)$-free then $APP_{\mathbb{N}} \vdash [(fr\phi) \rightarrow \phi]$.

Here (2) and (3) are proved by a simultaneous induction in order to take care of the case of implication, where we put $\tau_{(\phi \rightarrow \psi)} = k(\tau_\psi)$. Because of (2) we call τ_ϕ the canonical realizer of ϕ for css. $(\vee, \exists)$-free ϕ.

Remark. Formulas of the kind that we call essentially $(\vee, \exists)$-free are often called almost negative in the intuitionistic literature.

4. The scheme 'To assert is to realize'. This scheme consists of all formulas of the following form:

(A-r) $$\phi \leftrightarrow (\exists f)(fr\, \phi)$$

which expresses equivalence of the assertion of ϕ with its realizability. By (2), (3) of §3 each instance of (A-r) in which ϕ is ess.$(\vee, \exists)$-free is derivable in $APP_{\mathbb{N}}$. The scheme as a whole is itself realizable:

(1) for any formula ϕ we can find a $\dot{\tau}$ such that

$$APP_{\mathbb{N}} \vdash \tau r\, [\, \phi \leftrightarrow \exists f\, (fr\, \phi)\,].$$

19) The clause for $(\phi \rightarrow \psi)$ does not completely mirror the requirements for a constructive proof as expressed in II.4.2, which calls for constructive recognition of $\forall z[zr\phi \rightarrow (fz) r\, \phi]$ when $zr\phi$ is read 'z is a proof of ϕ'; similar remarks apply to the universal generalization cases.

For the proof one defines τ_1, τ_2 which realize each implication. Thus τ_1 is to be chosen so that $\forall z[zr\phi \to (\tau_1 z)r(\exists f(fr\phi))]$. This makes use of the fact from § 3 that $(zr\phi)$ is ess. $(\vee,\exists)$-free and so has a canonical realizer. The converse construction τ_2 is equally easy, and uses the fact (3) from § 3.

Remark. (A-r) is suggested by the basic tenet of constructive reasoning I.4.2, that a statement is to be asserted only if it is proved.

Note. It may be necessary to distinguish (A-r) for different realizability interpretations. For example we write (A-r_1) for that of §2.

5. Axioms of choice. The most general scheme considered here for the axiom of choice takes the form:

(AC) $$\forall x \in X\, \exists y\, \phi(x,y) \to \exists f\, \forall x \in X\, \phi(x,fx)$$

for variable X. Special cases of this can be formulated for each term $\mathbb{A}$ which denotes a class, e.g. $\mathbb{N}$, $\mathbb{N} \to \mathbb{N}$, etc. We write $(AC)_{\mathbb{A}}$ for the restriction of the scheme to $X = \mathbb{A}$. There is a close connection between the schemes (AC) and (A-r); we have:

(1) (A-r_1) implies (AC).

For suppose $\forall x[x \in X \to \exists y\, \phi(x,y)]$. By (A-r) for $r = r_1$ we can find g such that $\forall x,z[zr(x \in X) \to (gxz)r\, \exists y \phi(x,y)]$. But by clause (i) of the definition of r_1, $(x \in X) \to 0r\,(x \in X)$, so $p_1(gx0)r\,\phi(x, p_2(gx0))$. But then $\phi(x, p_2(gx0))$ holds, so $f = \lambda x(p_2(gx0))$ is a choice function. (Note that the proof just uses the APP axioms.) It may be of interest to the reader to see which instances of (A-r_1) are implied by (AC); they form a wide class.

While full (AC) will thus be realized in the r_1-interpretation, this will not hold for other r_i to be considered. A special consequence of (AC) which will be realized even when (AC) is not, is the axiom scheme of dependent choices:

(DC) $$\forall x \in X\, \exists y \in X\, \phi(x,y) \to \forall x_0 \in X\, \exists f \in X^{\mathbb{N}}[f0 = x_0 \wedge \forall n\, \phi(fn,fn')].$$

Using the axioms APP + $\mathbb{N}$ we can derive (DC) from (AC) in essentially the standard way. Namely, given g such that $\forall x \in X[gx \in X \wedge \phi(x,gx)]$ we define f by primitive recursion to satisfy $f0 \simeq x_0$, $fn' \simeq g(fn)$. Then it is proved by full induction on $\mathbb{N}$ that $\forall n[fn\downarrow \wedge\, \phi(fn, fn')]$.

6. The theory $T_0^{(-)}$. We do not have a soundness theorem for r_1-realizability of T_0 in itself. The problem arises with the elementary comprehension scheme CA_1(i.e. ECA). To realize $\exists X\{c_n(\underline{y},\underline{Z}) = X \wedge \forall x[x \in X \leftrightarrow \phi(x,\underline{y},\underline{Z})]\}$ for $n = \ulcorner\phi(x, \underline{y}, \underline{Z})\urcorner$ we have to show how to convert any u with $ur\,\phi(x,\underline{y},\underline{Z})$ into a w such that $wr(x \in X)$ where $x = c_n(y,\underline{Z})$ and conversely. But for $r = r_1$ we have $wr(x \in X) \leftrightarrow x \in X$. Thus we would have to obtain $\phi \leftrightarrow \exists u(ur\,\phi)$, which is only generally true for essentially $(\vee,\exists)$-free formulas. However, this difficulty

suggests an obvious modification of CA_1 to a scheme $CA_1^{(-)}$, where CA_1 is taken only for essentially $(\vee,\exists)$-free ϕ. By $EM_0^{(-)}$ we mean the axiom system $APP + CA_1^{(-)} + \mathbb{N}$, and by $EM_0^{(-)}\upharpoonright$, the same theory with induction on $\mathbb{N}$ restricted. We claim that $EM_0^{(-)}\upharpoonright$ serves to obtain the same mathematical consequences in BCM as $EM_0\upharpoonright$ (II.10), and similarly for $EM_0^{(-)}\upharpoonright + J$, $EM_0^{(-)} + J$ in place of $EM_0\upharpoonright + J$, $EM_0 + J$, resp. (II.11). The reason is very simple: in the formalization of BCM by following Bishop's official definitions, we never make essential use of $\vee$ or $\exists$ in defining properties of sets - since the witnessing information is always required to accompany the presentation of the elements of those sets (recall I.15). Hence $CA_1^{(-)}$ always suffices in place of CA_1. The only difference appears when we enter the theory of ordinals and Borel sets (II.12). Here one must make a slight modification in the IG axiom to achieve the same results. For example, previously we took $\mathcal{O}_1 = i(A,R)$ where $A = \{x | x=0 \vee x=(p_2x)^+ \vee x = \sup_{\mathbb{N}}(p_2^2 x)\}$, and $R = \{(y,x) | (x=(p_2x)^+ \wedge y = p_2x) \vee x = \sup_{\mathbb{N}}(p_2^2 x) \wedge \exists n(y = p_2^2 xn)\}$. Thus R is defined using $\exists$ in an essential way. We now modify IG to $IG^{(-)}$ by taking $i(A,S) = I$ to satisfy instead

$$\forall x \in A\{\forall y,z[(z,y,x) \in S \to y \in I] \to x \in I\}$$

as the closure axiom and then taking a corresponding induction principle. This has the same effect as the previous IG with $(y,x) \in R \leftrightarrow \exists z[(z,y,x) \in S]$. Let $T_0^{(-)} = EM_0^{(-)} + J + IG^{(-)}$. It is thus seen that $T_0^{(-)}$ serves to obtain the same mathematical consequences in BCM as T_0 (II.10 - II.12). (The theory $T_0^{(-)}$ was introduced in Feferman 1975, where the soundness result of the next section was outlined.)

7. Soundness theorem for r_1-realizability of $T_0^{(-)}$ in itself. It is usually a routine matter to verify soundness of the axioms and rules of intuitionistic logic for any reasonable realizability interpretation (cf. Troelstra 1973 Ch.III). For the present r_1 interpretation, soundness of the logical part of $T_0^{(-)}$ is easily verified using the APP axioms to provide the requisite constructions. Going on to the non-logical axioms, it is straightforward in each case to verify soundness of each axiom or scheme on the basis of the corresponding principles themselves. The reason in the case of $CA_1^{(-)}$ has already been explained in §6, by use of the properties of essentially $(\vee,\exists)$-free formulas from §3. In the case of the induction scheme on $\mathbb{N}$ we are required to give a τ which realizes $\phi(0) \wedge \forall x(\phi(x) \to \phi(x')) \to \forall x(x \in \mathbb{N} \to \phi(x))$. This must show how to convert any z_0, z_1 with $z_0 r \phi(0)$ and $\forall x,w[w r \phi(x) \to (z_1 xw) r \phi(x')]$ into some $u = \tau z_0 z_1$ with $\forall x[x \in \mathbb{N} \to (ux) r \phi(x)]$; u is defined by recursion (uniformly from z_0, z_1) to satisfy $u0 = z_0$ and $ux' \simeq z_1 x(ux)$ and the required conclusion is proved by induction on $\mathbb{N}$. One proceeds similarly for the realization of $IG^{(-)}$. In this way we conclude that the following holds for $r = r_1$:

(1) with each theorem ϕ of $T_0^{(-)}$ can be associated a term τ such that $T_0^{(-)} \vdash (\tau r \phi)$.

Corresponding results can be obtained for the various subtheories $EM_0^{(-)}\restriction$, $EM_0^{(-)}$, $EM_0^{(-)}+J$ which have been considered. In addition, if we take $CA_2^{(-)}$ to be the restriction of CA_2 to essentially $(\vee,\exists)$-free stratified formulas, we get:

(2) r_1-realizability is also sound for $T_0^{(-)}+CA_2^{(-)}$ in itself.

Now as we have seen in §4, the scheme (A-r) is itself r-realized in a trivial way; moreover (A-r) implies (AC) by §5. Finally, it is impossible to realize $\bot$ in a consistent theory, since $(fr_1\bot)\leftrightarrow\bot$. We may thus conclude that:

(3) r_1-realizability is sound for $T_0^{(-)}+(A\text{-}r)$ in $T_0^{(-)}$ and the same holds for $T_0^{(-)}+CA_2^{(-)}+(A\text{-}r)$ in $T_0^{(-)}+CA_2^{(-)}$; hence $T_0^{(-)}+CA_2^{(-)}+(A\text{-}r)$ and (therefore also) $T_0^{(-)}+CA_2^{(-)}+AC$ are consistent.

Coupled with the following we see now the metamathematical power of realizability interpretations applied to theories dealt with in this paper emerging.

8. Consistency of Church's Thesis (CT_0). Let $T=T_0^{(-)}+CA_2^{(-)}+(A\text{-}r)$ for $r=r_1$. Then T is consistent with CT_0, by the following argument. We read r_1-realizability in a model of T whose applicative part is the ordinary recursion-theoretic model of III.3. It is to be shown for any ϕ that $\forall n\exists m\,\phi(n,m)\to\exists e\forall n\,\phi(n,\{e\}(n))$ is realized in this model by suitable f. This is to convert any g realizing $\forall n\exists m\,\phi(n,m)$ into (fg) realizing $\exists e\forall n\,\phi(n,\{e\}(n))$. Now from the hypothesis we have $\forall n[(p_1(gn))r\,\phi(n,p_2(gn))]$. We take $\{e\}(n)=p_2(gn)$ for all n, from which description f is obtained very simply.

9. Closure properties of $T_0^{(-)}$ and related theories.

9.1 r_2-realizability. In order to obtain the disjunction and existence properties for $T_0^{(-)}$ we modify r_1 in a manner due to Kleene, (called q-realizability, cf. Troelstra 1973 p. 189); this is here denoted by r_2. Only the following clauses are varied:

(iii)' $[fr(\phi\vee\psi)]=[p_1f\in\mathbb{N}\wedge(p_1f=0\to\phi\wedge(p_2f)r\,\phi)]\wedge[p_1f\neq 0\to\psi\wedge(p_2f)r\,\psi]$

(iv)' $[fr(\phi\to\psi)]=\forall z[\phi\wedge(zr\,\phi)\to(fz)r\psi]$

(vi)' $[fr\,\exists x\,\phi(x)]=[\phi(p_2f)\wedge(p_1f)r\,\phi(p_2f)]$

(viii)' $[fr\,\exists X\,\phi(X)]=C\ell(p_2f)\wedge\phi(p_2f)\wedge(p_1f)r\,\phi(p_2f)$.

It is easily checked that the soundness theorem for r_2-realizability holds for each of the sub-theories of $T_0^{(-)}+CA_2^{(-)}$ considered. In addition, (A-r) is r_2-realized just as before.

9.2 The ED property. Suppose T is any theory for which we have soundness of r_2 - realizability e.g. any of the theories just indicated. Then if $T \vdash \exists x\, \phi(x)$ there is a term τ such that $T \vdash \tau r(\exists x\, \phi(x))$. Hence by clause (vi)' for r_2 we have $T \vdash \phi(p_2\tau)$. Thus T enjoys the existential definability property.

9.3 The DP and $ED_{\mathbb{N}}$ properties. For these we need a special argument due to Beeson 1977.[20] Let T be a subtheory of $T_0^{(-)} + CA_2^{(-)}$ for which we have soundness of r_2 - realizability of T in T. Assume also that $APP_{\mathbb{N}} \subseteq T$. It was stated in III.8.2 that if $T \vdash (\tau \in \mathbb{N})$ then for some n, $T \vdash (\tau = \bar{n})$. (The proof made use of the model NF of normal terms.) Now if $T \vdash \exists n\, \phi(n)$ it follows that for some τ, $T \vdash (p_2 \tau) \in \mathbb{N} \wedge \phi(p_2 \tau)$. Hence T has the $ED_{\mathbb{N}}$ property; it is a corollary that T has the DP property.)

Remark. It is a little more work to obtain closure under Church's rule. One way is to formalize the properties of r_2 - realizability of any finite subtheory of T within T.

10. Inconsistency of T_0 with AC. We next turn to the question of obtaining corresponding properties for T_0. This section shows that r_1-(or r_2-) realizability is not sound for T_0, since T_0 is inconsistent with AC. This will lead us to consider a new realizability interpretation (which does not verify full AC). The proof of contradiction of $T_0 + AC$ is by an argument due to Friedman (originally given for $T_0 + D_V + AC$). Let $X = \{x \mid \exists n(xx \in \mathbb{N} \to xx \neq n)\}$. Recall here that we are using the conventions $\exists n\, \phi(n) \leftrightarrow \exists x(x \in \mathbb{N} \wedge \phi(x))$ and $(xx \in \mathbb{N}) \leftrightarrow \exists y(xx \simeq y \wedge y \in \mathbb{N})$, i.e. $\exists n(xx \simeq n)$. Trivially by definition $\forall x \in X \exists n[xx \in \mathbb{N} \to xx \neq n]$. Hence if AC is assumed we can find an f such that $\forall x \in X[fx \in \mathbb{N} \wedge (xx \in \mathbb{N} \to xx \neq fx)]$. In particular, $f \in X \to ff \in \mathbb{N} \wedge ff \neq ff$ so $\neg(f \in X)$. Note that $\forall x[xx \in \mathbb{N} \to x \in X]$ holds since we can always find $n \neq xx$ using definition-by-cases on $\mathbb{N}$. Hence $\neg(x \in X) \to \neg(xx \in \mathbb{N})$. It follows that $\neg(ff \in \mathbb{N})$. But then by logic $ff \in \mathbb{N} \to ff \neq 0$ so $f \in X$ which is a contradiction to our original assumption, namely that AC_X holds. It is of course essential for this argument that X is existentially defined, which is not possible in $T_0^{(-)}$.

11. Realizability for T_0 via a refinement T_0^*.

11.1 The theory T_0^*. The language $\mathcal{L}^*$ of T_0^* is obtained by refining the language $\mathcal{L}$ of T_0 as follows: instead of the two-placed relation $(x \in A)$ of $\mathcal{L}$ we now have a 3-placed relation $(x \in_z A)$. In this language we take

$$x \in A \leftrightarrow_{\text{def}} \exists z\,(x \in_z A),$$

which gives a translation of $\mathcal{L}$ into $\mathcal{L}^*$. If ϕ is any formula of $\mathcal{L}$ we denote its translation by ϕ^*. If T is any theory in the language $\mathcal{L}$ of T_0, by T^* we mean the theory whose axioms are exactly the ϕ^* for all axioms ϕ of T.

20) It should be noted for comparison with Beeson 1977 that $T_0^{(-)}$ is there denoted EM and $EM_0^{(-)} + J$ is denoted EMN.

11.2 Translation of $\mathcal{L}^*$ into $\mathcal{L}$. This is accomplished in the following simple way. With each formula ψ of $\mathcal{L}^*$ is associated a formula $\tilde{\psi}$ of $\mathcal{L}$, which is obtained by replacing each atomic formula $(x \in_z A)$ by $[(x,z) \in A]$ and which, except for some changes of constants, is otherwise unaffected. Each of the combinatory constants $k, s, d, p, p_1, p_2, 0, s_{\mathbb{N}}, p_{\mathbb{N}}$ is unchanged, but the class formation constants c_k, j, i are replaced by new constants $\tilde{c}_k, \tilde{j}, \tilde{i}$ as will be explained in the next section.

11.3 $\tilde{r}_1$ - realizability. This is an interpretation of $\mathcal{L}^*$ (and thence of $\mathcal{L}$) which will make all of CA realizable. The idea to realize $\exists X[x \in X \leftrightarrow \phi(x)]$ as expressed in $\mathcal{L}^*$ and then translated back into $\mathcal{L}$ i.e. as $\exists X[\exists z((x,z) \in X) \leftrightarrow \tilde{\phi}(x)]$ is to produce an X such that from any z with $(x,z) \in X$ we can find a w with $w r \tilde{\phi}(x)$ and conversely. The simplest way to achieve this is to take $z = w$ and thus to take $X = \{(x,w) | w r\ \tilde{\phi}(x)\}$. It is here where the change of constants enters; if $c_n = \ulcorner\phi(x,\underline{y},\underline{Z})\urcorner$ we'll have $\tilde{c}_n = \ulcorner(p_2 x) r\ \tilde{\phi}(p_1 x, \underline{y}, \underline{Z})\urcorner$.

For this purpose $\tilde{r}_1$ - realizability is defined as follows; the translations of c_n, j, i are given by a simultaneous inductive definition. First we write down the clauses defining $f r \phi$ for ϕ in $\mathcal{L}^*$ exactly like those for r_1 in § 2. The only difference appears in the fact that we now have atomic formulas $(x \in_z A)$ in place of the old $(x \in A)$, so we read $[fr(x \in_z A)] = (x \in_z A)$. Then we take $\tilde{r}_1$ to be $\tilde{r}$ where

$$(f\,\tilde{r}\,\phi) = (\widetilde{fr\phi}),$$

i.e. we translate the realizability interpretation of $\mathcal{L}^*$ just described. Now $\tilde{c}_n, \tilde{j}, \tilde{i}$ are chosen to satisfy the following:

(i) if $n = \ulcorner\phi(x, \underline{y}, \underline{Z})\urcorner$, $\tilde{c}_n(\underline{y}, \underline{Z}) = \{(x,z) | z\,\tilde{r}\,\phi(x,\underline{y},\underline{Z})\}$

(ii) if $\forall x \in A[C\ell(fx)]$ then $\tilde{j}(A,f) = \{((x,y),(z,w)) | (x,z) \in A \wedge (y,w) \in fx\}$, and

(iii) $\tilde{i}(A,R) = i(A_1, R_1)$ where $A_1 = \{(x,(z,f)) | (x,z) \in A \wedge \forall y,w[((y,x),w) \in R \to f(y,w)\downarrow]\}$ and $R_1 = \{((y,fyw),(x,(z,f))) | ((y,x),w) \in R\}$.

The choice of the constants is made in such a way that for each of the axioms ϕ from CA_1 (or CA_2), J and IG as expressed in $\mathcal{L}^*$ we can find a τ such that $\tau\,\tilde{r}\,\phi$ is provable from the corresponding axiom (or axiom schema) as expressed in $\mathcal{L}$. It follows that for any theory T over APP which is based on some combination of these axioms and schemata plus the axioms $\mathbb{N}$ or $\mathbb{N}\restriction$, we have a soundness theorem for $\tilde{r}_1$ - realizability of T^* in T. Once more the scheme (A-r), as written out in $\mathcal{L}^*$, is trivially $\tilde{r}_1$ - realized. It is a corollary that

(1) $T_0^* + (CA_2)^* + (A\text{-}r)$ is consistent.

Hence any consequence of (A-r) in $\mathcal{L}$ is consistent with $T_0 + (CA_2)$ regarded as translated into $\mathcal{L}^*$. It is simpler to study such consequences in the latter language than to pass through $\tilde{r}$. Note: To get further consistency with Church's thesis CT_0 we simply couple this with the recursion-theoretic model as described in § 8.

12. Consequences of "to assert is to realize" in $\mathcal{L}^*$. We assume at least $EM_0^* + (A\text{-}r)$ where r is r_1 for $\mathcal{L}^*$ throughout this section.

12.1 Dependent choices. DC is derived from these assumptions in the following way. Suppose $\forall x \in A\, \exists y \in A\, \psi(x,y)$ holds, i.e. $\forall x,z[x \in_z A \rightarrow \exists y,w(y \in_w A \wedge \psi(x,y))]$. Then there exists g which realizes this statement, so for each x, z with $x \in_z A$, $g(x,z)$ provides us with a triple (y,w,u) such that $y \in_w A \wedge u\, r\, \psi(x,y)$. Given $x_0 \in A$ fix some z_0 with $x_0 \in_{z_0} A$. Then using g we define a sequence (x_n, z_n, u_n) by recursion such that $g(x_n, z_n) \simeq (x_{n+1}, z_{n+1}, u_n)$ and $x_n \in_{z_n} A$ and $u_n r\, \psi(x_n, x_{n+1})$ for each n. Let $f = \lambda n.x_n$. Then passing from the right to the left side of (A-r) we have $f0 = x_0 \wedge \forall n[fn \in A \wedge \psi(fn, fn')]$.

As a corollary of this and 11.3 we have that

(1) $T_0 + CA_2 + DC$ is consistent.

Remark. The argument here brings out the reason why DC can be dealt with constructively even where AC can't in the presence of full comprehension. From $\forall x \in A\, \exists y\, \phi(x,y)$ written in $\mathcal{L}^*$ as $\forall x,z[x \in_z A \rightarrow \exists y\, \phi(x,y)]$ we can merely conclude $\exists f\, \forall x,z[x \in_z A \rightarrow \phi(x, f(x,z))]$ from (A-r). This is the result which was referred to in I.4.7.

12.2 Canonically realizable classes (choice bases). We write $\mathbf{C}(A)$ for the following formula:

$$\exists g[\exists z(x \in_z A) \rightarrow x \in_{gx} A].$$

A is called canonically (or self-) realizable if $\mathbf{C}(A)$ holds. $\mathbf{C}(A)$ is equivalent to AC_A, i.e. the scheme of choice with base A. For suppose $\mathbf{C}(A)$ holds using g and that $\forall x \in A\, \exists y\, \phi(x,y)$. Then as just remarked we find f such that $\forall x,z[x \in_z A \rightarrow \phi(x, f(x,z))]$. It follows that $\forall x \in A\, \phi(x, f(x,gx))$. Conversely if AC_A holds then from $\forall x \in A\, \exists z(x \in_z A)$ (which is trivial by definition in $\mathcal{L}^*$) we conclude $\exists g\, \forall x \in A(x \in_{gx} A)$, i.e. $\mathbf{C}(A)$.

Now $\mathbf{C}(\mathbb{N})$ holds because $AC_{\mathbb{N}}$ is a consequence of DC. Also $\mathbf{C}(V)$ holds because $\forall x\, \exists y\, \phi(x,y) \rightarrow \exists f\, \forall x \phi(x, fx)$ by (A-r). Furthermore, the property $\mathbf{C}$ is closed under class constructions which can be defined by essentially $(\vee, \exists)$-free formulas, for which we have found canonical realizers by § 3. In particular, if $\mathbf{C}(A)$ and $\mathbf{C}(B)$ then $\mathbf{C}(A \times B)$ and $\mathbf{C}(A \rightarrow B)$ hold and if $\phi(x)$ is ess. $(\vee, \exists)$-free then also $\mathbf{C}(\{x \in A | \phi(x)\})$. This gives consistency of $T_0 + CA_2$ with an extensive collection of instances of AC. (The consistency of T_0 with $(AC)_{FT}$, i.e. AC_{N_σ} for all $\sigma \in FTS$ was noted by Beeson 1977.)

12.3 The presentation axiom (Aczel). Call (A^*,h) a presentation of A if $h : A^* \xrightarrow[\text{onto}]{} A$; this is called a full presentation of A if $\mathbf{C}(A^*)$ holds. Intuitively, in a presentation each element x of A is represented (in possibly more than one way) by $x^* \in A^*$ such that $h(x^*) = x$; x^* contains "additional information" that "verifies" $x \in A$. When we have a full presentation, no further information need be added. The presentation axiom PA is the statement that for every A there exists a full presentation (A^*,h) of A. This was introduced (in a slightly different form) by Aczel in unpublished notes; he observed that it serves to derive the various mathematical consequences of (A-r). In the present framework, PA is a trivial consequence of (A-r); we simply take $A^* = \{(x,z) \mid x \in_z A\}$ and $h(x,z) = x$.

12.4 Having your cake and eating it too with (A-r) as an implement. In the informal discussion of I.15.3 the attempt to have one's constructive cake and eat it too was taken to be a matter of being casual about showing the witnessing information required by the official definitions. Here we can provide a theoretical framework to justify such practices simply by assuming (A-r) for $r = r_1$ in $\mathcal{L}^*$. In effect, the informal definition of a class A in the form $A = \{x \mid \phi(x)\}$ gives rise to $A^* = \{(x,z) \mid z r\, \phi(x)\}$, which corresponds to the official definition. By $\phi(x) \leftrightarrow \exists z(z r\, \phi(x))$ we have $x \in A \leftrightarrow \exists z[(x,z) \in A^*]$. A realizable refinement of CA in $\mathcal{L}^*$ allows us to take $\forall x[x \in_z A \leftrightarrow z r\, \phi(x)]$, so that this A^* is exactly the same as the full presentation of A described in 12.3. For example, if we define

$\mathbb{R}^+ = \{x \mid x \in \mathbb{R} \wedge \exists n > 0 (x_n > \frac{1}{n})\}$ we than have $(\mathbb{R}^+)^* = \{(x,n) \mid x \in \mathbb{R} \wedge n > 0 \wedge (x_n > \frac{1}{n})\}$,

just as required by the official definition. Using AC in its weakened form $\forall x \in A\, \exists y\, \phi(x,y) \to \exists f\, \forall x,z[(x,z) \in A^* \to \phi(x,f(x,z))]$ we can conclude that an inverse function is defined on $(\mathbb{R}^+)^*$ knowing that $\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\, (x \cdot y = 1)$.

Remark. $T_0^* + $ (A-r) provides an alternative way of reading Bishop which is in some respects simpler than by T_0, since one can formalize the informal mathematical arguments more directly. (Note: The same ends can be achieved by the presentation axiom instead of (A-r).) It is not meant by this that $T_0^* + $ (A-r) is in direct accordance with Bishop's views; (that is open to discussion).

13. Closure properties of T_0. In order to obtain the ED and $ED_{\mathbb{N}}$ (and hence DP) properties for T_0, Beeson 1977 introduced a kind of combination of $\tilde{r}$ - and q-realizability. His definition (loc.cit.pp.281-282) is complicated by the requirement to have a doubling (X,X^*) of class variables, where the new variable X^* is to correspond to the class of all (x,z) such that $z r(x \in X)$. In addition, Beeson also doubled individual variables. This does not seem to be necessary, and the following simpler definition is proposed for the same purpose. With each formula $\phi(x_1,\ldots,x_n, Y_1,\ldots,Y_m)$ of $\mathcal{L} = \mathcal{L}(T_0)$ is associated a formula

$\mathrm{fr}\,\phi(x_1,\ldots,x_n, Y_1,\ldots,Y_m, Y_1^*,\ldots,Y_m^*)$ of $\mathcal{L}$ as follows:

(i) $[\mathrm{fr}(x \in Y)] = [(x \in Y) \wedge (x,f) \in Y^*]$ and $[\mathrm{fr}\ (x = Y)] = [x = (Y, Y^*)]$

$[\mathrm{fr}\,\phi] = \phi$ for the other atomic formulas

(ii) $[\mathrm{fr}(\phi \wedge \psi)] = [(p_1 f) r\, \phi \wedge (p_2 f) r \psi]$

(iii) $[\mathrm{fr}(\phi \vee \psi)] = [(p_1 f \in \mathbb{N}) \wedge (p_1 f = 0 \rightarrow \phi \wedge (p_2 f) r\, \phi) \wedge (p_1 f \neq 0 \rightarrow \psi \wedge (p_2 f) r\, \psi)]$

(iv) $[\mathrm{fr}(\phi \rightarrow \psi)] = \forall z[\phi \wedge (z r\, \phi) \rightarrow (fz) r\, \psi]$

(v) $[\mathrm{fr}\ \exists x\, \phi(x)] = [\phi(p_2 f) \wedge (p_1 f) r\, \phi(p_2 f)]$

(vi) $[\mathrm{fr}\ \exists X\, \phi(X)] = [\exists X, X^*((p_2 f) = (X,X^*) \wedge \phi(X) \wedge (p_1 f) r\, \phi(X,X^*)]$

(vii) $[\mathrm{fr}\ \forall x\, \phi(x)] = \forall x[(fx) r\, \phi(x)]$

(viii) $[\mathrm{fr}\ \forall X\, \phi(X)] = \forall X,X^*[f(X,X^*) r\, \phi(X, X^*)]$.

Remark (added in proof): Beeson has pointed out real difficulties with the proposed realizability of T_0 which are met when looking for suitable reinterpretations of the constants. It is thus not known whether his definition can be simplified in any essential way to serve the same purposes.

14. Applications to continuity properties. Beeson 1977 has used the consequences of realizability such as $AC_{\mathbb{N}}$ and $ED_{\mathbb{N}}$ for the theories T considered in the preceding sections [21] e.g. for $T = T_0^{(-)}$ and $T = T_0$, to prove local continuity rules of the following form, where $\mathbb{A}$, $\mathbb{B}$ are any closed terms for classes:

LCR($\mathbb{A}$, $\mathbb{B}$). If T proves $\mathbb{A}$ is a complete separable metric space and $\mathbb{B}$ is a separable metric space and ϕ is an extensional property and $\forall x \in \mathbb{A}\, \exists y \in \mathbb{B}\, \phi(x,y)$ then T proves that $\forall x \in \mathbb{A}\, \exists y \in \mathbb{B}\, [\phi(x,y) \wedge$ "y is stable for x" $]$.

Here "y is stable for x" stands for $\forall \epsilon > 0\ \exists \delta > 0\ \forall u \in N_\delta^{\mathbb{A}}(x)\ \exists v \in N_\epsilon^{\mathbb{B}}(y)[\phi(u,v)]$, where $N_\epsilon^{\mathbb{A}}(x) = \{u \in \mathbb{A} \mid d_{\mathbb{A}}(x,u) < \epsilon\}$ and $d_{\mathbb{A}}$ is the metric of $\mathbb{A}$. Equality in a metric space is defined by $x_1 =_{\mathbb{A}} x_2 \leftrightarrow d_A(x_1,x_2) = 0$; extensional properties are understood to be those $\phi(x,y)$ for which

$$\phi(x_1,y_1) \wedge x_1 =_{\mathbb{A}} x_2 \wedge y_1 =_{\mathbb{B}} y_2 \rightarrow \phi(x_2, y_2).$$

As a corollary of LCR($\mathbb{A}$, $\mathbb{B}$) one has: if T proves that F is a function from $\mathbb{A}$ to $\mathbb{B}$ under the same hypotheses on $\mathbb{A}$, $\mathbb{B}$ then T proves that F is continuous. (Note that the hypothesis means $F: \mathbb{A} \rightarrow \mathbb{B}$ and $x_1 =_{\mathbb{A}} x_2 \rightarrow F(x_1) =_{\mathbb{B}} F(x_2$

Beeson's method of proof of LCR($\mathbb{A}$, $\mathbb{B}$) uses the representation of complete metric spaces with countable dense subset D in the form of the Cauchy sequences from D.

21) The matter is actually more complicated: one must formalize within the T considered the corresponding results for all finite subtheories of T.

This allows one to push the problem back to verification of $LCR(\mathbb{N} \to \mathbb{N}, \mathbb{N})$. Roughly, the idea is that if $\forall x \in (\mathbb{N} \to \mathbb{N}) \exists m\, \phi(x,m)$ is proved, where ϕ is extensional, then for each specific $g \in (\mathbb{N} \to \mathbb{N})$ we can prove $\exists m\, \phi(g,m)$ from $T \cup Diag(g)$. This requires only a finite part $\langle g(0),\ldots,g(n-1)\rangle$ of g. Further by $ED_{\mathbb{N}}$ we can find an m s.t. $\phi(g,\bar{m})$ is proved from the same. By formalizing this argument one gets the desired result. Beeson also has results on local uniform continuity rules for compact spaces and a number of consistency and independence results concerning continuity statements. He has further extended these to other formalisms such as those of Myhill and Friedman, as presented in his contribution to this volume.

Discussion. In a sense, Beeson's results confirm Brouwer's ideas that we should be able to prove that every real function on $\mathbb{R}$ (resp. $[0,1]$) is continuous (uniformly continuous). But the present results have the advantage that the systems to which they apply also have a set-theoretic interpretation. So one can be sure that if an existence proof $\forall x \in \mathbb{A} \exists y \in \mathbb{B}\, \phi(x,y)$ can be formalized in BCM then it yields stability or continuity of solutions which are true in the classical sense. Often such results can be obtained directly by ad hoc arguments. But the continuity results described considered as a part of global (or systematic) constructivity may first point the way to what can be obtained for special problems. In other words, the global results can serve as the stimulus and point of departure for mathematically interesting local results. (Indeed this has been the case with Beeson's studies of stability phenomena in the Plateau problem.)

Question. The property $ED_{\mathbb{N}}$ depends essentially on not having D_V. But one doesn't see why the continuity results should be disturbed by its presence. Do Beeson's results on LCR extend to $T_0 + D_V$ by some other arguments?

V. Relations with subsystems of analysis.

1. Introduction and summary of results. In this part (except for the special §2) we describe results which establish the equivalence of certain subsystems of T_0 with subsystems of classical 2nd order analysis. It is assumed here that the reader is familiar with the designations of various of the latter such as $(\Pi^0_\infty\text{-CA})$, $(\Delta^1_1\text{-CA})$, $(\Sigma^1_1\text{-AC})$, $(\Delta^1_2\text{-CA})$, $(\Sigma^1_2\text{-AC})$, as well as with the principle (BI) of bar induction.[22] When '$\upharpoonright$' is used following designation of a theory we mean that the principle of full induction on $\mathbb{N}$ is replaced by the axiom of induction. We write $T_1 \leq T_2$ to mean that T_1 is proof-theoretically reducible to T_2 (i.e. if $Con(T_2)$ implies $Con(T_1)$ by a finitary argument) and $T_1 \equiv T_2$

22) For descriptions of these and some information about their interrelationships cf. Feferman 1977.

if $T_1 \leq T_2$ and $T_2 \leq T_1$. In connection with the following results one also has much information about which sentences are conserved in one direction or the other; however, for simplicity we do not mention such for the most part. PA denotes classical Peano's arithmetic, HA = Heyting's arithmetic.

(1) $EM_0\upharpoonright \equiv HA$, in fact $EM_0\upharpoonright$ is a conservative extension of HA.

(2) $EM_0\upharpoonright \pm J \equiv (\Sigma^1_1 - AC)\upharpoonright \equiv PA$

(3) $EM_0 + J \boxminus (\Sigma^1_1 - AC)$

(4) $EM_0\upharpoonright \pm J + IG\upharpoonright \equiv (\Sigma^1_2 - AC)\upharpoonright \equiv (\Pi^1_1 - CA)\upharpoonright$

(5) $EM_0 + J + IG\upharpoonright \equiv (\Sigma^1_2 - AC)$

(6) $T_0 = EM_0 + J + IG \leq (\Sigma^1_2 - AC) + (BI)$.

In all of these except the conservation result of (1), we can also include classical logic and the axiom D_V on the l.h.s. The exact relationship in (6) is unsettled.

Conjecture. $T_0 \boxminus (\Sigma^1_2 - AC) + (BI)$.

Credits. The conservation result in (1) is due to Beeson 1979, by a Kripke-model argument outlined in the next section. The $\equiv$ in (1) comes simply from (2) and the fact that $PA \equiv HA$. Conservation of $(\Sigma^1_1 - AC)\upharpoonright$ over PA has been established by Barwise-Schlipf 1975 using recursively saturated models; it is also stated by Friedman 1975 where conservation of $(\Sigma^1_2 - AC)\upharpoonright$ over $(\Pi^1_1 - CA)\upharpoonright$ (for a certain class of sentences) is announced as well. (The method of recursively saturated models has also been extended to prove the latter in unpublished notes by myself.) The proof-theoretical equivalences $(\Sigma^1_1 - AC)\upharpoonright \equiv PA$ and $(\Sigma^1_2 - AC)\upharpoonright \equiv (\Pi^1_1 - CA)\upharpoonright$ have been established by Sieg. The result (3) is due to Aczel (unpublished); a new method of proof was found by myself (Feferman 1976 c). This method was also used there to establish $\leq$ in (2) and (4)-(6). The relations $\geq$ in (4) and (5) are due to Sieg 1977. Only outlines of the various ideas involved are given in the following. Detailed presentations of the proofs of these and related results will be found in the chapter by Feferman and Sieg in the projected volume "Iterated inductive definitions and subsystems of analysis: recent proof-theoretical studies" (for the Lecture Notes in Mathematics Series) which is to consist of contributions by Buchholz, Feferman, Pohlers and Sieg.

2. $EM_0\upharpoonright$ is conservative over HA. We first describe the proof of an easier result from Feferman 1976 a : $EM_0\upharpoonright$ in classical logic is conservative over PA. To begin with, the axioms APP (even with D_V) are formally modelled in PA by taking $App(x,y,z) \leftrightarrow \{x\}(y) \simeq z$. Any model $\mathfrak{M}$ of PA thus determines an applicative structure $\mathfrak{A}$. This is used to build a model $(\mathfrak{A}, C\ell, \epsilon)$ of $EM_0\upharpoonright$

by the method of III.1, but leaving off the clauses for i and j ; now the process closes off at ω. Namely, $C\ell = \bigcup_{n<\omega} C\ell_n$, $\epsilon = \bigcup_{n<\omega} \epsilon_n$ where $C\ell_o = \{\mathbb{N}\}$ and $x \epsilon_o \mathbb{N} \leftrightarrow x = x$, $C\ell_{n+1} = C\ell_n \cup \{c_k(\underline{y}, \underline{a}) \mid k = \ulcorner\phi(x,\underline{u}, \underline{Z})\urcorner$ and ϕ is elementary and $a_1,\ldots,a_m \epsilon C\ell_n\}$, with $x \epsilon_{n+1} c_k(\underline{y}, \underline{a}) \leftrightarrow (\mathfrak{A}, C\ell_n, \epsilon_n) \models \phi(x,\underline{y},\underline{a})$. Note that $\mathfrak{M}$ may be non-standard and $\mathbb{N}$ is coextensive with the domain of $\mathfrak{M}$. It may be seen that for each $A \epsilon C\ell$ there exists a formula $\psi(x,\underline{u})$ of arithmetic such that for some choice of parameters $\underline{y}$ in $\mathfrak{M}$, $\forall x[x \epsilon A \leftrightarrow \mathfrak{M} \models \psi(x,\underline{y})]$. Hence the induction axiom $(\mathbb{N}\restriction)$ is verified, and we do indeed have $(\mathfrak{A}, C\ell, \epsilon)$ a model of $EM_o\restriction$. To conclude, conservation holds by the completeness theorem for the classical predicate calculus: if θ is a sentence of arithmetic such that $(EM_o\restriction) \vdash \theta$ but $PA \nvdash \theta$ we can choose $\mathfrak{M} \models \neg\theta$ and get a contradiction.

Now Beeson 1979 has shown $EM_o\restriction$ conservative over HA by an adaptation of this argument to Kripke models, using the completeness theorem for intuitionistic logic in terms of the latter. Given any Kripke model $\mathfrak{M} = \langle(\mathfrak{M}_p)_{p\epsilon P}, \leq\rangle$ of HA one modifies the construction of $(\mathfrak{A}, C\ell_n, \epsilon_n)$ as just described to a construction of $\langle(\mathfrak{A}_p, C\ell_{n,p}, \epsilon_{n,p})_p, \leq\rangle$ for each n and thence of a Kripke model $\langle(\mathfrak{A}_p, C\ell_p, \epsilon_p)_{p\epsilon P}, \leq\rangle$ of $EM_o\restriction$.

Discussion. The significance of this result is given by I.15.5, according to which $EM_o\restriction$ is adequate to essentially all of BCM except for the theory of ordinals and Borel sets. A corresponding result had previously been obtained by Friedman 1977 (conservation of $\underset{\sim}{B}$ over HA for Π^0_2 sentences, strengthened to full conservation by Beeson 1979). Thus this portion of BCM does not really take advantage of the strong constructive principles implicitly accepted by Bishop; on the other hand it is of foundational interest that it is justified by the most elementary of these.

3. $EM_o\restriction + J \leq (\Sigma^1_1\text{-}AC)\restriction$, $EM_o + J \leq \Sigma^1_1\text{-}AC$. The proofs of these results from Feferman 1976c are given by formal models which verify classical logic and D_V. We start again with the recursion-theoretic interpretation $App(x,y,z) \leftrightarrow \{x\}(y) \simeq z$. Now, instead of defining $C\ell$ in transfinite stages, one defines it simply to be the set of Δ^1_1 indices. That is, let $P^1_1(e,x)$ (e=0,1,2,...) be a standard Π^1_1-enumeration of all Π^1_1 sets (predicates of one argument x); then $S^1_1(e,x) \leftrightarrow \neg P^1_1(e,x)$ induces a Σ^1_1-enumeration of all Σ^1_1 sets. We put e in $C\ell$ if the pair of indices $(e)_o$, $(e)_1$ determines a Δ^1_1 set, i.e.

$$C\ell(e) \rightarrow_{def} \forall x[P^1_1((e)_o, x) \leftrightarrow S^1_1((e)_1, x)].$$

Put $x \epsilon a \leftrightarrow_{def} P^1_1((a)_o, x)$ for $C\ell(a)$. To prove closure under CA_1 in this model reduces to showing that if $\phi(x, \underline{y}, \underline{Z})$ is elementary and we substitute Δ^1_1 definable sets D_i for the Z_i, the result is also Δ^1_1 (with index e uniformly recursive in given indices d_i for the D_i); this is by the Δ^1_1-substitution theorem of Addison-Kleene-Schoenfield. Formalization of the latter makes use of $(\Sigma^1_1\text{-AC})$. So far, the argument serves to give a model of EM in $(\Sigma^1_1\text{-AC})$; next it is seen that only restricted $(\Sigma^1_1\text{-AC})\restriction$ suffices if one starts with $EM_0\restriction$. To complete the proof, J is verified as follows. Suppose $C\ell(a)$ and $\forall x \in a\, C\ell(\{f\}(x))$, i.e. that

$$\forall x[P^1_1((a)_0, x) \to \forall y[P^1_1((\{f\}(x))_0, y) \leftrightarrow S^1_1((\{f\}(x))_1, y)]] .$$

Then we easily obtain a Δ^1_1 index for $j(A, f)$ where a is the index of A.

<u>Remark</u>. By $(\Sigma^1_1\text{-AC})\restriction \leq HA$, this shows that J is really of no use without unrestricted induction. That was already noted informally in II.11.3, where transfinite types were shown to exist in $EM_0 + J$ - but not in $EM_0\restriction + J$.

4. <u>$EM_0\restriction + J + IG\restriction \leq (\Sigma^1_2\text{-AC})\restriction$, $EM_0 + J + IG\restriction \leq (\Sigma^1_2\text{-AC})$ and $T_0 \leq (\Sigma^1_2\text{-AC}) + (BI)$.</u>

The proofs (again from Feferman 1976c) all use the same idea, which simply follows that of §3 one level up. Let $P^1_2(x,e)$, $S^1_2(x,e)$ enumerate the Π^1_2, resp. Σ^1_2 sets. Take $C\ell(a) \leftrightarrow \forall x[P^1_2(x,(e)_0) \leftrightarrow S^1_2(x, (e)_1)]$ and $x \in a \leftrightarrow P^1_2(x,(a)_0)$. Now one applies the A-K-S substitution theorem for Δ^1_2 predicates, which is proved using Σ^1_2-AC. This serves to show $EM_0 + J$ modelled in $(\Sigma^1_2\text{-AC})$ and $EM_0\restriction + J$ in $(\Sigma^1_2\text{-AC})\restriction$. To verify $IG\restriction$ in this model we simply apply A-K-S again: if A, R are Δ^1_2 then the set $i(A,R)$ which is Π^1_1 in A, R is also Δ^1_2 (with index e uniformly recursive in the indices a, r of A, R resp.). The induction <u>axiom</u> of $IG\restriction$ follows immediately by definition of $i(A,R)$ as the least set satisfying the given closure conditions. To obtain the full principle of induction for IG one must apply full (BI), which gives the final result: $T_0 = EM_0 + J + IG \leq (\Sigma^1_2\text{-AC}) + (BI)$.

5. <u>$(\Sigma^1_1\text{-AC})\restriction \leq PA$, $(\Sigma^1_2\text{-AC})\restriction \leq (\Pi^1_1\text{-CA})\restriction$; consequent reductions into T_0.</u>

As was remarked in the survey of credits in §1, one has proof-theoretical arguments due to Sieg for these first two reductions, corresponding to earlier conservation results of Barwise-Schlipf and Friedman. Now $PA \leq HA$ by the negative $(\neg\neg)$ translation as is well known, and $HA \subseteq EM_0\restriction$ so this completes the relations in §1(2). Next $(\Pi^1_1\text{-CA})^{\upharpoonright}$ can be interpreted in the corresponding intuitionistic system $(\Pi^1_1\text{-CA})^{\upharpoonright (i)}$ by the negative translation, and the latter is directly contained in $EM_0\restriction + IG\restriction$. This completes the chain in §1(4).

6. $(\Sigma_1^1\text{-AC}) \leq EM_0 + J$. To begin with, $(\Sigma_1^1\text{-AC}) \leq (\Pi_1^0\text{-CA})_{<\varepsilon_0}$ by Friedman 1970. (That used a model-theoretic argument; a proof-theoretical one is outlined in Feferman 1977.) As is familiar, $(\Pi_1^0\text{-CA})_{<\varepsilon_0} \equiv RA_{<\varepsilon_0}$ (ramified analysis in levels $<\varepsilon_0$), and $RA_{<\varepsilon_0} \leq RA^{(i)}_{<\varepsilon_0}$ by the negative translation. Finally, $RA^{(i)}_{<\varepsilon_0}$ is contained in $EM_0 + J$, using Join to transfinitely iterate the ramified hierarchy up to each ordinal $\alpha < \varepsilon_0$ (full induction up to α follows from full induction on $\mathbb{N}$). This completes the $\equiv$ in §1(3).

7. $(\Sigma_2^1\text{-AC}) \leq EM_0 + J + IG\upharpoonright$. By Friedman 1970, $(\Sigma_2^1\text{-AC}) \leq (\Pi_1^1\text{-CA})_{<\varepsilon_0}$ and by Feferman 1970, $(\Pi_1^1\text{-CA})_{<\varepsilon_0} \leq ID_{<\varepsilon_0}$ where the latter is a classical theory of iterated first-order inductive definitions up to any $\alpha < \varepsilon_0$. The main next step is to show $ID_{<\varepsilon_0} \leq ID^{(i)}_{<\varepsilon_0}(\mathfrak{O})$, i.e. to the intuitionistic theory of the classes $\mathfrak{O}_\alpha$ for $\alpha < \varepsilon_0$. This has been established by Sieg 1977. Finally, $ID^{(i)}_{<\varepsilon_0}(\mathfrak{O})$ is contained directly in $EM_0 + J + IG\upharpoonright$. In this way the $\equiv$ in §1(5) is completed.

Remark. Results closely related to those of Sieg 1977 have been obtained independently by Pohlers and Buchholz, by more complicated methods, but which also give more detailed information. Presentation and comparison of all this work will be found in the forthcoming joint volume referred to in §1.

8. Questions and conjectures.

(i) The conjecture $(\Sigma_2^1\text{-AC}) + (BI) \leq T_0$ has already been stated in §1. What is missing up to now is the proof-theory analogous to that indicated in §7.

(ii) We have shown $EM_0 + IG + POW$ consistent in III.12. What is the strength of this system and various of its subsystems? It appears that POW cannot be used very effectively with these axioms. I conjecture that $EM_0\upharpoonright + POW \equiv HA$.

(iii) The set-theoretical model of T_0 in III.6 can be modified to give a model of $S_0 + POW$ by essentially using $\{0,1\}^A$ as a representative of a power set of A for any set A. Now presence of J makes the axiom POW much more effective. What is the strength of $S_0 + POW$? Further, is $S_0 + POW + CT_i$ consistent $(i = 0,1)$?

(iv) What are the strengths of the various theories considered when CA_2 is added?

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ON PARTIALLY CONSERVATIVE EXTENSIONS OF ARITHMETIC

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Introduction.

Let PA be Peano arithmetic. Following Guaspari [2] we call a sentence φ Π^0_k-conservative (over PA) if each Π^0_k-sentence ψ provable in (PA + φ) (i.e. in PA with the additional axiom φ) is provable in PA. Similarly for Π^0_k being replaced by Σ^0_k or any reasonable class Γ of sentences. We write Γ-con for "Γ-conservative" and Γ-non for "Γ-nonconservative". Obviously, if φ is a sentence provable in PA then φ is Γ-con for each Γ; but if φ is unprovable and, say, Π^0_k-con then φ is not a Π^0_k-formula.

The existence of partially conservative formulas was first noted by Kreisel; the prototypical example is $\neg$Con(PA) which is an independent Σ^0_1-sentence which is Π^0_1-con, see [7] for a "modal proof" and see [6] for a "model theoretic" one. The two principal techniques used to construct further partially conservative formulas are Gödel diagonal technique and (usually "together with") partial truth definitions, see [4], [5] : for each $k \geq 1$, there is a Π^0_k-formula $tr_k(x)$ such that for each Π^0_k-sentence φ , we have $PA \vdash \varphi \leftrightarrow tr_k(\overline{\varphi})$ and similarly for Σ^0_k in place of Π^0_k. Kreisel's example can be generalized and the sentence $\exists x(tr_k(x) \wedge \neg Con(PA + x))$ is Π^0_k-con for $k \geq 1$. On the other hand, if φ says "beneath each proof of me there is a proof of a false Σ^0_k-sentence", then φ is a true Σ^0_k-formula which is unprovable and Π^0_k-con. Similarly, if φ says "beneath each proof of me there is a proof from me of a false Σ^0_k-sentence" then φ is a true, unprovable and Σ^0_k-conservative Π^0_k-sentence. We shall omit the proof of this fact, even if it is not difficult; various proofs of similar and stronger facts are contained in this paper.

The paper is organized as follows : In section 1, we investigate the set $I_k = \{ \varphi ; \varphi \text{ is } \Pi^0_k\text{-con} \}$ and show that it is Π^0_2-complete. This partially answers a problem of Guaspari; very little is known for Π_k^0 being replaced by Σ_k^0. In section 2 we strengthen Guaspari's results concerning Δ_k^0 sentences : we find a Δ_{k+1}^0 sentence which is Γ-con for Γ being the set of Boolean combinations of Σ_k^0 (and Π_k^0) sentences. In section 3 we classify independent Σ_2^0 sentences. Each such sentence in classified according to (1) its truth/falsity, (2) how much it is conservative and (3) how much its negation is conservative. We exhibit 21 natural non-empty disjoint classes of independent Σ_2^0 sentences.

Main results of this paper were communicated at the Logic Colloquium 78. Thanks are due to K. McAloon for an illuminating discussion that helped to complete the results.

Section 1. Π_k^0-conservative sentences.

It is easy to see that a sentence φ is Π_1^0-con iff it is relatively interpretable, i.e. if (PA + φ) has a relative interpretation in PA in the sense of [10] . Guaspari characterized Π_k^0-conservative sentences in terms of interpretability. Before we formulate his result let us fix some notation. Each formula is identified with its Gödel number as usual. (1) PA_n is $PA \upharpoonright n$, i.e. the set of all axioms of PA less than n. (2) $Tr(\Pi_k^0)$ is the set of all true Π_k^0 sentences; PA_n^k is $PA_n \cup Tr(\Pi_k^0)$. (3) $tr_k(x)$ is the Π_k^0 formula formally defining $Tr(\Pi_k^0)$ and mentioned in the introduction, i.e., for each Π_k^0 sentence φ we have $PA \vdash \varphi \equiv tr_k(\overline{\varphi})$. Note that this schema is a particular case of a more general schema concerning all Π_k^0 formulas, not only sentences; namely, for each Π_k^0-formula $\varphi(x_1, \ldots, x_n)$ we have

$$(\star) \qquad PA \vdash \varphi(x_1, \ldots, x_n) \equiv tr_k(\overline{\varphi}(\dot{x}_1, \ldots, \dot{x}_n))$$

(cf. [5] schema 1.5). Similarly for Σ_k^0 formulas. (The Σ_k^0 truth definition for Σ_k^0 sentences will be denoted by $tr_n'(x)$.)

We shall call the schema ($\star$) the "it's snowing"-it's snowing schema remembering Tarski's definition saying that the sentence "it's snowing" is true iff it's snowing.

(4) $\pi(x)$ is the natural binumeration of PA in PA; $\pi_y(x)$ is $\pi(x)$ & $x < y$; $\pi_y^k(x)$ is $\pi_y(x) \vee tr_k(x)$; $\pi^k(x)$ is $\pi(x) \vee tr_k(x)$ etc. $(\pi_n^k + \overline{\varphi})$ abbreviates the formula $\pi_n^k(x) \vee x = \overline{\varphi}$. We write $Con(\pi)$ instead of Con_π for the formal consistency statement; thus e.g. $Con(\pi_n^1 + \overline{\varphi})$ says : there is no proof of a contradiction from the theory whose axioms are (1) sentences x such that $\pi_n^1(x)$ and (2) the sentence $\overline{\varphi}$.

A relative interpretation I of PA in PA is provably Π_k^0-faithful if for each Π_k^0 sentence ψ we have $PA \vdash \psi^I \rightarrow \psi$. Note that each interpretation of PA in PA is provably Π_1^0 faithful.

Lemma 1. [2] The following are equivalent (for each $k \geq 1$) :
(i) φ is Π_k^0-con,
(ii) $(PA + \varphi)$ has a provably Π_k^0 faithful relative interpretation in PA,
(iii) For each n, $PA \vdash Con(\pi_n^{k-1} + \bar{\varphi})$.

Solovay proved [9] that the set

$$\{\varphi \,;(PA + \varphi) \text{ has a relative interpretation in PA}\}$$

is Π_2^0-complete. Put

$$I_k = \{\varphi ; \varphi \text{ is } \Pi_k^0\text{-con}\} \quad \text{and} \quad J_k = \{\varphi ; \varphi \text{ is } \Sigma_k^0\text{-con}\} .$$

Then Solovay's set is I_1. Each I_k and each J_k is easily seen to be a Π_2^0 set. Guaspari asked in [2] whether each I_k and each J_k is Π_2^0-complete. In this section we prove the following

Theorem 1. Each I_k $(k \geq 1)$ is Π_2^0-complete.

This gives a partial answer to Guaspari's question. Some remarks concerning J_k are placed at the end of this section. We utilize one of Solovay's proofs of Π_2^0-completeness of I_1 which will be probably distinct from the proof to be given in [9]; the author is indebted to Professor Solovay for his permission to use his proof. Let Tot be the index set of all total recursive functions. We show that a function Solovay uses to reduce Tot to I_1 in fact reduces Tot to I_k for all $k \geq 1$.

Definition (Solovay). Let $T(e,n,y)$ be the usual Kleene predicate. Set

$$T_{e,n} = \{\varphi ; \varphi \text{ an axiom of PA and } (\forall y \leq \varphi) \neg T(e,n,y)\},$$

let $\tau_{e,z}(x)$ be the corresponding formula such that $\tau_{e,n}(x)$ binumerates $T_{e,n}$, i.e.

$$\tau_{e,z}(x) \equiv \pi(x) \;\&\; (\forall y \leq x) \neg \tau(e,z,y)$$

(where $\tau(-,-,-)$ binumerates T) and set

$$\phi_e^k \equiv (\forall z)(Con(\pi_z^k) \rightarrow Con(\tau_{e,z})).$$

<u>Remark</u> (Solovay) $e \in Tot$ iff $(\forall n)(PA \vdash Con(\tau_{e,n}))$.

<u>Claim 1</u> $\Phi^k_e \in I_{k+1}$ iff $e \in Tot$. The following sums up a relativized version of the Second Incompleteness Theorem and will be used to prove Claim 1, generelizing Solovay's proof of the Π^o_2-completeness of I_1.

<u>Fact 1</u> There exists n_o such that for all $n \geq n_o$,

(1) $tr_k(x)$ binumerates $Tr(\Pi^o_k)$ in PA^k_n.

(2) $\pi^k_n(x)$ binumerates PA^k_n in PA^k_n.

(3) If $PA \vdash \nu \equiv \neg(\exists x)\, Prf_{\pi^k_n}(\bar{\nu},x)$, then $PA^k_n \nvdash \nu$.

(4) $PA^k_n \vdash Con(\pi^k_n) \rightarrow \nu$.

<u>Proof of Claim 1</u> : Assume $\Phi^k_e \in I_{k+1}$. For each n, $PA \vdash Con(\pi^k_n)$; so for given n there is m such that $PA_m \vdash Con(\pi^k_n)$ and so $PA_m + \Phi^k_e \vdash Con(\tau_{e,n})$. Since $\Phi^k_e \in I_{k+1}$ we have $PA \vdash Con(\pi^k_m + \overline{\Phi}^k_e)$ by Lemma 1 and thus $PA \vdash Con(\pi^k_m + \overline{Con(\tau_{e,n})})$. This implies $PA \vdash Con(\tau_{e,n})$, since $PA \vdash \neg Con(\tau_{e,n}) \rightarrow Pr_{\pi^k}(\overline{\neg Con(\tau_{e,n})})$.

For the other direction, by Fact (1) as^{m}formalized in PA, we have $PA \vdash Con(\pi^k_n + \neg Con(\pi^k_n))$. Assume $e \in Tot$. Then for each m, $PA \vdash Con(\tau_{e,m})$. Fix an $n \geq n_o$. We prove $PA \vdash Con(\pi^k_n + \overline{\Phi}^k_e)$. Put $S_n = (PA^k_n + \neg Con(\pi^k_n))$; we prove $S_n \vdash \Phi^k_e$. Indeed, for each $i < n$, $S_n \vdash Con(\tau_{e,i})$, since $Con(\tau_{e,i})$ is a true Π^o_1 sentence; this together with $S_n \vdash \neg Con(\pi^k_n)$, gives $S_n \vdash \Phi^k_e$. This proof of $S_n \vdash \Phi^k_e$ formalizes in PA, since we have $PA \vdash tr_1(\overline{Con(\tau_{e,i})})$ for each i and PA proves the consistency of S_n by Fact (1). Thus $PA \vdash Con(\pi^k_n + \overline{\Phi}^k_e)$. This concludes the proof of our claim, thus I_{k+1} is Π^o_2-complete. ⊣

We have proved that each I_k is Π^o_2-complete. Concerning the "dual" sets $J_k = \{\varphi;\ \varphi \text{ is } \Sigma^o_k\text{-con}\}$, the following proposition is everything we know :

<u>Proposition 1</u>. J_1 is neither a Σ^o_1 set nor a Π^o_1 set.

<u>Proof</u> : (1) Let T be the Kleene predicate and let $K = \{n; (\exists p) T(n,n,p)\}$ be the usual Σ^o_1-complete set. We reduce $\overline{K}$ (the complement of K) to J_1. Let τ binumerate T in PA. Indeed, $n \in \overline{K}$ iff $N \models (\forall z) \neg \tau(\bar{n},\bar{n},z)$ iff $\varphi_n \in J_1$ where φ_n is the formula $(\forall z) \neg\, \tau(\bar{n},\bar{n},z)$. This is because a true sentence is Σ^o_1-con; a false Π^o_1 sentence is Σ^o_1-non by [2]. Thus J_1 is not Σ^o_1.

(2) We reduce K to J_1. Indeed, $n \in K$ iff $PA \vdash (\exists z)\tau(\bar{n},\bar{n},z)$; so $n \in K$ iff $(\exists z)\tau(\bar{n},\bar{n},z) \in J_1$. ⊣

Section 2. Δ_n^o sentences.

A sentence φ is Δ_k^o (over PA) iff there is a Σ_k^o sentence φ_1 and a Π_k^o sentence φ_2 such that $PA \vdash (\varphi \equiv \varphi_1)\ \&\ (\varphi \equiv \varphi_2)$. φ is essentially Δ_k^o if φ is Δ_k^o but is neither a Σ_{k-1}^o formula (over PA) nor a Π_{k-1}^o formula (over PA).

Guaspari shows [2] that for each $k \geqslant 1$ there is a φ which is essentially Δ_{k+1}^o and is Δ_k^o-con (and both Σ_k^o-non and Π_k^o-non). We are going to prove

Theorem 2. For each $k \geqslant 1$, there is a Δ_{k+1}^o sentence γ such that γ is true, $B(\Sigma_k^o)$-con (where $B(\Sigma_k^o)$ denotes the set of all Boolean combinations of Σ_k^o sentences) and $\neg\gamma$ is Π_k^o-con (but Σ_k^o-non).

Note that this implies the (known) fact that there is a Δ_{k+1}^o sentence (over PA) not equivalent in PA to any element of $B(\Sigma_k^o)$.

Proof of Theorem 2. Inside PA, saying "proof" we mean a proof from π, i.e. "y is a proof of x" is $Prf_\pi(x,y)$ in the notation of [1]. Saying "y is a proof of x from z" we mean $Prf_{(\pi+z)}(x,y)$, i.e. "y is a proof of x from π with the additional axiom z". Roughly, our γ will be such that $\neg\gamma$ says : there is a proof y of a false Π_k^o sentence from $\neg\gamma$ such that for each proof $t \leqslant y$ of a $B(\Sigma_k^o)$ sentence from γ, the sentence proved by t is true. More formally, $PA \vdash \neg\gamma \equiv (\exists\, x,y)(Prf_{(\pi+\overline{\neg\gamma})}(x,y)\ \&\ x \text{ is } \Pi_k^o\ \&\ \neg tr_k(x)\ \&\ (\forall\, p,q,t \leqslant y)\ (p \text{ is } \Pi_k^o\ \&\ q \text{ is } \Sigma_k^o\ \&\ Prf_{(\pi+\overline{\gamma})}(p \vee q,t) \rightarrow$
$\rightarrow tr_k(p) \vee tr_k'(q)))$.

(In this formal version, we restrict ourselves to disjunctions of a Σ_k^o sentence and a Π_k^o sentence observing that each $B(\Sigma_k^o)$ sentence is equivalent to a conjunction of sentences of the former form). Obviously, γ exists by the self-reference lemma.

(1) γ is Δ_{k+1}^o : as written, $\neg\gamma$ is seen to be a Σ_{k+1}^o sentence. But $\neg\gamma$ is equivalent to the following sentence :

$(\exists\, y)$ (y is a proof of a false Π_k^o sentence from $\neg\gamma$) &
$(\forall\, y)$ (y is the least proof of a false Π_k^o sentence from $\neg\gamma \rightarrow (\forall p,q,t \leqslant y)$ (... as above ...)). (*)

This is a conjunction whose first conjunct is Σ_k^o (since "y is a proof of a false Π_k^o sentence from $\neg\gamma$" is Σ_k^o). The formula "y is the least proof of a false Π_k^o sentence from $\neg\gamma$" is a conjunction of a Σ_k^o formula and a Π_k^o formula, thus Δ_{k+1}^o; and the part

$(\forall p,q,t \leq y)(\ldots)$ is also Δ^0_{k+1}. Thus the second conjunct of the formula $(*)$ is Π^0_{k+1} and so is the whole sentence $(*)$. Thus γ is Δ^0_{k+1}.

(2) γ is true. Suppose that $\neg\gamma$ is true. Then it is true that there is a proof of a false Π^0_k sentence φ from $\neg\gamma$. But all axioms of PA are true and $\neg\gamma$ is true so that φ must also be true, a contradiction.

(3) $\neg\gamma$ is Π^0_k-con : Suppose $(PA + \neg\gamma) \vdash \varphi$, φ is Π^0_k. Let d be the proof of φ in $(PA + \neg\gamma)$. We show $(PA + \gamma) \vdash \varphi$. Let us proceed in $(PA + \gamma)$. If $tr_k(\overline{\varphi})$, we are done. Thus assume $\neg tr_k(\overline{\varphi})$. Then by γ, there is a $t \leq \overline{d}$ such that t is a proof of a false disjunction $p \vee q$ from γ, p being Π^0_k and q being Σ^0_k. Arguing metamathematically, let $\rho_1 \vee \sigma_1, \ldots, \rho_h \vee \sigma_h$ be all disjunctions of the required syntactic form having a proof in $(PA + \gamma)$ of the length $\leq$ d. Then, in $(PA + \gamma)$, we have : $\bigwedge_{i\leq h} (\rho_i \vee \sigma_i)$ but $\bigvee_{i\leq h}(\neg tr_k(\rho_i)\ \&\ \neg tr'_k(\sigma_i))$, i.e. $\bigvee_{i\leq h}(\neg\rho_i\ \&\ \neg\sigma_i)$, a contradiction. We have proved φ in $(PA + \gamma)$.

(4) Obviously, $\neg\gamma$ is Σ^0_k-non : the first conjunct of $(*)$ is a Σ^0_k consequence of $\neg\gamma$ which is false by (3), hence unprovable in PA.

(5) γ is $B(\Sigma^0_k)$-con. Let $(PA + \gamma) \vdash \rho \vee \sigma$, ρ being Π^0_k and σ being Σ^0_k, let d be the corresponding proof. We prove $\rho \vee \sigma$ in $(PA + \neg\gamma)$. The sentence $\neg\gamma$ says that there is a proof y of a false Π^0_k sentence from $\neg\gamma$ such that something holds for all $t \leq y$. First observe that this y must be bigger than d (in fact, y must be non-standard). Indeed, if $\overline{c}$ is a numeral and if $\overline{c}$ is a proof of $\overline{\varphi}$ from $\overline{\neg\gamma}$ then we have φ (since we are assuming $\neg\gamma$), thus $tr_k(\overline{\varphi})$. Hence there is a y <u>bigger than</u> d such that, for each proof $t \leq y$ of a disjunction $p \vee q$ from γ where p is Π^0_k and q is Σ^0_k we have : $tr_k(p) \vee tr'_k(q)$. In particular, take d for t : we obtain $tr_k(\rho) \vee tr_k(\sigma)$, thus $\rho \vee \sigma$. We have proved $\rho \vee \sigma$ in $(PA + \neg\gamma)$, thus $PA \vdash \rho \vee \sigma$ and γ is $B(\Sigma^0_k)$-con. This concludes the proof. $\dashv$

This proof is analogous to the proof of Theorem 2.7 in [2], due to Solovay. We presented a detailed proof since various further proofs will be quite analogous; then we shall proceed more quickly.

Observe that Theorem 2 cannot be improved by requiring that both γ and $\neg\gamma$ should be $B(\Sigma^0_k)$-con. We have the following evident

Proposition 2. A false Π^0_{n+1} sentence is Σ^0_n-non.

Proof : Let $(\forall x)\varphi(x)$ be a false Π^0_{n+1} sentence; thus $(\exists x)\neg\varphi(x)$ is true and therefore there is a natural number a such that $\neg\varphi(a)$ is true. Thus $\varphi(a)$ is a false Σ^0_n consequence of $(\forall x)\varphi(x)$; being false, $\varphi(a)$ is not provable in PA. $\dashv$

Section 3. Σ^0_2 sentences.

In this section, we shall investigate independent Σ^0_2 sentences (or independent Π^0_2 sentences, i.e. negations of the former ones). Obviously, "independent" means here "neither provable nor refutable". There are at least two reasons for this investigation : first, independent Σ^0_2 sentences are the simplest sentences φ such that knowing merely that φ is independent we do not know whether φ is true or false : there are true independent Σ^0_2 sentences and false independent Σ^0_2 sentences, whereas each independent Σ^0_1 sentence is false. Second, there are independent Π^0_2 sentences of mathematical (combinatorial) content, not constructed using self-reference and not referring to formal proofs. Such a sentence was exhibited by Paris and simplified by Harrington, see [8]. Notably, their sentence is equivalent in PA to a consistency statement, namely, in the notation of Section 1, to $\mathrm{Con}(\pi^1)$ (where $\pi^1(x)$ is $\pi(x) \vee tr_1(x)$); thus the negation is a false independent Σ^0_2 sentence. Solovay's Π^0_2-conservative Σ^0_2 sentence (whose negation is Σ^0_2-con) is also false. This suggests the following question: Can there be a true Σ^0_2 sentence which is independent and Π^0_2-con ? (Such a sentence is essentially Σ^0_2, i.e. not equivalent to a Π^0_2 sentence). More generally, each independent Σ^0_2 sentence can be classified according to (1) its truth or falsity, (2) for what Γ it is Γ-conservative, (3) for what Γ its negation is Γ-conservative. Here e.g. the Paris-Harrington Σ^0_2 sentence obtains another classification than Solovay's sentence. We have the following

Proposition 3. The Σ^0_2 sentence $\neg\mathrm{Con}(\pi^1)$ is false and Π^0_2-con; its negation is Σ^0_1-con but Π^0_1-non.

Proof. Concerning the negation, i.e. the sentence $\mathrm{Con}(\pi^1)$, it is true and consequently Σ^0_1-con. But $\mathrm{Con}(\pi^1)$ is not interpretable in PA (since $\mathrm{Con}(\pi)$ is not, see [1]), and hence $\mathrm{Con}(\pi^1)$ is Π^0_1-non.

To prove that $\neg\mathrm{Con}(\pi^1)$ is Π^0_2-con it suffices to note that each countable model of PA has a Π^0_1-elementary end-extension to a model of $(PA + \neg\mathrm{Con}(\pi^1))$, see [2], Notes to 6.5 and 6.6 and/or [5], Theorem 1.7. ⊣

Theorem 3. There is a true Σ^0_2 sentence which is Π^0_2-con and whose negation is Π^0_1-con.

Proof. Our formula is a modification of the formula from the proof of Theorem 2. Let γ be such that, in PA,

$\neg\gamma \equiv (\exists y)$ (y is a proof of a false Π^0_1 sentence from $\neg\gamma$)
& $(\forall y)$ (y is the least proof of a false Π^0_1 sentence from $\neg\gamma$
$\rightarrow (\forall t \leq y)$($t$ is not a proof of a false Π^0_2 sentence from γ)).

This can be written concisely as follows : Let $\mathrm{Fls}[\Pi^0_1, \neg\gamma]$ stand for $(\exists y)$ (y is a proof of a false Π^0_1 sentence from $\neg\gamma$) and similarly for $\mathrm{Fls}[\Pi^0_2, \gamma]$. Then

$$\neg\gamma \equiv \mathrm{Fls}[\Pi^0_1, \neg\gamma] \prec \mathrm{Fls}[\Pi^0_2, \gamma]$$

in the notation of [2].

Now, $\neg\gamma$ is Π^0_2, i.e. γ is Σ^0_2. Similarly as in the proof of Theorem 2 we prove that γ is true, $\neg\gamma$ is Π^0_1-con but Σ^0_1-non;the proof of Π^0_2-conservativity of γ is fully analogous to part (5) of the proof of Theorem 2 but is simpler. ⊣

We can now turn to the classification of Σ^0_2 sentences mentioned above. To answer the question for which Γ a given Σ^0_2 or Π^0_2 sentence is conservative and for which it is not, we must choose some candidates. Let us choose simply the first levels of the arithmetical hierarchy, i.e. $\Gamma = \Sigma^0_1, \Pi^0_1, \Sigma^0_2, \Pi^0_2$. This gives five possibilities for an independent Σ^0_2 sentence :

(-) it is Π^0_1-non and Σ^0_1-non,
(Σ1) it is Σ^0_1-con but Π^0_1-non,
(Π1) it is Σ^0_1-non but Π^0_1-con,
(B) it is both Σ^0_1-con and Π^0_1-con but Σ^0_2-non and Π^0_2-non,
(Π2) it is Π^0_2-con.

If the sentence in question is true then it is Σ^0_1-con, thus (-) and (Π1) are impossible. Similarly, for an independent Π^0_2 sentence we have five possibilities, namely (-), (Σ1), (Π1), (B) and (Σ2); if such a sentence is true then (-) and (Π1) are impossible and if it is false then, by Proposition 2, (Σ1), (B) and (Σ2) are impossible.

Thus, with each independent Σ^0_2 sentence φ we can associate its characteristic $c(\varphi) = < c_1(\varphi), c_2(\varphi), c_3(\varphi) >$ where $c_1(\varphi) \in \{\text{true}, \text{false}\} = X_1$, $c_2(\varphi) \in \{(-),(\Sigma 1),(\Pi 1),(B),(\Pi 2)\} = X_2$ and $c_3(\varphi) \in \{(-),(\Sigma 1),(\Pi 1),(B),(\Sigma 2)\} = X_3$. For example, the characteristic of $\neg\mathrm{Con}(\pi^1)$ is $< \text{false},(\Pi 2),(\Sigma 1) >$ A <u>characteristic</u> is simply any triple from $X_1 \times X_2 \times X_3$. A characteristic is <u>admissible</u> if it is not impossible by the remarks above (e.g. $<\text{true},(-),(-)>$ is inadmissible). There are 21 admissible characteristics visualized by the following two tables

φ TRUE

$\neg\varphi$ \ φ	$\Sigma 1$	B	$\Pi 2$
-	×	×	×
$\Pi 1$	×	×	×

φ FALSE

$\neg\varphi$ \ φ	-	$\Sigma 1$	$\Pi 1$	B	$\Pi 2$
$\Sigma 1$	×	×	×	×	×
B	×	×	×	×	×
$\Sigma 2$	×	×	×	×	×

<u>Theorem 4</u>. There is an independent Σ^0_2 sentence of each admissible characteristic.

The proof is windy and we shall omit it.

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WEAKLY SEPARATED SUBSPACES AND NETWORKS

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Abstract

In this paper we give (consistent) solutions to two problems of M.G. Tkačenko [6], namely we produce under CH or by adding lots of Cohen reals a regular space X with $R(X)=\omega < nw(X) = 2^{\omega}$, and using $MA(\omega_1)+\diamondsuit_{\omega_2}(E)$, where $E=\{\alpha\in\omega_2 : cf(\alpha)=\omega\}$, a Hausdorff space X such that $nw(X) > \omega$ but $nw(Y)=\omega$ whenever $Y\subset X$, $|Y|=\omega_1$.

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Introduction

In [6], Problem 3, M.G. Tkačenko raised the problem whether there exists a T_3 space X with $R(X) < nw(X)$. In this paper we are going to show that a) under CH there is a 0-dimensional T_2 space X with $R(X)=\omega$ and $nw(X)=\omega_1$, and b) in a model of set theory obtained by adding κ Cohen-reals there is a 0-dimensional T_2 space X with $R(X)=\omega$ and $nw(X)=\kappa$, consequently the obvious inequality $nw(X) \leq 2^{R(X)}$ for T_3 spaces cannot be improved. We shall show that for Hausdorff spaces the consistency of the existence of a much stronger example is also provable, namely of a Hausdorff space X such that $nw(X) > \omega$ but $nw(Y)=\omega$ whenever $Y\subset X$ and $|Y|=\omega_1$. This answers Problem 2 of [6].

We recall that (cf. [6]) a space Y is called weakly separated if there is a map $y \to Uy$ associating a neighbourhood Uy with every point $y\in Y$ in such a way that if $y_1\neq y_2$ then either $y_1\notin Uy_2$ or $y_2\notin Uy_1$. Moreover

$$R(X) = \sup\{|Y| : Y\subset X \text{ is weakly separated}\}.$$

Since a right or left separated space is clearly weakly separated $R(X)=\kappa$ implies that X is both hereditary κ-Lindelöf and hereditarily κ-separable. Finally $nw(X)$ is the smallest cardinal of a network for X.

Our set theoretical notation and terminology are standard, as e.g. in [4].

The "graph" topology

Let X be a set and $f : [X]^2 \to 2$ be a map (i.e. a graph on X, where $\{x,y\}\in[X]^2$ is considered an edge iff $f(\{x,y\})=0$). We shall put

$$U_x^0 = \{y\in X : f(\{x,y\})=0\}\cup\{x\}$$

and

$$U_x^1 = \{y\in X : f(\{x,y\})=1\}.$$

(In accordance with this we shall write $f(\{x\})=0$ for each $x\in X$.) We are going to study the topology τ_f on X generated by the family

$$\{U_x^i : x\in X,\ i\in 2\}.$$

Then every U_x^i is τ_f-clopen, hence τ_f is 0-dimensional.

If $\varepsilon\in H(X)$ (=all finite functions from X to 2), then we put

$$U_\varepsilon = \cap\{U_x^{\varepsilon(x)} : x\in D(\varepsilon)\}.$$

Clearly, $\{U_\varepsilon : \varepsilon\in H(X)\}$ is a basis for τ_f.

Theorem 1.

Assume CH. Then there is a "graph" $f : [\omega_1]^2 \to 2$ such that $X=\langle\omega_1,\tau_f\rangle$ is T_2, $R(X)=\omega$ and $nw(X)=\omega_1$.

Proof.

Let us fix some notation first. L_1 denotes the set of all limit ordinals in ω_1. Using CH, we can put

$$[\omega_1]^2 = \{\{\alpha_\lambda,\ \beta_\lambda\} : \lambda\in L_1\}$$

and

$$E = \{E_\lambda : \lambda\in L_1\},$$

where E denotes the set of all maps of $\omega\times\omega$ into $H(\omega_1)$. We can assume that

$\{\alpha_\lambda, \beta_\lambda\}\subset\lambda$ and $E_\lambda(n,m)\in H(\lambda)$ holds for each $\lambda\in L_1$ and $\langle n,m\rangle\in\omega\times\omega$.

