COMBINATIONAL ARITHMETIC

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C O M B I N A T I O N A L

A R I T H M E T I C

By

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C O N T E N T S

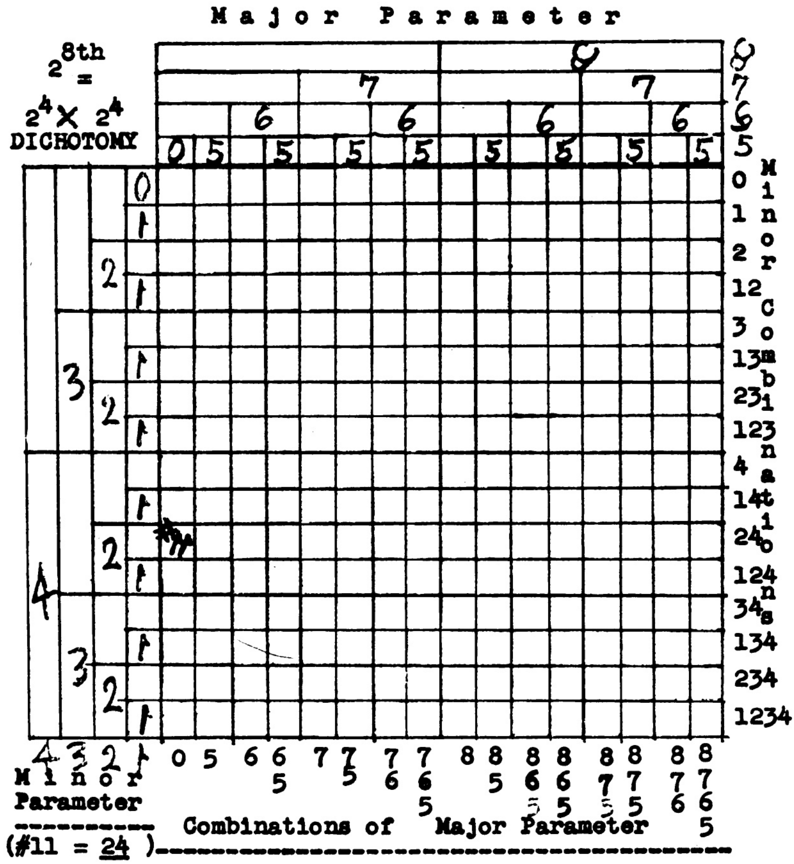


ILLUSTRATION OF TABULAR METHOD OF CONSTRUCTING & COUNTING COMBINATIONS (or Polygrams) CONSECUTIVELY.

C O M B I N A T I O N A L   A R I T H M E T I C

I. INTRODUCTION - In earlier books we have presented the algorithms of Permutations, such as the Plus-One Algorithm, the method of finding the number of a given perm, with its reverse which finds the perm corresponding to a given number. In Book PIH, as part of the analysis of point-sets in different dimensions we introduced Combinations, as ascribed to points of figures up to the sixth dimension. Here, in this new book we shall study Combinations in general, applying to figures of all dimensions, & set forth, for the first time the algorithms for performing all the operations of Arithmetic by means of combinations. Thus we shall show how to determine the number of any particular combination, of any size or degree & conversely, to discover its combination when there is given any number of any magnitude, integral, fractional, or mixed. Then we shall explain how to do addition, subtraction, multiplication, division, & extract roots, using combinations instead of their corresponding numbers as in common arithmetic.

II. DEFINITION - Let N stand for any number of things, elements, or points (#s); then we can select from N a number, T, in various ways, such that the value of T may = N, or any number less than N, or zero (0). Then, N minus T, (N-T), may = 0, 1, 2, . . . etc, up to N. E.g., if N = 4, then T may = 4, 3, 2, 1, or 0 & N-T may =, respectively, 0, 1, 2, 3, or 4.

Now, we can count the points of any set, ascribing some number to that set such that the integers from 1 to N have a one-one correspondence with the several #s of the set; thus, if we have a 4-# set (as, e.g. with the corners of a square), its constituent #s may be called #1, #2, #3, #4.

In the study of Permutations we observe a variable ascription of counters or digits, such that the N digits, (from 1 to N), may be applied to the #s of an N-# set in N! (N-factorial = 1 x 2 x 3 x . . . N) different ways constituting the N! possible permutations of N #s, taken all at a time. But in the study of Combinations we have the ascription invariable, such that the N digits may be applied to the N #s in only one way. Thus, while we can have 1 x 2 x 3 x 4 = 24 different permutations of 4 #s, we can have only one combination of 4 #s if we take them all at one time. Taking less than all we can get more than the one combination, according to how many (T) we choose, except when T=0, in which case we get also but one combination.

