COLLECTED PAPERS OF CHARLES SANDERS PEIRCE

VOLUME III

EXACT LOGIC

(Published Papers)

AND

VOLUME IV

THE SIMPLEST MATHEMATICS

EDITED BY

CHARLES HARTSHORNE

AND

PAUL WEISS

TWO VOLUMES IN ONE

THE BELKNAP PRESS OF
HARVARD UNIVERSITY PRESS

CAMBRIDGE, MASSACHUSETTS

1960

Second Printing

Volumes I–VI of the *Collected Papers of Charles Sanders Peirce* are now published by the Belknap Press of Harvard University Press, but responsibility for the content of these volumes and for the care of the entire corpus of Peirce Papers remains with the Department of Philosophy of Harvard University. The present volumes are reproduced by offset lithography from the first edition. Lists of errata are included.

DISTRIBUTED IN GREAT BRITAIN BY
OXFORD UNIVERSITY PRESS, LONDON

PRINTED IN THE UNITED STATES OF AMERICA
LIBRARY OF CONGRESS CATALOG CARD NUMBER 60–9172

VOLUME III

EXACT LOGIC

(Published Papers)

Errata

3.19n1 *for* vol. 9 *read* 8.1–6
3.88, line 11 for $p\infty$ read p^{∞}
3.88, line 12 for ∞x read ∞^{x}
3.97n (p. 60) prefix † and ‡ respectively to the last two lines of the page
3.97 (p. 61, lines 4–6) *for* II *read* Π *throughout*
3.110, line 13 for $x2$ read x^2
3.110 (p. 67, line 19) for $l\Delta' x - 1$ read $l^{\Delta x} - 1$
3.110 (p. 67, line 20) for $l\Delta x$ read $l^{\Delta x}$
3.110 (p. 67, line 22) for log $l\Delta x$ read log $l^{\Delta x}$
3.112 (p. 69, line 16) for $(1-{}^{u})$ read $^{(1-u)}$
3.112 (p. 69, line 18) for $(1-{}^{u})$ read $(1-u)$
3.114, line 26 for $\{^{l-1}\}$ read $\{l-1\}$
3.136n4 *for* contains *read* 'contains'
3.139 (p. 92, line 12) for $0x$ read 0^{x}
3.150, line 18 for $i_{11}i_{12}$ read i_{11}, i_{12}
3.181n1, line 2 *for* vol. 4 *read* vol. 10
3.184n1 (p. 117) *for* (1851), p. 104]. *read* (1851)], p. 104.
3.195, line 1 *for* contradiction *read* contraposition
3.200n* (p. 128) *for* vol. 3 *read* vol. 3 [of the *American Journal of Mathematics*]
*3.202, line 1 for $b = c = 0$ read $b = \infty$ $c = 0$
*3.202, line 2 for $a = d = \infty$ read $a = \infty$ $d = 0$
3.213, line 24 for $\bar{z} \times w -\!\!< v + y$ read $z \times \bar{w} -\!\!< v + y$
3.213, line 25 for $z \times \bar{w} -\!\!< x$ read $\bar{z} \times w -\!\!< x$
3.220–224 (pp. 141–143) *for* A=A : A+A : B+A : C+etc. *read* A = A:A + A:B + A:C + etc.
*3.238 (p. 149, lines 7–8) *for* also the first and second marks. *read* changing the first and second marks from | to ⌣ or from — to $\underline{\smile}$ or conversely and then interchanging the two resulting marks.

* This emendation departs from Peirce's own text, but it seems required for the reasoning.

*3.234, line 1 *for* first degree *read* zero degree
3.243, line 9 for l_8 read *ls*
3.243, line 11 for l_8 read *ˡs*
3.243, line 12 *for* servant *read* servant of —
3.243, line 13 for lo_8 read *los*
3.281, line 6 *for* as 'great *read* 'as great
3.288n1, line 3 *for* every number *read* every [other] number
3.289n* *for* Algebras *read* Algebra
3.292, line 12 for $= aJ +$ read $= aI +$
3.292n *precede footnote by* [1]
3.325 in four places, for a read α
3.331, line 13 *for* $0^{0^{(l)_{ij}} + {}^{(b)^{ij}}}$ *read* 0 to the power {0 to the power $[(l)_{ij} + (b)_{ij}]$}
3.372, line 8 *for* reverting to it. *read* being reverted to by it.
3.382, line 4 *for* things *read* thing's
3.393n1 *for* (At least 5/12) $(\bar{w} + \bar{d})$ *read* (At least 5/12) $\overline{\bar{\mathrm{w}} + \bar{\mathrm{d}}}$
3.403An* for *The American Journal of Mathematics* read the *American Journal of Mathematics*
3.403J, line 8, second parenthesis for $q_{\lambda i}$ read $q_{\lambda j}$
3.403J, line 14 for $\bar{q}_{\gamma i}$ read $\bar{q}_{\lambda i}$
3.403J, line 16 for Σ_q read Σ_k
3.403K (p. 246, line 2) for k read K
3.403L, line 21 for $\{r_{\gamma ab}$ (read $\{\bar{r}_{\gamma ab}$ (
3.415n* *for* 3416 *read* 3415
3.480 (p. 308, line 2) *for* 3 *read* 22
3.496 (p. 316, line 13) *for* with *read* with —
3.519, line 11 Transpose the variable x from the beginning of the second line of the equation to the end of the first line.
3.545, line 1 *for* % *read* Ↄ
3.558n1 for *associative algebras* read *Associative Algebra*
3.562G (p. 358, line 2) *for* beginnings *read* beginning
3.598, line 24 *for* however is *read* however, is
3.598n1 *for* in *read* am
3.603, line 7 for *Singly* read *singly*
3.603, line 8 for *Meerbesetzten* read *Mehrbesetzen*
3.603 (p. 385, line 1) for *Meerlückige-* read *Mehrlückige-*
3.610 (p. 390, line 3) *for* and no more *read* and to no more
3.616n* *for* pp. 24–27 *read* pp. 23–27
III, p. 417: Index s.n. Ladd-Franklin *for* 210n *read* 201n
III, p. 424: Index s.v. Implication, material *for* [443] *read* [441ff]

INTRODUCTION

In the editing of the present volume Peirce's punctuation and spelling have, wherever possible, been retained. Titles supplied by the editors have been marked ^{E}; and their remarks and additions are enclosed in light-faced brackets. The editors' footnotes are indicated by various typographical signs, while Peirce's footnotes are indicated by numbers. Paragraphs are numbered consecutively throughout the volume. The numbers at the top of each page signify the volume and the first paragraph of that page. All references in the text and in the indices are to the numbers of the paragraphs.

The department and the editors desire to express their gratitude to Dr. Henry S. Leonard, who has assisted with the proofs, references and editorial footnotes.

HARVARD UNIVERSITY
February, 1933

EDITORIAL NOTE

Charles Sanders Peirce was one of the most original and prolific logicians of the nineteenth century. His published papers contain important contributions to almost every phase of the subject. He radically modified, extended and transformed the Boolean algebra, making it applicable to propositions, relations, probability and arithmetic. Following De Morgan, he was one of the chief contributors to the logic of relatives. In addition to an analysis of "second intentional" relatives and a detailed classification of the main species of "first intentional" relatives, he supplied most of the fundamental theorems and distinctions in this branch of logic, providing, incidentally, two distinct algebras in terms of which they could be treated. He indicated how arithmetic, multiple algebras and quaternions could be derived from logic, made an independent discovery of the propositional function, of material and formal implication, and invented a new kind of logical diagram. Other discoveries which he did not publish will be found in volume four.

Peirce's symbolism and mode of procedure is somewhat antiquated and in many places his thought is difficult to grasp. The following selected list of important topics, together with the explanatory footnotes to the text, and the index at the end of the volume should, however, aid even the general reader to extract what is still living and important in Peirce's work. The items mentioned in the following list do not exhaust either the number of Peirce's contributions or the topics on which he will prove illuminating; they are offered solely as a guide through the mazes of his symbolism. For one unfamiliar with the history of the subject, or the technicalities of modern logic, the easiest approach is by way of paper No. XX, which consists of articles published in Baldwin's *Dictionary of Philosophy and Psychology*, to be followed by papers No. XIII to XV; after which the rest may be read in chronological order.

TOPICS OF HISTORICAL INTEREST

TOPICS OF GENERAL LOGICAL INTEREST

TOPICS OF GENERAL INTEREST

Bryn Mawr College
December, 1932.

CONTENTS

CONTENTS

EXACT LOGIC

(Previously published papers)

I

*ON AN IMPROVEMENT IN BOOLE'S CALCULUS OF LOGIC**

1. The principal use of Boole's Calculus of Logic lies in its application to problems concerning probability. It consists, essentially, in a system of signs to denote the logical relations of classes. The data of any problem may be expressed by means of these signs, if the letters of the alphabet are allowed to stand for the classes themselves. From such expressions, by means of certain rules for transformation, expressions can be obtained for the classes (of events or things) whose frequency is sought in terms of those whose frequency is known. Lastly, if certain relations are known between the logical relations and arithmetical operations, these expressions for events can be converted into expressions for their probability.

It is proposed, first, to exhibit Boole's system in a modified form, and second, to examine the difference between this form and that given by Boole himself.

2. Let the letters of the alphabet denote classes whether of things or of occurrences. It is obvious that an event may either be singular, as "this sunrise," or general, as "all sunrises." Let the sign of equality with a comma beneath it express numerical identity. Thus $a \underset{,}{=} b$ is to mean that a and b denote the same class — the same collection of individuals.

3. Let $a \underset{,}{+} b$ denote all the individuals contained under a and b together.† The operation here performed will differ from arithmetical addition in two respects: first, that it has reference to identity, not to equality; and second, that what

* *Proceedings of the American Academy of Arts and Sciences*, vol. 7, pp. 250–61, March 1867.

† I.e., $a \underset{,}{+} b$ is the class of those things which are a not-b, b not-a or both a and b.

is common to a and b is not taken into account twice over, as it would be in arithmetic. The first of these differences, however, amounts to nothing, inasmuch as the sign of identity would indicate the distinction in which it is founded; and therefore we may say that

(1) If No a is b $\quad a \mathbin{+\!\!,} b \mathrel{=\!\!,} a+b$.*

It is plain that

(2) $$a \mathbin{+\!\!,} a \mathrel{=\!\!,} a\dagger$$

and also, that the process denoted by $\mathbin{+\!\!,}$, and which I shall call the process of *logical addition*, is both commutative and associative. That is to say

(3) $$a \mathbin{+\!\!,} b \mathrel{=\!\!,} b \mathbin{+\!\!,} a$$

and

(4) $$(a \mathbin{+\!\!,} b) \mathbin{+\!\!,} c \mathrel{=\!\!,} a \mathbin{+\!\!,} (b \mathbin{+\!\!,} c).$$

4. Let a, b denote the individuals contained at once under the classes a and b; those of which a and b are the common species. If a and b were independent events, a, b would denote the event whose probability is the product of the probabilities of each. On the strength of this analogy (to speak of no other), the operation indicated by the comma may be called logical multiplication. It is plain that

(5) $$a, a \mathrel{=\!\!,} a.\dagger$$

Logical multiplication is evidently a commutative and associative process. That is,

(6) $$a,b \mathrel{=\!\!,} b,a$$

(7) $$(a,b),c \mathrel{=\!\!,} a,(b,c).$$

Logical addition and logical multiplication are doubly distributive, so that

* I. e., if $a,b=0$, $a+b=a \mathbin{+\!\!,} b$. Logical addition allows conjunction, arithmetic addition is exclusive.

† (2) and (5) embody the law of tautology — one of the features which distinguish the Boolian from the ordinary arithmetical calculus, limiting it to the numbers 1 and 0.

(8) $$(a +\!\!, b),c =\!\!, a,c +\!\!, b,c$$

and

(9) $$a,b +\!\!, c =\!\!, (a +\!\!, c),(b +\!\!, c).$$

Proof. Let $a =\!\!, a' + x + y + o$

$b =\!\!, b' + x + z + o$

$c =\!\!, c' + y + z + o$

where any of these letters may vanish. These formulæ comprehend every possible relation of a, b and c; and it follows from them, that

$$a +\!\!, b =\!\!, a' + b' + x + y + z + o \qquad (a +\!\!, b),c =\!\!, y + z + o.$$

But

$$a,c =\!\!, y + o \qquad b,c =\!\!, z + o \qquad a,c +\!\!, b,c =\!\!, y + z + o \qquad \therefore (8).$$

So

$$a,b =\!\!, x + o \qquad a,b +\!\!, c =\!\!, c' + x + y + z + o.$$

But

$$(a +\!\!, c) = a' + c' + x + y + z + o \qquad (b +\!\!, c) =\!\!, b' + c' + x + y + z + o$$
$$(a +\!\!, c),(b +\!\!, c) =\!\!, c' + x + y + z + o \qquad \therefore (9).$$

5. Let $-\!\!,$ be the sign of logical subtraction; so defined that

(10) If $b +\!\!, x =\!\!, a \qquad x =\!\!, a -\!\!, b.$

Here it will be observed that x is not completely determinate. It may vary from a to a with b taken away. This minimum may be denoted by $a - b$.* It is also to be observed that if the sphere of b reaches at all beyond a, the expression $a -\!\!, b$ is uninterpretable.† If then we denote the contradictory negative of a class by the letter which denotes the class itself, with a line above it,[1] if we denote by v a wholly indeterminate class, and if we allow $(0 -\!\!, 1)$ to be a wholly uninterpretable symbol, we have

(11) $$a -\!\!, b =\!\!, v,a,b + a,\bar{b} + [0 -\!\!, 1],\bar{a},b‡$$

* I.e., if x is a minimum, $a -\!\!, b =\!\!, a - b =\!\!, a - (a,b)$; if a maximum, $a -\!\!, b =\!\!, ab + a - b =\!\!, a$.

† I.e., Class b must be contained in Class a; b may of course be null.

[1] So that, for example, $\bar{a}$ denotes not-a.

‡ I.e., the class $a -\!\!, b$ is equal to the indeterminate class of that which is both a and b, plus the class of that which is a but not b, under the condition that there are no b's which are not a.

which is uninterpretable unless

$$\bar{a},b \underset{,}{=} 0.$$

If we define zero by the following identities, in which x may be any class whatever,

(12) $$0 \underset{,}{=} x \underset{,}{-} x \underset{,}{=} x - x$$

then, *zero* denotes the class which does not go beyond any class,* that is *nothing* or nonentity.

6. Let $a;b$ be read a logically divided by b, and be defined by the condition that

(13) $$\text{If } b,x \underset{,}{=} a \qquad x \underset{,}{=} a;b$$

x is not fully determined by this condition. It will vary from a to $a + \bar{b}$ and will be uninterpretable if a is not wholly contained under b.† Hence, allowing (1;0) to be some uninterpretable symbol,

(14) $$a;b \underset{,}{=} a,b + v,\bar{a},\bar{b} + (1;0)a,\bar{b}‡$$

which is uninterpretable unless

$$a,\bar{b} \underset{,}{=} 0.$$

7. Unity may be defined by the following identities in which x may be any class whatever.

(15) $$1 \underset{,}{=} x;x \underset{,}{=} x:x.§$$

Then *unity* denotes the class of which any class is a part; that is, *what is* or *ens*.

8. It is plain that if for the moment we allow $a:b$ to denote the maximum value of $a;b$, then

(16) $$\bar{x} \underset{,}{=} 1 - x \underset{,}{=} 0:x.¶$$

* As $[0 \underset{,}{=} (x \underset{,}{-} x)] \underset{,}{=} [0 \underset{,}{+} x \underset{,}{=} x]$, 0 added to a class is the class. 0 represents the minimum which results from subtracting a class from itself.

† If x is a minimum, $a;b = a,b = a$; if a maximum, $a;b, \underset{,}{=} \bar{a}, \bar{b} + ab$.

‡ The class $a;b$ is equal to the class of that which is both a and b plus the indeterminate class of what is neither a nor b on the condition that there are no a's which are not b.

§ As $a:b$ represents the maximum or upper limit of $a;b$ (see 8) unity represents the maximum which results from dividing any class by itself. As $(1 \underset{,}{=} x;x) \underset{,}{=} (x,1 \underset{,}{=} x)$ the result of multiplying a class by 1 is the class.

¶ As $a:b \underset{,}{=} a + \bar{b}$, $0:x \underset{,}{=} (0 + \bar{x} \underset{,}{=} \bar{x})$.

So that

(17) $x,(1-x) \mathrel{\overline{\overline{\,\,}}} 0 \qquad x \mathbin{+\!\!,} 0 : x \mathrel{\overline{\overline{\,\,}}} 1.$

9. The rules for the transformation of expressions involving logical subtraction and division would be very complicated. The following method is, therefore, resorted to.*

It is plain that any operations consisting solely of logical addition and multiplication, being performed upon interpretable symbols, can result in nothing uninterpretable. Hence, if $\varphi + \times x$ signifies such an operation performed upon symbols of which x is one, we have

$$\varphi + \times x \mathrel{\overline{\overline{\,\,}}} a,x + b,(1-x)\dagger$$

where a and b are interpretable.

It is plain, also, that all four operations being performed in any way upon any symbols, will, in general, give a result of which one term is interpretable and another not; although either of these terms may disappear. We have then

$$\varphi x \mathrel{\overline{\overline{\,\,}}} i,x + j,(1-x).$$[1]

We have seen that if either of these coefficients i and j is uninterpretable, the other factor of the same term is equal to nothing, or else the whole expression is uninterpretable. But

$$\varphi(1) \mathrel{\overline{\overline{\,\,}}} i \text{ and } \varphi(0) \mathrel{\overline{\overline{\,\,}}} j.$$

Hence

(18) $\varphi x \mathrel{\overline{\overline{\,\,}}} \varphi(1),x + \varphi(0),(1-x)$

$$\varphi(x \text{ and } y) \mathrel{\overline{\overline{\,\,}}} \varphi(1 \text{ and } 1),x,y + \varphi(1 \text{ and } 0),x,\bar{y} + \varphi(0 \text{ and } 1),\bar{x},y + \varphi(0 \text{ and } 0),\bar{x},\bar{y}.$$

(18′) $\varphi x \mathrel{\overline{\overline{\,\,}}} (\varphi(1) \mathbin{+\!\!,} \bar{x}),(\varphi(0) \mathbin{+\!\!,} x)$[2]

$$\varphi(x \text{ and } y) \mathrel{\overline{\overline{\,\,}}} (\varphi(1 \text{ and } 1) \mathbin{+\!\!,} \bar{x} \mathbin{+\!\!,} \bar{y}),(\varphi(1 \text{ and } 0) \mathbin{+\!\!,} \bar{x} \mathbin{+\!\!,} y), (\varphi(0 \text{ and } 1) \mathbin{+\!\!,} x \mathbin{+\!\!,} \bar{y}),(\varphi(0 \text{ and } 0) \mathbin{+\!\!,} x \mathbin{+\!\!,} y).$$

* See Lewis' *Survey of Symbolic Logic*, pp. 58–67, 82 and 132–174 for a very clear presentation and a development of these "transformations."

† $f(x) = x,a + \bar{x},b.$

[1] The proof offered for this is fallacious inasmuch as i and j have not been proved to be independent of x. — 1870. [Peirce follows this remark with a proof which is too long and of insufficient interest to be reproduced].

[2] Identity (18′) is reducible to (18) by development by second member by (18). — 1870.

Developing by (18) $x \mathbin{\bar{-}} y$, we have,

$$x \mathbin{\bar{-}} y \mathbin{\bar{=}} (1 \mathbin{\bar{-}} 1),x,y + (1 \mathbin{\bar{-}} 0),x,\bar{y} + (0 \mathbin{\bar{-}} 1),\bar{x},y + (0 \mathbin{\bar{-}} 0),\bar{x},\bar{y}.$$

So that, by (11),

(19) $(1 \mathbin{\bar{-}} 1) \mathbin{\bar{=}} v.\quad 1 \mathbin{\bar{-}} 0 \mathbin{\bar{=}} 1.\quad 0 \mathbin{\bar{-}} 1 \mathbin{\bar{=}} (0 \mathbin{\bar{-}} 1).\quad 0 \mathbin{\bar{-}} 0 \mathbin{\bar{=}} 0.$

10. Developing $x;y$ in the same way, we have[1]

$$x;y \mathbin{\bar{=}} 1;1,x,y + 1;0,x,\bar{y} + 0;1,\bar{x},y + 0;0,\bar{x},\bar{y}.$$

So that, by (14),

(20) $1;1 \mathbin{\bar{=}} 1 \qquad 1;0 \mathbin{\bar{=}} (1;0) \qquad 0;1 \mathbin{\bar{=}} 0 \qquad 0;0 \mathbin{\bar{=}} v.$

Boole gives (20),* but not (19).

In solving identities we must remember that

(21) $$(a \mathbin{\bar{+}} b) - b \mathbin{\bar{=}} a$$

(22) $$(a \mathbin{\bar{-}} b) \mathbin{\bar{+}} b \mathbin{\bar{=}} a.$$

From $a \mathbin{\bar{-}} b$ the value of b cannot be obtained.

(23) $$(a,b) \div b \mathbin{\bar{=}} a$$

(24) $$a;b,b \mathbin{\bar{=}} a.$$

From $a;b$ the value of b cannot be determined.

11. Given the identity $\varphi x \mathbin{\bar{=}} 0.$

Required to eliminate x.

$$\varphi(1) \mathbin{\bar{=}} x,\varphi(1) + (1-x),\varphi(1)$$
$$\varphi(0) \mathbin{\bar{=}} x,\varphi(0) + (1-x),\varphi(0).$$

Logically multiplying these identities, we get

$$\varphi(1),\varphi(0) \mathbin{\bar{=}} x,\varphi(1),\varphi(0) + (1-x),\varphi(1),\varphi(0).$$

For two terms disappear because of (17).

But we have, by (18),

$$\varphi(1),x + \varphi(0),(1-x) \mathbin{\bar{=}} \varphi x \mathbin{\bar{=}} 0.$$

Multiplying logically by x we get

$$\varphi(1),x \mathbin{\bar{=}} 0$$

and by $(1-x)$ we get

$$\varphi(0),(1-x) \mathbin{\bar{=}} 0.$$

[1] $a;b, c$ must always be taken as $(a;b), c$, not as $a;(b, c)$.

* *Laws of Thought*, vol. 2, p. 87ff.

Substituting these values above, we have

(25) $$\varphi(1),\varphi(0)\underset{,}{=}0 \text{ when } \varphi x\underset{,}{=}0.$$

12. Given $$\varphi x\underset{,}{=}1.$$

Required to eliminate x.

Let $$\varphi' x\underset{,}{=}1-\varphi x\underset{,}{=}0$$

$$\varphi'(1),\varphi'(0)\underset{,}{=}(1-\varphi(1)),(1-\varphi(0))\underset{,}{=}0$$

$$1-(1-\varphi(1)),(1-\varphi(0))\underset{,}{=}1.$$

Now, developing as in (18), only in reference to $\varphi(1)$ and $\varphi(0)$ instead of to x and y,

$$1-(1-\varphi(1)),(1-\varphi(0))\underset{,}{=}\varphi(1),\varphi(0)+\varphi(1),(1-\varphi(0)) \\ +\varphi(0),(1-\varphi(1)).$$

But by (18) we have also,

$$\varphi(1)\underset{,}{+}\varphi(0)\underset{,}{=}\varphi(1),\varphi(0)+\varphi(1),(1-\varphi(0))+\varphi(0),(1-\varphi(1)).$$

So that

(26) $$\varphi(1)\underset{,}{+}\varphi(0)\underset{,}{=}1 \text{ when } \varphi x\underset{,}{=}1.$$

Boole gives (25),* but not (26).

13. We pass now from the consideration of *identities* to that of *equations*.†

Let every expression for a class have a second meaning, which is its meaning in an equation. Namely, let it denote the proportion of individuals of that class to be found among all the individuals examined in the long run.

Then we have

(27) $$\text{If } a\underset{,}{=}b \qquad a=b$$

(28) $$a+b=(a\underset{,}{+}b)+(a,b).$$

14. Let b_a denote the frequency of b's among the a's. Then considered as a class, if a and b are events, b_a denotes the fact that if a happens b happens.

(29) $$ab_a=a,b.\ddagger$$

* *Ibid.*, p. 10.

† I.e., from the logical relations of class identity in extension to the "arithmetical" relations of numerical equality. Cf. 44.

‡ 'Arithmetical' multiplication is represented by juxtaposing the terms.

It will be convenient to set down some obvious and fundamental properties of the function b_a.

(30) $$ab_a = ba_b$$

(31) $$\varphi(b_a \text{ and } c_a) = (\varphi(b \text{ and } c))_a$$

(32) $$(1-b)_a = 1 - b_a$$

(33) $$b_a = \frac{b}{a} + b_{(1-a)}\left(1 - \frac{1}{a}\right)$$

(34) $$a_b = 1 - \frac{1-a}{b} b_{(1-a)}$$

(35) $$(\varphi a)_a = (\varphi(1))_a.$$

The application of the system to probabilities may best be exhibited in a few simple examples, some of which I shall select from Boole's work, in order that the solutions here given may be compared with his.

15. *Example* 1. Given the proportion of days upon which it hails, and the proportion of days upon which it thunders. Required the proportion of days upon which it does both.

Let $1 \mp$ days,

$p \mp$ days when it hails,

$q \mp$ days when it thunders,

$r \mp$ days when it hails and thunders.

$$p,q \mp r$$

Then by (29), $$r \mp p,q = pq_p = qp_q.$$

Answer. The required proportion is an unknown fraction of the least of the two proportions given.

By p might have been denoted the probability of the major, and by q that of the minor premiss of a hypothetical syllogism of the following form:

If a noise is heard, an explosion always takes place;

If a match is applied to a barrel of gunpowder, a noise is heard;

∴ *If a match is applied to a barrel of gunpowder, an explosion always takes place.*

In this case, the value given for r would have represented the probability of the conclusion. Now Boole (page 284) solves this problem by his unmodified method, and obtains the following answer:

$$r = pq + a(1-q)$$

where a is an arbitrary constant. Here, if $q=1$ and $p=0$, $r=0$. That is, his answer implies that if the major premiss be false and the minor be true, the conclusion must be false. That this is not really so is shown by the above example. Boole (page 286) is forced to the conclusion that "propositions which, when true, are equivalent, are not necessarily equivalent when regarded only as probable." This is absurd, because probability belongs to the events denoted, and not to forms of expression. The probability of an event is not altered by translation from one language to another.

Boole, in fact, puts the problem into equations wrongly (an error which it is the chief purpose of a calculus of logic to prevent), and proceeds as if the problem were as follows:

It being known what would be the probability of Y, if X were to happen, and what would be the probability of Z, if Y were to happen; what would be the probability of Z, if X were to happen?

But even this problem has been wrongly solved by him. For, according to his solution, where

$$p=Y_{\mathrm{x}} \qquad q=Z_{\mathrm{Y}} \qquad r=Z_{\mathrm{x}},$$

r must be at least as large as the product of p and q. But if X be the event that a certain man is a negro, Y the event that he is born in Massachusetts, and Z the event that he is a white man, then neither p nor q is *zero*, and yet r vanishes.

This problem may be rightly solved as follows:

$$\text{Let } p' \mathrel{\underset{,}{=}} Y_{\mathrm{p}} \mathrel{\underset{,}{=}} X,Y$$
$$q' \mathrel{\underset{,}{=}} Z_{\mathrm{q}} \mathrel{\underset{,}{=}} X,Z$$
$$r' \mathrel{\underset{,}{=}} Z_{\mathrm{r}} \mathrel{\underset{,}{=}} X,Z.$$

$$\text{Then, } r' \mathrel{\underset{,}{=}} p',q';p' \mathrel{\underset{,}{=}} p',q';q'.$$

Developing these expressions by (18) we have

$$r' \mathrel{\underset{,}{=}} p',q' + r'_{\mathrm{p}',\bar{\mathrm{q}}'}(p',\bar{q}') + r'_{\bar{\mathrm{p}}',\bar{\mathrm{q}}'}(\bar{p}',\bar{q}')$$
$$\mathrel{\underset{,}{=}} p',q' + r'_{\bar{\mathrm{p}}',\mathrm{q}'}(\bar{p}',q') + r'_{\bar{\mathrm{p}}',\bar{\mathrm{q}}'}(p',q').$$

The comparison of these two identities shows that

$$r' \mathrel{\underset{,}{=}} p',q' + r'_{\bar{\mathrm{p}}',\bar{\mathrm{q}}'}(\bar{p}',\bar{q}').$$

$$\text{Let } V \mathrel{\underset{,}{=}} r'_{\bar{\mathrm{p}}',\bar{\mathrm{q}}'} \mathrel{\underset{,}{=}} \frac{x,\bar{y},z}{\bar{x},y,\bar{z}+\bar{y}}$$

Now $$p',q' \mathrel{\underset{,}{=}} p' - p',\bar{q}' \mathrel{\underset{,}{=}} q' - q',\bar{p}'$$

$$\bar{p}',\bar{q}' \mathrel{\underset{,}{=}} \bar{q}' - p',\bar{q}' \mathrel{\underset{,}{=}} \bar{p}' - q',\bar{p}'$$

And $$p',\bar{q}' \mathrel{\underset{,}{=}} p' - p'_{q'}\, q' \mathrel{\underset{,}{=}} \bar{q}' - \bar{q}'_{\bar{p}'}\, \bar{p}'$$

$$\bar{p}',q' \mathrel{\underset{,}{=}} q' - q'_{p'}\, p' \mathrel{\underset{,}{=}} \bar{p}' - \bar{p}'_{\bar{q}'}\, \bar{q}'$$

Then let

$$A \mathrel{\underset{,}{=}} p'_{q'} \mathrel{\underset{,}{=}} \frac{x,y,z}{y,z}$$

$$B \mathrel{\underset{,}{=}} \bar{q}'_{\bar{p}'} \mathrel{\underset{,}{=}} \frac{\bar{x},y,\bar{z}+\bar{x},\bar{y},z+x,\bar{y},z+\bar{x},\bar{y},\bar{z}}{1-x,y}$$

$$C \mathrel{\underset{,}{=}} \bar{p}'_{\bar{q}'} \mathrel{\underset{,}{=}} \frac{\bar{x},y,\bar{z}+\bar{x},\bar{y},z+x,\bar{y},z+\bar{x},\bar{y},\bar{z}}{1-y,z}$$

$$D \mathrel{\underset{,}{=}} q'_{p'} \mathrel{\underset{,}{=}} \frac{x,y,z}{x,y}$$

And we have

$$r = \frac{Y}{Z}p + V\left(\frac{1}{Z}-q\right) - (1+V)\left(\frac{Y}{Z}p - Aq\right)$$

$$= \frac{Y}{Z}p + V\left(\frac{1}{Z}-q\right) - (1+V)\left(\frac{1}{Z}-q-B\left(\frac{1-Yp}{Z}\right)\right)$$

$$= q + V\left(\frac{1-Yp}{Z}\right) - (1+V)\left(\frac{1-Yp}{Z} - C\left(\frac{1}{Z}-q\right)\right)$$

$$= q + V\left(\frac{1-Yp}{Z}\right) - (1+V)\left(q - D\frac{Y}{Z}p\right)$$

16. *Example* 2. (See Boole, page 276.) Given r and q; to find p.

$p \mathrel{\underset{,}{=}} r;q \mathrel{\underset{,}{=}} r+v,(1-q)$ because p is interpretable.

Answer. The required proportion lies somewhere between the proportion of days upon which it both hails and thunders, and that added to one minus the proportion of days when it thunders.

17. *Example* 3. (See Boole, page 279.) Given, out of the number of questions put to two witnesses, and answered by *yes* or *no*, the proportion that each answers truly, and the proportion of those their answers to which disagree. Required,

out of those wherein they agree, the proportion they answer truly and the proportion they answer falsely.*

Let $1 \mathbin{\bar{=}}$ the questions put to both witnesses,
$p \mathbin{\bar{=}}$ those which the first answers truly,
$q \mathbin{\bar{=}}$ those which the second answers truly,
$r \mathbin{\bar{=}}$ those wherein they disagree,
$w \mathbin{\bar{=}}$ those which both answer truly,
$w' \mathbin{\bar{=}}$ those which both answer falsely.

$$w \mathbin{\bar{=}} p,q \qquad w' \mathbin{\bar{=}} \bar{p},\bar{q} \qquad r \mathbin{\bar{=}} p \mathbin{+\!\!,} q - w \mathbin{\bar{=}} \bar{p} \mathbin{+\!\!,} \bar{q} - w'.$$

Now by (28)

$$p \mathbin{+\!\!,} q = p + q - w \qquad \bar{p} \mathbin{+\!\!,} \bar{q} = p - p + 1 - q - w'.$$

Substituting and transposing,

$$2w = p + q - r \qquad 2w' = 2 - p - q - r.$$

$$\text{Now } w_{1-r} = \frac{w,(1-r)}{1-r} \qquad \text{but } w(1-r) \mathbin{\bar{=}} w.$$

$$w'_{1-r} = \frac{w',(1-r)}{1-r} \qquad \text{but } w'(1-r) = w'$$

$$\therefore w_{(1-r)} = \frac{p+q-r}{2(1-r)} \qquad w'_{(1-r)} = \frac{2-p-q-r}{2(1-r)}.$$

18. The differences of Boole's system, as given by himself, from the modification of it given here, are three.

First. Boole does not make use of the operations here termed logical addition and subtraction. The advantages obtained by the introduction of them are three, *viz.*, they give unity to the system; they greatly abbreviate the labor of working with it; and they enable us to express *particular* propositions. This last point requires illustration. Let i be a class only determined to be such that only some one individual of the class a comes under it. Then $a \mathbin{\bar{-}} i, a$ is the expression for some a. Boole cannot properly express some a.

Second. Boole uses the ordinary sign of multiplication for logical multiplication. This debars him from converting every logical identity into an equality of probabilities. Before the transformation can be made the equation has to be brought

* Cf. 2.674.

into a particular form, and much labor is wasted in bringing it to that form.

Third. Boole has no such function as a_b. This involves him in two difficulties. When the probability of such a function is required, he can only obtain it by a departure from the strictness of his system. And on account of the absence of that symbol, he is led to declare that, without adopting the principle that simple, unconditioned events whose probabilities are given are independent, a calculus of logic applicable to probabilities would be impossible.

19. The question as to the adoption of this principle is certainly not one of words merely. The manner in which it is answered, however, partly determines the sense in which the term "probability" is taken.

In the propriety of language, the probability of a fact either is, or solely depends upon, the strength of the argument in its favor, supposing all relevant relations of all known facts to constitute that argument. Now, the strength of an argument is only the frequency with which *such* an argument will yield a true conclusion when its premisses are true. Hence probability depends solely upon the relative frequency of a specific event (namely, that a certain kind of argument yields a true conclusion from true premisses) to a generic event (namely, that that kind of argument occurs with true premisses). Thus, when an ordinary man says that it is highly probable that it will rain, he has reference to certain indications of rain — that is, to a certain kind of argument that it will rain — and means to say that there is an argument that it will rain, which is of a kind of which but a small proportion fail. "Probability," in the untechnical sense, is therefore a vague word, inasmuch as it does not indicate what one, of the numerous subordinated and coördinated genera to which every argument belongs, is the one the relative frequency of the truth of which is expressed. It is usually the case, that there is a tacit understanding upon this point, based perhaps on the notion of an *infima species* of argument. But an *infima species* is a mere fiction in logic. And very often the reference is to a very wide genus.

The sense in which the term should be made a technical one is that which will best subserve the purposes of the calculus in question. Now, the only possible use of a calculation of a probability is security in the long run. But there can be no question that an insurance company, for example, which assumed that events were independent without any reason to think that they really were so, would be subjected to great hazard. Suppose, says Mr. Venn,* that an insurance company knew that nine tenths of the Englishmen who go to Madeira die, and that nine tenths of the consumptives who go there get well. How should they treat a consumptive Englishman? Mr. Venn has made an error in answering the question, but the illustration puts in a clear light the advantage of ceasing to speak of probability, and of speaking only of the relative frequency of this event to that.[1]

* *Logic of Chance*, ch. 9, section 24.

[1] See a notice, Venn's *Logic of Chance*, in the North American Review for July 1867 [vol. 9].

II

*UPON THE LOGIC OF MATHEMATICS**

Part I†

§1. THE BOOLIAN CALCULUS[E]

20. The object of the present paper is to show that there are certain general propositions from which the truths of mathematics follow syllogistically, and that these propositions may be taken as definitions of the objects under the consideration of the mathematician without involving any assumption in reference to experience or intuition. That there actually are such objects in experience or pure intuition is not in itself a part of pure mathematics.

21. Let us first turn our attention to the logical calculus of Boole. I have shown in a previous communication to the Academy,‡ that this calculus involves eight operations, *viz.*, Logical Addition, Arithmetical Addition, Logical Multiplication, Arithmetical Multiplication, and the processes inverse to these.

Definitions

1. *Identity.* $a =, b$ expresses the two facts that any a is b and any b is a.
2. *Logical Addition.* $a +, b$ denotes a member of the class which contains under it all the a's and all the b's, and nothing else.
3. *Logical Multiplication.* a,b denotes only whatever is both a and b.
4. *Zero* denotes *nothing*, or the class without extent, by which we mean that if a is any member of any class, $a +, 0$ is a.

* *Proceedings of the American Academy of Arts and Sciences*, vol. 7, pp. 402–412, September 1867.

† No other part seems to have been written.

‡ See Paper No. I.

5. *Unity* denotes *being*, or the class without content, by which we mean that, if a is a member of any class, a is a, 1.
6. *Arithmetical Addition.* $a+b$, if $a,b \underset{,}{=} 0$, is the same as $a \underset{,}{+} b$, but, if a and b are classes which have any extent in common, it is not a class.
7. *Arithmetical Multiplication.* ab represents an event when a and b are events only if these events are independent of each other, in which case $ab \underset{,}{=} a,b$. By the events being independent is meant that it is possible to take two series of terms, A_1, A_2, A_3, etc., and B_1, B_2, B_3, etc., such that the following conditions will be satisfied. (Here x denotes any individual or class, not nothing; A_m, A_n, B_m, B_n, any members of the two series of terms, and ΣA, ΣB, $\Sigma(A,B)$ logical sums of some of the A_n's, the B_n's, and the (A_n,B_n)'s respectively.)

Condition 1. No A_m is A_n.
Condition 2. No B_m is B_n.
Condition 3. $x \underset{,}{=} \Sigma(A,B)$
Condition 4. $a \underset{,}{=} \Sigma A$.
Condition 5. $b \underset{,}{=} \Sigma B$.
Condition 6. Some A_m is B_n.*

22. From these definitions a series of theorems follow syllogistically, the proofs of most of which are omitted on account of their ease and want of interest.

Theorems

I

23. If $a \underset{,}{=} b$, then $b \underset{,}{=} a$.

II

24. If $a \underset{,}{=} b$, and $b \underset{,}{=} c$, then $a \underset{,}{=} c$.

III

25. If $a \underset{,}{+} b \underset{,}{=} c$, then $b \underset{,}{+} a \underset{,}{=} c$.

* a and b are independent if they are summations of terms (4 and 5) each of whose members is distinct (1 and 2), so that there is a class of the terms of a and b together (3) and a term in a has a corresponding member in b (6). There are as many members of a,b as there are combinations of a member of a with one of b. Cf. 33.

IV

26. If $a\underset{,}{+}b\underset{,}{=}m$ and $b\underset{,}{+}c\underset{,}{=}n$ and $a\underset{,}{+}n\underset{,}{=}x$, then $m\underset{,}{+}c\underset{,}{=}x$.

Corollary. These last two theorems hold good also for arithmetical addition.

V

27. If $a+b\underset{,}{=}c$ and $a'+b\underset{,}{=}c$, then $a\underset{,}{=}a'$, or else there is nothing not b.

This theorem does not hold with logical addition. But from definition 6 it follows that

No a is b (supposing there is any a)
No a' is b (supposing there is any a')

neither of which propositions would be implied in the corresponding formulæ of logical addition. Now from definitions 2 and 6,

Any a is c
∴ Any a is c not b

But again from definitions 2 and 6 we have

Any c not b is a' (if there is any not b)
∴ Any a is a' (if there is any not b)

And in a similar way it could be shown that any a' is a (under the same supposition). Hence by definition 1,

$a\underset{,}{=}a'$ if there is anything not b.

Scholium. In arithmetic this proposition is limited by the supposition that b is finite.* The supposition here though similar to that is not quite the same.

VI

28. If $a,b\underset{,}{=}c$, then $b,a\underset{,}{=}c$.

VII

29. If $a,b\underset{,}{=}m$ and $b,c\underset{,}{=}n$ and $a,n\underset{,}{=}x$, then $m,c\underset{,}{=}x$.

VIII

30. If $m,n\underset{,}{=}b$ and $a\underset{,}{+}m\underset{,}{=}u$ and $a\underset{,}{+}n\underset{,}{=}v$ and $a\underset{,}{+}b\underset{,}{=}x$, then $u,v\underset{,}{=}x$.

* —because in transfinite arithmetic finite quantities can be added to infinites without affecting the total— $\aleph+x=\aleph+y=\aleph$ where x and y are finite and $\aleph$ is the smallest transfinite cardinal.

IX

31. If $m \mathbin{+_{,}} n \mathrel{=_{,}} b$ and $a,m \mathrel{=_{,}} u$ and $a,n \mathrel{=_{,}} v$ and $a,b \mathrel{=_{,}} x$, then $u \mathbin{+_{,}} v \mathrel{=_{,}} x$.

The proof of this theorem may be given as an example of the proofs of the rest.

It is required then (by definition 3) to prove three propositions, *viz.*

First. That any u is x.
Second. That any v is x.
Third. That any x not u is v.

First Proposition

Since $u \mathrel{=_{,}} a,m$, by definition 3
Any u is m,
and since $m \mathbin{+_{,}} n \mathrel{=_{,}} b$, by definition 2
Any m is b,
whence Any u is b,
But since $u \mathrel{=_{,}} a,m$, by definition 3
Any u is a,
whence Any u is both a and b,
But since $a,b \mathrel{=_{,}} x$, by definition 3
Whatever is both a and b is x
whence Any u is x.

Second Proposition

This is proved like the first.

Third Proposition

Since $a,m \mathrel{=_{,}} u$, by definition 3,
Whatever is both a and m is u.
or Whatever is not u is not both a and m.
or Whatever is not u is either not a or not m.
or Whatever is not u and is a is not m.
But since $a,b \mathrel{=_{,}} x$, by definition 3
Any x is a,
whence Any x not u is not u and is a,
whence Any x not u is not m.

But since $a,b \mathrel{\overline{\overline{\scriptstyle\vert}}} x$, by definition 3

Any x is b,

whence Any x not u is b,

Any x not u is b, not m.

But since $m \dotplus n \mathrel{\overline{\overline{\scriptstyle\vert}}} b$, by definition 2

Any b not m is n,

whence Any x not u is n,

and therefore Any x not u is both a and n.*

But since $a,n \mathrel{\overline{\overline{\scriptstyle\vert}}} v$, by definition 3

Whatever is both a and n† is v,

whence Any x not u is v.

32. *Corollary* 1. This proposition readily extends itself to arithmetical addition.

Corollary 2. The converse propositions produced by transposing the last two identities of theorems VIII and IX are also true.

Corollary 3. Theorems VI, VII, and IX hold also with arithmetical multiplication. This is sufficiently evident in the case of theorem VI, because by definition 7 we have an additional premiss, namely, that a and b are independent, and an additional conclusion which is the same as that premiss.

33. In order to show the extension of the other theorems, I shall begin with the following lemma. If a and b are independent, then corresponding to every pair of individuals, one of which is both a and b, there is just one pair of individuals one of which is a and the other b; and conversely, if the pairs of individuals so correspond, a and b are independent. For, suppose a and b independent, then, by definition 7, condition 3, every class (A_m, B_n) is an individual. If then A_a denotes any A_m which is a, and B_b any B_m which is b, by condition 6 (A_a, B_n) and (A_m, B_b) both exist, and by conditions 4 and 5 the former is any individual a, and the latter any individual b. But given this pair of individuals, both of the pair (A_a, B_b) and (A_m, B_n) exist by condition 6. But one individual of this pair is both a and b. Hence the pairs correspond, as stated above. Next,

* Originally m.

† Originally u.

suppose *a* and *b* to be any two classes. Let the series of A_m's be *a* and not-*a*; and let the series of B_m's be all individuals separately. Then the first five conditions can always be satisfied. Let us suppose, then, that the sixth alone cannot be satisfied. Then A_p and B_q may be taken such that (A_p, B_q) is nothing. Since A_p and B_q are supposed both to exist, there must be two individuals (A_p, B_n) and (A_m, B_q) which exist. But there is no corresponding pair (A_m, B_n) and (A_p, B_q). Hence, no case in which the sixth condition cannot be satisfied simultaneously with the first five is a case in which the pairs rightly correspond; or, in other words, every case in which the pairs correspond rightly is a case in which the sixth condition can be satisfied, provided the first five can be satisfied. But the first five can always be satisfied. Hence, if the pairs correspond as stated, the classes are independent.

34. In order to show that theorem VII may be extended to arithmetical multiplication, we have to prove that if *a* and *b*, *b* and *c*, and *a* and (*b*,*c*), are independent, then (*a*,*b*) and *c* are independent. Let *s* denote any individual. Corresponding to every *s* with (*a*,*b*,*c*), there is an *a* and (*b*,*c*). Hence, corresponding to every *s* with *s* and with (*a*,*b*,*c*) (which is a particular case of that pair), there is an *s* with *a* and with (*b*,*c*). But for every *s* with (*b*,*c*) there is a *b* with *c;* hence, corresponding to every *a* with *s* and with (*b*,*c*), there is an *a* with *b* and with *c*. Hence, for every *s* with *s* and with (*a*,*b*,*c*) there is an *a* with *b* and with *c*. For every *a* with *b* there is an *s* with (*a*,*b*); hence, for every *a* with *b* and with *c*, there is an *s* with (*a*,*b*) and *c*. Hence, for every *s* with *s* and with (*a*,*b*,*c*) there is an *s* with (*a*,*b*) and with *c*. Hence, for every *s* with (*a*,*b*,*c*) there is an (*a*,*b*) with *c*. The converse could be proved in the same way. Hence, etc.

35. Theorem IX holds with arithmetical addition of whichever sort the multiplication is. For we have the additional premiss that "No *m* is *n*"; whence since "any *u* is *m*" and "any *v* is *n*," "no *u* is *v*," which is the additional conclusion.

Corollary 2, so far as it relates to theorem IX, holds with arithmetical addition and multiplication. For, since no *m* is *n*, every pair, one of which is *a* and either *m* or *n*, is either a pair, one of which is *a* and *m*, or a pair, one of which is *a* and *n*, and is not both. Hence, since for every pair one of which is a and m, there is a pair one of which is a and the other m, and since for every pair one of which is a,n there is a pair one of which is a and the other n; for every pair one of which is a and either m or n, there is either a pair one of which is a and the other m, or a pair one of which is a and the other n, and not both; or, in other words, there is a pair one of which is a and the other either m or n.

(It would perhaps have been better to give this complicated proof in its full syllogistic form. But as my principal object is merely to show that the various theorems could be so proved, and as there can be little doubt that if this is true of those which relate to arithmetical addition it is true also of those which relate to arithmetical multiplication, I have thought the above proof (which is quite apodeictic) to be sufficient. The reader should be careful not to confound a proof which needs itself to be experienced with one which requires experience of the object of proof.)

X

36. If $ab \mathrel{\overline{\overline{,}}} c$ and $a'b \mathrel{\overline{\overline{,}}} c$, then $a \mathrel{\overline{\overline{,}}} a'$, or no b exists.

This does not hold with logical, but does with arithmetical multiplication.

For if a is not identical with a', it may be divided thus

$$a \mathrel{\overline{\overline{,}}} a,a' + a,\bar{a}'$$

if $\bar{a}'$ denotes not a'. Then

$$a,b \mathrel{\overline{\overline{,}}} (a,a'),b + (a,\bar{a}'),b$$

and by the definition of independence the last term does not vanish unless $(a,\bar{a}') \mathrel{\overline{\overline{,}}} 0$, or all a is a'; but since $a,b \mathrel{\overline{\overline{,}}} a',b \mathrel{\overline{\overline{,}}} (a,a'),b + (\bar{a},a'),b$, this term does vanish, and, therefore, only a is a', and in a similar way it could be shown that only a' is a.

XI

37. $1 \mathbin{+\!\!,} a \mathrel{\overline{\overline{,}}} 1$.

This is not true of arithmetical addition, for since by definition 7,

$$1x,1 \mathrel{\overline{\overline{,}}} x1$$

by theorem IX

$$x,(1+a) \mathrel{\overline{\overline{,}}} x(1+a) \mathrel{\overline{\overline{,}}} x1 + xa \mathrel{\overline{\overline{,}}} x + xa.$$

Whence $xa \mathrel{\overline{\overline{,}}} 0$, while neither x nor a is zero, which, as will appear directly, is impossible.

XII

38. $0,a \underset{,}{=} 0$.

Proof. For call $0, a \underset{,}{=} x$. Then by definition 3

x belongs to the class *zero*.

∴ by definition 4 $x \underset{,}{=} 0$.

Corollary 1. The same reasoning applies to arithmetical multiplication.

Corollary 2. From theorem X and the last corollary it follows that if $ab \underset{,}{=} 0$, either $a \underset{,}{=} 0$ or $b \underset{,}{=} 0$.

XIII

39. $a,a \underset{,}{=} a$.*

XIV

40. $a \underset{,}{+} a \underset{,}{=} a$.*

These do not hold with arithmetical operations.

41. *General Scholium.* This concludes the theorems relating to the direct operations. As the inverse operations have no peculiar logical interest, they are passed over here.

In order to prevent misapprehension, I will remark that I do not undertake to demonstrate the principles of logic themselves. Indeed, as I have shown in a previous paper, these principles considered as speculative truths are absolutely empty and indistinguishable.† But what has been proved is the *maxims* of logical procedure, a certain system of signs being given.

The definitions given above for the processes which I have termed arithmetical plainly leave the functions of these operations in many cases uninterpreted. Thus if we write

$$a+b \underset{,}{=} b+a$$
$$a+(b+c) \underset{,}{=} (a+b)+c$$
$$bc \underset{,}{=} cb$$
$$(ab)c \underset{,}{=} a(bc)$$
$$a(m+n) \underset{,}{=} am+an$$

we have a series of identities whose truth or falsity is entirely undeterminable. In order, therefore, *fully to define those*

* See 4n.

† See 2.467. Cf. Paul Weiss, *Erkenntnes* Bd. 2, H. 4, S. 242–8 where it is shown that all the logical propositions are variations of some such form as $PQ+\bar{P}Q+P\bar{Q}+\bar{P}\bar{Q}$.

operations, we will say that all propositions, equations, and identities which are in the general case left by the former definitions undetermined as to truth, shall be true, provided they are so in all interpretable cases.

§2. ON ARITHMETIC.*

42. *Equality* is a relation of which identity is a species.

If we were to leave equality without further defining it, then by the last scholium all the formal rules of arithmetic would follow from it. And this completes the central design of this paper, as far as arithmetic is concerned.

43. Still it may be well to consider the matter a little further. Imagine, then, a particular case under Boole's calculus, in which the letters are no longer terms of first intention, but terms of second intention, and that of a special kind. Genus, species, difference, property, and accident, are the well-known terms of second intention. These relate particularly to the *comprehension*† of first intentions; that is, they refer to different sorts of predication. Genus and species, however, have at least a secondary reference to the *extension*† of first intentions. Now let the letters, in the particular application of Boole's calculus now supposed, be terms of second intention which relate exclusively to the extension of first intentions.‡ Let the differences of the characters of things and events be disregarded, and let the letters signify only the differences of classes as wider or narrower. In other words, the only logical comprehension which the letters considered as terms will have is the greater or less divisibility of the classes. Thus, n in another case of Boole's calculus might, for example, denote "New England States"; but in the case now supposed, all the characters which make these States what they are being neglected, it would signify only what essentially belongs to a class which has the same relations to higher and lower classes

* The ideas here presented for the derivation of arithmetic from logic are somewhat similar to those employed in the *Principia Mathematica*, Whitehead and Russell, vol. 2, section A (1912).

† See vol. 2, bk. II, ch. 5. for a discussion of these terms.

‡ This leads to a somewhat similar definition of a cardinal number as that given in the *Principia Mathematica*. But see 4.333 where this paper is characterized as being the worst Peirce ever published.

which the class of New England States has, — that is, a collection of *six*.

44. In this case, the sign of identity will receive a special meaning. For, if m denotes what essentially belongs to a class of the rank of "sides of a cube," then $m \mathrel{=\!\!\!\!\!-} n$ will imply, not that every New England State is a side of a cube, and conversely, but that whatever essentially belongs to a class of the numerical rank of "New England States" essentially belongs to a class of the rank of "sides of a cube," and conversely. *Identity* of this particular sort may be termed *equality*, and be denoted by the sign $=$.[1] Moreover, since the numerical rank of a *logical sum* depends on the identity or diversity (in first intention) of the integrant parts, and since the numerical rank of a *logical product* depends on the identity or diversity (in first intention) of parts of the factors, logical addition and multiplication can have no place in this system. Arithmetical addition and multiplication, however, will not be destroyed. $ab=c$ will imply that whatever essentially belongs at once to a class of the rank of a, and to another independent class of the rank of b belongs essentially to a class of the rank of c, and conversely.* $a+b=c$ implies that whatever belongs essentially to a class which is the logical sum of two mutually exclusive classes of the ranks of a and b belongs essentially to a class of the rank of c, and conversely.* It is plain that from these definitions the same theorems follow as from those given above. *Zero* and *unity* will, as before, denote the classes which have respectively no extension and no comprehension; only the comprehension here spoken of is, of course, that comprehension which alone belongs to letters in the system now considered, that is, this or that degree of divisibility; and therefore *unity* will be what belongs essentially to a class of any rank independent of its divisibility. These two classes alone are common to the two systems, because the first intentions of these

[1] Thus, in one point of view, *identity* is a species of *equality*, and, in another, the reverse is the case. This is because the Being of the copula may be considered on the one hand (with De Morgan [*Formal Logic*, p. 59]) as a special description of "inconvertible, transitive relation," while, on the other hand, all relation may be considered as a special determination of Being. If a Hegelian should be disposed to see a contradiction here, an accurate analysis of the matter will show him that it is only a verbal one.

* Cf. *Principia Mathematica*, vol. 2, section A.

alone determine, and are determined by, their second intentions. Finally, the laws of the Boolian calculus, in its ordinary form, are identical with those of this other so far as the latter apply to *zero* and *unity*, because every class, in its first intention, is either without any extension (that is, is nothing), or belongs essentially to that rank to which every class belongs, whether divisible or not.

These considerations, together with those advanced [in 1.556], will, I hope, put the relations of logic and arithmetic in a somewhat clearer light than heretofore.

III

*DESCRIPTION OF A NOTATION FOR THE LOGIC OF RELATIVES, RESULTING FROM AN AMPLIFICATION OF THE CONCEPTIONS OF BOOLE'S CALCULUS OF LOGIC**

§1. DE MORGAN'S NOTATION[E]

45. Relative terms usually receive some slight treatment in works upon logic, but the only considerable investigation into the formal laws which govern them is contained in a valuable paper by Mr. De Morgan in the tenth volume of the *Cambridge Philosophical Transactions*.† He there uses a convenient algebraic notation, which is formed by adding to the well-known *spiculæ* of that writer the signs used in the following examples.

X . . LY signifies that X is some one of the objects of thought which stand to Y in the relation L, or is one of the L's of Y.

X . LMY signifies that X is not an L of an M of Y.

X . . (L,M)Y signifies that X is either an L or an M of Y.

LM′ an L of every M. L,M an L of none but M's.

L[$^{-1}$]Y something to which Y is L. l (small L) non-L.

This system still leaves something to be desired. Moreover, Boole's logical algebra has such singular beauty, so far as it goes, that it is interesting to inquire whether it cannot be extended over the whole realm of formal logic, instead of being restricted to that simplest and least useful part of the subject, the logic of absolute terms, which, when he wrote, was the only formal logic known. The object of this paper is to show that

* *Memoirs of the American Academy*, vol. 9, pp. 317–78 (1870). Reprinted separately by Welch, Bigelow and Company, Cambridge, Mass., 1870, pp.1–62. "In 1870 I made a contribution to this subject [logic] which nobody who masters the subject can deny was the most important excepting Boole's original work that ever has been made."— From the "Lowell Lectures," 1903.

† "On the Syllogism No. IV, and on the Logic of Relations," pp. 331–58, dated 1859.

an affirmative answer can be given to this question. I think there can be no doubt that a *calculus*, or art of drawing inferences, based upon the notation I am to describe, would be perfectly possible and even practically useful in some difficult cases, and particularly in the *investigation* of logic. I regret that I am not in a situation to be able to perform this labor, but the account here given of the notation itself will afford the ground of a judgment concerning its probable utility.

46. In extending the use of old symbols to new subjects, we must of course be guided by certain principles of analogy, which, when formulated, become new and wider definitions of these symbols. As we are to employ the usual algebraic signs as far as possible, it is proper to begin by laying down definitions of the various algebraic relations and operations. The following will, perhaps, not be objected to.

§2. GENERAL DEFINITIONS OF THE ALGEBRAIC SIGNS

47. *Inclusion in* or *being as small as* is a *transitive* relation. The consequence holds that[1]

If $x \prec y$,

and $y \prec z$,

then $x \prec z$.

48. *Equality* is the conjunction of being as small as and its converse. To say that $x = y$ is to say that $x \prec y$ and $y \prec x$.

[1] I use the sign $\prec$ in place of $\leqq$. My reasons for not liking the latter sign are that it cannot be written rapidly enough, and that it seems to represent the relation it expresses as being compounded of two others which in reality are complications of this. It is universally admitted that a higher conception is logically more simple than a lower one under it. Whence it follows from the relations of extension and comprehension, that in any state of information a broader concept is more simple than a narrower one included under it. Now all equality is inclusion in, but the converse is not true; hence inclusion in is a wider concept than equality, and therefore logically a simpler one. On the same principle, inclusion is also simpler than being less than. The sign $\leqq$ seems to involve a definition by enumeration; and such a definition offends against the laws of definition.

49. *Being less than* is being as small as with the exclusion of its converse. To say that $x < y$ is to say that $x \prec y$, and that it is not true that $y \prec x$.

50. *Being greater than* is the converse of being less than. To say that $x > y$ is to say that $y < x$.

51. Addition is an *associative* operation. That is to say,[1]

$$(x \underset{,}{+} y) \underset{,}{+} z = x \underset{,}{+} (y \underset{,}{+} z).$$

Addition is a *commutative* operation. That is,

$$x \underset{,}{+} y = y \underset{,}{+} x.$$

52. Invertible* addition is addition the corresponding inverse of which is determinative. The last two formulæ hold good for it, and also the consequence that

$$\begin{array}{ll} \text{If} & x+y=z, \\ \text{and} & x+y'=z, \\ \text{then} & y=y'.\dagger \end{array}$$

53. Multiplication is an operation which is *doubly distributive with reference to addition*. That is,

$$x(y \underset{,}{+} z) = xy \underset{,}{+} xz,$$
$$(x \underset{,}{+} y)z = xz \underset{,}{+} yz.$$

Multiplication is almost invariably an *associative* operation.‡

$$(xy)z = x(yz).$$

Multiplication is not generally commutative. If we write commutative§ multiplication with a comma,¶ we have

$$x,y = y,x.$$

[1] I write a comma below the sign of addition, except when (as is the case in ordinary algebra) the corresponding inverse operation (subtraction) is determinative, [i.e., except when the addition is arithmetical.]

* I.e., arithmetical.

† See 27.

‡ Cf. 55, 69f.

§ I.e., "non-relative," or what was before called "logical."

¶ Cf. 73, 74n.

54. Invertible* multiplication is multiplication whose corresponding inverse operation (division) is determinative. We may indicate this by a dot;† and then the consequence holds that

$$\begin{aligned} \text{If} \quad & x.y = z, \\ \text{and} \quad & x.y' = z, \\ \text{then} \quad & y = y'.\ddagger \end{aligned}$$

55. Functional multiplication§ is the application of an operation to a function. It may be written like ordinary multiplication; but then there will generally be certain points where the associative principle does not hold. Thus, if we write (sin *abc*) *def*, there is one such point. If we write (log (base *abc*) *def*) *ghi*, there are two such points. The number of such points depends on the nature of the symbol of operation, and is necessarily finite. If there were many such points, in any case, it would be necessary to adopt a different mode of writing such functions from that now usually employed. We might, for example, give to "log" such a meaning that what followed it up to a certain point indicated by a † should denote the base of the system, what followed that to the point indicated by a ‡ should be the function operated on, and what followed that should be beyond the influence of the sign "log." Thus log *abc* † *def* ‡ *ghi* would be (log *abc*) *ghi*, the base being *def*. In this paper I shall adopt a notation very similar to this, which will be more conveniently described further on.

56. The operation of involution obeys the formula[1]

$$(x^y)^z = x^{(yz)}.$$

* I.e., arithmetical. See 75.

† The symbolism of the earlier papers is here slightly modified: the simple conjunction of terms now represents relative, instead of arithmetical, multiplication, and the dot is introduced to represent arithmetical multiplication. The comma, however, is still retained for logical multiplication.

‡ See 36.

§ Cf. 71 and 72.

[1] In the notation of quaternions Hamilton has assumed

$$(x^y)^z = x^{(zy)} \quad \text{instead of} \quad (x^y)^z = x^{(yz)},$$

although it appears to make but little difference which he takes. Perhaps we should assume two involutions, so that

$$(x^y)^z = x^{(yz)}, \qquad {}^z({}^y x) = {}^{(zy)}x.$$

But in this paper only the former of these is required. [See 113ff. for the latter.]

Involution, also, follows the *indexical principle.*

$$x^{y +_, z} = x^y , x^z .$$

Involution, also, satisfies the *binomial theorem.**

$$(x +_, y)^z = x^z +_, \Sigma_p x^{z-p} , y^p +_, y^z ,$$

where Σ_p denotes that p is to have every value less than z, and is to be taken out of z in all possible ways, and that the sum of all the terms so obtained of the form x^{z-p}, y^p is to be taken.

57. Subtraction is the operation inverse to addition. We may write indeterminative† subtraction with a comma below the usual sign. Then we shall have that

$$(x -_, y) +_, y = x,$$
$$(x - y) + y = x,$$
$$(x + y) - y = x.$$

58. Division is the operation inverse to multiplication. Since multiplication is not generally commutative it is necessary to have two signs for division. I shall take

$$(x:y)y = x,$$
$$x\frac{y}{x} = y.$$

59. Division inverse to that multiplication which is indicated by a comma may be indicated by a semicolon. So that

$$(x;y),y = x.\text{‡}$$

60. Evolution and taking the logarithm are the operations inverse to involution.

$$(\sqrt[x]{y})^x = y,$$
$$x^{\log_x y} = y.$$

61. These conditions are to be regarded as imperative. But in addition to them there are certain other characters which it is highly desirable that relations and operations should possess, if the ordinary signs of algebra are to be applied to them. These I will here endeavour to enumerate.

* See 77.

† I.e., logical; see 10.

‡ See 10 (24).

1. It is an additional motive for using a mathematical sign to signify a certain operation or relation that the general conception of this operation or relation should resemble that of the operation or relation usually signified by the same sign. In particular, it will be well that the relation expressed by $-\!\!<$ should involve the conception of one member being in the other; addition, that of taking together; multiplication, that of one factor's being taken relatively to the other (as we write 3×2 for a triplet *of* pairs, and $D\varphi$ for the derivative of φ); and involution, that of the base being taken for every unit of the exponent.

2. In the second place, it is desirable that, in certain general circumstances, determinate numbers should be capable of being substituted for the letters operated upon, and that when so substituted the equations should hold good when interpreted in accordance with the ordinary definitions of the signs, so that arithmetical algebra should be included under the notation employed as a special case of it. For this end, there ought to be a number known or unknown, which is appropriately substituted in certain cases, for each one of, at least, some class of letters.

3. In the third place, it is almost essential to the applicability of the signs for addition and multiplication, that a *zero* and a *unity* should be possible. By a *zero* I mean a term such that

$$x +\!\!, 0 = x,$$

whatever the signification of x; and by a *unity* a term for which the corresponding general formula

$$x\mathbf{1} = x$$

holds good. On the other hand, there ought to be no term a such that $a^x = x$, independently of the value of x.

4. It will also be a strong motive for the adoption of an algebraic notation, if other formulæ which hold good in arithmetic, such as

$$x^z,y^z = (x,y)^z,$$

$$\mathbf{1}x = x,$$

$$x^{\mathbf{1}} = x,$$

$$x0 = 0,$$

continue to hold good; if, for instance, the conception of a differential is possible, and Taylor's Theorem holds, and ϐ* or $(1+i)^{\frac{1}{i}}$ plays an important part in the system, if there should be a term having the properties of ϑ* (3.14159), or properties similar to those of space should otherwise be brought out by the notation, or if there should be an absurd expression having the properties and uses of ȷ* or the square root of the negative.

§3. APPLICATION OF THE ALGEBRAIC SIGNS TO LOGIC

62. While holding ourselves free to use the signs of algebra in any sense conformable to the above absolute conditions, we shall find it convenient to restrict ourselves to one particular interpretation except where another is indicated. I proceed to describe the special notation which is adopted in this paper.

Use of the Letters

63. The letters of the alphabet will denote logical signs. Now logical terms are of three grand classes. The first embraces those whose logical form involves only the conception of quality, and which therefore represent a thing simply as "a —." These discriminate objects in the most rudimentary way, which does not involve any consciousness of discrimination. They regard an object as it is in itself as *such* (*quale*); for example, as horse, tree, or man. These are *absolute terms*. The second class embraces terms whose logical form involves the conception of relation, and which require the addition of another term to complete the denotation. These discriminate objects with a distinct consciousness of discrimination. They regard an object as over against another, that is as relative; as father of, lover of, or servant of. These are *simple relative terms*. The third class embraces terms whose logical form involves the conception of bringing things into relation, and which require the addition of more than one term to complete the denotation. They discriminate not only with consciousness of discrimination, but with consciousness of its origin. They

* The symbol ϐ represents the base of Naperian logarithms, the symbol ϑ represents π and ȷ represents the square root of the negative in Benjamin Peirce's *Linear Associative Algebras*, §15ff (1870).

regard an object as medium or third between two others, that is as conjugative; as giver of — to —, or buyer of — for — from —. These may be termed *conjugative terms*. The conjugative term involves the conception of *third*, the relative that of second or *other*, the absolute term simply considers *an* object.* No fourth class of terms exists involving the conception of *fourth*, because when that of *third* is introduced, since it involves the conception of bringing objects into relation, all higher numbers are given at once, inasmuch as the conception of bringing objects into relation is independent of the number of members of the relationship.† Whether this *reason* for the fact that there is no fourth class of terms fundamentally different from the third is satisfactory or not, the fact itself is made perfectly evident by the study of the logic of relatives. I shall denote absolute terms by the Roman alphabet, a, b, c, d, etc.; relative terms by italics, *a*, *b*, *c*, *d*, etc.; and conjugative terms by a kind of type called Kennerly, a, b, c, d, etc.

I shall commonly denote individuals by capitals, and generals‡ by small letters. General symbols for numbers will be printed in black-letter, thus, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, etc. The Greek letters will denote operations.

64. To avoid repetitions, I give here a catalogue of the letters I shall use in examples in this paper, with the significations I attach to them.

a.	animal.	p.	President of the United States Senate.
b.	black.	r.	rich person.
f.	Frenchman.	u.	violinist.
h.	horse.	v.	Vice-President of the United States.
m.	man.	w.	woman.

a.	enemy.	*h*.	husband.	*o*.	owner.
b.	benefactor.	*l*.	lover.	*s*.	servant.
c.	conqueror.	*m*.	mother.	*w*.	wife.
e.	emperor.	*n*.	not.		

g.	giver to — of —.	b.	betrayer to — of —.
w.	winner over of — to — from —.	t.	transferrer from — to —.

* Cf. the discussion on the categories in vol. 1, bk. III, and on signs in vol. 2, bk. II.

† Cf. 421 and 1.347.

‡ Cf. 69n.

Numbers Corresponding to Letters

65. I propose to use the term "universe" to denote that class of individuals *about* which alone the whole discourse is understood to run. The universe, therefore, in this sense, as in Mr. De Morgan's,* is different on different occasions. In this sense, moreover, discourse may run upon something which is not a subjective part of the universe; for instance, upon the qualities or collections of the individuals it contains.†

I propose to assign to all logical terms, numbers; to an absolute term, the number of individuals it denotes; to a relative term, the average number of things so related to one individual. Thus in a universe of perfect men (men), the number of "tooth of" would be 32. The number of a relative with two correlates would be the average number of things so related to a pair of individuals; and so on for relatives of higher numbers of correlates. I propose to denote the number of a logical term by enclosing the term in square brackets, thus $[t]$.

The Signs of Inclusion, Equality, Etc.

66. I shall follow Boole‡ in taking the sign of equality to signify identity. Thus, if v denotes the Vice-President of the United States, and p the President of the Senate of the United States,

$$\mathrm{v}=\mathrm{p}$$

means that every Vice-President of the United States is President of the Senate, and every President of the United States Senate is Vice-President. The sign "less than" is to be so taken that

$$\mathrm{f}<\mathrm{m}$$

means every Frenchman is a man, but there are men besides Frenchmen. Drobisch has used this sign in the same sense.[1] It will follow from these significations of $=$ and $<$ that the sign $-\!\!<$ (or $\leqq$, "as small as") will mean "is." Thus,

$$\mathrm{f}-\!\!<\mathrm{m}$$

* "On the Structure of the Syllogism," Section 1, *Cambridge Philosophical Transactions*, vol. 8 (1846); *Formal Logic*, p. 37 (1847).

† Cf. 2.518ff.

‡ *Laws of Thought*, p. 27.

[1] According to De Morgan, *Formal Logic*, p. 334. De Morgan refers to the first edition of Drobisch's Logic. The third edition contains nothing of the sort.

means "every Frenchman is a man," without saying whether there are any other men or not. So,

$$m -\!\!< l$$

will mean that every mother of anything is a lover of the same thing; although this interpretation in some degree anticipates a convention to be made further on. These significations of = and < plainly conform to the indispensable conditions. Upon the transitive character of these relations the syllogism depends, for by virtue of it, from

$$\mathrm{f} -\!\!< \mathrm{m}$$

and $$\mathrm{m} -\!\!< \mathrm{a},$$

we can infer that $$\mathrm{f} -\!\!< \mathrm{a};$$

that is, from every Frenchman being a man and every man being an animal, that every Frenchman is an animal. But not only do the significations of = and < here adopted fulfill all absolute requirements, but they have the supererogatory virtue of being very nearly the same as the common significations. Equality is, in fact, nothing but the identity of two numbers; numbers that are equal are those which are predicable of the same collections, just as terms that are identical are those which are predicable of the same classes.* So, to write $5<7$ is to say that 5 is part of 7, just as to write $\mathrm{f}<\mathrm{m}$ is to say that Frenchmen are part of men. Indeed, if $\mathrm{f}<\mathrm{m}$, then the number of Frenchmen is less than the number of men, and if $\mathrm{v}=\mathrm{p}$, then the number of Vice-Presidents is equal to the number of Presidents of the Senate; so that the numbers may always be substituted for the terms themselves, in case no signs of operation occur in the equations or inequalities.

The Signs for Addition

67. The sign of addition is taken by Boole,† so that

$$x+y$$

denotes everything denoted by x, and, *besides*, everything denoted by y. Thus

$$\mathrm{m}+\mathrm{w}$$

denotes all men, and, besides, all women. This signification

* Cf. 42–44. A class is the extension of a collection; see e.g., 537n.

† *Op. cit.*, p. 33.

for this sign is needed for connecting the notation of logic with that of the theory of probabilities. But if there is anything which is denoted by both the terms of the sum, the latter no longer stands for any logical term on account of its implying that the objects denoted by one term are to be taken *besides* the objects denoted by the other. For example,

$$\mathrm{f}+\mathrm{u}$$

means all Frenchmen besides all violinists, and, therefore, considered as a logical term, implies that all French violinists are *besides themselves.* For this reason alone, in a paper which is published in the Proceedings of the Academy for March 17, 1867,* I preferred to take as the regular addition of logic a non-invertible process, such that

$$\mathrm{m} \,+\!\!,\, \mathrm{b}$$

stands for all men and black things, without any implication that the black things are to be taken besides the men; and the study of the logic of relatives has supplied me with other weighty reasons for the same determination. Since the publication of that paper, I have found that Mr. W. Stanley Jevons, in a tract called *Pure Logic, or the Logic of Quality,* [1864]† had anticipated me in substituting the same operation for Boole's addition, although he rejects Boole's operation entirely and writes the new one with a + sign while withholding from it the name of addition.[1] It is plain that both the regular non-invertible addition and the invertible addition satisfy the absolute conditions. But the notation has other recommendations. The conception of *taking together* involved in these processes is strongly analogous to that of summation, the sum of 2 and 5, for example, being the number of a collection which consists of a collection of two and a collection of five. Any logical equation or inequality in which no operation but addition is involved may be converted into a numerical equation or inequality by substituting the numbers of the several terms for the terms themselves — provided all the terms summed

* In 3.

† Ch. 6, §63; ch. 15, §177ff.

[1] In another book [*Substitution of Similars* (1869) and subsequent works] he uses the sign ·|· instead of +.

are mutually exclusive. Addition being taken in this sense, *nothing* is to be denoted by *zero*, for then

$$x \dotplus 0 = x,$$

whatever is denoted by x; and this is the definition of *zero*.* This interpretation is given by Boole, and is very neat, on account of the resemblance between the ordinary conception of *zero* and that of nothing, and because we shall thus have

$$[0] = 0.$$

The Signs for Multiplication

68. I shall adopt for the conception of multiplication *the application of a relation*, in such a way that, for example, lw shall denote whatever is lover of a woman. This notation is the same as that used by Mr. De Morgan, although he appears not to have had multiplication in his mind. $s(\mathrm{m} \dotplus \mathrm{w})$ will, then, denote whatever is servant of anything of the class composed of men and women taken together. So that

$$s(\mathrm{m} \dotplus \mathrm{w}) = s\mathrm{m} \dotplus s\mathrm{w}.$$

$(l \dotplus s)\mathrm{w}$ will denote whatever is lover or servant to a woman, and

$$(l \dotplus s)\mathrm{w} = l\mathrm{w} \dotplus s\mathrm{w}.$$

$(sl)\mathrm{w}$ will denote whatever stands to a woman in the relation of servant of a lover, and

$$(sl)\mathrm{w} = s(l\mathrm{w}).$$

Thus all the absolute conditions of multiplication are satisfied

The term "identical with —" is a unity for this multiplication. That is to say, if we denote "identical with — " by **1** we have

$$x\mathbf{1} = x,$$

whatever relative term x may be. For what is a lover of something identical with anything, is the same as a lover of that thing.

69. A conjugative term like *giver* naturally requires two correlates, one denoting the thing given, the other the recipient of the gift. We must be able to distinguish, in our notation, the giver of A to B from the giver to A of B, and, therefore, I

* Cf. 82.

suppose the signification of the letter equivalent to such a relative to distinguish the correlates as first, second, third, etc., so that "giver of — to —" and "giver to — of — " will be expressed by different letters. Let g denote the latter of these conjugative terms. Then, the correlates or multiplicands of this multiplier cannot all stand directly after it, as is usual in multiplication, but may be ranged after it in regular order, so that

$$\mathrm{g}xy$$

will denote a giver to x of y. But according to the notation, x here multiplies y, so that if we put for x owner (o), and for y horse (h),

$$\mathrm{g}o\mathrm{h}$$

appears to denote the giver of a horse to an owner of a horse. But let the individual horses be H, H′, H″, etc. Then

$$\mathrm{h}^{*} = \mathrm{H} \dotplus \mathrm{H}' \dotplus \mathrm{H}'' \dotplus \text{ etc.}$$

$$\mathrm{g}o\mathrm{h} = \mathrm{g}o(\mathrm{H} \dotplus \mathrm{H}' \dotplus \mathrm{H}'' \dotplus \text{ etc.}) = \mathrm{g}o\mathrm{H} \dotplus \mathrm{g}o\mathrm{H}' \dotplus \mathrm{g}o\mathrm{H}'' \dotplus \text{ etc.}$$

Now this last member must be interpreted as a giver of a horse to the owner of *that* horse, and this, therefore, must be the interpretation of goh. This is always very important. *A term multiplied by two relatives shows that the same individual is in the two relations.* If we attempt to express the giver of a horse to a lover of a woman, and for that purpose write

$$\mathrm{g}l\mathrm{wh},$$

we have written giver of a woman to a lover of her, and if we add brackets, thus,

$$\mathrm{g}(l\mathrm{w})\mathrm{h},$$

we abandon the associative principle of multiplication. A little reflection will show that the associative principle must in some form or other be abandoned at this point. But while this principle is sometimes falsified, it oftener holds, and a notation must be adopted which will show of itself when it holds. We already see that we cannot express multiplication by writing the multiplicand directly after the multiplier; let us then affix subjacent numbers after letters to show where their correlates are to be found. The first number shall denote

* h is a variable designating an unspecified H. See 84, 94, 111.

how many factors must be counted from left to right to reach the first correlate, the second how many *more* must be counted to reach the second, and so on. Then, the giver of a horse to a lover of a woman may be written

$$\mathrm{g}_{12} l_1 \mathrm{wh} = \mathrm{g}_{11} l_2 \mathrm{hw} = \mathrm{g}_{2-1} \mathrm{h} l_1 \mathrm{w}.$$

70. Of course a negative number indicates that the former correlate follows the latter by the corresponding positive number. A subjacent *zero* makes the term itself the correlate. Thus,

$$l_0$$

denotes the lover of *that* lover or the lover of himself, just as $\mathrm{g}_o\mathrm{h}$ denotes that the horse is given to the owner of itself, for to make a term doubly a correlate is, by the distributive principle, to make each individual doubly a correlate, so that

$$l_0 = L_0 +\!, L_0{}' +\!, L_0{}'' +\!, \text{ etc.}$$

A subjacent sign of infinity may indicate that the correlate is indeterminate, so that

$$l_\infty$$

will denote a lover of something. We shall have some confirmation of this presently.*

If the last subjacent number is a *one* it may be omitted. Thus we shall have

$$l_1 = l,$$

$$\mathrm{g}_{11} = \mathrm{g}_1 = \mathrm{g}.$$

This enables us to retain our former expressions $l\mathrm{w}$, $\mathrm{g}_o\mathrm{h}$, etc.

71. The associative principle does not hold in this counting of factors. Because it does not hold, these subjacent numbers are frequently inconvenient in practice, and I therefore use also another mode of showing where the correlate of a term is to be found. This is by means of the marks of reference, † ‡ ‖ § ¶, which are placed subjacent to the relative term and before and above the correlate. Thus, giver of a horse to a lover of a woman may be written

$$\mathrm{g}_{\dagger\ddagger}\,{}^{\dagger}l_{\|}\,{}^{\|}\mathrm{w}\,{}^{\ddagger}\mathrm{h}.$$

* See 73.

The asterisk I use exclusively to refer to the last correlate of the last relative of the algebraic term.

72. Now, considering the order of multiplication to be: — a term, a correlate of it, a correlate of that correlate, etc., — there is no violation of the associative principle. The only violations of it in this mode of notation are that in thus passing from relative to correlate, we skip about among the factors in an irregular manner, and that we cannnot substitute in such an expression as g*o*h a single letter for *o*h. I would suggest that such a notation may be found useful in treating other cases of non-associative multiplication. By comparing this with what was said above* concerning functional multiplication, it appears that multiplication by a conjugative term is functional, and that the letter denoting such a term is a symbol of operation. I am therefore using two alphabets, the Greek and Kennerly, where only one was necessary. But it is convenient to use both.

73. Thus far, we have considered the multiplication of relative terms only. Since our conception of multiplication is the application of a relation, we can only multiply absolute terms by considering them as relatives. Now the absolute term "man" is really exactly equivalent to the relative term "man that is —," and so with any other. I shall write a comma after any absolute term to show that it is so regarded as a relative term. Then man that is black will be written

m,b.

But not only may any absolute term be thus regarded as a relative term, but any relative term may in the same way be regarded as a relative with one correlate more. It is convenient to take this additional correlate as the first one. Then

l,*s*w†

will denote a lover of a woman that is a servant of that woman. The comma here after *l* should not be considered as altering at all the meaning of *l*, but as only a subjacent sign, serving to alter the arrangement of the correlates. In point of fact, since a comma may be added in this way to any relative term, it may be added to one of these very relatives formed by a

* In 55.

† *l*, *s* is the logical product of *l* and *s*; *ls*, on the other hand, is the relative product of *l* and *s*. See 54n, 68.

comma, and thus by the addition of two commas an absolute term becomes a relative of two correlates. So

$$m,,b,r,$$

interpreted like $\mathit{g}o\mathrm{h},$

means a man that is a rich individual and is a black that is that rich individual. But this has no other meaning than

$$m,b,r\ ,$$

or a man that is a black that is rich. Thus we see that, after one comma is added, the addition of another does not change the meaning at all, so that whatever has one comma after it must be regarded as having an infinite number. If, therefore, *l*,,*s*w is not the same as *l*,*s*w (as it plainly is not, because the latter means a lover and servant of a woman, and the former a lover of and servant of and same as a woman), this is simply because the writing of the comma alters the arrangement of the correlates. And if we are to suppose that absolute terms are multipliers at all (as mathematical generality demands that we should), we must regard every term as being a relative requiring an infinite number of correlates to its virtual infinite series "that is — and is — and is — etc." Now a relative formed by a comma of course receives its subjacent numbers like any relative, but the question is, What are to be the implied subjacent numbers for these implied correlates? Any term may be regarded as having an infinite number of factors, those at the end being *ones*, thus,

$$l,s\mathrm{w} = l,s\mathrm{w},\mathit{1},\mathit{1},\mathit{1},\mathit{1},\mathit{1},\mathit{1},\mathit{1}, \text{ etc.}$$

A subjacent number may therefore be as great as we please. But all these *ones* denote the same identical individual denoted by w; what then can be the subjacent numbers to be applied to *s*, for instance, on account of its infinite "*that is*" 's? What numbers can separate it from being identical with w? There are only two. The first is *zero*, which plainly neutralizes a comma completely, since

$$s,_0\mathrm{w} = s\mathrm{w},^*$$

and the other is infinity; for as 1^∞ is indeterminate in ordinary algebra, so it will be shown hereafter to be here, so that to

*A servant of herself who is also a woman is the same as a servant of a woman?

remove the correlate by the product of an infinite series of *ones* is to leave it indeterminate. Accordingly,

$$\mathrm{m},_\infty$$

should be regarded as expressing *some* man. Any term, then, is properly to be regarded as having an infinite number of commas, all or some of which are neutralized by zeros.

"Something" may then be expressed by

$$1_\infty.*$$

I shall for brevity frequently express this by an antique figure one (1).

"Anything" by

$$1_0.\dagger$$

I shall often also write a straight 1 for *anything*.

74. It is obvious that multiplication into a multiplicand indicated by a comma is commutative,[1] that is,

$$s,l=l,s.$$

This multiplication is effectively the same as that of Boole in his logical calculus. Boole's unity is my 1, that is, it denotes whatever is.

75. The sum $x+x$ generally denotes no logical term. But $x,_\infty+x,_\infty$ may be considered as denoting some two x's. It is natural to write

$$x+x=2.x\ ,$$

and $$x,_\infty+x,_\infty=2.x,_\infty\,,$$

where the dot shows that this multiplication is invertible. We may also use the antique figures so that

$$2.x,_\infty=2x,$$

just as $$1_\infty=1.$$

Then 2 alone will denote some two things. But this multiplication is not in general commutative, and only becomes so when

* "Something" is whatever is identical with an undetermined thing.

† "Anything" is whatever is identical with itself.

[1] It will often be convenient to speak of the whole operation of affixing a comma and then multiplying, as a commutative multiplication, the sign for which is the comma. But though this is allowable, we shall fall into confusion at once if we ever forget that in point of fact it is not a different multiplication, only it is multiplication by a relative whose meaning — or rather whose syntax — has been slightly altered; and that the comma is really the sign of this modification of the foregoing term.

it affects a relative which imparts a relation such that a thing only bears it to *one* thing, and one thing *alone* bears it to a thing. For instance, the lovers of two women are not the same as two lovers of women, that is,

$$l2.\mathrm{w} \text{ and } 2.l\mathrm{w}$$

are unequal; but the husbands of two women are the same as two husbands of women, that is,

$$h2.\mathrm{w} = 2.h\mathrm{w},$$

and in general, $x,2.y = 2.x,y.$

76. The conception of multiplication we have adopted is that of the application of one relation to another. So, a quaternion being the relation of one vector to another, the multiplication of quaternions is the application of one such relation to a second. Even ordinary numerical multiplication involves the same idea, for 2×3 is a pair of triplets, and 3×2 is a triplet of pairs, where "triplet of" and "pair of" are evidently relatives.

If we have an equation of the form

$$xy = z,$$

and there are just as many x's per y as there are *per* things, things of the universe, then we have also the arithmetical equation,

$$[x]\,[y] = [z].$$

For instance, if our universe is perfect men, and there are as many teeth to a Frenchman (perfect understood) as there are to any one of the universe, then

$$[t]\,[\mathrm{f}] = [t\,\mathrm{f}]$$

holds arithmetically. So if men are just as apt to be black as things in general,

$$[\mathrm{m},]\,[\mathrm{b}] = [\mathrm{m},\mathrm{b}],$$

where the difference between [m] and [m,] must not be overlooked. It is to be observed that

$$[\mathbf{1}] = 1.$$

Boole was the first to show this connection between logic and probabilities.* He was restricted, however, to absolute terms. I do not remember having seen any extension of probability to relatives, except the ordinary theory of *expectation*.

* *Op. cit.*, p. 243ff.

Our logical multiplication, then, satisfies the essential conditions of multiplication, has a unity, has a conception similar to that of admitted multiplications, and contains numerical multiplication as a case under it.

The Sign of Involution

77. I shall take involution in such a sense that x^y will denote everything which is an x for every individual of y. Thus l^{w} will be a lover of every woman. Then $(s^l)^{\mathrm{w}}$ will denote whatever stands to every woman in the relation of servant of every lover of hers; and $s^{(l\mathrm{w})}$ will denote whatever is a servant of everything that is lover of a woman. So that

$$(s^l)^{\mathrm{w}} = s^{(l\mathrm{w})}.$$

A servant of every man and woman will be denoted by $s^{\mathrm{m}+\!\!,\,\mathrm{w}}$*, and $s^{\mathrm{m}}, s^{\mathrm{w}}$ will denote a servant of every man that is a servant of every woman. So that

$$s^{\mathrm{m}+\!\!,\,\mathrm{w}} = s^{\mathrm{m}},s^{\mathrm{w}}.$$

That which is emperor or conqueror of every Frenchman will be denoted by $(e+\!\!,\,c)^{\mathrm{f}}$, and $e^{\mathrm{f}} +\!\!,\, \Sigma_p e^{\mathrm{f}-p}, c^p +\!\!,\, c^{\mathrm{f}}$ will denote whatever is emperor of every Frenchman or emperor of some Frenchmen and conqueror of all the rest, or conqueror of every Frenchman. Consequently,

$$(e+\!\!,\,c)^{\mathrm{f}} = e^{\mathrm{f}} +\!\!,\, \Sigma_p\, e^{\mathrm{f}-p},c^p +\!\!,\, c^{\mathrm{f}}\ .$$

Indeed, we may write the binomial theorem so as to preserve all its usual coefficients; for we have

$$(e+\!\!,\,c)^{\mathrm{f}} = e^{\mathrm{f}} +\!\!,\, [\mathrm{f}].e^{\mathrm{f}-\dagger 1},c^{1\dagger} +\!\!,\, \frac{[\mathrm{f}].([\mathrm{f}]-1)}{2}\,.e^{\mathrm{f}-\ddagger 2},c^{2\ddagger} +\!\!,\ \text{etc.}\dagger$$

That is to say, those things each of which is emperor or conqueror of every Frenchman consist, first, of all those individuals each of which is a conqueror [emperor!] of every Frenchman; second, of a number of classes equal to the number of Frenchmen, each class consisting of everything which is an emperor of every Frenchman but some one and is a conqueror of that

* This is more accurately read as: a servant of all those who are either men or women.

† Cf. J. N. Lambert. "Sechs Versuche einer Zeichenkunst in der Vernunftlehre"; in *Logische u. Philosophische Abhandlungen*, vol. 1, ed. J. Bernouilli, Berlin, (1782) for the use of this "Newtonian Formula" in an intensional logic of absolute terms.

one; third, of a number of classes equal to half the product of the number of Frenchmen by one less than that number, each of these classes consisting of every individual which is an emperor of every Frenchman except a certain two, and is conqueror of those two, etc. This theorem holds, also, equally well with invertible addition, and either term of the binomial may be negative provided we assume

$$(-x)^y = (-)^{[y]}.x^y.$$

In addition to the above equations which are required to hold good by the definition of involution, the following also holds,

$$(s,l)^{w} = s^{w},l^{w},^{*}$$

just as it does in arithmetic.