We let X_k, denote the set of all disjoint members of $[[\omega_1]^k]^\omega$ and $X=\{X_k : k\in\omega\setminus\{0\}\}$. Using CH again we can write

$$X = \{Z_\alpha : \alpha\in\omega_1\}.$$

Next we put

$$X_k^{(\alpha)} = \{Z_\beta\in X_k : \beta < \alpha \ \ \& \ \cup Z_\beta\subset\alpha\}.$$

Before we can start the transfinite construction of $f : [\omega]^2 \to 2$ we need a technical lemma.

Lemma

Let α be a countably infinite ordinal, $h\in H(\alpha)$, $Z=\cup\{Z_k : k\in\omega\setminus\{0\}\}$, where each Z_k is a countable family of disjoint countably infinite subcollections of $[\alpha]^k$, and let S be a countable subfamily of $[H(\alpha)]^{\leq\omega}$ such that for every $S\in S$ and $g\in H(\alpha)$ some member of S is compatible with g. Then there is a map $F : \alpha \to 2$ such that

1) $h\subset F$;

2) every $S\in S$ has a member compatible with F;

3) if $k\in\omega\setminus\{0\}$, $Z\in Z_k$ and $\varepsilon\in 2^k$ then $|Y(Z, \varepsilon)|=\omega$, where $Y(Z, \varepsilon)$ denotes the set of $a\in Z$ such that $F(\eta_i)=\varepsilon(i)$ holds for each $i < k$, with η_i denoting the i^{th} member of a (we shall abbreviate this by writing $F\restriction a\simeq\varepsilon$).

Proof of the lemma.

We put

$$\cup\{Z_k\times 2^k : k\in\omega\setminus\{0\}\} = \{\langle Z_\ell, \varepsilon_\ell\rangle : \ell\in\omega\},$$

where every member of the left-hand side appears for infinitely many $\ell\in\omega$, moreover let $S=\{S_\ell : \ell\in\omega\}$. Then we define a sequence $\langle h_n : n\in\omega\rangle$ of members of $H(\alpha)$ by induction as follows. We put $h_0=h$. Now if h_n has been defined and $n=2\ell$, then consider $\langle Z_\ell, \varepsilon_\ell\rangle$; since h_n is finite and Z_ℓ is disjoint and infinite, there is $a\in Z_\ell$, $a=\{\eta_i : i < k\}$, with $a\cap D(h_n)=\emptyset$. Thus we can put

$$h_{n+1} = h_n\cup\{\langle \eta_i, \varepsilon_\ell(i)\rangle : i < k\}.$$

If, on the other hand, $n=2\ell+1$, we consider S_ℓ. By our assumption there is $\sigma\in S_\ell$ compatible with h_n. Consequently we can define $h_{n+1}\in H(\alpha)$ by

$$h_{n+1} = h_n\cup\sigma.$$

It is obvious then that if F is any extension of $\cup\{h_n : n\in\omega\}$ to α, then F is as required.

Now we return to the construction of the graph f. Suppose that $\lambda\in L_1$ and $f\upharpoonright[\lambda]^2$ has already been defined. We then first put $f(\{\alpha_\lambda, \lambda\})=0$ and $f(\{\beta_\lambda, \lambda\})=1$. Accordingly we define $h_\lambda\in H(\lambda)$ with $D(h_\lambda)=\{\alpha_\lambda, \beta_\lambda\}$, $h_\lambda(\alpha_\lambda)=0$ and $h_\lambda(\beta_\lambda)=1$. Next we put for each $n\in\omega$

$$S_n = \{E_\lambda(n,m) : m\in\omega\},$$

and

$$\mathcal{S}_\lambda = \{S_n : \forall g\in H(\lambda)\ \exists\sigma\in S_n \text{ s.t. } \sigma \text{ is compatible with } g\}.$$

Finally we put for $k\in\omega\setminus\{0\}$

$$Z_k^{(\lambda)} = \{Y(Z;\ \sigma_0,\dots,\sigma_{r-1};\ \eta_0,\dots,\eta_{r-1}) = Y : |Y| = \omega \ \&$$

$$\&\ \ Z\in X_k^{(\lambda)}\ \&\ \sigma_0,\dots,\sigma_{r-1}\in 2^k\ \&\ \eta_0,\dots,\eta_{r-1}\in\lambda\},$$

where $Y(Z;\ \vec{\sigma};\ \vec{\eta}) = \{a\in Z : \forall i < r\ \forall j < k\ f(\{\xi_j, \eta_i\}) = \sigma_i(j)\}$,

with ξ_j being the j^{th} member of a. Note that if $r=0$ then $Y(Z;-;-)=Z$. Then we can apply our lemma for $\alpha=\lambda$ with $h=h_\lambda$, $Z_k=Z_k^{(\lambda)}$, $S=\mathcal{S}_\lambda$ and obtain $F : \lambda\to 2$ described there. With this we put

$$f(\{\alpha,\lambda\}) = F(\alpha)$$

for each $\alpha\in\lambda$. We remark that this will imply $\lambda\in\cup\{U_\sigma : \sigma\in S_n\}$ whenever $S_n\in\mathcal{S}_\lambda$.

Next we consider those $n\in\omega$ for which $S_n\notin\mathcal{S}_\lambda$. For each such n there is $g_n\in H(\lambda)$ incompatible with every member of S_n.

Now if $m=n+1 > 0$, we apply our lemma again for the ordinal $\alpha=\lambda+m$ with $h= =g_n\cup\{\langle\lambda, 1\rangle\}$, if $S_n\notin\mathcal{S}_\lambda$ ($h=\emptyset$ otherwise), $S=\emptyset$, and

$$Z_k = Z_k^{(\alpha)} = \{Y(Z;\ \vec{\sigma};\ \vec{\eta}) = Y : |Y| = \omega\ \&\ Z\in X_k^{(\alpha)}\ \&\ \vec{\sigma}\in(2^k)^r\ \&\ \vec{\eta}\in\alpha^r\}.$$

With the thus obtained function $F : \lambda+m\to 2$ we put again $f(\{\beta,\lambda+m\})=F(\beta)$ for each $\beta\in\lambda+m$. Let us note that this will yield us $\lambda+m\in U_\lambda^1$ and $\lambda+m\notin\cup\{U_\sigma : \sigma\in S_n\}$ whenever $S_n\notin\mathcal{S}_\lambda$. This completes the construction of $f : [\omega_1]^2\to 2$.

Now we have to check that the topology τ_f on ω_1 is as required. First of all τ_f is T_2, since if $\{\alpha,\beta\}=\{\alpha_\lambda,\beta_\lambda\}\in[\omega_1]^2$, then we have $\alpha\in U_\lambda^0$ and $\beta\in U_\lambda^1$ and clearly $U_\lambda^0\cap U_\lambda^1=\emptyset$.

Next we show that $R(X)=\omega$. Suppose, on the contrary, that $\{\eta_\alpha : \alpha\in\omega_1\}$ – with $\eta_\alpha < \eta_\beta$ if $\alpha < \beta$ – is a weakly separated subspace with $U\varepsilon_\alpha$ a basic τ_f-

neighbourhood of η_α for each $\alpha\in\omega_1$ showing this. We can assume that $\eta_\alpha\in D(\varepsilon_\alpha)$ for each $\alpha\in\omega_2$, moreover that the $D(\varepsilon_\alpha)$ are pairwise disjoint, in fact $D(\varepsilon_\alpha) < D(\varepsilon_\beta)$ for $\alpha < \beta$, they have the same number of elements, say k, and finally that $\varepsilon_\alpha \approx \varepsilon$ for a fixed $\varepsilon\in 2^k$ and for all $\alpha\in\omega_1$. Thus $\varepsilon_\alpha(\eta_i^{(\alpha)})=\varepsilon(i)$ is the same for each $\alpha\in\omega_1$, where $\eta_i^{(\alpha)}$ denotes the i^{th} member of $D(\varepsilon_\alpha)$. It can also be assumed that $\eta_\alpha=\eta_j^{(\alpha)}$ for a fixed $j < k$ for each $\alpha\in\omega_1$.

With all these assumptions $Z=\{D(\varepsilon_n) : n\in\omega\}$ belongs to X_k, hence $Z=Z_\beta$ for some $\beta\in\omega_1$. Therefore we have $\alpha\in\omega_1\setminus\omega$ such that $Z\in Z_k^{(\eta_o^{(\alpha)})}$. Now consider for each $i < k$ the function $\sigma_i\in 2^k$ defined as follows:

$$\begin{cases} \sigma_i(n) = \varepsilon(i) & \text{for } n < k, \text{ if } i\neq j \\ \sigma_j(n) = \varepsilon(n) & \text{for } n < k. \end{cases}$$

Since $Z\in Z_k^{(\eta_o^{(\alpha)})}$, we have by our construction of f infinitely many $n\in\omega$ such that $f(\{\eta_m^{(n)}, \eta_o^{(\alpha)}\})=\sigma_o(m)$ for each $m < k$, consequently we have $Y(Z;\ \sigma_o;\ \eta_o^{(\alpha)})\in Z_k^{(\eta_1^{(\alpha)})}$ so infinitely many of these n also satisfy

$$f(\{\eta_m^{(n)}, \eta_1^{(\alpha)}\}) = \sigma_1(m) \quad \text{for every } m < k,$$

etc. Finally we get (infinitely many) $n\in\omega$ such that $f(\{\eta_m^{(n)}, \eta_\ell^{(\alpha)}\})=\sigma_\ell(m)$ for every $m,\ \ell < k$. But this implies then that $\eta_n=\eta_j^{(n)}\in U_{\varepsilon_\alpha}$ and $\eta_\alpha=\eta_j^{(\alpha)}\in U_{\varepsilon_n}$, contradicting that these neighbourhoods yield a weak separation of the set $\{\eta_\xi : \xi\in\omega_1\}$.

Since $R(X)=\omega$, we get that the topology τ_f is T_2, 0-dimensional and hereditarily Lindelöf. Now if we want to show that $nw(X) > \omega$, by the regularity of X it suffices to show that no countable family of closed sets forms a network in X. Moreover every open set in X is the union of countably many basic open sets of the form U_ε.

Consider a family $\{F_n : n\in\omega\}$ of closed sets in X, where for each $n\in\omega$ we have $X\setminus F_n=\cup\{U_{\varepsilon_m^n} : m\in\omega\}$, $\varepsilon_m^n\in H(\omega_1)$. Let us define $E\in E$ by $E(n,m)=\varepsilon_m^n$ for $n,m\in\omega$. Then $E=E_\lambda$ for some $\lambda\in L_1$. Consider the neighbourhood U_λ^o of this point λ. We claim that for no $n\in\omega$ do we have $\lambda\in F_n\subset U_\lambda^o$, hence $\{F_n : n\in\omega\}$ is indeed not a network for X.

Suppose that $\lambda\in F_n$, i.e. $\lambda\in F_n$, i.e. $\lambda\notin\cup\{U_{\varepsilon_m^n} : m\in\omega\}$. As was remarked in our construction this implies $S_n\in\{\varepsilon_m^n : m\in\omega\}\notin S_\lambda$ with the notation used there, and then by our construction again we have $(\lambda+n+1)\in U_\lambda^1$ and $(\lambda+n+1)\notin\cup\{U_{\varepsilon_m^n} : m\in\omega\}$, consequently $F_n \not\subset U^o_\lambda$. This completes the proof.

Now we turn to our next result.

Theorem 2.

Suppose that M is a countable standard model of ZFC, $\kappa\in M$ is a cardinal in M and $f:[\kappa]^2\to 2$ is Cohen-generic over M, i.e. $f=\cup G$, where $G\subset H([\kappa]^2)$ is Cohen-generic over M. Then for the space $X=\langle\kappa,\tau_f\rangle$ in $N=M[f]$ we have $R(X)=\omega$ and $nw(X)=\kappa$.

Proof.

Let us first show $R(X)=\omega$. Assume, on the contrary, that there is, in $M[f]$, an $Y\in[X]^{\omega_1}$ and a map $\varphi: Y\to H(\kappa)$ such that if $x,y\in Y$ and $x\neq y$ then $x\in U_{\varphi(x)}$ $y\in U_{\varphi(y)}$, but either $x\notin U_{\varphi(y)}$, or $y\notin U_{\varphi(x)}$. Let $p\in H([\kappa]^2)$ be a condition which forces all this about $\mathring{Y}$ and $\mathring{\varphi}$, a name for Y and φ, respectively.

Now, by an easy transfinite induction, we can define in M three sequences $\langle\alpha_\xi:\xi\in\omega_1\rangle$, $\langle\varepsilon_\xi:\xi\in\omega_1\rangle$ and $\langle p_\xi:\xi\in\omega_1\rangle$ such that $\alpha_\xi\in\kappa$, $\alpha_\xi\neq\alpha_\eta$ if $\xi\neq\eta$, $\varepsilon_\xi\in H(\kappa)$ and $p_\xi\in H([\kappa]^2)$ with $p_\xi\supset p$ such that for each $\xi\in\omega_1$

$$p_\xi \Vdash \check{\alpha}_\xi\in\mathring{Y}\ \&\ \mathring{\varphi}(\check{\alpha}_\xi)=\check{\varepsilon}_\xi .$$

By suitably extending the p_ξ's if necessary, we can assume that for each $\xi\in\omega_1$ we have $D(p_\xi)=[a_\xi]^2$ for some $a_\xi\in[\kappa]^{<\omega}$, moreover that $\alpha_\xi\in a_\xi$ and $D(\varepsilon_\xi)\subset a_\xi$. Then by a suitable "thinning out" of our sequences we can achieve that the a_ξ's form a Δ-system, i.e. each $a_\xi=a\cup b_\xi$, where the family $\{b_\xi:\xi\in\omega_1\}$ is disjoint. We can also assume that $\varepsilon_\xi\restriction a=\varepsilon_\eta\restriction a$ and $p_\xi\restriction[a]^2=p_\eta\restriction[a]^2$ whenever $\xi,\eta\in\omega_1$.

Note that since the α_ξ's are distinct we must have $\alpha_\xi\in b_\xi$ for all but finitely many $\xi\in\omega_1$. Thus let $\xi\neq\eta$ be such that $\alpha_\xi\in b_\xi$ and $\alpha_\eta\in b_\eta$. Now we define an extension q of $p_\xi\cup p_\eta$ as follows: for every $\nu\in b_\eta\cap D(\varepsilon_\eta)$ we put $q(\{\alpha_\xi,\nu\})=$ $=\varepsilon_\eta(\nu)$ and for $\mu\in b_\xi\cap D(\varepsilon_\xi)$ we put $q(\{\mu,\alpha_\eta\})=\varepsilon_\xi(\mu)$. Note that since $\varepsilon_\xi\restriction a=$ $=\varepsilon_\eta\restriction a$ these equalities will actually hold for all $\nu\in D(\varepsilon_\xi)$ and $\mu\in D(\varepsilon_\eta)$, respectively. That this is a valid definition follows from the fact that $p_\xi\Vdash\check{\alpha}_\xi\in U_{\check{\varepsilon}_\xi}$, hence if $\alpha_\xi\in D(\varepsilon_\xi)$ we must have $\varepsilon_\xi(\alpha_\xi)=0$, and similarly if $\alpha_\eta\in D(\varepsilon_\eta)$ then $\varepsilon_\eta(\alpha_\eta)=0$ as well. It is clear from all our assumptions that $q\Vdash\check{\alpha}_\xi\in U_{\check{\varepsilon}_\eta}\ \&\ \check{\alpha}_\eta\in U_{\check{\varepsilon}_\xi}$, hence obviously q forces that $\mathring{\varphi}$ is not a weak separation of $\mathring{Y}$, which is a contradiction, as p forces its negation.

Using that $R(X)=\omega$ and X is regular we show $nw(X)=\kappa$ hy proving that whenever $\mathcal{F}=\{F_\xi:\xi\in\lambda\}$ with $\lambda<\kappa$, is a family of closed subsets of X then $\mathcal{F}$ is not a network in X. Every F_ξ can be written as

$$F_\xi = X \setminus \cup\{U_{\varepsilon^\xi_m} : m\in\omega\}$$

with $\varepsilon^\xi_m \in H(\kappa)$. Consider the map $E : \lambda\times\omega \to H(\kappa)$ defined by $E(\xi, m)=\varepsilon^\xi_m$. It is well-known (cf e.g. [5]) that there is an $A\in[\kappa]^{\leq\lambda}$ such that we have already $E\in M[f\restriction[A]^2]=N'$. Since $H([\kappa]^2)=H([A]^2)\times H([\kappa]^2\setminus[A]^2)$, we can use the product theorem to force with $H([\kappa]^2\setminus[A]^2)$ over N'.

First we will show the following statement: whenever $\langle\varepsilon_m : m\in\omega\rangle\in N'$ is a sequence from $H(\kappa)$ and $F=\kappa\setminus\cup\{U_{\varepsilon_m} : m\in\omega\}$, then for every $\alpha\in\kappa\setminus A$ the set of conditions forcing "$\check\alpha\notin\check F \vee \check F\not\subset U^0_{\check\alpha}$" is dense in $H([\kappa]^2\setminus[A]^2)$.

Let $p\in H([\kappa]^2\setminus[A]^2)$. If p has an extension that forces $\check\alpha\notin\check F$ then we are done. If not then we have $p\Vdash\check\alpha\in\check F$, i.e. $p\Vdash\check\alpha\notin U_{\check\varepsilon_m}$ for all $m\in\omega$. But clearly this is only possible if for each $m\in\omega$ there is a $\nu_m\in D(\varepsilon_m)$ with $p(\{\alpha,\nu_m\})=1-\varepsilon_m(\nu_m)$. Let us note that since p is finite, there are only finitely many distinct ν_m. Now take any $\eta\in\kappa\setminus A$ which is not "mentioned" in p, i.e. $\eta\notin\cup D(p)$ (in particular, $\eta\neq\alpha$), and put $q(\{\eta,\nu_m\})=1-\varepsilon_m(\nu_m)$ for each $m\in\omega$, furthermore $q(\{\alpha,\eta\})=1$. Obviously, then $p\cup q\in H([\kappa]^2\setminus[A]^2)$ and $p\cup q\Vdash\check\eta\in\check F\cap U^1_{\check\alpha}$, consequently $p\cup q\Vdash\check F\not\subset U^0_{\check\alpha}$, and our statement is proven.

As a consequence of this we see that, in N, if $\alpha\in\kappa\setminus A$ then for each $\xi\in\lambda$ we have either $\alpha\notin F_\xi$ or $F_\xi\not\subset U^0_\alpha$, hence as U^0_α is a τ_f-neighbourhood of α the family $\mathcal{F}$ is not a network for X. This completes the proof of theorem 2.

Now we turn to a stronger form of Tkačenko's problem (cf [6], Problem 2). To show how it connects to the previous one let us recall that the property $R(X)=\omega$ of a space X is decided by its subspaces of size ω_1. Thus, in particular we obviously have $R(X)=\omega$ provided that $nw(Y)=\omega$ holds for each $Y\subset X$ with $|Y|=\omega_1$. Consequently a natural extension of Tkačenko's previous question is this: Does there exist a space X such that $nw(Y)=\omega$ for each $Y\in[X]^{\omega_1}$ but $nw(X)>\omega$? In what follows we show that it is at least consistent with the usual axioms of set theory that the answer to this question be affirmative within the class of Hausdorff spaces. It is interesting to note that if we replace the net-weight function nw with the weight w, then the situation changes completely: as we have recently shown in [3], for an *arbitrary* topological space X if $w(Y)=\omega$ whenever $Y\in[X]^{\leq\omega_1}$ then $w(X)=\omega$ as well (for regular X this was proved in [6]).

Before we formulate our result, however, we shall give a kind of "graph-theoretic" characterization of the property of having a countable network. For this we recall that a graph $G=\langle V, E\rangle$, where $E\subset[V]^2$, is said to be ω-chromatic iff there is a map $f : V\to\omega$ such that for each $n\in\omega$ the set $f^{-1}(\{n\})$ is independent, i.e.

$$[f^{-1}(\{n\})]^2 \cap E = \emptyset.$$

For an arbitrary topological space X with the open base $\mathcal{B}$ we define a graph $G(X, \mathcal{B})=G$ as follows: $G=\langle V,E\rangle$, where

$$V = \{\langle B,x\rangle : x\in B\in\mathcal{B}\}$$

and

$$E = \{\{\langle B,x\rangle, \langle C,y\rangle\}\in[V]^2 : x\notin C \text{ or } y\notin B\}.$$

Lemma.

For an arbitrary space X with any fixed open base $\mathcal{B}$ we have $nw(X)=\omega$ if and only if the graph $G(X,\mathcal{B})$ is ω-chromatic.

Proof.

Suppose first that $N=\{A_n : n\in\omega\}$ is a network for X. We can define the map $f : V \to \omega$ in such a way that if $\langle B,x\rangle\in V$ and $f(\langle B,x\rangle)=n$, then $x\in A_n\subset B$ be valid. But this f clearly shows that $G(X,\mathcal{B})$ is ω-chromatic.

Now we assume that $G(X,\mathcal{B})$ is ω-chromatic and $f : V \to \omega$ establishes this. Let us put for each $n\in\omega$

$$A_n = \cap\{B : \exists x \ [f(\langle B,x\rangle) = n]\}.$$

We claim that $\{A_n : n\in\omega\}$ is a network for X. Indeed, let $x\in B\in\mathcal{B}$. Then $\langle B,x\rangle\in V$, hence $f(\langle B,x\rangle)=n$ is defined. Since $A_n\subset B$ is trivial, it remains to show that $x\in A_n$, i.e. $f(\langle C,y\rangle)=n$ implies $x\in C$. But this is immediate from the fact that $\{\langle B,x\rangle, \langle C,y\rangle\}\notin E$.

Now let us turn to our promised result. As was mentioned above this is a consistency result, proved under the rather unusual combination of two well-known set-theoretic hypotheses, namely $MA(\omega_1)$ cf e.g. [5] and $\diamondsuit_{\omega_2}(E)$, where $E=\{\lambda\in\omega_2 : cf(\lambda)=\omega\}$ (see e.g. [1]). We do not go into proving the joint consistency of these two statements, but mention only that in the "usual" models of $MA(\omega_1)$ we also have $\diamondsuit_{\omega_2}(E)$. It is interesting though that W. Weiss has recently considered this same combination for a completely different purpose.

Theorem 3.

Suppose $MA(\omega_1)$ and $\diamondsuit_{\omega_2}(E)$ hold. Then there is a Hausdorff space X such that $nw(X) > \omega$ but $nw(Y)=\omega$ whenever $Y\subset X$, $|Y|=\omega_1$.

Proof.

Let us consider on ω_2 an arbitrary Hausdorff topology τ that has a countable base $\mathcal{B}$. We shall define a topology $\tau'\supset\tau$ on ω_2 and put $X=\langle\omega_2,\tau'\rangle$. We first define for each $\nu\in E$ an ω-type subset $S_\nu\subset\nu$ such that $\cup S_\nu=\nu$. To do this we use $\diamondsuit_{\omega_2}(E)$ by picking for each $\nu\in E$ a map $f_\nu:\nu\to\omega$ so that whenever $f:\omega_2\to\omega$ the set

$$E_f=\{\nu\in E : f\restriction\nu=f_\nu\}$$

is stationary in ω_2. Then for each $\nu\in E$ we pick the ω-type cofinal set $S_\nu\subset\nu$ in such a way that $S_\nu\cap f_\nu^{-1}(\{n\})\neq\emptyset$ be valid whenever the set $f_\nu^{-1}(\{n\})$ is cofinal in ν. This is clearly possible. Now, as a basis for τ' we choose all sets of the form $G\setminus\cup\{S_\nu : \nu\in a\}$, where $G\in\tau$ and $a\in[E]^{<\omega}$.

To show $nw(X)>\omega$ we will prove that $G(X,\tau')$ is not ω-chromatic. To see this let us first put $S_\nu=\emptyset$ for each $\nu\in\omega_2\setminus E$ and consider the subgraph H of $G(X,\tau')$ spanned by the vertices $\{\langle X\setminus S_\nu,\nu\rangle : \nu\in\omega_2\}$. We will actually show that already H fails to be ω-chromatic.

Indeed, ("identifying" $\langle X\setminus S_\nu,\nu\rangle$ with ν), let $f:\omega_2\to\omega$ be arbitrary. Let us put $f^{-1}(\{n\})=F_n$, moreover $I=\{n\in\omega : |F_n|=\omega_2\}$. There is an $\alpha\in\omega_2$ such that $\cup\{F_n : n\in\omega\setminus I\}\subset\alpha$. Now the set

$$C=\cap\{F'_n : n\in I\}\setminus\alpha$$

(where F'_n denotes the set of all limit points of F_n) is closed and unbounded in ω_2, hence there is a $\nu\in C\cap E_f$. Since $\nu\geq\alpha$, we have $f(\nu)=n\in I$, consequently $\nu\in F'_n$. But $f\restriction\nu=f_\nu$ implies then that $f_\nu^{-1}(\{n\})$ is cofinal in ν, therefore we must have a $\mu\in S_\nu\cap F_n$. In other words, although the vertices $\langle X\setminus S_\nu,\nu\rangle$ and $\langle X\setminus S_\mu,\mu\rangle$ have the same "color" under f, they are connected by an edge in H, since $\mu\notin X\setminus S_\nu$, hence f does not yield an ω-coloring of H.

Now, to show that the small subspaces of X do have a countable network, consider any $\alpha\in\omega_2$ and denote by τ'_α the subspace topology on α induced by τ'. Clearly, the family

$$\mathcal{B}_\alpha=\{(B\setminus\cup\{S_\nu : \nu\in a\})\cap\alpha=U(B,a) : B\in\mathcal{B}\ \&\ a\in[\alpha+1]^{<\omega}\}$$

is a basis for τ'_α, as for $\nu>\alpha$ the set $S_\nu\cap\alpha$ is finite. Thus it will suffice to prove that the graph $G(\alpha,\mathcal{B}_\alpha)$ is ω-chormatic. This will follow if we show that for every fixed $B\in\mathcal{B}$ the subgraph of $G(\alpha,\mathcal{B}_\alpha)$ spanned by the set of vertices $H_B=\{\langle U(B,a),\mu\rangle : a\in[\alpha+1]^{<\omega}\ \&\ \mu\in U(B,a)\}$ is ω-chromatic.

To this end we first define a partial order $\langle P,\leq\rangle$ as follows:

$$P = \{\langle a,b\rangle : a\in[\alpha+1]^{<\omega},\ b\in[\alpha]^{<\omega}\ \&\ b\cap S(a)=\emptyset\},$$

where $S(a) = \cup\{S_\nu : \nu\in a\}$, and $\langle a,b\rangle \le \langle a',b'\rangle$ iff $a\supset a'$ and $b\supset b'$. We claim that $\langle P, \le\rangle$ has the CCC. Indeed let $\{\langle a_\nu,b_\nu\rangle : \nu\in\omega_1\}\subset P$, and put for each $\nu\in\omega_1$ $F_\nu=S(a_\nu)$. (Let us note that the order type $tp(F_\nu) < \omega^2$.) Then what we have to show is that there are $\nu,\mu\in\omega_1$ with $\nu\neq\mu$ such that $\langle a_\nu\cup a_\mu, b_\nu\cup b_\mu\rangle\in P$, i.e. $F_\nu\cap b_\mu=\emptyset=F_\mu\cap b_\nu$. By the Δ-system lemma we can assume that the b_ν form a Δ-system, whose intersection then is disjoint from each F_ν, consequently we might assume that the b_ν are actually disjoint, and all have the same number of elements, say k. Then we write $b_\nu = \{\beta_i^{(\nu)} : i < k\}$, where $\beta_i^{(\nu)} < \beta_j^{(\nu)}$ if $i < j$. Now, using the Erdös-Dushnik-Miller theorem (see e.g. [4]) k times, we can also assume that for each fixed $i < k$ the sequence $\langle \beta_i^{(\nu)} : \nu\in\omega_1\rangle$ is strictly increasing in ν. Let us put then, for $\nu\in\omega_1$, $f(\nu)=\{\mu\in\omega_1 : F_\nu\cap b_\mu\neq\emptyset\}$. We claim that the order type $tp[f(\nu)] < \omega^3$ for each ν. Indeed we can write $f(\nu)=\bigcup_{i<k} f_i(\nu)$, where $f_i(\nu)= =\{\mu\in\omega_1 : \beta_i^{(\mu)}\in F_\nu\}$. Since the map $\mu \to \beta_i^{(\mu)}$ is strictly increasing, we see that $tp[f_i(\nu)] \le tp(F_\nu) < \omega^2$, hence obviously $tp[f(\nu)] < \omega^3$. But then for the set mapping $f : \omega_1 \to [\omega_1]^{<\omega^3}$ we can apply a result of Erdös and Specker from [2] which assures us an independent set of size ω_1, in particular then also two μ,ν that $\mu\notin f(\nu)$ and $\nu\notin f(\mu)$, which was to be shown.

Now, since $|P| \le \omega_1$ we can apply $MA(\omega_1)$, or rather its well-known consequence to obtain that $\langle P,\le\rangle$ is σ-centered, i.e. $P=\bigcup_{n\in\omega} P_n$ and for any fixed n if $\langle a,b\rangle, \langle a',b'\rangle\in P_n$ then $\langle a\cup a'\ b\in b'\rangle\in P$. Let us define the map $g : H_B \to \omega$ as follows:

$$g(\langle U(B,a),\mu\rangle) = \min\{n : \langle a, \{\mu\}\rangle\in P_n\}.$$

(Since $\mu\notin S(a)$, this is a good definition.) Now, if $g(\langle U(B,a), \mu\rangle) = = g(\langle U(B,a'), \mu'\rangle)$, then by the above we have $\langle a\cup a', \{\mu,\mu'\}\rangle\in P$, i.e. $\mu\in U(B,a')$ and $\mu'\in U(B,a)$, showing that the above two vertices of H_B are not connected by an edge. Consequently g establishes an ω-coloring of H_B. This completes the proof of theorem 3.

The problem whether a similar result could be proved for regular spaces seems to be very difficult.

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EQUIVALENCE RELATIONS, PROJECTIVE AND BEYOND

L. Harrington and R. Sami

Projective equivalence relations on $\mathbb{R}$ (= ${}^{\omega}\omega$) were first studied by Silver who showed in [Si] :
If E is a $\underset{\sim}{\Pi}^1_1$ equivalence relation on $\mathbb{R}$ then $\mathbb{R}/E$ is countable or else E admits a non-empty perfect set of pairwise inequivalent reals (hence $|\mathbb{R}/E| = 2^{\aleph_0}$.)

This was the paradigm for several subsequent results. Let us abbreviate the second alternative in Silver's theorem by : E has perfectly many classes, on the other hand call E coarse, if it fails to have this property. The following were proved :
$E \in \underset{\sim}{\Sigma}^1_1$ and coarse $\Rightarrow |\mathbb{R}/E| \leq \aleph_1$ (Burgess, [Bu])
$E \in \underset{\sim}{\Pi}^1_3$ and coarse $\Rightarrow |\mathbb{R}/E| < \underset{\sim}{\delta}^1_3$, assuming PD (Harrington, unpublished.)
$E \in \underset{\sim}{\Sigma}^1_2$ and coarse $\Rightarrow |\mathbb{R}/E| \leq \aleph_1$, assuming AD(L[$\mathbb{R}$]) (Kechris [Ke 2])

In § 1, using determinacy hypotheses we generalize these results to all levels of the projective hierarchy, this is Theorem 5, where also we prove : $E \in \underset{\sim}{\Pi}^1_2$ and coarse $\Rightarrow |\mathbb{R}/E| \leq \aleph_1$. In Theorem 4 we deal with inductive and co-inductive equivalence relations. An important tool is Theorem 1 where we prove a special case of a conjecture of Martin on co-λ-Souslin equivalence relations. In § 2, we study the related question of the size of A/E where $A \subseteq \mathbb{R}$, E is $\underset{\sim}{\Sigma}^1_{2n+2}$ or inductive, our results generalize results of J. Stern.

We wish to thank A.S. Kechris who pointed to us a simplification of the argument in Theorem 5, allowing us to state if from PD for all levels of the projective hierarchy. Originally we assumed stronger hypotheses beyond the fourth level.

§ 0. Preliminaries

E always denotes an equivalence relation on the Baire Space ${}^{\omega}\omega$ which we denote $\mathbb{R}$. $\mathbb{R}$ is equipped with the standard topology induced by the complete metric :

$$d(\alpha,\beta) = (n+1)^{-1}, \text{ where } \alpha \neq \beta \text{ and } n = \mu k\ (\alpha(k) \neq \beta(k)).$$

The class of absolutely inductive sets as defined in [Mo 2] will be denoted by IND, whereas HYP denotes the class of absolutely hyperprojective sets. The inductive and hyperprojective sets will be denoted by $\underset{\sim}{\text{IND}}$ and $\underset{\sim}{\text{HYP}}$ respectively.

We will derive our results from PD : all projective games are determined, AD(L [$\mathbb{R}$]) : all games in L[$\mathbb{R}$] are determined, and AD, the "demonstrably false" (see [Ma]) Axiom of Determinacy. Falsehood is best avoided by relativizing all statements to L [$\mathbb{R}$]. It will assumed throughout that DC holds in the universe, V, and that DC holds in L [$\mathbb{R}$].

A set of reals $A \subseteq \mathbb{R}$ is <u>λ-Souslin</u> (λ, an ordinal) if it is the projection of the set of branches of a tree T on $\omega \times \lambda$ (see [Ke 3] for the definitions). We write

$$A = p[T] = \{\alpha \in \mathbb{R} \mid \exists f\ \forall n\ \ (\alpha\restriction n, f\restriction n) \in T\}$$

The complement of such a set will be called <u>co-λ-Souslin</u>. If T is as above, $t \in {}^{n}\omega$, $\nu \in {}^{n}\lambda$ we set

$$T_{t,\nu} = \{(t',\nu') \in T \mid t' \supseteq t \ \&\ \nu' \supseteq \nu, \text{ or } t' \subseteq t \ \&\ \nu' \subseteq \nu\}$$

Similar definitions are made for subsets of $\mathbb{R}^k$ and trees on $\omega^k \times \lambda$.

It is a classical result that $\underset{\sim}{\Sigma}^1_1$ sets are ω-Souslin, $\underset{\sim}{\Sigma}^1_2$ sets are $\aleph_1$-Souslin (Shoenfield.). Assuming PD, $\underset{\sim}{\Sigma}^1_{2n+1}$ sets are λ_{2n+1}-Souslin where λ_{2n+1} is an ordinal $< \underset{\sim}{\delta}^1_{2n+1}$; $\underset{\sim}{\Sigma}^1_{2n+2}$ sets are $\underset{\sim}{\delta}^1_{2n+1}$-Souslin. Assuming Det ($\underset{\sim}{\text{HYP}}$), sets in $\underset{\sim}{\text{IND}}$ are $\underset{\sim}{\kappa}$-Souslin where :

$$\underset{\sim}{\kappa} = \sup\{\text{rank}(\lesssim) \mid \lesssim \text{ is a prewellordering of } \mathbb{R}, \lesssim \in \underset{\sim}{\text{HYP}}\}$$

(These results are due to Moschovakis, see [Ke - Mo 1], [Ke 2] and [Mo 2]).

The following well-known result is very useful (see [My]); it is true for arbitrary Polish spaces.

Proposition 0

If E is a meager subset of $\mathbb{R} \times \mathbb{R}$ then E has perfectly many classes.

Corollary 0

If E has the property of Baire and every class of E is meager then E has perfectly many classes.

Proof : Under the hypotheses E is meager. This follows from the Kuratowski-Ulam theorem (see [Ox]).

§ 1 The main theorems

The theorems of Silver and Burgess mentionned in the introduction render plausible the following conjecture of Martin:

If E is co-λ-Souslin (λ an ordinal $\geq \omega$) and coarse then $|\mathbb{R}/E| \leq \lambda$.

Theorem 1

AD $\Rightarrow$ If E is co-λ-Souslin, $\omega \leq \lambda \leq \underset{\sim}{\kappa}$, and E is coarse then $|\mathbb{R}/E| \leq \lambda$.

Before giving the proof we shall state a theorem of Harrington and Kechris which is instrumental here.

Let ζ be an ordinal, $W \subseteq \mathbb{R}$, $\varphi : W \overset{onto}{\to} \zeta$ a norm, given $A \subseteq {}^{\omega}\zeta$, the standard game G_A is defined as usual, whereas the coded game G_A^{φ} is defined as follows:

Players I and II alternate playing reals : $\alpha_0, \alpha_1, \alpha_2, \ldots$, the first player who plays $\alpha_i \notin W$, loses. If $\forall i : \alpha_i \in W$, then : I wins iff $\langle \varphi(\alpha_i) | i < \omega \rangle \in A$. Thus G_A^{φ} is none other than $G_{A^\star}$ for some $A^\star \subseteq {}^{\omega}\mathbb{R}$.

Note that if $\mathbb{R}$ is well-orderable then G_A is equivalent to G_A^{φ} i.e. : I (resp. II) wins G_A iff I (resp. II) wins G_A^{φ}.

Theorem 0 : (Harrington-Kechris [Ha - Ke 2])

(a) AD $\Rightarrow$ If $W \in \underset{\sim}{\text{IND}}$ and φ is an inductive norm then, for any $A \subseteq {}^{\omega}\zeta$, G_A^{φ} is determined.

(b) PD $\Rightarrow$ If W, $A^\star$ are projective and φ is a projective norm, then G_A^{φ} is determined.

The proof we now give traces back to [Ha 2] where a new proof of Silver's theorem is given; see also [L] for an elegant write up and see [St 3] for an account of Silver's original proof.
More immediately it is patterned after Harrington's proof for the $\underset{\sim}{\Pi}^1_3$ case.

Proof of the theorem 1

Let E be co-λ-Souslin, $\omega \leq \lambda \leq \underset{\sim}{\kappa}$.
A class Γ of subsets of the spaces $\lambda^n \times \mathbb{R}^m$ $(n,m \geq 0)$ is called an auxiliary class for E if :
(o) Γ contains $=$, $\neq$ and is closed under combinatorial substitutions (permutation and identification of variables, addition of "dummy variables")
(i) Γ is closed under substitution of constants from λ and contains the graph of a pairing function : $\lambda \times \lambda \to \lambda$
(ii) Γ is closed under $\wedge, \vee$
(iii) Γ is closed under $\forall^\lambda$, $\exists^\lambda$, $\forall^{\mathbb{R}}$
(iv) Γ is λ-parametrized and normed
(v) $E \in \Gamma$ and in fact, there is a tree T on $\omega \times \omega \times \lambda$ such that $\mathbb{R}^2 - E = p[T]$, and for any k, any $s,t \in {}^k\omega$, $\nu \in {}^k\lambda$, we have $p[T_{s,t,\nu}] \in \breve{\Gamma}$

Here $\breve{\Gamma}$ denotes the dual class of Γ. As usual $\Delta = \Gamma \cap \breve{\Gamma}$.
From these conditions one can deduce by standard arguments :
(vi) There are $I \subseteq \lambda$ in Γ; $D, \breve{D} \subseteq \lambda \times \mathbb{R}$ in Γ, $\breve{\Gamma}$ respectively such that :
- If $S \subseteq \mathbb{R}$ is in Δ then $S = D_\xi$ for some $\xi \in I$.
- If $\xi \in I$ then $D_\xi = \breve{D}_\xi$ (hence $D_\xi \in \Delta$)
(vii) $\breve{\Gamma}$ has the separation property.

Claim 1 : There is an auxiliary class for E.
Proof : Let T witness that E is co-λ-Souslin and let M be the smallest admissible set containing $\mathbb{R}$, λ, T. Let Γ consist of those subsets of $\lambda^n \times \mathbb{R}^m$ which are Σ_1-definable over M with parameters from $\{\mathbb{R}, T, \lambda\} \cup \lambda$.
That Γ posseses (o) through (v) is well-known (see for instance [Ba 1], as to (v) note that :

$$(\alpha,\beta) \notin p[T_{s,t,\nu}] \iff$$
$$(\alpha \supseteq s \ \& \ \beta \supseteq t \Rightarrow (T_{s,t,\nu}(\alpha,\beta), \supsetneq) \text{ is well-founded})$$

where, if $\alpha \supseteq s$, $\beta \supseteq t$:

$$T_{s,t,\nu}(\alpha,\beta) =^{df} \{\tau \in {}^{\omega}\lambda \mid \tau \supseteq \nu \text{ or } \tau \subseteq \nu \text{ \& if } n = \text{length}(\tau) \text{ then } (\alpha \upharpoonright n, \beta \upharpoonright n, \tau) \in T\}.$$

Now define :

$$Q = \cup \{S \subseteq \mathbb{R} \mid S \in \Delta \text{ and } S \text{ is contained in a single class of } E\}.$$

Case A : $Q = \mathbb{R}$.

In this case, it is easy to see that $|\mathbb{R}/E| \leqslant \lambda$, via the function $H : \mathbb{R} \to \lambda$ defined by :

$$H(\alpha) = \text{least } \xi \in \lambda : \xi \in I \text{ and } D_\xi \neq \emptyset \text{ and } D_\xi \text{ is contained in the equivalence class of } \alpha.$$

Case B : $\mathcal{U} = \mathbb{R} - Q \neq \emptyset$.

Here we will show that E has perfectly many classes. We note first the following.

Claim 2 : $\mathcal{U} \in \breve{\Gamma}$. Indeed :

$$Q(\alpha) \Leftrightarrow (\exists \xi)(\xi \in I \ \& \ \alpha \in D_\xi \ \& \ \forall \beta(\beta \in \breve{D}_\xi \Rightarrow \alpha \mathrel{E} \beta));$$

so that $Q \in \Gamma$.

Claim 3 : If $A \in \breve{\Gamma}$ and A is a non-empty subset of $\mathcal{U}$ then A meets more than one class of E. Indeed, suppose A is as described, if it were the case that $A \subseteq Z$ for some $Z \in \mathbb{R}/E$ then notice first that $Z \in \Gamma$, for :

$$\alpha \in Z \Leftrightarrow \forall \beta(\beta \in A \Rightarrow \beta \mathrel{E} \alpha),$$

next, by separation for $\breve{\Gamma}$ find $C \in \Delta$ such that $A \subseteq C \subseteq Z$, but then $C \subseteq Q = \mathbb{R} - \mathcal{U}$, a clear contradiction.

We now set :

$$\Sigma = \{A \subseteq \mathcal{U} \mid A \in \breve{\Gamma} \text{ and } A \neq \emptyset\}$$

and consider the game $G(E,\Gamma)$, as follows :

I	II	I	...
A_o	A_1	A_2	...
B_o	B_1	B_2	...

Players I and II alternate producing pairs (A_i, B_i) of members of Σ:
(1) The first player who fails to meet : $A_i \supseteq A_{i+1}$ & $B_i \supseteq B_{i+1}$, loses.
(2) If $\forall i$: $A_i \supseteq A_{i+1}$ & $B_i \supseteq B_{i+1}$, then II loses unless for all i

the diameters of A_{2i+1}, B_{2i+1} are $< 1/(i+1)$ (the diameter is relative to the standard metric).

(If a player loses by (1) or (2) we shall say : he loses for trivial reasons).

(3) If neither player has lost for trivial reasons then setting :

$$\{\alpha\} = \bigcap_i \overline{A}_i \quad , \quad \{\beta\} = \bigcap_i \overline{B}_i$$

I wins iff α E β. (The $\overline{}$ means topological closure).

We will show below that I has no winning strategy in $G(E,\Gamma)$. If II has a winning strategy in $G(E,\Gamma)$, then this will give rise to a perfect set of pairwise inequivalent reals.

Note that $\Sigma \subseteq \breve{\Gamma}$, which is λ-parametrized; hence $\Sigma \times \Sigma$ can be well ordered in type $\zeta \leq \lambda$. Thus the game $G(E,\Gamma)$ can literally be viewed as the standard game G_X, for some $X \subseteq {}^{\omega}\zeta$. Now since $\lambda \leq \underset{\sim}{\kappa}$, let $W \in \underset{\sim}{\text{IND}}$ and $\varphi : W \overset{\text{onto}}{\to} \zeta$ be an inductive norm.

Assume now AD, by the theorem of Harrington and Kechris mentionned above, G_X^{φ} is determined. At this point we should switch to this game, but to avoid a notational mess we keep to $G(E,\Gamma)$, do as if it were determined and ask the reader to make the translation (One can also invoke a familiar forcing argument whereby, there is a generic extension of V, with no new reals, and where $\mathbb{R}$ can be well ordered. In this extension G_X^{φ} is still determined hence $G(E,\Gamma)$ is determined, the conclusion to be derived from this fact easily relativizes to V).

<u>Claim 4</u> : I has no winning strategy in $G(E,\Gamma)$.

Proof : We plan to describe two runs of $G(E,\Gamma)$ in which player I follows a given strategy σ and things are so arranged that :

- player II does not lose for trivial reasons
- two pairs of reals are thus produced (as per (3)) say, (α,β) and $(\hat{\alpha},\hat{\beta})$ and $\alpha = \hat{\alpha}$, $\beta \not\!E\ \hat{\beta}$.

The contradiction now arises from player I's having followed σ : α E β and $\hat{\alpha}$ E $\hat{\beta}$, hence β E $\hat{\beta}$.

The two runs of the game are described in Diagram 1, subject to the following conventions :

- an arrow $X \to Y$ indicates $X = Y$
- $X - 1/2 \to Y$ indicates that Y is obtained in some standard way from X so as to have : $\emptyset \neq Y \subseteq X$ and $\text{Diam}(Y) \leq 1/2\ \text{Diam}(X)$. We will write $Y = 1/2\ X$.
- $X - \pi_1 \to Y$ indicates that :

  $Y = \pi_1(X) = 1^{st}$ projection of X.

  Similarly for π_2.

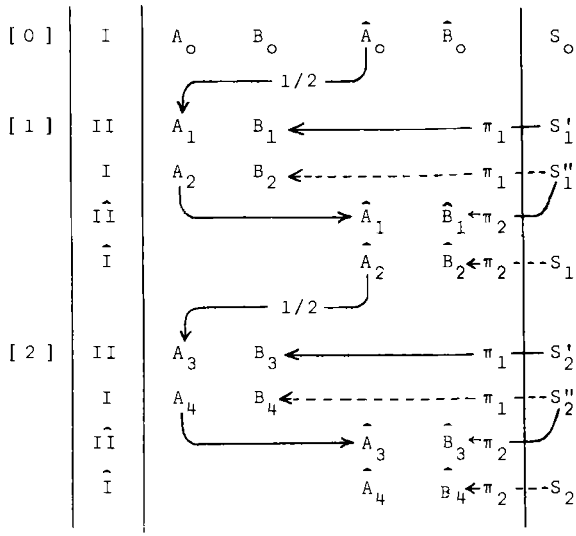

Diagram 1

Our description is in stages, the accent $\hat{}$ indicates moves on the 2nd board.

Stage [0] :

- Player I, using σ produces (A_o, B_o), player $\hat{I}$ similarly produces $(\hat{A}_o, \hat{B}_o)$. These pairs are identical.

- Set $S_o = (B_o \times B_o) \cap (\mathbb{R}^2 - E)$. Note here that by claim 3, B_o meets more than one class, hence $S_o \neq \emptyset$ in fact : $\pi_1(S_o) = \pi_2(S_o) = B_o$.

Inductively now, suppose that at the end of Stage [k] we have described :

- The moves up to (A_{2k}, B_{2k}), $(\hat{A}_{2k}, \hat{B}_{2k})$
- A set $S_k \subseteq \mathbb{R}^2$, sequences s_k, $t_k \in {}^k\omega$, $\nu_k \in {}^k\lambda$

such the following hold :

(a_k) II and $\hat{II}$ have played so as not to lose for trivial reasons.

(b_k) $\hat{A}_{2k} \subseteq A_{2k}$.

(c_k) $\emptyset \neq S_k \subseteq p[T_{s_k, t_k, \nu_k}]$.

(d_k) $\pi_1(S_k) \subseteq B_{2k}$, $\pi_2(S_k) = \hat{B}_{2k}$.

Stage [k+1] :

Find n, m, ξ such that :

$$S'_{k+1} \overset{df}{=} S_k \cap p[T_{s_{k+1}, t_{k+1}, \nu_{k+1}}] \neq \emptyset$$

where $s_{k+1} = s_k{}^\frown\langle n\rangle$, $t_{k+1} = t_k{}^\frown\langle m\rangle$, $\nu_{k+1} = \nu_k{}^\frown\langle \xi\rangle$

- Let II play $A_{2k+1} = 1/2\ \hat{A}_{2k}$, $B_{2k+1} = \pi_1(S'_{k+1})$.
- Let I reply (A_{2k+2}, B_{2k+2}) by σ.

 Set

$$S''_{k+1} = (B_{2k+2} \times \mathbb{R}) \cap \hat{S}'_k .$$

- Let II play : $\hat{A}_{2k+1} = \hat{A}_{2k+2}$, $B_{2k+1} = \pi_2(S''_{k+1})$.
- Let I reply $(\hat{A}_{2k+2}, \hat{B}_{2k+2})$, by σ.

 Set

$$S_{k+1} = (\mathbb{R} \times \hat{B}_{2k+2}) \cap S''_{k+1} .$$

It is straight forward to check that (a_{k+1}) through (d_{k+1}) now hold. We use of course the properties (O), (i), (ii), (v) of Γ to verify (a_{k+1}).

One thus gets $\{(A_i, B_i)\}$, $\{(\hat{A}_i, \hat{B}_i)\}$ setting $\alpha = \bigcap_i \overline{A}_i$, $\beta = \bigcap_i \overline{B}_i$ and similarly for $\hat{\alpha}$, $\hat{\beta}$ one sees :

$\alpha = \hat{\alpha}$, because $\forall k : A_{2k} \supseteq \hat{A}_{2k}$.

$\beta \not\mathrel{E} \hat{\beta}$, because $\forall k : \beta \upharpoonright k = s_k$, $\hat{\beta} \upharpoonright k = t_k$ and $(\alpha, \beta, f) \in [T]$ where $f = \bigcup_k \nu_k$.

To finish the proof of Theorem 1, it suffices now to show :

<u>Claim 5</u> : Suppose I has a winning strategy τ, in $G(E,\Gamma)$ then E has perfectly many classes.

Proof : Let $H_n = \{(s,t) \mid s,t \in {}^n2 \ \& \ s < t \text{ (lex.)}\}$

let $h_n : H_n \to k_n$ be a bijection and set : $H = \bigcup_n H_n$, $h = \bigcup_n h_n$.

Denote by $|s|$ the length of s. We are going to define a family $(A^i_s \mid s \in {}^{\omega}2,\ 0 \leq i \leq k_{|s|})$ of members of Σ such that

(a) $s \subsetneq t \Rightarrow A^i_s \supseteq A^j_t$, for all relevant i,j.

(b) $i < j \Rightarrow A^i_s \supseteq A^j_s$, for all $j \leq k_{|s|}$.

(c) whenever $|s| = |t| = n$, say, and $s < t$ (lex.) then letting

 $m = \text{least } k : (s(k) \neq t(k))$

and setting $s_j = s \upharpoonright (m+j)$, $t_j = t \upharpoonright (m+j)$

$$i_j = h(s_j, t_j)$$
$$P_j = (A^{i_j}_{s_j}, A^{i_j}_{t_j}), \quad \text{for } 0 < j \leq n-m$$

we have

$$\tau(P_1, \ldots, P_{j-1}) \supseteq P_j \quad , \quad \text{for } 1 < j \leq n-m.$$

We proceed by induction on $|s|$.

$k_o = 0$, define $A^o_\emptyset = \mathcal{U}$.

Having defined A^i_s for $|s| < n$, $i \leq k_{|s|}$, define A^i_u for $u \in {}^n2$ by induction on i as follows.

Set first $(s^i, t^i) = h(i)$ for $i < k_n$.

- $A^o_u = A^{k_{n-1}}_{u \upharpoonright n-1}$
- Given $i+1 \leq k_n$;

- If u is one of s^i, t^i then A_u^{i+1} is given by

$$(A_{s^i}^{i+1}, A_{t^i}^{i+1}) = \tau(P_1, \ldots, P_{n-m-1}, (A_{s^i}^i, A_{t^i}^i))$$

where m and $P_1, \ldots, P_{n-m-1}$ are defined as in (c) above with s,t replaced by s^i, t^i.

- If $u \neq s^i, t^i$ set :

$$A_u^{i+1} = A_u^i .$$

It is now straightforward to verify (a), (b), (c).
Define now, for $f \in {}^{\omega}2$:

$$F(f) = \text{the unique member of } \bigcap_n \overline{A_{f\restriction n}^{\circ}}$$

(This will make sense in a moment).

If $f, g \in {}^{\omega}2$ are distinct, say $f < g$ (lex.) then, if one defines $P_1, P_2, \ldots$ as in (c) above with f,g replacing t,s it is apparent that these can be viewed as I's moves in a run of $G(E,\Gamma)$, where II follows τ and I does not lose for trivial reasons. Further, the reals F(f), F(g) will be the end products of this run of the game, hence $F(f) \not{E}\, F(g)$.

F is continuous, clearly, and thus $F({}^{\omega}2)$ is a perfect set of pairwise inequivalent reals. ⊣

We postpone for a moment deriving conclusions from Theorem 1, to prove a coding theorem which generalizes, for the projective classes, results of [Mo 1] .

Given E, a set $A \subseteq \mathbb{R}$ is called <u>E-invariant</u> (or just, <u>invariant</u>) if : $\alpha \in A \;\&\; \alpha\, E\, \beta \Rightarrow \beta \in A$,
similarly, if $A \subseteq \mathbb{R}^n$. A function f is called <u>invariant</u>, if its domain is invariant and $f(\alpha)$ depends only on α/E.

A set $A \subseteq \mathbb{R}$ is called <u>E-thin</u> if it does not have a perfect subset consisting of pairwise inequivalent elements.

We introduce the notation : If Γ is a class, $A \subseteq \mathbb{R}^n$
$B(\Gamma,A)$ denotes the closure of $\Gamma \cup \{A\}$ under finite boolean combinations and continuous pre-images.
If $A = \emptyset$, we just write $B(\Gamma)$ for $B(\Gamma,A)$.

<u>Theorem 2</u> :

Let Γ be one of the classes Π^1_{2n+1}, Σ^1_{2n+2}, IND.
Suppose $E \in \utilde{\Gamma}$ and let $A \subseteq \mathbb{R}$ be E-thin and invariant then :

$$\text{Det} - B(\utilde{\Gamma}, A) \Rightarrow A \in \utilde{\Gamma} .$$

Proof : A game G is played :

I	II
n_o	k_o
n_1	k_1
$\vdots$	$\vdots$
α	$<\beta,\gamma>$

II wins iff : $\alpha \in A \Leftrightarrow \gamma \in \Delta(\beta)$.

Clearly if II has a winning strategy τ then $A \in \Gamma(\tau)$, for then :

$$\alpha \in A \Leftrightarrow (\tau \bullet [\alpha])_1 \in \Delta((\tau \bullet [\alpha])_o).$$

Claim : I has no winning strategy in G.

Suppose towards a contradiction, σ were such an object. Without loss of generality assume $E \in \Gamma$ and define :

$$F(\gamma) = \sigma \star [<\sigma,\gamma>]$$

Note that $\{\gamma \mid F(\gamma) \notin A\}$ is countable for, otherwise a real γ_o can be found with : $\gamma_o \notin \Delta(\sigma)$ and $F(\gamma_o) \notin A$ and II playing to produce $<\sigma,\gamma_o>$ would defeat I playing according to σ.

Define : $\alpha \hat{E} \beta \Leftrightarrow F(\alpha)\, E\, F(\beta)$. $\hat{E} \in \Gamma(\sigma)$, clearly. $\hat{E}$ is coarse because if there is P, a perfect set of pairwise $\hat{E}$-inequivalent reals, then by the previous observation we can find such a P with $F(P) \subseteq A$, F being continuous this easily contradicts that A is E-thin.

Thus, by Corollary 0, E has non-meager classes, let :

$$Z = \{\gamma \mid \gamma/\hat{E} \text{ is non-meager}\}$$

then :

(i) $Z \in \Gamma(\sigma)$, this follows from the results in [Ke 1]
(for Γ = IND, a direct computation is easy).

(ii) $\gamma \in Z \Rightarrow F(\gamma) \in A$, indeed A being E-invariant $F^{-1}(A)$ is $\hat{E}$-invariant and hence from the argument above $F(\gamma) \notin A \Rightarrow \gamma/\hat{E}$ is countable, hence meager.

Clearly Z is non-meager, hence it contains a real γ_o in $\Delta(\sigma)$ (see [Ke 1] for $\Gamma = \Pi^1_{2n+1}$, the other cases are just the ordinary basis theorems for Γ).

Again II playing to produce $<\sigma,\gamma_o>$ defeats I playing according to σ-a contradiction. $\dashv$

This argument was inspired from [Ha - Ke 1]. We will apply

Theorem 2 in § 2, meanwhile the following is useful to notice

Corollary 3 :

Suppose E is coarse and Γ is as above,

(a) If $E \in \underset{\sim}{\Gamma}$ and $A \subseteq \mathbb{R}^n$ is E-invariant then : Det - $B(\underset{\sim}{\Gamma}, A) \Rightarrow A \in \underset{\sim}{\Delta}$.

(b) Det - $B(\underset{\sim}{\text{IND}}) \Rightarrow$ If $E \in \underset{\sim}{\text{IND}}$ then $E \in \underset{\sim}{\text{HYP}}$.

(c) PD $\Rightarrow$ If E is in $\underset{\sim}{\Pi}^1_{2n+1}$ [resp. $\underset{\sim}{\Sigma}^1_{2n+2}$] then E is in $\underset{\sim}{\Delta}^1_{2n+1}$ [resp. $\underset{\sim}{\Delta}^1_{2n+2}$] .

Proof : (b) and (c) follow from (a) by taking there A = E.
To prove (a), take n = 2 for simplicity, define

$(\alpha,\alpha')\ E^2\ (\beta,\beta') \Leftrightarrow \alpha\ E\ \beta\ \&\ \alpha'\ E\ \beta'$

Claim : E^2 is coarse.

If not let $H : {}^{\omega}2 \to \mathbb{R}^2$ be continuous and such that :
$\alpha \neq \beta \Rightarrow H(\alpha)\ \not E^2\ H(\beta)$. Define, for $\alpha,\beta \in {}^{\omega}2$

$$\alpha\ E_1\ \beta \Leftrightarrow \pi_1(H(\alpha))\ E\ \pi_1(H(\beta)).$$

Since E is coarse, E_1 is coarse hence by Corollary 0, E_1 has a non-meager class, whence a perfect $P \subseteq {}^{\omega}2$ can be found (a subset of this class) such that :

$$\alpha,\beta \in P \Rightarrow \pi_1(H(\alpha))\ E\ \pi_1(H(\beta)).$$

We now repeat this argument with P, $H \restriction P$, π_2 replacing ${}^{\omega}2$, H, π_1 to get a perfect $P' \subseteq P$ such that

$$\alpha,\beta \in P' \Rightarrow \pi_2(H(\alpha))\ E\ \pi_2(H(\beta)).$$

Hence $\alpha,\beta \in P' \Rightarrow H(\alpha)\ E^2\ H(\beta)$ - a contradiction.
Now $A \subseteq \mathbb{R}^2$ is E^2-thin, hence by the previous theorem $A \in \underset{\sim}{\Gamma}$; similarly, $\mathbb{R}^2 - A \in \underset{\sim}{\Gamma}$. ⊣

We will mainly use (a) above with $A \subseteq \mathbb{R}^2$, a prewellordering of $\mathbb{R}$, inducing E.

We are now in a position to prove :

Theorem 4 :

Assume Det(L[$\mathbb{R}$]).

(a) If $E \in \underset{\sim}{\text{IND}}$ is coarse, then $|\mathbb{R}/E| < \underset{\sim}{\kappa}$ and E is induced by a $\underset{\sim}{\text{HYP}}$ prewellordering of $\mathbb{R}$.

(b) If $E \in$ co-$\underset{\sim}{\text{IND}}$ is coarse, then $|\mathbb{R}/E| \leq \underset{\sim}{\kappa}$.

Proof : Work in $L[\mathbb{R}]$, AD is thus available.
(b) is an immediate consequence of Theorem 1 (recall that co-$\underset{\sim}{\text{IND}}$ sets are co-$\underset{\sim}{\kappa}$-Souslin).
(a) follows from (b), indeed a coarse E in $\underset{\sim}{\text{IND}}$ will be in fact in $\underset{\sim}{\text{HYP}}$ by Corollary 3.b.. Using (b) E is induced by a prewellordering $\lesssim$ of $\mathbb{R}$ invoking now Corollary 3.a. we get $\lesssim \in \underset{\sim}{\text{HYP}}$ (being an E-invariant subset of $\mathbb{R}^2$) and hence the length of $\lesssim$ is $< \underset{\sim}{\kappa}$. ⊣

The determinacy hypothesis is a bit extravagant here, for instance an examination of the proofs (including that of Theorem 0) would show that Det-B($\underset{\sim}{\text{IND}}$) suffices for (a) above. This is not very satisfactory. In the next theorem we consider projective equivalence relations and though the conclusions are easy consequences of our previous results, some work is needed to weaken the hypothesis to PD.

Theorem 5 :

Assume PD : Let E be a coarse equivalence relation.
(a) If E is $\underset{\sim}{\Pi}^1_{2n+1}$ [resp. $\underset{\sim}{\Sigma}^1_{2n+2}$] then $|\mathbb{R}/E| < \underset{\sim}{\delta}^1_{2n+1}$ [resp. $\leq \underset{\sim}{\delta}^1_{2n+1}$] and E is induced by a $\underset{\sim}{\Delta}^1_{2n+1}$ [resp. $\underset{\sim}{\Delta}^1_{2n+2}$] prewellordering of $\mathbb{R}$.
(b) If E is $\underset{\sim}{\Pi}^1_{2n+2}$ then $|\mathbb{R}/E| \leq \underset{\sim}{\delta}^1_{2n+1}$.

Proof : Assume AD, momentarily :
If $E \in \underset{\sim}{\Pi}^1_{2n+2}$ then E is co-$\underset{\sim}{\delta}^1_{2n+1}$-Souslin and thus we get from Thm 1 that E being coarse is induced by a prewellordering of $\mathbb{R}$ of length $\leq \underset{\sim}{\delta}^1_{2n+2}$, this with Corollary 3.a yields :
(b') If $E \in \underset{\sim}{\Pi}^1_{2n+2}$ is coarse then $|\mathbb{R}/E| \leq \underset{\sim}{\delta}^1_{2n+1}$ and E is induced by a projective prewellordering of $\mathbb{R}$.
Reverting to PD one can derive (a) from (b') :
Suppose $E \in \underset{\sim}{\Pi}^1_{2n+1}$ and is coarse then from (b') E is induced by a projective prewellordering of $\mathbb{R}$, by Corollary 3.a this prewellordering is in fact $\underset{\sim}{\Delta}^1_{2n+1}$ hence has length $< \underset{\sim}{\delta}^1_{2n+1}$ hence, $|\mathbb{R}/E| < \underset{\sim}{\delta}^1_{2n+1}$.
If $E \in \underset{\sim}{\Sigma}^1_{2n+2}$ and is coarse then by Corollary 3.b $E \in \underset{\sim}{\Delta}^1_{2n+2}$, again we quote (b') above and Corollary 3.b to get $|\mathbb{R}/E| \leq \underset{\sim}{\delta}^1_{2n+2}$ and E is induced by a $\underset{\sim}{\Delta}^1_{2n+2}$ prewellordering of $\mathbb{R}$.

It remains to prove (b') from PD. Which we assume from here on.

For notational simplicity suppose $E \in \Pi^1_{2n+2}$ and set $\delta = \underset{\sim}{\delta}^1_{2n+1}$.

E is co-δ-Souslin.

We follow very closely the proof ot Theorem 1, the <u>only</u> difference being the construction of a "better" auxiliary class for E.

Fix W a complete Π^1_{2n+1} set, $\varphi : W \overset{onto}{\to} \delta$ a Π^1_{2n+1} norm.

Let Λ be a point class. Given $S \subseteq \delta^k \times \mathbb{R}^m$, set code$(S,\varphi)$ $= \{(u_1, \ldots, u_k, \alpha_1, \ldots, \alpha_m) \mid u_i \in W \ \& \ \alpha_j \in \mathbb{R}$ and $(\varphi(u_1), \ldots, \varphi(u_k), \alpha_1, \ldots, \alpha_m) \in S\}$.

S is called <u>Λ in the codes</u> (relative to φ) if code$(S,\varphi) \in \Lambda$.

Set : $\Gamma' = \{S \mid S$ is Π^1_{2n+3} in the codes$\}$ finally Γ is defined to consist of those subsets $S \subseteq \delta^k \times \mathbb{R}^m$ such that for some $\mathcal{U} \subseteq \delta^{k+1} \times \mathbb{R}^m$, some $\xi_0 < \delta$ one has :

$$S(\overline{\xi},\overline{\alpha}) \Leftrightarrow \mathcal{U}(\xi_0,\overline{\xi},\overline{\alpha}).$$

<u>Claim</u> : Γ is an auxiliary claim.

Proof : (Sketch)

The main technical result of [Ha - Ke 1] can be stated as follows :

(*) If $S \subseteq \delta \times \mathbb{R}$ is projective in the codes and $\forall\xi < \delta \ \exists\alpha : S(\xi,\alpha)$ then there is a $\underset{\sim}{\Delta}^1_{2n+1}$ partial function F, such that :

- Dom $F \supseteq W$
- $\forall u \in W : S(\varphi(u),F(u))$

From this it is not too difficult to show :

(**) If $V \in \Pi^1_{2n+1}$ and $\psi : V \to \delta$ is a Π^1_{2n+1} norm then : $\{(u,v) \mid u \in W \ \& \ v \in V \ \& \ \varphi(u) = \psi(v)\}$ is Δ^1_{2n+2}.

We now go through (0) to (v) of the definition:(o), (i) and (ii) are trivial except for the pairing function : one takes $p : \delta \times \delta \to \delta$ a bijection à la Gödel and uses (**) to verify that graph p is Δ^1_{2n+2} in the codes.

(iii) Closure of Γ' (hence, of Γ) under $\forall^\lambda$, $\forall^{\mathbb{R}}$ is obvious. It suffices now to verify closure of $\check{\Gamma}'$ under $\forall^\lambda$. This is essentially in [Ha - Ke 1] where it is shown :

Let $\psi : S \overset{onto}{\to} \eta$ be a Δ^1_{2n+1} norm, if $\mathcal{U} \subseteq \eta \times \mathbb{R}$ is Σ^1_{2n+2} in the codes, relative to ψ, then $(\forall\xi < \eta)\ S(\xi,.)$ is Σ^1_{2n+2}. (For a proof see Theorem 37 of [Be]).

The same proof, mutatis mutandis, yields closure of $\check{\Gamma}'$ under $\forall^\lambda$.

(iv) We follow [Ke 4] . We are going to show that Γ' is ω-parametrized and normed. This will suffice. Let $\mathcal{U}'' \subseteq \omega \times \mathbb{R} \times \mathbb{R}$ be ω-universal for Π^1_{2n+3} subsets of $\mathbb{R} \times \mathbb{R}$, set

$$\mathcal{U}'(n,u,\alpha) \Leftrightarrow \mathcal{U} \in W \ \& \ \forall v(v \in W \ \& \ \varphi(v) = \varphi(u) \Rightarrow \mathcal{U}''(n,v,\alpha)).$$

A theorem of Solovay states that : If a Π^1_{2n+3} set is invariant for a Σ^1_{2n+3} equivalence relation on $\mathbb{R}$ then it admits a Π^1_{2n+3} norm, invariant for this relation (see [Ke 4], Theorem 2.1). This yields a Π^1_{2n+3} norm : $\theta' : U' \to \delta$ such that :

$$U'(n,u,\alpha) \ \& \ \varphi(u) = \varphi(v) \Rightarrow \theta'(n,u,\alpha) = \theta'(n,v,\alpha).$$

U', θ' give rise to $U \subseteq \omega \times \lambda \times \mathbb{R}$, $\theta : U \to \delta$, an ω-universal Γ' set and a Γ'-norm respectively.

(v) For notational simplicity we show that, given $A \subseteq \mathbb{R}$ in Π^1_{2n+2} there is a tree T on $\omega \times \delta$ such that whenever $t \in {}^k\omega$, $\nu \in {}^k\delta$, then : $p[T_{t,\nu}] \in \breve{\Gamma}$.

Say $\sim A(\alpha) \Leftrightarrow \exists\beta\ B(\alpha,\beta)$ with $B \in \Pi^1_{2n+1}$ let $\{\psi_n\}$ be a Π^1_{2n+1} scale on B, let T' be the tree associated to $\{\psi_n\}$ (see [Ke - Mo 1]). Define T a tree on $\omega \times \delta$ by :

$$T = \{((n_o,\ldots,n_k),(p(m_o,\xi_o),\ldots,p(m_k,\xi_k))) \mid (\bar{n},\bar{m},\bar{\xi}) \in T'\}$$

Set $Q(\alpha,u) \Leftrightarrow \forall i[(u)_i \in W \ \& \ (\alpha \restriction i,(\varphi((u)_o),\ldots,\varphi((u)_{i-1}))) \in T]$.

One can now verify using (**) that $Q \in \Delta^1_{2n+3}$.

Now if $\nu = (\xi_o, \ldots, \xi_{k-1})$ then

$$\alpha \in p[T_{t,\nu}] \Leftrightarrow \alpha \supseteq t \ \& \ (\exists u)(Q(\alpha,u) \ \& \ (u)_o = \xi_o, \ldots, (u)_{k-1} = \xi_{k-1})$$

Thus $p[T_{t,\nu}]$ is in Δ.

Referring again to the proof of Theorem 1, it is straightforward to check that if Case A holds, the function H defined there : $H : \mathbb{R} \to \delta$, is projective in the codes, hence E is induced by a projective prewellordering of $\mathbb{R}$. If Case B holds then we are in presence of the game $G(E,\Gamma)$ as in our discussion there this can be viewed as the standard game G_X for some $X \subseteq {}^\omega\delta$. Again it is straightforward (but rather tedious) to verify that if this is done in the obvious way then the set :

$$\tilde{X}(u) \Leftrightarrow^{df} \forall i : (u)_i \in W \text{ and } <\varphi((u)_i) | i < \omega> \in X$$

is projective.

We can now invoke (b) of Theorem 0 to see that PD is enough to ascertain the determinacy of the coded game G^φ_X. ⊣

Corollary 6 :

Assume the Axiom of Choice and PD, let E be coarse : $E \in \underset{\sim}{\Pi}^1_3 \Rightarrow |\mathbb{R}/E| \leqslant \aleph_2$

$E \in \underset{\sim}{\Sigma}^1_4$ or $\underset{\sim}{\Pi}^1_4 \Rightarrow |\mathbb{R}/E| \leq \aleph_3$.

Proof : By a theorem of Martin (see [Ke 3]), under the hypotheses, $\underset{\sim}{\delta}^1_3 \leq \aleph_3$.

§ 2. Sundry questions :

Let $A \subseteq \mathbb{R}$, by definition A/E is $\{\alpha/E \mid \alpha \in A\}$. In [St 1], J. Stern proves the following : PD $\Rightarrow$ If $E \in \underset{\sim}{\Pi}^1_1$ and A is a projective E-thin set then $|A/E| \leq \aleph_1$. We generalize and strengthen this, here.

Theorem 7 :

Assume PD. Let $E \in \underset{\sim}{\Sigma}^1_{2n+2}$ and suppose A is projective and E-thin then $|A/E| \leq \underset{\sim}{\delta}^1_{2n+1}$.

Proof : Consider first the case where A is invariant then by Theorem 2, $A \in \underset{\sim}{\Sigma}^1_{2n+2}$. Further more what is shown in effect in the proof of Corollary 3.a is that : A is E-thin $\Rightarrow$ $A \times A$ is E^2-thin. Hence $(A \times A) \cap (\mathbb{R}^2 - E)$ is E^2-thin, so again by Theorem 1, this set is in $\underset{\sim}{\Sigma}^1_{2n+2}$.
If we now define

$$\alpha E^A \beta \Leftrightarrow \alpha \notin A \ \& \ \beta \notin A \quad \text{or} \quad \alpha E \beta$$

then clearly E^A is coarse and the previous considerations show $E^A \in \underset{\sim}{\Pi}^1_{2n+2}$, hence by Theorem 4 : $|\mathbb{R}/E^A| \leq \underset{\sim}{\delta}^1_{2n+1}$.
But $|A/E| + 1 = |\mathbb{R}/E^A|$, hence the conclusion.
If now A is not E-invariant consider :

$$\tilde{A} =^{df} \{\alpha \mid \exists \beta \in A : \alpha E \beta\}.$$

An easy argument using uniformization shows that $\tilde{A}$ is E-thin. Thus $|A/E| = |\tilde{A}/E| \leq \underset{\sim}{\delta}^1_{2n+1}$. ⊣

Assuming AD, A could be taken to be an arbitrary invariant subset of $\mathbb{R}$. A similar result can be formulated for $E \in \underset{\sim}{\mathrm{IND}}$:

Theorem 8 :

AD $\Rightarrow$ If $E \in \underset{\sim}{\mathrm{IND}}$ and $A \subseteq \mathbb{R}$ is E-invariant, then $|A/E| \leq \underset{\sim}{\kappa}$.

Theorem 7 can in fact be derived from the following :

Theorem 9 :

(a) Assume PD. If E is a coarse projective equivalence relation and every class of E is Σ^1_{2n+2} then $|\mathbb{R}/E| \leq \delta^1_{2n+1}$.
(b) Assume AD(L[$\mathbb{R}$]). If E is a coarse inductive or coinductive equivalence relation with Σ^1_{2n+2} classes then $|\mathbb{R}/E| \leq \delta^1_{2n+1}$.

Proof : (b) Work in L[$\mathbb{R}$]. A theorem of Kechris in [Ke 2] can be stated as : AD ⇒ If $(A_\xi \mid \xi < \eta)$ is a 1-1 sequence of pairwise disjoint Σ^1_{2n+2} sets then $\eta < \delta^1_{2n+2}$. Now if E is as given then $\mathbb{R}/E$ is well-orderable (by Theorem 4), hence the conclusion. We shall not prove (a) here, we just mention that Kechris'theorem can be proved, assuming only PD, for those sequences $(A_\xi \mid \xi < \eta)$ which are "projective in the codes". This uses the techniques of [Ke 5].

To conclude we list a few open questions and conjectures.

(1) Conjecture (Martin) : If E is co-λ-Souslin and coarse then $|\mathbb{R}/E| \leq \lambda \quad (\omega \leq \lambda)$
(2) Conjecture (Solovay) : AD ⇒ If E is coarse then $\mathbb{R}/E$ is well-orderable.
(3) What are the minimal determinacy hypotheses needed say, for Theorem 5. One should be able to derive the results for Π^1_{2n+1}, Σ^1_{2n+1} (and perhaps Π^1_{2n+2}) from Det (Δ^1_{2n}). Note that Σ^1_{2n+2} is different; from the results of [Ha 1] one can quote relative consistency results to the effect that a Σ^1_2 equivalence relation "can" have any number of classes $\leq 2^{\aleph_0}$
(4) Generalize the following result of Stern [St 2] $\forall\alpha : \aleph_1^{L[\alpha]} < \aleph_1 \Rightarrow$ If $E \in \Delta^1_2$ and all the classes of E (except possibly a countable number) are Borel of bounded rank then $|\mathbb{R}/E| \leq \aleph_0$.

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PROJECTIONS OF LAWLESS SEQUENCES II

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Contents

1. INTRODUCTION

1.1. This paper may be regarded as a sequel to [T3] and [D,T], which explains the title.

The conceptual basis of choice sequences has been discussed at length in earlier publications, to which we refer the reader for detailed information (see e.g. [T4], especially appendices B and C). We are indebted to D. van Dalen and G. Kreisel for criticism of earlier drafts.

1.2. We recall motivation and results of [D,T] and [T2]. The simplest concept of choice sequence is that of a lawless sequence; analysis of this concept leads to a convincing axiomatization ($\underset{\sim}{\mathrm{LS}}$ in [T4], Chapters 2 and 3; originally due to Kreisel) which via an elimination theorem completely characterizes the properties of lawless sequences LS relative to (the properties of) lawlike objects and relative to a suitable language. Lawless sequences however exhibit two peculiarities which might be regarded as limitations (though in certain contexts they are an asset!):

(a) LS is not closed under any non-trivial continuous operation such as e.g. $\alpha \to \lambda x.2\alpha x$;

(b) no element of LS satisfies any non-trivial overall-restriction on its values, not even if the restriction does not contain choice parameters. For example, no lawless sequence satisfies $R\alpha \equiv \exists x \forall y{>}x \exists z(\alpha y = 2z)$, and a fortiori the lawlike sequences are not even extensionally contained in LS, i.e. $\forall\alpha\forall a(\alpha \neq a)$.

In order to discover universes of sequences not subject to such limitations, we may either

(1) consider more general *primitive* concepts of choice sequence, or

(2) construct universes with the desired properties from lawless sequences and lawlike objects. (Examples: the universes $\mathcal{U}$, $\mathcal{U}^*_\beta$ in Chapter 4 of [T4]).

Here and in the sequel the term "universe" refers, rather loosely, either to a collection of sequences which corresponds to some specific (informally described) concept, or to a collection of sequences which can serve as "choice sequences" in some interpretation of a formal theory of choice sequences (such as $\underset{\sim}{\mathrm{CS}}$ or $\underset{\sim}{\mathrm{LS}}$), i.e. the objects that the choice variables range over. Thus e.g. we may speak about a $\underset{\sim}{\mathrm{CS}}$-universe for a collection of choice sequences satisfying the axioms of $\underset{\sim}{\mathrm{CS}}$.

Since continuity is, historically as well as conceptually, a key notion in the theory and practice of choice sequences, we will be primarily interested in universes satisfying certain continuity schemata.

Possibility (1) mentioned above has been investigated for example in [T1],[T2] (cf. also [T4], Appendix B), and mainly concerns fairly straightforward generalizations of LS where a choice sequence is viewed as a process of choosing at each stage a value and a (lawlike) restriction on future choices of values (lawlike means that the restrictions do not contain choice parameters); the restrictions are taken from a fixed collection $\mathcal{R}$.