EVERY SET, OF T #s EACH, TAKEN FROM N #s, WITH NO ATTENTION PAID TO THE ORDER OF THE #s IN THE SET, IS CALLED A COMBINATION FROM THE N #s.

Thus, when N = 3, the number of #s is 3 - viz- #1, #2, #3, & the possible combinations, in this instance, are eight, viz - 0, 1, 2, 12, 3, 13, 23, & 123. Note that the combinations, 13 & 31, are identical, therefore we include but one of them in an exhaustive inventory; similarly, while 123, itself, can have six different permutations, i.e.- 123, 132, 213, 231, 312, 321, all these six are the same one combination, i.e., 123.

For uniformity we shall usually write the terms of a combination in ascending numerical sequence, without inversions, the lower number or digit preceding the higher, but if inversions should occur in our calculations they are not to be counted as combinations different from the number one permutation in the given case.

III - HOW TO CALCULATE THE NUMBER OF COMBINATIONS POSSIBLE - The student should be familiar with this from ordinary arithmetic or algebra. Given N as the number of things, T as the number chosen from N, (! as the factorial sign), & C as the number of possible combinations of N #s taken T at a time, then, $C = \frac{N!}{T!(N-T)!}$ Suppose N=5 & T=3, meaning we are choosing #s from a 5-# set, 3 at a time, viz- making 3-# sets outof a 5-# set, then, the application of the formula becomes $= \frac{5!}{3! \times (5-3)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)\times(2 \times 1)} = \frac{120}{12} = 10.$ In practice, this can be simplified by carrying the numerator down only to T terms, while we carry the denominator up from 1 to T, thus, e.g. - $\frac{5 \times 4 \times \not{3}}{1 \times 2 \times 3} = \frac{60}{6} = 10.$ This shows how to get the number of combinations possible from any number of things taken any particular number at a time. Now, to find the total number of possible combinations from N #s when T may vary throughout its range, the formula is found by adding together the values of C for all the values of T. Thus, if N = 4, then T may = 0, 1, 2, 3, &/or 4 & S, or the sum of all the combs. = C(N=4,T=0)+C(N=4,T=1)+C(N=4,T=2) + C(N=4,T=3) + C(N=4,T=4) = 1 + 4 + 6 + 4 + 1 = 16, or the sum of the coefficients in the expansion of (1 + 1) to the 4th power by the binomial theorem, which is the same as the expansion of 2 to the 4th power. Consequently, the formula for the total number of combinations from N #s = 2 Nth power. E.g., let N = 7, then the total combs. = 2^7 = 2x2x2x2x2x2x2 = 128. Here we differ slightly from the result taught in common arithmetic or algebra, which neglects the combination where T = 0; for, neglecting the combination where we take them none at a time, the formula for the total number is $2^N - 1$. Our own work is of the utmost generality, hence we must include the value, T= 0.

IV - COMBINATIONS AS POLYGRAMS - Now, we begin to advance beyond ordinary arithmetic & algebra & continue the investigations already originated in previous works of ours such as Book PIH, TA-KHU, N-W, TROPERMIC CALCULUS, GRAMMAR OF CHANGES, etc. A <u>polygram</u> is a figure made up of grams, or straight lines (unbroken & broken) placed, one above the other, each grammic place, called a principal, representing a dimension of the geometrical figure of which the polygram, itself, stands for one point. When the geometrical figure is of the second degree, i. e., has two poles or sides in each dimension, then, each gram must have two forms, so that either pole or side may be adequately represented. These two forms, in this case, are called, respectively, YANG (———) & YIN (—— ——), also expressed by the numbers 1 & 2. We can have figures of any number of dimensions, so we can have polygrams of any number of principals. Consequently, (this book considers only 2nd degree polygrams), any polygram (in this book) consists of a heap of yangs &/or

yins, or a series of the numbers, 1,2, etc, as,e g..the number 3, is represented by the Digram, — — , or 12, if we read up & 21, if we read down. The ——— principals of a polygram are ordered & counted from the bottom up, thus the first principal is represented by the bottom gram, the second by the one immediately above it & so on.

Now, since only yangs & yins, or 1's & 2's will occupy our attention in this connection, any polygram can be uniquely represented simply by numbering its principals which are <u>yins</u> & omitting the numbers of those principals which are filled with yangs. Thus the above digram has a yin in the 2nd principal,hence can be uniquely represented by the number 2, whereas the following Trigram, — — , which has yins in all three principal places can be = = uniquely represented by the series of numbers,123. The tetragram, 1212, which is #11,can be adequately expressed by the combination of numbers <u>24</u>, since its 2nd & 4th principals are occupied by yins & it is to be understood that the principals not mentioned are filled with yangs. Similarly, any <u>combination</u>, of digits or numbers, no two of which are alike, stands uniquely for some polygram (of the second degree) & any polygram of any number of principals (2nd deg.) can be uniquely expressed by some combination of numbers.