78. The application of involution to conjugative terms presents little difficulty after the explanations which have been given under the head of multiplication. It is obvious that betrayer to every enemy should be written

$$\mathfrak{b}^{a},$$

just as lover of every woman is written

$$l^{w},$$

but $\mathfrak{b} = \mathfrak{b}_{11}$ and therefore, in counting forward as the subjacent numbers direct, we should count the exponents, as well as the factors, of the letter to which the subjacent numbers are attached. Then we shall have, in the case of a relative of two correlates, six different ways of affixing the correlates to it, thus:

$\mathfrak{b}a\mathrm{m}$ betrayer of a man to an[1] enemy of him;
$(\mathfrak{b}a)^{\mathrm{m}}$ betrayer of every man to some enemy of him;
$\mathfrak{b}a^{\mathrm{m}}$ betrayer of each man to an enemy of every man;
$\mathfrak{b}^{a^{\mathrm{m}}}$ betrayer of a[1] man to all[1] enemies of all men;
$\mathfrak{b}^{a}\mathrm{m}$ betrayer of a man to every enemy of him;
$\mathfrak{b}^{a\mathrm{m}}$ betrayer of every man to every enemy of him.

* I.e., a servant and lover of every woman is a servant of every woman and a lover of every woman.

[1] "The same" substituted for "an"; "some" for "a," "every" for "all" — ink correction on C. S. P.'s own copy; cf. 145.

If both correlates are absolute terms, the cases are

$\mathfrak{b}\mathrm{mw}$ betrayer of a woman to a man;
$(\mathfrak{b}\mathrm{m})^{\mathrm{w}}$ betrayer of each woman to some man;
$\mathfrak{b}\mathrm{m}^{\mathrm{w}}$ betrayer of all women to a man;
$\mathfrak{b}^{\mathrm{m}^{\mathrm{w}}}$ betrayer of a woman to every man;[1]
$\mathfrak{b}^{\mathrm{m}}\mathrm{w}$ betrayer of a woman to all men;
$\mathfrak{b}^{\mathrm{mw}}$ betrayer of every woman to every man.

These interpretations are by no means obvious, but I shall show that they are correct further on.†

79. It will be perceived that the rule still holds here that

$$(\mathfrak{b}^{a})^{\mathrm{m}} = \mathfrak{b}^{(a\mathrm{m})},$$

that is to say, that those individuals each of which stand to every man in the relation of betrayer to every enemy of his are identical with those individuals each of which is a betrayer to every enemy of a man of that man.

80. If the proportion of lovers of each woman among lovers of other women is equal to the average number of lovers which single individuals of the whole universe have, then

$$[l^{\mathrm{w}}] = [l\mathrm{W}'_{,}]\,[l\mathrm{W}''_{,}]\,[l\mathrm{W}'''_{,}] \text{ etc.} = [l]^{[\mathrm{w}]}.$$

Thus arithmetical involution appears as a special case of logical involution.

§4. GENERAL FORMULÆ

81. The formulæ which we have thus far obtained, exclusive of mere explanations of signs and of formulæ relating to the numbers of classes, are:

(1) If $x \prec y$ and $y \prec z$, then $x \prec z$.
(2) $(x +\!\!, y) +\!\!, z = x +\!\!, (y +\!\!, z)$. (Jevons)
(3) $x +\!\!, y = y +\!\!, x$. (Jevons)
(4) $(x +\!\!, y)z = xz +\!\!, yz$.
(5) $x(y +\!\!, z) = xy +\!\!, xz$.
(6) $(xy)z = x(yz)$.
(7) $x,(y +\!\!, z) = x,y +\!\!, x,z$. (Jevons)

[1] "Follows from last because [it is] negative of $\bar{\mathfrak{b}}m^{\mathrm{w}}$" — marginal note.
† See 145.

(8) $(x,y),z=x,(y,z)$. (Boole)

(9) $x,y,=y,x$. (Boole)

(10) $(xy)^z=x(y^z)$.

(11) $xy+\!\!,^{z}=xy,\ x^z$.

(12) $(x+\!\!,y)^z=x^z+\!\!,\Sigma_p(x^{x-p},y^p)+\!\!,y^p$

$$=x^z+\!\!,[z].x^{z-\dagger 1},y^{\dagger 1}+\!\!,\frac{[z].[z-1]}{2}.x^{z-\ddagger 2},y^{\ddagger 2}$$

$$+\!\!,\frac{[z].[z-1].[z-2]}{2.3}.x^{z-\|3},y^{\|3}+\!\!,\text{ etc.}$$

(13) $(x,y)^z=x^z,y^z$.

(14) $x+0=x$. (Boole)

(15) $x\mathbf{1}=x$.

(16) $(x+y)+z=x+(y+z)$. (Boole)

(17) $x+y=y+x$. (Boole)

(18) $x+y-y=x$. (Boole)

(19) $x,(y+z)=x,y+x,z$. (Boole)

(20) $(x+y)^z=x+[z].x^{z-\dagger 1},y^{\dagger 1}+$ etc.

We have also the following, which are involved implicitly in the explanations which have been given.

(21) $x\prec x+\!\!,y$.*

This, I suppose, is the principle of identity, for it follows from this that $x=x$.†

(22) $x+\!\!,x=x$. (Jevons)

(23) $x,x=x$. (Boole)

(24) $x+\!\!,y=x+y-x,y$.

The principle of contradiction is

(25) $x,n^x=0$,

where n stands for "not." The principle of excluded middle is

(26) $x+\!\!,n^x=1$.

It is an identical proposition, that, if φ be determinative, we have

(27) If $x=y$ $\quad \varphi x=\varphi y$.

* This proposition is the source of the famous so-called paradoxes of material implication.

† This sentence seems to have been misplaced and should have appeared after (22) or (23).

The six following are derivable from the formulæ already given:

(28) $(x +\!\!, y),(x +\!\!, z) = x +\!\!, y,z.$

(29) $(x-y) +\!\!, (z-w) = (x +\!\!, z) - (y +\!\!, w) + y,z,(1-w)$
$+ x,(1-y),w.$

In the following, φ is a function involving only the commutative operations and the operations inverse to them.

(30) $\varphi x = (\varphi 1),x + (\varphi 0),(1-x)$. (Boole)

(31) $\varphi x = (\varphi 1 +\!\!, (1-x)),(\varphi 0 +\!\!, x).$

(32) If $\varphi x = 0 \quad (\varphi 1),(\varphi 0) = 0.$ (Boole)

(33) If $\varphi x = 1 \quad \varphi 1 +\!\!, \varphi 0 = 1.$

The reader may wish information concerning the proofs of formulæ (30) to (33). When involution is not involved in a function nor any multiplication except that for which $x,x = x$, it is plain that φx is of the first degree, and therefore, since all the rules of ordinary algebra hold, we have as in that

$$\varphi x = \varphi 0 + (\varphi 1 - \varphi 0),x.$$

We shall find, hereafter, that when φ has a still more general character, we have,

$$\varphi x = \varphi 0 + (\varphi 1 - \varphi 0)x.$$

The former of these equations by a simple transformation gives (30).

If we regard $(\varphi 1)$, $(\varphi 0)$ as a function of x and develop it by (30), we have

$$(\varphi 1),(\varphi 0) = x,(\varphi 1),(\varphi 0) + (\varphi 1),(\varphi 0),(1-x).$$

Comparing these terms separately with the terms of the second member of (30), we see that

$$(\varphi 1),(\varphi 0) -\!\!< \varphi x.$$

This gives at once (32), and it gives (31) after performing the multiplication indicated in the second member of that equation and equating φx to its value as given in (30). If $(\varphi 1 +\!\!, \varphi 0)$ is developed as a function of x by (31), and the factors of the second member are compared with those of the second member of (31), we get

$$\varphi x -\!\!< \varphi 1 +\!\!, \varphi 0,$$

from which (33) follows immediately.

Properties of Zero and Unity

82. The symbolical definition of zero is

$$x+0=x,$$

so that by (19) $\quad x,a=x,(a+0)=x,a+x,0.$

Hence, from the invertible character of this addition, and the generality of (14), we have

$$x,0=0.$$

By (24) we have in general,

$$x+\!\!,\,0=x+0-x,0=x,$$

or $\quad x+\!\!,\,0=x.$

By (4) we have $\quad ax=(a+\!\!,\,0)x=ax+\!\!,\,0x.$

But if a is an absurd relation, $\quad ax=0,$

so that $\quad 0x=0,$

which must hold invariably.

From (12) we have $\quad a^x=(a+\!\!,\,0)^x=a^x+\!\!,\,0^x+\!\!,\,\text{etc.},$

whence by (21) $\quad 0^x -\!\!< a^x.$*

But if a is an absurd relation, and x is not zero,

$$a^x=0.$$

And therefore, unless $x=0$, $\quad 0^x=0.$

83. Any relative x may be conceived as a sum of relatives X, X', X'', etc., such that there is but one individual to which anything is X, but one to which anything is X', etc. Thus, if x denote "cause of," X,X',X'' would denote different kinds of causes, the causes being divided according to the differences of the things they are causes of. Then we have

$$Xy=X(y+\!\!,\,0)=Xy+\!\!,\,X0,$$

whatever y may be. Hence, since y may be taken so that

$$Xy=0,$$

we have $\quad X0=0;$

and in a similar way,

$$X'0=0, \qquad X''0=0, \qquad X'''0=0, \text{ etc.}$$

We have, then,

$$x0=(X+\!\!,\,X'+\!\!,\,X''+\!\!,\,X'''+\!\!,\,\text{etc.})0$$
$$=X0+\!\!,\,X'0+\!\!,\,X''0+\!\!,\,X'''0+\!\!,\,\text{etc.}=0.$$

* I.e., by (21) $0^x -\!\!< a^x+\!\!,\,0^x$ and as by (12) $a^x+0^x=a^x$, $0^x -\!\!< a^x$.

84. If the relative x be divided in this way into X, X', X'', X''', etc., so that x is that which is either X or X' or X'' or X''', etc., then non-x is that which is at once non-X and non-X' and non-X'', etc.; that is to say,

$$\text{non-}x = \text{non-}X, \text{non-}X', \text{non-}X'', \text{non-}X''', \text{etc.};$$

where non-X is such that there is something (Z) such that everything is non-X to Z; and so with non-X', non-X'', etc. Now, non-x may be any relative whatever. Substitute for it, then, y; and for non-X, non-X', etc., Y, Y', etc. Then we have

$$y = Y, Y', Y'', Y''', \text{etc.};$$

and $$Y'Z' = 1, \quad Y''Z'' = 1, \quad Y'''Z''' = 1, \quad \text{etc.},$$

where Z', Z'', Z''' are individual terms which depend for what they denote on Y', Y'', Y'''

Then we have

$$1 = Y'Z' = Y'^{Z'} = Y'^{(Z' \dagger 0)} = Y'^{Z'}, Y'^{0} = Y'^{Z'}, Y'^{0},$$

or $$Y'^{0} = 1, \quad Y''^{0} = 1, \quad Y'''^{0} = 1, \quad \text{etc.}$$

Then $$y^{0} = (Y', Y'', Y''', \text{etc.})^{0} = Y'^{0}, Y''^{0}, Y'''^{0}, \text{etc.} = 1.$$

We have by definition, $$x\mathbf{1} = x.$$

Hence, by (6), $$ax = (a\mathbf{1})x = a(\mathbf{1}x).$$

Now a may express any relation whatever, but things the same way related to everything are the same. Hence,

$$x = \mathbf{1}x.$$

We have by definition, $1 = \mathbf{1}_0$.

Then if X is any individual $X,1 = X,\mathbf{1}_0 = X,\mathbf{1}X$.

But $\mathbf{1}X = X$.

Hence $X,1 = X,X$;

and by (23) $X,1 = X$;

whence if we take $x = X + X' + X'' + X''' +$ etc.,

where X, X' etc., denote individuals (and by the very meaning of a general term this can always be done, whatever x may be)

$$x,1 = (X + X' + X'' + \text{etc.}),1 = X,1 + X',1 + X'',1 + \text{etc.}$$
$$= X + X' + X'' + \text{etc.} = x,$$

or $$x,1 = x.$$

We have by (24) $x\dagger 1 = x + 1 - x{,}1 = x + 1 - x = 1,$

or $$x\dagger 1 = 1.$$

85. We may divide all relatives into limited and unlimited. Limited relatives express such relations as nothing has to everything. For example, nothing is knower of everything. Unlimited relatives express relations such as something has to everything. For example, something is as good as anything. For limited relatives, then, we may write

$$p^1 = 0.$$

The converse of an unlimited relative expresses a relation which everything has to something. Thus, everything is as bad as something. Denoting such a relative by q,

$$q1 = 1.$$

These formulæ remind one a little of the logical algebra of Boole; because one of them holds good in arithmetic only for *zero*, and the other only for *unity*.

We have by (10) $1^x = (q^0)^x = q^{(0x)} = q^0 = 1,$

or $$1^x = 1.$$

We have by (4) $1x = (a\dagger 1)x = ax \dagger 1x,$

or by (21) $$ax \mathrel{-\!\!<} 1x.$$

But everything is somehow related to x unless x is 0; hence unless x is 0, $1x = 1.$

If a denotes "what possesses," and y "character of what is denoted by x,"

$$x = ay = a(y\mathit{1}) = (ay)\mathit{1} = x\mathit{1},$$

or $$x\mathit{1} = x.$$

Since $\mathit{1}$ means "identical with," $l{,}\mathit{1}$w denotes whatever is both a lover of and identical with a woman, or a woman who is a lover of herself. And thus, in general,

$$x{,}\mathit{1} = x_{0},.$$

86. Nothing is identical with every one of a class; and therefore $\mathit{1}^x$ is zero, unless x denotes only an individual when $\mathit{1}^x$ becomes equal to x. But equations founded on interpretation may not hold in cases in which the symbols have no rational interpretation.

Collecting together all the formulæ relating to *zero* and *unity*, we have

(34) $x +\!\!\!, 0 = x.$ (Jevons)
(35) $x +\!\!\!, 1 = 1.$ (Jevons)
(36) $x0 = 0.$
(37) $0x = 0.$
(38) $x,0 = 0.$ (Boole)
(39) $x^0 = 1.$
(40) $0^x = 0$, provided $x > 0$.*
(41) $\mathit{1}x = x.$
(42) $x,\mathit{1} = x_{0},.$
(43) $x^{\mathit{1}} = x.$
(44) $\mathit{1}^x = 0$, unless x is individual, when $\mathit{1}^x = x$.
(45) $q1 = 1$, where q is the converse of an unlimited relative.
(46) $1x = 1$, provided $x > 0$.*
(47) $x,1 = x.$ (Boole)
(48) $p^1 = 0$, where p is a limited relative.
(49) $1^x = 1.$

These, again, give us the following:

(50) $0 +\!\!\!, 1 = 1$
(51) $0 +\!\!\!, \mathit{1} = \mathit{1}$
(52) $00 = 0$
(53) $0,0 = 0$
(54) $0^0 = 1$
(55) $\mathit{1}0 = 0$
(56) $0\mathit{1} = 0$
(57) $0,\mathit{1} = 0$
(58) $0^{\mathit{1}} = 0$
(59) $\mathit{1}^0 = 1$
(60) $01 = 0$
(61) $10 = 0$
(62) $0,1 = 0$
(63) $1^0 = 1$
(64) $0^1 = 0$
(65) $\mathit{1}\mathit{1} = \mathit{1}$
(66) $\mathit{1},\mathit{1} = \mathit{1}$
(67) $\mathit{1}^{\mathit{1}} = \mathit{1}$
(68) $11 = 1$
(69) $1,1 = 1$
(70) $1^1 = 1$
(71) $1\mathit{1} = 1$
(72) $\mathit{1}1 = 1$
(73) $\mathit{1},1 = \mathit{1}$
(74) $1^{\mathit{1}} = 1$
(75) $\mathit{1}^1 = 0$
(76) $1, = \mathit{1}$

From (64) we may infer that 0 is a limited relative, and from (60) that it is not the converse of an unlimited relative. From (70) we may infer that 1 is not a limited relative, and from (68) that it is the converse of an unlimited relative.

* On his own copy, Peirce substitutes the condition "x is an unlimited relative," for "$x > 0$."

Formulæ Relating to the Numbers of Terms

87. We have already seen that

(77) If $x -\!\!< y$, then $[x] -\!\!< [y]$.

(78) When $x,y=0$, then $[x +\!, y]=[x] +\!, [y]$.

(79) When $[xy]:[n^x y]=[x]:[n^x]$, then $[xy]=[x][y]$.

(80) When $[x\mathfrak{n}y]=[x][\mathfrak{n}y][1]$, then $[x^y]=[x]^{[y]}$.

It will be observed that the conditions which the terms must conform to, in order that the arithmetical equations shall hold, increase in complexity as we pass from the more simple relations and processes to the more complex.

88. We have seen that

(81) $[0]=\mathit{0}$.

(82) $[\mathbf{1}]=\mathit{1}$.

Most commonly the universe is unlimited, and then

(83) $[1]=\infty$;*

and the general properties of 1 correspond with those of infinity. Thus,

$x +\!, 1=1$	corresponds to	$x+\infty=\infty$,
$q1=1$	corresponds to	$q\infty=\infty$,
$1x=1$	corresponds to	$\infty x=\infty$,
$p^1=0$	corresponds to	$p^\infty=0$,
$1^x=1$	corresponds to	$\infty^x=\infty$.

The formulæ involving commutative multiplication are derived from the equation $1,=\mathit{1}$. But if 1 be regarded as infinite, it is not an absolute infinite; for $10=0$. On the other hand, $\mathbf{1}^1=0$.

It is evident, from the definition of the number of a term, that

(84) $[x,]=[x]:[1]$.

We have, therefore, if the probability of an individual being x to any y is independent of what other y's it is x to, and if x is independent of y,

(85) $[x^y,]=[x,]^{[y]}$.

* See 198.

§5. GENERAL METHOD OF WORKING WITH THIS NOTATION

89. Boole's logical algebra contains no operations except our invertible addition and commutative multiplication, together with the corresponding subtraction and division. He has, therefore, only to expand expressions involving division, by means of (30), so as to free himself from all non-determinative operations, in order to be able to use the ordinary methods of algebra, which are, moreover, greatly simplified by the fact that

$$x,x = x.$$

90. Mr. Jevons's modification* of Boole's algebra involves only non-invertible addition and commutative multiplication, without the corresponding inverse operations. He is enabled to replace subtraction by multiplication, owing to the principle of contradiction, and to replace division by addition, owing to the principle of excluded middle. For example, if x be unknown, and we have

$$\mathrm{x} +\!\!, \mathrm{m} = \mathrm{a},$$

or what is denoted by x together with men make up animals, we can only conclude, with reference to x, that it denotes (among other things, perhaps) all animals not men; that is, that the x's not men are the same as the animals not men. Let $\bar{\mathrm{m}}$ denote non-men; then by multiplication we have

$$\mathrm{x}\bar{\mathrm{m}}, +\!\!, \mathrm{m},\bar{\mathrm{m}} = \mathrm{x},\bar{\mathrm{m}} = \mathrm{a},\bar{\mathrm{m}},$$

because, by the principle of contradiction,

$$\mathrm{m},\bar{\mathrm{m}} = 0.$$

Or, suppose, x being again unknown, we have given

$$\mathrm{a},\mathrm{x} = \mathrm{m}.$$

Then all that we can conclude is that the x's consist of all the m's and perhaps some or all of the non-a's, or that the x's and non-a's together make up the m's and non-a's together. If, then, $\bar{\mathrm{a}}$ denote non-a, add $\bar{\mathrm{a}}$ to both sides and we have

$$\mathrm{a},\mathrm{x} +\!\!, \bar{\mathrm{a}} = \mathrm{m} +\!\!, \bar{\mathrm{a}}.$$

Then by (28) $\quad (\mathrm{a} +\!\!, \bar{\mathrm{a}}),(\mathrm{x} +\!\!, \bar{\mathrm{a}}) = \mathrm{m} +\!\!, \bar{\mathrm{a}}.$

* In his *Pure Logic*.

But by the principle of excluded middle,

$$a +\!\!, \bar{a} = 1$$

and therefore $$x +\!\!, \bar{a} = m +\!\!, \bar{a}.$$

I am not aware that Mr. Jevons actually uses this latter process, but it is open to him to do so. In this way, Mr. Jevons's algebra becomes decidedly simpler even than Boole's.

It is obvious that any algebra for the logic of relatives must be far more complicated. In that which I propose, we labor under the disadvantages that the multiplication is not generally commutative, that the inverse operations are usually indeterminative, and that transcendental equations, and even equations like

$$a^{b^x} = cd^{e^x} + f^x + x,$$

where the exponents are three or four deep, are exceedingly common. It is obvious, therefore, that this algebra is much less manageable than ordinary arithmetical algebra.

91. We may make considerable use of the general formulæ already given, especially of (1), (21), and (27), and also of the following, which are derived from them:

(86) If $a -\!\!< b$ then there is such a term x that $a +\!\!, x = b$.
(87) If $a -\!\!< b$ then there is such a term x that $b,x = a$.
(88) If $b,x = a$ then $a -\!\!< b$.
(89) If $a -\!\!< b$ $\quad c +\!\!, a -\!\!< c +\!\!, b$.
(90) If $a -\!\!< b$ $\quad ca -\!\!< cb$.
(91) If $a -\!\!< b$ $\quad ac -\!\!< bc$.
(92) If $a -\!\!< b$ $\quad c^b -\!\!< c^a$*
(93) If $a -\!\!< b$ $\quad a^c -\!\!< b^c$.
(94) $a,b -\!\!< a$

There are, however, very many cases in which the formulæ thus far given are of little avail.

92. Demonstration of the sort called mathematical is founded on suppositions of particular cases. The geometrician draws a figure; the algebraist assumes a letter to signify a single quantity fulfilling the required conditions. But while the mathematician supposes an individual case, his hypothesis is yet perfectly general, because he considers no characters of the individual case but those which must belong to every such

* Where c=not, this becomes the formula for contraposition. See 142 and 2.550. In 186, however, a different formula for contraposition is given.

case. The advantage of his procedure lies in the fact that the logical laws of individual terms are simpler than those which relate to general terms, because individuals are either identical or mutually exclusive, and cannot intersect or be subordinated to one another as classes can. Mathematical demonstration is not, therefore, more restricted to matters of intuition than any other kind of reasoning. Indeed, logical algebra conclusively proves that mathematics extends over the whole realm of formal logic; and any theory of cognition which cannot be adjusted to this fact must be abandoned. We may reap all the advantages which the mathematician is supposed to derive from intuition by simply making general suppositions of individual cases.

93. In reference to the doctrine of individuals,* two distinctions should be borne in mind. The logical atom, or term not capable of logical division, must be one of which every predicate may be universally affirmed or denied. For, let A be such a term. Then, if it is neither true that all A is X nor that no A is X, it must be true that some A is X and some A is not X; and therefore A may be divided into A that is X and A that is not X, which is contrary to its nature as a logical atom. Such a term can be realized neither in thought nor in sense. Not in sense, because our organs of sense are special — the eye, for example, not immediately informing us of taste, so that an image on the retina is indeterminate in respect to sweetness and non-sweetness. When I see a thing, I do not see that it is not sweet, nor do I see that it is sweet; and therefore what I see is capable of logical division into the sweet and the not sweet. It is customary to assume that visual images are absolutely determinate in respect to color, but even this may be doubted. I know no facts which prove that there is never the least vagueness in the immediate sensation. In thought, an absolutely determinate term cannot be realized, because, not being given by sense, such a concept would have to be formed by synthesis, and there would be no end to the synthesis because there is no limit to the number of possible predicates. A logical atom, then, like a point in space, would involve for its precise determination an endless process. We can only say, in a general way, that a term, however determi-

* Cf. the definition of individuals in 611–13.

nate, may be made more determinate still, but not that it can be made absolutely determinate. Such a term as "the second Philip of Macedon" is still capable of logical division — into Philip drunk and Philip sober, for example; but we call it individual because that which is denoted by it is in only one place at one time. It is a term not *absolutely* indivisible, but indivisible as long as we neglect differences of time and the differences which accompany them. Such differences we habitually disregard in the logical division of substances. In the division of relations, etc., we do not, of course, disregard these differences, but we disregard some others. There is nothing to prevent almost any sort of difference from being conventionally neglected in some discourse, and if I be a term which in consequence of such neglect becomes indivisible in that discourse, we have in that discourse,

$$[I]=1.$$

This distinction between the absolutely indivisible and that which is one in number from a particular point of view is shadowed forth in the two words *individual* (τὸ ἄτομον) and *singular* (τὸ καθ' ἕκαστον); but as those who have used the word *individual* have not been aware that absolute individuality is merely ideal, it has come to be used in a more general sense.[1]

94. The old logics distinguish between *individuum signatum* and *individuum vagum*. "Julius Cæsar" is an example of the former; "a certain man," of the latter. The *individuum vagum*, in the days when such conceptions were exactly investigated, occasioned great difficulty from its having a certain generality, being capable, apparently, of logical division. If we include under the *individuum vagum* such a term as "any

[1] The absolute individual can not only not be realized in sense or thought, but cannot exist, properly speaking. For whatever lasts for any time, however short, is capable of logical division, because in that time it will undergo some change in its relations. But what does not exist for any time, however short, does not exist at all. All, therefore, that we perceive or think, or that exists, is general. So far there is truth in the doctrine of scholastic realism. But all that exists is infinitely determinate, and the infinitely determinate is the absolutely individual. This seems paradoxical, but the contradiction is easily resolved. That which exists is the object of a true conception. This conception may be made more determinate than any assignable conception; and therefore it is never so determinate that it is capable of no further determination.

individual man," these difficulties appear in a strong light, for what is true of any individual man is true of all men. Such a term is in one sense not an individual term; for it represents every man. But it represents each man as capable of being denoted by a term which is individual; and so, though it is not itself an individual term, it stands for any one of a class of individual terms. If we call a thought about a thing in so far as it is denoted by a term, a *second intention*, we may say that such a term as "any individual man" is individual by second intention. The letters which the mathematician uses (whether in algebra or in geometry) are such individuals by second intention. Such individuals are one in number, for any individual man is one man; they may also be regarded as incapable of logical division, for any individual man, though he may either be a Frenchman or not, is yet altogether a Frenchman or altogether not, and not some one and some the other. Thus, all the formal logical laws relating to individuals will hold good of such individuals by second intention, and at the same time a universal proposition may at any moment be substituted for a proposition about such an individual, for nothing can be predicated of such an individual which cannot be predicated of the whole class.

95. There are in the logic of relatives three kinds of terms which involve general suppositions of individual cases. The first are *individual* terms, which denote only individuals*; the second are those relatives whose correlatives are individual: I term these *infinitesimal relatives*†; the third are *individual infinitesimal* relatives, and these I term *elementary* relatives.‡

Individual Terms

96. The fundamental formulæ relating to individuality are two. Individuals are denoted by capitals.

(95) If $x>0 \quad x = X +\!\!, X' +\!\!, X'' +\!\!, X''' +\!\!,$ etc.

(96) $y^X = yX.$

* See 96ff.
† See 100ff.
‡ See 121ff.

We have also the following which are easily deducible from these two:

(97) $(y,z)X = (yX),(zX)$. (99) $[X] = 1$.

(98) $X,y_0 = X,yX$. (100) $1^X = X$.

We have already seen that

$$1^x = 0, \text{ provided that } [x] > 1.$$

97. As an example of the use of the formulæ we have thus far obtained, let us investigate the logical relations between "benefactor of a lover of every servant of every woman," "that which stands to every servant of some woman in the relation of benefactor of a lover of him," "benefactor of every lover of some servant of a woman," "benefactor of every lover of every servant of every woman," etc.

In the first place, then, we have by (95)

$$s\mathrm{w} = s(\mathrm{W}' +\!, \mathrm{W}'' +\!, \mathrm{W}''' +\!, \text{etc.}) = s\mathrm{W}' +\!, s\mathrm{W}'' +\!, s\mathrm{W}''' +\!, \text{ etc.}$$
$$s^{\mathrm{w}} = s^{\mathrm{W}' +, \mathrm{W}'' +, \mathrm{W}'' +, \text{etc.}} = s^{\mathrm{W}'}, s^{\mathrm{W}''}, s^{\mathrm{W}''}, \text{ etc.}$$

From the last equation we have by (96)

$$s^{\mathrm{w}} = (s^{\mathrm{W}'}),(s^{\mathrm{W}''}),(s^{\mathrm{W}'''}), \text{ etc.}$$

Now by (31) $x' +\!, x'' +\!, \text{etc.} = x',x'',x''', \text{ etc.} +\!, \text{ etc.}$,

or

(101) $\Pi' \prec \Sigma'$,

where Π' and Σ' signify that the addition and multiplication with commas* are to be used. From this it follows that

(102) $s^{\mathrm{w}} \prec s\mathrm{w}$.†

If w vanishes, this equation fails, because in that case (95) does not hold.

From (102) we have

(103) $(ls)^{\mathrm{w}} \prec ls\mathrm{w}$.‡

Since $\mathrm{a} = \mathrm{a,b} +\!, \text{ etc.}$,

$\mathrm{b} = \mathrm{a,b} +\!, \text{ etc.}$,

we have $l\mathrm{a} = l(\mathrm{a,b} +\!, \text{etc.}) = l(\mathrm{a,b}) +\!, l\,(\text{etc.})$,

$l\mathrm{b} = l(\mathrm{a,b} +\!, \text{etc.}) = l(\mathrm{a,b}) +\!, l\,(\text{etc.})$.

* Π' signifies logical multiplication and Σ' signifies logical addition.

A servant of every woman is a servant of a woman.

A lover-of-a-servant of every woman is a lover-of-a-servant of a woman.

Multiplying these two equations commutatively we have

$$(la),(lb) = l(a,b) \dotplus \text{etc.}$$

or

(104) $$l\Pi' \prec \Pi' l.^{*}$$

Now $(ls)^{w} = (\ s)^{W' \dotplus W'' \dotplus W'' \dotplus \text{etc.}} = \Pi'(ls)^{W} = \Pi' ls^{W}$,

$ls^{w} = ls^{W' \dotplus W'' \dotplus W''' \dotplus \text{etc.}} = l\Pi' s^{W'} = l\Pi' s^{W}$.

Hence,

(105) $$ls^{w} \prec (ls)^{w},$$

or every lover of a servant of all women stands to every woman in the relation of lover of a servant of hers.

From (102) we have

(106) $$l^{s^{w}} \prec ls^{w}.\dagger$$

By (95) and (96) we have

$$l^{s}\mathrm{w} = l^{s}(W' \dotplus W'' \dotplus W''' \text{ etc.}) = l^{s}W' \dotplus l^{s}W'' \dotplus l^{s}W''' \dotplus \text{etc.}$$
$$= l^{s}W' \dotplus l^{s}W'' \dotplus l^{s}W''' + \text{etc.}$$

Now $s^{w} = s^{W' \dotplus W'' \dotplus W''' \dotplus \text{etc.}} = s^{W'}, s^{W''}, s^{W'''}$, etc.

So that by (94) $s^{w} \prec s^{W'} \prec s^{W'}$.

Hence by (92)

$$l^{s}W' \prec l^{s^{w}}, \quad l^{s}W'' \prec l^{s^{w}} \quad l^{s}W''' \prec l^{s^{w}}.$$

Adding, $l^{s}W' \dotplus l^{s}W'' \dotplus l^{s}W''' \prec l^{s^{w}}$;

or

(107) $$l^{s}\mathrm{w} \prec l^{s^{w}}.$$

That is, every lover of every·servant of any particular woman is a lover of every servant of all women.

By (102) we have

(108) $$l^{s\mathrm{w}} \prec l^{s}\mathrm{w}.\ddagger$$

Thus we have

$$l^{s\mathrm{w}} \prec l^{s}\mathrm{w} \prec l^{s^{w}} \prec ls^{w} \prec (ls)^{w} \prec ls\mathrm{w}.^{1}$$

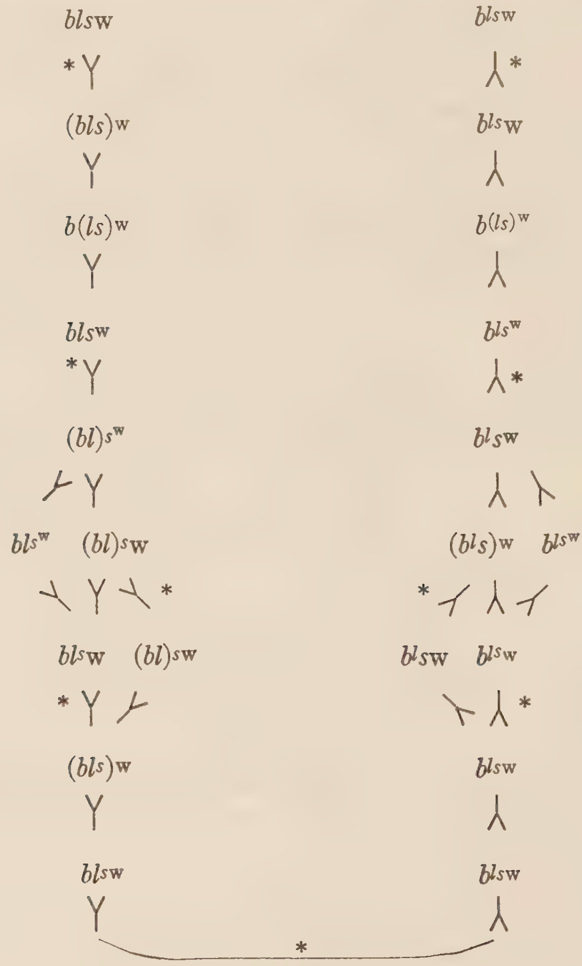
* See Lewis' *Survey of Symbolic Logic*, p. 87n for a proof of this theorem.

† A lover of every servant of all women is a lover of a servant of every woman.

‡ A lover of every servant-of-a-woman is to a woman a lover of all her servants.

[1] $l^{s}\mathrm{w} \prec l^{s^{w}}$ and $ls^{w} \prec (ls)^{w}$ invariably holds — marginal note.

98. By similar reasoning we can easily make out the relations shown in the following table. It must be remembered that the formulæ do not generally hold when exponents vanish.



99. It appears to me that the advantage of the algebraic notation already begins to be perceptible, although its powers are thus far very imperfectly made out. At any rate, it seems

* On Peirce's copy a line was drawn through the vinculum in each of these cases with the comment, "Crossed are not universally true."

to me that such a *prima facie* case is made out that the reader who still denies the utility of the algebra ought not to be too indolent to attempt to write down the above twenty-two terms in ordinary language with logical precision. Having done that, he has only to disarrange them and then restore the arrangement by ordinary logic, in order to test the algebra so far as it is yet developed.

Infinitesimal Relatives

100. We have by the binomial theorem by (49) and by (47),

$$(1+x)^n = 1 + \Sigma_p x^{n-p} + x^n.$$

Now, if we suppose the number of individuals to which any one thing is x to be reduced to a smaller and smaller number, we reach as our limit

$$x^2 = 0,$$

$$\Sigma_p x^{n-p} = [n].1^{n-\dagger 1}, x^{\dagger 1} = xn,$$

$$(1+x)^n = 1 + xn.$$

101. If, on account of the vanishing of its powers, we call x an infinitesimal here and denote it by i, and if we put

$$xn = in = y,$$

our equation becomes

(109) $$(1+i)^{\frac{y}{i}} = 1 + y.$$

Putting $y=1$, and denoting $(1+i)^{\frac{1}{i}}$ by $\mathfrak{S}$, we have

(110) $$\mathfrak{S} = (1+i)^{\frac{1}{i}} = 1 + 1.$$

102. In fact, this agrees with ordinary algebra better than it seems to do; for $\mathit{1}$ is itself an infinitesimal, and $\mathfrak{S}$ is $\mathfrak{S}\mathit{1}$. If the higher powers of $\mathit{1}$ did not vanish, we should get the ordinary development of $\mathfrak{S}$.

103. Positive powers of $\mathfrak{S}$ are absurdities in our notation. For negative powers we have

(111) $$\mathfrak{S}^{-x} = 1 - x.$$

104. There are two ways of raising $\mathfrak{S}^{-x}$ to the y^{th} power. In the first place, by the binomial theorem,

$$(1-x)^y = 1-[y].1^{y-\dagger 1},x^{\dagger 1} + \frac{[y].[y-1]}{2}.1^{y-\ddagger 2},x^{\ddagger 2} - \text{etc.};$$

and, in the second place, by (111) and (10).

$$\mathfrak{S}^{-xy} = 1-xy.^*$$

It thus appears that the sum of all the terms of the binomial development of $(1-x)^y$, after the first, is $-xy$.† The truth of this may be shown by an example. Suppose the number of y's are four, viz. Y', Y'', Y''', and Y''''. Let us use x', x'', x''', and x'''' in such senses that

$$xY' = x', \quad xY'' = x'', \quad xY''' = x''', \quad xY'''' = x''''.$$

Then the negatives of the different terms of the binomial developement are,

$$[y].1^{y-\dagger 1},x^{\dagger 1} = x'+x''+x'''+x''''.$$

$$-\frac{[y].[y-1]}{2}.1^{y-\ddagger 2},x^{\ddagger 2} = -x',x''-x',x'''-x',x''''-x'',x'''$$
$$-x'',x''''-x''',x''''.\ddagger$$

$$+\frac{[y].[y-1].[y-2]}{2.3}.1^{-\|3}x^{\|3} = x',x'',x'''+x',x'',x''''$$
$$+x',x''',x''''+x'',x''',x''''.$$

$$x^y = -x',x'',x''',x''''.\S$$

Now, since this addition is invertible, in the first term, x' that is x'', is counted over twice, and so with every other pair. The second term subtracts each of these pairs, so that it is only counted once. But in the first term the x' that is x'' that is x''' is counted in three times only, while in the second term it is subtracted three times; namely, in (x',x''), in (x',x''') and in (x'',x'''). On the whole, therefore, a triplet would not be represented in the sum at all, were it not added by the third term. The whole quartette is included four times in the first term, is subtracted six times by the second term, and is added

* I.e., $1-(xy)$.

† I.e., $-(xy)$.

‡ I.e., $-(x', x'')-(x', x''')$ etc.

§ I.e., $-(x', x'', x''', x'''')$.

four times in the third term. The fourth term subtracts it once, and thus in the sum of these negative terms each combination occurs once, and once only; that is to say the sum is

$$x' \dotplus x'' \dotplus x''' \dotplus x'''' = x(Y' \dotplus Y'' \dotplus Y''' \dotplus Y'''') = xy.$$

105. If we write $(ax)^3$ for $[x].[x-1].[x-2].1^{x-\dagger 3},a^{\dagger 3}$, that is for whatever is a to any three x's, regard being had for the order of the x's; and employ the modern numbers as exponents with this signification generally, then

$$1 - ax + \frac{1}{2!}(ax)^2 - \frac{1}{3!}(ax)^3 + \text{etc.}$$

is the development of $(1-a)^x$ and consequently it reduces itself to $1-ax$. That is,

(112) $$x = x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \text{etc.}$$

106. $1-x$ denotes everything except x, that is, whatever is other than every x; so that $\mathfrak{S}-$ means "not." We shall take log x in such a sense that

$$\mathfrak{S}^{\log x} = x.$$ [1]

107. I define the first difference of a function by the usual formula,

(113) $$\Delta\varphi x = \varphi(x \dotplus \Delta x) - \varphi x,$$

where Δx is an indefinite relative which never has a correlate in common with x. So that

(114) $$x,(\Delta x) = 0 \qquad x + \Delta x = x \dotplus \Delta x.$$

Higher differences may be defined by the formulæ

(115) $$\Delta^{\mathfrak{n}} \cdot x = 0 \quad \text{if } \mathfrak{n} > 1$$

$$\Delta^2 . \varphi x = \Delta\Delta x = \varphi(x + 2.\Delta x) - 2.\varphi(x + \Delta x) + \varphi x,$$

$$\Delta^3 \cdot \varphi x = \Delta\Delta^2 . x = \varphi(x + 3.\Delta x) - 3.\varphi(x + 2.\Delta x) + 3.\varphi(x + \Delta x) - \varphi x.$$

(116) $$\Delta^{\mathfrak{n}} \cdot \varphi x = \varphi(x + \mathfrak{n}.\Delta x) - \mathfrak{n}.\varphi(x + (\mathfrak{n}-1).\Delta x) + \frac{\mathfrak{n}.(\mathfrak{n}-1)}{2}.\varphi(x + (\mathfrak{n}-2).\Delta x) - \text{etc.}$$

108. The exponents here affixed to Δ denote the number of times this operation is to be repeated, and thus have quite a different signification from that of the numerical coefficients

[1] It makes another resemblance between 1 and infinity that log 0 = −1.

in the binomial theorem. I have indicated the difference by putting a period after exponents significative of operational repetition. Thus, m^2 may denote a mother of a certain pair, $m^{2.}$ a maternal grandmother.

109. Another circumstance to be observed is, that in taking the second difference of x, if we distinguish the two increments which x successively receives as $\Delta' x$ and $\Delta'' x$, then by (114)

$$(\Delta' x),(\Delta'' x) = 0$$

If Δx is relative to so small a number of individuals that if the number were diminished by one $\Delta^{n.}\varphi x$ would vanish, then I term these two corresponding differences *differentials*, and write them with $\mathbf{d}$ instead of Δ.

110. The difference of the invertible sum of two functions is the sum of their differences; for by (113) and (18),

$$(117) \quad \Delta(\varphi x + \psi x) = \varphi(x+\Delta x) + \psi(x+\Delta x) - \varphi x - \psi x$$
$$= \varphi(x+\Delta x) - \varphi x + \psi(x+\Delta x) - \psi x = \Delta\varphi x + \Delta\psi x.$$

If a is a constant, we have

$$(118) \quad \Delta a\varphi x = a(\varphi x \,+\!\!,\, \Delta\varphi x) - a\varphi x = a\Delta\varphi x - (a\Delta\varphi x),a\varphi x,$$
$$\Delta^{2.}a\varphi x = -\Delta a\varphi x,a\Delta x, \text{ etc.}$$
$$\Delta(\varphi x)a = (\Delta\varphi x)a - ((\Delta\varphi x)a),\varphi xa,$$
$$\Delta^{2.}(\varphi x)a = -\Delta(\varphi x)a, \text{ etc.}$$

$$(119) \quad \Delta(a,\varphi x) = a,\Delta\varphi x.$$

Let us differentiate the successive powers of x. We have in the first place,

$$\Delta(x_2) = (x+\Delta x)^2 - x^2 = 2.x^{2-\dagger 1},(\Delta x)^{\dagger 1} + (\Delta x)^2.$$

Here, if we suppose Δx to be relative to only one individual, $(\Delta x)^2$ vanishes, and we have, with the aid of (115),

$$\mathbf{d}(x^2) = 2.x^1,\mathbf{d}x\,.$$

Considering next the third power, we have, for the first differential,

$$\Delta(x^3) = (x+\Delta x)^3 - x^3 = 3.x^{3-\dagger 1},(\Delta x)^{\dagger 1} + 3.x^{3-\ddagger 2},(\Delta x)^{\ddagger 2} + (\Delta x)^3,$$
$$\mathbf{d}(x^3) = 3.x^2,\mathbf{d}(x).$$

To obtain the second differential, we proceed as follows:

$$\begin{aligned}\Delta^2.(x^3) &= (x+2.\Delta x)^3 - 2.(x+\Delta x)^3 + x^3 \\ &= x^3 + 6.x^{3-\dagger 1},(\Delta x)^{\dagger 1} + 12.x^{3-\ddagger 2},(\Delta x)^{\ddagger 2} + 8.(\Delta x)^3 \\ &\quad - 2.x^3 - 6.x^{3-\|1},(\Delta x)^{\|1} - 6.x^{3-\S 2},(\Delta x)^{\S 2} - 2.(\Delta x)^3 \\ &\quad + x^3 \\ &= 6.x^{3-\ddagger 2},(\Delta x)^{\ddagger 2} + 6.(\Delta x)^3.\end{aligned}$$

Here, if Δx is relative to less than two individuals, $\Delta\varphi x$ vanishes. Making it relative to two only, then, we have

$$d_2.(x^3) = 6.x^1,(dx)_2.$$

These examples suffice to show what the differentials of $x^{\mathfrak{n}}$ will be. If for the number $\mathfrak{n}$ we substitute the logical term n, we have

$$\begin{aligned}\Delta(x^n) &= (x+\Delta x)^n - x^n = [n].x^{n-\dagger 1},(\Delta x)^{\dagger 1} + \text{etc.} \\ d(x^n) &= [n].x^{n-1},(dx).\end{aligned}$$

We should thus readily find

(120) $d^{\mathfrak{m}}\cdot(x^n) = [n].[n-1].[n-2]....[n-\mathfrak{m}+1].x^{n-\dagger\mathfrak{m}},(dx)^{\dagger\mathfrak{m}}.$

Let us next differentiate l^x. We have, in the first place,

$$\Delta l^x = l^{x +\!\!, \Delta x} - l^x = l^x,l^{\Delta x} - l^x = l^x,(l^{\Delta x} - 1).$$

The value of $l^{\Delta' x} - 1$ is next to be found.

We have by (111)	$\mathfrak{S}^{l^{\Delta x}} - 1 = l^{\Delta x}.$
Hence,	$l^{\Delta x} - 1 = \log l^{\Delta x}.$
But by (10)	$\log l^{\Delta x} = (\log l)\Delta x.$

Substituting this value of $l^{dx} - 1$ in the equation lately found for dl^x we have

(121) $dl^x = l^x,(\log,l)\ dx = l^x,(l-1)\ dx = -l^x,(1-l)\ dx.$

111. In printing this paper, I here make an addition which supplies an omission in the account given above * of involution in this algebra. We have seen that every term which does not vanish is conceivable as logically divisible into individual terms. Thus we may write

$$s = S' +\!\!, S'' +\!\!, S''' +\!\!, \text{etc.}$$

where not more than one individual is in any one of these relations to the same individual, although there is nothing to prevent the same person from being so related to many individ-

* In 56.

uals.* Thus, "bishop of the see of" may be divided into first bishop, second bishop, etc., and only one person can be n^{th} bishop of any one see, although the same person may (where translation is permitted) be n^{th} bishop of several sees. Now let us denote the converse of x by Kx; thus, if s is "servant of," Ks is "master or mistress of." Then we have

$$Ks = KS' \dotplus KS'' \dotplus KS''' \dotplus \text{etc.};$$

and here each of the terms of the second member evidently expresses such a relation that the same person cannot be so related to more than one, although more than one may be so related to the same. Thus, the converse of "bishop of the see of —" is "see one of whose bishops is —," the converse of "first bishop of —" is "see whose first bishop is —," etc. Now, the same see cannot be a see whose n^{th} bishop is more than one individual, although several sees may be so related to the same individual. Such relatives I term infinitesimal on account of the vanishing of their higher powers. Every relative has a converse, and since this converse is conceivable as divisible into individual terms, the relative itself is conceivable as divisible into infinitesimal terms. To indicate this we may write

$$\text{(122)} \qquad \text{If } x > 0 \quad x = X_{,} \dotplus X_{,,} \dotplus X_{,,,} \dotplus \text{etc.}$$

112. As a term which vanishes is not an individual, nor is it composed of individuals, so it is neither an infinitesimal nor composed of infinitesimals.

As we write $\quad lS', lS'', lS''', \text{etc.} = l^s,$

so we may write

$$\text{(123)} \qquad L_{,}s, L_{,,}s, L_{,,,}s, \text{etc.} = {}^{l}s.$$

But as the first formula is affected by the circumstance that *zero* is not an individual, so that l^sw does not vanish on account of no woman having the particular kind of servant denoted by S'', l^sw denoting merely every lover of whatever servant there is of any woman; so the second formula is affected in a similar

* I.e., each correlate has only one relate though a given relate may have many correlates.

way, so that the vanishing of L,s does not make ${}^l s$ to vanish, but this is to be interpreted as denoting everything which is a lover, *in whatever way it is a lover at all*, of a servant.* Then just as we have by (112), that

(124) $$l^s = 1-(1-l)s;\dagger$$

so we have

(125) $${}^l s = 1 - l(1-s).\ddagger$$

Mr. De Morgan denotes l^s and ${}^l s$ by $LS,$ and L,S respectively,§ and he has traced out the manner of forming the converse and negative of such functions in detail. The following table contains most of his results in my notation.¶ For the converse of m, I write ɯ; and for that of n, u.

x	Kx
mn $m^n = {}^{(1-m)}(1-n)$ ${}^m n = (1-m)^{(1-n)}$	$u\text{ɯ}$ ${}^u\text{ɯ} = (1-u)^{(1-\text{ɯ})}$ $u^{\text{ɯ}} = {}^{(1-u)}(1-\text{ɯ})$
$\text{ϭ}{-x}$	$K\text{ϭ}{-x}$
$(1-m)^n = {}^m(1-n)$ $(1-m)n$ $m(1-n)$	${}^{\text{ɯ}}(1-u) = (1-\text{ɯ})^u$ $\text{ɯ}(1-u)$ $(1-m)n$

113. I shall term the operation by which w is changed to ${}^l\mathrm{w}$, *backward involution*. All the laws of this but one are the same as for ordinary involution, and the one exception is of that kind which is said to prove the rule. It is that whereas with ordinary involution we have,

$$(l^s)^{\mathrm{w}} = l^{(s\mathrm{w})};$$

in backward involution we have

(126) $${}^l({}^s\mathrm{w}) = {}^{({}^l s)}\mathrm{w};$$

that is, the things which are lovers to nothing but things that are servants to nothing but women are the things which are lovers of servants to nothing but women.

* I.e., as a lover of none but servants.

† A lover of all servants is not a non-lover of a servant.

‡ A lover of none but servants is not a lover of a non-servant. Cf. 116.

§ *Formal Logic*, p. 341.

¶ Cf. *ib.*, p. 343; see also 244.

114. The other fundamental formulæ of backward involution are as follows:

(127) $${}^{l\mathbin{+\!\!,}s}\mathrm{w} = {}^{l}\mathrm{w},{}^{s}\mathrm{w},$$

or, the things which are lovers or servants to nothing but women are the things which are lovers to nothing but women and servants to nothing but women.

(128) $${}^{l}(\mathrm{f},\mathrm{u}) = {}^{l}\mathrm{f},{}^{l}\mathrm{u},$$

or, the things which are lovers to nothing but French violinists are the things that are lovers to nothing but Frenchmen and lovers to nothing but violinists. This is perhaps not quite axiomatic. It is proved as follows. By (125) and (30)

$${}^{l}(\mathrm{f},\mathrm{u}) = \mathfrak{G}^{-l(1-\mathrm{f},\mathrm{u})} = \mathfrak{G}^{-(l(1-\mathrm{f}) \mathbin{+\!\!,} l(1-\mathrm{u}))}$$

By (125), (13), and (7),

$${}^{l}\mathrm{f},{}^{l}\mathrm{u} = \mathfrak{G}^{-l(1-\mathrm{f})},\mathfrak{G}^{-l(1-\mathrm{u})} = \mathfrak{G}^{-(l(1-\mathrm{f}) \mathbin{+\!\!,} l(1-\mathrm{u}))}.$$

Finally, the binomial theorem holds with backward involution. For those persons who are lovers of nothing but Frenchmen and violinists consist first of those who are lovers of nothing but Frenchmen; second, of those who in some ways are lovers of nothing but Frenchmen and in all other ways of nothing but violinists, and finally of those who are lovers only of violinists. That is,

(129) $${}^{l}(\mathrm{u} \mathbin{+\!\!,} \mathrm{f}) = {}^{l}\mathrm{u} \mathbin{+\!\!,} \Sigma_p {}^{l-p}\mathrm{u},{}^{p}\mathrm{f} \mathbin{+\!\!,} {}^{l}\mathrm{f}.$$

In order to retain the numerical coefficients, we must let $\{l\}$ be the number of persons that one person is lover of. We can then write

$${}^{l}(\mathrm{u}+\mathrm{f}) = {}^{l}\mathrm{u} + \{l\}\, {}^{l-\dagger 1}\mathrm{u},{}^{\dagger 1}\mathrm{f} + \frac{\{l\}.\{l-1\}}{2}\, {}^{l-\ddagger 2}\mathrm{u},{}^{\ddagger 2}\mathrm{f} + \text{etc.}$$

115. We have also the following formula which combines the two involutions:

(130) $${}^{l}(s^{\mathrm{w}}) = ({}^{l}s)^{\mathrm{w}};$$

that is, the things which are lovers of nothing but what are servants of all women are the same as the things which are related to all women as lovers of nothing but their servants.

116. It is worth while to mention, in passing, a singular proposition derivable from (128). Since, by (124) and (125)

$${}^{x}y = (1-x)^{(1-y)},\text{*}$$

* An x of none but y's is a non-x of all non-y's; i.e., ${}^{x}y = -x^{-y}$.

and since

$$1-(u \dagger f) = \mathfrak{S}^{-(u \dagger f)} = \mathfrak{S}^{-u}, \mathfrak{S}^{-f} = (1-u),(1-f),$$

(128) gives us,

$$(1-l)^{(1-u),(1-f)} = (1-l)^{(1-u)} \dagger \Sigma_p (1-(l-p))^{(1-u)},$$
$$(1-p)^{(1-f)} \dagger (1-l)^{(1-f)}.$$

This is, of course, as true for u and f as for $(1-u)$ and $(1-f)$. Making those substitutions, and taking the negative of both sides, we have, by (124)

(131) $$l(u,f) = (lu), \Pi'_p((l-p)u \dagger pf),(lf),$$

or, the lovers of French violinists are those persons who, in reference to every mode of loving whatever, either in that way love some violinists or in some other way love some Frenchmen. This logical proposition is certainly not self-evident, and its practical importance is considerable. In a similar way, from (12) we obtain

(132) $$(e,c)f = \Pi'_p(e(f-p) \dagger cp),$$

that is, to say that a person is both emperor and conqueror of the same Frenchman is the same as to say that, taking any class of Frenchmen whatever, this person is either an emperor of some one of this class, or conqueror of some one among the remaining Frenchmen.

117. The properties of zero and unity, with reference to backward involution, are easily derived from (125). I give them here in comparison with the corresponding formulæ for forward involution.

(133) $\quad {}^0x = 1 \qquad x^0 = 1.$

(134) $\quad {}^q0 = 0 \qquad 0^r = 0,$

where q is the converse of an unlimited relative, and r is greater than zero.

(135) $\quad {}^1x = x \qquad x^1 = x.$

(136) $\quad {}^y1 = y \qquad 1^z = z,$

where y is infinitesimal, and z is individual. Otherwise, both vanish.

(137) $\quad {}^1s = 0 \qquad p^1 = 0,$

where s is less than unity and p is a limited relative.

(138) $\quad {}^x1 = 1 \qquad 1^x = 1.$

118. In other respects the formulæ for the two involutions are not so analogous as might be supposed; and this is owing to the dissimilarity between individuals and infinitesimals. We have, it is true, if X' is an infinitesimal and X' an individual,

(139) $X_{,}(y,z) = X_{,}y;X_{,}z$ like $(y,z)X' = yX',zX'$;

(140) $X_{,}y_0 = X_{,},X_{,}y$ " $X',y_0 = X',yX'$;

(141) $\{X_{,}\} = 1$ " $[X'] = 1$.

We also have

(142) $$X_{,}y -\!\!< {}^{X_{,}}y.$$

But we have *not* ${}^{X'}y = X'y$, and consequently we have *not* ${}^{s}\mathrm{w} -\!\!< s\mathrm{w}$, for this fails if there is anything which is not a servant at all, while the corresponding formula $s^{\mathrm{w}} -\!\!< s\mathrm{w}$ only fails if there is not anything which is a woman. Now, it is much more often the case that there is something which is not x, than that there is not anything which is x. We have with the backward involution, as with the forward,* the formulæ

(143) If $x -\!\!< y$ ${}^{y}z -\!\!< {}^{x}z$;*

(144) If $x -\!\!< y$ ${}^{z}x -\!\!< {}^{z}y$.*

The former of these gives us

(145) $${}^{ls}\mathrm{w} -\!\!< {}^{(l^s)}\mathrm{w},$$

or, whatever is lover to nothing but what is servant to nothing but women† stands to nothing but a woman in the relation of lover of every servant of hers. The following formulæ can be proved without difficulty.

(146) $$l^{s}\mathrm{w} -\!\!< {}^{ls}\mathrm{w},$$

or, every lover of somebody who is servant to nothing but a woman stands to nothing but women in the relation of lover of nothing but a servant of them.

(147) $${}^{l}s\mathrm{w} -\!\!< {}^{l}(s\mathrm{w}),$$

or, whatever stands to a woman in the relation of lover of nothing but a servant of hers is a lover of nothing but servants of women.

* Cf. (92) (93) and 332.

† Or: Whatever is a lover-of-a-servant to nothing but women . . . etc.

The differentials of functions involving backward involution are

(148) $$d\,{}^{n}x = \{n\}\,{}^{n-1}x,dx.$$

(149) $$d\,{}^{x}l = {}^{x}l,dx\ \log.x.$$

In regard to powers of $\mathfrak{S}$ we have

(150) $${}^{x}\mathfrak{S} = \mathfrak{S}^{x}.$$

Exponents with a dot may also be put upon either side of the letters which they affect.

119. The greater number of functions of x in this algebra may be put in the form

$$\varphi x = \Sigma_p\ \Sigma_q\ {}_p\mathrm{A}\ \ {}^{p}x^{q}\ {}_p\mathrm{B}_q.$$

For all such functions Taylor's and Maclaurin's theorems hold good in the form,

(151) $$\boxed{\frac{y}{dx}}\ \boxed{\frac{0}{y}}\ \sum_{\mathfrak{p}=0}^{\infty}\frac{1}{\mathfrak{p}!}.d^{\mathfrak{p}\cdot} = 1.$$

The symbol $\boxed{\frac{\mathrm{a}}{\mathrm{b}}}$ is used to denote that a is to be substituted for b in what follows. For the sake of perspicuity, I will write Maclaurin's theorem at length.

$$\varphi x = \boxed{\frac{x}{dx}}\ \boxed{\frac{0}{x}}\left(\frac{1}{0!}.d^{0\cdot} + \frac{1}{1!}.d^{1\cdot} + \frac{1}{2!}.d^{2\cdot} + \frac{1}{3!}.d^{3\cdot} + \text{etc.}\right)\varphi x.$$

The proof of these theorems is very simple. The $(p+q)^{\text{th}}$ differential of ${}^{p}x^{q}$ is the only one which does not vanish when x vanishes. This differential then becomes $[p+q]!.{}^{p}(dx)^{q}$. It is plain, therefore, that the theorems hold when the coefficients ${}^{p}\mathrm{A}^{q}$ and ${}^{p}\mathrm{B}^{q}$ are 1. But the general development, by Maclaurin's theorem, of $a\varphi x$ or $(\varphi x)a$ is in a form which (112) reduces to identity. It is very likely that the application of these theorems is not confined within the limits to which I have restricted it. We may write these theorems in the form

(152) $$\boxed{\frac{y}{dx}}\ \boxed{\frac{0}{y}}\ \mathfrak{S}^{d} = 1,$$

provided we assume that when the first differential is positive

$$\circledS^{d} = \frac{1}{0!}d^{0} + \frac{1}{1!}d^{1} + \frac{1}{2!}d^{2} + \text{etc.},$$

but that when the first differential is negative this becomes by (111),

$$\circledS^{d} = 1 + d.$$

120. As another illustration of the use which may be made of differentiation in logic, let us consider the following problem. In a certain institution all the officers (x) and also all their common friends (*f*) are privileged persons (y). How shall the class of privileged persons be reduced to a minimum? Here we have

$$y = x + f^{x},$$

$$dy = dx + df^{x} = dx - f^{x},(1-f)dx.$$

When y is at a minimum it is not diminished either by an increase or diminution of x. That is,

$$[dy] \succ 0,$$

and when [x] is diminished by one,

$$[dy] \prec 0.$$

When x is a minimum, then

$$[dx - f^{x},(1-f)dx] \succ 0 \qquad [dx - f^{x-1},(1-f)dx] \prec 0$$

(A) $$[dx] - [f^{x},(1-f)dx \succ 0 \qquad [dx] - [f^{x-1},(1-f)dx] \prec 0.$$

Now we have by (30)

$$f^{x},(1-f)dx = f^{x} - (0;0),(1-f)dx.$$

Hence,

$$[f^{x}] \prec [dx] + [0;0,].[(1-f)dx]$$

$$[f^{x-1}] \succ [dx] + [0;0,].[(1-f)dx].$$

But [0;0,] lies between the limits 0 and 1, and

(153) $$[dx] = 1.$$

We have, therefore,

$$[f^{x}] \prec 1 + [(1-f)1.] \qquad [f^{x-1}] \succ 1.$$

This is the general solution of the problem. If the event of a person who may be an official in the institution being a friend of a second such person is independent of and equally probable with his being a friend of any third such person, and if we take p, or the whole class of such persons, for our universe, we have,

$$p = 1;$$

$$[f^{x},] = \frac{[f^{x}]}{[p]} = \left(\frac{[f]}{[p]}\right)^{[x]},$$

$$[(1-f)dx] = [1-f].[dx] = ([p]-[f]).[dx],$$

$$[f^{x},(1-f)\,dx] = \left(\frac{[f]}{[p]}\right)^{[x]}.([p]-[f]).[dx]$$

Substituting these values in our equations marked (A) we get, by a little reduction,

$$[x] \succ \frac{\log([p]-[f])}{\log[p]-\log[f]},$$

$$[x] \prec \frac{\log([p]-[f])}{\log[p]-\log[f]} + 1.$$

The same solution would be reached through quite a different road by applying the calculus of finite differences in the usual way.

Elementary Relatives*

121. By an elementary relative I mean one which signifies a relation which exists only between mutually exclusive pairs (or in the case of a conjugative term, triplets, or quartettes, etc.) of individuals, or else between pairs of classes in such a way that every individual of one class of the pair is in that relation to every individual of the other. If we suppose that in every school, every teacher teaches every pupil (a supposition which I shall tacitly make whenever in this paper I speak of a school), then *pupil* is an elementary relative. That every relative may be conceived of as a logical sum of elementary relatives is plain, from the fact that if a relation is sufficiently determined it can exist only between two individuals. Thus, a *father* is either father in the first ten years of the Christian era, or father in the second ten years, in the third ten years, in the first ten years, B. C., in the second ten years, or the third ten years, etc. Any one of these species of father is father for

* Cf. 602ff.

the first time or father for the second time, etc. Now such a relative as "father for the third time in the second decade of our era, of —" signifies a relation which can exist only between mutually exclusive pairs of individuals, and is therefore an elementary relative; and so the relative *father* may be resolved into a logical sum of elementary relatives.

122. The conception of a relative as resolvable into elementary relatives has the same sort of utility as the conception of a relative as resolvable into infinitesimals or of any term as resolvable into individuals.

123. Elementary simple relatives are connected together in systems of four. For if A:B be taken to denote the elemen tary relative which multiplied into B gives A, then this relation existing as elementary, we have the four elementary relatives

A:A   A:B   B:A   B:B.

An example of such a system is — colleague: teacher: pupil: schoolmate. In the same way, obviously, elementary conjugatives are in systems the number of members in which is $(\mathfrak{n}+1)^{\mathfrak{n}+1}$ where $\mathfrak{n}$ is the number of correlates which the conjugative has. At present, I shall consider only the simple relatives.

124. The existence of an elementary relation supposes the existence of mutually exclusive pairs of classes. The first members of those pairs have something in common which discriminates them from the second members, and may therefore be united in one class, while the second members are united into a second class. Thus *pupil* is not an elementary relative unless there is an absolute distinction between those who teach and those who are taught. We have, therefore, two general absolute terms which are mutually exclusive, "body of teachers in a school," and "body of pupils in a school." These terms are general because it remains undetermined what school is referred to. I shall call the two mutually exclusive absolute terms which any system of elementary relatives supposes, the *universal extremes* of that system. There are certain characters in respect to the possession of which both members of any one of the pairs, between which there is a certain elementary relation, agree. Thus, the body of teachers and the body of pupils in any school agree in respect to the country and age in which

they live, etc., etc. Such characters I term *scalar characters* for the system of elementary relatives to which they are so related; and the relatives written with a comma which signify the possession of such characters, I term *scalars* for the system. Thus, supposing French teachers have only French pupils and *vice versa*, the relative

$$\mathrm{f},$$

will be a scalar for the system "colleague: teacher: pupil: schoolmate." If r is an elementary relative for which s is a scalar,

(154) $$\mathrm{s},r = r\mathrm{s},.$$

125. Let c, t, p, s, denote the four elementary relatives of any system; such as colleague, teacher, pupil, schoolmate; and let a,, b,, c,, d,, be scalars for this system. Then any relative which is capable of expression in the form

$$\mathrm{a},c+\mathrm{b},t+\mathrm{c},p+\mathrm{d},s$$

I shall call a *logical quaternion*. Let such relatives be denoted by q, q', q'', etc. It is plain, then, from what has been said, that any relative may be regarded as resolvable into a logical sum of logical quaternions.

126. The multiplication of elementary relatives of the same system follows a very simple law. For if u and v be the two universal extremes of the system c, t, p, s, we may write

$$c=\mathrm{u:u} \quad t=\mathrm{u:v} \quad p=\mathrm{v:u} \quad s=\mathrm{v:v},$$

and then if w and w′ are each either u or v, we have

(155) $$(\mathrm{w}':\mathrm{w})\,\mathfrak{G}^{-\mathrm{w}}=0.$$

This gives us the following multiplication-table, where the multiplier is to be entered at the side of the table and the multiplicand at the top, and the product is found in the middle:

(156)

	c	t	p	s
c	c	t	0	0
t	0	0	c	t
p	p	s	0	0
s	0	0	p	s

The sixteen propositions expressed by this table are in ordinary language as follows:*

The colleagues of the colleagues of any person are that person's colleagues;

The colleagues of the teachers of any person are that person's teachers;

There are no colleagues of any person's pupils;

There are no colleagues of any person's schoolmates;

There are no teachers of any person's colleagues;

There are no teachers of any person's teachers;

The teachers of the pupils of any person are that person's colleagues;

The teachers of the schoolmates of any person are that person's teachers;

The pupils of the colleagues of any person are that person's pupils;

The pupils of the teachers of any person are that person's schoolmates;

There are no pupils of any person's pupils;

There are no pupils of any person's schoolmates;

There are no schoolmates of any person's colleagues;

There are no schoolmates of any person's teachers;

The schoolmates of the pupils of any person are that person's pupils;

The schoolmates of the schoolmates of any person are that person's schoolmates.

This simplicity and regularity in the multiplication of elementary relatives must clearly enhance the utility of the conception of a relative as resolvable into a sum of logical quaternions.

127. It may sometimes be convenient to consider relatives each one of which is of the form

$$\mathrm{a},i+\mathrm{b},j+\mathrm{c},k+\mathrm{d},l+\mathrm{etc.}$$

where a,, b,, c,, d,, etc. are scalars, and i, j, k, l, etc. are each of the form

$$\mathrm{m},u+\mathrm{n},v+\mathrm{o},w+\mathrm{etc.}$$

* See Lewis, *Survey of Symbolic Logic*, p. 103, for a symbolic and analytic account of some of these propositions.

where m,, n,, o,, etc. are scalars, and u, v, w, etc. are elementary relatives. In all such cases (155) will give a multiplication-table for i, j, k, l, etc. For example, if we have three classes of individuals, u_1, u_2, u_3, which are related to one another in pairs, we may put

$$u_1:u_1=i \qquad u_1:u_2=j \qquad u_1:u_3=k$$
$$u_2:u_1=l \qquad u_2:u_2=m \qquad u_2:u_3=n$$
$$u_3:u_1=o \qquad u_3:u_2=p \qquad u_3:u_3=q$$

and by (155) we get the multiplication-table

	i	*j*	*k*	*l*	*m*	*n*	*o*	*p*	*q*
i	*i*	*j*	*k*	0	0	0	0	0	0
j	0	0	0	*i*	*j*	*k*	0	0	0
k	0	0	0	0	0	0	*i*	*j*	*k*
l	*l*	*m*	*n*	0	0	0	0	0	0
m	0	0	0	*l*	*m*	*n*	0	0	0
n	0	0	0	0	0	0	*l*	*m*	*n*
o	*o*	*p*	*q*	0	0	0	0	0	0
p	0	0	0	*o*	*p*	*q*	0	0	0
q	0	0	0	0	0	0	*o*	*p*	*q*

128. If we take

$$i=u_1:u_2+u_2:u_3+u_3:u_4,$$

$$j=u_1:u_3+u_2:u_4,$$

$$k=2.u_1:u_4,$$

we have

	i	j	k
i	j	k	0
j	k	0	0
k	0	0	0

129. If we take

$$i = u_1{:}u_2 + u_2{:}u_3 + u_3{:}u_4 + u_5{:}u_6 + u_7{:}u_8,$$

$$j = u_1{:}u_3 + u_2{:}u_4,$$

$$k = 2.u_1{:}u_4,$$

$$l = u_6{:}u_8 + \mathfrak{a}.u_5{:}u_7 + 2\mathfrak{b}.u_1{:}u_9 + u_9{:}u_4 + \mathfrak{c}.u_5{:}u_6,$$

$$m = u_5{:}u_8,$$

we have

	i	j	k	l	m
i	j	k	0	m	0
j	k	0	0	0	0
k	0	0	0	0	0
l	$\mathfrak{a}.m$	0	0	$\mathfrak{b}.k+$ $\mathfrak{c}.m$	0
m	0	0	0	0	0

130. These multiplication-tables have been copied from Professor Peirce's monograph on Linear Associative Algebras.[1] I can assert, upon reasonable inductive evidence, that all such algebras can be interpreted on the principles of the present

[1] *Linear Associative Algebras*. By Benjamin Peirce. 4to, lithographed. Washington. 1870. [Published with notes and addenda by C. S. Peirce in *The American Journal of Mathematics*, vol. 4, pp. 97–229 (1881); see paper no. VIII of this volume.]

notation in the same way as those given above. In other words, all such algebras are complications and modifications of the algebra of (156). It is very likely that this is true of all algebras whatever. The algebra of (156), which is of such a fundamental character in reference to pure algebra and our logical notation, has been shown by Professor Peirce* to be the algebra of Hamilton's quaternions.† In fact, if we put

$$1 = i + l.$$

$$i' = \sqrt{1-b^2}\,\mathfrak{J}i - (\sqrt{1-a^2}\,b + ab\,\mathfrak{J})j + (\sqrt{1-a^2}\,b - ab\,\mathfrak{J})k - \sqrt{1-b^2}\,\mathfrak{J}l.$$

$$j' = -b\sqrt{1-c^2}\,\mathfrak{J}i + (ac - \sqrt{1-a^2}\sqrt{1-b^2}\sqrt{1-c^2} - (\sqrt{1-a^2}\,c + a\sqrt{1-b^2}\sqrt{1-c^2})\,\mathfrak{J})j - (ac - \sqrt{1-a^2}\sqrt{1-b^2}\sqrt{1-c^2} + (\sqrt{1-a^2}\,c + a\sqrt{1-b^2}\sqrt{1-c^2})\,\mathfrak{J})k + b\sqrt{1-c^2}\,\mathfrak{J}l.$$

$$k' = bc\,\mathfrak{J}i + (\sqrt{1-a^2}\sqrt{1-b^2}\,c + a\sqrt{1-c^2} + (a\sqrt{1-b^2}\,c - \sqrt{1-a^2}\sqrt{1-c^2})\,\mathfrak{J})j - (\sqrt{1-a^2}\sqrt{1-b^2}\,c + a\sqrt{1-c^2} - (a\sqrt{1-b^2}\,c - \sqrt{1-a^2}\sqrt{1-c^2})\,\mathfrak{J})k - bc\,\mathfrak{J}l.$$

where a, b, c, are scalars, then 1, i', j', k' are the four fundamental factors of quaternions, the multiplication-table of which is as follows:

	1	i'	j'	k'
1	1	i'	j'	k'
i'	i'	-1	k'	$-j'$
j'	j'	$-k'$	-1	i'
k'	k'	j'	$-i'$	-1

* *Ibid.*, (161²).

† See Appendix to this volume.

131. It is no part of my present purpose to consider the bearing upon the philosophy of space of this occurrence, in pure logic, of the algebra which expresses all the properties of space; but it is proper to point out that one method of working with this notation would be to transform the given logical expressions into the form of Hamilton's quaternions (after representing them as separated into elementary relatives), and then to make use of geometrical reasoning. The following formulæ will assist this process. Take the quaternion relative

$$q = xi + yj + zk + wl,$$

where x, y, z, and w are scalars. The conditions of q being a *scalar*, *vector*, etc. (that is, being denoted by an algebraic expression which denotes a scalar, a vector, etc., in geometry), are

(157) Form of a scalar: $x(i+l)$.

(158) Form of a vector: $xi+yi+zk-xl$.

(159) Form of a versor:

$$\frac{x}{y}\left(\frac{x}{z}-1\right)^{-\frac{1}{2}}i+\frac{y}{x}\left(\frac{x}{z}-1\right)^{-\frac{1}{2}}j+\frac{z}{y}\left(\frac{z}{x}-1\right)^{-\frac{1}{2}}k+\frac{y}{z}\left(\frac{z}{x}-1\right)^{-\frac{1}{2}}l.$$

(160) Form of zero: $xi+xyj+\frac{z}{y}k+zl$.

(161) Scalar of q: $\mathrm{S}q=\frac{1}{2}(x+w)(i+l)$.

(162) Vector of q: $\mathrm{V}q=\frac{1}{2}(x-w)i+yj+zk+\frac{1}{2}(w-x)l$.

(163) Tensor of q: $\mathrm{T}q=\sqrt{xw-yz}(i+l)$.

(164) Conjugate of q: $\mathrm{K}q=wi-yj-zk+xl$.

132. In order to exhibit the logical interpretations of these functions, let us consider a universe of married monogamists, in which husband and wife always have country, race, wealth, and virtue, in common. Let i denote "man that is —," j "husband of —," k "wife of —," and l "woman that is —"; x "negro that is —," y "rich person that is —," z "American that is —," and w "thief that is —." Then, q being defined as above, the q's of any class will consist of so many individuals of that class as are negro-men or women-thieves together with all persons who are rich husbands or American wives of persons of this class. Then, 2Sq denotes, by (160),* all the

* By (161).

negroes and besides all the thieves, while Sq is the indefinite term which denotes half the negroes and thieves. Now, those persons who are self-q's of any class (that is, the q's of themselves among that class) are $xi+wl$; add to these their spouses and we have 2Sq. In general, let us term $(j+k)$ the "correspondent of —." Then, the double scalar of any quaternion relative, q, is that relative which denotes all self-q's, and, besides, "all correspondents of self-q's of —." $(\mathrm{T}q)^2$ denotes all persons belonging to pairs of corresponding self-q's *minus* all persons belonging to pairs of corresponding q's of each other.

133. As a very simple example of the application of geometry to the logic of relatives, we may take the following. Euclid's axiom concerning parallels corresponds to the quaternion principle that the square of a vector is a scalar. From this it follows, since by (157) $yz+zk$ [?] is a vector, that the rich husbands and American wives of the rich husbands and American wives of any class of persons are wholly contained under that class, and can be described without any discrimination of sex. In point of fact, by (156), the rich husbands and American wives of the rich husbands and American wives of any class of persons, are the rich Americans of that class.

Lobatchewsky* has shown that Euclid's axiom concerning parallels may be supposed to be false without invalidating the propositions of spherical trigonometry. In order, then, that corresponding propositions should hold good in logic, we need not resort to elementary relatives, but need only take S and V in such senses that every relative of the class considered should be capable of being regarded as a sum of a scalar and a vector, and that a scalar multiplied by a scalar should be a scalar, while the product of a scalar and a vector is a vector. Now, to fulfill these conditions we have only to take Sq as "self-q of," and Vq as "alio-q of" (q of another, that other being —), and q may be any relative whatever. For, "lover," for example, is divisible into self-lover and alio-lover; a self-lover of a self-benefactor of persons of any class is contained under that class, and neither the self-lover of an alio-benefactor of any persons nor the alio-lover of the self-benefactor of any

* In his "New Elements of Geometry with a Complete Theory of Parallels," *Gelehrte Schriften der Universität*, Kasan, 1836–1838.

persons are among those persons. Suppose, then, we take the formula of spherical trigonometry,

$$\cos a = \cos b \cos c + \cos A \sin b \sin c.$$

In quaternion form, this is,

$$(165) \qquad S(pq) = (Sp)(Sq) + S\big((Vp)(Vq)\big).$$

Let p be "lover," and q be "benefactor." Then this reads, lovers of their own benefactors consist of self-lovers of self-benefactors together with alio-lovers of alio-benefactors of themselves. So the formula

$$\sin b \cos p\, b' = -\sin a \cos c \cos p\, a'$$
$$-\sin c \cos a \cos p\, c' + \sin a \sin c \sin b \cos p\, b,$$

where A′, B′, C′, are the positive poles of the sides a, b, c, is in quaternions

$$(166) \qquad V(pq) = (Vp)(Sq) + (Sp)(Vq) + V\big((Vp)(Vq)\big),$$

and the logical interpretation of this is: lovers of benefactors of others consist of alio-lovers of self-benefactors, together with self-lovers of alio-benefactors, together with alio-lovers of alio-benefactors of others. It is a little striking that just as in the non-Euclidean or imaginary geometry of Lobatchewsky the axiom concerning parallels holds good only with the ultimate elements of space, so its logical equivalent holds good only for elementary relatives.

134. It follows from what has been said that for every proposition in geometry there is a proposition in the pure logic of relatives. But the method of working with logical algebra which is founded on this principle seems to be of little use. On the other hand, the fact promises to throw some light upon the philosophy of space.[1]

[1] The researches of Lobatchewsky furnish no solution of the question concerning the apriority of space. For though he has shown that it is conceivable that space should have such properties that two lines might be in a plane and inclined to one another without ever meeting, however far produced, yet he has not shown that the facts implied in that supposition are inconsistent with supposing space to retain its present [Euclidean] nature and the properties only of the things in it to change. For example, in Lobatchewsky's geometry a star at an infinite distance has a finite parallax. But suppose space to have its

§6. PROPERTIES OF PARTICULAR RELATIVE TERMS

Classification of Simple Relatives*

135. Any particular property which any class of relative terms may have may be stated in the form of an equation, and affords us another premiss for the solution of problems in which such terms occur. A good classification of relatives is, therefore, a great aid in the use of this notation, as the notation is also an aid in forming such a classification.

136. The first division of relatives is, of course, into simple relatives and conjugatives. The most fundamental divisions

present properties, and suppose that there were one point in the universe towards which anything being moved should expand, and away from which being moved should contract. Then this expansion and contraction might obey such a law that a star, the parallax of which was finite, should be at an infinite distance measured by the number of times a yard-stick must be laid down to measure off that distance. I have not seen Beltrami's investigations, ["Saggio di interpretazione della geometrica non-euclidea," *Giorn. di Matem.*, 6, 1868.] but I understand that they do show that something of this sort is possible. Thus, it may be that, make what suppositions you will concerning phenomena, they can always be reconciled to our present geometry or be shown to involve implicit contradictions. If this is so — and whether it is or not is a completely open question — then the principles of geometry are necessary, and do not result from the specialities of any object cognized, but from the conditions of cognition in general. In speaking of the conditions of cognition, in general, I have in view no psychological conception, but only a distinction between principles which, if the facts should present a sufficient difficulty, I may always logically doubt, and principles which it can be shown cannot become open to doubt from any difficulty in my facts, as long as they continue to be supposed in all logical procedure.

But, waiving this point, Lobatchewsky's conclusions do not positively overthrow the hypothesis that space is *a priori*. For he has only shown that a certain proposition, *not usually believed to be axiomatical*, is conceivably false. That people may be doubtful or even mistaken about *a priori* truth does not destroy all important practical distinction between the two kinds of necessity. It may be said that if Lobatchewsky's geometry is the true one, then space involves an arbitrary constant, which value cannot be given *a priori*. This may be; but it may be that the general properties of space, with the general fact that there is such a constant, are *a priori*, while the value of the constant is only empirically determined.

It appears to me plain that no geometrical speculations will settle the philosophy of space, which is a logical question. If space is *a priori*, I believe that it is in some recondite way involved in the logic of relatives.

* Cf. 225 and 585.

of simple relatives are based on the distinction between elementary relatives of the form (A:A), and those of the form (A:B). These are divisions in regard to the amount of opposition between relative and correlative.

a. Simple relatives are in this way primarily divisible into relatives all of whose elements are of the form (A:A) and those which contain elements of the form (A:B). The former express a mere agreement among things, the latter set one thing over against another, and in that sense express an opposition (ἀντικεῖσθαι); I shall therefore term the former *concurrents*,[1] and the latter *opponents*. The distinction appears in this notation as between relatives with a comma, such as (w,), and relatives without a comma, such as (w); and is evidently of the highest importance. The character which is signified by a concurrent relative is an absolute character, that signified by an opponent is a relative character, that is, one which cannot be prescinded from reference to a correlate.

b. The second division of simple relatives with reference to the amount of opposition between relative and correlative is into those whose elements may be arranged in collections of squares, each square like this,

A:A	A:B	A:C
B:A	B:B	B:C
C:A	C:B	C:C

and those whose elements cannot be so arranged.* The former (examples of which are, "equal to —," "similar to —") may be called *copulatives*,[2] the latter *non-copulatives*. A copulative multiplied into itself gives itself. Professor Peirce calls letters having this property, *idempotents*.† The present distinction is of course very important in pure algebra. All concurrents are copulatives.

c. Third, relatives are divisible into those which for every element of the form (A:B) have another of the form (B:A), and

[1] "As we speak of *self-loving*, etc., the former of these classes should be called self-relatives" — marginal note.

* Cf. 230.

[2] "The idea of a copula is different. These should be called *assimilative*."— Marginal note. See 592.

† *Op. cit.*, §25; see also 593.

those which want this symmetry. This is the old division into *equiparants*[1] and *disquiparants*,[2] or in Professor De Morgan's language, convertible and inconvertible relatives.* Equiparants are their own correlatives. All copulatives are equiparant.

d. Fourth, simple relatives are divisible into those which contain elements of the form (A:A) and those which do not. The former express relations such as a thing may have to itself, the latter (as cousin of —, hater of —) relations which nothing can have to itself. The former may be termed *self-relatives*,[3] the latter *alio-relatives*. All copulatives are self-relatives.

e. The fifth division is into relatives some power (*i.e.* repeated product) of which contains[4] elements of the form (A:A), and those of which this is not true.† The former I term *cyclic*, the latter *non-cyclic*[4] relatives. As an example of the former, take

$$(A:B) \mathbin{+\!\!,} (B:A) \mathbin{+\!\!,} (C:D) \mathbin{+\!\!,} (D:E) \mathbin{+\!\!,} (E:C).$$

The product of this into itself is

$$(A:A) \mathbin{+\!\!,} (B:B) \mathbin{+\!\!,} (C:E) \mathbin{+\!\!,} (D:C) \mathbin{+\!\!,} (E:D).$$

The third power is

$$(A:B) \mathbin{+\!\!,} (B:A) \mathbin{+\!\!,} (C:C) \mathbin{+\!\!,} (D:D) \mathbin{+\!\!,} (E:E).$$

The fourth power is

$$(A:A) \mathbin{+\!\!,} (B:B) \mathbin{+\!\!,} (C:D) \mathbin{+\!\!,} (D:E) \mathbin{+\!\!,} (E:C).$$

[1] "If such reciprocation is admissible but not necessary they may be called *reciprocal*."— Marginal note.

[2] "Quædam sunt relatione equiparantiæ, quædam disquiparantiæ. Primæ sunt relationes similium nominum, secundæ relationes dissimilium nominum. Exemplum primi est quando idem nomen ponitur in recto et in obliquo, sicut simile simili est simile. . . . Exemplum secundi est quando unum nomen ponitur in recto sed aliud in obliquo, sicut pater est filii pater et non oportet quod sit patris pater." Ockham *Quodlibetum* 6, qu.20. See also his *Summa Logices*, pars 1, cap. 52. "Relativa equiparantiæ: quæ sunt synonyma cum suis correlativis. . . . Relativa diquiparantiæ: quæ non sunt synonyma cum suis correlativis." Pschlacher in Petr. Hisp. The same definitions substantially may be found in many late mediæval logics.

* *Formal Logic*, p. 345.

[3] "Instead of self-relatives, better concurrents." — Marginal note.

[4] "Insert 'only' between contains and 'elements'; for 'non-cyclic' substitute 'acyclic.' "— Marginal note.

† Cf. 233.

The fifth power is

$$(A:B) +\!\!, (B:A) +\!\!, (C:E) +\!\!, (D:C) +\!\!, (E:D).$$

The sixth power is

$$(A:A) +\!\!, (B:B) +\!\!, (C:C) +\!\!, (D:D) +\!\!, (E:E).$$

where all the terms are of the form (A:A). Such relatives, as *cousin of* —, are cyclic. All equiparants are cyclic.

f. The sixth division is into relatives no power of which is zero, and relatives some power of which is zero. The former may be termed *inexhaustible*, the latter *exhaustible*. An example of the former is "spouse of —," of the latter, "husband of —." All cyclics are inexhaustible.

g. Seventh, simple relatives may be divided into those whose products into themselves are not zero, and those whose products into themselves are zero. The former may be termed *repeating*, the latter, *non-repeating* relatives. All inexhaustible relatives are repeating.

h. Repeating relatives may be divided (after De Morgan) into those whose products into themselves are contained under themselves, and those of which this is not true. The former are well named by De Morgan* *transitive*, the latter *intransitive*. All transitives are inexhaustible; all copulatives are transitive; and all transitive equiparants are copulative. The class of transitive equiparants has a character, that of being self-relatives, not involved in the definitions of the terms.†

i. Transitives are further divisible into those whose products by themselves are equal to themselves, and those whose products by themselves are less than themselves; the former may be termed *continuous*,[1] the latter *discontinuous*. An example of the second is found in the pure mathematics of a continuum, where if a is greater than b it is greater than something greater than b; and as long as a and b are not of the same magnitude, an intervening magnitude always exists. All concurrents are continuous.

j. Intransitives may be divided into those the number of the powers (repeated products) of which not contained in the

* *Op. cit.*, p. 346.

† For if R is transitive then $aRb.bRc \mathrel{-\!\!<} aRc$; if R is also equiparant then $aRb.bRa$ is true and aRa is a necessary consequent.

[1] "Should be called concatenated" — marginal note.

first is infinite, and those some power of which is contained in the first. The former may be called *infinites*, the latter, *finites*. Infinite inexhaustibles are cyclic.

In addition to these, the old divisions of relations into relations of reason and real relations, of the latter into aptitudinal and actual, and of the last into extrinsic and intrinsic, are often useful.[1]

"NOT"

137. We have already seen that "not," or "other than," is denoted by $\mathfrak{S}^{-1}$. It is often more convenient to write it, n. The fundamental property of this relative has been given above (111). It is that,

$$\mathfrak{S}^{-x} = 1 - x.$$

Two other properties are expressed by the principles of contradiction and excluded middle. They are,

$$x,\mathfrak{S}^{-x} = 0;\text{*}$$
$$x \dotplus \mathfrak{S}^{-x} = 1.\text{*}$$

The following deduced properties are of frequent application:

(167) $$\mathfrak{S}^{-(x,y)} = \mathfrak{S}^{-x} \dotplus \mathfrak{S}^{-y};\dagger$$

(168) $$\mathfrak{S}^{-x^y} = \mathfrak{S}^{-x}y.$$

The former of these is the counterpart of the general formula, $z^{x \dotplus y} = z^x, z^y\dagger$. The latter enables us always to bring the exponent of the exponent of $\mathfrak{S}-$ down to the line, and make it a

[1] "Duplex est relatio: scilicet rationis et realis. Unde relatio rationis est quæ fit per actum comparativum intellectus, ut sunt secundæ intentiones; sed relatio realis est duplex, scilicet aptitudinalis et actualis. Aptitudinalis est quæ non requirit terminum actu existere sed solum in aptitudine; cujusmodi sunt omnes propriæ passiones, omnes aptitudines, et omnes inclinationes; et tales sunt in illo prædicamento reductive in quo sunt illa quorum sunt propriæ passiones. Sed relatio actualis est duplex, scilicet, intrinsecus adveniens, et extrinsecus adveniens. Intrinsecus adveniens est quæ necessario ponitur positis extremis in quacunque etiam distantia ponantur, ut similitudo, paternitas, equalitas. Extrinsecus adveniens est quæ necessario non ponitur, positis extremis, sed requiritur debita approximatio extremorum; cujusmodi sunt sex ultima prædicamenta, scilicet, actio, passio, quando, ubi, situs, et habitus." Tartaretus.

* Cf. 8 and 81 (25), (26.)

† These two together contain De Morgan's Theorem to the effect that the negative of a logical product is the logical sum of the negatives of its factors; and that the negative of a sum (z=not) is the product of the negatives of the summands.

factor. By the former principle, objects not French violinists consist of objects not Frenchmen, together with objects not violinists; by the latter, individuals not servants of all women are the same as non-servants of some women.

Another singular property of $\mathfrak{S}-$ is that,

$$\text{If } [x]>1 \quad \mathfrak{S}^{-1}x=1.$$

"CASE OF THE EXISTENCE OF —," AND "CASE OF THE NON-EXISTENCE OF —."