Universes of this type do permit non-trivial overall restrictions, but are still subject to limitation (a), and might therefore rightly be termed *individualistic* ("anti-social" in [D,T]).

If the collection $\mathcal{R}$ of restrictions of an individualistic universe $\mathcal{A}$ is enumerable, then it seems natural to attempt to imitate elements of $\mathcal{A}$ by picking numerical values and restrictions in a lawless way. To be precise, if α is a lawless sequence, we use the value αx to code both a numerical value and a restriction from $\mathcal{R}$, such that if $\bar{\alpha}x$ codes an admissible sequence σ of values and restrictions (i.e. a possible initial segment for a sequence from $\mathcal{A}$), then

A) for each y $\bar{\alpha}x * \hat{y}$ codes an admissible continuation $\sigma * \langle\langle x, R\rangle\rangle$ of σ, and

B) each admissible continuation of σ is coded by $\bar{\alpha}x * \hat{y}$ for some y.

Let Π be the mapping assigning to each $\alpha \in$ LS the sequence of *values* coded by α; Π is called the projection mapping, and $\mathcal{A}' = \{\Pi\alpha : \alpha \in \mathrm{LS}\}$ the set of projections imitating $\mathcal{A}$. The study of such projections constitutes the subject of [D,T] and [T3].

Note that thus approach (1) leads to the study of a very special case of approach (2). To extend the imitation by projections to cases where $\mathcal{R}$ cannot be indexed by N, one needs to consider lawless sequences with values in other domains (e.g. lawlike sequences).

As noted in [D,T], the universe of projections A' will as a rule be a satisfactory imitation of A only if we restrict ourselves to a suitable language (e.g. the language of LS, with A' as the interpretation of the lawless variables). The need for such a restriction stems from the fact that a single initial segment of an A-sequence, (consisting of pairs of values and restrictions) may be coded by distinct initial segments of lawless sequences, thereby introducing inessential "intensional information". Because of the relevance of this phenomenon for the rest of this paper, we recall a very simple illustration: as a projection model imitating lawless 0-1 sequences (variables ε,η), we may take

$$A = \{\pi\alpha : \alpha \in \mathrm{LS}\} \text{ with } \pi = \lambda\alpha\lambda x.\mathrm{sg}(\alpha x).$$

We have $\forall\varepsilon\exists x\exists\alpha(\varepsilon=\Pi\alpha \wedge \alpha 0=x)$ in A', but x cannot be found continuously in ε. On the other hand, for the primitive concept of lawless 0-1 sequences $\forall\varepsilon\exists x$-continuity is always valid. However, relative to the language of $\underset{\sim}{\mathrm{LS}}^{\leq 1}$ (the theory of lawless 0-1 sequences) A' satisfies all axioms of $\underset{\sim}{\mathrm{LS}}^{\leq 1}$ (cf. [T4], 2.18, 3.23; [T3]).

1.3. Individualistic universes are still more or less direct generalizations of LS, but *social* (= non-individualistic) universes which are not subject to limitation (a), present new problems. An example of such a concept, intended as a model for the choice sequences of $\underset{\sim}{\mathrm{CS}}$ of [K,T] is sketched in appendix C of [T4] (and before in [T1], [T2]).

We cannot a priori expect to find a perspicuous axiomatization together with a convincing conceptual motivation for the axioms, as in the case of LS.

This presents us with a strong pragmatic argument for using the technique of projections of LS. The properties of the projections are derivable from $\underset{\sim}{\mathrm{LS}}$, and inasmuch the projections truly imitate the primitive concept studied, we can use them to back up our analysis of the primitive concept.

Note that establishing the correctness of the imitation can be delicate or difficult. Either one has a strong conceptual insight into the notion under consideration, in which case it also becomes possible to judge the success of the imitation, or one has a number of plausible key properties (axioms) which the imitation has to satisfy. In this case one may hope that the study of projections enables us to improve on the conceptual analysis of the concept to be imitated.

The project of imitating primitive notions of sequence by means of projections of LS, and thus in a sense *reducing* these primitive notions to lawless sequences may be compared to a similar reduction in set-theoretic foundations, where it is shown that the structures of actual mathematics (which do not a priori present themselves as set-theoretic structures) can be isomorphically represented by means of sets. The analogy is not complete: the set-theoretic representation is regarded as fully isomorphic (of course w.r.t. extensional properties only) to the original structure, without restriction to properties expressible in a particular language, though obviously limited to extensional properties. Restrictions to a particular language play a role in set-theoretic descriptions of e.g. hyperarithmetic sets, where one considers e.g. Δ^1_1-comprehension.

1.4. We now turn to the task of actually representing social notions of choice sequence by means of projections of LS. Let us first describe the concept studied in this paper, which may be regarded as a typical example.

Let $\{F^*_n : n \in N\}$ be a countable set of "local" continuous mappings in the form

$$(F^*_n \varepsilon)(x) = F_n(\bar{\varepsilon}x, x)$$

(F_n a lawlike two-place function determining F^*_n).
Relative to $\{F^*_n : n \in N\}$ we now describe the universe of choice sequences U as follows. An arbitrary element ε of U is generated in stages: at stage 0, we know that $\varepsilon = F^*_{n_0}\varepsilon_0$; we choose values $\varepsilon_0 0$, $\varepsilon_0 1, \ldots$ at stage $1, 2, \ldots$; at a certain stage x_1+1 we may decide to make ε_0 dependent on another sequence ε_1 by choosing further values of ε_0 according to

$$\varepsilon_0(x_1+y) = F_{n_1}(\bar{\varepsilon}_1(x_1+y), x_1+y) ,$$

i.e. for stages $x_1+y+1 \geq x_1+1$ ε_0 behaves like $F^*_{n_1}\varepsilon_1$; at stage x_1+1 , ε_1 itself has not yet been made dependent on another sequence, but at a later stage $x_2+1 > x_1+1$ ε_1 may in turn be made to behave like $F^*_{n_2}\varepsilon_2$ for all stages $x_3 \geq x_2+1$, etc. etc.

If we now attempt to generalize the method of projections from the individualistic to the social case we meet with an obstacle: in order to imitate the generation of an $\varepsilon \in U$, described above, by a projection of a lawless sequence α, α must not only tell us what values to choose for $\varepsilon_0 0, \varepsilon_0 1, \ldots$, but α must also inform us when further choices are to be made dependent on ε_1, and when ε_1 in turn is made dependent on $\varepsilon_2, \ldots$ etc. Since α should contain all the information on ε, α must necessarily also contain all information on $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$.

In fact, *any* two sequences ε, η of U at some stage may become dependent, either ε on η, or η on ε, or both ε and η on a third sequence ε', and thus a lawless α used in the projection should code information about all possible projected sequences, i.e. a single α should be capable of generating the whole projected universe.

The solution of the difficulty is suggested by the observation that it is possible to construct a countable $\underset{\sim}{\text{LS}}$-universe from a single lawless sequence α; this universe is $\{n*(\alpha)_n : n \in N\}$, where for $(\alpha)_n$ we may take $\lambda y.\alpha j(n,y)$ or $\lambda y.(\alpha y)_n$ (cf. [T4], 3.19-22). In this universe it is possible to refer to individual sequences "by name" : the index n serves as the name of $n*(\alpha)_n$.

This suggests that we look for projections π_n such that for any single lawless α the universe $U_\alpha = \{\pi_n\alpha : n \in N\}$ imitates the social universe U sketched above.

1.5. Principal results.

For the universe $\mathcal{U}$ described in the preceding subsection it is indeed possible to construct such projection models $\mathcal{U}_\alpha$, which can be described in a very "pictorial" way (see section 2) and which can be shown to be adequate in the following sense:

(i) it is intuitively evident that the $\mathcal{U}_\alpha$ really imitate $\mathcal{U}$;

(ii) relative to statements in the language $\mathcal{L}$ of $\underset{\sim}{LS}$, $\mathcal{U}_\alpha$ can be shown (in $\underset{\sim}{LS}$) to satisfy a set of axioms which permits elimination of variables ranging over $\mathcal{U}_\alpha$.

More precisely, let $\varepsilon, \varepsilon', \eta$ be used for choice variables of $\mathcal{L}$, and let α, β be distinct variables (not in $\mathcal{L}$) ranging over LS. Let A be any sentence of $\mathcal{L}$, and A^α its interpretation in $\mathcal{U}_\alpha$ (i.e. the choice variables of $\mathcal{L}$ are interpreted as ranging over $\mathcal{U}_\alpha$). The set of axioms determines a theory $\underset{\sim}{T}$ over $\mathcal{L}$, for which an elimination mapping σ may be defined similar to the elimination mappings for $\underset{\sim}{LS}$ and $\underset{\sim}{CS}$ (cf. [T4], 3.13, page 40 and 5.3, page 78) such that

$$\underset{\sim}{T} \vdash \sigma(A) \leftrightarrow A.$$

Then also $\underset{\sim}{LS} \vdash \sigma(A) \leftrightarrow A^\alpha$ (because $\underset{\sim}{T}$ was valid for $\mathcal{U}_\alpha$) and therefore by the elimination theorem for $\underset{\sim}{LS}$: $\underset{\sim}{IDB}_1 \vdash \sigma(A) \leftrightarrow \tau(\forall\alpha A^\alpha)$ (τ the elimination mapping for $\underset{\sim}{LS}$), hence $\underset{\sim}{T} \vdash A \Rightarrow \underset{\sim}{IDB}_1 \vdash \sigma(A)$. (This method of proof is at least as simple as a direct proof of the elimination theorem for $\underset{\sim}{T}$ would have been.) Thus, relative to $\mathcal{L}$, the properties of $\mathcal{U}_\alpha$ are fully characterized relative to the properties of lawlike objects.

There is one schema shown to be valid for $\mathcal{U}_\alpha$ which is not immediately obvious for $\mathcal{U}$, an analogue of $\forall\alpha\exists\beta$-continuity for $\underset{\sim}{CS}$:

$$(1) \qquad \forall\varepsilon\exists\eta A(\varepsilon,\eta) \rightarrow \exists e\in K \forall\varepsilon A(\varepsilon, F^*_{e(\varepsilon)}(\varepsilon)) \ .$$

If we think of a universe closed under some set of continuous operations $\{\Gamma_i : i\in I\}$ the analogue would become

$$(2) \qquad \forall\varepsilon\exists\eta A(\varepsilon,\eta) \rightarrow \exists e\in K \exists\xi\in N\to I \forall\varepsilon A(\varepsilon, \Gamma_{\xi(e(\varepsilon))}(\varepsilon)) \ .$$

If $\{\Gamma_i : i\in I\}$ contains *all* inductively defined continuous functionals, the schema (2) is easily seen to be equivalent to $\forall\alpha\exists\beta$-continuity.

The proof of the validity of (1) lends support to the remark in [T4](A6, page 137) on the inconclusiveness of Myhill's counterexample to $\forall\alpha\exists\beta$-continuity : the counterexample tacitly assumed the universe of choice sequences to be closed under certain highly non-extensional operations (introduced by the actions of the "creative subject"), whereas our result seems to point to the conclusion that all conditions of closure under certain continuous operations really have to be built a priori into the construction of the universe. Furthermore we observe

(iii) The collection of models $\{\mathcal{U}_\alpha : \alpha \in LS\}$ can be made into a sheaf model over Baire space (see 8.7).

(iv) It looks as if the proofs given in this paper do not readily generalize to more complex cases without losing their intuitive appeal, because of the increasing technical complexities. However, the intuitive directness of the present approach and the paradigmatic value of $\mathcal{U}$ make an independent treatment of this special case appear worthwhile. (The first author is at present working on an alternative treatment which is less direct but promises much wider applicability.)

(v) Our proofs do not essentially depend on bar-induction for LS; the extension principle ([T4], 2.11, page 20) is enough. (The extension principle states that each continuous operation defined on LS is also defined on all sequences.)

$\mathcal{U}_\alpha$ is described in the next section; in 2.8 , the reader will find an outline of the rest of the paper.

2. DESCRIPTION OF THE MODEL

2.1. The models $\mathcal{U}_\alpha = \{\pi_n \alpha : n \in N\}$ to be described below are obtained by applying lawlike projections π_n to a single lawless sequence α. As a matter of convenience however, we shall use three parameters α, β, γ instead; we may think of α, β, γ either as distinct lawless sequences or as three independent sequences extracted from a single lawless sequence δ (e.g. according to $\delta = \nu_3(\alpha, \beta, \gamma) = \lambda x.\nu_3(\alpha x, \beta x, \gamma x)$, ν_3 coding N^3 onto N) ; relative to the language considered this yields equivalent theories. Thus we write $\mathcal{U}_{\alpha,\beta,\gamma}$ ($= \mathcal{U}_\delta$) for the universe of projections to be constructed from α, β, γ.

2.2. In the construction of $\mathcal{U}_{\alpha,\beta,\gamma}$, α serves as a source of numerical values, β governs the introduction of relations of dependence between different elements of the universe, and γ regulates the application of continuous mappings from a set $\{F_n^* : n \in N\}$.

Below we shall first introduce the "carriers", and describe in pictorial terms their behaviour. The carriers themselves behave like a model of the "free sequences" ([T4], C6, p.158), except that we have to guarantee that all initial segments can occur; this may be done as follows : if $\{\sigma_n : n \in N\}$ is the set of carriers (defined from α, β, γ) below, we take as our imitation (by projection) of the free sequences the collection $\{\Delta_n \sigma_m : n, m \in N\}$, where Δ_n is the operation given by

$$(\Delta_n \varepsilon)(x) = \begin{cases} (n)_x & \text{if } \mathrm{lth}(n) > x \\ \varepsilon x & \text{otherwise} \end{cases}$$

We abbreviate

$$n|\varepsilon \equiv_{def} \Delta_n \varepsilon$$

Free sequences are such that individually they behave like lawless sequences; however, they can have a non-trivial relationship to each other, namely coinciding from a certain point onwards. Thus we think of a free sequence ε as a process of generating values $\varepsilon 0$, $\varepsilon 1$, $\varepsilon 2, \ldots$ while at any stage (x+1) we may decide to let $\varepsilon(x+y)$ coincide with $\eta(x+y)$ for all y , η another free sequence, and so on.

Later, by introducing "dressed carriers" constructed from the carriers, we guarantee closure under the operations of $\{F^*_n : n \in N\}$.

2.3. Carriers.

"val" is a function which serves to extract many values from a single number, e.g. we may take

$$\mathrm{val}(x,y) = (y)_x$$

("value of y at x"). Any sequence ξ gives rise to an infinite sequence of sequences via "val" :

$$(\xi)_x \equiv \lambda y.(\xi y)_x \equiv \lambda y.\mathrm{val}(x,\xi y).$$

We call the sequences $(\alpha)_n$, $n \in N$ basis sequences. Below we shall introduce a function r ; relative to this function we define

$$\sigma_n \equiv \lambda x.\mathrm{val}(r(n,\bar{\beta}(x+1)),\alpha x) \ , \ n \in N \ ;$$

the sequences σ_n are called the *carriers*; σ_n is the n^{th} carrier, carrier n or carrier with index n.

2.4. Root, root-function, bundle. Let p be a pairing function from N^2 onto $N-\{0\}$, with inverses p_1,p_2 (e.g. $p(x,y) = 2^x(2y+1)$). The action of the *root-function* r may now be described as follows

$$r(n,0) = n$$
$$r(n,m*\langle x\rangle) = \begin{cases} r(p_2x,m) & \text{if } x \neq 0 \wedge r(p_1x,m)=r(n,m) \\ r(n,m) & \text{otherwise.} \end{cases}$$

Intuitively $r(n,\bar{\beta}x) = m$ expresses that at stage x and at all later stages the values of σ_n are going to be identified with the values of σ_m. Let us try to develop an intuitive picture of the behaviour of the carriers first.

(a) At any stage x+1, the value of σ_n at x coincides with the value of a certain basis sequence, namely the basis sequence $(\alpha)_{r(n,\bar{\beta}(x+1))}$.

The carrier with index $r(n,\bar{\beta}x)$ is called the *root* of carrier n *at stage* x; the carriers with the same root at stage x form an *x-bundle*.

Note:

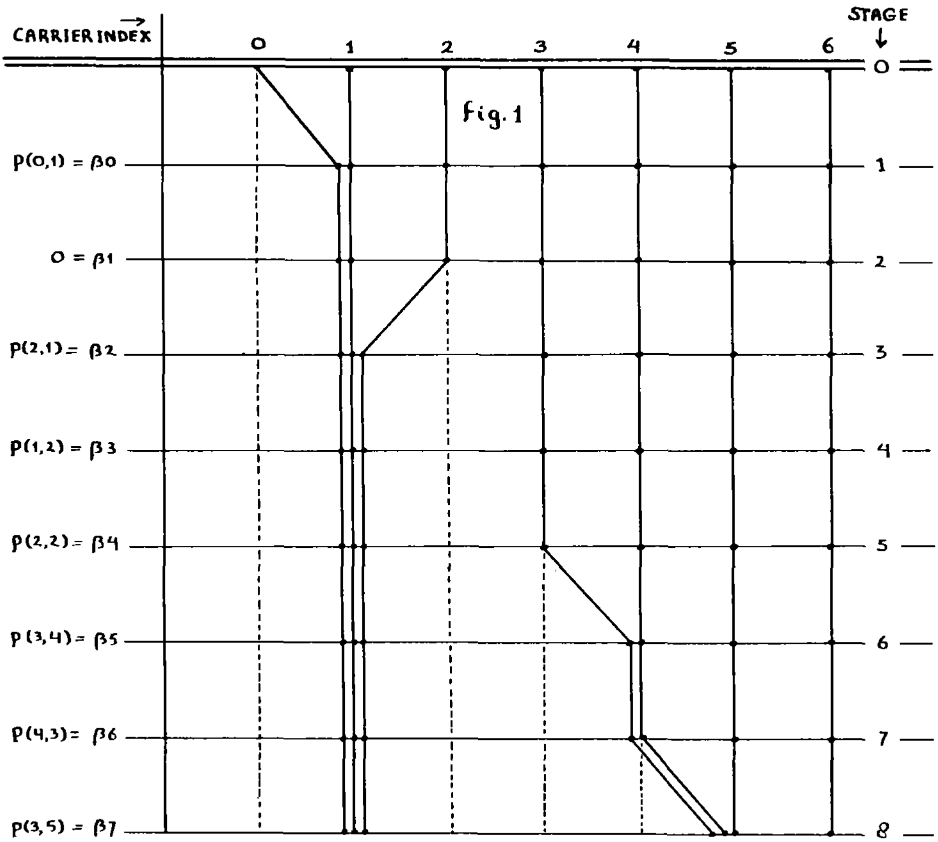
(b) If $\beta x \neq 0$, then at stage x+1 the x-bundle containing $\sigma_{p_1\beta x}$ will be joined to the x-bundle containing $\sigma_{p_2\beta x}$; the root of the resulting (x+1)-bundle is the root of $\sigma_{p_2\beta x}$ at stage x.

(c) With each passage of a stage to the next, the set of indices of roots can only *diminish* . With each passage, at most *one* bundle gets assigned to the root of another bundle.

As a result, it is now easy to develop some pictorial intuition as to how a typical identification pattern over a number of stages proceeds. Take e.g. a β starting with

$$\bar{\beta}8 = \langle p(0,1),0,p(2,1),p(1,2),p(2,2),p(3,4),p(4,3),p(3,5)\rangle .$$

We can now make a picture (fig. 1) in which the vertical lines correspond to basis sequences; horizontal lines represent stages; basis sequences, where not coinciding with a carrier are indicated by interrupted vertical lines.



The carriers coincide over parts of their length with basis sequences, but may jump from one basis sequence to another. No carrier ever jumps back to a former basis, nor does it jump to a basis deserted before by another carrier; the carriers once in the same bundle stay together.

In our picture, at stage 8 the carriers with indices 0,1,2,...,6 are grouped in bundles $\{0,\underline{1},2\}$, $\{3,4,\underline{5}\}$, $\{\underline{6}\}$ (the root-index of each bundle is underlined).

2.5. Closure under continuous functionals.

$\{F^*_n : n\in N\}$ is a set of functionals which acts pointwise:

$$F_n^*\alpha = \lambda x.F_n(\alpha x, x) \quad ,$$

contains the identity and which is closed under composition:

$$F_{n*m}^*\alpha \;=\; F_n^* \circ F_m^*(\alpha) \;=\; F_n^*(F_m^*\alpha).$$

We want $\mathcal{U}_{\alpha,\beta,\gamma}$ to be closed under the F_n^* and the Δ_n. Of course we might have absorbed the Δ_n simply among the F_n^*, but since the Δ_n are of a special nature (they change initial segments only) we found it slightly more convenient to deal with them separately.

2.6. <u>The function d ; dressed carriers</u>.

$$d(m,n,0,0) = m,$$
$$d(m,n,v*\hat{x},w*\hat{y}) = \begin{cases} d(m,n,v,w) & \text{if } r(n,v) = r(n,v*\hat{x}), \\ d(m,n,v,w)*\hat{y} & \text{otherwise} . \end{cases}$$

d serves to determine the index of the continuous functional from the set $\{F_n^* : n\epsilon N\}$ to be applied to the σ_n ; the resulting sequences $\rho_{n,m}$ are called *dressed carriers* and are given by

$$\rho_{j(n,m)}(\alpha,\beta,\gamma)(x) \equiv \rho_{n,m}(\alpha,\beta,\gamma)(x) \equiv$$
$$\equiv F_{d(n,m,\bar{\beta}(x+1),\bar{\gamma}(x+1))}(\mathrm{val}(r(n,\bar{\beta}(x+1)),\alpha x),x).$$

We shall often simply write ρ_n or $\rho_{n,m}$ instead of $\rho_n(\alpha,\beta,\gamma)$, $\rho_{n,m}(\alpha,\beta,\gamma)$.

Two dressed carriers $\rho_{n,m}$ and $\rho_{n,m'}$ belong to the same carrier σ_n. At each stage we know that further values of $\rho_{n,m}$ are going to be obtained by application of a certain F_k^* to the carrier σ_n ; when the carrier n jumps at a later stage to a basis sequence n', the F_k^* is extended to $F_{k*k'}^* = F_k^* \circ F_{k'}^*$ and values of $\rho_{n,m}$ now are obtained by application of $F_{k*k'}^*$ to $\sigma_{n'}$, until the next jump.
As long as there is no jump, the F_k cannot be extended! Note that

(d) at stage 0 it is already specified that F_m^* has to be applied in any case - this is done in order to ensure closure of $\mathcal{U}_{\alpha,\beta,\gamma}$ under F_n^*.

We can make a picture of the history of dressed carriers in a similar way as before; now we label the nodes with the index of the operations to be applied at that stage, the "dressing".
The dressing is governed by both β and γ and regulated via the d-function.

A typical diagram, representing the history over 4 stages of ρ_{n_i,m_i} (i = 0,1,2,3) is given in fig. 2 .

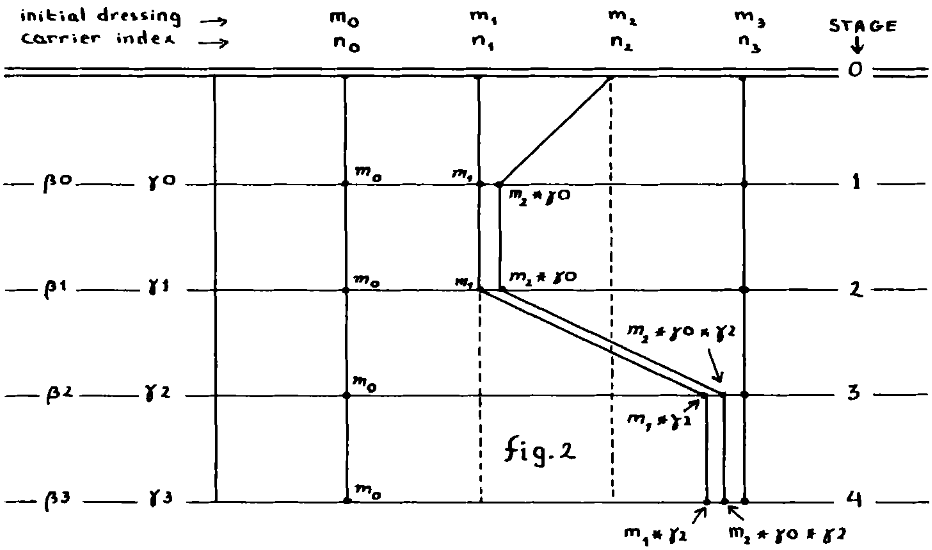


fig. 2

At stage 4 we have a grouping in bundles

$\{n_0\}$ $\{n_1,n_2,n_3\}$

with corresponding dressings

$\{m_0\}$ $\{m_1{*}\gamma 2,\ m_2{*}\gamma 0{*}\gamma 2,\ m_3\}$.

The dressings of sequences in the same bundle may differ because the various sequences joined the bundle at different stages.

2.7. The projections.
The elements $\pi_n(\alpha,\beta,\gamma)$ (π_n for short) of $\mathcal{U}_{\alpha,\beta,\gamma}$ are given by

$$\pi_n \equiv \Delta_{j_1 n}(\rho_{j_2 n}(\alpha,\beta,\gamma)).$$

This guarantees that all initial segments will occur among the π_n. Of course, we might have assumed the Δ_n to occur among the F^*_n ; then if $\Delta_n = F^*_{f(n)}$, any dressed carrier $\rho_{u,f(n)*m}$ would automatically have an initial segment n. On the other hand, as remarked before, the Δ_n play a special rôle.

2.8. Outline of the rest of the paper. The reader should compare the universe $\{\pi_n : n\in N\}$ thus constructed with our description of the universe in the introduction (1.4); the universeof projections may indeed be said to imitate the universe described there. In order to give a precise meaning to the term "imitate", we need first of all for n-tuples of projections a suitable concept of *restriction* or *available information* at some stage.

Consider a sequence of projections $\vec{\pi}(\delta) = \langle \pi_{n_1}(\delta),\ldots,\pi_{n_p}(\delta)\rangle$ at stage x, and assume for simplicity $x \geq \max\{lth(j_1 n_1),\ldots,lth(j_1 n_p)\}$, i.e. x is greater than or equal to the maximum length of initial segments specified in advance. Then the available information on possible continuations of $\vec{\pi}$ at stage x consists of

(1°) the grouping of the carriers σ_{m_i} ($m_i = j_1 j_2 n_i$ for $1 \leq i \leq p$) underlying π_{n_i} into bundles, and

(2°) the dressings, i.e. the continuous functionals to be applied from stage x onwards.

Note that $\bar{\pi}_n x$, and the restriction at stage x on the future of π_n is completely determined by $\bar{\alpha}x, \bar{\beta}x, \bar{\gamma}x$, or $\bar{\delta}x$ (with $\delta = \nu_3(\alpha,\beta,\gamma)$). Therefore we can use the expression "the *value of* $\bar{\pi}_n x$ *(or* $\pi_n y$ *for* $y < x$*) at* $v = \bar{\delta}x$" without ambiguity; similarly, "*the restriction of* π_n *at* v", meaning the restriction on (continuations of) $\pi_n(\delta)$ at stage lth(v) for any $\delta \in v$.

Precise definitions of identification patterns, restrictions and equality between restrictions are given in section 3. Section 4 introduces a natural ordering between restrictions : $R_1 \leq R_2$ (R_2 is a stronger restriction, i.e. contains more information, than R_1), and it is shown that for a sequence $\vec{\pi}$ with restriction R_1 at v it is always possible to continue v so as to let $\vec{\pi}$ reach R_2 at some $v*w$.

If finite sequences $\vec{\pi}$ and $\vec{\pi}'$ of projections obey the same restrictions at v and v' respectively, then to each $v*\hat{x}$ there is a y such that the restriction of $\vec{\pi}$ at $v*\hat{x}$ coincides with the restriction of $\vec{\pi}'$ at $v'*\hat{y}$. Also, to each extension $\vec{\pi} \cup \{\pi_n\}$ we can find an extension $\vec{\pi}' \cup \{\pi_m\}$ such that the restrictions of $\vec{\pi} \cup \{\pi_n\}$ at v and $\vec{\pi}' \cup \{\pi_m\}$ at v' coincide, and $\bar{\pi}_n(lth\ v)$ at $v = \bar{\pi}_m(lth\ v')$ at v'.

The next crucial property is the "overtake-property" in section 5. Let us call $R=R0,R1,R2,\ldots$ an admissible sequence if there is some sequence ξ such that for some π_n R_x is the restriction at $\bar{\xi}(x+1)$ for all x.

If $R \le \mathcal{R}0$, we can find another admissible sequence $\mathcal{R}'$ with $\forall x(\mathcal{R}'x \le \mathcal{R}x)$, $\mathcal{R}'0=R$, $\mathcal{R}'z= \mathcal{R}z$ for all sufficiently large z : $\mathcal{R}'$ starts at a weaker restriction but overtakes $\mathcal{R}$ (the overtake-property).

Section 6 deals with the assignment of values to the projections. In section 7, the restriction on the language considered becomes essential. Let $\vec{\pi},\vec{\pi}'$ be n-tuples of projections from δ,δ' respectively. Suppose: (a) $\vec{\pi},\vec{\pi}'$ satisfy the same restriction at stage x, x greater than the length of any initial segment specified in advance for $\vec{\pi}$ or $\vec{\pi}'$, (b) if π_n, $\pi_{n'}$ are corresponding members from $\vec{\pi},\vec{\pi}'$ respectively then $\overline{\pi_n(\delta)}(x) = \overline{\pi_{n'}(\delta')}(x)$; under these assumptions $\vec{\pi}$ and $\vec{\pi}'$ are permutable, i.e. for all formulae A of the language considered

$$\forall\alpha\epsilon\bar{\delta}x\ A(\vec{\pi}(\alpha)) \leftrightarrow \forall\alpha'\epsilon\bar{\delta}'x\ \ A(\vec{\pi}'(\alpha')).$$

Compared with the primitive notion being modelled, the projections may contain extra (intensional) information at stage x since an initial segment of values $\bar{\pi}_n(x)$ and a sequence $\bar{\mathcal{R}}x = \langle \mathcal{R}0,\mathcal{R}1,\dots,\mathcal{R}(x-1)\rangle$ can be obtained by projection from *different* initial segments of lawless sequences; in general, operations on the projections might possibly refer to this intensional information (i.e. to the lawless sequence being projected). Also, for the primitive concept as well as for the projections imitating this concept, operations which can be applied at stage x might depend not only on the final restriction $\mathcal{R}(x-1)$, but on the whole sequence $\bar{\mathcal{R}}x$. The theorem on permutable n-tuples shows that relative to a suitable restricted language both possible intensional effects do not occur.

The final section 8 is then devoted to the derivation of several schemata for the model described above; the essential tools being the "overtake property" and the theorem on permutability.

The preceding sketch of the contents of sections 3-7 should enable the reader to follow the arguments in section 8 (granting a few details) - so if he is interested in the main ideas only, he may continue directly with the final section.

2.9. <u>Connection with forcing</u>. For a reader familiar with forcing but not with the theory of lawless sequences, it might be helpful to keep the following connection in mind. Lawless sequences are closely related with strong forcing, where "$\forall\alpha\epsilon vA\alpha$"

corresponds to "the finite information (on a sequence α), coded by v, forces A", and individual lawless sequences play the rôle of generic sequences. The study of projections then amounts to a reduction of certain other types of forcing to strong forcing. Note however, that our treatment is entirely intuitionistic.

3. PATTERNS, DRESSINGS AND RESTRICTIONS

3.1. Definition of pattern.

A *pattern* P of *length* n is (given by) an equivalence relation on $\{0,1,\ldots,n-1\}$. We shall write $E_P(i)$ for the equivalence class to which i belongs. We use $P,P',P'',P_1,\ldots,Q,Q'\ldots$ for patterns. If P is a pattern, we define "n *has pattern* P" by

$$n \in P \equiv_{def} \text{lth}(P)=\text{lth}(n) \wedge \forall i,j<\text{lth}(n)(j \in E_P(i) \leftrightarrow (n)_i=(n)_j).$$

Each finite sequence n has a pattern denoted by P(n).

We say that Q *is coarser than* P, or $P \leq Q$ if each equivalence class of P is contained in an equivalence class of Q, and P,Q are of equal length.

Let X be a finitely indexed set of carriers, with indices $v_0,v_1,\ldots,v_n$; let $v = \langle v_0,v_1,\ldots,v_n\rangle$. At each stage x, X has a pattern, the pattern of the sequence

$$\hat{r}(v,\bar{\beta}x) = \langle r(v_0,\bar{\beta}x),\ldots,r(v_n,\bar{\beta}x)\rangle \ ;$$

for this pattern we write $P(v,\bar{\beta}x)$. We put

$$e(\bar{\beta}x,i,v) = \{v_j : r(v_i,\bar{\beta}x)=r(v_j,\bar{\beta}x)\},$$
$$E(\bar{\beta}x,i,v) = \{j : r(v_i,\bar{\beta}x)=r(v_j,\bar{\beta}x)\}.$$

3.2. Definition of dressing and restriction.

A *dressing* D is nothing but a finite sequence of (indices of) elements of $\{F^*_n : n \in N\}$.

A *restriction* is a pair $\langle P,D\rangle$, P a pattern, D a dressing of the same length; the length of P and D is called the *length* of the restriction.

The "intensional" information at stage x concerning possible continuations of finite sequences of dressed carriers is completely given by a restriction.

For a finite sequence of dressed carriers

$$\rho_{v_0,w_0},\ldots,\rho_{v_n,w_n}$$

with $\langle v_0,\ldots,v_n\rangle = v$, $\langle w_0,\ldots,w_n\rangle = w$, the pattern at stage x is given by $P(v,\bar{\beta}x)$, the dressing is specified by a finite sequence $D(v,w,\bar{\beta}x,\bar{\gamma}x)$ satisfying

$$D(v,w,\bar{\beta}x,\bar{\gamma}x)_i = d(v_i,w_i,\bar{\beta}x,\bar{\gamma}x)$$

Equality of restrictions is given by

$$\langle P,D\rangle = \langle P',D'\rangle \equiv_{def} (P=P') \wedge (D=D')$$

We shall use $D,D',D_1,\ldots$ for dressings, $R,R_1,\ldots,R',\ldots$ for restrictions.

3.2. Successors of patterns and restrictions.

A pattern P' is said to be a *successor* of $P(v,\bar{\beta}x)$, if $P' = P(v,\bar{\beta}x*\hat{y})$ for some y.

P' is either equal to P, or is obtained from P' by identification of two equivalence classes. If at the transition from stage x to stage x+1 there is no jump, or a bundle X with elements in v jumps to a bundle outside v, the pattern is unchanged; if on the other hand a bundle with elements in $e(\bar{\beta}x,i,v)$ jumps to a bundle with elements in $e(\bar{\beta}x,j,v)$, the corresponding equivalence classes are identified.

A *successor* of a restriction $\langle P(v,\bar{\beta}x), D(v,w,\bar{\beta}x,\bar{\gamma}x)\rangle$ consists of a pair $\langle P',D'\rangle$, P' a successor of P, and D' related to $D = D(v,w,\bar{\beta}x,\bar{\gamma}x)$ as follows: if the elements of $e(\bar{\beta}x,i,v)$ jump on the transition from stage x to stage x+1, there is a z such that $(D')_j = (D)_j*z$ for all $j \in E(\bar{\beta}x,i,v)$, $(D')_j = (D)_j$ for $j \notin E(\bar{\beta}x,i,v)$; if there is no jump, $D' = D$.

In fig.3 we have illustrated two possibilities for successors to a pattern $\langle P,D\rangle$, $D = \langle m_0,m_1,\ldots,m_5\rangle$.

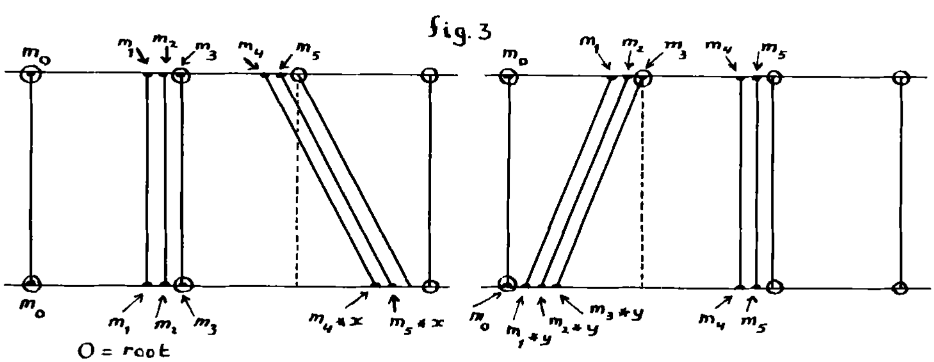


fig.3

Note: The restriction R at stage x for a finite sequence X of dressed carriers (of length n say) generated from α,β,γ is determined by $\bar{\beta}x,\bar{\gamma}x$; if another sequence of dressed carriers Y of length n is generated from α',β',γ', and for the restriction R' at stage x we have R = R', then the successor-restrictions of R and R' are in one-to-one correspondence, and to a continuation $\bar{\beta}x*\hat{u}$, $\bar{\gamma}x*\hat{v}$ determining a successor R_1 of R for X we can always find effectively u',v' such that $\bar{\beta}'x*\hat{u}'$, $\bar{\gamma}'x*\hat{v}'$ determine R_1 at stage x+1 for Y (in fact, one can take v' = v).

3.3 Extendability.

Assume

$$\langle P(v,t),D(w,v,t,s)\rangle = \langle P(v',t'),D(w',v',t',s')\rangle$$

with

$$v = \langle n_0,\ldots,n_p\rangle \ , \ v' = \langle n_0',\ldots,n_p'\rangle \ ,$$
$$w = \langle w_0,\ldots,w_p\rangle \ , \ w' = \langle w_0',\ldots,w_p'\rangle \ ,$$
$$\mathrm{lth}(t) = \mathrm{lth}(t'),$$

and let

$$R = \langle P(v*\hat{z}_1,t) \ , \ D(w*\hat{z}_2,v*\hat{z}_1,t,s)\rangle.$$

Then we can find u_1,u_2 such that

$$R = \langle P(v'*\hat{u}_1,t') \ , \ D(w'*\hat{u}_2,v'*\hat{u}_1,t',s')\rangle$$

as follows.

a) Assume that ρ_{z_1,z_2} at stage lth(t) does not belong to a bundle with elements in v. Then choose u_1 such that u_1 does not belong to any bundle with elements in v', and such that $r(u_1,t') = u_1$; let u_2 be equal to the dressing of z_1 at stage lth(t), i.e. $u_2 = d(z_2,z_1,t,s)$.

b) If on the other hand ρ_{z_1,z_2} at stage lth(t) *does* belong to a bundle with elements in v, say $r(z_1,t) = r(v_i,t)$, take $u_1 = r(v_i',t)$ and again $u_2 = d(z_2,z_1,t,s)$.

Thus given two finite sets of dressed carriers with equal restrictions, possible extensions of the sets correspond one-to-one w.r.t. restrictions.

4. COMPARISON OF RESTRICTIONS

4.1. Definition of expansion.

Let D_1, D_2 be two dressings. D_1 *expands to* D_2 (D_1 exp D_2) if D_1 is of the form $<d_0,\ldots,d_p>$, and D_2 of the form $<d_0*d'_0,\ldots,d_p*d'_p>$. In this case

$$D_2\backslash D_1 = <d'_0,\ldots,d'_p>$$

is called the *difference* of D_2 and D_1.

4.2. Definition. (Comparability of restrictions).

We say that the restriction R_1 *is weaker than* R_2 ($R_1 \leq R_2$, or R_2 *stronger than* R_1) iff

a) $R_1 = <P_1,D_1>$, $R_2 = <P_2,D_2>$, $P_1 \leq P_2$,

b) D_1 exp D_2, and

c) $P_1 \leq P(D_2\backslash D_1)$.

4.3. Convention. We shall use script letters $\mathcal{P},\mathcal{D},\mathcal{R}$ for infinite sequences of patterns, dressings, restrictions respectively.

Finite sequences of patterns,dressings or restrictions may then be indicated as $\bar{\mathcal{P}}z,\bar{\mathcal{D}}z,\bar{\mathcal{R}}z$.

4.4. Definition. $\bar{\mathcal{P}}z$ ($\bar{\mathcal{R}}z$) is said to be *admissible* if (effectively) we can obtain $\mathcal{P}(y+1)$ ($\mathcal{R}(y+1)$) as a successor to $\mathcal{P}y$ ($\mathcal{R}$ y) for each y such that $0\leq y<z$. A sequence $\mathcal{P}$($\mathcal{R}$) is said to be *admissible* if $\bar{\mathcal{P}}z$ ($\bar{\mathcal{R}}$ z) is admissible for all z.

4.4. Lemma. If $R_1 \leq R_2$ there is an admissible $\bar{\mathcal{R}}(z+1)$ with $z \leq 2\mathrm{lth}(R_1)$, $\mathcal{R}0=R_1$, $\mathcal{R}z=R_2$.

Proof. We describe the construction informally. The idea is to start from R_1 = (P_1,D_1) and to expand *first* D_1 to D_2 in a number of steps, *then* to transform (P_1,D_2) in a number of steps to $(P_2,D_2) = R_2$. In detail, we proceed as follows.

Assume $D_1 \neq D_2$, and let i be the least j such that $(D_2\backslash D_1)_j \neq 0$. Let P_1 = $P(v,t)$, and construct a successor by letting the carrier with index $(v)_i$ jump to a root outside v, and add $(D_2\backslash D_1)_i$ to the dressing $(D_1)_i$ of this carrier, and similarly for all carriers $(v)_j$ with $j \in E_{P_1}(i)$. The resulting restriction has the same pattern, but its dressing differs from D_2 at most for a $j > i$.

We may thus continue until we have reached a position with (P_1, D_2); this requires at most $lth(P_1)$ steps, since D_2 can differ from D_1 at most at $lth(P_1)$ places.

Now we keep the dressing constant and start changing the pattern. Let

$$j = \min_i [E_{P_1}(i) \neq E_{P_2}(i)]$$

$$k = \min_i [E_{P_1}(i) \neq E_{P_1}(j) \wedge E_{P_2}(i) = E_{P_2}(j)]$$

If $P_1 = P(v,s)$, we construct a successor P' by letting the bundle of carriers with elements in $E_{P_1}(j)$ jump to the bundle with carriers in $E_{P_1}(k)$; P' has less equivalence classes than P_1, the dressing is kept constant. Thus we reach (P_2, D_2) from (P_1, D_2) in at most $lth(R_1)$ steps.□

5. THE OVERTAKE-PROPERTY

Our next aim is the following

5.1. Proposition. Let $\mathcal{R}$ be admissible, R a restriction such that $R \leq \mathcal{R}0$, then we may construct, primitively recursively in R, an admissible sequence $\mathcal{R}'$ such that

a) $\mathcal{R}'0 = R$

b) $\forall x(\mathcal{R}'x \leq \mathcal{R}x)$

c) $\mathcal{R}'z = \mathcal{R}z$ for all $z \geq 3(lth\ \mathcal{R}0)^2 = 3(lth\ R)^2$.

This is called the "overtake-property". $\mathcal{R}'$ starts "below" $\mathcal{R}$, but overtakes $\mathcal{R}$ in at most $3(lth\ R)^2$ steps. We need a number of definitions and lemmata.

5.2. Definitions. We put

$ch(\mathcal{R},m) \equiv m>0 \wedge \mathcal{R}(m \dot{-} 1) \neq \mathcal{R}m$

("$\mathcal{R}$ changes at m")

$CH(\bar{\mathcal{R}}z) \equiv$ cardinal of $\{m : ch(\mathcal{R},m) \wedge m<z\}$

("changeability of $\bar{\mathcal{R}}z$")

$cl(\mathcal{R},n,z) \equiv n<lth(\mathcal{R}) \wedge (\mathcal{D}0)_n = (\mathcal{D}z)_n$

where $\mathcal{R}x = \langle \mathcal{P}x, \mathcal{D}x \rangle$, for all x. $cl(\mathcal{R},n,z)$, "n is closed up to stage z", means that no dressing is added between stage 0 and stage z at the n^{th} place. If $\neg ch(\mathcal{R},m)$, we say that $\mathcal{R}$ has a *repetition* at m.

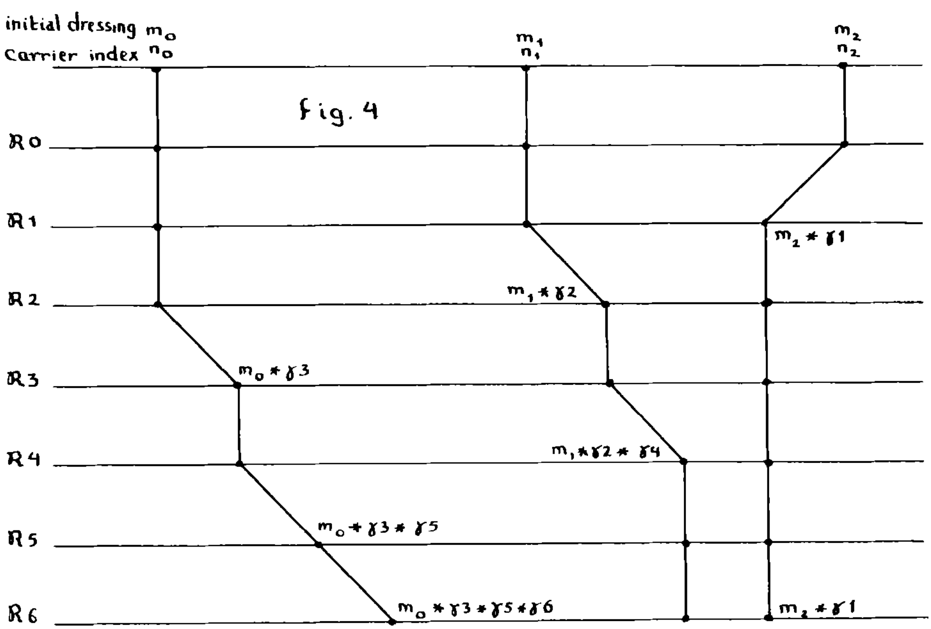
5.3. <u>Lemma</u> (lowering of CH). Let R have a constant pattern, and let $\bar{R}(z+1)$ admissible. Then there is an R' such that

a) $R'0 = R0$, $R'z = Rz$, $\forall x \le z(R'x \le Rx)$,

b) $\bar{R}'(z+1)$ admissible ,

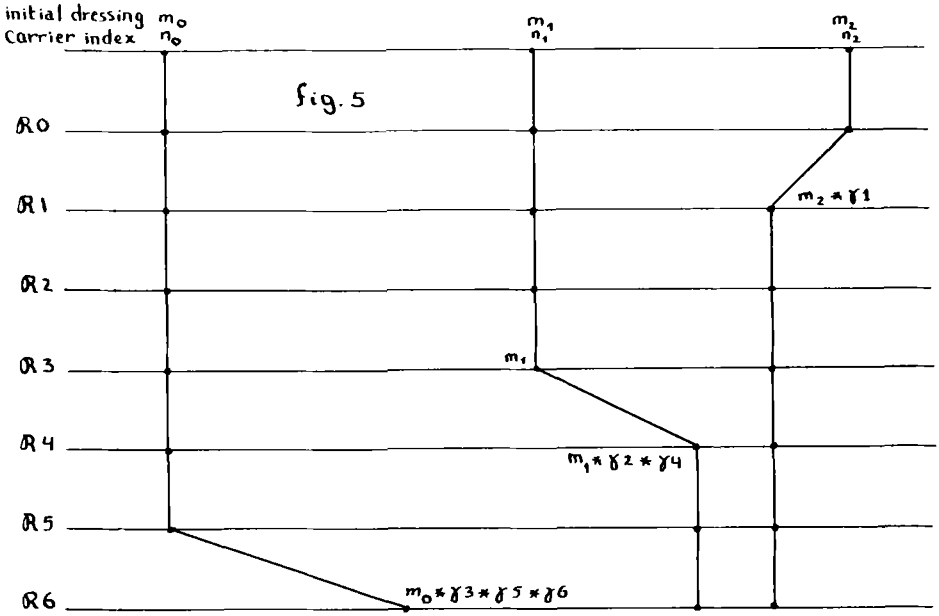
c) $CH(\bar{R}'(z+1)) \le \mathrm{lth}(R0) - \mathrm{cardinal}\{n : \mathrm{cl}(R,n,z)\}$.

<u>Proof</u>. For each $n < \mathrm{lth}R$, all changes in $\bar{R}(z+1)$, which only occur in the dressing, are postponed till the last change in the dressing at the n^{th} index occurs, and then the combined additions in the dressing are made all at once; the resulting sequence is $\bar{R}'(z+1)$. We illustrate the idea pictorially in figs. 4 and 5. Let $\bar{R}7$ describe a piece of the history of the dressed carriers ρ_{n_0,m_0}, ρ_{n_1,m_1}, ρ_{n_2,m_2}; see fig. 4.



Now we look for carrier-index with the highest change number; this is n_0, with a final change at stage 7. Replace all changes in the dressing of n_0 by a single change, adding $\gamma 3 * \gamma 5 * \gamma 6$ at stage 7.

Then look at the next highest change number, stage 5 for index n_1, and combine all changes into a single change at stage 5 (adding $\gamma 2 * \gamma 4$). The result is pictured in fig. 5. □



5.4. Definition. Let m be a change in $\bar{R}(z+1)$, i.e. $ch(R,m) \wedge m \le z$. Then

$$depth(m,\bar{R}(z+1)) \equiv \text{number of repetitions } k \text{ in } \bar{R}(z+1) \text{ with } m<k\le z, \text{ i.e.}$$

$$depth(m,\bar{R}(z+1)) \equiv cardinal\{k : \neg ch(R,k) \wedge m<k\le z\}.$$

We put

$$depth(\bar{R}(z+1)) \equiv \Sigma\{depth(m,\bar{R}(z+1)) : m\le z \wedge ch(R,m)\} .$$

As an example see fig. 6.
* indicates the points of change,
o the repetitions.
$\text{depth}(\bar{R}6) = \text{depth}(1,\bar{R}6) + \text{depth}(3,\bar{R}6) = 3+2 = 5$.

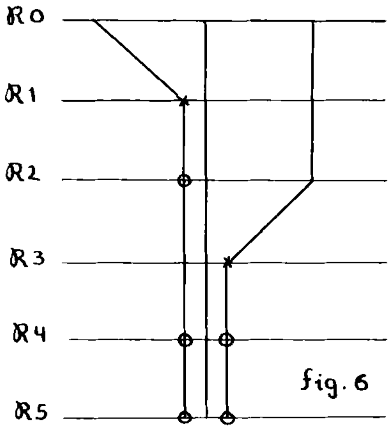


fig. 6

5.5. Lemma (shifting changes upwards). Let $\bar{R}(z+1)$ be admissible. We may construct primitive recursively in $\bar{R}(z+1)$, an admissible $\bar{R}'(z+1)$ with

a) $R0 = R'0$, $R'z = Rz$,

b) $CH(\bar{R}'(z+1)) = CH(\bar{R}(z+1))$

c) $\forall x \le z (R'x \le Rx)$

d) depth $\bar{R}'(z+1) = 0$.

Proof. We transform R via a number of intermediates $R=R''_1$, $R''_2,\ldots,R''_n=R'$ into R' ; all intermediates have the same number of changes up to stage z, but with ever lower depth. Assume R''_i to have been constructed; determine the maximal change $m<z$ in $R''_i(z+1)$, followed by a non-change $m+1$. Postponing the change made at m to $m+1$ yields $\bar{R}''_{i+1}(z+1)$ with lower depth but the same number of changes.□

5.6. Lemma. Let R be admissible and let $R0 = \langle P,\mathcal{D}0\rangle$, and let n be the number of equivalence classes in P (the *width* of P), and assume $z \ge mn$. Then $\bar{R}(z+1)$ contains a subsequence of length $(m+1)$, $\langle Rk,\ldots,R(k+m)\rangle$ with constant pattern.

Proof. Any change of pattern diminishes the width. Assume all subsequences with constant pattern to have length $\le m$, then there can be at most nm elements in $\bar{R}(z+1)$ (maximum number of changes in width multiplied by the maximum length of subsequences with constant pattern), and this is contradictory.□

5.7. Proof of proposition 5.1.

Put $z_0 = 3(\mathrm{lth}\ R)^2$. R' is constructed in the following stages:

a) determine (lemma 5.6) a subsequence $\langle Rk,\dots,R(k+m)\rangle$ of $\bar{R}(z+1)$ with constant pattern, $m \geq 3(\mathrm{lth}R)$.

b) lower the number of changes in this subsequence; this yields (lemma 5.3) R'' with a subsequence

$$Rk=R''k,\ R''(k+1),\dots,R''(k+m)=R(k+m)$$

with at least $2(\mathrm{lth}\ R)+1$ repetitions.

Then put $R''x = Rx$ for $x < k$, $x > k+m$.

c) shift changes upwards (5.5); this yields an R''' with $R'''0 = R''0 = R0$, $R'''z = R''z = Rz$, with repetitions at the first $2(\mathrm{lth}\ R0)+1$ arguments.

d) using the lemma of section 4, determine $R^*(y+1)$ with $R^*0 = R0 = R$, $R^*y = R0$, $y \leq 2\mathrm{lth}(R)$.

e) since $R^*y = R0 = R'''(y+1)$, we may take $R^*0,\dots,R^*y$, $R'''(y+1),\dots,R'''(z)$ for $\bar{R}'(z+1)$.□

6. ASSIGNMENT OF VALUES

This section is completely straightforward.

6.1. Definition. A p-tuple of functions $\chi_1,\dots,\chi_p$ *conforms to* a sequence of patterns P iff

$$\forall x(Px \leq P(\langle\chi_1 x,\dots,\chi_p x\rangle)).$$

$\xi_1,\dots,\xi_p$ *meet* a sequence of restrictions R with $Rx = \langle Px, Dx\rangle$ iff there are $\chi_1,\dots,\chi_p$ (we say that $\xi_1,\dots,\xi_p$ meet R *via* $\chi_1,\dots,\chi_p$) which conform to P, and such that

$$\forall x(\xi_i x = F_{(Dx)_i}(\chi_i x, x)) \quad \text{for } 1 \leq i \leq p.$$

These definitions can also be extended to p-tuples of finite sequences of the same length in an obvious way.

Note that carriers $\sigma_{n_1},\dots,\sigma_{n_p}$ generated from α,β conform to

(1) $\quad P \equiv \lambda x.P(\langle n_1,\dots,n_p\rangle, \bar{\beta}(x+1))$,

and that the dressed carriers $\rho_{n_1,m_1},\dots,\rho_{n_p,m_p}$ generated from α,β,γ meet

(2) $\mathcal{R} \equiv \lambda x.\langle \mathcal{P}x, D(n,m,\bar\beta(x+1),\bar\gamma(x+1))\rangle$

$(n = \langle n_1,\dots,n_p\rangle,\ m = \langle m_1,\dots,m_p\rangle)$ via $\sigma_{n_1},\dots,\sigma_{n_p}$.

6.2. Lemma . Let (1),(2) be as in 6.1, and let $\chi_1,\dots,\chi_p$ conform to $\mathcal{P}$, and let $\xi_1,\dots,\xi_p$ meet $\mathcal{R}$ via $\chi_1,\dots,\chi_p$; then there is an θ such that

$$\chi_i = \lambda x.\mathrm{val}(r(n_i,\bar\beta(x+1)),\theta x) = \sigma_{n_i}(\theta,\beta), \text{ and}$$

$$\xi_i = \rho_{n_i,m_i}(\theta,\beta,\gamma) \text{ for } 1 \le i \le p.$$

Proof. Observe that if $P(n,s) \le P(\langle x_1,\dots,x_p\rangle)$, it is easy to indicate a t such that

$$\mathrm{val}(r(n_i,s),t) = (t)_{r(n_i,s)} = x_i \text{ for } 1 \le i \le p :$$

let $k = \max\{r(n_i,s) : 1\le i\le p\}$, $t = \langle y_0,\dots,y_k\rangle$ with $y_{r(n_i,s)} = x_i$, $y_j = 0$ for $j \notin \{r(n_i,s) : 1\le i\le p\}$.

Now observe $P(n,\bar\beta(x+1)) \le P(\langle \chi_1 x,\dots,\chi_p x\rangle)$, and thus with $\langle \chi_1 x,\dots,\chi_p x\rangle = \langle x_1,\dots,x_p\rangle$ we may take θx equal to the t indicated above. □

6.3. Lemma. Let $\delta = \lambda x.\langle \alpha x,\beta x,\gamma x\rangle$, $n = \langle n_1,\dots,n_p\rangle$, $m = \langle m_1,\dots,m_p\rangle$ and similarly for $\delta',\alpha',\beta',\gamma',n',m'$, and let

$$R(n,m,\bar\beta x,\bar\gamma x) = R(n',m',\bar\beta' x,\bar\gamma' x).$$

There is a primitive recursive χ such that if we put $y = \langle y_1,y_2,y_3\rangle$, $\chi y = \langle \chi_1 y,\ \chi_2 y,\ \chi_3 y\rangle$, then

$$R(n,m,\bar\beta x * \hat y_2,\bar\gamma x * \hat y_3) = R(n',m',\bar\beta' x * \langle\chi_2 y\rangle,\bar\gamma' x * \langle\chi_3 y\rangle)$$

and

$$\rho_{n_i,m_i}(\bar\delta x * \hat y * \xi)(x) = \rho_{n_i',m_i'}(\bar\delta' x * \langle\chi y\rangle * \xi')(x)$$

for $1 \le i \le p$ and arbitrary ξ,ξ'.

Proof. Combine the remark at the end of 3.2 with the observation in the proof of the preceding lemma.□

6.4. Lemma. Let $\mathcal{R}_1,\mathcal{R}_2$ be sequences of restrictions such that $\forall x(\mathcal{R}_1 x \le \mathcal{R}_2 x)$, and let $\rho_1,\dots,\rho_p$ meet $\mathcal{R}_2$, then $\rho_1,\dots,\rho_p$ also meet $\mathcal{R}_1$.

Proof. Let $\rho_1,\dots,\rho_p$ meet $\mathcal{R}_2 = \lambda x.\langle \mathcal{P}_2 x,\mathcal{D}_2 x\rangle$, *via* $\chi_1,\dots,\chi_p$, i.e.

$$\rho_i x = F_{(\mathcal{D}_2 x)_i}(\chi_i x, x) \quad \text{for} \quad 1 \le i \le p.$$

Let $\mathcal{D}_3 = \lambda x.(\mathcal{D}_2 x \backslash \mathcal{D}_1 x)$, and put

$$\chi'_i \equiv \lambda x.F_{(\mathcal{D}_3 x)_i}(\chi_i x, x) \ , \ 1 \le i \le p$$

then

$$P_1 x \le P(\mathcal{D}_3 x) \ ,$$

$$P_1 x \le P_2 x \le P(\langle \chi_1 x, \ldots, \chi_p x \rangle),$$

hence $P_1 x \le P(\langle \chi'_1 x, \ldots, \chi'_p x \rangle)$, and the ρ_i meet R_1 via the χ'_i . □

7. PERMUTABILITY CONDITIONS

7.1. As already mentioned in the introduction, equality of restrictions was meant to guarantee that two n-tuples of projections satisfy the same formulae of a suitably restricted language, i.e. can be interchanged with respect to this language. Since our definition of "equal restrictions" primarily applied to dressed carriers, not to the projections (i.e. specifications of initial segments were disregarded) we have to make a slight adaptation in order to obtain a "permutability condition" on pairs of n-tuples of projections.

First of all we need to specify the language. Since our theory concerns sequences which are not closed under all continuous operations, but only under some, the language of CS (of [K,T]) is not suitable, since $e|\alpha$ will in general not be a legitimate choice function. We therefore prefer the language L of LS, which is noncommittal as to closure under continuous operations. We can certainly express in L closure under an operation $e|$: $\forall\alpha\exists\beta\forall x\exists y(e(\hat{x} * \bar{\alpha}y) = \beta x + 1)$, and may introduce $e|\alpha = \beta$ as an abbreviation.

Also, if the need arises, one may add functional constants F_i to the language of LS.

From now on, we shall use α, α' (instead of δ, δ' as in the preceding sections) to indicate the lawless sequences encoding all information on which the projections depend.

7.2. Definition. Let α be a sequence coding a triple β, γ, δ and let $\{\pi_n(\alpha) : n \in N\}$

indicate our universe of projections generated from β,γ,δ.

Let $\phi_0,\phi_1,\phi_2,\ldots$ enumerate the choice variables of L, and let $A(\phi_{i_1},\ldots,\phi_{i_p})$ be any formula of L with choice variables among $\phi_{i_1},\ldots,\phi_{i_p}$, and only odd-numbered numerical variables $v_1,v_3,\ldots$. Then A^α is obtained from A by

a) replacing each occurrence of ϕ_n in A by $\pi_{v_{2n}}(\alpha)$, and

b) replacing quantifiers $\forall\phi_n,\exists\phi_n$ by $\forall v_{2n},\exists v_{2n}$ respectively.

As a tacit convention, we shall use, whenever $A(\phi_{i_1},\ldots,\phi_{i_p})$ has been introduced, $A^\alpha(\underline{n}_1,\ldots,\underline{n}_p)$ for the formula obtained from A^α by substituting n_k for v_{2i_k}.

7.3. Definition. The four-place predicate $\mathrm{Per}(u_1,v;u_2,w)$ is said to be a permutability condition for the $\pi_n\alpha$ if

$$(1)\qquad \mathrm{Per}(\langle n_1,\ldots,n_p\rangle,v;\langle m_1,\ldots,m_p\rangle,w) \rightarrow \forall\alpha\in v\ A^\alpha(\underline{n}_1,\ldots,\underline{n}_p) \leftrightarrow \forall\alpha\in w\ A^\alpha(\underline{m}_1,\ldots,\underline{m}_p)$$

For the set of projections we are studying, we may define a predicate "Per" as follows.

Let β,γ,δ be coded by α, such that $\alpha x = \nu_3(\beta x,\gamma x,\delta x)$, and let β',γ',δ' be similarly coded by α'; let $\pi_{n_1},\ldots,\pi_{n_p}$ be projected from β,γ,δ , and $\pi'_{m_1},\ldots,\pi'_{m_p}$ from β',γ',δ' (i.e. $\pi_{n_i} = \pi_{n_i}(\beta,\gamma,\delta)$, $\pi'_{m_j} = \pi'_{m_j}(\beta',\gamma',\delta')$).

Then $\mathrm{Per}(\langle n_1,\ldots,n_p\rangle,v;\langle m_1,\ldots,m_p\rangle,w)$ holds if

(a) $\mathrm{lth}(v) = \mathrm{lth}(w)$, say $\mathrm{lth}(v) = x$;

(b) the restrictions at stage x determined respectively from $n_1,\ldots,n_p,\bar{\alpha}x$ and $m_1,\ldots,m_p,\bar{\alpha}'x$ are the same;

(c) $\bar{\pi}_{n_i}x = \bar{\pi}'_{m_i}x$ for $1 \le i \le p$;

(d) $\mathrm{lth}(v) = \mathrm{lth}(w) \ge \mathrm{lth}(n_i),\mathrm{lth}(m_i)$ for $1 \le i \le p$.

Now we shall first state and prove a theorem stating sufficient conditions for a predicate Per to be a permutability condition.

7.4. Theorem. Let $\mathrm{Per}(u_1,v;u_2,w)$ be a four-place predicate, $\phi(u_1,v,u_2,w,x)$ a function from N^5 to N ($\phi(x)$ for short), $\mathrm{Ess}(n,v)$ ("v is essential for π_n") a two-place predicate, $\pi(n,v)$ a finite sequence (i.e. a natural number) such that

(i) $\forall n \exists e \epsilon K \forall v(ev \neq 0 \rightarrow Ess(n,v))$,

(ii) $\forall \alpha \epsilon v(\pi_n \alpha \in \pi(n,v))$,

(iii) $\forall n \forall x \exists e \epsilon K \forall v(ev \neq 0 \rightarrow lth\ \pi(n,v) > x)$,

(iv) $Per(u_1,v;u_2,w) \leftrightarrow Per(u_2,w;u_1,v)$,

(v) $Per(u_1,v;u_2,w) \rightarrow \forall x(Ess(x,v) \rightarrow \exists y\ Per(u_1*\hat{x},v;u_2*\hat{y},w))$,

(vi) $Per(u_1,v;u_2,w) \rightarrow \forall x\ Per(u_1,v*\langle\phi(x)\rangle;u_2,w*\hat{x})$,

(vii) $Per(u_1,v;u_2,w) \rightarrow \forall i < lth(u_1)(\pi((u_1)_i,v) = \pi((u_2)_i,w))$;

then Per is a permutability condition.

Note that because of (i) it is no restriction to assume Ess(n,v) to be decidable (simply replace in (i)-(vii) Ess(n,v) by $ev \neq 0$, for an $e \in K$ validating (i)); and also note that (vi) yields a $\psi \equiv \psi[u_1,v,u_2,w,\chi]$ such that

$$Per(u_1,v;u_2,w) \rightarrow \forall \chi \forall x\ Per(u_1,v*\bar{\psi}x;u_2,w*\bar{\chi}x),$$

and a $\psi'[u_1,v,u_2,w]$ such that

$$Per(u_1,v;u_2,w) \rightarrow \forall w'\ Per(u_1,v*\psi'w';u_2,w*w')$$

(Take $\psi x = \phi(u_1,v*\bar{\psi}x,u_2,w*\bar{\chi}x,\chi x)$, and $\psi'(m*\hat{x}) = \psi'm*\phi(u_1,v*\psi'm,u_2,w*m,x))$.

Proof. We have to establish (1) in 7.3 by induction on the logical complexity of A. Let A have its choice parameters among $\varepsilon_1,\ldots,\varepsilon_p$, i.e. $A \equiv A(\varepsilon_1,\ldots,\varepsilon_p)$. At each step our hypotheses are

(2) $Per(\langle n_1,\ldots,n_p\rangle,v,\langle m_1,\ldots,m_p\rangle,w)$; we let

$\langle n_1,\ldots,n_p\rangle = u_1$, $\langle m_1,\ldots,m_p\rangle = u_2$.

(3) $\forall \alpha \epsilon v\ A^{\alpha}(\underline{n}_1,\ldots,\underline{n}_p)$

(4) Induction hypothesis : Per satisfies (1) of 7.3 for all formulae of complexity less than A.

We shall assume our formulae to be rewritten such that all prime formulae containing variables ε_i are of the form $\varepsilon_i t = s$, t and s not containing choice variables.

Case 1. $A \equiv (\varepsilon_i t=s)$. Hypothesis (3) becomes in this case

(3.1) $\forall \alpha \epsilon v(\pi_{n_i}(\alpha)(t)=s)$.

Let $\alpha \in w$, $\psi \equiv \psi[u_1,v,u_2,w,\lambda z.\alpha(lth(w) + z)]$.

With (2) and (vi)

(5) $\forall x\ Per(u_1,v*\bar{\psi}x;u_2,\bar{\alpha}(lth(w) + x))$.

(iii) guarantees the existence of an x_0 with

(6) $\pi(n_i, v*\bar{\psi}x_0) > t,\ \pi(m_i, \bar{\alpha}(\mathrm{lth}(w) + x_0)) > t \quad (1 \le i \le p)$.

Apply (vii) to (5) with $x = x_0$ then

$$\pi(n_i, v*\bar{\psi}x_0) = \pi(m_i, \bar{\alpha}(\mathrm{lth}(w) + x_0)) \quad (1 \le i \le p),$$

and then with (3.1), (ii), (6)

$$\pi_{m_i}(\alpha)(t) = \pi(m_i, \bar{\alpha}(\mathrm{lth}(w) + x_0))_t = \pi(n_i, v*\bar{\psi}x_0)_t = s.$$

<u>Case 2.</u> $A \equiv B \wedge C$: trivial.

<u>Case 3.</u> $A \equiv B \to C$. Hypothesis (3) now becomes

(3.3) $\forall\alpha\epsilon v(B^\alpha(\underline{n}_1, \ldots, \underline{n}_p) \to C^\alpha(\underline{n}_1, \ldots, \underline{n}_p))$.

We have to show

$$\forall\alpha\epsilon w*w'\ B^\alpha(\underline{m}_1, \ldots, \underline{m}_p) \to \forall\alpha\epsilon w*w'\ C^\alpha(\underline{m}_1, \ldots, \underline{m}_p);$$

therefore assume

(7) $\forall\alpha\epsilon w*w'\ B^\alpha(\underline{m}_1, \ldots, \underline{m}_p)$.

By (2) and (vi) we find a v' such that

$$\mathrm{Per}(u_1, v*v'; u_2, w*w'),$$

and thus with (4),(7) $\forall\alpha\epsilon v*v'\ B^\alpha(\underline{n}_1, \ldots, \underline{n}_p)$; with (3.3) $\forall\alpha\epsilon v*v'\ C^\alpha(\underline{n}_1, \ldots, \underline{n}_p)$, hence (again with (4)) $\forall\alpha\epsilon w*w'\ C^\alpha(\underline{m}_1, \ldots, \underline{m}_p)$.

<u>Case 4.</u> $A \equiv B \vee C$ is reduced to other cases by the use of

$A \leftrightarrow \exists x((x=0 \to B) \wedge (x \ne 0 \to C))$.

<u>Case 5.</u> $A \equiv \forall x\, Bx,\ \forall a\, Ba,\ \forall e\, Be$; trivial.

<u>Case 6.</u> $A \equiv \exists x\, Bx,\ \exists a\, Ba,\ \exists e\, Be$. Similar to, but simpler than Case 8 treated below.

<u>Case 7.</u> $A \equiv \forall\epsilon B\epsilon$. Hypothesis (3) becomes (3.7)

(3.7) $\forall\alpha\epsilon v \forall n\ B^\alpha(\underline{n}_1, \ldots, \underline{n}_p, \underline{n})$.

We have to show $\forall m \forall\alpha\epsilon w\ B^\alpha(\underline{m}_1, \ldots, \underline{m}_p, \underline{m})$.

Let m, $\alpha\epsilon w$ be arbitrary, and let ψ be as before. By (i) there is an x with

$\mathrm{Ess}(m, \bar{\alpha}(\mathrm{lth}(w)+x))$.

With (vi),(iv),(v) we find successively

$$\mathrm{Per}(u_1, v*\bar{\psi}x; u_2, \bar{\alpha}(\mathrm{lth}(w)+x));$$

$$\mathrm{Per}(u_2, \bar{\alpha}(\mathrm{lth}(w)+x); u_1, v*\bar{\psi}x);$$

$$\mathrm{Per}(u_2*\hat{m}, \bar{\alpha}(\mathrm{lth}(w)+x); u_1*\hat{n}, v*\bar{\psi}x) \quad \text{for suitable } n;$$

hence with (iv), (3.7) and (4)

$$B^{\alpha}(\underline{m}_1,\ldots,\underline{m}_p,\underline{m})$$

Case 8. $A \equiv \exists\varepsilon B\varepsilon$. Hypothesis (3) becomes

(3.8) $\forall\alpha\in v\exists n\ B^{\alpha}(\underline{n}_1,\ldots,\underline{n}_p,\underline{n})$.

We have to show $\forall\alpha\in w\exists n\ B^{\alpha}(\underline{m}_1,\ldots,\underline{m}_p,\underline{n})$. There is an $f \in K$ such that

(8) $\forall v'(fv'\neq 0 \rightarrow \forall\alpha\in v*v'\ B^{\alpha}(\underline{n}_1,\ldots,\underline{n}_p,\underline{fv' \dot{-} 1}))$.

We may assume that f also satisfies

$$\forall v'(fv'\neq 0 \rightarrow Ess(fv' \dot{-} 1, v*v')).$$

(To see this, let the e_n be elements of K satisfying

$$e_n v\neq 0 \rightarrow Ess(n,v)$$

as postulated in (i) ; replace an f satisfying (8) by

$$\lambda v'.fv'.sg(e_{f\ v' \dot{-} 1}(v*v'))$$

which is easily seen to be an element of K.)

Let ξ be such that ((v) of 7.4)

$$f(\psi'w')\neq 0 \rightarrow$$

$$Per(u_1*\langle f(\psi'w')\dot{-}1\rangle,\ v*\psi'w';\ u_2*\langle\xi(f\ (\psi'w')\dot{-}1)\rangle,\ w*w');$$

then with (8)

$$\forall w'(f(\psi'w') \neq 0 \rightarrow \forall\alpha\in v*(\psi'w')B^{\alpha}(\underline{n}_1,\ldots,\underline{n}_p,\underline{f(\psi'w')\dot{-}1})),$$

hence by (4)

$$\forall w'(f(\psi'w')\neq 0 \rightarrow \exists n\forall\alpha\in w*w'\ B^{\alpha}(\underline{m}_1,\ldots,\underline{m}_p,\underline{n})),$$

and thus $\forall\alpha\in w\exists n\ B^{\alpha}(\underline{m}_1,\ldots,\underline{m}_p,\underline{n})$.□

Remark. Assuming our language to be extended with other sorts of choice variables, which are thought of as sequences completely unrelated to the range of the lawless variables, we can still extend our theorem carrying such parameters along; elements of K then have to be interpreted as elements of K lawlike relative to such parameters.

7.5. Theorem. The relation Per in 7.3 defined by the stipulations (a)-(d) fulfills all the conditions (i)-(vii) for appropriate $\pi(n,v),\phi,Ess$ and is hence a permutability condition for the projections studied in this paper.

Proof

(A). $Ess(n,v) \equiv lth(v) \geq lth(j_1 n)$; i.e., if we think of v as a code of $\bar{\beta}x, \bar{\gamma}x, \bar{\delta}x$, and $\pi_n = j_1 n|\rho_{j_2 n}$, then $lth(\bar{\beta}x) = x = lth(v) > lth(j_1 n)$ guarantees that values of π_n for $y \geq x$ are determined by α, i.e. $\Delta_{j_1 n}$ has no longer any effect.

(B). For $\pi(n,v)$ we take

$$\pi(n,v) = \begin{cases} \overline{\pi_n}(lth\ v) & \text{if } lth(v) \geq lth(j_1 n) , \\ j_1 n & \text{otherwise.} \end{cases}$$

(C). Let $Per(u_1,v;u_2,w)$. Then ϕ is a function which assigns to any continuation $w*\hat{x}$ a continuation $v*\langle\phi x\rangle$, such that the restrictions on the projections with indices $u_2 = \langle m_1,\dots,m_p\rangle$, at $w*\hat{x}$, are the same as the restrictions on the projections with indices $u_1 = \langle n_1,\dots,n_p\rangle$ at $v*\langle\phi x\rangle$, and such that the values remain corresponding as well. The existence of such a ϕ follows from 6.3.

(D) With these specifications, the properties (i)-(iv),(vi),(vii) become obvious. Property (v) expresses that provided v is essential for π_x , i.e. $lth(v) \geq lth(j_1 x)$, we can find to each extension $u_1*\hat{x}$ of u_1 a corresponding extension $u_2*\hat{y}$ of u_2 , for a suitable y , such that the permutability property is preserved. To see this, let

(1) $Per(\langle n_1,\dots,n_p\rangle,v;\langle m_1,\dots,m_p\rangle,w)$,

(2) $lth(v) \geq lth(j_1 x)$.

Then certainly ((2), definition of $\pi(x,v)$)

(3) $lth(\pi(x,v)) = lth(v) = lth(w)$;

also

(4) $y=j(\pi(x,v),m) \rightarrow Ess(y,w)$,

(5) $y=j(\pi(x,v),m) \rightarrow \pi(x,v)=\pi(y,w)$.

The dressed carriers underlying $\pi_{n_1},\dots,\pi_{n_p},\pi_x$ have indices $j_2 n_1,\dots,j_2 n_p,j_2 x$; to this there corresponds at v (stage $lth(v)$) a particular restriction R. As noted in 3.3 we can find a y' such that $j_2 m_1,\dots,j_2 m_p,y'$ at stage $lth(v) = lth(w)$ obey the same restriction R; if we take $y = j(\pi(x,v),y')$, then $\pi_{m_1},\dots,\pi_{m_p},\pi_y$ at w ($lth(w) = lth(v)$) are permutable with $\pi_{n_1},\dots,\pi_{n_p},\pi_x$ at v.□

8. SCHEMATA VALID IN THE MODEL

8.1. This section is devoted to establishing the validity of various schemata in our model. We shall assume the F^*_n to contain the Δ_m , and in particular, we may assume $F^*_0 = \Delta_0$ = identity.

In order to simplify the notation, we shall often use $\underline{n}$ for the projection π_n , and $\Gamma, \Gamma', \Gamma_i \ldots$ as syntactical variables for variables ranging over $\{F^*_n : n \in N\}$. Almost self-evident is the following result

Theorem In our models $\mathcal{U}_\alpha$

$$\forall \varepsilon \forall \Gamma \exists \eta (\Gamma \varepsilon = \eta).$$

8.2. Theorem. In our models $\mathcal{U}_\alpha$

$$A\varepsilon \rightarrow \exists \Gamma (\exists \eta (\varepsilon = \Gamma \eta) \wedge \forall \varepsilon' A(\Gamma \varepsilon'))$$

and more generally

$$A(\varepsilon_1, \ldots, \varepsilon_p) \rightarrow \exists \Gamma_1 \ldots \Gamma_p \bigvee_\sigma (\exists \eta'_1 \ldots \eta'_p \bigwedge_{1 \le i \le p} \varepsilon_i = \Gamma_i \eta'_{\sigma(i)} \wedge$$
$$\wedge \forall \eta_1 \ldots \eta_p A(\Gamma_1 \eta_{\sigma(1)}, \ldots, \Gamma_p \eta_{\sigma(p)})),$$

where σ ranges over mappings from $\{1, \ldots, p\}$ into $\{1, \ldots, p\}$.