V - POLYGRAMS &/OR COMBINATIONS IN CONSECUTIVE SEQUENCE -

The array of points which make up a geometrical figure, as,e.g., a cube, tetrahedron, tetrak, etc, have a one-one correspondence with a set of polygrams, each # ascribed to a different polygram. We can arrange the #s in consecutive sequence, as 1, 2, 3, 4 .., similarly, with their polygrams,or combinations. This consecutive sequence has definite characteristics such that we can construct it without considering specifically the ascribed numbers. This is done empirically by a method of successive dichotomy from the highest principal down.

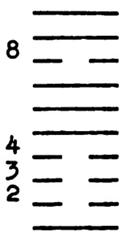
```
              1         2
   '      1     2     1     2
(Index) 1   2 1   2 1  2  1  2
   '    1 2 1 21 21 2121 21 21 2
---i--- ------------------------
    pg  1 1 1 11 11 1222 22 22 2
    or  1 1 1 12 22 2111 12 22 2
    la  1 1 2 21 12 2112 21 12 2
    ym  1 2 1 21 21 2121 21 21 2
_____s  ------------------------
```

The first # has a yang(1) in each principal,regardless of the number of dimensions & is ,here, 1111; the #2 changes the first principal to a yin(2) & is thus(reading up) 2111;the #3 has a yin in the 2nd principal only & hence is 1211 & the remaining #s have yins in the principal or principals where you see a 2; e.g. the 12th, = #12, = 2212,(reading up). Study of this system reveals that in the first principal, the sex of the grams alternates, or changes "one at a time"- viz - 12 12 12 12, etc.Thus in a series of numbers consecutively,the odd numbers all begin with a yang & the evens with a yin. In the 2nd principal they(yangs & yins)alternate by two's or two at a time,thus- 11 22 11 22 11 & so on. In the 3rd principal they alternate by fours, or four at a time, thus - 1111 2222. In the 4th, they alternate by eights, as 11111111 22222222,etc. In the 5th, they alternate by sixteens, 1111111111111111 2222222222222222, etc; in the 6th by thirty-twos & so on, as far as you may wish to carry it. Let the principal in order be represented by P, then the alternation formula = 2^{P-1}; i.e., if we raise 2 to the (P-1)th power we shall have the number of grams of one sex to put together to alternate with the same number of the opposite sex in the P principal. Empirically, then, we construct a consecutive sequence of polygrams & determine the ordinal number of each polygram or combination by using the above described scheme of dichotomy.

VI - HOW TO CALCULATE THE # OF ANY POLYGRAM. There are many ways to do this. There is, e.g., the tabular method, in which they are arrayed consecutively in columns of a table, each particular polygram (or particle) located at the junction of a major (file) row & a minor (rank) row. Such a table is indexed by margins, across the top (or bottom) & down the side (right or left), having a separate or individual index for each row, vertical or horizontal & the series of individual indices (major, or minor) is called a parameter. In such a table the total number or sum of particles tabulated must be a power of 2, such, e.g. as 64 = 2^{6}th power. Then the number of major rows, as well as that of the minor rows will also (must also) be a power of 2, such that the sum of the major power plus the minor power = the total power; the distribution may be any such as to fulfil this rule; e.g. with a table of 64 particles, we may have 2^5= 32 major rows with 2^1= 2 minors; or we may have 2^4= 16 majors with 2^2= 4 minors; or 2^3 majors with 2^3 minors; etc. Then, to find the # of any particle in the table, given its major (or Mth row) & its minor (or mth row), SUBTRACT 1 FROM THE MAJOR, MULTIPLY THE REMAINDER BY THE NUMBER OF MINORS IN EACH MAJOR OF THE TABLE, & ADD TO THIS PRODUCT THE NUMBER OF THE MINOR ROW WHICH CONTAINS THE PARTICLE. Thus, the formula = (M-1)x the number of minor rows in the table, + m. To illustrate, with a 64 = 8x8 table: let M = 2 & m = 3; then, (2-1)x 8, + 3 = #11. Suppose we should redistribute the rows so as to have 64 = 4 x 16, then, (with 16 majors & 4 minors), #11 = (3-1)x 4, +3; i.e. #11 is that particle in the 3rd major & 3rd minor. In a table of this sort where the particles are numbered consecutively, the same consecutive particle always has the same #, regardless of the distribution of major & minor rows & also regardless of the size of the table, viz- total number of particles. Thus, the polygram, or combination which is #11 in a 64 table, which may be of any of the above mentioned distributions, or some other, will also be #11 in any other similarly constructed table, such as one containing 128 particles, or 256, or 512, or any other power of 2. This empirical method of counting can be rationalised into a formula, by dividing the polygram which is to be counted or enumerated into a series of smaller polygrams, but the rule involves some exceptional clauses & is too complicated to be of universal interest, hecne we skip it & proceed to explain other rules which are simpler & more interesting.