138. That which first led me to seek for the present extension of Boole's logical notation was the consideration that as he left his algebra, neither hypothetical propositions nor particular propositions could be properly expressed. It is true that Boole was able to express hypothetical propositions in a way which answered some purposes perfectly. He could, for example, express the proposition, "Either the sun will shine, or the enterprise will be postponed," by letting x denote "the truth of the proposition that the sun will shine," and y "the truth of the proposition that the enterprise will be postponed"; and writing,

$$x \mathbin{+\!\!,} y=1,$$

or, with the invertible addition,

$$x+(1-x),y=1.$$

But if he had given four letters denoting the four terms, "sun," "what is about to shine," "the enterprise," and "what is about to be postponed," he could make no use of these to express his disjunctive proposition, but would be obliged to assume others. The imperfection of the algebra here was obvious. As for particular propositions, Boole could not accurately express them at all. He did undertake to express them and wrote

Some Y's are X's:	$v,y=v,x;$
Some Y's are not X's:	$v,y=v,(1-x).$

The letter v is here used, says Boole, for an "indefinite class symbol." * This betrays a radical misapprehension of the nature of a particular proposition. To say that some Y's are X's, is not the same as saying that a logical species of Y's are

* *Laws of Thought*, p. 62.

X's. For the logical species need not be the name of anything existing. It is only a certain description of things fully expressed by a mere definition, and it is a question of fact whether such a thing really exist or not. St. Anselm wished to infer existence from a definition, but that argument has long been exploded. If, then, v is a mere logical species in general, there is not necessarily any such thing, and the equation means nothing. If it is to be a logical species, then, it is necessary to suppose in addition that it exists, and further that *some v is y*. In short, it is necessary to assume concerning it the truth of a proposition, which, being itself particular, presents the original difficulty in regard to its symbolical expression. Moreover, from

$$\mathrm{v},\mathrm{y}=\mathrm{v},(1-\mathrm{x})$$

we can, according to algebraic principles, deduce successively

$$\mathrm{v},\mathrm{y}=\mathrm{v}-\mathrm{v},x$$
$$\mathrm{v},x=\mathrm{v}-\mathrm{v},\mathrm{y}=\mathrm{v},(1-\mathrm{y}).$$

Now if the first equation means that some Y's are not X's, the last ought to mean that some X's are not Y's; for the algebraic forms are the same, and the question is, whether the algebraic forms are adequate to the expression of particulars. It would appear, therefore, that the inference from Some Y's are not X's to Some X's are not Y's, is good; but it is not so, in fact.

139. What is wanted, in order to express hypotheticals and particulars analytically, is a relative term which shall denote "case of the existence of —," or "what exists only if there is any —"; or else "case of the non-existence of —," or "what exists only if there is not —." When Boole's algebra is extended to relative terms, it is easy to see what these particular relatives must be. For suppose that having expressed the propositions "it thunders," and "it lightens," we wish to express the fact that "if it lightens, it thunders." Let

$$\mathrm{A}=0 \quad \text{and} \quad \mathrm{B}=0,$$

be equations meaning respectively, it lightens and it thunders. Then, if φx vanishes when x does not and *vice versa*, whatever x may be, the formula

$$\varphi\mathrm{A} \prec \varphi\mathrm{B}$$

expresses that if it lightens it thunders; for if it lightens, A vanishes; hence φA does not vanish, hence φB does not vanish, hence B vanishes, hence it thunders. It makes no difference what the function φ is, provided only it fulfills the condition mentioned. Now, 0^x is such a function, vanishing when x does not, and not vanishing when x does. *Zero*, therefore, may be interpreted as denoting "that which exists if, and only if, there is not —." Then the equation

$$0^0=1$$

means, everything which exists, exists only if there is not anything which does not exist. So,

$$0x=0$$

means that there is nothing which exists if, and only if, *some* x does not exist. The reason of this is that *some* x means some existing x.

"It lightens" and "it thunders" might have been expressed by equations in the forms

$$A=1, \qquad B=1.$$

In that case, in order to express that if it lightens it thunders, in the form

$$\varphi A \prec \varphi B,$$

it would only be necessary to find a function, φx, which should vanish unless x were 1, and should not vanish if x were 1. Such a function is ${}^1x$. We must therefore interpret 1 as "that which exists if, and only if, there is —," ${}^1x$ as "that which exists if, and only if, there is nothing but x," and $1x$ as "that which exists if, and only if, there is some x." Then the equation

$$1^x=1,$$

means everything exists if, and only if, whatever x there is exists.

140. Every hypothetical proposition may be put into four equivalent forms, as follows:

If X, then Y.
If not Y, then not X.
Either not X or Y.
Not both X and not Y.

If the propositions X and Y are A=1 and B=1, these four forms are naturally expressed by

$$^{1}A \prec {}^{1}B,$$
$$1(1-A) \prec 1(1-B),^*$$
$$1(1-A) \dotplus B = 1,$$
$$^{1}A, 1(1-B) = 0.$$

For ^{1}x we may always substitute $0^{(1-x)}$.

141. Particular propositions are expressed by the consideration that they are contradictory of universal propositions. Thus, as h,(1—b)=0 means every horse is black, so $0^{h, (1-b)} = 0$ means that some horse is not black; and as h,b=0 means that no horse is black, so $0^{h,b} = 0$ means that some horse is black. We may also write the particular affirmative 1(h,b)=1, and the particular negative $1(h, n^{b}) = 1$.

142. Given the premisses, every horse is black, and every horse is an animal; required the conclusion. We have given

$$h \prec b;$$
$$h \prec a.$$

Commutatively multiplying, we get

$$h \prec a,b.$$

Then, by (92) or by (90),

$$0^{a \dotplus b} \prec 0^{h}, \quad \text{or} \quad 1h \prec 1(a,b).$$

Hence, by (40) or by (46),

$$\text{If } h > 0 \quad 0^{a, b} = 0, \quad \text{or} \quad 1(a,b) = 1;$$

or if there are any horses, some animals are black. I think it would be difficult to reach this conclusion, by Boole's method unmodified.

143. Particular propositions may also be expressed by means of the signs of inequality. Thus, some animals are horses, may be written

$$a,h > 0;$$

and the conclusion required in the above problem might have been obtained in this form, very easily, from the product of the premisses, by (1) and (21).

* This should be $1(1-B) \prec 1(1-A)$

We shall presently see* that conditional and disjunctive propositions may also be expressed in a different way.

Conjugative Terms

144. The treatment of conjugative terms presents considerable difficulty, and would no doubt be greatly facilitated by algebraic devices. I have, however, studied this part of my notation but little.

A relative term cannot possibly be reduced to any combination of absolute terms, nor can a conjugative term be reduced to any combination of simple relatives; but a conjugative having more than two correlates can always be reduced to a combination of conjugatives of two correlates. Thus for "winner over of —, from —, to —," we may always substitute u, or "gainer of the advantage — to —," where the first correlate is itself to be another conjugative v, or "the advantage of winning over of — from —." Then we may write,

$$w = uv.$$

It is evident that in this way all conjugatives may be expressed as production of conjugatives of two correlates.

145. The interpretation of such combinations as $b^{a\mathrm{m}}$, etc., is not very easy. When the conjugative and its first correlative can be taken together apart from the second correlative, as in $(ba)\mathrm{m}$ and $(ba)^{\mathrm{m}}$ and $(b^{a})\mathrm{m}$ and $(b^{a})^{\mathrm{m}}$, there is no perplexity, because in such cases (ba) or (b^{a}) is a simple relative. We have, therefore, only to call the betrayer to an enemy an inimical betrayer, when we have

$(ba)\mathrm{m}$ = inimical betrayer of a man = betrayer of a man to an enemy of him,

$(ba)^{\mathrm{m}}$ = inimical betrayer of every man = betrayer of every man to an enemy of him.

And we have only to call the betrayer to every enemy an unbounded betrayer, in order to get

$(b^{a})\mathrm{m}$ = unbounded betrayer of a man = betrayer of a man to every enemy of him,

$(b^{a})^{\mathrm{m}}$ = unbounded betrayer of every man = betrayer of every man to every enemy of him.

* In 146.

The two terms $\mathbf{b}a^{\mathrm{m}}$ and $\mathbf{b}^{a^{\mathrm{m}}}$ are not quite so easily interpreted. Imagine a separated into infinitesimal relatives, $A_{,}, A_{,,}, A_{,,,}$, etc., each of which is relative to but one individual which is m. Then, because all powers of $A_{,}, A_{,,}, A_{,,,}$, etc., higher than the first, vanish, and because the number of such terms must be [m,] we have,

$$a^{\mathrm{m}} = (A_{,} \dotplus A_{,,} \dotplus A_{,,,} \dotplus \text{etc.})^{\mathrm{m}} = (A_{,}\mathrm{m}),(A_{,,}\mathrm{m}),(A_{,,,}\mathrm{m}), \text{ etc.}$$

or if M′, M″, M‴, etc., are the individual m's,

$$a^{\mathrm{m}} = (A_{,}\mathrm{M}'),(A_{,,}\mathrm{M}''),(A_{,,,}\mathrm{M}'''), \text{ etc.}$$

It is evident from this that $\mathbf{b}a^{\mathrm{m}}$ is a betrayer to an $A_{,}$ of M′, to an $A_{,,}$ of M″, to an $A_{,,,}$ of M‴, etc., in short of all men to some enemy of them all. In order to interpret $\mathbf{b}^{a^{\mathrm{m}}}$ we have only to take the negative of it. This, by (124), is $(1-\mathbf{b})a^{\mathrm{m}}$, or a non-betrayer of all men to some enemy of them. Hence, $\mathbf{b}^{a^{\mathrm{m}}}$, or that which is *not* this, is a betrayer of some man to each enemy of all men. To interpret $\mathrm{b}(a\mathrm{m})$ we may put it in the form $(1-\mathbf{b})^{(1-a)^{\mathrm{m}}}$. This is "non-betrayer of a man to all non-enemies of all men." Now, a non-betrayer of some X to every Y, is the same as a betrayer of all X's to nothing but what is not Y; and the negative of "non-enemy of all men," is "enemy of a man." Thus, $\mathrm{b}(a\mathrm{m})$ is, "betrayer of all men to nothing but an enemy of a man." To interpret $\mathrm{b}a\mathrm{m}$ we may put it in the form $(1-\mathbf{b})^{(1-a)}\mathrm{m}$, which is, "non-betrayer of a man to every non-enemy of him." This is a logical sum of terms, each of which is "non-betrayer of an individual man M to every non-enemy of M." Each of these terms is the same as "betrayer of M to nothing but an enemy of M." The sum of them, therefore, which is $\mathrm{b}a\mathrm{m}$ is "betrayer of some man to nothing but an enemy of him." In the same way it is obvious that $^{b}a\mathrm{m}$ is "betrayer of nothing but a man to nothing but an enemy of him." We have $\mathbf{b}^{a}\mathrm{m} = \mathbf{b}(1-a)^{(1-\mathrm{m})}$, or "betrayer of all non-men to a non-enemy of all non-men." This is the same as "that which stands to something which is an enemy of nothing but a man in the relation of betrayer of nothing but men to what is not it." The interpretation of $\mathbf{b}^{a}\mathrm{m}$ is obviously "betrayer of nothing but a man to an enemy of him." It is equally plain that $\mathrm{b}a^{\mathrm{m}}$ is "betrayer of no man to anything but an enemy of him," and that $\mathrm{b}^{a}\mathrm{m}$ is "betrayer of nothing but a man to every enemy of him." By putting $b^{a\mathrm{m}}$ in the form $b^{(1-a)(1-\mathrm{m})}$ we find that it denotes "betrayer of something besides a man to all things which are enemies of nothing but men." When an absolute term is put in place of *a*, the interpretations are obtained in the same way, with greater facility.

146. The sign of an operation is plainly a conjugative term. Thus, our commutative multiplication might be denoted by the conjugative

$$\mathbf{1},.$$

For we have

$$l,\mathrm{sw} = \mathbf{1},l,\mathrm{sw}.$$

As conjugatives can all be reduced to conjugatives of two correlates, they might be expressed by an operative sign (for which a Hebrew letter might be used) put between the symbols for the two correlates. There would often be an advantage in doing this, owing to the intricacy of the usual notation for conjugatives. If these operational signs happened to agree in their properties with any of the signs of algebra, modifications of the algebraic signs might be used in place of Hebrew letters. For instance, if Υ were such that

$$\Upsilon x \Upsilon yz = \Upsilon_{13} \Upsilon yz,\text{*}$$

then, if we were to substitute for Υ the operational sign ℸ we have

$$x ℸ (y ℸ z) = (x ℸ y) ℸ z,$$

which is the expression of the associative principle. So, if

$$\Upsilon xy = \Upsilon yx$$

we may write,

$$x ℸ y = y ℸ x$$

which is the commutative principle. If both these equations held for any conjugative, we might conveniently express it by a modified sign +. For example, let us consider the conjugative "what is denoted by a term which either denotes — or else —." For this, the above principles obviously hold, and we may naturally denote it by +. Then, if p denotes Protestantism, r Romanism, and f what is false,

* The second half of this equation should be:

$$\Upsilon_{13} \Upsilon xyz.$$

$$p \dotplus r \prec f$$

means either all Protestantism or all Romanism is false. In this way it is plain that all hypothetical propositions may be expressed. Moreover, if we suppose any term as "man" (m) to be separated into its individuals, M′, M″, M‴, etc., then,

$$M' \dotplus M'' \dotplus M''' \dotplus \text{etc.},$$

means "some man." This may very naturally be written

$$\text{'m'}$$

and this gives us an improved way of writing a particular proposition; for

$$\text{'x'} \prec y$$

seems a simpler way of writing "Some X is Y" than

$$0^{x,\, y} = 0.$$

Converse

147. If we separate *lover* into its elementary relatives, take the reciprocal of each of these, that is, change it from

$$A:B \quad \text{to} \quad B:A,$$

and sum these reciprocals, we obtain the relative *loved by*. There is no such operation as this in ordinary arithmetic, but if we suppose a science of discrete quantity in quaternion form (a science of equal intervals in space), the sum of the reciprocals of the units of such a quaternion will be the conjugate-quaternion. For this reason, I express the conjugative term "what is related in the way that to — is —, to the latter" by $\mathcal{K}$. The fundamental equations upon which the properties of this term depend are

(169) $$\mathcal{K}\mathcal{K} = 1.$$

(170) If $x < y^z$ then $z \prec (\mathcal{K}y)^x$,

or $$1(x, yz) = 1(z, \mathcal{K}yx)$$

We have, also,

(171) $$\mathcal{K}\Sigma = \Sigma\mathcal{K},$$

(172) $$\mathcal{K}\Pi = \Pi\mathcal{K},$$

where Π denotes the product in the reverse order. Other equations will be found in Mr. De Morgan's table, given above.*

Conclusion

148. If the question is asked, What are the axiomatic principles of this branch of logic, not deducible from others? I reply that whatever rank is assigned to the laws of contradiction and excluded middle belongs equally to the interpretations of all the general equations given under the head of "Application of the Algebraic Signs to Logic," together with those relating to backward involution, and the principles expressed by equations (95), (96), (122), (142), (156), (25), (26), (14), (15).

149. But these axioms are mere substitutes for definitions of the universal logical relations, and so far as these can be defined, all axioms may be dispensed with. The fundamental principles of formal logic are not properly axioms, but definitions and divisions; and the only *facts* which it contains relate to the identity of the conceptions resulting from those processes with certain familiar ones.

* In 112.

IV

*ON THE APPLICATION OF LOGICAL ANALYSIS TO MULTIPLE ALGEBRA.**

150. The letters of an algebra express the relation of the product to the multiplicand. Thus, iA expresses the quantity which is related to A in the manner denoted by i. This being the conception of these algebras, for each of them we may imagine another "absolute" algebra, as we may call it, which shall contain letters which can only be products and multiplicands, not multipliers. Let the general expression of the absolute algebra be $aI+bJ+cK+dL+$ etc. Multiply this by any letter i of the relative algebra, and denote the product by

$$\begin{aligned}&(A_1a+A_2b+A_3c+\text{ etc.})I.\\+&(B_1a+B_2b+B_3c+\text{ etc.})J.\\+&\text{ etc.}\end{aligned}$$

Now we may obviously enlarge the given relative algebra, so that

$$\begin{aligned}i=&A_1i_{11}+A_2i_{12}+A_3i_{13}+\text{ etc.}\\+&B_1i_{21}+B_2i_{22}+B_3i_{23}+\text{ etc.}\\+&\text{ etc.}\end{aligned}$$

where $i_{11}i_{12}$ etc., are such that the product of either of them into any letter of the absolute algebra shall equal some letter of that algebra. That there is no self-contradiction involved in this supposition seems axiomatic.†

151. In this way each letter of the given algebra is resolved into a sum of terms of the form $aA:B$, a being a scalar, and $A:B$ such that

$$(A:B)(B:C)=A:C.$$
$$(A:B)(C:D)=0.$$

The actual resolution is usually performed with ease, but in some cases a good deal of ingenuity is required. I have not found the process facilitated by any general rules. I have actually resolved all the Double, Triple, and Quadruple alge-

* *Proceedings of the American Academy of Arts and Sciences*, pp. 392–94, vol. 10, (1875).

† See 294 for another approach to this same problem.

bras, and all the Quintuple ones, that appeared to present any difficulty. I give a few examples.

bi_5.*

	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	$j+al$	0	0	0	$bj+cl$
l	0	0	0	0	0
m	$a'j+b'l$	0	$c'j+d'l$	0	l

$i=cd'A:B+b'B:C+b'D:E.$ $\qquad j=b'cd'A:C.$
$k=cd'A:B+acd'D:B+b'c^2d'D:F+cd'E:C+bb'cd'A:F.$
$l=b'cd'D:C.$
$m=a'cd'A:B+b'c'A:E+b'cd'D:B+b'd'D:E+b'cd'D:F+F:C.$

bd_5.†

	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	$j+rl$	0	$i+m$	0	$-j-rl$
l	0	0	j	0	0
m	$(r^2-1)j$	0	$-l$	0	$-r^2j$

$i=A:D+D:F+B:E+C:F.$ $\qquad j=A:F.$

$k=rA:B+rB:C+D:E-\frac{1}{r}D:F+E:F.$

$l=A:E-\frac{1}{r}A:F+B:F.$ $\qquad m=r^2A:C-A:D-B:E-C:F.$

* See Benjamin Peirce's *Linear Associative Algebras*, *op. cit.*, p. 195.
† *Ibid.*, p. 188.

bh_6.*

	i	j	k	l	m	n
i	i	j	k	l	m	
j	j	k				
k	k					
l	l	ak		k		
m						k
n	n					

$i = A{:}A + B{:}B + C{:}C + D{:}D.$ $l = aA{:}B + A{:}D + D{:}C.$
$j = A{:}B + B{:}C.$ $m = A{:}E.$
$k = A{:}C.$ $n = E{:}C.$

br_5.†

	i	j	k	l	m
i	j				
j					
k	l		m		
l					
m					

$i = A{:}B + B{:}C.$ $l = D{:}C.$
$j = A{:}C.$ $m = D{:}F.$
$k = D{:}B + D{:}E + E{:}F.$

* *Ibid.*, p. 209.
† *Ibid.*, p. 202.

V

NOTE ON GRASSMANN'S CALCULUS OF EXTENSION.*

152. The last *Mathematische Annalen* contains a paper by H. Grassmann, on the application of his calculus of extension to mechanics.†

He adopts the quaternion addition of vectors. But he has two multiplications, internal and external, just as the principles of logic require.

The *internal* product of two vectors, v_1 and v_2, is simply what is written in quaternions as $-S.v_1v_2$. He writes it $[v_1|v_2]$. So that

$$[v_1 \mid v_2] = [v_2 \mid v_1],$$

$$v_2 = (Tv)^2.$$

The *external* product of two vectors is the parallelogram they form, account being taken of its plane and the direction of running round it, which is equivalent to its *aspect*. We therefore have:

$$[v_1v_2] = v_1v_2 \sin \angle_{v_2}^{v_1}. \mathrm{I}.$$

$$[v_1v_2] = -[v_2v_1], \qquad v^2 = o,$$

where I is a new unit. This reminds me strongly of what is written in quaternions as $-V(v_1v_2)$. But it is not the same thing in fact, because $[v_1v_2]v_3$ is a solid, and therefore a new kind of quantity. In truth, Grassmann has got hold (though he did not say so) of an eight-fold algebra, which may be written in my system as follows:

* *Proceedings of the American Academy of Arts and Sciences*, vol. 13, pp. 115–16, (1877).

† "Die mechanik nach den Principien der Ausdehnungslehre," Bd. 12, H. 2, S. 222–40, (1877).

*Three Rectangular Versors**

$$i = M:A - B:Z + C:Y + X:N$$
$$j = M:B - C:X + A:Z + Y:N$$
$$k = M:C - A:Y + B:X + Z:N$$

Three Rectangular Planes

$$I = M:X + A:N$$
$$J = M:Y + B:N$$
$$K = M:Z + C:N$$

One Solid

$$V = M:N$$

Unity

$$1 = M:M + A:A + B:B + C:C + N:N + X:X + Y:Y + Z:Z$$

This unity might be omitted.

153. The recognition† of the two multiplications is exceedingly interesting. The system seems to me more suitable to three dimensional space, and also more natural than that of quaternions. The simplification of mechanical formulæ is striking, but not more than quaternions would effect, that I see.

By means of eight rotations through two-thirds of a circumference, around four symmetrically placed axes, together with unity, all distortions of a particle would be represented linearly. I have therefore thought of the nine-fold algebra thus resulting.

* Originally 'vectors'; corrected by Peirce.
† Originally 'relation'; corrected by Peirce.

VI

ON THE ALGEBRA OF LOGIC*

PART I.† SYLLOGISTIC[1]

§1. DERIVATION OF LOGIC

154. In order to gain a clear understanding of the origin of the various signs used in logical algebra and the reasons of the fundamental formulæ, we ought to begin by considering how logic itself arises.

155. Thinking, as cerebration, is no doubt subject to the general laws of nervous action.

156. When a group of nerves are stimulated, the ganglions with which the group is most intimately connected on the whole are thrown into an active state, which in turn usually occasions movements of the body. The stimulation continuing, the irritation spreads from ganglion to ganglion (usually increasing meantime). Soon, too, the parts first excited begin to show fatigue; and thus for a double reason the bodily activity is of a changing kind. When the stimulus is withdrawn, the excitement quickly subsides.

It results from these facts that when a nerve is affected, the reflex action, if it is not at first of the sort to remove the irritation, will change its character again and again until the irritation is removed; and then the action will cease.

* *American Journal of Mathematics*, vol. 3, pp. 15–57 (1880), with Peirce's marginal corrections, and the printed corrections of September 15, 1880, in which he says, "The manuscript left my hands in April last before I had seen several important publications — Mr. McColl's third paper, Prof. Wundt's *Logik*, etc."

† The editors have changed 'Chapter' to 'Part.'

[1] "The whole of these two parts is bad, first, because it does not treat the subject from the point of view of pure mathematics, as it should have done; and second because the fundamental propositions are not made out. I follow too much in the footsteps of ordinary numerical algebra, and the sketch of the algebra of the copula is very insufficient." — from the Lowell Lectures, 1903.

157. Now, all vital processes tend to become easier on repetition. Along whatever path a nervous discharge has once taken place, in that path a new discharge is the more likely to take place.

Accordingly, when an irritation of the nerves is repeated, all the various actions which have taken place on previous similar occasions are the more likely to take place now, and those are most likely to take place which have most frequently taken place on those previous occasions. Now, the various actions which did not remove the irritation may have previously sometimes been performed and sometimes not; but the action which removes the irritation must have always been performed, because the action must have every time continued until it was performed. Hence, a strong habit of responding to the given irritation in this particular way must quickly be established.

158. A habit so acquired may be transmitted by inheritance.

One of the most important of our habits is that one by virtue of which certain classes of stimuli throw us at first, at least, into a purely cerebral activity.

159. Very often it is not an outward sensation but only a fancy which starts the train of thought. In other words, the irritation instead of being peripheral is visceral. In such a case the activity has for the most part the same character; an inward action removes the inward excitation. A fancied conjuncture leads us to fancy an appropriate line of action. It is found that such events, though no external action takes place, strongly contribute to the formation of habits of really acting in the fancied way when the fancied occasion really arises.*

160. A cerebral habit of the highest kind, which will determine what we do in fancy as well as what we do in action, is called a *belief*. The representation to ourselves that we have a specified habit of this kind is called a *judgment*. A belief-habit in its development begins by being vague, special, and meagre; it becomes more precise, general, and full, without limit. The process of this development, so far as it takes place in the imagination, is called *thought*. A judgment is formed;

* Cf. 2.146, 2.148.

and under the influence of a belief-habit this gives rise to a new judgment, indicating an addition to belief. Such a process is called an *inference;* the antecedent judgment is called the *premiss;* the consequent judgment, the *conclusion;* the habit of thought, which determined the passage from the one to the other (when formulated as a proposition), the *leading principle.*[1]

161. At the same time that this process of inference, or the spontaneous development of belief, is continually going on within us, fresh peripheral excitations are also continually creating new belief-habits. Thus, belief is partly determined by old beliefs and partly by new experience. Is there any law about the mode of the peripheral excitations? The logician maintains that there is, namely, that they are all adapted to an end, that of carrying belief, in the long run, toward certain predestinate conclusions which are the same for all men. This is the faith of the logician. This is the matter of fact, upon which all maxims of reasoning repose. In virtue of this fact, what is to be believed at last is independent of what has been believed hitherto, and therefore has the character of *reality.* Hence, if a given habit, considered as determining an inference, is of such a sort as to tend toward the final result, it is correct; otherwise not. Thus, inferences become divisible into the valid and the invalid; and thus logic takes its reason of existence.

§2. SYLLOGISM AND DIALOGISM*

162. The general type of inference is

$$\begin{array}{c} P \\ \therefore C, \end{array}$$

where ∴ is the sign of illation.

163. The passage from the premiss (or set of premisses) P to the conclusion C takes place according to a habit or rule active within us. All the inferences which that habit would

[1] *Deductive logic,* perhaps, does not involve the principle that there is any special character in the peripheral excitation but only that reasoning proceeds by habits that are consistent. *Deductive* — consistency of thought with itself. *Inductive* — consistency of the world (Uniformity of Nature).— marginal note, c. 1882.

* Cf. vol. 2, bk. III, ch. 1, §§ 1, 2, and ch. 2, Part I.

determine when once the proper premisses were admitted, form a class. The habit is logically good provided it would never (or in the case of a probable inference, seldom) lead from a true premiss to a false conclusion; otherwise it is logically bad. That is, every possible case of the operation of a good habit would either be one in which the premiss was false or one in which the conclusion would be true; whereas, if a habit of inference is bad, there is a possible case in which the premiss would be true, while the conclusion was false. When we speak of a *possible* case, we conceive that from the general description of cases we have struck out all those kinds which we know how to describe in general terms but which we know never will occur; those that then remain, embracing all whose non-occurrence we are not certain of, together with all those whose non-occurrence we cannot explain on any general principle, are called possible.

164. A habit of inference may be formulated in a proposition which shall state that every proposition c, related in a given general way to any true proposition p, is true. Such a proposition is called the *leading principle* of the class of inferences whose validity it implies. When the inference is first drawn, the leading principle is not present to the mind, but the habit it formulates is active in such a way that, upon contemplating the believed premiss, by a sort of perception the conclusion is judged to be true.[1] Afterwards, when the inference is subjected to logical criticism, we make a new inference, of which one premiss is that leading principle of the former inference, according to which propositions related to one another in a certain way are fit to be premiss and conclusion of a valid inference, while another premiss is a fact of observation, namely, that the given relation does subsist between the premiss and conclusion of the inference under criticism; whence it is concluded that the inference was valid.

165. Logic supposes inferences not only to be drawn, but also to be subjected to criticism; and therefore we not only require the form $P \therefore C$ to express an argument, but also a form, $P_i \prec C_i$, to express the truth of its leading principle. Here P_i denotes any one of the class of premisses, and C_i the

[1] Though the leading principle itself is not present to the mind, we are generally conscious of inferring on some general principle. [Cf. 2.186 ff.]

corresponding conclusion. The symbol $-\!\!<$ is the copula, and signifies primarily that every state of things in which a proposition of the class P_i is true is a state of things in which the corresponding propositions of the class C_i are true. But logic also supposes some inferences to be invalid, and must have a form for denying the leading premiss [?principle]. This we shall write $P_i \overline{-\!\!<} C_i$, *a dash over any symbol signifying in our notation the negative of that symbol.*[1]

Thus, the form $P_i -\!\!< C_i$ implies

either, 1, that it is impossible that a premiss of the class P_i should be true,

or, 2, that every state of things in which P_i is true is a state of things in which the corresponding C_i is true.

The form $P_i \overline{-\!\!<} C_i$ implies

both, 1, that a premiss of the class P_i is possible,

and, 2, that among the possible cases of the truth of a P_i there is one in which the corresponding C_i is not true.

This acceptation of the copula differs from that of other systems of syllogistic in a manner which will be explained below in treating of the negative.

166. In the form of inference $P \therefore C$ the leading principle is not expressed; and the inference might be justified on several separate principles. One of these, however, $P_i -\!\!< C_i$, is the formulation of the habit which, in point of fact, has governed the inferences. This principle contains all that is necessary besides the premiss P to justify the conclusion. (It will generally assert more than is necessary.) We may, therefore, construct a new argument which shall have for its premisses the two propositions P and $P_i -\!\!< C_i$ taken together, and for its conclusion, C. This argument, no doubt, has, like every other, its leading principle, because the inference is governed by some habit; but yet the substance of the leading principle must already be contained implicitly in the premisses, because the proposition $P_i -\!\!< C_i$ contains by hypothesis all that is requisite to justify the inference of C from P. Such a leading principle, which contains no fact not implied or observ-

[1] This dash was used by Boole, but not over other than class-signs.

able in the premisses, is termed a *logical* principle, and the argument it governs is termed a *complete*, in contradistinction to an *incomplete*, argument, or *enthymeme*.

The above will be made clear by an example. Let us begin with the enthymeme,

Enoch was a man,
∴ Enoch died.

The leading principle of this is, "All men die." Stating it, we get the complete argument,

All men die,
Enoch was a man;
∴ Enoch was to die.

The leading principle of this is *nota notae est nota rei ipsius*. Stating this as a premiss, we have the argument,

Nota notae est nota rei ipsius,
Mortality is a mark of humanity, which is a mark of Enoch;
∴ Mortality is a mark of Enoch.

But this very same principle of the *nota notae* is again active in the drawing of this last inference, so that the last state of the argument is no more complete than the last but one.

167. There is another way of rendering an argument complete, namely, instead of adding the leading principle $P_i \prec C_i$ conjunctively to the premiss P, to form a new argument, we might add its denial disjunctively to the conclusion; thus,

$$P$$
$$\therefore \text{Either } C \text{ or } \overline{P_i \prec C_i}.$$

168. A logical principle is said to be an *empty* or merely formal proposition, because it can add nothing to the premisses of the argument it governs, although it is relevant; so that it implies no fact except such as is presupposed in all discourse, as we have seen in §1 that certain facts are implied. We may here distinguish between *logical* and *extralogical* validity; the former being that of a *complete*, the latter that of an *incomplete* argument. The term *logical leading principle* we may take to mean the principle which must be supposed true in order to sustain the logical validity of any argument. Such a principle

states that among all the states of things which can be supposed without conflict with logical principles, those in which the premiss of the argument would be true would also be cases of the truth of the conclusion. Nothing more than this would be relevant to the *logical leading principle*, which is, therefore, perfectly determinate and not vague, as we have seen an extralogical leading principle to be.

169. A complete argument, with only one premiss, is called an *immediate* inference. *Example:* All crows are black birds; therefore, all crows are birds. If from the premiss of such an argument everything redundant is omitted, the state of things expressed in the premiss is the same as the state of things expressed in the conclusion, and only the form of expression is changed. Now, the logician does not undertake to enumerate all the ways of expressing facts: he supposes the facts to be already expressed in certain standard or canonical forms. But the equivalence between different ones of his own standard forms is of the highest importance to him, and thus certain immediate inferences play the great part in formal logic. Some of these will not be reciprocal inferences or logical equations, but the most important of them will have that character.

170. If one fact has such a relation to a different one that, if the former be true, the latter is necessarily or probably true, this relation constitutes a determinate fact; and therefore, since the leading principle of a complete argument involves no matter of fact (beyond those employed in all discourse), it follows that every complete and *material* (in opposition to a merely *formal*) argument must have at least two premisses.

171. From the doctrine of the leading principle it appears that if we have a valid and complete argument from more than one premiss, we may suppress all premisses but one and still have a valid but incomplete argument. This argument is justified by the suppressed premisses; hence, from these premisses alone we may infer that the conclusion would follow from the remaining premisses. In this way, then, the original argument

$$P\ Q\ R\ S\ T$$
$$\therefore C$$

is broken up into two, namely, 1st,

$$\begin{array}{c} P\ Q\ R\ S \\ \therefore T -\!\!< C \end{array}$$

and, 2d,

$$\begin{array}{c} T -\!\!< C \\ T \\ \therefore C. \end{array}$$

By repeating this process, any argument may be broken up into arguments of two premisses each. A complete argument having two premisses is called a *syllogism.*[1]

172. An argument may also be broken up in a different way by substituting for the second constituent above, the form

$$\begin{array}{c} T -\!\!< C \\ \therefore \text{Either } C \text{ or not } T. \end{array}$$

In this way, any argument may be resolved into arguments, each of which has one premiss and two alternative conclusions. Such an argument, when complete, may be called a *dialogism.*

§3. FORMS OF PROPOSITIONS

173. In place of the two expressions $A -\!\!< B$ and $B -\!\!< A$ taken together we may write $A = B$;[2] in place of the two expres-

[1] The general doctrine of this section is contained in my paper, *On the Natural Classification of Arguments*, 1867 [vol. 2, Bk. III, ch. 2].

[2] There is a difference of opinion among logicians as to whether $-\!\!<$ or $=$ is the simpler relation. But in my paper on the *Logic of Relatives* [47n.], I have strictly demonstrated that the preference must be given to $-\!\!<$ in this respect. The term *simpler* has an exact meaning in logic; it means that whose logical depth is smaller; that is, if one conception implies another, but not the reverse, then the latter is said to be the simpler. Now to say that $A = B$ implies that $A -\!\!< B$, but not conversely. *Ergo*, etc. It is to no purpose to reply that $A -\!\!< B$ implies $A = (A$ that is $B)$; it would be equally relevant to say that $A -\!\!< B$ implies $A = A$. Consider an analogous case. Logical sequence is a simpler conception than causal sequence, because every causal sequence is a logical sequence but not every logical sequence is a causal sequence; and it is no reply to this to say that a logical sequence between two facts implies a causal sequence between some two facts whether the same or different. The idea that $=$ is a very simple relation is probably due to the fact that the discovery of such a relation teaches us that instead of two objects we have only one, so that it simpli-

sions $A \mathrel{-\!\!<} B$ and $B \mathrel{\overline{-\!\!<}} A$ taken together we may write $A<B$ or $B>A$; and in place of the two expressions $A \mathrel{\overline{-\!\!<}} B$ and $B \mathrel{\overline{-\!\!<}} A$ taken together [disjunctively] we may write $A \asymp B$.*

174. De Morgan, in the remarkable memoir with which he opened his discussion of the syllogism (1846, p. 380,†) has pointed out that we often carry on reasoning under an implied restriction as to what we shall consider as possible, which restriction, applying to the whole of what is said, need not be expressed. The total of all that we consider possible is called the *universe* of discourse, and may be very limited. One mode of limiting our universe is by considering only what actually occurs, so that everything which does not occur is regarded as impossible.

175. The forms $A \mathrel{-\!\!<} B$, or A implies B, and $A \mathrel{\overline{-\!\!<}} B$, or A does not imply B‡, embrace both hypothetical and categorical propositions. Thus, to say that all men are mortal is the same as to say that if any man possesses any character whatever then a mortal possesses that character. To say, 'if

fies our conception of the universe. On this account the existence of such a relation is an important fact to learn; in fact, it has the sum of the importances of the two facts of which it is compounded. It frequently happens that it is more convenient to treat the propositions $A \mathrel{-\!\!<} B$ and $B \mathrel{-\!\!<} A$ together in their form $A=B$; but it also frequently happens that it is more convenient to treat them separately. Even in geometry we can see that to say that two figures A and B are equal is to say that when they are properly put together A will cover B and B will cover A; and it is generally necessary to examine these facts separately. So, in comparing the numbers of two lots of objects, we set them over against one another, each to each, and observe that for every one of the lot A there is one of the lot B, and for every one of the lot B there is one of the lot A.

In logic, our great object is to analyse all the operations of reason and reduce them to their ultimate elements; and to make a calculus of reasoning is a subsidiary object. Accordingly, it is more philosophical to use the copula $\mathrel{-\!\!<}$ apart from all considerations of convenience. Besides, this copula is intimately related to our natural logical and metaphysical ideas; and it is one of the chief purposes of logic to show what validity those ideas have. Moreover, it will be seen further on that the more analytical copula does in point of fact give rise to the easiest method of solving problems of logic.

* I.e., $-(A=B)$.

† "On the Structure of the Syllogism, and on the Application of the Theory of Probabilities to Questions of Argument and Authority." *Transactions, Cambridge Philosophical Society*, vol. 8, pp. 379–408, (1849). The paper was read and dated 1846.

‡ I.e., it is false that $A \mathrel{-\!\!<} B$.

A, then B' is obviously the same as to say that from A, B follows, logically or extralogically. By thus identifying the relation expressed by the copula with that of illation, we identify the proposition with the inference, and the term with the proposition. This identification, by means of which all that is found true of term, proposition, or inference is at once known to be true of all three, is a most important engine of reasoning, which we have gained by beginning with a consideration of the genesis of logic.[1]

176. Of the two forms $A \prec B$ and $A \mathrel{\overline{\prec}} B$, no doubt the former is the more primitive, in the sense that it is involved in the idea of reasoning, while the latter is only required in the criticism of reasoning. The two kinds of proposition are essentially different, and every attempt to reduce the latter to a special case of the former must fail. Boole* attempts to express 'some men are not mortal,' in the form 'whatever men have a certain unknown character *v* are not mortal.' But the propositions are not identical, for the latter does not imply that some men have that character *v;* and, accordingly, from Boole's proposition we may legitimately infer that 'whatever mortals have the unknown character *v* are not men';† yet we cannot reason from 'some men are not mortal' to 'some mortals are not men.'[2] On the other hand, we can rise to a more general form under which $A \prec B$ and $A \mathrel{\overline{\prec}} B$ are both included. For this purpose we write $A \mathrel{\overline{\prec}} B$ in the form $\breve{A} \prec \bar{B}$,‡ where $\breve{A}$ is *some*-A and $\bar{B}$ is *not*-B. This more general form is equivocal in so far as it is left undetermined whether the proposition would be true if the subject were impossible. When the subject is general this is the case, but when the subject is particu-

[1] In consequence of the identification in question, in $S \prec P$, I speak of S indifferently as *subject, antecedent*, or *premiss*, and of P as *predicate, consequent*, or *conclusion*.

* See *Laws of Thought*, p. 62f.

† See 138.

[2] Equally unsuccessful is Mr. Jevons' attempt to overcome the difficulty by omitting particular propositions, 'because we can always substitute for it [*some*] more definite expressions if we like.' The same reason might be alleged for neglecting the consideration of *not*. But in fact the form $A \mathrel{\overline{\prec}} B$ is required to enable us to simply deny $A \prec B$.

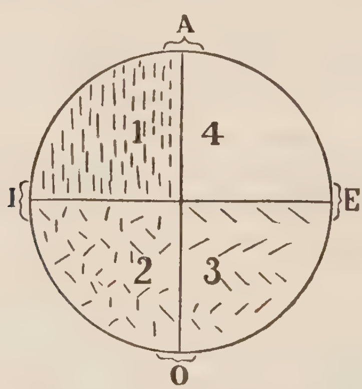
‡ To express such a particular proposition disjunctively, change the quantity and quality of the antecedent and the consequent and deny their disjunction. Cf. 196.

lar (i.e., is subject to the modification *some*) it is not.* The general form supposes merely inclusion of the subject under the predicate. The short curved mark over the letter in the subject shows that some part of the term denoted by that letter is the subject, and that that is asserted to be in possible existence.

177. The modification of the subject by the curved mark and of the predicate by the straight mark gives the old set of propositional forms, viz.:

A.	$a \prec b$	Every *a* is *b*.	Universal affirmative.
E.	$a \prec \bar{b}$	No *a* is *b*.	Universal negative.
I.	$\breve{a} \prec b$	Some *a* is *b*.	Particular affirmative.
O.	$\breve{a} \prec \bar{b}$	Some *a* is not *b*.	Particular negative.

178. There is, however, a difference between the senses in which these propositions are here taken and those which are traditional; namely, it is usually understood that affirmative propositions imply the existence of their subjects, while negative ones do not. Accordingly, it is said that there is an immediate inference from A to I and from E to O. But in the sense assumed in this paper, universal propositions do not, while particular propositions do, imply the existence of their subjects. The following figure illustrates the precise sense here assigned to the four forms A, E, I, O.



179. In the quadrant marked 1 there are lines which are all vertical; in the quadrant marked 2 some lines are vertical

* See 178.

and some not; in quadrant 3 there are lines none of which are vertical; and in quadrant 4 there are no lines. Now, taking *line* as subject and *vertical* as predicate,

> A is true of quadrants 1 and 4 and false of 2 and 3.
> E is true of quadrants 3 and 4 and false of 1 and 2.
> I is true of quadrants 1 and 2 and false of 3 and 4.
> O is true of quadrants 2 and 3 and false of 1 and 4.

Hence, A and O precisely deny each other, and so do E and I. But any other pair of propositions may be either both true or both false or one true while the other is false.*

180. De Morgan ("On the Syllogism," No. I., 1846, p. 381) has enlarged the system of propositional forms by applying the sign of negation which first appears in $A \overline{\prec} B$ to the subject and predicate. He thus gets

$A \prec B$.	Every A is B.†	A is species of B.‡
$A \overline{\prec} B$.	Some A is not B.	A is exient of B.
$A \prec \bar{B}$.	No A is B.	A is external of B.
$A \overline{\prec} \bar{B}$.	Some A is B.	A is partient of B.
$\bar{A} \prec B$.	Everything is either A or B.	A is complement of B.
$\bar{A} \overline{\prec} B$.	There is something besides A and B.	A is coinadequate of B.
$\bar{A} \prec \bar{B}$.	A includes all B.	A is genus of B.
$\bar{A} \overline{\prec} \bar{B}$.	A does not include all B.	A is deficient of B.

De Morgan's table of the relations of these propositions must be modified to conform to the meanings here attached to $\prec$ and to $\overline{\prec}$.

181. We might confine ourselves to the two propositional forms $S \prec P$ and $S \overline{\prec} P$. If we once go beyond this and adopt the form $S \prec \bar{P}$, we must, for the sake of completeness, adopt the whole of De Morgan's system. But this system, as we shall see in the next section, is itself incomplete, and requires to complete it the admission of particularity in the predicate. This has already been attempted by Hamilton, with an incompetence which ought to be extraordinary.§ I

* See vol. 2, Bk. III, ch. 1, §3 for a later analysis of this quadrant.

† The readings in this column are not precise.

‡ The terms in this column are taken from De Morgan's later papers.

§ Cf. vol. 2, Bk. III, ch. 3, §3.

shall allude to this matter further on, but I shall not attempt to say how many forms of propositions there would be in the completed system.[1]

§4. THE ALGEBRA OF THE COPULA

182. From the identity of the relation expressed by the copula with that of illation, springs an algebra. In the first place, this gives us

$$x \prec x \qquad (1)$$

the principle of identity, which is thus seen to express that what we have hitherto believed we continue to believe, in the absence of any reason to the contrary. In the next place, this identification shows that the two inferences

$$\begin{array}{ccc} x & & \\ y & \text{and} & x \\ \therefore z & & \therefore y \prec z \end{array} \qquad (2)$$

are of the same validity. Hence we have

$$\{x \prec (y \prec z)\} = \{y \prec (x \prec z)\}.^{2} \qquad (3)$$

183. From (1) we have

$$(x \prec y) \prec (x \prec y),$$

whence by (2)

$$\begin{array}{cc} x \prec y & x \\ \multicolumn{2}{c}{\therefore y} \end{array} \qquad (4)$$

is a valid inference.

184. By (4), if x and $x \prec y$ are true y is true; and if y and $y \prec z$ are true z is true. Hence, the inference is valid

$$\begin{array}{ccc} x & x \prec y & y \prec z \\ \multicolumn{3}{c}{\therefore z.} \end{array}$$

By the principle of (2) this is the same as to say that

$$\begin{array}{cc} x \prec y & y \prec z \\ \multicolumn{2}{c}{\therefore x \prec z} \end{array} \qquad (5)$$

[1] In this connection see De Morgan, "On the Syllogism," No. V., 1862. [*Transactions, Cambridge Philosophical Society*, vol. 4, p. 467, (1864), read and dated 1863.]

[2] Mr. Hugh McColl (*Calculus of Equivalent Statements*, Second Paper, 1878, [*Proceedings, London Mathematical Society*, vol. 9, p. 183 (1877)]), makes use of the sign of inclusion several times in the same proposition. He does not, however, give any of the formulæ of this section.

is a valid inference. This is the canonical form of the syllogism, *Barbara*. The statement of its validity has been called the *dictum de omni*, the *nota notae*, etc.; but it is best regarded, after De Morgan,[1] as a statement that the relation signified by the copula is a transitive one.[2] It may also be considered as implying that in place of the subject of a proposition of the form A —< B, any subject of that subject may be substituted, and that in place of its predicate any predicate of that predicate may be substituted.[3] The same principle may be algebraically conceived as a rule for the elimination of y from the two propositions x —< y and y —< z.[4]

185. It is needless to remark that any letters may be substituted for x, y, z; and that, therefore, $\bar{x}$, $\bar{y}$, $\bar{z}$, some or all, may be substituted. Nevertheless, after these purely extrinsic changes have been made, the argument is no longer called *Barbara*, but is said to be some other universal mood of the *first figure*. There are evidently eight such moods.

186. From (5) we have, by (2), these two forms of valid immediate inference:

$$\begin{gathered} \mathrm{S} \prec \mathrm{P} \\ \therefore (x \prec \mathrm{S}) \prec (x \prec \mathrm{P}) \end{gathered} \tag{6}$$

and

$$\begin{gathered} \mathrm{S} \prec \mathrm{P} \\ \therefore (\mathrm{P} \prec x) \prec (\mathrm{S} \prec x). \end{gathered} \tag{7}$$

The latter may be termed the inference of *contraposition*.*

[1] "On the Syllogism," No. II., 1850, [*Transactions, Cambridge Philosophical Society*, vol. 9, (1851), p. 104].

[2] That the validity of syllogism is not deducible from the principles of identity, contradiction, and excluded middle, is capable of strict demonstration. The transitiveness of the copula is, however, implied in the identification of the copula-relation with illation, because illation is obviously transitive.

[3] The conception of substitution (already involved in the mediæval doctrine of descent), as well as the word, was familiar to logicians before the publication of Mr. Jevons's *Substitution of Similars*. [see vol. 8] This book argues, however, not only that inference is substitution, but that it and induction in particular consist in the substitution of similars. This doctrine is allied to Mill's theory of induction.

[4] This must have been in Boole's mind from the first. De Morgan ("On the Syllogism," No. II., 1850, p. 83) goes too far in saying that "what is called elimination in algebra is called inference in logic," if he means, as he seems to do, that all inference is elimination. [Cf. 2.442f.]

* See 91n.

187. From the transitiveness of the copula, the following inference is valid:

$$(S \multimap M) \multimap (S \multimap P)$$
$$(S \multimap P) \multimap x$$
$$\therefore (S \multimap M) \multimap x.$$

But, by (6), from $(M \multimap P)$ we can infer the first premiss immediately; hence the inference is valid

$$\begin{array}{c} M \multimap P \\ (S \multimap P) \multimap x \\ \therefore (S \multimap M) \multimap x. \end{array} \qquad (8)$$

This may be called the *minor indirect syllogism*. The following is an example:

All men are mortal,
If Enoch and Elijah were mortal, the Bible errs;
∴ If Enoch and Elijah were men, the Bible errs.

188. Again we may start with this syllogism in *Barbara*

$$(M \multimap P) \multimap (S \multimap P),$$
$$(S \multimap P) \multimap x;$$
$$\therefore (M \multimap P) \multimap x.$$

But by the principle of contraposition (7), the first premiss immediately follows from $(S \multimap M)$, so that we have the inference valid

$$\begin{array}{c} S \multimap M, \\ (S \multimap P) \multimap x; \\ \therefore (M \multimap P) \multimap x. \end{array} \qquad (9)$$

This may be called the *major indirect syllogism*.

Example: All patriarchs are men,
If all patriarchs are mortal, the Bible errs;
∴ If all men are mortal, the Bible errs.

189. In the same way it might be shown that (6) justifies the syllogism

$$\begin{array}{c} M \multimap P, \\ x \multimap (S \multimap M); \\ \therefore x \multimap (S \multimap P). \end{array} \qquad (10)$$

And (7) justifies the inference

$$\begin{aligned} &S \prec M, \\ &x \prec (M \prec P); \\ \therefore\ &x \prec (S \prec P). \end{aligned} \qquad (11)$$

But these are only slight modifications of *Barbara*.

190. In the form (10), x may denote a limited universe comprehending some cases of S. Then we have the syllogism

$$\begin{aligned} &M \prec P, \\ &S \,\overline{\prec}\, \overline{M}; \\ \therefore\ &S \,\overline{\prec}\, \overline{P}. \end{aligned} \qquad (12)$$

This is called *Darii*. A line might, of course, be drawn over the S. So, in the form (11), x may denote a limited universe comprehending some P. Then we have the syllogism

$$\begin{aligned} &S \prec M, \\ &\overline{M} \,\overline{\prec}\, P; \\ \therefore\ &\overline{S} \,\overline{\prec}\, P. \end{aligned} \qquad (13)$$

Here a line might be drawn over the P. But the forms (12) and (13) are deduced from (10) and (11) only by principles of interpretation which require demonstration.

191. On the other hand, if in the *minor indirect syllogism* (8), we put "what does not occur" for x, we have by definition

$$\{(S \prec P) \prec x\} = (S \,\overline{\prec}\, P)$$

and we then have

$$\begin{aligned} &M \prec P, \\ &S \,\overline{\prec}\, P; \\ \therefore\ &S \,\overline{\prec}\, M, \end{aligned} \qquad (14)$$

which is the syllogism *Baroko*. If a line is drawn over P, the syllogism is called *Festino;* and by other negations eight essentially identical forms are obtained, which are called minor-particular moods of the second figure.[1] In the same way the major indirect syllogism (9) affords the form

$$\begin{aligned} &S \prec M, \\ &S \,\overline{\prec}\, P; \\ \therefore\ &M \,\overline{\prec}\, P. \end{aligned} \qquad (15)$$

[1] De Morgan, *Syllabus*, 1860, p. 18.

This form is called *Bocardo*. If P is negatived, it is called *Disamis*. Other negations give the eight major-particular moods of the third figure.

192. We have seen that $S \mathrel{\overline{\prec}} P$ is of the form $(S \prec P) \prec x$. Put A for $S \prec P$, and we find that $\bar{A}$ is of the form $A \prec x$. Then the principle of contraposition (7) gives the immediate inference

$$S \prec P \qquad (16)$$
$$\therefore \bar{P} \prec \bar{S}.$$

Applying this to the universal moods of the first figure justifies six moods. These are two in the second figure,

$$x \prec \bar{y} \qquad z \prec y \qquad \therefore x \prec \bar{z} \text{ (\textit{Camestres})}$$
$$\bar{x} \prec \bar{y} \qquad z \prec y \qquad \therefore \bar{x} \prec \bar{z};$$

two in the third figure,

$$y \prec x \qquad \bar{y} \prec z \qquad \therefore \bar{x} \prec z$$
$$y \prec x \qquad \bar{y} \prec \bar{z} \qquad \therefore \bar{x} \prec \bar{z};$$

and two others which are said to be in the fourth figure,

$$x \prec y \qquad y \prec z \qquad \therefore \bar{z} \prec \bar{x}$$
$$x \prec \bar{y} \qquad \bar{y} \prec z \qquad \therefore \bar{z} \prec \bar{x}.$$

But the negative has two other properties not yet taken into account. These are

$$x \prec \bar{\bar{x}} \qquad (17)$$

or x is not not-X, which is called the *principle of contradiction;* and

$$\bar{\bar{x}} \prec x \qquad (18)$$

or what is not not-X is x, which is called the *principle of excluded middle.**

193. By (17) and (16) we have the immediate inference

$$S \prec \bar{\bar{P}} \qquad (19)$$
$$\therefore P \prec \bar{S}$$

which is called the conversion of E. By (18) and (16) we have

$$\bar{S} \prec P \qquad (20)$$
$$\therefore \bar{P} \prec S.$$

* Cf. 2.597-8.

By (17), (18), and (16), we have

$$\bar{S} \prec \bar{P} \qquad (21)$$
$$\therefore P \prec S.$$

Each of the inferences (19), (20), (21), justifies six universal syllogisms; namely, two in each of the figures, second, third, and fourth. The result is that each of these figures has eight universal moods; two depending only on the principle that $\bar{A}$ is of the form $A \prec x$,[1] two depending also on the principle of contradiction, two on the principle of excluded middle, and two on all three principles conjoined.

194. The same formulæ (16), (19), (20), (21), applied to the minor-particular moods of the second figure, will give eight minor-particular moods of the first figure; and applied to the major-particular moods of the third figure, will give eight major-particular moods of the first figure.[2]

195. The principle of contradiction in the form (19) may be further transformed thus:

$$\text{If } (P \therefore \bar{C}) \text{ is valid, then } (C \therefore \bar{P}) \text{ is valid.* } \qquad (22)$$

Applying this to the minor-particular moods of the first figure, will give eight minor-particular moods of the third figure; and applying it to the major-particular moods of the first figure will give eight major-particular moods of the second figure.

It is very noticeable that the corresponding formula,

$$\text{If } (\bar{P} \therefore C) \text{ is valid, then } (\bar{C} \therefore P) \text{ is valid,* } \qquad (23)$$

has no application in the existing syllogistic, because there are no syllogisms having a particular premiss and universal conclusion. In the same way, in the Aristotelian system an affirmative conclusion cannot be drawn from negative premisses, the reason being that negation is only applied to the

[1] An oversight has here been committed. For from $\bar{A} = (A \prec x)$ follows not merely (16) but also (19), (20), and (21), and thus all the properties of the negative which concern syllogistic. But this does not affect the view taken of the subject, nor the division of the moods according to the properties of the negative on which they depend; for whatever is shown in the text to be deducible from $\bar{A} = (A \prec x)$ is in fact deducible from (16). — Sept., 1880.

[2] Aristotle and De Morgan have particular conclusions from two universal premisses. These are all rendered illogical by the significations which I attach to $\prec$ and $\overline{\prec}$.

* $\bar{P}$ and $\bar{C}$ here represent some such forms as $S \overline{\prec} M$ and $S \overline{\prec} P$.

predicate. So in De Morgan's system the subject only is made particular, not the predicate.

196. In order to develop a system of propositions in which the predicate shall be modified in the same way in which the subject is modified in particular propositions, we should consider that to say $S \prec P$ is the same as to say $(S \overline{\prec} x) \prec (P \overline{\prec} x)$, whatever x may be. That

$$(S \prec P) \prec \{(S \overline{\prec} x) \prec (P \overline{\prec} x)\}$$

follows at once from *Bokardo* (15) by means of (2). Moreover, since $\bar{A}$ may be put in the form $A \prec x$, it follows that $\bar{\bar{A}}$ may be put in the form $A \overline{\prec} x$, so that by the principles of contradiction and excluded middle, A may be put in the form $A \overline{\prec} x$. On the other hand, to say $S \overline{\prec} \bar{P}$ is the same as to say $(S \prec \bar{x}) \prec (P \overline{\prec} x)$, whatever x may be; for

$$(S \overline{\prec} \bar{P}) \prec \{(S \prec \bar{x}) \prec (P \overline{\prec} x)\}$$

is the principle of *Ferison*, a valid syllogism of the third figure; and if for x we put $\bar{S}$, we have

$$(S \prec \bar{\bar{S}}) \prec (P \overline{\prec} \bar{S}),$$

which is the same as to say that $P \overline{\prec} \bar{S}$ is true if the principle of contradiction is true. So that it follows that $P \overline{\prec} \bar{S}$ is $S \overline{\prec} \bar{P}$ from the principle of contradiction. Comparing

$$S \prec P \text{ or } (S \overline{\prec} x) \prec (P \overline{\prec} x)$$

with

$$S \overline{\prec} \bar{P} \text{ or } (S \prec \bar{x}) \prec (P \overline{\prec} x),$$

we see that they differ by a modification of the subject. Denoting this by a short curve over the subject, we may write $\breve{S} \prec P$ for $S \overline{\prec} \bar{P}$. We see then that while for A we may write $A \overline{\prec} x$, where x is anything whatever, so for $\breve{A}$ we may write $A \prec \bar{x}$. If we attach a similar modification to the predicate also, we have

$$\breve{S} \prec \breve{P} \quad \text{or} \quad (S \prec \bar{x}) \prec (P \prec \bar{x}),$$

which is the same as to say that you can find an S which is any P you please. We thus have

$$(S \prec P) \prec (\breve{P} \prec \breve{S}), \tag{24}$$

a formula of contraposition, similar to (16).

It is obvious that

$$(\breve{S} -\!\!< P) -\!\!< (\check{P} -\!\!< S); \tag{25}$$

for, negating both propositions, this becomes, by (16),

$$(P -\!\!< \bar{S}) -\!\!< (S -\!\!< \bar{P}),$$

which is (19). The inference justified by (25) is called the conversion of I. From (25) we infer

$$\breve{\check{x}} -\!\!< x,^{*} \tag{26}$$

which may be called the principle of particularity. This is obviously true, because the modification of particularity only consists in changing $(A \overline{-\!\!<} x)$ to $(A -\!\!< \bar{x})$, which is the same as negating the copula and predicate, and a repetition of this will evidently give the first expression again. For the same reason we have

$$x -\!\!< \check{\breve{x}},^{*} \tag{27}$$

which may be called the principle of individuality. This gives

$$(S -\!\!< \check{P}) -\!\!< (P -\!\!< \breve{S}), \tag{28}$$

and (26) and (27) together give

$$(\breve{S} -\!\!< \check{P}) -\!\!< (P -\!\!< S). \tag{29}$$

It is doubtful whether the proposition $S -\!\!< \check{P}$ ought to be interpreted as signifying that S and P are one sole individual, or that there is something besides S and P. I here leave this branch of the subject in an unfinished state.

197. Corresponding to the formulæ which we have obtained by the principle (2) are an equal number obtained by the following principle:

(2′) The inference

$$x$$

$$\therefore \text{Either } y \text{ or } z$$

has the same validity as

$$x \overline{-\!\!<} y$$

$$\therefore z.$$

[Add the formula (3′) $\{(x \overline{-\!\!<} y) \overline{-\!\!<} z\} = \{(x \overline{-\!\!<} z) \overline{-\!\!<} y\}$.
—1880.]

* See 2.458.

From (1) we have

$$(x \,\overline{-\!\!<}\, y) -\!\!< (x \,\overline{-\!\!<}\, y),$$

whence, by (2),

(4′)
$$\begin{array}{c} x \\ \therefore \text{Either } (x -\!\!< y) \text{ or } y.^{*} \end{array}$$

This gives

$$\begin{array}{c} x \\ \therefore \text{Either } x \,\overline{-\!\!<}\, y \text{ or } y \,\overline{-\!\!<}\, z \text{ or } z. \end{array}$$

Then, by (2),

(5′)
$$\begin{array}{c} x \,\overline{-\!\!<}\, z \\ \therefore x \,\overline{-\!\!<}\, y \text{ or } y \,\overline{-\!\!<}\, z, \end{array}$$

which is the canonical form of dialogism. The minor indirect dialogism is

(8′)
$$\begin{array}{c} x \,\overline{-\!\!<}\, (\mathrm{M} \,\overline{-\!\!<}\, \mathrm{P}) \\ \therefore \text{Either } x \,\overline{-\!\!<}\, (\mathrm{S} \,\overline{-\!\!<}\, \mathrm{P}) \text{ or } \mathrm{S} \,\overline{-\!\!<}\, \mathrm{M}. \end{array}$$

The major indirect dialogism is

$$\begin{array}{c} x \,\overline{-\!\!<}\, (\mathrm{S} \,\overline{-\!\!<}\, \mathrm{M}) \\ \therefore \text{Either } x \,\overline{-\!\!<}\, (\mathrm{S} \,\overline{-\!\!<}\, \mathrm{P}) \text{ or } \mathrm{M} \,\overline{-\!\!<}\, \mathrm{P}. \end{array}$$

We have also

(12′)
$$\begin{array}{c} (\mathrm{S} \,\overline{-\!\!<}\, \mathrm{P}) \,\overline{-\!\!<}\, x \\ \therefore \text{Either } (\mathrm{S} \,\overline{-\!\!<}\, \mathrm{M}) \text{ or } (\mathrm{M} \,\overline{-\!\!<}\, \mathrm{P}) \,\overline{-\!\!<}\, x \end{array}$$

and

(13′)
$$\begin{array}{c} (\mathrm{S} \,\overline{-\!\!<}\, \mathrm{P}) \,\overline{-\!\!<}\, x \\ \therefore \text{Either } (\mathrm{M} \,\overline{-\!\!<}\, \mathrm{P}) \text{ or } (\mathrm{S} \,\overline{-\!\!<}\, \mathrm{M}) \,\overline{-\!\!<}\, x. \end{array}$$

We have A of the form $x \,\overline{-\!\!<}\, \overline{\mathrm{A}}$. And we have the inferences

$$\begin{array}{cccc} \mathrm{S} \,\overline{-\!\!<}\, \mathrm{P} & \mathrm{S} \,\overline{-\!\!<}\, \overline{\mathrm{P}} & \overline{\mathrm{S}} \,\overline{-\!\!<}\, \mathrm{P} & \overline{\mathrm{S}} \,\overline{-\!\!<}\, \overline{\mathrm{P}} \\ \therefore \overline{\mathrm{P}} \,\overline{-\!\!<}\, \overline{\mathrm{S}}. & \therefore \mathrm{P} \,\overline{-\!\!<}\, \overline{\mathrm{S}}. & \therefore \overline{\mathrm{P}} \,\overline{-\!\!<}\, \mathrm{S}. & \therefore \mathrm{P} \,\overline{-\!\!<}\, \mathrm{S}. \end{array}$$

* This should be: either $(x \,\overline{-\!\!<}\, y)$ or y.

PART II. THE LOGIC OF NON-RELATIVE TERMS

§1. THE INTERNAL MULTIPLICATION AND THE ADDITION OF LOGIC

198. We have seen that the inference

$$x \text{ and } y$$
$$\therefore z$$

is of the same validity with the inference

$$x$$
$$\therefore \text{Either } \bar{y} \text{ or } z$$

and the inference

$$x$$
$$\therefore \text{Either } y \text{ or } z,$$

with the inference

$$x \text{ and } \bar{y}$$
$$\therefore z.$$

In like manner,

$$x \prec y$$

is equivalent to

$$(\text{The possible}) \prec \text{Either } \bar{x} \text{ or } y,$$

and to

$$x \text{ which is } \bar{y} \prec (\text{The impossible}).$$

To express this algebraically, we need, in the first place, symbols for the two terms of second intention, the possible and the impossible. Let ∞ and 0 be the terms; then we have the definitions

$$x \prec \infty \qquad 0 \prec x \tag{1}$$

whatever x may be.[1]

199. We need also two operations which may be called non-relative addition and multiplication. They are defined as follows:[2]

[1] The symbol 0 is used by Boole; the symbol ∞ replaces his 1, according to a suggestion in my *Logic of Relatives*, 1870 [88].

[2] These forms of definition are original. The algebra of non-relative terms was given by Boole (*Mathematical Analysis of Logic*, 1847). Boole's addition was not the same as that in the text, for with him whatever was common to the

If $a -\!\!< x$ and $b -\!\!< x$,	If $x -\!\!< a$ and $x -\!\!< b$, (2)
then $a+b -\!\!< x$;	then $x -\!\!< a \times b$;
and conversely,	and conversely,
if $a+b -\!\!< x$,	if $x -\!\!< a \times b$, (3)
then $a -\!\!< x$ and $b -\!\!< x$.	then $x -\!\!< a$ and $x -\!\!< b$.

200. From these definitions we at once deduce the following formulæ:

two terms added was taken twice over in the sum. The operations in the text were given as complements of one another, and with appropriate symbols, by De Morgan ("On the Syllogism," No. III., 1858 [*Cambridge Philosophical Transactions*, vol. 10], p. 185). For addition, sum, parts, he uses aggregation, aggregate, aggregants; for multiplication, product, factors, he uses composition, compound, components. Mr. Jevons (*Formal Logic* [*Pure Logic*?], 1864) — I regret that I can only speak of this work from having read it many years ago, and therefore cannot be sure of doing it full justice — improved the algebra of Boole by substituting De Morgan's aggregation for Boole's addition. The present writer, not having seen either De Morgan's or Jevons's writings on the subject, again recommended the same change (*On an Improvement in Boole's Calculus of Logic*, 1867 [3]), and showed the perfect balance existing between the two operations. In another paper, published in 1870 [47], I introduced the sign of inclusion into the algebra.

In 1872, Robert Grassmann, brother of the author of the *Ausdehnungslehre*, published a work entitled '*Die Formenlehre oder Mathematik*,' the second book of which gives an algebra of logic identical with that of Jevons. The very notation is reproduced, except that the universe is denoted by T instead of U, and a term is negatived by drawing a line over it, as by Boole, instead of by taking a type from the other case, as Jevons does. Grassmann also uses a sign equivalent to my $-\!\!<$. In his third book, he has other matter which he might have derived from my paper of 1870. Grassmann's treatment of the subject presents inequalities of strength; and most of his results had been anticipated. Professor Schröder, of Karlsruhe, in the spring of 1877, produced his *Operationskreis des Logikkalkuls*. He had seen the works of Boole and Grassmann, but not those of De Morgan, Jevons, and me. He gives a fine development of the algebra, adopting the addition of Jevons, and he exhibits the balance between $+$ and $\times$ by printing the theorems in parallel columns, thus imitating a practice of the geometricians. Schröder gives an original, interesting, and commodious method of working with the algebra. Later in the same year, Mr. Hugh McColl, apparently having known nothing of logical algebra except from a jejune account of Boole's work in Bain's *Logic*, published several papers on a *Calculus of Equivalent Statements*, [*Proceedings, London Mathematical Society*, Series 1, vol. IX, pp. 177–186], the basis of which is nothing but the Boolian algebra, with Jevons's addition and a sign of inclusion. Mr. McColl adds an exceedingly ingenious application of this algebra to the transformation of definite integrals.

A.[1] $a \prec a+b$ $\quad a\times b \prec a$ (Peirce, 1870)[2] (4)

$b \prec a+b$ $\quad a\times b \prec b.$

These are proved by substituting $a+b$ and $a\times b$ for x in (3).

B. $x=x+x$ $\quad x\times x=x$ (Jevons, 1864). (5)

By substituting x for a and b in (2), we get

$$x+x \prec x \qquad x \prec x\times x;$$

and, by (4),

$$x \prec x+x \qquad x\times x \prec x.$$

C. $a+b=b+a$ $\quad a\times b=b\times a$ (Boole, Jevons). (6)

These formulæ are examples of the *commutative principle*. From (4) and (2),

$$b+a \prec a+b \qquad a\times b \prec b\times a$$

and interchanging a and b we get the reciprocal inclusion implied in (6).

D. $(a+b)+c=a+(b+c)$ $\quad a\times(b\times c)=(a\times b)\times c$

(Boole, Jevons). (7)

These are cases of the *associative principle*. By (4), $c \prec b+c$ and $b\times c \prec c$; also $b+c \prec a+(b+c)$ and $a\times(b\times c) \prec b\times c$; so that $c \prec a+(b+c)$ and $a\times(b\times c) \prec c$. In the same way, $b \prec a+(b+c)$ and $a\times(b\times c) \prec b$, and, by (4), $a \prec a+(b+c)$ and $a\times(b\times c) \prec a$. Hence, by (2), $a+b \prec a+(b+c)$ and $a\times(b\times c) \prec a\times b$. And, again by (2), $(a+b)+c \prec a+(b+c)$ and $a\times(b\times c) \prec (a\times b)\times c$. In a similar way we should prove the converse propositions to these and so establish (7).

E. $(a+b)\times c=(a\times c)+(b\times c)$ $\quad (a\times b)+c=(a+c)\times(b+c)$.[3] (8)

[1] Remark that the proofs of the lettered propositions follow the enunciations —1880.

[2] *Logic of Relatives* (§4) gives $a\times b \prec a$. The other formulæ, equally obvious, I do not find anywhere.

[3] The first of these given by Boole for his addition, was retained by Jevons in changing the addition. The second was first given by me (1867) [See 4].

These are cases of the *distributive principle*. They are easily proved by (4) and (2), but the proof is too tedious to give.*

F. $(a+b)+c=(a+c)+(b+c)$ $(a\times b)\times c=(a\times c)\times(b\times c)$. (9)

These are other cases of the distributive principle. They are proved by (5), (6) and (7). These formulæ, which have hitherto escaped notice, are not without interest.

G. $a+(a\times b)=a$ $a\times(a+b)=a$ (Grassmann, Schröder). (10)

By (4), $a\prec a+(a\times b)$ $a\times(a+b)\prec a$.

Again, by (4), $(a\times b)\prec a$ and $a\prec a+b$; hence, by (2)

$$a+(a\times b)\prec a \qquad a\prec a\times(a+b).$$

* "It seems that $(a+b)\times c\prec(a\times c)+(b\times c)$ cannot be proved from the definitions. The propositions L are needed" — a marginal note prompted apparently by Schröder's criticisms in his *Vorlesungen über die Algebra der Logik*, Bd. 1, Kap. 6.

On February 14, 1904, Peirce wrote Prof. E. V. Huntington of Harvard as follows:

"Dear Mr. Huntington: Should you decide to print the proof of the distributive principle (and this would not only relieve me from a long procrastinated duty, but would have a certain value for exact logic, as removing the eclipse under which the method of developing the subject followed in my paper in vol. 3 has been obscured) I should feel that it was incumbent upon me, in decency, to explain its having been so long withheld. The truth is that the paper aforesaid was written during leisure hours gained to me by my being shut up with a severe influenza. In writing it, I omitted the proof, as there said, because it was 'too tedious' and because it seemed to me very obvious. Nevertheless, when Doctor Schröder questioned its possibility, I found myself unable to reproduce it, and so concluded that it was to be added to the list of blunders, due to the grippe, with which that paper abounds — a conclusion that was strengthened when Schröder thought he demonstrated the indemonstrability of the law of distributiveness. (I must confess that I never carefully examined his proof, having my table loaded with logical books for the perusal of which life was not long enough.) It was not until many years afterwards that, looking over my papers of 1880 for a different purpose, I stumbled upon this proof written out in full for the press, though it was eventually cut out, and, at first, I was inclined to think that it employed the principle that *all* existence is individual, which my method, in the paper in question, did not permit me to employ at that stage. I venture to opine that it fully vindicates my characterisation of it as 'too tedious'. But this is how I have a new apology to make to exact logicians."

This letter and the proof were used by Professor Huntington in his "Sets of Independent Postulates for the Algebra of Logic," *Transactions, American Mathematical Society*, vol. 5, p. 300 n. (1904), and in proof of proposition 22a. The proof is also to be found in Lewis's *Survey of Symbolic Logic*, p. 128 (5.5). A more elegant proof is to be found in the *Principia Mathematica*. See 384n.

H. $$(a+b \prec a) = (b \prec a \times b). \qquad (11)$$

This proposition is a transformation of Schröder's two propositions 21 (p. 25), one of which was given by Grassmann. By (3)

$$(a+b \prec a) \prec (b \prec a) \quad (b \prec a \times b) \prec (b \prec a).$$

Hence, since $b \prec b$, $a \prec a$

we have, by (2),

$$(a+b \prec a) \prec (b \prec a \times b) \quad (b \prec a \times b) \prec (a+b \prec a).$$

I. $$\left.\begin{array}{l}(a \prec b) \times (x \prec y) \prec (a+x \prec b+y) \\ (a \prec b) \times (x \prec y) \prec (a \times x \prec b \times y)\end{array}\right\} \text{(Peirce, 1870).} \qquad (12)$$

Readily proved from (2) and (4).

J. $$(a \prec b+x) \times (a \times x \prec b) = (a \prec b). \qquad (13)$$

This is a generalization of a theorem by Grassmann. In stating it, he erroneously unites the first two propositions by + instead of ×. By (12), (5), and (8),

$$(a \prec b+x) \prec \{ a \prec (a \times b) + (a \times x) \}$$
$$(a \times x \prec b) \prec \{ (a+b) \times (x+b) \prec b \}.$$

But by (4)

$$a \prec a+b \qquad a \times b \prec b.$$

Hence, by (2), it is doubly proved that

$$(a \prec b+x) \times (a \times x \prec b) \prec (a \prec b).$$

The demonstration of the converse is obvious.

We have immediately, from (2) and (3),

K. $$(a+b \prec c) = (a \prec c) \times (b \prec c)$$
$$(c \prec a \times b) = (c \prec a) \times (c \prec b) \qquad (14)$$

L. $$(c \prec a+b) = \Sigma \{ (p \prec a) \times (q \prec b) \} \text{ where } p+q=c$$
$$(a \times b \prec c) = \Sigma \{ (a \prec p) \times (b \prec q) \} \text{ where } c = p \times q. \qquad (15)$$

The propositions (15) are new. By (12)

$$\{ (p \prec a) \times (q \prec b) \} \prec (c \prec a+b) \text{ where } p+q=c$$
$$\{ (a \prec p) \times (b \prec q) \} \prec (a \times b \prec c) \text{ where } c = p \times q.$$

And, since these are true for any set of values of p and q, we have by (2)

$$\Sigma\{(p \prec a)\times(q \prec b)\} \prec (c \prec a+b), \text{ where } p+q=c.$$
$$\Sigma\{(a \prec p)\times(b \prec q)\} \prec (a\times b \prec c), \text{ where } c=p\times q.$$

By (4) and (8), we have

$$(c \prec a+b) \prec \{(a\times c)+(b\times c)=c\}$$
$$(a\times b \prec c) \prec \{(c+a)\times(c+b)=c\}.$$

Hence, putting

$$a\times c=p \qquad b\times c=q, \qquad \text{where } p+q=c$$
$$a+c=p \qquad b+c=q, \qquad \text{where } p\times q=c,$$

we have

$$(c \prec a+b) \prec (p \prec a)\times(q \prec b), \text{ where } p+q=c$$
$$(a\times b \prec c) \prec (a \prec p)\times(b \prec q), \text{ where } c=p\times q,$$

whence, by (4)

$$(c \prec a+b) \prec \Sigma\{(p \prec a)\times(q \prec b)\} \text{ where } p+q=c$$
$$(a\times b \prec c) \prec \Sigma\{(a \prec p)\times(b \prec q)\} \text{ where } c=p\times q.$$

A formula analogous to (15) will be found below (35).

201. From (1) and (2) and (4) we have

$$x+0=x \qquad x=x\times\infty. \tag{16}$$

From (1) and (4),

$$x+\infty=\infty \qquad 0=x\times 0. \tag{17}$$

The definition of the negative has as we have seen three clauses: first, that $\bar{a}$ is of the form $a \prec x$; second, $a \prec \bar{\bar{a}}$; third, $\bar{\bar{a}} \prec a$.

From the first we have that if

$$\begin{matrix} c & a \\ & \therefore b \end{matrix}$$

is valid, then

$$\begin{matrix} c & \bar{b} \\ & \therefore \bar{a} \end{matrix}$$

is valid. Or

$$(c\times a \prec b) \prec (c\times\bar{b} \prec \bar{a}).^* \tag{18}$$

* Cf. Mrs. Ladd-Franklin's *Antilogism* in her "On the Algebra of Logic," *Studies in Logic*, edited by C. S. Peirce, Little, Brown & Co., Boston, 1883.

Also, that if

$$b$$
$$\therefore \text{Either } c \text{ or } a$$

is valid, then

$$\bar{a}$$
$$\therefore \text{Either } c \text{ or } \bar{b}$$

is valid; or

$$(b \prec c+a) \prec (\bar{a} \prec c+\bar{b}). \qquad (19)$$

Combining (18) and (19), we have

$$(a \times b \prec c+d) \prec (a \times \bar{d} \prec c+\bar{b}). \qquad (20)$$

By the principles of contradiction and excluded middle, this gives

$$(a \times \bar{d} \prec c+\bar{b}) \prec (a \times b \prec c+d). \qquad (21)$$

Thus the formula

$$(a \times b \prec c+d) = (a \times \bar{d} \prec c+\bar{b}) \qquad (22)$$

embodies the essence of the negative.

202. If in (22) we put, first, $a=d \quad b=c=0$, and then $a=d=\infty \quad b=c$, we have from the formula of identity

$$a \times \bar{a} = 0 \qquad a+\bar{a}=\infty. \qquad (23)$$

We have

$$p=(p \times x)+(p \times \bar{x}) \qquad p=(p+x) \times (p+\bar{x}), \qquad (24)$$

by the distributive principle and (23). If we write

$$i=p+(a \times \bar{x}) \quad j=p+(b \times x) \quad k=p \times (c+x) \quad l=p \times (d+\bar{x}),$$

we equally have

$$p=(i \times x)+(j \times \bar{x}) \qquad p=(l+x) \times (k+\bar{x}). \qquad (25)$$

Now p may be a function of x, and such values may perhaps be assigned to a, b, c, d, that i, j, k, l, shall be free from x. It is obvious that if the function results from any complication of the operations $+$ and $\times$, this is the case. Supposing, then, i, j, k, l, to be constant, we have, putting successively, ∞, and 0, for x.

$$\phi\infty = i = k$$
$$\phi 0 = j = l$$

so that

$$\phi x = (\phi\infty \times x)+(\phi 0 \times \bar{x}) \quad \phi x = (\phi 0 + x) \times (\phi\infty + \bar{x}). \qquad (26)$$

The first of these formulæ was given by Boole for his addition. I showed (1867)* that both hold for the modified addition. These formulæ are real analogues of mathematical developments; but practically they are not convenient. Their connection suggests the general formula

$$(a+x)\times(b+\bar{x})=(a\times\bar{x})+(b\times x) \tag{27}$$

a formula of frequent utility.

The distributive principle and (3) applied to (26) give

$$\phi 0\times\phi\infty \prec \phi x \qquad \phi x \prec \phi\infty+\phi 0. \tag{28}$$

Hence

$$(\phi x=0)\prec(\phi 0\times\phi\infty=0) \quad (\phi x=\infty)\prec(\phi 0+\phi\infty=\infty). \tag{29}$$

Boole gave the former, and I (1867)† the latter. These formulæ are not convenient for elimination.

203. The following formulæ (probably given by De Morgan) are of great importance:

$$\overline{a\times b}=\bar{a}+\bar{b} \qquad \overline{a+b}=\bar{a}\times\bar{b}.\ddagger \tag{30}$$

By (23)

$$(a\times b)\times(\overline{a\times b})\prec 0 \qquad \infty\prec(a+b)+(\overline{a+b}),$$

whence by (22) and the associative principle

$$b\times(\overline{a\times b})\prec\bar{a} \qquad \bar{a}\prec b+(\overline{a+b})$$

$$\overline{a\times b}\prec\bar{a}+\bar{b} \qquad \bar{a}\times\bar{b}\prec\overline{a+b}.$$

By (4) and (22)

$$\bar{a}\prec\overline{a\times b} \qquad \overline{a+b}\prec\bar{a}$$

$$\bar{b}\prec\overline{a\times b} \qquad \overline{a+b}\prec\bar{b},$$

whence by (2)

$$\bar{a}+\bar{b}\prec\overline{a+b}\S \qquad \overline{a+b}\prec\bar{a}\times\bar{b}.$$

The application of (22) gives from (11)

$$(b\ \overline{\prec}\ a\times b)=(a+b\ \overline{\prec}\ a); \tag{31}$$

from (12)

$$(a+x\ \overline{\prec}\ b+y)\prec(a\ \overline{\prec}\ b)+(x\ \overline{\prec}\ y)$$

$$(a\times x\ \overline{\prec}\ b\times y)\prec(a\ \overline{\prec}\ b)+(x\ \overline{\prec}\ y); \tag{32}$$

* In 9.

† In 12.

‡ These two embody De Morgan's principle of duality.

§ This should be: $\bar{a}+\bar{b}\prec\overline{a\times b}$.

from (13)

$$(a \overline{\prec} b) = (a \overline{\prec} b+x) + (a \times x \overline{\prec} b); \qquad (33)$$

from (14)

$$(a+b \overline{\prec} c) = (a \overline{\prec} c) + (b \overline{\prec} c);\ (c \overline{\prec} a \times b) = (c \overline{\prec} a) + (c \overline{\prec} b); \qquad (34)$$

from (15)

$$(c \overline{\prec} a+b) = \Pi\{(p \overline{\prec} a) + (q \overline{\prec} b)\} \text{ where } p+q=c$$
$$(a \times b \overline{\prec} c) = \Pi\{(a \overline{\prec} p) + (b \overline{\prec} q)\} \text{ where } p \times q = c; \qquad (35)$$

from (22)

$$(a \times b \overline{\prec} c+d) = (a \times \bar{d} \overline{\prec} c+\bar{b}). \qquad (36)$$

§2. THE RESOLUTION OF PROBLEMS IN NON-RELATIVE LOGIC

204. Four different algebraic methods of solving problems in the logic of non-relative terms have already been proposed by Boole, Jevons, Schröder, and McColl. I propose here a fifth method which perhaps is simpler and certainly is more natural than any of the others. It involves the following processes:

205. *First Process.* Express all the premisses with the copulas $\prec$ and $\overline{\prec}$, remembering that $A = B$ is the same as $A \prec B$ and $B \prec A$.

206. *Second Process.* Separate every predicate into as many factors and every subject into as many aggregant terms as is possible without increasing the number of different letters used in any subject or predicate.

207. An expression might be separated into such factors or aggregants (let us term them *prime* factors and *ultimate* aggregants) by one or other of these formulæ:

$$\phi x = (\phi\infty \times x) + (\phi 0 \times \bar{x})$$
$$\phi x = (\phi\infty + \bar{x}) \times (\phi 0 + x).$$

But the easiest method is this. To separate an expression into its $\left\{\begin{matrix}\text{ultimate aggregants}\\ \text{prime factors}\end{matrix}\right\}$ take any $\left\{\begin{matrix}\text{product}\\ \text{sum}\end{matrix}\right\}$ of all the different letters of the expression, each taken either positively or negatively (that is, with a dash over it). By means of the fundamental formulæ

$$X \times Y \prec Y \prec Y + Z,$$

examine whether the $\left\{\begin{matrix}\text{product}\\ \text{sum}\end{matrix}\right\}$ taken is a $\left\{\begin{matrix}\text{subject}\\ \text{predicate}\end{matrix}\right\}$ of every $\left\{\begin{matrix}\text{factor}\\ \text{aggregant}\end{matrix}\right\}$ of the given expression. If so, it is a $\left\{\begin{matrix}\text{ultimate aggregant}\\ \text{prime factor}\end{matrix}\right\}$ of that expression; otherwise not. Proceed in this way until as many $\left\{\begin{matrix}\text{ultimate aggregants}\\ \text{prime factors}\end{matrix}\right\}$ have been found as the expression possesses. This number is found in the case of a $\left\{\begin{matrix}\text{product of sums}\\ \text{sum of products}\end{matrix}\right\}$ of letters, as follows. Let m be the number of *different* letters in the expression (a letter and its negative not being considered different); let n be the total number of letters whether the same or different, and let p be the number of $\left\{\begin{matrix}\text{factors}\\ \text{terms}\end{matrix}\right\}$. Then the number of $\left\{\begin{matrix}\text{ultimate aggregants}\\ \text{prime factors}\end{matrix}\right\}$ is

$$2^m + n - mp - p.$$

208. For example, let it be required to separate $x + (y \times z)$ into its prime factors. Here $m = 3$, $n = 3$, $p = 2$. Hence the number of factors is three. Trying $x + y + z$, we have

$$x \prec x + y + z \qquad y \times z \prec x + y + z,$$

so that this is a factor. Trying $x + y + \bar{z}$, we have

$$x \prec x + y + \bar{z} \qquad y \times z \prec x + y + \bar{z},$$

so that this is also a factor. It is, also, obvious that $x + \bar{y} + z$ is the third factor. Accordingly,

$$x + (y \times z) = (x + y + z) \times (x + y + \bar{z}) \times (x + \bar{y} + z).$$

Again, let us develop the expression

$$(\bar{a} + b + c) \times (a + \bar{b} + \bar{c}) \times (a + b + c).$$

Here $m = 3$, $n = 9$, $p = 3$; so that the number of ultimate aggregants is five. Of the eight possible products of three letters, then, only three are excluded, namely: $(a \times \bar{b} \times \bar{c})$, $(\bar{a} \times b \times c)$ and $(\bar{a} \times \bar{b} \times \bar{c})$. We have, then,

$$(\bar{a} + b + c) \times (a + \bar{b} + \bar{c}) \times (a + b + c) =$$
$$(a \times b \times c) + (a \times b \times \bar{c}) + (a \times \bar{b} \times c) + (\bar{a} \times b \times \bar{c}) + (\bar{a} \times \bar{b} \times c).$$

209. *Third Process.* Separate all complex propositions into simple ones by means of the following formulæ from the definitions of $+$ and $\times$:

$$(X+Y-\!\!<Z)^{*}=(X-\!\!<Z)\times(Y-\!\!<Z)$$
$$(X-\!\!<Y\times Z)^{\dagger}=(X-\!\!<Y)\times(X-\!\!<Z)$$
$$(X+Y\overline{-\!\!<}Z)^{\ddagger}=(X\overline{-\!\!<}Z)+(Y\overline{-\!\!<}Z)$$
$$(X\overline{-\!\!<}Y\times Z)^{\S}=(X\overline{-\!\!<}Y)+(X\overline{-\!\!<}Z).$$

In practice, the first three operations will generally be performed off-hand in writing down the premisses.

210. *Fourth Process.* If we have given two propositions, one of one of the forms

$$a-\!\!<b+x \qquad\qquad a\times\bar{x}-\!\!<b,$$

and the other of one of the forms

$$c-\!\!<d+\bar{x} \qquad\qquad c\times x-\!\!<d,$$

we may, by the transitiveness of the copula, eliminate x, and so obtain

$$a\times c-\!\!<b+d.$$

211. *Fifth Process.* We may transpose any term from subject to predicate or the reverse, by changing it from positive to negative or the reverse, and at the same time its mode of connection from addition to multiplication or the reverse. Thus,

$$(x\times y-\!\!<z)=(x-\!\!<\bar{y}+z).$$

We may, in this way, obtain all the subjects and predicates of any letter; or we may bring all the letters into the subject, leaving the predicate 0, or all into the predicate, leaving the subject ∞.

212. *Sixth Process.* Any number of propositions having a common $\left\{\begin{matrix}\text{subject}\\ \text{predicate}\end{matrix}\right\}$ are, taken together, equivalent to their $\left\{\begin{matrix}\text{product}\\ \text{sum}\end{matrix}\right\}$.

* I.e., $\left\{(x+y)-\!\!<z\right\}$.

† I.e., $\left\{x-\!\!<(y\times z)\right\}$.

‡ I.e., $\left\{(x+y)\overline{-\!\!<}z\right\}$.

§ I.e., $\left\{x\overline{-\!\!<}(y\times z)\right\}$.

213. As an example of this method, we may consider a well-known problem given by Boole.* The data are

$$\bar{x}\times\bar{z} \prec v\times(y\times\bar{w}+\bar{y}\times w)$$
$$\bar{v}\times x\times w \prec (y\times z)+(\bar{y}\times\bar{z})$$
$$(x\times y)+(v\times x\times\bar{y})=(z\times\bar{w})+(\bar{z}\times w).$$

The quæsita are: first, to find those predicates of x which involve only y, z, and w; second, to find any relations which may be implied between y, z, w; third, to find the predicates of y; fourth, to find any relation which may be implied between x, z, and w. By the first three processes, mentally performed, we resolve the premisses as follows: the first into

$$\bar{x}\times\bar{z} \prec v$$
$$\bar{x}\times\bar{z} \prec y+w$$
$$\bar{x}\times\bar{z} \prec \bar{y}+\bar{w};$$

the second into

$$\bar{v}\times x\times w \prec y+\bar{z}$$
$$\bar{v}\times x\times w \prec \bar{y}+z;$$

the third into

$$x\times y \prec z+w$$
$$x\times y \prec \bar{z}+\bar{w}$$
$$v\times x\times\bar{y} \prec z+w$$
$$v\times x\times\bar{y} \prec \bar{z}+\bar{w}$$
$$z\times\bar{w} \prec x$$
$$\bar{z}\times w \prec v+y$$
$$z\times\bar{w} \prec x$$
$$\bar{z}\times w \prec v+y.$$

We must first eliminate v, about which we want to know nothing. We have, on the one hand, the propositions

$$v\times x\times\bar{y} \prec z+w$$
$$v\times x\times\bar{y} \prec \bar{z}+\bar{w};$$

and, on the other, the propositions

$$\bar{x}\times\bar{z} \prec v$$
$$\bar{v}\times x\times w \prec y+\bar{z}$$
$$\bar{v}\times x\times w \prec \bar{y}+z$$
$$z\times\bar{w} \prec v+y$$
$$\bar{z}\times w \prec v+y.$$

* *The Laws of Thought*, pp. 146–9.