Proof. Consider any sequence of projections $X = \langle \underline{n}_1, \ldots, \underline{n}_p \rangle$ at v , and let $n_i = j(n'_i, j(n''_i, n'''_i))$. Assume also $\mathrm{lth}(v) \ge \mathrm{lth}(n'_i)$ for $1 \le i \le p$, i.e. v is essential for X. $\underline{n}_i$ has carrier $\sigma_{n''_i}$ with root $r(v, n''_i) = n^*_i$ at v ; if we put $m_i = j(0, j(n^*_i, 0))$, then $\underline{m}_i$ is completely unrestricted at v , and there are p_i, q_i such that $\pi_{n_i} = (\Delta_{p_i} \circ F^*_{q_i})(\pi_{m_i})$ at v ; here p_i is the initial segment $\bar{\pi}_{n_i}(\mathrm{lth}(v))$ at v , and $F^*_{q_i}$ is the dressing of π_{n_i} at v. There are Γ_i such that $\Gamma_i = \Delta_{p_i} \circ F^*_{q_i}$, $1 \le i \le p$, and thus

$$\forall \beta \in v (\underline{n}_i = \Gamma_i \underline{m}_i) \quad , \quad 1 \le i \le p.$$

Now let $A^\alpha(\underline{n}_1, \ldots, \underline{n}_p)$. By the axiom of open data for lawless sequences, there is a v such that

$$\alpha \in v \ , \ \forall \beta \in v \ A^\beta(\underline{n}_1, \ldots, \underline{n}_p),$$

v essential for $\underline{n}_1, \ldots, \underline{n}_p$. By the preceding remark, we can find σ, $m_1, \ldots, m_q$, $\Gamma_1, \ldots, \Gamma_p$ such that $\{\sigma(1), \ldots, \sigma(p)\} = \{1, \ldots, q\}$, and

$$\forall\beta\in v(\underline{n}_i(\beta) = \Gamma_i\underline{m}_{\sigma(i)}(\beta))$$

and $\underline{m}_1,\ldots,\underline{m}_q$ completely unrestricted at v , and thus

(1) $\forall\beta\in v\ A^{\beta}(\Gamma_1\underline{m}_{\sigma(1)},\ldots,\Gamma_p\underline{m}_{\sigma(p)})$

Consider any set $\underline{r}_1,\ldots,\underline{r}_q$ and let s(i) be the initial segment of $\underline{m}_i$ at v ($1 \le i \le q$) . Since $\underline{m}_1,\ldots,\underline{m}_q$ are completely unrestricted at v , the restriction on $\underline{m}_1,\ldots,\underline{m}_q$ at v is weaker than the restriction on $\Delta_{s(1)}\underline{r}_1,\ldots,\Delta_{s(q)}\underline{r}_q$ at v ; and thus we can apply the overtake-property to find x,v' such that

(i) $\mathrm{lth}(v*v') = x$,

(ii) $\bar{\alpha}x$ essential for $\underline{r}_1,\ldots,\underline{r}_q$, and

(iii) the restriction of $\underline{m}_1,\ldots,\underline{m}_q$ at v*v' is equal to the restriction of

$\Delta_{s(1)}\underline{r}_1,\ldots,\Delta_{s(q)}\underline{r}_q$ at $\bar{\alpha}x$.

Therefore $\underline{m}_1,\ldots,\underline{m}_q$ at v*v' and $\underline{r}_1,\ldots,\underline{r}_q$ at $\bar{\alpha}x$ are permutable, and thus from (1)

$$A^{\alpha}(\Gamma_1\ \Delta_{s(\sigma(1))}(\underline{r}_{\sigma(1)}),\ldots,\Gamma_p\ \Delta_{s(\sigma(p))}(\underline{r}_{\sigma(p)})).$$

Since $\Gamma_i\underline{r}_{\sigma(i)} = \Gamma_i\ \Delta_{s(\sigma(i))}(\underline{r}_{\sigma(i)})$, we also have

$$A^{\alpha}(\Gamma_1\underline{r}_{\sigma(1)},\ldots,\Gamma_p\underline{r}_{\sigma(p)}).$$ □

8.3. <u>Theorem</u>. In our models $\mathcal{U}_{\alpha}$

$$\forall\varepsilon_1\ldots\varepsilon_p\exists x\ A(\varepsilon_1,\ldots,\varepsilon_p,x) \to \exists e\forall\varepsilon_1\ldots\varepsilon_p\ A(\varepsilon_1,\ldots,\varepsilon_p,e(\varepsilon_1,\ldots,\varepsilon_p)).$$

Similarly

$$\forall\varepsilon_1\ldots\varepsilon_p\exists a\ A(\varepsilon_1,\ldots,\varepsilon_p,a) \to$$
$$\to \exists e\exists b\forall\varepsilon_1\ldots\varepsilon_p A(\varepsilon_1,\ldots,\varepsilon_p,(b)_{e(\varepsilon_1,\ldots,\varepsilon_p)}).$$

<u>Proof</u>. Assume $\forall n_1\ldots n_p\exists x\ A^{\alpha}(\underline{n}_1,\ldots,\underline{n}_p,x)$, then by the axiom of open data there is a v such that

$$\alpha \in v,\quad \forall\beta\in v\forall n_1\ldots n_p\ \exists x\ A^{\beta}(\underline{n}_1,\ldots,\underline{n}_p,x)\ ,$$

and thus

$$\forall n_1\ldots n_p\ \exists e\ \forall\beta\in v\ A^{\beta}(\underline{n}_1,\ldots,\underline{n}_p,e(\beta)\dot{-}1).$$

Let $\mathrm{lth}(m) = \mathrm{lth}(v) = y$, $m_i = k_i^p(m)$. There are fixed primitive recursive functions f_i ($1 \le i \le p$) of v and m such that

$$^{\pi}f_1(m,v),\ldots,{}^{\pi}f_p(m,v)$$

are completely free at v, and $\overline{\pi}_{f_i(m,v)}(y) = m_i$. Let e_m be the element of K such that

(1) $\quad \forall\beta\epsilon v\ A^{\beta}(\underline{f_1(m,v)},\ldots,\underline{f_p(m,v)},e_m(\beta)\dot{-}1).$

and let $\xi(m,n)$ be a primitive recursive function such that at $v*\xi(m,n)$ $\underline{f_1(m,v)},\ldots,\underline{f_p(m,v)}$ are still completely unrestricted, while

$$\overline{\pi}_{f_i(m,v)}(\mathrm{lth}(v)+\mathrm{lth}(n)) = m_i*k_i^p(n) \quad (1 \le i \le p),$$

$$\mathrm{lth}(n) = \mathrm{lth}(\xi(m,n)).$$

Determine $f \in K$ as follows:

$$fn = 0 \quad \text{if} \quad \mathrm{lth}(n) < \mathrm{lth}(v),$$

$$f(m*n) = e_m(v*\xi(m,n)).$$

Quite obviously, $f \in K$. We shall prove

$$fn'\neq 0 \to \forall\underline{n}_1\epsilon k_1^p n' \ldots \forall\underline{n}_p\epsilon k_p^p n'\ A^{\alpha}(\underline{n}_1,\ldots,\underline{n}_p,fn'\dot{-}1).$$

Assume $fn' \neq 0$, $\underline{n}_i \in k_i^p n'$ $(1 \le i \le p)$. Since $fn' \neq 0$, $\mathrm{lth}(n') \ge \mathrm{lth}(v)$, and $n' = m*n$, $\mathrm{lth}(m) = \mathrm{lth}(v)$ for certain m,n. Consider the $\underline{f_i(m,v)}$; they satisfy $f_i(m,v) \in k_i^p n' = k_i^p(m*n)$ at $v*\xi(m,n)$, and $e_m(v*\xi(m,n)) = fn'$ (by the definition of f). Continue $v*\xi(m,n)$ to $v*v'$ such that (overtake-property) the restriction of the $\underline{f_i(m,v)}$ at $v*v'$ is the same as the restriction of the $\underline{n}_i$ at $\overline{\alpha}(\mathrm{lth}(v*v'))$. Then $\underline{n}_1,\ldots,\underline{n}_p$ at $\overline{\alpha}(\mathrm{lth}(v*v'))$ and $\underline{f_1(m,v)},\ldots,\underline{f_p(m,v)}$ at $v*v'$ are permutable, and thus from

$$\forall\beta\epsilon v*v'\ A^{\beta}(\underline{f_1(m,v)},\ldots,\underline{f_p(m,v)},fn'\dot{-}1)$$

we may conclude

$$A^{\alpha}(\underline{n}_1,\ldots,\underline{n}_p,fn'\dot{-}1). \qquad \square$$

8.4. <u>Theorem</u>. In our models $\mathcal{U}_\alpha$

$$\forall\varepsilon_1\ldots\varepsilon_p\exists\eta A(\varepsilon_1,\ldots,\varepsilon_p,\eta) \to \forall\varepsilon_1\ldots\varepsilon_p\exists ij(0\le j\le p \wedge A(\varepsilon_1,\ldots,\varepsilon_p,F_i^*\varepsilon_j)).$$

<u>Corollaries</u>

(i) $\forall\varepsilon\exists\eta A(\varepsilon,\eta) \to \exists e\forall n(en\neq 0 \to \forall\alpha\epsilon n\ A(\varepsilon,F^*_{en\dot{-}1}\varepsilon))$,

(ii) $\forall\varepsilon_1\ldots\varepsilon_p\exists\eta A(\varepsilon,\eta) \to \exists e\forall n(en\neq 0 \to$

$$\forall\varepsilon_1\epsilon k_1^p n\ldots\forall\varepsilon_p\epsilon k_p^p n\ A(\varepsilon_1,\ldots,\varepsilon_p,F^*_{j_1(en\dot{-}1)}\varepsilon_{j_2(en\dot{-}1)}).$$

Proof. We first consider a v such that

$$\forall\beta\epsilon v\exists m\ A^{\beta}(\underline{n}_1,\ldots,\underline{n}_p,\underline{m}) ,$$

for $\underline{n}_1,\ldots,\underline{n}_p$ which are completely unrestricted at v , v essential for $\underline{n}_1,\ldots,\underline{n}_p$. There is an $e\epsilon K$ such that

$$\forall n(en\neq 0 \rightarrow \forall\beta\epsilon v*n\ A^{\beta}(\underline{n}_1,\ldots,\underline{n}_p,\underline{en\dot{-}1})).$$

Let $v*v'$ be a continuation such that $\underline{n}_1,\ldots,\underline{n}_p$ are still completely unrestricted at $v*v'$, and such that $ev'\neq 0$. Then there are two possibilities.

(a) $n' = ev'\dot{-}1$ and $\underline{n}'$ has the same carrier as $\underline{n}_j$ at $v*v'$ for some j. In this case there is an i such that

$$\forall\beta\epsilon v*v'(F_i^*\underline{n}_j(\beta) = \underline{n}'(\beta)).$$

(b) $\underline{n}'$ has a carrier with root different from one of the carrier-indices of $\underline{n}_1,\ldots,\underline{n}_p$ at $v*v'$; say $\underline{n}' = F_i\underline{n}''$, $\underline{n}''$ not subject to any restriction at $v*v'$. Now extend $v*v'$ to $v*v'*\hat{x}$ such that (the carrier of) $\underline{n}''$ jumps to $\underline{n}_1$ say ,no dressing added, and thus $\forall\beta\epsilon v*v'*\hat{x}(\underline{n}'=F_i^*\ \underline{n}_1)$.

Therefore in both cases we find some continuation $v*v''$ of v such that

$$\exists ij\forall\beta\epsilon v*v''[A^{\beta}(\underline{n}_1,\ldots,\underline{n}_p,F_i^*\underline{n}_j) \wedge 1\leq j\leq p].$$

We shall use this observation below. Assume $\forall\varepsilon_1,\ldots,\varepsilon_p\exists\eta A(\varepsilon_1,\ldots,\varepsilon_p,\eta)$ in our model, i.e.

$$\forall n_1\ldots n_p\exists m\ A^{\alpha}(\underline{n}_1,\ldots,\underline{n}_p,\underline{m}).$$

By "open data" there is a v such that

$$\alpha\epsilon v,\ \forall n_1\ldots n_p\forall\beta\epsilon v\exists m\ A^{\beta}(\underline{n}_1,\ldots,\underline{n}_p,\underline{m}).$$

Consider an arbitrary set $\underline{m}_1(\alpha),\ldots,\underline{m}_p(\alpha)$, and determine $n_1,\ldots,n_p$ such that $\underline{n}_1,\ldots,\underline{n}_p$ are not subject to any restrictions at v , $\bar{\pi}_{n_i}(\mathrm{lth}(v)) = \bar{\pi}_{m_i}(\mathrm{lth}(v))$ at v for $1 \leq i \leq p$.

Let $v*v'$ be a continuation of v such that $\bar{\pi}_{n_i}\mathrm{lth}(v*v')$ at $v*v' = \bar{\pi}_{m_i}\mathrm{lth}(v*v')$ at $\bar{\alpha}(\mathrm{lth}(v*v')) = v*u'$, $\underline{n}_i$ completely unrestricted at $v*v'$, and such that for some i, j $(1 \leq j \leq p)$

$$\forall\beta\epsilon v*v'\ A^{\beta}(\underline{n}_1,\ldots,\underline{n}_p,F_i^*(\underline{n}_j))$$

Extend $v*v'$ to $v*v'*v''$ such that the restrictions on $\underline{n}_1,\ldots,\underline{n}_p$ at $v*v'*v''$ coincide with the restrictions on $\underline{m}_1,\ldots,\underline{m}_p$ at $v*u'*u'' = \bar{\alpha}\mathrm{lth}(v*v'*v'')$ (overtake-property). Then $\underline{n}_1,\dots,\underline{n}_p$ at $v*v'*v''$ and $\underline{m}_1,\dots,\underline{m}_p$ at $v*u'*u''$ are permutable, and thus

$$\forall\beta\in v*u'*u''\ A^{\beta}(\underline{m}_1,\dots,\underline{m}_p,\Gamma_i^*\underline{m}_j)$$

and hence also

$$A^{\alpha}(\underline{m}_1,\dots,\underline{m}_p,\Gamma_i^*\underline{m}_j).$$

8.5. All schemata derived above concerned formulae A in our standard language ; in contrast, we note the following

Theorem. Assume $\{\Gamma_n^* : n\in N\}$ to include operations with constant (lawlike) values, then we have in our model, for any X such that

$$\varepsilon=\varepsilon' \wedge x=x' \wedge X(\varepsilon,x) \rightarrow X(\varepsilon',x')$$

the validity of

$$\forall\varepsilon\exists!x\ X(\varepsilon,x) \rightarrow \exists e\forall\varepsilon X(\varepsilon,e(\varepsilon))$$

We omit the proof here ; it is similar to the proofs of $\forall\varepsilon\exists!x$-continuity for projections in [D,T] , and for choice sequences in [T2].

8.6. Elimination. The schemata derived in 8.1.-4 for our projections suffice to define an elimination mapping similar to the ones given for LS and CS (see e.g. [T4]).

For example, the schema of 8.2. is used in the form

$$\forall\varepsilon_1\dots\varepsilon_p(A(\varepsilon_1,\dots,\varepsilon_p) \rightarrow B(\varepsilon_1,\dots,\varepsilon_p)) \leftrightarrow$$
$$\forall\Gamma_1\dots\Gamma_p \bigwedge_\sigma(\forall\varepsilon_1\dots\varepsilon_p A(\Gamma_1\varepsilon_{\sigma(1)}\dots\Gamma_p\varepsilon_{\sigma(p)}) \rightarrow$$
$$\forall\varepsilon_1\dots\varepsilon_p B(\Gamma_1\varepsilon_{\sigma(1)}\dots\Gamma_p\varepsilon_{\sigma(p)})) ,$$

and the schema of 8.4. is combined with 8.3; the schemata then enable us to push strings of quantifiers $\forall\varepsilon_1\dots\varepsilon_p$ inwards, to be replaced ultimately by lawlike quantifiers in front of prime formulae. (Cf. 1.5, (ii) in the introduction to this paper.)

8.7. Extension to higher types, sheaf model.

It is possible to include sets and relations in our language and comprehension axioms in our theory, as follows. We simply put down as an extra axiom

(1) $$\forall X \forall n_1 \ldots n_p \forall m_1 \ldots m_p [\mathrm{Per}(\langle n_1,\ldots,n_p\rangle, v; \langle m_1,\ldots,m_p\rangle, w) \rightarrow$$
$$\rightarrow (\forall \alpha \in v\ X(\underline{n}_1,\ldots,\underline{n}_p) \rightarrow \forall \alpha \in w\ X(\underline{m}_1,\ldots,\underline{m}_p))];$$

let us abbreviate this axiom as $\forall X\ \mathrm{Per}(X)$. The theory with (1) added can be reduced to the theory without (1), simply by relativizing the quantifiers $\forall X, \exists X$ to $\{X : \mathrm{Per}(X)\}$. In the theory with axiom (1), all our arguments for the various schemata given above remain valid. Similarly for the theory extended to a language with relations of all finite types.

This opens the possibility of re-interpreting our collections of models as a sheaf model over Baire space (topological model) for a theory with finite types, since the model depends ultimately on a single lawless parameter, which may be interpreted as ranging over Baire space (cf. [T4], 7.16, 4.17;[F,H]). Thus we obtain an example of a sheaf model satisfying $\forall \alpha \exists x$-continuity. Seen from the outside, the model is very thin: there are only countably many functions.

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Linear orders in $(\omega)^\omega$ under eventual dominance

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Let $(\omega)^\omega$, the set of functions $f : \omega \to \omega$, be ordered by eventual dominance. Under this ordering, $(\omega)^\omega$ embeds every linear ordering of power $\leq \aleph_1$. In the absence of CH, which linear orderings of power $\leq 2^{\aleph_0}$ are embeddable in $(\omega)^\omega$? It is well known that $2^{\aleph_0}$ can be arbitrarily large with ω_2 not embeddable in $(\omega)^\omega$ (for example start with a ground model of CH and add Cohen reals, Sacks reals, or Solovay reals). See also Solovay ([4]) for a stronger result in this direction. Now MA implies that every well ordering or converse well ordering of power $\leq 2^{\aleph_0}$, and every linear ordering of power $< 2^{\aleph_0}$, is embeddable in $(\omega)^\omega$, but Kunen [2] has shown that MA is consistent with the existence of a linear ordering of power $2^{\aleph_0}$ which is not embeddable in $(\omega)^\omega$. The problem with an attempt to inductively embed, using MA, any linear ordering of power $2^{\aleph_0}$ into $(\omega)^\omega$, is the possibility of creating in the course of the constriction a "Hausdorff gap" ([1]), that is, a cut in a linearly ordered subset of $(\omega)^\omega$, of left and right character ω_1, which cannot be filled in V or in any extension of V which preserves ω_1.

In this paper it is proved consistent that $2^{\aleph_0} > \aleph_1$ and every linear ordering of cardinality $\leq 2^{\aleph_0}$ is embeddable in $(\omega)^\omega$. This question was raised by Solovay (unpublished) in connection with his and Woodin's results on homomorphisms of Banach algebras ([4], [5]).

For $f,g \in (\omega)^\omega$, let $f < g$ mean that $\lim_n(g(n) - f(n)) = +\infty$. Let $\mathfrak{M}$ be a countable transitive model of ZFC in which κ is a regular uncountable cardinal satisfying $2^{<\kappa} = \kappa$. It is shown that there is a ccc forcing extension of $\mathfrak{M}$ in which $2^{\aleph_0} = 2^{<\kappa} = \kappa$ and the saturated linear ordering of power $2^{\aleph_0}$ is embeddable in $((\omega)^\omega, <)$, which will prove the theorem.

The extension is by ccc iterated forcing ([3]). There are κ steps in the iteration; for $\alpha \leq \kappa$ let $\mathfrak{M}_\alpha$ be the model obtained after the first α steps. To go from $\mathfrak{M}_\alpha$ to $\mathfrak{M}_{\alpha+1}$ a partial ordering $Q_\alpha \in \mathfrak{M}_\alpha$ is forced with, and from the resulting generic set an $f_\alpha \in (\omega)^\omega$ is defined. The final set $\{f_\alpha : \alpha < \kappa\}$ in $\mathfrak{M}_\kappa$, ordered by $<$, is the saturated linear ordering of the theorem.

Let $\mathfrak{M}_0 = \mathfrak{M}$. Suppose $\mathfrak{M}_\alpha$ is defined, in which $\{f_\beta : \beta < \alpha\}$ is linearly ordered by $<$. In $\mathfrak{M}_\alpha$, define Q_α as follows. If $\alpha = 0$, Q_α is the standard partial ordering of finite conditions for adding a generic $g : \omega \to \omega$ If $\alpha > 0$, pick a cut C_α in $(\{f_\beta : \beta < \alpha\},<)$. Then a condition in Q_α

is a $q = \langle L,R,s,m\rangle$, such that

(i) $L,R \in [\alpha]^{<\aleph_0}$, $f_\gamma < C_\alpha$ $(\gamma \in L)$, $C_\alpha < f_\delta$ $(\delta \in R)$

(ii) $s \in (\omega)^{<\omega}$, $m < \omega$, and, letting $\ell(s)$ be the greatest member of $\mathrm{dom}(s)$,

(iii) $f_\gamma(n) + 2m \leq f_\delta(n)$ $(\gamma \in L , \delta \in R , \ell(s) < n)$.

Order Q_α by the rule: $\langle L,R,s,m\rangle \leq \langle L',R',s',m'\rangle$ if and only if $L \subseteq L'$, $R \subseteq R'$, $s \leq s'$ ($\leq$ means initial segment here), $m \leq m'$, and

$$f_\gamma(n) + m \leq s'(n) \leq f_\delta(n) - m \quad (\gamma \in L , \delta \in R , \ell(s) < n \leq \ell(s')) .$$

Let G_α be $(\mathcal{M}_\alpha , Q_\alpha)$ - generic. Then $f_\alpha =_{\mathrm{def}} \bigcup_{q\in G_\alpha} s_q$ is by genericity a member of $(\omega)^\omega$ satisfying $f_\gamma < f_\alpha < f_\delta$ $(\gamma,\delta < \alpha ,\ f_\gamma < C_\alpha , C_\alpha < f_\delta)$.

The theorem will be proved upon showing that for any sequence of choices of cuts C_α in the course of the iteration, the associated orderings Q_α have the ccc . Namely, when this is done, the usual ccc bookkeeping method allows the C_α's to be chosen so that in M_κ , if $A,B \in [\{f_\alpha : \alpha < \kappa\}]^{<\kappa}$ and $A < B$ then there is an $\alpha < \kappa$ with $A < f_\alpha < B$.

The orderings $\mathcal{P}_\alpha$, $\alpha > 0$, such that $\mathcal{P}_\alpha$ gives the extension $\mathcal{M} \to \mathcal{M}_\alpha$, may be inductively characterized as follows. $\mathcal{P}_\alpha$ is a set of functions with domain α . $\mathcal{P}_1 = \{p : p(0) \in Q_0\}$ ordered by $p \leq q \Leftrightarrow p(0) \leq q(0)$. $\mathcal{P}_{\alpha+1} = \{p : p\restriction\alpha \in \mathcal{P}_\alpha , p(\alpha)$ a term in the forcing language of $\mathcal{P}_\alpha , \Vdash p(\alpha) \in Q_\alpha\}$, ordered by $p \leq q \Leftrightarrow p\restriction\alpha \leq q\restriction\alpha$ and $q\restriction\alpha \Vdash p(\alpha) \leq q(\alpha)$. Finally, for λ a limit ordinal, $\mathcal{P}_\lambda = \{p : p\restriction\alpha \in \mathcal{P}_\alpha$ for all $\alpha < \lambda$, and for all but finitely many $\alpha < \lambda$, $p(\alpha)$ is the canonical term for $\emptyset\}$, ordered by $p \leq q \Leftrightarrow$ for all $\alpha < \lambda$, $p\restriction\alpha \leq q\restriction\alpha$. We prove by induction on γ a statement which implies that $\mathcal{P}_\gamma$ has the ccc $(1 \leq \gamma \leq \kappa)$, which will prove the theorem.

For $p \in \mathcal{P}_\alpha$, $\mathrm{supp}(p)$ is the finite set $\{\beta < \alpha : p(\beta)$ is not the canonical term for $\emptyset\}$. If $p \in \mathcal{P}_\alpha$ and $\beta \in \mathrm{supp}\ (p)$ then $p(\beta) = \langle \dot{L}_{\beta,p}, \dot{R}_{\beta,p}, \dot{s}_{\beta,p}, \dot{m}_{\beta,p}\rangle$, written $\langle \dot{L}_\beta,\dot{R}_\beta,\dot{s}_\beta,\dot{m}_\beta\rangle$ if the context permits. Call p <u>determined</u> if for each $\beta \in \mathrm{supp}\ (p)$, the denotation of $p(\beta)$ has been decided, that is, there is an $\langle L_\beta,R_\beta,s_\beta,m_\beta\rangle \in \mathcal{M}$ such that $p\restriction\beta \Vdash \langle \dot{L}_\beta,\dot{R}_\beta,\dot{s}_\beta,\dot{m}_\beta\rangle = \langle \hat{L}_\beta,\hat{R}_\beta,\hat{s}_\beta,\hat{m}_\beta\rangle$. A determined condition p is <u>uniform</u> if there is an ℓ_p such that for all $\beta \in \mathrm{supp}\ (p)$, $\ell(s_\beta) = \ell_p$. A determined condition p has <u>closed support</u> if for every $\beta \in \mathrm{supp}\ (p)$, $L_\beta \cup R_\beta \subseteq \mathrm{supp}\ (p)$. Let $\mathcal{U}_\gamma$ be the set of determined, uniform $p \in \mathcal{P}_\gamma$ with closed support, and let $\mathcal{U}_{\gamma,n} = \{p \in \mathcal{U}_\gamma : \ell_p \geq n\}$.

<u>Lemma 1</u>. Each $\mathcal{U}_{\gamma,n}$ is dense in $\mathcal{P}_\gamma$.

Proof. By induction on γ. At limit ordinals γ the lemma follows from the induction hypothesis and the finite support property of the conditions. Suppose now that $\gamma = \alpha + 1$ and that $p \in \mathcal{P}_\gamma$. Let $q = p \restriction \alpha$. Choose $\langle L_\alpha, R_\alpha, s_\alpha, m_\alpha \rangle \in \mathcal{M}$ and $q_1 \geq q$ such that $q_1 \Vdash p(\alpha) = \langle \hat{L}_\alpha, \hat{R}_\alpha, \hat{s}_\alpha, \hat{m}_\alpha \rangle$. Choose $q_2 \geq q_1$ so that $L_\alpha \cup R_\alpha \subseteq \operatorname{supp}(q_2)$. Let $q_3 \geq q_2$ be a member of $\mathcal{U}_\alpha$. Pick $\bar{n} \geq n$, ℓ_{q_3}. Let $q_4 \geq q_3$ be a member of $\mathcal{U}_{\alpha,\bar{n}}$. We claim there is an $s \in \mathcal{M}$ with $\ell_s = \ell_{q_4}$, such that $q_4 \frown \langle \hat{L}_\alpha, \hat{R}_\alpha, \hat{s}_\alpha, \hat{m}_\alpha \rangle \leq q_4 \frown \langle \hat{L}_\alpha, \hat{R}_\alpha, \hat{s}, \hat{m}_\alpha \rangle$; then $q_4 \frown \langle \hat{L}_\alpha, \hat{R}_\alpha, \hat{s}, \hat{m}_\alpha \rangle$ will be an extension of p in $\mathcal{U}_{\alpha+1,n}$ as desired. The existence of such an s is guaranteed by the facts that $L_\alpha \cup R_\alpha \subseteq \operatorname{supp} q_4$, $q_4 \in \mathcal{U}_\alpha$, $\ell_{q_4} \geq \ell_s$, and $q_4 \frown \langle \hat{L}_\alpha, \hat{R}_\alpha, \hat{s}_\alpha, \hat{m}_\alpha \rangle$ is a condition (extending p).

We work only with determined conditions from now on, and suppress the $\hat{}$ notation.

For $p, q \in \mathcal{U}_\gamma$, $p \leq^* q$ means that $p \leq q$, $\operatorname{supp}(p) = \operatorname{supp}(q)$, and for each $\beta \in \operatorname{supp} p$, $s_{\beta,p} = s_{\beta,q}$ (thus q is obtained from p by raising some of the $m_{\beta,p}$'s and increasing some of the $L_{\beta,p}$'s, $R_{\beta,p}$'s to include new members of $\operatorname{supp}(p)$).

Lemma 2. Let $p \in \mathcal{U}_\gamma$. Then

(i) For every $\bar{m}$ there is an r with $p \leq^* r$ such that for each $\beta \in \operatorname{supp}(r)$, $m_{\beta,r} \geq \bar{m}$.

(ii) If $\beta, \delta \in \operatorname{supp}(p)$, $\beta < \delta$, and $p \Vdash f_\beta < f_\delta$ ($p \Vdash f_\delta < f_\beta$) then there is an r with $p \leq^* r$ such that $\beta \in L_{r,\delta}$ ($\beta \in R_{r,\delta}$).

Proof. We prove (i) and (ii) simultaneously, by induction on γ. If γ is a limit ordinal the lemma follows from the induction hypothesis. Assume $\gamma = \alpha + 1$.

(i) We are given $p \in \mathcal{U}_\gamma$. Let $q = p \restriction \alpha$. If $\beta \in L_\alpha$ and $\delta \in R_\alpha$ then $p \restriction \alpha \Vdash f_\beta < f_\delta$. Apply, then, the lemma for α, $\operatorname{Card} L_\alpha \cdot \operatorname{Card} R_\alpha$ times to get a q_1 with $q \leq^* q_1$ such that whenever $\beta \in L_\alpha$, $\delta \in R_\alpha$, then either $\beta \in L_{q_1,\delta}$ or $\delta \in R_{q_1,\beta}$, depending on whether $\beta < \delta$ or $\delta < \beta$. Now pick a q_2 with $q_1 \leq^* q_2$ such that for each $\beta \in \operatorname{dom} q_2$, $m_{q_2,\beta} \geq 2\bar{m}$. Thus q_2 forces that if $\beta \in L_\alpha$, $\delta \in R_\alpha$, and $n > \ell_p$, then $f_\beta(n) + 2\bar{m} \leq f_\delta(n)$. Then (i) is satisfied by taking $r = q_2 \frown \langle L_\alpha, R_\alpha, s_\alpha, \bar{m} \rangle$.

(ii) By symmetry it is enough to consider the case that $p \Vdash f_\beta < f_\delta$. Then $p \restriction \delta \Vdash f_\beta < f_\delta$, since the cut C_δ is determined by $\mathcal{P}_\delta$. We may as well assume $\delta = \alpha$, lest we be done by induction. Let $q = p \restriction \alpha$. For every $\sigma \in R_\alpha$, $q \Vdash f_\beta < f_\sigma$ (since $q \Vdash f_\beta < f_\alpha$ and $\sigma \in R_\alpha$). Apply the lemma for α, $\operatorname{Card} R_\alpha$ times, to get a q_1 with $q \leq^* q_1$ such that for all

$\sigma \in R_\alpha$, $\beta \in L_{q_1\sigma}$ or $\sigma \in L_{q_1\beta}$, depending on whether $\beta < \sigma$ or $\sigma < \beta$. Pick q_2 with $q_1 \leq^* q_2$ such that for each $\lambda \in \text{supp}\ (q_2)$, $m_{q_2,\lambda} \geq 2m_\alpha$. Then for each $\sigma \in R_\alpha$, $q_2 \Vdash \forall n > \ell(p) f_\beta(n) + 2m_\alpha \leq f_\sigma(n)$. Then (ii) is satisfied by taking $r = q_2 \frown \langle L_\alpha \cup \{\beta\}, R_\alpha, s_\alpha, m_\alpha\rangle$.

Lemma 3. If $p,q \in \mathcal{U}_\gamma$, $\ell_p = \ell_q$, and for every $\beta \in \text{supp}\ (p) \cap \text{supp}\ (q)$, $s_{p,\beta} = s_{q,\beta}$, then there is an $r \in \mathcal{U}_\gamma$ with $p \leq r$, $q \leq r$, $\ell_p = \ell_r$.

Proof. By induction on γ . Again, the induction hypothesis gives the result for γ a limit ordinal, so assume $\gamma = \alpha + 1$. Assuming $\text{supp}(p \restriction \alpha)$, $\text{supp}\ (q \restriction \alpha) \neq \emptyset$, let $r_1 \in \mathcal{U}_\alpha$ be given by applying the lemma to $p \restriction \alpha$, $q \restriction \alpha$ (if one of $p \restriction \alpha$, $q \restriction \alpha$ has empty support, let r_1 be the other). If, say, $\alpha \notin \text{supp}\ (p)$ then $r = r_1 \frown q(\alpha)$ satisfies the lemma, and symmetrically for q , so assume $\alpha \in \text{supp}\ (p) \cap \text{supp}\ (q)$. We have that $r_1 \Vdash \forall \beta \in L_{p,\alpha} \cup L_{q,\alpha}\ \forall \delta \in R_{p,\alpha} \cup R_{q,\alpha}\ f_\beta < f_\delta$ (since r_1 forces $f_\beta(f_\delta)$ to be left (right) of the cut C_α). By repeated applications of Lemma 2(ii) there is an r_2 with $r_1 \leq^* r_2$ such that for all $\beta \in L_{p,\alpha} \cup L_{q,\alpha}$, all $\delta \in R_{p,\alpha} \cup R_{q,\alpha}$, $\beta \in L_{r_2,\delta}$ or $\delta \in R_{r_2,\beta}$. By Lemma 2(i) there is an r_3 with $r_2 \leq^* r_3$ such that for all $\beta \in \text{supp}\ r_3$, $m_{r_3,\beta} \geq 2m_{p,\alpha}$, $2m_{q,\alpha}$. Take $r = r_3 \frown \langle L_{p,\alpha} \cup L_{q,\alpha}, R_{p,\alpha} \cup R_{q,\alpha}, s_{p,\alpha}, \max\{m_{p,\alpha}, m_{q,\alpha}\}\rangle$. Then r is as desired, giving the lemma.

The lemmas imply that $\mathcal{P}_\gamma$ has the ccc $(1 \leq \gamma \leq \kappa)$. Namely, given $\{p_\alpha : \alpha < \omega_1\} \subseteq \mathcal{P}_\gamma$, choose, by Lemma 1 , $\{q_\alpha : \alpha < \omega_1\} \subseteq \mathcal{U}_\gamma$ with $p_\alpha \leq q_\alpha$, then the usual Δ - system argument gives $\aleph_1$ q_α's which pairwise satisfy the hypothesis of Lemma 3 , and which are thus pairwise compatible.

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HYPERMEASURABLE CARDINALS

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If μ is a normal measure on κ then $L(\mu)$ is an inner model in which κ is measurable; and the existence of this L-like model has been vital to understanding measurable cardinals. For larger cardinals this vital tool is missing: even for a κ^+-supercompact cardinal κ, the weakest basic large cardinal stronger than a measurable cardinal, no such inner model is known. The problem is that if μ is a normal measure on $P_\kappa(\kappa^+)$ then $\mu(P_\kappa(\kappa^+) \cap L) = 0$ so $L(\mu)$ is just L. A possible approach to finding an inner model for a κ^+-supercompact cardinal would be to define large cardinal properties intermediate in strength between measurability and κ^+-supercompactness and find inner models for these "hypermeasurable" cardinals, and then to work up through these models to construct the needed members of $P_\kappa(\kappa^+)$. This paper is a first step in this program.

We will say that a cardinal κ is μ-measurable if there is an elementary embedding $j:V \to M$ such that M is a transitive class, κ is the first ordinal moved, and $U_j = \{x \subset \kappa : \kappa \in j(x)\} \in M$. A cardinal κ is $P^2(\kappa)$-measurable if there is such an embedding with $P^2(\kappa) \subset M$. Although these cardinals are introduced as steppingstones to supercompactness, we hope that they will prove interesting in themselves. Many of the definitions given here should be regarded as provisional: more study may show how they can be modified to be more general and more informative.

The paper is divided into 3 sections. The first section defines hypermeasures and their properties. The second section is the major part of the paper: it defines and studies inner models for μ-measurable and $P^2(\kappa)$-measurable cardinals. This section is selfcontained but a knowledge of [4] will be helpful. The final section discusses some actual and possible extensions of the earlier material. Some of this section assumes some knowledge of [5].

1. Hypermeasures

Unlike measurable cardinals and κ^+-supercompact cardinals, the hypermeasurable cardinals do not seem to have an ultrafilter characterization which is, in general, equivalent. We present a notion which seems to be related and which is equivalent in the models to be presented in the next section.

Hypermeasures are motivated by the following approximation of an arbitrary elementary embedding $j:V \to M$ by an iterated ultrapower.

Define X_α and κ_α by induction on α:

$$X_\alpha = \{j(f)(x): f \in V \text{ and } x \text{ is a finite subset of } \{\kappa_{\alpha'}:\alpha'<\alpha\}\} \quad .$$

$$\kappa_\alpha = \bigcap (ON - X_\alpha) \quad .$$

The definition continues for all $\alpha \in ON$ or until $ON \subset X_\alpha$. Then $X_\alpha \prec M$ for each α. Now let $k_\alpha : M_\alpha \cong X_\alpha$ be the transitive collapse of X_α and $U_\alpha = \{x \subset \kappa_\alpha : x \in M_\alpha \text{ and } \kappa_\alpha \in k_\alpha(x)\}$. Then U_α is a normal ultrafilter on $P(\kappa_\alpha) \cap M_\alpha$ and $M_{\alpha+1} = M_\alpha^{\kappa_\alpha}/U_\alpha$. Indeed we have $M_0 = V$ and $M_\alpha = \text{ult}_\alpha(M_0,(U_{\alpha'}:\alpha'<\alpha))$. If j_α is the elementary embedding, $j_\alpha : M_0 \to M_\alpha$ then $j = \kappa_\alpha j_\alpha$. If we take j_∞ to be the limit of the j_α's then $j = k_\infty j_\infty$ and $k \restriction ON = id$. Hence $j | ON = j_\infty | ON$ and if there is a definable well ordering of the universe (as would be the case, for example, in an inner model) then $j = j_\infty$.

This is at first sight somewhat surprising. One expects, for example, an iterated ultrapower M_α to have ordinals which are regular in M_α but really of cofinality ω, but this does not happen. The reason is that unlike in the classical iterated ultrapowers (see [1] and [3]) the ultrafilters are not in general M_α-ultrafilters (indeed, κ_α may be a successor cardinal in M_α) and are not images of M_0-ultrafilters. Thus, in contrast to a classical iterated ultrapower, where the individual ultrapowers are totally independent, it is possible to have one ultrafilter U_α in the sequence isomorphic to $\prod(U_{\alpha'}:\alpha'<\alpha)$. Thus the ultrapowers are so interrelated as to form a hybrid between a single ultrapower and a classical iterated ultrapower. Our definition of a hypermeasure is an attempt to capture this interrelationship.

We follow Kunen [3] in representing an iterated ultrapower as a single ultrapower by an ultrafilter on the set of subsets of ${}^\delta\kappa$ with finite support, where δ is the number of iterations and the ultrafilter is on κ. A subset Y of ${}^\delta\kappa$ has support $x \subset \delta$ if there is a set $Y' \subset {}^x\kappa$ such that $Y = \{a \in {}^\delta\kappa : a \restriction x \in Y'\}$. We will not generally distinguish between Y and Y'. For example if $a \in {}^{x'}\kappa$ for some x', $x \subset x' \subset \delta$, we will write "$a \in Y$" for "$a \restriction x \in Y'$". Functions on ${}^\delta\kappa$ are treated similarly. $P_f({}^\delta\kappa)$ is the set of subsets of ${}^\delta\kappa$ having a finite support.

An iterated ultrapower of length δ on α is defined to be an ultrafilter F on the Boolean algebra $P_f({}^\delta\alpha)$ which satisfies the following 4 conditions:

(1) $\forall \nu < \nu' < \delta \quad \{a : a_\nu < a_{\nu'}\} \in F$ and
$\forall \lambda < \alpha \quad \{a : \lambda < a_0\} \in F$

(2) (normality) for all ν and all f, if $\{a : f(a) \in a_\nu\} \in F$ then there is an f' with support in ν such that $\{a : f(a) = f'(a)\} \in F$.

(3) (nontriviality) for all ν and all f with support in ν, $\{a : f(a) = a_\nu\} \notin F$.

(4) (countable completeness) if $\{X_n : n \in \omega\} \subset F$ then $\bigcap_{n\in\omega} X_n \neq \phi$.

One example of an iterated ultrafilter is, of course, that given by a classical iterated ultrapower. If F is an iterated ultrafilter of length δ on α then for $x \subset \delta$ define $F[x] = F \cap P_f({}^x\delta)$. We attempt to express the desired interrelationship by defining a hypermeasure of length δ on α to be an iterated ultrafilter such that

(5) for all $\nu < \delta$, $F[\nu] \in \mathrm{ult}(V,F)$.

If F is an iterated ultrafilter then we write i^F for the elementary embedding $V \to \mathrm{ult}(V,F)$ and, if f is a function with finite support, $[f]_F$ for the equivalence class of f in the ultrapower by F. Let $b^F : \delta \to i^F(\alpha)$ be defined by $b^F(\nu) = [\lambda a \ a_\nu]_F$. Then for all X, $X \in F \equiv b^F(i^F)^{-1} \in i^F(X)$. This apparent dependence on $(i^F)^{-1}$ can be removed if necessary by taking a finite support x for X, a map $\sigma : n \cong x$, and $X' = \{a \in {}^n\alpha : a\sigma^{-1} \in X\}$. Then $X \in F$ iff $b^F\sigma \in i^F(X')$.

<u>1.1 Proposition</u>: If F is an iterated ultrafilter of length δ and $\nu < \delta$ then $b^{F[\nu]} = b^F \restriction \nu$.

<u>Proof</u>: If λ is any ordinal less than b^F_ν then by (2) $\lambda = [f]_F$ for some function f with support in ν. It follows that for any such f, $[f]_F = [f]_{F[\nu]}$. In particular, this holds for the functions $\lambda a \ a_\eta$ for $\eta \in \nu$ so $b^{F[\nu]} = b^F \restriction \nu$. $\square$

<u>1.2 Proposition</u>: If F is a hypermeasure of length δ on σ then $\mathrm{ult}(V,F)$ is closed under sequences of length less than $\inf(\alpha^+, \mathrm{cf}(\delta))$.

<u>Proof</u>: Let $([f_\eta] : \eta < \lambda)$ be a sequence of members of $\mathrm{ult}(V,F)$, with $\lambda \leq \alpha$, and let x_η be the support of f_η. Then $(i^F(f_\eta) : \eta < \lambda)$ and $(i^F(x_\eta) : \eta < \lambda)$ are in $\mathrm{ult}(U,F)$, as is $i^F \restriction \bigcup_{\eta<\lambda} x_\eta$. If $\lambda < \mathrm{cf}(\delta)$ then there is ν such that $\bigcup_{\eta<\lambda} x_\eta \subset \nu$. We have $F[\nu] \in \mathrm{ult}(V,F)$ and, since all functions involved are coded by subsets of α, $b^{F[\nu]}$ is the same in $\mathrm{ult}(V,F)$ as in the real world. Then

$$\begin{aligned} ([f_\eta]_F : \eta < \lambda) &= (i^F(f_\eta)((b^F \restriction x_\eta) i^F) : \eta < \lambda) \\ &= (i^F(f_\eta)((b^{F[\nu]} \restriction x_\eta) i^F) : \eta < \lambda) \\ &\in \mathrm{ult}(V,F) \quad . \quad \square \end{aligned}$$

Clearly, if F is a hypermeasure on σ of length more than 1 then α is μ-measurable. We do not know whether the other direction holds, or if either direction holds for larger hyper-measures.

2. The Inner Models

The models we are considering are of the form $L(\mathcal{F})$, where $\mathcal{F}$ is an ultrafilter sequence.

2.1 definition: An ultrafilter sequence $\mathcal{F}$ is a function with domain of the form $\{(\alpha,\beta):\alpha<\ell^{\mathcal{F}}$ and $\beta<0^{\mathcal{F}}(\alpha)\}$, where $\ell^{\mathcal{F}} \in ON$ and $0^{\mathcal{F}}:\ell^{\mathcal{F}} \to ON$, such that for each $(\alpha,\beta) \in$ domain $\mathcal{F}$, $F = \mathcal{F}(\alpha,\beta)$ is an ultrafilter on $P_f(\delta_\alpha)$ for some $\delta = \delta(\alpha,\beta)$ and F satisfies (1) - (5) below.

(1) $\forall\lambda < \alpha\{a:\lambda<a_0\} \in F$ and if $\nu < \nu' < \delta$ then $\{a:a_\nu < a_{\nu'} < 0^{\mathcal{F}}(a_0)\} \in F$.

(2) (normality) If $\nu < \delta$ and $\{a:f(a)<a_\nu\} \in F$ then for some $f' \in L(\mathcal{F}\restriction(\alpha,\beta))$ with support in ν, $\{a:f(a)=f'(a)\} \in F$.

(3) (nontriviality) If $f \in L(\mathcal{F}\restriction(\alpha,\beta))$ has support in ν then $\{a:f(a)=a_\nu\} \notin F$.

(4) (countable completeness) If $\{X_n:n\in\omega\} \subset F$ then $\bigcap_{n\in\omega}X_n \neq \phi$.

(5) (coherence) $i^F(\mathcal{F})\restriction(\alpha+1) = \mathcal{F}\restriction(\alpha,\beta)$, and if $\{a:f(a)<0^{\mathcal{F}}(a_0)\} \in F$ then there is $f' \in L(\mathcal{F}\restriction(\alpha,\beta))$ such that $\{a:f(a)=f'(a)\} \in F$.

Here, $\mathcal{F}\restriction(\alpha,\beta) = \mathcal{F}\restriction\{(\alpha',\beta'):(\alpha'<\alpha)$ or $(\alpha'=\alpha$ and $\beta'<\beta)\}$ and $\mathcal{F}\restriction\gamma = \mathcal{F}\restriction(\gamma,0)$. Hence (5) implies in particular that $0^{i^F(\mathcal{F})}(\alpha) = \beta$. The second part of (5) and the strengthening of (2) and the weakening of (3) from the definition of an iterated ultrafilter are included to ensure that the property of being an ultrafilter sequence is absolute in the sense that if $\mathcal{F}$ is an ultrafilter sequence then any $\mathcal{F}\restriction(\alpha,\beta)$ is an ultrafilter sequence in any model M containing $\mathcal{F}\restriction(\alpha,\beta)$. Because of the weakening of (3) the $\mathcal{F}(\alpha,\beta)$'s need not be iterated ultrafilters but we do have:

2.2 proposition: If $\mathcal{F}$ is an ultrafilter sequence then every $\mathcal{F}(\alpha,\beta)$ is isomorphic to an iterated ultrafilter.

Proof: Let $F = \mathcal{F}(\alpha,\beta)$ and $\delta = \delta(\alpha,\beta)$. Let $y = \{\nu \in \delta$: for all f with support in ν $\{a:f(a)=a_\nu\} \notin F\}$. Then there is a sequence $(f_\nu:\nu \in \delta-y)$ such that for each $\nu \in \delta - y$ f_ν has support in $\nu \cap y$ and $\{a:a_\nu = f_\nu(a)\} \in F$. Let $\sigma:\delta' \cong y$ and set $F^* = \{x \subset^{\delta'} \alpha:\{a\sigma^{-1}:a \in x\} \in F\}$. Then F^* is an iterated ultrafilter and F^* is isomorphic to F by σ and $(f_\nu:\nu \in \delta-y)$. $\square$

For the rest of this paper $\mathcal{F}$ or $\mathcal{G}$ will always be an ultrafilter sequence in $L(\mathcal{F})$ or $L(\mathcal{G})$ but not necessarily, unless stated otherwise, in the real world. In addition we will assume throughout that for all $\alpha < \ell^{\mathcal{F}}$, $0^{\mathcal{F}}(\beta) = 0$ for every

$\beta \le 0^F(\alpha)$. We will in fact be primarily concerned with sequences F such that $0^F(\alpha) \le \alpha^{++} + 1$ in $L(F)$. Such sequences will give models for μ-measurable and $P^2(\kappa)$-measurable cardinals. The more general case is discussed further in section 3.

As in [4], the next lemma is the key to the whole theory of ultrafilter sequences.

2.3 Main Lemma: For all F and F' there are iterated ultrapowers $i:L(F) \to L(i(F))$ and $i':L(F') \to L(i'(F'))$ such that either $i(F) = i'(F') \restriction \gamma$ or $i'(F') = i(F) \restriction \gamma$ for some $\gamma \in ON$.

Proof: The proof is a modification of that of lemma 2.3 of [4]. As in [4] we define a sequence of iterated ultrapowers

$$i_\lambda:L(F) \to L(F_\lambda),\ F_\lambda = i_\lambda(F)$$

$$i'_\lambda:L(F') \to L(F'_\lambda),\ F'_\lambda = i'_\lambda(F'_\lambda)\ ,$$

by induction on λ. For each λ, $(\alpha_\lambda,\beta_\lambda)$ is the least pair (α,β) such that $\beta < \sup(0^{F_\lambda}(\alpha),0^{F'_\lambda}(\alpha))$ and either $\beta = \inf(0^{F_\lambda}(\alpha),0^{F'_\lambda}(\alpha))$ or $F_\lambda(\alpha,\beta) \ne F'_\lambda(\alpha,\beta)$. Then $i_{\lambda\lambda+1}$ = id if $\beta_\lambda = 0^{F_\lambda}(\alpha_\lambda)$ and otherwise $i_{\lambda\lambda+1}:L(F_\lambda)\to \mathrm{ult}(L(F_\lambda),F_\lambda(\alpha_\lambda,\beta_\lambda))$; $i'_{\lambda\lambda+1}$ is defined similarly. If for some λ either $(\alpha_\lambda,\beta_\lambda)$ is undefined or $\alpha_\lambda \ge \inf(\ell^{F_\lambda},\ell^{F'_\lambda})$ then $i = i_\lambda$, $i' = i'_\lambda$ and $\gamma = \alpha_\lambda$ are as required. We will complete the proof by showing that the assumption that $(\alpha_\lambda,\beta_\lambda)$ is defined and $\alpha_\lambda < \inf(\ell^{F_\lambda},\ell^{F'_\lambda})$ for all ordinals λ leads to a contradiction. The proof will be based on the following proposition, which was suggested by the referee of [4].

2.4 Proposition: Suppose $x \in L(F)$, $x' \in L(F')$, Γ is a stationary class, and for each $\lambda \in \Gamma$ $d_\lambda \in i_\lambda(x) \cap i'_\lambda(x')$. Then there is a stationary subclass of Γ (which we will still call Γ) such that for $\lambda < \nu$ in Γ we have $i_{\lambda\nu}(d_\nu) = i'_{\lambda\nu}(d_\nu) = d_\nu$.

Proof: At any limit λ in Γ there is a $\lambda' < \lambda$ such that $d_\lambda = i_{\lambda'\lambda}(\bar{d})$ for some $\bar{d} \in i_{\lambda'}(x)$. Then we can shrink Γ to a stationary class so that the same λ' works for all $\lambda \in \Gamma$. Then we can shrink Γ further so that $\bar{d}$ is the same for all $\lambda \in \Gamma$. For $\lambda < \nu$ in Γ we now have $i_{\lambda\nu}(d_\lambda) = i_{\lambda\nu}(i_{\lambda'\lambda}(\bar{d})) = i_\nu(\bar{d}) = d_\nu$. Repeating the argument with i' instead of i completes the proof. ☐ 2.4

Proof of 2.3, concluded: First we apply proposition 2.4 with $d_\lambda = (\alpha_\lambda,\beta_\lambda)$. Then for $\lambda, \nu \in \Gamma$, $i_{\lambda\nu}(\alpha_\lambda) = i'_{\lambda\nu}(\alpha_\lambda) = \alpha_\nu > \alpha_\lambda$ so $i_{\lambda\lambda+1} \ne$ id and $i'_{\lambda\lambda+1} \ne$ id. Thus $\beta_\lambda < \inf(0^{F_\lambda}(\alpha_\lambda),0^{F'_\lambda}(\alpha_\lambda))$ and $F_\lambda(\alpha_\lambda,\beta_\lambda) \ne F'_\lambda(\alpha_\lambda,\beta_\lambda)$. Let $F_\lambda = F_\lambda(\alpha_\lambda,\beta_\lambda)$ and $F'_\lambda = F'_\lambda(\alpha_\lambda,\beta_\lambda)$. We will first show that the class of $\lambda \in \Gamma$ such that $b^{F_\lambda} \ne b^{F'_\lambda}$ is nonstationary; then we will conclude the proof by showing that our assumption that $F_\lambda \ne F'_\lambda$ for $\lambda \in \Gamma$ leads to a contradiction.

Suppose we can shrink Γ to a stationary subclass such that $b^{F_\lambda} \neq b^{F'_\lambda}$ for $\lambda \in \Gamma$. Let η_λ be least such that $b^{F_\lambda} \restriction \eta_\lambda + 1 \neq b^{F'} \restriction \eta_\lambda + 1$. At least one of δ^{F_λ} and $\delta^{F'_\lambda}$ is greater than η_λ. We can assume (possibly shrinking Γ) that $\delta^{F'_\lambda} > \eta_\lambda$ and $b^{F'_\lambda}_{\eta_\lambda} < b^{F_\lambda}_{\eta_\lambda}$ if $\delta^{F_\lambda} > \eta_\lambda$. Hence by definition 2.1 (2) and (5), $b^{F'_\lambda}_{\eta_\lambda}$ can be represented in the form $[g_\lambda]_{F_\lambda}$ where $g_\lambda \in L(\mathcal{F}_\lambda \restriction (\alpha_\lambda, \beta_\lambda)) = L(\mathcal{F}'_\lambda \restriction (\alpha_\lambda, \beta_\lambda))$ and g_λ has support in η_λ. Now apply proposition 2.4 with $d_\lambda = (\eta_\lambda, g_\lambda)$ and let λ, ν be in Γ with $\lambda < \nu$. We have $i_{\lambda\nu}(g_\lambda)((b^{F_\lambda})(i^{-1}_{\lambda\nu})) = [g_\lambda]_{F_\lambda} = b^{F'_\lambda}_{\eta_\lambda}$. But $i_{\lambda\nu}(g_\lambda) = g_\nu = i'_{\lambda\nu}(g_\lambda)$. Also $i_{\lambda\nu}(\text{support } g_\lambda) = \text{support } g_\nu = i'_{\lambda\nu}(\text{support } g_\lambda)$ and so, since support $(g_\lambda) \subset \eta_\lambda$, $(b^{F_\lambda})(i_{\lambda\nu})^{-1} \restriction \text{support}(g_\nu) = (b^{F'_\lambda})(i'_{\lambda\nu})^{-1} \restriction \text{support}(g_\nu)$. Hence $i'_{\lambda\nu}(g_\lambda)((b^{F'_\lambda})(i'_{\lambda\nu})^{-1}) = b^{F'_\lambda}_{\eta_\lambda}$, so $\{a : a_{\eta_\lambda} = g_\lambda(a \restriction \eta_\lambda)\} \in F'_\lambda$, contrary to definition 2.1 (3).

Hence we can shrink Γ to a stationary subclass so that $b^{F_\lambda} = b^{F'_\lambda}$ for $\lambda \in \Gamma$. Pick X_λ so that F_λ and F'_λ disagree on X_λ and apply proposition 2.4 with $d_\lambda = X_\lambda$. Then for $\lambda < \nu$ in Γ we have $X_\lambda \in F_\lambda$ iff $b^{F_\lambda} i^{-1}_{\lambda\nu} \in i_{\lambda\nu}(X_\lambda)$ and $X'_\lambda \in F'_\lambda$ iff $(b^{F'_\lambda})(i'_{\lambda\nu})^{-1} \in i'_{\lambda\nu}(X_\lambda)$. But $i_{\lambda\nu}(X_\lambda) = X_\nu = i'_{\lambda\nu}(X_\nu)$, $b^F_\lambda = b^{F'}_\lambda$, and $i_{\lambda\nu} \restriction \text{support}(X_\lambda) = i'_{\lambda\nu} \restriction \text{support}(X_\lambda)$, so we get $X_\lambda \in F_\lambda$ iff $X_\lambda \in F'_\lambda$, contrary to the choice of X_λ. $\square$ 2.3

<u>2.5 Definition</u>: A sequence $\mathcal{F}$ is ϕ-minimal if ϕ is a sentence such that $L(\mathcal{F}) \models \phi$ but $L(\mathcal{F} | \delta) \not\models \phi$ for all proper initial segments $\mathcal{F} \restriction \delta$ of $\mathcal{F}$.

This property is important because if $\mathcal{F}$ and $\mathcal{F}'$ are ϕ-minimal for the same ϕ then the iterated ultrapowers $i(\mathcal{F})$ and $i(\mathcal{F}')$ given by lemma 2.3 are also ϕ-minimal, so $i(\mathcal{F}) = i(\mathcal{F}')$. Hence we have, as in [4]:

<u>2.6 Proposition</u>: Suppose $\mathcal{F}$ is ϕ-minimal and α is any ordinal such that $\alpha' < \alpha$ implies $0^{\mathcal{F}}(\alpha') < \alpha$. Then for any $x \subset \alpha$ in $L(\mathcal{F})$ and any class Γ of ordinals there is a set y definable in $L(\mathcal{F})$ from parameters in $\alpha \cup \Gamma$ such that $x = y \cap \alpha$.

<u>Proof</u>: Let $\pi : L(\mathcal{F}') \cong X \prec L(\mathcal{F})$ where X is minimal such that $\alpha \cup \Gamma \subset X$. Then $L(\mathcal{F}')$ is also ϕ minimal, so we have iterated ultrapowers $i : L(\mathcal{F}) \to L(\mathcal{G})$ and $i' : L(\mathcal{F}') \to L(\mathcal{G})$ with $i \restriction \alpha = i' \restriction \alpha = \text{id}$. Then $x \in L(\mathcal{G})$, so $x \in L(\mathcal{F}')$ and we can take $y = \pi(x)$. $\square$

<u>2.7 Theorem</u>: $\mathcal{F}$ is unique in $L(\mathcal{F})$: If $F \in L(\mathcal{F})$ and F satisfies 2.1 (1) - (5) for some (α, β) then $\beta < 0^{\mathcal{F}}(\alpha)$ and $F = \mathcal{F}(\alpha, \beta)$.

<u>Proof</u>: If not we can assume $\mathcal{F}$ is ϕ-minimal where ϕ asserts that the theorem fails. Then we have iterated ultrapowers i and i'

$$\begin{array}{ccccc} & & \text{ult}(L(\mathcal{F}), F) & & \\ & \overset{j}{\nearrow} & & \overset{i}{\searrow} & \\ L(\mathcal{F}) & & \xrightarrow{\quad i' \quad} & & L(\mathcal{G}) \end{array} \tag{1}$$

We can assume F is minimal in the ordering of $L(F)$ such that the theorem fails. Thus F is definable in $L(F)$, as are the maps in (1). But then if there are any x such that $ij(x) \neq i'(x)$ then the least such x is definable in $L(F)$. This is impossible since $ij(x) = i(x)$ for any definable x, so $ij = i'$.

Since F satisfies 2.1 (1)-(5), $\beta \leq 0^F(\alpha)$. If $\beta = 0^F(\alpha)$ then $j(F) \restriction \alpha + 1 = F \restriction (\alpha, 0^F(\alpha) = F \restriction (\alpha+1)$ so $\alpha = i(\alpha) = ij(\alpha) > \alpha$. Hence $\beta < 0^F(\alpha)$ and $F = F(\alpha,\beta)$ by the same argument as in the end of the proof of lemma 2.4. $\square$

The most important application of theorem 2.7 is given by the following corollary. Note that if $0^F(\kappa) \leq \kappa^{++} + 1$ in $L(F)$ then $\delta(\alpha,\beta) \leq \kappa^{++}$ for all (α,β) and the only sets X with the stated condition are the ordinals $\eta \in \delta(\alpha,\beta)$.

<u>2.9 corollary</u>: Suppose $F = F(\alpha,\beta)$, $\delta = \delta(\alpha,\beta)$, and $X \subset \delta$ are such that if $g \in L(F)$ has support in X and $[g]_F = b^F_\nu$ then $\nu \in X$. Then $F[X]$ is isomorphic to some $F(\alpha,\beta')$. In particular for every $F(\alpha,\beta)$ the iterated ultrapower isomorphic to $F(\alpha,\beta)$ by proposition 2.2 is a hypermeasure.

<u>Proof</u>: First note that $i^{F[X]}(F) \restriction \alpha + 1 = F \restriction (\alpha, \beta')$ for some β'; otherwise we could let $\bar{\beta}$ be least such that $i^{F[X]}(F)(\alpha, \bar{\beta}) \neq F(\alpha,\bar{\beta})$ and reach a contradiction with theorem 2.8. Now let $\sigma:\eta \cong X$ for some ordinal η and let $\bar{F}$ be isomorphic to $F[X]$ by σ. It is easy to check that $\bar{F}$ satisfies 2.1 (1)-(5) for (α,β') if we know that $P(\kappa) \cap L(F\restriction(\alpha,\beta')) = P(\kappa) \cap L(F\restriction(\alpha,\beta))$. But if $y \subset \kappa$ is in $L(F \restriction(\alpha,\beta))$ then $\{a:y \cap a_o \in L(F\restriction a_o+1)\} \in F$. This set has support in X so it is in $F[X]$ and hence in $\bar{F}$. Then $y \in L(F\restriction(\alpha,0(\bar{F})) = L(F\restriction(\alpha,\beta'))$. $\square$

<u>2.9 Theorem</u>: The GCH holds in $L(F)$, $\diamondsuit_\alpha$ holds in $L(F)$ for all α, and there is a Δ^1_3 well ordering of the reals.

<u>Proof</u>: We will only prove the GCH; the rest of the theorem uses the same idea. We will work inside $L(F)$. The proof divides into two cases

<u>Case 1</u>: $\forall \nu < \alpha \;\; 0^F(\nu) < \alpha$.

If $0^F(\alpha) \neq 0$, then $P(\alpha) \cap L(F) = P(\alpha) \cap \mathrm{ult}(L(F),F(\alpha,0))$ so we can assume $0^F(\alpha) = 0$. Every subset x of κ is in some $L\,(F)$ with η regular and by collapsing we get $x \in L_\gamma(G)$ for some γ and G such that $L_\gamma(G) \models (ZF^- + G$ is an ultrafilter sequence), $|L_\gamma(G)| = \alpha$, $G \restriction \alpha = F \restriction \alpha$ and $0^G(\alpha) = 0$. If $x,x' \subset \alpha$ then we say $x < x'$ if for any such $L_\gamma(G)$ with $x' \in L_\gamma(G)$, $x \in L_\gamma(G)$ and $x < x'$ in the order of construction in $L_\gamma(G)$. Since each $|L_\gamma(G)| = \alpha$, any x has at most α predecessors so the order type of $<$ is at most α^+. We have to show that $<$ is a linear order; that is, that if $L_{\gamma_1}(G_1)$ and $L_{\gamma_2}(G_2)$ are two such models then the ordering of $P(\alpha)$ in one of them is an initial segment of that in the other. By lemma 2.3 there are iterated ultrapowers $i_1: L_{\gamma_1}(G_1) \to L_{\bar{\gamma}_1}(\bar{G}_1)$ and $i_2: L_{\gamma_2}(G_2) \to L_{\bar{\gamma}_2}(\bar{G}_2)$ such that (say) $\bar{G}_1 = \bar{G}_2 \mid \bar{\gamma}_1$. Then i_1 and i_2 preserve the ordering of $P(\alpha)$ and $L_{\bar{\gamma}_1}(\bar{G}_1)$ is an initial segment of $L_{\bar{\gamma}_2}(\bar{G}_2)$. This completes the proof of case 1.

Case 2: $0^F(\nu) \geq \alpha$ for some $\nu < \alpha$.

There is no problem if $0^F(\nu) < \alpha^+$. If $\beta < \alpha^+$ and $i: L(F) \to \mathrm{ult}(L(F), F(\nu,\beta))$ then $|i(\nu)| \leq 2^\nu \cdot \delta(\nu,\beta) \leq \nu^+ \cdot \alpha \leq \alpha$ by case 1. Hence $|P(\alpha) \cap \mathrm{ult}(L(F),F(\nu,\beta))| < i(\nu) < \alpha^+$, and lemma 2.10, which is of some independent interest, will complete the proof.

2.10 lemma: If $\nu < \alpha$ and $\alpha^+ \leq 0^F(\nu)$ then every subset of α is in $\mathrm{ult}(L(F), F(\alpha,\beta))$ for some $\beta < \alpha^+$.

Proof: We first show that every $x \subset \alpha$ is in some $L_\gamma(G)$ such that $L_\gamma(G) \models (ZF^- + G$ is an ultrafilter sequence), $G \restriction \alpha + 1 = F \restriction (\nu,\beta)$ for some $\beta < \alpha^+$, and every member of $L_\gamma(G)$ is definable in $L_\gamma(G)$ from parameters in $\alpha \cup \{x\}$. Let η be a regular cardinal such that $P(\alpha) \subset L_\eta(F)$ and take $L_\gamma(G) \cong X \prec L_\eta(F)$ where X is the smallest such set with $\alpha \subset X$ and $x \in X$. Since $\nu^+ \subset \alpha \subset X$, $P(\nu) \subset X$ and so for any $\lambda \in X \cap 0^F(\nu)$, $X \cap \delta(\nu,\lambda)$ satisfies the hypothesis of corollary 2.8. Thus $G \restriction \alpha + 1 = F \restriction (\nu,\beta)$ where $\beta = \bigcup(X \cap 0^F(\nu))$, and $|X| = \alpha$ so $\beta < \alpha^+$.

To finish proving the lemma, we show that $L_\gamma(G) \in \mathrm{ult}(L(F),F(\nu,\beta))$. Use lemma 2.3 to define iterated ultrapowers i of $L_\gamma(G)$ and j of $L(\bar{F}) = \mathrm{ult}(L(F),F(\nu,\beta))$ so that one of $i(G)$ and $j(\bar{F})$ is an initial segment of the other. $j(\bar{F})$ cannot be a proper initial segment of $i(G)$; if it were we could use an extra measurable cardinal in $i(G)$ to generate indiscernibles for $L(i(\bar{F}))$ and hence for $L(\bar{F})$ and $L(F)$. This is impossible since the entire construction is inside of $L(F)$. Hence $L_{i(\gamma)}(i(G)) \in L(j(\bar{F}))$ and so $L_\gamma(G) \in L(i(\bar{F}))$ since $L_\gamma(G)$ is the transitive collapse of the members to $L_{i(\gamma)}(i(G))$ which are definable from parameters in $\alpha \cup \{x\}$. But $|L_\gamma(G))| = \alpha$ and $j \restriction P(\alpha) = \mathrm{id}$, so $L_\gamma(G) \in L(\bar{F})$, as was to be shown. $\square$ 2.10,2.9

The next two results show that, as claimed, the models $L(F)$ give inner models for μ-measurable and $P^2(\kappa)$-measurable cardinals.

2.11 Proposition: (i) If $\delta(\alpha,\beta) > 1$ and $P(\alpha) \cap L(F) \subset L(F \restriction (\alpha,\beta))$ then is μ-measurable in $L(F)$. (ii) If $0^F(\alpha) > \alpha^{++}$ in $L(F)$ then α is $P^2(\alpha)$-measurable in $L(F)$.

Proof: Part (i) is clear. Because the GCH holds in $L(F)$, $P^2(\alpha)$-measurability is equivalent to $P(\alpha^+)$-measurability. By lemma 2.10, any subset of α^+ is in $ult(L(F),F(\alpha,\beta)) \subset ult(L(F),F(\alpha,\alpha^{++}))$ for some $\beta < \alpha^{++}$, so $P(\alpha^+) \subset ult(L(F), (\alpha,\alpha^{++}))$ and α is $P(\alpha^+)$-measurable.

2.12 Theorem: Suppose κ is μ-measurable or $P^2(\kappa)$-measurable. Then there is a sequence G such that κ has the same property in $L(G)$.

Proof: First let $j:V \to M$ be an arbitrary elementary embedding with critical point κ. We will use j to construct a sequence F and prove a general lemma which will lead to the desired result when j witnesses that κ has one of the stated properties.

$F \restriction \kappa$ is defined by induction on (α,β): if $F \restriction (\alpha,\beta)$ is defined, there is an F satisfying 2.1 (1)-(5), and $\beta \leq \alpha^{++}$ in $L(F\restriction(\alpha,\beta))$, then $F(\alpha,\beta)$ is set equal to some such F. $0^F(\alpha)$ is the least β such that one of these conditions fails. Once $F \restriction \kappa$ is defined F is set equal to $j(F \restriction \kappa) \restriction \kappa + 1$. Now define sequences b_ν and M_ν by induction on ν:

$$M_\nu = \{j(f)((b\restriction\nu)j^{-1}): f \text{ has finite support}\}$$
$$b_\nu = \bigcap (0^F(\kappa)-M_\nu) \quad .$$

Let δ be least such that $0^F(\kappa)-M_\delta = 0$ and set $F_j = \{x \in P_f({}^\delta M): bj^{-1} \in j(x)\}$.

2.13 lemma: If $F_j \in M$ then κ is $P^2(\kappa)$-measurable in $L(F)$.

Proof: It can easily be verified that F_j satisfies 2.1 (1) - (5) for $(\kappa,0^F(\kappa))$. Since F is defined at κ in M the way $F \restriction \kappa$ was defined in V, either $F_j \notin M$ or $0^F(\kappa) > \kappa^{++}$ in $L(F)$. Thus if $F_j \in M$ then $0^F(\kappa) = \kappa^{++} + 1$ in $L(F)$ and κ is $P^2(\kappa)$-measurable in $L(F)$ by proposition 2.11. □ 2.13

For the rest of the proof we can assume $F_j \notin M$. Let $F = F_j$, $\delta = \delta(F)$ and $b = b^F$ and define F' with domain (F') = domain $(F) \cup \{(\kappa,0^F(\kappa)\}$ by

$$F' \restriction (\kappa,0^F(\kappa)) = F$$
$$F'(\kappa,0^F(\kappa)) = F \quad .$$

Now suppose that $U_j \in M$, so j witnesses that κ is μ-measurable. Since $F \notin M$ and $U_j = F[1] \in M$, $\delta \geq 2$. If $\delta = 2$ then κ is μ-measurable in $L(F')$ unless $F \cong F[1]$ in $L(F)$. But then $F[1] = U_j$ and the isomorphism is in M, since it is given by a function from κ into κ so $F \in M$, contrary to assumption. If $\delta > 2$ then by proposition 2.11 $F(\kappa,b_2) = F[2]$ shows that κ is μ-measurable in $L(F)$ unless $P(\kappa) \cap L(F) \not\subset L(F\restriction(\kappa,b_2))$. But if $x \in P(\kappa) \cap L(F)$ and $x \notin L(F\restriction(\kappa,b_1))$ then $\{a: x \cap a_0 \in L(F(a_0+1))\} \in F$ but $\{a_0: x \cap a_0 \in L(F|(a_0+1))\} \notin F[1]$, which is absurd. Hence if $\delta > 2$ then α is μ measurable in $L(F)$.

Now suppose $P^2(\kappa) \subset M$, so κ is $P^2(\kappa)$-measurable. If $\delta < \kappa^{++}$ then $F = \bigcup\{F[x] : x \in [\delta]^{<\omega}\}$, so F is essentially a κ^+-sequence of members of $P(\kappa)$. But this would imply $F \in M$. Hence $\delta = \kappa^{++} = \kappa^{++L(\mathcal{F})}$ so κ is $P^2(\kappa)$-measurable in $L(\mathcal{F}')$ by proposition 2.11. $\square$

We now know that $L(\mathcal{F})$ is an inner model for μ-measurable or $P^2(\kappa)$-measurable cardinals, but in what sense is there a unique minimal such model? There is no minimal model M in the sense that $M_k \models (\text{ZFC} + \kappa \text{ is } P^2(\kappa)\text{-measurable})$ and $M \subset M'$ for every other such M', since the first measurable cardinal can be arbitrarily large below κ. However by 2.11 we can restrict ourselves to models $M = L(\mathcal{F})$, with $\mathcal{F}$ ϕ-minimal for the desired property. The next theorem shows that there is such a model which is minimal in the same sense as the model $L(\mu)$ is minimal when μ is a measure on the smallest possible ordinal.

Theorem 2.13: If there is a sequence $\mathcal{F}$ such that $L(\mathcal{F}) \models \phi$ then there is a ϕ-minimal sequence $\mathcal{F}$ such that for any ϕ-minimal sequence $\mathcal{F}'$ there is an iterated ultrapower $i : L(\mathcal{F}) \to L(\mathcal{F}')$.

Proof: Let $\overline{\mathcal{F}}$ be an arbitrary ϕ-minimal sequence and let $\lambda = \ell^{\overline{\mathcal{F}}}$. If Γ is a class then X_Γ is the smallest class $X \prec L(\overline{\mathcal{F}})$ such that $\Gamma \subset X$. Call Γ closed if Γ is a proper class and for some ordinal δ, if cf $\gamma > \delta$ and Γ is cofinal in γ then $\gamma \in \Gamma$. Any intersection of closed classes is a closed class so there is a closed Γ such that $X_\Gamma \cap L_{\lambda^{++}}(\overline{\mathcal{F}}) \subset X_{\Gamma'}$, for any closed class Γ'. Fix such a Γ and let $\pi : L(\mathcal{F}) \cong X_\Gamma$. We will show that $\mathcal{F}$ is the required sequence.

Let $\mathcal{F}'$ be any ϕ-minimal sequence. Then by lemma 1.3 there are iterated ultrafilters

$$\begin{array}{l} i : L(\mathcal{F}) \searrow \\ \qquad\qquad\qquad L(\mathcal{G}) \\ i' : L(\mathcal{F}') \nearrow \end{array} .$$

We have to show that $i' = \text{id}$ and hence (taking the notation of the proof of lemma 1.3) $\beta_\nu = 0^{\mathcal{F}}(\alpha_\nu)$ for all ν. Let ν be least such that $\beta_\nu < 0^{\mathcal{F}'}(\alpha_\nu)$ and set $F = \mathcal{F}'(\alpha_\nu, \beta_\nu)$. Since $L(\mathcal{F}_\nu)$ and $L(\mathcal{F}')$ have the same subsets of α_ν, F is an ultrafilter on $L(\mathcal{F}_\nu)$. We claim that $\text{ult}(L(\mathcal{F}_\nu), F)$ is well founded. If not, neither is $\text{ult}(L(\mathcal{G}), F)$. But then by absoluteness the statement "there is an iterated ultrapower $L(\mathcal{F}^*)$ of $L(\mathcal{F}')$ such that $\text{ult}(L(\mathcal{F}^*), F)$ is not well founded" is true in $L(\mathcal{F}')$. This is impossible because F is countably complete in $L(\mathcal{F}')$.

Now consider the following triangle, where j and j' are given by lemma 1.3:

$$\begin{array}{ccccc} & & \text{ult}(L(\mathcal{F}_\nu), F) & & \\ & {\scriptstyle k} \nearrow & & \searrow {\scriptstyle j} & \\ L(\mathcal{F}_\nu) & & \xrightarrow{\quad j' \quad} & & L(\mathcal{H}) \end{array} .$$

We claim that $jk \upharpoonright L_{\bar\lambda}(F_\nu) = j' \upharpoonright L_{\bar\lambda}(F_\nu)$, where $\bar\lambda = i_\nu(\lambda^{++})$. This follows from the fact that every member of $L_{\bar\lambda}(F_\nu)$ has the form $i_\nu(f)(\xi)$ where $\xi \in \alpha_\nu$ and $f:\alpha_\nu \to L_{\lambda^{++}}(F_\nu)$ is in $L(F)$. If $\Gamma' = \{\gamma : i_\nu(\gamma) = j\kappa(\gamma) = j'(\gamma) = \gamma\}$ then every such f is definable in $L(F)$ from parameters in Γ'. Thus any member of $L_{\bar\lambda}(F_\nu)$ is definable from members of $\Gamma' \cup \alpha_\nu$ so $j\kappa \upharpoonright L_{\bar\lambda}(F_\nu) = j' \upharpoonright L_{\bar\lambda}(F_\nu)$. In particular $j'(\alpha_\nu) = jk(\alpha_\nu) > \alpha_\nu$ so $0^{F_\nu}(\alpha_\nu) > \beta_\nu$, as well. But now we can use the argument at the end of the proof of lemma 1.3 to show that $F'(\alpha_\nu,\beta_\nu) = F = F_\nu(\alpha_\nu,\beta_\nu)$, contradicting the definition of (α_ν,β_ν). $\square$

<u>2.14 Corollary</u>: If F and F' are ϕ-minimal for the same ϕ and $0^F(\alpha) = 0^{F'}(\alpha)$ for all α then $F = F'$. $\square$

The corollary is not true without the assumption of ϕ-minimality, even if $0^F(\alpha) \le 1$ for all α (see [5]).

<u>2.15 Corollary</u>: Every elementary embedding $j:L(F) \to M$ definable in $L(F)$ is an iterated ultrapower.

<u>Proof</u>: If not, then j is defined by some formula ϕ with parameter η. We can take F ϕ-minimal for the assertion that there is an η such that ϕ defines a Σ_1-elementary embedding. We can also assume η is the smallest parameter that works, so j is definable in $L(F)$. Now $M = L(F')$ for some F', and F' is also ϕ-minimal. By theorem 2.14, applied in $L(F)$, there is an iterated ultrapower $i:L(F) \to L(F')$ in $L(F)$. Then $j = i$: otherwise the least x such that $i(x) \neq j(x)$ would be definable in $L(F)$ and certainly $i(x) = j(x)$ for all definable x. $\square$

This corollary lets us characterize the countably complete, uniform ultrafilters in $L(F)$:

<u>2.16 Theorem</u> In $L(F)$

(i) For all $(\alpha,\beta) \in$ domain (F) if $\delta(\alpha,\beta) = \nu + 1$ then $F(\alpha,\beta) \cong F(\alpha,\beta)[\{\nu\}]$, a uniform ultrafilter on α.

(ii) The only countably complete, uniform ultrafilters are the ultrafilters $F(\alpha,\beta)[\{\nu\}]$ and finite iterations of them.

<u>Proof</u>: Let $F = F(\alpha,\beta)$ and suppose $\delta = \delta(\alpha,\beta) = \nu + 1$. Then $b^F_\nu < \alpha^{++}$ in $L(F\upharpoonright(\alpha,\beta))$ and there are functions f_1 and f_2 with support in $\{\nu\}$ such that $[f_1]_F = \alpha$ and $[f_2]_F: \alpha^+ \cong b^F_\nu$. But for each $\nu' < \nu$ there is $\gamma < \alpha^+$ such that $b^F_{\nu'} = [f_2]_F(\gamma)$ and there is a g with support in $\{0\}$ such that $\gamma = [g_\gamma]_F$. Set $h_{\nu'}(a_\nu) = f_2(a_\nu)(g_\gamma(f_1(a_\nu)))$; then $\{a : a_{\nu'} = h_{\nu'}(a_\nu)\} \in F$. Since ν' was arbitrary, $F[\{\nu\}] \cong F$ and we have proved (i).