VII - THE PLUS-ONE ALGORITHM APPLIED TO COMBINATIONS -
This has been thoroughly explained in our other books so we shall touch upon it here, but lightly, so as to link up with what follows. It treats the polygram as an imprimitive

permutation. We will give an example; writing from the highest principal down we have the polygram, say, 1121112221, a decigram, to find its #. This will be the combination, 2348, (where the yin places are counted up). Let us write the combination (or polygram) as a series of yangs & yins, across from left to right corresponding to the sequence from top to bottom:-



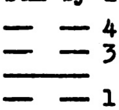
2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0	= MODULI
(512	256	128	64	32	16	8	4	2	1)	"
(1)	(1)	(2)	(1)	(1)	(1)	(2)	(2)	(2)	(1)	= TERMS
0	0	1	0	0	0	1	1	1	0	= ORDINAL REMAINDERS

<u>O-R x Mod</u> = 0 + 0 + 128 + 0 + 0 + 0 + 8 + 4 + 2 + 0 = 142
+ 1
= #143

We see that only those terms corresponding to yin places count in making the final sum which obtains the correct number of the polygram. Thus the product of the ordinal remainder x the modulus, respectively, must always be 1 x 2 (raised to a power 1 less than the place value of the yin, counting up). Therefore, to get the # of the polygram (or its corresponding combination), all we have to do is to obtain a series of powers of 2, sum them & add 1; this series of exponents is obtained by subtracting 1 from the place value of each yin. E.g. (see above), here, the combination = series of yin-place values (reading up) = 2 3 4 8, from which we subtract 1 (from each term) & get 1 2 3 7; then the # = $2^1 + 2^2 + 2^3 + 2^7 + 1 = 2 + 4 + 8 + 128 + 1$ = #143. Consequently, we have the formula, called the PERMIC METHOD = SUBTRACT 1 FROM EACH TERM OF THE COMBINATION; CONSTRUCT A SERIES OF POWERS OF 2, USING THESE REMAINDERS, RESPECTIVELY, AS THE EXPONENTS, SUM THESE POWERS (after expansion) & ADD 1.

VIII - THE DIRECT METHOD TO GET THE # OF A COMBINATION -

Now, we can eliminate the operation of subtracting 1 from each exponent, if we take simply the combinational terms (place values), themselves, as the exponents, then divide the sum by 2 before we add 1. E.g., to get the # of the comb <u>134</u>;



by the Permic Method (so-called) we get -

$2^{1-1} + 2^{3-1} + 2^{4-1} + 1 = 1 + 4 + 8 + 1$ = #14.

By the Direct Method (so-called) we get -

$\frac{2^1 + 2^3 + 2^4}{2} = \frac{2 + 8 + 16}{2} = \frac{26}{2} = 13$, + 1 = #14.

This latter method is to be preferred to the former because it involves less operations, but when large numbers are involved, we prefer to use a method even simpler which we shall distinguish as the QUICK METHOD.

IX - THE QUICK METHOD OF ENUMERATING ANY COMBINATION - For example - to find the # of the comb, <u>45</u>, corresponding to a polygram whose 4th & 5th places are filled with yins, the remaining places being occupied, all, with yangs: 11122.

```
___   ___ 5
___   ___ 4
_________
_________
_________
```

Now, we know from (V) that an odd-numbered combination must have a yang for its first, or bottom gram, consequently no ordinal number or term for the first place & conversely, if the term, 1, does not appear in the combination, the first gram must be a yang. This fact is the clue which leads to the discovery of the Quick Method (so-called).

WRITE THE COMBINATION DOWN, LEAVING SPACES FOR THE YANGS, thus-

NEXT PUT 1 OPPOSITE THE FINAL PLACE(bottom), then, for the next place above multiply the occupant of the immediately lower place by 2, but if a yin, i.e. a term, occupies this place add 1 to your doubling; if the place is unoccupied, that is, if a yang is here in the polygram, leave the product even; then the # of the combination will be one more than the top or final doubling.

```
24 .
12 .
 6 .
 3 4
 1 5
```