The conclusions from these propositions are obtained by taking one from each set, multiplying their subjects, adding their predicates, and omitting v. The result will be a merely empty proposition if the same letter in the same quality as to being positive or negative be found in the subject and in the predicate, or if it be found twice with opposite qualities either in the subject or in the predicate. Thus, it will be useless to combine the proposition $v \times x \times \bar{y} -\!\!< z + w$ with any which contains $\bar{x}$, y, z, or w, in the subject. But all of the second set do this, so that nothing can be concluded from this proposition. So it will be useless to combine $v \times x \times \bar{y} -\!\!< \bar{z} + \bar{w}$ with any which contains $\bar{x}$, y, $\bar{z}$, $\bar{w}$ in the subject, or z in the predicate. This excludes every proposition of the second set except $\bar{v} \times x \times w -\!\!< y + \bar{z}$, which, combined with the proposition under discussion, gives

$$x \times w -\!\!< y + \bar{z} + \bar{w}$$

or

$$x \times w -\!\!< y + \bar{z},$$

which is therefore to be used in place of all the premisses containing v.

One of the other propositions, namely, $\bar{x} \times \bar{z} -\!\!< \bar{y} + \bar{w}$ is obviously contained in another, namely: $\bar{z} \times w -\!\!< x$. Rejecting it, our premisses are reduced to six, namely:

$$\bar{x} \times \bar{z} -\!\!< y + w$$
$$x \times y -\!\!< z + w$$
$$x \times y -\!\!< \bar{z} + \bar{w}$$
$$z \times \bar{w} -\!\!< x$$
$$\bar{z} \times w -\!\!< x$$
$$x \times w -\!\!< y + \bar{z}$$

The second, third, and sixth of these give the predicates of x. Their product is

$$x -\!\!< (\bar{y} + z + w) \times (\bar{y} + \bar{z} + \bar{w}) \times (y + \bar{z} + \bar{w})$$

or

$$x -\!\!< y \times z \times \bar{w} + y \times \bar{z} \times w + \bar{y} \times z \times \bar{w} + \bar{y} \times \bar{z} \times w + \bar{y} \times \bar{z} \times \bar{w}$$

or

$$x -\!\!< z \times \bar{w} + \bar{z} \times w + \bar{y} \times \bar{z} \times \bar{w}.$$

To find whether any relation between y, z, and w can be obtained by the elimination of x, we find the subjects of x by combining the first, fourth, and fifth premisses. Thus we find

$$\bar{y} \times \bar{z} \times \bar{w} + z \times \bar{w} + \bar{z} \times w -\!\!< x.$$

It is obvious that the conclusion from the last two propositions is a merely identical proposition, and therefore no independent relation is implied between y, z, and w.

To find the predicates of y we combine the second and third propositions. This gives

$$y -\!\!< (\bar{x}+z+w)\times(\bar{x}+\bar{z}+\bar{w})$$

or

$$y -\!\!< x\times z\times\bar{w}+x\times\bar{z}\times w+\bar{x}.$$

Two relations between x, z, and w are given in the premisses, namely: $z\times\bar{w} -\!\!< x$ and $\bar{z}\times w -\!\!< x$. To find whether any other is implied, we eliminate y between the above proposition and the first and sixth premisses. This gives

$$\bar{x}\times\bar{z} -\!\!< x\times z\times\bar{w}+w+\bar{x}$$

$$x\times w -\!\!< x\times z\times\bar{w}+\bar{x}+\bar{z}.$$

The first conclusion is empty. The second is equivalent to $x\times w -\!\!< \bar{z}$, which is a third relation between x, z, and w.

Everything implied in the premisses in regard to the relations of x, y, z, w may be summed up in the proposition

$$\infty -\!\!< x+z\times w+y\times\bar{z}\times\bar{w}.$$

Part III. The Logic of Relatives

§1. INDIVIDUAL AND SIMPLE TERMS

214. Just as we had to begin the study of Logical Addition and Multiplication by considering ∞ and 0, terms which might have been introduced under the Algebra of the Copula, being defined in terms of the copula only, without the use of $+$ or $\times$, but which had not been there introduced, because they had no application there, so we have to begin the study of relatives by considering the doctrine of individuals and simples, — a doctrine which makes use only of the conceptions of non-relative logic, but which is wholly without use in that part of the subject, while it is the very foundation of the conception of a relative, and the basis of the method of working with the algebra of relatives.

215. The germ of the correct theory of individuals and simples is to be found in Kant's *Critic of the Pure Reason*, "Appendix to the Transcendental Dialectic," where he lays it down as a regulative principle, that, if

$$a -\!\!< b \qquad b\,\overline{-\!\!<}\,a,$$

then it is always possible to find such a term x, that

$$a \prec x \qquad x \prec b$$
$$x \overline{\prec} a \qquad b \overline{\prec} x.$$

Kant's distinction of regulative and constitutive principles is unsound, but this *law of continuity*,* as he calls it, must be accepted as a fact. The proof of it, which I have given elsewhere,† depends on the continuity of space, time, and the intensities of the qualities which enter into the definition of any term. If, for instance, we say that Europe, Asia, Africa and North America are continents, but not all the continents, there remains over only South America. But we may distinguish between South America as it now exists and South America in former geological times; we may, therefore, take x as including Europe, Asia, Africa, North America, and South America as it exists now, and every x is a continent, but not every continent is x.

216. Just as in mathematics we speak of infinitesimals and infinites, which are fictitious limits of continuous quantity, and every statement involving these expressions has its interpretation in the doctrine of limits,‡ so in logic we may define an *individual*, A, as such a term that

$$\mathrm{A} \overline{\prec} 0,$$

but such that if

$$x < \mathrm{A}$$

then $$x \prec 0.$$

And in the same way, we may define the *simple*, α, as such a term that

$$\infty \overline{\prec} \alpha,$$

but such that if

$$\alpha < x$$

then $$\infty \prec x.$$

The individual and the simple, as here defined, are ideal limits, and every statement about either is to be interpreted by the doctrine of limits.

217. Every term may be conceived as a limitless logical

* Cf. 4.121.

† See 93, 2.646 and 4.121–22.

‡ Cf. 4.118–9.

sum of individuals,* or as a limitless logical product of simples; thus,

$$a = A_1 + A_2 + A_3 + A_4 + A_5 + \text{ etc.}$$
$$\bar{a} = \bar{A}_1 \times \bar{A}_2 \times \bar{A}_3 \times \bar{A}_4 \times \bar{A}_5 \times \text{ etc.}$$

It will be seen that a simple is the negative of an individual.

§2. RELATIVES

218. A *relative* is a term whose definition describes what sort of a system of objects that is whose first member (which is termed the *relate*) is denoted by the term; and names for the other members of the system (which are termed the *correlates*) are usually appended to limit the denotation still further. In these systems the order of the members is essential; so that (A, B, C) and (A, C, B) are different systems. As an example of a relative, take 'buyer of — for — from'; we may append to this three correlates, thus, 'buyer of every horse of a certain description in the market for a good price from its owner.'

219. A relative of only one correlate, so that the system it supposes is a pair, may be called a *dual* relative; a relative of more than one correlate may be called *plural*. A non-relative term may be called a term of *singular reference*.

220. Every relative, like every term of singular reference, is general; its definition describes a system in general terms; and, as general, it may be conceived either as a logical sum of individual relatives, or as a logical product of simple relatives.[1] An individual relative refers to a system all the members of which are individual. The expressions

(A : B)   (A : B : C)

may denote individual relatives. Taking dual individual relatives, for instance, we may arrange them all in an infinite block, thus,

A : A	A : B	A : C	A : D	A : E	etc.
B : A	B : B	B : C	B : D	B : E	etc.
C : A	C : B	C : C	C : D	C : E	etc.
D : A	D : B	D : C	D : D	D : E	etc.
E : A	E : B	E : C	E : D	E : E	etc.
etc.	etc.	etc.	etc.	etc.	

* Cf. 2.356.

[1] In my *Logic of Relatives*, 1870 [§6], I have used this expression ['simple relatives'] to designate what I now call *dual relatives*.

In the same way, triple individual relatives may be arranged in a cube, and so forth. The logical sum of all the relatives in this infinite block will be the relative universe, ∞, where

$$x \prec \infty,$$

whatever dual relative x may be. It is needless to distinguish the dual universe, the triple universe, etc., because, by adding a perfectly indefinite additional member to the system, a dual relative may be converted into a triple relative, etc. Thus, for *lover of a woman*, we may write *lover of a woman coexisting with anything*. In the same way, a term of single reference is equivalent to a relative with an indefinite correlate; thus, *woman* is equivalent to *woman coexisting with anything*. Thus, we shall have

$$A = A:A + A:B + A:C + A:D + A:E + \text{etc.}$$
$$A:B = A:B:A + A:B:B + A:B:C + A:B:D + \text{etc.}$$

221. From the definition of a simple term given in the last section, it follows that every simple relative is the negative of an individual term. But while in non-relative logic negation only divides the universe into two parts, in relative logic the same operation divides the universe into 2^n parts, where n is the number of objects in the system which the relative supposes; thus,

$$\infty = A + \bar{A}$$
$$\infty = A:B + \bar{A}:B + A:\bar{B} + \bar{A}:\bar{B}$$
$$\infty = (A:B:C) + (\bar{A}:B:C) + (A:\bar{B}:C) + (A:B:\bar{C}) + (\bar{A}:\bar{B}:\bar{C}) + (A:\bar{B}:\bar{C}) + (\bar{A}:B:\bar{C}) + (\bar{A}:\bar{B}:C).$$

Here, we have

$$A = A:B + A:\bar{B};\quad \bar{A} = \bar{A}:B + \bar{A}:\bar{B};$$
$$A:B = A:B:C + A:B:\bar{C};\quad A:\bar{B} = A:\bar{B}:C + A:\bar{B}:\bar{C};$$
$$\bar{A}:B = \bar{A}:B:C + \bar{A}:B:\bar{C};\quad \bar{A}:\bar{B} = \bar{A}:\bar{B}:C + \bar{A}:\bar{B}:\bar{C}.$$

It will be seen that a term which is individual when considered as n-fold is not so when considered as more than n-fold; but an n-fold term when made $(m+n)$-fold, is individual as to n members of the system, and indefinite as to m members.

222. Instead of considering the system of a relative as con-

sisting of non-relative individuals, we may conceive of it as consisting of relative individuals. Thus, since

$$A = A : A + A : B + A : C + A : D + \text{etc.},$$

we have

$$A : B = (A : A) : B + (A : B) : B + (A : C) : B + (A : D) : B + \text{etc.}$$

But

$$B = B : A + B : B + B : C + B : D + \text{etc.};$$

so that

$$A : B = A : (B : A) + A : (B : B) + A : (B : C) + A : (B : D) + \text{etc.}^*$$

§3. RELATIVES CONNECTED BY TRANSPOSITION OF RELATE AND CORRELATE

223. Connected with every dual relative, as

$$l = \Sigma(A : B) = \Pi(\alpha : \beta),\dagger$$

is another which is called its *converse*,

$$k\text{-}l = \Sigma(B : A) = \Pi(\beta : \alpha),$$

in which the relate and correlate are transposed. The converse, k, is itself a relative, being

$$k = \Sigma[(A : B) : (B : A)];$$

that is, it is the first of any pair which embraces two individual dual relatives, each of which is the converse of the other. The converse of the converse is the relation itself, thus

$$k\text{-}k\text{-}l = l,$$

or say

$$kk = 1.$$

We have also

$$k\text{-}\bar{l} = \overline{k\text{-}l}$$

$$k\Sigma = \Sigma k$$

$$k\Pi = \Pi k.$$

* Four lines of formula are here deleted, in accordance with Peirce's subsequent marginal comment. They involved the invalid use of the law of association in connection with triple relatives.

† $\Sigma(A : B)$ represents the logical sum of individual relatives; $\Pi(\alpha : \beta)$, represents the logical product of simple relatives.

224. In the case of triple relatives there are five transpositions possible. Thus, if

$$b=\Sigma[(A:B):C]$$

we may write

$$Ib=\Sigma[(B:A):C]$$
$$Jb=\Sigma[(A:C):B]$$
$$Kb=\Sigma[(C:B):A]$$
$$Lb=\Sigma[(C:A):B]$$
$$Mb=\Sigma[(B:C):A].$$

Here we have $LM^{*}=ML=1$

$$II=JJ=KK=1$$
$$IJ=JK=KI=L$$
$$JI=KJ=IK=M$$
$$IL=MI=J=KM=LK$$
$$JL=MJ=K=IM=LI$$
$$KL=MK=I=JM=LJ.$$

If we write $a:b$ to express the operation of putting A in place of B in the original relative

$$b=\Sigma[(A:B):C]$$

then we have

$$I=a:b+b:a+c:c\dagger$$
$$J=a:a+b:c+c:b$$
$$K=a:c+b:b+c:a$$
$$L=a:b+b:c+c:a$$
$$M=a:c+b:a+c:b$$
$$1=a:a+b:b+c:c.$$

Then we have

$$I+J+K=1+L+M,$$

which does not imply

$$(I+J+K)l=(1+L+M)l.$$

* LM indicates the application of L on M. L says that any formula of the form (1 : 2): 3 is to be changed to (3 : 1): 2. As M means that (1 : 2): 3 is to be changed to (2 : 3): 1, the application of L on M yields the original. The rest of the formulæ are to be understood in a similar way.

† I.e., I results by substituting a for b, b for a and c for c in the equation for b; and so on with the rest.

In a similar way the *n*-fold relative will have $(n! - 1)$ transposition-functions.

§4. CLASSIFICATION OF RELATIVES

225. Individual relatives are of one or other of the two forms

$$A : A \qquad A : B,$$

and simple relatives are negatives of one or other of these two forms.

226. The forms of general relatives are of infinite variety, but the following may be particularly noticed.

Relatives may be divided into those all whose individual aggregants are of the form A : A and those which contain individuals of the form A : B. The former may be called *concurrents*,* the latter *opponents*. Concurrents express a mere agreement among objects. Such, for instance, is the relative '*man that is* —,' and a similar relative may be formed from any term of singular reference. We may denote such a relative by the symbol for the term of singular reference with a comma after it; thus (*m*,) will denote 'man that is —' if (*m*) denotes 'man.' In the same way a comma affixed to an *n*-fold relative will convert it into an $(n+1)$-fold relative. Thus, (*l*) being 'lover of —,' (*l*,) will be 'lover that is — of — .'

The negative of a concurrent relative will be one each of whose simple components is of the form $\overline{A : A}$, and the negative of an opponent relative will be one which has components of the form $\overline{A : B}$.

We may also divide relatives into those which contain individual aggregants of the form A : A and those which contain only aggregants of the form A : B. The former may be called *self-relatives*,† the latter *alio-relatives*. We also have negatives of self-relatives and negatives of alio-relatives.

227. These different classes have the following relations. Every negative of a concurrent and every alio-relative is both an opponent and the negative of a self-relative. Every concurrent and every negative of an alio-relative is both a self-

* See 136*a*n.

† See 136*d*n.

relative and the negative of an opponent. There is only one relative which is both a concurrent and the negative of an alio-relative; this is 'identical with —.' There is only one relative which is at once an alio-relative and the negative of a concurrent; this is the negative of the last, namely, 'other than —.'[1] The following pairs of classes are mutually exclusive, and divide all relatives between them:

Alio-relatives and self-relatives,
Concurrents and opponents,
Negatives of alio-relatives and negatives of self-relatives,
Negatives of concurrents and negatives of opponents.

No relative can be at once either an alio-relative or the negative of a concurrent, and at the same time either a concurrent or the negative of an alio-relative.

228. We may append to the symbol of any relative a semicolon to convert it into an alio-relative of a higher order. Thus $(l;)$ will denote a 'lover of — that is not —.'

229. This completes the classification of dual relatives founded on the difference of the fundamental forms A : A and A : B. Similar considerations applied to triple relatives would give rise to a highly complicated development, inasmuch as here we have no less than five fundamental forms of individuals, namely,

(A : A) : A (A : A) : B (A : B) : A (B : A) : A (A : B) : C.

The number of individual forms for the $(n+2)$-fold relative is

$$2+(2^n-1)\,.\,3+\frac{1}{2!}\left\{(3^n-1)-2(2^n-1)\right\}.\,4+\frac{1}{3!}\left\{(4^n-1)-3(3^n-1)+3(2^n-1)\right\}.\,5+\frac{1}{4!}\left\{(5^n-1)-4(4^n-1)+6(3^n-1)-4(2^n-1)\right\}.\,6+\frac{1}{5!}\left\{(6^n-1)-5(5^n-1)+10(4^n-1)-10(3^n-1)+5(2^n-1)\right\}.\,7+\text{ etc.}$$

[1] The relative 0 ought to be considered as at once a concurrent and an alio-relative, and the relative ∞ as at once the negative of a concurrent and the negative of an alio-relative. [Cf. 585–6.] The statements in the text require to be modified to this extent. [This note, apparently a correction made after receiving proofs, was published at the end of the original paper.]

If this number be called fn, we have

$$\Delta^n f0 = f(n-1)$$

$$f0 = 1.$$

The form of calculation is

1					
2	1				
5	3	2			
15	10	7	5		
52	37	27	20	15	
203	151	114	87	67	52

where the diagonal line is copied number by number from the vertical line, as fast as the latter is computed.[1]

230. Relatives may also be classified according to the general amount of filling up of the above-mentioned block, cube, etc., they present. In the first place, we have such relatives in whose block, cube, etc., every line in a certain direction in which there is a single individual is completely filled up. Such relatives may be called *complete in regard to* the relate, or first, second, third, etc., correlate. The dual relatives which are equivalent to terms of singular reference are complete as to their correlate.

231. A relative may be incomplete with reference to a certain correlate or to its relate, and yet every individual of the universe may in some way enter into that correlate or relate. Such a relative may be called *unlimited* in reference to the correlate or relate in question. Thus, the relative

A : A+A : B+C : C+C : D+E : E+E : F+G : G+G : H + etc.

is unlimited as to its correlate. The negative of an unlimited relative will be unlimited unless the relative has as an integrant a relative which is complete with regard to every other relate and correlate than that with reference to which the given relative is unlimited.

[1] These numbers are every fifth of the series:
$\overline{1}$*, −7, 10, −3, $\overline{2}$*, −5, 5, 2, 3, $\overline{5}$*, 0, 5, 7, 10, $\overline{15}$*, 15, 20, 27, 37, $\overline{52}$*, where $u_x + u_{x+3} = u_{x+4}$. But only holds up to 203 and is therefore valueless.—marginal note.

232. A totally unlimited relative is one which is unlimited in reference to the relate and all the correlates. A totally unlimited relative in which each letter enters only once into the relate and once into the correlate is termed a *substitution*.

233. Certain classes of relatives are characterized by the occurrence or non-occurrence of certain individual aggregants related in a definite way to others which occur. A set of individual dual relatives each of which has for its relate the correlate of the last, the last of all being considered as preceding the first of all, may be called a *cycle*. If there are n individuals in the cycle it may be called a cycle of the n^{th} order. For instance,

$$A:B \quad B:C \quad C:D \quad D:E \quad E:A$$

may be called the cycle of the fifth order. Now, if a certain relative be such that of any cycle of the n^{th} order of which it contains any m terms, it also contains the remaining $(n - m)$ terms, it may be called a cyclic relative of the n^{th} order and m^{th} degree. If, on the other hand, of any cycle of the n^{th} order of which it contains m terms the remaining $(n - m)$ are wanting, the relative may be called an anticyclic relative of the n^{th} order and m^{th} degree.

234. A cyclic relative of the first order and first degree contains all individual components of the form A : A. A cyclic relative of the second order and first degree is called an *equiparant** in opposition to a *disquiparant*.

235. A highly important class of relatives is that of *transitives;* that is to say, those which for every two individual terms of the forms (A : B) and (B : C) also possess a term of the form (A : C).†

§5. THE COMPOSITION OF RELATIVES

236. Suppose two relatives either individual or simple, and having the relate or correlate of the first identical with the relate or correlate of the second or of its negative. This pair of relatives will then be of one or other of sixteen forms, viz.:

* See 136cn.

† See 136*h*n.

$(A : B)(B : C)$	$\overline{(A : B)}(B : C)$	$(A : B)\overline{(B : C)}$	$\overline{(A : B)}\,\overline{(B : C)}$
$(A : B)(C : B)$	$\overline{(A : B)}(C : B)$	$(A : B)\overline{(C : B)}$	$\overline{(A : B)}\,\overline{(C : B)}$
$(B : A)(B : C)$	$\overline{(B : A)}(B : C)$	$(B : A)\overline{(B : C)}$	$\overline{(B : A)}\,\overline{(B : C)}$
$(B : A)(C : B)$	$\overline{(B : A)}(C : B)$	$(B : A)\overline{(C : B)}$	$\overline{(B : A)}\,\overline{(C : B)}$

Now we may conceive an operation upon any one of these sixteen pairs of relatives of such a nature that it will produce one or other of the four forms $(A : C)$, $\overline{(A : C)}$, $(C : A)$, $\overline{(C : A)}$. Thus, we shall have sixty-four operations in all.

237. We may symbolize them as follows: Let

$$A : B\ (|||)B : C = A : C;$$

that is, let $(|||)$ signify such an operation that this formula necessarily holds. The three lines in the sign of this operation are to refer respectively to the first relative operated upon, the second relative operated upon, and to the result. When either of these lines is replaced by a hyphen (–), let the operation signified be such that the negative of the corresponding relative must be substituted in the above formula. Thus,

$$\overline{A : B}(-||)B : C = A : C.$$

In the same way, let a semicircle $(\smile)$ signify that the converse of the corresponding relative is to be taken. The hyphen and the semicircle may be used together. If, then, we write the symbol of a relative with a semicircle or curve over it to denote the converse of that relative, we shall have, for example,

$$\overset{\smile}{A : B}(\smile||)B : C = A : C.$$

238. Then any combination of the relatives a and e, in this order, is equivalent to others formed from this by making any of the following changes:

First. Putting a straight or curved mark over a and changing the first mark of the sign of operation in the corresponding way; that is,

for $\breve{a}$, from $|$ to $\smile$ or from $-$ to $\overset{\smile}{-}$ or conversely,
for $\bar{a}$, from $|$ to $-$ or from $\smile$ to $\overset{\smile}{-}$ or conversely,
for $\breve{\bar{a}}$, from $|$ to $\overset{\smile}{-}$ or from $-$ to $\smile$ or conversely.

Second. Making similar simultaneous modifications of *e* and of the second mark.

Third. Changing the third mark from | to – or from ⌣ to ≚ or conversely, and simultaneously writing the mark of negation over the whole expression.

Fourth. Changing the third mark from | to ⌣ or from – to ≚ or conversely, and interchanging *a* and *e* and also the first and second marks.

239. We have thus far defined the effect of the sixty-four operations when certain members of the individual relatives operated upon are identical. When these members are not identical, we may suppose either that the operation ||| produces either the first or second relative or 0. We cannot suppose that it produces ∞ for a reason which will appear further on. Let us elect the formula

$$\mathrm{A} : \mathrm{B}(|||)\mathrm{C} : \mathrm{D} = 0.$$

The other excluded operations will be included in a certain manner, as we shall see below. From this formula, by means of the rules of equivalence, it will follow that all operations in whose symbol there is no hyphen in the third place will also give 0 in like circumstances, while all others will give $\bar{0}$ or ∞.

240. We have thus far only defined the effect of the sixty-four operations upon individual or simple terms. To extend the definitions to other cases, let us suppose first that the rules of equivalence are generally valid, and second, that

$$\text{If } a \prec b \text{ and } c \prec d \text{ then } a(|||)c \prec b(|||)d$$

or

$$(a \prec b) \times (c \prec d) \prec \{a(|||)c \prec b(|||)d\}.$$

Then, this rule will hold good in all operations in whose symbols the first and second places agree with the third in respect to having or not having hyphens. For operations, in whose symbols the $\left\{\begin{matrix}\text{first}\\ \text{second}\end{matrix}\right\}$ mark disagrees with the third in this respect we must write $\left\{\begin{matrix} b \prec a \\ d \prec c \end{matrix}\right\}$ instead of $\left\{\begin{matrix} a \prec b \\ c \prec d \end{matrix}\right\}$ in this rule. Thus, the sixty-four operations are divisible into four classes according to which one of the four rules so produced they follow.

241. It now appears that only the hyphens and not the curved marks are of significance in reference to the rule which an operation follows. Let us accordingly reject all operations whose symbols contain curved marks, and there remain only eight. For these eight the following formulæ hold:

$$\mathrm{A:B}\ (|||)\ \mathrm{B:C} = \mathrm{A:C} \qquad \mathrm{A:B}\ (||-)\ \mathrm{B:C} = \overline{\mathrm{A:C}}$$
$$\overline{\mathrm{A:B}}\ (-||)\ \mathrm{B:C} = \mathrm{A:C} \qquad \overline{\mathrm{A:B}}\ (-|-)\ \mathrm{B:C} = \overline{\mathrm{A:C}}$$
$$\mathrm{A:B}\ (|-|)\ \overline{\mathrm{B:C}} = \mathrm{A:C} \qquad \mathrm{A:B}\ (|--)\ \overline{\mathrm{B:C}} = \overline{\mathrm{A:C}}$$
$$\overline{\mathrm{A:B}}\ (--|)\ \overline{\mathrm{B:C}} = \mathrm{A:C} \qquad \overline{\mathrm{A:B}}\ (---)\ \overline{\mathrm{B:C}} = \overline{\mathrm{A:C}}$$
$$\mathrm{A:B}\ (|||)\ \mathrm{C:D} = 0 \qquad \mathrm{A:B}\ (||-)\ \mathrm{C:D} = \infty$$
$$\overline{\mathrm{A:B}}\ (-||)\ \mathrm{C:D} = 0 \qquad \overline{\mathrm{A:B}}\ (-|-)\ \mathrm{C:D} = \infty$$
$$\mathrm{A:B}\ (|-|)\ \overline{\mathrm{C:D}} = 0 \qquad \mathrm{A:B}\ (|--)\ \overline{\mathrm{C:D}} = \infty$$
$$\overline{\mathrm{A:B}}\ (--|)\ \overline{\mathrm{C:D}} = 0 \qquad \overline{\mathrm{A:B}}\ (---)\ \overline{\mathrm{C:D}} = \infty$$

$$(a -\!\!< b)\times(c -\!\!< d) -\!\!< \{a\ (|||)\ c -\!\!< b\ (|||)\ d\}$$
$$(a -\!\!< b)\times(c -\!\!< d) -\!\!< \{a(---)c -\!\!< b(---)d\}$$

$$(b -\!\!< a)\times(c -\!\!< d) -\!\!< \{a\ (-||)\ c -\!\!< b\ (-||)\ d\}$$
$$(b -\!\!< a)\times(c -\!\!< d) -\!\!< \{a\ (|--)\ c -\!\!< b\ (|--)\ d\}$$

$$(a -\!\!< b)\times(d -\!\!< c) -\!\!< \{a\ (|-|)\ c -\!\!< b\ (|-|)\ d\}$$
$$(a -\!\!< b)\times(d -\!\!< c) -\!\!< \{a\ (-|-)\ c -\!\!< b\ (-|-)\ d\}$$

$$(b -\!\!< a)\times(d -\!\!< c) -\!\!< \{a\ (--|)\ c -\!\!< b\ (--|)\ d\}$$
$$(b -\!\!< a)\times(d -\!\!< c) -\!\!< \{a\ (||-)\ c -\!\!< b\ (||-)\ d\}.$$

242. As it is inconvenient to consider so many as eight distinct operations, we may reject one-half of these so as to retain one under each of the four rules. We may reject all those whose symbols contain an odd number of hyphens (as being negative). We then retain four, to which we may assign the following names and symbols:

$a\ (|||)\ e = ae$ *Relative or external multiplication.*
$a(|--)e = {}^{a}e$ *Regressive involution.*
$a(-|-)e = a^{e}$ *Progressive involution.*
$a(--|)e = a \circ e$ *Transaddition.*[1]

[1] The first three of these were studied by De Morgan ("On the Syllogism," No. IV.); the last is new. The above names for the first three (except the adjective *external* suggested by Grassmann's operation) are given in my *Logic of Relatives.*

243. We have then the following table of equivalents, negatives, and converses:[1]

x	$\bar{x}$	$\breve{x}$	$\breve{\bar{x}}$
$ae = \bar{a} \circ \bar{e}$	$\bar{a}^{e} = {}^{a}\bar{e}$	$\breve{e}\breve{a} = \breve{\bar{e}} \circ \breve{\bar{a}}$	$\breve{\bar{e}}^{a} = {}^{\bar{e}}\breve{\bar{a}}$
$a^{e} = {}^{\bar{a}}\bar{e}$	$\bar{a}e = a \circ \bar{e}$	${}^{\bar{e}}\breve{a} = \breve{\bar{e}}^{\breve{\bar{a}}}$	$\breve{e}\breve{\bar{a}} = \breve{\bar{e}} \circ \breve{a}$
${}^{a}e = \bar{a}^{\bar{e}}$	$a\bar{e} = \bar{a} \circ e$	$\breve{e}^{a} = {}^{\breve{\bar{e}}}\breve{\bar{a}}$	$\breve{\bar{e}}\breve{a} = \breve{e} \circ \breve{\bar{a}}$
$a \circ e = \bar{a}\bar{e}$	$a^{\bar{e}} = {}^{\bar{a}}e$	$\breve{e} \circ \breve{a} = \breve{\bar{e}}\breve{\bar{a}}$	$\breve{e}^{\breve{\bar{a}}} = {}^{\breve{\bar{e}}}\breve{a}$

244. If l denote 'lover' and s 'servant,' then

l_s denotes whatever is lover of a servant of —,

l^s whatever is lover of every servant of —,

l_s whatever is in every way (in which it loves at all) lover of a servant,

$l \circ_s$ whatever is not a non-lover only of a servant of —
or whatever is not a lover of everything but servants of —
or whatever is some way a non-lover of some thing besides servants of —.

§6. METHODS IN THE ALGEBRA OF RELATIVES

245. The universal method in this algebra is the method of limits. For certain letters are to be substituted an infinite sum of individuals or product of simples; whereupon certain transformations become possible which could not otherwise be effected.

246. The following theorems are indispensable for the application of this method:

First $$l^{A:B} = l(A:B) + k\bar{B}.$$

Since $\bar{B}$ is equivalent to the relative term which comprises all individual relatives whose relates are not B, so $k\bar{B}$ may be conveniently used, as it is here, to express the aggregate of all individual relatives whose correlate is $\bar{B}$. To prove this proposition, we observe that

$$l^{A:B} = \overline{\bar{l}(A:B)}.$$

[1] A similar table is given by De Morgan. Of course, it lacks the symmetry of this, because he had not the fourth operation. [Cf. 112.]

Now $\bar{l}(A : B)$ contains only individual relatives whose correlate is B, and of these it contains those which are not included in $l(A : B)$. Hence the negative of $\bar{l}(A : B)$ contains all individual relatives whose correlates are not B, together with all contained in $l(A : B)$. Q. E. D.

Second $$^{A:B}l = (A : B)l + \bar{A}.$$

Here $\bar{A}$ is used to denote the aggregate of all individual relatives whose relates are not A. This proposition is proved like the last.

Third $$\overline{A : B}^{l} = (A : B)\bar{l} + \bar{A}.$$

This is evident from the second proposition, because

$$\overline{A : B}^{l} = {}^{(A:B)}\bar{l}.$$

Fourth $$^{l}\overline{A : B} = \bar{l}(A : B) + k\bar{B}.$$

Another method of working with the algebra is by means of negations. This becomes quite indispensable when the operations are defined by negations, as in this paper.

247. To illustrate the use of these methods, let us investigate the relations of ^{l}b and l^{b} to lb when l and b are totally unlimited relatives.

Write $$l = \Sigma_i(L_i : M_i) \qquad b = \Sigma_j(B_j : C_j).$$

Then, by the rules of the last section,

$$^{l}b \prec {}^{L:M}b \qquad l^{b} \prec l^{B:C};$$

whence, by the second and third propositions above,

$$^{l}b \prec (L_i : M_i)b + \bar{L}_i \qquad l^{b} \prec l\,(B_j : C_j) + k\bar{B}_j.$$

But by the first rule of the last section

$$(L_i : M_i)b \prec lb \qquad l(B_j : C_j) \prec lb;$$

hence,

$$^{l}b \prec lb + \bar{L}_i \qquad l^{b} \prec lb + k\bar{B}_j.$$

There will be propositions like these for all the different values of i and j. Multiplying together all those of the several sets, we have

$$^{l}b \prec lb + \Pi_i\bar{L}_i \qquad l^{b} \prec lb + \Pi_j k\bar{B}_j.$$

But

$$\Pi_i \bar{L}_i = \overline{\Sigma_i L_i} \qquad \Pi_j k\bar{B}_j = \overline{\Sigma_j k B_j},$$

and since the relatives are unlimited,

$$\Sigma_i L_i = \infty \qquad \Sigma_j k B_j = \infty$$

$$\overline{\Sigma_i L_i} = 0 \qquad \overline{\Sigma_j k B_j} = 0;$$

hence

$$^l b \prec lb \qquad l^b \prec lb.$$

In the same way it is easy to show that, if the negatives of l and b are totally unlimited,

$$l^b \prec l\circ b \qquad ^l b \prec l\circ b.$$

§7. THE GENERAL FORMULÆ FOR RELATIVES

248. The principal formulæ of this algebra may be divided into *distribution formulæ* and *association formulæ*. The distribution formulæ are those which give the equivalent of a relative compounded with a sum or product of two relatives in such terms as to separate the latter two relatives. The association formulæ are those which give the equivalent of a relative A compounded with a compound of B and C in terms of a compound of A and B compounded with C.

249. I. DISTRIBUTION FORMULÆ

1. Affirmative

i. *Simple Formulæ*

$$(a+b)c = ac+bc \qquad a(b+c) = ab+ac$$

$$(a\times b)^c = a^c \times b^c \qquad a^{b+c} = a^b \times a^c$$

$$^{a+b}c = {}^a c \times {}^b c \qquad ^a(b\times c) = {}^a b \times {}^a c$$

$$(a\times b)\circ c = (a\circ c)+(b\circ c) \qquad a\circ(b\times c) = (a\circ b)+(a\circ c)$$

ii. *Developments*

$$(a\times b)c = \Pi_p\{\, a(c\times p)+b(c\times\bar{p})\,\}$$
$$(a+b)^c = \Sigma_p\{\, a^{c\times p}\times b^{c\times\bar{p}}\,\}$$
$${}^{(a\times b)}c = \Sigma_p\{\, {}^a(c+p)\times{}^b(c+\bar{p})\,\}$$
$$(a+b)\circ c = \Pi_p\{\, a\circ(c+p)+b\circ(c+\bar{p})\,\}$$
$$a(b\times c) = \Pi_p\{\, (a\times p)b+(a\times\bar{p})c\,\}$$
$$a^{b\times c} = \Sigma_p\{\, (a+p)^b\times(a+\bar{p})^c\,\}$$
$${}^a(b+c) = \Sigma_p\{\, {}^{a\times p}b\times{}^{a\times\bar{p}}c\,\}$$
$$a\circ(b+c) = \Pi_p\{\, (a+p)\circ b+(a+\bar{p})\circ c\,\}$$

2. Negative

i. *Simple Formulæ*

$$\overline{(a+b)c} = \overline{ac}\times\overline{bc}$$
$$\overline{(a\times b)^c} = \overline{a^c}+\overline{b^c}$$
$$\overline{{}^{a+b}c} = \overline{{}^ac}+\overline{{}^bc}$$
$$\overline{(a\times b)\circ c} = \overline{a\circ c}\times\overline{b\circ c}$$
$$\overline{a(b+c)} = \overline{ab}\times\overline{ac}$$
$$\overline{a^{b+c}} = \overline{a^b}+\overline{a^c}$$
$$\overline{{}^a(b\times c)} = \overline{{}^ab}+\overline{{}^ac}$$
$$\overline{a\circ(b\times c)} = \overline{a\circ b}\times\overline{a\circ c}$$

ii. *Developments*

$$\overline{(a\times b)c} = \Sigma_p\{\, \overline{a(c\times p)}\times\overline{b(c\times\bar{p})}\,\}$$
$$\overline{(a+b)^c} = \Pi_p\{\, \overline{a^{c\times p}}+\overline{b^{c\times\bar{p}}}\,\}$$
$$\overline{{}^{(a\times b)}c} = \Pi_p\{\, \overline{{}^a(c+p)}+\overline{{}^b(c+\bar{p})}\,\}$$
$$\overline{(a+b)\circ c} = \Sigma_p\{\, \overline{a\circ(c+p)}\times\overline{b\circ(c+\bar{p})}\,\}$$
$$\overline{a(b\times c)} = \Sigma_p\{\, \overline{(a\times p)b}\times\overline{(a\times\bar{p})c}\,\}$$
$$\overline{a^{b\times c}} = \Pi_p\{\, \overline{(a+p)^b}+\overline{(a+\bar{p})^c}\,\}$$
$$\overline{{}^a(b+c)} = \Pi_p\{\, \overline{{}^{a\times p}b}+\overline{{}^{a\times\bar{p}}c}\,\}$$
$$\overline{a\circ(b+c)} = \Sigma_p\{\, \overline{(a+p)\circ b}\times\overline{(a+\bar{p})\circ c}\,\}$$

250. II. ASSOCIATION FORMULÆ

1. Affirmative

i. *Simple Formulæ*

$$\overline{{}^{a}\overline{(bc)}} = a(bc) = (ab)c = \overline{\overline{(ab)}^{c}}$$

$$\overline{a\circ\overline{(b\circ c)}} = a^{(b\circ c)} = {}^{(a\circ b)}c = \overline{\overline{(a\circ b)}\circ c}$$

$$\overline{a\circ\overline{(bc)}} = a^{(bc)} = (a^{b})^{c} = \overline{\overline{(a^{b})}c}$$

$$\overline{{}^{a}\overline{(b\circ c)}} = a(b\circ c) = ({}^{a}b)\circ c = \overline{{}^{\overline{(ab)}}c}$$

$$\overline{a\overline{(b^{c})}} = {}^{a}(b^{c}) = ({}^{a}b)^{c} = \overline{\overline{({}^{a}b)}c}$$

$$\overline{a^{\overline{({}^{b}c)}}} = a\circ({}^{b}c) = (a^{b})\circ c = \overline{{}^{\overline{(a^{b})}}c}$$

$$\overline{a\overline{({}^{b}c)}} = {}^{a}({}^{b}c) = {}^{(ab)}c = \overline{\overline{(ab)}\circ c}$$

$$\overline{a^{\overline{(b^{c})}}} = a\circ(b^{c}) = (a\circ b)c = \overline{\overline{(a\circ b)}^{c}}$$

ii. *Developments*

(A and E are individual aggregants, and α and ϵ simple components of a and e. The summations and products are relative to all such aggregants and components. The formulæ are of four classes; and for any relative c either all formulæ of Class 1 or all of Class 2, and also either all of Class 3 or all of Class 4 hold good.)

Class 1.

$$\overline{a\overline{(bc)}} = {}^{a}(bc) = \Pi\{({}^{\mathrm{A}}b)c\} = \Pi\{\overline{\overline{({}^{\mathrm{A}}b)}^{c}}\}$$

$$\overline{a^{\overline{(bc)}}} = a\circ(bc) = \Sigma\{(\alpha\circ b)^{c}\} = \Sigma\{\overline{\overline{(\alpha\circ b)}c}\}$$

$$\overline{a\circ\overline{(b^{c})}} = a^{(b^{c})} = \Pi\{(\alpha^{b})c\} = \Pi\{\overline{\overline{(\alpha^{b})}^{c}}\}$$

$$\overline{{}^{a}\overline{(b^{c})}} = a(b^{c}) = \Sigma\{(\mathrm{A}b)^{c}\} = \Sigma\{\overline{\overline{(\mathrm{A}b)}c}\}$$

Class 2.

$$\overline{\overline{(c\circ d)}e} = (c\circ d)^{e} = \Pi\{c\circ(d\mathrm{E})\} = \Pi\{\overline{c^{\overline{(d\mathrm{E})}}}\}$$

$$\overline{{}^{\overline{(c\circ d)}}e} = (c\circ d)\circ e = \Sigma\{c^{(d_{e})}\} = \Sigma\{\overline{c\circ\overline{({}^{d}\epsilon)}}\}$$

$$\overline{\overline{(c^{d})}\circ e} = {}^{(c^{d})}e = \Pi\{c\circ(d\circ\epsilon)\} = \Pi\{\overline{c^{\overline{(d\circ\epsilon)}}}\}$$

$$\overline{\overline{(c^{d})}^{e}} = (c^{d})e = \Sigma\{c^{(d^{\mathrm{E}})}\} = \Sigma\{\overline{c\circ\overline{(d^{\mathrm{E}})}}\}$$

CLASS 3.

$$\overline{a^{\overline{(b\circ c)}}} = a\circ(b\circ c) = \Sigma\left\{({}^{(a^b)}c\right\} = \Sigma\left\{\overline{\overline{(a^b)}c}\right\}$$
$$\overline{a\overline{(b\circ c)}} = {}^a(b\circ c) = \Pi\left\{(\mathrm{A}b)\circ c\right\} = \Pi\left\{\overline{{}^{\overline{(\mathrm{A}b)}}c}\right\}$$
$$\overline{{}^a\overline{({}^bc)}} = a({}^bc) = \Sigma\left\{{}^{(\mathrm{A}_b)}c\right\} = \Sigma\left\{\overline{\overline{({}^{\mathrm{A}}b)}\circ c}\right\}$$
$$\overline{a\circ\overline{({}^bc)}} = a^{({}^bc)} = \Pi\left\{(a\circ b)\circ c\right\} = \Pi\left\{\overline{{}^{\overline{(a\circ b)}}c}\right\}$$

CLASS 4.

$$\overline{{}^{\overline{(cd)}}e} = (cd)\circ e = \Sigma\left\{{}^c(d\circ\epsilon)\right\} = \Sigma\left\{\overline{c\overline{(d\circ\epsilon)}}\right\}$$
$$\overline{\overline{(cd)}e} = (cd)^e = \Pi\left\{c(d^{\mathrm{E}})\right\} = \Pi\left\{\overline{{}^c\overline{(d^{\mathrm{E}})}}\right\}$$
$$\overline{\overline{({}^cd)}^e} = ({}^cd)e = \Sigma\left\{{}^c(d\mathrm{E})\right\} = \Sigma\left\{\overline{c\overline{(d\mathrm{E})}}\right\}$$
$$\overline{\overline{({}^cd)}\circ e} = {}^{({}^cd)}e = \Pi\left\{c({}^d\epsilon)\right\} = \Pi\left\{\overline{{}^c\overline{({}^d\epsilon)}}\right\}$$

The negative formulæ are derived from the affirmative by simply drawing or erasing lines over the whole of each member of every equation.

251. In order to see the general rules which these formulæ follow, we must imagine the operations symbolized by three marks, as in the commencement of this part [237]. We may term the operation uniting the two letters within the parenthesis the *interior* operation, and that which unites the whole parenthesis to the third letter the *exterior* operation. By *junction-marks* will be meant, in case the parenthesis $\left\{\begin{matrix}\text{follows}\\ \text{precedes}\end{matrix}\right\}$ the third letter, the $\left\{\begin{matrix}\text{first}\\ \text{second}\end{matrix}\right\}$ mark of the symbol of the interior operation and the $\left\{\begin{matrix}\text{second}\\ \text{first}\end{matrix}\right\}$ mark of the symbol of the exterior operation. Using these terms, we may say that the exterior junction-mark and the third mark of the interior operation may always be changed together. When they are the same there is a simple association formula. This formula consists in the possibility of simultaneously interchanging the junction-marks, the third marks, and the exteriority or interiority of the two operations. When the exterior junction-mark and the third mark of the interior operation are unlike, there is a developmental association formula. The

general term of this formula is obtained by making the same interchanges as in the simple formulæ, and then changing a to A when after these interchanges ab or ${}^{a}b$ occurs in parenthesis, changing a to α when a^b or $a{\circ}b$ occurs in parenthesis, changing e to E when de or d^e occurs in parenthesis, and changing e to ϵ when ${}^{d}e$ or $d{\circ}e$ occurs in parenthesis. When the third mark in the symbol of the exterior operation is affirmative the development is a summation; when this mark is negative there is a continued product.

In the first half of the formulæ, the second mark in the sign of the interior operation is a line in Class 1 and a hyphen in Class 3. In the second half, the first mark in the sign of the interior operation is a hyphen in Class 2 and a line in Class 4.

VII

*ON THE LOGIC OF NUMBER**

§1. DEFINITION OF QUANTITY[E]

252. Nobody can doubt the elementary propositions concerning number: those that are not at first sight manifestly true are rendered so by the usual demonstrations. But although we see they *are* true, we do not so easily see precisely *why* they are true; so that a renowned English logician† has entertained a doubt as to whether they were true in all parts of the universe. The object of this paper is to show that they are strictly syllogistic consequences from a few primary propositions. The question of the logical origin of these latter, which I here regard as definitions, would require a separate discussion. In my proofs I am obliged to make use of the logic of relatives, in which the forms of inference are not, in a narrow sense, reducible to ordinary syllogism. They are, however, of that same nature, being merely syllogisms in which the objects spoken of are pairs or triplets. Their validity depends upon no conditions other than those of the validity of simple syllogism, unless it be that they suppose the existence of singulars, while syllogism does not.

The selection of propositions which I have proved will, I trust, be sufficient to show that all others might be proved with like methods.

253. Let r be any relative term, so that one thing may be said to be r of another, and the latter r'd by the former. If in a certain system of objects, whatever is r of an r of anything is itself r of that thing, then r is said to be a transitive relative in that system. (Such relatives as "lover of everything loved by —" are transitive relatives.) In a system in which r is transitive, let the q's of anything include that thing

* *The American Journal of Mathematics*, vol. 4, pp. 85–95 (1881), as subsequently corrected.

† J. S. Mill, e.g. in *Logic*, bk. II, ch. 6, §2–3.

itself, and also every *r* of it which is not *r*'d by it. Then *q* may be called a fundamental relative of quantity; its properties being, first, that it is transitive; second, that everything in the system is *q* of itself, and, third, that nothing is both *q* of and *q*'d by anything except itself. The objects of a system having a fundamental relation of quantity are called quantities, and the system is called a system of quantity.

254. A system in which quantities may be *q*'s of or *q*'d by the same quantity without being either *q*'s of or *q*'d by each other is called multiple;[1] a system in which of every two quantities one is a *q* of the other is termed simple.

§2. SIMPLE QUANTITY

255. In a simple system every quantity is either "as great as" or "as small as" every other; whatever is as great as something as great as a third is itself as great as that third, and no quantity is at once as great as and as small as anything except itself.

256. A system of simple quantity is either continuous, discrete, or mixed. A continuous system is one in which every quantity greater than another is also greater than some intermediate quantity greater than that other.* A discrete system is one in which every quantity greater than another is next greater than some quantity (that is, greater than without being greater than something greater than). A mixed system is one in which some quantities greater than others are next greater than some quantities, while some are continuously greater than some quantities.

§3. DISCRETE QUANTITY

257. A simple system of discrete quantity is either limited, semi-limited, or unlimited. A limited system is one which has an absolute maximum and an absolute minimum quantity; a semi-limited system has one (generally considered a minimum) without the other; an unlimited has neither.

[1] For example, in the ordinary algebra of imaginaries two quantities may both result from the addition of quantities of the form a^2+b^2i to the same quantity without either being in this relation to the other.

* But see 4.121.

258. A simple, discrete, system, unlimited in the direction of increase or decrement, is in that direction either infinite or super-infinite. An infinite system is one in which any quantity greater than x can be reached from x by successive steps to the next greater (or less) quantity than the one already arrived at. In other words, an infinite, discrete, simple, system is one in which, if the quantity next greater than an attained quantity is itself attained, then any quantity greater than an attained quantity is attained; and by the class of attained quantities is meant any class whatever which satisfies these conditions. So that we may say that an infinite class is one in which if it is true that every quantity next greater than a quantity of a given class itself belongs to that class, then it is true that every quantity greater than a quantity of that class belongs to that class. Let the class of numbers in question be the numbers of which a certain proposition holds true. Then, an infinite system may be defined as one in which from the fact that a certain proposition, if true of any number, is true of the next greater, it may be inferred that that proposition if true of any number is true of every greater.*

259. Of a super-infinite system this proposition, in its numerous forms, is untrue.

§4. SEMI-INFINITE QUANTITY

260. We now proceed to study the fundamental propositions of semi-infinite,† discrete, and simple quantity, which is ordinary number.‡

Definitions

261. The minimum number is called one.

262. By $x+y$ is meant, in case $x=1$, the number next greater than y; and in other cases, the number next greater than $x'+y$, where x' is the number next smaller than x.

263. By $x \times y$ is meant, in case $x=1$, the number y, and in other cases $y+x'y$, where x' is the number next smaller than x.

264. It may be remarked that the symbols $+$ and $\times$ are triple relatives, their two correlates being placed one before and the other after the symbols themselves.

* See 564.

† I.e., semi-limited and infinite; see 4.107.

‡ Cf. 4.150ff.

THEOREMS

265. The proof in each case will consist in showing, first, that the proposition is true of the number one, and second, that if true of the number n it is true of the number $1+n$, next larger than n. The different transformations of each expression will be ranged under one another in one column, with the indications of the principles of transformation in another column.

266. (1) To prove the associative principle of addition, that

$$(x+y)+z=x+(y+z)$$

whatever numbers x, y, and z, may be. First it is true for $x=1$; for

$(1+y)+z$

$=1+(y+z)$ by the definition of addition, second clause. Second, if true for $x=n$, it is true for $x=1+n$; that is, if $(n+y)+z=n+(y+z)$ then $((1+n)+y)+z=(1+n)+(y+z)$.

For

$((1+n)+y)+z$	
$=(1+(n+y))+z$	by the definition of addition:
$=1+((n+y)+z)$	by the definition of addition:
$=1+(n+(y+z))$	by hypothesis:
$=(1+n)+(y+z)$	by the definition of addition,

267. (2) To prove the commutative principle of addition that

$$x+y=y+x$$

whatever numbers x and y may be. First, it is true for $x=1$ and $y=1$, being in that case an explicit identity. Second if true for $x=n$ and $y=1$, it is true for $x=1+n$ and $y=1$. That is, if $n+1=1+n$, then $(1+n)+1=1+(1+n)$. For $(1+n)+1$

$=1+(n+1)$	by the associative principle:
$=1+(1+n)$	by hypothesis.

We have thus proved that whatever number x may be $x+1=1+x$, or that $x+y=y+x$ for $y=1$. It is now to be shown that if this be true for $y=n$, it is true for $y=1+n$;

that is, that if $x+n=n+x$, then $x+(1+n)=(1+n)+x$. Now

$$\begin{aligned}
&x+(1+n) \\
&=(x+1)+n && \text{by the associative principle:} \\
&=(1+x)+n && \text{as just seen:} \\
&=1+(x+n) && \text{by the definition of addition:} \\
&=1+(n+x) && \text{by hypothesis:} \\
&=(1+n)+x && \text{by the definition of addition.}
\end{aligned}$$

Thus the proof is complete.

268. (3) To prove the distributive principle, first clause. The distributive principle consists of two propositions:

$$\text{First,} \quad (x+y)z=xz+yz$$
$$\text{Second,} \quad x(y+z)=xy+xz.$$

We now undertake to prove the first of these. First it is true for $x=1$. For

$$\begin{aligned}
&(1+y)z \\
&=z+yz && \text{by the definition of multiplication:} \\
&=1.z+yz && \text{by the definition of multiplication.}
\end{aligned}$$

Second, if true for $x=n$ it is true for $x=1+n$; that is, if $(n+y)z=nz+yz$ then $((1+n)+y)z=(1+n)z+yz$. For

$$\begin{aligned}
&((1+n)+y)z \\
&=(1+(n+y))z && \text{by the definition of addition:} \\
&=z+(n+y)z && \text{by the definition of multiplication:} \\
&=z+(nz+yz) && \text{by hypothesis:} \\
&=(z+nz)+yz && \text{by the associative principle of addition:} \\
&=(1+n)z+yz && \text{by the definition of multiplication.}
\end{aligned}$$

269. (4) To prove the second proposition of the distributive principle, that

$$x(y+z)=xy+xz.$$

First, it is true for $x=1$; for

$$\begin{aligned}
&1(y+z) \\
&=y+z && \text{by the definition of multiplication;} \\
&=1y+1z && \text{by the definition of multiplication.}
\end{aligned}$$

Second, if true for $x=n$, it is true for $x=1+n$; that is, if $n(y+z)=ny+nz$, then $(1+n)(y+z)=(1+n)y+(1+n)z$. For

$(1+n)(y+z)$

$=(y+z)+n(y+z)$ by the definition of multiplication:

$=(y+z)+(ny+nz)$ by hypothesis:

$=(y+ny)+(z+nz)$ by the principles of addition:

$=(1+n)y+(1+n)z$ by the definition of multiplication.

270. (5) To prove the associative principle of multiplication; that is, that

$$(xy)z=x(yz)$$

whatever numbers x, y, and z, may be. First, it is true for $x=1$, for

$(1y)z$

$=yz$ by the definition of multiplication:

$=1.yz$ by the definition of multiplication.

Second, if true for $x=n$, it is true for $x=1+n$; that is, if $(ny)z=n(yz)$ then $((1+n)y)z=(1+n)(yz)$. For

$((1+n)y)z$

$=(y+ny)z$ by the definition of multiplication:

$=yz+(ny)z$ by the distributive principle:

$=yz+n(yz)$ by hypothesis:

$=(1+n)(yz)$ by the definition of multiplication.

271. (6) To prove the commutative principle of multiplication; that

$$xy=yx,$$

whatever numbers x and y may be. In the first place we prove that it is true for $y=1$. For this purpose, we first show that it is true for $y=1$ $x=1$; and then that if true for $y=1$, $x=n$ it is true for $y=1$, $x=1+n$. For $y=1$ and $x=1$, it is an explicit identity. We have now to show that if $n1=1n$ then $(1+n)1=1(1+n)$. Now

$(1+n)1$

$=1+n1$ by the definition of multiplication:

$=1+1n$ by hypothesis:

$=1+n$ by the definition of multiplication:

$=1(1+n)$ by the definition of multiplication.

Having thus shown the commutative principle to be true for $y=1$, we proceed to prove that if it is true for $y=n$, it is true for $y=1+n$; that is, if $xn=nx$, then $x(1+n)=(1+n)x$. For

$(1+n)x$	
$=x+nx$	by the definition of multiplication:
$=x+xn$	by hypothesis:
$=1x+xn$	by the definition of multiplication:
$=x1+xn$	as already seen:
$=x(1+n)$	by the distributive principle.

§5. DISCRETE SIMPLE QUANTITY INFINITE IN BOTH DIRECTIONS

272. A system of number infinite in both directions has no minimum, but a certain quantity is called *one* and the numbers as great as this constitute a partial system of semi-infinite number, of which this one is a minimum. The definitions of addition and multiplication require no change except that the *one* therein is to be understood in the new sense.

273. To extend the proofs of the principles of addition and multiplication to unlimited number, it is necessary to show that if true for any number $(1+n)$ they are also true for the next smaller number n. For this purpose we can use the same transformations as in the second clauses of the former proof; only we shall have to make use of the following lemma.

274. If $x+y=x+z$ then $y=z$ whatever numbers x, y, and z, may be. First this is true in case $x=1$ for then y and z are both next smaller than the same number. Therefore, neither is smaller than the other, otherwise it would not be next smaller to $1+y=1+z$. But in a simple system, of any two different numbers one is smaller. Hence, y and z are equal. Second, if the proposition is true for $x=n$, it is true for $x=1+n$. For if $(1+n)+y=(1+n)+z$, then by the definition of addition $1+(n+y)=1+(n+z)$; whence it would follow that $n+y=n+z$, and, by hypothesis, that $y=z$. Third, if the proposition is true for $x=1+n$ it is true for $x=n$. For if $n+y=n+z$, then $1+n+y=1+n+z$ because the system is simple. The proposition has thus been proved to be true of 1 of every

greater and of every smaller number and therefore to be universally true.

275. An inspection of the above proofs of the principles of addition and multiplication for semi-infinite number will show that they are readily extended to doubly infinite number by means of the proposition just proved.

276. The number next smaller than one is called naught, 0. This definition in symbolic form is $1+0=1$. To prove that $x+0=x$, let x' be the number next smaller than x. Then

$x+0$	
$=(1+x')+0$	by the definition of x':
$=(1+0)+x'$	by the principles of addition:
$=1+x'$	by the definition of naught:
$=x$	by the definition of x'.

277. To prove that $x0=0$. First, in case $x=1$ the proposition holds by the definition of multiplication. Next, if true for $x=n$ it is true for $x=1+n$. For

$(1+n)0$	
$=1.0+n.0$	by the distributive principle:
$=1.0+0$	by hypothesis:
$=1.0$	by the last theorem:
$=0$	as above.

Third, the proposition, if true for $x=1+n$, is true for $x=n$. For, changing the order of the transformations,

$$1.0+0=1.0=0=(1+n)0=1.0+n.0.$$

Then by the above lemma, $n.0=0$ so that the proposition is proved.

278. A number which added to another gives naught is called the negative of the latter. To prove that every number greater than naught has a negative. First, the number next smaller than naught is the negative of one; for, by the definition of addition, one plus this number is naught. Second, if any number n has a negative then the number next greater than n has for its negative the number next smaller than the

negative of n. For let m be the number next smaller than the negative of n. Then $n+(1+m)=0$.

But $n+(1+m)$

$=(n+1)+m$ by the associative principle of addition.

$=(1+n)+m$ by the commutative principle of addition.

So that $(1+n)+m=0$. *Q. E. D.* Hence every number greater than 0 has a negative, and naught is its own negative.

To prove that $(-x)y=-(xy)$. We have

$0=x+(-x)$ by the definition of the negative:

$0=0y=(x+(-x))y$ by the last proposition but one:

$0=xy+(-x)y$ by the distributive principle:

$-(xy)=(-x)y$ by the definition of the negative.

279. The negative of the negative of a number is that number. For $x+(-x)=0$. Whence by the definition of the negative $x=-(-x)$.

§6. LIMITED DISCRETE SIMPLE QUANTITY

280. Let such a relative term, c, that whatever is a c of anything is the only c of that thing, and is a c of that thing only, be called a relative of simple correspondence.* In the notation of the logic of relatives

$$c\breve{c} -\!\!< 1, \quad \breve{c}c -\!\!< 1.†$$

281. If every object, s, of a class is in any such relation being c'd by‡ a number of a semi-infinite discrete simple system, and if further every number smaller than a number c of§ an s is itself c of§ an s, then the numbers c of§ the s's are said to count them,¶ and the system of correspondence is called a count. In logical notation, putting g for as 'great as,' and n for a positive integral number,

$$s -\!\!< \breve{c}n \quad \breve{g}cs -\!\!< cs.$$

* I.e., c is a one-one relation.

† $\breve{c}$ is the converse of c.

‡ Originally 'c'*d* with.'

§ Originally 'c'*d* by.'

¶ I.e., counting involves the establishment of a one-one correlation between the members of a given class and the natural numbers.

If in any count there is a maximum counting number the count is said to be finite, and that number is called the number of the count. Let [*s*] denote the number of a count of the *s*'s, then

$$[s] \prec cs \quad \bar{g}cs \prec \overline{[s]}.$$

282. The relative "identical with" satisfies the definition of a relative of simple correspondence, and the definition of a count is satisfied by putting "identical with" for *c*, and "positive integral number as small as *x*" for *s*. In this mode of counting, the number of numbers as small as *x* is *x*.

283. Suppose that in any count the number of numbers as small as the minimum number, one, is found to be *n*. Then, by the definition of a count, every number as small as *n* counts a number as small as one. But by the definition of one there is only one number as small as one. Hence, by the definition of single correspondence, no other number than one counts one. Hence, by the definition of one, no other number than one counts any number as small as one. Hence, by the definition of the count, one is, in every count, the number of numbers as small as one.

284. If the number of numbers as small as *x* is in some count *y*, then the number of numbers as small as *y* is in some count *x*. For if the definition of a simple correspondence is satisfied by the relative *c*, it is equally satisfied by the relative *c*'d by.

285. Since the number of numbers as small as *x* is in some count *y*, we have, *c* being some relative of simple correspondence,

First. Every number as small as *x* is *c*'d by a number.

Second. Every number as small as a number that is *c* of a number as small as *x* is itself *c* of a number as small as *x*.

Third. The number *y* is *c* of a number as small as *x*.

Fourth. Whatever is not as great as a number that is *c* of a number as small as *x* is not *y*.

286. Now let c_1 be the converse of *c*. Then the converse of c_1 is *c*; whence, since *c* satisfies the definition of a relative of simple correspondence, so also does c_1. By the third proposition above, every number as small as *y* is as small as a number that is *c* of a number as small as *x*. Whence, by the second

proposition every number as small as y is c of a number as small as x; and it follows that every number as small as y is c_1'd by a number. It follows further that every number c_1 of a number as small as y is c_1 of something c_1'd by (that is, c_1 being a relative of simple correspondence, is identical with) some number as small as x. Also, "as small as" being a transitive relative, every number as small as a number c of a number as small as y is as small as x. Now by the fourth proposition y is as great as any number that is c of a number as small as x, so that what is not as small as y is not c of a number as small as x; whence whatever number is c'd by a number not as small as y is not a number as small as x. But by the second proposition, every number as small as x not c'd by a number not as small as y is c'd by a number as small as y. Hence, every number as small as x is c'd by a number as small as y. Hence, every number as small as a number c_1 of a number as small as y is c_1 of a number as small as y. Moreover, since we have shown that every number as small as x is c_1 of a number as small as y, the same is true of x itself. Moreover, since we have seen that whatever is c_1 of a number as small as y is as small as x, it follows that whatever is not as great as a number c_1 of a number as small as y is not as great as a number as small as x; *i.e.* ("as great as" being a transitive relative) is not as great as x, and consequently is not x.[1] We have now shown:

First, that every number as small as y is c_1'd by a number;

Second, that every number as small as a number that is c_1 of a number as small as y is itself c_1 of a number as small as y;

Third, that the number x is c_1 of a number as small as y; and

Fourth, that whatever is not as great as a number that is c_1 of a number as small as y is not x.

These four propositions taken together satisfy the definition of the number of numbers as small as y counting up to x.

Hence, since the number of numbers as small as one cannot in any count be greater than one, it follows that the number of numbers as small as any number greater than one cannot in any count be one.

[1] This long proof is quite unnecessary. The whole thing depends not on Fermat's mode of reasoning but on De Morgan's.— marginal note. [I.e., it depends, not on mathematical induction but on the syllogism of transposed quantity.]

287. Suppose that there is a count in which the number of numbers as small as $1+m$ is found to be $1+n$, since we have just seen that it cannot be 1. In this count, let m' be the number which is c of $1+n$, and let n' be the number which is c'd by $1+m$. Let us now consider a relative, e, which differs from c only in excluding the relation of m' to $1+n$ as well as the relation of $1+m$ to n' and in including the relation of m' to n'. Then e will be a relative of single correspondence; for c is so, and no exclusion of relations from a single correspondence affects this character, while the inclusion of the relation of m' to n' leaves m' the only e of n' and an e of n' only. Moreover, every number as small as m is e of a number, since every number except $1+m$ that is c of anything is e of something, and every number except $1+m$ that is as small as $1+m$ is as small as m. Also, every number as small as a number e'd by a number is itself e'd by a number; for every number c'd is e'd except $1+m$, and this is greater than any number e'd. It follows that e is the basis of a mode of counting by which the numbers as small as m count up to n. Thus we have shown that if in any way $1+m$ counts up to $1+n$, then in some way m counts up to n. But we have already seen that for $x=1$ the number of numbers as small as x can in no way count up to other than x. Whence it follows that the same is true whatever the value of x.[1]

288. If every S is a P, and if the P's are a finite lot counting up to a number as small as the number of S's, then every P is an S. For if, in counting the P's, we begin with the S's (which are a part of them), and having counted all the S's arrive at the number n, there will remain over no P's not S's. For if there were any, the number of P's would count up to more than n. From this we deduce the validity of the following mode of inference:

Every Texan kills a Texan,
Nobody is killed by but one person,
Hence, every Texan is killed by a Texan,

[1] It may be remarked that when we reason that a certain proposition, if false of any number, is false of some smaller number, and since there is no number (in a semi-limited system) smaller than every number, the proposition must be true, our reasoning is a mere logical transformation of the reasoning that a proposition, if true for n, is true for $1+n$, and that it is true for 1.

supposing Texans to be a finite lot. For, by the first premiss, every Texan killed by a Texan is a Texan killer of a Texan, By the second premiss, the Texans killed by Texans are as many as the Texan killers of Texans. Whence we conclude that every Texan killer of a Texan is a Texan killed by a Texan, or, by the first premiss, every Texan is killed by a Texan. This mode of reasoning* is frequent in the theory of numbers.

* This mode of reasoning was uncovered by De Morgan and called the syllogism of transposed quantity. See 402 and 4.103f.

VIII

*ASSOCIATIVE ALGEBRAS**E

§1. ON THE RELATIVE FORMS OF THE ALGEBRAS

289. Given an associative algebra whose letters are i, j, k, l, etc., and whose multiplication table is

$$i^2 = a_{11}i + b_{11}j + c_{11}k + \text{etc.}$$ [1]
$$ij = a_{12}i + b_{12}j + c_{12}k + \text{etc.}$$
$$ji = a_{21}i + b_{21}j + c_{21}k + \text{etc.},$$
$$\text{etc., etc.}$$

I proceed to explain what I call the relative form of this algebra.

290. Let us assume a number of new units, A, I, J, K, L, etc., one more in number than the letters of the algebra, and every one except the first, A, corresponding to a particular letter of the algebra. These new units are susceptible of being multiplied by numerical coefficients and of being added together;[2] but they cannot be multiplied together, and hence are called *non-relative* units.

291. Next, let us assume a number of operations each denoted by bracketing together two non-relative units separated by a colon. These operations, equal in number to the square of the number of non-relative units, may be arranged as follows:

$$\begin{matrix} (A:A) & (A:I) & (A:J) & (A:K), \text{etc.} \\ (I:A) & (I:I) & (I:J) & (I:K), \text{etc.} \\ (J:A) & (J:I) & (J:J) & (J:K), \text{etc.} \end{matrix}$$

* *The American Journal of Mathematics*, vol. 4, pp. 221–29, (1881), an addendum to Benjamin Peirce's *Linear Associative Algebras* published posthumously in the same volume with notes by C. S. P. These notes throw considerable light on the significance and relationship of algebra to the logic of relatives.

[1] I have used a_{11} etc., in place of the a_1, etc., used by my father in his text.

[2] Any one of them multiplied by 0 gives 0.

292. Any one of these operations performed upon a polynomial in non-relative units, of which one term is a numerical multiple of the letter following the colon, gives the same multiple of the letter preceding the colon. Thus, $(I : J)(aI + bJ + cK) = bI$.[1] These operations are also taken to be susceptible of associative combination. Hence $(I : J)(J : K) = (I : K)$; for $(J : K)K = J$ and $(I : J)J = I$, so that $(I : J)(J : K)K = I$. And $(I : J)(K : L) = 0$; for $(K : L)L = K$ and $(I : J)K = (I : J)(0.J + K) = 0.I = 0$. We further assume the application of the distributive principle to these operations; so that, for example,

$$\{(I : J) + (K : J) + (K : L)\}(aJ + bL) = aJ + (a+b)K.$$

293. Finally, let us assume a number of complex operations denoted by i', j', k', l', etc., corresponding to the letters of the algebra and determined by its multiplication table in the following manner:

$$\begin{aligned} i' = (I : A) &+ a_{11}(I : I) + b_{11}(J : I) + c_{11}(K : I) + \text{etc.} \\ &+ a_{12}(I : J) + b_{12}(J : J) + c_{12}(K : J) + \text{etc.} \\ &+ a_{13}(I : K) + b_{13}(J : K) + c_{13}(K : K) + \text{etc.} + \text{etc.} \end{aligned}$$

$$\begin{aligned} j' = (J : A) &+ a_{21}(I : I) + b_{21}(J : I) + c_{21}(K : I) + \text{etc.} \\ &+ a_{22}(I : J) + b_{22}(J : J) + c_{22}(K : J) + \text{etc.} \\ &+ a_{23}(I : K) + b_{23}(J : K) + c_{23}(K : K) + \text{etc.} + \text{etc.} \end{aligned}$$

$k' = \text{etc.}$

294. Any two operations are equal which, being performed on the same operand, invariably give the same result. The ultimate operands in this case are the non-relative units. But any operations compounded by addition or multiplication of the operations i', j', k', etc., if they give the same result when performed upon A, will give the same result when performed upon any one of the non-relative units. For suppose $i'j'A = k'l'A$. We have

$$i'j'A = i'J = a_{12}I + b_{12}J + c_{12}K + \text{etc.}$$

$$k'l'A = k'L = a_{34}I + b_{34}J + c_{34}K + \text{etc.}$$

[1] If $B = 0$, of course the result is 0.

so that $a_{12}=a_{34}$, $b_{12}=b_{34}$, $c_{12}=c_{34}$, etc., and in our original algebra $ij=kl$. Hence, multiplying both sides of the equation into any letter, say m, $ijm=klm$. But

$$ijm=i(a_{25}i+b_{25}j+c_{25}k+\text{etc.})=(a_{11}a_{25}+a_{12}b_{25}+a_{13}c_{25}+\text{etc.})i \\ +(b_{11}a_{25}+b_{12}b_{25}+b_{13}c_{25}+\text{etc.})j+\text{etc.}$$

But we have equally

$$i'j'm'A=(a_{11}a_{25}+a_{12}b_{25}+a_{13}c_{25}+\text{etc.})I+(b_{11}a_{25}+b_{12}b_{25}+b_{13}c_{25} \\ +\text{etc.})J+\text{etc.}$$

So that $i'j'm'A=k'l'm'A$. Hence, $i'j'M=k'l'M$. It follows, then, that if $i'j'A=k'l'A$, then $i'j'$ into any non-relative unit equals $k'l'$ into the same unit, so that $i'j'=k'l'$. We thus see that whatever equality subsists between compounds of the accented letters i', j', k', etc., subsists between the same compounds of the corresponding unaccented letters i, j, k, so that the multiplication tables of the two algebras are the same.[1] Thus, what has been proved is that any associative algebra can be put into relative form, *i. e.* (see my *brochure* entitled *A Brief Description of the Algebra of Relatives*)* that every such algebra may be represented by a matrix.

Take, for example, the algebra (bd_5).† It takes the relative form

$$i=(I\ :A)+(J\ :I)+(L:K),\ j=(J:A),$$

$$k=(K\ :A)+(J\ :I)+\mathfrak{r}(L\ :I)+(I\ :K)+(M\ :K)+\mathfrak{r}(J\ :L) \\ -(J:M)-\mathfrak{r}(L:M),$$

$$l=(L\ :A)+(J\ :K),\ m=(M\ :A)+(\mathfrak{r}^2-1)(J\ :I)-(L:K) \\ -\mathfrak{r}^2(J:M).$$

This is the same as to say that the general expression $xi+yj+zk+ul+vm$ of this algebra has the same laws of multiplication as the matrix

[1] A brief proof of this theorem, perhaps essentially the same as the above, was published by me in the *Proceedings of the American Academy of Arts and Sciences*, for May 11, 1875. [150–51.]

* Paper No. IX.

† The reference is to the *Linear Associative Algebras*, p. 188.

0,	0,	0,	0,	0,	0,
x,	0,	0,	z,	0,	0,
y,	$x+z$, $+(\mathfrak{r}^2-1)v$,	0,	u,	$\mathfrak{r}z$,	$-z-\mathfrak{r}^2v$,
z,	0,	0,	0,	0,	0,
u,	$\mathfrak{r}z$,	0,	$x-v$,	0,	$-\mathfrak{r}z$,
v,	0,	0,	z,	0,	0.

295. Of course, every algebra may be put into relative form in an infinity of ways; and simpler ways than that which the rule affords can often be found. Thus, for the above algebra, the form given in the foot-note is simpler, and so is the following:

$$i=(B:A)+(C:B)+(F:D)+\ (C:E),\quad j=(C:A),$$

$$k=(D:A)+(E:D)+(C:B)+\mathfrak{r}(F:B)+\mathfrak{r}(C:F),$$

$$l=(F:A)+(C:D),\ m=(E:A)+(\mathfrak{r}^2-1)(C:B)-(B:A)$$
$$-(F:D)-(C:E).$$

These different forms will suggest transformations of the algebra. Thus, the relative form in the foot-note to (bd_5) suggests putting

$$i_1=i+m,\ j_1=\mathfrak{r}^2j,\ k_1=k+\mathfrak{r}^{-1}i+\mathfrak{r}^{-1}m,\ l_1=\mathfrak{r}l+j,\ m_1=-m,$$

when we get the following multiplication table, where ρ is put for $\mathfrak{r}^{-1}$:

	i	j	k	l	m
i	0	0	0	0	j
j	0	0	0	0	0
k	0	0	i	j	l
l	0	0	ρj	0	0
m	$\rho^2 j$	0	ρl	0	j

296. Ordinary algebra with imaginaries, considered as a double algebra, is, in relative form,

$$1=(X:X)+(Y:Y),\quad \text{J}=(X:Y)-(Y:X).$$

This shows how the operation J turns a vector through a right angle in the plane of X, Y. Quaternions in relative form is

$$\begin{aligned}1&=(W:W)+(X:X)+(Y:Y)+(Z:Z),\\ i&=(X:W)-(W:X)+(Z:Y)-(Y:Z),\\ j&=(Y:W)-(Z:X)-(W:Y)+(X:Z),\\ k&=(Z:W)+(Y:X)-(X:Y)-(W:Z).\end{aligned}$$

We see that we have here a reference to a space of four dimensions corresponding to X, Y, Z, W.

§2. ON THE ALGEBRAS IN WHICH DIVISION IS UNAMBIGUOUS

297. (1) In the *Linear Associative Algebra*, the coefficients are permitted to be imaginary. In this note they are restricted to being real. It is assumed that we have to deal with an algebra such that from $AB=AC$ we can infer that $A=0$ or $B=C$. It is required to find what forms such an algebra may take.

298. (2) If $AB=0$, then either $A=0$ or $B=0$. For if not, $AC=A(B+C)$, although A does not vanish and C is unequal to $B+C$.

299. (3) The reasoning of §40 [of the *Linear Associative Algebra*] holds, although the coefficients are restricted to being real. It is true, then, that since there is no expression (in the algebra under consideration) whose square vanishes, there must be an expression, i, such that $i^2=i$.

300. (4) By §41, it appears that for every expression in the algebra we have

$$iA=Ai=A.$$

301. (5) By the reasoning of §53, it appears that for every expression A there is an equation of the form

$$\Sigma_m(a_mA^m)+bi=0.$$

But i is virtually arithmetical unity, since $iA = Ai = A$; and this equation may be treated by the ordinary theory of equations. Suppose it has a real root, α; then it will be divisible by $(A - \alpha)$, and calling the quotient B we shall have

$$(A - \alpha i)B = 0.$$

But $A - \alpha i$ is not zero, for A was supposed dissimilar to i. Hence a product of finites vanishes, which is impossible. Hence the equation cannot have a real root. But the whole equation can be resolved into quadratic factors, and some one of these must vanish. Let the irresoluble vanishing factor be

$$(A - s)^2 + t^2 = 0.$$

Then

$$\left(\frac{A - s}{t}\right)^2 = -1,$$

or, every expression, upon subtraction of a real number (*i. e.* a real multiple of i), can be converted, in one way only, into a quantity whose square is a negative number. We may express this by saying that every quantity consists of a scalar and a vector part. A quantity whose square is a negative number we here call a *vector*.

302. (6) Our next step is to show that the vector part of the product of two vectors is linearly independent of these vectors and of unity. That is, i and j being any two vectors,[1] if

$$ij = s + v$$

where s is a scalar and v a vector, we cannot determine three real scalars a, b, c, such that

$$v = a + bi + cj.$$

This is proved, if we prove that no scalar subtracted from ij leaves a remainder $bi + cj$. If this be true when i and j are any unit vectors whatever, it is true when these are multiplied by real scalars, and so is true of every pair of vectors. We will, then, suppose i and j to be unit vectors. Now,

$$ij^2 = -i.$$

[1] The idempotent basis having been shown to be arithmetical unity, we are free to use the letter i to denote another unit.

If therefore we had

$$ij = a + bi + cj,$$

we should have

$$-i = ij^2 = aj + bij - c = ab - c + b^2 i + (a + bc)j;$$

whence, i and j being dissimilar,

$$-i = b^2 i, \qquad b^2 = -1,$$

and b could not be real.

303. (7) Our next step is to show that, i and j being any two vectors, and

$$ij = s + v,$$

s being a scalar and v a vector, we have

$$ji = r(s - v),$$

where r is a real scalar. It will be obviously sufficient to prove this for the case in which i and j are unit vectors. Assuming them such, let us write

$$ji = s' + v', \qquad vv' = s'' + v'',$$

where s' and s'' are scalars, while v' and v'' are vectors. Then

$$ij . ji = (s + v)(s' + v') = ss' + sv' + s'v + v'' + s''.$$

But we have

$$ij . ji = ij^2 i = -i^2 = 1.$$

Hence,

$$v'' = 1 - ss' - s'' - sv' - s'v.$$

But v'' is the vector of vv', so that by the last paragraph such an equation cannot subsist unless v'' vanishes. Thus we get

$$0 = 1 - ss' - s'' - sv' - s'v,$$

or

$$sv' = 1 - ss' - s'' - s'v.$$

But a quantity can only be separated in one way into a scalar and a vector part; so that

$$sv' = -s'v.$$

That is,

$$ji = \frac{s'}{s}(s - v). \quad Q.\,E.\,D.$$

304. (8) Our next step is to prove that $s=s'$; so that if $ij=s+v$ then $ji=s-v$. It is obviously sufficient to prove this when i and j are unit vectors. Now from any quantity a scalar may be subtracted so as to leave a remainder whose square is a scalar. We do not yet know whether the sum of two vectors is a vector or not (though we do know that it is not a scalar). Let us then take such a sum as $ai+bj$ and suppose x to be the scalar which subtracted from it makes the square of the remainder a scalar. Then, C being a scalar,

$$(-x+ai+bj)^2=C.$$

But developing the square we have

$$(-x+ai+bj)^2=x^2-a^2-b^2+abs+abs'-2axi+2bxj$$
$$+ab\left(1-\frac{s'}{s}\right)v=C;$$

i. e.

$$ab\left(1-\frac{s'}{s}\right)v=C-x^2+a^2+b^2-abs-abs'+2axi+2bxj.$$

But v being the vector of ij, by the last paragraph but one the equation must vanish. Either then $v=0$ or $1-\frac{s'}{s}=0$. But if $v=0$, $ij=s$, and multiplying into j,

$$-i=sj,$$

which is absurd, i and j being dissimilar. Hence $1-\frac{s'}{s}=0$ and

$$ji=s-v. \quad Q.\,E.\,D.$$

305. (9) The number of independent vectors in the algebra cannot be two. For the vector of ij is independent of i and j. There may be no vector, and in that case we have the ordinary algebra of reals; or there may be only one vector, and in that case we have the ordinary algebra of imaginaries.

Let i and j be two independent vectors such that

$$ij=s+v.$$

Let us substitute for j

$$j_1=si+j.$$

Then we have

$$ij_1=v, \quad j_1i=-v,$$
$$j_1v=j_1ij_1=-j_1^2i=i, \quad vj_1=ij_1^2=-i,$$
$$iv=i^2j_1=-j_1, \quad vi=ij_1i=-j_1i^2=j_1.$$

Thus we have the algebra of real *quaternions*. Suppose we have a fourth unit vector, k, linearly independent of all the others, and let us write

$$j_1k=s'+v',$$
$$ki=s''+v''.$$

Let us substitute for k

$$k_1=s''i+s'j_1+k,$$

and we get

$$j_1k_1=-s''v+v', \quad k_1j_1=s''v-v',$$
$$k_1i=-s'v+v'', \quad ik_1=s'v-v''.$$

Let us further suppose

$$(ij_1)k_1=s''+v'''.$$

Then, because ij_1 is a vector,

$$k_1(ij_1)=s'''-v'''.$$

But

$$k_1j_1=-j_1k_1, \quad k_1i=-ik_1,$$

because both products are vectors.

Hence

$$i.j_1k_1=-i.k_1j_1=-ik_1.j_1=k_1i.j_1=k_1.ij_1.$$

Hence

$$s'''+v'''=s'''-v'''$$

or $v'''=0$, and the product of the two unit vectors is a scalar. These vectors cannot, then, be independent, or k cannot be independent of $ij=v$. Thus it is proved that a fourth independent vector is impossible, and that ordinary real algebra, ordinary algebra with imaginaries, and real quaternions are the only associative algebras in which division by finites always yields an unambiguous quotient.

IX

*BRIEF DESCRIPTION OF THE ALGEBRA OF RELATIVES**

306. Let A, B, C, etc., denote objects of any kind. These letters may be conceived to be finite in number or innumerable. The sum of them, each affected by a numerical coefficient (which may equal 0), is called an *absolute term.* Let x be such a term; then we write

$$x=(x)_aA+(x)_bB+(x)_cC+\text{etc.}=\Sigma_i(x)_iI.$$

Here (x), etc., are numbers, which may be permitted to be imaginary or restricted to being real or positive, or to being roots of any given equation, algebraic or transcendental.[1] By ϕx, any mathematical function of the absolute term x, we mean such an absolute term that

$$(\phi x)_i=\phi(x)_i.$$

That is, each numerical coefficient of ϕx is the function, ϕ, of the corresponding coefficient of x. In particular,

$$(x+y)_i=(x)_i+(y)_i,$$
$$(x\times y)_i=(x)_i\times(y)_i.$$

Otherwise written,

$$x+y=\{(x)_a+(y)_a\}A+\{(x)_b+(y)_b\}B+\text{etc.}$$
$$x\times y=\{(x)_a\times(y)_a\}A+\{(x)_b\times(y)_b\}B+\text{etc.}$$

307. Two peculiar absolute terms are suggested by the logic of the subject. I call them terms of second intention. The first is zero, 0, and is defined by the equation

$$(0)_i=0$$

or

$$0=0.A+0.B+0.C+\text{etc.}$$

* Dated, Baltimore, January 7, 1882, pp. 1–6, with a postscript of January 16, 1882. To judge from a search through the technical journals and from Peirce's reference in 294, this paper was privately printed.

[1] I have usually restricted the coefficients to one or other of two values: but the more general view was distinctly recognized in my paper of 1870.

The other is *ens* (or non-relative unity), $\overline{0}$, and is defined by the equation

$$(\overline{0})_i = 1,$$

or

$$\overline{0} = A + B + C + \text{etc.}$$

308. The symbol $(A : B)$ is called an *individual dual relative*. It signifies simply a pair of individual objects, $(A : B)$ and $(B : A)$ being different. An aggregate of such symbols, each affected by a numerical coefficient, is called a *general dual relative*. The totality of pairs of letters arrange themselves with obvious naturalness in the block,

$A : A$	$A : B$	$A : C$	etc.
$B : A$	$B : B$	$B : C$	etc.
$C : A$	$C : B$	$C : C$	etc.
etc.	etc.	etc.	etc.

309. If l denotes any general dual relative, then the coefficient of the pair $I : J$ in l is written $(l)_{ij}$. These coefficients are thus each referred to a place in the above block, and may themselves be arranged in the block

$(l)_{aa}$	$(l)_{ab}$	$(l)_{ac}$	etc.
$(l)_{ba}$	$(l)_{bb}$	$(l)_{bc}$	etc.
$(l)_{ca}$	$(l)_{cb}$	$(l)_{cc}$	etc.
etc.	etc.	etc.	etc.

310. Every relative term, x, is separable into a part called 'self-x,' Sx, such that

$$\mathrm{S}x = \Sigma_i (x)_{ii} (I : I)$$

and the remaining part, called 'alio-x,' Vx; comprising all the terms in x not in the principal diagonal of the block; so that we write

$$x = \mathrm{S}x + \mathrm{V}x.^*$$

311. Each absolute term is considered to be equivalent to a certain relative term; namely,

$$A = (A : A) + (A : B) + (A : C) + \text{etc.}$$

or, if x be an absolute term,

$$(x)_{ij} = (x)_i.$$

* Cf. 133.

The self-part of the relative equivalent to an absolute term is denoted by writing a comma after the term. Accordingly,

$$(x,)_{ii}=(x)_i, \qquad (x,)_{ij}=0.$$

312. Besides 0 and $\overline{0}$, two other dual relative terms have been called terms of second intention. These are simply $S\overline{0}$ and $V\overline{0}$. The relative $S\overline{0}$ or $(\overline{0},)$ is also written 1, and is called unity, or 'identical with.' It is defined by the equations

$$(1)_{ii}=1, \qquad (1)_{ij}=0.$$

That is,

$$1=(A:A)+(B:B)+(C:C)+\text{etc.}$$

The relative $V\overline{0}$ is written $\overline{1}$ or $\mathfrak{n}$, and is called 'not,' or 'the negative of.' It is defined by the equations

$$(\overline{1})_{ij}=0,^* \qquad (\overline{1})_{ij}=1.$$

313. By an absolute function of a relative term is meant that function taken according to the rule for taking the function of an absolute term. That is,

$$(\phi x)_{ij}=\phi(x)_{ij}.$$

In particular,

$$(x+y)_{ij}=(x)_{ij}+(y)_{ij}$$
$$(x\times y)_{ij}=(x)_{ij}\times(y)_{ij}$$

314. Of the various external or relative combinations that have been employed the following may be particularly specified.† (1), External multiplication, defined by the equation

$$(xy)_{ij}=\Sigma_n(x)_{in}(y)_{nj}$$

(2), External progressive involution, defined by the equation

$$(x^y)_{ij}=\Pi_n=(x)_{in}^{(y)_{nj}}$$

(3), External regressive involution, defined by the equation

$$({}^xy)_{ij}=\Pi_n(y)_{nj}^{(x)_{in}}.$$

In general, using Miss Ladd's notation[1] for the different orders of multiplication,

$$(x\underset{p}{\times}y)_{ij}=\underset{p-1}{\Pi_n}\left\{(x)_{in}\underset{p}{\times}(y)_{nj}\right\}.$$

* This should be: $(\overline{1})_{ii}=0$.

† Cf. 242.

[1] "On De Morgan's Extension of the Algebraic Processes," *American Journal of Mathematics*, vol. 3, no. 3.

Other modes of external combination have been used, but they are believed to have only a special utility. Division does not generally yield an unambiguous quotient. Indeed, I have shown that it does so only in the cases of ordinary real algebra, of imaginary algebra, and of real quaternions.*

315. Besides the *mathematical* functions of relatives, there are various modes in which one relative may *logically* depend upon another. Thus, Sx and Vx may be said to be logical functions of x. The most important of such operations is that of taking the *converse* of a relative. The converse of x, written $\breve{x}$ or Kx, is defined by the equation

$$(\breve{x})_{ij} = (x)_{ji}.$$

316. The algebraical laws of all these combinations are obtained with great facility by a method of which the following are examples:

Example 1. $\{(xy)z\}_{ij} = \Sigma_n (xy)_{in} (z)_{nj} = \Sigma_n \Sigma_m (x)_{im} (y)_{mn} (z)_{nj}$

$$\{x(yz)\}_{ij} = \Sigma_m (x)_{im} (yz)_{mj} = \Sigma_m \Sigma_n (x)_{im} (y)_{mn} (z)_{nj}$$

$$\therefore \ (xy)z = x(yz).$$

Example 2. $\{(x+y)z\}_{ij} = \Sigma_n (x+y)_{in} (z)_{nj} = \Sigma_n \{(x)_{in} + (y)_{in}\} (z)_{nj}$

$$= \Sigma_n (x)_{in} (z)_{nj} + \Sigma_n (y)_{in} (z)_{nj} = (xz)_{ij} + (yz)_{ij}$$

$$\therefore (x+y)z = xz + yz.$$

The following are some of the elementary formulæ so obtained. Non-relative multiplication is indicated by a comma, relative multiplication by writing the factors one after the other, without the intervention of any sign.

$$(x+y)+z = x+(y+z), \qquad x+y = y+x,$$

$$(x,y),z = x,(y,z), \qquad x,y = y,x,$$

$$(x+y),z = (x,z)+(y,z),$$

$$(xy)z = x(yz),$$

$$(x+y)z = xz+yz, \qquad x(y+z) = xy+xz,$$

$$(x^y)^z = x^{(yz)}, \quad {}^x({}^y z) = {}^{(xy)}z, \qquad {}^x(y^z) = ({}^x y)^z,$$

$$(x,y)^z = (x^z),(y^z), \qquad {}^x(y,z) = ({}^x y),({}^x z),$$

$$x^{y+z} = (x^y),(x^z), \qquad {}^{x+y}z = ({}^x z),({}^y z),$$

$$kkx = x$$

* See 297–305.

$$k(x+y)=kx+ky, \qquad k(x,y)=kx,ky$$

$$k(xy)=(ky)(kx), \qquad k(x^y)={}^{(ky)}(kx)$$

$$0+x=0, \quad 0,x=0x=x0=0, \qquad x^0={}^0x=\bar{0},$$

$$\bar{0}\times x=x, \qquad \bar{0}^x={}^x\bar{0}=\bar{0},$$

$$1x=x1=x^1={}^1x=x,$$

$$({}^x\bar{1})_{ij}=(\dot{\bar{1}}^x)_{ij}=\begin{cases} 0, \text{ if } x_{ij}\neq 0. \\ 1, \text{ if } x_{ij}=0. \end{cases}$$

317. Just as the different pairs of letters, A, B, C, etc., have been conceived to be arranged in a square block, so the different triplets of them may be conceived to be arranged in a cube, and the algebraical sum of all such triplets, each affected with a numerical coefficient, may be called a *triple relative.*

Every dual relative may be regarded as equivalent to a triple relative, just as every absolute term is equivalent to a dual relative.