Suppose U is a countably complete uniform ultrafilter on λ and let $i:L(F) \to \text{ult}(L(F),U)$. Then if $\overline{\lambda} = [\text{id}_\lambda]_u$, then $X \in U \equiv \overline{\lambda} \in i(X)$. Now by 2.15 i is an iterated ultrapower. Let F be the associated iterated ultrafilter, let f be such that $[f]_F = \overline{\lambda}$ and let x be a support for f. Then x is also a support for all of F: Suppose $w \in \text{ult}(L(F),F) = \text{ult}(L(F),U)$, then for some $g \in L(F)$ $\omega = [g]_U = i(g)(\overline{\lambda}) = i(g)[f]_F = [g(f)]_F$. Hence as claimed F is a finite iteration and only includes $F(\alpha,\beta)$'s with $\delta(\alpha,\beta)$ a successor. Also, $X \in U \equiv \{a:f(a)\in X\} \in F$ so $F \cong U$. □

2.17 Corollary: The only cardinals with uniform, countably complete ultrafilters in $L(F)$ are measurable cardinals and limits of measurable cardinals. Hence there are no κ^+-strongly compact cardinals in $L(F)$. □

We will see in section 3 that $L(F)$ is not even close to being an inner model for a κ^+-strongly compact cardinal.

If F is a sequence of measures then it was shown in [4] that every countably complete uniform ultrafilter is isomorphic to a member of the class Γ of finite iterations of normal measures. The existence of a μ-measurable cardinal, however, implies that there is a uniform ultrafilter not in Γ.

2.18 Problem: Does it follow from the existence of a countably complete, uniform ultrafilter not in Γ that there is an inner model with a μ-measurable cardinal?

3. Loose Ends

In this section we will discuss the fine structure of $L(F)$ and the extension of $L(F)$ to larger cardinals. Since our intention is more to point out loose ends than to tie them up we will at best sketch proofs. Indeed no detailed proofs of several of the results mentioned in this section have been written out, so these results should be regarded as preliminary.

The models in this paper extend well beyond $P^2(\kappa)$-measurable cardinals: They can include, for example, $P^3(\kappa)$-measurable cardinals, $P^\kappa(\kappa)$-measurable cardinals and $P^\lambda(\kappa)$-measurable cardinals κ where λ is the least weakly compact or Ramsey cardinal greater than κ. They can, in fact, include anything short of a $P^\lambda(\kappa)$-measurable cardinal where λ is the next measurable cardinal. In particular theorem 2.12 extends to these cardinals: If κ is one of these cardinals, then there is a sequence F such that κ has the same property in $L(F)$. The strength of these cardinals is not so clear as for $P^2(\kappa)$-measurable cardinals. If κ is $P^\lambda(\kappa)$-supercompact then of course κ is $P^{\lambda+1}(\kappa)$-measurable, but on the other hand it is easy to see that the least $P(\kappa)$-supercompact cardinal is not even $P^3(\kappa)$-measurable. It is still open whether $P(\kappa)$-supercompactness implies the existence of $P^3(\kappa)$-measurable cardinals, but it is known that it implies the existence of inner models with all of these hypermeasurable cardinals. The proof, which is outlined below, only assumes that κ is κ^+-strongly compact and requires the use of the covering lemma for the core model $K(F)$. No direct proof from κ^+-supercompactness is known.

The core model $K(F)$ is defined for the ultrafilter sequences F described in this paper just as it is defined in [5] for sequences of measures. It is the class of sets constructible from F together with approximations, called mice, of extensions of F. We have been unable to check details, but the basic results about $K(F)$ seem to generalize straightforwardly to hypermeasures. The covering lemma is somewhat more difficult but we are confident the following theorem is true:

<u>3.1 theorem</u>: If there is no model $L(F)$ with a $P^\lambda(\kappa)$-measurable cardinal, where λ is the next measurable cardinal above κ in $L(F)$, then there is a sequence F such that

(i) If δ is a cardinal and $\lambda = \delta^+$ in $K(F)$ then $cf(\lambda) \geq \delta$.

(ii) Any elementary embedding $i:K(F) \to M$, with M a transitive class, is an iterated ultrapower of $K(F)$.

(iii) Everything that was proved for $L(F)$ in section 2 is true of $K(F)$.

Theorem 3.1 can be used to show that many of the conditions which are proved in [5] to imply the existence of a model with $O(\kappa) = \kappa^{++}$ are much stronger.

<u>3.2 theorem</u>: Any of the following imply the existence of a model $L(F)$ in which κ is $P^\lambda(\kappa)$-measurable, where λ is the next measurable cardinal in $L(F)$:

(i) κ is κ^+-strongly compact.

(ii) There is an $\aleph_2$-saturated ideal on $\aleph_1$.

(iii) (dependent choice only) The closed unbounded filter C on $\aleph_1$ is an ultrafilter and $i^C(\aleph_1) = \aleph_2$.

<u>Proof</u>: We will prove (i); the others are similar. Suppose the conclusion fails. We will let $K(F)$ be given by theorem 3.1 and prove a contradiction.

Since κ is κ^+-strongly compact there is an elementary embedding $i:K(F) \to M$ such that $i(\kappa^+) > \bigcup i''\kappa^+$. Let $\lambda = \kappa^+$ in $K(F)$. Then by 3.1 (i) either $cf\ \lambda = \kappa$ or $\lambda = \kappa^+$; in either case $i(\lambda) > \bigcup i''\lambda$. But by 3.1 (ii) i is an iterated ultrapower of $K(F)$ and so $i(\lambda) = \bigcup i''\lambda$ since $\lambda = \kappa^+$ in $K(F)$. This contradiction completes the proof. $\square$

It should be noted that when appropriate models become available for larger cardinals this proof should extend to give (at least for (i) and (ii)) the best possible result, which is probably a κ^+-supercompact cardinal for (i) and (see [2, §17]) an almost huge cardinal for (ii). In contrast, we have been unable to strengthen

3.3 theorem: If there is a measurable cardinal κ with $2^\kappa > \kappa^+$ then there is an $L(F)$ with a $\beta < 0^F(\kappa)$ such that $\beta = \kappa^{++}$ in $ult(L(F), F(\kappa,\beta))$.

While 3.3 is unlikely to be the best possible, the following problem is worth considering:

3.4 problem: Does the consistency of κ measurable and $2^\kappa > \kappa^+$ follow from that of a $P^2(\kappa)$-measurable cardinal?

It does, of course, follow from that of a κ^+-supercompact cardinal (Silver, see [2, §25]). In one case we are unable to improve the result of [4] and the following seems quite plausible:

3.5 problem: If $0^F(\kappa) = \kappa^{++}$ in $L(F)$, then is it true in $L(F)$ that every κ-complete filter on κ can be extended to an κ-complete ultrafilter?

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ON THE NUMBER OF EXPANSIONS OF THE MODELS OF ZFC-SET THEORY TO MODELS OF KM-THEORY OF CLASSES

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Assume that $\underline{M} = \langle M,E \rangle \models ZF$. We say that $\underline{M}$ is expandable to a model for KM iff there is a family $F \subseteq P(M)$ such that $\underline{F} = \langle F,M,E^* \rangle$, the model obtained by adding a new class universe, is a model for KM; where by KM we mean the Kelley-Morse class theory with full comprehension scheme and with no form of the axiom of choice.

The following theorem is well known: If M is a countable, KM-expandable model for ZF, then M has continuum many expansions which are models for KM.
This theorem is a direct consequence of the Mansfield Perfect Set Theorem (see [5], ch. 8, 8B), which states that every Σ_1^1-relation included in $P(M)$ has the power $2^{\text{card}(M)}$ or contains a hyper-elementary element. It suffices to observe that inside every model $\langle F,M,E^* \rangle$ for KM, we can define a class which is not hyper-elemen-tary with respect to M.

In the case of uncountable models the situation is different. Basing ourselves on the proof of the Chang-Makai theorem ([2],ch.5, §3) we can show that the Mansfield Theorem is true for special models. I do not know, however, if it is valid for a wider class of models, in particular for Kleene structures (i.e. such $\underline{M}$ that the completeness theorem is true for $HYP_{\underline{M}}$). Neither do I know if it is true for at least one structure of the form R_α, where cf $\alpha = \omega$, $\alpha \geqslant \omega+\omega$.

By adapting the construction used in the proof of Mansfield theorem we show here that:

If $\underline{M}$ is a KM-expandable model for ZFC whose height has the cofinality ω, then $\underline{M}$ has at least $\text{card}(M)^{\aleph_0}$ expansions to models for KM+$\forall X$ r.a (X) where the sentence $\forall X$ r.a (X) states that all classes are ramified analytical (for a reference see [4]).

Marek and Mostowski [4] have observed that if $cf(On^{\underline{M}}) > \omega$ and M is transitive then each expansion to a model for KM is a β-model.
From this it follows that for the least α such that R_α is KM-expandable and $cf(\alpha) = \omega_1$, R_α has exactly one expansion to a model for KM+ X r.a (X). Hence the conclusion of our theorem is not true for $\underline{M} = R_\alpha$. This shows that the assumption $cf(On^{\underline{M}}) = \omega$ is essential.

In the first section we shall show a useful partition theorem which is applied in the next section. In section 2-d, the use of recursive closed game formulas and their infinite approximations (see Barwise [1], ch. VI, §6) is essential for our purposes.

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1. A partition theorem

Erdos, Hajnal and Rado considered, in [3], the following partition property:

We say, that $a \to (b,c)$, where a,b,c are cardinals, if for any set S such that card(S) = a and for any two-element partition $[S]^2 = I \cup J$ there exists a set $X \subseteq S$ such that card(X) = b and $[X]^2 \subseteq I$ or card(X) = c and $[X]^2 \subseteq J$.

The following theorem, among others, is proved in [3]:
ZFC $\vdash$ If a is a strongly inaccessible cardinal, then

$$(\forall b)_{<a} \quad a \to (a,b).$$

Here we shall prove a corresponding theorem, which informally, can be expressed as follows:

$$(ZFC) \vdash (\forall \alpha) \ On \to (On,\alpha)$$

To do this we shall adjust to our needs the ramification lemma, which is the key point in the proof of the above mentioned theorem of [3]. Before doing this, however, let us equip our theorem with a precise and formal shape:

The theory ZFC(R) is constructed as follows: we add to the language of ZFC a new relation symbol R. ZFC(R) is then ZFC plus the replacement schema for all formulae of the extended language.

Theorem 1.1. For all $k \in \omega$ there exists an $\ell \in \omega$ such that the following is provable in ZFC(R):

$$(\forall\alpha)(\exists x)_{\subseteq On}\ \text{card}\ x > \alpha\ \&\ (\forall y,z)_x [y \in z \Rightarrow \sim\varphi(y,z)]\ \vee$$

$$* \qquad (\exists x)\{(\forall y)[(\psi^\ell(x,y) \Rightarrow Ord(y))\ \&\ (\exists z)(rank(y) \in z\ \&$$

$$\psi^\ell(x,z))\ \&\ (\forall y,z)[\psi^\ell(x,y)\ \&\ \psi^\ell(x,z)\ \&\ y \in z \Rightarrow \varphi(y,z)]\}$$

where: $\psi^\ell(x,z)$ is a universal formula for formulas of classes Σ_ℓ with one free variable and one parameter, and runs through all formulas of class Σ_k.

Proof. Let us fix the number k and try to prove the schema *. The number ℓ will result from the proof. Hence, assume that the second part of the disjunction in * does not hold, i.e.

$$(\forall x)\{(\forall y)[(\psi(x,y) \Rightarrow Ord(y))\ \&\ (ord\ y \Rightarrow (\exists z)(y \in z\ \&$$

$$\psi^\ell(x,z)))] \Rightarrow (\exists y,z)[\psi^\ell(x,y)\ \&\ \psi^\ell(x,z)\ \&\ y \in z\ \&$$

$$\sim\varphi(y,z)]\}.$$

Let κ be an ordinal. Now we shall define an object, called by Erdos, Hajnal and Rado "the ramification system". We shall use definable ZFC classes.

Let $[On]^2 = \{\{\alpha,\beta\} : \alpha,\beta \in On\ \&\ \alpha \neq \beta\}$.

Let us denote by I,J the partition $[On]^2$ described by the formula φ, $I = \{\{\alpha,\beta\} : \alpha \in \beta \in On\ \&\ \varphi(\alpha,\beta)\}$, $J = [On]^2 \setminus I$.

We shall denote the elements of the class $\bigcup_{\alpha\leqslant} On^\alpha$ by σ. Now, by induction with respect to $\beta < \kappa$, we define sets $N_\beta \subseteq On^\beta$ such that:

(*i*) $(\forall\gamma \leqslant \beta)(\forall\sigma)[\sigma \in N_\beta \Rightarrow \sigma\restriction\gamma \in N_\gamma]$

(*ii*) $(\forall\gamma \leqslant \beta)[Lim(\lambda)\ \&\ (\forall\alpha < \lambda)(\sigma\restriction\alpha \in N_\alpha) \Rightarrow \sigma \in N_\gamma]$

and mappings S and F such that

(*iii*) $(\forall\gamma < \kappa)(\forall\sigma \in N_\gamma)[S(\sigma) = F(\sigma) \cup \bigcup\{S(\sigma^\frown\alpha) : \sigma^\frown\alpha \in N_{\gamma+1}\}]$

(*iv*) $Lim(\beta) \Rightarrow (\forall\sigma \in N_\beta)[S(\sigma) = \bigcap_{\alpha<\beta} S(\sigma^\frown\alpha)$

where $F(\sigma)$ is the union of all maximal subsets $x \subseteq S(\sigma)$ of minimal rank such, that $[x]^2 \subseteq I$, if such maximal subsets exist, otherwise $F(\sigma)$ is not defined.

(v) $(\forall\gamma < \beta)[N_{\gamma+1} = \{\sigma^\frown\alpha : \alpha \in F(\sigma) \;\&\; \sigma \in N_\gamma\}]$

(vi) $(\forall\gamma < \beta)(\forall\sigma \in N_\gamma)(\forall\alpha \in F(\sigma))[S(\sigma^\frown\alpha) = \{\beta \in S(\sigma) : \{\alpha,\beta\} \in J\}]$

Note, that the only mapping S satisfying conditions (iv) and (vi) is the mapping defined by the formula

$$S(\sigma) = \{\beta \in On : (\forall\alpha \in Dom\,\sigma)\ \{\beta,\sigma(\alpha)\} \in J\}$$

Substituting the above definition for $S(\sigma)$ we see that the conditions (i) → (vi) give us a definition by transfinite induction in set theory, provided we show the following:

$$(\forall\beta < \kappa)\ [N_\beta \text{ is defined } \Rightarrow N_\beta \text{ is a set}].$$

We shall obtain this by proving, by simultaneous induction, that

$$(\forall\beta < \kappa)\ [N_\beta \text{ is a set } \&\ \forall\sigma \in N_\beta\ F(\sigma) \text{ is defined}].$$

Thus, we shall also show that induction defined by the conditions (i) → (vi) can be continued up to κ.

First, we prove, that $(\forall\sigma \in N_\beta)$ $F(\sigma)$ is defined. This can be reduced to the proof of the formula:

$$(\forall\sigma \in N_\beta)(\exists x)[x \subseteq S(\sigma) \;\&\; [x]^2 \subseteq I \;\&\; x \text{ is the maximal set possessing this property}].$$

Assume the contrary, i.e. that there exists N_β such, that

(vii) $(\forall x)[x \subseteq S(\sigma) \;\&\; [x]^2 \subseteq I \Rightarrow (\exists y)(x \subsetneq y \subseteq S(\sigma) \;\&\; [y]^2 \subseteq I)]$.

We define by induction the function $G : On \to V$ such that $G(0)$ is any set x such that $x \subseteq S(\sigma) \;\&\; [x]^2 \subseteq I$.

$$G(\alpha) = \bigcup_{\beta<\alpha} G(\beta) \cup \{\mu\gamma(\gamma > \cup \bigcup_{\beta<\alpha}(\beta) \;\&\; \gamma \in S(\sigma) \;\&\; [\bigcup_{\beta<\alpha}(\beta) \cup \{\gamma\}]^2 \subseteq I)\}.$$

From this definition we infer, that for every α, $G(\alpha)$ is the maximal subset of $\cup\, G(\alpha)$ such that $[G(\alpha)]^2 \subseteq I$.
From this and from (vii) it follows that

$$(\forall\alpha)_{On}(tp\ G(\alpha) \geq \alpha).$$

Assuming $G = \bigcup_{\alpha\in} G(\alpha)$ we obtain a proper class such that $[G]^2 \subseteq I$.

Let ℓ be the smallest ℓ' such that the formula defining G is in $\Sigma_{\ell'}$. Clearly, ℓ depends only on k.

So we obtain a contradiction with our initial assumption.

Now we shall prove that N_β is a set. The non-limit case is trivial, whereas in the case of lim(β) it is sufficient to employ the fact that

$$N_\beta = \{\sigma \in On^\beta : (\forall\alpha < \beta)\ \sigma\restriction\alpha \in N_\alpha\}$$

We have, hereby, shown that for all β, N_β is defined and, moreover, that $(\forall\sigma)[\sigma \in \bigcup_{\beta<\kappa} N_\beta \Rightarrow F(\sigma)$ is a set]. Hence, as a corollary, we obtain

(viii) $\bigcup\{F(\sigma) : \sigma \in \bigcup_{\beta<\kappa} N_\beta\}$ is a set.

In the sequel we shall show that there exists a $\sigma \in On^\kappa$ such, that

$$(\forall\beta < \kappa)\ \sigma\restriction\beta \in N_\beta \quad \text{and} \quad \bigcap_{\beta<\kappa} S(\sigma\restriction\beta) \neq \emptyset$$

Let γ be an ordinal number not in the set (viii) and let L be a maximal chain in the set

$\{\sigma \in \bigcup_{\beta<\kappa} N_\beta : \gamma \in S(\sigma)\}$, partially ordered by inclusion.

Let $\sigma = \bigcup L$. As is easily seen, Dom $\sigma \in$ On, Dom $\sigma \leqslant \kappa$. To end our reasoning we must show that Dom $\sigma = \kappa$. Let us assume the converse, i.e. that Dom $\sigma = \beta \in \kappa$. Since $\gamma \in S(\sigma)\setminus F(\sigma) \subseteq \bigcup_{\alpha\in F(\sigma)} S(\sigma^\frown\alpha)$, there exists an $\alpha \in F(\sigma)$ such that $\gamma \in S(\sigma^\frown\alpha)$. This contradicts the choice of L, since $\sigma^\frown\alpha$ is the upper bound of elements of L not beloning to L.

To conclude the proof, we shall show, that card(Rng σ) = κ and that $[\text{Rng}\,\sigma]^2 \subseteq J$, where σ is the element whose existence we have just proved. To do this, let us note that from (iii) and (v) we get

(ix) $(\forall\beta < \kappa)(\forall\sigma \in N_\beta)(\forall\gamma < \beta)[\sigma(\gamma) \in S(\sigma\restriction\gamma)]$,

and from (vi)

(x) $(\forall\beta < \kappa)(\forall\sigma \in N_\beta)(\forall\gamma \leqslant \beta)(\forall\alpha \in S(\sigma\restriction\gamma+1))[\{\alpha,\sigma(\gamma)\} \in J]$

It is easily observed, that $\sigma : \kappa \xrightarrow[\text{onto}]{1-1} \text{Rng}\,\sigma$.

To show that $[\text{Rng}\,\sigma]^2 \subseteq J$ let us assume that $\alpha,\beta \in \text{Rng}\,\sigma, \alpha \neq \beta$. Hence, $\alpha = \sigma(\gamma)$, $\beta = \sigma(\delta)$, where $\gamma,\delta < \kappa$ (we can assume that $\gamma < \delta$).

Thus, by (ix), $\beta = \sigma(\delta) \in S(\sigma\restriction\delta) \subseteq S(\sigma\restriction\gamma)$, which, by (x), implies that $\{\alpha,\beta\} = \{\sigma(\gamma),\beta\} \in J$.

Remark: Note that in the proof, the fact that I,J is a partition of $[On]^2$ was used in following cases: when defining the function G (it was significant for the class to be well ordered) and when choosing the element γ (it was significant only that On be a proper class). Thus we obtain the following:

Corollary. ZFC(R). If X is a well-orderable proper class (definable with a definable well-ordering) then for any definable partition: $[X]^2 = I+J$ there exists a definable proper class $S \subseteq X$ such that $[S]^2 \subseteq I$ or

$$\forall\kappa(\exists x \subseteq X)[\mathrm{card}(x) \geqslant \kappa \ \& \ [x]^2 \subseteq J].$$

2. The perfect set theorem for the case cf = ω

The main goal of this section is the adaption of the construction of the Perfect set theorem to the case of cf = ω.

Here we restrict the class of Σ_1^1-predicates under consideration to those characterizing the expandability to KM.

A total open problem, not investigated here, is the generalization of the Perfect Set Theorem to the case of arbitrary Σ_1^1-predicates over models considered by Nyberg [6].

One of the essential properties of KM-expandability in the case of cf = ω, which we intend to employ here, is the possibility of characterizing it with the help of games expressed by recursive game formulas (see Barwise [1], Ch. VI, §6): a recursive closed game formulas is an infinite formula $G\vec{x}\ \varphi(\vec{x}) = Q_1x_1\ldots Q_nx_n\ldots \ldots\bigwedge_{i\in\omega} R_i(x_1,\ldots,x_n)$ where each Q_n is either $\exists$ or $\forall$, and $i \to R_i$ is a recursive sequence of a formulas of a language $K_{\omega\omega}$.

Definition. The notion of a winning plan: assume that the recursive formula $Q_1x_1\ldots Q_2x_2\ldots\bigwedge_{i\in\omega} R_i(x_1\ldots x_{n_i})$ describes a closed game over model $\underline{M}$.

A subset Σ of $\bigcup_{n\in\omega} M^n$ is called a plan of the 1st player also called the $\exists$-player if the following conditions are satisfied:

(1) $s \in \Sigma \ \& \ \mathrm{lh}(s) = k \ \& \ Q_{k+1} = \forall \Rightarrow (\forall x)_M(s^\frown x \in \Sigma)$

(2) $s \in \Sigma \ \& \ lh(s) = k \ \& \ Q_{k+1} = \exists \Rightarrow (\exists x)_M(s^\frown x \in \Sigma)$

A plan Σ of the first player is called a winning plan, for him, if

$$(\forall f)_{M^\omega}[(\forall n)_\omega(f\restriction n \in \Sigma \Rightarrow \underline{M} \models \bigwedge_{i\in\omega} R_i[x_1/f(1),\dots,x_i/f(n_i)]]$$

From the proof of the Gale Stewart theorem it follows that if player ($\exists$) possesses a winning strategy (i.e. the 2^{nd} player ($\forall$) does not have a winning strategy) then $\Sigma = \emptyset \cup \{\langle a_1,\dots ,a_n\rangle : \underline{M} \models Q_{n+1}x_{n+1}\dots\bigwedge_{i\in\omega} R_i[a_1/x_1,\dots,a_n/x_n,x_{n+1},\dots,x_{n_i}], n \in \omega\}$ is a maximal winning plan of player ($\exists$).

Moschovakis proves in [5] that if $\underline{M}$ is acceptable, then Σ (or, more precisely, the set of codes of the sequences belonging to Σ) is a coinductive set,

For our needs it will be more convenient to define the set Σ using approximation formulas σ_α^n, and the following theorem proved in Barwise [1], Ch. VI, §6:

there exists a Δ_1^{KP} $^{\mathrm{Inf}}$-operation

$\sigma : \omega\times On \to K_{\omega\omega}$ such that

(1) for every n and $p \in \omega$, $\sigma_p^n \in K_{\omega\omega}$, and if $\underline{M}$ is a structure for K, then

(2) $$\underline{M} \models Q_{n+1}x_{n+1}\dots\bigwedge_{i\in\omega} R_i[a_1/x_1,\dots,a_n/x_n,x_{n+1},\dots,x_{n_i}] \Rightarrow \underline{M} \models \bigwedge_\alpha \sigma_\alpha^n$$

(3) $$K^{\underline{M}} \models O(HYP_{\underline{M}}) \ \& \ \underline{M} \models \bigwedge_{\alpha<\kappa^{\underline{M}}} \sigma_\alpha^n[a_1/x_1,\dots,a_n/x_n] \Rightarrow \underline{M} \models Q_{n+1}x_{n+1}\dots\bigwedge_{i\in\omega} R_i[a_1/x_1,\dots,a_n/x_n,x_{n+1},\dots,x_{n_i}]$$

Now the maximal winning plan of the first player ($\exists$) can be defined as follows:

$$\Sigma = \{\langle a_1,\dots,a_n\rangle : \underline{M} \models \bigwedge_{\alpha<\kappa^M} \sigma_\alpha^n[a_1/x_1,\dots,a_n/x_n], n \in \omega\}$$

For $\alpha < \kappa^{\underline{M}}$ let us denote

$$\Sigma_\alpha = \{\langle a_1,\dots,a_n\rangle : \underline{M} \models \sigma_\alpha^n[a_1/x_1,\dots,a_n/x_n], n \in \omega\}$$

Clearly, $\Sigma = \bigcap_{\alpha<\kappa} \Sigma_\alpha$. In the sequel, the investigation of plan Σ can be reduced to the investigations of sets Σ_α which, as it should be noted, are elements of $HYP_{\underline{M}}$ (if $\kappa^{\underline{M}} > \omega$).

Lemma 2.1. Assume, that $\underline{M}$ is a model for ZF.

(*i*) $(\forall F)[\underline{F} = \langle F,M,E^*\rangle \models KM+\forall Xr.a(X) \Rightarrow \langle Def(\underline{F}),M,E^*\rangle \models KM]$,

where $Def(\underline{F})$ denotes the family of all classes of the model $\underline{F}$ definable by formulas of the language $\mathcal{L}_{KM}$ with parameters from M.

(*ii*) there exists a recursive closed game formula $G\vec{x}\ \varphi(\vec{x})$ in the language containing only the symbols from $\mathcal{L}_{ZF}$ such that

(*a*) $\underline{M}$ is KM expandable $\Rightarrow \underline{M} \models G\vec{x}\ \varphi(\vec{x})$.

If $\underline{M} \models G\vec{x}\ \varphi(\vec{x})$ and $cf(On^{\underline{M}}) = \omega$, then $\underline{M}$ has an expansion $\underline{F}$ satisfying $\underline{F} = Def(\underline{F})$, and $KM+\forall X\ r.a(X)$.

Proof. In [8] we find the proof of the following theorem:

$KM_n+\forall Xr.a(X) \vdash$ The Basis Theorem for Σ^1_n-formulas.

In any model $\underline{F}^n \models KM_n+\forall Xr.a\ X$ we can express this as follows: the family of all classes, Δ^1_n-definable by formulas with parameters from M, denoted by $Def^n(\underline{F}^n)$, is a base in $\underline{F}^n$ for Σ^1_n-formulea without class parameters.

If $\langle F,M,E^*\rangle \models KM+\ Xr.a(X)$, then by the reflection principle, see [4], there exists a tower

$\underline{F}^1 \prec^1_1 \underline{F}^2 \prec^1_2 \ldots$ such that $\bigcup_{n\in\omega} F^n \subseteq F$ and for every $n \in \omega$, $\underline{F}^n \models KM_n+\forall Xr.a(X)$.

It is easy to see that $Def^n(\underline{F}) = Def^n(\underline{F}^n)$. Hence $Def(\underline{F}) = \bigcup_{n\in\omega} Def^n(\underline{F})$ is a base for formulas without class parameters, which implies that $Def(\underline{F}) \prec \underline{F}$. This concludes the proof of (i).

To prove (ii), we note first that for every $\underline{F} = \langle F,M,E^*\rangle \models$ $\models KM+\forall Xr.a(X)$, $Def^n(\underline{F}) \prec^1_n \underline{F}$; This follows easily from the basis theorem.

Moreover, let us note that for $2 \leqslant n \in \omega$, $Def^n(\underline{F}) \models$ $\models KM_{n-2}+\forall Xr.a(X)$ and that

$Def^n(\underline{F}) \models (\forall X)(\exists x)(\exists y)(\forall z)[z \in X \Leftrightarrow \varphi^n(x,z) \Leftrightarrow \sim\varphi^n(y,z)]$, where φ^n is a universal formula for Σ^1_n-formulas without class parameters (in short, $Def^n(\underline{F}) \models \Delta^1_n$-Def).

Arguing as in the proof of lemma 1.1, [7], we find a direct system

$g_i : \langle F_i,R_i,E^*\rangle \prec^1_i \langle F_{i+1},R_{i+1},E^*\rangle,\ i \in \omega,$

the elements of which are of the form $a_E = \{b \in M : bEa\}$, where

$a \in M$. In particular, $\exists \alpha_i \in On^{\underline{M}}$ $(R_i = R_{\alpha_i}^{\underline{M}})$. If cf $\underline{M} = \omega$, it can be chosen so that its limit is a certain expansion of $\underline{M}$ to a model of $KM+\forall Xr.a(X)$.

It differs from the system of lemma 1.1 [7] only in that $\langle F_i, R_i, E^* \rangle = KM_{i-2}+\forall Xr.a(X)+\Delta_i^1\text{-Def}$.

Moreover, we can assume, that $g_i''(F_i) = \{X \in F_{i+1} : X$ is Δ_i^1-definable in $\langle F_{i+1}, R_{i+1}, E^* \rangle$ by formulae with parameters belonging to $R_i\}$ for $i \in \omega$.

Thus consider the following game between $(\exists)$ and $(\forall)$: the n^{th} move of $(\forall)$ is an ordinal α_n of M; the n^{th} move of $(\exists)$ is a set of M of the form $\langle F_n, \beta_n, h_{n-1} \rangle$. Let $F_n = (F_n)_E$, $R_n = (R_{\beta_n}^{\underline{M}})_E$, $g_n = (h_n)_E$; then $(\exists)$ wins if for all $n \in \omega$, $\alpha_n \leqslant \beta_n$, $\langle F_n, R_n, E^* \rangle$ is a model of $KM_{n-2}+\forall Xr.a(X)+\Delta_n^1\text{-Def}$, and $g_n : \langle F_n, R_n, E^* \rangle \prec_n^1 \langle F_{n+1}, R_{n+1}, E^* \rangle$. Clearly this game is expressible by a recursive closed game formula $G\vec{x}\ \varphi(\vec{x})$ on $\underline{M}$; and the above remarks imply that if $\underline{M}$ is expandable to a model of KM, then $(\exists)$ has a winning strategy. Moreover, if $cf(On^{\underline{M}}) = \omega$ and $(\forall)$ chooses his sequence of moves, α_n, to be cofinal in $On^{\underline{M}}$, then the limit of the direct system constructed by $(\exists)$ applying a winning strategy is an expansion of $\underline{M}$ satisfying $F = \text{Def}(\underline{F})$ and $KM+\forall Xr.a(X)$. This shows (ii) of the Lemma (comp. proof of Th. 2.1 in [7]).

Theorem 2.2. If $\underline{M} = \langle M, E \rangle$ is a KM-expandable model for ZFC such that cf $On^{\underline{M}} = \omega$, then the family

$$\{F : \langle F, M, E^* \rangle \models KM+\forall Xr.a(X),\ \text{card}(F) = \text{card}(M)\}$$

has power $\geqslant \text{card}(M)^{\aleph_0}$.

Proof. Let $G\vec{x}\ \varphi(\vec{x})$ be the recursive closed game formula of the above lemma.

In view of the assumption, $\underline{M} \models G\vec{x}\ \varphi(\vec{x})$. Thus, in the game defined by $G\vec{x}\ \varphi(\vec{x})$ the first player possesses a winning strategy.

Let Σ be the maximal winning plan of the first player. To estimate the number of expansions of model M to models for $KM+\forall Xr.a(X)$, we shall in the first place attempt to estimate the number of possible moves of the first player, consistent with the plan and defining different expansions.

Assume, that $s \in \Sigma$, $s = (\alpha_1, F_1, \beta_1; \alpha_2, F_2, h_1, \beta_2; \ldots$ $\ldots; \alpha_n, F_n, h_{n-1}, \beta_n)$, where $\alpha_1, F_1; \alpha_2, F_2, h_1; \ldots \alpha_n, F_n, h_{n-1}$ are moves of the first player, which constitute the direct system:

$(h_1)_E : \langle (F_1)_E, (R^{\underline{M}}_{\alpha_1})_E, E^* \rangle \prec^1_1$
$\langle (F_2)_E, (R^{\underline{M}}_{\alpha_2})_E, E^* \rangle, \ldots, (h_{n-1})_E : \langle (F_{n-1})_E, (R^{\underline{M}}_{\alpha_{n-1}})_E, E^* \rangle$
$\prec^1_{n-1} \langle (F_n)_E, (R^{\underline{M}}_{\alpha_n})_E, E^* \rangle$; $\beta_1, \ldots, \beta_n$ are moves of the second player, which determine the level which has to be passed by the first player.

Let $\delta < \kappa^{\underline{M}}$ be a fixed ordinal. Let us consider two cases:

Case 1. $\underline{M}$ is an ω-model, so $\underline{M}$ is acceptable.

Hence $HYP_{\underline{M}} \models KPU^+$ + Infinity. In view of the definition of approximation formula, the sequence of formulas $\{\sigma^n_\delta(x_1, \ldots, x_n)\}_{n \in \omega}$ belongs to $HYP_{\underline{M}}$, thus Σ_δ as the union $\bigcup_{n\in\omega} \sigma^n_\delta(\underline{M})$ belongs to $HYP_{\underline{M}}$.

As is well-known, $HYP_{\underline{M}}$ (under the above assumption) is closed with respect to the operation Def. Hence $Def(\underline{M};\Sigma_\delta) \overset{df}{=} \bigcup_{n\in\omega} Def(\underline{M}; \Sigma_\delta \cap M^n) \in HYP_{\underline{M}}$. Let H be an element of $HYP_{\underline{M}} \cap P(M) \setminus Def(\underline{M};\Sigma_\delta)$ – the existence of such an H follows from the fact that $HYP_{\underline{M}} \models$ "P(M) does not exist", as is well-known from the projectibility of $HYP_{\underline{M}}$ and the diagonal argument. From Moschovakis [5] ch. 7 theorem 7.F.2 it follows that there exist a Σ^1_1-formula Φ (in the language $\mathcal{L}_{KM}(\{\bar{a} : a \in M\})$ defining the above set H in any model for Δ^1_1-comprehension (Δ^1_1-CA) extending the model $\langle \underline{M}; a \rangle_{a\in M}$. Let $\bar{a}_1, \ldots, \bar{a}_m$ be the sequence of all constant occuring in the formula Φ.

Let $\gamma = \max^{\underline{M}}(\mathrm{rank}^{\underline{M}}(a_1), \ldots, \mathrm{rank}^{\underline{M}}(a_m))$.

Let $R = \{\{\langle \alpha,\beta \rangle, \langle \alpha',\beta' \rangle\}^{\underline{M}} : (\exists F,h,F_1,h_1)$
$[s^\frown\langle \alpha,F,h \rangle \in \Sigma_\delta \ \& \ s^\frown\langle \alpha',F_1,h_1 \rangle \in \Sigma_\delta \ \& \ \underline{M} \models (\beta = osp(F) \ \&$
$\beta' = osp(F_1) \ \& \ (\exists g)(g : \langle F,R_\alpha,\epsilon \rangle \prec^1_1 \langle F_1,R_{\alpha_1},\epsilon \rangle \ \& \ g \restriction R_\alpha = id))$
$\& \ \langle \alpha,\beta \rangle^{\underline{M}} \neq \langle \alpha',\beta' \rangle^{\underline{M}}] \}$.

Note, that if $s^\frown\langle \alpha_{n+1},F_{n+1},h_n \rangle \in \Sigma_\delta$, then from the definition of approximation formula (see Barwise [1], ch. VI §6) we get $s^\frown\langle \alpha_{n+1},F_{n+1},h_n \rangle \in \Sigma_0$ and this means that

$M \models h_n : \langle F_n,R_{\alpha_n},\epsilon \rangle \prec^1_n \langle F_{n+1},R_{\alpha_{n+1}},\epsilon \rangle$ and
$\underline{M} \models (\langle F_n,R_{\alpha_n},\epsilon \rangle = \forall Xr.a(X) + \Delta^1_n\text{-Def})$.

Moreover we may assume, (cf. Remark in the proof of lemma 2.1) that $\underline{M} \models [h_n''(F_n) = \{X \in F_{n+1} : X$ is Δ^1_n-definable in $\langle F_{n+1},R_{\alpha_{n+1}},\epsilon \rangle$ by formulas with parameters from $R_{\alpha_n}\}]$.

Thus, h_n is uniquely determined by $\alpha_n, F_n, \alpha_{n+1}, F_{n+1}$. Indeed,

if $\varphi_1(X),\varphi_2(X)$ is a pair consisting of a Σ^1_n and Π^1_n formula, respectively and defines $h'_n(X)$ in F_{n+1}, then the same pair defines X in $\langle F_n, R_{\alpha_n}, \epsilon\rangle$, which implies that the Σ^1_n-formula $(\forall x)(x \in X \Rightarrow \varphi_1(x))$ & $(\forall x)(\varphi_2(x) \Rightarrow x \in X)$ is satisfied in $\langle F_n, R_{\alpha_n}, \epsilon\rangle$.
So, by Σ^1_n-elementary of h_n, it is satisfied also in $\langle F_{n+1}, R_{\alpha_{n+1}}, \epsilon\rangle$. This consequently means that $h_n(X) = \varphi_1(\langle F_{n+1}, R_{\alpha_{n+1}}, \epsilon\rangle) =$ $= h'_n(X)$ i.e. h_n is uniquely determined.

Finally, let us note that F_{n+1} is uniquely determined by α_{n+1} and $osp(F_{n+1})$ -this is a consequence of the fact that F_{n+1} may be assumed to be standard and that ramified analysis is absolute with respect to β-models (see [4]).

Let $X = Dom^{\underline{M}}R$. Obviously $X \in HYP_{\underline{M}}$. Moreover, X is an unbounded subset of $(On^2)^{\underline{M}}$ in the sense or ordering: $\langle\alpha,\beta\rangle^{\underline{M}} \prec \langle\alpha_1,\beta_1\rangle^{\underline{M}} \Leftrightarrow \alpha <^{\underline{M}} \alpha_1$ & $\beta <^{\underline{M}} \beta_1)$, since it is a superset of the set of all winning moves of the first player.

Since R and X are hyperelementary sets, they belong to every model for KM expanding model $\underline{M}$, which implies $\langle \underline{M}; R, X\rangle \models ZFC(\bar{R},\bar{X})$.

From the corollary of Theorem 1.2, by the fact that X is unbounded in $(On^2)^{\underline{M}}$, we obtain: there exists a formula $\varphi \in \mathcal{L}_{ZF}(\bar{R},\bar{X})$ such, that $\varphi(\langle \underline{M}; R, X\rangle)$ is an unbounded subset of X and $(\forall\alpha)_{On}(\exists x)_M[x_E \subseteq X$ & $(\underline{M} \models card(x) \geq \alpha)$ & $(\forall y,z)_{x_E}(y \neq z \Rightarrow \sim R(y,z))]$ v v $\varphi(\langle \underline{M}; R, X\rangle) \subseteq X$ & $(\forall y,z)[y,z \in \varphi(\langle \underline{M}; R, X\rangle)$ & $y \neq z \Rightarrow R(y,z)]$.

Now we shall show that the second component of the above disjunction is not true. Suppose the contrary, i.e. that $(\forall y,z)[y,z \in \varphi(\langle \underline{M}; R, X\rangle)$ & $y \prec z \Rightarrow R(y,z)]$. Assume that $\langle\alpha,\beta\rangle^{\underline{M}} \in \varphi(\langle \underline{M}; R, X\rangle)$, and let F be an element such that $(\exists h)(s^\frown\langle\alpha,F,h\rangle \in \Sigma_\delta$ & $osp^{\underline{M}}(F) = \beta)$. Let $\langle\alpha,\beta\rangle^{\underline{M}}, \langle\alpha_1,\beta_1\rangle^{\underline{M}}, \langle\alpha_2,\beta_2\rangle^{\underline{M}}, \ldots$ .. be a fixed sequence of elements of $\varphi(\langle \underline{M}; R, X\rangle)$, unbounded and increasing for $\prec$ in $(On^2)^{\underline{M}}$.

From the definition of R we infer that there exists a direct system:

$$g_i : \langle F_i, R_i, E^*\rangle \prec^1_1 \langle F_{i+1}, R_{i+1}, E^*\rangle;\ i \in \omega$$

such that $F_1 = (F)_E$, $R_1 = (R^{\underline{M}}_\alpha)_E$, $g_i \upharpoonright R_i = id_{R_i}$ and for every $i \in \omega$, $R_i = (R^{\underline{M}}_{\alpha_i})_E$.

The direct limit of the above system is a relational system of the form $\langle F, M, E^*\rangle$. By $g_{i\infty}$ let us denote the natural monomorphism from the i-th model $\underline{F}_i = \langle F_i, R_i, E^*\rangle$ into $\underline{F} = \langle F, M, E^*\rangle$. Obviously

$g_{i\infty}$ is a Σ^1_1-elementary monomorphis. Now we shall prove that $\underline{F} \models \Delta^1_1$-CA.

Let then $\varphi(x,A)$ be a Σ^1_1-formula, $\psi(x,A)$- a π^1_1-formula, respectively, where A is a parameter belonging to F.

Assume also that $\underline{F} \models (\forall x)\varphi[(x,A) \Leftrightarrow \psi(x,A)]$ and that $A = g_{i_0\infty}(B)$, where $i_0 \in \omega$, $B \in F_{i_0}$.
The property of being Σ^1_1-elementary extends to the family of boolean combinations of Σ^1_1-formulas. Moreover, adding the universal quantifier does not lead beyond the class of formulas invariant with respect to the passing to a fixed submodel. Hence

$$\langle F_{i_0}, R_{i_0}, E^*\rangle \models (\forall x)[\varphi(x,B) \Leftrightarrow \psi(x,B)]$$

Thus, also the formula $(\exists X)(\forall x)[x \in X \Leftrightarrow \varphi(x,B)]$ is satisfied in $\langle F_{i_0}, R_{i_0}, E^*\rangle$.

Since $g_{i_0\infty}$ is a Σ^1_1-elementary monomorphism, the same formula will be satisfied in $\langle F, M, E^*\rangle$ and this means that $\underline{F} \models \Delta^1_1$-CA.

Summing up, for every $\langle \alpha,\beta\rangle^{\underline{M}} \in \varphi(\langle \underline{M}; R; X\rangle)$ and for every F such that $(\exists h)(s^\frown\langle \alpha,F,h\rangle \in \Sigma_\delta \ \&\ osp^{\underline{M}}(F) = \beta)$ there exists an extension $\langle F,M,E^*\rangle$ of $\underline{M}$ which is a model for Δ^1_1-CA and such that $(\forall \phi \in \Sigma^1_1)(\forall a_1,\dots,a_s)(R_\alpha^{\underline{M}})_E\langle (F)_E, (R_\alpha)^{\underline{M}}_E, E^*\rangle \models$

$$\phi(a_1/x_1,\dots,a_s/x_s) \Leftrightarrow \langle F,M,E^*\rangle \models \phi(a_1/x_1,\dots,a_s/x_s).$$

Since the set H is definable in every model for Δ^1_1-CA (extending $\underline{M}$) by a Σ^1_1-formula $\phi(x)$ with parameters $a_1,\dots,a_k$ belonging to $(R_\gamma^{\underline{M}})_E$, the following holds for every $\alpha > \gamma$ (α as above):

$$H \cap (R_\gamma^{\underline{M}})_E = \{a \in M : \langle F,M,E^*\rangle \models \phi(a/x, a_1/x_1,\dots,a_k/x_k)\} =$$
$$= \{a \in M : a \in (R_\gamma^{\underline{M}})_E \ \&\ \langle (F)_E, (R_\alpha^{\underline{M}})_E, E^*\rangle \models$$
$$\models \phi(a/x, a_1/x_1,\dots,a_k/x_k)\}$$

Hence

$$H = \{a \in M : \langle \underline{M}, X, R, \Sigma_\delta\rangle = (\exists\alpha)_{>\gamma}(\exists\beta)(\exists F)(\exists h)[\varphi(\alpha,\beta)\ \&$$
$$s^\frown\langle \alpha,F,h\rangle \in \Sigma_\delta \ \&\ osp(F) = \beta \ \&\ \langle F, R_\alpha, \epsilon\rangle \models \phi(x/a, x_1/a_1,\dots$$
$$\dots, x_n/a_n)]\}.$$

Since $X,R \in Def(\underline{M},\Sigma_\delta)$, then also $H \in Def(\underline{M},\Sigma_\delta)$ which contradicts the choice of H.

Thus, we have proved the second component of the disjunction to be false.

The first component, on the other hand, can be interpreted as follows:

$$\langle \underline{M}; a\rangle_{a\in M} \models (\forall\alpha)_{On}(\exists x)_{\subseteq On^2}\{\bar{\bar{x}} \geq \alpha \ \&\ (\forall\beta,\beta',\gamma,\gamma')[\langle\beta,\beta'\rangle,\langle\gamma,\gamma'\rangle \in x \Rightarrow (\exists F,h,F_1,h_1)[s^\frown\langle\beta,F,h\rangle \in \Sigma_\delta \ \&\ s^\frown\langle\gamma,F_1,h_1\rangle \in \Sigma_\delta \ \&\ osp(F) = \beta' \ \&\ osp(F_1) = \gamma' \ \&\ \sim(\exists g)[g : \langle F,R_\beta,\epsilon\rangle \prec^1_1 \langle F_1,R_\gamma,\epsilon\rangle \ \&\ g\restriction R_\beta = id_{R_\beta}]]]\}$$

where Σ_δ should be replaced by an appropriate approximation formula.

Let us denote the formula appearing in the curly brackets by $\varphi_\delta(\alpha,x) = \varphi(\alpha,x,\Sigma_\delta)$. By Barwise's theorem in [1] quoted before 2.1, the mapping $\delta \to \varphi_\delta(\alpha,x)$ is Σ_1-definable over $HYP_{\underline{M}}$.

Let $\alpha \in On^{\underline{M}}$ be a fixed ordinal in $\underline{M}$. Hence

$$(\forall\delta)_{<\kappa^{\underline{M}}} \langle \underline{M}; a\rangle_{a\in M} \models (\exists x)_{\subseteq On^2}\ \varphi_\delta(\alpha,x).$$

We shall show that the quantifiers $\forall\delta,\exists x$ can be interchanged.

Let us assume the contrary, i.e.

$$(\forall x)_M(\underline{M} \models x \subseteq On^2 \Rightarrow (\exists\delta)_{\kappa^{\underline{M}}}\langle \underline{M}; a\rangle_{a\in M} \models \sim\varphi_\delta(\alpha,x))$$

Using Σ_1-collection in $HYP_{\underline{M}}$, we obtain that there exists an ordinal number $\pi < \kappa^{\underline{M}}$ such that

$$(\forall x)_M[\underline{M} \models x \subseteq On^2 \Rightarrow (\exists\delta)_{<\pi}\langle \underline{M}; a\rangle_{a\in M} \models \sim\varphi_\delta(\alpha,x)]$$

On the other hand, since the formula defining the set Σ_δ is positively included in the fomula φ_δ,

$$(\forall\delta,\delta_1)[\delta < \delta_1 \Rightarrow \langle \underline{M}; a\rangle_{a\in M} \models (\forall\alpha)(\forall x)_{\subseteq On^2}(\varphi_{\delta_1}(\alpha,x) \Rightarrow \varphi_\delta(\alpha,x))]$$

and by $\langle \underline{M}; a\rangle_{a\in M} \models (\exists x)_{\subseteq On^2}\ \varphi_\pi(\alpha,x)$ we have:

$$\langle \underline{M}; a\rangle_{a\in M} \models (\exists x)_{\subseteq On^2} \bigwedge_{\delta<\pi} \varphi_\delta(\alpha,x).$$

This contradicts the previous conclusion. Thus, the possibility of interchanging the quantifiers has been shown (note that the above "trick" is closely connected with the notion of α-recursive saturation appearing in Barwise [1].

By the same argument, the quantifier $(\forall\delta)_{<\kappa^{\underline{M}}}$ can be interchanged with the remaining existential quantifiers, hence, also with $(\exists F,H,F_1,h_1)$. Thus, we finally obtain the following property:

There exists a set x of $\underline{M}$ such that $\mathrm{Card}^{\underline{M}}(x) \geq \alpha$ and such that for all $\beta,\gamma,\beta',\gamma' \in On^{\underline{M}}; F,h,F_1,h_1 \in M$ if $\langle \beta,\beta'\rangle^{\underline{M}}, \langle \gamma,\gamma'\rangle^{\underline{M}} \in x_E$, $\langle \beta,\beta'\rangle^{\underline{M}} \neq \langle \gamma,\gamma'\rangle^{\underline{M}}$ and $osp^{\underline{M}}(F) = \beta'$, $osp^{\underline{M}}(F_1) = \gamma'$ and $s^\frown\langle \beta,F,h\rangle \in \Sigma$, $s^\frown\langle \gamma,F_1,h_1\rangle \in \Sigma$ then there does not exist

$$g : \langle (F)_E, (R^{\underline{M}}_\beta)_E, E^*\rangle \prec^1_1 \langle (F_1)_E, (R^{\underline{M}}_\gamma)_E, E^*\rangle \text{ such that}$$

$g\restriction(R^{\underline{M}}_\beta)_E = \mathrm{id}$.

Let us assume that $s_1 = s^\frown\langle \beta,F,h\rangle \in \Sigma$, $\beta < \gamma$ $s_2 = s^\frown\langle \gamma,F_1,h_1\rangle \in \Sigma$. Moreover, assume that there exist complete games (compatible with plan Σ) extending respectively s_1 and s_2, such that the models for KM determined by those games are identical and equal to $\underline{F} = \langle F,M,E^*\rangle$. Then there exists

$$g : \underline{F}_1 = \langle (F)_E, (R^{\underline{M}}_\beta)_E, E^*\rangle \prec^1_1 \langle (F_1)_E, (R^{\underline{M}}_\gamma)_E, E\rangle = \underline{F}_2.$$

To show this, let us observe that by the assumption and lemma 2.1, $\underline{F} = \mathrm{Def}(\underline{F})$, $\underline{F}$ is a model for KM and there exist mappings:

$$h_1 : \underline{F}_1 \prec^1_{n+1} \underline{F}, \quad h_1\restriction (R^{\underline{M}}_\beta)_E = \mathrm{id}_{(R^{\underline{M}}_\beta)_E}$$

$$h_2 : \underline{F}_2 \prec^1_{n+1} \underline{F}, \quad h_2\restriction (R^{\underline{M}}_\gamma)_E = \mathrm{id}_{(R^{\underline{M}}_\gamma)_E} \quad \text{such, that}$$

$h_1 : = \{X \in F : X$ is Δ^1_{n+1}-definable in $\underline{F}$ by formulas with parameters belonging to $(R^{\underline{M}}_\beta)_E\}$, $h_2''(F_2) = \{X \in F : X$ is Δ^1_{n+1}-definable in $\underline{F}$ by formulas with parameters belonging to $(R^{\underline{M}}_\gamma)_E\}$. Hence, by $\underline{M} \models \beta < \gamma$, $h_1''(F_1) \subseteq h_2''(F_2)$.

Let $h = h_2^{-1} \circ h_1$.

From the Σ^1_{n+1}-elementarity of h_1 and h_2 it follows that h is a Σ^1_{n+1}-elementary monomorphism. Of course, also $h\restriction(R^{\underline{M}}_\beta)_E = \mathrm{id}_{(R^{\underline{M}}_\beta)_E}$ holds.

In view of the above fact, the previously obtained property can be formulated as follows:

For any $s \in \Sigma$ and for any $\alpha \in On^{\underline{M}}$ there exists a set x in $\underline{M}$ (one such set let us denote by $m(s,\alpha)$) such that

(i) the elements of set x_E are the moves of the first player extending the game s and compatible with plan Σ.

(ii) $\mathrm{card}^{\underline{M}}(x) \geq \alpha$

(*iii*) for any $\langle\beta,F,h\rangle^{\underline{M}}$, $\langle\beta_1,F_1,h_1\rangle^{\underline{M}} \in x_E$ the models for KM determined by any two games (compatible with plan Σ) extending respectively $s^\frown\langle\beta,F_1,h\rangle$, $s^\frown\langle\beta_1,F_1,h_1\rangle$ are different.

To end the proof of the theorem for this case, we choose a sequence $\alpha_1,\alpha_2,\dots$ of ordinals from $\underline{M}$ cofinal with the height of $\underline{M}$.

Let $T \subseteq \bigcup_{n\in\omega} M^n$ be the least transitive relation generated by the relation:

$$s \in \Sigma \ \& \ t \in \Sigma \ \& \ (\exists x)[x \in m(s,\alpha_{\ell h(s)})_E \ \& \ t = s^\frown x^\frown \alpha_{\ell h(s)}]$$

Evidently, T is a tree of height ω. Every maximal branch of this tree is a game consistent with the winning strategy of the first player and determining, in effect, a certain extension of the model $\underline{M}$ to a model for KM of the form $\langle F,M,E^*\rangle$.

As is easily noted, different branches determine different extensions. Hence the power of the set of extensions of a model $\underline{M}$ to models of KM+$\forall X r.a(X)$ is greater or equal to the power of the set of branches of the tree T i.e. it is $\geqslant \prod_{n\in\omega}(\alpha_n)_E$.

Thus it suffices to show, that $\prod_{n\in\omega}(\overline{\overline{\alpha_n}})_E = \overline{\overline{M}}^{\aleph_0}$.

If there exists an $\alpha \in On^{\underline{M}}$ such that $\overline{\overline{\alpha_E}} = \overline{\overline{M}}$, then, obviously, $\prod_{n\in\omega}(\overline{\overline{\alpha_n}})_E = (\overline{\overline{\alpha}}_E)^{\aleph_0}$. So, let us assume that such an α does not exist. We have $(\overline{\overline{M}})^{\aleph_0} = (\sum_{n\in\omega}(\overline{\overline{\alpha_n}})_E)^{\aleph_0} = 2^{\aleph_0}\cdot\prod_{n\in\omega}(\overline{\overline{\alpha_n}})_E^{\aleph_0} = \prod_{n\in\omega}(\overline{\overline{\alpha_n}})_E$.

Case 2. $\underline{M}$ is not an ω-model. Since $\underline{M}$ is KM-expandable, there exists a class of satisfaction for $\underline{M}$ and all the formulas belonging to $\underline{M}$ which can be used as a parameter in the induction scheme. Hence follows that $\underline{M}$ is recursively saturated. Thus (see [1]) $\kappa^{\underline{M}} = \omega$.

Putting

$$R = \{\langle\alpha,\beta\rangle^{\underline{M}} : (\exists F,h,F_1,h_1)[s^\frown\langle\alpha,F,h\rangle \in \Sigma_n \ \& \ s^\frown\langle\beta,F_1,h_1\rangle \in \Sigma_n \ \& \ \underline{M} \models (R_\alpha \prec R_\beta)]\}$$

$X = \mathrm{Dom}\, R$, where $n < \omega$, we obtain

(*i*) $R, X \in \mathrm{Def}(\underline{M})$

(*ii*) $(\forall\alpha)_{On^{\underline{M}}}(\exists x)_M$

$$[x_E \subseteq X \ \& \ \mathrm{card}^{\underline{M}}(x) \geqslant \alpha \ \& \ \forall(\beta,\gamma)_{x_E}(\beta < \gamma \Rightarrow \underline{M} \models R_\beta \prec R_\gamma)]$$

Indeed, by Th. 1.1, the negation of (ii) would mean that there exists an unbounded and definable subclass $\varphi(\underline{M})$ of $On^{\underline{M}}$, such that

$$(\forall\alpha,\beta)_{\varphi(\underline{M})} \quad \underline{M} \models (\alpha \in \beta \Rightarrow R_\alpha \prec R_\beta)$$

When summing all the classes of satisfaction for $(R_\alpha^{\underline{M}})_E$ with respect to all $\alpha \in \varphi(\underline{M})$, we would obtain a definable class of satisfaction for model $\underline{M}$, which contradicts Tarski's theorem on the non-definability of the notion of truth.

By the same argument as in the previous case we show that in (ii) the subformula $x_E \subseteq X$ can be replaced by

$$(\forall\beta)_{x_E}(\exists F,h)(s^\frown\langle\beta,F,h\rangle \in \Sigma).$$

Let $m(s,\alpha)$ be such a set x of $\underline{M}$ that $card^M(x) \geqslant \alpha$ and

$$(\forall\beta)_{x_E}(\exists F,h)(s^\frown\langle\beta,F,h\rangle \in \Sigma)$$

and

$$(\forall\beta,\gamma)_{x_E}(\beta \neq \gamma \Rightarrow \underline{M} \models \sim R_\beta \prec R\gamma)$$

If $\beta,\gamma \in m(s,\alpha)_E, \beta \neq \gamma$ and $\langle F_1,M,E^*\rangle, \langle F_2,M,E^*\rangle$ are two expansions determined by complete games (compatible with plan Σ) extending respectively $s^\frown\langle\beta,F,h\rangle, s^\frown\langle\gamma,F_1,h_1\rangle$, then these expansions have different classes of satisfaction for $\underline{M}$. To prove this, let S_1,S_2 be classes of satisfaction for $\underline{M}$ in these expansions. Assume that $S_1 = S_2$ and let $\beta < \gamma$. Hence if $\nu E(R_\beta^\omega)^{\underline{M}}$, then $\underline{M} \models (R_\beta \models \varphi[\nu])$ iff $\langle\varphi,\nu\rangle \in S_1$ iff $\langle\varphi,\nu\rangle \in S_2$ iff $\underline{M} \models (R_\gamma \models \varphi[\nu])$. This proves that $\underline{M} \models R_\beta \prec R_\gamma$, which contradicts our assumption.

Constructing an identical tree as in Case 1 we obtain the required "lower" estimation of the number of extensions to models of KM+$\forall$Xr.a(X).

<u>Corollary.</u> If $\underline{M}$ is a KM-expandable model for ZFC such that cf $On^{\underline{M}} = \omega$, then the power of the set of expansions of this model to models $\underline{F}$ satisfying $\underline{F}$ = Def($\underline{F}$) and KM+ Xr.a(X) is equal to $\bar{\bar{M}}^{\aleph_0}$ (in particular, for $\underline{M}$ a standard model of the form R_α with cf $\alpha = \omega$, this power $\bar{\bar{M}}^{\aleph_0}$ equals to $2^{\bar{\bar{M}}}$).

Proof. From the proof of the previous theorem we infer that $\bar{\bar{M}}^{\aleph_0} \leqslant$ the power of the family of expansions F of models $\underline{M}$ satisfying $\underline{F} = \mathrm{Def}(\underline{F})$ and $\mathrm{KM}+\forall X r.a(X)$.

Let $\underline{F}$ be a member of this family; $F = \bigcup_{n\in\omega} F_n$, where the F_n's are codable in F, as follows from the proof of lemma 2.1. Using the fact that $\mathrm{cf}\ \mathrm{On}^{\underline{M}} = \omega$ we can represent F as a countable sum of subsets of F, codable in $\underline{F}$ by classes with a set domain. From the proof of lemma 2.1 (see also the proof of lemma 1,[7]), we deduce that there exists a direct system of power $\aleph_0$, with elements belonging to M and the limit being equal to $\underline{F}$. Thus we obtain the required estimation of the power.

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A FINE STRUCTURE GENERATED BY REFLECTION FORMULAS OVER PRIMITIVE RECURSIVE ARITHMETIC

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In the following we shall describe a fine structure on hierarchies generated by reflection formulas over primitive recursive arithmetic (PRA) and some of its extensions.

Let T be some rec. enum. extension of PRA and $C_n[T]$ a formula expressing the sentence "every Π^0_{n+1}-formula provable in T is true". Then for each natural number n hierarchies of theories $\binom{n}{\alpha}$ can be defined by

$$\binom{n}{0} = \text{PRA}$$

$$\binom{n}{\alpha+1} = \binom{n}{\alpha} + C_n[\binom{n}{\alpha}]$$

$$\binom{n}{\lambda} = \bigcup_{\alpha<\lambda}\binom{n}{\alpha} \quad \text{for limit ordinals } \lambda,$$

where the ordinals are supposed to be elements of a well order definable by a p.r. order predicate <.

The main result of this paper, the fine structure theorem on the hierarchies $\binom{n}{\alpha}$, reads: For all natural numbers n, p and all

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[1]) The results reported here are part of the author's doctoral dissertation, Heidelberg 1978. Thanks are due to J.-Y. GIRARD (Paris), G.H.MÜLLER (Heidelberg), H.SCHWICHTENBERG (München), and C.SMORYNSKI (Chicago).

ordinals α, β

$$\binom{n+p}{\alpha} \equiv_n \binom{n}{\omega_p(\alpha)} \qquad (p,\alpha \neq 0)$$

$$\binom{n}{\beta+1} \cup \binom{n+p}{\alpha} \equiv_n \binom{n}{\beta+\omega_p(\alpha)} \qquad (p \neq 0) ,$$

where $\equiv_n$ means provability of the same Π^o_{n+1}-formulas. - These relations had been conjectured by GIRARD.

The comparison of the hierarchies $\binom{n}{\alpha}$ or the iterated reflection formulas involved with other proof theoretical concepts and the application of the fine structure theorem yield a lot of results; some of them are new, others, already known, are much easier to obtain by the fine structure than by their original proofs. It turns out, that this method is a useful tool for dealing with problems concerning consistency and reflection principles, transfinite induction, proofs of restricted complexity, and perhaps some other concepts.

Although in the sequel only systems based on classical logic are considered, it seems evident that the fine structure relations apply to intuitionistic systems too.
For the sake of shortness we shall not give complete proofs; some are sketched, most of them have been dropped and are somewhat technical anyway.

Let $<$ be a p.r. order predicate describing some total well order on the natural numbers (in fact, well order properties will not even be used). For this order, p.r. functions like ' (successor with respect to $<$), $+$, $\cdot$, $\exp_\omega$, ε etc. should be defined and their usual properties should be provable in PRA. In addition, let $\{\ \}$ be a two-placed p.r. function which assigns to each non-zero ordinal α a certain series $(\{\alpha\}n)_{n<\omega}$ of ordinals converging to α if α is a limit ordinal or the constant series $\{\alpha\}n = \beta$ if α is the successor of β.

Extensive use will be made of the following notion characterizing provability in the formal systems we consider here: Let T be a r.e. extension of PRA and let $Pr_T(x)$ be a canonical (Σ^o_1-) provability predicate for T. Then a formula $\varphi(x)$ is called reflexively progressive (in x) with respect to T if

$$\forall x[\forall y<x Pr_T(\ulcorner\varphi(\dot{y})\urcorner) \to \varphi(x)]$$

is provable in T. This characterizes provability in T:

Theorem (reflexive progressiveness and provability):

A formula is provable in T iff it is reflexively progressive with respect to T.

Proof: One direction is trivial; for the other, suppose $\varphi(x)$ is reflexively progressive with respect to T. Then

$$T \vdash Pr_T(\ulcorner\forall y<x\varphi(y)\urcorner) \to \forall y<x Pr_T(\ulcorner\varphi(\dot{y})\urcorner) \to$$

$$[z<x \to \forall y<z Pr_T(\ulcorner\varphi(\dot{y})\urcorner)] \overset{hyp}{\to} [z<x \to \varphi(z)] \quad ,$$

i.e. $$T \vdash Pr_T(\ulcorner\forall y<x\varphi(y)\urcorner) \to \forall y<x\varphi(y).$$

By LÖB's theorem [6] follows: $T \vdash \forall y<x\varphi(y)$. But then $Pr_T(\ulcorner\forall y<x\varphi(y)\urcorner)$, and hence $\forall y<x Pr_T(\ulcorner\varphi(\dot{y})\urcorner)$, is also provable in T and another application of the hypothesis shows the provability of $\varphi(x)$ in T. - We owe this simple and short proof to a hint by GIRARD (it replaces a much longer one we had before).

We can now give an exact definition of the iterated reflection formulas needed for the construction of the hierarchies $\binom{n}{\alpha}$: By KLEENE's Primitive Recursion Theorem [4] a primitive recursive function F_n exists such that

$$F_n(0) = \ulcorner\forall x[\Pi_{n+1}(x) \wedge Pr(x) \to T_{n+1}(x)]\urcorner$$

$$F_n(\alpha+1) = \ulcorner\forall x[\Pi_{n+1}(x) \wedge Pr(F_n(\alpha)\dot{\to}x) \to T_{n+1}(x)]\urcorner$$

$$F_n(\lambda) = \ulcorner\forall y T_{n+1}(F_n(\{\lambda\}\dot{y}))\urcorner \quad \text{for limit ordinals } \lambda$$

(here Pr means "provable in PRA", $\Pi_{n+1}(x)$ is a p.r. predicate for Gödel numbers of Π^0_{n+1}-formulas and $T_{n+1}(x)$ is a partial truth definition for these formulas). Apparently $F_n(\alpha)$ is a Gödel number of a certain formula $C_n(\alpha)$: For every ordinal α, let

$$C_n(\alpha) \overset{def}{\leftrightarrow} T_{n+1}(F_n(\alpha)) \; .$$

The C_n are iterated reflection formulas having the following basic properties:

Proposition (properties of the C_n): The following is provable in PRA:

(1) $\alpha\neq 0$: $C_n(\alpha) \leftrightarrow \forall x[\Pi_{n+1}(x)\wedge\exists y<\alpha Pr(\ulcorner C_n(\dot{y})\urcorner \dot{\rightarrow} x) \rightarrow T_{n+1}(x)]$

(2) $\alpha\leqq\beta \rightarrow [C_n(\beta) \rightarrow C_n(\alpha)]$

(3) $C_n(0) \leftrightarrow \forall x[\Pi_{n+1}(x)\wedge Pr(x) \rightarrow T_{n+1}(x)]$

(4) $C_n(\alpha') \leftrightarrow \forall x[\Pi_{n+1}(x)\wedge Pr(\ulcorner C_n(\alpha)\urcorner \dot{\rightarrow} x) \rightarrow T_{n+1}(x)]$

(5) $Lim(\lambda) \rightarrow [C_n(\lambda) \leftrightarrow \forall y<\lambda C_n(y)]$

(6) $C_{n+1}(0) \rightarrow C_n(0)$

Proof: (1) by showing the reflexive progressiveness of the formula; (2) follows from (1); (3), (4), (5) follow from (1) and the definition of the C_n; (6) is trivial.

Now let, for all natural numbers n and ordinals α of the pre-fixed order

$$\binom{n}{0} \equiv PRA \quad \text{and} \quad \binom{n}{\alpha} = PRA \cup\{C_n(\beta):\ \beta<\alpha\} .$$

For abbreviation we note

$$Pr_{\binom{n}{\alpha}}(x) \leftrightarrow Pr(x)\vee\exists y<\alpha Pr(\ulcorner C_n(\dot{y})\urcorner \dot{\rightarrow} x)$$

$$Pr_{\binom{n}{\alpha}\cup\binom{m}{\rho}}(x) \leftrightarrow Pr_{\binom{n}{\alpha}}(x)\vee\exists y<\rho Pr_{\binom{n}{\alpha}}(\ulcorner C_m(\dot{y})\urcorner \dot{\rightarrow} x) .$$

The fine structure theorem stated in the introductory remark is just a straightforward generalization of the following

Theorem (fine structure theorem): For all n and ordinals α, β

(1) $$\binom{n+1}{\alpha} \equiv_n \binom{n}{\omega^\alpha} \qquad (\alpha\neq 0)$$

(2) $$\binom{n}{\beta+1} \cup \binom{n+1}{\alpha} \equiv_n \binom{n}{\beta+\omega^\alpha}$$

Sketch of proof: Apparently (2) implies (1). In order to prove (2), two directions have to be shown:

(a) For every Π^0_{n+1}-formula φ we have: If $\binom{n}{\beta+1}\cup\binom{n+1}{\alpha} \vdash \varphi$, then $\binom{n}{\beta+\omega^\alpha} \vdash \varphi$

(b) For every $\gamma<\beta+\omega^{\alpha}$ $\quad \binom{n}{\beta+1}\cup\binom{n+1}{\alpha} \vdash C_n(\gamma)$.

To (a): Consider the following formalization of (a):

$$\forall\gamma\forall x[\Pi_{n+1}(x)\wedge \Pr_{\binom{n}{\gamma+1}\cup\binom{n+1}{\alpha}}(x) \rightarrow \Pr_{\binom{n}{\gamma+_{\omega}\alpha}}(x)] \quad (\overset{\text{def}}{\leftrightarrow} \varphi_n(\alpha)\ ).$$

Then (a) can be proven by showing that $\varphi_n(x)$ is reflexively progressive in x with respect to $\binom{0}{1}$ (this will be explained later); clearly, if $\varphi_n(\alpha)$ holds for all α, then (a) holds.

$$\binom{0}{1} \vdash \forall y<0\Pr_{\binom{0}{1}}(\ulcorner\varphi_n(\dot{y})\urcorner) \rightarrow \varphi_n(0)$$

is trivially true; even

$$\binom{0}{1} \vdash \mathrm{Lim}(\lambda)\wedge\forall y<\lambda\Pr_{\binom{0}{1}}(\ulcorner\varphi_n(\dot{y})\urcorner) \rightarrow \varphi_n(\lambda)$$

is obvious. The successor case would be done, if

$$\binom{0}{1} \vdash \Pr_{\binom{0}{1}}(\ulcorner\varphi_n(\alpha)\urcorner) \rightarrow \varphi_n(\alpha+1)$$

could be shown for arbitrary α. We shall give an informal proof of this by pointing out that under the assumption $\binom{0}{1} \vdash \varphi_n(\alpha)$ the following holds for every ordinal γ and every Π^{o}_{n+1}-formula ψ:

$$\text{If } \binom{n}{\gamma+1}\cup\binom{n+1}{\alpha+1} \vdash \psi, \text{ then } \binom{n}{\gamma+_{\omega}\alpha+1} \vdash \psi .$$

So suppose $\binom{n}{\gamma+1}\cup\binom{n+1}{\alpha+1} \vdash \psi$. Then there exists a Gentzen-type formal PRA-deduction (the induction rule is restricted to Π^{o}_{o}-formulas) with an endsequent

$$C_{n+1}(\alpha),\ C_n(\gamma) \vdash \psi$$

and from which all cuts more complicated than Π^{o}_{o} have been eliminated. As it is easy to see that there are a Π^{o}_{n}-formula $\theta^{n}_{\alpha}(x,y)$ and a Σ^{o}_{n}-formula $\chi^{n}_{\gamma}(x)$ such that

$$C_n(\gamma) \leftrightarrow \forall x\chi^{n}_{\gamma}(x),$$

$$C_{n+1}(\alpha) \leftrightarrow \forall x\exists y\theta^{n}_{\alpha}(x,y) \quad\text{and}\quad \forall x\Pr_{\binom{n+1}{\alpha}}(\ulcorner\exists y\theta^{n}_{\alpha}(\dot{x},y)\urcorner)$$

are provable in PRA, we may demand without loss of generality that all occurrences of $C_{n+1}(\alpha)$ and $C_n(\gamma)$ in the given deduction are in the form $\forall x\exists y\theta^{n}_{\alpha}(x,y)$ and $\forall x\chi^{n}_{\gamma}(x)$. We are now going to prove that for each sequent $\Gamma\vdash\Delta$ in the deduction there is a natural number k such that the sequent $C_n(\gamma+_{\omega}\alpha\cdot k),\ \Gamma^{*} \vdash \Delta$ is also provable in PRA, where Γ^{*} is obtained from Γ by deleting all occurrences of the formulas $\forall x\chi^n_\gamma(x)$, $\forall x\exists y\theta^n_\alpha(x,y)$, $\exists y\theta^n_\alpha(t,y)$ (t an arbitrary term). Note that then $\bigwedge\Gamma^* \rightarrow \bigvee\Delta$ is a Π^0_{n+1}-formula: This is true first for the endsequent; but since the deduction has no non-trivial cuts, it holds for all sequents in it. The proof proceeds by an induction on the height of the sequents $\Gamma \vdash \Delta$ in the given deduction.

- Suppose $h(\Gamma\vdash\Delta)=0$. Then $\Gamma\vdash\Delta$ is a logical axiom or a proper PRA-axiom and, trivially, $C_n(\gamma),\Gamma^*\vdash\Delta$ is provable in PRA.

- Suppose the allegation proven for all sequents of height $\leq$ p. In order to prove it for sequents of height p+1, we distinguish different cases according to the last inference rule. Checking all possible inference rules reveals that there is only one non-trivial case, viz. when the last inference is of the form

$$\frac{\theta^n_\alpha(t,y),\ \Gamma\vdash\ \Delta \qquad \text{(height p)}}{\exists y\theta^n_\alpha(t,y),\ \Gamma\vdash\ \Delta \qquad \text{(height p+1)}}$$

with a free variable y not occurring in Γ, Δ. Let $x_1,\ldots,x_r$ be the free variables occurring in t, Γ, Δ. By the induction hypothesis, there exists k such that

$$C_n(\gamma+\omega^\alpha\cdot k),\ (\theta^n_\alpha(t,y),\ \Gamma)^*\ \vdash\ \Delta$$

is provable in PRA. As $\theta^n_\alpha(t,y)$ is not one of the "prohibited" formulas, $(\theta^n_\alpha(t,y),\ \Gamma)^*$ is identical with $\theta^n_\alpha(t,y),\Gamma^*$, i.e. the sequent

$$C_n(\gamma+\omega^\alpha\cdot k),\ \theta^n_\alpha(t,y),\ \Gamma^*\ \vdash\ \Delta$$

is provable in PRA. An inference "$\exists$-left" gives

$$C_n(\gamma+\omega^\alpha\cdot k),\ \exists y\theta^n_\alpha(t(x_1\ldots x_r),y),\ \Gamma^*(x_1\ldots x_r)\ \vdash\ \Delta(x_1\ldots x_r)$$

and therefore

$$\forall x_1\ldots\forall x_r Pr(\ulcorner C_n(\gamma+\omega^\alpha\cdot k),\exists y\theta^n_\alpha(t(\dot{x}_1..\dot{x}_r),y),\Gamma^*(\dot{x}_1..\dot{x}_r)\vdash\Delta(\dot{x}_1..\dot{x}_r)\urcorner)$$

is provable in PRA too. With

$$\forall x Pr_{\binom{n+1}{\alpha}}(\ulcorner\exists y\theta^n_\alpha(\dot{x},y)\urcorner)$$

the formula $\exists y\theta^n_\alpha(t(\dot{x}_1\ldots\dot{x}_r),y)$ can be cut off and we get

$$\forall x_1\ldots\forall x_r Pr_{\binom{n+1}{\alpha}\cup\binom{n}{\gamma+\omega^\alpha\cdot k+1}}(\ulcorner\Gamma^*(\dot{x}_1\ldots\dot{x}_r)\ \vdash\ \Delta(\dot{x}_1\ldots\dot{x}_r)\urcorner).$$

$\Pi_{n+1}(\ulcorner\Gamma^* \vdash \Delta\urcorner)$ is provable in PRA, so the assumption $\binom{0}{1} \vdash \varphi_n(\alpha)$ can be applied and gives

$$\binom{0}{1} \vdash \forall x_1 \ldots \forall x_r Pr_{(\gamma+\omega^\alpha \cdot k+\omega^\alpha)}^{n}(\ulcorner\Gamma^*(\dot{x}_1 \ldots \dot{x}_r) \vdash \Delta(\dot{x}_1 \ldots \dot{x}_r)\urcorner),$$

from what the provability of

$$C_0(0),\ C_n(\gamma+\omega^\alpha \cdot (k+1)), \Gamma^*(x_1 \ldots x_r) \vdash \Delta(x_1 \ldots x_r)$$

follows and hence

$$C_n(\gamma+\omega^\alpha \cdot (k+1)),\ \Gamma^* \vdash \Delta \text{ ; q.e.d.}$$

The formalization of this proof needs an induction on some Π_1^0-formula ψ. This is the reason why the reflexive progressiveness of $\varphi_n(x)$ has to be proven in PRA+$C_0(0)$ (and not in PRA): If for a Π_1^0-formula ψ $\psi(0) \wedge \forall x[\psi(x) \rightarrow \psi(Sx)]$ is provable in PRA, then $\forall x \psi(x)$ is provable in PRA+$C_0(0)$.

The proof of direction (b) is much easier: It suffices to show the reflexive progressiveness of the formula

$$C_{n+1}(x) \rightarrow \forall z[C_n(z) \rightarrow C_n(z+\omega^x)]$$

in x with respect to PRA; this goes straightforward.

The usefulness of iterated reflection formulas and the fine structure becomes more evident by comparing the C_n to other proof theoretical concepts and by showing some applications. Here some examples:

1. Induction and iterated reflection formulas

Definition: For every formula $\varphi(x)$ let Ind(φ) be the formula $\varphi(0) \wedge \forall x[\varphi(x) \rightarrow \varphi(Sx)] \rightarrow \forall x \varphi(x)$ and let n-Ind be $\{\text{Ind}(\varphi) : \varphi \in \Pi_{n+1}^0\}$.

Proposition (Π_{n+1}^0-induction and the C_n):

(1) PRA+n-Ind $\vdash C_n(0)$

(2) PRA+$C_{n+2}(0) \vdash$ n-Ind

Proof: (1) Consider the formula

$$\varphi_n(z) \overset{\text{def}}{\Leftrightarrow} \forall x[\Pi_{n+1}(x) \wedge \exists y Prf^*(y,x) \wedge h(y) \leq z \rightarrow T_{n+1}(x)] ,$$

where Prf*(y,x) stands for "y is a number of a PRA-proof of the formula with number x and this proof has no cuts more complicated than Π^o_o" and h is a p.r. function assigning to each number of a proof its height. Then clearly $C_n(0) \leftrightarrow \forall z \varphi_n(z)$ is provable in PRA. Furthermore, we have

$$\mathrm{PRA} \vdash \varphi_n(0) \quad \text{and} \quad \mathrm{PRA} \vdash \forall z[\varphi_n(z) \rightarrow \varphi_n(Sz)] \ .$$

As $\varphi_n(z)$ is a Π^o_{n+1}-formula, an application of Π^o_{n+1}-induction gives PRA + n-Ind $\vdash \forall z \varphi_n(z)$.