Evidently, where there are comb. terms the number will be odd & where they are absent it will be even. The lowest place will always have "1" to begin with. Take another example: the comb. 1267.

```
#25=24+1
--------
12=6x2
 6=3x2
 3=1x2+1
 1=1
```

```
Write: #100 = 99 + 1   (the # of the combination)
  1     99  = 49 x 2 + 1   (to make it odd)
  2     49  = 24 x 2 + 1   (to make it odd)
  .     24  = 12 x 2  (even)
  .     12  = 6 x 2   (even)
  .      6  = 3 x 2   (even)
  6      3  = 1x2 +1  (to make it odd)
  7      1  = 1       (odd)
```

X - TO FIND THE COMBINATION WHEN ITS # IS GIVEN - The formulae here will reverse the rules explained above. By the PERMIC method: subtract 1 from the #; then subtract successively the highest possible powers of 2, until there is no remainder; then to the exponents of these powers used add 1, respectively to get the terms of the combination required. E.g., given #14. 14-1=13. 13 - 8 = 5; 5- 4 = 1 1- 1 = 0; but 8 = 2^3; 4 = 2^2 & 1 = 2^0; hence, (3+1), (2+1), (0+1) = 4 3 1, the comb.

By the DIRECT method: subtract 1 from the #; multiply the remainder by 2, then from this product successively subtract the highest powers of 2 until there is no remainder; then the exponents of these subtrahends as powers of 2 are the terms of the required combination; e.g., with #14. 14-1 = 13, x 2 = 26

```
(4) 16
    10
(3)  8
     2
(1)  2
```

By the QUICK method: subtract 1 from the #, (14 - 1 = 13); then divide successively by 2, disregarding the remainders, considering only the successive quotients; count them down, if they are odd, number the place (counting down), if even omit to number; thus, e.g. we do not count the 6, here, because it is an even number. Take another example, #63. (63 - 1) = 62, then divide successively by 2, getting - , the first place (62) is even, so we do not count it, the remaining places are all odd, so we ascribe their ordinals as places to them, getting as the required combination - 2 3 4 5 6

```
#14
 13  (1)
  6  ---------
  3  (3)
  1  (4)
```

```
31 (2)
15 (3)
 7 (4)
 3 (5)
 1 (6)
```

The student should work out several more examples by all three mthods so as thoroughly to understand them.

XI - ADDITION - Having learned how to resolve a combination into its number & conversely, the next operation is to add combinations & get a combination whose corresponding # will be the sum of the numbers which correspond to the combs added. E.g., #8 + #15 = #23. But #8 = comb 123 & #15 = 234; whereas #23 = 235; hence 123 + 234 = 235. The problem then is to find an algorithm which takes the two combs,123 & 234 & by some legitimate process,transforms them into 235 & will work generally. This process can be discovered by reducing the combs, (using the direct method) thus --

$$\#8 = \frac{2^1 + 2^2 + 2^3}{2} + 1 \qquad \#15 = \frac{2^2 + 2^3 + 2^4}{2} + 1 \qquad \text{Hence } \#23 =$$

$$\frac{2^1 + 2^2 + 2^3 + 2^2 + 2^3 + 2^4 + 2^1}{2} + 1 .$$

Therefore, since this sum is in the same form as the addends, it must correspond to the combination - 1 2 3 2 3 4 1, which puts the two addends together as one combination with the addition of the extra term, 1. This then constitutes the method of addition. But in order that we may avoid having duplicate terms we must invent a method of reducing the result. This is done by considering that since the terms represent powers of 2 & whereas any two identical powers of 2 added together = a power just one higher, two 2's make a 3, two 3's make a 4 & so on. This system of combining duplicate powers or terms of the comb is called REDUCTION. Then, in the example above, 1232341 we say - 1 & 1 is 2, & 2 is 3, & 3 is 4, & 4 is 5. This leaves, then 2 & 3 which are not duplicated which with the 5 make 235, the required sum. (Q.E.D.) The rule, then = WRITE DOWN THE COMBINATIONS TO BE ADDED: PUT DOWN WITH THEM ALSO(the term) 1, FOR EACH COMB TO BE ADDED,EXCEPT THE LAST ONE, THEN "REDUCE" WHAT YOU HAVE TO GET RID OF DUPLICATES & THE FINAL RESULT IS THE SUM REQUIRED AS ONE COMB. E.g.- to find the sum of #2 + #3 + #5; the combs are 1,2, 3; so their sum will be 1 1 2 1 3 (as one comb). Reduce this = "1 & 1 is 2, & 2 is 3, & 3 is 4; leaving 1 4 the answer = #10.