Every triple relative may be regarded as a sum of five parts, each being a linear expression in terms of one of the five forms,

$$(A:A):A \quad (A:B):A \quad (A:A):B \quad (B:A):A \quad (A:B):C$$

The sign of a dual relative followed by a comma denotes that part of the equivalent triple relative which consists of terms in one of the forms

$$(A:A):(A:A) \qquad (A:B):(A:B).$$

The multiplication of triple relatives is not perfectly associative and the multiplication of two triple relatives yields a quadruple relative.

The modes of combination of a triple relative followed by two dual relatives are the same as the modes of combination of three dual relatives. This ceases to be true for quadruple and higher relatives.

Corresponding to the operation of taking the converse of a dual relative, there are five operations upon triple relatives. They are defined as follows:

$$(Ix)_{ijk}=(x)_{jik}, \quad (Jx)_{ijk}=(x)_{ikj}, \quad (Kx)_{ijk}=(x)_{kji}, \quad (Lx)_{ijk}=(x)_{jki},$$
$$(Mx)_{ijk}=(x)_{kij}.$$

Every quadruple or higher relative may be conceived as a product of triple relatives.

318. Thus, the essential characteristics of this algebra are (1) that it is a multiple algebra depending upon the addition

of square blocks or cubes of numbers, (2) that in the external multiplication the rows of the block of the first factor are respectively multiplied by the columns of the block of the second factor, and (3) that the multiplication so resulting is, for the two-dimensional form of the algebra, always associative. I have proved in a paper presented to the American Academy of Arts and Sciences, May 11, 1875,* that this algebra necessarily embraces every associative algebra.

319. I have here described the algebra apart from the logical interpretation with which it has been clothed. In this interpretation a letter is regarded as a name applicable to one or more objects. By a name is usually meant something representative *of* an object *to* a mind. But I generalize this conception and regard a name as merely something in *a conjoint* relation to a second and a third, that is as a triple relative.† A sum of different individual names is a name for each of the things named severally by the aggregant letters. A name multiplied by a positive integral coefficient is the aggregate of so many different senses in which that name may be taken. The individual relative $A : B$ is the name of A considered as the first member of the pair $A : B$. The signification of the external multiplication is then determined by its algebraical definition.

320. Professor Sylvester, in his "New Universal Multiple Algebra,"‡ appears to have come, by a line of approach totally different from mine, upon a system which coincides, in some of its main features, with the Algebra of Relatives, as described in my four papers upon the subject,[1] and in my lectures on logic.

* 150–51.

† Cf. the analysis of signs, 2.274 and in the present volume, 359f.

‡ There is no paper by this title known to have been published by Sylvester, though a paper entitled 'Lectures on the Principles of Universal Algebra,' was published in the *American Journal of Mathematics*, vol. 6, pp. 270–86, (1884). A number of papers on algebra published in 1881–82 are to be found in the *Collected Mathematical Papers of J. J. Sylvester*, ed. by H. F. Baker, Cambridge University Press, vol. 3 (1904–12).

[1] "Description of a Notation for the Logic of Relatives." *Memoirs, American Academy of Arts and Sciences*, vol. 9, 1870. [III.] "On the Application of Logical Analysis to Multiple Algebra." *Proceedings* of the same Academy, 1875, May 11. [IV.] "Note on Grassman's Calculus of Extension." *Ibid.* 1877, Oct. 10. [V.] "On the Algebra of Logic." *American Journal of Mathematics*, vol. 3. [VI.]

I am unable to judge, from my unprofessional acquaintance with pure mathematics, how much of novelty there may be in my conceptions; but as the researches of the illustrious geometer who has now taken up the subject must draw increased attention to this kind of algebra, I take occasion to redescribe the outlines of my own system, and at the same time to declare my modest conviction that the logical interpretation of it, far from being in any degree special, will be found a powerful instrument for the discovery and demonstration of new algebraical theorems.

321. *Postscript.*— I have this day had the delight of reading for the first time Professor Cayley's *Memoir on Matrices*, in the *Philosophical Transactions* for 1858. The algebra he there describes seems to me substantially identical with my long subsequent algebra for dual relatives. Many of his results are limited to the very exceptional cases in which division is a determinative process.

322. My own studies in the subject have been logical not mathematical, being directed toward the essential elements of the algebra, not towards the solution of problems.

X

*ON THE RELATIVE FORMS OF QUATERNIONS**

323. If X, Y, Z denote the three rectangular components of a vector, and W denote numerical unity (or a fourth rectangular component, involving space of four dimensions), and $(Y:Z)$ denote the operation of converting the Y component of a vector into its Z component, then

$$1=(W:W)+(X:X)+(Y:Y)+(Z:Z)$$
$$i=(X:W)-(W:X)-(Y:Z)+(Z:Y)$$
$$j=(Y:W)-(W:Y)-(Z:X)+(X:Z)$$
$$k=(Z:W)-(W:Z)-(X:Y)+(Y:X).$$

In the language of logic $(Y:Z)$ is a relative term whose relate is a Y component, and whose correlate is a Z component. The law of multiplication is plainly $(Y:Z)(Z:X)=(Y:X)$, $(Y:Z)(X:W)=0$, and the application of these rules to the above values of $1, i, j, k$ gives the quaternion relations

$$i^2=j^2=k^2=-1,\ ijk=-1, \text{ etc.}$$

The symbol $a(Y:Z)$ denotes the changing of Y to Z and the multiplication of the result by a. If the relatives be arranged in the block

$$\begin{matrix} W:W & W:X & W:Y & W:Z \\ X:W & X:X & X:Y & X:Z \\ Y:W & Y:X & Y:Y & Y:Z \\ Z:W & Z:X & Z:Y & Z:Z, \end{matrix}$$

then the quaternion $w+xi+yj+zk$ is represented by the matrix of numbers

$$\begin{matrix} w & -x & -y & -z \\ x & w & -z & y \\ y & z & w & -x \\ z & -y & x & w. \end{matrix}$$

* *Johns Hopkins University Circulars*, No. 13, p. 179, (1882).

The multiplication of such matrices follows the same laws as the multiplication of quaternions. The determinant of the matrix = the fourth power of the tensor of the quaternion.

The imaginary $x+y\sqrt{-1}$ may likewise be represented by the matrix

$$\begin{matrix} x & y \\ -y & x, \end{matrix}$$

and the determinant of the matrix = the square of the modulus.

XI

*ON A CLASS OF MULTIPLE ALGEBRAS**

324. The object of this paper is to show what algebras express all the substitutions of two, of three, and of four letters; and to put these algebras into familiar forms.†

It is evident that every substitution is a relative term. Thus, the transposition of AB to BA is in relative form $(A:B)+(B:A)$, and the circular substitution $\binom{BCA}{ABC}$ is $(B:A)+(C:B)+(A:C)$. In this point of view, we see that substitutions may be added and multiplied by scalars, although the results will usually no longer be substitutions. A group of substitutions may, then, be linear expressions in an associative multiple algebra of a lower order than that of the group.

325. Of two letters, there are two substitutions $(X:X)+(Y:Y)$ and $(X:Y)+(Y:X)$. We may denote these by α and β respectively, so that taking A and B as indeterminate coefficients, the general expression of the algebra is $A\alpha+B\beta$, or in the form of a matrix is

$$\begin{matrix} AB \\ BA. \end{matrix}$$

Assume i and j such that $i=\frac{1}{2}(\alpha+\beta)$

$$j=\tfrac{1}{2}(\alpha-\beta).$$

Then the multiplication table of i and j is as follows:

	i	j
i	i	0
j	0	j

The algebra is, therefore, a mixture of two ordinary simple algebras of B. Peirce's form (a_1).

* *Johns Hopkins University Circulars*, No. 19, pp. 3–4, (1882), read before the University Mathematical Society, October 18, 1882.

† The first and second paragraph were originally transposed.

326. Of three letters, there are *six* substitutions. Let A, B, Γ, Δ, E, Z be indeterminate coefficients. Then, the general expression of the algebra is equivalent to

$$\begin{matrix} A & B & \Gamma \\ \Gamma & A & B \\ B & \Gamma & A \end{matrix} + \begin{matrix} \Delta & E & Z \\ E & Z & \Delta \\ Z & \Delta & E \end{matrix}$$

Or, denoting the six substitutions by α, β, γ, δ, ϵ, ζ, we may write the general expression as $A\alpha+B\beta+\Gamma\gamma+\Delta\delta+E\epsilon+Z\zeta$. There is an equation between these substitutions, namely:

$$\alpha+\beta+\gamma=\delta+\epsilon+\zeta.$$

Assume 5 relatives h, i, j, k, l, such that

$$\begin{aligned} -h&=\tfrac{1}{3}\ (\alpha+\beta+\gamma) \\ i&=\tfrac{1}{2}\ (\alpha-\epsilon) \\ j&=\tfrac{1}{3}\ (\beta-\gamma+\delta-\zeta) \\ k&=\tfrac{1}{4}\ (-\beta+\gamma+\delta-\zeta) \\ l&=\tfrac{1}{6}\ (2\alpha-\beta-\gamma-\delta+2\epsilon-\zeta). \end{aligned}$$

In matricular form,

$$h=\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} \quad 2i=\begin{matrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \quad 3j=\begin{matrix} 1 & 1 & -2 \\ -1 & -1 & +2 \\ 0 & 0 & 0 \end{matrix} \quad 4k=\begin{matrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ -2 & +2 & 0 \end{matrix}$$

$$6l=\begin{matrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & +4. \end{matrix}$$

The multiplication table of h, i, j, k, l, is as follows:

	h	i	j	k	l
h	h	0	0	0	0
i	0	i	j	0	0
j	0	0	0	i	j
k	0	k	l	0	0
l	0	0	0	k	l

The algebra is a mixture of ordinary single algebra (a_1) with the algebra of Hamilton's biquaternions (g_4).

327. In six letters, there are twenty-four substitutions. Using the capital Greek letters for indeterminate coefficients, the general linear expression in these substitutions is equivalent to

$$
\begin{array}{cccc} \mathrm{A} & \Lambda & \Gamma & \mathrm{B} \\ \Gamma & \mathrm{B} & \mathrm{A} & \Delta \\ \mathrm{B} & \Gamma & \Delta & \mathrm{A} \\ \Delta & \mathrm{A} & \mathrm{B} & \Gamma \end{array}
+
\begin{array}{cccc} \mathrm{E} & \mathrm{Z} & \Theta & \mathrm{H} \\ \mathrm{Z} & \mathrm{E} & \mathrm{H} & \Theta \\ \Theta & \mathrm{H} & \mathrm{E} & \mathrm{Z} \\ \mathrm{H} & \Theta & \mathrm{Z} & \mathrm{E} \end{array}
+
\begin{array}{cccc} \mathrm{I} & \Lambda & \mathrm{K} & \mathrm{M} \\ \mathrm{M} & \mathrm{K} & \Lambda & \mathrm{I} \\ \Lambda & \mathrm{I} & \mathrm{M} & \mathrm{K} \\ \mathrm{K} & \mathrm{M} & \mathrm{I} & \Lambda \end{array}
$$

$$
+
\begin{array}{cccc} \mathrm{N} & \mathrm{O} & \Pi & \Xi \\ \mathrm{O} & \mathrm{N} & \Xi & \Pi \\ \Xi & \Pi & \mathrm{O} & \mathrm{N} \\ \Pi & \Xi & \mathrm{N} & \mathrm{O} \end{array}
+
\begin{array}{cccc} \mathrm{P} & \Upsilon & \Sigma & \mathrm{T} \\ \Sigma & \mathrm{T} & \mathrm{P} & \Upsilon \\ \Upsilon & \mathrm{P} & \mathrm{T} & \Sigma \\ \mathrm{T} & \Sigma & \Upsilon & \mathrm{P} \end{array}
+
\begin{array}{cccc} \Phi & \mathrm{X} & \Psi & \Omega \\ \Omega & \Psi & \mathrm{X} & \Phi \\ \Psi & \Omega & \Phi & \mathrm{X} \\ \mathrm{X} & \Phi & \Omega & \Psi \end{array}
$$

Using the twenty-four small Greek letters to denote the twenty-four substitutions, so that the general linear expression is $A\alpha+B\beta+$, etc., an attentive observation of the above scheme will show that the following equations subsist:

$$\alpha+\beta+\gamma+\delta=\epsilon+\zeta+\eta+\theta=\iota+\kappa+\lambda+\mu$$
$$=\nu+\xi+o+\pi=\rho+\sigma+\tau+\upsilon=\phi+\chi+\psi+\omega.$$

Also,

$$
\begin{array}{lll}
\alpha+\beta=\nu+\xi & \epsilon+\zeta=\nu+o & \iota+\kappa=\nu+\pi \\
\alpha+\gamma=\rho+\sigma & \epsilon+\eta=\rho+\tau & \iota+\lambda=\rho+\upsilon \\
\alpha+\delta=\phi+\chi & \epsilon+\theta=\phi+\psi & \iota+\mu=\phi+\omega.
\end{array}
$$

It is plain that these equations are all independent, and not difficult to see that there are no more. Since they are fourteen in number, a ten-fold algebra is required to express the twenty-four substitutions.

Assume the ten relatives $h, i, j, k, l, m, n, o, p, q$, such that

$$
\begin{aligned}
h&=\tfrac{1}{4}\ (\alpha+\beta+\gamma+\delta)\\
i&=\tfrac{1}{4}\ (\epsilon+\zeta-\eta-\theta)\\
j&=\tfrac{1}{4}\ (\alpha-\beta+\gamma-\delta)\\
k&=\tfrac{1}{4}\ (\iota-\kappa-\lambda+\mu)\\
l&=\tfrac{1}{4}\ (\iota-\kappa+\lambda-\mu)\\
m&=\tfrac{1}{4}\ (\epsilon-\zeta-\eta+\theta)\\
n&=\tfrac{1}{4}\ (\alpha+\beta-\gamma-\delta)\\
o&=\tfrac{1}{4}\ (\alpha-\beta-\gamma+\delta)\\
p&=\tfrac{1}{4}\ (\iota+\kappa-\lambda-\mu)\\
q&=\tfrac{1}{4}\ (\epsilon-\zeta+\eta-\theta)
\end{aligned}
$$

In matricular form, these are as follows (where + is written for +1 and − for −1):

$$4h = \begin{matrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{matrix}$$

$$4i = \begin{matrix} + & + & - & - \\ + & + & - & - \\ - & - & + & + \\ - & - & + & + \end{matrix} \qquad 4j = \begin{matrix} + & - & + & - \\ + & - & + & - \\ - & + & - & + \\ - & + & - & + \end{matrix} \qquad 4k = \begin{matrix} + & - & - & + \\ + & - & - & + \\ - & + & + & - \\ - & + & + & - \end{matrix}$$

$$4l = \begin{matrix} + & + & - & - \\ - & - & + & + \\ + & + & - & - \\ - & - & + & + \end{matrix} \qquad 4m = \begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix} \qquad 4n = \begin{matrix} + & - & - & + \\ - & + & + & - \\ + & - & - & + \\ - & + & + & - \end{matrix}$$

$$4o = \begin{matrix} + & + & - & - \\ - & - & + & + \\ - & - & + & + \\ + & + & - & - \end{matrix} \qquad 4p = \begin{matrix} + & - & + & - \\ - & + & - & + \\ - & + & - & + \\ + & - & + & - \end{matrix} \qquad 4q = \begin{matrix} + & - & - & + \\ - & + & + & - \\ - & + & + & - \\ + & - & - & + \end{matrix}$$

The multiplication table is as follows:

	h	*i*	*j*	*k*	*l*	*m*	*n*	*o*	*p*	*q*
h	*h*	0	0	0	0	0	0	0	0	0
i	0	*i*	*j*	*k*	0	0	0	0	0	0
j	0	0	0	0	*i*	*j*	*k*	0	0	0
k	0	0	0	0	0	0	0	*i*	*j*	*k*
l	0	*l*	*m*	*n*	0	0	0	0	0	0
m	0	0	0	0	*l*	*m*	*n*	0	0	0
n	0	0	0	0	0	0	0	*l*	*m*	*n*
o	0	*o*	*p*	*q*	0	0	0	0	0	0
p	0	0	0	0	*o*	*p*	*q*	0	0	0
q	0	0	0	0	0	0	0	*o*	*p*	*q*

The algebra is, therefore, a mixture of ordinary algebra with that of my nonions.

*Of course, the representation of quaternions as linear expressions in substitutions of three letters, the sum of the coefficients being zero, is equivalent to finding *a theorem of plane geometry corresponding to each theorem of solid geometry expressible by quaternions.* For instance, let the three letters which are interchanged be the coördinates x, y, z, of a point in space. Then, as above, let

$$\alpha=\begin{pmatrix}x,y,z\\x,y,z\end{pmatrix}\qquad\beta=\begin{pmatrix}z,x,y\\x,y,z\end{pmatrix}\qquad\gamma=\begin{pmatrix}y,z,x\\x,y,z\end{pmatrix}$$

$$\delta=\begin{pmatrix}x,z,y\\x,y,z\end{pmatrix}\qquad\epsilon=\begin{pmatrix}y,x,z\\x,y,z\end{pmatrix}\qquad\zeta=\begin{pmatrix}z,y,x\\x,y,z\end{pmatrix}.$$

Thus, β and γ represent the operations of rotation through one-third of a circumference, the one forward, the other backward, about an axis passing through the origin and the point (1, 1, 1); while δ, ϵ, ζ represent three perversions with reference to axes passing through the origin and the points $(0, 1, -1)$, $(1, -1, 0)$, and $(1, 0, -1)$, respectively. Quaternions may be represented thus:

$$1=\tfrac{1}{3}\ (2\alpha-\beta-\gamma)$$

$$i=-\tfrac{1}{3}\ (2\epsilon-\delta-\zeta)\sqrt{-1}$$

$$j=\frac{1}{\sqrt{3}}(\beta-\gamma)$$

$$k=\frac{1}{\sqrt{3}}(\delta-\zeta)\sqrt{-1}.$$

We have here a new geometrical interpretation of quaternions. Since the sum of the coefficients of the substitutions is equal to zero in the values of every one of the quaternion elements, it follows that under this interpretation any quaternion operating upon any point brings it into the plane.

$$x+y+z=0.$$

Hence, every quaternion equation has an interpretation relating to points in this plane. The reason why a quaternion,

* The rest of this paper was added on October 30.

which has a four-fold multiplicity, is no more than adequate to expressing operations upon points in space, is that the operations are of such a nature that different ones may have the same effect upon single points. But a *real* quaternion has no greater multiplicity than the real and imaginary points of a plane; and the geometrical effects of different real quaternions upon points in the plane $x+y+z=0$ under the new interpretation are different upon all points except the origin.

For the axes of x, y, z, in trilinear coördinates, take three lines meeting in one point and equally inclined to one another. To plot the point $x=a+b\sqrt{-1}$, $y=c+d\sqrt{-1}$, $z=c+f\sqrt{-1}$, plot the point $x=a$, $y=c$, $z=e$ in *blue*, and the point $x=b$, $y=d$, $d=f$ in *red*. Then the effects of the different quaternion elements upon points in the plane $x+y+z=0$ are as follows: 1 leaves every point unchanged. The vector i reverses the position of a blue point with reference to the line $z=0$ and changes it to red, and reverses the position of a red point with reference to the line $x=y$ and changes it to blue. The vector j rotates every point through a quadrant round the origin in the direction from $x=0$ to $y=z$, without changing the color. The vector k reverses the position of a blue point with reference to the line through the origin that bisects the angle between $y=0$ and $y=z$ and changes it to red, and reverses the position of a red point with reference to the line through the origin that bisects the angle between $x=0$ and $x=z$ and changes it to blue.

XII

*THE LOGIC OF RELATIVES**

328. A dual relative term, such as "lover," "benefactor," "servant," is a common name signifying a pair of objects. Of the two members of the pair, a determinate one is generally the first, and the other the second; so that if the order is reversed, the pair is not considered as remaining the same.

329. Let A, B, C, D, etc., be all the individual objects in the universe; then all the individual pairs may be arrayed in a block, thus:

A : A	A : B	A : C	A : D	etc.
B : A	B : B	B : C	B : D	etc.
C : A	C : B	C : C	C : D	etc.
D : A	D : B	D : C	D : D	etc.
etc.	etc.	etc.	etc.	etc.

A general relative may be conceived as a logical aggregate of a number of such individual relatives. Let l denote "lover"; then we may write

$$l = \Sigma_i \Sigma_j (l)_{ij} (I : J)$$

where $(l)_{ij}$ is a numerical coefficient, whose value is 1 in case I is a lover of J, and 0 in the opposite case, and where the sums are to be taken for all individuals in the universe.

330. Every relative term has a negative (like any other term) which may be represented by drawing a straight line over the sign for the relative itself. The negative of a relative includes every pair that the latter excludes, and vice versa. Every relative has also a *converse*, produced by reversing the order of the members of the pair. Thus, the converse of "lover" is "loved." The converse may be represented by drawing a curved line over the sign for the relative, thus: $\breve{l}$. It is defined by the equation

$$(\breve{l})_{ij} = (l)_{ji}.$$

* *Note B*, pp. 187–203, *Johns Hopkins Studies in Logic*, ed. by C. S. Peirce, Little, Brown & Co., Boston, 1883.

The following formulæ are obvious, but important:

$$\bar{\bar{l}} = l \qquad \breve{\breve{l}} = l$$

$$\bar{\breve{l}} = \breve{\bar{l}}$$

$$(l -\!\!< b) = (\bar{b} -\!\!< \bar{l}) \qquad\qquad (l -\!\!< b) = (\breve{l} -\!\!< \breve{b}).$$

331. Relative terms can be aggregated and compounded like others. Using + for the sign of logical aggregation, and the comma for the sign of logical composition (Boole's multiplication, here to be called non-relative or internal multiplication), we have the definitions

$$(l+b)_{ij} = (l)_{ij} + (b)_{ij}$$

$$(l,b)_{ij} = (l)_{ij} \times (b)_{ij}.$$

The first of these equations, however, is to be understood in a peculiar way: namely, the + in the second member is not strictly addition, but an operation by which

$$0+0=0 \qquad 0+1=1+0=1+1=1.$$

Instead of $(l)_{ij}+(b)_{ij}$, we might with more accuracy write

$$0^{0^{(l)_{ij}+(b)_{ij}}}$$

The main formulæ of aggregation and composition are

$$\left\{ \begin{array}{l} \text{If } l -\!\!< s \text{ and } b -\!\!< s, \text{ then } l+b -\!\!< s. \\ \text{If } s -\!\!< l \text{ and } s -\!\!< b, \text{ then } s -\!\!< l,b. \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{If } l+b -\!\!< s, \text{ then } l -\!\!< s \text{ and } b -\!\!< s. \\ \text{If } s -\!\!< l,b, \text{ then } s -\!\!< l \text{ and } s -\!\!< b. \end{array} \right\}$$

$$\left\{ \begin{array}{l} (l+b),s -\!\!< l,s+b,s. \\ (l+s),(b+s) -\!\!< l,b+s. \end{array} \right\}$$

The subsidiary formulæ need not be given, being the same as in non-relative logic.

332. We now come to the combination of relatives. Of these, we denote two by special symbols; namely, we write

lb for lover of a benefactor,

and

$l\dagger b$ for lover of everything but benefactors.*

The former is called a particular combination, because it implies the *existence* of something *loved by* its relate and a *benefactor of* its correlate. The second combination is said to be *universal*, because it implies the *non-existence* of anything except what is either loved by its relate or a benefactor of its correlate. The combination lb is called a relative product, $l\dagger b$ a relative sum. The l and b are said to be undistributed in both, because if $l -\!\!< s$, then $lb -\!\!< sb$ and $l\dagger b -\!\!< s\dagger b$; and if $b -\!\!< s$, then $lb -\!\!< ls$ and $l\dagger b -\!\!< l\dagger s$.†

333. The two combinations are defined by the equations

$$(lb)_{ij} = \Sigma_x (l)_{ix}(b)_{xj}$$
$$(l\dagger b)_{ij} = \Pi_x \ \{(l)_{ix} + (b)_{xj}\}\ddagger$$

The sign of addition in the last formula has the same signification as in the equation defining non-relative multiplication.§

334. Relative addition and multiplication are subject to the associative law. That is,

$$l\dagger(b\dagger s) = (l\dagger b)\dagger s,$$
$$l(bs) = (lb)s.$$

Two formulæ so constantly used that hardly anything can be done without them are

$$l(b\dagger s) -\!\!< lb\dagger s,$$
$$(l\dagger b)s -\!\!< l\dagger bs.$$

The former asserts that whatever is lover of an object that is benefactor of everything but a servant, stands to everything but servants in the relation of lover of a benefactor. The latter

* As this is not intended to exclude being a lover of a benefactor, but only being a non-lover of a non-benefactor, the following alternative expressions may be clearer: "lover of all non-benefactors"; "a non-lover only of benefactors," or "Either a lover of X or X is a benefactor." Relative addition is the denial of transaddition (243), so that the following are equivalent: $-(\overline{lb})$, $-l_b$, l^{-b}, $l\dagger b$, and $-(l \circ b)$. In 473 it is shown that $\check{l} -\!\!< b$ is also equivalent to the above.

† Cf. 118 (143) (144).

‡ Cf. 352.

§ I.e., it is not exclusive.

asserts that whatever stands to any servant in the relation of lover of everything but its benefactors, is a lover of everything but benefactors of servants. The following formulæ are obvious and trivial:

$$ls+bs \prec (l+b)s^*$$
$$l,b\dagger s \prec (l\dagger s),(b\dagger s).^\dagger$$

Unobvious and important, however, are these:

$$(l+b)s \prec ls+bs^*$$
$$(l\dagger s),(b\dagger s) \prec l,b\dagger s.^\dagger$$

335. There are a number of curious development formulæ.‡ Such are

$$(l,b)s = \Pi_p \{ l(s,p)+b(s,\bar{p}) \}$$
$$l(b,s) = \Pi_p \{ (l,p)b+(l,\bar{p})s \}$$
$$(l+b)\dagger s = \Sigma_p \{ [l\dagger(s+p)],[b\dagger(s+\bar{p})] \}$$
$$l\dagger(b+s) = \Sigma_p \{ [(l+p)\dagger b],[(l+\bar{p})\dagger s] \}.$$

The summations and multiplications denoted by Σ and Π are to be taken non-relatively, and all relative terms are to be successively substituted for p.

336. The negatives of the combinations follow these rules:

$$\overline{l+b} = \bar{l},\ \bar{b} \qquad \overline{l,b} = \bar{l}+\bar{b}$$
$$\overline{l\dagger b} = \bar{l}\bar{b} \qquad \overline{lb} = \bar{l}\dagger\bar{b}$$

337. The converses of combinations are as follows:

$$\overset{\smile}{l+b} = \breve{l}+\breve{b} \qquad \overset{\smile}{l,b} = \breve{l},\ \breve{b}$$
$$\overset{\smile}{l\dagger b} = \breve{b}\dagger\breve{l} \qquad \overset{\smile}{lb} = \breve{b}\breve{l}.$$

338§. Individual dual relatives are of two types,

A : A and A : B.

Relatives containing no pair of an object with itself are called *alio-relatives* as opposed to *self-relatives*. The negatives of alio-

* These two formulæ together show that lovers of servants or benefactors of servants are the same as lovers or benefactors of servants.

† These two formulæ together show that lovers and benefactors of all non-servants are the same as lovers of all non-servants and benefactors of all non-servants.

‡ Cf. 249, 1ii.

§ Cf. 136 and 227.

relatives pair every object with itself. Relatives containing no pair of an object with anything but itself are called *concurrents* as opposed to *opponents*. The negatives of concurrents pair every object with every other.

339. There is but one relative which pairs every object with itself and with every other. It is the aggregate of all pairs, and is denoted by ∞. It is translated into ordinary language by "coexistent with." Its negative is 0. There is but one relative which pairs every object with itself and none with any other. It is

$$(A : A) + (B : B) + (C : C) + \text{etc.};$$

is denoted by 1, and in ordinary language is "identical with —." Its negative, denoted by $\mathfrak{n}$, is "other than —," or "not."

340. No matter what relative term x may be, we have

$$0 \prec x \qquad\qquad x \prec \infty.$$

341. Hence, obviously

$$x + 0 = x \qquad\qquad x, \infty = x$$
$$x + \infty = \infty \qquad\qquad x, 0 = 0.$$

The last formulæ hold for the relative operations; thus,

$$x \dagger \infty = \infty \qquad\qquad x0 = 0.$$
$$\infty \dagger x = \infty \qquad\qquad 0x = 0.$$

The formulæ

$$x + 0 = x \qquad\qquad x, \infty = x$$

also hold if we substitute the relative operations, and also 1 for ∞, and $\mathfrak{n}$ for 0; thus,

$$x \dagger \mathfrak{n} = x \qquad\qquad x\,1 = x.$$
$$\mathfrak{n} \dagger x = x \qquad\qquad 1\,x = x.$$

We have also

$$l + \bar{l} = \infty \qquad\qquad l, \bar{l} = 0.$$[1]

[1] Sometimes important. $1 \prec l + \check{l}$ [i.e., identity implies to be either a lover or not loved by] and $l, \check{l} \prec \mathfrak{n}$ [i.e., to love and not be loved by implies otherness] — marginal note.

To these partially correspond the following pair of highly important formulæ:

$$1 \prec l \dagger \breve{l} \text{*} \qquad l\breve{l} \prec \mathfrak{n}. \dagger$$

342. The logic of relatives is highly multiform; it is characterized by innumerable immediate inferences, and by various distinct conclusions from the same sets of premisses. An example of the first character is afforded by Mr. Mitchell's F_{1v} following from $F_{1v'}$.‡ As an instance of the second, take the premisses,

Every man is a lover of an animal;

and

Every woman is a lover of a non-animal.

From these we can equally infer that

Every man is a lover of something which stands to each woman in the relation of not being the only thing loved by her, and that

Every woman is a lover of something which stands to each man in the relation of not being the only thing loved by him.

The effect of these peculiarities is that this algebra cannot be subjected to hard and fast rules like those of the Boolian calculus; and all that can be done in this place is to give a general idea of the way of working with it. The student must at the outset disabuse himself of the notion that the chief instruments of algebra are the inverse operations. General algebra hardly knows any inverse operations. When an inverse operation is identical with a direct operation with an inverse quantity (as subtraction is the addition of the negative, and as division is multiplication by the reciprocal), it is useful; otherwise it is almost always useless. In ordinary algebra, we speak of the "principal value" of the logarithm, etc., which is a direct operation substituted for an indefinitely ambiguous inverse operation. The elimination and transposition in this algebra really does depend, however, upon formulæ quite analogous to the

$$x+(-x)=0 \qquad x\times\frac{1}{x}=1,$$

* I.e., Identity implies to love all that is loved by.

† I.e., to love something that is not loved by implies otherness.

‡ "On a New Algebra of Logic," *Studies in Logic*, pp. 87, 88.

of arithmetical algebra. These formulæ are

$$l,\bar{l}=0 \qquad l\breve{l} \prec \mathfrak{n}$$
$$l+\bar{l}=\infty \qquad 1 \prec l\dagger\breve{l}.$$

For example, to eliminate s from the two propositions

$$1 \prec l\bar{s} \qquad 1 \prec \breve{s}b,$$

we relatively multiply them in such an order as to bring the two s's together, and then apply the second of the above formulæ, thus:

$$1 \prec l\bar{s}\breve{s}b \prec l\mathfrak{n}b.$$

This example shows the use of the association formulæ in bringing letters together. Other formulæ of great importance for this purpose are

$$(b\dagger l)s \prec b\dagger ls \qquad b(l\dagger s) \prec bl\dagger s.$$

The distribution formulæ are also useful for this purpose.

343. When the letter to be eliminated has thus been replaced by one of the four relatives — $0, \infty, 1, \mathfrak{n}$ — the replacing relative can often be got rid of by means of one of the formulæ

$$l+0=l \qquad l,\infty=l$$
$$l\dagger\mathfrak{n}=\mathfrak{n}\dagger l=l \qquad l1=1l=l.$$

344. When we have only to deal with universal propositions, it will be found convenient so to transpose everything from subject to predicate as to make the subject 1. Thus, if we have given $l \prec b$, we may relatively add $\breve{l}$ to both sides; whereupon we have

$$1 \prec l\dagger\breve{l} \prec b\dagger\breve{l}.$$

Every proposition will then be in one of the forms

$$1 \prec b\dagger l \qquad 1 \prec bl.$$

With a proposition of the form $1 \prec b\dagger l$, we have the right (1) to transpose the terms, and (2) to convert the terms. Thus, the following are equivalent:

$$1 \prec b\dagger l$$
$$1 \prec l\dagger b \qquad 1 \prec \breve{b}\dagger\breve{l}$$
$$1 \prec \breve{l}\dagger\breve{b}.$$

With a proposition of the form $1 \prec bl$, we have only the right to convert the predicate giving $1 \prec \breve{l}\breve{b}$.

With three terms, there are four forms of universal propositions, namely:

$$1 -\!\!< l \dagger b \dagger s \qquad 1 -\!\!< l(b \dagger s) \qquad 1 -\!\!< lb \dagger s \qquad 1 -\!\!< lbs.$$

Of these, the third is an immediate inference from the second.

345. By way of illustration, we may work out the syllogisms whose premisses are the propositions of the first order referred to in Note A.* Let a and c be class terms, and let β be a group of characters. Let p be the relative "possessing as a character." The non-relative terms are to be treated as relatives — a, for instance, being considered as "a coexistent with" and $\breve{a}$ as "coexistent with a that is." Then, the six forms of affirmative propositions of the first order are

$$1 -\!\!< \breve{a} \dagger p \dagger \beta$$
$$1 -\!\!< \breve{a}(p \dagger \beta) \qquad 1 -\!\!< (\breve{a} \dagger p)\beta$$
$$1 -\!\!< \breve{a}p \dagger \beta \qquad 1 -\!\!< \breve{a} \dagger p\beta$$
$$1 -\!\!< \breve{a}p\beta.\dagger$$

346. The various kinds of syllogism‡ are as follows:

1. *Premisses:* $1 -\!\!< \breve{a} \dagger p \dagger \beta \qquad 1 -\!\!< \breve{c} \dagger p \dagger \bar{\beta}.$

Convert one of the premisses and multiply,

$$1 -\!\!< (\breve{a} \dagger p \dagger \beta)(\check{\breve{\beta}} \dagger \not{p} \dagger c) -\!\!< \breve{a} \dagger p \dagger \beta\check{\breve{\beta}} \dagger \not{p} \dagger c$$
$$-\!\!< \breve{a} \dagger p \dagger \mathfrak{n} \dagger \not{p} \dagger c -\!\!< \breve{a} \dagger p \dagger \not{p} \dagger c.$$

The treatment would be the same if one or both of the premisses were negative: that is, contained $\bar{p}$ in place of p.

* 2. 521ff.

† The present paper was rewritten to serve as the thirteenth chapter of the Grand Logic of 1893–94, where the above six propositions are replaced by the following five:

$\check{\breve{a}} \dagger p \dagger \bar{\beta}$	Every a has every β
$\breve{a}(p \dagger \bar{\beta})$	Some a has all β
$\breve{a}p \dagger \bar{\beta}$	Every β is some a or other
$(\check{\breve{a}} \dagger p)\beta$	Some β is all a
$\check{\breve{a}} \dagger p\beta$	Some β is in some a;

and the various syllogisms are modified accordingly. There are no other changes of consequence. These are reduced to four by Schröder, *Algebra der Logik*, III, 1, 470.

‡ For a reduction in the number of these forms see Schröder, *ibid.*, 470f.

2. *Premisses:* $1 \prec \breve{a} \dagger p \dagger \beta \qquad 1 \prec \breve{c}(p \dagger \bar{\beta})$.

We have

$$1 \prec (\breve{a} \dagger p \dagger \beta)(\breve{\beta} \dagger \not{p})c \prec (\breve{a} \dagger p \dagger \not{p})c.$$

The same with negatives.

3. *Premisses:* $1 \prec \breve{a}(p \dagger \beta) \qquad 1 \prec \breve{c}(p \dagger \bar{\beta})$.

$$1 \prec \breve{a}(p \dagger \beta)(\breve{\beta} \dagger \not{p})c \prec \breve{a}(p \dagger \not{p})c.$$

The same with negatives.

4. *Premisses:* $1 \prec \breve{a} \dagger p \dagger \beta \qquad 1 \prec (\breve{c} \dagger p)\bar{\beta}$.

$$1 \prec (\breve{a} \dagger p \dagger \beta)\breve{\beta}(\not{p} \dagger c) \prec (\breve{a} \dagger p \dagger \beta\breve{\beta})(\not{p} \dagger c) \prec (\breve{a} \dagger p)(\not{p} \dagger c).$$

If one of the premisses, say the first, were negative, we should obtain a similar conclusion —

$$1 \prec (\breve{a} \dagger \bar{p})(\not{p} \dagger c);$$

but from this again p could be eliminated, giving

$$1 \prec \breve{a} \dagger c, \quad \text{or} \quad \bar{a} \prec c.$$

5. *Premisses:* $1 \prec \breve{a}(p \dagger \beta) \qquad 1 \prec (\breve{c} \dagger p)\bar{\beta}$.

$$1 \prec \breve{a}(p \dagger \beta)\beta(\not{p} \dagger c) \prec \breve{a}p(\not{p} \dagger c).$$

If either premiss were negative, p could be eliminated, giving $1 \prec \breve{a}\, c$, or some a is c.

6. *Premisses:* $1 \prec (\breve{a} \dagger p)\beta \quad 1 \prec (\breve{c} \dagger p)\bar{\beta}$.

$$1 \prec (\breve{a} \dagger p)\beta\breve{\beta}(\not{p} \dagger c) \prec (\breve{a} \dagger p)\mathfrak{n}(\not{p} \dagger c).$$

7. *Premisses:* $1 \prec \breve{a} \dagger p \dagger \beta \quad 1 \prec \breve{c}p \dagger \bar{\beta}$.

$$1 \prec (\breve{a} \dagger p \dagger \beta)(\breve{\beta} \dagger \not{p}c) \prec \breve{a} \dagger p \dagger \not{p}c.$$

8. *Premisses:* $1 \prec \breve{a}(p \dagger \beta) \quad 1 \prec \breve{c}p \dagger \bar{\beta}$.

$$1 \prec \breve{a}(p \dagger \beta)(\breve{\beta} \dagger \not{p}c) \prec \breve{a}(p \dagger \not{p}c).$$

9. *Premisses:* $1 \prec (\breve{a} \dagger p)\beta \quad 1 \prec \breve{c}p \dagger \bar{\beta}$.

$$1 \prec (\breve{a} \dagger p)\beta(\breve{\beta} \dagger \not{p}c) \prec (\breve{a} \dagger p)\not{p}c.$$

If one premiss is negative, we have the further conclusion $1 \prec \breve{a}c$.

10. *Premisses:* $1 \prec \breve{a}p \dagger \beta \quad 1 \prec \breve{c}p \dagger \bar{\beta}$.

$$1 \prec (\breve{a}p \dagger \beta)(\breve{\beta} \dagger \not{p}c) \prec \breve{a}p \dagger \not{p}c.$$

11. *Premisses:* $1 -\!\!< \breve{a}\dagger p\dagger\beta \qquad 1 -\!\!< \breve{c}\dagger p\bar{\beta}$.

$$1 -\!\!< (\breve{a}\dagger p\dagger\beta)(\check{\beta}\not{p}\dagger c) -\!\!< (\breve{a}\dagger p)\not{p}\dagger c.$$

We might also conclude

$$1 -\!\!< \breve{a}\dagger p\dagger\mathfrak{n}\not{p}\dagger c;$$

but this conclusion is an immediate inference from the other; for

$$(\breve{a}\dagger p)\not{p}\dagger c -\!\!< (\breve{a}\dagger p)(1\dagger\mathfrak{n})\not{p}\dagger c -\!\!< (\breve{a}\dagger p)1\dagger\mathfrak{n}\not{p}\dagger c -\!\!< \breve{a}\dagger p\dagger\mathfrak{n}\not{p}\dagger c.$$

If one premiss is negative, we have the further conclusion $1 -\!\!< \breve{a}\dagger c$.

12. *Premisses:* $1 -\!\!< \breve{a}(p\dagger\beta) \qquad 1 -\!\!< \breve{c}\dagger p\bar{\beta}$.

$$1 -\!\!< \breve{a}(p\dagger\beta)(\check{\beta}\not{p}\dagger c) -\!\!< \breve{a}(p\not{p}\dagger c).$$

If one premiss is negative, we have the further inference $1 -\!\!< \breve{a}c$.

13. *Premisses:* $1 -\!\!< (\breve{a}\dagger p)\beta \qquad 1 -\!\!< \breve{c}\dagger p\bar{\beta}$.

$$1 -\!\!< (\breve{a}\dagger p)\beta(\check{\beta}\not{p}\dagger c) -\!\!< (\breve{a}\dagger p)(\mathfrak{n}\not{p}\dagger c).$$

14. *Premisses:* $1 -\!\!< \breve{a}p\dagger\beta \qquad 1 -\!\!< \breve{c}\dagger p\bar{\beta}$.

$$1 -\!\!< (\breve{a}p\dagger\beta)(\check{\beta}\not{p}\dagger c) -\!\!< \breve{a}p\not{p}\dagger c.$$

If one premiss is negative, we have the further spurious* inference $1 -\!\!< \breve{a}\mathfrak{n}\dagger c$.

15. *Premisses:* $1 -\!\!< \breve{a}\dagger p\beta \qquad 1 -\!\!< \breve{c}\dagger p\bar{\beta}$.

$$1 -\!\!< (\breve{a}\dagger p\beta)(\check{\beta}\not{p}\dagger c) -\!\!< \breve{a}\dagger p(\mathfrak{n}\not{p}\dagger c).$$

We can also infer $1 -\!\!< (\breve{a}\dagger p\mathfrak{n})\not{p}\dagger c$.

16. *Premisses:* $1 -\!\!< \breve{a}\dagger p\dagger\beta \qquad 1 -\!\!< \breve{c}p\bar{\beta}$.

$$1 -\!\!< (\breve{a}\dagger p\dagger\beta)\check{\beta}\not{p}c -\!\!< (\breve{a}\dagger p)\not{p}c.$$

If one premiss is negative, we can further infer $1 -\!\!< \breve{a}c$.

17. *Premisses:* $1 -\!\!< \breve{a}(p\dagger\beta) \qquad 1 -\!\!< \breve{c}p\bar{\beta}$.

$$1 -\!\!< \breve{a}(p\dagger\beta)\check{\beta}\not{p}c -\!\!< \breve{a}p\not{p}c.$$

If one premiss is negative, we have the further spurious conclusion $1 -\!\!< \breve{a}\mathfrak{n}c$.

18. *Premisses:* $1 -\!\!< (\breve{a}\dagger p)\beta \qquad 1 -\!\!< \breve{c}p\bar{\beta}$.

$$1 -\!\!< (\breve{a}\dagger p)\beta\check{\beta}\not{p}c -\!\!< (\breve{a}\dagger p)\mathfrak{n}\not{p}c.$$

* See 2.526n for definition of this term.

19. *Premisses:* $1 \prec \breve{a} p \dagger \beta \qquad 1 \prec \breve{c} p \bar{\beta}.$

$$1 \prec (\breve{a} p \dagger \beta) \breve{\bar{\beta}} \breve{p} c \prec \breve{a} p \breve{p} c.$$

If one premiss is negative, we further conclude $1 \prec \breve{a} \mathfrak{n} c$.

20. *Premisses:* $1 \prec \breve{a} \dagger p \beta \qquad 1 \prec \breve{c} p \bar{\beta}.$

$$1 \prec (\breve{a} \dagger p \beta) \breve{\bar{\beta}} \breve{p} c \prec (\breve{a} \dagger p \mathfrak{n}) \breve{p} c.$$

21. *Premisses:* $1 \prec \breve{a} p \beta \qquad 1 \prec \breve{c} p \bar{\beta}.$

$$1 \prec \breve{a} p \beta \breve{\bar{\beta}} \breve{p} c \prec \breve{a} p \mathfrak{n} \breve{p} c.$$

347. When we have to do with particular propositions, we have the proposition $\infty \overline{\prec} 0$, or "something exists"; for every particular proposition implies this. Then every proposition can be put into one or other of the four forms

$$\infty \prec 0 \dagger l \dagger 0$$
$$\infty \prec (0 \dagger l) \infty$$
$$\infty \prec 0 \dagger l \infty$$
$$\infty \prec \infty l \infty .$$

Each of these propositions immediately follows from the one above it. The *enveloped* expressions which form the predicates have the remarkable property that each is either 0 or ∞. This fact gives extraordinary freedom in the use of the formulæ. In particular, since if anything not zero is included under such an expression, the whole universe is included, it will be quite unnecessary to write the $\infty \prec$ which begins every proposition.

348. Suppose that f and g are general relatives signifying relations of things to times. Then, Dr. Mitchell's* six forms of two dimensional propositions appear thus:

$$F_{11} = 0 \dagger f \dagger 0$$
$$F_{1v} = 0 \dagger f \infty$$
$$F_{u1} = \infty f \dagger 0$$
$$F_{1v'} = (0 \dagger f) \infty$$
$$F_{u'1} = \infty (f \dagger 0)$$
$$F_{uv} = \infty f \infty .$$

It is obvious that $l \dagger 0 \prec l$, for

$$l \dagger 0 \prec (l \dagger 0) \infty \prec l \dagger 0 \, \infty l \prec l \dagger \mathfrak{n} \prec l.$$

* *Op. cit.*, p. 87.

If then we have $0 \dagger f \dagger 0$ as one premiss, and the other contains g, we may substitute for g the product (f, g).

$$g \prec g, \infty \prec g, (0 \dagger f \dagger 0) \prec g, f.$$

349. From the two premisses

$$\infty\,(f \dagger 0) \qquad \text{and} \qquad 0 \dagger g \infty\,,$$

by the application of the formulæ

$$ls, (b \dagger \bar{s}) \prec (l, b)s$$
$$sl, (\bar{s} \dagger b) \prec s(l, b),$$

we have

$$\{\,\infty\,(f \dagger 0)\,\}, (0 \dagger g \infty) \prec \infty\,\{\,(f \dagger 0), g \infty\,\} \prec \infty\,(f, g)\infty\,.$$

These formulæ give the first column of Dr. Mitchell's rule on page 90.

350. The following formulæ may also be applied:

1. $(0 \dagger f \dagger 0), (0 \dagger g \dagger 0) = 0 \dagger (f, g) \dagger 0.$
2. $(0 \dagger f)\infty \;\; (0 \dagger \breve{g} \dagger 0) \prec (0 \dagger f)(\breve{g} \dagger 0).$
3. $(0 \dagger f)\infty \;\; \infty\,(\breve{g} \dagger 0) = (0 \dagger f)(\breve{g} \dagger 0) + (0 \dagger f)\,\mathfrak{n}\,(\breve{g} \dagger 0).$
4. $(0 \dagger f)\infty \;\; (0 \dagger \breve{g})\infty \prec (0 \dagger f)\breve{g}\infty\,.$
5. $(0 \dagger f \dagger 0)\;(0 \dagger \breve{g}\infty) = 0 \dagger (\breve{g}f, f) \dagger 0.$
6. $(0 \dagger f)\infty \;\; (0 \dagger \breve{g}\infty) = (0 \dagger \breve{g}f, f)\infty\,.$
7. $(0 \dagger f)\infty, (0 \dagger g\infty) = (0 \dagger f, g\infty)\infty\,.$
8. $(0 \dagger f\infty)\;(0 \dagger g\infty) = 0 \dagger (f\breve{g}, \breve{g}f)\infty\,.$
9. $(0 \dagger f\infty), (0 \dagger g\infty) = 0 \dagger f\infty, g\infty\,.$
10. $(0 \dagger f \dagger 0)\infty\,\breve{g}\infty = 0 \dagger (f\breve{g}f, f) \dagger 0.$
11. $(0 \dagger f)\infty \;\; \infty\,\breve{g}\infty = (0 \dagger f)\breve{g}\infty + (0 \dagger f)\mathfrak{n}\breve{g}\infty\,.$
12. $(0 \dagger f\infty) \;\; \infty\,\breve{g}\infty = (0 \dagger f\breve{g}\infty) + (0 \dagger f\mathfrak{n}\breve{g}\infty).$
13. $\infty f\infty \;\; \infty\,\breve{g}\infty = \infty f\,\breve{g}\infty + \infty f\mathfrak{n}\breve{g}\infty\,.$

351. When the relative and non-relative operations occur together, the rules of the calculus become pretty complicated. In these cases, as well as in such as involve *plural* relations (subsisting between three or more objects), it is often advantageous to recur to the numerical coefficients mentioned in 329. Any proposition whatever is equivalent to saying that some complexus of aggregates[1] and products of such numerical

[1] The sums of 331.

coefficients is greater than zero. Thus,

$$\Sigma_i \Sigma_j l_{ij} > 0$$

means that something is a lover of something; and

$$\Pi_i \Sigma_j l_{ij} > 0$$

means that everything is a lover of something. We shall, however, naturally omit, in writing the inequalities, the >0 which terminates them all; and the above two propositions will appear as

$$\Sigma_i \Sigma_j l_{ij} \quad \text{and} \quad \Pi_i \Sigma_j l_{ij}.$$

352. The following are other examples:

$$\Pi_i \Sigma_j (l)_{ij} (b)_{ij}$$

means that everything is at once a lover and a benefactor of something.

$$\Pi_i \Sigma_j (l)_{ij} (b)_{ji}$$

means that everything is a lover of a benefactor of itself.

$$\Sigma_i \Sigma_k \Pi_j (l_{ij} + b_{jk})$$

means that there is something which stands to something in the relation of loving everything except benefactors of it.*

353. Let α denote the triple relative "accuser to — of —," and ϵ the triple relative "excuser to — of —. Then,

$$\Sigma_i \Pi_j \Sigma_k (\alpha)_{ijk} (\epsilon)_{jki}$$

means that an individual i can be found, such, that taking any individual whatever, j, it will always be possible so to select a third individual, k, that i is an accuser to j of k, and j an excuser to k of i.

354. Let π denote "preferrer to — of —." Then,

$$\Pi_i \Sigma_j \Sigma_k (\alpha)_{ijk} (\epsilon_{jki} + \pi_{kij})$$

means that, having taken any individual i whatever, it is always possible so to select two, j and k, that i is an accuser to j of k, and also is either excused by j to k or is something to which j is preferred by k.

* More clearly: For some i and k, i is a lover of all non-benefactors of k; or for some i and some k every j is such that either i loves j or j is a benefactor of k.

355. When we have a number of premisses expressed in this manner, the conclusion is readily deduced by the use of the following simple rules. In the first place, we have

$$\Sigma_i \Pi_j \prec \Pi_j \Sigma_i.$$

In the second place, we have the formulæ

$$\{\Pi_i \varphi(i)\}\{\Pi_j \psi(j)\} = \Pi_i\{\varphi(i)\cdot\psi(i)\}.$$
$$\{\Pi_i \varphi(i)\}\{\Sigma_j \psi(j)\} \prec \Sigma_i\{\varphi(i)\cdot\psi(i)\}.$$

In the third place, since the numerical coefficients are all either *zero* or *unity*, the Boolian calculus is applicable to them.

356. The following is one of the simplest possible examples. Required to eliminate *servant* from these two premisses:

First premiss. There is somebody who accuses everybody to everybody, unless the latter is loved by some person that is servant of all not accused to him.*

Second premiss. There are two persons, the first of whom excuses everybody to everybody, unless the unexcused be benefited by, without the person to whom he is unexcused being a servant of, the second.

These premisses may be written thus:

$$\Sigma_h \Pi_i \Sigma_j \Pi_k (a_{hik} + s_{jk} l_{ji}).$$
$$\Sigma_u \Sigma_v \Pi_x \Pi_y (\epsilon_{uyx} + \bar{s}_{yv} b_{vx}).$$

The second yields the immediate inference,

$$\Pi_x \Sigma_u \Pi_y \Sigma_v (\epsilon_{uyx} + \bar{s}_{yv} b_{vx}).$$

Combining this with the first, we have

$$\Sigma_x \Sigma_u \Sigma_y \Sigma_v (\epsilon_{uyx} + \bar{s}_{yv} b_{vx})(a_{xuv} + s_{yv} l_{yu}).$$

Finally, applying the Boolian calculus, we deduce the desired conclusion

$$\Sigma_x \Sigma_u \Sigma_y \Sigma_v (\epsilon_{uyx} a_{xuv} + \epsilon_{uyx} l_{yu} + a_{xuv} b_{vx}).$$

The interpretation of this is that either there is somebody excused by a person to whom he accuses somebody, or somebody excuses somebody to his (the excuser's) lover, or somebody accuses his own benefactor.

* This sentence is slightly different from the original, in accordance with Peirce's correction.

357. The procedure may often be abbreviated by the use of operations intermediate between Π and Σ. Thus, we may use Π', Π'', etc. to mean the products for all individuals except one, except two, etc. Thus,

$$\Pi_i{}'\Pi_j{}''(l_{ij} + b_{ji})$$

will mean that every person except one is a lover of everybody except its benefactors, and at most two non-benefactors. In the same manner, Σ', Σ'', etc., will denote the sums of all products of two, of all products of three, etc. Thus,

$$\Sigma_i{}''(l_{ii})$$

will mean that there are at least three things in the universe that are lovers of themselves. It is plain that if $m<n$, we have

$$\Pi^m \prec \Pi^n \qquad \Sigma^n \prec \Sigma^m.$$

$$(\Pi_i{}^m \varphi i)(\Sigma_j{}^n \psi j) \prec \Sigma_i{}^{n-m}(\varphi i \cdot \psi i)$$

$$(\Pi_i{}^m \varphi i)(\Pi_j{}^n \psi j) \prec \Pi_i{}^{m+n}(\varphi i \cdot \psi i)$$

358. Mr. Schlötel has written to the London Mathematical Society,* accusing me of having, in my *Algebra of Logic*, plagiarized from his writings. He has also written to me to inform me that he has read that Memoir with "heitere Ironie," and that Professor Drobisch, the Berlin Academy, and I constitute a "liederliche Kleeblatt," with many other things of the same sort. Up to the time of publishing my Memoir, I had never seen any of Mr. Schlötel's writings; I have since procured his *Logik*,† and he has been so obliging as to send me two cuttings from his papers, thinking, apparently, that I might be curious to see the passages that I had appropriated. But having examined these productions, I find no thought in them that I ever did, or ever should be likely to put forth as my own.

* No record of this communication has been found in the official *Proceedings* of the London Mathematical Society; nor has any letter from Mr. Schlötel come to hand in a search through Peirce's correspondence.

† *Die Logik, neu bearbeitet*, 1854.

XIII

ON THE ALGEBRA OF LOGIC

A CONTRIBUTION TO THE PHILOSOPHY OF NOTATION*

§1. THREE KINDS OF SIGNS†

359. Any character or proposition either concerns one subject, two subjects, or a plurality of subjects. For example, one particle has mass, two particles attract one another, a particle revolves about the line joining two others. A fact concerning two subjects is a dual character or relation; but a relation which is a mere combination of two independent facts concerning the two subjects may be called *degenerate*, just as two lines are called a degenerate conic. In like manner a plural character or conjoint relation is to be called degenerate if it is a mere compound of dual characters.

360. A sign is in a conjoint relation to the thing denoted and to the mind. If this triple relation is not of a degenerate species, the sign is related to its object only in consequence of a mental association, and depends upon a habit. Such signs are always abstract and general, because habits are general rules to which the organism has become subjected. They are, for the most part, conventional or arbitrary. They include all general words, the main body of speech, and any mode of conveying a judgment. For the sake of brevity I will call them *tokens*.‡

361. But if the triple relation between the sign, its object, and the mind, is degenerate, then of the three pairs

sign	object
sign	mind
object	mind

* *The American Journal of Mathematics*, vol. 7, No. 2, pp. 180–202, (1885); reprinted pp. 1–23.

† See vol. 2, bk. II, for a detailed analysis of signs.

‡ More frequently called 'symbols'; the word 'token' is later (in 4.537) taken to apply to what in 2.245 is called a 'sinsign.'

two at least are in dual relations which constitute the triple relation. One of the connected pairs must consist of the sign and its object, for if the sign were not related to its object except by the mind thinking of them separately, it would not fulfill the function of a sign at all. Supposing, then, the relation of the sign to its object does not lie in a mental association, there must be a direct dual relation of the sign to its object independent of the mind using the sign. In the second of the three cases just spoken of, this dual relation is not degenerate, and the sign signifies its object solely by virtue of being really connected with it. Of this nature are all natural signs and physical symptoms. I call such a sign an *index*, a pointing finger being the type of the class.

The index asserts nothing; it only says "There!" It takes hold of our eyes, as it were, and forcibly directs them to a particular object, and there it stops. Demonstrative and relative pronouns are nearly pure indices, because they denote things without describing them; so are the letters on a geometrical diagram, and the subscript numbers which in algebra distinguish one value from another without saying what those values are.

362. The third case is where the dual relation between the sign and its object is degenerate and consists in a mere resemblance between them. I call a sign which stands for something merely because it resembles it, an *icon*. Icons are so completely substituted for their objects as hardly to be distinguished from them. Such are the diagrams of geometry. A diagram, indeed, so far as it has a general signification, is not a pure icon; but in the middle part of our reasonings we forget that abstractness in great measure, and the diagram is for us the very thing. So in contemplating a painting, there is a moment when we lose the consciousness that it is not the thing, the distinction of the real and the copy disappears, and it is for the moment a pure dream — not any particular existence, and yet not general. At that moment we are contemplating an *icon*.

363. I have taken pains to make my distinction[1] of icons, indices, and tokens clear, in order to enunciate this proposi-

[1] See *Proceedings, American Academy of Arts and Sciences*, vol. 7, p. 294, May 14, 1867. [1.558.]

tion: in a perfect system of logical notation signs of these several kinds must all be employed. Without tokens there would be no generality in the statements, for they are the only general signs; and generality is essential to reasoning. Take, for example, the circles by which Euler represents the relations of terms. They well fulfill the function of icons, but their want of generality and their incompetence to express propositions must have been felt by everybody who has used them.* Mr. Venn† has, therefore, been led to add shading to them; and this shading is a conventional sign of the nature of a token. In algebra, the letters, both quantitative and functional, are of this nature. But tokens alone do not state what is the subject of discourse; and this can, in fact, not be described in general terms; it can only be indicated. The actual world cannot be distinguished from a world of imagination by any description. Hence the need of pronoun and indices, and the more complicated the subject the greater the need of them. The introduction of indices into the algebra of logic is the greatest merit of Mr. Mitchell's system.[1] He writes F_1 to mean that the proposition F is true of every object in the universe, and F_u to mean that the same is true of some object.‡ This distinction can only be made in some such way as this. Indices are also required to show in what manner other signs are connected together. With these two kinds of signs alone any proposition can be expressed; but it cannot be reasoned upon, for reasoning consists in the observation that where certain relations subsist certain others are found, and it accordingly requires the exhibition of the relations reasoned within an icon. It has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. Various have been the attempts to solve the paradox by breaking down one or other of these assertions, but without success. The truth,

* See 4.356.

† "On the Diagrammatic and Mechanical Representations of Propositions and Reasoning." *Philosophical Magazine*, ser. 5, vol. 10, pp. 1–15 (1880).

[1] *Studies in Logic*, by members of the Johns Hopkins University; Boston, Little, Brown and Co., 1883.

‡ *Ibid.*, p. 74.

however, appears to be that all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts. For instance, take the syllogistic formula,

All M is P
S is M
$\therefore$ S is P.

This is really a diagram of the relations of S, M, and P. The fact that the middle term occurs in the two premisses is actually exhibited, and this must be done or the notation will be of no value. As for algebra, the very idea of the art is that it presents formulæ which can be manipulated, and that by observing the effects of such manipulation we find properties not to be otherwise discerned. In such manipulation, we are guided by previous discoveries which are embodied in general formulæ. These are patterns which we have the right to imitate in our procedure, and are the *icons par excellence* of algebra. The letters of applied algebra are usually tokens, but the x, y, z, etc., of a general formula, such as

$$(x+y)z=xz+yz,$$

are blanks to be filled up with tokens, they are indices of tokens. Such a formula might, it is true, be replaced by an abstractly stated rule (say that multiplication is distributive); but no application could be made of such an abstract statement without translating it into a sensible image.

364. In this paper, I purpose to develop an algebra adequate to the treatment of all problems of deductive logic, showing as I proceed what kinds of signs have necessarily to be employed at each stage of the development. I shall thus attain three objects. The first is the extension of the power of logical algebra over the whole of its proper realm. The second is the illustration of principles which underlie all algebraic notation. The third is the enumeration of the essentially different kinds of necessary inference; for when the notation which suffices for

exhibiting one inference is found inadequate for explaining another, it is clear that the latter involves an inferential element not present to the former. Accordingly, the procedure contemplated should result in a list of categories of reasoning, the interest of which is not dependent upon the algebraic way of considering the subject. I shall not be able to perfect the algebra sufficiently to give facile methods of reaching logical conclusions: I can only give a method by which any legitimate conclusion may be reached and any fallacious one avoided. But I cannot doubt that others, if they will take up the subject, will succeed in giving the notation a form in which it will be highly useful in mathematical work. I even hope that what I have done may prove a first step toward the resolution of one of the main problems of logic, that of producing a method for the discovery of methods in mathematics.

§2. NON-RELATIVE LOGIC

365. According to ordinary logic, a proposition is either true or false, and no further distinction is recognized. This is the descriptive conception, as the geometers say; the metric conception would be that every proposition is more or less false, and that the question is one of amount. At present we adopt the former view.

366. Let propositions be represented by quantities. Let **v** and **f** be two constant values, and let the value of the quantity representing a proposition be **v** if the proposition is true and be **f** if the proposition is false. Thus, x being a proposition, the fact that x is either true or false is written

$$(x-\mathbf{f})(\mathbf{v}-x)=0.*$$

So

$$(x-\mathbf{f})(\mathbf{v}-y)=0$$

will mean that either x is false or y is true. This may be said to be the same as 'if x is true, y is true.' A hypothetical proposition, generally, is not confined to stating what actually happens, but states what is invariably true throughout a universe of possibility. The present proposition is, however, limited to that one individual state of things, the Actual.

* If this proposition be added to the postulates of Boolean algebra and if the terms of that algebra be interpreted as propositions, a propositional calculus is secured. From an historical standpoint this is of tremendous significance.

367. We are, thus, already in possession of a logical notation, capable of working syllogism. Thus, take the premisses, 'if x is true, y is true,' and 'if y is true, z is true.' These are written

$$(x-\mathbf{f})(\mathbf{v}-y)=0$$
$$(y-\mathbf{f})(\mathbf{v}-z)=0.$$

Multiply the first by $(\mathbf{v}-z)$ and the second by $(x-\mathbf{f})$ and add. We get

$$(x-\mathbf{f})(\mathbf{v}-\mathbf{f})(\mathbf{v}-z)=0,$$

or dividing by $\mathbf{v}-\mathbf{f}$, which cannot be 0,

$$(x-\mathbf{f})(\mathbf{v}-z)=0;$$

and this states the syllogistic conclusion, "if x is true, z is true."

368. But this notation shows a blemish in that it expresses propositions in two distinct ways, in the form of quantities, and in the form of equations; and the quantities are of two kinds, namely those which must be either equal to $\mathbf{f}$ or to $\mathbf{v}$, and those which are equated to *zero*. To remedy this, let us discard the use of equations, and perform no operations which can give rise to any values other than $\mathbf{f}$ and $\mathbf{v}$.

369. Of operations upon a simple variable, we shall need but one. For there are but two things that can be said about a single proposition, by itself; that it is true and that it is false,

$$x=\mathbf{v} \quad \text{and} \quad x=\mathbf{f}.$$

The first equation is expressed by x itself, the second by any function, ϕ, of x, fulfilling the conditions

$$\phi\mathbf{v}=\mathbf{f} \quad \phi\mathbf{f}=\mathbf{v}.$$

The simplest solution of these equations is

$$\phi x=\mathbf{f}+\mathbf{v}-x.$$

A product of n factors of the two forms $(x-\mathbf{f})$ and $(\mathbf{v}-y)$, if not zero, equals $(\mathbf{v}-\mathbf{f})^n$. Write P for the product. Then $\mathbf{v}-\dfrac{P}{(\mathbf{v}-\mathbf{f})^{n-1}}$ is the simplest function of the variables which becomes $\mathbf{v}$ when the product vanishes and $\mathbf{f}$ when it does not. By this means any proposition relating to a single individual can be expressed.

370. If we wish to use algebraical signs with their usual significations, the meanings of the operations will entirely depend upon those of **f** and **v**. Boole* chose $\mathbf{v}=1$, $\mathbf{f}=0$. This choice gives the following forms:

$$\mathbf{f}+\mathbf{v}-x=1-x$$

which is best written $\bar{x}$.

$$\mathbf{v}-\frac{(x-\mathbf{f})(\mathbf{v}-y)}{\mathbf{v}-\mathbf{f}}=1-x+xy=\overline{x\bar{y}}.$$

$$\mathbf{v}-\frac{(\mathbf{v}-x)(\mathbf{v}-y)}{\mathbf{v}-\mathbf{f}}=x+y-xy\dagger$$

$$\mathbf{v}-\frac{(\mathbf{v}-x)(\mathbf{v}-y)(\mathbf{v}-z)}{(\mathbf{v}-\mathbf{f})^2}=x+y+z-xy-xz-yz+xyz$$

$$\mathbf{v}-\frac{(x-\mathbf{f})(y-\mathbf{f})}{\mathbf{v}-\mathbf{f}}=1-xy=\overline{xy}\ddagger$$

371. It appears to me that if the strict Boolian system is used, the sign + ought to be altogether discarded. Boole and his adherent, Mr. Venn (whom I never disagree with without finding his remarks profitable), prefer to write $x+\bar{x}y$ in place of $\overline{\bar{x}\,\bar{y}}$. I confess I do not see the advantage of this, for the distributive principle holds equally well when written

$$\overline{\bar{x}\,\bar{y}}z=\overline{\overline{xz}\;\overline{yz}}\,\S$$

$$\overline{\overline{xy}\;\bar{z}}\qquad=\overline{\bar{x}\,\bar{z}}.\overline{\bar{y}\,\bar{z}}\,.\P$$

The choice of $\mathbf{v}=1$, $\mathbf{f}=0$, is agreeable to the received measurement of probabilities. But there is no need, and many times no advantage, in measuring probabilities in this way. I presume that Boole, in the formation of his algebra, at first considered the letters as denoting propositions or events. As he presents the subject, they are class-names; but it is not necessary so to regard them. Take, for example, the equation

$$t=n+hf,$$

* *The Laws of Thought*, pp. 47ff.

† I.e., $\overline{\bar{x}\bar{y}}$.

‡ I.e., $-(xy)$ or $\bar{x}+\bar{y}$.

§ I.e., $-(\bar{x}\bar{y})z=-\left\{-(xz)-(yz)\right\}$

¶ I.e., $-\left\{-(xy)\,\bar{z}\right\}=-(\bar{x}\,\bar{z})-(\bar{y}\,\bar{z})$.

which might mean that the body of taxpayers is composed of all the natives, together with householding foreigners. We might reach the signification by either of the following systems of notation, which indeed differ grammatically rather than logically.

Sign.	Signification. 1st System.	Signification. 2d System.
t	Taxpayer.	He is a Taxpayer.
n	Native.	He is a Native.
h	Householder.	He is a Householder.
f	Foreigner.	He is a Foreigner.

There is no *index* to show who the "He" of the second system is, but that makes no difference. To say that he is a taxpayer is equivalent to saying that he is a native or is a householder and a foreigner. In this point of view, the constants 1 and 0 are simply the probabilities, to one who knows, of what is true and what is false; and thus unity is conferred upon the whole system.

372. For my part, I prefer for the present not to assign determinate values to **f** and **v**, nor to identify the logical operations with any special arithmetical ones, leaving myself free to do so hereafter in the manner which may be found most convenient. Besides, the whole system of importing arithmetic into the subject is artificial, and modern Boolians do not use it. The algebra of logic should be self-developed, and arithmetic should spring out of logic instead of reverting to it. Going back to the beginning, let the writing of a letter by itself mean that a certain proposition is true. This letter is a *token*. There is a general understanding that the actual state of things or some other is referred to. This understanding must have been established by means of an *index*, and to some extent dispenses with the need of other indices. The denial of a proposition will be made by writing a line over it.

373. I have elsewhere* shown that the fundamental and primary mode of relation between two propositions is that

* In paper No. VI, part I.

which we have expressed by the form

$$\mathbf{v}-\frac{(x-\mathbf{f})(\mathbf{v}-y)}{\mathbf{v}-\mathbf{f}}.$$

We shall write this $x \prec y$,

which is also equivalent to $(x-\mathbf{f})(\mathbf{v}-y)=0$.

It is stated above that this means "if x is true, y is true." But this meaning is greatly modified by the circumstance that only the actual state of things is referred to.

374. To make the matter clear, it will be well to begin by defining the meaning of a hypothetical proposition, in general. What the usages of language may be does not concern us; language has its meaning modified in technical logical formulæ as in other special kinds of discourse. The question is what is the sense which is most usefully attached to the hypothetical proposition in logic? Now, the peculiarity of the hypothetical proposition is that it goes out beyond the actual state of things and declares what *would* happen were things other than they are or may be. The utility of this is that it puts us in possession of a rule, say that "if A is true, B is true," such that should we hereafter learn something of which we are now ignorant, namely that A is true, then, by virtue of this rule, we shall find that we know something else, namely, that B is true. There can be no doubt that the Possible, in its primary meaning, is that which may be true for aught we know, that whose falsity we do not know.* The purpose is subserved, then, if throughout the whole range of possibility, in every state of things in which A is true, B is true too. The hypothetical proposition may therefore be falsified by a single state of things, but only by one in which A is true while B is false. States of things in which A is false, as well as those in which B is true, cannot falsify it. If, then, B is a proposition true in every case throughout the whole range of possibility, the hypothetical proposition, taken in its logical sense, ought to be regarded as true, whatever may be the usage of ordinary speech. If, on the other hand, A is in no case true, throughout the range of possibility, it is a matter of indifference whether the hypothetical be understood to be true or not, since it is useless. But it will be more

* Cf. 527.

simple to class it among true propositions, because the cases in which the antecedent is false do not, in any other case, falsify a hypothetical. This, at any rate, is the meaning which I shall attach to the hypothetical proposition in general, in this paper.

375. The range of possibility is in one case taken wider, in another narrower; in the present case it is limited to the actual state of things. Here, therefore, the proposition

$$a -\!\!< b$$

is true if a is false or if b is true, but is false if a is true while b is false. But though we limit ourselves to the actual state of things, yet when we find that a formula of this sort is true by logical necessity, it becomes applicable to any single state of things throughout the range of logical possibility. For example, we shall see that from $x \overline{-\!\!<} y$ we can infer $z -\!\!< x$. This does not mean that because in the actual state of things x is true and y false, therefore in every state of things either z is false or x true; but it does mean that in whatever state of things we find x true and y false, in that state of things either z is false or x is true. In that sense, it is not limited to the actual state of things, but extends to any single state of things.

376. The *first icon* of algebra is contained in the formula of identity

$$x -\!\!< x.$$

This formula does not of itself justify any transformation, any inference. It only justifies our continuing to hold what we have held (though we may, for instance, forget how we were originally justified in holding it).

377. The *second icon* is contained in the rule that the several antecedents of a *consequentia* may be transposed; that is, that from

$$x -\!\!< (y -\!\!< z)$$

we can pass to

$$y -\!\!< (x -\!\!< z).$$

This is stated in the formula

$$\{x -\!\!< (y -\!\!< z)\} -\!\!< \{y -\!\!< (x -\!\!< z)\}.$$

Because this is the case, the brackets may be omitted, and we may write

$$y -\!\!< x -\!\!< z.$$

By the formula of identity

$$(x \prec y) \prec (x \prec y);$$

and transposing the antecedents

$$x \prec \{(x \prec y) \prec y\}$$

or, omitting the unnecessary brackets

$$x \prec (x \prec y) \prec y.$$

This is the same as to say that if in any state of things x is true, and if the proposition "if x, then y" is true, then in that state of things y is true. This is the *modus ponens* of hypothetical inference, and is the most rudimentary form of reasoning.*

378. To say that $(x \prec x)$ is generally true is to say that it is so in every state of things, say in that in which y is true; so that we may write

$$y \prec (x \prec x),$$

and then, by transposition of antecedents,

$$x \prec (y \prec x),$$

or from x we may infer $y \prec x$.

379. The *third icon* is involved in the principle of the transitiveness of the copula, which is stated in the formula

$$(x \prec y) \prec (y \prec z) \prec x \prec z.\dagger$$

According to this, if in any case y follows from x and z from y, then z follows from x.‡ This is the principle of the syllogism in *Barbara*.

380. We have already seen that from x follows $y \prec x$. Hence, by the transitiveness of the copula, if from $y \prec x$ follows z, then from x follows z, or from

$$(y \prec x) \prec z$$

follows

$$x \prec z,$$

or

$$\{(y \prec x) \prec z\} \prec x \prec z.$$

* Strictly speaking, the *modus ponens* is expressed as $\{(x \prec y).x\} \prec y$. The given proposition states that if x is true then provided "x then y" is true, y is true.

† I.e. $(x \prec y) \prec \{(y \prec z) \prec (x \prec z)\}$, which is often called the *nota notæ* or the *dictum de omni*. See 383, and vol. 2, Bk. III, ch. 4, §14, and cf. Joseph's *An Introduction to Logic*, p. 296n and 308n.

‡ This is not an exact reading of the given formula even as modified in the last note. The formula for this expression and thus for Barbara is:

$$\{(x \prec y)\overline{\prec}(y \overline{\prec} z)\} \prec (x \prec z).$$

381. The original notation $x \prec y$ served without modification to express the pure formula of identity. An enlargement of the conception of the notation so as to make the terms themselves complex was required to express the principle of the transposition of antecedents; and this new *icon* brought out new propositions. The third *icon* introduces the image of a chain of consequence. We must now again enlarge the notation so as to introduce negation. We have already seen that if a is true, we can write $x \prec a$, whatever x may be. Let b be such that we can write $b \prec x$ whatever x may be. Then b is false. We have here a *fourth icon*, which gives a new sense to several formulæ. Thus the principle of the interchange of antecedents is that from

$$x \prec (y \prec z)$$

we can infer $$y \prec (x \prec z).$$

Since z is any proposition we please, this is as much as to say that if from the truth of x the falsity of y follows, then from the truth of y the falsity of x follows.

382. Again the formula

$$x \prec \{(x \prec y) \prec y\}$$

is seen to mean that from x, we can infer that anything we please follows from that things following from x, and *a fortiori* from everything following from x. This is, therefore, to say that from x follows the falsity of the denial of x; which is the principle of contradiction.

383. Again the formula of the transitiveness of the copula, or

$$\{x \prec y\} \prec \{(y \prec z) \prec (x \prec z)\}$$

is seen to justify the inference

$$x \prec y$$
$$\therefore \bar{y} \prec \bar{x}.$$

The same formula justifies the *modus tollens*,

$$x \prec y$$
$$\bar{y}$$
$$\therefore \quad \bar{x}.$$

So the formula $\{(y \prec x) \prec z\} \prec (x \prec z)$ shows that from the falsity of $y \prec x$ the falsity of x may be inferred.

All the traditional moods of syllogism can easily be reduced to *Barbara* by this method.

384. A *fifth icon* is required for the principle of excluded middle and other propositions connected with it. One of the simplest formulæ of this kind is

$$\{(x \prec y) \prec x\} \prec x.$$

This is hardly axiomatical. That it is true appears as follows. It can only be false by the final consequent x being false while its antecedent $(x \prec y) \prec x$ is true. If this is true, either its consequent, x, is true, when the whole formula would be true, or its antecedent $x \prec y$ is false. But in the last case the antecedent of $x \prec y$, that is x, must be true.[1]

From the formula just given, we at once get

$$\{(x \prec y) \prec a\} \prec x,$$

where the a is used in such a sense that $(x \prec y) \prec a$ means that from $(x \prec y)$ every proposition follows. With that understanding, the formula states the principle of excluded middle, that from the falsity of the denial of x follows the truth of x.

[1] It is interesting to observe that this reasoning is dilemmatic. In fact, the dilemma involves the fifth icon. The dilemma was only introduced into logic from rhetoric by the humanists of the *renaissance;* and at that time logic was studied with so little accuracy that the peculiar nature of this mode of reasoning escaped notice. I was thus led to suppose that the whole non-relative logic was derivable from the principles of the ancient syllogistic, and this error [But it was not an error!!! See my original demonstration. — marginal note. [See 200n].] is involved in chapter 2 of my paper in the third volume of this Journal [No. VI]. My friend, Professor Schröder, detected the mistake and showed that the distributive formulæ

$$(x+y)z \prec xz+yz$$
$$(x+z)(y+z) \prec xy+z$$

could not be deduced from syllogistic principles. [This matter is discussed at length by Schröder in his *Vorlesungen über die Algebra der Logik*, Bd. 1, §12, (1890)]. I had myself independently discovered and virtually stated the same thing. (*Studies in Logic*, p. 189 [331].) There is some disagreement as to the definition of the dilemma (see Keynes's excellent *Formal Logic*, p. 241); but the most useful definition would be a syllogism depending on the above distribution formulæ. The distribution formulæ

$$xz+yz \prec (x+y)z$$
$$xy+z \prec (x+z)(y+z)$$

are strictly syllogistic. DeMorgan's added moods are virtually dilemmatic depending on the principle of excluded middle.

385. The logical algebra thus far developed contains signs of the following kinds:

First, tokens; signs of simple propositions, as *t* for 'He is a taxpayer,' etc.

Second, the single operative sign $-\!\!<$; also of the nature of a token.

Third, the juxtaposition of the letters to the right and left of the operative sign. This juxtaposition fulfils the function of an index, in indicating the connections of the tokens.

Fourth, the parentheses, subserving the same purpose.

Fifth, the letters α, β, etc. which are indices of no matter what tokens, used for expressing negation.

Sixth, the indices of tokens, x, y, z, etc., used in the general formulæ.

Seventh, the general formulæ themselves, which are *icons*, or exemplars of algebraic proceedings.

Eighth, the fourth *icon* which affords a second interpretation of the general formulæ.

386. We might dispense with the fifth and eighth species of signs — the devices by which we express negation — by adopting a second operational sign $\overline{-\!\!<}$, such that

$$x \,\overline{-\!\!<}\, y$$

should mean that $x=\mathbf{v}$, $y=\mathbf{f}$. With this, we should require new indices of connections, and new general formulæ. Possibly this might be the preferable notation. We should thus have two operational signs but no sign of negation. The forms of Boolian algebra hitherto used, have either two operational signs and a special sign of negation, or three operational signs. One of the operational signs is in that case superfluous. Thus, in the usual notation we have

$$\overline{x+y}=\bar{x}\;\bar{y}$$
$$\bar{x}+\bar{y}=\overline{xy}$$

showing two modes of writing the same fact. The apparent balance between the two sets of theorems exhibited so strikingly by Schröder, arises entirely from this double way of writing everything. But while the ordinary system is not so analytically fitted to its purpose as that here set forth, the character of superfluity here, as in many other cases in algebra, brings with it great facility in working.

387. The general formulae given above are not convenient in practice. We may dispense with them altogether, as well as with one of the indices of tokens used in them, by the use of the following rules. A proposition of the form

$$x \prec y$$

is true if $x = \mathbf{f}$ or $y = \mathbf{v}$. It is only false if $y = \mathbf{f}$ and $x = \mathbf{v}$. A proposition written in the form

$$x \overline{\prec} y$$

is true if $x = \mathbf{v}$ and $y = \mathbf{f}$, and is false if either $x = \mathbf{f}$ or $y = \mathbf{v}$. Accordingly, to find whether a formula is necessarily true substitute **f** and **v** for the letters and see whether it can be supposed false by any such assignment of values. Take, for example, the formula

$$(x \prec y) \prec \{(y \prec z) \prec (x \prec z)\}.$$

To make this false we must take

$$(x \prec y) = \mathbf{v}$$
$$\{(y \prec z) \prec (x \prec z)\} = \mathbf{f}.$$

The last gives

$$(y \prec z) = \mathbf{v}, \quad (x \prec z) = \mathbf{f}, \quad x = \mathbf{v}, \quad z = \mathbf{f}.$$

Substituting these values in

$$(x \prec y) = \mathbf{v} \quad (y \prec z) = \mathbf{v}$$

we have $\quad (\mathbf{v} \prec y) = \mathbf{v} \quad (y \prec \mathbf{f}) = \mathbf{v},$

which cannot be satisfied together.

388. As another example, required the conclusion from the following premisses: Anyone I might marry would be either beautiful or plain; anyone whom I might marry would be a woman; any beautiful woman would be an ineligible wife; any plain woman would be an ineligible wife. Let

m be anyone whom I might marry,
b, beautiful,
p, plain,
w, woman,
i, ineligible.

Then the premisses are

$$m -\!\!< (b -\!\!< \mathbf{f}) -\!\!< p,$$
$$m -\!\!< w,$$
$$w -\!\!< b -\!\!< i,$$
$$w -\!\!< p -\!\!< i.$$

Let x be the conclusion. Then,

$$[m -\!\!< (b -\!\!< \mathbf{f}) -\!\!< p] -\!\!< (m -\!\!< w) -\!\!< (w -\!\!< b -\!\!< i) -\!\!< (w -\!\!< p -\!\!< i) -\!\!< x$$

is necessarily true. Now if we suppose $m = \mathbf{v}$, the proposition can only be made false by putting $w = \mathbf{v}$ and either b or $p = \mathbf{v}$. In this case the proposition can only be made false by putting $i = \mathbf{v}$. If, therefore, x can only be made $\mathbf{f}$ by putting $m = \mathbf{v}$, $i = \mathbf{f}$, that is if $x = (m -\!\!< i)$ the proposition is necessarily true.

In this method, we introduce the two special tokens of second intention $\mathbf{f}$ and $\mathbf{v}$, we retain two indices of tokens x and y, and we have a somewhat complex *icon*, with a special prescription for its use.

389. A better method may be found as follows. We have seen that $x -\!\!< (y -\!\!< z)$

may be conveniently written $x -\!\!< y -\!\!< z$;

while $(x -\!\!< y) -\!\!< z$

ought to retain the parenthesis. Let us extend this rule, so as to be more general, and hold it necessary *always* to include the antecedent in parenthesis.

Thus, let us write $(x) -\!\!< y$

instead of $x -\!\!< y$. If now, we merely change the external appearance of two signs; namely, if we use the vinculum instead of the parenthesis, and the sign $+$ in place of $-\!\!<$, we shall have

$$x -\!\!< y \text{ written } \bar{x}+y$$
$$x -\!\!< y -\!\!< z \text{ written } \bar{x}+\bar{y}+z$$
$$(x -\!\!< y) -\!\!< z \text{ written } \overline{\bar{x}+\bar{y}}+z,\text{* etc.}$$

We may further write for $\overline{x -\!\!< y}$, $\overline{\bar{x}+y}$ implying that $x+y$† is

* This should be: $\overline{\bar{x}+y}+z$.

† This should be: $\bar{x}+y$.

an antecedent for whatever consequent may be taken, and the vinculum becomes identified with the sign of negation. We may also use the sign of multiplication as an abbreviation, putting $$xy=\overline{\bar{x}+\bar{y}}=\overline{x \prec \bar{y}}.$$

390. This subjects addition and multiplication to all the rules of ordinary algebra, and also to the following:

$$y+x\bar{x}=y \qquad y(x+\bar{x})=y$$
$$x+\bar{x}=\mathbf{v} \qquad \bar{x}x=\mathbf{f}$$
$$xy+z=(x+z)(y+z).$$

391. To any proposition we have a right to add any expression at pleasure; also to strike out any factor of any term. The expressions for different propositions separately known may be multiplied together. These are substantially Mr. Mitchell's rules of procedure.* Thus the premisses of Barbara are

$$\bar{x}+y \text{ and } \bar{y}+z.$$

Multiplying these, we get $(\bar{x}+y)(\bar{y}+z)=\bar{x}\bar{y}+yz$. Dropping $\bar{y}$ and y we reach the conclusion $\bar{x}+z$.

§3. FIRST-INTENTIONAL LOGIC OF RELATIVES

392. The algebra of Boole affords a language by which anything may be expressed which can be said without speaking of more than one individual at a time. It is true that it can assert that certain characters belong to a whole class, but only such characters as belong to each individual separately. The logic of relatives considers statements involving two and more individuals at once. Indices are here required. Taking, first, a degenerate form of relation, we may write $x_i y_j$ to signify that x is true of the individual i while y is true of the individual j. If z be a relative character z_{ij} will signify that i is in that relation to j. In this way we can express relations of considerable complexity. Thus, if

$$\begin{matrix} 1, & 2, & 3, \\ 4, & 5, & 6, \\ 7, & 8, & 9, \end{matrix}$$

* *Op. cit.*, p. 80f.

are points in a plane, and l_{123} signifies that 1, 2, and 3 lie on one line, a well-known proposition of geometry* may be written

$$l_{159} \prec l_{267} \prec l_{348} \prec l_{147} \prec l_{258} \prec l_{369} \prec l_{123} \prec l_{456} \prec l_{789}.$$

In this notation is involved a *sixth icon.*

393. We now come to the distinction of *some* and *all*, a distinction which is precisely on a par with that between truth and falsehood; that is, it is descriptive.

All attempts to introduce this distinction into the Boolian algebra were more or less complete failures until Mr. Mitchell† showed how it was to be effected. His method really consists in making the whole expression of the proposition consist of two parts, a pure Boolian expression referring to an individual and a Quantifying part saying what individual this is. Thus, if k means 'he is a king,' and h, 'he is happy,' the Boolian

$$(\bar{k}+h)$$

means that the individual spoken of is either not a king or is happy. Now, applying the quantification, we may write

$$\text{Any } (\bar{k}+h)$$

to mean that this is true of any individual in the (limited) universe, or

$$\text{Some } (\bar{k}+h)$$

to mean that an individual exists who is either not a king or is happy. So

$$\text{Some } (kh)$$

means some king is happy, and

$$\text{Any } (kh)$$

means every individual is both a king and happy. The rules for the use of this notation are obvious. The two propositions

$$\text{Any } (x) \quad \text{Any } (y)$$

are equivalent to

$$\text{Any } (xy).$$

* If the six vertices of a hexagon lie three and three on two straight lines, the three points of intersection of the opposite sides lie on a straight line.

† *Op. cit.*, p. 79.

From the two propositions

$$\text{Any } (x) \quad \text{Some } (y)$$

we may infer $\quad$ Some (xy).[1]

Mr. Mitchell has also a very interesting and instructive extension of his notation for *some* and *all*, to a two-dimensional universe, that is, to the logic of relatives. Here, in order to render the notation as iconical as possible we may use Σ for *some*, suggesting a sum, and Π for *all*, suggesting a product. Thus $\Sigma_i x_i$ means that x is true of some one of the individuals denoted by i or

$$\Sigma_i x_i = x_i + x_j + x_k + \text{etc.*}$$

In the same way, $\Pi_i x_i$ means that x is true of all these individuals, or

$$\Pi_i x_i = x_i x_j x_k, \text{ etc.†}$$

If x is a simple relation, $\Pi_i \Pi_j x_{ij}$ means that every i is in this relation to every j, $\Sigma_i \Pi_j x_{ij}$ that some one i is in this relation to every j, $\Pi_j \Sigma_i x_{ij}$ that to every j some i or other is in this relation, $\Sigma_i \Sigma_j x_{ij}$ that some i is in this relation to some j. It is to be remarked that $\Sigma_i x_i$ and $\Pi_i x_i$ are only *similar* to a sum and a product; they are not strictly of that nature, because the individuals of the universe may be innumerable.