(2) For every formula ψ, $\forall z \mathrm{Pr}(\ulcorner \psi(0) \wedge \forall x[\psi(x) \rightarrow \psi(Sx)] \rightarrow \psi(\dot{z}) \urcorner)$ is provable in PRA. If ψ is Π^o_{n+1}, then $\psi(0) \wedge \forall x[\psi(x) \rightarrow \psi(Sx)] \rightarrow \psi(z)$ is Π^o_{n+3} and therefore PRA $\vdash C_{n+2}(0) \rightarrow \mathrm{Ind}(\psi)$.

The above proposition enables us to compare Peano arithmetic PA with the PRA hierarchies. We have the following

<u>Theorem (Peano arithmetic and the PRA hierarchies)</u>:

For each natural number n

(1) PRA + n-Ind $\equiv_n \binom{n}{\omega^{\omega}}$ (2) PA $\equiv_n \binom{n}{\varepsilon_0}$.

<u>Proof</u>: (1) If PRA+n-Ind $\vdash \varphi$, then by the above proposition PRA + $C_{n+2}(0) \vdash \varphi$, i.e. $\binom{n+2}{1} \vdash \varphi$. If φ is a Π^o_{n+1}-formula, then two applications of the fine structure theorem yield $\binom{n}{\omega^{\omega}} \vdash \varphi$. On the other side, we have PRA + n-Ind $\vdash C_n(0)$ and, by an induction on k, PRA + n-Ind $\vdash \forall z[C_n(z) \rightarrow C_n(z+\omega^k)]$, for all finite k. Hence for all $\gamma < \omega^{\omega}$ PRA + n-Ind $\vdash C_n(\gamma)$.

(2) This results from (1) and by iterated applications of the fine structure theorem.

Note that the uniform reflection schema over PRA, i.e. the set

$$\mathrm{RFN(PRA)} = \{\forall x \mathrm{Pr}(\ulcorner \varphi(\dot{x}) \urcorner) \rightarrow \forall x \varphi(x) : \varphi \text{ formula with only x free}\}$$

is equivalent over PRA to the set $\{C_n(0): n < \omega\}$. This, together with the proposition , gives a very short proof of the result by KREISEL and LEVY [5] : PRA + RFN(PRA) $\equiv$ PA.

2. Transfinite induction and iterated reflection formulas

Definition: Using a partial truth definition, define $TI_n(\alpha)$ to be a formula expressing transfinite induction (with respect to the prefixed order) on Π^0_{n+1}-formulas up to the ordinal α. Let

$$[TI]^n_\alpha = PRA \cup \{TI_n(\beta): \beta<\alpha\}$$

$$[TI]\alpha = \bigcup_{n<\omega}[TI]^n_\alpha .$$

Note that because of the provability of

$$TI_n(\alpha) \rightarrow TI_n(\alpha^k) \quad \text{for all finite } k$$

and

$$\alpha\geq\omega \rightarrow [TI_{n+1}(\alpha) \rightarrow TI_n(\omega^\alpha)]$$

in PRA, we have for all $\alpha>\omega$

$$[TI]^n_\alpha \equiv [TI]^n_{\omega_2(\beta)} \quad \text{for some ordinal } \beta$$

and

$$[TI]\alpha \equiv [TI]\varepsilon_\beta \quad \text{for some } \varepsilon\text{-number } \varepsilon_\beta$$

(if $\alpha\leq\omega$, then of course $[TI]^n_\alpha \equiv [TI]\alpha \equiv PRA$). So it is convenient to designate the $[TI]^n_\alpha$ only with ω_2-numbers and the $[TI]\alpha$ only with ε-numbers.

The following proposition demonstrates the close relationship between transfinite induction and iterated reflection formulas.

Proposition (TI_n and C_n): The following is provable in PRA:

(1) $$\alpha\geq\omega\wedge TI_n(\alpha) \rightarrow C_n(\alpha)$$

(2) $$C_{n+2}(\alpha) \rightarrow TI_n(\omega_2(\alpha)) .$$

For the proof we need a lemma which connects progressiveness to reflexive progressiveness. A formula $\varphi(x)$ with free variable x is said to be progressive in x if $\forall x[\forall y<x\varphi(y) \rightarrow \varphi(x)]$ (= $Prog_x\varphi$) holds. There are the following relations between progressiveness and reflexive progressiveness (and hence provability):

Lemma: Let $\varphi(x)$ be a Π^0_{n+1}-formula. Then

(1) If it is provable in PRA that φ is progressive, then $C_n(x)\rightarrow\varphi(x)$ is reflexively progressive with respect to PRA.

(2) If it is provable in PRA + $C_n(\alpha)$ that φ is progressive, then $C_n(\alpha+x) \to \varphi(x)$ is reflexively progressive with respect to PRA.

Proof: obvious.

Proof of proposition: (1) Using an analogous argument as in the proposition on Π^o_{n+1}-induction and the C_n, we show that

$$\text{PRA} \vdash TI_n(\omega) \to C_n(0)$$

$$\text{PRA} \vdash TI_n(\omega) \to \forall\alpha[C_n(\alpha) \to C_n(\alpha')] \ .$$

Hence, with regard to $Lim(\lambda)\wedge\forall y<\lambda C_n(y) \to C_n(\lambda)$, $TI_n(\omega) \to Prog_x C_n$ is provable in PRA. C_n is Π^o_{n+1} and so TI_n can be applied to C_n; this proves (1).

(2) The progressiveness of $TI_n(\omega_2(x))$ is provable in PRA + $C_{n+2}(0)$. From the preceding lemma follows the provability of $C_{n+2}(\alpha) \to TI_n(\omega_2(\alpha))$ for all α.

The proposition yields that, concerning provability of Π^o_{n+1}-formulas, the transfinite induction theories $[TI]^n_\alpha$ can be embedded in the PRA-hierarchies and thus the fine structure relations for the latter induce the same structure on the $[TI]^n_\alpha$. We have the following

Theorem ($[TI]^n_\alpha$ and the hierarchies $\binom{n}{\alpha}$): Let α be an ω_2-number different from ω. Then for all natural numbers n

(1) $$[TI]^n_\alpha \equiv_n \binom{n}{\alpha}$$

(2) $$[TI]^{n+1}_\alpha \equiv_n [TI]^n_{\omega^\alpha}$$

(3) $$[TI]\varepsilon_\gamma \equiv_n \binom{n}{\varepsilon_\gamma} \quad (\varepsilon_\gamma \text{ an arbitrary } \varepsilon\text{-number}).$$

Proof: (1) As $\alpha>\omega$ we have for all $\gamma<\alpha$: $[TI]^n_\alpha \vdash C_n(\gamma)$. On the other side, let φ be a Π^o_{n+1}-formula. We distinguish two cases:
(a) $\alpha=\omega_2(\gamma+1)$. If $[TI]^n_\alpha \vdash \varphi$, then $\exists k<\omega$ PRA $\vdash TI_n(\omega_2(\gamma)^k) \to \varphi$ and so, by the above mentioned remark, PRA $\vdash TI_n(\omega_2(\gamma)) \to \varphi$; from the proposition follows PRA $\vdash C_{n+2}(\gamma) \to \varphi$, hence $\binom{n+2}{\gamma+1} \vdash \varphi$. Two applications of the fine structure theorem give $\binom{n}{\omega_2(\gamma+1)} = \binom{n}{\alpha} \vdash \varphi$.
(b) $\alpha=\omega_2(\lambda)$, where λ is a limit ordinal. Then $[TI]^n_\alpha \vdash \varphi$ implies $\exists\gamma<\lambda$ such that PRA $\vdash TI_n(\omega_2(\gamma)) \to \varphi$, hence $\exists\gamma<\lambda$ PRA $\vdash C_{n+2}(\gamma) \to \varphi$, i.e. $\binom{n+2}{\lambda} \vdash \varphi$ and again $\binom{n}{\omega_2(\lambda)} \vdash \varphi$.

(2) and (3) follow from (1) and the fine structure.

The relations between transfinite induction and the iterated reflection formulas together with fine structure properties allow a very simple proof of the well-known GENTZEN result in [1] , that is the answer to the question how far transfinite induction can be proven by a deduction of bounded complexity.
By a TI-derivation we shall understand a Peano derivation with additional initial sequents of the form $\forall y{<}t\varphi(y) \vdash \varphi(t)$ with an arbitrary term t and whose endsequent is of the form $\vdash \varphi(\alpha)$, α an ordinal of the pre-fixed order. The complexity of a TI-derivation is the least k such that all formulas occurring in it are Π^0_{k+1}. Then the - slightly modified and generalized - Gentzen result reads:

Theorem (Gentzen): Transfinite induction on all Π^0_{k+1}-formulas up to the ordinal α is derivable by a TI-derivation of complexity n iff $\alpha < \omega_{n+3-k}$.

Proof: First note that there is a TI-derivation of complexity n for a formula φ up to the ordinal α iff PRA + n-Ind $\vdash TI_n(\varphi,\alpha)$. So it suffices to show that

$$\text{for every } \varphi\in\Pi^0_{k+1},\ \text{PRA + n-Ind} \vdash TI(\varphi,\alpha) \quad \text{iff} \quad \alpha<\omega_{n+3-k}.$$

(a) Let $n\geq k$; as SCHÜTTE in [8], we define

$$\varphi_0(x) \leftrightarrow \varphi(x) \quad \text{and} \quad \varphi_{m+1}(x) \leftrightarrow \forall z[\forall y{<}z\varphi_m(y) \rightarrow \forall y{<}z+\omega^x\varphi_m(y)]$$

and show that PRA + (m+k)-Ind $\vdash Prog_x\varphi_m \rightarrow Prog_x\varphi_{m+1}$,

and therefore PRA + (m+k)-Ind $\vdash TI(\varphi_{m+1},\alpha) \rightarrow TI(\varphi_m,\omega^\alpha)$

hence PRA + (m+k)-Ind $\vdash TI(\varphi_{m+1},\alpha) \rightarrow TI(\varphi,\omega_{m+1}(\alpha))$.

Trivially, PRA $\vdash TI(\varphi_m,p)$ for all m and finite p. Thus we have PRA + (m+k)-Ind $\vdash TI(\varphi,\omega_{m+1}(p))$ with $p<\omega$, or PRA+(m+k)-Ind $\vdash TI(\varphi,\alpha)$ for all $\alpha<\omega_{m+3}$. With m=n-k, this gives one half of the proof.

(b) $\forall\varphi\in\Pi^0_{k+1}$ PRA + n-Ind $\vdash TI(\varphi,\alpha)$ only for $\alpha<\omega_{n+3-k}$:
Suppose PRA + n-Ind $\vdash TI(\varphi,\omega_{n+3-k})$ for all formulas $\varphi\in\Pi^0_{k+1}$; then PRA + n-Ind $\vdash C_k(\omega_{n+3-k})$. But we also have

$$\text{PRA + n-Ind} \equiv_n \binom{n}{\omega_3} \equiv_{n-1} \binom{n-1}{\omega_4} \equiv_{n-2} \cdots \equiv_k \binom{k}{\omega_{n+3-k}}$$

and so $\binom{k}{\omega_{n+3-k}} \vdash C_k(\omega_{n+3-k})$. This is a contradiction!

Remark: In [7] MINTS observed that transfinite induction up to ω_{n+3} is not only not provable by a derivation of complexity n, there exists even a quantifierfree formula provable in PRA + TI up to ω_{n+3}, but not provable by a deduction of complexity n. This result is quite easy to obtain by using iterated reflection formulas:

- By the above proposition: PRA + $TI_0(\omega_{n+3}) \vdash C_0(\omega_{n+3})$

- By the fine structure theorem:

$$\text{PRA + n-Ind} \equiv_n \binom{n}{\omega_3} \equiv_{n-1} \cdots \equiv_0 \binom{0}{\omega_{n+3}},$$

i.e. PRA + n-Ind $\nvdash C_0(\omega_{n+3})$. $C_0(\omega_{n+3})$ is a Π_1^0-formula; so by dropping the universal quantifier of its prenex form, one obtains a quantifierfree formula (with a free variable) provable in PRA + $TI_0(\omega_{n+3})$ but not provable by a derivation of complexity n.

3. Fine structure relations for other theories than PRA

It is of course quite natural to ask whether a fine structure like that over PRA does exist for other theories too. The definition of iterated reflection formulas $C_n^T(\alpha)$ over a theory T only requires (the Gödel number of) a formula $Pr_T(x)$ characterizing provability in T. Once defined the $C_n^T(\alpha)$, hierarchies $\binom{n}{\alpha}_T$ over T can be defined by

$$\binom{n}{\alpha}_T = T \cup \{C_n^T(\beta) : \beta < \alpha\}.$$

As a first example we consider hierarchies over basic theories of the form $\binom{n}{\alpha}$; we already defined $Pr_{\binom{n}{\alpha}}(x)$. These will be noted $\binom{m}{\beta}\binom{n}{\alpha}$. The corresponding reflection formulas $C_m^{\binom{n}{\alpha}}(\beta)$ are linked to the ordinary C_n by the following

Lemma: The following formulas are reflexively progressive with respect to PRA:

(1) $$C_n^{\binom{n+p}{\gamma}}(x) \leftrightarrow C_n(\omega_p(\gamma)(1+x)) \qquad (p \neq 0)$$

(2) $$C_n^{\binom{n}{\gamma}}(x) \leftrightarrow C_n(\gamma+x)$$

(3) $$C_{n+p}^{\binom{n}{\gamma}}(x) \leftrightarrow C_n(\gamma) \wedge C_{n+p}(x) \qquad (p \neq 0)$$

From this lemma we easily infer the

<u>Fine structure theorem for mixed hierarchies</u>:

For all natural numbers n, p and ordinals α, β

(1) $$\binom{n}{\beta}\binom{n+p}{\alpha} \equiv_n \binom{n}{\omega_p(\alpha)(1+\beta)} \qquad (\alpha, p \neq 0)$$

(2) $$\binom{n+p}{\beta}\binom{n}{\alpha} \equiv_n \binom{n}{\alpha+\omega_p(\beta)} \qquad (\beta \neq 0)\ .$$

A more serious example is Peano arithmetic. We already saw, that for all n $PA \equiv_n \binom{n}{\varepsilon_0}$. So we may use as a characterization of provability in Peano arithmetic the following formula

$$Pr^*_{PA}(x) \overset{def}{\leftrightarrow} \exists y<\varepsilon_0 Pr(\ulcorner C_{d(x)}(\dot{y})\urcorner \dot{\to} x)\ ,$$

where d is a p.r. function which assigns to each Gödel number x of a formula φ the least k such that φ is Π^0_{k+1}. With $Pr^*_{PA}(x)$ the $C^{PA}_n(\alpha)$ and the hierarchies $\binom{n}{\alpha}_{PA}$ over PA can be defined. The proof of the following lemma is rather technical; so we omit it.

<u>Lemma</u>: For all n the formulas $C^{PA}_n(x) \leftrightarrow C_n(\varepsilon_0\cdot(1+x))$ are reflexively progressive with respect to PRA.

The lemma shows that - concerning provability of Π^0_{n+1}-formulas - the PA-hierarchies are embeddable in the PRA-hierarchies. An immediate consequence is that the fine structure relations for the PRA-hierarchies can be transferred correspondingly to the PA-hierarchies. Thus we get the

<u>Embedding and fine structure theorem for PA-hierarchies</u>:

For all natural numbers n and ordinals α

(1) $$\binom{n}{\alpha}_{PA} \equiv_n \binom{n}{\varepsilon_0\cdot(1+\alpha)} \qquad \text{(embedding)}$$

(2) $$\binom{n+1}{\alpha}_{PA} \equiv_n \binom{n}{\varepsilon_0^{\alpha}}_{PA} \quad (\alpha \neq 0) \quad \text{(fine structure)}$$

<u>Proof</u>: (1) from the lemma;

(2) $$\binom{n+1}{\alpha}_{PA} \equiv_{n+1} \binom{n+1}{\varepsilon_0\cdot(1+\alpha)} \equiv_n \binom{n}{\omega^{\varepsilon_0\cdot(1+\alpha)}} = \binom{n}{\varepsilon_0\cdot(1+\varepsilon_0^{\alpha})} \equiv_n \binom{n}{\varepsilon_0^{\alpha}}_{PA} \qquad (\alpha \neq 0)\ .$$

As in the case of PRA, the uniform reflection schema for Peano arithmetic, RFN(PA), is equivalent (over PRA) to $\{C_n^{PA}(0): n<\omega\}$. It follows that

$$PA + RFN(PA) \equiv_n \binom{n}{\varepsilon_1} \equiv_n [TI]^n_{\varepsilon_1}, \text{ and hence}$$

$$PA + RFN(PA) \equiv [TI]\varepsilon_1.$$

Since $[TI]\varepsilon_1 \equiv [TI]\varepsilon_0+1 \equiv PRA + \{TI_n(\varepsilon_0): n<\omega\}$, we have the KREISEL and LEVY result from 1968: Over PA, uniform reflection is equivalent to transfinite induction up to ε_0 [5].

Embedding and fine structure relations similar to those of Peano arithmetic hold for theories of type $[TI]\varepsilon_\gamma$ as basic theories. We state the corresponding facts without proof:

(1) $$\binom{n}{\alpha}_{[TI]\varepsilon_\gamma} \equiv_n \binom{n}{\varepsilon_\gamma(1+\alpha)}$$

(2) $$\binom{n+1}{\alpha}_{[TI]\varepsilon_\gamma} \equiv_n \binom{n}{\varepsilon_\gamma^\alpha}_{[TI]\varepsilon_\gamma} \qquad (\alpha\neq 0)$$

(3) $$[TI]\varepsilon_\gamma + RFN([TI]\varepsilon_\gamma) \equiv [TI]\varepsilon_{\gamma+1}.$$

These theories $[TI]\varepsilon_\gamma$ are to some extent the most general example of theories comparable to the hierarchies $\binom{n}{\alpha}$ over PRA: If a theory T is for each Π-class equivalent to some $\binom{n}{\alpha_n}$ - i.e. if for all n $T \equiv_n \binom{n}{\alpha_n}$ - then the fine structure theorem implies that the series of the α_n is constantly 0 or constantly some ε-number ε_γ. Therefore $T \equiv PRA$ or $\forall n\ T \equiv_n \binom{n}{\varepsilon_\gamma}$ and thus $T \equiv [TI]\varepsilon_\gamma$.

As a final example we consider the "filled up" theories T_k+PRA, where T_k is the set of all true Π^0_{k+1}-sentences. It turns out that the fine structure on the PRA-hierarchies is invariant under such a partial filling up with true sentences; in fact, the hierarchies $\binom{n}{\alpha}_{T_k+PRA}$ erected over the filled up theory T_k+PRA behave for all n>k as if this hierarchy had first been erected over PRA and filled up afterwards, i.e. we have for all n>k

$$\binom{n}{\alpha}_{T_k+PRA} \equiv_n T_k + \binom{n}{\alpha} \qquad \text{and therefore}$$

$$\binom{n+1}{\alpha}_{T_k+PRA} \equiv_n \binom{n}{\omega^\alpha}_{T_k+PRA} \qquad (\alpha\neq 0)$$

- the fine structure relations for PRA still hold. The reason is that $C_n^{T_k+PRA}(x) \leftrightarrow C_n(x)$ is provable in PRA. So T_k does not really

strengthen PRA concerning provability of Π^{o}_{n+1}-formulas if n>k. For instance, T_k+PRA has not even the strength to prove $C_{k+1}(0)$ which is only one Π-class higher than T_k. Although T_k+PRA is complete for the provability of Π^{o}_{k+1}-formulas, it is already incomplete in the next upper Π-class.
The proof of $C_n^{T_k+PRA}(x) \leftrightarrow C_n(x)$ is technical and lengthy; so we drop it.

Of course, exactly the same things happen if Peano arithmetic or a theory $[TI]\varepsilon_\gamma$ is partially filled up with T_k. We only have to replace PRA by the corresponding theory and ω by ε_0 or ε_γ.

4. k-Consistency and iterated reflection formulas

The search for relations between the C_n and the notion of k-consistency or ω-consistency of Peano arithmetic leads to an answer to an open problem which was announced by SMORYNSKI in [9]. He asked for "something of the form

$$\omega\text{-CON}^G(PA) \leftrightarrow \ldots\varepsilon_1\ldots$$

$$k\text{-CON }(PA) \leftrightarrow \ldots\varepsilon_0\ldots$$

$$\text{or} \quad \leftrightarrow \ldots\alpha_k\ldots \text{"}$$

and he continues "exactly what the dots represent is not clear - perhaps a mixture of recursion and induction principles. Until such equivalences are established, the situation regarding k-consistency remains open". Using iterated reflection formulas, the following answer can be given to SMORYNSKI's problem:

Theorem [1]) : The following is provable in PRA

$$\omega\text{-CON}^G(PA) \leftrightarrow C_2(\varepsilon_1) \qquad \omega\text{-CON}^G(PRA) \leftrightarrow C_2(\varepsilon_0)$$

$$1\text{-CON }(PA) \leftrightarrow C_1(\varepsilon_0) \qquad 1\text{-CON }(PRA) \leftrightarrow C_1(0)$$

$$2\text{-CON }(PA) \leftrightarrow C_2(\varepsilon_0) \qquad 2\text{-CON }(PRA) \leftrightarrow C_2(0)$$

$$(k+3)\text{-CON }(PA) \leftrightarrow C_2(\omega_k(\varepsilon_0 \cdot 2)), (k+2)\text{-CON }(PRA) \leftrightarrow C_2(\omega_k)$$

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[1]) For notations see SMORYNSKI's article [9]

Proof: This results essentially from SMORYNSKI's theorem 1.1 in [9], which reads in our notations: If T is PA or PRA, then

$$\omega\text{-CON}^G(T) \leftrightarrow C_2^{T+\{C_n^T(0):\ n<\omega\}}(0)$$

$$1\text{-CON }(T) \leftrightarrow C_1^T(0)$$

$$2\text{-CON }(T) \leftrightarrow C_2^T(0)$$

$$(k+2)\text{-CON }(T) \leftrightarrow C_2^{T+C_{k+1}^T(0)}(0) \ .$$

All the rest - except some minor technical lemmata - is a consequent application of the fine structure properties.

Remark: GOLDFARB and SCANLON in [3] and GOLDFARB in [2] give ordinal bounds α such that functionals of k-consistency are α-recursive. These bounds coincide for 1- and 2-consistency of PA with the ordinals given above; for k-CON(PA), $k\geq 3$, the latter are two ω-powers less.

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LOGIC COLLOQUIUM 78
M. Boffa, D. van Dalen, K. McAloon (eds.)

LOGIC AND THE AXIOM OF CHOICE

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We shall prove the following:

(1) $\forall x \exists y\ \varphi(x,y) \to \exists f \forall x\ \varphi(x,fx)$ is conservative over classical (first order) logic.

(2) $\forall x \exists y\ \varphi(x,y) \to \exists f \forall x\ \varphi(x,fx)$ is conservative over intuitionistic logic without equality.

(3) $\forall x \exists y\ \varphi(x,y) \to \exists f \forall x\ \varphi(x,fx)$ is conservative over intuitionistic logic with decidable equality.

(4) $\forall x \exists y\ \varphi(x,y) \to \exists f \forall x\ \varphi(x,fx)$ is not conservative over intuitionistic logic.

(5) $\bigwedge\limits_n \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n [\bigwedge\limits_i \varphi(x_i,y_i) \wedge \bigwedge\limits_{i,j} (x_i = x_j \to y_i = y_j) \to$
$\to \exists f \forall x\ \varphi(x,fx)$

is conservative over classical and intuitionistic logic.

More precisely: Addition of finitely many instances of the respective schema with all (number and function) parameters generalized is conservative over any first order theory in the respective logic.

None of these results is new. (1) is already in Hilbert-Bernays 1939 (p. 141 in the second edition). (2) - (4) are due to Minc 1966, 1974, 1966; note that (3) is an immediate consequence of (5). As to (4), a simpler counterexample is in Osswald 1975 and probably the simplest (which is reproduced here) in Smorynski 1978. (5) is due to Minc and Smorynski; it was first announced in Minc 1977. Proofs are in Smorynski 1978 and (in a generalized form dealing with "simultaneous Skolem functors") in Luckhardt A.

The proofs given here are relatively simple. For (5) the proof consists in a procedure which transform a derivation of a first order formula involving the axiom of choice into a derivation not involving it. The main technical tool is the use of a new type of

function variables: Whenever terms $r_1,\ldots,r_n$, $s_1,\ldots,s_n$ are introduced, then $f^{\underset{\sim}{s}}_{\underset{\sim}{r}}$ is a function variable. The intended meaning of $f^{\underset{\sim}{s}}_{\underset{\sim}{r}}$ is that it should range over all functions mapping $\underset{\sim}{r}$ in $\underset{\sim}{s}$ (provided $\underset{\sim}{r}$, $\underset{\sim}{s}$ determine a finite function, i.e. $_{i,j}(r_i = r_j \to s_i \square s_j)$).
As already noted, (3) and also (1) are easy consequences of (5), since the premiss of the implication in (5) is under the assumption $\forall x,y\ (x = y \vee x \neq y)$ equivalent to $\forall x \exists y\ \varphi(x,y)$. So we start with a proof of (2), then give Smorynski's counterexample to prove (4), and finally extend the method for proving (2) to a proof of (5); only this last step involves the function variables $f^{\underset{\sim}{s}}_{\underset{\sim}{r}}$.

Note: (1) - (5) remain valid - with essentially the same proofs - when all variables x, y, f, ... are replaced by finite sequences $\underset{\sim}{x}$, $\underset{\sim}{y}$, $\underset{\sim}{f}$, ... of variables; $\underset{\sim}{f}\underset{\sim}{x}$ then means $f_1\underset{\sim}{x},\ldots,f_n\underset{\sim}{x}$. However, for simplicity we only deal with single variables here.

Note: (2), (4) and (5) also hold for minimal logic. This is seen easily from the proofs.

Proof of (2):

In this section we only consider first order intuitionistic logic without equality. We shall work with a Gentzen sequent calculus as described in Kleene 1952, p. 481 (there it is called G3). For simplicity we modify it to include $\bot$ (falsum) as a propositional constant and treat $\neg\varphi$ as defined by $\varphi \to \bot$. First note that (2) can be reduced to

(2)' If $\vdash \forall x\ \varphi(x,fx), \Delta \to \Psi$ with Δ,Ψ of first order and without f, then $\vdash \forall x \exists y\ \varphi(x,y), \Delta \to \Psi$.

Proof of (2) from (2)': Let a cut-free derivation of $\Gamma,\Delta \to \Psi$ be given with Δ,Ψ of first order and Γ a list of generalizations of instances the schema $\forall x \exists y\ \varphi(x,y) \to \exists f \forall x\ \varphi(x,fx)$. By induction on the length of this derivation we construct a derivation of $\Delta \to \Psi$. It suffices to consider an inference

$$\frac{\forall x\exists y \to \exists f\forall x,\ \Gamma',\ \Theta \to \forall x\exists y\ \varphi(x,y) \qquad \exists f\forall x\ \varphi(x,fx),\ \forall x\exists y \to \exists f\forall x,\ \Gamma',\Theta \to \chi}{\forall x\exists y \to \exists f\forall x,\ \Gamma',\ \Theta \to \chi}$$

First the leftmost $\exists f$ in the right hand subderivation can be cancelled by an inversion lemma. Then the occurences of $\forall x\exists y \to \exists f\forall x$, Γ' in the antecedent of both subderivations can be cancelled by induction hypothesis. Then by (2)' $\forall x\ \varphi(x,fx)$ in the antecedent of the right hand subderivation can be replaced by $\forall x \exists y\ \varphi(x,y)$. A cut then gives the desired derivation of $\Theta \to \chi$.

Proof of (2)': Let a cut-free derivation of $\forall x\ \varphi(x,fx),\ \Delta \to \Psi$ with Δ,Ψ of first order and without f be given. By induction on the length of this derivation we construct a derivation of $\forall x \exists y\ \varphi(x,y),\ \Delta \to \Psi$. It suffices to consider

Case 1 $\dfrac{\varphi(t,ft),\ \forall x\ \varphi(x,fx),\ \Gamma \to \chi}{\forall x\ \varphi(x,fx),\ \Gamma \to \chi}$

Replace all occurrences of ft in this derivation by a new variable w. This gives a derivation of $\varphi(t,w),\ \forall x\ \varphi(x,fx),\ \Gamma \to \chi$. By induction hypothesis we obtain a derivation of $\varphi(t,w)$, $\forall x \exists y\ \varphi(x,y),\ \Gamma \to \chi$. Application of $(\exists\!\to)$ and $(\forall\!\to)$ gives the desired derivation of $\forall x \exists y\ \varphi(x,y),\ \Gamma \to \chi$.

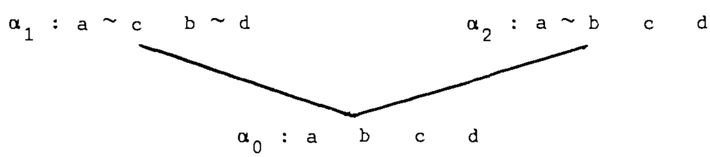
Case 2 $\dfrac{\forall x\ \varphi(x,fx),\ \Gamma \to\ \chi(r(ft_1,\dots,ft_n))}{\forall x\ \varphi(x,fx),\ \Gamma \to\ \exists z\ \chi(z)}$

where $ft_1,\dots,ft_n$ are all outermost occurrences of f-terms in $r(ft_1,\dots,ft_n)$. Replace again all outermost occurrences of $ft_1,\dots,ft_n$ in this derivation by new variables $w_1,\dots,w_n$. This gives a derivation of
$\varphi(t_1,w_1),\dots,\varphi(t_n,w_n),\ \forall x\ \varphi(x,fx),\ \Gamma \to \chi(r(w_1,\dots,w_n))$. Then apply the induction hypothesis, $(\to\!\exists)$, n times $(\exists\!\to)$ and n times $(\forall\!\to)$.

Proof of (4):

In this section we only consider first order intuitionistic logic (with equality). It suffices to prove
$\vdash \forall x \exists y\ (x \neq y) \to \forall x_1 \exists y_1 \forall x_2 \exists y_2\ (x_1 \neq y_1 \wedge x_2 \neq y_2 \wedge (x_1 = x_2 \to y_1 = y_2))$.

Consider the Kripke model



This is obviously a model of the equality axioms. Furthermore, $\alpha_0 \Vdash \forall x \exists y\ (x \neq y)$ since $\alpha_i \Vdash a \neq d,\ b \neq c$ for all i. But
$\alpha_0 \Vdash \forall x_1 \exists y_1 \forall x_2 \exists y_2\ (x_1 \neq y_1 \wedge x_2 \neq y_2 \wedge\ (x_1 = x_2 \to y_1 = y_2))$. To see this assume the contrary. Choose a for x_1. Then y_1 must be d. Choose

b for x_2. Then y_2 must be c. Hence $\alpha_0 \Vdash (a = b \to c = d)$. But this is a contradiction, since $\alpha_2 \Vdash a = b$, $\alpha_2 \Vdash c$ ◘ d. (This counterexample is due to Smorynski 1978)

Proof of (5):

We only consider first order intuitionistic logic; it is easily seen that the same proof also applies to classical logic. First note that (5) can be reduced to

(5)' If $\vdash \forall x\, \varphi(x,fx), \Delta \to \Psi$ with Δ,Ψ of first order and without f, then there is an n such that
$\vdash \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n [\ _i\ \varphi(x_i,y_i) \wedge \mathrm{Fct}(x;y)], \Delta \to \Psi$,where Fct(x,y) abbreviates $_{i,j}(x_i = x_j \to y_i = y_j)$.

(5) can be proved from (5)' exactly as we proved (2) from (2)' above. To prove (5)' we cannot proceed as simply as in the proof of (2)'. For, the replacement of ft by a new variable w in case 1 would not lead to a derivation anymore, since an equality axiom t=s   ft=fs would be transformed into an underivable formula t=s   w=fs. The idea now is to replace f by a new variable f_w^t with the intended meaning that it should range over functions mapping w into t. To make this precise we first extend our language. Variables and terms and now generated simultaneously with the additional clause

> If $r_1,\ldots,r_n$ $s_1,\ldots,s_n$ (short: r, s) are terms, then f_r^s is a function variable (where f is any of the countably many symbols reserved for function variables).

Corresponding to the intended meaning of f_r^s we add the following axioms to our logical formalism:

$$\mathrm{Fct}(r;s) \quad f_r^s r_i = s_i \quad \text{for all } i.$$

We now formulate a generalization of (5)' involving these new function variables, which can then be proved by induction.

(5)" If $\vdash \forall x\, \varphi(x, f_{\tilde{r}}^{\tilde{s}} x), \mathrm{Fct}(\underset{\sim}{r};\underset{\sim}{s}), \Delta \to \Psi$ with Δ,Ψ of first order and without $f_{\tilde{r}}^{\tilde{s}}$, then there is an n such that $\Vdash \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n$ $[\ _i\ \varphi(x_i,\tilde{y}_i) \wedge \mathrm{Fct}(r,x;s,y)], \Delta \to \Psi$.

Proof of (5)": For simplicity we only write out the case for $\underset{\sim}{r}$, $\underset{\sim}{s}$ empty. The general case can be dealt with in exactly the same manner. We use induction on the length of the given derivation, which we may assume to be cut-free. It suffices to consider

Case 1 $\dfrac{\varphi(t,ft),\ \forall x\varphi(x,fx),\ \Gamma \rightarrow \chi}{\forall x\varphi(x,fx\ ,\ \Gamma \rightarrow \chi}$

We first describe the well-known technique of "extracting f-subterms from ft". Let $ft_1,\dots,ft_n \equiv ft$ be all f-subterms of ft ordered by increasing depth of nesting of f. Let $w_1,\dots,w_n$ be new variables. For any subterm s of ft denote by s* the result of replacing all outermost occurrences of f-subterms $ft_{i_1},\dots,ft_{i_k}$ in s by $w_{i_1},\dots,w_{i_k}$. Using the new axioms on $f^{s}_{\underset{\sim}{r}}$ one can prove easily by induction on s

$$(*)\qquad Fct(\underset{\sim}{t}^*;\underset{\sim}{w}) \rightarrow s(f^{w}_{\underset{\sim}{t}*}) \equiv s^*.$$

$\underset{\sim}{t}^*$ denotes of course $t_1,\dots,t_n$; note t_i contains only w_j with $j < i$. - Now replace all occurrences of f in the given derivation by f^{w}_{t} . Writing t(f) for t one obtains a derivation of

$$\varphi(t(f^{w}_{\underset{\sim}{t}*})\ ,\ f^{w}_{\underset{\sim}{t}*}t(f^{w}_{\underset{\sim}{t}*}))\ ,\ \forall x\varphi(x,f^{w}_{\underset{\sim}{t}*}x)\ ,\Gamma \rightarrow \chi$$

Using (*) this derivation can easily be transformed into a derivation of

$$\varphi(t^*,w_n)\ ,\ \forall x\varphi(x,f^{w}_{\underset{\sim}{t}*}x)\ ,\ Fct(\underset{\sim}{t}^*;\underset{\sim}{w}),\Gamma \rightarrow \chi$$

of the same lenght (it is necessary her to allow as axioms all quasi-tautologics, i.e. all tautological consequences of the equality axioms including the new axioms on $f^{s}_{\underset{\sim}{t}}$).

By induction hypothesis we then obtain a derivation of

$$\varphi(t^*,w_n)\ ,\ \forall x_1\exists y_1\dots\forall x_m\exists y_m\ [\ _i\ \varphi(x_i,y_i) \wedge Fct(\underset{\sim}{t}^*,\underset{\sim}{x};\underset{\sim}{w},\underset{\sim}{y})],\Gamma \rightarrow \chi.$$

Now $\varphi(t^*,w_n)$ can be cancelled since it follows from the second member of the antecedent (we may assume $m \geqslant 1$). Then using the rules $(\exists\rightarrow)$, $(\forall\rightarrow)$ we obtain

$$\forall u_1\exists w_1\dots\forall u_n\exists w_n\forall x_1\exists y_1\dots\forall x_m\exists y_m\ [\ _i\ \varphi(x_i,y_i) \wedge Fct(\underset{\sim}{u},\underset{\sim}{x};\underset{\sim}{w},\underset{\sim}{y})],\Gamma \rightarrow \chi.$$

Case 2 $\dfrac{\forall x\ \varphi(x,fx)\ ,\ \Gamma \rightarrow \chi(t)}{\forall x\ \varphi(x,fx)\ ,\ \Gamma \rightarrow \exists z\chi(z)}$

Here again we extract the f-subterms from t and then replace f by $f^{\tilde{w}}_{\underset{\sim}{t}*}$. This gives as above a derivation of

$$\forall x \varphi(x, f^{\tilde{w}}_{\underset{\sim}{t}*} x) \quad Fct(\underset{\sim}{t}*; \underset{\sim}{w}) \ , \ \Gamma \rightarrow \chi(t*) \ .$$

By induction hypothesis we then obtain a derivation of

$$\forall x_1 \exists y_1 \ldots \forall x_m \exists y_m \ [\ {}_{i} \ \varphi(x_i, y_i) \wedge Fct(\underset{\sim}{t}*, \underset{\sim}{x}; \underset{\sim}{w}, \underset{\sim}{y}) \ , \Gamma \rightarrow \chi(t*) \ .$$

Now apply (→∃) and then proceed as in case 1 above.

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ON SUCCESSORS OF SINGULAR CARDINALS

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Introduction :

We will clarify the situation for the successor of a strong limit singular cardinal λ. We find a special subset $S^*(\lambda^+)$, from which we can find which stationary subsets of λ^+ can be stopped from being stationary by μ-complete forcing (Baumgartner has done this for successor λ^+ of regular $\lambda = \lambda^{<\lambda}$).
For $\lambda = \aleph_{\omega+1}$ we succeed in continuing an induction construction done for a λ^+-free not λ^{++} (abelian) group, and similar things for transversals; on those problems see history and references in [Sh 2].
A solution of a related problem - which stationary subsets of λ^+ can be "killed" by a forcing not adding bounded subsets of λ^+-will appear in a paper by U. Avraham, J. Stavi and the author.
We also prove a result related to the title but not to the rest of the paper, improving a result of Gregory [Gr] : assuming G.C.H., for $\lambda \neq \aleph_0$, $\diamondsuit^*_S$ holds, where $S = \{\delta < \lambda^+;\ \mathrm{cf}\delta \neq \mathrm{cf}\lambda\}$; hence $\diamondsuit_{S_1}$ holds for any stationary $S_1 \subseteq S$.

For a reader interested only with the GCH, he can simplify for himself the part up to section 13. A reader interested in more general cases than those discussed in the main part has to go to the end. There we also show that the special set $S^*(\aleph_{\omega+1})$ can be stationary (even with the GCH).

The main results were announced in the AMS Notices [Sh 3].

Notation: We shall denote infinite cardinals by λ,μ,κ,χ, ordinals by i,j, $\alpha,\beta,\gamma,\xi,\zeta$ limit ordinals by δ, natural numbers by m,n,r,p,q.

Let $\bar N$ denote a sequence $\langle N_i : i < \lambda \rangle$ where for some $\mu,\chi \leqslant \mu$, $N_i \prec (H(\mu),\in)$; $i \subseteq N_i$, $\|N_i\| < \lambda$, $i < j \Rightarrow N_i \prec N_j$, and for limit $\delta, N_\delta = \bigcup_{i<\delta} N_i$. We call this a λ-approximating sequence (for μ).

We denote by d a two-place function from one cardinal to another; $cf\delta$ is the cofinality of δ; $cf^*\delta$ is $cf\delta$ if $cf\delta < \delta$ and is ∞ otherwise. D_δ is the filter over δ generated by the closed unbounded subsets of δ (so we assume $cf\delta > \aleph_0$). If D is a filter over I, $A \subseteq B$ mod D means $I - (A - B) \in D$; similarly $A \equiv B$ mod D means $I - (A - B) \cup (B - A) \in D$. If $A \not\equiv \phi$ mod D, D + A is the filter $\{B : B \cup (I - A) \in D\}$.

Let $CF(\delta,\kappa) = \{i < \delta : cf i = \kappa\}$, similarly $CF(\delta,<\kappa) = \bigcup_{\mu<\kappa} CF(\delta,\mu)$ $CF(\delta, \leqslant \kappa) = \bigcup_{\mu\leqslant\kappa} CF(\delta,\mu)$ $D_{\delta,\kappa} = D_\delta + CF(\delta,\kappa)$ etc.

1. Definition : 1) We say κ is good for λ if $\lambda = \lambda^{<\lambda}, \kappa = \infty$ or there is a family $\underline{P}^o_{\lambda,\kappa}$ such that

a) $|\underline{P}^o_{\lambda,\kappa}| = \lambda$

b) every member of $\underline{P}^o_{\lambda,\kappa}$ is a subset of λ of cardinality κ

c) every subset of λ of cardinality κ contains a member of $\underline{P}^o_{\lambda,\kappa}$

2) We call κ a good cofinality for λ if $\lambda = \lambda^{<\lambda}, \kappa$ is ∞ or if λ and κ are regular and there is a family $\underline{P}_{\lambda,\kappa}$ such that

a) $|\underline{P}_{\lambda,\kappa}| = \lambda$

b) every member of $\underline{P}_{\lambda,\kappa}$ is a subset of λ of cardinality $< \kappa$

c) every subset of λ of cardinality κ has a subset $\{\alpha_i : i < \kappa\}$ such that α_i is increasing and for every $j < \kappa$, $\{\alpha_i : i < j\} \in \underline{P}_{\lambda,\kappa}$

d) $\lambda = \lambda^{<\kappa}$ or $2^\mu < \lambda$ for every $\mu < \kappa$

2. Definition : 1) $Gcf(\lambda) = \{\kappa : \kappa$ is a good cofinality for $\lambda\}$

$G(\lambda) = \{\kappa : \kappa$ is good for $\lambda\}$

2) $gcf(\lambda) = \{i < \lambda : cf^* i \in Gcf(\lambda)\}$ (note that we use cf^* not cf)

3) $D^g_\lambda = D_\lambda + gcf(\lambda)$

3. Claim : 1) If $\lambda^\kappa = \lambda$ then κ is good for λ

2) If $\kappa < \infty$ is good for λ then κ is good for λ^+

3) If $\lambda = \sum_{i<\mu} \lambda_i$, $cf\mu \neq cf\kappa$, $\lambda_i (i < \mu)$ increasing and $\kappa < \infty$ is good for each λ_i then κ is good for λ

4) If $(\forall \mu < \aleph_\alpha)\mu^\kappa < \aleph_\alpha, \beta < cf\kappa$, $cf\aleph_\alpha \neq cf\kappa$ then κ is good for $\aleph_{\alpha+\beta}$ [in fact $(\forall \mu < \aleph_\alpha)\mu^\kappa \leqslant \aleph_{\alpha+\beta}$ suffice]

5) if λ,κ are regular, κ good for λ then κ is a good cofinality for λ, provided that $2^{<\kappa} \leqslant \lambda$

6) If λ,κ are regular $\lambda^{<\kappa} = \lambda$ then κ is a good cofinality for λ

7) If $\kappa < \infty$ is a good cofinality for λ then κ is a good cofinality for λ^+

8) If $\lambda = \sum_{i<\mu} \lambda_i$, $cf\mu \neq cf\kappa$, $\kappa \in Gcf(\lambda_i)$ for every $i < \mu$, λ_i increasing, and $\kappa < \infty$ then $\kappa \in Gcf(\lambda)$

9) If $(\forall \mu < \aleph_\alpha)\mu^{<\kappa} < \aleph_\alpha$, $cf\aleph_\alpha \neq \kappa$, κ regular, $\beta < \kappa$ then $\kappa \in Gcf(\aleph_{\alpha+\beta+1})$ [in fact, $(\forall \mu < \aleph_\alpha)\mu^{<\kappa} \leqslant \aleph_{\alpha+\beta+1}$ suffice].

4. Definition : For d a two-place function from δ into $\kappa (cf\delta > \aleph_0)$ we let $S_1(d) = \{\xi : \xi < \delta$, ξ a limit ordinal such that there is an unbounded $A \subseteq \xi$ on which d is constant$\}$

$S_0(d) = \{\xi : \xi < \delta$, ξ a limit ordinal such that there are unbounded subsets A, B of ξ, such that $(\forall b \in B)(\exists \alpha < \kappa)(\forall a \in A)[a < b \rightarrow d(a,b) \leqslant \alpha]\}$

Remark : Note that d determines δ (as Dom d) but not κ (as d is into κ, not necessarily onto κ), so if the value of κ is not clear we shall write $S_0(d,\kappa)$. In the definition of $S_1(d)$, κ has no role.

5. Claim : For d a two-place function from δ to κ:

1) $S_1(d) \subseteq S_0(d)$,

2) in the definition of $S_\ell(d)$ ($\ell = 0,1$) we can assume A,B have order type $\mathrm{cf}\xi$ (and generally replace them by unbounded subsets),

3) $CF(\delta, \leq \kappa) \subseteq S_0(d)$,

4) If $\ell = 0,1$, $\xi \in S_\ell(d)$, $\mathrm{cf}\,\xi > \aleph_0$, then there is $C \in D_\xi$ such that $C \subseteq S_\ell(d)$.

6. Definition : For a λ-approximating sequence $\overline{N}$ (see notation) let $S_2(\overline{N}) = \{\xi : \xi < \lambda,\ \xi$ a limit such that there is an unbounded $A \subseteq \xi$ of order type $\mathrm{cf}\xi$ such that $(\forall i < \xi)\,[A \cap i \in N_\xi]$ and $N_\xi \cap \lambda = \xi\}$

7. Claim : 1) If λ is regular, $\overline{N}^0$, $\overline{N}^1$ are λ-approximating sequences for μ_0, μ_1 respectively, and $\mu_\ell > \lambda$, then $S_2(\overline{N}^1) = S_2(\overline{N}^0) \bmod D^g_\lambda$.

Proof : Let $\overline{N}^\ell = \langle N^\ell_i : i < \lambda\rangle$, where $N^\ell_i \prec (H(\mu_\ell), \in)$, and let $C = \{\alpha < \lambda : N^0_\alpha \cap (\bigcup_{j<\lambda} N^1_j) = (\bigcup_{j<\lambda} N^0_j) \cap N^1_\alpha = N^0_\alpha \cap N^1_\alpha$ and $N^\ell_\alpha \cap \lambda = \alpha\}$ (we do not distinguish strictly between a model N and its universe). It is easy to check that C is a closed unbounded subset of λ.

By transitivity of equality we can assume $N^0_\alpha \prec N^1_\alpha$.

Now suppose $\xi \in C$, and $\mathrm{cf}^*\xi \in \mathrm{Gcf}(\lambda)$. We shall prove $\xi \in S_2(\overline{N}^0)$ iff $\xi \in S_2(\overline{N}^1)$, thus completing the proof. The "only if" part is now trivial, so we concentrate on the "if" part. Also the case $\mathrm{cf}^*\xi = \infty$ is easy, so we assume $\mathrm{cf}^*\xi = \mathrm{cf}\xi < \xi$.

Let $\kappa = \mathrm{cf}\xi < \xi$. We have just assumed $\kappa \in \mathrm{Gcf}(\lambda)$, so the appropriate $\underline{P}_{\lambda,\kappa}$ (as in Definition 1.2) exists, hence belongs to $H(\mu_1)$, hence w.l.o.g it belongs to N^0_0, and hence, by assumption, to N^1_0.

If $\xi \in S_2(\overline{N}^1)$, then (by definition) there is an unbounded $A \subseteq \xi$ of order-type $\mathrm{cf}\xi$, such that for every $\zeta < \xi$, $A \cap \zeta \in N^1_\xi$.

If $\lambda = \lambda^{<\kappa}$, we can assume $\underline{P}_{\lambda,\kappa} = \{B \subseteq \lambda : |B| < \kappa\} = \{B_i : i < \lambda\}$ (since $|\underline{P}_{\lambda,\kappa}| = \lambda$), and so $\underline{P}_{\lambda,\kappa} \cap N^0_\xi = \underline{P}_{\lambda,\kappa} \cap N^1_\xi = \{B_i : i < \xi\}$, hence $\zeta < \xi \Rightarrow A \cap \zeta \in N^0_\xi$, hence A witnesses that $\xi \in S_2(\overline{N}^0)$. Thus finishing.

So we are left with the case $\lambda < \lambda^{<\kappa}$. Then, by d) of Definition 1.2, $(\forall \mu < \kappa) 2^\mu < \lambda$. So, as $N^\ell_\xi \cap \lambda = \xi$, and A has order-type κ, every subset of A of power $< \kappa$ is included in a set from N^1_ξ of cardinality $<\kappa$, hence it belongs to N^1_ξ. So we can replace A by any subset of it which is unbounded in ξ. In particular, by the choice of $\underline{P}_{\lambda,\kappa}$ (see Definition 2), we can assume $A = \{\alpha_i : i < \kappa\}$, and for $j < \kappa$, $\{\alpha_i : i < j\} \in \underline{P}_{\lambda,\kappa}$ and, as mentioned above, $\{\alpha_i : i < j\} \in N^1_\xi$. But as $|\underline{P}_{\lambda,\kappa}| = \lambda$, $\underline{P}_{\lambda,\kappa} \in N^0_0$, clearly $\underline{P}_{\lambda,\kappa} \subseteq \bigcup_{i<\lambda} N^0_i$, hence (as $\xi \in C$) $\underline{P}_{\lambda,\kappa} \cap N^0_\xi = \underline{P}_{\lambda,\kappa} \cap N^1_\xi$, hence for every $j < i$, $\{\alpha_i : i < j\} \in N^0_i$. So $\{\alpha_i : i < \kappa\}$ witnesses that $\xi \in S_2(\overline{N}^0)$, and this finishes the proof of the theorem.

<u>8. Definition</u> : $S^*(\lambda) \subseteq \lambda$ is defined as $(\lambda - S_2(\overline{N})) \cap \mathrm{gcf}(\lambda)$ for $\overline{N}$ any λ-approximating sequence for λ^+, where λ is regular. (so S^* is uniquely defined mod D_λ only).

<u>9. Definition</u> : For λ singular, a two-place function d from λ^+ to $\kappa = \mathrm{cf}\lambda$ is called <u>normal</u> if for every $i < \kappa, \alpha < \lambda^+$, the set $\{\beta < \alpha : d(\beta,\alpha) \leq i\}$ has cardinality $<\lambda$. It is called subadditive if for $\gamma < \beta < \alpha < \lambda^+$, $d(\gamma,\alpha) \leq \max\{d(\gamma,\beta), d(\beta,\alpha)\}$.

<u>10. Claim</u> : For every singular λ, there is a normal subadditive two-place function d from λ^+ to $\mathrm{cf}\lambda$; moreover, if $\lambda = \sum_{i<\mathrm{cf}\lambda} \lambda_i$ (λ_i increasing), then $|\{\beta < \alpha : d(\beta,\alpha) \leq i\}| \leq \lambda_i$.

<u>Proof</u> : Easy.

11. Claim : 1) Suppose λ is singular, $\kappa = \mathrm{cf}\lambda$, $(\forall\mu < \lambda)(\mu^{<\chi} \leq \lambda)$, and d is a normal two-place function from λ^+ to κ. Then for some λ^+-approximating sequence $\overline{N}$ for λ^{++},

$$CF(\lambda^+, \leq \chi) \cap S_o(d) \subseteq S_2(\overline{N}) \bmod D_\lambda.$$

2) Suppose λ is singular, $\kappa = \mathrm{cf}\lambda$, χ is regular and is a good cofinality for λ^+,and d is a normal two-place function from λ^+ to κ. Then for some λ^+-approximating sequence $\overline{N}$ for λ^{++}, $CF(\lambda^+,\chi) \cap S_o(d) \subseteq S_2(\overline{N})$.

Proof : 1) Choose a λ^+-approximate sequence $\overline{N}$ for λ^{++} such that $d \in N_o$, $N_i \in N_{i+1}$. Clearly $= \{\delta < \lambda^+ : N_\delta \cap \lambda = \delta\}$ is closed and unbounded. So for every $\alpha < \lambda^+$, $i < \kappa$, the set $A^* = \{\beta < \alpha : d(\beta,\alpha) \leq i\}$ belongs to N_{i+1} and has cardinality $< \lambda$. Hence $P_i^\alpha = \{A : B \subseteq A^*, |B| < \chi\}$ belongs to N_{i+1} and has cardinality $< \lambda$, hence $P_i^\alpha \subseteq N_{i+1}$. So suppose $\delta \in S_o(d)$, and $A,B \subseteq \delta$ are witness to it (i.e. they are unbounded in δ and have order-type $\mathrm{cf}\delta$, and for every $b \in B$, for some $i(b) < \kappa$, $(\forall a \in A)(a < b \rightarrow d(a,b) \leq i(b))$). Suppose further $\delta \in C$, $\mathrm{cf}\delta \leq \chi$. Then $A,B \subseteq N_\delta$ (as $\delta \subseteq N_\delta$) and for every $b \in B$, $\{a : a \in A, a < b\}$ belongs to $\underline{P}^b_{i(b)}$, hence to N_{i+1}, hence to N_δ. So A witnesses that $\delta \in S_2(\overline{N})$. We have just proved $\delta \in CF(\lambda^+, \leq \chi) \cap S_o(d) \Rightarrow \delta \in S_2(\overline{N})$, thus finishing the proof of the claim.

2) A similar proof.

12. Claim : Suppose λ is regular, $\kappa < \chi$, $\kappa < \lambda$, χ is a good cofinality for λ and $(\forall\mu < \chi)2^\mu < \lambda$ or $\chi = \infty$. Then for every two-place function d from λ to κ and for some λ-approximate sequence $\overline{N}$ for λ^+,

$$S_2(\overline{N}) \cap CF(\lambda,\chi) \subseteq S_1(d).$$

Proof : Choose $\overline{N}$ as λ-approximate sequence for λ^+ such that $d \in N_o$. Suppose $\delta \in S_2(\overline{N}) \cap CF(\lambda,\chi)$. We shall prove $\delta \in S_1(d)$. The case

$\chi = \infty$ is easy, so assume $\chi < \infty$.

As $\delta \in S_2(\bar{N})$, there is a set $\{\alpha_i : i < \chi\} \subseteq \delta$, unbounded in δ, such that for every $j < \chi$, $\{\alpha_i : i < j\} \in N_\delta$. Let h be the function with domain χ, $h(i) = \alpha_i$. Clearly for $j < \chi$, $h|j \in N_\delta$.

Now we define by induction on $i < \chi$ an element x_i and function f_i as follows :

$f_i(j) = d(x_j, \delta)$ for $j < i$ (so Dom $f_i = i$)

x_i is the first ordinal which is bigger than α_i and $x_j (j < i)$ and is such that $(\forall j < i)\, [\, d(x_j, x_i) = f_i(j)\,]$.

This can be carried out in $H(\lambda^+)$. But now as $\mu < \chi \Rightarrow 2^\mu < \chi$, and $\mu < \chi = \mathrm{cf}\delta \leqslant \delta$, clearly each f_i is in N_δ.

Note also that x_i depends only on f_i and $\{\alpha_j : j \leqslant i\}$ (as for $j < i$, $f_j = f_i|j$). So $x_i \in N_\delta$ for each $i < \chi$.

Now there is an unbounded $S \subseteq \chi$ and $i_o < \kappa$ such that $j \in S \Rightarrow d(x_j, \delta) = i_o$. It is easy to check that $\{x_j : j \in S\}$ witnesses that $\delta \in S_1(d)$.

From now on we concentrate on successors of strong limit singular cardinals. We can conclude e.g.

<u>13. Conclusion</u> : Suppose λ is a singular strong limit. Then for every normal two place function d from λ^+ to $\kappa = \mathrm{cf}\lambda$, the following holds :

$$S_o(d) \equiv S_1(d) \cup CF(\lambda^+, \leqslant \kappa) \equiv \lambda^+ - S^*(\lambda^+) \bmod_{D_{\lambda^+}} .$$

(So in particular $S_o(d)$ does not depend on d (when d is normal) up to equivalence $\mathrm{mod}_{D_{\lambda^+}}$).

<u>Proof</u> : Trivial by 5.1, 5.3, 11 and 12.

<u>14. Claim</u> : If λ is regular, $\kappa < \lambda$ and $(\forall \mu < \lambda)\ \mu^{<\kappa} < \lambda)$, then $CF(\lambda, \leqslant \kappa) \leqslant \lambda - S^*(\lambda) \bmod D_{\lambda^+}$.

<u>Proof</u> : We can find a λ-approximating sequence $\langle N_i : i < \lambda \rangle$ to λ^+ such that every subset of N_i of cardinality $< \kappa$ belongs to N_{i+1}. So $CF(\lambda, \leqslant\kappa) \subseteq S_2(\bar{N})$.

15. Claim : If $\delta \in \lambda - S_1(d)$, d a two-place function from λ to $\kappa < \mathrm{cf}\delta$, then $\mathrm{cf}\delta$ is not weakly compact.

Proof : If $\mathrm{cf}\delta$ is weakly compact then $\mathrm{cf}\delta \to (\mathrm{cf}\delta)^2_\kappa$.

16. Definition : 1) For a set $S \subseteq \lambda$ let

$$F(S) \;\square\; \{\delta < \lambda : S \cap \delta \text{ is a stationary subset of } \delta\}$$

2) Define $F^n(S)$ by induction on n:

$$F^o(S) = S,\ F^{n+1}(S) = F(F^n(S)).$$

17. Claim : 1) $FF(S) \subseteq F(S)$.

2) $F(S^*(\lambda)) \subseteq S^*(\lambda)$, hence $F^n(S^*(\lambda)) \subseteq F^m(S^*(\lambda))$ if $n > m \geqslant 0$.

3) $\delta \in F^n(S)$ implies $\mathrm{cf}\delta \geqslant \aleph_n$; moreover, if $\aleph_\alpha = \min\{\mathrm{cf}\delta : \delta \in S\}$, then $\delta \in F^n(S)$ implies $\mathrm{cf}\delta \geqslant \aleph_{\alpha+n}$.

4) If $\alpha \leqslant \min\{\mathrm{cf}\delta : \delta \in \bigcup_{i<\alpha} S_i\}$, $S_i \subseteq \lambda$ then

$$F(\bigcup_{i<\alpha} S_i) = \bigcup_{i<\alpha} F(S_i) \bmod D_\lambda.$$

Proof : 1) Easy

2) By 5.4 (and second part-by induction)

3), 4) Easy.

18. Lemma : Suppose λ is a singular strong limit of cofinality κ. Then for some $C \in D_{\lambda^+}$, for every $\delta \in C$, letting $\langle \alpha_i : i < \mathrm{cf}\delta \rangle$ be increasing, continuous and converging to δ, the following holds :

$$\{i : \alpha_i \in S^*(\lambda)\} \supseteq S^*(\mathrm{cf}\delta) \bmod D_{\mathrm{cf}\delta}$$

Proof : Let d be as in 10. Then by 13, for some $C \in D_{\lambda^+}$, $S^*(\lambda^+) \cap C = S_o(d) \cap C$, so we need only deal with $S_o(d)$.

Now define a two-place function d^* from $\mathrm{cf}\delta$ to κ by : $d^*(i,j) = d(\alpha_i, \alpha_j)$. It is easy to check that

$$\{\alpha_i : i \in S_o(d^*)\} \subseteq S_o(d).$$

But by 10, $S_o(d^*) \subseteq \mathrm{cf}\delta - S^*(\mathrm{cf}\delta)$ (remember $\kappa < \mathrm{cf}\delta$), so we are finished.

<u>19. Conclusion</u> : 1) Suppose λ is a singular strong limit, χ,μ regular, $\chi\mu < \lambda$ and $(\forall \mu_1 < \mu)\mu_1^{\chi} < \mu$. Then $F[S^*(\lambda^+) \cap CF(\lambda^+,\chi)] \cap CF(\lambda^+,\mu)$ is not stationary.

2) If $n < \omega$ and $2^{\aleph_k} \leqslant \aleph_{k+n}$ for every $k < \omega$, then $F^n(S^*(\aleph_{\omega+1})) \equiv \phi \bmod D_{\aleph_{\omega+1}}$.

3) If $\aleph_\omega$ is a strong limit and $S^*(\aleph_{\omega+1})$ is stationary, then for some stationary $S \subseteq \aleph_{\omega+1}$, $F(S) = \phi$

<u>Proof</u> : 1) By 14 and 18.

2) Suppose $F^n(S^*(\aleph_{\omega+1}))$ is stationary. Then by 17.4 for some $k < \omega$, $F^n[S^*(\aleph_{\omega+1}) \cap CF(\aleph_{\omega+1},\aleph_k)]$ is stationary. Hence for some $\ell < \omega$, $F^n[S^*(\aleph_{\omega+1}) \cap CF(\aleph_{\omega+1},\aleph_k)] \cap CF(\aleph_{\omega+1},\aleph_\ell)$ is stationary. If $\ell \leqslant k+n$, this contradicts 19.3. But if $\ell > k+n$, then $(\forall \mu < \aleph_\ell)\mu^{\aleph_k} < \aleph_\ell$ (since $2^{\aleph_k} \leqslant \aleph_{k+n}$), hence we get a contradiction by 19.1. So in all cases we get a contradiction; hence $F^n(S^*(\aleph_{\omega+1}))$ is not stationary.

3) Since $S^*(\aleph_{\omega+1})$ is stationary, for some $k < \omega$, $S^*(\aleph_{\omega+1}) \cap CF(\aleph_{\omega+1},\aleph_k)$ is stationary. Let $2^{\aleph_k} = \aleph_{k+n}$ ($n < \omega$ since $\aleph_\omega$ is a strong limit). So $k+n < \ell < \omega$ implies $(\forall \mu < \aleph_\ell)\mu^{\aleph_k} < \aleph_\ell$; hence, by 19.1, $F(S) \subseteq CF(\aleph_{\omega+1}, \leqslant \aleph_{k+n})$, where $S = S^*(\aleph_{\omega+1}) \cap CF(\aleph_{\omega+1},\aleph_k)$. But by 17.1, $F^{n+1}(S) \subseteq F(S)$, hence $\delta \in F^{n+1}(S)$ implies $cf\delta \leqslant \aleph_{k+n}$, and by 17.2 $\delta \in F^{n+1}(S)$ implies $cf\delta \geqslant \aleph_{k+n+1}$ (since $\delta \in S \Rightarrow cf\delta = \aleph_k$), so we get that there is no $\delta \in F^{n+1}(S)$, i.e. $F^{n+1}(S) = \phi$. Since $F^{o}(S) = S$ is stationary, for some ℓ, $F^{\ell}(S)$ is stationary but $F(F^{\ell}(S))$ ■ $F^{\ell+1}(S)$ is not; $F^{\ell}(S)$ is as required.

<u>Theorem 20</u> : Suppose $S \subseteq \lambda$ is stationary, and $S \subseteq gcf(\lambda) - S^*(\lambda)$, $S \subseteq CF(\lambda,\mu)$. <u>If</u> P is a μ^+-complete forcing (i.e. if $\langle p_i : i < \mu \rangle$ is an increasing sequence of elements of P then some $p \in P$ is $\geqslant p_i$ for every i), <u>then</u> S is stationary even in the universe V^P.

Remark : Remember that λ-complete forcing forces the stationariness of any $S \subseteq \lambda$.

Proof : Let $\bar{N}$ be a λ'-approximate sequence for some $\lambda' > \lambda$, such that a P-name $\underset{\sim}{C}$ of a closed unbounded subset of λ, a $p \in P$, are in N_0. So trivially there is $\delta \in S$, $A \subseteq \delta$ such that $\delta = N_\delta \cap \lambda$ and A has order type $cf\delta$, and for every $\zeta < \delta$, $A \cap \zeta \in N_\delta$. Let $f : cf\delta \to A$ enumerate A, hence $\zeta < cf\delta$ implies $f|\zeta \in N_\delta$.
We want to prove that not : $p \Vdash$ "$\underset{\sim}{C}$ is disjoint from S". For this it suffices to find $q \in P$ such that $p \leq q$ and $q \Vdash$ "$\delta \in \underset{\sim}{C}$" (since $\delta \in S$). We can assume that a well-ordering $<^*$ of $P \cup P \times \lambda$ belongs to N_0. Now we define by induction on $i < cf\delta$, $p_i \in N_\delta$.
We let $p_0 = p$, and for i a limit, p_i is the $<^*$-first p' which is $\geq p_j$ for every j (which exists since P is μ^+-complete).
We let p_{i+1}, β_i be such that (p_{i+1}, β_i) is the $<^*$-first pair (p', β') such that $p' \geq p_i$, $\beta' \geq f(i)$ and $p' \Vdash \beta' \in \underset{\sim}{C}$. There is such (p', β') since $\underset{\sim}{C}$ was a P-name of an unbounded subset of λ. It is easy to check that $p_i, \beta_i \in P \cap N_\delta$, so $\beta_i < \delta$. Hence $\delta = \sup\{\beta_i : i < cf\delta\}$. Since P is μ^+-complete, there is $q \in P$, $p_i \leq q$ for every $i < cf\delta$. So q force $\underset{\sim}{C} \cap \delta$ to be unbounded below δ. But $\underset{\sim}{C}$ was a P-name of a closed subset of δ. Hence $q \Vdash$ "$\delta \in \underset{\sim}{C}$". So we are finished.

21. Theorem : Suppose $\mu < \lambda$, μ regular. Then there is a μ-complete forcing P, such that in V^P $S^*(\lambda)$ is not stationary.

Proof : First assume $\lambda = \lambda^{<\lambda}$, so $\underline{P} = \{B \subseteq \lambda : |B| < \lambda\} = \{B_i : i < \lambda\}$, each $B \in \underline{P}$ appearing in $\{B_i : i < \lambda\}$ λ times, and let $\bar{B} = \langle B_i : i < \lambda \rangle$. Clearly there is a λ-approximating sequence $\bar{N}$ of λ^+, with $\bar{B} \in N_0$; and then $\underline{P} \cap N_\delta = \{B_i : i < \delta\}$ for a closed unbounded set of δ's.

So (w.l.o.g.) $S^*(\lambda) \subseteq \{\delta < \lambda : N_\delta \cap \underline{P} = \{B_i : i < \delta\}\}$.

$P = \{\eta = \langle \alpha_i : i \leqslant \zeta \rangle$, an increasing, continuous sequence, where $B_{\alpha_{i+1}} = \{\alpha_j : j \leqslant i\}\}$. The order on P is : $\eta_1 < \eta_2$ iff η_1 is an initial segment of η_2.

It is obvious that P is μ-complete; and if $G \subseteq P$ is generic, let $C[G] = \{\alpha_\delta : \delta$ limit, and $\langle \alpha_i : i \leqslant \xi \rangle \in G, \zeta \geqslant \delta\}$. Clearly in $V[G]$, $C[G]$ is a closed unbounded subset of λ. Now we have to prove only : $C[G] \cap S^* = \phi$, where $S^* = S^*(\lambda)^V$. Suppose, in V, for some $p \in P$, $p \Vdash$ "$\delta \in C[\underset{\sim}{G}]$" where $\delta \in S^*$. Let $p = \langle \alpha_j : j \leqslant \zeta \rangle$, so clearly for some limit $i \leqslant \zeta$, δ ▪ α_i. Since $\delta \in S^*$, $N_\delta \cap \{B_i : i < \lambda\} = \{B_i : i < \delta\}$, and there is no unbounded $A \subseteq \delta$ of order type $cf\delta$, such that $\xi < \delta \Rightarrow A \cap \xi \in N_\delta$. But there is such an A namely $\{\alpha_j : j < i\}$ ($\{\alpha_j : j < j_o < i\}$ belongs to N_δ since it is $B_{j_o+1} - \{j_o\}$), contradiction. So we are finished when $\lambda = \lambda^{<\lambda}$.

If $\lambda < \lambda^{<\lambda}$, let Q be the collapsing of 2^λ to λ, i.e. $P = \{f : \text{Dom } f = \xi < \lambda, \text{Range } f \subseteq 2^\lambda\}$. Note that V^P may have a different $gcf(\lambda)$, but $S^*(\lambda)^{V^Q} \cap gcf(\lambda)^V = S^*(\lambda)^V$. Now in V^Q define P as before, and Q * P (the composition)is as required.

<u>22. Conclusion</u> : Suppose λ is regular, $\mu < \lambda$ regular, $S \subseteq gcf(\lambda)$. There is a μ-complete forcing P such that in V^P, S is not stationary iff $(S - S^*(\lambda)) \cap CF(\lambda, <\mu)$ is stationary.

<u>23. Lemma</u> : Suppose λ is regular, $S \subseteq \lambda$ stationary, but $F(S) = \phi$ and for every $\alpha \in S$, A_α is an unbounded subset of α of order-type $cf\alpha$.

Then for every $S' \subseteq S$ with $|S'| < \lambda$, the family $\{A_\alpha : \alpha \in S'\}$ has a transversal (=one-to-one choice function). Moreover we can find $A'_\alpha \subseteq A_\alpha$ $(\alpha \in S')$, $|A'_\alpha| < cf\alpha$, such that the sets $A_\alpha - A'_\alpha$ $(\alpha \in S')$ are pairwise disjoint.

However $\{A_\alpha : \alpha \in S\}$ does not have a transversal.

Proof : See [Sh 1].

24. Lemma : Suppose λ is singular strong limit, $\kappa = cf\lambda$, $S^*(\lambda^+) = \phi \bmod D_{\lambda^+}$, and let

$$S = \{\delta < \lambda^+ : cf\delta \neq \kappa, \aleph_0 \text{ , and } \lambda\omega \text{ divides } \delta\}$$

Then we can define $A_\alpha \subseteq \alpha$ $(\alpha \in S)$, A_α unbounded in α and with order-type $\kappa(cf\alpha)$ (ordinal multiplication), such that

A) $\{A_\alpha : \alpha \in S\}$ has no transversal

B) For every $S' \subseteq S$ with $|S'| < \lambda^+$, $\{A_\alpha : \alpha \in S'\}$ has a transversal. Moreover

B') For every $S' \subseteq S$ with $|S'| < \lambda^+$, there are $A'_\alpha \subseteq A_\alpha (\alpha \in S')$ such that :

(i) they are pairwise disjoint,

(ii) A'_α is a big [and even very big] subset of A_α, which means that there is a closed (in A_α) unbounded [resp. cobounded] $C \subseteq A^*_\alpha$ so that

$$(\forall\delta \in C)(\exists\zeta < \kappa)(\forall\xi)(\delta + \zeta \leqslant \xi < \delta + \kappa \rightarrow \xi \in A'_\alpha).$$

Proof : Stage A :

There is a normal $d : \lambda^+ \rightarrow \kappa$, $\lambda = \sum_{i<\kappa} \lambda_i, \lambda_i < \lambda$, $|\{\beta < \alpha : d(\alpha,\beta) \leqslant i\}| \leqslant \lambda_i$, such that for every $\delta < \lambda^+$, $cf\delta \neq \kappa$, there is $A \subseteq \delta$, $\sup A = \delta$, $d|A$ bounded, and each $i \in A$ is a successor.

Pf : Let d be from 10, then $S_1(d) \equiv \phi \bmod D_{\lambda^+}$, hence there is a closed unbounded $C \subseteq \lambda^+$, $C \cap S_0(d) = \phi$. Let $C = \{\alpha_i : i < \lambda^+\}$, α_i increasing and continuous, $\alpha_0 = 0$. For each $i < \lambda^+$, we can find $A^i_\zeta \subseteq (\alpha_i,\alpha_{i+1})(\zeta < \kappa)$ such that : $|A^i_\zeta| = \lambda_\zeta$, A^i_ζ is closed (in the interval), if $\delta \in A^i_\zeta$ is a limit then $\delta = \sup(\delta \cap A^i_\zeta)$, $\alpha_{i+1} = \sup A^i_\zeta$, for some ζ.

A^i_ζ increases with ζ and $(\alpha_i,\alpha_{i+1}) = \bigcup_{\zeta<\kappa} A^i_\zeta$. Now we define d' by :

if $\alpha < \beta$ then $d'(\beta,\alpha) = d(\beta,\alpha)$ if $(\exists i)(\beta \geq \alpha_i > \alpha)$, and otherwise $d'(\beta,\alpha) = \min\{d(\beta,\alpha), \min\{\zeta : \alpha,\beta \in A^i_\zeta\}\}$. It is easy to check that d' is as required. For showing that every $i \in A$ is a successor, use subadditivity.

Stage B :

For any $\alpha < \lambda^+$ the family

$$\underline{P}_\alpha = \{A \subseteq \alpha : |A| < \lambda,\ d|A \text{ is bounded, } \mathrm{cf}(\sup A) \neq \kappa\}$$

has cardinality $\leq \lambda$.

Pf : Let $\alpha = \bigcup_{i<\kappa} B_i$, $|B_i| < \lambda$, B_i increasing, and let, for $i < \kappa$, $\zeta < \kappa$, $\underline{P}^\zeta_{\alpha,i} = \{A \in \underline{P}_\alpha : A \cap B_i$ unbounded in A, $d|A$ bounded by $\zeta\}$. Since $A \in P_\alpha \Rightarrow [\mathrm{cf}(\sup A) \neq \kappa$ and $d|A$ bounded$]$, and by the choice of the B_i's, $\underline{P}_\alpha = \bigcup_{\zeta,i<\kappa} \underline{P}^\zeta_{\alpha,i}$, it suffices to prove $|\underline{P}^\zeta_{\alpha,i}| \leq \lambda$ (for given $i,\zeta < \lambda$). Let $B^\zeta_i = B_i \cup \bigcup_{\beta\in B_i} \{\gamma : \gamma < \beta,\ d(\beta,\gamma) \leq \zeta\}$. Clearly $|B^\zeta_i| \leq |B_i| + \lambda_\zeta < \lambda$, and $A \in \underline{P}^\zeta_{\alpha,i}$ implies $A \subseteq B^\zeta_i$. So $|\underline{P}^\zeta_{\alpha,i}| \leq 2^{|B_i| + \lambda_i} < \lambda$, so we have proved stage B.

Stage C :

If P is a family of subsets of A each of cardinality $< \lambda$, but $|\underline{P}| \leq |A| = \lambda$, then there is a set $C \subseteq A$ such that

(i) $|C| = \kappa$,

(ii) $(\forall A \in P)\ |A \cap C| < \kappa$.

This is trivial.

Stage D :

We define the A_α's by induction on α for $\alpha \in S$. Suppose we arrive at α. Let $\langle \gamma_i : i < \mathrm{cf}\alpha \rangle$ be increasing with limit α, $\gamma_i + \lambda \leq \gamma_{i+1}$.

For a set A of ordinals, let $\mathrm{acc}(A) = \{\delta : \delta$ a limit, $\delta = \sup(A \cap \delta)\}$ (= the set of accumulation points of A). By stage B, $|\underline{P}_\alpha| \leq \lambda$, so by stage C we can find $c^i_\alpha \subseteq (\gamma_i, \gamma_i + \lambda)$, of power κ such that :

(*) for every $A \in P_\alpha \cup \{\cup\{A_\gamma : \gamma < \alpha, \gamma \in acc(A)\} : A \in P_\alpha\}$, its intersection with c^i_α has power $<\kappa$.

In fact we have to check that $|\cup \{A_\gamma : \gamma < \alpha, \gamma \in acc(A)\}| < \lambda$ (for $A \in \underline{P}_\alpha$), but this is easy : $\lambda \in acc(A) \Rightarrow cf\lambda \leqslant |A| \Rightarrow |A_\gamma| \leqslant \kappa + cf\gamma = \kappa + |A|$, hence the set has power $\leqslant (\kappa + A)\ |A| < \lambda$. We let $A_\alpha = \bigcup_{i<cf\alpha} c^i_\alpha$.

Stage E :

$\{A_\alpha : \alpha \in S\}$ has no transversal.

Because $A_\alpha \subseteq \alpha$, by Fodor's theorem.

Stage E :

We prove (A^*) from the lemma. We prove by induction on α that there are big $A'_\beta \subseteq A_\beta$ $(\beta \leqslant \alpha, \beta \in S)$, pairwise disjoint. This will clearly suffice.

Case 1 : For α a successor ordinal, it follows from the induction hypothesis on $\alpha-1$.

Case 2 : For α such that $(\exists\beta < \alpha)\ \beta + \lambda\omega > \alpha$: proof as in the first case.

Case 3 : For α a limit, $cf\alpha = \aleph_0$. Choose ordinals $\alpha_n < \alpha$, $\alpha_n < \alpha_{n+1}$, $\alpha = \cup\, \alpha_n$, $\alpha_0 = 0$. For each n, by the induction hypothesis there are big $A^n_\beta \subseteq A_\beta$ $(\beta \leqslant \alpha_n)$, pairwise disjoint.

Define A'_β , for $\beta \leqslant \alpha$, $\beta \in S$ (hence $\beta \neq 0$), by :

$$A'_\beta = A^{n+1}_\beta - (\alpha_n + \lambda), \text{ where } \alpha_n < \beta \leqslant \alpha_{n+1}$$

It is easy to check that $A'_\beta \subseteq A_\beta$ is still big, and obviously the A'_β are pairwise disjoint. Note that $\alpha \in S$, so we do not have to define A'_α.

Case 4 : For α limit, not case 2, $cf\alpha > \aleph_0$. There is $E \subseteq \alpha$, unbounded, of order type $cf\alpha$ (hence $<\lambda$) and $E = \{\beta_{i+1} : i < cf\alpha\}$ (the β_i increasing), such that $d|E_i$ is unbounded for $i < cf\alpha$, where

$E_i = \{\beta_{j+1} : j < i\}$, and each β_{i+1} is a successor ordinal. (For $cf\alpha \leq \kappa$, any unbounded A of order type $cf\alpha$ is as required). (Remember d is from stage A).

We can define for limit $\delta \leq cf\alpha$, $\beta_\delta = \sup\{\beta_{i+1} : i < \delta\}$.

Since $\beta_i + \lambda < \alpha$, we can assume w.l.o.g. $\beta_i + \lambda < \beta_{i+1}$ (by making deletions if necessary). Let $A^i_\beta \subseteq A_\beta$ be big, pairwise disjoint, for $\beta \leq \beta_i$ (possible by the induction hypothesis).