XII - SUBTRACTION - Rule - WRITE THE COMBS WITH A PLUS SIGN ON THE MINUEND & A MINUS ON THE SUBTRAHEND,PUTTING AN EXTRA "1" IN THE SUBTRAHEND; THEN REDUCE, SUBTRACTING OR CANCELING WHERE POSSIBLE ; thus, if a b is the minuend & x y, the subtrahend, then the remainder = +a + b - x - y - 1, as one comb. E.G. - #67 (= comb 27) - #22 (135) = #45 (346), thus --

+ 2 7   Here, in the subtrahend, 1 & 1 is 2,which can-
- 1 3 5 1   cels the 2 of the minuend, leaving   + 7
Now, since 7 = 66 & 66 = 655, we can cancel the   - 3 5
5 below & the 5,now, above, leaving -   + 65   , but,since
5 = 433, 3 from 433 = 43; hence   - 3   the final
remainder will be 3 4 6   3 4 6.

Again, to subtract the comb 1 from the comb 4,e.g., we have +4 - 11 (or 2), then 2 from 4 = 23. Proof:1 = #2, 4 = #9; 9-2=#7= 23, the comb. In all this we give examples to clarify the rules; the student should invent & work out many more for practice.

XIII - MULTIPLICATION - Rule:- Set down the two factors as combs; add each term of the multiplier to each term of the multiplicand,arithmetically,& subtract one from each sum,putting the remainders each in the product as terms of one combination, to which as further terms of the same comb add all the terms of both factors individually; then reduce the result. Example:- Combs- 123 x 245; viz- #8 x #27 =#216 = comb, 123578.

```
                            245 = multiplicand
                            123 = multiplier
(First Operation)(1+2-1) (1+4-1) (1+5-1)=  2  4  5
                 (2+2-1) (2+4-1) (2+5-1)=  3  5  6
                 (3+2-1) (3+4-1) (3+5-1)=  4  6  7
                 ----------------------
(Second Operation)------------------------ 2  4  5
Then reduce as follows:                    1  2  3
```

2 & 2 is 3, & 3 is 4, & 4 is 5, & 5 is 6, & 6 is 7, & 7 is 8; 5 & 5 is 6, & 6 is (7; 4 & 4 is 5; leaving 1 2 3 5 7 8.

XIV - DIVISION - Rule:- Set down the divisor & the dividend as in long division of ordinary arithmetic; then estimate each term of the quotient (or partial quotient),such that the highest term of the divisor minus 1 shll be not more than the highest term of the dividend; then multiply the quotient term times the divisor & subtrast the product from the dividend; but in this multiplication bring down the duplicate of the whole divisor only the first time; after that bring down the duplicate of only the partial quotient; thus the requirement that in a product the duplicates of both factors shall be put in the product shall be fulfilled by putting the terms of the divisor down all at once & of the quotient a term at a time. Also, when there is a final remainder less than the divisor, subtract the comb term 1, from it,before making with it a fraction,with the divisor as denominator.

Example:

```
#14 / #56   / #4
134 / 12356 / 111 = 12
    134 134 1
        345
      134 1
        234
      134 1
```

Here we take 1 as the first term of the quotient,although we might try 2; then 1 x 134 = 134 134 1, which reduces to 1245,which from 12356 = 345; then the next partial quotient x 134,the divisor = 134 1, for we do not bring down 134 the second time; then this reduces to 234 & from 345 = 234; then, the last partial quotient brings down another 134 1, which = 234 & leaves no remainder. The 111 reduces to 12, the answer,which is the comb. of #4.

EXAMPLES OF DIVISION WHERE THERE IS A REMAINDER -

```
#17 / #342  / #20-2/17              #19 / #193  / #10-3/19
  5 / 13579 / 125- 1/5              25/   7 8   / 1 4- 2/25
155 = 16                      25 25 1 =  1 3 6
      3569                               12458
        26                                 458
       259                                  12 ( - 1 = 2)
        59
         2 (-1 = 1)
```

XV - SQUARE ROOT - Example - to get the sq. rt of #529 = the comb, 5 10

√5 10 / 2 3 5 = #23

(comb) 2 x 2 = 4

4 10

4 6

6 7 8 9

6 7 8 9

2 - 3
x 2 - 3
- 4
- 4 5
- 3 3
- 4 6

2 3 - 5
2 3 - 5
- 6 7
- 6 7
- 9 5 5
- 6 7 8 9

Here we try 2 as the first term of the root, then square it (2 x 2) for the first trial product; next we try 3 & set down the result of squaring 2 3, but minus the product of 2 x 2; finally we have 2 3 5 to square, but setting down only the product minus 23 x 23.