[1] I will just remark, quite out of order, that the quantification may be made numerical; thus producing the numerically definite inferences of DeMorgan and Boole. Suppose at least $\frac{2}{3}$ of the company have white neckties and at least $\frac{3}{4}$ have dress coats. Let w mean 'he has a white necktie,' and d 'he has a dress coat.' Then, the two propositions are

$$\tfrac{2}{3}\ (w) \text{ and } \tfrac{3}{4}\ (d).$$

These are to be multiplied together. But we must remember that xy is a mere abbreviation for $\overline{\bar{x}+\bar{y}}$, and must therefore write

$$\overline{\overline{\tfrac{2}{3}w}+\overline{\tfrac{3}{4}d}}.$$

Now $\overline{\frac{2}{3}w}$ is the denial of $\frac{2}{3}w$, and this denial may be written $(>\frac{1}{3})\bar{w}$, or more than $\frac{1}{3}$ of the universe (the company) have not white neckties. So $\overline{\frac{3}{4}d}=(>\frac{1}{4})\bar{d}$. The combined premisses thus become

$$\overline{(>\tfrac{1}{3})\bar{w}+(>\tfrac{1}{4})\bar{d}}.$$

Now $(>\frac{1}{3})\bar{w}+(>\frac{1}{4})\bar{d}$ gives $\quad$ May be $(\frac{1}{3}+\frac{1}{4})(\bar{w}+\bar{d})$.

Thus we have $\quad$ May be $\overline{(\frac{7}{12})(\bar{w}+\bar{d})}$,

and this is $\quad$ (At least $\frac{5}{12})(\overline{\bar{w}+\bar{d}})$,

which is the conclusion.

* This is the seventh icon?

† This is the eighth icon?

394. At this point, the reader would perhaps not otherwise easily get so good a conception of the notation as by a little practice in translating from ordinary language into this system and back again. Let l_{ij} mean that i is a lover of j, and b_{ij} that i is a benefactor of j. Then

$$\Pi_i \Sigma_j l_{ij} b_{ij}$$

means that everything is at once a lover and a benefactor of something; and $\Pi_i \Sigma_j l_{ij} b_{ji}$

that everything is a lover of a benefactor of itself.

$$\Sigma_i \Sigma_k \Pi_j (l_{ij} + b_{jk})$$

means that there are two persons, one of whom loves everything except benefactors of the other (whether he loves any of these or not is not stated). Let g_i mean that i is a griffin, and c_i that i is a chimera, then

$$\Sigma_i \Pi_j (g_i l_{ij} + \bar{c}_j)$$

means that if there be any chimeras there is some griffin that loves them all; while $\Sigma_i \Pi_j g_i (l_{ij} + \bar{c}_j)$

means that there is a griffin and he loves every chimera that exists (if any exist). On the other hand,

$$\Pi_j \Sigma_i g_i (l_{ij} + \bar{c}_j)$$

means that griffins exist (one, at least), and that one or other of them loves each chimera that may exist; and

$$\Pi_j \Sigma_i (g_i l_{ij} + \bar{c}_j)$$

means that each chimera (if there is any) is loved by some griffin or other.

395. Let us express: every part of the world is either sometimes visited with cholera, and at others with small-pox (without cholera), or never with yellow fever and the plague together.

Let c_{ij} mean the place i has cholera at the time j.
s_{ij} mean the place i has small-pox at the time j.
y_{ij} mean the place i has yellow fever at the time j.
p_{ij} mean the place i has plague at the time j.

Then we write $\Pi_i \Sigma_j \Sigma_k \Pi_l (c_{ij} \bar{c}_{ik} s_{ik} + \bar{y}_{il} + \bar{p}_{il})$.

Let us express this: one or other of two theories must be admitted, first, that no man is at any time unselfish or free, and some men are always hypocritical, and at every time some men are friendly to men to whom they are at other times inimical, or second, at each moment all men are alike either angels or fiends. Let

u_{ij} mean the man i is unselfish at the time j,
f_{ij} mean the man i is free at the time j,
h_{ij} mean the man i is hypocritical at the time j,
a_{ij} mean the man i is an angel at the time j,
d_{ij} mean the man i is a fiend at the time j,
p_{ijk} mean the man i is friendly at the time j, to the man k,
e_{ijk} the man i is an enemy at the time j to the man k;
1_{jm} the two objects j and m are identical.

Then the proposition is

$$\Pi_i \Sigma_h \Pi_j \Sigma_k \Sigma_l \Sigma_m \Pi_n \Pi_p \Pi_q (\bar{u}_{ij} \bar{f}_{ij} h_{hj} p_{kjl} e_{kml} \bar{1}_{jm} + a_{pn} + d_{qn})$$

396. We have now to consider the procedure in working with this calculus. It is far from being true that the only problem of deduction is to draw a conclusion from given premisses. On the contrary, it is fully as important to have a method for ascertaining what premisses will yield a given conclusion. There are besides other problems of transformation, where a certain system of facts is given, and it is required to describe this in other terms of a definite kind. Such, for example, is the problem of the fifteen young ladies,* and others relating to synthemes.† I shall, however, content myself here with showing how, when a set of premisses are given, they can be united and certain letters eliminated. Of the various methods which might be pursued, I shall here give the one which seems to me the most useful on the whole.

* See *Lady's and Gentleman's Diary* for 1850, p. 48 for the original presentation of 'Kirkman's School-Girls Problem.' See also W. W. R. Ball, *Mathematical Recreations and Essays*, Ch. IX, MacMillan & Co., 5 ed. (1911), where a reference is given to Benjamin Peirce's method of solution and his enunciation of a corresponding problem.

† Peirce defines a syntheme in the *Century Dictionary*, p. 6139, ed. 1889, as "a system of groups of objects comprising every one of a larger set just once, twice or other given number of times. The groups may be divided into subgroups subject to various conditions."

First, the different premisses having been written with distinct indices (the same index not used in two propositions) are written together, and all the Π's and Σ's are to be brought to the left. This can evidently be done, for

$$\Pi_i x_i \,.\, \Pi_j x_j = \Pi_i \Pi_j x_i x_j$$
$$\Sigma_i x_i \,:\, \Pi_j x_j = \Sigma_i \Pi_j x_i x_j$$
$$\Sigma_i x_i \,.\, \Sigma_j x_j = \Sigma_i \Sigma_j x_i x_j.$$

Second, without deranging the order of the indices of any one premiss, the Π's and Σ's belonging to different premisses may be moved relatively to one another, and as far as possible the Σ's should be carried to the left of the Π's. We have

$$\Pi_i \Pi_j x_{ij} = \Pi_j \Pi_i x_{ij}$$
$$\Sigma_i \Sigma_j x_{ij} = \Sigma_j \Sigma_i x_{ij}$$

and also $$\Sigma_i \Pi_j x_i y_j = \Pi_j \Sigma_i x_i y_j.$$

But this formula does not hold when the *i* and *j* are not separated. We do have, however,

$$\Sigma_i \Pi_j x_{ij} \multimap \Pi_i \Sigma_j x_{ij}.^*$$

It will, therefore, be well to begin by putting the Σ's to the left, as far as possible, because at a later stage of the work they can be carried to the right but not to the left. For example, if the operators of the two premisses are $\Pi_i \Sigma_j \Pi_k$ and $\Sigma_x \Pi_y \Sigma_z$, we can unite them in either of the two orders

$$\Sigma_x \Pi_y \Sigma_z \Pi_i \Sigma_j \Pi_k$$
$$\Sigma_x \Pi_i \Sigma_j \Pi_y \Sigma_z \Pi_k,$$

and shall usually obtain different conclusions accordingly. There will often be room for skill in choosing the most suitable arrangement.

Third, it is next sometimes desirable to manipulate the Boolian part of the expression, and the letters to be eliminated can, if desired, be eliminated now. For this purpose they are replaced by relations of second intention, such as "other than," etc. If, for example, we find anywhere in the expression

$$a_{ijk} \bar{a}_{xyz},$$

this may evidently be replaceable by

$$(n_{ix} + n_{jy} + n_{kz})$$

* Obviously a misprint for $\Sigma_i \Pi_j x_{ij} \multimap \Pi_j \Sigma_i x_{ij}$.

where, as usual, n means not or other than. This third step of the process is frequently quite indispensable, and embraces a variety of processes; but in ordinary cases it may be altogether dispensed with.*

Fourth, the next step, which will also not commonly be needed, consists in making the indices refer to the same collections of objects, so far as this is useful. If the quantifying part, or Quantifier, contains Σ_x, and we wish to replace the x by a new index i, not already in the Quantifier, and such that every x is an i, we can do so at once by simply multiplying every letter of the Boolian having x as an index by x_i. Thus, if we have "some woman is an angel" written in the form $\Sigma_w a_w$ we may replace this by $\Sigma_i(a_i w_i)$. It will be more often useful to replace the index of a Π by a wider one; and this will be done by adding $\bar{x}_i$ to every letter having x as an index. Thus, if we have "all dogs are animals, and all animals are vertebrates" written thus

$$\Pi_d \alpha_d \qquad \Pi_a v_a,$$

where a and α alike mean animal, it will be found convenient to replace the last index by i, standing for any object, and to write the proposition

$$\Pi_i(\bar{a}_i + v_i).$$

Fifth,† the next step consists in multiplying the whole Boolian part, by the modification of itself produced by substituting for the index of any Π any other index standing to the left of it in the Quantifier. Thus, for

$$\Sigma_i \Pi_j l_{ij},$$

we can write

$$\Sigma_i \Pi_j l_{ij} l_{ii}.$$

Sixth, the next step consists in the re-manipulation of the Boolian part, consisting, first, in adding to any part any term we like; second, in dropping from any part any factor we like, and third, in observing that

$$x\bar{x} = \mathbf{f}, \qquad x + \bar{x} = \mathbf{v},$$

so that

$$x\bar{x}y + z = z, \quad (x + \bar{x} + y)z = z.$$

Seventh, Π's and Σ's in the Quantifier whose indices no longer appear in the Boolian are dropped.

* See 403A and B at end of article.

† See 403C and M.

The fifth step will, in practice, be combined with part of the sixth and seventh. Thus, from $\Sigma_i \Pi_j l_{ij}$ we shall at once proceed to $\Sigma_i l_{ii}$ if we like.

397. The following examples will be sufficient.

From the premisses $\Sigma_i a_i b_i$ and $\Pi_j(\bar{b}_j + c_j)$, eliminate b. We first write

$$\Sigma_i \Pi_j a_i b_i(\bar{b}_j + c_j).$$

The distributive process gives

$$\Sigma_i \Pi_j a_i(b_i \bar{b}_j + b_i c_j).$$

But *we always have a right to drop a factor or insert an additive term.* We thus get

$$\Sigma_i \Pi_j a_i(b_i \bar{b}_j + c_j).$$

By the third process, we can, if we like, insert n_{ij} for $b_i \bar{b}_j$. In either case, we identify j with i and get the conclusion

$$\Sigma_i a_i c_i.$$

Given the premisses:

$$\Sigma_h \Pi_i \Sigma_j \Pi_k (\alpha_{hik} + s_{jk} l_{ji})$$
$$\Sigma_u \Sigma_v \Pi_x \Pi_y (\epsilon_{uyx} + \bar{s}_{yv} b_{vx}).$$

Required to eliminate s. The combined premiss is

$$\Sigma_u \Sigma_v \Sigma_h \Pi_i \Sigma_j \Pi_x \Pi_k \Pi_y (\alpha_{hik} + s_{jk} l_{ji})(\epsilon_{uyx} + \bar{s}_{yv} b_{vx}).$$

Identify k with v and y with j, and we get

$$\Sigma_u \Sigma_v \Sigma_h \Pi_i \Sigma_j \Pi_x (\alpha_{hiv} + s_{jv} l_{ji})(\epsilon_{ujx} + \bar{s}_{jv} b_{vx}).$$

The Boolian part then reduces, so that the conclusion is

$$\Sigma_u \Sigma_v \Sigma_h \Pi_i \Sigma_j \Pi_x (\alpha_{hiv} \epsilon_{ujx} + \alpha_{hiv} b_{vx} + \epsilon_{ujx} l_{ji}).$$

§4. SECOND-INTENTIONAL LOGIC*

398. Let us now consider the logic of terms taken in collective senses. Our notation, so far as we have developed it, does not show us even how to express that two indices, i and j, denote one and the same thing. We may adopt a special token of second intention, say 1, to express identity, and may write 1_{ij}. But this relation of identity has peculiar properties. The first is that if i and j are identical, whatever is true of i is true of j. This may be written

$$\Pi_i \Pi_j \{ \bar{1}_{ij} + \bar{x}_i + x_j \}.$$

* See 403H.

The use of the general index of a token, x, here, shows that the formula is iconical. The other property is that if everything which is true of i is true of j, then i and j are identical. This is most naturally written as follows: Let the token, q, signify the relation of a quality, character, fact, or predicate to its subject. Then the property we desire to express is

$$\Pi_i \Pi_j \Sigma_k (1_{ij} + \bar{q}_{ki} q_{kj}).$$

And identity is defined thus

$$1_{ij} = \Pi_k (q_{ki} q_{kj} + \bar{q}_{ki} \bar{q}_{kj}).$$

That is, to say that things are identical is to say that every predicate is true of both or false of both. It may seem circuitous to introduce the idea of a quality to express identity; but that impression will be modified by reflecting that $q_{ki}q_{jk}$ merely means that i and j are both within the class or collection k. If we please, we can dispense with the token q, by using the index of a token and by referring to this in the Quantifier just as subjacent indices are referred to. That is to say, we may write

$$1_{ij} = \Pi_x (x_i x_j + \bar{x}_i \bar{x}_j).$$

399. The properties of the token q must now be examined. These may all be summed up in this, that taking any individuals i_1, i_2, i_3, etc., and any individuals, j_1, j_2, j_3, etc., there is a collection, class, or predicate embracing all the i's and excluding all the j's except such as are identical with some one of the i's. This might be written

$$(\Pi_\alpha \Pi_{i_\alpha})(\Pi_\beta \Pi_{j_\beta}) \Sigma_k (\Pi_\alpha \Sigma_{i'_\alpha}) \Pi_l \, q_{ki_\alpha} (\bar{q}_{kj_\beta} + q_{li'_\alpha} q_{lj_\beta} + \bar{q}_{li'_\alpha} \bar{q}_{lj_\beta}),$$

where the i's and the i''s are the same lot of objects. This notation presents indices of indices. The $\Pi_\alpha \Pi_{i_\alpha}$ shows that we are to take any collection whatever of i's, and then any i of that collection. We are then to do the same with the j's. We can then find a quality k such that the i taken has it, and also such that the j taken wants it unless we can find an i that is identical with the j taken. The necessity of some kind of notation of this description in treating of classes collectively appears from this consideration: that in such discourse we are neither speaking of a single individual (as in the non-relative logic) nor of a small number of individuals considered each for itself, but of a whole class, perhaps an infinity of individuals. This suggests a relative term with an indefinite series of indices as x_{ijkl} . . . . Such a relative will, however, in most, if not in all cases, be of a degenerate kind and is consequently expressible as above. But it seems preferable to attempt a partial decomposition of this definition. In the first place, any individual may be considered as a class. This is written

$$\Pi_i \Sigma_k \Pi_j \; q_{ki}(\bar{q}_{kj} + 1_{ij}).$$

This is the *ninth icon.** Next, given any class, there is another which includes all the former excludes and excludes all the former includes. That is,

$$\Pi_l \Sigma_k \Pi_i (q_{li}\bar{q}_{ki} + \bar{q}_{li}q_{ki}).$$

This is the *tenth icon.* Next, given any two classes, there is a third which includes all that either includes and excludes all that both exclude. That is

$$\Pi_l \Pi_m \Sigma_k \Pi_i (q_{li}q_{ki} + q_{mi}q_{ki} + \bar{q}_{li}\bar{q}_{mi}\bar{q}_{ki}).$$

This is the *eleventh icon.* Next, given any two classes, there is a class which includes the whole of the first and any one individual of the second which there may be not included in the first and nothing else. That is,

$$\Pi_l \Pi_m \Pi_i \Sigma_k \Pi_j \{ q_{li} + \bar{q}_{mi} + q_{ki}(q_{kj} + \bar{q}_{lj}) \}.$$

This is the *twelfth icon.*

400. To show the manner in which these formulæ are applied let us suppose we have given that everything is either true of i or false of j. We write

$$\Pi_k (q_{ki} + \bar{q}_{kj}).\dagger$$

The tenth icon gives

$$\Pi_l \Sigma_k (q_{li}\bar{q}_{ki} + \bar{q}_{li}q_{ki})(q_{lj}\bar{q}_{kj} + \bar{q}_{lj}q_{kj}).$$

Multiplication of these two formulæ give

$$\Pi_l \Sigma_k (q_{ki}\bar{q}_{li} + q_{lj}\bar{q}_{kj}),$$

or, dropping the terms in k

$$\Pi_l (\bar{q}_{li} + q_{lj}).$$

* See 403I.

† See 403J.

Multiplying this with the original datum and identifying l with k, we have

$$\Pi_k(q_{ki}q_{kj}+\bar{q}_{ki}\bar{q}_{kj}).$$

No doubt, a much more direct method of procedure could be found.

401. Just as q signifies the relation of predicate to subject, so we need another token, which may be written r, to signify the conjoint relation of a simple relation, its relate and its correlate. That is, r_{jai} is to mean that i is in the relation α to j. Of course, there will be a series of properties of r similar to those of q. But it is singular that the uses of the two tokens are quite different. Namely, the chief use of r is to enable us to express that the number of one class is at least as great as that of another. This may be done in a variety of different ways. Thus, we may write that for every a there is a b, in the first place, thus:

$$\Sigma_\alpha\Pi_i\Sigma_j\Pi_h\{\,\bar{a}_i+b_jr_{jai}(\bar{r}_{jah}+\bar{a}_h+1_{ih})\,\}.$$

But, by an icon analogous to the eleventh, we have

$$\Pi_\alpha\Pi_\beta\Sigma_\gamma\Pi_u\Pi_v(r_{u\alpha v}r_{u\gamma v}+r_{u\beta v}r_{u\gamma v}+\bar{r}_{u\alpha v}\bar{r}_{u\beta v}\bar{r}_{u\gamma v}).$$

From this, by means of an icon analogous to the *tenth*, we get the general formula

$$\Pi_\alpha\Pi_\beta\Sigma_\gamma\Pi_u\Pi_v\{\,r_{u\alpha v}r_{u\beta v}r_{u\gamma v}+\bar{r}_{u\gamma v}(\bar{r}_{u\alpha v}+\bar{r}_{u\beta v})\,\}.$$

For $r_{u\beta v}$ substitute a_u and multiply by the formula the last but two. Then, identifying u with h and v with j, we have

$$\Sigma_\alpha\Pi_i\Sigma_h\Pi_h\{\,\bar{a}_i+b_jr_{jai}(\bar{r}_{jah}+1_{ih})\,\}$$

a somewhat simpler expression. However, the best way to express such a proposition is to make use of the letter c as a token of a one-to-one correspondence. That is to say, c will be defined by the three formulae,*

$$\Pi_\alpha\Pi_u\Pi_v\Pi_w(\bar{c}_\alpha+\bar{r}_{u\alpha v}+\bar{r}_{u\alpha w}+1_{vw})$$
$$\Pi_\alpha\Pi_u\Pi_v\Pi_w(\bar{c}_\alpha+\bar{r}_{u\alpha w}+r_{v\alpha w}\dagger+1_{uv})$$
$$\Pi_\alpha\Sigma_u\Sigma_v\Sigma_w(c_\alpha+r_{u\alpha v}r_{u\alpha w}\bar{1}_{vw}+r_{u\alpha w}r_{v\alpha w}\bar{1}_{uv}).\ddagger$$

* Cf. 21 (7).

† $\bar{r}_{v\alpha w}$.

‡ I.e., any two terms which are relates in any one-one correspondence to any term are identical; any two terms which are the correlates by one-one correspondence to any term are identical; and any case of terms related one to one is always a one-one correspondence.

Making use of this token, we may write the proposition we have been considering in the form

$$\Sigma_a \Pi_i \Sigma_j \ c_a(\bar{a}_i + b_j r_{jai}).$$

402. In an appendix to his memoir on the logic of relatives,* De Morgan enriched the science of logic with a new kind of inference, the syllogism of transposed quantity. De Morgan was one of the best logicians that ever lived and unquestionably the father of the logic of relatives. Owing, however, to the imperfection of his theory of relatives, the new form, as he enunciated it, was a down-right paralogism, one of the premisses being omitted. But this being supplied, the form furnishes a good test of the efficacy of a logical notation. The following is one of De Morgan's examples:†

Some X is Y,
For every X there is something neither Y nor Z;
Hence, something is neither X nor Z.

The first premiss is simply $\Sigma_a x_a y_a$.

The second may be written

$$\Sigma_a \Pi_i \Sigma_j \ c_a(\bar{x}_i + r_{jai} \bar{y}_j \bar{z}_j).$$

From these two premisses, little can be inferred. To get the above conclusion it is necessary to add that the class of X's is a finite collection‡; were this not necessary the following reasoning would hold good (the limited universe consisting of numbers); for it precisely conforms to De Morgan's scheme.

Some odd number is prime;
Every odd number has its square, which is neither prime nor even;
Hence, some number is neither odd nor even.[1]

* *Transactions Cambridge Philosophical Society*, vol. 10, pp. *355–*358, (1864).

†*Ibid.*, p. *356.

‡ See 564 and 4.103f.

[1] Another of De Morgan's examples [*Formal Logic*, p. 168] is this: "Suppose a person, on reviewing his purchases for the day, finds, by his counterchecks, that he has certainly drawn as many checks on his banker (and maybe more) as he has made purchases. But he knows that he paid some of his purchases in money, or otherwise than by checks. He infers then that he has drawn checks

Now, to say that a lot of objects is finite, is the same as to say that if we pass through the class from one to another we shall necessarily come round to one of those individuals already passed; that is, if every one of the lot is in any one-to-one relation to one of the lot, then to every one of the lot some one is in this same relation. This is written thus:

$$\Pi_\beta \Pi_u \Sigma_v \Sigma_s \Pi_t \{ \bar{c}_\beta + \bar{x}_u + x_v r_{u\beta v} + x_s(\bar{x}_t + \bar{r}_{t\beta s}) \}$$

Uniting this with the two premisses and the second clause of the definition of c, we have

$$\Sigma_a \Sigma_\alpha \Pi_\beta \Pi_u \Sigma_v \Sigma_s \Pi_i \Sigma_j \Pi_t \Pi_\gamma \Pi_e \Pi_f \Pi_g x_a y_a c_\alpha (\bar{x}_i + r_{j\alpha i} \bar{y}_j \bar{z}_j)$$
$$\{ \bar{c}_\beta + \bar{x}_u + x_v r_{u\beta v} + x_s(\bar{x}_t + \bar{r}_{t\beta s}) \} (\bar{c}_\gamma + \bar{r}_{e\gamma g} + \bar{r}_{f\gamma w} + 1_{ef}).$$

We now substitute α for β and for γ, a for u and for e, j for t and for f, v for g. The factor in i is to be repeated, putting first s and then v for i. The Boolian part thus reduces to

$$(\bar{x}_s + r_{j\alpha s} \bar{y}_j \bar{z}_j) c_\alpha x_a y_a r_{a\alpha v} x_v r_{j\alpha v} \bar{y}_j \bar{z}_j 1_{aj} + r_{j\alpha s} \bar{y}_j \bar{z}_j x_s \bar{x}_j$$
$$(\bar{x}_v + r_{j\alpha v} \bar{y}_j \bar{z}_j)(\bar{r}_{a\alpha v} + \bar{r}_{j\alpha v} 1_{aj}),$$

which, by the omission of factors, becomes

$$y_a \bar{y}_j 1_{aj} + \bar{x}_j \bar{z}_j.$$

Thus we have the conclusion $\Sigma_j \bar{x}_j \bar{z}_j$.

403. It is plain that by a more iconical and less logically analytical notation this procedure might be much abridged. How minutely analytical the present system is, appears when we reflect that every substitution of indices of which nine were used in obtaining the last conclusion is a distinct act of inference. The annulling of $(y_a \bar{y}_j 1_{aj})$ makes ten inferential steps between the premisses and conclusion of the syllogism of transposed quantity.

for something else except that day's purchases. He infers rightly enough." Suppose, however, that what happened was this: He bought something and drew a check for it; but instead of paying with the check, he paid cash. He then made another purchase for the same amount, and drew another check. Instead, however, of paying with that check, he paid with the one previously drawn. And thus he continued without cessation, or *ad infinitum*. Plainly the premisses remain true, yet the conclusion is false.

§5. NOTE*

403A. Under the third step [396], an example was given which is really a general formula of elimination. Namely, we have

$$a_{ijk \text{ etc.}}\ \bar{a}_{xyz \text{ etc.}} \prec \bar{1}_{ix}+\bar{1}_{jy}+\bar{1}_{kz \text{ etc.}}$$

and conversely

$$1_{ix}1_{jy}1_{zk \text{ etc.}} \prec a_{ijk \text{ etc.}} + \bar{a}_{xyz \text{ etc.}}.$$

The principle of contradiction and of excluded middle might be considered as mere special cases of these formulæ. Namely, the latter give

$$a_i\bar{a}_i \prec \bar{1}_{ii}$$
$$1 \prec a_i+\bar{a}_i$$

But the definition of identity is

$$\mathbf{v} \prec 1_{ii} \quad \bar{1}_{ii} \prec \mathbf{f};$$

whence

$$a_i\bar{a}_i \prec \mathbf{f} \quad \mathbf{v} \prec a_i+\bar{a}_i.$$

403B. Under the head of the third step belongs the frequently necessary development of the Boolian by means of distribution formulæ. (This proceeding, and indeed the whole of the third step, might more properly have been made to follow the fourth.) The fundamental formulæ of distribution are

$$x(y+z)=xy+xz$$
$$x+yz=(x+y)(x+z).$$

The general development formulæ thence resulting are

$$Fx=xF1+\bar{x}F0$$
$$=(x+F0)(\bar{x}+F1).$$

The following formulæ, which find continual application, are deducible from the above:

$$a+b=a+\bar{a}b$$
$$xy+\bar{x}\bar{y}=(x+\bar{y})(\bar{x}+y)$$
$$(a+b)c \prec a+bc$$
$$(a+x)(b+y) \prec a+b+xy.$$

* This undated note seems to have been written for publication in some issue of *The American Journal of Mathematics*, shortly subsequent to that in which the previous article appeared. Why it was not published is unknown.

403C. The *fifth step* is composed of two kinds of operations, the involution of the whole expression, and the identification and discrimination of indices.

403D. It is plain that *any algebraical expression of a proposition may be multiplied into itself any number of times without ceasing to be true;* and it will be found that such involution is essentially necessary in all difficult modes of inference. Consider, for example, the last formula but one of those last given,

$$(a+b)c \prec a+bc.$$

This is not itself a distribution formula, but only an association formula, and therefore the deduction of a distribution formula from it is not a matter of the very utmost simplicity. If, however, we square the antecedent we have

$$(a+b)c=(a+b)c(a+b)c=c(a+b)c,$$

and then by the application of the association formula twice over we get

$$c(a+b)c \prec (ca+b)c \prec ca+bc;$$

so that we have proved

$$(a+b)c \prec ac+bc,$$

which is the main distribution formula, with the aid of which all the others are readily obtained.

Other much more striking examples of the utility of involution will present themselves in the course of this paper.

403E. In multiplying a proposition by itself, we have the right to choose any point from the beginning to the end of the quantifier and to identify in two factors all the indices to the left of this point while diversifying all to the right of it. When there is but one index, this is plainly true (see the formulæ in first step); for

$$\Pi_i a_i=(\Pi_i a_i)(\Pi_i a_i)=(\Pi_i a_i)(\Pi_j a_j)$$

$$\Sigma_i a_i=(\Sigma_i a_i)(\Sigma_i a_i)=(\Sigma_i a_i)(\Sigma_j a_j).$$

Now $$(\Pi_i a_i)(\Pi_j a_j)=(a_1 a_2 a_3 \text{ etc.})(a_1 a_2 a_3 \text{ etc.})$$

while $$\Pi_i \Pi_j a_i a_j = a_1 a_1 \cdot a_1 a_2 \cdot a_1 a_3 \text{ etc.}$$
$$\times a_2 a_1 \cdot a_2 a_2 \cdot a_2 a_3 \text{ etc.}$$
$$\times a_3 a_1 \cdot a_3 a_2 \cdot a_3 a_3 \text{ etc.}$$
$$\times \text{etc.}$$

and $$\Pi_i a_i a_i = a_1 a_1 \cdot a_2 a_2 \cdot a_3 a_3 \cdot \text{etc.}$$

But since $aa=a$, all these are equivalent.

Also

$$(\Sigma_i a_i)(\Sigma_j a_j) = (a_1 + a_2 + a_3 + \text{etc.})(a_1 + a_2 + a_3 + \text{etc.})$$

while

$$\begin{aligned}\Sigma_i \Sigma_j a_i a_j = & a_1 a_1 + a_1 a_2 + a_1 a_3 + \text{etc.} \\ & + a_2 a_1 + a_2 a_2 + a_2 a_3 + \text{etc.} \\ & + a_3 a_1 + a_3 a_2 + a_3 a_3 + \text{etc.} \\ & + \text{etc.}\end{aligned}$$

and $$\Sigma_i a_i a_i = a_1 a_1 + a_2 a_2 + a_3 a_3 + \text{etc.}$$

The first two expressions are equivalent by the distribution principle. The third is equivalent to $\Sigma_i a_i$ because $aa = a$, and thus all three are equivalent to one another.

The theorem enunciated having thus been proved true of every proposition having a single index, it only remains to show that if it be true of every proposition having n indices, it is true for every proposition having $n+1$ indices.

Now let Φ and $*$ stand either for Σ and $+$ or for Π and $\times$ respectively, so that

$$\Phi_i u_i = u_1 {}^* u_2 {}^* u_3 {}^* \text{ etc.}$$

This may represent any proposition in $n+1$ indices of which i is the first. When i is fixed in a certain individual as in u_1, u_2, etc., u becomes a proposition in n indices. Let u^2 represent any legitimate square of u. Then the square of the whole expression with identification of i in the two factors is

$$\Phi_i u_i^2 = u_1^{2*} u_2^{2*} u_3^{2*} \text{ etc.}$$

and since by hypothesis

$$u_1 -\!\!< u_1^2 \qquad u_2 -\!\!< u_2^2 \qquad u_3 -\!\!< u_3^2 \text{ etc.}$$

it follows that

$$\Phi_i u_i -\!\!< \Phi_i u_i^2.$$

Thus it is shown that the rule holds for every proposition in $n+1$ indices, if the first of them is identified in the two factors. But according to the rule, the first cannot be diversified unless all are diversified; and all may be diversified, by an obvious extension of the formulæ relating to propositions in a single index. The theorem is therefore proved.

If the proposition is raised to a higher power, the diversifications toward the right may be represented as branchings from a stem, thus,

After the involution has once been performed, further identification and diversification of indices may be effected by applying the following formulæ.

For identifications	*For diversifications*
$\Pi_i \Pi_j l_{ij} -\!\!< \Pi_i l_{ii}$	$\Pi_i l_{ii} -\!\!< \Pi_i \Sigma_j l_{ij}$
$\Sigma_i \Pi_j l_{ij} -\!\!< \Sigma_i l_{ii}$	$\Sigma_i l_{ii} -\!\!< \Sigma_i \Sigma_j l_{ij}$

403F. Besides these formulæ, the following are occasionally useful

$$\Pi_i m_i = \Pi_i \Pi_j m_i m_j \qquad \Sigma_i m_i = \Sigma_i \Sigma_j (m_i + m_j).$$

The following example shows the utility of changing the indices, even without involution. From the premiss

$$\Pi_i \Pi_j (m_i l_{ij} + \overline{m}_i \bar{l}_{ij})$$

let it be required to eliminate m. The immediate dropping of this token would only yield an identical proposition. But by separating the Boolian into factors and then changing the indices, we have

$$\begin{aligned} & \Pi_i \Pi_j (m_i l_{ij} + \overline{m}_i \bar{l}_{ij}) \\ = \ & \Pi_i \Pi_j (m_i + \bar{l}_{ij})(l_{ij} + \overline{m}_i) \\ = \ & \Pi_i \Pi_j (m_i + \bar{l}_{ij}) \Pi_j (l_{ij} + \overline{m}_i) \\ = \ & \Pi_i \Pi_j (m_i + \bar{l}_{ij}) \Pi_k (l_{ik} + \overline{m}_i) \\ = \ & \Pi_i \Pi_j \Pi_k (m_i + \bar{l}_{ij})(l_{ik} + \overline{m}_i) \\ = \ & \Pi_i \Pi_j \Pi_k (m_i l_{ik} + \overline{m}_i \bar{l}_{ij}) \end{aligned}$$

Now dropping m we have

$$\Pi_i \Pi_j \Pi_k (l_{ik} + \bar{l}_{ij})$$

i.e. whatever is in the relation l to anything is in that relation to everything.

403G. An artifice which I have not included among the regular steps of the inferential procedure but which is occasionally useful, consists in taking the latter part of the quantifier away from its position at the head of the proposition and putting it before the part which alone it concerns. The formulæ governing this operation are

$$\Pi_j a_i b_{ij} = a_i \Pi_i b_{ij} \qquad \Sigma_j a_i b_{ij} = a_i \Sigma_j b_{ij}$$
$$\Pi_j (a_i + b_{ij}) = a_i + \Pi_j b_{ij} \qquad \Sigma_j (a_i + b_{ij}) = a_i + \Sigma_j b_{ij}.$$

This transformation will generally be used in connection with the formulæ

$$a \prec a+b \text{ and } a+xb = a\bar{x}+xb.*$$

403H. The special peculiarity of ordinary algebra has given us the false notion that inverse operations are the general means of solving algebraical problems. But the study of general algebra shows that inverse operations lead to determinate results only in special cases — that what is called "the general case" is in truth a mere form of speciality — and that the truly general method of elimination is by performing a direct operation which will give a constant result whatever the value of the variable. Thus, in ordinary algebra, it happens to be the case that every quantity has a reciprocal so that

$$\frac{1}{x} \cdot x = 1.$$

So in logical algebra, the only way of eliminating any token is by means of the properties of special terms of second intention, such as

$$a_{ij}\bar{a}_{xy} \prec \bar{1}_{ix} + 1_{jy} \qquad a_i\bar{a}_i \prec 0.$$

In order to bring these formulæ to bear it may be necessary to multiply the premiss into itself, to manipulate the indices, and use processes of association and of development by distribution; but whatever cannot be eliminated by these means cannot be eliminated at all.

403I. The universes of marks† to which the tokens q, r and others like these refer are, in reference to the combinations of objects to which they are attached, unlimited universes.

* The second formula is incorrect. The second half should read:

$$a\bar{x} + xb + ax\bar{b}; \text{ or } a\bar{b} + xb + a\bar{x}b.$$

† Cf. 2.517–27.

(Compare the Thomist doctrine of angelic natures.) That is to say, every lot of objects has some quality common and peculiar to the objects composing it; every lot of pairs has some relation subsisting between the first and second members of each one of those pairs and between no others. In other regards, the universes are not unlimited; of characters familiar to us there is quite a limited number; of colours definable by Newton's diagram not all can [co-?]exist. In like manner, the universes of quantity, position, etc., of mathematics are unlimited universes. Of the objects of such a universe everything is true, which can be true; every proposition is true from which the unlimited universe cannot be eliminated without yielding a true proposition.

As a general rule, every proposition in $\Sigma_a q_a$ and $\Sigma_a r_a$ is true; but there are many exceptions.

To illustrate the application of this principle, consider the ninth icon.* This is written

$$\Pi_a \Sigma_\kappa \Pi_b q_{\kappa a}(\bar{q}_{\kappa b} + 1_{ab})$$

Now κ can only be eliminated from this without involution in two ways; first, eliminating $q_{\kappa a}$ and $q_{\kappa b}$ independently, we have

$$\Pi_a \Pi_b(\bar{1}_{ab} + 1_{ab}),$$

an identical proposition; second, identifying b with a, we get

$$\Pi_a 1_{aa},$$

another identical proposition. We next proceed to square the proposition; and we have

$$\Pi_a \Sigma_\kappa \Pi_b \Pi_c q_{\kappa a}(\bar{q}_{\kappa b} + 1_{ab})(\bar{q}_{\kappa c} + 1_{ac}).$$

We must identify κ in the factors or we should reach no result; and this forces us to identify a. But it is plain that we cannot from this square eliminate in any new way, and therefore we could not from any higher power, and consequently the original proposition is proved true.

Suppose, however, we had written that proposition

$$\Sigma_\kappa \Pi_a \Pi_b q_{\kappa a}(\bar{q}_{\kappa b} + 1_{ab}).$$

Squaring this we have

$$\Sigma_\kappa \Pi_a \Pi_b \Pi_c \Pi_d q_{\kappa a} q_{\kappa c}(\bar{q}_{\kappa b} + 1_{ab})(\bar{q}_{\kappa d} + 1_{cd}).$$

* 399.

Now identify d with a, c with b, and we have

$$\Pi_a \Pi_b 1_{ab},$$

which is absurd.*

403J. By the same principle, we can if we please solve the example of 400 as follows. Given the premiss

$$\Pi_\kappa (q_{\kappa i} + \bar{q}_{\kappa j}):$$

We multiply the square of this by an identical proposition, thus,

$$\Pi_\lambda \Pi_\kappa \Pi_\mu (q_{\lambda i} + \bar{q}_{\lambda i})(q_{\lambda j} + \bar{q}_{\lambda j})(q_{\mu i} + \bar{q}_{\mu j})(q_{\kappa i} + \bar{q}_{\kappa j}).$$

Identifying μ with λ, we get

$$\Pi_\lambda \Pi_\kappa (q_{\lambda i} + \bar{q}_{\lambda i})(q_{\lambda i} + \bar{q}_{\lambda j})(q_{\lambda i} + \bar{q}_{\lambda j})(q_{\kappa i} + \bar{q}_{\kappa j}).$$

We now introduce $\bar{q}_{\kappa i}$ and $q_{\kappa j}$ into the identical proposition wherever we please so long as they cannot be eliminated without making the proposition otherwise than identical; and this condition will be fulfilled so long as we do not introduce $q_{\kappa i}$ or $\bar{q}_{\kappa j}$. Only Π_κ must be changed to Σ_κ. We get then

$$\Pi_\lambda \Sigma_\kappa (q_{\lambda i}\bar{q}_{\kappa i} + \bar{q}_{\gamma i})(q_{\lambda j} + q_{\kappa j}\bar{q}_{\lambda i})(q_{\lambda i} + \bar{q}_{\lambda j})(q_{\kappa i} + \bar{q}_{\kappa j}).$$

This gives successively

$$\Pi_\lambda \Sigma_q (q_{\lambda i}\bar{q}_{\kappa i}q_{\lambda j}\bar{q}_{\kappa j} + \bar{q}_{\lambda i}q_{\kappa j}\bar{q}_{\lambda j}q_{\kappa i})$$

and

$$\Pi_\lambda (q_{\lambda i}q_{\lambda j} + \bar{q}_{\lambda i}\bar{q}_{\lambda j}).$$

But the same conclusion can be reached much more easily by identifying q_κ with 1_j, which we have a right to do on account of the universal quantifier. Thus from

$$\Pi_i \Sigma_j \Pi_\kappa (q_{\kappa i} + \bar{q}_{\kappa j})$$

we get

$$\Pi_i \Sigma_j (1_{ji} + \bar{1}_{jj}) = \Pi_i \Sigma_j 1_{ji}.$$

Hence the proposition

$$\Pi_\lambda (q_{\lambda i}q_{\lambda j} + \bar{q}_{\lambda i}\bar{q}_{\lambda j})$$

holds, because it becomes identical when i is substituted for j.

403K. Let us now consider some examples of a somewhat more difficult kind. Given the proposition

$$\Sigma_\kappa \Pi_a \Sigma_b \Pi_c \{ \bar{x}_a + r_{\kappa ab} y_b (\bar{r}_{\kappa cb} + 1_{ac}) \},$$

* I.e., every a is identical with every b.

that is, there is a relation κ such that whatever object be taken, either this is not an x or it stands in the relation k to some object which is a y and to which nothing except that x stands in that relation. Required to find all the propositions deducible from this by elimination of κ. The simple omission of factors gives

$$\Pi_a \Sigma_b (\bar{x}_a + y_b),$$

that is, there either is no x or there is a y. We get nothing further by identifying c with a. There is therefore no further conclusion without involution. Squaring, with identification of κ (without which we should plainly reach nothing new) we have

$$\Sigma_\kappa \Pi_a \Sigma_b \Pi_c \Pi_d \Sigma_e \Pi_f \left\{ \bar{x}_a + r_{\kappa ab} y_b (\bar{r}_{\kappa cb} + 1_{ac}) \right\} \left\{ \bar{x}_d + r_{\kappa de} y_e (\bar{r}_{\kappa fe} + 1_{df}) \right\}.$$

The application to the Boolian of the common formula

$$(a+x)(b+y) \prec a+b+xy$$

gives

$$\bar{x}_a + \bar{x}_d + r_{\kappa ab} r_{\kappa de} y_b y_e (\bar{r}_{\kappa cb} + 1_{ac})(\bar{r}_{\kappa fe} + 1_{df}).$$

We now identify f with a; when the first and last factors of the last term become

$$r_{\kappa ab}(\bar{r}_{\kappa ae} + 1_{ad}).$$

But we have

$$r_{\kappa ab} \bar{r}_{\kappa ae} \prec \bar{1}_{be}.$$

Thus, we reach the conclusion

$$\Pi_a \Sigma_b \Pi_d \Sigma_e (\bar{x}_a + \bar{x}_d + 1_{ad} + y_b y_e \bar{1}_{be});$$

that is, there are either not two x's different from one another or there are two y's different from one another. A little examination will show that no other conclusion could be reached by elimination from the square.

Cubing the original proposition, we reach in a similar way

$$\Sigma_\kappa \Pi_a \Sigma_b \Pi_c \Pi_d \Sigma_e \Pi_f \Pi_g \Sigma_h \Pi_i \bar{x}_a + \bar{x}_d + \bar{x}_g + y_b y_e y_h r_{\kappa ab} r_{\kappa de} r_{\kappa gh} (\bar{r}_{\kappa cb} + 1_{ac}) (\bar{r}_{\kappa fe} + 1_{df})(\bar{r}_{\kappa ih} + 1_{gi})$$

Identifying f with a, i with d, g with c, the last term of the Boolian becomes

$$y_b y_e y_h r_{\kappa ab} r_{\kappa de} r_{\kappa ch} (\bar{r}_{\kappa cb} + 1_{ac})(\bar{r}_{\kappa ae} + 1_{ad})(\bar{r}_{\kappa dh} + 1_{cd})$$

whence

$$y_b y_e y_h (\bar{1}_{bh} + 1_{ac})(\bar{1}_{be} + 1_{ad})(\bar{1}_{eh} + 1_{cd}),$$

whence again

$$y_b y_e y_h \bar{1}_{bh} \bar{1}_{be} 1_{ch}{}^* + 1_{ac} + 1_{ad} + 1_{cd},$$

so that we have the conclusion

$$\Pi_a \Sigma_b \Pi_c \Pi_d \Sigma_e \Sigma_h$$
$$\bar{x}_a + \bar{x}_d + \bar{x}_c + 1_{ac} + 1_{ad} + 1_{cd} + y_b y_e y_h \bar{1}_{be} \bar{1}_{bh} \bar{1}_{eh},$$

that is, there either are not three x's all different from one another or there are three y's all different from one another.

It is plain that by raising to the $n!$ power we should get any proposition of this form, and that no others could be obtained.

403L. Let us now seek all the propositions deducible, by elimination of κ, from

$$\Pi_\kappa \Sigma_a \Pi_b \Sigma_c \; u_a(\bar{w}_b + \bar{r}_{\kappa ab} + r_{\kappa cb} \bar{1}_{ac}).$$

We have at once

$$\Sigma_a u_a.$$

The following proposition is universally true:

$$\Pi_\alpha \Pi_\beta \Sigma_\gamma \Pi_i \Pi_j \left\{ \bar{r}_{\gamma ij}(\bar{r}_{\alpha ij} + \bar{r}_{\beta ij}) + r_{\gamma ij} r_{\alpha ij} r_{\beta ij} \right\}.$$

To prove this, note first that if we eliminate γ at once, we have an identical proposition. If we raise the whole to a power, there will be no additional mode of elimination, unless i, j and γ be identified in different factors; but then all the indices must be alike and nothing will be changed. As a special case of this formula we put u_i for $r_{\beta ij}$. This we can do, because

$$\Sigma_\beta \Pi_i \Pi_j \left\{ u_i r_{\beta ij} + \bar{u}_i \bar{r}_{\beta ij} \right\}$$

which is true by the same reasoning as that just used. The product of these two formulæ gives

$$\Pi_\alpha \Sigma_\gamma \Pi_i \Pi_j \left\{ \bar{r}_{\gamma ij}(\bar{r}_{\alpha ij} + \bar{u}_i) + r_{\gamma ij} r_{\alpha ij} u_i \right\}.$$

Multiplying this twice into the first proposition, identifying κ with γ, i with a, and j with b, in one factor, and κ with γ, i with c, j with b in the other, we have

$$\Pi_\alpha \Sigma_\gamma \Sigma_a \Pi_b \Sigma_c \; u_a(\bar{w}_b + \bar{r}_{\gamma ab} + r_{\gamma cb} \bar{1}_{ac}) \left\{ r_{\gamma ab}(\bar{r}_{\alpha ab} + \bar{u}_a) + r_{\gamma ab} r_{\alpha ab} u_a \right\}$$
$$\left\{ \bar{r}_{\gamma cb}(\bar{r}_{\alpha cb} + \bar{u}_c) + r_{\gamma cb} r_{\alpha cb} u_c \right\}.$$

This gives

$$\Pi_\alpha \Sigma_a \Pi_b \Sigma_c \; u_a(\bar{w}_b + \bar{r}_{\alpha ab} + r_{\alpha cb} u_c \bar{1}_{ac}).$$

* This should be $\bar{1}_{eh}$.

We have universally

$$\Sigma_\delta \Pi_m \Pi_n (r_{\delta mn} 1_{mn} + \bar{r}_{\delta mn} \bar{1}_{mn}).*$$

Multiplying this twice into the last proposition, identifying α with δ, and n with b, in both factors, m with a in one and with c in the other, we have

$$\Sigma_a \Pi_b \Sigma_c \ \dot{u}_a(\bar{w}_b + \bar{1}_{ab} + 1_{cb} u_c \bar{1}_{ac});$$

or since

$$1_{cb} \bar{1}_{ac} \prec \bar{1}_{ab}$$

$$\Sigma_a \Pi_b \ u_a(\bar{w}_b + \bar{1}_{ab});$$

and identifying b with a

$$\Sigma_a u_a \bar{w}_a.$$

Again, we have universally

$$\Sigma_\delta \Pi_m \Pi_n r_{\delta mn}.\dagger$$

Multiplying this twice into the proposition

$$\Pi_\alpha \Sigma_a \Pi_b \Sigma_c \ u_a(\bar{w}_b + \bar{r}_{\alpha ab} + r_{\alpha cb} u_c \bar{1}_{ac})$$

with the same identifications as before, we get

$$\Sigma_a \Pi_b \Sigma_c \ u_a(\bar{w}_b + u_c \bar{1}_{ac}),$$

that is, there is a u, and if there be a w there is a second u. This last proposition shows, by the formula

$$\Sigma_a \Sigma_c \ x_a x_c \bar{1}_{ac} \prec \Pi_a \Sigma_c (\bar{x}_a + x_c \bar{1}_{ac})$$

that the original proposition may be written in the form

$$\Pi_\alpha \Sigma_a \Pi_b \Sigma_c \ u_a \left\{ \bar{w}_b + u_c \bar{1}_{ac} (\bar{r}_{\alpha ab} + r_{\alpha cb}) \right\}$$

403M. Sixth step. The step numbered fifth [396] may more conveniently be separated into two. The first of these somewhat resembles the last; it is a sort of development or setting forth in detail the premiss, but instead of being founded upon a distribution formula it consists in raising the whole premiss to a power or multiplying it into itself. At each such multiplication any of the indices may be changed to new ones, their order in the quantifier being determined by the rules of

* I.e., any two terms are identical and related or they are not identical and the said relation does not relate them.

† I.e., every two terms are related by some relation.

the second step. To prove that this can be done we begin by confining our attention to the first index of the quantifier. The proposition is then either of the form $\Pi_i a_i$ or $\Sigma_i a_i$. We have obviously

$$\Pi_i a_i = a_1 a_2 a_3 \text{etc.} = \Pi_i \Pi_j a_i a_j$$
$$\Sigma_i a_i = a_1 + a_2 + a_3 + \text{etc.}$$
$$= (a_1 + a_2 + a_3 + \text{etc.})(a_1 + a_2 + a_3 + \text{etc.})$$
$$= \Sigma_i \Sigma_j a_i a_j.$$

Next consider the first two indices. The proposition is of one of the four forms

$$\Pi_i \Pi_j a_{ij} \qquad \Sigma_i \Pi_j a_{ij} \qquad \Pi_i \Sigma_j a_{ij} \qquad \Sigma_i \Sigma_j a_{ij}$$

For the first and last of these, we have only to apply the formulæ just obtained, with the first two [of step two in 396]. That is,

$$\Pi_i \Pi_j a_{ij} = \Pi_j \Pi_i a_{ij} = \Pi_i \Pi_k \Pi_j a_{ij} a_{kj}$$
$$= \Pi_j \Pi_i \Pi_k a_{ij} a_{kj} = \Pi_j \Pi_l \Pi_i \Pi_k a_{ij} a_{kl}.$$
$$\Sigma_i \Sigma_j a_{ij} = \Sigma_j \Sigma_i a_{ij} = \Sigma_i \Sigma_k \Sigma_j a_{ij} a_{kj}$$
$$= \Sigma_i \Sigma_j \Sigma_k \Sigma_l a_{ij} a_{kl}.$$

For the other two forms the proceeding is not more difficult. By the formulæ for a single index

$$\Sigma_i (\Pi_j a_{ij}) = \Sigma_i \Sigma_k (\Pi_j a_{ij})(\Pi_j a_{kj}) = \Sigma_i \Sigma_k \Pi_j \, a_{ij} a_{kj}$$
$$\Pi_i (\Sigma_j a_{ij}) = \Pi_i \Pi_k (\Sigma_j a_{ij})(\Sigma_j a_{kj})$$

XIV

*THE CRITIC OF ARGUMENTS**

§1. EXACT THINKING

404. "Critic" is a word used by Locke in English, by Kant in German, and by Plato in Greek, to signify the art of judging, being formed like "logic." I should shrink from heading my papers *Logic*, because logic, as it is set forth in the treatises, is an art far worse than useless, making a man captious about trifles and neglectful of weightier matters, condemning every inference really valuable and admitting only such as are really childish.

> It is naughty to do what mamma forbids;
> Now, mamma forbids me to cut off my hair:
> Therefore, it would be naughty for me to cut off my hair.

This is the type of reasoning to which the treatises profess to reduce all the reasonings which they approve. Reasoning from authority does, indeed, come to that, and in a broad sense of the word authority, such reasoning only. This reminds us that the logic of the treatises is, in the main, a heritage from the ages of faith and obedience, when the highest philosophy was conceived to lie in making everything depend upon authority. Though few men and none of the less sophisticated minds of the other sex ever, nowadays, plunge into the darkling flood of the medieval commentaries, and fewer still dive deep enough to touch bottom, everybody has received the impression they are full of syllogistic reasoning; and this impression is correct. The syllogistic logic truly reflects the sort of reasoning in which the men of the middle ages sincerely put their trust; and yet it is not true that even scholastic theology was sufficiently prostrate before its authorities to have possibly been, in the main, a product of ordinary syllogistic thinking. Nothing can be imagined more strongly marked in its distinctive character than the method of discussion of the old doctors. Their one

* *The Open Court*, vol. 6, pp. 3391–4 (1892).

recipe for any case of difficulty was a distinction. That drawn, they would proceed to show that the difficulties were in force against every member of it but one. Therein all their labor of thinking lies, and thence comes all that makes their philosophy what it is. Without pretending, then, to pronounce the last word on the character of their thought, we may, at least, say it was not, in their sense, syllogistic; since in place of syllogisms it is rather characterised by the use of such forms as the following:

Everything is either P or M,
S is not M;
$\therefore S$ is P.

This is commonly called disjunctive reasoning; but, for reasons which it would be too long to explain in full, I prefer to term it dilemmatic reasoning. Such modes of inference are, essentially, of the same character as the dilemma. Indeed, the regular stock example of the dilemma (for the logicians, in their gregariousness, follow their leader even down to the examples), though we find it set down in the second-century commonplace-book of Aulus Gellius, has quite the ring of a scholastic disquisition. The question, in this example, is, ought one to take a wife? In answering it, we first distinguish in regard to wives (and I seem to hear the Doctor subtillissimus saying: *primo distinguendum est de hoc nomine uxor*). A wife may mean a plain or a pretty wife. Now, a plain wife does not satisfy her husband; so one ought not to take a plain wife. But a pretty wife is a perpetual source of jealousy; so one ought still less to take a pretty wife. In sum, one ought to take no wife, at all. It may seem strange that the dilemma is not mentioned in a single medieval logic. It first appears in the *De Dialectica* of Rudolph Agricola.* But it should surprise nobody that the most characteristic form of demonstrative reasoning of those ages is left unnoticed in their logical treatises. The best of such works, at all epochs, though they reflect in some measure contemporary modes of thought, have always been considerably behind their times. For the methods of thinking that are living activities in men are not objects of reflective consciousness.

* Or possibly in some other Renaissance writing. My memory may deceive me; and my library is precious small.

They baffle the student, because they are a part of himself.

"Of thine eye I am eye-beam,"

says Emerson's sphynx. The methods of thinking men consciously admire are different from, and often, in some respects, inferior to those they actually employ. Besides, it is apparent enough, even to one who only knows the works of the modern logicians, that their predecessors can have been little given to seeing out of their own eyes, since, had they been so, their sequacious successors would have been religiously bound to follow suit.

405. One has to confess that writers of logic-books have been, themselves, with rare exceptions, but shambling reasoners. How wilt thou say to thy brother, Let me pull out the mote out of thine eye; and behold, a beam is in thine own eye? I fear it has to be said of philosophers at large, both small and great, that their reasoning is so loose and fallacious, that the like in mathematics, in political economy, or in physical science, would be received in derision or simple scorn. When, in my teens, I was first reading the masterpieces of Kant, Hobbes, and other great thinkers, my father, who was a mathematician, and who, if not an analyst of thought, at least never failed to draw the correct conclusion from given premisses, unless by a mere slip, would induce me to repeat to him the demonstrations of the philosophers, and in a very few words would usually rip them up and show them empty. In that way, the bad habits of thinking that would otherwise have been indelibly impressed upon me by those mighty powers, were, I hope, in some measure, overcome. Certainly, I believe the best thing for a fledgling philosopher is a close companionship with a stalwart practical reasoner.

406. How often do we hear it said that the study of philosophy requires *hard thinking!* But I am rather inclined to think a man will never begin to reason well about such subjects, till he has conquered the natural impulse to making spasmodic efforts of mind. In mathematics, the complexity of the problems renders it often a little difficult to hold all the different elements of our mental diagrams in their right places. In a certain sense, therefore, hard thinking *is* occasionally requisite in that discipline. But metaphysical philosophy does not

present any such complications, and has no work that *hard thinking* can do. What is needed above all, for metaphysics, is thorough and mature thinking; and the particular requisite to success in the critic of arguments is exact and diagrammatic thinking.

407*. To illustrate my meaning, and at the same time to justify myself, in some degree, for conceding all I have to the prejudice of logicians, I will devote the residue of the space which I can venture to occupy today, to the examination of a statement which has often been made by logicians, and often dissented from, but which I have never seen treated otherwise than as a position quite possible for a reputable logician. I mean the statement that the principle of identity is the necessary and sufficient condition of the validity of all affirmative syllogisms, and that the principles of contradiction and excluded middle, constitute the additional necessary and sufficient conditions for the validity of negative syllogisms. The principle of identity, expressed by the formula "*A* is *A*," states that the relation of subject to predicate is a relation which every term bears to itself. The principle of contradiction, expressed by the formula "*A* is not not *A*," might be understood in three different senses; first, that any term is in the relation of negation to whatever term is in that relation to it, which is as much as to say that the relation of negation is its own converse; second, that no term is in the relation of negation to itself; third, that every term is in the relation of negation to everything but itself. But the first meaning is the best, since from it the other two readily follow as corollaries. The principle of excluded middle, expressed by the formula "Not not *A* is *A*," may also be understood in three senses; first, that every term, *A*, is predicable of anything that is in the relation of negation to a term which is in the same relation to it, *A*; second, that the objects of which any term, *A*, is predicable together with those of which the negative of *A* is predicable together make up all the objects possible; third, that every term, *A*, is predicable of whatever is in the relation of negation to everything but *A*. But, as before, the first meaning is to be preferred, since from it the others are immediately deducible.

* Cf. 2.593ff

408. There is but one mood of universal affirmative syllogism. It is called *Barbara*, and runs thus:

Any M is P,
Any S is M;
∴ Any S is P.

Now the question is, what one of the properties of the relation of subject to predicate is it, with the destruction of which alone this form of inference ceases invariably to yield a true conclusion from true premisses? To find that out the obvious way is to destroy all the properties of the relation in question, so as to make it an entirely different relation, and then note what condition this relation must satisfy in order to make the inference valid. Putting *loves* in place of *is*, we get:

M loves P,
S loves M;
∴ S loves P.

That this should be universally true, it is necessary that every lover should love whatever his beloved loves. A relation of which the like is true is called a *transitive* relation. Accordingly, the condition of the validity of *Barbara* is that the relation expressed by the copula should be a transitive relation. This statement was first accurately made by De Morgan*; but it is in substantial agreement with the doctrine of Aristotle. The analogue of the principle of identity, when *loves* is the copula of the proposition, is that everybody loves himself. This would plainly not suffice of itself to make the inferential form valid; nor would its being false prevent that form from being valid, provided loving were a transitive relation. Thus, by a little exact thinking, the principle of identity is clearly seen to be neither a sufficient nor a necessary condition for the truth of *Barbara*.

409. Let us now examine the negative syllogisms. The simplest of these is *Celarent*, which runs as follows:

Any M is not P,
Any S is M;
∴ Any S is not P

* "On the Syllogism," II, *Transactions, Cambridge Philosophical Society*, vol. 9, p. 104, (1851).

Let us substitute *injures* for *is not*. Then the form becomes

Every M injures P,
Every S is M;
∴ Every S injures P.

This is a good inference, still, no matter what sort of relation injuring is. Consequently, this syllogism is dependent upon no property of negation, except that it expresses a relation. Let us, in the last form, substitute *loves* for *is*. Then, we get

M injures P,
S loves M;
∴ S injures P.

In order that this should hold good irrespective of the nature of the relation of injuring, it is necessary that nobody should love anybody but himself. A relation of that sort is called a *sibi-relation* or *concurrency*.* The necessary and sufficient condition of the validity of *Celarent* is, then, that the copula should express a *sibi-relation*. This is *not* what the principle of identity expresses. Of course, every *sibi-relation* is transitive.

410. The next simplest of the universal negative syllogisms is *Camestres*, which runs thus:

Any M is P,
Any S is not P;
∴ Any S is not M.

Substitute *injures* for *is not*, and we get,

Every M is a P,
Every S injures every P;
∴ Every S injures every M.

This obviously holds because the injuring is to *every* one of the class injured. It would not do to reason,

Every M is a P,
Every S injures a P;
∴ Every S injures an M.

We see, then, that the principal reason of the validity of *Camestres* is that by *not*, we mean *not any*, and not *not some*. In logical lingo, this is expressed by saying that negative predicates are distributed. But the condition that the copula expresses a *sibi-relation* is also involved.

* See 136a.

411. The remaining universal negative syllogisms of the old enumeration, *Celantes* and *Cesare*, depend upon one principle. They are:

Celantes	*Cesare*
Any M is not P,	Any M is not P,
Any S is M;	Any S is P;
∴ Any P is not S.	∴ Any S is not M.

Substituting *fights* for *is not*, we get

Every M fights every P,
Every S is M;
∴ Every P fights every S.

Every M fights every P,
Every S is P;
∴ Every S fights every M.

What is requisite to the validity of these inferences is plainly that the relation expressed by *fights* should be its own converse, or that everything should fight whatever fights it. This is the analogue of the principle of contradiction.

412. We see, then, that the principles of universal syllogism of the ordinary sort are that the copula expresses a *sibi-relation*, not that it expresses an agreement, which is what the principle of identity states, and that the negative is its own converse, which is the law of contradiction.

413. The authors who say that the principle of identity governs affirmative syllogism give no proof of what they allege. We are expected to see it by "hard thinking." I fancy I can explain what this process of "hard thinking" is. By a spasm induced by self-hypnotisation you throw yourself into a state of mental vacancy. In this state the formula "A is A" loses its definite signification and seems quite empty. Being empty it is regarded as wonderfully lofty and precious. Fired into enthusiasm by the contemplation of it, the subject, with one wild mental leap, throws himself into the belief that it must rule all human reason. Consequently, it is the principle of syllogism. If this is, as I suspect, what hard thinking means, it is of no use in philosophy.

414. As for the principle of excluded middle, the only syllogistic forms it governs are the dilemmatic ones.

Any not P is M,
Any S is not M;
$\therefore$ Any S is P.

Putting *admiring* for *not*, we have:

Everything admiring every P is an M,
Every S admires every M;
$\therefore$ Every S is a P.

To make this good, it must be that the only person who admires everybody that admires a given person is that person. This is the analogue of "everything not not A is A," which is the principle of excluded middle.

§2. THE READER IS INTRODUCED TO RELATIVES*

415. There is a melancholy book entitled *Astronomy Without Mathematics*. The author, an F. R. A. S., presumably knew something of astronomy; therefore, I pity him. I think I hear his groans and maledictions, as he wrote the book, over the initial lie to which he had committed himself, that it is possible to convey any idea of the science of astronomy without making use of mathematics. He could tell roughly how the planets go round the sun, and make his readers think they knew what the error of the ancient system was (namely, that all went round the earth — really, no error), and could set down surprising figures about the stars (beaten, however, by Buddhistic numbers both in magnitude and in intellectual value). A book so made might well have been called "The Story of the Heavens" (in anticipation of Dr. Ball's splendid volume, which, promising little, performs much), but it was not the "astronomy" stipulated for in the title page. When, in a neighbor's house yesterday, my eye lit upon that book, I shuddered. For I too have engaged myself by the title of these papers to produce something of solid value to my readers; but, thank God, I have not agreed to do it without the use of mathematics. I came home and pondered; and have decided

* *The Open Court*, vol. 6, pp. 3416–8, (1892).

that, in order to fulfill legitimate expectations, I must begin with a few chapters upon certain dry and somewhat technical matters that underlie the more interesting questions concerning reasoning. Do not fear a repetition of matter to be found in common textbooks. I shall suppose the reader to be acquainted with what is contained in Dr. Watts's *Logick*, a book very cheap and easily procured, and far superior to the treatises now used in colleges, being the production of a man distinguished for good sense. I mean to bring out a reprint of it, with extensive annotations, whenever I can find an eligible publisher.* Though a life-long student of reasonings, I know no way of giving the reader the benefit of what I ought to have learned, without asking him to go through with some irksome preliminary thinking about relations.

For this subject, although always recognised as an integral part of logic, has been left untouched on account of its intricacy. It is as though a geographer, finding the whole United States, its topography, its population, its industries, etc., too vast for convenient treatment, were to content himself with a description of Nantucket. This comparison hardly, if at all, exaggerates the inadequacy of a theory of reasoning that takes no account of relative terms.

416. A *relation* is a fact about a number of things. Thus the fact that a locomotive blows off steam constitutes a relation, or more accurately a relationship (the *Century Dictionary*, under *relation*, 3, gives the terminology.† See also *relativity*, etc.) between the locomotive and the steam. In reality, every fact is a relation. Thus, that an object is blue consists of the peculiar regular action of that object on human eyes. This is what should be understood by the "relativity of knowledge."

417. Not only is every fact really a relation, but your thought of the fact *implicitly* represents it as such. Thus, when you think "this is blue," the demonstrative "this" shows you are thinking of something just brought up to your notice; while the adjective shows that you recognise a familiar idea as applicable to it. Thus, your thought, when explicated, develops into the thought of a fact concerning this thing and concerning the

* Nothing of the kind seems to have been published by Peirce, and there is no record of any relevant manuscript.

† Page 5057, ed. of 1889; see also 571f.

character of blueness. Still, it must be admitted that, antecedently to the unwrapping of your thought, you were not actually thinking of blueness as a distinct object, and therefore were not thinking of the relation as a relation.[1] There is an aspect of every relation under which it does not appear as a relation. Thus, the blowing off of steam by a locomotive may be regarded as merely an action of the locomotive, the steam not being conceived to be a thing distinct from the engine. This aspect we enphrase in saying, "the engine blows."

418. Thus, the question whether a fact is to be regarded as referring to a single thing or to more is a question of the form of proposition under which it suits our purpose to state the fact. Consider any argument concerning the validity of which a person might conceivably entertain for a moment some doubt. For instance, let the premiss be that from either of two provinces of a certain kingdom it is possible to proceed to any province by floating down the only river the kingdom contains, combined with a land-journey within the boundaries of one province; and let the conclusion be that the river, after touching every province in the kingdom, must again meet the one which it first left. Now, in order to show that this inference is (or that it is not) absolutely necessary, it is requisite to have something analogous to a diagram with different series of parts, the parts of each series being evidently related as those provinces are said to be, while in the different series something corresponding to the course of the river has all the essential variations possible; and this diagram must be so contrived that it is easy to examine it and find out whether the course of the river is in truth in every case such as is here proposed to be inferred. Such a diagram has got to be either auditory or visual, the parts being separated in the one case in time, in the other in space. But in order completely to exhibit the analogue of the conditions of the argument under examination, it will be necessary to use signs or symbols repeated in different places and in different juxtapositions, these signs being subject to certain "rules," that is, certain general relations associated with them by the mind. Such a method of forming a diagram

[1] In this connection, see James's, *Principles of Psychology*, vol. 1, pp. 237–271; *Briefer Course*, pp. 160 et seq. James is no logician, but it is not difficult to trace a connection between the points he makes and the theory of inference.

is called *algebra*. All speech is but such an algebra, the repeated signs being the words, which have relations by virtue of the meanings associated with them. What is commonly called *logical algebra* differs from other formal logic only in using the same formal method with greater freedom. I may mention that unpublished studies* have shown me that a far more powerful method of diagrammatisation than algebra is possible, being an extension at once of algebra and of Clifford's method of graphs; but I am not in a situation to draw up a statement of my researches.

419. Diagrams and diagrammatoidal figures are intended to be applied to the better understanding of states of things, whether experienced, or read of, or imagined. Such a figure cannot, however, show what it is to which it is intended to be applied; nor can any other diagram avail for that purpose. The where and the when of the particular experience, or the occasion or other identifying circumstance of the particular fiction to which the diagram is to be applied, are things not capable of being diagrammatically exhibited. Describe and describe and describe, and you never can describe a date, a position, or any homaloidal quantity. You may object that a map is a diagram showing localities; undoubtedly, but not until the law of the projection is understood, nor even then unless at least two points on the map are somehow previously identified with points in nature. Now, how is any diagram ever to perform that identification? If a diagram cannot do it, algebra cannot: for algebra is but a sort of diagram; and if algebra cannot do it, language cannot: for language is but a kind of algebra. It would, certainly, in one sense be extravagant to say that we can never tell what we are talking about; yet, in another sense, it is quite true. The meanings of words ordinarily depend upon our tendencies to weld together qualities and our aptitudes to see resemblances, or, to use the received phrase, upon associations by *similarity;* while experience is bound together, and only recognisable, by forces acting upon us, or, to use an even worse chosen technical term, by means of associations by *contiguity*. Two men meet on a country road. One says to the other, "that house is on fire." "What house?" "Why, the house about a mile to my right." Let this

* Paper no. XVI and see vol. 4, bk. II.

speech be taken down and shown to anybody in the neighboring village, and it will appear that the language by itself does not fix the house. But the person addressed sees where the speaker is standing, recognises his *right* hand side (a word having a most singular mode of signification) estimates a *mile* (a length having no geometrical properties different from other lengths), and looking there, sees a house. It is not the language alone, with its mere associations of similarity, but the language taken in connection with the auditor's own experiential associations of contiguity, which determines for him what house is meant. It is requisite then, in order to show what we are talking or writing about, to put the hearer's or reader's mind into real, active connection with the concatenation of experience or of fiction with which we are dealing, and, further, to draw his attention to, and identify, a certain number of particular points in such concatenation. If there be a reader who cannot understand my writings, let me tell him that no straining of his mind will help him: his whole difficulty is that he has no personal experience of the world of problems of which I am talking, and he might as well close the book until such experience comes. That the diagrammatisation is one thing and the application of the diagram quite another, is recognised obscurely in the structure of such languages as I am acquainted with, which distinguishes the *subjects* and *predicates* of propositions. The subjects are the indications of the things spoken of, the predicates, words that assert, question, or command whatever is intended.* Only, the shallowness of syntax is manifest in its failing to recognise the impotence of mere words, and especially of common nouns, to fulfil the function of a grammatical subject. Words like *this*, *that*, *lo*, *hallo*, *hi there*, have a direct, forceful action upon the nervous system, and compel the hearer to look about him; and so they, more than ordinary words, contribute towards indicating what the speech is about. But this is a point that grammar and the grammarians (who, if they are faithfully to mirror the minds of the language-makers, can hardly be scientific analysts) are so far from seeing as to call demonstratives, such as *that* and *this*,

* Cf. discussion in vol. 2, bk. II, ch. 4, on the nature of propositions. The previous chapters of the same book should clarify what follows.

pronouns — a literally preposterous designation, for nouns may more truly be called pro-demonstratives.*

420. If upon a diagram we mark two or more points to be identified at some future time with objects in nature,[1] so as to give the diagram at that future time its meaning; or if in any written statement we put dashes in place of two or more demonstratives or pro-demonstratives, the professedly incomplete representation resulting may be termed a *relative rhema*. It differs from a relative *term* only in retaining the "copula," or signal of assertion. If only one demonstrative or pro-demonstrative is erased, the result is a *non-relative rhema*. For example, "— buys — from — for the price —," is a relative rhema; it differs in a merely secondary way from

"— is bought by — from — for —,"
from "— sells — to — for —,"
and from "— is paid by — to — for —."

On the other hand, "— is mortal" is a non-relative rhema.

421. A rhema is somewhat closely analogous to a chemical atom or radicle with unsaturated bonds.† A non-relative rhema is like a univalent radicle; it has but one unsaturated bond. A relative rhema is like a multivalent radicle. The blanks of a rhema can only be filled by terms, or, what is the same thing, by "something which" (or the like) followed by a rhema; or, two can be filled together by means of "itself" or the like. So, in chemistry, unsaturated bonds can only be saturated by joining two of them, which will usually, though not necessarily, belong to different radicles. If two univalent radicles are united, the result is a saturated compound. So, two non-relative rhemas being joined give a complete proposition. Thus, to join "— is mortal" and "— is a man," we have "X is mortal and X is a man," or some man is mortal. So likewise, a saturated compound may result from joining two bonds of a bivalent radicle;[2] and, in the same way, the two blanks of a

* Cf. 2.287n.

[1] *Nature*, in connection with a picture, copy, or diagram, does not necessarily denote an object not fashioned by man, but merely the object represented, as something existing apart from the representation.

† Cf. 469f, 1.289f, 1.346.

[2] Thus, *CO*, which appears as such a radicle in formic acid, makes of itself a saturated compound.

dual rhema may be joined to make a complete proposition. Thus, "— loves —," "*X* loves *X*," or something loves itself. A univalent radicle united to a bivalent radicle gives a univalent radicle (as H-O-); and, in like manner, a non-relative rhema, joined to a dual rhema, gives a non-relative rhema. Thus, "— is mortal" joined to "— loves —" gives "— loves something that is mortal," which is a non-relative rhema, since it has only one blank. Two, or any number of bivalent radicles united, give a bivalent radicle (as-O-O-S-O-O-), and so two or more dual rhemata give a dual rhema; as "— loves somebody that loves somebody that serves somebody that loves —." Non-relative and dual rhemata only produce rhemata of the same kind, so long as the junctions are by twos; but junctions of triple rhemata (or junctions of dual rhemata by threes), will produce all higher orders. Thus, "— gives — to —" and "— takes — from —," give "— gives — to somebody who takes — from —," a quadruple rhema. This joined to another quadruple rhema, as "— sells — to — for —," gives the sextuple rhema "— gives — to somebody who takes — from somebody who sells — to — for —." Accordingly, all rhemata higher than the dual may be considered as belonging to one and the same order; and we may say that all rhemata are either singular, dual, or plural.*

422. Such, at least, is the doctrine I have been teaching for twenty-five years, and which, if deeply pondered, will be found to enwrap an entire philosophy.† Kant taught that our fundamental conceptions are merely the ineluctable ideas of a system of logical forms; nor is any occult transcendentalism requisite to show that this is so, and must be so. Nature only appears intelligible so far as it appears rational, that is, so far as its processes are seen to be like processes of thought. I must take this for granted, for I have no space here to argue it. It follows that if we find three distinct and irreducible forms of rhemata, the ideas of these should be the three elementary conceptions of metaphysics. That there are three elementary forms of categories is the conclusion of Kant, to which Hegel subscribes; and Kant seeks to establish this from the analysis of formal logic. Unfortunately, his study of that subject was

* Cf. 63 and 1.347.

† Cf. vol. 1, bk. III.

so excessively superficial that his argument is destitute of the slightest value. Nevertheless, his conclusion is correct; for the three elements permeate not only the truths of logic, but even to a great extent the very errors of the profounder logicians. I shall return to them next week.* I will only mention here that the ideas which belong to the three forms of rhemata are firstness, secondness, thirdness; firstness, or spontaneity; secondness, or dependence; thirdness, or mediation.

423. But Mr. A. B. Kempe, in his important memoir on the "Theory of Mathematical Forms,"[1] presents an analysis which amounts to a formidable objection to my views. He makes diagrams of spots connected by lines; and it is easy to prove that every possible system of relationship can be so represented, although he does not perceive the evidence of this. But he shows (§ 68) that every such form can be represented by spots indefinitely varied, some of them being connected by lines, all of the same kind. He thus represents every possible relationship by a diagram consisting of only *two* different kinds of elements, namely, spots and lines between pairs of spots. Having examined this analysis attentively, I am of opinion that it is of extraordinary value. It causes me somewhat to modify my position, but not to surrender it. For, in the first place, it is to be remarked that Mr. Kempe's conception depends upon considering the diagram purely in its self-contained relations, the idea of its representing anything being altogether left out of view; while my doctrine depends upon considering how the diagram is to be connected with nature. It is not surprising that the idea of thirdness, or mediation, should be scarcely discernible when the representative character of the diagram is left out of account. In the second place, while it is not in the least necessary that the spots should be of different kinds, so long as each is distinguishable[2] from the others, yet it is necessary that the connections between the spots should be of two different kinds, which, in Mr. Kempe's diagrams, appear as lines and as the absence of lines. Thus,

* This is the last paper on logic to be published in *The Open Court*.

[1] *Philosophical Transactions* for 1886 [pp. 1–70]. No logician should fail to study this memoir.

[2] I use this word in its proper sense, and not to mean unlike, as Mr. Kempe does.

Mr. Kempe has, and must have, three kinds of elements in his diagrams, namely, one kind of spots, and two kinds of connections of spots. In the third place, the spots, or units, as he calls them, involve the idea of firstness; the two-ended lines, that of secondness; the attachment of lines to spots, that of mediation.

424. My position has been modified by the study of Mr. Kempe's analysis. For, having a perfect algebra for dual relations, by which, for instance, I could express that "*A* is at once lover of *B* and servant of *C*," I declared that this was inadequate for the expression of plural relations; since to say that *A* gives *B* to *C* is to say more than that *A* gives something to *C*, and gives to somebody *B*, which is given to *C* by somebody. But Mr. Kempe (§ 330) virtually shows that my algebra is perfectly adequate to expressing that *A* gives *B* to *C*; since I can express each of the following relations:

In a certain act, *D*, something is given by *A*;
In the act, *D*, something is given to *C*;
In the act, *D*, to somebody is given *B*.

This is accomplished by adding to the universe of concrete things the abstraction "this action." But I remark that the diagram fails to afford any formal representation of the manner in which this abstract idea is derived from the concrete ideas. Yet it is precisely in such processes that the difficulty of all difficult reasoning lies. We have an illustration of this in the circumstance that I was led into an error about the capability of my own algebra for want of just the idea that process would have supplied. The process consists, psychologically, in catching one of the transient elements of thought upon the wing and converting it into one of the resting places of the mind. The difference between setting down spots in a diagram to represent recognised objects, and making new spots for the creations of logical thought, is huge. To include this last as one of the regular operations of logical algebra is to make an intrinsic transmutation of that algebra. What that mutation was I had already shown before Mr. Kempe's memoir appeared.

XV

*THE REGENERATED LOGIC**

425. The appearance of Schröder's *Exact Logic*[1] has afforded much gratification to all those homely thinkers who deem the common practice of designating propositions as "unquestionable," "undoubtedly true," "beyond dispute," etc., which are known to the writer who so designates them to be doubted, or perhaps even to be disputed, by persons who with good mental capacities have spent ten or more years of earnest endeavor in fitting themselves to judge of matters such as those to which the propositions in question relate, to be no less heinous an act than a trifling with veracity, and who opine that questions of logic ought *not* to be decided upon philosophical principles, but on the contrary, that questions of philosophy ought to be decided upon logical principles, these having been themselves settled upon principles derived from the only science in which there has never been a prolonged dispute relating to the proper objects of that science. Among those homely thinkers the writer of this review is content to be classed.

Why should we be so much gratified by the appearance of a single book? Do we anticipate that this work is to convince the philosophical world? By no means; because we well know that prevalent philosophical opinions are not formed upon the above principles, nor upon any approach to them. A recent little paper by an eminent psychologist concludes with the remark that the verdict of a majority of four of a jury, provided the individual members would form their judgments independently, would have greater probability of being true than the unanimous verdict now is. Certainly, this may be

* *The Monist*, vol. 7, pp. 19–40, (1896).

[1] *Vorlesungen über die Algebra der Logik*, (Exakte Logik). Von Dr. Ernst Schröder, Ord. Professor der Mathematik an der technischen Hochschule zu Karlsruhe in Baden. Dritter Band. *Algebra und Logik der Relative*. Leipsic: B. G. Teubner. 1895. Price, 16M.

assented to; for the present verdict is not so much an opinion as a resultant of psychical and physical forces. But the remark seemed to me a pretty large concession from a man imbued with the idea of the value of modern opinion about philosophical questions formed according to that scientific method which the Germans and their admirers regard as the method of modern science — I mean, that method which puts great stress upon coöperation and solidarity of research even in the early stages of a branch of science, when independence of thought is the wholesome attitude, and gregarious thought is really sure to be wrong. For, as regards the verdict of German *university professors*, which, excepting at epochs of transition, has always presented a tolerable approach to unanimity upon the greater part of fundamental questions, it has always been made up as nearly as possible in the same way that the verdict of a jury is made up. Psychical forces, such as the spirit of the age, early inculcations, the spirit of loyal discipline in the general body, and that power by virtue of which one man bears down another in a negotiation, together with such physical forces as those of hunger and cold, are the forces which are mainly operative in bringing these philosophers into line; and none of these forces have any direct relation to reason. Now, these men write the larger number of those books which are so thorough and solid that every serious inquirer feels that he is obliged to read them; and his time is so engrossed by their perusal that his mind has not the leisure to digest their ideas and to reject them. Besides, he is somewhat overawed by their learning and thoroughness. This is the way in which certain opinions — or rather a certain verdict — becomes prevalent among philosophical thinkers everywhere; and reason takes hardly the leading part in the performance. It is true, that from time to time, this prevalent verdict becomes altered, in consequence of its being in too violent opposition with the changed spirit of the age; and the logic of history will usually cause such a change to be an advance toward truth in some respect. But this process is so slow, that it is not to be expected that any rational opinion about logic will become prevalent among philosophers within a generation, at least.

Nevertheless, hereafter, the man who sets up to be a logician without having gone carefully through Schröder's *Logic* will be tormented by the burning brand of *false pretender* in his conscience, until he has performed that task; and that task he cannot perform without acquiring habits of exact thinking which shall render the most of the absurdities which have hitherto been scattered over even the best of the German treatises upon logic impossible for him. Some amelioration of future treatises, therefore, though it will leave enough that is absurd, is to be expected; but it is not to be expected that those who form their opinions about logic or philosophy rationally, and therefore not gregariously, will ever comprise the majority even of philosophers. But opinions thus formed, and among such those formed by thoroughly informed and educated minds, are the only ones which need cause the homely thinker any misgiving concerning his own.

426. It is a remarkable historical fact that there is a branch of science in which there has never been a prolonged dispute concerning the proper objects of that science. It is the mathematics. Mistakes in mathematics occur not infrequently, and not being detected give rise to false doctrine, which may continue a long time. Thus, a mistake in the evaluation of a definite integral by Laplace, in his *Mécanique céleste*, led to an erroneous doctrine about the motion of the moon which remained undetected for nearly half a century. But after the question had once been raised, all dispute was brought to a close within a year. So, several demonstrations in the first book of Euclid, notably that of the sixteenth proposition, are vitiated by the erroneus assumption that a part is necessarily less than its whole. These remained undetected until after the theory of the non-Euclidean geometry had been completely worked out; but since that time, no mathematician has defended them; nor could any competent mathematician do so, in view of Georg Cantor's,* or even of Cauchy's discoveries. Incessant disputations have, indeed, been kept up by a horde of undisciplined minds about quadratures, cyclotomy, the theory of parallels, rotation, attraction, etc. But the disputants are one and all men who cannot discuss any mathematical problem without betraying their want of mathematical power and their gross ignorance of mathematics at every step. Again, there have been prolonged disputes among real mathe-

* See 2.30.

maticians concerning questions which were not mathematical or which had not been put into mathematical form. Instances of the former class are the old dispute about the measure of force, and that lately active concerning the number of constants of an elastic body; and there have been sundry such disputes about mathematical physics and probabilities. Instances of the latter class are the disputes about the validity of reasonings concerning divergent series, imaginaries, and infinitesimals. But the fact remains that concerning strictly mathematical questions, and among mathematicians who could be considered at all competent, there has never been a single prolonged dispute.

It does not seem worth while to run through the history of science for the sake of the easy demonstration that there is no other extensive branch of knowledge of which the same can be said.

Nor is the reason for this immunity of mathematics far to seek. It arises from the fact that the objects which the mathematician observes and to which his conclusions relate are objects of his mind's own creation. Hence, although his proceeding is not infallible — which is shown by the comparative frequency with which mistakes are committed and allowed — yet it is so easy to repeat the inductions upon new instances, which can be created at pleasure, and extreme cases can so readily be found by which to test the accuracy of the processes, that when attention has once been directed to a process of reasoning suspected of being faulty, it is soon put beyond all dispute either as correct or as incorrect.

427. Hence, we homely thinkers believe that, considering the immense amount of disputation there has always been concerning the doctrines of logic, and especially concerning those which would otherwise be applicable to settle disputes concerning the accuracy of reasonings in metaphysics, the safest way is to appeal for our logical principles to the science of mathematics, where error can only long go unexploded on condition of its not being suspected.

This double assertion, first, that logic ought to draw upon mathematics for control of disputed principles, and second that ontological philosophy ought in like manner to draw upon logic, is a case under a general assertion which was made by

Auguste Comte,* namely, that the sciences may be arranged in a series with reference to the abstractness of their objects, and that each science draws regulating principles from those superior to it in abstractness, while drawing data for its inductions from the sciences inferior to it in abstractness. So far as the sciences can be arranged in such a scale, these relationships must hold good. For if anything is true of a whole genus of objects, this truth may be adopted as a principle in studying every species of that genus. While whatever is true of a species will form a datum for the discovery of the wider truth which holds of the whole genus. Substantially the following scheme of the sciences† is given in the *Century Dictionary*‡:

MATHEMATICS

Philosophy { Logic, Metaphysics.

Science of Time	Geometry	
Nomological Psychics	Nomological Physics	{ Molar, Molecular, Ethereal
Classificatory Psychics	Classificatory Physics	{ Chemistry, Biology, or the chemistry of protoplasms
Descriptive Psychics	Descriptive Physics	

PRACTICAL SCIENCE.

Perhaps each psychical branch ought to be placed above the corresponding physical branch. However, only the first three branches concern us here.

428. Mathematics is the most abstract of all the sciences. For it makes no external observations, nor asserts anything as a real fact. When the mathematician deals with facts, they become for him mere "hypotheses"; for with their truth he refuses to concern himself. The whole science of mathematics is a science of hypotheses; so that nothing could be more completely abstracted from concrete reality. Philosophy is not quite so abstract. For though it makes no *special* observations,

* *La philosophie positive*, deuxième leçon.

† Cf. the classification of sciences in vol. 1, Bk. II.

‡ By Peirce, p. 5397, ed. of 1889.

as every other positive science does, yet it does deal with reality. It confines itself, however, to the universal phenomena of experience; and these are, generally speaking, sufficiently revealed in the ordinary observations of every-day life. I would even grant that philosophy, in the strictest sense, confines itself to such observations as *must* be open to every intelligence which can learn from experience. Here and there, however, metaphysics avails itself of one of the grander generalisations of physics, or more often of psychics, not as a governing principle, but as a mere datum for a still more sweeping generalisation. But logic is much more abstract even than metaphysics. For it does not concern itself with any facts not implied in the supposition of an unlimited applicability of language.

Mathematics is not a positive science; for the mathematician holds himself free to say that *A* is *B* or that *A* is not *B*, the only obligation upon him being, that as long as he says *A* is *B*, he is to hold to it, consistently. But logic begins to be a positive science; since there are some things in regard to which the logician is not free to suppose that they are or are not; but acknowledges a compulsion upon him to assert the one and deny the other. Thus, the logician is forced by positive observation to admit that there is such a thing as doubt, that some propositions are false, etc. But with this compulsion comes a corresponding responsibility upon him not to admit anything which he is not forced to admit.

429. Logic may be defined as the science of the laws of the stable establishment of beliefs. Then, *exact* logic will be that doctrine of the conditions of establishment of stable belief which rests upon perfectly undoubted observations and upon mathematical, that is, upon *diagrammatical*, or, *iconic*, thought. We, who are sectaries of "exact" logic, and of "exact" philosophy, in general, maintain that those who follow such methods will, so far as they follow them, escape all error except such as will be speedily corrected after it is once suspected. For example, the opinions of Professor Schröder and of the present writer diverge as much as those of two "exact" logicians well can; and yet, I think, either of us would acknowledge that, however serious he may hold the errors of the other to be, those errors are, in the first place, trifling in comparison

with the original and definite advance which their author has, by the "exact" method, been able to make in logic, that in the second place, they are trifling as compared with the errors, obscurities, and negative faults of any of those who do not follow that method, and in the third place, that they are chiefly, if not wholly, due to their author not having found a way to the application of diagrammatical thought to the particular department of logic in which they occur.

430. "Exact" logic, in its widest sense, will (as I apprehend) consist of three parts.* For it will be necessary, first of all, to study those properties of beliefs which belong to them as beliefs, irrespective of their stability. This will amount to what Duns Scotus† called *speculative grammar*. For it must analyse an assertion into its essential elements, independently of the structure of the language in which it may happen to be expressed. It will also divide assertions into categories according to their essential differences. The second part will consider to what conditions an assertion must conform in order that it may correspond to the "reality," that is, in order that the belief it expresses may be stable. This is what is more particularly understood by the word *logic*. It must consider, first, *necessary*, and second, *probable* reasoning. Thirdly, the general doctrine must embrace the study of those general conditions under which a problem presents itself for solution and those under which one question leads on to another. As this completes a triad of studies, or trivium, we might, not inappropriately, term the last study *Speculative rhetoric*. This division was proposed in 1867‡ by me, but I have often designated this third part as *objective logic*.

431. Dr. Schröder's Logic is not intended to cover all this ground. It is not, indeed, as yet complete; and over five hundred pages may be expected yet to appear. But of the seventeen hundred and sixty-six pages which are now before the public, only an introduction of one hundred and twenty-five pages rapidly examines the speculative grammar, while all the rest, together with all that is promised, is restricted to the

* The first and second parts are the topics of bks. II and III of vol. 2; the third is discussed in vol. 5 and 6.

† *Opera Omnia Collecta*, T. 1, pp. 45–76. L. Durand.

‡ 1.559.

deductive branch of logic proper. By the phrase "exact logic" upon his title-page, he means logic treated algebraically. Although such treatment is an aid to exact logic, as defined on the last page, it is certainly not synonymous with it. The principal utility of the algebraic treatment is stated by him with admirable terseness: it is "to set this discipline free from the fetters in which language, by force of custom, has bound the human mind."* Upon the algebra may, however, be based a calculus, by the aid of which we may in certain difficult problems facilitate the drawing of accurate conclusions. A number of such applications have already been made; and mathematics has thus been enriched with new theorems. But the applications are not so frequent as to make the elaboration of a facile calculus one of the most pressing desiderata of the study. Professor Schröder has done a great deal in this direction; and of course his results are most welcome, even if they be not precisely what we should most have preferred to gain.