We now define A'_β, if $\beta \notin \bigcup_{i<cf\alpha} [\beta_i, \beta_i + \lambda) \cup \{\alpha\}$, by :

$A'_\beta = A^i_\beta - (\beta_i + \lambda)$, where $\beta_i + \lambda < \beta \leq \beta_{i+1}$.

Clearly, the $A'_\beta \subseteq A_\beta$ are big, pairwise disjoint and disjoint from $D =^{df} \bigcup_{i<cf\alpha} [\beta_i, \beta_{i+1} + \lambda)$. For which β's have we still not defined A'_β ? For $\beta = \beta_i$ $(i \leq cf\delta)$ i.e., $\beta = \beta_j$, for which $\beta \in S$, hence $cfj \neq \aleph_o, \kappa, 1$. Checking definitions we can see that for each such β, $A_\beta \cap D \subseteq A_\beta$ is big. So it suffices to find pairwise disjoint big $A'_{\beta_j} \subseteq A_{\beta_j}$ $(j \leq cf\delta$, j a limit). This we do by induction on j. Suppose we have defined these for every $j' < j$. For j a successor among $\{i \leq cf\delta : i$ a limit$\}$ or $\beta_j \notin S$, there is no problem. (Remember for j a successor, β_j is a successor, hence $\notin S$). Otherwise, note that $cfj \neq \kappa$, hence $cf(\sup(E_j)) \neq \kappa$, hence $E_j \in P_\alpha$ (see stage B). Now look at Stage D, for β_j. We chose there an increasing continuous sequence of ordinals $\langle \gamma_i : i < cf\,\beta_j \rangle$ converging to β_j. Since $cf\,\beta_j \neq \aleph_o$, there is a closed unbounded $C \subseteq cf\,\beta_j$, such that $i \in C \Rightarrow \gamma_i \in \{\beta_\xi : \xi < j\}$. We then defined $A_{\beta_j} = \bigcup_{i<cf\beta_j} c^i_{\beta_j}$, where $c^i_{\beta_j} \subseteq (\gamma_i, \gamma_i + \lambda)$, has order type κ, and in particular

$$[\cup\{A_\zeta : \zeta \in \delta, \zeta \in acc(E_j)\}] \cap c^i_{\beta_j} \quad \text{has power } < \kappa.$$

But what is $acc(E_j)$? It is just $\{\beta_{j(o)} : j(o) < j, j(o)$ a limit$\}$. So $c^i_{\beta_j} \cap [\cup\{A_{j(o)} : j(o) < j, j(o)$ a limit, $A_{j(o)}$ defined$\}]$ has power $< \kappa$.

Let $A'_{\beta_j} = \cup\{c^i_{\beta_j} - \cup\{A_\zeta : \zeta \in S, \zeta \in acc(E_j)\} : i \in C\}$.

It is easy to check that it is a big subset of A_{β_j}, and obviously it is disjoint from $A_{\beta_{j(o)}}$, where $j(o) < j$ is a limit. So we have finished the proof.

Stage E : Suppose λ singular strong limit, $cf\lambda = \kappa$, S a stationary subset of λ^+, and every member of S divisible by $\lambda\omega$. Suppose further $A_\alpha \subseteq \alpha$, $|A_\alpha| \leqslant \kappa cf\alpha$ for $\alpha \in S$, and for any $\alpha_o < \lambda^+$, $\{A_\alpha : \alpha < \alpha_o\}$ has a transversal. Then we can find $A^*_\alpha \subseteq \alpha$ for $\alpha \in S$, so that $A^*_\alpha = \{\gamma(\alpha,i) : i < \kappa(cf\alpha)\}$, where $\gamma(\alpha,i)$ increase with i, (hence $|A^*_\alpha| \leqslant cf\alpha + \kappa$ $(< \lambda)$) and for every $\alpha_o < \lambda^+$ there are pairwise disjoint $A'_\alpha \subseteq A_\alpha$ (for $\alpha < \alpha_o$, $\alpha \in S$), such that for each α for some $i_o < cf\alpha$

$$(\forall i < cf\alpha)\ (\exists \zeta < \kappa)(\forall \xi)(\zeta \leqslant \xi < \kappa\ \&\ i_o < i \rightarrow \gamma(\alpha,\kappa i+\xi) \in A'_\alpha).$$

Proof : For every α, choose $B^\xi_\alpha \subseteq \alpha$, B^ξ_α increase with ξ, $\alpha = \bigcup_{\xi<\kappa} B^\xi_\alpha$ and $|B^\xi_\alpha| < \lambda$. We can define functions h_o, h_1, Dom $h_\ell = \lambda^+$, so that for any β_o, $\beta_1 \leqslant \beta < \lambda^+$, $\xi < \kappa$, $A \subseteq B^\xi_{\beta_o}$, there are λ β^*'s, $\beta \leqslant \beta^* < \beta + \lambda$, such that $h_1(\beta^*) = \beta_1$, $h_2(\beta^*) = A$. (We define $h_\ell \upharpoonright [\lambda i, \lambda(i+1))$ for each i; the number of possible tuples $< \beta_1, A, \beta, \xi, \beta_o >$ is $\leqslant \lambda$, so there is no problem).

For each $\alpha \in S$ choose an increasing sequence $\beta(\alpha,i)$ $(i < cf\alpha)$ converging to it.

First note that $(\forall \alpha_o < \alpha)\ \alpha_o + \lambda < \alpha$ (since $\alpha \in S$) hence w.l.o.g. $\beta(\alpha,i) + \lambda < \beta(\alpha,i+1)$, and $\beta(\alpha,i)$ is divisible by λ.

Now we define by induction on $j = i\kappa + \xi$ $(i < cf\alpha,\ \xi < \kappa)$ an ordinal $\gamma(\alpha,j)$, increasing with j, such that

(i) $\beta(\alpha,i) < \gamma(\alpha,j) < \beta(\alpha,i) + \lambda$,

(ii) $h_1(\gamma(\alpha,j)) = cf\alpha$,

(iii) $h_2(\gamma(\alpha,j)) = A_\alpha \cap B^\xi_{\beta(\alpha,i)}$, and

(iv) $\gamma(\alpha,j) \notin \{A^*_{\alpha(o)} : \alpha(o) \in B^\xi_\alpha\}$.

The last condition excludes $< \lambda$ γ's, and the conditions (ii), (iii)

are satisfied by λ γ's, $\beta(\alpha,i) < \gamma < \beta(\alpha,i) + \lambda$.
So we can define $A^*_\alpha = \{\gamma(\alpha,i) : i < \kappa(cf\alpha)\}$, and $\gamma(\alpha,i)$ increase with i and converge to α.

Now we are given $\alpha(o) < \lambda^+$ and have to find $A'_\alpha \subseteq A^*_\alpha$ as required. By hypothesis, there is a transversal f of $\{A_\alpha : \alpha < \alpha(o)\}$.
Define $A^1_\alpha = \{\gamma(\alpha,\kappa i + \xi) : i < cf\alpha,\ f(A_\alpha) \in A_\alpha \cap B^\xi_{\beta(\alpha,i)}\}$.

Clearly it is a very big subset of A_α.
On $S \cap \alpha(o)$ we define a graph : (α_1,α_2) is an edge iff $A^1_{\alpha_1} \cap A^1_{\alpha_2} \neq \phi$.
Note :
(a) If (α_1,α_2) is an edge then $cf\alpha_1 = cf\alpha_2$ (because $\gamma \in A_{\alpha_\ell}$ implies $h_1(\gamma) = cf\alpha_\ell$).
(b) The valency of any α_1 ($= |\{\alpha_2 : (\alpha_1,\alpha_2) \text{ is an edge }\}|$) is $\leq |A^*_\alpha|$.

As f is one-to-one, it suffices to prove that $f(A_{\alpha_2}) \in A_{\alpha_1}$ whenever $A_{\alpha_2} \cap A_{\alpha_1} \neq \phi$. If $\gamma = \gamma(\alpha_1,\kappa i_1 + \xi_1) = \gamma(\alpha_2,\kappa i_2 + \xi_2) \in A^1_{\alpha_1} \cap A^1_{\alpha_2}$, then $\beta = \beta(\alpha_1,i_1) = \beta(\alpha_2,i_2)$ (it is the biggest ordinal $\leq \gamma$ divisible by λ), so $A_{\alpha_1} \cap B^{\xi_1}_{\beta(\alpha_1,i_1)} = h_2(\gamma) = A_{\alpha_2} \cap B^{\xi_2}_{\beta(\alpha_2,i_2)}$, but $f(A_{\alpha_2}) \in A_{\alpha_2} \cap B^{\xi_2}_{\beta(\alpha_2,i_2)}$ (since $\gamma \in A^1_{\alpha_2}$) hence $f(A_{\alpha_2}) \in A_{\alpha_1} \cap B^{\xi_1}_{\beta(\alpha_1,i_1)} \subseteq A_{\alpha_1}$, as required.

Now we deal with each component C of the graph separately. By (a), all $\alpha \in C$ have the same cofinality, say μ, and by b), $|C| \leq \kappa + \mu$. If $\mu > \kappa$ note that each A^1_α has order type μ and is unbounded below α, hence $\alpha_1 \neq \alpha_2 = C \Rightarrow |A^1_{\alpha_1} \cap A^1_{\alpha_2}| < \mu$.
So let $C = \{\alpha_\zeta : \zeta < \mu\}$, and we can define $A^*_{\alpha_\zeta} = A^1_{\alpha_\zeta} - \bigcup_{\xi<\zeta} A^1_{\alpha_\xi}$, which are as required. If $\mu \leq \kappa$, we give a similar treatment to each $\{\gamma(\alpha,\kappa i + \xi) : \xi < \kappa\}$ for $i < \mu$, $\alpha \in C$.

<u>25. Conclusion</u> :

1) Suppose $\aleph_\omega$ is a strong limit.

a) There is a family of $\aleph_{\omega+1}$ countable subsets of $\aleph_{\omega+1}$ which does not have a transversal, but every subfamily of cardinality $< \aleph_{\omega+1}$ has a transversal.

b) There is an abelian group [group] of power $\aleph_{\omega+1}$, which is not free, but every subgroup of cardinality $< \aleph_{\omega+1}$ is.

2) Suppose $\aleph_{\omega\ell}$ is strong limit for $\ell \leqslant n$. Then a), b) hold for $\aleph_{\omega n+1}$.

<u>Proof</u> : 1 a), 2 a). It is easy to see this after reading Milner and Shelah [MS].

1 b), 2 b) are easy to see.

<u>26. Claim</u> : Suppose λ is strong limit, $cf\lambda = \aleph_0$, $\mu < \kappa$, μ regular and : P is μ-complete or among any μ members of P there are μ which are pairwise compatible.

If in V^P λ is still a strong limit cardinal, then

$$S^*(\lambda^+)^V \cap CF(\lambda,\mu)^V,\ S^*(\lambda^+)^{V^P} \cap CF(\lambda,\mu)^{V^P}$$

are equal (i.e., for some representation they are equal).

<u>Proof</u> : Let $d : \lambda^+ \to \kappa$ be normal. Clearly it is still normal in V^P. By 13 it suffices to prove that the truth value of "$\alpha \in S_1(d)$" is not changed, which is quite easy.

<u>27. Claim</u> : If χ is supercompact, $\lambda > \chi$, $cf\lambda < \chi$, then $S^*(\lambda^+)$ is stationary.

<u>Proof</u> : Let $d : \lambda^+ \to cf\lambda$ be normal and subadditive, and suppose $C \subseteq \lambda^+$ is closed and unbounded.

Suppose $N \prec (H(\lambda^{++}), \in)$, $cf\lambda + 1 \subsetneq N$, $C,d \in N$, $\|N\| < \chi$ and every subset of $N \cap \lambda^+$ belongs to N (this is possible as χ is supercompact). Let $\delta^* = \sup(N \cap \lambda^+)$. Clearly $cf\delta^*$ is the successor of a singular cardinal of cofinality $cf\lambda$ so $cf\delta^* > cf\lambda$. Clearly $C \cap N$ is unbounded, hence $\delta^* \in C$; so it suffices to prove $\delta^* \notin S_0(d)$.

So suppose $A \subseteq \delta^*$ is unbounded, and $d|A$ is bounded by ζ. Let $A = \{\beta_i : i < \delta^*\}$, β_i increasing. We may assume, w.l.o.g., for each i there is γ_i, $\beta_i < \gamma_i < \beta_{i+1}$, $\gamma_i \in N$. Let $\zeta_i = \text{Max}\{\zeta, d(\beta_{i+1},\gamma_i), d(\gamma_i,\beta_i)\} < cf\lambda < cf\delta^*$. So (w.l.o.g.) $\zeta_i = \zeta^*$ for every i. Now if $i < j$, then by the subadditivity :

$$d(\gamma_i,\gamma_j) \leq \max\{d(\gamma_j,\beta_{j+1}), d(\beta_{j+1},\beta_{i+1}), d(\beta_{i+1},\gamma_i)\} \leq \zeta^*$$

So $d|\{\gamma_i : i < cf\delta^*\}$ is bounded, but the set necessarily belongs to N, and, as $N \prec (H(\lambda^{++}), \in)$, there is an unbounded $B \subseteq \lambda^+$ on which d is bounded, giving an easy contradiction to normality.

28. Remark : We in fact prove that if d is a subadditive function, with domain α^*, $\alpha \leq \alpha^*$, and d is bounded on some unbounded $A \subseteq \alpha$, then every unbounded $A' \subseteq \alpha$ has an unbounded subset $A'' \subseteq A' \subseteq \alpha$ such that $d|A''$ is bounded.

29. Conclusion : If ZFC + "$\exists$ a supercompact" is consistent then the following is consistent :

$$\text{ZFC + GCH + "}S^*(\aleph_{\omega+1})\text{ is stationary".}$$

Proof : Suppose χ is supercompact, and also (w.l.o.g.) GCH holds. Let λ be the first singular cardinal $> \chi$. By 27 we can choose a regular $\mu < \chi$ such that $S^*(\lambda^+) \cap CF(\lambda^+,\mu)$ is stationary. We use Levy collapsing P to collapse every $\mu' < \mu$ to $\aleph_0$ (by finite conditions). So now, in V^P, μ is $\aleph_1$. By 26, in V^P, $S^*(\lambda^+)^{V^P} \supseteq S^*(\lambda^+)^V \cap CF(\lambda^+,\mu)^V$, and the latter obviously remains stationary. Now collapse χ to $\aleph_1$ by a Q which is $\aleph_1$-complete. Again $S^*(\lambda^+)^V \cap CF(\lambda^+,\mu)^V$ remains stationary and is still included in $S^*(\lambda^+)^{P*Q}$.

$\diamond_\lambda$ is not a strong requirement

30. Definition : Let λ be a regular cardinal and $E \subseteq \lambda$ a stationary

set in it.

(1) $\diamondsuit^*_\lambda(E)$. There is $< W_\alpha : \alpha \in E >$ such that for every α, W_α is a family of subsets of α with $|W_\alpha| \leq |\alpha|$, and for every $X \subseteq \lambda$ there is a closed and unbounded $C \subseteq \lambda$ such that $X \cap \alpha \in W_\alpha$ for all $\alpha \in C \cap E$.

(2) $\diamondsuit_\lambda(E)$. There is $< S_\alpha : \alpha \in E >$ such that $S_\alpha \subseteq \alpha$, and for every $X \subseteq \lambda$, $\{\alpha : X \cap \alpha = S_\alpha\}$ is stationary in λ.

<u>31. Theorem</u> : (Kunen) : (1) For stationary $E \subseteq \lambda$, $\diamondsuit^*_\lambda(E)$ implies $\diamondsuit_\lambda(E)$.

(2) For $E_1 \subseteq E_2 \subseteq \lambda$, $\diamondsuit_\lambda(E_1)$ implies $\diamondsuit_\lambda(E_2)$ and $\diamondsuit^*_\lambda(E_2)$ implies $\diamondsuit^*_\lambda(E_1)$.

<u>32. Theorem</u> : Suppose $\lambda = 2^\mu = \mu^+$ and for some regular $\kappa < \mu$, either

(i) $\mu^\kappa = \mu$, or

(ii) μ is singular $\kappa \neq cf\mu$ and for every $\delta < \mu, |\delta|^\kappa < \mu$

<u>Then</u> $\diamondsuit^*_\lambda(E(\kappa))$ where $E(\kappa)$ is the stationary subset $\{\alpha < \lambda : cf\alpha = \kappa\}$.

<u>Remark</u> : Case (i) is due to Gregory [Gr].

<u>Proof</u> : Let $<A_\alpha : \alpha < \lambda >$ be a list of all bounded subsets of λ each appearing λ times (there are λ such subsets as $\lambda = 2^\mu = \mu^+$)

<u>Case (i)</u> : For $\alpha \in E(\kappa)$ let W_α be the set of all unions of no more than κ subsets of α belonging to $< A_\beta : \beta < \alpha>$.

$(W_\alpha = \{\cup Y : |Y| \leq \kappa, x \in Y \rightarrow x \subseteq \alpha, x \in \{A_\beta : \beta < \alpha\}\})$.

Given $X \subseteq \lambda$, let C be $\{\alpha_i \mid i < \lambda\}$ where α_0 is any successor less than $\lambda, \alpha_\delta = \bigcup_{\beta<\delta} \alpha_\beta$ for limit δ, and α_{i+1} is the least $\alpha > \alpha_i$ such that for some $\gamma < \alpha$, $A_\gamma = X \cap \alpha_i$.

Now $C' = \{\delta : \delta = \cup \{\alpha_i : \alpha_i < \delta\}\}$ is closed unbounded, and for $\delta \in C \cap E(\kappa)$ there are $i(j)$ and $\gamma_j < \delta$ $(j < \kappa)$ such that

$\bigcup_{j<\kappa} \alpha_{i(j)} \square \delta$, $X \cap \alpha_{i(j)} = A_{\gamma_j}$. So $X \cap \delta = \bigcup_{j<\kappa} A_{\gamma_j} \in W_\delta$.

Case (ii) : For δ such that $cf\delta = \kappa$, let $\delta = \bigcup_{j<\mu} V^\delta_j$, where $\langle V^\delta_j : j < \mu \rangle$ is increasing and for $j < \mu$, $|V^\delta_j| < \mu$.

Let W_δ be $\{\bigcup_{\alpha\in Q} A_\alpha : (\exists j < \mu)\ Q \subseteq V^\delta_j,\ |Q| \leqslant \kappa\}$.

Given $X \subseteq \lambda$ let $f : \lambda \to \lambda$ be such that $X \cap \alpha = A_{f(\alpha)}$ $f(\alpha) > \sup f(\beta)$.

There exists a closed unbounded $C \subseteq \lambda$ such that for $\alpha \in C$, $\beta < \alpha$ implies $f(\beta) < \alpha$.

Let $\delta \in C \cap E(\kappa)$, and for increasing $\langle \delta_i : i < \kappa \rangle$, $\delta = \bigcup_{i<\kappa} \delta_i$.

There exists j such that

$$\kappa = |V^\delta_j \cap \{f(\delta_i) : i < \kappa\}| \text{ hence } X \cap \delta = \bigcup \{X \cap \delta_i : i < \kappa, f(\delta_i) \in V^\delta_j\} \in W_\delta.$$

33. Conclusion : (GCH) If $\lambda > \aleph_0$, then $\diamondsuit^*_{\lambda^+}(E(\kappa))$ holds, whenever $\kappa \neq cf\lambda$. In particular $\diamondsuit_\lambda$ holds.

Final comments

1) The restriction "λ strong limit" in most cases can be weakened at the expense of complicating the results : assuming $(\forall \mu < \lambda)$ $\mu^{<\chi} < \lambda$, and restricting ourselves to $CF(\lambda^+, <\chi)$ or $CF(\lambda^+, \leqslant \chi)$.

2) A more serious question is whether we can, in 7, replace D^g_λ by D_λ. This remains open.

Note that the natural notion is $S_2(\overline{N})$, and that for regular λ, $I^+(\lambda) = \{A \subseteq \lambda$: for some λ-approximating sequence $\overline{N}$, $A \subseteq S_2(\overline{N})\}$ is always a normal ideal. Similarly $I^-(\lambda) = \{A \subseteq \lambda : A \cap B \equiv \phi \bmod D_\lambda$ for every $B \in I^+(\lambda)\}$ is a normal ideal. The meaning of claim 7 is that $I^+(\lambda)$ is $\{A : A \subseteq A_0 \bmod D_\lambda\}$ for some A_0, when $gcf(\lambda) = \lambda$. Another formulation of our question is whether this always holds.

However, we can meanwhile just formulate the later theorems in terms of $I^+(\lambda)$ instead of $S^*(\lambda)$ (and the changes in the proofs are minor). By the way it may be more natural to use $S_3(\overline{N}) = \{\delta$: there is a function h, Dom h = cfδ, Range h an unbounded subset of δ, $(\forall i < \mathrm{cf}\delta)$ $h|i \in N_\delta$, and $N_\delta \cap \lambda = \delta\}$ (in gcf(λ) it does not matter).

3) Why were we interested mainly in $\aleph_{\omega+1}$ and not in e.g. $\aleph_{\omega+2}$? The answer is that several inductive proofs work for successors of regular cardinal, and it was not clear whether they fail at successors of singulars. (But see remarks 5 and 6 below).

4) It may be of interest to mention our original line of thought, which is not so transparent from the present paper.

We want to prove that $S_2(\overline{N})$ is quite "big", where $\overline{N}$ is an $\aleph_{\omega+1}$-approximating sequence for $\aleph_{\omega+1}$, assuming GCH. So we let $d : \aleph_{\omega+1} \to \aleph_0$ be normal, and using the Erdös-Rado theorem $(2^{\aleph_n})^+ \to (\aleph_{n+1})^2_{\aleph_0}$, prove that if $C \subseteq \aleph_{\omega+1}$ is closed of order type $(2^{\aleph_n})^+$ then it contains C_1 of order type $\aleph_{n+1}$, with d constant on C_1. C_1' (the set of accumulation points of C_1) is $\subseteq S_2(\overline{N})$ and is a closed subset of C of order type $\aleph_{n+1}$. This proves that $S_2(\overline{N})$ is in some sense big.

5) We can try to generalize 4) to other cardinals.

Let $\kappa = \mathrm{cf}\,\aleph_\alpha < \aleph_\alpha$.

<u>Definition</u> : Call an (n+1)-place function d from $\aleph_{\alpha+n}$ to κ <u>normal</u> if for every $\alpha_0 < \ldots < \alpha_n < \aleph_{\alpha+n}$ there is $k < n$ such that

$$\{\alpha < \aleph_{\alpha+n} : d(\alpha_0,\alpha_1,\ldots,\alpha_{k-1},\alpha,\alpha_{k+1},\ldots,\alpha_n) = d(\alpha_0,\ldots,\alpha_k,\ldots,\alpha_n)\}$$

has cardinality $< \aleph_\alpha$.

<u>Claim</u> : There is a normal function $d : \aleph_{\alpha+n} \to \kappa$.

<u>Proof</u> : By induction on n.

<u>Lemma</u> : Let $\overline{N}$ be an $\aleph_{\alpha+n}$-approximating sequence for $\aleph_{\alpha+n+1}$, C a closed subset of $\aleph_{\alpha+n}$ of order type $\beth_{n+1}(\kappa + \mu)^+$, where $\mu < \lambda$.

Then C has a closed subset of order type μ^+ which is included in $S_2(\overline{N})$.

Proof : Let $d \in N_o$, $d : \aleph_{\alpha+n} \to \kappa$, d normal. By the Erdös-Rado theorem $(\beth_{n+1}(\kappa+\mu)^+ \to (\mu^+)^{n+1}_{\kappa})$ there is $C_1 \subseteq C$ of order type μ^+ on which d is constant. If $\delta \in C_1'$, then $C_1 \cap \delta$ witnesses that $\delta \in S_2(\overline{N})$.

6) Suppose $\aleph_\alpha$ is strong limit, $\kappa \blacksquare \mathrm{cf}\, \aleph_\alpha$, γ a successor ordinal, $\kappa \leqslant \mu < \aleph_\alpha$ and $\beth_\gamma(\mu) < \aleph_\alpha$. If $\overline{N}$ is a $\aleph_{\alpha+\gamma}$-approximating sequence for $\aleph_{\alpha+\gamma+1}$, and $C \subseteq \aleph_{\alpha+\gamma}$ has order type $\beth_\gamma(\mu)^+$, then C has a closed subset C_1 of order type μ^+ which is included in $S_2(\overline{N})$.

Proof : We prove a somewhat stronger statement :
If $C \subseteq \aleph_{\alpha+\beta}$, $\beta \leqslant \gamma$ a successor ordinal, and C has order type $\geqslant \beth_\beta(\mu)^+$, then there is $C_1 \subseteq C \cap S_2(\overline{N})$ of order type μ^+, such that for some $\ell < n$, if $\alpha_o < \ldots < \alpha_n \in C_1$ then
$(H(\aleph_{\alpha+\gamma+1}),\in) \models \varphi(\alpha_o,\ldots,\alpha_n)$ & $|\{x : \varphi(\alpha_o,\ldots,\alpha_{\ell-1},x,\alpha_\ell,\ldots,\alpha_n)\}| < \aleph_\alpha$
(This imples $C_1' \subseteq S_2(\overline{N})$).

We prove this by induction on β. For finite β this was done above, and the induction step is easy.

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PAUL BERNAYS

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Paul Bernays was born on October 17th 1888 in London; he died after a short illness on September 18th 1977 in Zürich. He was the son of Julius and Sara Bernays, née Brecher. His father was a businessman and - as he states in the curriculum vitae appended to his thesis - he was of jewish confession and a citizen of Switzerland. (1) Soon after the birth of Paul, the family moved to Paris and from there to Berlin. It is in Berlin that he attended school, from 1895 to 1907. He seems to have been quite happy at school, a gifted, well adapted child accepting the prevailing cultural values in literature as well as in music. It was indeed his musical talent that first attracted attention; he tried his hand at composing, but being never quite satisfied with what he achieved, he decided on a scientific career. He studied engineering at the Technische Hochschule Charlottenburg for one semester, then realizing (and convincing his parents) that pure mathematics was what he wanted to do, he transferred to the University of Berlin. His main teachers were: Schur, Landau, Frobenius and Schottky in mathematics; Riehl, Stumpf and Cassirer in philosophy, Planck in physics. After four semesters, he moved to Göttingen; there he attended lectures on mathematics by Hilbert, Landau, Weyl and Klein, on physics by Born, and on philosophy by Leonard Nelson. Nelson was the center of the Neu-Friessche Schule - Bernays was quite an active member of the group and stayed in contact with it all his life. His first publication - in 1910 - was "Das Moralprinzip bei Sidgwick und bei Kant", published in the Abhandlungen der Friesschen Schule. (2) There were two further publications in 1913 in the same Abhandlungen, one "Ueber den transzendentalen Idealismus", the other "Ueber die Bedenklichkeiten der neueren Relativitätstheorie". (3,4) Though we no longer share the difficulties discussed by Bernays, it is remarkable how calmly he takes part in otherwise rather heated controversies. There is no doubt that Bernays was deeply influenced by Nelson - by his liberal socialism as well as by his revised version of Kant's imperative demanding the permanent readiness to act according to duty (Nelson lived from 1882 to 1927).

In the spring of 1912 Bernays received his doctorate with a dissertation (written with Landau) on analytic number theory - the exact title being: "Ueber die Darstellung von positiven, ganzen Zahlen durch die primitiven, binären quadratischen Formen einer nicht-quadratischen Diskriminante." (1)

At the end of the same year he obtained his Habilitation at the University of Zürich where Zermelo was professor. His Habilitationsschrift was on function theory: "Zur elementaren Theorie der Landauschen Funktion $\phi(\alpha)$". (5)

From 1912 to 1917, Bernays was Privatdozent in Zürich. There are no publications (except the one's already cited of 1913) in this period. Bernays must have passed some crisis in these years. In the short biography published in the book "Sets and classes" (6), he states:
"At the beginning of the First World War, I worked on a reply to a critique by Alfred Kastil of the Fries philosophy. This reply was not published - by the time there was an opportunity to have it published I no longer agreed with all of it." He even considered giving up mathematics at this time - but did not see anything he felt he could do better. Therefore, Hilbert's proposal to be his collaborator in Göttingen must have come as a relief to Bernays. That he found his way back to mathematics is shown by his Göttinger Habilitationsschrift, a brilliant piece of work written in a short time. Bernays left Zürich in 1917; his paper was submitted in 1918. The title is "Beiträge zur axiomatischen Behandlung des Logikkalküls". (7) In retrospect it is hard to understand that the paper was not published at that time - parts of it appeared in 1926 in the Mathematische Zeitschrift. (8) Bernays explained this long delay once in the following way:
"To be sure, the paper was of definite mathematical character, but investigations inspired by mathematical logic were not taken quite seriously - they were thought of as amusing, half-way part of recreational mathematics. I myself had this tendency, and therefore did not take pains to publish it in time. It has appeared only much later, and strictly speaking not quite complete, only certain parts. Many things I had in the paper have therefore not been recorded accordingly in descriptions of the devlopment of mathematical logic". (9)

An analysis of the content of the paper will be given in some detail - the library at the ETH in Zürich is in possession of Bernays' copy.

As to his activity in Göttingen in the period 1917 - 1934, Bernays describes it as follows: My work with Hilbert consisted on the one hand in helping him to prepare his lectures and making notes of some of them, and on the other hand in talking over his research, which gave rise to a lot of discussions. (6)

Bernays also gave lectures on various subjects of mathematics and became an extraordinary professor of mathematics in 1922. Besides a series of papers written as a collaborator of Hilbert (a typical title being: Die Bedeutung Hilbert's für die Philosophie der Mathematik (10)), there are two papers where he is on his own: One with Schönfinkel (Zum Entscheidungsproblem der mathematischen

Logik (11)), where a case of the decision problem is treated, the other: Zur mathematischen Grundlegung der kinetischen Gastheorie (12) (where a special case of the ergodic theorem is proved). That his activity is only partially reflected by his publiations is shown by references in the literature of students of Hilbert's. We learn e.g. from Ackermann in 1928 that the axiomatization of predicate calculus based on the rules

$$\frac{\varphi \to \psi}{(\exists x)\varphi \to \psi} \qquad \frac{\psi \to \varphi}{\psi \to (\forall x)\varphi}$$

(with the well known condition on ψ) is due to Bernays. (20, p. 54)

In 1933 Bernays, as a "non-aryan", lost his position at the University of Göttingen. In 1934 he moved to Zürich - to call it a return would give a wrong impression; still in the fifties he could say: "bei uns in Göttingen" (we in Göttingen). In the same year 1934, the first volume of the Grundlagen der Mathematik appeared. (13) It was hailed from the beginning as a masterpiece on a par with the works of Frege, Peano, Russell-Whitehead.

From 1934 to his death, the home town of Bernays was Zürich. Being single, he first lived with his mother and two sisters, in the last years with his sister Martha, who survives him.

Twice he spent a year at the Institute for Advanced Study in Princeton (1935/36 and 1959/60), and three times he was visiting-professor at the University of Pennsylvania in Philadelphia. In Zürich, he was first a Privatdozent, then a professor till his retirement in 1958.

During his first stay in Princeton, he lectured on mathematical logic and on axiomatic set theory. His lectures on logic have been published as notes under the title: "Logical calculus" (14) - much of the material is taken up in the second volume of the Grundlagenbuch (1939). (15) The consistency theorem (a central theorem of the second volume) is contained in the Princeton notes.

In set theory he lectured on his own axiomatization. He had presented this system already in Göttingen in a lecture of 1929/30, but hesitated to publish it because he felt that the axiomatization was, to a certain extent, artificial. As Bernays records, he expressed this feeling to Alonzo Church, who replied with a consoling smile: That cannot be otherwise. (6) This persuaded him to publish - the work appeared in seven parts in the years 1937/1954 (16) and (almost unchanged) in the volume "Sets and classes". (6) (The same volume contains an English translation of a paper published in the anniversary volume of Fraenkel: "Schemata of infinity in axiomatic set theory". (17)

From the first volume of the Journal of Symbolic Logic up to volume 40, he published about a hundred reviews - it is he who reviewed there the fundamental papers of Gödel, Church, Gentzen, and many others. His last review is on the first volume of Schröder's algebra of logic - an essay of 6 pages in small print.

But it was not only through reviews that he reacted to the development - his correspondence was immense. The list of his correspondents counts up to a thousand persons, there are preserved up to 6000 copies of letters, many of them rather an essay than a letter.

In the second part, the work of Bernays shall be discussed in more detail. His main works may perhaps be grouped under three headings:

(1) Logic; (2) Set theory; (3) Philosophy.

There are, of course, papers which do not fall within these groups - papers on theoretical physics, calculus of variation, and especially on elementary geometry.

The first (and most detailed) analysis is on his Habilitationsschrift of 1918. The paper is on propositional calculus - following Russell and Whitehead, it is (except in studies of independence) based on the connectives of negation and disjunctions; implication is considered an abbreviation. The starting point of Bernays are the following axioms (slight variations of the axioms of PM):

$$(p \vee p) \to p$$

$$p \to (p \vee q)$$

$$(p \vee q) \to (q \vee p)$$

$$p \vee (q \vee r) \to (p \vee q) \vee r$$

$$(q \to r) \to ((p \vee q) \to (p \vee r))$$

Rules of inference are substitution and modus ponens. Bernays then introduces the following two notions: Derivable formula (ableitbare Formel) and identity (allgemeingültige Formel), the latter being defined by the truth table method. He shows that a formula is derivable if and only if it is an identity. The non trivial part is based on the following lemma: If a non derivable formula is added to the axioms, then every formula is derivable. The lemma is proved via the normal form theorem: Every formula is equivalent to a conjunction of "simple disjunctions". Bernays goes on to remark: This consideration gives furthermore a uniform procedure to decide whether a formula is derivable or not.

In the next paragraph is stated (though not proved) what is known as "Post-completeness":
"With respect to the logical interpretation of our calculus (which was at the

origin of this study), we obtain the result that the totality of provable formulas coincides with the totality of identical formulas. And this means that our calculus contains a formal systematization of those laws of logic which concern relations of truth and falsity of propositions subsisting independently of their structure and content. Indeed, all relations between truth and falsity of propositions may be expressed with the help of conjunction, disjunction and negation, and therefore also with the help of the symbols of our calculus, and insofar these relations hold for arbitrary propositions, the corresponding formal expressions must be identical formulas in the sense defined."

The next question considered is the problem which connectives form a basis. Besides giving complete answers (for the classical cases), partial systems are introduced and the following theorem is stated: If α is any formula (say, in $\neg, \vee$), either α or $\neg\alpha$ (and not both) is equivalent to a formula in $\vee, \wedge, \rightarrow$ alone.
(It is added that there exists a finite system of axioms for the identical formulas in $\vee, \wedge, \rightarrow$.)

In the next paragraph, the independence of the five formulas is studied. First of all, it is shown that the formula expressing associativity is derivable from the 4 others.

Next, it is shown that none of the four others are derivable from the rest. - It is here (as far as I know), that "many-valued" logic occurs for the first time.

Bernays describes the method as follows:
"In each one of the following proofs of independence, the calculus is reduced to a finite system for the elements of which a composition ("symbolic product") and a "negation" is defined, and this reduction is carried out in such a way that the variables of the calculus are related to the elements of the system as their values. The "correct" formulas shall be characterized by the fact that they assume only values of a certain given subsystem."

The most complicated system,introduced for these independence proofs, is one with 4 elements and a subsystem of 2 elements.

The last paragraph contains a detailed study on the possibility of replacing axioms by rules. There is given e.g. a system containing the only axiom $p \rightarrow p$ and six rules.

Bernays' best known contribution to mathematical logic is the work "Grundlagen der Mathematik", published under Hilbert's and his name, but written by Bernays alone. It is unique because of the wealth of material it contains -

published there for the first time (as much of what had been achieved by the Hilbert school in proof theory) or published in a more detailed form than in the original papers. It was - for a long time - a standard reference on mathematical logic, proof theory, arithmetization of metamathematics, recursion theory. But it is unique also in another respect - "foundation" is taken quite literally, it does not reduce mathematics to logic, or logic to mathematics - both are developed at the same time and (to some extent) the philosophy of mathematics along with it. Bernays once was asked why he had preferred mathematics to music as a career. One of the reasons he gave was that he had difficulty in following three tunes simultaneously - the Grundlagenbuch shows that he hadn't this difficulty in foundation.

The work contains, of course, much material original to Bernays, but as he chose not to divide his work in definitions, theorems, proofs, it is perhaps best not to single out results which might be given the name "theorem of Bernays".

A further major contribution of Bernays is his system of set theory, first presented in the year 1929/30 in Göttingen, and then elaborated in a series of papers in the years 1937/54. The basic result is, of course, known to everybody - how the fact that predicate logic can be based on, say, disjunction, negation and existential quantification is transformed into an axiomatization of the concept of class in such a way that a finite number of class axioms suffice to prove a general scheme of class existence. What is perhaps less well known is the careful study of basing parts of mathematics on subsystems of the full system of axioms - there are discussed e.g. three systems, sufficient to develop analysis. It is here that Bernays introduces his weakened forms of the axiom of choice (as the axiom of dependent choice) and shows that they are sufficient for analysis in a wide sense - including, say, Lebesgue measure theory and the theory of function spaces. If a general tendency of these studies is to be mentioned, the most distinctive feature is that Bernays tries to get along without sum axiom and power set axiom as long as possible. This is shown to be feasable in number theory, analysis, as well as in "general set theory".

In the book "Axiomatic set theory" (18) of 1958, a somewhat different version of the system is given - in stating the axioms, the existential form is replaced by the use of primitive symbols. Furthermore, the succession of steps in the development of the theory is different.

About half of the papers of Bernays' may be classified as philosophical. A collection of fourteen, referring to mathematics, has been edited by the Wissenschaftliche Buchgesellschaft under the title "Abhandlungen zur Philosophie der

Mathematik". (19) The first article published there dates from 1927, the last from 1971.

The thinking of Bernays is characterized by his constant effort to do justice to all aspects of the problems he considered. He believed in their inherent complexity and always resisted the temptation of explaining away. A typical example is geometry - for him it had not vanished in an abstract structure and/or a part of physics, nor did he adhere to reductionism so common in foundational studies. It is clear that such an attitude excludes short answers to almost all problems. Nevertheless, in order to give some impression of Bernays' way of philosophical thinking, a short text is presented. (The German contains a word - "Sachhaltigkeit" which no dictionary lists.It has been translated by "reality".)(The text is from an essay on philosophy of mathematics, presented at the International Congress of Philosophy in 1969. (19, p. 174,175)
"It seems appropriate to attribute to mathematics a reality which, however, is different from that of material world. That there exist other types of objectivity than that of material world is shown by objectivity in the domain of phenomena. Mathematics is insofar phenomenological as it is concerned predominantly with the study of idealized structures and is furthermore governed by the method of deduction. In the process of idealizing,the phenomenological and the conceptual come into contact. (It is therefore inappropriate to oppose these two to such an extent as is done in Kantian philosophy.) The specific character of mathematics as opposed to empirical science does not mean that we have in mathematics knowledge a priori. It seems necessary to concede that we have to learn also in the domain of mathematics and that we have there a kind of experience sui generis. (We may call it mental experience.) This is not prejudical to the rationality of mathematics. Rather it seems a prejudice that rationality is necessarily linked to certainty. Certain knowledge in the simple and full sense is given us almost nowhere. This is the old insight of Socrates."

For those who have not known Bernays personally, a few words on his personality may be added.

As his immense correspondence, his friendliness to visitors, his acceptance of invitations to congresses until the last years of his life, clearly show, he liked the contact with other human beings. He was extremely benevolent, helping many an author with his papers - from Hilbert to a high-school teacher having made some small discovery. On the other hand, he lived in an aura of detachment. He was unique in his refusal to judge other people; he never spoke badly of anybody - there is every reason to assume that he did not even think badly of others. When, once, reference was made to a statesman almost universally recognized as one of the villains of this century, in order to induce him to a negative judgement, he replied: "My situation is so different from his, that it is not for me to pass judgement". There is no doubt that his gift of seeing everywhere the best and refraining from judgement where he could not see anything good, helped a great deal to free foundational studies from the situation where different schools are expected to fight one another.

In the name of all those who have known Bernays personally, it certainly may be said: We are grateful for the privilege to have been in contact with Bernays.

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LOGIC COLLOQUIUM 78
M. Boffa, D. van Dalen, K. McAloon (eds.)

ABSTRACT LOGIC AND SET THEORY. I. DEFINABILITY

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Definability in an abstract logic is compared with definability in set theory. This leads to set theoretical characterizations of implicit definability, Löwenheim-numbers and Hanf-numbers of various abstract logics. A new logic, sort logic, is introduced as the ultimate limit of abstract logics definable in set theory.

§ 0. Introduction

The aim of this paper is to bring together, in a coherent framework, both old and new results about unbounded abstract logics (a logic is unbounded if it is able to characterize the notion of well-ordering). Typical problems that can be asked about any logic are:

(1) Which model classes are implicitly (with extra predicates and sorts) definable?
(2) Which classes of cardinals are spectra?
(3) What is the Löwenheim-number?
(4) What is the Hanf-number?

In the case of unbounded logics these problems are particularly relevant as such logics fail to be axiomatizable and mostly lack workable model theory. An attempt to shed light on (1)-(4) is the main purpose of this paper.

Out method is to build, right from the beginning, a close connection between abstract logic and set theory.

The basic notion of the whole paper is that of symbiosis. We say that an abstract logic L^* and a predicate P of set theory are symbiotic if, roughly speaking, the family of $\Delta(L^*)$-definable model classes coincides with the family of model classes which are $\Delta_1(P)$. For example, second order logic L^{II} is symbiotic with the power-set operation, or, what amounts to the same,

$$\Delta(L^{II}) = \{K \mid \text{the model class } K \text{ is } \Delta_2\}.$$

In Chapter 2 we give a new proof of the following result (essentially due to

Oikkonen [10]): If L^* and P are symbiotic, then

$$\Delta_n(L^*) = \{K \mid \text{the model class } K \text{ is } \Delta_n(P)\}.$$

As a corollary we get for $n > 1$:

$$\Delta_n(L_{\omega\omega}) = \{K \mid \text{the model class } K \text{ is } \Delta_n\}.$$

Consideration of the logics $\Delta_n(L_{\omega\omega})$ leads very naturally to what we call sort logic. To grasp the idea of sort logic, let us consider a typical many-sorted structure

$$\mathbf{M} = \langle M_1,\ldots,M_n;R_1,\ldots,R_m;a_1,\ldots,a_k\rangle.$$

$\mathbf{M}$ consists of three kinds of objects: universes M_i, relations R_i and individuals a_i. To quantify over the individuals we have first order logic; to quantify over relations we have second order logic; but to quantify over universes (i.e. sorts) we need a new logic. Accordingly, let sort logic L^s be the many-sorted logic which allows quantification over individuals, relations and sorts. It is clearly impossible to define the semantics of sort logic in set theory, but it can be done, for example, in MKM (Morse-Kelley-Mostowski) theory of classes.

It follows readily from the above analysis of $\Delta_n(L_{\omega\omega})$ that

$$L^s = \{K \mid \text{the model class } K \text{ is definable in set theory}\}$$

(stated in [8] p. 174).

The rest of Chapter 2 is devoted to an analysis of the non-syntactic nature of the Δ-operation. We show, for example, that the set of L^{II}-sentences which give rise to $\Delta(L^{II})$-definitions, is Π_3- but not Σ_3-definable in set theory. This result reflects the difficultness of finding a simple syntax for $\Delta(L^{II})$.

Chapter 3 is concerned with a restricted Δ-operation, Δ_1^1, which does not allow the use of new sorts (or universes). This operation is clearly related to L^{II} as we may think of L^{II} as $\Delta_{(\omega)}^1(L_{\omega\omega})$. The key notion of this chapter is that of a flat formula of set theory. We obtain the following characterization of generalized second order logic: If L^* and P satisfy a strengthend symbiosis assumption, then

$$\Delta_{(\omega)}^1(L^*) = \{K \mid \text{the model class } K \text{ is defined by a flat formula of the language } \{\varepsilon,P\}\}.$$

In particular

$$L^{II} = \{K \mid \text{the model class } K \text{ is defined by a flat formula of set theory}\}.$$

These results are proved in a level-by-level form.

In Chapter 4 we extend the analysis of the set theoretic nature of model theoretic definability to spectra and Löwenheim-numbers $l(L^*)$. We characterize the spectra of symbiotic logics and prove for L^*, symbiotic with P,

$$l(\Delta_n(L^*_A)) = \sup\{\alpha \mid \alpha \text{ is } \Pi_n(P)\text{-definable with parameters in } A\}$$

$$l(\Delta_n(L_A)) = \sup\{\alpha \mid \alpha \text{ is } \Delta_n\text{-definable with parameters in } A\}\ (n > 1).$$

A similar analysis of Hanf-numbers $h(L^*)$ is carried out in Chapter 5. The non-preservation of Hanf-numbers under Δ necessiates the introduction of a bounded Δ-operation Δ^B, and respective set theoretical notions Σ^B_1, Π^B_1 and Δ^B_1. The main result says: If L^* and P are symbiotic in a sufficiently bounded way, then

$$h(L^*_A) = \sup\{\alpha \mid \alpha \text{ is } \Sigma^B_1(P)\text{-definable with parameters in } A\}$$

and for $n > 1$,

$$h(\Delta_n(L_A)) = \sup\{\alpha \mid \alpha \text{ is } \Sigma_n(P)\text{-definable with parameters in } A\}.$$

In the rest of Chapter 5 we consider the numbers

$$l_n = \sup\{\alpha \mid \alpha \text{ is } \Pi_n\text{-definable}\}$$

$$h_n = \sup\{\alpha \mid \alpha \text{ is } \Sigma_n\text{-definable}\}.$$

Note that $l_n = l(\Delta_n(L_{\omega\omega}))$ and $h_n = h(\Delta(L_{\omega\omega}))$ (for $n > 1$). It turns out that for $n > 1$,

$$l_n = \sup\{\alpha \mid \alpha \text{ is } \Delta_n\text{-definable}\}$$

and

$$l_n < h_n = l_{n+1}.$$

In particular, we get

$$l(L^s) = h(L^s) = \sup\{\alpha \mid \alpha \text{ is definable in set theory}\}.$$

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§ 1. Preliminaries

We give at first a rough sketch of the preliminaries, which should be enough for a casual reader familiar with [2] and [8]. More detailed preliminaries then follow.

Our abstract logics are defined roughly as in [2]. If Q is a generalized quantifier LQ is like $L_{\omega\omega}[Q]$ in [8]. I is the Härtig-quantifier, W the well-ordering quantifier and Q_H the Henkin-quantifier. L^{II} is second order logic. All logics are understood to be many-sorted. The logic which is obtained from L^{II} by adding quantification over sorts is called sort logic and denoted L^s. If L^* is an abstract logic, $\sum(L^*)$ is the family of PC-classes of L^* in the sense of [8]. $\Pi(L^*)$ consists of the complements of PC-classes of L^*. $\sum_n(L^*)$ and $\Pi_n(L^*)$ are obtained by iterations of the $\sum$- and Π-operations. $\Delta_n(L^*)$ refers to the intersection of $\sum_n(L^*)$ and $\Pi_n(L^*)$. The families $\sum_n^1(L^*)$, $\Pi_n^1(L^*)$ and $\Delta_n^1(L^*)$ are defined similarly but the PC-definitions are not allowed to introduce new sorts. This ends the sketch.

1.1. Abstract logics

For many-sorted logic we refer to [5]. Types are sets of sorts, relation-symbols and constant-symbols. If L is a type, the class of all structures of type L is denoted $Str(L)$. If $M \in Str(L)$ and K is a type such that $K \subset L$, then $M|_K$ denotes the reduct of M to K. If $x \in L$, then x^M denotes the interpretation of x in M. $|M|$ denotes the union of the universes of M.

A quasilogic is a pair $L^* = \langle Stc^*, \models^* \rangle$ such that

(L1) If $\varphi \in L^*$ (that is $Stc^*(L,\varphi)$), then L is a type and φ is a set called an L^*-sentence,

(L2) If $M \models^* \varphi$ (that is $\models^* (M,\varphi)$), then there is a type L such that $M \in Str(L)$ and $\varphi \in L^*$,

(L3) If $M \models^* \varphi$ and $M \cong N$, then $N \models^* \varphi$.

This definition is somewhat weaker than the definition of a system of logics in [2], and substantially weaker than the definition of a logic in [8].

The quasilogic $L_{\infty\omega}$ is defined as usual. If $Q^1,\dots,Q^n$ are generalized quantifiers, we let $L_{\infty\omega}(Q^1,\dots,Q^n)$ denote the quasilogic which is obtained from $L_{\infty\omega}$ by addition of the new quantifiers $Q^1\dots Q^n$. Second order infinitary logic, which is denoted by $L^{II}_{\infty\omega}$, is obtained from $L_{\infty\omega}$ by addition of quantification over (finitary) relations. The following generalized quantifiers play a special role in this paper:

Härtig-quantifier: $IxyA(x)B(y) \leftrightarrow card(A) = card(B)$, Well-ordering-quantifier: $WxyA(x,y) \leftrightarrow A$ well-orders its domain,

Regularity-quantifier: $RxyA(x,y) \leftrightarrow A$ orders its domain in the type of a regular cardinal,

Henkin-quantifier: $Q_H xyuvA(x,y,u,v) \leftrightarrow \forall f \forall g \exists x \exists y A(x,y,f(x),g(y))$.

Note that our Henkin-quantifier is the dual of the original one.

If L^* is a quasilogic we let L^*_A be the quasilogic the sentences of which are those $\varphi \in L^*$ for which $L \in A$ and $\varphi \in A$, and the semantics of which follows that of L^*. For example, $(L_{\infty\omega})_{H(\kappa)}$ will be $L_{\kappa\omega}$ if the syntax of $L_{\infty\omega}$ is defined in the usual set theoretical way (see e.g. [3]). We denote L^*_{HF} by $L^*_{\omega\omega}$ and in general $L^*_{H(\kappa)}$ by $L^*_{\kappa\omega}$. $L_{\omega\omega}(Q^1,\dots,Q^n)$ and $L^{II}_{\omega\omega}$ are shortend to $L(Q^1,\dots,Q^n)$ and L^{II}. As usual, L_A denotes $(L_{\infty\omega})_A$. For $\omega < \lambda \in A$, we use $L_{A\lambda}$ to denote $(L_{\infty\lambda})_A$. Similarly L_{AG} denotes $(L_{\infty G})_A$. If $\kappa < \lambda$, then $L_{\kappa\lambda}$ does not make much sense, but we redefine it as $L_{\kappa\kappa}$ added with the weak second order quantifiers $\exists X(|X| \leq \alpha \wedge \dots)$ for $\alpha < \lambda$. The obvious set theoretic definition gives $L_{\kappa\lambda} \subset H(\kappa + |\alpha|)$ whenever $\lambda = \aleph_\alpha$.

A class of structures of the same type is called a <u>model class</u> if it is closed under isomorphisms. If $K \subseteq Str(L)$ is a model class, then the model class $Str(L) - K$ is denoted by $\overline{K}$. A model class K is L^*-definable if there are L and $\varphi \in L^*$ such that $K = Mod(\varphi) = \{M \in Str(L) | M \models^* \varphi\}$. We say, as usual, that a quasilogic L^* is a sublogic of another quasilogic L^+, $L^* \leq L^+$, if every L^*-definable model class is L^+-definable. L^* and L^+ are equivalent, $L^* \sim L^+$, if they are soblogics of each other.

An <u>abstract logic</u> is a quasilogic L^* such that

(L4) If L and L' are types such that $L \subseteq L'$, then $L^* \subseteq L'^*$ and for $\varphi \in L^*$, $M \in Str(L')$,

$$M \models^* \varphi \text{ if and only if } M|_L \models^* \varphi.$$

(L5) For every rudimentary set A, type $L \in A$ and $\varphi, \psi \in L^*_A$, there are $\varphi \wedge \psi$ and $\varphi \vee \psi$ in L^*_A such that $Mod(\varphi\wedge\psi) = Mod(\varphi) \cap Mod(\psi)$ and $Mod(\varphi\wedge\psi) = Mod(\varphi) \cap Mod(\psi)$.

(L6) For every rudimentary set A, types $L, L' \in A$ and $\varphi \in L'^*$ there are $\exists c\varphi$ and $\forall c\varphi$ in L^*_A such that if $L' - L$ consists of the constant-symbol c, then

$$Mod(\exists c\varphi) = \{M | \exists N \in Str(L')(N|_L = M \ \& \ N \models^* \varphi)\}$$

$$Mod(\forall c\varphi) = \{M | \forall N \in Str(L')(N|_L = M \rightarrow N \models^* \varphi)\}.$$

(L7) $L_{\omega\omega} \leq L^*_{\omega\omega}$.

It is obvious that $L_{\infty\omega}$, $L_{\infty\omega}(Q^1,\ldots,Q^n)$, $L^{II}_{\infty\omega}$, and their fragments are abstract logics. If the analogue of (L5) and (L6) holds for the negation, we call L^* a Boolean logic.

When we are only interested in the definable model classes of an abstract logic L^*, we sometimes write

$$L^* = \{K \mid \text{the model class } K \text{ is}\ldots\}$$

meaning that an arbitrary model class K is L^*-definable if and only if K is... .

1.2. Sort logic

The class of formulae of infinitary sort logic $L^s_{\infty\omega}$ is obtained if the following formation rule is added to the recursive definition of $L^{II}_{\infty\omega}$-formulae:

> If φ is a formula and s is a sort, then $\exists s\varphi$ and $\forall s\varphi$ are formulae.

To define the semantics of $L^s_{\infty\omega}$ we have to work in the MKM theory of classes or in any other theory in which satisfaction for formulae of set theory is definable. If L is a type, s a sort, $s \notin L$ and $L' = L \cup \{s\}$, then for any $M \in Str(L)$ we define

$$M \models \exists s\varphi \text{ if and only if } \exists N \in Str(L')(N|_L = M \;\&\; N \models \varphi)$$

$$M \models \forall s\varphi \text{ if and only if } \forall N \in Str(L')(N|_L = M \rightarrow N \models \varphi).$$

This defines $L^s_{\infty\omega}$ as an abstract logic. We denote L^s_{HF} by L^s. It appears that sort logic has not been singled out as a logic before, although it has been studied in a semantical form in [10].

Let $\sum_n(L_{\infty\omega})$, $n \geq 1$, be the sublogic of $L^s_{\infty\omega}$ the formulae of which have the form

$$\exists s_1 \forall s_2 \ldots \exists(\forall) s_n \varphi$$

where $s_1,\ldots,s_n$ are sorts and $\varphi \in L^{II}_{\infty\omega}$. Let $\Pi_n(L_{\infty\omega})$ be the sublogic of $L^s_{\infty\omega}$ consisting of formulae of the form

$$\forall s_1 \exists s_2 \ldots \forall(\exists) s_n \varphi$$

where $s_1,\ldots,s_n$ are sorts and $\varphi \in L^{II}_{\infty\omega}$. For each $n < \infty$, the abstract logics $\sum_n(L_{\infty\omega})$ and $\Pi_n(L_{\infty\omega})$ are definable in ZF. Note that $\sum_n(L_{\infty\omega})$ and $\Pi_n(L_{\infty\omega})$ are closed under second order quantifiers.

Let $\Delta_n(L_{\infty\omega})$ be the sublogic of $\sum_n(L_{\infty\omega})$ the formulae of which are equivalent to $\Pi_n(L_{\infty\omega})$-formula. $\Delta_n(L_{\infty\omega})$ is an abstract logic but it does not seem to have such a simple syntax as $\sum_n(L_{\infty\omega})$ and $\Pi_n(L_{\infty\omega})$. In fact the class of $\Delta_n(L_{\infty\omega})$-formulae reflects to a certain extent the properties of the underlying model of set theory and changes when the model is changed. Note that $\Delta_1(L_{\infty\omega})$ is just $\Delta(L_{\infty\omega})$ in the sense of [8]. More generally, $\Delta(\sum_n(L_{\infty\omega}) \cup \Pi_n(L_{\infty\omega})) \sim \Delta_{n+1}(L_{\infty\omega})$. The fragments $\sum_n(L_A)$, $\Pi_n(L_A)$ and $\Delta_n(L_A)$ are defined similarly.

1.3. Extension-operations

We review the definition of the Δ-operation from [8] because the definition naturally leads to both more general and more restricted operations.

If $\varphi \in L^*$, $L' \subset L$ and $M \in Str(L')$, let

$$E(M,\varphi) = \{N \in Str(L) | N|_{L'} = M \ \& \ N \models^* \varphi\}.$$

A model class K of type L' is *$\sum$-defined* by φ if $L - L'$ is finite and

$$K = \{M \in Str(L') | E(M,\varphi) \neq \emptyset\}.$$

K is *Π-defined* by φ is $L - L'$ is finite and

$$K = \{M \in Str(L') | \forall N \in Str(L)(N|_{L'} = M \rightarrow N \models^* \varphi)\}.$$

K is *$\sum(L^*)$-definable* (*$\Pi(L^*)$-definable*) if it is $\sum$-defined (Π-defined) by some $\varphi \in L^*$. Finally, K is *$\Delta(L^*)$-definable* if it is both $\sum(L^*)$- and $\Pi(L^*)$-definable. $\Delta(L^*)$ gives rise to the semantics of an abstract logic, but to find a syntax for that logic seems as difficult as finding a syntax for $\Delta_n(L_{\infty\omega})$. Note however, that in special cases $\Delta(L^*)$ has a beautiful syntax (see e.g. [8] §4). To be specific let us agree that $\Delta(L^*)$ is the abstract logic the sentences of which are 4-tuples $\langle\varphi,L,\psi,L'\rangle$ where $\varphi \in L^*$, $\psi \in L'^*$ and φ $\sum$-defines the same model class as ψ Π-defines. One of our results will imply that $\Delta(L^*)$ hardly has a less artificial syntax. The families $\sum(L^*)$ and $\Pi(L^*)$ can also be made into abstract logics if, for example, a model class which is $\sum$-defined by $\varphi \in L^*$ is associated an artificial sentence $\langle\varphi,L\rangle$. Note that this syntax for $\sum(L^*)$ depends only on the syntax of L^* and not on the underlying set theory. We say that L^* is *unbounded* if W is $\Delta(L^*)$-definable.

By induction on $n < \omega$ we define $\sum_{n+1}(L^*) = \sum(\Pi_n(L^*))$, $\Pi_{n+1}(L^*) = \Pi(\sum_n(L^*))$ and $\Delta_{n+1}(L^*) = \Delta(\sum_n(L^*) \cup \Pi_n(L^*))$. Now we have two interpretations for $\sum_n(L_{\infty\omega})$, either as a $\sum_n$-extension of $L_{\infty\omega}$ or as a fragment of $L^s_{\infty\omega}$, but it is obvious that the two interpretations are essentially equivalent. All standard logics are sublogics of $\Delta_3(L_{\infty\omega})$ and therefore the operations Δ_n, $n > 2$, are relatively uninteresting, apart from their relation to sort logic.

If the above definition of $\sum(L^*)$ and $\Pi(L^*)$ is modified by requiring that

L has no new sorts over and above those of L', the essentially weaker notions of $\Pi_1^1(L^*)$- and $\Sigma_1^1(L^*)$-definability are obtained. Let $\Delta_1^1(L^*)$ be defined as $\Delta(L^*)$ above. More generally we define $\Sigma_{n+1}^1(L^*) = \Sigma_1^1(\Pi_n^1(L^*))$, $\Pi_{n+1}^1(L^*) = \Pi_1^1(\Sigma_n^1(L^*))$ and $\Delta_{n+1}^1(L^*) = \Delta_1^1(\Sigma_n^1(L^*) \cup \Pi_n^1(L^*))$.

1.4. Set theory

Our set theoretical notation follows mostly that of [4]. However, we write R_α for the α'th level of the ramified hierarchy. Cd(x) is the predicate "x is a cardinal number (initial ordinal)", Rg(x) is the predicate "x is a regular cardinal" and Pw(x,y) is the predicate $x = P(y)$, where P is the power-set operation. Card(x) is the least ordinal which has the same power as x. HC(x) = max(card(TC(x)), $\aleph_0$). We sometimes use $\aleph_n$ as a predicate, meaning the predicate "$x \in \aleph_n$", of course. $P_\kappa(y)$ is the set $\{x \subseteq y \mid |x| < \kappa\}$ and $Pw_\kappa(x,y)$ is the predicate "$y = P_\kappa(x)$". The sets of $\Sigma_n(P)$- and $\Pi_n(P)$-formulae are defined as usual. A predicate is $\Sigma_n(P)$ $(\Pi_n(P))$ w.p.i. (= with parameters in) A if it is definable with a $\Sigma_n(P)$ $(\Pi_n(P))$-formula w.p.i. A. A predicate is $\Delta_n(P)$ w.p.i. A if it is both $\Sigma_n(P)$ and $\Pi_n(P)$ w.p.i. A. An ordinal α is $\Sigma_n(P)$ $(\Pi_n(P), \Delta_n(P))$-definable w.p.i. A if the predicate "$x \in \alpha$" is.

§ 2. The basic representations

In this chapter we define the symbiosis of a logic and a predicate of set theory, and prove the main result about symbiosis (Theorem 2.4). The chapter ends with some remarks on the non-absolute nature of the Δ-operation.

By its very definition an abstract logic determines two predicates of set theory: Stc and $\models$. It is convenient for our purposes to establish a converse relation, that is, associate every predicate with a generalized quantifier.

Suppose $P = P(x_1,\ldots,x_n)$ is a predicate of set theory. Let

$$K[P] = \{M \mid M \cong \langle M,\in,a_1,\ldots,a_n\rangle \text{ such that } M \text{ is transitive and } P(a_1,\ldots,a_n)\}.$$

Let Q_P be the generalized quantifier associated with K[P] and

$$L[P] = L_{\infty\omega}(Q_P).$$

Lemma 2.1.

(1) $\models_{L[P]}$ *is* $\Delta_1(P)$.

(2) K[P] *is* $\Delta(L[P]_{\omega\omega})$*-definable.*

Proof. An elaboration of the proof of the well-known fact that $\models_{L_{\infty\omega}}$ is Δ_1 (see e.g. [3] p. 83) gives (1). (2) is trivial as K[P] is even $L[P]_{\omega\omega}$-definable. □

The above lemma shows that L[P] and P are in a sense definable from each other. We take a slight weakening of this property as the definition of symbiosis.

Definition 2.2. *Suppose* L* *is an abstract logic,* P *a predicate of set theory and* A *a transitive class. We say that* L* *and* P *are* symbiotic on A *if the following two conditions hold:*

(S1) If $\varphi \in L^*$, *then* $\mathrm{Mod}(\varphi)$ *is* $\Delta_1(P)$ *w.p.i.* $\{\varphi, L\}$

(S2) K[P] *is* $\Delta(L^*_A)$*-definable.*

The logic L* *is* symbiotic on A *if there is a predicate* $P \neq \emptyset$ *such that* L* *and* P *are symbiotic on* A. *If* A = HF, *we omit the clause "on* A".

Note that symbiosis on A implies symbiosis on any transitive $A' \supseteq A$.

Examples 2.3. *The following pairs are symbiotic on* HF *for any rudimentary* A:

(1) $L[P]_A$ *and* P,

(2) LQ_A *and* Q, *if* Q *is a generalized quantifier and* LQ *is unbounded,*

(3) LW_A *and* On,

(4) LI_A *and* Cd,

(5) LR_A *and* Rg,

(6) L^{II}_A *and* Pw,

(7) $L_{\omega\omega_n}$ *and* Pw_{ω_n}.

The following pairs are symbiotic on $H(\kappa)$ *for any rundimentary* $A \supseteq H(\kappa)$ *and* $\kappa = \lambda^+$, $\lambda \geq \omega$:

(8) $L_{A\kappa}$ *and* Pw_κ,

(9) L_{AG} *and* On.

Proof. The proof of (S1) is similar to the proof of 2.1(1) in any of (1)-(9). As a typical example of the proof of (S2), let us consider (4). Recall that W is $\sum(LI)$. Now

$$\langle M,E,a\rangle \in K[d] \leftrightarrow$$

$$\langle M,E,a\rangle \models Wxy(xEy) \wedge \forall xy(yEx \wedge xEa \rightarrow yEa) \wedge$$

$$\wedge \forall xyz(zEy \wedge yEx \wedge xEa \rightarrow zEx) \wedge$$

$$\wedge \forall z(zEa \rightarrow \neg Ixy(xEz)(yEa)) \wedge$$

$$\wedge \forall xy(\forall z(zEx \leftrightarrow zEy) \rightarrow x=y). \quad □$$

Note that, if L^* is symbiotic with P ($\neq \emptyset$) on A then L^*_A is unbounded, in fact, as $K[P]$ is $\Delta(L^*_A)$-definable, it suffices to observe that

$$Wxy(xAy) \leftrightarrow \text{there are } M,\ E \text{ and } a_1,\ldots,a_n \text{ such that}$$
$$\langle M,E,a_1,\ldots,a_n\rangle \in K[P] \text{ and}$$
$$\langle M,E,a_1,\ldots,a_n\rangle \models \forall xy(xAy \rightarrow xEy).$$

Conversely, it is by no means the case that every unbounded logic is symbiotic on some A. We shall indicate later why the logic $L(W,Q_1,\ldots,Q_n,\ldots)_{n<\omega}$ is not symbiotic.

The next theorem is the basic result about symbiosis and about relation between logic and set theory in general. It was proved in the author's Ph.D. thesis [13] and appeared later, but independently in [10]. We repeat the proof here for completeness. The proof clearly owes a great deal to [2].

<u>Theorem 2.4.</u> *Suppose* A *and* $A_o \subseteq A$ *are transitive classes,* P *a predicate, and* L^* *an abstract logic extending* L_A *and symbiotic with* P *on* A_o. *Then the following are equivalent for any model class* K *of type* $L \in A$:

(1) K *is* $\sum(L^*_A)$*-definable,*
(2) K *is* $\sum_1(P)$ *w.p.i.* A.

<u>Proof</u>. (1) → (2): Suppose $\varphi \in L^*_A$ $\sum$-defines K. Let $L_o \in A$ such that $\varphi \in L^*_o$. Now

$$K = \{M \in Str(L) \mid \exists N \in Str(L_o)(N|_L = M \ \&\ N \models^* \varphi)\}.$$

Hence K is $\sum_1(P)$ w.p.i. A.

(2) → (1): Suppose

$$\forall x(x \in K \leftrightarrow \varphi(x,a))$$

where $\varphi(x,y)$ is $\sum_1(P)$ and $a \in A$. Let $a' = TC(\{a\})$ $(\in A)$. Let E be a binary predicate symbol not in L and i a sort not in L. For any formula $\psi(x_1,\ldots,x_n)$ of set theory let $[\psi(x_1,\ldots,x_n)]_E$ be obtained from $\psi(x_1,\ldots,x_n)$ by replacing atomic formulae $t\epsilon u$ by tEu (i.e. $E(t,u)$) and changing all bound variables to variables of sort i. Let L_o be L extended with E and i. As L^* is symbiotic, there is an $L_1 \supseteq L_o$ and $\theta \in L^*_{1A}$ such that θ $\sum$-defines the class of well-founded extensional structures $\langle M_i,E\rangle$. By (S2) there is an $L_2 \supseteq L_1$ and $\eta \in L^*_{2A}$ such that η $\sum$-defines the class $K[P]$. Suppose $P = P(x_1,\ldots,x_n)$ and $c_1,\ldots,c_n$ are the constant symbols in the type of $K[P]$. Let ζ be the L^*_{2A}-sentence obtained from

$$[P(c_1,\ldots,c_n)]_E \leftrightarrow \eta(c_1,\ldots,c_n)$$

by universally quantifying over $c_1,\dots,c_n$ using variables of sort i. Let $\varphi_b(x)$ be the formula $\forall y(yEx \leftrightarrow \bigvee\!\!\!\bigvee_{c\in b} \varphi_c(x))$ for every element b of a'. Let $\psi(x)$ be the L^*_0-sentence which says, using E instead of ε, that x is a structure of type L in which any atomic $R(x_1,\dots,x_n)$ is satisfied by elements $a_1,\dots,a_m$ if and only if $R(a_1,\dots,a_m)$ is true. Finally, let ξ be the conjunction of

$$\theta \ \& \ \zeta$$

$$\bigwedge\!\!\!\bigwedge_{b\in a} \exists x\varphi_b(x) \ \& \ \varphi_a(c_1)$$

$$\exists x^i([\varphi(x^i,c_1]_E \ \& \ \psi(x^i)).$$

Now $\xi \in L^*_{2A}$ and we prove that it $\sum$-defines K. Suppose at first that $M \in K$. Hence $\varphi(M,a)$ is true. Let N be a transitive set which reflects $\varphi(M,a)$ and P. M can be expanded to a model of ξ by letting N serve as the universe of sort i elements. For the converse, suppose $N \models \xi$. Let $N \cong M = \langle\dots,M_i,\in,\dots\rangle$ such that M_i is a transitive set. As $M \models \zeta$, M_i reflects P. Clearly c_1 is interpreted as a in M. Let $A \in M_i$ such that

$$M \models [\varphi(A,c_1)]_E \ \& \ \psi(A).$$

Then $A \in Str(L)$ and $\varphi(A,a)$, whence $A \in K$. As K is closed under isomorphism, $M|_L \in K$. ◘

<u>Corollary 2.5.</u> *For any rudimentary class* A:

(1) $\Delta(LW_A) = \{K|$ *the model class* K *is* Δ_1 *w.p.i.* A$\}$,
$= \Delta(L_{AG})$

(2) $\Delta(LI_A) = \{K|$ *the model class* K *is* $\Delta_1(Cd)$ *w.p.i.* A$\}$,

(3) $\Delta(L^{II}_A) = \{K|$ *the model class* K *is* Δ_2 *w.p.i.* A$\}$.

(4) $\Delta(L_{\omega\omega_n}) = \{K|$ *the model class* K *is* $\Delta_1(Pw_{\omega_n})\}$,

If $A \supseteq H(\kappa)$, $\kappa = \lambda^+$, $\lambda \geq \omega$, *then:*

(5) $\Delta(L_{A\kappa}) = \{K|$ *the model class* K *is* $\Delta_1(Pw_\kappa)$ *w.p.i.* A$\}$.

In [11] a predicate P(x) is called <u>local</u> if it is of the form $\exists\alpha(R_\alpha \models \varphi(x))$ for some formula $\varphi(x)$ of set theory, and a proof is sketched to the effect that a predicate is local if and only if it is equivalent to a $\sum_2$-predicate. Combining this with Theorem 2.4 yields:

<u>Corollary 2.6.</u> *A model class is* $\sum(L^{II}_A)$*-definable if and only if it is defined by a local property w.p.i.* A.

Theorem 2.4 can be immediately iterated to yield a result about $\sum_n$-defina-

bility. A form of the following corollary was first proved by J. Oikkonen in [10] with a different proof.

Corollary 2.7. *Suppose* A *and* $A_0 \subseteq A$ *are transitive classes,* P *a predicate, and* L_A^* *a Boolean logic extending* L_A *and symbiotic with* P *on* A_0. *Then the following are equivalent for any model class* K *of type* $L \in A$ *and for any* $n < \omega$:

(1) K *is* $\Sigma_{n+1}(L_A^*)$*-definable,*

(2) K *is* $\Sigma_{n+1}(P)$ *w.p.i.* A.

Proof. We use induction on n. If $n = 0$, the claim follows from 2.4. Suppose then K is $\sum_{n+1}(L_A^*)$-definable and $n > 0$. Let $\varphi \in \Pi_n(L_A^*)$ $\sum$-define K. By induction hypothesis $\mathrm{Mod}(\varphi)$ is $\Pi_n(P)$ w.p.i. A. Now

$$K = \{M \in \mathrm{Str}(L) \mid \exists N \in \mathrm{Mod}(\varphi)(N|_L = M \wedge N \models \varphi)\},$$

and therefore K is $\sum_{n+1}(P)$ w.p.i. A.

For the converse, suppose (2) holds. Let S be a $\Pi_n(P)$-predicate such that K is $\sum_1(S)$ w.p.i. A. By 2.4 K is $\sum$-defined by some $\varphi \in L[S]_A$. By 2.1 $\mathrm{Mod}(\varphi)$ is $\Delta_1(S)$ w.p.i. A, and therefore $\Delta_{n+1}(L_A^*)$-definable. Hence K is $\Sigma_{n+1}(L_A^*)$-definable. ◻

Corollary 2.8. *For* $n > 0$ *and for any rudimentary class* A:

(1) $\Delta_n(LW_A) = \{K \mid$ *the model class* K *is* Δ_n *w.p.i.* $A\}$,

(2) $\Delta_n(L_A^{II}) = \{K \mid$ *the model class* K *is* Δ_{n+1} *w.p.i.* $A\}$.

Note that $K[Pw]$ is $\Pi(L_{\omega\omega})$-definable, whence $L[Pw] \leq \Delta_2(L_{\omega\omega})$ and therefore $\Delta(L_A^{II}) \leq \Delta_2(L_A)$. On the other hand $\Delta_2(L_A) \leq \Delta_2(LW_A) \leq \Delta(L_A^{II})$. Hence in fact $\Delta(L_A^{II}) \sim \Delta_2(L_A)$, and therefore $\Delta_n(L_A^{II}) \sim \Delta_{n+1}(L_A)$ for all $n > 0$. If this is combined with 2.8, the following obtains:

Corollary 2.9. *For* $n > 1$,

$$\Delta_n(L_A) = \{K \mid \textit{the model class } K \textit{ is } \Delta_n \textit{ w.p.i. } A\}.$$

Therefore in MKM:

$$L_A^s = \{K \mid \textit{the model class } K \textit{ is definable in set theory w.p.i. } A\}.$$

The second part of the above corollary was stated on page 174 of [8].

As $\Delta_1(L_{\omega\omega})$ is just the usual first order logic $L_{\omega\omega}$, and $\Delta_2(L_{\omega\omega})$ is $\Delta(L^{II})$, that is, essentially second order logic, it would be tempting to conjecture that $\Delta_3(L_{\omega\omega})$ is essentially third order logic. This is not the case, however. By familiar methods (see e.g. [9]) one can prove that for any analytical (this can be improved, see [9]) ordinal α the α'th order logic in Δ-equivalent to second order logic. It seems plausible to put $\Delta_3(L_A)$ above the whole notion of higher order logic and consider it rather as a fragment of a quite new powerful logic, sort logic. Similarly it seems implausible to call $\sum$ a second order quantifier, or even a generalized second order quantifier, as $\sum$ leads to something far beyond second and higher order logic, viz. set theory. We return to the problematics of second order logic in the next chapter.

The Δ-operation can be used to give a very near characterization of symbiosis:

Proposition 2.10. *Suppose* L_A^* *is a Boolean logic extending* L_A *and* P *a predicate. Then* $\Delta(L_A^*) \sim \Delta(L[P]_A)$ *if and only if (S2) and* $(S1)_A$*: If* $\phi \in L_A^*$*, then* $\mathrm{Mod}(\phi)$ *is* $\Delta_1(P)$ *w.p.i.* A.

Proof. Suppose at first that $(S1)_A$ and (S2) hold. By Theorem 2.1 every $L[P]_A$-definable model class is $\Delta_1(P)$ w.p.i. A, whence by 2.4, $L[P]_A \leq \Delta(L_A^*)$. Therefore $\Delta(L[P]_A) \leq \Delta(L_A^*)$. On the other hand, if K is L_A^*-definable, then by $(S1)_A$ K is $\Delta_1(P)$ w.p.i. A, whence by 2.4 K is $\Delta(L[P]_A)$-definable. Hence $\Delta(L_A^*) \sim \Delta(L[P]_A)$. The converse is immediate in view of 2.4. ◻

We can use 2.10 to show that the logic $L^* = L(W,Q_1,\ldots,Q_n,\ldots)_{n<\omega}$ is not symbiotic on any A. Indeed, suppose L_A^* is symbiotic with P on A. As $L_A^* = L_{HF}^*$, we may assume A = HF. By 2.10, $\Delta(L^*) \sim \Delta(L[P]_{\omega\omega})$. Let $n < \omega$ such that K[P] is Δ-definable in $L^+ = L(W,Q_1,\ldots,Q_n)$. Now $\Delta(L^*) \sim \Delta(L[P]_{\omega\omega}) \sim \Delta(L^+)$, a contradiction.

The existence on non-symbiotic unbounded logics may seem to limit the applicability of Theorem 2.4. However, if L^* is the union (in the obvious sense) of the logics L^{+n} $(n < \omega)$, where L^{+n} is symbiotic with P_n on A_n, then for $A = \bigcup_n A_n$,

$$\Delta(L_A^*) = \{K \mid \text{the model class } K \text{ is } \Delta_1(P_n) \text{ w.p.i. } A_n \text{ for some } n < \omega\}.$$

Thus the range of Theorem 2.4 extends to many non-symbiotic logics. For example:

$$\begin{aligned}\Delta(L(W,Q_1,\ldots,Q_n,\ldots)_{n<\omega}) &= \{K \mid \text{the model class } K \text{ is } \Delta_1(\aleph_1,\ldots,\aleph_n) \text{ for some } n < \omega\}\\ &= \{K \mid \text{the model class } K \text{ is } \Delta_1 \text{ w.p.i. } \{\aleph_1,\ldots,\aleph_n,\ldots\}\}.\end{aligned}$$

Note however, that $L_{\omega_1\omega}(W,Q_1,\ldots,Q_n,\ldots)_{n<\omega}$ is symbiotic on HC.

In the next results we investigate the absoluteness of symbiotic logics. Let us consider the following three properties of an abstract logic L^* and a predicate P:

(A1) Stc^* is $\sum_1(P)$ and $\varphi \vee \psi$, $\varphi \wedge \psi$, $\exists c\varphi$ and $\forall c\varphi$ in (L5) and (L6) can be found with $\sum_1(P)$-functions.

(A2) There is a $\Delta_1(P)$-predicate S such that if $\varphi \in L^*$, then

$$\forall M \in Str(L)(M \models^* \varphi \leftrightarrow S(M,\varphi)).$$

The conditions (A1) and (A2) together form a natural notion of <u>P-absoluteness</u> of L^* generalizing the notion of an absolute logic in [2]. Note that (A2) → (S1). The following lemma is obvious:

<u>Lemma 2.11.</u>

(1) If L^* *and* P *are symbiotic on* A *and* $L^+ \le \Delta(L^*_A)$, *then* L^+ *and* P *satisfy* $(S1)_A$.