ANOTHER EXAMPLE:- √#4 7 10 12 13 16 18 / 3 4 5 8 9 = #413

x #413
=#170569 *

3 x 3 =335 — 3 3 5

3-4 x 3-4 - 3 x 3 =√ 5 6 10 12 13 16 18

6 6 7 4 4

6 8 9 12 13 16 18

34-5 x 34-5 - 34x34 =7 8 7 8 9 5 5

9 10 11 13 16 18

$345-8 \times 345-8-345^2$ = 10 11 12 10 11 12 15 8 8

10 12 13 14 18

$3458-9 \times 3458-9-3458^2$ =11 12 13 16 11 12 13 16 17 9 9

XVI - CUBE ROOT - Example - ∛#245 / 1 1 = 2 = #3x3x3 =#27*

Try first 1, then cube it = 1

123
125
125

1
111 = 12
1
12112 = 123

Then try 1 again & cube it, neglecting the first part, thus-

1-1
x 1-1
1-1
- 11
11- 11
1 2- 1 3
x 1- 1
1 2- 1 3
- 12 13
112- 113
123- 1 2 5

ANOTHER: get the cube rt. of #729, = 4 5 7 8 10 / 3 3

3 4 5 6 7
3 4 5 7 10
3 4 5 7 10

Here,

3- 3
x 3- 3
5- 5
- 5 5
3 3- 3 3
45- 4 5 6
x 3- 3
6 7- 6 7 8
- 6 7 6 7 8
3 4 5 - 3 4 5 6
3 4 5 6 7 - 3 4 5 7 10

The process is similar to that used for square root & can be extended & carried out for any root.

XVII - CONCERNING DECIMALS - To change any # with a decimal into a combination, simply disregard the decimal point, change it as though it were a whole #, then in the comb point off the same number of terms as in the original #. Thus, e.g., # 1.5 (1-1/2) would have the same comb as #15 but with one term or place pointed-off, thus - 23.4. Then, to work with this, use as though the point were not there, but always point-off the result properly. To transform a comb with a decimal point into a mixed #, i.e. an integer & a fraction, divide it by the power of ten (as a comb) indicated by the decimal part; e.g. 23.4 as a comb = 234/14, for 14 = #10, viz - #1.5 = 0-3/14, the comb. To get the sq.rt. of #2 to two decimal places, using combs, then, we should multiply the comb for #2 which = 1, by the comb for #10000 = 1 2 3 4 9 10 11 14, then get the sq.rt. of the product which is 1 2 3 4 5 10 11 12 15, which would be 3.48 +, i.e. #1.41 + carrying it to but two decimal points; or we would better multiply #2 by #10,000 getting #20,000; find the comb of that & extract its sq.rt,point off properly & have the same result. Similarly with other roots & all decimals.

XVIII - TEST PROBLEMS - WITH ANSWERS - Answer.
NUMERATION - What is the comb. which = #64? 123456.
#1094 = 1 3 7 11 (Work it out by all three methods).
#1,000,000 = 2 4 5 6 10 15 17 18 19 20. (Prove it.)
The comb. 12.67 = #1.00. 1 2 3 4 9. 10 11 14 = #10.000(Prove).

ADDITION - Add the following combinations:-
0 + 1 + 2 + 12 + 3 + 13 + 23 + 123 + 4 + 14 & get 2 3 5 6.
Whatis the sum of a + b (as combs)? Ans = + a b 1.(one comb.)
SUBTRACTION - Subtrast the comb. 247 from 1568: ans = 2367.
Subtract b from a; ans: = +a - b - 1. (three c.)

MULTIPLICATION - Multiply combs. 23 x 2345 & get 3 4 7 10.
Multiply combs a + b x combs a + b & get (those in parentheses are arithmetical,not combinational sums;those not, are,each, whole combinations)-1,2,(a+2),(b+2),(a+b),(2a-1),(2b-1); e.g. if a = 1 & b=2,then (1+2)as a comb = 121 = 3 & 1+2 x 1+2 = 3 x 3 = 45 = 1 2 3 4 3 1 3 4 3.

DIVISION - Divide the comb 7 8 12 by 1 59& get 321-56/ 159.
" " " 236 9 10 by 6 & get 1235- 234/6.

SQUARE ROOT - Get the sq.rt. of #8 to four decimal places, using combs. Ans in comb = 1234567 10. 11 12 14 15.
OTHER ROOTS - Get the cube rt. of #144,using comb. to three decimals- ans = 456.7 11 13.
Find the 5th rt. of #2 to two decimals,by combs.Ans=135. 67.
" " 5th rt. of (a) (2a)(a+1)(a+2)(2a-1)(3a-2)(3a+1)(4a-1) (4a-3)(5a-4): Ans = (a).
Question: Why do we subtract the comb 1 from the last remainder to make a fraction left over, in division? (Ans- let the student answer this question,without prompting.)
Finally, work out the combination for the number = one nonillion: answer = 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 34 36 38 39 40 41 43 44 46 47 48 51 53 54 55 58 59 63 69 71 72 75 76 77 78 79 80 81 82 83 84 85 86 87 89 91 94 95 96 100. By the direct method this should take about two hours (or more) but by the quick method it can be done in half an hour.