432. The introduction, which relates to first principles, while containing many excellent observations, is somewhat fragmentary and wanting in a unifying idea; and it makes logic too much a matter of feeling.† It cannot be said to belong to exact logic in any sense. Thus, under β (Vol. I., p. 2) the reader is told that the sciences have to suppose, not only that their objects really exist, but also that they are knowable and that for every question there is a true answer and but one. But, in the first place, it seems more exact to say that in the discussion of one question nothing at all concerning a wholly unrelated question can be implied. And, in the second place, as to an inquiry presupposing that there is some one truth, what can this possibly mean except it be that there is one destined upshot to inquiry with reference to the question in hand — one result, which when reached will never be overthrown? Undoubtedly, we hope that this, *or something approximating to this*, is so, or we should not trouble ourselves to make the inquiry. But we do not necessarily have much confidence that it *is* so. Still less need we think it is so about the *majority* of the questions with which we concern ourselves. But in so exaggerating the presupposition, both in regard to its universality,

* Bd. 1, S. 118.

† Cf. 2.19.

its precision, and the amount of belief there need be in it, Schröder merely falls into an error common to almost all philosophers about all sorts of "presuppositions." Schröder (under ϵ, p. 5) undertakes to define a contradiction in terms without having first made an ultimate analysis of the proposition. The result is a definition of the usual peripatetic type; that is, it affords no analysis of the conception whatever. It amounts to making the contradiction in terms an ultimate unanalysable relation between two propositions — a sort of blind reaction between them. He goes on (under ζ, p. 9) to define, after Sigwart, logical consequentiality, as *a compulsion of thought.* Of course, he at once endeavors to avoid the dangerous consequences of this theory, by various qualifications. But all that is to no purpose. Exact logic will say that *C*'s following logically from *A* is a state of *things* which no impotence of thought can alone bring about, unless there is also an impotence of existence for *A* to be a fact without *C* being a fact. Indeed, as long as this latter impotence exists and can be ascertained, it makes little or no odds whether the former impotence exists or not. And the last anchor-hold of logic he makes (under ι) to lie in the correctness of a feeling! If the reader asks *why* so subjective a view of logic is adopted, the answer seems to be (under β, p. 2), that in this way Sigwart escapes the necessity of founding logic upon the theory of cognition. By the theory of cognition is usually meant an explanation of the possibility of knowledge drawn from principles of psychology. Now, the only sound psychology being a special science, which ought itself to be based upon a well-grounded logic, it is indeed a vicious circle to make logic rest upon a theory of cognition so understood. But there is a much more general doctrine to which the name theory of cognition might be applied. Namely, it is that speculative grammar, or analysis of the nature of assertion, which rests upon observations, indeed, but upon observations of the rudest kind, open to the eye of every attentive person who is familiar with the use of language, and which, we may be sure, no rational being, able to converse at all with his fellows, and so to express a doubt of anything, will ever have any doubt. Now, proof does not consist in giving superfluous and superpossible certainty to that which nobody ever did or ever will doubt, but in removing doubts which do, or at least might at some time, arise. A man first comes to the study of logic with an immense multitude of opinions upon a vast variety of topics; and they are held with a degree of confidence, upon which, after he has studied logic, he comes to look back with no little amusement. There remains, however, a small minority of opinions that logic never shakes; and among these are certain observations about assertions. The student would never have had a desire to learn logic if he had not paid some little attention to assertion, so as at least to attach a definite signification to assertion. So that, if he has not thought more accurately about assertions, he must at least be conscious, in some out-of-focus fashion, of certain properties of assertion. When he comes to the study, if he has a good teacher, these already dimly recognised facts will be placed before him in accurate formulation, and will be accepted as soon as he can clearly apprehend their statements.

433. Let us see what some of these are. When an assertion is made, there really is some speaker, writer, or other sign-maker who delivers it; and he supposes there is, or will be, some hearer, reader, or other interpreter who will receive it. It may be a stranger upon a different planet, an æon later; or it may be that very same man as he will be a second after. In any case, the deliverer makes signals to the receiver. Some of these signs (or at least one of them) are supposed to excite in the mind of the receiver familiar images, pictures, or, we might almost say, *dreams* — that is, reminiscences of sights, sounds, feelings, tastes, smells, or other sensations, now quite detached from the original circumstances of their first occurrence, so that they are free to be attached to new occasions. The deliverer is able to call up these images at will (with more or less effort) in his own mind; and he supposes the receiver can do the same. For instance, tramps have the habit of carrying bits of chalk and making marks on the fences to indicate the habits of the people that live there for the benefit of other tramps who may come on later. If in this way a tramp leaves an assertion that the people are stingy, he supposes the reader of the signal will have met stingy people before, and will be able to call up an image of such a person attachable to a person whose acquaintance he has not yet made. Not only is the outward significant word or mark a sign, but the image which it is

expected to excite in the mind of the receiver will likewise be a sign — a sign by resemblance, or, as we say, an *icon* — of the similar image in the mind of the deliverer, and through that also a sign of the real quality of the thing. This icon is called the *predicate* of the assertion. But instead of a single *icon*, or sign by resemblance of a familiar image or "dream," evocable at will, there may be a complexus of such icons, forming a composite image of which the whole is not familiar. But though the whole is not familiar, yet not only are the parts familiar images, but there will also be a familiar image of its mode of composition. In fact, two types of complication will be sufficient. For example, one may be conjunctive and the other disjunctive combination. Conjunctive combination is when two images are both to be used at once; and disjunctive when one or other is to be used. (This is not the most scientific selection of types; but it will answer the present purpose.) The sort of idea which an icon embodies, if it be such that it can convey any positive information, being applicable to some things but not to others, is called a *first intention*. The idea embodied by an icon which cannot of itself convey any information, being applicable to everything or to nothing, but which may, nevertheless, be useful in modifying other icons, is called a *second intention*.

434. The assertion which the deliverer seeks to convey to the mind of the receiver relates to some object or objects which have forced themselves upon his attention; and he will miss his mark altogether unless he can succeed in forcing those very same objects upon the attention of the receiver. No icon can accomplish this, because an icon does not relate to any particular thing; nor does its idea strenuously force itself upon the mind, but often requires an effort to call it up. Some such sign as the word *this*, or *that*, or *hullo*, or *hi*, which awakens and directs attention must be employed. A sign which denotes a thing by forcing it upon the attention is called an *index*. An index does not describe the qualities of its object. An object, in so far as it is denoted by an index, having *thisness*, and distinguishing itself from other things by its continuous identity and forcefulness, but not by any distinguishing characters, may be called a *hecceity*. A hecceity in its relation to the assertion is a *subject* thereof. An assertion may have a multitude of subjects; but to that we shall return presently.

435. Neither the predicate, nor the subjects, nor both together, can make an *assertion.* The assertion represents a compulsion which experience, meaning the course of life, brings upon the deliverer to attach the predicate to the subjects as a sign of them taken in a particular way. This compulsion strikes him at a certain instant; and he remains under it forever after. It is, therefore, different from the temporary force which the hecceities exert upon his attention. This new compulsion may pass out of mind for the time being; but it continues just the same, and will act whenever the occasion arises, that is, whenever those particular hecceities and that first intention are called to mind together. It is, therefore, a permanent conditional force, or *law.* The deliverer thus requires a kind of sign which shall signify a law that to objects of indices an icon appertains as sign of them in a given way. Such a sign has been called a *symbol.* It is the *copula* of the assertion.

436. Returning to the subjects, it is to be remarked that the assertion may contain the suggestion, or request, that the receiver *do* something with them. For instance, it may be that he is first to take any one, no matter what, and apply it in a certain way to the icon, that he is then to take another, perhaps this time a suitably chosen one, and apply that to the icon, etc. For example, suppose the assertion is: "Some woman is adored by all catholics." The constituent icons are, in the probable understanding of this assertion, three, that of a woman, that of a person, *A*, adoring another, *B*, and that of a non-catholic. We combine the two last disjunctively, identifying the non-catholic with *A*; and then we combine this compound with the first icon conjunctively, identifying the woman with *B*. The result is the icon expressed by, "*B* is a woman, and moreover, either *A* adores *B* or else *A* is a non-catholic." The subjects are all the things in the real world past and present. From these the receiver of the assertion is suitably to choose one to occupy the place of *B*; and then it matters not what one he takes for *A*. A suitably chosen object is a woman, and any object, no matter what, adores her, unless that object be a non-catholic. This is forced upon the deliverer by experience; and it is by no idiosyncrasy of his; so that it will be forced equally upon the receiver.

437. Such is the meaning of one typical assertion. An

assertion of *logical necessity* is simply one in which the subjects are the objects of any collection, no matter what. The consequence is, that the icon, which can be called up at will, need only to be called up, and the receiver need only ascertain by experiment whether he can distribute any set of indices in the assigned way so as to make the assertion false, in order to put the truth of the assertion to the test. For example, suppose the assertion of logical necessity is the assertion that from the proposition, "Some woman is adored by all catholics," it logically follows that "Every catholic adores some woman." That is as much as to say that, for every imaginable set of subjects, either it is false that some woman is adored by all catholics or it is true that every catholic adores some woman. We try the experiment. In order to avoid making it false that some woman is adored by all catholics, we must choose our set of indices so that there shall be one of them, *B*, such that, taking any one, *A*, no matter what, *B* is a woman, and moreover either *A* adores *B* or else *A* is a non-catholic. But that being the case, no matter what index, *A*, we may take, either *A* is a non-catholic or else an index can be found, namely, *B*, such that *B* is a woman, and *A* adores *B*. We see, then, by this experiment, that it is impossible so to take the set of indices that the proposition of consecution shall be false. The experiment may, it is true, have involved some blunder; but it is so easy to repeat it indefinitely, that we readily acquire any desired degree of certitude for the result.

438. It will be observed that this explanation of logical certitude depends upon the fact of speculative grammar that the predicate of a proposition, being essentially of an ideal nature, can be called into the only kind of existence of which it is capable, at will.

439. A not unimportant dispute has raged for many years as to whether hypothetical propositions (by which, according to the traditional terminology, I mean any compound propositions, and not merely those *conditional* propositions to which, since Kant, the term has often been restricted) and categorical propositions are one in essence. Roughly speaking, English logicians maintain the affirmative, Germans the negative. Professor Schröder is in the camp of the latter, I in that of the former.

440. I have maintained since 1867 that there is but one primary and fundamental logical relation, that of illation, expressed by *ergo*. A proposition, for me, is but an argumentation divested of the assertoriness of its premiss and conclusion. This makes every proposition a conditional proposition at bottom. In like manner a "term," or class-name, is for me nothing but a proposition with its indices or subjects left blank, or indefinite. The common noun happens to have a very distinctive character in the Indo-European languages. In most other tongues it is not sharply discriminated from a verb or participle. "Man," if it can be said to mean anything by itself, means "what I am thinking of is a man." This doctrine, which is in harmony with the above theory of signs, gives a great unity to logic; but Professor Schröder holds it to be very erroneous.

441. Cicero* and other ancient writers† mention a great dispute between two logicians, Diodorus and Philo, in regard to the significance of conditional propositions.‡ This dispute has continued to our own day. The Diodoran view seems to be the one which is natural to the minds of those, at least, who speak the European languages. How it may be with other languages has not been reported. The difficulty with this view is that nobody seems to have succeeded in making any clear statement of it that is not open to doubt as to its justice, and that is not pretty complicated. The Philonian view has been preferred by the greatest logicians. Its advantage is that it is perfectly intelligible and simple. Its disadvantage is that it produces results which seem offensive to common sense.

442. In order to explain these positions, it is best to mention that *possibility* may be understood in many senses; but they may all be embraced under the definition that that is pos-

* *Acad. Quaest.* II, 143.

† E.g. Sextus Empiricus, *Adv. Math.* VIII, 113–17.

‡ The Diodorians in opposition to the Philonians deny that material implication expresses what is usually meant by "if . . . then . . . ." For contemporary discussion see G. E. Moore, *Philosophical Studies*, p. 276ff; C. I. Lewis, *A Survey of Symbolic Logic*, esp. p. 324ff, and Paul Weiss, "Relativity in Logic," *The Monist*, October, 1928; "The Nature of Systems," *The Monist*, April/July, 1929; "Entailment and the Future of Logic," *Proceedings*, Seventh International Congress of Philosophy; E. J. Nelson, "Intensional Relations," *Mind*, October, 1930.

sible which, in a certain state of information, is not known to be false. By varying the supposed state of information all the varieties of possibility are obtained. Thus, *essential* possibility is that which supposes nothing to be known except logical rules. *Substantive* possibility, on the other hand, supposes a state of omniscience. Now the Philonian logicians have always insisted upon beginning the study of conditional propositions by considering what such a proposition means in a state of omniscience; and the Diodorans have, perhaps not very adroitly, commonly assented to this order of procedure. Duns Scotus* terms such a conditional proposition a "*consequentia simplex de inesse*." According to the Philonians, "If it is now lightening it will thunder," understood as a consequence *de inesse*, means "It is either not now lightening or it will soon thunder." According to Diodorus, and most of his followers (who seem here to fall into a logical trap), it means "It is now lightening and it will soon thunder."

443. Although the Philonian views lead to such inconveniences as that it is true, as a consequence *de inesse*, that if the Devil were elected president of the United States, it would prove highly conducive to the spiritual welfare of the people (because he will not be elected), yet both Professor Schröder and I prefer to build the algebra of relatives upon this conception of the conditional proposition. The inconvenience, after all, ceases to seem important, when we reflect that, no matter what the conditional proposition be understood to mean, it can always be expressed by a complexus of Philonian conditionals and denials of conditionals. It may, however, be suspected that the Diodoran view has suffered from incompetent advocacy, and that if it were modified somewhat, it might prove the preferable one.

444. The consequence *de inesse*, "if A is true, then B is true," is expressed by letting i denote the actual state of things, A_i mean that in the actual state of things A is true, and B_i mean that in the actual state of things B is true, and then saying "If A_i is true then B_i is true," or, what is the same thing, "Either A_i is not true or B_i is true." But an *ordinary* Philonian conditional is expressed by saying, "In *any* possible state of things, i, either A_i is not true, or B_i is true."

* *Quæstiones in Octo libror Physicorum Aristotelis*, L. 1, qu. II.

445. Now let us express the categorical proposition, "Every man is wise." Here, we let m_i mean that the individual object i is a man, and w_i mean that the individual object i is wise. Then, we assert that, "taking any individual of the universe, i, no matter what, either that object, i, is not a man or that object, i, is wise"; that is, whatever is a man is wise. That is, "whatever i can indicate, either m_i is not true or w_i is true. The conditional and categorical propositions are expressed in precisely the same form; and there is absolutely no difference, to my mind, between them. The *form* of relationship is the same.

446. I find it difficult to state Professor Schröder's objection to this, because I cannot find any clear-cut, unitary conception governing his opinion. More than once in his first volume promises are held out that § 28, the opening section of the second volume, shall make the matter plain. But when the second volume was published, all we found in that section was, as far as repeated examination has enabled me to see, as follows. First, hypothetical propositions, unlike categoricals, essentially involve the idea of time. When this is eliminated from the assertion, they relate only to two possibilities, what always is and what never is. Second, a categorical is always either true or false; but a hypothetical is either true, false, or meaningless. Thus, "this proposition is false" is meaningless; and another example is, "the weather will clear as soon as there is enough sky to cut a pair of trousers." Third, the supposition of negation is forced upon us in the study of hypotheticals, never in that of categoricals. Such are Schröder's arguments, to which I proceed to reply.

As to the idea of time, it *may* be introduced; but to say that the range of possibility in hypotheticals is always a unidimensional continuum is incorrect. "If you alone trump a trick in whist, you take it." The possibilities are that each of the four players plays any one of the four suits. There are 2^{16} different possibilities. Certainly, the universe in hypotheticals is far more frequently finite than in categoricals. Besides, it is an *ignoratio elenchi* to drag in time, when no logician of the English camp has ever alleged anything about propositions involving time. That is not the question.

Every proposition is either true or false, and something not a proposition, when considered as a proposition, is, from the Philonian point of view, true. To be objectionable, a proposition must assert something; if it is merely neutral, it is not positively objectionable, that is, it is not false. "This proposition is false," far from being meaningless, is self-contradictory. That is, it means two irreconcilable things. That it involves contradiction (that is, leads to contradiction if supposed true), is easily proved. For if it be true, it is true; while if it be true, it is false. Every proposition besides what it explicitly asserts, tacitly implies its own truth. The proposition is not true unless *both*, what it explicitly asserts and what it tacitly implies, are true. This proposition, being self-contradictory, is false; and hence, what it explicitly asserts is true. But what it tacitly implies (its own truth) is false.* The difficulty about the proposition concerning the piece of blue sky is not a logical one, at all. It is no more senseless than any proposition about a "red odor" which might be a term of a categorical.

The fact stated about negation is only true of the sorts of propositions which are commonly put into categorical and hypothetical shapes, and has nothing to do with the essence of the propositions. In a paper "On the Validity of the Laws of Logic" in the *Journal of Speculative Philosophy*, Vol. II.,† I have given a sophistical argument that black is white, which shows in the domain of categoricals the phenomena to which Professor Schröder refers as peculiar to hypotheticals.

The *consequentia de inesse* is, of course, the extreme case where the conditional proposition loses all its proper signification, owing to the absence of any range of possibilities. The conditional proper is, "In any possible case, i, either A_i is not true, or B_i is true." In the consequence *de inesse* the meaning sinks to, "In the true state of things, i, either A_i is not true or B_i is true."

447. My general algebra of logic (which is not that algebra of dual relations, likewise mine, which Professor Schröder prefers, although in his last volume he often uses this general algebra) consists in simply attaching indices to the letters of an expression in the Boolian algebra, making what I term a

* Cf. 2.352 and 2.618.

† Vol. 5, bk. II, ch. 3.

Boolian, and prefixing to this a series of "quantifiers," which are the letters Π and Σ, each with an index attached to it. Such a quantifier signifies that every individual of the universe is to be substituted for the index the Π or Σ carries, and that the non-relative product or aggregate of the results is to be taken.

448. Properly to express an ordinary conditional proposition the quantifier Π is required. In 1880, three years before I developed that general algebra, I published a paper containing a chapter on the algebra of the copula* (a subject I have since worked out completely in manuscript.)† I there noticed the necessity of such quantifiers properly to express conditional propositions; but the algebra of quantifiers not being at hand, I contented myself with considering consequences *de inesse*. Some apparently paradoxical results were obtained. Now Professor Schröder seems to accept these results as holding good in the general theory of hypotheticals; and then, since such results are in strong contrast with the doctrine of categoricals, he infers, in § 45 of his Vol. II., a great difference between hypotheticals and categoricals. But the truth simply is that such hypotheticals want the characteristic feature of conditionals, that of a range of possibilities.

449. In connexion with this point, I must call attention to a mere algebraical difference between Schröder and me. I retain Boole's idea that there are but two *values* in the system of logical quantity. This harmonises with my use of the general algebra. Any two numbers may be selected to represent those values. I prefer 0 and a positive logarithmic ∞. To express that something is A and something is not A, I write:

$$\infty = \Sigma_i A_i \qquad \infty = \Sigma_j \bar{A}_j$$

or, what is the same thing:

$$\Sigma_i A_i > 0 \qquad \Sigma_j \bar{A}_j > 0.$$

I have no objection to writing, *as a mere abbreviation*, which may, however, lead to difficulties, if not *interpreted:*

$$A > 0 \qquad \bar{A} > 0.$$

* 182–197.

† The editors have not considered it worth publishing. But see 4.277ff. and vol. 4, Bk. II.

But Professor Schröder understands these formulæ literally, and accordingly *rejects* Boole's conception of two values. He does not seem to understand my mode of apprehending the matter; and hence considers it a great limitation of my system that I restrict myself to two values. In fact, it is a mere difference of algebraical form of conception. I very much prefer the Boolian idea as more simple, and more in harmony with the general algebra of logic.

450. Somewhat intimately connected with the question of the relation between categoricals and hypotheticals is that of the quantification of the predicate. This is the doctrine that identity, or equality, is the fundamental relation involved in the copula. Holding as I do that the fundamental relation of logic is the illative relation, and that only in special cases does the premiss follow from the conclusion, I have in a consistent and thoroughgoing manner opposed the doctrine of the quantification of the predicate.* Schröder seems to admit some of my arguments; but still he has a very strong *penchant* for the equation.

Were I not opposed to the quantification of the predicate, I should agree with Venn† that it was a mistake to replace Boole's operation of [arithmetical] addition by the operation of [logical] aggregation, as most Boolians now do. I should consider the "principle of duality"‡ rather an argument *against* than *for* our modern practice. The algebra of dual relatives would be almost identical with the theory of matrices were addition retained; and this would be a great advantage.

451. It is Schröder's predilection for equations which motives his preference for the algebra of dual relatives, namely, the fact that in that algebra, even a simple undetermined inequality can be expressed as an equation. I think, too, that that algebra has merits; it certainly has uses to which Schröder seldom puts it. Yet, after all, it has too much formalism to greatly delight me — too many bushels of chaff *per* grain of wheat. I think Professor Schröder likes algebraic formalism better, or dislikes it less, than I.

* See 472 and 2.532–5.

† *Symbolic Logic*, p. 39ff.

‡ The principle of duality is expressible in the formulæ: $-(a+b)=(\bar{a}\bar{b})$ and $-(ab)=(\bar{a}+\bar{b})$.

He looks at the problems of logic through the spectacles of equations, and he formulates them, from that point of view, as he thinks, with great generality; but, as I think, in a narrow spirit. The great thing, with him, is to solve a proposition, and get a *value* of x, that is, an equation of which x forms one member without occurring in the other. How far such equation is *iconic*, that is, has a meaning, or exhibits the constitution of x, he hardly seems to care. He prefers general values to particular roots. Why? I should think the particular root alone of service, for most purposes, unless the general expressions were such that particular roots could be deduced from it — particular instances, I mean, *showing* the constitution of x. In most instances, a profitable solution of a mathematical problem must consist, in my opinion, of an exhaustive examination of special cases; and quite exceptional are those fortunate problems which mathematicians naturally prefer to study, where the enumeration of special cases, together with the pertinent truths about them, flow so naturally from the general statement as not to require separate examination.

I am very far from denying the interest and value of the problems to which Professor Schröder has applied himself; though there are others to which I turn by preference. Certainly, he has treated his problems with admirable power and clearness. I cannot in this place enter into the elementary explanations which would be necessary to illustrate this for more than a score of readers.

452. In respect to individuals, both non-relative and pairs, he has added some fundamental propositions to those which had been published. But he is very much mistaken in supposing that I have expressed contrary views. He simply mistakes my meaning.

453. In regard to algebraical signs, I cannot accept any of Professor Schröder's proposals except this one. While it would be a serious hindrance to the promulgation of the new doctrine to insist on new types being cut, and while I, therefore, think my own course in using the dagger as the sign of relative addition must be continued, yet I have always given that sign in its cursive form a scorpion-tail curve to the left; and it would be finical to insist on one form of curve rather than another. In almost all other cases, in my judgment, Professor Schröder's

signs can never be generally received, because they are at war with a principle, the general character of which is such that Professor Schröder would be the last of all men to wish to violate it, a principle which the biologists have been led to adopt in regard to their systematic nomenclature. It is that priority must be respected, or all will fall into chaos. I will not enter further into this matter in this article.*

454. Of what use does this new logical doctrine promise to be? The first service it may be expected to render is that of correcting a considerable number of hasty assumptions about logic which have been allowed to affect philosophy. In the next place, if Kant has shown that metaphysical conceptions spring from formal logic, this great generalisation upon formal logic must lead to a new apprehension of the metaphysical conceptions which shall render them more adequate to the needs of science. In short, "exact" logic will prove a stepping-stone to "exact" metaphysics. In the next place, it must immensely widen our logical notions. For example, a class consisting of a lot of things jumbled higgledy-piggledy must now be seen to be but a degenerate form of the more general idea of a *system*. Generalisation, which has hitherto meant passing to a larger class, must mean taking in the conception of the whole system of which we see but a fragment, etc., etc. In the next place, it is already evident to those who know what has already been made out, that that speculative rhetoric, or objective logic, mentioned at the beginning of this article, is destined to grow into a colossal doctrine which may be expected to lead to most important philosophical conclusions. Finally, the calculus of the new logic, which is applicable to everything, will certainly be applied to settle certain logical questions of extreme difficulty relating to the foundations of mathematics. Whether or not it can lead to any method of discovering methods in mathematics it is difficult to say. Such a thing is conceivable.

455. It is now more than thirty years since my first published contribution to "exact" logic.† Among other serious studies, this has received a part of my attention ever since. I have contemplated it in all sorts of perspectives and have

* See vol. 2, bk. II, ch. 1, for a discussion of the "ethics of terminology."

† Peirce's first contribution to 'exact' logic is published in the Appendix to vol. 2.

often reviewed my reasons for believing in its importance. My confidence that the key of philosophy is here, is stronger than ever after reading Schröder's last volume. One thing which helps to make me feel that we are developing a living science, and not a dead doctrine, is the healthy mental independence it fosters, as evidenced, for example, in the divergence between Professor Schröder's opinions and mine. There is no bovine nor ovine gregariousness here. But Professor Schröder and I have a common method which we shall ultimately succeed in applying to our differences, and we shall settle them to our common satisfaction; and when that method is pouring in upon us new and incontrovertible positively valuable results, it will be as nothing to either of us to confess that where he had not yet been able to apply that method he has fallen into error.

XVI

*THE LOGIC OF RELATIVES**

§1. THREE GRADES OF CLEARNESS†

456. The third volume of Professor Schröder's *Exact Logic*,[1] which volume bears separately the title I have chosen for this paper, is exciting some interest even in this country. There are in America a few inquirers into logic, sincere and diligent, who are not of the genus that buries its head in the sand — men who devote their thoughts to the study with a view to learning something that they do not yet know, and not for the sake of upholding orthodoxy, or any other foregone conclusion. For them this article is written as a kind of popular exposition of the work that is now being done in the field of logic. To them I desire to convey some idea of what the new logic is, how two "algebras," that is, systems of diagrammatical representation by means of letters and other characters, more or less analogous to those of the algebra of arithmetic, have been invented for the study of the logic of relatives, and how Schröder uses one of these (with some aid from the other and from other notations) to solve some interesting problems of reasoning. I also wish to illustrate one other of several important uses to which the new logic may be put. To this end I must first clearly show what a relation is.

457. Now there are three grades of clearness in our apprehensions of the meanings of words. The first consists in the connexion of the word with familiar experience. In that sense, we all have a clear idea of what *reality* is and what *force* is — even those who talk so glibly of mental force being correlated with the physical forces. The second grade consists in the abstract definition, depending upon an analysis of just what it is that makes the word applicable. An example of defective

* *The Monist*, vol. 7, pp. 161–217, (1897).

† Cf. vol. 5, bk. II, ch. 5.

[1] *Algebra und Logik der Relative.* Leipsic: B. G. Teubner. 1895. Price, 16 M.

apprehension in this grade is Professor Tait's holding (in an appendix to the reprint of his Britannica article, *Mechanics*) that energy is "objective" (meaning it is a substance), because it is permanent, or "persistent." For independence of time does not of itself suffice to make a substance; it is also requisite that the aggregant parts should always preserve their identity, which is not the case in the transformations of energy. The third grade of clearness consists in such a representation of the idea that fruitful reasoning can be made to turn upon it, and that it can be applied to the resolution of difficult practical problems.

§2. OF THE TERM RELATION IN ITS FIRST GRADE OF CLEARNESS

458. An essential part of speech, the Preposition, exists for the purpose of expressing relations. Essential it is, in that no language can exist without prepositions, either as separate words placed before or after their objects, as case-declensions, as syntactical arrangements of words, or some equivalent forms. Such words as "brother," "slayer," "at the time," "alongside," "not," "characteristic property" are relational words, or *relatives*, in this sense, that each of them *becomes a general name when another general name is affixed to it as object.* In the Indo-European languages, in Greek, for example, the so-called genitive case (an inapt phrase like most of the terminology of grammar) is, very roughly speaking, the form most proper to the attached name. By such attachments, we get such names as "brother *of* Napoleon," "slayer *of* giants," "*ἐπὶ* Ἐλλισσαίου, at the time of Elias," "*παρὰ ἀλλήλων*, alongside *of* each other," "not guilty," "a characteristic property *of* gallium." *Not* is a relative because it means "other than"; *scarcely*, though a relational word of highly complex meaning, is not a relative. It has, however, to be treated in the logic of relatives. Other relatives do not become general names until two or more names have been thus affixed. Thus, "giver to the city" is just such a relative as the preceding; for "giver to the city of a statue of himself" is a complete general name (that is, there might be several such humble admirers of themselves, though there be but one, as yet); but "giver" requires *two* names to be attached to it, before it becomes a

complete name. The dative case is a somewhat usual form for the second object. The archaic, instrumental, and locative cases were serviceable for third and fourth objects.

459. Our European languages are peculiar in their marked differentiation of common nouns from verbs. *Proper* nouns must exist in all languages; and so must such "pronouns," or indicative words, as *this*, *that*, *something*, *anything*. But it is probably true that in the great majority of the tongues of men, distinctive common nouns either do not exist or are exceptional formations. In their meaning as they stand in sentences, and in many comparatively widely-studied languages, common nouns are akin to participles, as being mere inflexions of verbs. If a language has a verb meaning "is a man," a noun "man" becomes a superfluity. For all men are mortals is perfectly expressed by "Anything either is-a-man not or is-a-mortal." Some man is a miser is expressed by "Something both is-a-man and is-a-miser." The best treatment of the logic of relatives, as I contend, will dispense altogether with class names and only use such verbs. A verb requiring an object or objects to complete the sense may be called a *complete relative*.

A verb by itself signifies a mere dream, an imagination unattached to any particular occasion. It calls up in the mind an *icon*. A *relative* is just that, an icon, or image, without attachments to experience, without "a local habitation and a name," but with indications of the need of such attachments.

460. An indexical word, such as a proper noun or demonstrative or selective pronoun, has force to draw the attention of the listener to some hecceity common to the experience of speaker and listener. By a hecceity, I mean, some element of existence which, not merely by the likeness between its different apparitions, but by an inward force of identity, manifesting itself in the continuity of its apparition throughout time and in space, is distinct from everything else, and is thus fit (as it can in no other way be) to receive a proper name or to be indicated as *this* or *that*. Contrast this with the signification of the verb, which is sometimes in my thought, sometimes in yours, and which has no other identity than the agreement between its several manifestations. That is what we call an abstraction or idea. The nominalists say it is a *mere* name. Strike out the "mere," and this opinion is approximately true.

The realists say it *is* real. Substitute for "is," *may be*, that is, *is* provided experience and reason shall, as their final upshot, uphold the truth of the particular predicate, and the natural existence of the law it expresses, and this is likewise true. It is certainly a great mistake to look upon an idea, merely because it has not the mode of existence of a hecceity, as a lifeless thing.

461. The proposition, or sentence, signifies that an eternal fitness, or truth, a permanent conditional force, or law, attaches certain hecceities to certain parts of an idea. Thus, take the idea of "buying by — of — from — in exchange for —." This has four places where hecceities, denoted by indexical words, may be attached. The proposition "A buys B from C at the price D," signifies an eternal, irrefragable, conditional force gradually compelling those attachments in the opinions of inquiring minds.

462. Whether or not there be in the reality any definite separation between the hecceity-element and the idea-element is a question of metaphysics, not of logic. But it is certain that in the expression of a fact we have a considerable range of choice as to how much we will denote by the indexical and how much signify by iconic words. Thus, we have stated "all men are mortal" in such a form that there is but one index. But we may also state it thus: "Taking anything, either it possesses not humanity or it possesses mortality." Here "humanity" and "mortality" are really proper names, or purely denotative signs, of familiar ideas. Accordingly, as here stated, there are three indices. Mathematical reasoning largely depends on this treatment of ideas as things*; for it aids in the iconic representation of the whole fact. Yet for some purposes it is disadvantageous. These truths will find illustration in §13 below.

463. Any portion of a proposition expressing ideas but requiring something to be attached to it in order to complete the sense, is in a general way relational. But it is only a *relative* in case the attachment of indexical signs will suffice to make it a proposition, or, at least, a complete general name. Such a word as *exceedingly* or *previously* is relational, but is not a relative, because significant words require to be added to it to make complete sense.

* See 42–44, 1.83, 2.227, 2.364 and 4.235.

§3. OF RELATION IN THE SECOND GRADE OF CLEARNESS

464. Is relation anything more than a connexion between two things? For example, can we not state that A gives B to C without using any other relational phrase than that one thing is connected with another? Let us try. We have the general idea of *giving*. Connected with it are the general ideas of *giver, gift,* and "*donée*." We have also a particular transaction connected with no general idea except through that of giving. We have a first party connected with this transaction and also with the general idea of giver. We have a second party connected with that transaction, and also with the general idea of "donée." We have a subject connected with that transaction and also with the general idea of gift. A is the only hecceity directly connected with the first party; C is the only hecceity directly connected with the second party, B is the only hecceity directly connected with the subject. Does not this long statement amount to this, that A gives B to C?

In order to have a distinct conception of Relation, it is necessary not merely to answer this question but to comprehend the reason of the answer. I shall answer it in the negative. For, in the first place, if relation were nothing but connexion of two things, all things would be connected. For certainly, if we say that A is unconnected with B, that non-connexion is a relation between A and B. Besides, it is evident that any two things whatever make a pair. Everything, then, is equally related to everything else, if mere connexion be all there is in relation. But that which is equally and necessarily true of everything is no positive fact, at all. This would reduce relation, considered as simple connexion between two things, to nothing, unless we take refuge in saying that relation *in general* is indeed nothing, but that *modes* of relation are something. If, however, these different modes of relation are different modes of connexion, relation ceases to be simple bare connexion. Going back, however, to the example of the last paragraph, it will be pointed out that the peculiarity of the mode of connexion of A with the transaction consists in A's being in connexion with an element connected with the transaction, which element is connected with the peculiar general idea of a *giver*. It will, therefore, be said, by those who attempt to defend an affirmative answer to our question, that the peculiarity of a mode of connexion consists in this, that that connexion is indirect and takes place through something which is connected with a peculiar general idea. But I say that is no answer at all; for if all things are equally connected, nothing can be more connected with one idea than with another. This is unanswerable. Still, the affirmative side may modify their position somewhat. They may say, we grant that it is necessary to recognise that relation is something more than connexion; it is *positive* connexion. Granting that all things are connected, still all are not positively connected. The various modes of relationship are, then, explained as above. But to this I reply: you propose to make the peculiarity of the connexion of A with the transaction depend (no matter by what machinery) upon that connexion having a positive connexion with the idea of a giver. But "positive connexion" is not enough; the relation of the general idea is quite peculiar. In order that it may be characterised, it must, on your principles, be made indirect, taking place through something which is itself connected with a general idea. But this last connexion is again more than a mere general positive connexion. The same device must be resorted to, and so on *ad infinitum*. In short, you are guilty of a *circulus in definiendo*. You make the relation of any two things consist in their connexion being connected with a general idea. But that last connexion is, on your own principles, itself a *relation*, and you are thus defining relation by relation; and if for the second occurrence you substitute the definition, you have to repeat the substitution *ad infinitum*.

The affirmative position has consequently again to be modified. But, instead of further tracing possible tergiversations, let us directly establish one or two positive positions. In the first place, I say that every relationship concerns some definite number of correlates. Some relations have such properties that this fact is concealed. Thus, any number of men may be brothers. Still, brotherhood is a relation between pairs. If A, B, and C are all brothers, this is merely the consequence of the three relations, A is brother of B, B is brother of C, C is brother of A. Try to construct a relation which shall exist either between two or between three things such as "— is either a brother or betrayer of — to —." You can only make sense of it by somehow interpreting the dual relation as a triple one. We may express this as saying that every relation has a definite number of blanks to be filled by indices, or otherwise. In the case of the majority of relatives, these blanks are qualitatively different from one another. These qualities are thereby communicated to the connexions.

465. In a complete proposition there are no blanks. It may be called a *medad*, or *medadic relative*, from *μηδαμός*, none, and -*άδα* the accusative ending of such words as *μόνας*, *δυάς*, *τριάς*, *τετράς* etc.[1] A non-relative name with a substantive verb, as "—is a man," or "man that is —," or "—'s manhood" has one blank; it is a *monad*, or *monadic relative*. An ordinary relative with an active verb as "— is a lover of —" or "the loving by — of —" has two blanks; it is a *dyad*, or *dyadic relative*. A higher relative similarly treated has a plurality of blanks. It may be called a *polyad*. The rank of a relative among these may be called its *adinity*, that is, the peculiar quality of the number it embodies.

466. A *relative*, then, may be defined as the equivalent of a word or phrase which, either as it is (when I term it a *complete* relative), or else when the verb "is" is attached to it (and if it wants such attachment, I term it a *nominal* relative), becomes a sentence with some number of proper names left blank. A *relationship*, or *fundamentum relationis*, is a fact relative to a number of objects, considered apart from those objects, as if, after the statement of the fact, the designations of those objects had been erased. A *relation* is a relationship considered as something that may be said to be true of one of the objects, the others being separated from the relationship yet kept in view. Thus, for each relationship there are as many relations as there are blanks. For example, corresponding to the relationship which consists in one thing loving another there are two relations, that of loving and that of being loved by. There is a nominal relative for each of these relations, as "lover of —," and "loved by —." These nominal relatives

[1] The Pythagoreans, who seem first to have used these words, probably attached a patronymic signification to the termination. A *triad* was derivative of *three*, etc.

belonging to one relationship, are in their relation to one another termed *correlatives*. In the case of a dyad, the two correlatives, and the corresponding relations are said, each to be the *converse* of the other. The objects whose designations fill the blanks of a complete relative are called the *correlates*. The correlate to which a nominal relative is attributed is called the *relate*.

467. In the statement of a relationship, the designations of the correlates ought to be considered as so many *logical subjects* and the relative itself as the *predicate*. The entire set of logical subjects may also be considered as a *collective subject*, of which the statement of the relationship is *predicate*.

§4. OF RELATION IN THE THIRD GRADE OF CLEARNESS*

468. Mr. A. B. Kempe has published in the *Philosophical Transactions*† a profound and masterly "Memoir on the Theory of Mathematical Form," which treats of the representation of relationships by "Graphs," which is Clifford's name for a diagram, consisting of spots and lines, in imitation of the chemical diagrams showing the constitution of compounds. Mr. Kempe seems to consider a relationship to be nothing but a complex of bare connexions of pairs of objects, the opinion refuted in the last section. Accordingly, while I have learned much from the study of his memoir, I am obliged to modify what I have found there so much that it will not be convenient to cite it; because long explanations of the relation of my views to his would become necessary if I did so.

469. A chemical atom is quite like a relative in having a definite number of loose ends or "unsaturated bonds," corresponding to the blanks of the relative.‡ In a chemical molecule, each loose end of one atom is joined to a loose end, which it is assumed must belong to some other atom, although in the vapor of mercury, in argon, etc., two loose ends of the same atom would seem to be joined; and why pronounce such hermaphroditism impossible? Thus the chemical molecule is a

* In this section Peirce presents his "Entitative Graphs." The "Existential Graphs" are to be found in vol. 4, bk. II.

† Part 1, 1886, pp. 1–70.

‡ Cf. 1.289f., 1.346 and 421.

medad, like a complete proposition. Regarding proper names and other indices, after an "is" has been attached to them, as monads, they, together with other monads, correspond to the two series of chemical elements, H, Li, Na, K, Rb, Cs, etc., and Fl, Cl, Br, I. The dyadic relatives correspond to the two series, Mg, Ca, Sr, Ba, etc., and O, S, Se, Te, etc. The triadic relatives correspond to the two series B, Al, Zn, In, Tl, etc., and N, P, As, Sb, Bi, etc. Tetradic relatives are, as we shall see, a superfluity; they correspond to the series C, Si, Ti, Sn, Ta, etc. The proposition "John gives John to John" corresponds in

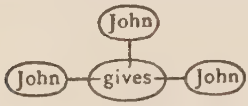


Figure 1

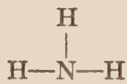


Figure 2

its constitution, as Figs. 1 and 2 show, precisely to ammonia.

470. But beyond this point the analogy ceases to be striking. In fact, the analogy with the ruling theory of chemical compounds quite breaks down. Yet I cannot resist the temptation to pursue it. After all, any analogy, however fanciful, which serves to focus attention upon matters which might otherwise escape observation is valuable. A chemical compound might be expected to be quite as much like a proposition as like an algebraical invariant; and the brooding upon chemical graphs has hatched out an important theory in invariants.* Fifty years ago, when I was first studying chemistry, the theory was that every compound consisted of two oppositely electrified atoms or radicles; and in like manner every compound radicle consisted of two opposite atoms or radicles. The argument to this effect was that chemical attraction is evidently between things unlike one another and evidently has a saturation point; and further that we observe that it is the elements the most extremely unlike which attract one another. [Julius] Lothar Meyer's curve having for its ordinates the atomic volumes of the elements and for its abscissas their atomic weights tends to support the opinion that elements

* See J. J. Sylvester, "Chemistry and Algebra," *Mathematical Papers*, vol. III, No. 14; W. K. Clifford, "Remarks on the Chemico-Algebraic Theory," *Mathematical Papers*, No. 28.

strongly to attract one another must have opposite characters*; for we see that it is the elements on the steepest downward slopes of that curve which have the strongest attractions for the elements on the steepest upward inclines. But when chemists became convinced of the doctrine of valency, that is, that every element has a fixed number of loose ends, and when they consequently began to write graphs for compounds, it seems to have been assumed that this necessitated an abandonment of the position that atoms and radicles combine by opposition of characters, which had further been weakened by the refutation of some mistaken arguments in its favor. But if chemistry is of no aid to logic, logic here comes in to enlighten chemistry. For in logic, the medad must always be composed of one part having a negative, or antecedental, character, and another part of a positive, or consequental, character; and if either of these parts is compound its constituents are similarly related to one another. Yet this does not, at all, interfere with the doctrine that each relative has a definite number of blanks or loose ends. We shall find that, in logic, the negative character is a character of reversion in this sense, that if the negative part of a medad is compound, *its* negative part has, on the whole, a positive character. We shall also find, that if the negative part of a medad is compound, the bond joining its positive and negative parts has its character reversed, just as those relatives themselves have.†

471. Several propositions are in this last paragraph stated about logical medads which now must be shown to be true. In the first place, although it be granted that every relative has a definite number of blanks, or loose ends, yet it would seem, at first sight, that there is no need of each of these joining no more than one other. For instance, taking the triad

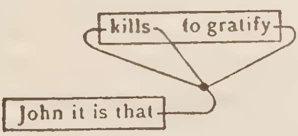


Figure 3

* Meyer used the volumes as abscissæ and the weights as ordinates. See *Das Natürliche System der Chemischen Elemente*, Meyer u. Mendejeff, Leipzig, 1895.

† See 475f.

"— kills — to gratify —," why may not the three loose ends all join in one node and then be connected with the loose end of the monad "John is —" as in Figure 3 making the proposition "John it is that kills what is John to gratify what is John"? The answer is, that a little exercise of generalising power will show that such a four-way node is really a tetradic relative, which may be expressed in words thus, "— is identical with — and with — and with —"; so that the medad is

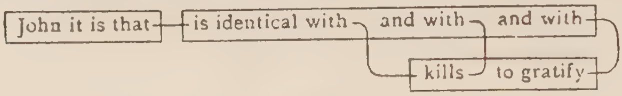


Figure 4

really equivalent to that of Figure 4, which corresponds to prussic acid as shown in Figure 5.

Figure 5

Thus, it becomes plain that every node of bonds is equivalent to a relative; and the doctrine of valency is established for us in logic.

472. We have next to inquire into the proposition that in every combination of relatives there is a negative and a positive constituent. This is a corollary from the general logical doctrine of the illative character of the copula, a doctrine precisely opposed to the opinion of the quantification of the predicate. A satisfactory discussion of this fundamental question would require a whole article. I will only say in outline that it can be positively demonstrated in several ways that a proposition of the form "man = rational animal," is a compound of propositions each of a form which may be stated thus: "*Every* man (if there be any) is a rational animal" or "Men are *exclusively* (if anything) rational animals." Moreover, it must be acknowledged that the illative relation (that expressed by "therefore") is the most important of logical relations, the be-all and the end-all of the rest. It can be demonstrated that formal logic needs no other elementary logical relation than this; but that

with a symbol for this and symbols of relatives, including monads, and with a mode of representing the attachments of them, all syllogistic may be developed, far more perfectly than any advocate of the quantified predicate ever developed it, and in short in a way which leaves nothing to be desired. This in fact *will* be virtually shown in the present paper. It can further be shown that no other copula will of itself suffice for all purposes. Consequently, the copula of equality ought to be regarded as merely derivative.

473. Now, in studying the logic of relatives we must sedulously avoid the error of regarding it as a highly specialised doctrine. It is, on the contrary, nothing but formal logic generalised to the very tip-top. In accordance with this view, or rather with this theorem (for it is susceptible of positive demonstration), we must regard the *relative copula*, which is the bond between two blanks of relatives, as only a generalisation of the ordinary copula, and thus of the "*ergo*." When we say that from the proposition A the proposition B necessarily follows, we say that "the truth of A in *every way* in which it can exist at all is the truth of B," or otherwise stated "A is true *only* in so far as B is true." This is the very same relation which we express when we say that "*every* man is mortal," or "men are *exclusively* mortals." For this is the same as to say, "Take anything whatever, M; then, if M is a man, it follows necessarily that M is mortal." This mode of junction is essentially the same as that between the relatives in the compound relative "lover, in *every* way in which it may be a lover at all, of a servant," or, otherwise expressed, "lover (if at all) *exclusively* of servants." For to say that "Tom is a lover (if at all) only of servants of Dick," is the same as to say "Take anything whatever, M; then, if M is loved by Tom, M is a servant of Dick," or "everything there may be that is loved by Tom is a servant of Dick."*

474. Now it is to be observed that the illative relation is not simply convertible; that is to say, that "from A necessarily follows B" does not necessarily imply that "from B necessarily follows A." Among the vagaries of some German logicians of some of the inexact schools, the convertibility of illation (like almost every other imaginable absurdity) has been main-

*I.e., $\bar{l} \dagger s = \check{l} \prec s$.

tained; but all the other inexact schools deny it, and exact logic condemns it, at once. Consequently, the copula of inclusion, which is but the *ergo* freed from the accident of asserting the truth of its antecedent, is equally inconvertible. For though "men include only mortals," it does not follow that "mortals include *only* men," but, on the contrary, what follows is "mortals include *all* men." Consequently, again, the fundamental *relative copula* is inconvertible. That is, because "Tom loves (if anybody) only a servant (or servants) of Dick," it does not follow that "Dick is served (if at all) only by somebody loved by Tom," but, on the contrary, what follows is "Dick is master of *every* person (there may be) who is loved by Tom." * We thus see, clearly, first, that, as the fundamental relative copula, we must take that particular mode of junction; secondly, that that mode is at bottom the mode of junction of the *ergo*, and so joins a relative of antecedental character to a relative of consequental character; and, thirdly, that that copula is inconvertible, so that the two kinds of constituents are of opposite characters. There are, no doubt, convertible modes of junction of relatives, as in "lover of a servant[1]; but it will be shown below that these are complex and indirect in their constitution.

* i.e., $\bar{l} \dagger s \,-\!\!< \breve{s} \dagger \check{\bar{l}}$ but not $\bar{\breve{s}} \dagger \check{l}$; or perhaps more clearly $(\check{l} -\!\!< s) -\!\!< (\bar{s} -\!\!< \check{\bar{l}})$ but not $s -\!\!< \check{l}$.

[1] Professor Schröder proposes to substitute the word "symmetry" for *convertibility*, and to speak of *simply convertible* modes of junction as "symmetrical." Such an example of wanton disregard of the admirable traditional terminology of logic, were it widely followed, would result in utter uncertainty as to what any

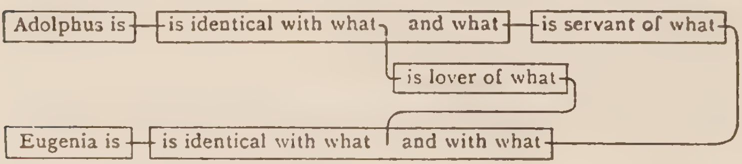


Figure 6

writer on logic might mean to say, and would thus be utterly fatal to all our efforts to render logic exact. Professor Schröder denies that the mode of junction in "lover of a servant" is "symmetrical," which word in practice he makes synonymous with "commutative," applying it only to such junctions as that between "lover" and "servant" in "Adolphus is at once lover and servant of Eugenia." Commutativity depends on one or more polyadic relatives having two like blanks as shown in Fig. 6.

475. It remains to be shown that the antecedent part of a medad has a negative, or reversed, character, and how this, in case it be compound, affects both its relatives and their bonds. But since this matter is best studied in examples, I will first explain how I propose to draw the logical graphs.

It is necessary to use, as the sign of the relative copula, some symbol which shall distinguish the antecedent from the consequent; and since, if the antecedent is compound (owing to the very character which I am about to demonstrate, namely, its reversing the characters of the relatives and the bonds it contains), it is very important to know just how much is included in that antecedent, while it is a matter of comparative indifference how much is included in the consequent (though it is simply everything not in the antecedent), and since further (for the same reason) it is important to know how many antecedents, each after the first a part of another, contain a given relative or copula, I find it best to make the line which joins antecedent and consequent encircle the whole of the former. Letters of the alphabet may be used as abbreviations of complete relatives; and the proper number of bonds may be attached to each. If one of these is encircled, that circle must have a bond corresponding to each bond of the encircled letter. Chemists sometimes write above atoms Roman numerals to indicate their *adinities;* but I do not think this necessary. Figure 7 shows, in a complete medad, my sign of the

Figure 7

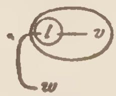


Figure 8

relative copula. Here, *h* is the monad "— is a man," and *d* is the monad "— is mortal." The antecedent is completely enclosed, and the meaning is "Anything whatever, if it be a man, is mortal." If the circle encloses a dyadic or polyadic relative, it must, of course, have a tail for every bond of that relative. Thus, in Figure 8, *l* is the dyad "— loves —," and it is important to remark that the bond to the left is the lover and that to the right is the loved. Monads are the only relatives for which we need not be attentive to the positions of attachment of the bonds. In this figure, *w* is the monad "— is wise," and *v* is the monad "— is virtuous." The *l* and *v* are enclosed in a large common circle. Had this not been done, the medad could not be read (as far as any rules yet given show), because it would not consist of antecedent and consequent. As it is, we begin the reading of the medad at the bond connecting antecedent and consequent. Every bond of a logical graph denotes a hecceity; and every unencircled bond (as this one is) stands for any hecceity the reader may choose from the universe. This medad evidently refers to the universe of men. Hence the interpretation begins: "Let M be any man you please." We proceed along this bond in the direction of the antecedent, and on entering the circle of the antecedent we say: "If M be." We then enter the inner circle. Now, entering a circle means a relation to *every*. Accordingly we add "whatever." Traversing *l* from left to right, we say "lover." (Had it been from right to left we should have read it "loved.") Leaving the circle is the mark of a relation "only to," which words we add. Coming to *v* we say "what is virtuous." Thus our antecedent reads: "Let M be any man you please. If M be whatever it may that is lover only to the virtuous." We now return to the consequent and read, "M is wise." Thus the whole means, "Whoever loves only the virtuous is wise."

As another example, take the graph of Figure 9, where *l* has

Figure 9

the same meaning as before and *m* is the dyad "— is mother of —." Suppose we start with the left hand bond. We begin with saying "Whatever." Since cutting this bond does not sever the medad, we proceed at once to read the whole as an unconditional statement and we add to our "whatever" "there is." We can now move round the ring of the medad either clockwise or counter-clockwise. Taking the last way, we come to *l* from the left hand and therefore add "is a lover." Moving on, we enter the circle round *m*; and entering a circle is a sign that we must say "of *everything* that." Since we pass through *m* backwards we do not read "is mother" but "is mothered" or "has for mother." Then, since we pass *out* of the circle we should have to add "only"; but coming back, as we do, to the starting point, we need only say "that same thing." Thus, the interpretation is "Whatever there is, is lover of everything that has for mother that same thing," or "Every woman loves everything of which she is mother." Starting at the same point and going round the other way, the reading would be "Everybody is mother (if at all) only of what is loved by herself." Starting on the right and proceeding clockwise, "Everything is loved by every mother of itself." Proceeding counter-clockwise, "Everything has for mothers only lovers of itself."

476. Triple relatives afford no particular difficulty. Thus, in Figure 10, w and v have the same significations as before; r is the monad, "— is a reward," and g is the triad "— gives | to —." It can be read either

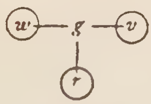


Figure 10

"Whatever is wise gives every reward to every virtuous person," or "Every virtuous person has every reward given to him by everybody that is wise," or "Every reward is given by everybody who is wise to every virtuous person."

477. A few more examples will be instructive. Figure 11, where A is the proper name "Alexander" means "Alexander loves only the virtuous," i.e., "Take anybody you please; then, if he be Alexander and if he loves anybody, this latter is virtuous."

Figure 11

Figure 12

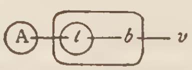


Figure 13

If you attempt, in reading this medad, to start to the right of l, you fall into difficulty, because your antecedent does not then consist of an antecedent and consequent, but of two circles joined by a bond, a combination to be considered below. But Figure 12 may be read with equal ease on whichever side of l you begin, whether as "whoever is wise loves everybody that is virtuous," or "whoever is virtuous is loved by everybody that is wise." If in Figure 13 –*b*– be the dyad "— is a benefactor of —," the medad reads, "Alexander stands only to virtuous persons in the relation of loving only their benefactors."

Figure 14, where –*s*– is the dyad "— is a servant of —" may be read, according to the above principles, in the several ways following:

"Whoever stands to any person in the relation of lover to none but his servants benefits him."

"Every person stands only to a person benefited by him in the relation of a lover only of a servant of that person."

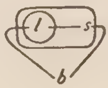


Figure 14

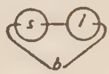


Figure 15

"Every person, M, is benefactor of everybody who stands to M in the relation of being served by everybody loved by him."

"Every person, N, is benefited by everybody who stands to N in the relation of loving only servants of him."

"Every person, N, stands only to a benefactor of N in the relation of being served by everybody loved by him."

"Take any two persons, M and N. If, then, N is served by every lover of M, N is benefited by M."

Figure 15 represents a medad which means, "Every servant of any person, is a benefactor of whomever may be loved by that person." Equivalent statements easily read off from the graphs are as follows:

"Anybody, M, no matter who, is servant (if at all) only of somebody who loves (if at all) only persons benefited by M."

"Anybody, no matter who, stands to every master of him in the relation of benefactor of whatever person may be loved by him."

"Anybody, no matter who, stands to whoever loves him in the relation of being benefited by whatever servant he may have."

"Anybody, N, is loved (if at all) only by a person who is served (if at all) only by benefactors of N."

"Anybody, no matter who, loves (if at all) only persons benefited by all servants of his."

"Anybody, no matter who, is served (if at all) only by benefactors of everybody loved by him."

478. I will now give an example containing triadic relatives, but no monads. Let p be "— prevents — from communicating with —," the second blank being represented by a bond from the right of p and the third by a bond from below p. Let β mean "— would betray — to —," the arrangement of bonds being the same as with p. Then, Figure 16 means that "whoever loves only persons who prevent every servant

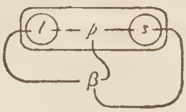


Figure 16

of any person, A, from communicating with any person, B, would betray B to A." I will only notice one equivalent statement, viz.: "Take any three persons, A, B, C, no matter who. Then, either C betrays B to A, or else two persons, M and N, can be found, such that M does not prevent N from communicating with B, although M is loved by C and N is a servant of A."

479. This last interpretation is an example of the method which is, by far, the plainest and most unmistakable of any in complicated cases. The rule for producing it is as follows:

1. Assign a letter of the alphabet to denote the hecceity represented by each bond.[1]

2. Begin by saying: "Take any things you please, namely," and name the letters representing bonds not encircled; then add, "Then suitably select objects, namely," and name the letters representing bonds each once encircled; then add, "Then take any things you please, namely," and name the letters representing bonds each twice encircled. Proceed in

[1] In my method of graphs, the spots represent the relatives, their bonds the hecceities; while in Mr. Kempe's method, the spots represent the objects, whether individuals or abstract ideas, while their bonds represent the relations. Hence, my own exclusive employment of bonds between pairs of spots does not, in the least, conflict with my argument that in Mr. Kempe's method such bonds are insufficient.

this way until all the letters representing bonds have been named, no letter being named until all those encircled fewer times have been named; and each hecceity corresponding to a letter encircled odd times is to be suitably chosen according to the intent of the assertor of the medad proposition, while each hecceity corresponding to a bond encircled even times is to be taken as the interpreter or the opponent of the proposition pleases.*

3. Declare that you are about to make statements concerning certain propositions, to which, for the sake of convenience, you will assign numbers in advance of enunciating them or stating their relations to one another. These numbers are to be formed in the following way. There is to be a number for each letter of the medad (that is for those which form spots of the graph, not for the letters assigned by clause 1 of this rule to the bonds), and also a number for each circle round more than one letter; and the first figure of that number is to be a 1 or a 2, according as the letter or the circle is in the principal antecedent or the principal consequent; the second figure is to be 1 or 2, according as the letter or the circle belongs to the antecedent or the consequent of the principal antecedent or consequent, and so on.

Declare that one or other of those propositions whose numbers contain no 1 before the last figure is true. Declare that each of those propositions whose numbers contain an odd number of 1's before the last figure consists in the assertion that *some one* or another of the propositions whose numbers commence with its number is true. For example, 11 consists in the assertion that either 111 or 1121 or 1122 is true, supposing that these are the only propositions whose numbers commence with 11. Declare that each of those propositions whose numbers contain an even number of 1's (or none) before the last figure consists in the assertion that *every one of* the propositions whose numbers commence with its number is true. Thus, 12 consists in the assertion that 121, 1221, 1222 are all true, provided those are the only propositions whose numbers commence with 12. The process described in this clause will be abridged except in excessively complicated cases.

*I.e., evenly encircled bonds have π as their quantifier, while the others have Σ.

4. Finally, you are to enunciate all those numbered propositions which correspond to single letters. Namely, each proposition whose number contains an even number of 1's, will consist in affirming the relative of the spot-letter to which that number corresponds after filling each blank with that bond-letter which by clause 1 of this rule was assigned to the bond at that blank. But if the number of the proposition contains an odd number of 1's, the relative, with its blanks filled in the same way, is to be denied.

480. In order to illustrate this rule, I will restate the meanings of the medads of Figures 7–16, in all the formality of the rule; although such formality is uncalled for and awkward, except in far more complicated cases.

Figure 7. Let A be anything you please. There are two propositions, 1 and 2, one of which is true. Proposition 1 is, that A is not a man. Proposition 2 is, that A is mortal. More simply, Whatever A may be, either A is not a man or A is mortal.

Figure 8. Let A be anybody you please. Then, I will find a person, B, so that either proposition 1 or proposition 2 shall be true. Proposition 1 asserts that both propositions 11 and 12 are true. Proposition 11 is that A loves B. Proposition 12 is that B is not virtuous. Proposition 2 is that A is wise. More simply, Take anybody, A, you please. Then, either A is wise, or else a person, B, can be found such that B is not virtuous and A loves B.

Figure 9. Let A and B be any persons you please. Then, either proposition 1 or proposition 2 is true. Proposition 1 is that A is not a mother of B. Proposition 2 is that A loves B. More simply, whatever two persons A and B may be, either A is not a mother of B or A loves B.

Figure 10. Let A, B, C be any three things you please. Then, one of the propositions numbered, 1, 21, 221, 222 is true. Proposition 1 is that A is not wise. Proposition 21 is that B is not a reward. Proposition 221 is that C is not virtuous. Proposition 222 is that A gives B to C. More simply, take any three things, A, B, C, you please. Then, either A is not wise, or B is not a reward, or C is not virtuous, or A gives B to C.

Figure 11. Take any two persons, A and B, you please.

Then, one of the propositions 1, 21, 22 is true. 1 is that A is not Alexander. 21 is that A does not love B. Proposition 3 is that B is virtuous.

Figure 12. Take any two persons, A and B. Then, one of the propositions 1, 21, 22 is true. 1 is that A is not wise. 21 is that B is not virtuous. 22 is that A loves B.

Figure 13. Take any two persons, A and C. Then a person, B can be found such that one of the propositions 1, 21, 22 is true. Proposition 21 asserts that both 211 and 212 are true. Proposition 1 that A is not Alexander. Proposition 211 is that A loves B. Proposition 212 is that B does not benefit C. Proposition 22 is that C is virtuous. More simply, taking any two persons, A and C, either A is not Alexander, or C is virtuous, or there is some person, B, who is loved by A without benefiting C.

Figure 14. Take any two persons, A and B, and I will then select a person C. Either proposition 1 or proposition 2 is true. Proposition 1 is that both 11 and 12 are true. Proposition 11 is that A loves C. Proposition 12 is that C is not a servant of B. Proposition 2 is that A benefits B. More simply, of any two persons, A and B, either A benefits the other, B, or else there is a person, C, who is loved by A but is not a servant of B.

Figure 15. Take any three persons, A, B, C. Then one of the propositions 1, 21, 22 is true. 1 is that A is not a servant of B; 21 is that B is not a lover of C; 22 is that A benefits C.

Figure 16. Take any three persons, A, B, C. Then I can so select D and E, that one of the propositions 1 or 2 is true. 1 is that 11 and 121 and 122 are all true. 11 is that A loves D, 121 is that E is a servant of C, 122 is that D does not prevent E from communicating with B. 2 is that A betrays B to C.

I have preferred to give these examples rather than fill my pages with a dry abstract demonstration of the correctness of the rule. If the reader requires such a proof, he can easily construct it. This rule makes evident the reversing effect of the encirclements, not only upon the "quality" of the relatives as affirmative or negative, but also upon the selection of the hecceities as performable by advocate or opponent of the proposition, as well as upon the conjunctions of the propositions as disjunctive or conjunctive, or (to avoid this absurd grammatical terminology) as alternative or simultaneous.

481. It is a curious example of the degree to which the thoughts of logicians have been tied down to the accidents of the particular language they happened to write (mostly Latin), that while they hold it for an axiom that two *nots* annul one another, it was left for me to say as late as 1867[1] that *some* in formal logic ought to be understood, and could be understood, so that *some-some* should mean *any*. I suppose that were ordinary speech of any authority as to the forms of logic, in the overwhelming majority of human tongues two negatives intensify one another. And it is plain that if "not" be conceived as less than anything, what is less than that is *a fortiori* not. On the other hand, although *some* is conceived in our languages as *more than none*, so that two "somes" intensify one another, yet what it ought to signify for the purposes of syllogistic is that, instead of the selection of the instance being left — as it is, when we say "any man is not good" — to the opponent of the proposition, when we say "some man is not good," this selection is transferred to the opponent's opponent, that is to the defender of the proposition. Repeat the some, and the selection goes to the opponent's opponent's opponent, that is, to the opponent again, and it becomes equivalent to *any*. In more formal statement, to say "Every man is mortal," or "Any man is mortal," is to say, "A man, as suitable as any to prove the proposition false, is mortal," while "Some man is mortal" is equivalent to "A man, as suitable as any to prove the proposition *not* false, is mortal." "Some-some man is mortal" is accordingly "A man, as suitable as any to prove the proposition *not not*-false, is mortal."

482. In like manner, encircled 2N+1 times, a disjunctive conjunction of propositions becomes a copulative conjunction. Here, the case is altogether similar. Encircled even times, the statement is that some one (or more) of the propositions is true; encircled odd times, the statement is that any one of the propositions is true. The negative of "lover of every servant" is "non-lover of some servant." The negative of "lover every way (that it is a lover) of a servant" is "lover some way of a non-servant."

The general nature of a relative and of a medad has now

[1] "On the Natural Classification of Arguments." *Proceedings of the American Academy of Arts and Sciences.* [2.477.]

been made clear. At any rate, it will become so, if the reader carefully goes through with the explanations. We have not, however, as yet shown how every kind of proposition can be graphically expressed, nor under what conditions a medad is necessarily true. For that purpose it will be necessary to study certain special logical relatives.

§5. TRIADS, THE PRIMITIVE RELATIVES

483. That out of triads all polyads can be constructed is made plain by Figure 17.

Figure 17

484. Figure 18 shows that from two triads a dyad can be made. Figure 19 shows that from one triad a monad can be made. Figure 20 shows that from any even number of triads

Figure 18

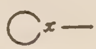


Figure 19

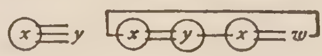


Figure 20

a medad can be made. In general, the union of a μ-ad and a ν-ad gives a $(\mu+\nu-2\lambda)$-ad, where λ is the number of bonds of union. This formula shows that *artiads*, or even-ads, can produce only *artiads*. But any perissid, or odd-ad (except a monad), can by repetition produce a relative of any *adinity*.

485. Since the principal object of a notation for relatives is not to produce a handy *calculus* for the solution of special logical problems, but to help the study of logical principles, the study of logical graphs from that point of view must be postponed to a future occasion. For present purposes that notation is best which carries analysis the furthest, and presents the smallest number of unanalyzed forms. It will be best, then, to use single letters for relatives of some one definite and odd number of blanks. We naturally choose three as the smallest number which will answer the purpose.

486. We shall, therefore, substitute for such a dyad as "— is lover of —" some such triad as "— is coexistent with and a lover of —." If, then, we make *-w-* to signify "— is

coexistent with | and with —," that which we have hitherto written as in Figure 12 will be written as in Figure 21. But having once recognised that such a mode of writing is possible,

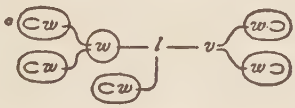


Figure 21

we can continue to use our former methods, provided we now consider them as abbreviations.

487. The logical doctrine of this section, must, we may remark, find its application in metaphysics, if we are to accept the Kantian principle that metaphysical conceptions mirror those of formal logic.

§6. RELATIVES OF SECOND INTENTION

488. The general method of graphical representation of propositions has now been given in all its essential elements, except, of course, that we have not, as yet, studied any truths concerning special relatives; for to do so would seem, at first, to be "extralogical." Logic in this stage of its development may be called *paradisaical logic*, because it represents the state of Man's cognition before the Fall. For although, with this apparatus, it is easy to write propositions necessarily true, it is absolutely impossible to write any which is necessarily false, or, in any way which that stage of logic affords, to find out that anything is false. The mind has not as yet eaten of the fruit of the Tree of Knowledge of Truth and Falsity. Probably it will not be doubted that every child in its mental development necessarily passes through a stage in which he has some ideas, but yet has never recognised that an idea may be erroneous; and a stage that every child necessarily passes through must have been formerly passed through by the race in its adult development. It may be doubted whether many of the lower animals have any clear and steady conception of falsehood; for their instincts work so unerringly that there is little to force it upon their attention. Yet plainly without a knowledge of falsehood no development of discursive reason can take place.

489. This paradisaical logic appears in the study of non-relative formal logic. But *there* no possible avenue appears by which the knowledge of falsehood could be brought into this Garden of Eden except by the arbitrary and inexplicable introduction of the Serpent in the guise of a proposition necessarily false. The logic of relatives, affords such an avenue, and *that*, the very avenue by which in actual development, this stage of logic supervenes. It is the avenue of experience and logical reflexion.

490. By *logical* reflexion, I mean the observation of thoughts in their expressions. Aquinas remarked that this sort of reflexion is requisite to furnish us with those ideas which, from lack of contrast, ordinary external experience fails to bring into prominence. He called such ideas *second intentions*. It is by means of *relatives of second intention* that the general method of logical representation is to find completion.

491. Let ⤙ signify that "— is {neither —* / nor —." Then Figure 22 means

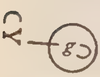


Figure 22

Figure 23

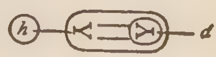


Figure 24

that taking any two things whatever, either the one is neither itself nor the other (putting it out of the question as an absurdity), or the other is a non-giver of something to that thing. That is, nothing gives all things, each to itself. Thus, the existence of any general description of thing can be denied.

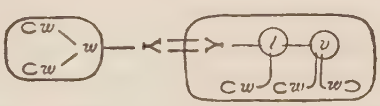


Figure 25

Either medad of Figure 23 means no wise men are virtuous. Figure 24 is equivalent to Figure 7. Figure 25 means "each wise man is a lover of something virtuous." Thus we see that this mode of junction — lover of some virtuous — which seems

* The use of one such logical constant is shown by Peirce to be sufficient for the development of Boolian Algebra. See e.g., 4.1ff.

so simple — is really complex. Figure 26 means "some one

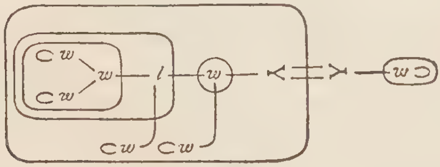


Figure 26

thing is loved by all wise men." Figure 27 means that every man is either wise or virtuous. Figure 28 means that every man is both wise and virtuous.

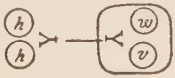


Figure 27

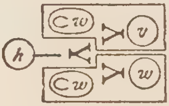


Figure 28

These explanations need not be carried further to show that we have here a perfectly efficient and highly analytical method of representing relations.

§7. THE ALGEBRA OF DYADIC RELATIVES

492. Although the primitive relatives are triadic, yet they may be represented with but little violence by means of dyadic relatives, provided we allow several attachments to one blank. For instance, A gives B to C, may be represented by saying A is the first party in the transaction D, B is subject of D, C is second party of D, D is a giving by the first party of the subject to the second party. Triadic relatives cannot conveniently be represented on one line of writing. These considerations led me to invent the algebra of dyadic relatives as a tolerably convenient substitute in many cases for the graphical method of representation. In place of the one "operation," or mode of conjunction of graphical method, there are in this algebra four operations.

493. For the purpose of this algebra, I entirely discard the idea that every compound relative consists of an antecedent and a consequent part. I consider the circle round the antecedent as a mere sign of negation, for which in the algebra I

substitute an *obelus* over that antecedent. The line between antecedent and consequent, I treat as a sign of an "operation" by itself. It signifies that anything whatever being taken as correlate of the first written member — antecedent or consequent — and as first relate of the second written member, either the one or the other is to be accepted. Thus in place of the relative of Figure 29 signifying that "taking anything whatever, M, either — is not a lover of M, or M is a benefactor of —," that is "— is a lover only of a benefactor of —," I write

$$\bar{l} \dagger b.$$

Or if it happens to be read the other way, putting a short mark over any letters to signify that relate and correlate are interchanged, I write the same thing

$$\breve{b} \dagger \breve{l}.$$

This operation, which may, at need, be denoted by a dagger in print, to which I give a scorpion-tail curve in its cursive form, I call *relative addition*.*

494. The relative "— stands to everything which is a benefactor of — in the relation of servant of every lover of his," shows,

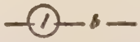


Figure 29

Figure 30

as written in Figure 30, an unencircled bond between *s* and *l*. The junction of the *l* and the *b* may therefore be regarded as direct. Stating the relative so as to make this direct junction prominent, it is "— is servant of everything that is a lover of a benefactor of —." In the algebra, as far as already explained, "lover of a benefactor" would be written

$$\overline{\bar{l} \dagger \bar{b}}$$

that is, not a non-lover of every benefactor, or not a lover only of non-benefactors. This mode of junction, I call, in the algebra, the operation of *relative multiplication*, and write it

$$lb.$$

* Cf. 332–4.

We have, then, the purely formal, or meaningless, equation

$$lb = \overline{\bar{l} \,\text{ɟ}\, \bar{b}}.$$

And in like manner, as a consequence of this,

$$l \,\text{ɟ}\, b = \overline{\bar{l}\bar{b}}.$$

That is to say, "To say that A is a lover of everything but benefactors of B," or "A is a non-lover only of benefactors of B," is the same as to say that A is not a non-lover of a non-benefactor of B.

495. To express in the algebra the relative of Figure 31.

Figure 31

or "— is both a lover and a benefactor of —," I write

$$l \cdot b,$$

calling this "the operation of *non-relative multiplication*." To express "— is either a lover or a benefactor of —," which might be written

$$\overline{\bar{l} \cdot \bar{b}},$$

I write

$$l \,{+\!\!\!,}\, b,$$

calling this the operation of *non-relative addition*, or more accurately, of *aggregation*. These last two operations belong to the Boolian algebra of non-relative logic. They are De Morgan's operations of composition and aggregation. Boole himself did not use the last, but in place of it an operation more properly termed addition which gives no interpretable result when the aggregants have any common aggregant. Mr. Venn* still holds out for Boole's operation, and there are weighty considerations in its favor. In my opinion, the decision between the two operations should depend upon whether the quantified predicate is rejected (when aggregation should be used), or accepted (when Boole's strict addition should be used).

496. The use of these four operations necessitates continual resort to parentheses, brackets, and braces to show how

Symbolic Logic, p. 39ff.

far the different compound relatives extend. It also becomes desirable to have a "copula of inclusion," or the sign of "is exclusively (if anything)." For this purpose I have since 1870* employed the sign $-\!\!<$ (intended for an improved $\leqq$). It is easily made in the composing room from a dash followed by $<$, and in its cursive form is struck off in two rapid strokes, thus $\prec$. Its meaning is exemplified in the formula

$$w \prec v$$

"anybody who is wise (if any there be) is exclusively found among the virtuous." We also require in this algebra the signs of relatives of second intention

0, "—is inconsistent with—," ∞, "— is coexistent with—,"
T, "—is other than—," I, "— is identical with."

497. The algebra has a moderate amount of power in skilful hands; but its great defect is the vast multitude of purely formal propositions which it brings along. The most significant of these are

$$s(l\,\text{ʒ}\,b) \prec sl\,\text{ʒ}\,b$$

and

$$(l\,\text{ʒ}\,b)s \prec l\,\text{ʒ}\,bs.$$

That is, whatever is a servant of something which is a lover of everything but benefactors is a servant-of-a-lover to everything but benefactors, etc.

498. Professor Schröder attaches, as it seems to me, too high a value to this algebra. That which is in his eyes the greatest recommendation of it is to me scarcely a merit, namely that it enables us to express in the outward guise of an equation propositions whose real meaning is much simpler than that of an equation.

§8. GENERAL ALGEBRA OF LOGIC

499. Besides the algebra just described, I have invented another which seems to me much more valuable. It expresses with the utmost facility everything which can be expressed by a graph, and frequently much more clearly than the unabridged graphs described above. The method of using it in the

* See 47n.

solution of special problems has also been fully developed by me.

500. In this algebra every proposition consists of two parts, its quantifiers and its Boolian. The Boolian consists of a number of relatives united by a non-relative multiplication and aggregation. No relative operations are required (though they can be introduced if desired). Each elementary relative is represented by a letter on the line of writing with subjacent indices to denote the hecceities which fill its blanks. An obelus is drawn over such a relative to deny it.

501. To the left of the Boolian are written the quantifiers. Each of these is a Π or a Σ with one of the indices written subjacent to it, to signify that in the Boolian every object in the universe is to be imaged substituted successively for that index and the non-relative product (if the quantifier is Π) or the aggregate (if the quantifier is Σ) of the results taken. The order of the quantifiers is, of course, material. Thus

$$\Pi_i \Sigma_j l_{ij} = (l_{11} + l_{12} + l_{13} + \text{etc.}) \cdot (l_{21} + l_{22} + l_{23} + \text{etc.}) \cdot \text{etc.}$$

will mean anything loves something. But

$$\Sigma_j \Pi_i l_{ij} = l_{11} \cdot l_{21} \cdot l_{31} \cdot \text{etc.} + l_{12} \cdot l_{22} \cdot l_{32} \cdot \text{etc.} + l_{13} \cdot l_{23} \cdot l_{33} \cdot \text{etc.} + \text{etc.}$$

will mean something is loved by all things.

502. This algebra, which has but two operations, and those easily manageable, is, in my opinion, the most convenient apparatus for the study of difficult logical problems, although the graphical method is capable of such modification as to render it substantially as convenient on the average. Nor would I refuse to avail myself of the algebra of dyadic relatives in the simpler cases in which it is easily handled.

§9. METHOD OF CALCULATING WITH THE GENERAL ALGEBRA

503. My rules for working this algebra, the fruit of long experience with applying it to a great variety of genuine inquiries, have never been published.* Nor can I here do more than state such as the beginner will be likely to require.

* But see 396.

504. A number of premisses being given, it is required to know the most important conclusions of a certain description which can be drawn from them. The first step will be to express the premisses by means of the general algebra, taking care to use entirely different letters *as indices* in the different premisses.

505. These premisses are then to be copulated (or, in Whewell's phrase, colligated), *i.e.*, non-relatively multiplied together, by multiplying their Boolians and writing before the product all the quantifiers. The relative order of the quantifiers of each premiss must (in general) be undisturbed; but the relative order of quantifiers of different premisses is arbitrary. The student ought to place Σ's as far to the left and Π's as far to the right as possible. Different arrangements of the quantifiers will lead to different conclusions from the premisses. It sometimes happens that each of several arrangements leads to a conclusion which could not easily be reached from any other arrangement.

506. The premisses, being so copulated, become one copulated premiss. This copulated premiss is next to be logically multiplied into itself any number of times, the indices being different in all the different factors. For there will be certain conclusions which I call conclusions of the first order, which can be drawn from the copulated premiss without such involution, certain others, which I call inferences of the second order, which can be drawn from its square, etc. But after involution has been carried to a certain point, higher powers will only lead to inferences of subsidiary importance. The student will get a just idea of this matter by considering the rise and decline of interest in the theorems of any mathematical theory, such as geometry or the theory of numbers, as the fundamental hypotheses are applied more and more times in the demonstrations. The number of factors in the copulated premiss, which embraces *all* the hypotheses that either theory assumes, is not great. Yet from this premiss many thousand conclusions have already been drawn in the case of geometry and hundreds in the case of the theory of numbers. New conclusions are now coming in faster than ever before. From the nature of logic they can never be exhausted. But as time goes on the conclusions become more special and less important. It is true

that mathematics, as a whole, does not become more special nor its late discoveries less important, because there is a growth of the hypotheses. Up to a certain degree, the importance of the conclusions increases with their "order." Thus, in geometry, there is nothing worth mention of the first order, and hardly of the second. But there is a great falling off in the importance of conclusions in the theories mentioned long before the fiftieth order has been reached.

507. This involution having been performed, the next step will be the identification (occasionally the diversification) of certain indices. The rule is, that any index quantified with a Π can be transmitted, throughout the Boolian, into any other index whose quantifier stands to the left of its own, which now becomes useless, since it refers to nothing in the Boolian. For example, in

$$\Sigma_i \Pi_j l_{ij}$$

which in the Algebra of Dyadic Relatives would be written $\infty(l \dagger 0)$, we can identify j with i and write

$$\Sigma_i l_{ii}$$

which in the other algebra becomes $\infty(l \cdot \mathsf{I})\infty$.

508. That done, the Boolian is to be manipulated according to any of the methods of non-relative Boolian algebra, and the conclusion is read off.

509. But it is only in the simplest cases that the above operations suffice. Relatives of second intention will often have to be introduced; and their peculiar properties must be attended to. Those of 0 and ∞ are covered by the rules of non-relative Boolian algebra; but it is not so with I and T. We have, for example, to observe that

$$\Pi_i x_i \dotplus y_i = \Pi_i \Pi_j x_i \dotplus \mathsf{T}_{ij} \dotplus y_i{}^*$$
$$\Sigma_i x_i \cdot y_i = \Sigma_i \Sigma_j x_i \cdot \mathsf{I}_{ji} \cdot y_j.$$

Exceedingly important are the relatives signifying "— is a quality of —" and "— is a relation of — to —." It may be said that mathematical reasoning (which is the only deductive reasoning, if not absolutely, at least eminently) almost entirely turns on the consideration of abstractions as if they were objects. The protest of nominalism against such hypostatisation, although, if it knew how to formulate itself, it would be

* This should be y_j.

justified as against much of the empty disputation of the medieval Dunces, yet, as it was and is formulated, is simply a protest against the only kind of thinking that has ever advanced human culture. Nobody will work long with the logic of relatives — unless he restricts the problems of his studies very much — without seeing that this is true.

§10. SCHRÖDER'S CONCEPTION OF LOGICAL PROBLEMS

510. Of my own labors in the logic of relatives since my last publication in 1884,* I intend to give a slight hint in §13. But I desire to give some idea of a part of the contents of Schröder's last volume. In doing so, I shall adhere to my own notation; for I cannot accept Professor Schröder's proposed innovations. I shall give my reasons in detail for this dissent in the *Bulletin of the American Mathematical Society*.† I will here only indicate their general nature. I have no objection whatever to the creation of a new system of signs *ab ovo*, if anybody can propose such a system sufficiently recommending itself. But *that* Professor Schröder does not attempt. He wishes his notation to have the support of existing habits and conventions, while proposing a measure of reform in the present usage. For that he must obtain general consent. Now it seems to me quite certain that no such general agreement can be obtained without the strictest deference to the principle of priority. Without that, new notations can only lead to confusion thrice confounded. The experience of biologists in regard to the nomenclature of their genera and other groups shows that this is so. I believe that their experience shows that the only way to secure uniformity in regard to conventions of this sort, is to accept for each operation and relative the sign definitively recommended by the person who introduced that operation or relative into the Boolian algebra, unless there are the most *substantial* reasons for dissatisfaction with the meaning of the sign. Objections of lesser magnitude may justify slight modifications of signs; as I modify Jevons's ·|· to ⊹, by uniting the two dots by a connecting line, and as I so far yield to Schröder's objections to using ∞ for the sign of whatever is,

* 1885, see No. XIII.

† This does not seem to have been done.

as to resort to the similarly shaped sign of Aries ♈ (especially as a notation of some power is obtained by using all the signs of the Zodiac in the same sense, as I shall show elsewhere). In my opinion, Professor Schröder alleges no sufficient reason for a single one of his innovations; and I further consider them as *positively* objectionable.

511. The volume consists of thirty-one long sections filling six hundred and fifty pages. I can, therefore, not attempt to do more than to exemplify its contents by specimens of the work selected as particularly interesting. Professor Schröder chiefly occupies himself with what he calls "solution-problems," in which it is required to deduce from a given proposition an *equation* of which one member consists in a certain relative determined in advance, while the other member shall not contain that relative. He rightly remarks that such problems often involve problems of elimination.

512. While I am not at all disposed to deny that the so-called "solution-problems," consisting in the ascertainment of the general forms of relatives which satisfy given conditions, are often of considerable importance, I cannot admit that the interest of logical study centres in them. I hold that it is usually much more to the purpose to express in the simplest way what a given premiss discloses in regard to the constitution of a relative, whether that simplest expression is of the nature of an equation or not. Thus, one of Schröder's problems is, "Given $x \prec a$, required x," — for instance, knowing that an opossum is a marsupial, give a description of the opossum.* The so-called solution is $\sum_u (x = a \cdot u)$,† or opossums embrace precisely what is common to marsupials and to some other class. In my judgment $x \prec a$ might with great propriety be called the solution of $\sum_u (x = a \cdot u)$.† When the information contained in a proposition is not of the nature of an equation, why should we, by circumlocutions, insist upon expressing it in the form of an equation?

513. Professor Schröder attaches great importance to the generality of solutions. In my opinion, this is a mistake. It is

* See S. 296.

† Peirce wrote $\sum_u = x\ u \cdot a$ which conforms neither to Schröder nor to the illustration in the text.

not merely that he insists that solutions shall be *complete*, as for example when we require *every root* of a numerical equation, but further that they shall all be embraced under one algebraical expression. Upon that he insists and with that he is satisfied. Whether or not the "solution" is such as to exhibit anything of the real constitution of the relative which forms the first member of the equation he does not seem to care; at least, there is no apparent consideration of the question of how such a result can be secured.

514. Pure mathematics always selects for the subjects of its studies manifolds of perfect homogeneity; and thence it comes that for the problems which first present themselves general solutions are possible, which notwithstanding their generality, guide us at once to all the particular solutions. But even in pure mathematics the class of problems which are capable of solutions at once general and useful is an exceedingly limited one. All others have to be treated by subdivision of cases. That is what meets us everywhere in higher algebra. As for general solutions, they are for the most part trivial — like the well-known and obvious test for a prime number that the continued product of all lesser numbers increased by 1 shall be divisible by that number. Only in those cases in which a general solution points the way to the particular solutions is it valuable; for it is only the particular solutions which picture to the mind the solution of a problem; and a form of words which fails to produce a definite picture in the mind is meaningless.*

515. Professor Schröder endeavors to give the most general formula of a logical problem. It is in dealing with such very general and fundamental matters that the exact logician is most in danger of violating his own principles of exactitude. To seek a formula for all logical problems is to ask what it is, in general terms, that men inquire. To answer that question, my own logical proceeding would be to note that it asks what the essence of a question, in general, is. Now a question is a rational contrivance or device, and in order to understand any rational contrivance, experience shows that the best way is to begin by considering what circumstances of need prompted the contrivance, and then upon what general principle its

* Cf. 2.316.

action is designed to fill that need. Applying this general experience to the case before us, we remark that every question is prompted by some need — that is, by some unsatisfactory condition of things, and that the object of asking the question is to fill that need by bringing reason to bear upon it and to do this by a hypnotically suggestive indication of that to which the mind has to apply itself. I do not know that I have ever, before this minute, considered the question what is the most general formulation of a problem in general; for I do not find much virtue in general formulæ. Nor do I think my answer to this question affords any particularly precious suggestion. But its ordinary character makes it all the better an illustration of the manner — or one of the manners — in which an exact logician may attack, off-hand, a suddenly sprung question. A question, I say, is an indication suggestive (in the hypnotic sense) of what has to be thought about in order to satisfy some more or less pressing want. Ideas like those of this statement, and not talk about φx, and "roots," and the like, must, in my opinion, form the staple of a logical analysis and useful description of a problem, in general. I am none the less a mathematical logician for that. If of two students of the theory of numbers one should insist upon considering numbers as expressed in a system of notation like the Arabic (though using now one number as base of the numeration, and now another), while the other student should maintain that all that was foreign to the theory of numbers, which ought not to consider upon what system the numbers with which it deals are expressed, those two students would, to my apprehension, occupy positions analogous to that of Schröder and mine in regard to this matter of the formulation of the problems of logic; and supposing the student who wished to consider the forms of expression of numbers were to accuse the other of being wanting in the spirit of an arithmetician, that charge would be unjust in quite the same way in which it would be unjust to charge me with deficiency in the mathematical spirit on account of my regarding the conceptions of "values," and "roots," and all that as very special ideas, which can only lumber up the field of consciousness with such hindrances as it is the very end and aim of that diagrammatic method of thinking that characterises the mathematician to get rid of.

516. But different questions are so very unlike that the only way to get much idea of the nature of a problem is to consider the different cases separately. There are in the first place questions about needs and their fulfillment which are not directly affected by the asking of the questions. A very good example is a chess problem. You have only to experiment in the imagination just as you would do on the board if it were permitted to touch the men, and if your experiments are intelligently conducted and are carried far enough, the solution required must be discovered. In other cases, the need to which the question relates is nothing but the intellectual need of having that question answered. It may happen that questions of this kind can likewise be answered by imaginary experimentation; but the more usual case requires real experimentation. The need is of one or other of two kinds. In the one class of cases we experience on several occasions to which our own deliberate action gave a common character, an excitation of one and the same novel idea or sensation, and the need is that a large number of propositions having the same novel consequent but different antecedents, should be replaced by one proposition which brings in the novel element, so that the others shall appear as mere consequences of everyday facts with a single novel one. We may express this intellectual need in a brief phrase as the need of synthetising a multitude of subjects. It is the need of *generalisation*. In another class of cases, we find in some new thing, or new situation, a great number of characters, the same as would naturally present themselves as consequences of a hypothetical state of things, and the need is that the large number of novel propositions with one subject or antecedent should be replaced by a single novel proposition, namely that the new thing or new occasion belongs to the hypothetical class, from which all those other novelties shall follow as mere consequences of matters of course. This intellectual need, briefly stated, is the need of synthetising a multitude of predicates. It is the need of *theory*. Every problem, then, is either a problem of consequences, a problem of generalisation, or a problem of theory.* This statement illustrates how special solutions are the only ones which directly mean

* Compare the physiological explanation of deductions, inductions and abductions in 2.643.

anything or embody any knowledge; and general solutions are only useful when they happen to suggest what the special solutions will be.

517. Professor Schröder entertains very different ideas upon these matters. The general problem, according to him,* is, "Given the proposition $Fx=0$, required the 'value' of x_0," that is, an expression not containing x which can be equated to x. This "value" must be the "general root," that is, it must, under one general description, cover every possible object which fulfils a given condition. This, by the way, is the simplest explanation of what Schröder means by a "solution-problem"; it is the problem to find that form of relative which necessarily fulfils a given condition and in which every relative that fulfils that condition can be expressed. Schröder shows that the solution of such a problem can be put into the form $(\sum_u x=fu)$, which means that a suitable logical function (f) of *any* relative, u, no matter what, will satisfy the condition $Fx=0$; and that nothing which is not equivalent to such a function will satisfy that condition. He further shows, what is very significant, that the solution may be required to satisfy the "adventitious condition" $fx=x$.† This fact about the adventitious condition is all that prevents me from rating the value of the whole discussion as far from high.