(2) If L^*_A *and* P *satisfy (A2) and* $L^+ \le \Delta(L^*_A)$, *then* L^+ *and* P *satisfy* $(S1)_A$.

The next result generalizes a theorem by Burgess (Theorem 2.2 in [8]) which says that no unbounded absolute logic is Δ-closed. The proof remains almost the same.

<u>Theorem 2.12.</u> *Suppose* L^* *is a Boolean logic symbiotic with* P *on* A *and* $L^+ \sim \Delta(L^*_A)$. *Then* L^+ *and* P *satisfy* $(S1)_A$ *but not (A2).*

<u>Proof</u>. Suppose S is a $\Delta_1(P)$-predicate such that if $\varphi \in L^+$, then $\forall M \in Str(L)(M \models^+ \varphi \leftrightarrow S(M,\varphi))$. Let

$$K = \{M | M \cong \langle TC(\{a\}),\in\rangle \text{ for some } a \text{ such that } \neg S(M,a)\}.$$

K is clearly $\Delta_1(P)$. By 2.4 there is a $\varphi \in L^+$ such that $K = Mod(\varphi)$. Let $M = \langle TC(\{\varphi\}),\in\rangle$. Then $M \in K \leftrightarrow M \models^+ \varphi \leftrightarrow \neg S(M,\varphi) \leftrightarrow M \notin K$, a contradiction. ◻

<u>Corollary 2.13.</u> *Suppose* L^* *is a Boolean logic,* L^* *and* P *are symbiotic on* A *and they satisfy (A2). Then* L^*_A *is not Δ-closed.*

<u>Corollary 2.14.</u> *The following logics are not Δ-closed:*

(1) $L[P]_A$ *where* P *is a predicate of set theory,*

(2) LQ_A *where* Q *is a generalized quantifier such that* LQ *is unbounded.*

(3) LW_A, LI_A, LR_A, L^{II}_A, $L_{\omega_1\omega_1}$, $L_{\omega_1 G}$.

Hence, if L^*_A *is a symbiotic logic extending* L_A, *there are no* $Q^1,\ldots,Q^n$ *such that*

$$\Delta(L^*_A) \sim LQ^1\ldots Q^n_A.$$

The following theorem gives another aspect of the failure of syntactical methods in constructing Δ-extensions. Recall the definition of $Stc_{\Delta(L^*)}$ in §1.

Theorem 2.15. *Suppose* A *and* $A_o \subseteq A$ *are transitive classes,* P *a predicate and* L_A^* *an abstract logic such that there is a* $\sum_1(P)$*-function embedding* L_A *into* L_A^* *and* L_A^* *is symbiotic with* P *on* A_o*. Then the predicate* $Stc_{\Delta(L_A^*)}$ *is* $\Pi_2(P)$ *but not* $\sum_2(P)$*. Therefore* $\Delta(L_A^*)$ *and* P *do not satisfy (A1).*

Proof. By definition

$$Stc_{\Delta(L_A^*)}(L,x) \leftrightarrow \exists\phi\psi L_0 L_1 \in TC(x)$$

$$(Stc^*(L_o,\varphi) \ \& \ Stc^*(L_1,\psi) \ \& \ x = \langle\varphi,L_o,\psi,L_1\rangle \ \&$$

$$\forall M \in Str(L)(\exists N \in Str(L_o)(N|_L = M \ \& \ N \models^* \varphi) \leftrightarrow$$

$$(\forall N \in Str(L_1)(N|_L = M \to N \models^* \psi)))).$$

This proves that $Stc_{\Delta(L_A^*)}$ is $\Pi_2(P)$. To prove that $Stc_{\Delta(L_A^*)}$ is not $\sum_2(P)$, let $R(x,y)$ be a $\Pi_2(P)$-predicate which is not $\sum_2(P)$. We construct $\sum_1(P)$-functions f and g such that

$$(*) \qquad \forall x\forall y(R(x,y) \leftrightarrow Stc_{\Delta(L_A^*)}(f(x,y),g(x,y))).$$

From this it follows that $Stc_{\Delta(L_A^*)}$ is not $\sum_2(P)$.

Let $T(x,y,z)$ be a $\sum_1(P)$-predicate such that

$$\forall x\forall y(R(x,y) \leftrightarrow \forall z T(x,y,z)).$$

$K[T]$ is $\sum_1(P)$ whence by 2.4 there is an L_A^*-sentence $\varphi(c_1,c_2,c_3)$ which $\sum$-defines $K[T]$. For any x let $\psi_x(y,E)$ be the $L_{\infty\omega}$-formula (see the proof of 2.4)

$$\bigwedge\!\!\!\bigwedge_{a\in TC(\{x\})} \exists z\varphi_a(z) \ \& \ \varphi_x(y).$$

Let $\theta(E)$ be an L_A^*-sentence which $\sum$-defines the class of models $\langle dom(E),E,c_3\rangle$ where E is well-founded and extensional. For any x and y let η_{xy} be the sentence

$$(\theta(E) \ \& \ \psi_x(c_1,E) \ \& \ \psi_y(c_2,E)) \to \varphi(c_1,c_2,c_3).$$

By (A1) we may assume there is a type L'_{xy} such that $\eta_{xy} \in L'^*_{xyA}$ and the predicates $z = \eta_{xy}$ and $z = L'_{xy}$ are $\sum_1(P)$. Let L_{xy} be the subtype of L'_{xy} associated with c_3,c_2,c_1 and E. Let ξ be an arbitrary valid L^*-sentence. We define

$$f(x,y) = L_{xy}$$

$$g(x,y) = \langle \eta_{xy}, L'_{xy}, \xi, L \rangle.$$

Now (*) holds as is not too difficult to see. ◻

It follows, for example, that there is no Σ_3-formula which decides whether a given $L^{II}_{\infty\omega}$-formula Σ-defines the same model class as another given $L^{II}_{\infty\omega}$-sentence Π-defines. So, although $L^{s}_{\infty\omega}$ has a primitive recursive syntax, it seems unlikely that any similar syntax can be found for its fragment $\Delta(L^{II}_{\infty\omega})$ or for $\Delta_2(L_A)$.

We end this chapter with some remarks on decision problems of symbiotic logics. For simplicity we only consider logics of the form $L^*_{\omega\omega}$. The <u>decision problem</u> of $L^*_{\omega\omega}$ is the set $Val(L^*_{\omega\omega}) = \{\varphi \in HF | \varphi \in L^*_{\omega\omega}$ and φ is valid$\}$. It is known (see [13] and [12]) that $Val(L^{II}_{\omega\omega})$ is the complete Π_2-subset of HF. More generally, if L^* is symbiotic with P and L^* is sufficiently syntactic (e.g. $L^* = LQ$ for some Q), then $Val(L^*_{\omega\omega})$ is the complete $\Pi_1(P)$-subset of HF. A proof of this can be found in [13]. For results about $Val(LI_{\omega\omega})$ and $Val(LR_{\omega\omega})$ see [14].

§ 3. <u>Flat definability and second order logic</u>

In this chapter we construct the part of set theory which coincides with second order logic in the same way as the whole set theory coincides with sort logic.

<u>Definition 3.1.</u> *Quantifiers of the form*

(1) $\exists x(HC(x) \leq HC(y_1 \cup \ldots \cup y_n) \;\&\; \varphi(x,y_1,\ldots,y_n))$

(2) $\forall x(HC(x) \leq HC(y_1 \cup \ldots \cup y_n) \rightarrow \varphi(x,y_1,\ldots,y_n))$

are called <u>flat</u> <u>quantifiers</u>. The set of <u>flat</u> formulae of set theory is the smallest set containing Σ_0-formulae and closed under $\&, \vee, \neg$ *and flat quantification.*

The $\Sigma^b_n(P)$- and $\Pi^b_n(P)$-formulae are defined by induction on n as follows: $\Sigma^b_0(P)$- and $\Pi^b_0(P)$-formulae are just the $\Sigma_0(P)$-formulae. $\Sigma^b_{n+1}(P)$-formulae are formulae of the form (1) where $\varphi(x,y_1,\ldots,y_n)$ is $\Pi^b_n(P)$. $\Pi^b_{n+1}(P)$-formulae are formulae of the form (2) where $\varphi(x,y_1,\ldots,y_n)$ is $\Sigma^b_n(P)$. $\Sigma^b_n(P)^{ZFC}$- and $\Pi^b_n(P)^{ZFC}$-formulae are defined as usual.

It is easy to see that the set of $\Sigma^b_n(P)^{ZFC}$-formulae is closed under $\&, \vee, \neg, \exists x\in y, \forall x \in y$ and (1) above. Note that by Levy's theorem ([4] p. 104) every Σ_1-formula is Σ^{bZFC}_1.

The whole point of flat formulae is the following reflection principle:

Lemma 3.2. *Suppose* $\varphi(x_1,\ldots,x_n)$ *is a flat formula and* a *an arbitrary set. Then there is a transitive set* M *such that* $a \in M$, $HC(M) = HC(a)$ *and* M *reflects* $\varphi(x_1,\ldots,x_n)$.

Proof. By the usual reflection principle ([4] p. 99) there is a transitive set N containing a such that N reflects $\varphi(x_1,\ldots,x_n)$. For any subformula $\psi(y_1,\ldots,y_m)$ of $\varphi(x_1,\ldots,x_n)$ and for any $b_2,\ldots,b_m \in N$ let $f = f_{\psi(y_1,\ldots,y_m)}(b_2,\ldots,b_m) \in N$ such that $HC(f) \leq HC(b_2 \cup \ldots \cup b_m)$ and

$$N \models \exists y_1(HC(y_1) \leq HC(b_2,\ldots,b_m) \ \& \ \psi(y_1,b_2,\ldots,b_m)) \to \psi(f,b_2,\ldots,b_m).$$

Choose M to be the smallest transitive set containing a and closed under the functions $f_{\psi(y_1,\ldots,y_m)}$, where $\psi(y_1,\ldots,y_m)$ runs through the subformulae of $\varphi(x_1,\ldots,x_n)$. □

The above lemma shows, among other things, that every flat formula is Δ_2^{ZFC} (using Theorem 3.7.2 of [4]).

Definition 3.3. *Suppose* L* *is an abstract logic,* A *a transitive class and* P *a predicate of set theory.* L* *and* P *are* <u>*strongly symbiotic on A*</u> *if the following two conditions are satisfied*

(SS1) If $\varphi \in L^*$, *then* $Mod(\varphi)$ *is* $\Delta_1^b(P)$ *w.p.i.* $\{\varphi, L\}$.

(SS2) $K[P]$ *is* $\Delta_1^1(L_A^*)$.

Strong symbiosis is harder to come by than symbiosis. For example, W is not Δ_1^1(LI)-definable (essentially because in countable domains I is redudent and Theorem 7.3 of [3] can be used), whence LI is not strongly symbiotic. This failure can be regarded as an indication of the incompleteness of the definition of LI, rather than as a characteristic property of LI. The situation is different with second order logic which seems to resist strong symbiosis in an essential way, as we shall prove in a moment.

Examples 3.4. *The following pairs are strongly symbiotic on* HF *for any rudimentary* A:

(1) $L[P]_A$ *and* P,

(2) $L_A(W,Q)$ *and* Q, *if* Q *is any generalized quantifier,*

(3) $L_A(W)$ *and* On,

(4) $L_A(W,I)$ *and* Cd,

(5) $L_A(W,R)$ *and* Rg.

Theorem 3.5. *Suppose* $A \subseteq HC$ *and* $A_0 \subseteq A$ *are transitive sets,* P *a predicate, and* L_A^* *an abstract logic extending* L_A *and strongly symbiotic with*

P *on* A_o. *Then the following are equivalent for any model class* K *of type* $L \in A$:

(1) K *is* $\sum_1^1(L_A^*)$*-definable,*

(2) K *is* $\sum_1^b(P)$ *w.p.i.* A.

Proof. We follow the proof of 2.4. The implication (1) → (2) is obvious. For (2) → (1), suppose $\varphi(x,y)$ is $\sum_1^b(P)$, $a \in A$ and

$$\forall x(x \in K \leftrightarrow \varphi(x,a)).$$

Let $a' = TC(\{a\})$. Let μ be the conjunction of ξ (as it is defined in the proof of 2.4) and the first order sentence which says that there is a bijection which maps all elements of the sorts in L_3 one-one to elements of the sorts in L. Using Lemma 3.2 one can prove that μ still $\sum$-defines K. But every model of μ has the same power as its L-reduct. Hence the new universes introduced by L_3 can be dispensed with in favour of new predicates, and therefore μ can be converted into a $\Delta_1^1(L_A^*)$-definition of K. □

The proof of Corollary 2.7 carries over immediately and we have:

Corollary 3.6. *Suppose* $A \subseteq HC$ *and* $A_o \subseteq A$ *are transitive sets,* P *a predicate, and* L_A^* *a Boolean logic extending* L_A *and strongly symbiotic with* P *on* A_o. *Then the following are equivalent for any model class* K *of type* $L \in A$ *and for any* $n < \omega$:

(1) K *is* $\sum_{n+1}^1(L_A^*)$*-definable,*

(2) K *is* $\sum_{n+1}^b(P)$ *w.p.i.* A.

Corollary 3.7. *For* $A \subseteq H(\omega_1)$:

(1) $\Delta_n^1(L_A(W)) = \{K|$ *the model class* K *is* Δ_n^b *w.p.i.* A$\}$,

(2) $\Delta_n^1(L_A(W,I)) = \{K|$ *the model class* K *is* $\Delta_n^b(Cd)$ *w.p.i.* A$\}$.

The following corollary is proved mutatis mutandis as Proposition 2.10:

Corollary 3.8. *Suppose* L_A^* *is a Boolean logic extending* L_A *and* P *a predicate. Then* $\Delta_1^1(L_A^*) \sim \Delta_1^1(L[P]_A)$ *if and only if (SS2) and* $(SS1)_A$: *If* $\phi \in L_A^*$, *then* $Mod(\phi)$ *is* $\Delta_1^b(P)$ *w.p.i.* A.

It follows that second order logic is not strongly symbiotic, because there is no Q such that L^{II} $(\sim \Delta_1^1(L^{II})) \sim \Delta_1^1(LQ)$. Second order logic is rather the closure of first order logic under the $\sum_1^1$-operation. More exactly, let us define for any abstract logic L^*:

$$\Delta_{(\omega)}^1(L^*) = \{K| \text{ the model class } K \text{ is } \Delta_n^1(L^*)\text{-definable for some } n < \omega\}.$$

Clearly, $L^{II} = \Delta^1_{(\omega)}(L_{\omega\omega}) = \Delta^1_{(\omega)}(LW) = \Delta^1_{(\omega)}(LI)$. Note that $\Delta^1_{(\omega)}(L^*)$ satisfies the single-sorted interpolation theorem. The following characterization of $\Delta^1_{(\omega)}(L^*)$ and second order logic follows from 3.6:

<u>Corollary 3.9.</u> *Suppose* L^*_A *is a Boolean logic extending* L_A *and* P *a predicate such that* L^*_A *and* P *are strongly symbiotic on* $A \subseteq H(\omega_1)$. *Then the following hold:*

(1) $\Delta^1_{(\omega)}(L^*_A) = \{K |$ *the model class* K *is definable with a flat formula of the language* $\{\varepsilon, P\}$ *w.p.i.* A$\}$.

(2) $L^{II}_A = \{K |$ *the model class* K *is definable with a flat formula w.p.i.* A$\}$.

To sum up, second order definability corresponds to flat definability in set theory, implicit second order (that is $\Delta(L^{II})$-) definability corresponds to Δ_2-definability in set theory, and finally, definability in sort logic corresponds to definability in set theory. Recall that by Theorem 3.7 of [2], first order definability corresponds to Δ^{KP}_1-definability.

It is well-known (see e.g. [9]) that the Π^1_1-part of second order logic has already the whole implicit strength of second order logic. Another way of saying the same is $L^{II} \leq \Delta(LQ_H)$, because Q_H is Π^1_1-definable (see [7]). This fact has the following more general analogue:

<u>Proposition 3.10.</u> *Suppose* $A \subseteq HC$ *and* $A_0 \subseteq A$ *are transitive classes and* L^* *is a Boolean logic symbiotic on* A_0. *If* $\Pi^1_1(L_{\omega\omega}) \leq \Delta(L^*_A)$, *then* $\Delta^1_{(\omega)}(L^*_A) \subseteq \Delta(L^*_A)$.

<u>Proof</u>. Suppose L^* is symbiotic with P on A_0. In view of 3.9 it suffices to prove that if K is definable by a flat formula of set theory in the language $\{\varepsilon, P\}$ w.p.i. A, then K is $\Delta_1(P)$ w.p.i. A. Suppose $\varphi(x,y)$ is a flat formula in the language $\{\varepsilon, P\}$ and $a \in A$. Then $\varphi(a,b)$ holds if and only if there is a strong limit α such that $R(\alpha)$ reflects P, a and b are in $R(\alpha)$, and $R(\alpha) \models \varphi(a,b)$. As the assumption $\Pi^1_1(L_{\omega\omega}) \leq \Delta(L^*_A)$ implies that Pw is $\Delta_1(P)$ w.p.i. A, the above equivalence shows that $\varphi(a,y)$ is $\sum_1(P)$ w.p.i. A. Similarly $\neg\varphi(a.y)$ is $\sum_1(P)$ w.p.i. A. ◻

Proposition 3.10 can be improved by considering suitably defined $\Delta^{\alpha}_{(\omega)}$-operations, where α is an ordinal definable in finite order logic or $\alpha \in A$ (see [9]).

The results about absoluteness of symbiotic logics in the previous chapter carry over to strongly symbiotic logics as follows: Let us consider the following properties:

(SA1) Stc* is $\sum^b_1(P)$ and $\varphi\vee\psi$, $\varphi\wedge\psi$, $\exists c\varphi$ and $\forall c\varphi$ in (L5) and (L6) can be found with $\sum^b_1(P)$-functions.

(SA2) There is a $\Delta_1^b(P)$-predicate S such that if $\varphi \in L^*$, then

$$\forall M \in \mathrm{Str}(L)(M \models^* \varphi \leftrightarrow S(M,\varphi)).$$

The condition (SA1) and (SA2) together from a notion of strong P-absoluteness of L^*. Note that if P is omitted, strong absoluteness coincides with the notion of absoluteness, because every Σ_1-predicate is Σ_1^b. So the difference comes only when some non-trivial predicates P are considered. For example, second order logic is Pw-absolute but not strongly Pw-absolute (see the remarks after 3.8).

The following theorem is proved as 2.12:

Theorem 3.11. *Suppose Boolean* L^* *and* P *are strongly symbiotic on* A *and* $L^+ \le \Delta_1^1(L_A^*)$. *Then* L^+ *and* P *satisfy* $(SS1)_A$ *but not (SA2).*

Corollary 3.12. *Suppose Boolean* L^* *and* P *are strongly symbiotic on* A *and satisfy (SA2). Then* L_A^* *is not* Δ_1^1*-closed.*

Corollary 3.13. *The following logics are not* Δ_1^1*-closed:*

(1) $L[P]_A$, *where* P *is a predicate of set theory,*

(2) $L_A(W,Q)$, *where* Q *is a generalized quantifier.*

Hence, if L_A^* *is a strongly symbiotic logic on* A *extending* L_A, *there are no generalized quantifiers* $Q^1 \ldots Q^n$ *such that*

$$\Delta_1^1(L_A^*) \sim L_A(Q^1,\ldots,Q^n).$$

Also the proof of Theorem 2.15 carries over:

Theorem 3.14. *Suppose* L^* *and* P *are strongly symbiotic on* V, *satisfy (SA1), and there is a* $\Sigma_1^b(P)$*-function which embedds* $L_{\infty\omega}$ *into* L^*. *Then the predicate* $\mathrm{Stc}_{\Delta_1^1(L^*)}$ *is* $\Pi_2^b(P)$ *but not* $\Sigma_2^b(P)$.

This theorem shows how difficult it is to find a syntax for Δ_1^1-extensions, whereas the full $\Delta_{(\omega)}^1$-extension has a simple primitive recursive syntax. The situation is hence similar as in the case of Δ-extension.

§ 4. Löwenheim numbers

The purpose of this chapter is to transfer the definability results of § 2 from the level of model classes to the level of spectra and in particular minima of spectra, that is Löwenheim numbers.

Definition 4.1. *Suppose* L^* *is an abstract logic and* $\varphi \in L^*$. *The spectrum* *of* φ, $Sp(\varphi)$, *is the class*

$$\{card(M) | M \models^* \varphi\}.$$

The indexed family

$$Sp(L^*) = \{Sp(\varphi) | \varphi \in L^*\}$$

is called the family of L^**-spectra.*

<u>Examples 4.2.</u>

(1) The class of successor cardinals and the class of limit cardinals are LI*-spectra.*

(2) $\{\lambda | \exists\kappa(\kappa^+ < \lambda \leq 2^\kappa)\}$ *is an* LI*-spectrum.*

(3) The class of regular cardinals and the class of weakly inaccessible cardinals are LR*-spectra.*

(4) $\{2^\kappa | \kappa$ *a cardinal*$\}$ *is an* L^{II}*-spectrum.*

(5) $\{2^\kappa | \kappa$ *is measurable*$\}$ *is an* L^{II}*-spectrum.*

For other examples of spectra see [13] *and* [14].

The following problem is called the <u>spectrum problem for</u> <u>L^*</u>: Is the complement of an arbitrary L^*-spectrum again an L^*-spectrum? The spectrum problem for LW, for example, has a negative solution because $\{\kappa | \kappa \leq \aleph_0\}$ is an LW-spectrum but $\{\kappa | \kappa > \aleph_0\}$ is not. The spectrum problem for LI can have a negative answer - this will be discussed later. The spectrum problem for L^{II} has a positive solution for a rather trivial reason: if C is an L^{II}-spectrum, then $C = Sp(\varphi)$ for some identity-sentence φ and the complement of C is just $Sp(\neg\varphi)$. This fact has a more general analogue. At first we note the following trivial lemma:

<u>Lemma 4.3.</u> *Suppose* C *is a class of cardinals and* C' *is the class of structures* $\langle A \rangle$*, where* $card(A) \in C$*. Then* C *is an* L^**-spectrum if and only if* C' *is* $\Sigma^1_1(L^*)$*-definable.*

If this is combined with Theorem 3.5 and Corollary 3.9, the following characterization of spectra yields:

<u>Theorem 4.4.</u> *Suppose* $A \subseteq HC$ *and* $A_0 \subseteq A$ *are transitive classes,* P *a predicate, and* L^*_A *a Boolean logic extending* L_A *and strongly symbiotic with* P *on* A_0*. Then*

$$Sp(L^*_A) = \{C \subseteq Cd | C \text{ is } \Sigma^b_1(P) \text{ w.p.i. } A\},$$

$$Sp(\Delta^1_{(\omega)}(L_A)) = \{C \subseteq Cd | C \text{ is definable by a flat formula in the language } \{\varepsilon, P\} \text{ w.p.i. } A\},$$

$Sp(L_A^{II}) = \{C \subseteq Cd \mid C$ *is definable with a flat formula of set theory w.p.i.* $A\}$.

Definition 4.5. *Suppose* L^* *is an abstract logic. The Löwenheim-number* $l(L^*)$ *of* L^* *is the least cardinal* κ *such that* $\min(C) \leq \kappa$ *for every* $C \in Sp(L^*)$*, if any such* κ *exist. Equivalently,* $l(L^*)$ *is the least cardinal* κ *such that if* $\varphi \in L^*$ *has a model, then* φ *has a model power* $\leq \kappa$.

It is well-known that $l(L_A^*)$ exists if A is a set.

Theorem 4.6. *Suppose* A *and* $A_0 \subseteq A$ *are transitive classes,* P *a predicate, and* L_A^* *a Boolean logic extending* L_A *and symbiotic with* P *on* A_0*. Then for any* $n < \omega$:

$$l(\Delta_n(L_A^*)) = sup\ \{\kappa \mid \kappa \text{ is } \Pi_n(P)\text{-definable w.p.i. } A\}.$$

If $l(\Delta_n(L_A^*))$ *is a limit cardinal (e.g.* $n > 1$ *or* $LI \leq \Delta(L_A^*)$*), then moreover*

$$l(\Delta_n(L_A^*)) = sup\ \{\alpha \mid \alpha \text{ is } \Pi_n(P)\text{-definable w.p.i. } A\}.$$

Proof. Suppose at first that α is $\Pi_n(P)$-definable w.p.i. A. Suppose $\varphi(x,y)$ is a $\sum_n(P)$-formula and $a \in A$ such that

$$\forall\beta(\beta \geq \alpha \leftrightarrow \varphi(\beta,a)).$$

Let K be the class of linearly ordered structures the ordertype of which is an ordinal $\geq \alpha$. K is clearly $\sum_n(P)$ w.p.i. A, whence K is $\sum_n(L_A^*)$-definable. But every model of K has power $\geq card(\alpha)$. Hence $\alpha \leq \min\{card(M) \mid M \in K\} < l(\Delta_n(L_A^*))$. For the converse, suppose $\kappa < l(\Delta_n(L_A^*))$. Let $\varphi \in L_A^*$ such that $\kappa \leq \lambda = \min(Sp(\varphi)) < l(\Delta_n(L_A^*))$. Now

$$\forall\beta(\beta > \lambda \leftrightarrow \exists\gamma \leq \beta\exists M(|M| = \gamma \ \&\ M \models^* \varphi))$$

whence λ is $\Pi_n(P)$-definable w.p.i. A. □

Corollary 4.7. *For any rudimentary set* A *and* $n > 1$:

(1) $l(LI_A) = sup\ \{\alpha \mid \alpha$ *is* $\Pi_1(Cd)$*-definable w.p.i.* $A\}$,

(2) $l(L_A^{II}) = sup\ \{\alpha \mid \alpha$ *is* Π_2*-definable w.p.i.* $A\}$,

(3) $l(\Delta_n(L_A)) = sup\ \{\alpha \mid \alpha$ *is* Π_n*-definable w.p.i.* $A\}$,

(4) $l(L_A^S) = sup\ \{\alpha \mid \alpha$ *is definable in set theory w.p.i.* $A\}$ *(in MKM).*

Part (2) of the above corollary was proved earlier but independently in [6].

Löwenheim-numbers can also be characterized in terms of a notion of describability. This notion is related to the notion of indiscribability (see [4] p. 268) but differs mainly in that less parameters are allowed.

Definition 4.8. *Let* D *be a set of formulae of set theory. An ordinal* α *is* D-*describable w.p.i.* A *if there are a* $\varphi(x) \in D$ *and an* $a \in R_\alpha \cap A$ *such that*

$$R_\beta \models \varphi(a) \quad \textit{for} \quad \beta \geq \alpha$$

and

$$R_\beta \not\models \varphi(a) \quad \textit{for} \quad rk(a) \leq \beta < \alpha.$$

The predicate P *is* R-*absolute if every* R_α *reflects* P.

Lemma 4.9. *Suppose* P *is* R-*absolute and* α *is* $\Pi_1(P)$-*definable w.p.i.* A. *Then there is a* $\beta > \alpha$ *such that* β *is* $\sum_1(P)$-*describable w.p.i.* A.

Proof. Suppose $\varphi(x,y)$ is $\sum_1(P)$-formula and $a \in A$ such that

$$\forall\beta(\beta \geq \alpha \leftrightarrow \varphi(\beta,a)).$$

Let $\psi(y)$ be the $\sum_1(P)$-formula $\exists x\varphi(x,y)$ and β the least β such that $R_\beta \models \psi(a)$. Then $\gamma \geq \beta \rightarrow R_\gamma \models \psi(a)$. Hence $\psi(a)$ describes β. $R_\beta \models \psi(a)$ clearly implies $\beta > \alpha$. □

Lemma 4.10. *Suppose* P *is* R-*absolute and* α *is* $\sum_1(P)$-*describable w.p.i.* A. *Then* $\alpha + 1$ *is* $\Pi_1(P)$-*definable w.p.i.* A.

Proof. Suppose $\varphi(x)$ is a $\sum_1(P)$-formula and $a \in A$ such that $R_\beta \models \varphi(a)$ if and only if $\beta \geq \alpha$. Let $\psi(y,x)$ be the $\sum_1(P)$-formula which says that $\varphi(x)$ is true in a transitive set which reflects P and the ordinal of which is $< y$. If $\psi(\beta,a)$, then (because P reflects) for some $\gamma < \beta$ $R_\gamma \models \varphi(a)$, whence $\beta > \alpha$. On the other hand, if $\beta > \alpha$, then $\psi(\beta,a)$ as one can choose R_α as the required transitive set. □

Corollary 4.11. *Suppose* A *and* $A_o \subseteq A$ *are transitive sets,* P *an* R-*absolute predicate, and* L_A^* *an abstract logic extending* L_A *and symbiotic with* P *on* A_o, *and* $l(L_A^*)$ *is a limit cardinal. Then*

$$l(L_A^*) = \sup \{\alpha | \alpha \textit{ is } \textstyle\sum_1\textit{-describable w.p.i. } A\}.$$

Proof. The claim follows immediately from 4. 9, 4.10 and 4.6. □

Lemma 4.12. *Suppose* α *is first order describable (that is described by* *some formula of set theory) w.p.i.* A. *Then* α *is* Π_2*-definable w.p.i.* A.

Proof. Suppose $\varphi(x)$ is a formula and $a \in A$ such that for $\beta \geq rk(a)$

$$R_\beta \models \varphi(a) \text{ if and only if } \beta \geq \alpha.$$

Let $\psi(y,x)$ be the $\sum_2$-formula "$R_y \models \varphi(a)$". Then $\psi(\beta,a)$ if and only if $\beta \geq \alpha$.

Corollary 4.13. *For any set* A:

$$l(L_A^{II}) = sup\ \{\alpha|\alpha \text{ is first order describable w.p.i. } A\}.$$

Another way of formulating Corollary 4.11 is the following:

Proposition 4.14. *Suppose* A *and* $A_o \subseteq A$ *are rudimentary sets,* P *an* R-*absolute predicate, and* L_A^* *an abstract logic extending* L_A *and symbiotic with* P *on* A_o*, and* $l(L_A^*)$ *is a limit cardinal. Then*

$$l(L_A^*) = \text{the least } \alpha \text{ such that } <R_\alpha,\in,a>_{a\in A\cap R_\alpha} \equiv_{\sum_1(P)} <V,\in,a>_{a\in A}.$$

In particular,

$$l(L_A^{II}) = \text{the least } \alpha \text{ such that } <R_\alpha,\in,a>_{a\in A\cap R_\alpha} \equiv_{\sum_2} <V,\in,a>_{a\in A}.$$

The above result suggest the study of ordinals α such that

$$(*) \qquad <R_\alpha,\in> \prec_{\sum_n} <V,\in>.$$

Let us denote the predicate (*) of α by $D_n(\alpha)$. The following lemma will be most useful:

Lemma 4.15. *The predicate* $D_n(\alpha)$ *is* Π_n*, for* $n > 1$.

Proof. Let $S(x,y)$ be the $\sum_n$-predicate which is universal for $\sum_n$-formulae with one free variable y (see e.g. [4] p. 272). Let $F(z)$ by the Δ_1-predicate "z is a $\sum_n$-formula with one free variable y". If $F(z)$, let $f(z,\alpha)$ be the relativization of z to R_α. f is clearly Δ_2. Let $S_o(x,y)$ be the Δ_1-predicate which is universal for $\sum_o$-formulae with one free variable y. Now we have:

$$D_n(\alpha) \leftrightarrow \forall y \in R_\alpha \forall z \in \omega(F(z) \to (S_o(f(z,\alpha),y) \vee \neg S(z,y))),$$

and therefore $D_n(\alpha)$ is Π_n. □

Proposition 4.16. *If* α *is* Π_n*-definable w.p.i.* A, *then there is a* $\beta \geq \alpha$ *such that* β *is* Δ_n*-definable w.p.i.* A $(n > 1)$.

Proof. Let $\varphi(x,y)$ be a Π_n-formula and $a \in A$ such that

$\forall\beta(\beta \in \alpha \leftrightarrow \varphi(\beta,a))$.

Let $\psi(u,v)$ be the Δ_n-predicate "$D_{n-1}(u)$ & $R_u \models \exists x - \varphi(x,v)$". Note that $D_1(\alpha)$ may not be Π_1 but by the proof of 4.15 it is Δ_2. Let $\theta(w,v)$ be the Δ_n-predicate $\exists u \le w\psi(u,v)$. We claim that $\neg\theta(w,a)$ defines an ordinal $\ge \alpha$. By reflection there is an ordinal β such that $\psi(\beta,a)$. Let β be the least such β. If $\gamma \ge \beta$ then $\theta(\gamma,a)$. On the other hand, if $\theta(\gamma,a)$, then $\psi(\delta,a)$ for some $\delta \le \gamma$, whence $\beta \le \delta \le \gamma$. □

Corollary 4.17. *For any rudimentary set* A *and* $n > 1$:

$l(L_A^{II}) = \sup\{\alpha | \alpha$ *is* Δ_2*-definable w.p.i.* $A\}$.

$l(\Delta_n(L_A)) = \sup\{\alpha | \alpha$ *is* Δ_n*-definable w.p.i.* $A\}$.

The predicate $D_n(\alpha)$ is actually equivalent to a Löwenheim-Skolem-theorem, as the following theorem shows:

Theorem 4.18. *The following are equivalent for any* $n > 1$:

(1) $l(\Delta_n(L_{\kappa\omega})) = \kappa$,

(2) $\langle R,\in\rangle \prec_{\Sigma_n} \langle V,\in\rangle$.

Proof. Note that both (1) and (2) imply $\kappa = \beth_\kappa$. If (2) holds and $\varphi \in \Delta_n(L_{\kappa\omega})$ has a model, then $R_\kappa \models$ "φ has a model", whence φ has a model of power $< \beth_\kappa = \kappa$. So (2) implies (1). Suppose then (1) holds. We may assume thet $D_{n-1}(\kappa)$ holds because if $n = 2$, it follows from $\kappa = \beth_\kappa$, and if $n > 2$, it follows from a suitable induction hypothesis. Suppose $\varphi(x)$ is a Σ_n-formula and $a \in R_\kappa$ such that $\varphi(a)$ holds. Let K be the class of ordinals α such that $D_{n-1}(\alpha)$ and $R_\alpha \models \varphi(a)$. By Theorem 2.4 and (1), there is a $\beta \in K$ such that $\beta \in \kappa$. As $D_{n-1}(\kappa)$, we have $R_\kappa \models \varphi(a)$, as required. □

Corollary 4.19. *If* κ *is supercompact, then* $l(\Delta_2(L_{\kappa\omega})) = \kappa$. *If* κ *is extendible, then* $l(\Delta_3(L_{\kappa\omega})) = \kappa$.

Proof. If κ is supercompact, then $D_2(\kappa)$; if κ is extendible, then $D_3(\kappa)$. These facts are proved in [11]. □

§ 5. Hanf-numbers

Hanf-numbers can be characterized in the same way as Löwenheim-numbers. One has to bear in mind, however, that Δ does not preserve Hanf-numbers (see [15]). Therefore we introduce a new notion of definability, bounded definability, which is neat enough to preserve Hanf-numbers but still almost as powerful as Δ- or Δ_1-

definability. This notion was first studied in [15]. The main result of this chapter is Theorem 5.6. The chapter ends with a discussion on definable ordinals and sort logic.

Definition 5.1. *Let* P *be a predicate of set theory. A predicate* $S(x_1,\ldots,x_n)$ *of set theory is* $\sum_1^B(P)$ *w.p.i.* A *if there are a* $\sum_0(P)$-*formula* $\varphi(x_1,\ldots,x_n,x,z)$ *and* $a \in A$ *such that*

$$\forall x_1 \ldots \forall x_n (S(x_1,\ldots,x_n) \leftrightarrow \exists x \varphi(x_1,\ldots,x_n,x,a))$$

and

$$\forall x_1 \ldots \forall x_n (\{x | \varphi(x_1,\ldots,x_n,x,a)\} \text{ is a set}).$$

$S(x_1,\ldots,x_n)$ *is* $\Pi_1^B(P)$ *w.p.i.* A *if* $\neg S(x_1,\ldots,x_n)$ *is* $\sum_1^B(P)$ *w.p.i.* A. S *is* Δ_1^B *w.p.i.* A *if* S *is both* $\sum_1^B(P)$ *and* $\Pi_1^B(P)$ *w.p.i.* A.

An example of a Δ_1(Cd)-predicate which is not (provably) Δ_1^B(Cd) is given in [15]. Note that every $\sum_1$-predicate is $\sum_1^B$ by Levy's theorem. From the fact

$$\exists x \varphi(x) \leftrightarrow \exists x(\varphi(x) \ \& \ \forall y \& rk(y) < rk(x) \rightarrow \neg \varphi(y))$$

it follows that every $\sum_1$(P)-predicate is $\sum_1^B$(P,Pw). Therefore there is no need to define $\sum_n^B$(P)-predicates for $n > 1$ - they would coincide with the $\sum_n$(P)-predicates.

Now we define the model theoretic analogues of the above notions.

Definition 5.2. *Let* L* *be an abstract logic. A model class* K *is* $\sum^B(L^*)$-*definable if it is* $\sum$-*defined by an* L*-*sentence* φ *such that*

$$\forall A \ \exists \kappa \forall B \in E(A,\varphi)(\mathrm{card}(B) \leq \kappa).$$

K *is* $\Pi^B(L^*)$-*definable if* $\overline{K}$ *is* $\sum^B(L^*)$-*definable.* K *is* $\Delta^B(L^*)$-*definable if it is both* $\sum^B(L^*)$- *and* $\Pi^B(L^*)$-*definable.*

Δ^B is a natural operation on logics and resembles Δ-operation so much that it is in fact not at all obvious that there is any difference between them. For a treatment of Δ^B see [15]. We pick up some of the results of [15] to the following lemma (note that (1) below fails for Δ):

Lemma 5.3.

(1) Δ^B *preserves Löwenheim- and Hanf-numbers.*

(2) $\sum^B(L^*) \sim \sum(L^*)$ *if* $L^{II} \leq \Delta^B(L^*)$ *or* L* *is one of the following logics (or a fragment of one)* $L_{\infty\omega}$, $L_{\infty\omega}(W)$, $L_{\infty\omega}(Q_\alpha)$, $L_{\infty\omega}(Q^{MM(n)})$, $L_{\infty G}$. *Hence* $\Delta^B(L^*) \sim \Delta(L^*)$ *for such* L*.

(3) $V = L$ *implies* $\Delta^B(LI) \sim \Delta(LI)$.

(4) If Con(ZF), *then* Con(ZFC + $\Delta^B(LI) \not\to \Delta(LI)$).

Related to the bounded notions of definability is a new notion of symbiosis as well:

Definition 5.4. *Suppose* L* *is an abstract logic,* P *a predicate of set theory and* A *a transitive class.* L* *and* P *are* boundedly symbiotic *on* A *if the following two conditions are satisfied:*
(BS1) If $\varphi \in L^*$, *then* $\mathrm{Mod}(\varphi)$ *is* $\Delta_1^B(P)$ *w.p.i.* $\{\varphi, L\}$
(BS2) K[P] *is* $\Delta^B(L^*_A)$*-definable.*
The pairs of example 2.3 are all boundedly symbiotic.

We omit the proof of the following theorem because the proof would be mutatis mutandis as that of 2.4.

Theorem 5.5. *Suppose* A *and* $A_o \subseteq A$ *are transitive classes,* P *a predicate, and* L^*_A *an abstract logic extending* L_A *and boundedly symbiotic with* P *on* A_o. *Then the following are equivalent:*
(1) K *is* $\sum^B(L^*_A)$*-definable,*
(2) K *is* $\sum_1^B(P)$ *w.p.i.* A.

Theorem 5.6. *Suppose* A *and* $A_o \subseteq A$ *are transitive classes,* P *a predicate, and* L^*_A *a Boolean logic extending* L_A *and boundedly symbiotic with* P *on* A_o. *Then*

$$h(L^*_A) = \sup \{\alpha \mid \alpha \text{ is } \textstyle\sum_1^B(P)\text{-definable w.p.i. } A\}$$

and for $n > 1$:

$$h(\Delta_n(L^*_A)) = \sup \{\alpha \mid \alpha \text{ is } \textstyle\sum_n(P)\text{- definable w.p.i. } A\}.$$

Proof. In order to prove the two claims simultaneously, let us agree that $\sum_n^B(P)$ for $n > 1$ means $\sum_n(P)$. Now, let $n > 0$. Suppose that α is $\sum_n^B(P)$-definable w.p.i. A. Let K be the class of linearly ordered structures the order type of which is $< \alpha$. K is $\sum_n^B(P)$ w.p.i. A and therefore $\sum_n^B(L^*_A)$-definable (using 2.4 and 5.4, $\sum_n^B(L^*_A)$ for $n > 1$ means $\sum_n(L^*_A)$). Hence K is $\sum(\Delta_n^B(L_A))$-definable. If $n > 1$, then $L^{II} \leq \Delta_n^B(L^*_A)$ whence by 5.3 (2) K is $\sum^B(\Delta_n^B(L^*_A))$-definable. If $n = 1$, the same conclusion follows trivially. Hence there is a $\psi \in \Delta_n^B(L^*_A)$ which $\sum^B$-defines K. As K has models of power $\leq \mathrm{card}(\alpha)$ only, ψ does not have arbitrary large models. But ψ has a model of power $\geq \kappa$ for every $\kappa < \alpha$. Hence $\mathrm{card}(\alpha) < h(\Delta_n^B(L^*_A))$. It follows easily that $\alpha < h(\Delta_n^B(L^*_A))$. For the converse, suppose $\kappa < h(\Delta_n^B(L^*_A))$. Let φ be in $\Delta_n^B(L_A)$ such that $\kappa \leq \lambda = \sup \mathrm{Sp}(\varphi)$. Now

$$\alpha < \lambda \leftrightarrow \exists\beta\exists M(\alpha \leq \beta \;\&\; |M| = \beta \;\&\; M \models^* \varphi).$$

Hence λ is $\sum_n^B(P)$-definable w.p.i. A. □

Corollary 5.7. *For any rudimentary set* A:

(1) $h(L_A^{II}) = sup\ \{\alpha | \alpha$ *is* $\sum_2$*-definable w.p.i.* A}.

(2) $h(\Delta_n(L_A)) = sup\ \{\alpha|\alpha$ *is* $\sum_n$*-definable w.p.i.* A} $(n > 1)$.

(3) *In* MKM: $h(L_A^s) = sup\ \{\alpha|\alpha$ *is definable in set theory w.p.i.* A}
$= l(L_A^s)$.

Part (1) of the above corollary was proved earlier, but independently, in [6] (see also [1]).

Let us write

l_n for $\sup\ \{\alpha|\alpha$ is Π_n-definable},

h_n for $\sup\ \{\alpha|\alpha$ is $\sum_n$-definable}.

By what we have already proved: (for $n > 1$)

$$l_n = l(\Delta_n^*(L_{HF})) = \sup\ \{\alpha|\alpha \text{ is } \Delta_n\text{-definable}\},$$

$$h_n = h(\Delta_n(L_{HF})).$$

In the next few lemmas we shall establish the mutual relations of the ordinals $l_n, h_n, n < \omega$. It turns out that the following notation is helpful:

$$\begin{aligned} t_n &= \text{the least } \alpha \text{ such that } D_n(\alpha) \\ &= \text{the least } \alpha \text{ such that } \langle R_\alpha, \in\rangle \prec_{\sum_n} \langle V, \in\rangle \\ &= \text{the least } \alpha \text{ such that } l(\Delta_n(L_{\alpha\omega})) = \alpha. \end{aligned}$$

Trivially $l_n \leq t_n \leq h_n$ for $n > 1$.

Lemma 5.8. *For* $n > 0$, $l_n < t_n$.

Proof. Let $S(x,y)$ be the $\sum_n$-formula which is universal for $\sum_n$-formulae with the free variable y. Let

$$a = \{\varphi(y)|\varphi(y) \text{ is a } \textstyle\sum_n\text{-formula such that } \neg\varphi(y) \text{ defines an ordinal}\}.$$

$a \in R_{\omega+1}$ and therefore $a \in R_{t_n}$. Let $\psi(x,y)$ be a $\sum_n$-formula equivalent to $\forall u \in xS(u,y)$. Now $\psi(a,y)$ is true for some y, whence $\psi(a,y)$ is true for some $y \in R_{t_n}$. This y is an ordinal which is greater than any Π_n-definable ordinal. Therefore $l_n \leq y < t_n$. □

Lemma 5.9. *If* $n > 1$, *then* t_n *is* $\sum_n$*-definable, and hence* $t_n < h_n$.

Proof. Recall from 4.15 that D_n is Π_n. Hence the claim follows from

$$\forall\alpha(\alpha < t_n \leftrightarrow \neg D_n(\alpha) \ \& \ \forall\beta < \alpha \neg D_n(\beta)). \quad \square$$

Lemma 5.10. *If* $n > 1$, *then* $h_n = l_{n+1}$.

Proof. Suppose α is Π_{n+1}-definable and $\varphi(x,y)$ is a $\sum_n$-formula such that

$$(*) \qquad \forall\beta \ \beta < \alpha \leftrightarrow \forall x\varphi(x,\beta)).$$

Let $\psi(x)$ be a $\sum_n$-formula saying that x is an ordinal and $\varphi(y,\beta)$ holds for all $y \in R_x$ and $\beta < x$. If $\forall x\psi(x)$, then $\forall x\varphi(x,\alpha)$, a contradiction. Therefore there are δ such that $-\psi(\delta)$. Let δ be the least of them. Hence if $\beta < \delta$ then $\psi(\beta)$. On the other hand, if $\psi(\beta)$ and $\gamma \leq \beta$ then $\psi(\gamma)$, whence $\gamma \neq \delta$. Therefore $\psi(x)$ $\sum_n$-defines δ. Hence it suffices to prove that $\alpha \leq \delta$. Suppose the contrary, that is $\delta < \alpha$. If $y \in R_\delta$ and $\beta < \delta$, then by (*) $\varphi(y,\beta)$. Hence $\psi(\delta)$ holds, a contradiction. Therefore $\alpha \leq \delta$. $\square$

Corollary 5.11. $l_2 < h_2 = l_3 < h_3 = l_4 < h_4 = l_5 < \ldots$

If the proofs of 5.8-5.10 are carried out with parameters, the following theorem yields:

Theorem 5.12. *Suppose* A *is a rudimentary set and* $n > 1$. *Then*

$$l(\Delta_n(L_A)) < h(\Delta_n(L_A)) = l(\Delta_{n+1}(L_A)).$$

Corollary 5.13. *(MKM)* $l(L^s_{HF}) = h(L^s_{HF})$ = *the least* α *such that* $R_\alpha \prec V$.

Proof. Suppose $\varphi(x_1,\ldots,x_n)$ is a formula of set theory and $a_1,\ldots,a_n$ sets in R_α, $\alpha = l(L^s_{HF})$, such that $\varphi(a_1,\ldots,a_n)$. Let $m < \omega$ such that $\varphi(x_1,\ldots,x_n)$ is equivalent to a $\sum_m$-formula $\varphi(x_1,\ldots,x_n)$. Now $R_{t_k} \models \psi(a_1,\ldots,a_n)$ for a sufficiently large $k < \omega$. We may assume $D_{m-1}(\alpha)$. Therefore $R_\alpha \models \psi(a_1,\ldots,a_n)$. Hence $R_\alpha \models \varphi(a_1,\ldots,a_n)$. For the converse, suppose R_α V. Then every definable ordinal must be $< \alpha$. Therefore $\alpha \geq l(L^s_{HF})$. $\square$

Corollary 5.14. *If the required cardinals exist, then*

1st *measurable* $< l_2 <$ 1st *supercompact* $< h_2 = l_3 <$ 1st *extendible* $< h_3$.

Proof. The predicate "α is measurable" is $\sum_2$. Hence the 1st measurable is Π_2-definable and therefore $< l_2$. If κ is supercompact, then $D_2(\kappa)$ (see [11] p. 86), and hence $l_2 < t_2 \leq \kappa$. The predicate "α is supercompact" is Π_2. Hence the 1st supercompact is $\sum_2$-definable and therefore $< h_2$. If κ is extendible, then $D_3(\kappa)$ (see [11] p. 103), and hence $l_3 < t_3 \leq \kappa$. The predicate

"α is extendible" is Π_3 and therefore the 1st extendible is Σ_3-definable. Hence the 1st extendible $< h_3$. □

So we see that the Löwenheim- and Hanf-numbers of even the lowest levels of sort logic exhaust a wide range of large cardinals. This would seem to suggest that the logics $\Delta_n(L_A)$ are rather strong indeed. In connection with 5.14, note that the ordinals $l_n, h_n, n < \omega$ exist even if there are no large cardinals; they exist in L, for example.

It seems to be a rather common phenomenon that the Löwenheim-number of a logic is smaller (often substantially) than the Hanf-number (see e.g. 5.12). However, in the second part of this paper we shall construct a model of set theory where the Hanf-number of LI is smaller than the Löwenheim-number of LI. In that model the spectrum problem for LI has a negative solution, because there is a cardinal κ between l(LI) and h(LI) such that $\{\lambda | \lambda \geq \kappa\}$ is a spectrum, but $\{\lambda | \lambda < \kappa\}$ is (obviously) not.

We end this chapter with a remark on another way of characterizing $h(L_A^{II})$.

Definition 5.15. *An ordinal* α *is weakly first order describable w.p.i.* A *if there are a formula* $\varphi(x)$ *of set theory and an* $a \in R_\alpha \cap A$ *such that*

$$R_\beta \models \varphi(a) \quad \textit{for} \quad \beta \geq \alpha$$

and

$$R_\beta \not\models \varphi(a) \quad \textit{for arbitrary large} \quad \beta < \alpha,\ \beta \geq rk(a).$$

Theorem 5.16. *Suppose* A *is a rudimentary set.*
$h(L_A^{II}) = \sup \{\alpha | \alpha$ *is weakly first order describable w.p.i.* A$\}$.

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LATTICE PRODUCTS

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A basic technique in algebra is the representation of an algebraic structure A as a 'compound' of 'simpler' structures. One way of doing this is to represent the elements of A by global or partial functions; more specifically, one considers representations of A as a subdirect or 'partial' subdirect product of structures $\{A_i\}_{i\in I}$. The usefulness of this procedure depends on two points:

1. How much information is available on the factors A_i ?
2. How much information is transferable from the A_i to A ?

Since arbitrary subdirect product representations yield very little on point 2, a number of concepts has been proposed to improve this situation by specifying more precisely, how 'thick' or how 'thin' A is in the (global or partial) direct product of the A_i. In this connection, representations of first-order structures by sections in sheaves have received growing attention in last 15 years - first from algebraists and then also from model theorists.While the algebraic papers centre on representability and characterization of factors (point 1), the model theoretical ones deal mostly with the transfer of properties from factors to functions (point 2). The concept of a (global or partial) lattice product is suited for both purposes. It reduces representations to two steps:

1. Expansion of a given structure by a distributive lattice and certain 'truth-valuations' to an abstract lattice product.
2. Representations of distributive lattices by rings of sets.

Most of the model theoretical information contained in a representation is already captured in the first step. As a consequence, model theoretical transfer principles (point 2) can be formulated and proved in a (largely effective) syntactical manner for abstract lattice products.

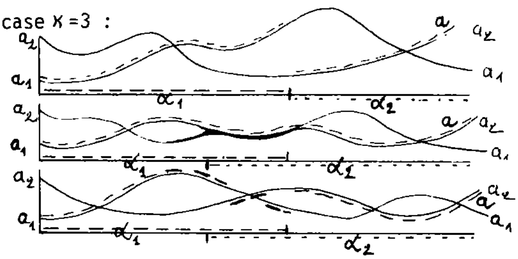
The following sketch is intended to outline some concepts and results in the theory of lattice products. A detailed treatment with complete proofs will appear elsewhere. For simplicity, we restrict ourselves here to global functions; similar results hold for partial functions.

A lattice space (LS) is a pair $(I,\mathcal{L})$, where I is a non-empty set, $\mathcal{L}$ is a sublattice of P(I) with $I\in\mathcal{L}, \bigcap\mathcal{L}=\emptyset$. $(I,\mathcal{L})$ is called boolean if $\mathcal{L}$ is a boolean algebra. Topological notions such as compactness, the separation axioms T_0,T_1,T_2, continuous maps, homeomorphisms are defined as in a topological space $(I,\mathcal{O})$, where $\mathcal{O}$ is the lattice of open sets. Let $A \subset \prod_{i\in I} A_i$ be a subdirect product of structures A_i for a first-order language L, and suppose $(I,\mathcal{L})$ is a LS. Then the truth-value of an L-formula $\varphi(\bar{x})$ at $\bar{a}\in A^n$ is defined by $[\varphi(\bar{a})] = \{i\in I \mid A_i \models \varphi(\bar{a}(i))\}$. A is a global lattice product (GLP) over $(I,\mathcal{L})$ if $[\varphi(\bar{a})] \in \mathcal{L}$ for all atomic φ and $\bar{a}\in A^n$. So this is a condition restricting the 'thickness' of A in $\prod A_i$ from above. Corresponding conditions bounding this thickness from below are given by three types of 'patchwork principles' for A: The κ-disjoint global patchwork principle (κ-DGPW), the κ-global patchwork principle (κ-GPW), and the κ-strong global patchwork principle (κ-SGPW), where κ is a cardinal or ∞. Each of these principles says that certain families of functions $\{a_k\}_{k\in K}$ in A can be patched up to a function a in A with respect to a corresponding family $\{\alpha_k\}_{k\in K}$ of elements of $\mathcal{L}$ (or of a specified sublattice of $\mathcal{L}$), provided card(K) < κ . Instead of giving the exact definitions, we illustrate the case $\kappa=3$:

(3-DGPW)

(3-GPW)

(3-SGPW)



Then the following relations hold:

1.(i) κ-SGPW $\Rightarrow$ κ-GPW $\Rightarrow$ κ-DGPW.

(ii) If $(I,\mathcal{L})$ is compact, then ω-GPW $\Rightarrow$ ∞-GPW and similar for ω-DGPW, ω-SGPW.

(iii) If $(I,\mathcal{L})$ is compact and boolean, then 3-DGPW $\Rightarrow$ ∞-SGPW.

The significance of these principles is the following: ω-DGPW and ω-GPW are most often encountered in 'natural' algebraic representations. ∞-GPW characterizes the structure of global sections of a full sheaf of L-structures (comp. Macintyre [1973]). ∞-SGPW is the most important principle for model theoretic transfer theorems, since it allows to infer global existence from local existence in the following sense: $[\exists\bar{x}\varphi(\bar{x},\bar{a})] \geq \alpha \implies \exists\bar{x}([\varphi(\bar{x},\bar{a})] \geq \alpha)$ for $\bar{a}\in A^n$, $\alpha\in\mathcal{L}$. (In the case of ∞-GPW this does not hold in general, unless the existence $\exists\bar{x}\varphi(\bar{x},\bar{a}(i))$ is unique in each factor A_i.)

As indicated above, the representation of an L-structure A by a GLP proceeds via the intermediate concept of an abstract global lattice product (AGLP). Let $\mathcal{B}$ be the language of lattices with 1, and let L^* be the two-sorted language (L, $\mathcal{B}$, $\{ [\varphi(\bar{x})] \mid \varphi(\bar{x})$ atomic L-formula $\}$), where $[\varphi(\bar{x})]$ is a function-symbol having arguments of sort L and values of sort $\mathcal{B}$. If now A happens to be a GLP over $(I,\mathcal{L})$, then the natural L^*-expansion $A^* = (A, \mathcal{L}, \{[\varphi(\bar{x})]\})$ of A apparently satisfies the following axioms for every atomic L-formula $\varphi(\bar{x})$:

2.(i) $\varphi(\bar{x}) \leftrightarrow [\varphi(\bar{x})] = 1$;(ii) $\cap_{i=1}^{n}[x_i=y_i] \cap [\varphi(\bar{x})] \leq [\varphi(\bar{y})]$.

Using this fact, we now define an AGLP as an L^*-structure B (with L-part B_L, $\mathcal{B}$-part $B_{\mathcal{B}}$) satisfying 2(i),(ii) and the axioms for distributive lattices with 1. Then a strong converse to the above can be proved:

3.REPRESENTATION THEOREM. Let B be an AGLP. Then any representation of $B_{\mathcal{B}}$ by a lattice space $(I,\mathcal{L})$ extends in an essentially unique way to a representation of B by A^* ,where A is a GLP over $(I,\mathcal{L})$.

In principle, every representation of a first-order structure A by global sections in a sheaf can be obtained from this theorem by suitable choice of an expansion of A to an AGLP B and a representation of $B_{\mathcal{B}}$. For the known representations of rings and lattice-ordered rings and groups in Hofmann [1972] and Keimel [1971] this method is in most cases simpler than the original one. A crucial point here is, that the AGLP B that comes up 'naturally' is sometimes not the one that yields the sheaf representation. Instead one passes to a suitable meet-homomorphic image of $B_{\mathcal{B}}$ to obtain a new AGLP B'. Another method covering these representations which overlaps with special cases of 3. has been found independently by Krauss and Clark [1979].

Call the representation of an AGLP B induced by the Stone representation of $B_{\mathcal{B}}$ the canonical representation of B, and call a LS $(I,\mathcal{L})$ a Stone LS if it is T_0 and if for all $D,E\subset\mathcal{L}$, $\cap D \leq \cup E$ implies that there exist finite sets $D'\subset D$, $E'\subset E$ with $\cap D' \leq \cup E'$. Observe moreover that ω-DGPW, ω-GPW, ω-SGPW make sense for AGLP's, too. Then 1. and 3. yield:

4.COROLLARY. Let B be an AGLP satisfying ω-DGPW, ω-GPW, ω-SGPW, respectively, and let B be canonically represented by A^*,where A is a GLP over $(I,\mathcal{L})$. Then A satisfies ∞-DGPW, ∞-GPW, ∞-SGPW, respectively.

With naturally defined concepts of morphisms for GLP's and AGLP's this canonical representation becomes an equivalence between the category of AGLP's and the category of GLP's over Stone lattice spaces. If B is an AGLP canonically represented by A^* , we denote A by $\underline{B}$, its factors by $\underline{B}_i$, and the underlying space by $(Sp_B,\mathcal{L}_B)$.

Call an AGLP A an abstract boolean product (ABP), if $A_{\mathcal{B}}$ is a boolean algebra. The best results concerning the transfer of model theoretic properties from the canonical factors $\underline{A}_i$ of A to A are obtained for ABP's A and properties defined in terms of universal and existential formulas. Other properties require certain 'maximum principles' for A. The method of proof consists in an effective reduction of existential L^*-formulas to certain normal forms. It involves besides Feferman-Vaught-type arguments (see Volger [1976]) a combinatorial theorem related to P. Hall's theorem on distinct representatives. One obtains in this way e.g. a characterization of existentially complete ABP's:

Let K be an $\forall\exists$-theory in L, and let Σ be the class of all ABP's for the language

L^* such that A satisfies 3-DGPW and all canonical factors $\underline{A}_i$ are models of K. Then Σ is an inductive, elementary class. Moreover, $\forall\exists$-axioms for Σ can be effectively constructed from K. Denote the class of existentially complete structures in a class Δ by $E(\Delta)$. Let $A\in\Sigma$, $\varphi(\bar{x})$ an existential L-formula, $\bar{a}\in \underline{A}^n$. Then we define the potential truth-value of $\varphi(\bar{a})$ in Σ by $\overline{[\![\varphi(\bar{a})]\!]} = \{i\in Sp_A \mid \text{ex. } \underline{A}_i \subset B_i \models K$ with $B_i \models \varphi(\bar{a}(i))\}$. In terms of this concept, $E(\Sigma)$ can be characterized as follows:

5. THEOREM. Let $A\in\Sigma$. Then $A\in E(\Sigma)$ iff $A_{\mathcal{B}}$ is atomless and for all existential L-formulas $\varphi(\bar{x})$ and all $\bar{a}\in \underline{A}^n$, $[\![\varphi(\bar{a})]\!] \geq \operatorname{int}\overline{[\![\varphi(\bar{a})]\!]}$ and $\operatorname{cl}[\![\varphi(\bar{a})]\!] \geq \overline{[\![\varphi(\bar{a})]\!]}$ (where int, cl are taken in the topological space Sp_A with $\mathcal{L}_A$ as a basis of open sets).

At first glance, the second condition in the theorem appears to be just slightly weaker than the corresponding condition $[\![\varphi(\bar{a})]\!] \geq \overline{[\![\varphi(\bar{a})]\!]}$ which means that $\underline{A}_i \in E(\text{Mod } K)$ for all $i\in Sp_A$. In case K has a model companion, the two are in fact equivalent:

6. COROLLARY. Suppose E(Mod K) is elementary. Then $E(\Sigma) = \{A\in\Sigma \mid A$ atomless, $\underline{A}_i \in E(\text{Mod } K)$ for all $i\in Sp_A\}$ is also elementary.

This can be used to reprove the existence of a model completion for commutative regular rings, commutative regular f-rings (comp. Macintyre [1973], Weispfenning [1975]), and to prove this fact for lattice-ordered abelian groups with projector and weak unit. The situation changes radically, if K has no model companion: There exists such a K and $A\in E(\Sigma)$ with no canonical factor $\underline{A}_i \in E(\text{Mod } K)$. More generally, for any countable universal theory K such that Mod K has the amalgamation property, and any $n<\omega$ there exists $A\in E(\Sigma)$ with at least n factors $\underline{A}_i \notin E(\text{Mod } K)$. This applies in particular to (non-commutative) strongly regular rings, where K is the theory of skewfields. The best general information we have on the factors $\underline{A}_i$ is the following:

7. COROLLARY. Suppose $A\in E(\Sigma)$. Then $\underline{A}_i \models Th_{\forall\exists}(E(\text{Mod } K))$ for all $i\in Sp_A$, and $\underline{A}_i \in E(\text{Mod } K)$ if $\{i\}\in\mathcal{L}_A$.

A characterization similar to theorem 5 can be given for 1-extensions in Σ. Other properties tranferable by our methods include elimination of quantifiers, $\aleph_0$-categorical model companions, model theoretic resultants, n-completeness, n-decidability, $\aleph_0$-categoricity, prime model extensions.

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SOME σ-FIELDS OF SUBSETS OF REALS

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The origin of the present considerations lies in two following problems of Ulam:

(I) *Let* $\mathcal{B}$ *be a σ-field of subsets of the real line* $\mathbb{R}$, *which contains all Lebesgue measurable sets. Suppose that, for every uncountable partition* V *of* $\mathbb{R}$ *such that* $V \subseteq \mathcal{B}$ *and each member of* V *is uncountable, there is a selector of* V *in* $\mathcal{B}$. *Does* $\mathcal{B} = P(\mathbb{R})$? (See [10], Problem 34).

(II) *Let* $\mathcal{B}$ *be a σ-field of subsets of the real line* $\mathbb{R}$, *which contains all Borel sets. Suppose that, for every partition* V *of* $\mathbb{R}$ *into two-elements sets there is a selector of* V *in* $\mathcal{B}$. *Does* $\mathcal{B} = P(\mathbb{R})$? (See [11], p.15 and [12]).

In 1975, E. Grzegorek and I solved completely both problems by showing the following theorem.

THEOREM. There exists a σ-complete field $\mathcal{B}$ of subsets of the real line $\mathbb{R}$ such that:

(a) all Lebesgue measurable sets are in $\mathcal{B}$;

(b) all subsets of $\mathbb{R}$ of the cardinality less than 2^{ω} are in $\mathcal{B}$;

(c) for every family of pairwise disjoint two-elements subsets of $\mathbb{R}$ there exists a selector of V in $\mathcal{B}$; and

(d) $\mathcal{B} \neq P(\mathbb{R})$, i.e. $\mathcal{B}$ is proper.

See [4] and [5].

Since the formulation of problems (I) and (II) suggested a positive answer rather than the negative one, and our Theorem above gives a negative answer to both (I) and (II), we were trying to "save" anything from Ulam problems by adding some extra assumptions on the field $\mathcal{B}$ in the above formulation of (I) and (II). The first attempt in this direction has been formulated in our paper [5]. We were asking there the following problem.

(III) *Let* $\mathcal{B}$ *be a* σ*-field of subsets of the real line* $\mathbb{R}$, *which contains all Lebesgue measurable sets. Suppose that, for every partition* $V \subseteq \mathcal{B}$ *of* $\mathbb{R}$ *there exists a selector of* V *in* $\mathcal{B}$. *Does* $\mathcal{B} = P(\mathbb{R})$?

We conjectured that the answer is NO, at least in ZFC + CH. (Notice that, if e.g. 2^{ω} is singular and each subset of $\mathbb{R}$ of the cardinality less than 2^{ω} is Lebesgue measurable, then the answer for (III) is YES).

Another approach in giving some extra assumptions to (I) and (II) had been suggested to us by Professor S. Gładysz. Namely, he drove our attention on to the fact that real line $\mathbb{R}$ is a group, consequently it seems to be natural to ask in all questions (I), (II) and (III) about fields invariant under translations. Unfortunately, it is quite easy to show the following proposition.

PROPOSITION. Let $\mathcal{B}$ be a proper λ-complete field of subsets of the real line $\mathbb{R}$, which contains all one-element subsets of $\mathbb{R}$. Suppose $\mathcal{B}$ is invariant under translations. Then for each $2 \leq \delta < \lambda$ there is a partition V of $\mathbb{R}$ into at least δ elements sets without any selector in $\mathcal{B}$. (For more detailed discussion see [15]).

This Proposition shows that there only remains the following question (being an "invariant" version of (I)).

(IV) *Let* $\mathcal{B}$ *be an invariant under translation* σ*-field of subsets of the real line* $\mathbb{R}$, *which contains all Lebesgue measurable sets. Suppose that, for every partition* V *of* $\mathbb{R}$ *into uncountable sets there is a selector of* V *in* $\mathcal{B}$. *Does* $\mathcal{B} = P(\mathbb{R})$?

It turned out that assumming something like CH in both questions (III) and (IV) the answers are again NO.

The last question discussed here, which is also closely related with Ulam problems, arose when Grzegorek and I were trying to solve (I) and (II). Namely, our way of constructing fields required in (I) and (II), was the following. We were searching for $\mathcal{B}$ to be a field generated by the field of all Borel subsets of $\mathbb{R}$ and by a σ-ideal which has some combinatorial properties and extends the ideal of all sets having the Lebesgue measure zero. This has led us to the following class of ideals.

DEFINITION. An ideal I on κ is an Ulam ideal iff for every partition U of κ into at least two-elements sets, there is a selector of U in I.

It has turned out that the class of Ulam ideals plays quite an important rôle in investigations of structural properties of ideals. (For more informations about Ulam ideals see [5], also [13], [1] and [2]) Nevertheless, until Autumn 1977, the following problem had been open.

(V) *Let J be a λ-complete ideal on κ. Does there exist a λ-complete Ulam ideal I on κ such that $J \subseteq I$?*

Extending some partial results of Grzegorek and me [5], and Taylor [9], and, in fact, using Taylor's technique, we prove that the answer for (V) is YES. (For more informations see [9] and [14]).

The aim of this paper is to give larger or shorter outlines of the proofs of three theorems answering the questions (III), (IV) and (V).

Let begin from the end, i.e. begin with the problem (V). To simplify the formulations, recall the following notion introduced by A. Taylor (see e.g. [9]).

DEFINITION. An ideal I is friendly with respect to a class $\mathcal{K}$ of ideals on κ, if for each $K \in \mathcal{K}$ there is a permutation π of κ such that the ideals K and $\pi * I$ are compatible, i.e. the ideal generated by K and $\pi * I$ is proper.

Notice that, if I is friendly with respect to $\mathcal{K}$ and I is Ulam, then each ideal from $\mathcal{K}$ can be extended to an Ulam ideal. Thus, to solve (V) it is sufficient to find a λ-complete Ulam ideal which is friendly with respect to the class $\mathcal{K}_\lambda$ of all λ-complete ideals on κ. (As a matter of fact, this point may be quite delicate, because no proper extension of an Ulam ideal is friendly with respect to that class - see [1] and [13]). In fact all partial results in solving (V) were obtained using that way.

The first one has been obtained by Grzegorek and me (see [5]), and it was just sufficient to solve problems (I) and (II). Namely, we had proved the following theorem.

THEOREM. The ideal NS_κ of all nonstationary subsets of a regular uncountable cardinal κ is friendly with respect to the class $\mathcal{K}_p$ of all ideals satisfying the following combinatorial property:

$J \in \mathcal{K}_p$ iff there exists a partition $\mathcal{U}$ of κ into κ sets, each of which has the cardinality κ, such that no selector of $\mathcal{U}$ is in J.

Then, a much better result, for κ-complete ideals on κ, was obtained by A. Taylor [9].

<u>THEOREM</u>. Let κ be a successor cardinal. Then the ideal NS_κ of all nonstationary subsets of κ is friendly with respect to the class of all κ-complete ideals on κ.

The proof uses the following technique. Let J be a given ideal on κ. Consider the set S of all functions $g: \kappa \to \kappa$ such that for each $\xi < \kappa$, we have $|g^{-1}(\{\xi\})| < \kappa$. Introduce the relation $<_J$ on S by

$$f <_J g \quad \text{iff} \quad \{\xi: g(\xi) \leq f(\xi)\} \in J.$$

It is easy to see that $<_J$ is well-founded whenever J is ω_1-complete. Moreover, if f is a $<_J$-minimal element of S then $f * J$ and NS_κ are compatible. Thus using the κ-completeness of J and the fact that κ is a successor cardinal, we can extend J by adding a new set A such that f is one-one on A, and the ideals $f * J(A)$ and NS_κ are compatible. (For more informations and details see [2] or [9]).

Unfortunately this proof does not work when κ is inaccessible. More precisely, one can show (see e.g. [1]) that NS_κ is friendly with respect to $\mathcal{K}_\kappa$ (the class of all κ-complete ideals on κ) iff κ is a successor cardinal. Another defect of this proof lies in the fact that it is not too easy to see what can we do whenever J is not a κ-complete ideal on κ. To omit all those problems let introduce the following ideals on κ. Let $\lambda \leq \kappa$ be regular.

$X \in I_\kappa^\lambda$ iff *there is a cardinal* $\theta < \lambda$ *and a regressive function* $f: X \to \kappa$ *such that, for each* $\xi < \kappa$ *we have* $|f^{-1}(\{\xi\})| < \theta$.

It is not too difficult to check that if $\lambda = \kappa$ and κ is a successor cardinal then $I_\kappa^\kappa = NS_\kappa$ and that $I_\kappa^\lambda \subsetneq NS_\kappa$ otherwise. Now, repeat Taylor's proof. Namely, let J be a given λ-complete ideal on κ. Consider the set S_λ of all functions $g: \kappa \to \kappa$ such that there is a cardinal $\theta_g < \lambda$ that for each $\xi < \kappa$, $|g^{-1}(\{\xi\})| < \theta_g$. As before introduce the relation $<_J$ on S_λ. Again $<_J$ is well-founded whenever $\lambda \geq \omega_1$. Moreover, if f is a $<_J$-minimal element of S_λ then $f * J$ and I_κ^λ are compatible. Thus, using the fact that, for some $\theta_f < \lambda$ we have $(\forall\xi<\kappa)|f^{-1}(\{\xi\})| < \theta_f$ and J is λ-complete, we can extend our ideal J by adding a new set A such that f is one-one on A and the ideals $f * J(A)$ and I_κ^λ are compatible. (For more informations and details see [14]). This construction yields the following theorem which solves the problem (V).

THEOREM. A. Let J be a λ-complete ideal on κ. Then there exists a λ-complete Ulam ideal I on κ such that $J \subseteq I$.

To solve (III) and (IV) we use an old notion introduced by Lusin and Sierpiński (see properties L and S in [7], pages 36, 80, 81 and 82).

DEFINITION. A subset $X \subseteq \mathbb{R}$ is a Lusin set if for each set N of Lebesgue measure zero, we have $|X \cap N| \leq \omega$

A key to solve (IV) is the following theorem.

THEOREM. (Assume CH). There exists an invariant under translations σ-complete ideal I on $\mathbb{R}$ which contains all sets of Lebesgue measure zero such that for every partition V of $\mathbb{R}$ into uncountable sets there is a selector of V in I.

To prove it, we construct (with a small modification of Sierpiński's construction - see e.g. [8]) a Hamel basis E for $\mathbb{R}$ which is a Lusin set. Let $E = \{e_\alpha : \alpha < \omega_1\}$, and let, for $\alpha < \omega_1$, E_α be the linear subspace of $\mathbb{R}$ spaned by the set $\{e_\xi : \xi \leq \alpha\}$. Put $F_\alpha = E_\alpha - \bigcup\{E_\xi : \xi < \alpha\}$, and define a function $r: \mathbb{R} \to \omega_1$, by $r(x) = \alpha$ iff $x \in F_\alpha$.

The crucial point of the proof is the following fact.

FACT. If N has Lebesgue measure zero then r(N) is a nonstationary subset of ω_1.

Indeed, suppose that for some N of Lebesgue measure zero, we have that r(N) is stationary.

First remark that without loss of generality we can assume that r is one-one on N. Since $\mathbb{R}$ is treated as a linear space over the countable field of rationals $\mathbb{Q}$, we can assume without loss of generality that there is some $n < \omega$ and non-zero rationals $s_0, \ldots, s_n$ such that each $x \in N$ has the following form

$$x = s_0 e_{\alpha_0} + \ldots + s_n e_{\alpha_n}, \quad \text{where} \quad \alpha_0 > \ldots > \alpha_n.$$

Then, using n times Fodor Theorem, we can assume, again without loss of generality, that there are countable ordinals $\beta_1 > \ldots > \beta_n$, such that for each $x \in N$, we have

$$x = s_0 e_{\alpha_0} + s_1 e_{\beta_1} + \ldots + s_n e_{\beta_n}$$

But then the set $C = \frac{1}{s_0}(N - (s_1 e_{\beta_1} + \ldots + s_n e_{\beta_n}))$ has Lebesgue measure zero and $r(C) = r(N)$, consequently $r(C)$ is uncountable, but $C \subseteq E$ and E is a Lusin set which is impossible. This contradiction proves our Fact.

To prove the theorem, define the required ideal I on $\mathbb{R}$ by:

$$X \in I \quad \text{iff} \quad r(X) \in NS_{\omega_1}$$

It is easy to see that I is a σ-complete ideal on $\mathbb{R}$ containing all sets of Lebesgue measure zero.

To see that I is invariant under translations it suffices to notice that if $x,y \in \mathbb{R}$ are such that $r(x) < r(y)$, then $r(x+y) = r(y)$. Consequently, for each $a \in \mathbb{R}$ and each $X \subseteq \mathbb{R}$ we have $r(X) \Delta r(X+a) \subseteq$ $\subseteq r(a)$.

Finally, let $V = \{V_\alpha : \alpha < \omega_1\}$ be a partition of $\mathbb{R}$ into uncountable sets. Since, for each $\xi < \omega_1$, the set $r^{-1}(\{\xi\})$ is countable, the family $\{r(V_\alpha): \alpha < \omega_1\}$ consists of uncountable sets. By Sierpiński Rifining Theorem (see [7]), there is a family $\{U_\alpha : \alpha < \omega_1\}$ of pairwise disjoint uncountable sets such that, for each $\alpha < \omega_1$, $U_\alpha \subseteq r(V_\alpha)$. It is easy to see that there is a selector G of $\{U_\alpha : \alpha < \omega_1\}$ in NS_{ω_1}. But then $r^{-1}(G) \in I$ and $r^{-1}(G) \cap V_\alpha \neq 0$, for all $\alpha < \omega_1$. Consequently, there is a selector of V in I. This finishes the proof of our Theorem.

Using this Theorem and Corollary 3 from [5], we can see that the field generated by our ideal I and the field of all Borel subsets of $\mathbb{R}$ is a proper field which gives the answer NO for the problem (IV). Thus we have the following theorem.

THEOREM. B. Assume CH. There exists a proper σ-field $\mathcal{B}$ of subsets of $\mathbb{R}$ such that

(a) $\mathcal{B}$ contains all Lebesgue measurable sets;

(b) $\mathcal{B}$ is invariant under translations of $\mathbb{R}$;

(c) for every partition V of $\mathbb{R}$ into uncountable sets there is a selector of V in $\mathcal{B}$.

We do not know if the assumption of CH in Theorem B is redundant.

To solve (III), we need a stronger concept than just Lusin sets. Call, after T.G. McLaughlin (see [6]), a subset $X \subseteq \mathbb{R}$ a strongly Lusin set, if for each Lebesgue measurable set N we have $|X \cap N| \leq \omega$ iff N has Lebesgue measure zero.

Assume CH. Then notice that if X is a strongly Lusin set then we can order X in the type ω_1 such that for each set N of positive Lebesgue measure, the set $N \cap X$ is a stationary subset of X. This fact yields some consequences. First of all, repeating the arguments used in the proof of Theorem B, with a Hamel basis being a strong Lusin set, we can get the following strengthening of Theorem B.

THEOREM. B′. Assume CH. Then there exists a σ-additive invariant measure m on $\mathbb{R}$ such that, for each Lebesgue measurable set X, its Lebesgue measure is just m(X), and the σ-field of m-measurable sets satisfies the thesis of Theorem B.

Another application of strong Lusin sets is the following theorem.

THEOREM. C. (Brzuchowski - Cichoń). Assume CH. Then there exists a σ-field $\mathcal{B}$ of subsets of the real line $\mathbb{R}$ such that for each partition $\mathcal{V} \subseteq \mathcal{B}$ of $\mathbb{R}$ there exists a selector of $\mathcal{V}$ in $\mathcal{B}$.

Indeed, let X be a strong Lusin set. Let $\mathcal{I}$ be the ideal of all nonstationary subsets of X. Let $\mathcal{B}$ be the σ-field generated by $\mathcal{I}$ and the field of all Lebesgue measurable sets. Then $\mathcal{B}$ satisfies all requirements of Theorem C.

This theorem solves (III). In fact, to prove Theorem C, much less than CH is neccessary. On the other hand, J. Cichoń has informed me, that the regularity of 2^{ω} is not enough to prove Theorem C. For more detailed discussion and other applications of strong Lusin sets see [3].

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