XIX - HOW TO BEGIN A TABULATION WHICH WILL SERVE TO FIND COMBINATIONS CORRESPONDING TO #s WHICH ARE INTEGRAL POWERS OF #10. Here,(see next page) the numbers in parentheses are terms of the combination for the power of #10 reading across the top parameter ending at the column just preceding the given combination read down the column.

#1 0	0	0	0	0	0	0	0	0	0
9 (1)	9 (1)	9 (1)	9 (1)	(9) (1)	9 (1)	9 (1)	9 (1)	9	-----
4	9 (2)	9 (2)	9 (2)	9 (2)	9 (2)	9 (2)	9 (2)	9	-----
2	4	9 (3)	9 (3)	9 (3)	9 (3)	9 (3)	9 (3)	9	-----
1 (4)	2	4	9 (4)	9 (4)	9 (4)	9 (4)	9 (4)	9	
	6	2	4	9 (5)	9 (5)	9 (5)	9 (5)	9	
	3 (6)	1 (6)	2	4	9 (6)	9 (6)	9 (6)	9	
	1 (7)	5 (7)	6	2	4	9 (7)	9 (7)	9	
		7 (8)	8	1 (8)	2	4	9 (8)	9	
		3 (9)	9 (9)	0	6	2	4	9	
		1 (10)	9 (10)	5 (10)	3 (10)	1 (10)	2	4	
			9 (11)	7 (11)	6	5 (11)	6	2	
			4	8	8	2	8	1	
			2	4	4	1 (13)	4	0	
			1 (14)	2	2	0	7 (14)	0	
				6	1 (15)	0	3 (15)	5	
				3 (16)	0	5 (16)	1 (16)	7	
				1 (17)	5 (17)	2	5 (17)	8	
					7 (18)	6	2	9	
					3 (19)	8	1 (19)	4	
					1 (20)	9 (20)	0	7	
						9 (21)	5 (21)	3	
						4 (	7 (22)	6	
						2	3 (23)	8	
						1 (24)	1 (24)	9	
							5 (25)	9	
							2	9	
							1 (27)	4	
								7	

Explanation: first we subtract 1 from the # on the major parameter; leaving a # all of whose digits are 9's as far as we carry it. Then, we divide successively by 2, treating each rank as a whole as far as we can go, & so on - it is merely a repetition of the quick method, tabulated. Then for the combs. which go with the #s we put a term, viz- count only the figures which are odd, not writing down the count of those which are even. Thus, eg., in #10000, which reads down beneath its final zero as (reading down), 99994268999421, the odd digits are only those as 1234 911 14 indicated by the corresponding numbers 10 in parentheses to the right on the same rank, viz - (1)(2)(3)(4)(9)(10)(11)(14) which is the combination = to #10,000. The student should make up this table & carry it somewhat further for practice & for reference.

XX - CONCLUSION - In this book we have shown how any ordinary number can be represented uniquely by a specific combination of numbers & conversely & how to do the ordinary operations of arithmetic, such as addition, subtraction, multiplication & division, & extraction of roots, using these corresponding <u>combinations</u>, instead of the common #s, getting results in combinations which when translated back into #s again = the same results which would have been obtained had we used the regular #s throughout the computations.

In this it has been necessary to employ polygrams of the second degree, i.e. with two poles in each principal. Thus when we make up any particular combination which will be written as a series of numbers consecutively, from 1 to N, in each consecutive, ordinal place we write a number or we do not write one; hence there is a choice of two alternatives. If we had a choice of three alternatives then we should require polygrams of the third degree, which supposition can be studied further by the earnest investigator. This book is an original contribution to the theory of numbers.

Thanks to everything! C.F.RUSSELL 1 November 1944

GETTING THE COMBINATION FROM ITS NUMBER		
#156 −1 155 −128 (7+P=(8) 27 16 (4 (5) 11 8 (3 (4) 3 2 (P (2) P (0 (P)	#156 −1 155 ×2 310 −256 = (8) 54 32 (5) 22 16 (4) 6 4 (2) 2 (P)	#156 −1 2 | 155 odd (P) 77 o (2) 38 even 19 o (4) 9 o (5) 4 e 2 e P o (8)
PERMIC	**DIRECT**	**QUICK**
(P) −1 = 0 = P (2) P 2 (4) 3 8 (5) 4 16 (8) 7 128 155 + P #156	(P) 2 (2) 4 (4) 16 (5) 32 (8) 256 2 | 310 155 + P #156	odd (8) P even — 2 e — 4 o (5) 9 o (4) 19 e — 38 o (2) 77 o (P) 155 + P #156
GETTING THE NUMBER FROM ITS COMBINATION		

ILLUSTRATING THE THREE METHODS OF ENUMERATION