518. Professor Schröder next produces what he calls "the rigorous solution" of the general question. This promises something very fine — the rigorously correct resolution of everything that ever could (but for this knowledge) puzzle the human mind. It is true that it supposes that a particular relative has been found which shall satisfy the condition $Fx=0$. But that is seldom difficult to find. Either 0, or ∞, or some other trivial solution commonly offers itself. Supposing, then, that a be this particular solution, that is, that $Fa=0$, the "rigorous solution" is

$$x=fu=a\cdot\infty(Fu)\infty\dagger u\cdot(0\mathfrak{J}\overline{Fu}\mathfrak{J}0).‡$$

That is, it is such a function of u that when u satisfies the condition $Fu=0$, $fu=u$; but when u does not satisfy this condition $fu=a$. Now $Fa=0$.

* See S.161ff.

† See S.163.

‡ See S.165.

519. Since Professor Schröder carries his algebraicity so very far, and talks of "roots," "values," "solutions," etc., when, even in my opinion, with my bias towards algebra, such phrases are out of place, let us see how this "rigorous solution" would stand the climate of numerical algebra. What should we say of a man who professed to give rigorous general solutions of algebraic equations of every degree (a problem included, of course, under Professor Schröder's general problem)? Take the equation: $x^5+Ax^4+Bx^3+Cx^2+Dx+E=0$. Multiplying by $x-a$ we get

$$x^6+(A-a)x^5+(B-aA)x^4+(C-aB)x^3+(D-aC)x^2+(E-aD)x-aE=0$$

The roots of this equation are precisely the same as those of the proposed quintic together with the additional root $x=a$. Hence, if we solve the sextic we thereby solve the quintic. Now our Schröderian solver would say, "There is a certain function, fu, every value of which, no matter what be the value of the variable, is a root of the sextic. And this function is formed by a direct operation. Namely, for all values of u which satisfy the equation

$$u^6+(A-a)u^5+(B-aA)u^4+(C-aB)u^3+(D-aC)u^2+(E-aD)u-aE=0$$

$fu=u$, while for all other values, $fu=a$. Then, $x=fu$ is the expression of every root of the sextic and of nothing else. It is safe to say that Professor Schröder would pronounce a pretender to algebraical power who should talk in that fashion to be a proper subject for *surveillance* if not for confinement in an asylum. Yet he would only be applying Professor Schröder's "rigorous solution," neither more nor less. It is true that Schröder considers this solution as somewhat unsatisfactory; but he fails to state any principle according to which it should be so. Nor does he hold it too unsatisfactory to be frequently resorted to in the course of the volume. The *invention* of this solution exhibits in a high degree that very effective ingenuity which the *solution itself* so utterly lacks, owing to its resting on no correct conception of the nature of problems in general and of their solutions and of the meaning of a proposition.

§11. PROFESSOR SCHRÖDER'S PENTA-GRAMMATICAL NOTATION

520. Professor Schröder's greatest success in the logic of relatives, is due precisely to his having, in regard to certain questions, proceeded by the separation of cases, quite abandoning the glittering generalities of the algebra of dyadic relatives. As his greatest success, I reckon his solutions of "inverse row and column problems" in §16, resting upon an investigation in §15 of the relations of various compound relatives which end in 0, ∞, I, and T. The investigations of §15 might perfectly well have been carried through without any other instrument than the algebra of dyadic relatives. This course would have had certain advantages, such as that of exhibiting the principles on which the formulæ rest. But directness of proof would not have been of the number of those advantages; this is on the contrary decidedly with the notation invented and used by Professor Schröder. This notation may be called *pentagrammatic*, since it denotes a relative by a row of five characters. Imagine a list to be made of all the objects in the universe. Second, imagine a switchboard, consisting of a horizontal strip of brass for each object (these strips being fastened on a wall at a little distance one over another according to the order of the objects in the list) together with a vertical strip of brass for each object (these strips being fastened a little forward of the others, and being arranged in the same order), with holes at all the intersections, so that when a brass plug is inserted in any hole, the object corresponding to the horizontal brass strip can act in some way upon the object corresponding to the vertical brass strip. In order then, by means of this switchboard, to get an analogue of any dyadic relative, a lover of —," we insert plugs so that A and B, being any two objects, A can act on B, if and only if A is a lover of B. Now in Professor Schröder's pentagrammatic notation, the first of the five characters denoting any logical function of a primitive relative, *a*, refers to those horizontal strips, all whose holes are plugged in the representation of *a* (or, as we may say for short, "in *a*"), the second refers to those horizontal strips, each of which has in *a* every hole plugged but one. This one, not necessarily the same for all such strips, may be denoted by A. The third character refers to those horizontal strips which in *a* have

several holes plugged, and several empty. The full holes (different, it may be, in the different horizontal strips) may be denoted by β. The fourth character refers to those horizontal strips which in a have, each of them, but one hole plugged, generally a different hole in each. This one plugged hole may be denoted by Γ. The fifth character will refer to those rows each of which in a has all its holes empty. Then, a will be denoted by $\infty\bar{\mathrm{A}}\beta\Gamma 0$; and $\bar{a}$ by $0\mathrm{A}\bar{\beta}\bar{\Gamma}\infty$;* for in $\bar{a}$, all the holes must be filled that are void in a, and *vice versa*. Consequently $\bar{a}\mathsf{T}=0\bar{\mathrm{A}}\infty\infty\infty$. This shall be shown as soon as we have first examined the pentagrammatic symbol for a. This symbol divides a into four aggregants, viz.:

$$a=(a\jmath 0)+\!\!,\, a\cdot[(a\jmath\mathsf{1})\cdot\bar{a}]\mathsf{T}+\!\!,\, a\cdot a\mathsf{T}\cdot(\bar{a}\cdot\bar{a}\mathsf{T})\mathsf{T}+\!\!,\, a\cdot(\bar{a}\jmath\mathsf{1})$$

In order to prove, by the algebra itself that this equation holds, we remark that $a=a\cdot b+\!\!,\, a\cdot\bar{b}$, whatever b may be. For b, substitute $(a\jmath 0)$. Then, $a\jmath 0\prec a\jmath\mathsf{T}$; but $a\jmath\mathsf{T}=a$. Hence, $a\cdot b=a\jmath 0$. $a\cdot\bar{b}=a\cdot\bar{a}\infty=a\cdot\bar{a}(\mathsf{1}+\!\!,\,\mathsf{T})=a\cdot(\bar{a}\mathsf{1}+\!\!,\,\bar{a}\mathsf{T})$. But $\bar{a}\mathsf{1}=\bar{a}$, and $a\cdot\bar{a}=0$. Hence $a\cdot\bar{b}=a\cdot\bar{a}\mathsf{T}$. Thus $a=a\jmath 0+\!\!,\, a\cdot\bar{a}\mathsf{T}$. Now, in $\bar{a}=\bar{a}\cdot c+\!\!,\,\bar{a}\cdot\bar{c}$, substitute for c, $a\jmath\mathsf{1}$. This gives $\bar{a}=(a\jmath\mathsf{1})\cdot\bar{a}+\!\!,\,\bar{a}\mathsf{T}\cdot\bar{a}$; and thus, $a=a\jmath 0+\!\!,\, a\cdot[(a\jmath\mathsf{1})\cdot\bar{a}]+\!\!,\, a\cdot(\bar{a}\mathsf{T}\cdot\bar{a})\mathsf{T}$. Finally, $a=a\cdot a\mathsf{T}+\!\!,\, a\cdot(\bar{a}\jmath\mathsf{1})$. But $a\cdot(\bar{a}\jmath\mathsf{1})=a\cdot(\bar{a}\jmath\mathsf{1})\cdot(\bar{a}\mathsf{T}\cdot\bar{a})\mathsf{T}+\!\!,\, a\cdot(\bar{a}\jmath\mathsf{1})\cdot\{[(a\jmath\mathsf{1})+\!\!,\, a]\jmath\mathsf{1}\}$.

And

$a\cdot(\bar{a}\jmath\mathsf{1})\cdot\{[(a\jmath\mathsf{1})+\!\!,\, a]\jmath\mathsf{1}\}$

$=a\cdot\{\bar{a}\cdot[(a\jmath\mathsf{1})+\!\!,\, a]\jmath\mathsf{1}\}$	(by distribution)
$=a\cdot[\bar{a}\cdot(a\jmath\mathsf{1})\jmath\mathsf{1}]$	(since $\bar{a}\cdot a=0$)
$=a\cdot(\bar{a}\jmath\mathsf{1})\cdot(a\jmath\mathsf{1}\jmath\mathsf{1})$	(by distribution)
$=a\cdot(\bar{a}\jmath\mathsf{1})\cdot(a\jmath 0)$	(if more than 2 things exist)
$=a\cdot(\bar{a}\jmath\mathsf{1})\cdot(a\jmath\mathsf{1}\cdot\mathsf{T})$	(since $0=\mathsf{1}\cdot\mathsf{T}$)
$=a\cdot(\bar{a}\jmath\mathsf{1})\cdot(a\jmath\mathsf{1})\cdot(a\jmath\mathsf{T})$	(by distribution)
$=a\cdot(\bar{a}\jmath\mathsf{1})\cdot(a\jmath\mathsf{1})$	(since $a\jmath\mathsf{T}=a$)
$=a\cdot(\bar{a}\cdot a\jmath\mathsf{1})$	(by distribution)
$=a\cdot(0\jmath\mathsf{1})$	(since $\bar{a}\cdot a=0$)
$=a\cdot 0$	(if more than 1 object exists)
$=0.$	

* This notation is not exactly that of Schröder, who writes $a=1\alpha\beta\gamma 0$, and $\bar{a}=0\,\bar{\alpha}\bar{\beta}\bar{\gamma}1$. See S. 205.

So that $a\cdot(\bar{a}\jmath 1)=a\cdot(\bar{a}\jmath 1)\cdot(\bar{a}\mathsf{T}\cdot\bar{a})\,\mathsf{T}$ and thus

$$a=a\jmath 0\dotplus a\cdot[(a\jmath 1)\cdot\bar{a}]\mathsf{T}\dotplus a\cdot a\mathsf{T}\,(\bar{a}\mathsf{T}\cdot\bar{a})\,\mathsf{T}\dotplus a\cdot(\bar{a}\jmath 1).$$

This is the meaning of the symbol $\text{♈}\bar{\mathrm{A}}\beta\Gamma 0$.

521. We, now, at length, return, as promised to the examination of $\bar{a}\mathsf{T}$. First, $a\jmath 0\prec\overline{\bar{a}\mathsf{T}}\jmath 0$. For $\overline{\bar{a}\mathsf{T}}=a\jmath 1$ and $a\jmath 1\jmath 0=a\jmath(1\jmath 0)=a\jmath 0$. Hence the first character in the pentagrammatic symbol for $\bar{a}\mathsf{T}$ must be 0. Second $a\cdot[(a\jmath 1)\cdot\bar{a}]\mathsf{T}\prec\bar{a}\mathsf{T}\cdot[(\bar{a}\mathsf{T}\jmath 1)\cdot\overline{\bar{a}\mathsf{T}}]\mathsf{T}$. For it is plain that $a\cdot[(a\jmath 1)\cdot\bar{a}]\mathsf{T}\prec[(a\jmath 1)\cdot\bar{a}]\mathsf{T}\prec\bar{a}\mathsf{T}$. Also $\bar{a}\prec\bar{a}\text{♈}\prec\bar{a}(\mathsf{T}\jmath 1)\prec\bar{a}\mathsf{T}\jmath 1$. Hence $[(a\jmath 1)\cdot\bar{a}]\mathsf{T}\prec[(a\jmath 1)\cdot(\bar{a}\mathsf{T}\jmath 1)]\mathsf{T}$. But $a\jmath 1=\overline{\bar{a}\mathsf{T}}$. Hence, $a\cdot[(a\jmath 1)\cdot\bar{a}]\mathsf{T}\prec\bar{a}\mathsf{T}\cdot[(\bar{a}\mathsf{T}\jmath 1)\cdot\overline{\bar{a}\mathsf{T}}]\mathsf{T}$. Hence, the second character in the pentagrammatic sign for $\bar{a}\mathsf{T}$, is the same as that of a. Thirdly $a\cdot a\mathsf{T}\cdot(\bar{a}\mathsf{T}\cdot\bar{a})\mathsf{T}\prec\bar{a}\mathsf{T}\jmath 0$. For $\bar{a}\prec\bar{a}1\prec\bar{a}(\mathsf{T}\jmath 1)\prec\bar{a}\mathsf{T}\jmath 1$. Hence $(\bar{a}\cdot\bar{a}\mathsf{T})\mathsf{T}\prec[(\bar{a}\mathsf{T}\jmath 1)\cdot(\bar{a}\mathsf{T}\jmath\mathsf{T})]\mathsf{T}\prec(\bar{a}\mathsf{T}\jmath 1\cdot\mathsf{T})\mathsf{T}\prec(\bar{a}\mathsf{T}\jmath 0)\mathsf{T}\prec\bar{a}\mathsf{T}\jmath 0\mathsf{T}\prec\bar{a}\mathsf{T}\jmath 0$. Consequently, the third character of the pentagrammatic symbol of $\bar{a}\mathsf{T}$ must be ♈. Fourthly, $a\cdot(\bar{a}\jmath 1)\prec\bar{a}\mathsf{T}\jmath 0$. For we have just seen that $\bar{a}\prec\bar{a}\mathsf{T}\jmath 1$. Hence $\bar{a}\jmath 1\prec\bar{a}\mathsf{T}\jmath 1\jmath 1$. But $1\jmath 1=0$ if there is more than one object in the universe. Hence $\bar{a}\jmath 1\prec\bar{a}\mathsf{T}\jmath 0$. Consequently, the fourth character of the pentagrammatic formula for $\bar{a}\mathsf{T}$ is ♈. Finally, $\bar{a}\jmath 0\prec\bar{a}\mathsf{T}\jmath 0$. For $\bar{a}\jmath 0\prec\bar{a}\jmath 0\jmath 0\prec\bar{a}\jmath 1\cdot\mathsf{T}\jmath 0\prec(\bar{a}\jmath 1)\cdot(\bar{a}\jmath\mathsf{T})\jmath 0\prec\bar{a}\jmath 1\jmath 0\prec\bar{a}\mathsf{T}\jmath 0$. Hence the fifth character of the pentagram of $\bar{a}\mathsf{T}$ is ♈. In fine, that pentagram is $0\bar{\mathrm{A}}$ ♈ ♈ ♈. Professor Schröder obtains this result more directly by means of a special calculus of the pentagrammatic notation. In that way, he obtains, in §15, a vast number of formulæ, which in §16 are applied in the first place with great success to the solution of such problems as this: Required a form of relation in which everything stands to something but nothing to everything. The author finds instantaneously that every relative signifying such a relation must be reducible to the form $\bar{u}\text{♈}\cdot u\dotplus 1\cdot(u\jmath 0\dotplus\bar{u}\jmath 0)$.* In fact, the first term of this expression $\bar{u}\text{♈}\cdot u$, for which $\bar{u}\text{♈}\cdot u\text{♈}$ might as well be written, embraces all the relatives in question. For let $\bar{u}$ be any such relative. Then, $u=\bar{u}\text{♈}\cdot u$. The second term is added, curiously enough, merely to *exclude other relations*. For if u is such a relative that something is u to everything or to nothing, then that something would be in the relation $\bar{u}\text{♈}\cdot u$ to nothing. To give it a correlate the second term is added; and since all the relatives are

* See S. 231.

already included, it matters not what that correlate be, so long as the second term does not exclude any of the required relatives which are included under the first term. Let v be any relative of the kind required, then $v \cdot (u ʒ 0 ✝ \bar{u} ʒ 0)$ will answer for the second term. If we had no letter expressing a relation known to be of the required kind, the problem would be impossible. Fortunately, both I and T are of that kind. Of course, the negative of such a relative is itself such a relative; so that

$$(u ʒ 0\ \bar{u} ʒ 0) \cdot (v ✝ u \infty \cdot \bar{u} \infty)$$

would be an equivalent form, equally with

$$(u ʒ 0 ✝ \bar{u} ʒ 0) \cdot v ✝ u\ .\ \infty\ \bar{u}\ \infty.$$

522. §16 concludes with some examples of eliminations of great apparent complexity.* In the first of these we have given $x = (\bar{u} ʒ \mathsf{I}) \infty ✝ u$; and it is required to eliminate u. We have, however, instantly $u ⤙ x$

$$(\bar{u} ʒ \mathsf{I}) \infty ⤙ x$$

Whence, immediately,

$$(\bar{x} ʒ \mathsf{I}) \infty ⤙ x,$$

or

$$\infty ⤙ (x \cdot x\mathsf{T}) \infty.$$

The next example, the most complicated, requires u to be eliminated from the equation

$$x = \bar{u} ʒ 0 ✝ (u ʒ \mathsf{I}) \infty \cdot \bar{u}\mathsf{T} ✝ (u ʒ \mathsf{I}) \cdot \bar{u} ✝ (\bar{u} ʒ \mathsf{I}) \cdot u$$
$$✝ (u\mathsf{T} \cdot \bar{u}\mathsf{T} ʒ 0) \cdot \bar{u},$$

He performs the elimination by means of the pentagrammatic notation very easily as follows: Putting $u = \infty \bar{\mathrm{A}} \beta \Gamma 0$

$\bar{u} ʒ 0$	$= 0000\infty$
$(u ʒ \mathsf{I}) \infty \cdot \bar{u}\mathsf{T}$	$= 0\bar{\mathrm{A}}000$
$(u ʒ \mathsf{I}) \cdot \bar{u}$	$= 0\mathrm{A}000$
$(\bar{u} ʒ \mathsf{I}) \cdot u$	$= 000\Gamma 0$
$(u\mathsf{T}\ .\ \bar{u}\mathsf{T} ʒ 0) \cdot \bar{u}$	$= 00\bar{\beta}00$
sum	$0\infty\bar{\beta}\Gamma\infty$

Thus, x is of the form $\infty\bar{\beta}\Gamma 0$, which has been found in former problems to imply $x ʒ \mathsf{I} ⤙ x$.

Without the pentagrammatical notation this elimination would prove troublesome, although with that as a guide it could easily be obtained by the algebra alone.

* See S. 239.

§12. PROFESSOR SCHRÖDER'S ICONIC SOLUTION OF $x \prec \varphi x$.

523. Another valuable result obtained by Professor Schröder is the solutions of the problem

$$x \prec \varphi x.$$

Namely, he shows that

$$x = f^{\infty} u$$

where

$$fu = u \cdot \varphi u$$

(Of course, by contraposition, this gives for the solution of $\varphi x \prec x$, $x = f^{\infty} u$ where $fu = u + \varphi u$.) The correctness of this solution will appear upon a moment's reflexion; and nearly all the useful solutions in the volume are cases under this.

524. It happens very frequently that the iteration of the functional operation is unnecessary, because it has no effect.

Suppose, for example, that we desire the general form of a "transitive" relative, that is, such a one, x, that

$$xx \prec x.$$

In this case, since $\mathbf{1} \prec l \dagger \breve{\bar{l}}$ whatever l may be, we have

$$x \prec x\mathbf{1} \prec x(x \dagger \breve{\bar{x}}) \prec xx \dagger \breve{\bar{x}} \prec x \dagger \breve{\bar{x}},$$

or

$$x \prec x \dagger \breve{\bar{x}}$$

If, then,

$$fu = u \cdot (u \dagger \breve{\bar{u}}),$$

we have

$$x = f^{\infty} u.$$

Here,

$$fu \prec u;$$

so that

$$f^{\infty} u \prec fu.$$

Also,

$$f^2 u = fu \cdot (fu \dagger \breve{\overline{fu}}) = u \cdot (u \dagger \breve{\bar{u}}) \cdot [u \cdot (u \dagger \breve{\bar{u}}) \dagger (\breve{\bar{u}} + u\breve{\bar{u}})]$$
$$= u \cdot (u \dagger \breve{\bar{u}}) \cdot [uf(\mathbf{1} + u)\breve{\bar{u}}] \cdot [u \dagger \breve{\bar{u}} \dagger (\mathbf{1} + u)\breve{\bar{u}}].$$

Now

$$fu = u \cdot (u \dagger \breve{\bar{u}}) = u \cdot (u \dagger \breve{\bar{u}}) \cdot (u \dagger \breve{\bar{u}}) \cdot (u \dagger \breve{\bar{u}})$$
$$= u \cdot (u \dagger \breve{\bar{u}}) \cdot (u \dagger \mathbf{1}\breve{\bar{u}}) \cdot (u \dagger \mathbf{1}\breve{\bar{u}})$$
$$\prec u \cdot (u \dagger \breve{\bar{u}}) \cdot [u \dagger (\mathbf{1} + u)\breve{\bar{u}}] \cdot [u \dagger (\breve{\bar{u}} \dagger u)\breve{\bar{u}}] \prec$$
$$\prec u \cdot (u \dagger \breve{\bar{u}}) \cdot [u \dagger (\mathbf{1} + u)\breve{\bar{u}}] \cdot (u \dagger \breve{\bar{u}} \dagger u\breve{\bar{u}})$$
$$\prec u \cdot (u \dagger \breve{\bar{u}}) \cdot [u \dagger (\mathbf{1} + u)\breve{\bar{u}}] \cdot [u \dagger \breve{\bar{u}} \dagger (\mathbf{1} + u)\breve{\bar{u}}] \prec f^2 u.$$

Thus $fu=f^{\times}u$; and

$$x=\sum_{u} u\cdot(u\,ɟ\,\breve{u})$$

This is a truly iconic result; that is, it shows us what the constitution of a transitive relative really is. It shows us that transitiveness always depends upon inclusion; for to say that A is $l\,ɟ\,\breve{l}$ of B is to say that the things loved by B are included among those loved by A.* The factor $u\,ɟ\,\breve{u}$ is transitive by itself; for

$$(u\,ɟ\,\breve{u})(u\,ɟ\,\breve{u})\prec u\,ɟ\,\breve{u}u\,ɟ\,\breve{u}\prec u\,ɟ\,T\,ɟ\,\breve{u}\prec u\,ɟ\,\breve{u}.$$

The effect of the other factor, u, of the form for the general transitive is merely in certain cases to exclude universal identity, and thus to extend the class of relatives represented by $u\,ɟ\,\breve{u}$ so as to include those of which it is not true that $\mathbf{1}\prec x$. Here we have an instance of restriction having the effect of extension, that is, restriction of special relatives extends the class of relatives represented. This does not take place in all cases, but only where certain relatives can be represented in more than one way.

525. Indicating, for a moment, the copula by a dash, the typical and fundamental syllogism is

$$\text{A}-\text{B} \quad \text{B}-\text{C}$$
$$\therefore \text{A}-\text{C}.$$

That is to say, the principle of this syllogism enters into every syllogism. But to say that this is a valid syllogism is merely to say that the copula expresses a transitive relation. Hence, when we now find that transitiveness always depends upon inclusion, the initial analysis by which the copula of inclusion was taken as the general one is fully confirmed. For the chief end of formal logic is the representation of the syllogism.

§13. INTRODUCTION TO THE LOGIC OF QUANTITY†

526. The great importance of the idea of quantity in demonstrative reasoning seems to me not yet sufficiently explained. It appears, however, to be connected with the circumstance that the relations of being greater than and of being at least

* For $l_1\,ɟ\,\breve{l}_2, =\breve{l}_1\prec\breve{l}_2, =\breve{l}_2,\prec\breve{l}_1$.

† See vol. 4, bk. I, ch. 4.

as great as are transitive relations. Still, a satisfactory evolutionary logic of mathematics remains a desideratum. I intend to take up that problem in a future paper.* Meantime the development of projective geometry and of geometrical topics has shown that there are at least two large mathematical theories of continuity into which the idea of continuous *quantity*, in the usual sense of that word, does not enter at all. For projective geometry Schubert† has developed an algebraical calculus which has a most remarkable affinity to the Boolian algebra of logic. It is, however, imperfect, in that it only gives imaginary points, rays, and planes, without deciding whether they are real or not. This defect cannot be remedied until topology — or, as I prefer to call it, mathematical topics — has been further developed and its logic accurately analysed.‡ To do this ought to be one of the first tasks of exact logicians. But before that can be accomplished, a perfectly satisfactory logical account of the conception of continuity is required. This involves the definition of a certain kind of infinity; and in order to make that quite clear, it is requisite to begin by developing the logical doctrine of infinite multitude. This doctrine still remains, after the works of Cantor, Dedekind, and others, in an inchoate condition. For example, such a question remains unanswered as the following: Is it, or is it not, logically possible for two collections to be so multitudinous that neither can be put into a one-to-one correspondence with a part or the whole of the other? To resolve this problem demands, not a mere *application* of logic, but a further *development* of the conception of logical possibility.

527§. I formerly defined the possible as that which in a given state of information (real or feigned) we do not know not to be true.¶ But this definition today seems to me only a twisted phrase which, by means of two negatives, conceals an anacoluthon. We know in advance of experience that certain things are not true, because we see they are impossible. Thus, if a chemist tests the contents of a hundred bottles for fluorine, and finds it present in the majority, and if another chemist

* See vol. 4, bk. I, ch. 7.

† "*Kalkül der Abzählenden Geometrie,*" 1879.

‡ See 4.219ff.

§ See 6.450.

¶ See e.g. 374 and 442.

tests them for oxygen and finds it in the majority, and if each of them reports his result to me, it will be useless for them to come to me together and say that they know infallibly that fluorine and oxygen cannot be present in the same bottle; for I see that such infallibility is *impossible*. I know it is not true, because I satisfy myself that there is no room for it even in that ideal world of which the real world is but a fragment. I need no sensible experimentation, because ideal experimentation establishes a much broader answer to the question than sensible experimentation could give. It has come about through the agencies of development that man is endowed with intelligence of such a nature that he can by ideal experiments ascertain that in a certain universe of logical possibility certain combinations occur while others do not occur. Of those which occur in the ideal world some do and some do not occur in the real world; but all that occur in the real world occur also in the ideal world.[1] For the real world is the world of sensible experience, and it is a part of the process of sensible experience to locate its facts in the world of ideas. This is what I mean by saying that the sensible world is but a fragment of the ideal world. In respect to the ideal world we are virtually omniscient; that is to say, there is nothing but lack of time, of perseverance, and of activity of mind to prevent our making the requisite experiments to ascertain positively whether a given combination occurs or not. Thus, every proposition about the ideal world can be ascertained to be either true or false. A description of thing which occurs in that world is *possible, in the substantive logical sense*. Very many writers assert that everything is logically possible which involves no contradiction. Let us call that sort of logical possibility, *essential*, or *formal*, logical possibility. It is not the only logical possibility; for in this sense, two propositions contradictory of one another may both be severally possible, although their combination is not possible.[2] But in the *substantive* sense, the contradictory of a

[1] For the simple reason that the real world is a part of the ideal world. namely, that part which sufficient experience would tend ultimately (and therefore definitively), to compel Reason to acknowledge as having a being independent of what he may arbitrarily, or willfully, create.—*Marginal note*, 1908.

[2] That is to say each is *vaguely*, not *distinctly*, possible.—*Marginal note*, 1908.

possible proposition is impossible, because we are virtually omniscient in regard to the ideal world. For example, there is no contradiction in supposing that only four, or any other number, of independent atoms exist. But it is made clear to us by ideal experimentation, that five atoms are to be found in the ideal world. Whether all five are to be found in the sensible world or not, to say that there are only four in the ideal world is a proposition absolutely to be rejected, notwithstanding its involving no contradiction.

528. It would be a great mistake to suppose that ideal experimentation can be performed without danger of error; but by the exercise of care and industry this danger may be reduced indefinitely. In sensible experimentation, no care can always avoid error. The results of induction from sensible experimentation are to afford some ratio of frequency with which a given consequence follows given conditions in the existing order of experience. In induction from ideal experimentation, no particular order of experience is forced upon us; and consequently no such numerical ratio is deducible. We are confined to a dichotomy: the result either is that some description of thing occurs or that it does not occur. For example, we cannot say that one number in every three is divisible by three and one in every five is divisible by five. This is, indeed, so if we choose to arrange the numbers in the order of counting; but if we arrange them with reference to their prime factors, just as many are divisible by one prime as by another. I mean, for instance, when they are arranged [in blocks] as follows:

1, 2, 4, 8, etc.	5, 10, 20, 40, etc.	7, 14, 28, 56, etc.	35, 70, etc.
3, 6, 12, 24, etc.	15, 30, 60, 120, etc.	21, 42, 84, 168, etc.	105, 210, etc.
9, 18, 36, 72, etc.	45, 90, 180, 360, etc.	etc.	etc.[1]
27, 54, 108, 216, etc.	135, 270, 540, 1080, etc.		
etc.	etc.		

529. Thus, dichotomy rules the ideal world. Plato, therefore, for whom that world alone was real, showed that insight into concepts but dimly apprehended that has always characterised philosophers of the first order, in holding dichotomy

[1] Or linearly [by taking the diagonals] as follows. But there the small primes come earlier: 1; 2, 3; 5; 4, 6, 9; 10, 15; 7; 8, 12, 18, 27; 20, 30, 45; 14, 21; 35; 16, 24, 36, 54, 81; 40, 60, 90, 135; 28, 42, 63; 70, 105; 11, etc." — *Marginal note*, 1908.

to be the only truthful mode of division. Lofty moral sense consists in regarding, not indeed *the*, but yet *an*, ideal world as in some sense the only real one; and hence it is that stern moralists are always inclined to dual distinctions.*

530. Ideal experimentation has one or other of two forms of results. It either proves that $\Sigma_i m_i$, a particular proposition true of the ideal world, and going on, finds $\Sigma_j \bar{m}_j$ also true; that is, that m and $\bar{m}$ are both possible, or it succeeds in its induction and shows the universal proposition $\Pi_i \bar{m}_i$ to be true of the ideal world; that is that $\bar{m}$ is *necessary* and *m impossible*.

531. Every result of an ideal induction clothes itself, in our modes of thinking, in the dress of a *contradiction*. It is an anacoluthon to say that a proposition is impossible *because* it is self-contradictory. It rather is thought so as to appear self-contradictory because the ideal induction has shown it to be impossible. But the result is that in the absence of any interfering contradiction every particular proposition is possible in the substantive logical sense, and its contradictory universal proposition is impossible. But where contradiction interferes this is reversed.

532. In former publications† I have given the appellation of *universal* or *particular* to a proposition according as its *first* quantifier is Π or Σ. But the study of substantive logical possibility has led me to substitute the appellations *negative* and *affirmative* in this sense, and to call a proposition *universal* or *particular* according as its *last* quantifier is Π or Σ‡. For letting l be any relative, one or other of the two propositions

$$\Pi_i \Sigma_j l_{ij} \qquad \Sigma_i \Pi_j \bar{l}_{ij}$$

and one or other of the two propositions

$$\Pi_j \Sigma_i \bar{l}_{ij} \qquad \Sigma_j \Pi_i l_{ij}$$

are true, while the other one of each pair is false. Now, in the absence of any peculiar property of the special relative l, the two similar forms $\Sigma_i \Pi_i \bar{l}_{ij}$§ and $\Sigma_j \Pi_i l_{ij}$ must be equally possible in the substantive logical sense. But these two propositions cannot both be true. Hence, both must be false in the

* Cf. 1.61, 1.591ff, 2.156.

† Where?

‡ See 4.552n.

§ Obviously a misprint for: $\Sigma_i \Pi_j \bar{l}_{ij}$.

ideal world, in the absence of any constraining contradiction. Accordingly, these ought to be regarded as universal propositions, and their contradictions, $\Pi_i \Sigma_j l_{ij}$ and $\Pi_j \Sigma_i \bar{l}_{ji}$, as particular propositions.*

533. There are two opposite points of view, each having its logical value, from one of which, of two quantifiers of the same proposition, the preceding is more important than the following, while from the other point of view the reverse is the case. Accordingly, we may say that an affirmative proposition is particular in a secondary way, and that a particular proposition is affirmative in a secondary way.

534. If an index is not quantified at all, the proposition is, with reference to that index, *singular*. To ascertain whether or not such a proposition is true of the ideal world, it must be shown to depend upon some universal or particular proposition.

535. If some of the quantifiers refer not to hecceities, having in themselves no general characters except the logical characters of identity, diversity, etc., but refer to *characters*, whether non-relative or relative, these alone are to be considered in determining the "quantity" of an ideal proposition as universal or particular. For anything whatever is true of *some* character, unless that proposition be downright absurd; while nothing is true of *all* characters except what is formally necessary.† Consider, for example, a dyadic relation. This is nothing but an aggregation of pairs. Now any two hecceities may in either order form a pair; and any aggregate whatever of such pairs will form *some* dyadic relation. Hence, we may totally disregard the manner in which the hecceities are connected in determining the possibility of a hypothesis about *some* dyadic relation.

536. Characters have themselves characters, such as importance, obviousness, complexity, and the like. If some of the

* (1)$\Sigma_i \Pi_j \bar{l}_{ij} \prec \Pi_j \Sigma_i l_{ij}$ (2); (3)$\Sigma_j \Pi_i l_{ij} \prec \Pi_i \Sigma_j l_{ij}$(4). (2) and (3) are contradictories; (1) and (4) are contradictories. If (1) were true (2) would be true. If (3) were true (4) would be true. If (1) and (3) were true, (2) and (3), which are contradictories, would both be true. Though (1) and (3) thus cannot both be true, they may both be false. They are related to one another as an A and an E; their contradictories (4) and (2) must be related to one another as an I and an 0 — both can be and one at least must be true.

† See 2.517ff.

quantified indices denote such characters of characters, they will, in reference to a purely ideal world be paramount in determining the quantity of the proposition as universal or particular.

537. All quantitative comparison depends upon a *correspondence*. A correspondence is a relation which every subject[1] of one collection bears to a subject of another collection, to which no other is in the same relation. That is to say, the relative "corresponds to" has

$$\sum_u u \cdot (1 \dagger \bar{u})$$

not merely as its *form*, but as its *definition*. This relative is transitive; for its relative product into itself is

$$[\sum_u u \cdot (1 \dagger \bar{u})][\sum_v v \cdot (1 \dagger \bar{v})] \prec \sum_u \sum_v uv \cdot (1 \dagger \bar{u})(1 \dagger \bar{v})$$

$$\prec \sum_u \sum_v uv \cdot (1 \dagger \bar{u} \dagger \bar{v}) \prec \sum_u \sum_v uv \cdot (1 \dagger \overline{uv}) \prec \sum_w w \cdot (1 \dagger \bar{w}).$$

But it is to be observed that if the P's, the Q's, and the R's are three collections, it does not follow because every P corresponds to an R, and every Q corresponds to an R that every object of the aggregate collection P+Q corresponds to an R. The *dictum de omni* in external appearance fails here. For P may be $[u \cdot (1 \dagger \bar{u})]$R and Q may be $[v \cdot (1 \dagger \bar{v})]$R; but the aggregate of these is not $[(u + v) \cdot (1 \dagger \overline{u + v})]$R, which equals $[(u + v) \cdot (1 \dagger \bar{u}) \cdot (1 \dagger \bar{v})]$R. The aggregate of the two first is $\{(u \dagger v) \cdot [v \cdot (1 \dagger \bar{v}) + 1 \dagger \bar{u}] \cdot [u \cdot (1 \dagger \bar{u}) + 1 \dagger \bar{v}]\}$R, which is obviously too broad to be necessarily included under the other expression. Correspondence is, therefore, not a relation between the subjects of one collection and those of another, but between the collections themselves. Let $q_{\alpha i}$ mean that i is a subject of the collection, α, and let $r_{\beta jk}$ mean that j stands in the relation β to k. Then, to say that the collection P corresponds to the collection Q, or, as it is sometimes expressed, that "*for* every subject of Q there is a subject of P," is to make the assertion expressed by

[1] I prefer to speak of a member of a collection as a *subject* of it rather than as an *object* of it; for in this way I bring to mind the fact that the collection is virtually a quality or class-character. [A collection is a rhema or propositional function. Its members are those subjects which make it a true proposition. See 66.]

$$\Sigma_\beta\Pi_i\Sigma_j\Pi_k\,\bar{q}_{\mathrm{P}i}\maltese r_{\beta ij}\cdot(\mathrm{I}_{ik}\maltese\bar{r}_{\beta kj})\cdot q_{\mathrm{Q}j}.^*$$

In the algebra of dual relatives this may be written

$$\Sigma_\beta\mathrm{P}\prec\bar{q}\,\mathfrak{J}\,[r_\beta\cdot(\mathrm{I}\,\mathfrak{J}\,\bar{r}_\beta)]\breve{q}\mathrm{Q}.$$

The transitivity is evident; for

$$\Sigma_\beta\Sigma_\gamma\bar{q}\,\mathfrak{J}[r_\beta\cdot(\mathrm{I}\,\mathfrak{J}\,\bar{r}_\beta)]\breve{q}\,\{\bar{q}\,\mathfrak{J}[r_\gamma\cdot(\mathrm{I}\,\mathfrak{J}\,\bar{r}_\gamma)]\breve{q}\,\}$$
$$\prec\Sigma_\beta\Sigma_\gamma\bar{q}\,\mathfrak{J}[r_\beta\cdot(\mathrm{I}\,\mathfrak{J}\,\bar{r}_\beta)]\{\,\breve{q}\bar{q}\,\mathfrak{J}[r_\gamma\cdot(\mathrm{I}\,\mathfrak{J}\,\bar{r}_\gamma)]\breve{q}\,\}$$
$$\prec\Sigma_\beta\Sigma_\gamma\bar{q}\,\mathfrak{J}[r_\beta\cdot(\mathrm{I}\,\mathfrak{J}\,\bar{r}_\beta)]\,\{\top\,\mathfrak{J}[r_\gamma\cdot(\mathrm{I}\,\mathfrak{J}\,\bar{r}_\gamma)]\breve{q}\,\}$$
$$\prec\Sigma_\beta\Sigma_\gamma\bar{q}\,\mathfrak{J}[r_\beta\cdot(\mathrm{I}\,\mathfrak{J}\,\bar{r}_\beta)][r_\gamma\cdot(\mathrm{I}\,\mathfrak{J}\,\bar{r}_\gamma)]\breve{q}$$
$$\prec\Sigma_\beta\Sigma_\gamma\bar{q}\,\mathfrak{J}[r_\beta r_\gamma\cdot(\mathrm{I}\,\mathfrak{J}\,\bar{r}_\beta\,\mathfrak{J}\,\bar{r}_\gamma)]\breve{q}$$
$$\prec\Sigma_\delta\,\bar{q}\,\mathfrak{J}[r_\delta\cdot(\mathrm{I}\,\mathfrak{J}\,\bar{r}_\delta)]\breve{q}.^1$$

538. Not only is the relative of correspondence transitive but it also possesses what may be called *antithetic transitivity*. Namely, if c be the relative, not only is $cc\prec c$ but also $c\prec c\,\mathfrak{J}\,c$. To demonstrate this very important proposition is, however, far from easy. The quantifiers of the assertion that for every subject of one character there is a subject of another are $\Sigma_\beta\Pi_i\Sigma_j\Pi_k$. Hence, the proposition is particular† and will be true in the ideal world, except in case a positive contradiction is involved.

539. Let us see how such contradiction can arise. The assertion that for every subject of P there is a subject of Q is

$$\Sigma_\beta\Pi_i\Sigma_j\Pi_k\,\bar{q}_{\mathrm{P}i}\maltese r_{\beta ij}\cdot(\mathrm{I}_{ik}\maltese\bar{r}_{\beta kj})\cdot q_{\mathrm{Q}j}.$$

This cannot vanish if the first aggregant term does not vanish, that is, if $\Pi_i q_{\mathrm{P}i}$ or there is no subject of P. It cannot vanish if everything is a subject of Q. For in that case, the last factor of the latter aggregant disappears, and substituting I for r_β the second aggregant becomes ∞. The expression cannot vanish if every subject of P is a subject of Q. For when I is substituted for r_β, we get

$$\Pi_i\,\bar{q}_{\mathrm{P}i}\maltese q_{\mathrm{Q}i}.$$

If P has but a single individual subject and Q has a subject, for every P there is a Q. For in this case we have only to take for

* I.e., all the *i's* which are members of P, are related to the *j's* which are members of Q, and there is no *k* distinct from an *i* which has the same relation to the *j's* which the *i's* have to the *j's*.

[1] It must be remembered that to a person familiar with the algebra all such series of steps become evident at first glance.

† Cf. 532.

β the relation of the subject of P to any one of the subjects of Q. But if P has more than one subject, and Q has but one, the expression above vanishes. For let 1 and 2 be the two subjects of P. Substituting 1 for i, we get

$$\Pi_k\, r_{\beta 1j} \cdot (\mathrm{l}_{1k} \mathbin{+\!\!,} \bar{r}_{\beta kj}) \cdot q_{Qj}$$

Substituting 2 for i we get

$$\Pi_k\, r_{\beta 2j} \cdot (\mathrm{l}_{2k} \mathbin{+\!\!,} \bar{r}_{\beta kj}) \cdot q_{Qj}$$

Multiplying these

$$\Pi_k \Pi_k\, r_{\beta 1j} \cdot r_{\beta 2j} \cdot (\mathrm{l}_{1k} \mathbin{+\!\!,} \bar{r}_{\beta kj}) \cdot (\mathrm{l}_{2k'} \mathbin{+\!\!,} \bar{r}_{\beta k'j}) \cdot q_{Qi}$$

Substituting 2 for k and 1 for k', this gives

$$r_{\beta 1j} \cdot r_{\beta 2j} \cdot \bar{r}_{\beta 2j} \cdot {}_{\beta 1j} \cdot q_{Qi}$$

which involves two contradictions.

540. It is to be remarked that although if every subject of P is a subject of Q, then for every subject of P there is a subject of Q, yet it does not follow that if the subjects of P are a part only of the subjects of Q, that there is then not a subject of P for every subject of Q. For example, numbering 2, 4, 6, etc., as the first, second, third, etc., of the even numbers, there is an even number for every whole number, although the even numbers form but a part of the whole numbers.

541. It is now requisite, in order to prove that $c \prec c \dagger c$, to draw three propositions from the doctrine of substantive logical possibility. The first is that given any relation, there is a possible relation which differs from the given relation only in excluding any of the pairs we may choose to exclude. Suppose, for instance, that for every subject of P there is a subject of Q, that is that

$$\Sigma_\beta\, \breve{q}\mathrm{P} \prec [r_\beta \cdot (\mathrm{l} \dagger \bar{r}_\beta)]\, \breve{q}\mathrm{Q}.$$

The factor $(\mathrm{l} \dagger \bar{r}_\beta)$ here has the effect of allowing each correlate but one relate. Each relate is, however, allowed any number of correlates. If we exclude all but one of these, the one retained being, if possible, a subject of Q, we have a possible relation, β', such that

$$\Sigma_{\beta'}\, \breve{q}\mathrm{P} \prec [r_{\beta'} \cdot (\mathrm{l} \dagger \bar{r}_{\beta'}) \cdot (\bar{r}_{\beta'} \dagger \mathrm{l})]\, \breve{q}\mathrm{Q}.$$

542. The second proposition of substantive logical possibility is that whatever is true of *some* of a class is true of the

whole of *some* class. That is, if we accept a proposition of the form $\Sigma_i a_i \cdot b_i$, we can write

$$\Sigma_\gamma \Pi_i \bar{q}_{\gamma i} \pmb{+} \bar{a}_i \pmb{+} b_i,$$

though this will generally fail positively to assert, in itself, what is implied, that the collection γ excludes whatever is a but not b, and includes something in common with a. There are, however, cases in which this implication is easily made plain.

Applying these two principles to the relation of correspondence, we get a new statement of the assertion that for every P there is a Q. Namely, if we write $a_{\alpha i}$ to signify that i is a relate of the relative r_α to some correlate, that is if $a_{\alpha i} = (i \prec r_\alpha \infty)$, if we write $b_{\alpha j}$ to signify that j is a correlate of the relative r_α to some relate, that is if $b_{\alpha j} = (j \prec \breve{r}_\alpha \infty)$, and if we write $p_{c\alpha}$ to signify that r_α is an aggregate of the relative r_c, that is, if $p_{c\alpha} = (r_\alpha \prec r_c)$, then the proposition that for every subject of P there is a subject of Q may be put in the form,

$$\Sigma_c \Sigma_\gamma \Pi_x \Pi_y \Sigma_\delta \Sigma_\epsilon \Pi_\alpha \Sigma_i \Sigma_j \Pi_\beta \Pi_u \Pi_v$$
$$[\bar{p}_{c\alpha} \pmb{+} a_{\alpha i} \cdot q_{\mathrm{P}i} \cdot b_{\alpha j} \cdot q_{\mathrm{Q}j} \cdot q_{\gamma j} \cdot (\bar{a}_{\alpha u} \pmb{+} \mathbf{l}_{iu}) \cdot (\bar{b}_{\alpha v} \pmb{+} \mathbf{l}_{jv}) \cdot (\bar{p}_{c\beta} \pmb{+} \mathbf{l}_{\alpha\beta}$$
$$\pmb{+} \bar{a}_{\beta i} \cdot b_{\beta j})] \cdot (\bar{q}_{\mathrm{P}x} \pmb{+} a_{\delta x} \cdot p_{c\delta}) \cdot (\bar{q}_{\mathrm{Q}y} \pmb{+} \bar{q}_{\gamma y} \pmb{+} b_{\epsilon y} \cdot p_{c\epsilon}).$$

This states that there is a collection of pairs, c, any single pair of which, α, has for its sole first subject a subject of P, and for its sole second subject a subject of Q which is at the same time a subject of a collection, j, and that no two pairs of the collection, c, have the same first subject or the same second subject, and that every subject of P is a first subject of some pair of this collection, c, and every subject of Q which is at the same time a subject of γ is a second subject of some pair of the same collection, c.

543. The third proposition of the doctrine of substantive logical possibility of which we have need is that all hecceities are alike in respect to their capacity for entering into possible pairs. Consequently, all the objects of any collection whatever may be severally and distinctly paired with all the objects of a collection which shall either be wholly contained in, or else shall entirely contain, any other collection whatever. Consequently,

$$\Pi_{\mathrm{P}} \Pi_{\mathrm{Q}} \Sigma_c \Sigma_\delta \Pi_x \Sigma_\delta \Pi_y \Sigma_\delta \Pi_\alpha \Sigma_i \Sigma_j \Pi_u \Pi_v \Pi_\beta \Pi_m \Pi_n$$
$$[\bar{p}_{c\alpha} \pmb{+} a_{\alpha i} \cdot q_{\mathrm{P}i} \cdot b_{\alpha j} \cdot q_{\delta j} \cdot (\bar{a}_{\alpha u} \pmb{+} \mathbf{l}_{iu}) \cdot (\bar{b}_{\alpha v} \pmb{+} \mathbf{l}_{vj}) \cdot (\bar{p}_{c\beta} \pmb{+} \mathbf{l}_{\alpha\beta} \pmb{+} \bar{a}_{\beta i} \cdot$$
$$b_{\beta j})] \cdot (\bar{q}_{\mathrm{P}x} \pmb{+} a_{\delta x} \cdot p_{c\delta}) \cdot (\bar{q}_{\delta y} \pmb{+} b_{\epsilon y} \cdot p_{c\epsilon}) \cdot (\bar{q}_{\delta m} \pmb{+} q_{\mathrm{Q}m} \pmb{+} \bar{q}_{\mathrm{Q}n} \pmb{+} q_{\delta n}).$$

544. Although the above three propositions belong to a system of doctrine not universally recognised, yet I believe their truth is unquestionable. Suppose, now, that it is not true that for every subject of P there is a subject of Q. Then, in the last formula, $\Pi_m \bar{q}_{\delta m} +\!\!, q_{Qm} \prec 0$. This leaves for the last factor $\Pi_n \bar{q}_{Qn} +\!\!, q_{\delta n}$, and then the formula expresses that for every subject of Q there is a subject of P. In other words, we have demonstrated the important proposition that *two collections cannot be disparate in respect to correspondence*, but that for every subject of the one there must be a subject of the other.

545. The theorem $c \prec c \%_0 c$ is now established; for since of any two collections one corresponds to the other, we have $\infty \prec c +\!\!, \breve{c}$ or (non-relatively multiplying by $\breve{c}$) $\breve{c} \prec c$. Hence, $c \prec ! c \prec (\breve{c} \mathfrak{z} c) c \prec \breve{\breve{c}} \mathfrak{z} cc \prec c \mathfrak{z} cc$; and, by the transitive principle $cc \prec c$, we finally obtain $c \prec c \mathfrak{z} c$.

546. Thus is established the conception of *multitude*. Namely, if for every subject of P there is a subject of *Q*, while there is not for every subject of Q a subject of P, the *multitude* of Q is said to be *greater* that that of P. But if for every subject of each collection there is a subject of the other, the *multitudes* of the two collections are said to be *equal* the one to the other. We may create a scale of objects, one for every group of equal collections. Calling these objects *arithms*, the first arithm will belong to 0 considered as a collection, the second to individuals, etc. Calling a collection the counting of which can be completed an *enumerable* collection, the multitude of any enumerable collection equals that of the arithms that precede its arithm. Calling a collection whose multitude equals that of all the arithms of enumerable collection a *denumerable* collection (because its subjects can all be distinguished by ordinal numbers, though the counting of it cannot be completed), the arithms preceding the arithm of denumerable collections form a denumerable collection. More multitudinous collections are greater than the collections of arithms which precede their arithm.

547. Let there be a denumerable collection, say the cardinal numbers; and let there be two houses. Let there be a collection of children, each of whom wishes to have those numbers placed in some way into those houses, no two children wishing for the same distribution, but every distribution being wished

for by some child. Then, as Dr. George Cantor has proved,* the collection of children is greater in multitude than the collection of numbers. Let a collection equal in multitude to that collection of children be called an *abnumeral* collection *of the first dignity*. The real numbers (surd and rational) constitute such a collection.

548. I now ask, suppose that for every way of placing the subjects of one collection in two houses, there is a way of placing the subjects of another collection in two houses, does it follow that for every subject of the former collection there is a subject of the latter? In order to answer this, I first ask whether the multitude of possible ways of placing the subjects of a collection in two houses can equal the multitude of those subjects. If so, let there be such a multitude of children. Then, each having but one wish, they can among them wish for every possible distribution of themselves among two houses. Then, however they may actually be distributed, some child will be perfectly contented. But ask each child which house he wishes himself to be in, and put every child in the house where he does not want to be. Then, no child would be content. Consequently, it is absurd to suppose that any collection can equal in multitude the possible ways of distributing its subjects in two houses.

549. Accordingly, the multitude of ways of placing a collection of objects abnumeral of the first dignity into two houses is still greater in multitude than that multitude, and may be called abnumeral of the second dignity. There will be a denumerable succession of such dignities. But there cannot be any multitude of an infinite dignity; for if there were, the multitude of ways of distributing it into two houses would be no greater than itself.[1]

* See "Ueber eine elementare Frage der Mannigfaltigkeitslehre," (1890–1), *Georg Cantor Gesammelte Abhandlung*, herausg. E. Zermelo, S. 278–81, Berlin, (1932) where Cantor shows that 2^n is always greater than n.

[1] Inasmuch as the above theorem is, as I believe, quite opposed to the opinion prevalent among students of Cantor, and they may suspect that some fallacy lurks in the reasoning about wishes, I shall here give a second proof of a part of the theorem, namely that there is an endless succession of infinite multitudes related to one another as above stated, a relation entirely different, by the way, from those of the orders of infinity used in the calculus. I shall not be able to prove by this second method, as is proved in the text, that there are no higher multitudes, and in particular no maximum multitude.

The ways of distributing a collection into two houses are equal to the possible

550. We thus not only answer the question proposed, and show that of two unequal multitudes the multitude of ways of distributing the greater is the greater; but we obtain the entire scale of collectional quantity, which we find to consist of two equal parts (that is two parts whose multitudes of grades are equal), the one finite, the other infinite. Corresponding to the

combinations of members of that collection (including zero); for these combinations are simply the aggregates of individuals put into either one of the houses in the different modes of distribution. Hence, the proposition is that the combinations of whole numbers are more multitudinous than the whole numbers, that the combinations of combinations of whole numbers are still more multitudinous, the combinations of combinations of combinations again more multitudinous, and so on without end.

I assume the previously proved proposition that of any two collections there is one which can be placed in one-to-one correspondence with a part or the whole of the other. This obviously amounts to saying that the members of any collection can be arranged in a linear series such that of any two different members one comes later in the series than the other.

A part may be equal to the whole; as the even numbers are equal in multitude to all the numbers (since every number has a double distinct from the doubles of all other numbers, and that double is an even number). Hence, it does not follow that because one collection can be placed in one-to-one correspondence to a part of another, it is less than that other, that is, that it cannot also, by a rearrangement, be placed in one-to-one correspondence with the whole. This makes an inconvenience in reasoning which can be overcome in a manner I proceed to describe.

Let a collection be arranged in a linear series. Then, let us speak of a *section* of that series, meaning the aggregate of all the members which are later than (or as late as) one *assignable* member and at the same time earlier than (or as early as) a second *assignable* member. Let us call a series *simple* if it cannot be severed into sections each equal in multitude to the whole. A series not simple itself may be conceivably severed into *simple sections*, or it may be so arranged that it cannot be so severed (for example the series of rational fractions arranged in the order of their magnitudes). But suppose two collections to be each ranged in a linear series, and suppose one of them, A, is in one-to-one correspondence with a part of the other B. If now the latter series, B, can be severed into simple sections, in each of which it is possible to find a member at least as early in the series as any member of that section that is in correspondence with a member of the other collection A, and also a member at least as late in the series as any member of that section that is in correspondence with any member of the other collection, and if it is also possible to find a section of the series, B, equal to the whole series, B, in which it is possible to find a member *later* than any member that is in correspondence with any member of the collection, A, then I say that the collection, B, is greater than the collection, A. This is so obvious that I think the demonstration may be omitted.

Now, imagine two infinite colections, the α's and the β's, of which the β's are the more multitudinous. I propose to prove that the possible combinations of β's are more multitudinous than the possible combinations of α's. For let the multitude of 0 on the finite scale is the abnumeral of 0 dignity, which is the denumerable, on the infinite scale, etc.*

551. So much of the general logical doctrine of quantity has been here given, in order to illustrate the power of the logic of relatives in enabling us to treat with unerring confidence the most difficult conceptions, before which mathematicians have heretofore shrunk appalled.

552. I had been desirous of examining Professor Schröder's developments concerning individuals and individual pairs; but owing to the length this paper has already reached, I must remit that to some future occasion.†

pairs of conjugate combinations (meaning by conjugate combinations a pair each of which includes every member of the whole collection which the other excludes) of the β's be arranged in a linear series; and those of the α's in another linear series. Let the order of the pairs in each of the two series be subject to the rule that if of two pairs one contains a combination composed of fewer members than either combination of the other pair, it shall precede the latter in the series. Let the order of the pairs in the series of pairs of combinations of β's be further determined by the rule that where the first rule does not decide, one of two pairs shall precede the other whose smaller combination (this rule not applying where one [?] combinations are equal) contains fewer β's which are in correspondence with α's in one fixed correspondence of all the α's with a part of the β's.

In this fixed correspondence each α has its β, while there is an infinitely greater multitude of β's without α's than with. Let the two series of pairs of combinations be so placed in correspondence that every pair of unequal combinations of α's is placed in correspondence with that pair of combinations of β's of which the smaller contains only the β's corresponding in the fixed correspondence to the smaller combination of α's; and let every pair of equal combinations of α's be put into correspondence with a pair of β's of which the smaller contains only the β's belonging in the fixed correspondence to one of the combinations of α's,

Then it is evident that each series will generally consist of an infinite multitude of simple sections. In none of these will the combinations be more multitudinous than those of the β's. In some, the combinations of α's will be equal to those of the β's; but in an infinitely greater multitude of such simple sections and each of these infinitely more multitudinous, the combinations of β's will be infinitely more multitudinous than those of the α's. Hence it is evident that the combinations of the β's will on the whole be infinitely more multitudinous than those of the α's.

That is if the multitude of finite numbers be α, and $2^a=b$, $2^b=c$, $2^c=d$, etc. $a<b<c<d<$ etc. *ad infinitum.*

It may be remarked that the finite combinations of finite whole numbers form no larger a multitude than the finite whole numbers themselves; i.e. they are at least enumerable. But there are infinite collections of finite whole numbers; and it is these which are infinitely more numerous than those numbers themselves.

* Cf. 4 113.

† This is the last of the published papers on Schröder.

XVII

THE LOGIC OF MATHEMATICS IN RELATION TO EDUCATION*

§1. OF MATHEMATICS IN GENERAL

553. In order to understand what number is, it is necessary first to acquaint ourselves with the nature of the business of mathematics in which number is employed.

554. I wish I knew with certainty the precise origin of the definition of mathematics as the science of quantity. It certainly cannot be Greek, because the Greeks were advanced in projective geometry, whose problems are such as these: whether or not four points obtained in a given way lie in one plane; whether or not four planes have a point in common; whether or not two rays (or unlimited straight lines) intersect, and the like — problems which have nothing to do with quantity, as such. Aristotle† names, as the subjects of mathematical study, quantity and continuity. But though he never gives a formal definition of mathematics, he makes quite clear, in more than a dozen places, his view that mathematics ought not to be defined by the things which it studies but by its peculiar mode and degree of abstractness. Precisely what he conceives this to be it would require me to go too far into the technicalities of his philosophy to explain; and I do not suppose anybody would today regard the details of his opinion as important for my purpose. Geometry, arithmetic, astronomy, and music were, in the Roman schools of the fifth century[1] and earlier, recognized as the four branches of mathematics. And we find Boëthius (A.D. 500) defining them as the arts which relate, not to quantity, but to *quantities*, or *quanta*. What this would seem to imply is, that mathematics is the foundation of the minutely exact sciences; but really it is not worth

* *Educational Review*, pp. 209–16, (1898).

† *Metaphysica* 1061*a* 28–1061*b* 3; 1061*b* 21–25.

[1] Davidson, *Aristotle and the ancient educational ideals*. Appendix: The Seven Liberal Arts. (New York: Charles Scribner's Sons.)

our while, for the present purpose, to ascertain what the schoolmasters of that degenerate age conceived mathematics to be.

555. In modern times projective geometry was, until the middle of this century, almost forgotten, the extraordinary book of Desargues[1] having been completely lost until, in 1845, Chasles came across a MS. copy of it; and, especially before imaginaries became very prominent, the definition of mathematics as the science of quantity suited well enough such mathematics as existed in the seventeenth and eighteenth centuries.

556. Kant, in the *Critique of Pure Reason* (Methodology, chapter I, section 1), distinctly rejects the definition of mathematics as the science of quantity. What really distinguishes mathematics, according to him, is not the subject of which it treats, but its method, which consists in studying constructions, or diagrams. That such is its method is unquestionably correct; for, even in algebra, the great purpose which the symbolism subserves is to bring a skeleton representation of the relations concerned in the problem before the mind's eye in a schematic shape, which can be studied much as a geometrical figure is studied.

557. But Rowan Hamilton and De Morgan, having a superficial acquaintance with Kant, were just enough influenced by the *Critique* to be led, when they found reason for rejecting the definition as the science of quantity, to conclude that mathematics was the science of pure time and pure space. Notwithstanding the profound deference which every mathematician must pay to Hamilton's opinions and my own admiration for De Morgan, I must say that it is rare to meet with a careful definition of a science so extremely objectionable as this. If Hamilton and De Morgan had attentively read what Kant himself has to say about number, in the first chapter of the *Analytic of principles* and elsewhere, they would have seen that it has no more to do with time and space than has every conception. Hamilton's intention probably was, by means of this definition, to throw a slur upon the introduction of imaginaries into geometry, as a false science; but what De Morgan, who was a student of multiple algebra, and whose own formal

[1] Brouillon, *Proiet d'une atteinte aux événemens des rencontres du cône avec son plan*, 1639.

logic is plainly mathematical, could have had in view, it is hard to comprehend, unless he wished to oppose Boole's theory of logic. Not only do mathematicians study hypotheses which, both in truth and according to the Kantian epistemology, no otherwise relate to time and space than do all hypotheses whatsoever, but we now all clearly see, since the non-Euclidean geometry has become familiar to us, that there *is* a real science of space and a real science of time, and that these sciences are positive and experiential — branches of physics, and so not mathematical except in the sense in which thermotics and electricity are mathematical; that is, as calling in the aid of mathematics. But the gravest objection of all to the definition is that it altogether ignores the veritable characteristics of this science, as they were pointed out by Aristotle and by Kant.

558. Of late decades philosophical mathematicians have come to a pretty just understanding of the nature of their own pursuit. I do not know that anybody struck the true note before Benjamin Peirce, who, in 1870,[1] declared mathematics to be "the science which draws necessary conclusions," adding that it must be defined "subjectively" and not "objectively." A view substantially in accord with his, though needlessly complicated, is given in the article "Mathematics," in the ninth edition of the *Encyclopædia Britannica*. The author, Professor George Chrystal, holds that the essence of mathematics lies in its making pure hypotheses, and in the character of the hypotheses which it makes. What the mathematicians mean by a "hypothesis" is a proposition imagined to be strictly true of an ideal state of things. In this sense, it is only about hypotheses that necessary reasoning has any application; for, in regard to the real world, we have no right to presume that any given intelligible proposition is true in absolute strictness. On the other hand, probable reasoning deals with the ordinary course of experience; now, nothing like *a course of experience* exists for ideal hypotheses. Hence to say that mathematics busies itself in drawing necessary conclusions, and to say that it busies itself with hypotheses, are two statements which the logician perceives come to the same thing.

559. A simple way of arriving at a true conception of the

[1] In his *Linear associative algebras*, [p. 97, published in the *American Journal of Mathematics*, vol. 4, (1881), pp. 97–229; see No. VIII.]

mathematician's business is to consider what service it is which he is called in to render in the course of any scientific or other inquiry. Mathematics has always been more or less a trade. An engineer, or a business company (say, an insurance company), or a buyer (say, of land), or a physicist, finds it suits his purpose to ascertain what the necessary consequences of possible facts would be; but the facts are so complicated that he cannot deal with them in his usual way. He calls upon a mathematician and states the question. Now the mathematician does not conceive it to be any part of his duty to verify the facts stated. He accepts them absolutely without question. He does not in the least care whether they are correct or not. He finds, however, in almost every case that the statement has one inconvenience, and in many cases that it has a second. The first inconvenience is that, though the statement may not at first sound very complicated, yet, when it is accurately analyzed, it is found to imply so intricate a condition of things that it far surpasses the power of the mathematician to say with exactitude what its consequence would be. At the same time, it frequently happens that the facts, as stated, are insufficient to answer the question that is put. Accordingly, the first business of the mathematician, often a most difficult task, is to frame another simpler but quite fictitious problem (supplemented, perhaps, by some supposition), which shall be within his powers, while at the same time it is sufficiently like the problem set before him to answer, well or ill, as a substitute for it.[1] This substituted problem differs also from that which was first set before the mathematician in another respect: namely, that it is highly abstract. All features that have no bearing upon the relations of the premisses to the conclusion are effaced and obliterated. The skeletonization or diagrammatization of the problem serves more purposes than one; but its principal purpose is to strip the significant relations of all disguise. Only one kind of concrete clothing is permitted — namely, such as, whether from habit or from the constitution of the mind, has become so familiar that it decidedly aids in tracing the consequences of the hypothesis. Thus, the mathematician does two very different things: namely, he first frames a pure hypothesis stripped of all features which do not concern the drawing of

[1] See this well put in Thomson and Tait's *Natural Philosophy*, §447.

consequences from it, and this he does without inquiring or caring whether it agrees with the actual facts or not; and, secondly, he proceeds to draw necessary consequences from that hypothesis.

560. Kant is entirely right in saying that, in drawing those consequences, the mathematician uses what, in geometry, is called a "construction," or in general a diagram, or visual array of characters or lines. Such a construction is formed according to a precept furnished by the hypothesis. Being formed, the construction is submitted to the scrutiny of observation, and new relations are discovered among its parts, not stated in the precept by which it was formed, and are found, by a little mental experimentation, to be such that they will always be present in such a construction. Thus, the necessary reasoning of mathematics is performed by means of observation and experiment, and its necessary character is due simply to the circumstance that the subject of this observation and experiment is a diagram of our own creation, the conditions of whose being we know all about.

But Kant, owing to the slight development which formal logic had received in his time, and especially owing to his total ignorance of the logic of relatives, which throws a brilliant light upon the whole of logic, fell into error in supposing that mathematical and philosophical necessary reasoning are distinguished by the circumstance that the former uses constructions. This is not true. All necessary reasoning whatsoever proceeds by constructions; and the only difference between mathematical and philosophical necessary deductions is that the latter are so excessively simple that the construction attracts no attention and is overlooked. The construction exists in the simplest syllogism in Barbara. Why do the logicians like to state a syllogism by writing the major premiss on one line and the minor below it, with letters substituted for the subject and predicates? It is merely because the reasoner has to notice that relation between the parts of those premisses which such a diagram brings into prominence. If the reasoner makes use of syllogistic in drawing his conclusion, he has such a diagram or construction in his mind's eye, and observes the result of eliminating the middle term. If, however, he trusts to his unaided reason, he still uses some kind of a diagram which is familiar to him personally. The true difference between the necessary logic of philosophy and mathematics is merely one of degree. It is that, in mathematics, the reasoning is frightfully intricate, while the elementary conceptions are of the last degree of familiarity; in contrast to philosophy, where the reasonings are as simple as they can be, while the elementary conceptions are abstruse and hard to get clearly apprehended. But there is another much deeper line of demarcation between the two sciences. It is that mathematics studies nothing but pure hypotheses, and is the only science which never inquires what the actual facts are; while philosophy, although it uses no microscopes or other apparatus of special observation, is really an experimental science, resting on that experience which is common to us all; so that its principal reasonings are not mathematically necessary at all, but are only necessary in the sense that all the world knows beyond all doubt those truths of experience upon which philosophy is founded. This is why the mathematician holds the reasoning of the metaphysician in supreme contempt, while he himself, when he ventures into philosophy, is apt to reason fantastically and not solidly, because he does not recognize that he is upon ground where elaborate deduction is of no more avail than it is in chemistry or biology.

561. I have thus set forth what I believe to be the prevalent opinion of philosophical mathematicians concerning the nature of their science. It will be found to be significant for the question of number. But were I to drop this branch of the subject without saying one word more, my criticism of the old definition, "mathematics is the science of quantity," would not be quite just. It must be admitted that quantity is useful in almost every branch of mathematics. Jevons wrote a book entitled *Pure logic, the science of quality*, which expounded, with a certain modification, the logical algebra of Boole. But it is a mistake to regard that algebra as one in which there is no system of quantity. As Boole rightly holds, there is a quadratic equation which is fundamental in it. The meaning of that equation may be expressed as follows: Every proposition has one or other of two *values*, being either *true* (which gives it one value) or *false* (which gives it the other). So stated, we see that the algebra of Boole is nothing but the algebra of that

system of quantities which has but two values — the simplest conceivable system of quantity. The widow of the great Boole has lately written a little book[1] in which she points out that, in solving a mathematical problem, we usually introduce some part or element into the construction which, when it has served our purpose, is removed. Of that nature is a scale of quantity, together with the apparatus by which it is transported unchanged from one part of the diagram to another, for the purpose of comparing those two parts. Something of this general description seems to be indispensable in mathematics. Take, for example, the Theorem of Pappus concerning ten rays in a plane. The demonstration of it which is now usual, that of von Staudt, introduces a third dimension; and the utility of that arises from the fact that a ray, or unlimited straight line, being the intersection of two planes, these planes show us exactly where the ray runs, while, as long as we confine ourselves to the consideration of a single plane, we have no easy method of describing precisely what the course of the ray is. Now this is not precisely a system of quantity; but it is closely analogous to such a system, and that it serves precisely the same purpose will appear when we remember that that same theorem can easily (though not *so* easily) be demonstrated by means of the barycentric calculus. Although, then, it is not true that all mathematics is a science of quantity, yet it is true that all mathematics makes use of a scaffolding altogether *analogous* to a system of quantity; and quantity itself has more or less utility in every branch of mathematics which has as yet developed into any large theory.

562. I have only to add that the hypotheses of mathematics may be divided into those *general hypotheses* which are adhered to throughout a whole branch of mathematics, and the *particular hypotheses* which are peculiar to different special problems.*

§2. OF PURE NUMBER

562A. The system of pure number is the general hypothesis of arithmetic — at any rate, of scientific arithmetic (or, the

[1] *The Mathematical Psychology of Boole and Gratry.*

* No further articles were published in the *Educational Review.* The following, however, was part of the original article and seems to have been omitted only because of lack of space. What follows is taken from paginated page proofs.

theory of numbers) — for whether it is best to say anything about it in vulgar arithmetic (or, the art of computing with the Arabic figures)[1] is a question of educational theory to be considered after studying Counting and Dating.

562B. Preparatory to showing what this system is, it will be well to describe a still more general hypothesis — that of a *sequence*. A sequence is a multitude of objects connected with a relation, which we may call the "relation of sequence," or R, in a manner I proceed to define. It will be convenient to use this following locution: I shall say, A is R to B, meaning that A stands to B in the relation R, and with the same meaning I shall also say, B is R'd by A. Then the sequence is defined by these two precepts:

Precept I. If A is any object of the sequence whatever and B is any object of the sequence whatever, either A is not R to B or else B is not R to A.

Precept II. If A is any object of the sequence whatever, B is any such object, and C is any such object, then, so far as Precept I permits, either A is R to B, or B is R to C, or C is R to A.

Certain corollaries are easily deduced from these precepts. First, if, in applying Precept I, we choose for B the same object represented by A, that precept leads to the conclusion that no object of the sequence is R itself. This, then, shows the limitation which Precept I imposes upon Precept II. Namely, in the latter, A, B, and C cannot, all three, represent the same object. Secondly, if in Precept II we take C to be the same as B, that precept shows that of any two different objects of the sequence, A and B, either A is R to B or B is R to A. For by the first corollary, B cannot be R to C; that is, to itself. Thirdly, if A, B, and C represent objects of the sequence (the same or different), then if A is R'd by B, it follows that either B is R to C or C is R to A. For if A is R'd by B, then by Precept I, A is not R to B, and then, by Precept II, either B is R to C or C is R to A, or else A, B, and C are

[1] I notice that writers of school arithmetics shrink from accepting the correct name of their art, Vulgar Arithmetic. It is a pity we have lost Chaucer's word "augrim," which, etymologically meaning "the art of the Chorasmian," is free from all objection. Still, we cannot find fault with these writers who adopt no more high sounding title for their subject than Practical Arithmetic.

identical. But the last cannot be the case by the first corollary, if A is R'd by B. Fourthly, whatever objects of the sequence A, B, and C may be, if A is R'd by B and B is R'd by C, then A is R'd by C. For, by Precept I, if B is R'd by C, C is not R to B.* Hence, by the third corollary, if A is R'd by B and B is R'd by C, C is R to A, that is, A is R'd by C. Fifthly, whatever objects of the sequence A, B, and C may be, either A is not R'd by B or B is not R'd by C, or C is not R'd by A. For, by the fourth corollary, either A is not R'd by B, or B is not R'd by C, or A is R'd by C. But in the last case, by Precept I, C is not R'd by A.†

* This should be: C is not *R'd by* B.

† Given the precepts

$$\text{I.}\quad (\Pi a \Pi b) -(aRb) \maltese -(bRa)$$
$$\text{II.}\quad (\Pi a \Pi b \Pi c)\ aRb \maltese bRc \maltese cRa$$

through the use of the propositions of logic and the principles of substitution and inference, the following are some of the theorems that can be derived.

A. $(\Pi b) -(bRb)$

$$\left[\text{I} : \frac{b}{a}\right] -(bRb) \maltese -(bRb) \qquad (1)$$
$$(1) -\!\!< -(bRb)$$

B. $(\Pi a \Pi b)\ aRb \maltese bRa$

$$\left[\text{II} : \frac{b}{c}\right] aRb \maltese bRb \maltese bRa \qquad (1)$$
$$(A) \cdot (1) -\!\!< aRb \maltese bRa$$

C. $(\Pi a \Pi b \Pi c)\ bRa -\!\!< (bRc \maltese cRa)$

$$\text{I} = bRa -\!\!< -(aRb) \qquad (1)$$
$$\text{II} = -(aRb) -\!\!< (bRc \maltese cRa) \qquad (2)$$
$$(1) \cdot (2) -\!\!< [bRa -\!\!< (bRc \maltese cRa)]$$

D. $(\Pi a \Pi b \Pi c)(bRa \cdot cRb) -\!\!< cRa$

$$\left[\text{I} : \frac{c}{a}\right] -(cRb) \maltese -(bRc) \qquad (1)$$
$$= cRb -\!\!< -(bRc) \qquad (2)$$
$$\text{C} = bRa -\!\!< [-(bRc) -\!\!< cRa] \qquad (3)$$
$$= [bRa\ .\ -(bRc)] -\!\!< cRa \qquad (4)$$
$$(2) \cdot (4) -\!\!< (bRa \cdot cRb) -\!\!< cRa$$

E. $(\Pi a \Pi b \Pi c) -(bRa) \maltese -(cRb) \maltese -(aRc)$

$$\text{D} = -(bRa) \maltese -(cRb) \maltese cRa \qquad (1)$$
$$\left[\text{I} : \frac{c}{b}\right] -(aRc) \maltese -(cRa) \qquad (2)$$
$$(1) \cdot (2) -\!\!< -(bRa) \maltese -(cRb) \maltese -(aRc)$$

This defines a sequence in general. The description of the relation R agrees with that of a relation which is familiar to us, namely, that of *following* in sequence. Of course, it equally agrees with the relation of *preceding*. Substituting "following" for R, Precept I becomes, that whatever members of the sequence A and B may be; they do not both follow one another. Precept II becomes that, whatever members of the sequence A, B, and C may be, provided they are not all three identical, either A follows B, or B follows C, or C follows A. The first corollary becomes, that no member of the sequence follows itself. The second becomes, that of any two different members of the sequence one follows the other. The third becomes, that whatever members A, B, and C, may be, if A is followed by B, either B follows C or C follows A. The fourth becomes, that whatever members A, B, and C may be, if A is followed by B and B by C, then A is followed by C. The fifth becomes, that whatever members A, B, and C may be, either A is not followed by B, or B is not followed by C, or C is not followed by A.

A sequence may nor may not have an absolute end; that is, a member which is followed by no member; and it may or may not have an absolute beginning; that is, a member which follows no member. These distinctions are not mathematically important in most cases.

562C. A highly important distinction, however, arises in a way which I proceed to describe. We know from the fourth corollary that, taking any member of the sequence M, and any member N, which does not follow M, then whatever member X, we may select, either X does not follow M or is not followed by N.* Now M may be such that it *is followed by* a member N, of which this same thing is true; namely, that there is no member of the sequence which at once follows M and is followed by N. The member N may, in this case, be said *hardly* to follow M, since it has this property which belongs generally to members which do not follow M. We usually say that N follows *next* after M.[1] Now a sequence

* $4 = -(cRa) \prec -(bRa) \curlyvee -(cRb)$, which by $\left[\frac{n}{c}, \frac{m}{a}, \frac{x}{b}\right]$ is $-(nRm) \prec -(xRm) \curlyvee -(nRx)$.

[1] The word hardly is in older English "hard." N follows *hard* upon M, that is, solidly up against M, with nothing between them.

may either be such that no member has another that *hardly*, or *next*, follows it, or such that some members do and some do not, or that all have such members hardly following them.

562D. If a sequence be such that some member has a member that hardly follows it, the sequence is such that some member has a member that hardly precedes it. For, if N follows hard after M, then M hardly precedes N. But, nevertheless, a sequence may be such that *every* member has a member hardly following it without being such that every member has a member hardly preceding it. An accurate logic will show that this is quite admissible. Suppose, for example, the sequence is composed of all the whole numbers, together with whole numbers each added to a vulgar fraction having 1 for its numerator. And suppose these to follow one another in the reverse order of their values, thus:

$$4, 3\tfrac{1}{2}, 3\tfrac{1}{3}, 3\tfrac{1}{4}, 3\tfrac{1}{5}, \ldots 3, 2\tfrac{1}{2}, 2\tfrac{1}{3}, 2\tfrac{1}{4}, 2\tfrac{1}{5}, \ldots 2, 1\tfrac{1}{2}, 1\tfrac{1}{3}, 1\tfrac{1}{4}, 1\tfrac{1}{5}, \ldots 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \ldots$$

Then every member, as $p+\frac{1}{q}$ has $p+\frac{1}{q+1}$ following hard after it. But the whole numbers have no members which *they* follow hard after.

562E. If a sequence be such that there is some member of it, not its absolute end, which has no member hardly following it, then a part of that sequence forms a sequence with the same, R (or, relation of sequence), which sequence has no absolute beginning. Namely, if M be a member of the total sequence such that some members are R'd by M, but such that taking any member P, either P is not R to M, or else there is a member X, such that P is R to X and X is R to M, in that case the partial sequence composed of all the members of the total sequence that are R'd by M, contains no member that is not R to another member of the same partial sequence. We might express the same thing by saying that if a member of a sequence not its absolute beginning follows hard upon no member, then all the members it follows form a sequence with no absolute end. But the converse of this proposition is not true. That is to say, it is not true that a sequence, every member of which *has* another following hard after it and itself following hard upon another, *cannot* be cut up into sequences having neither beginning nor end. For instance, take all numerals of the forms $p+\frac{1}{q}$ where p and q are whole numbers; and let them be arranged in the order of their magnitudes, thus:

$$\tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \ldots 1\tfrac{1}{5}, 1\tfrac{1}{4}, 1\tfrac{1}{3}, 1\tfrac{1}{2}, 1\tfrac{2}{3}, 1\tfrac{3}{4}, 1\tfrac{4}{5}, \ldots 2\tfrac{1}{5}, 2\tfrac{1}{4}, 2\tfrac{1}{3}, 2\tfrac{1}{2}, 2\tfrac{2}{3}, 2\tfrac{3}{4}, 2\tfrac{4}{5}.$$

Here it is plain that every member has an assignable one next after and another next before it; and yet the sequence can be broken up into partial sequences, each without beginning or end.

562F. Let us call a sequence of which every member follows hard after a second, and is itself followed hard after by a third, a *sparse* sequence; and let us call a sparse sequence which cannot be broken into parts which want either beginning or end (except so far as they may retain the infinity of the total sequence), a *simple* sparse sequence.

Then the system of pure number may be defined as a simple sparse sequence, having an absolute beginning, called *zero*, but no absolute end. Of course, *zero* might be dropped from the system, or zero and unity might both be dropped, and so on *ad infinitum*. But slightly simpler definitions of addition and multiplication are obtained by retaining the *zero*.

562G. Arithmetic begins with the following fundamental theorem:

Whatever character belongs to zero, and belongs to the number following hard after any number to which it belongs, belongs to all numbers.

Demonstration. Let X be a character which zero possesses. And suppose that, whatever number N may be, either N does not possess X, or else N^1, the number that follows hard after N, possesses X. Then I say that every number possesses X.

For, since zero possesses X, and zero is followed by every other number (by the second corollary above, since zero, as absolute beginning, follows no number) there are numbers (or, at least, a number) which are followed by whatever numbers there may be which do not possess X. Let us consider, then, the sequence composed of those numbers which are smaller than whatever numbers there may be which do not possess X; these numbers being taken in their order, that is, with the same R as the total sequence. (Even a single number may be regarded as a sequence of which the absolute beginnings and end are identical.) This sequence may either be the total sequence of numbers, in which case it will have no end, or it may be a partial sequence. In the latter case, every number of this sequence will be followed by every number which does not belong to it, since the latter is not followed by some number not possessing X which does follow every number of this partial sequence. Hence, since the sequence of pure number is a simple sparse sequence, it follows that the partial sequence (if it be partial) must have an absolute end. Let N be this absolute end. Then N is followed by whatever number there may be which does not possess X. But by the first corollary, no number is followed by itself. Hence N possesses X. Hence, by hypothesis, N^1, the number that follows hard after N, possesses X. Hence, by the definition of "following hard after," N^1 is followed by every number except itself, and follows N; that is, by whatever number there may be which does not possess X. But this is contrary to the hypothesis that N was the absolute end of the partial series; that is, was not followed by any number followed by whatever number does not possess X. Hence, the supposition that the series is a partial one is absurd. Hence, every number is followed by whatever number there may be that does not possess X. But, by the first corollary, no number is followed by itself. Hence, every number possesses X. Q. E. D.

562H. By means of this theorem all the elementary propositions of arithmetic as to the associativeness and distributiveness of addition and multiplication, as well as the fundamental principles of subtraction and division of whole numbers are easily deduced from the following definitions of addition and multiplication. Of course subtraction and division are to be defined as inverse operations:

Definition of addition. The *sum* of two numbers, M and N, is a number depending upon M and N according to the following precepts.

Precept III. The Sum of O and O is O.

Precept IV. The sum of any number, M, and of the number N^1, next following any number, N, is the number next following the sum of M and N.

5621. *Definition of multiplication.* The *product* of two numbers, M and N, is a number depending upon M and N according to the following precepts:

Precept V. The product of O and O is O.

Precept VI. The product of any number, M, and of the number, N^1, next following any number, N, is the number which is the sum of M and the product of M and N.

This gives a sufficient idea of pure number. But I will remark that, when this method is applied to division, it leads so easily into the theory of numbers that it is difficult to restrain the pupil from that line of thought. But I need not say that I should not teach the doctrine of pure number to young children. At present, I am merely collecting the conceptions of the subject. I shall pass next to the application of number to the counting of collections.*

* The proof sheet ends here with the note "to be continued." The ms. of this paper has not been uncovered but similar papers are published in vol. 4, bk. I.

XVIII

*INFINITESIMALS**

563. Will you kindly accord me space for a few remarks about Infinity and Continuity which I seem called upon to make by several notes to Professor Royce's Supplementary Essay in his strong work *The World and the Individual?* I must confess that I am hardly prepared to discuss the subject as I ought to be, since I have never had an opportunity sufficiently to examine the two small books by Dedekind,† nor two memoirs by Cantor,‡ that have appeared since those contained in the second volume of the *Acta Mathematica*. I cannot even refer to Schröder's *Logic*.

564. (1) There has been some question whether Dedekind's definition of an infinite collection or that which results from negativing my definition of a finite collection is the best. It seems to me that two definitions of the same conception, not subject to any conditions, as a figure in space, for example, is subject to geometrical conditions, must be substantially the same. I pointed out (*Am. Journ. Math.* IV. 86,§ but whether I first made the suggestion or not I do not know) that a finite collection differs from an infinite collection in nothing else than that the syllogism of transposed quality [quantity] is applicable to it (and by the consequences of this logical property). For that reason, the character of being finite seemed to me a positive extra determination which an infinite collection does not possess. Dr. Dedekind defines an infinite collection as one of which every *echter Theil* is similar to the whole collection.¶ It obviously would not do to say a *part*, simply, for every collection, even if it be infinite, is composed of individuals; and

* A letter to the Editor of *Science*, vol. 2, pp. 430–33, March 16, 1900.

† *Stetigkeiten u. Irrationalen Zahlen*, 2te Auf., 1892; *Was sind u. was sollen die Zahlen*, 1888.

‡ "Beiträge zur Begründung der transfiniten Mengenlehre," *Georg Cantor Gesammelte Abhandlung*, herausg. E. Zermelo, S. 282–351, Berlin, (1932).

§ 258, 281ff., 402.

¶ But see *Was sind u. was sollen die Zahlen*, §64.

these individuals are parts of it, differing from the whole in being indivisible. Now I do not believe that it is possible to define an *echter Theil* without substantially coming to my definition. But, however that may be, Dedekind's definition is not of the kind of which I was in search. I sought to define a finite collection in logical terms. But a "part," in its mathematical, or collective, sense, is not a logical term, and itself requires definition.

565. (2) Professor Royce remarks that my opinion that differentials may quite logically be considered as true infinitesimals, if we like, is shared by no mathematician "*outside of Italy.*"* As a logician, I am more comforted by corroboration in the clear mental atmosphere of Italy than I could be by any seconding from a tobacco-clouded and bemused land (if any such there be) where no philosophical eccentricity misses its champion, but where sane logic has not found favor. Meantime, I beg leave briefly to submit certain reasons for my opinion.

566. In the first place, I proved in January, 1897, in an article in the *Monist* (VII 215),† that the multitude of possible collections of members of any given collection whatever is greater than the multitude of the latter collection itself. That demonstration is so simple, that, with your permission, I will here repeat it. If there be any collection as great as the multitude of possible collections of its members, let the members of one such collection be called the A's. Then, by Cantor's definition of the relation of multitude, there must be some possible relation, r, such that every possible collection of A's is r to some A, while no two possible collections of A's are r to the same A. But now I will define a certain possible collection of A's, which I will call the collection of B's, as follows: Whatever A there may be that is not included in any collection of A's that is r to it, shall be included in the collection of B's, and whatever A there may be that is included in a collection of A's that is r to it, shall not be included in the collection of B's. If

* "Mr. Charles Peirce, as I understand his statements in the *Monist*, appears to stand almost alone amongst recent mathematical logicians outside of Italy, in still regarding the Calculus as properly to be founded upon the conception of the actually infinite and infinitesimal." *The World and the Individual*, 1st Series, p. 562n.

† 548.

there is any A to which no collection of A's stands in the relation r, I do not care whether it is included among the B's or not. Now I say the collection of B's is not in the relation r to any A. For every A is either an A to which no collection of A's stands in the relation r, or it is included in a collection of A's that is r to it, or it is excluded from every collection of A's that is r to it. Now the collection of B's, being a collection of A's, is not r to any A to which no collection of A's is r; and it is not r to any A that is included in a collection of A's that is r to it, since only one collection of A's is r to the same A, so that were that the case the A in question would be a B, contrary to the definition which makes the collection of B's exclude every A included in a collection that is r to it; and finally, the collection of B's is not r to any A not included in any collection of A's that is r to it, since by definition every such A is a B, so that, if the collection of B's were r to that A, that A would be included in a collection of A's that was r to it. It is thus absurd to say that the collection of B's is r to any A; and thus there is always a possible collection of A's not r to any A; in other words, the multitude of possible collections of A's is greater than the multitude of the A's themselves. That is, every multitude is less than a multitude; or, there is no maximum multitude.

567. In the second place I postulate that it is an admissible hypothesis that there may be a something, which we will call a *line*, having the following properties: first, points may be determined in a certain relation to it, which relation we will designate as that of "lying on" that line; second, four different points being so determined, each of them is separated from one of the others by the remaining two; third, any three points, A, B, C, being taken on the line, any multitude whatever of points can be determined upon it so that every one of them is separated from A by B and C.

568. In the third place, the possible points so determinable on that line cannot be distinguished from one another by being put into one-to-one correspondence with any system of "assignable quantities." For such assignable quantities form a collection whose multitude is exceeded by that of another collection, namely, the collection of all possible collections of those "assignable quantities." But points are, by our postu-

late, determinable on the line in excess of that or of any other multitude. Now, those who say that two different points on a line must be at a finite distance from one another, virtually assert that the points are distinguishable by corresponding (in a one-to-one correspondence) to different individuals of a system of "assignable quantities." This system is a collection of individual quantities of very moderate multitude, being no more than the multitude of all possible collections of integral numbers. For by those "assignable quantities" are meant those toward which the values of fractions can indefinitely approximate. According to my postulate, which involves no contradiction, a line may be so conceived that its points are not so distinguishable and consequently can be at infinitesimal distances.

Since, according to this conception, any multitude of points whatever are determinable on the line (not, of course, by us, but of their own nature), and since there is no maximum multitude, it follows that the points cannot be regarded as constituent parts of the line, existing on it by virtue of the line's existence. For if they were so, they would form a collection; and there would be a multitude greater than that of the points determinable on a line. We must, therefore, conceive that there are only so many points on the line as have been marked, or otherwise determined, upon it. Those do form a collection; but ever a greater collection remains determinable upon the line. *All* the determinable points cannot form a collection, since, by the postulate, if they did, the multitude of that collection would not be less than another multitude. The explanation of their not forming a collection is that all the determinable points are not individuals, distinct, each from all the rest. For individuals can only be distinct from one another in three ways: First, by acts of reaction, immediate or mediate, upon one another; second, by having *per se* different qualities; and third, by being in one-to-one correspondence to individuals that are distinct from one another in one of the first two ways. Now the points on a line not yet actually determined are mere potentialities, and, as such, cannot react upon one another actually; and, *per se*, they are all exactly alike; and they cannot be in one-to-one correspondence to any collection, since the multitude of that collection would require to be a maximum multitude. Consequently, all the possible points are not distinct from one another; although any possible multitude of points, once determined, become so distinct by the act of determination. It may be asked, "If the totality of the points determinable on a line does not constitute a collection, what shall we call it?" The answer is plain: the possibility of determining more than any given multitude of points, or, in other words, the fact that there is room for any multitude at every part of the line, makes it *continuous*. Every point actually marked upon it breaks its continuity, in one sense.

569. Not only is this view admissible without any violation of logic, but I find — though I cannot ask the space to explain this here — that it forms a basis for the differential calculus preferable, perhaps, at any rate, quite as clear, as the doctrine of limits. But this is not all. The subject of topical geometry has remained in a backward state because, as I apprehend, nobody has found a way of reasoning about it with demonstrative rigor. But the above conception of a line leads to a definition of continuity very similar to that of Kant. Although Kant confuses continuity with infinite divisibility, yet it is noticeable that he always defines a continuum as that of which every part (not every *echter Theil*) has itself parts. This is a very different thing from infinite divisibility, since it implies that the continuum is not composed of points, as, for example, the system of rational fractions, though infinitely divisible, is composed of the individual fractions. If we define a continuum as that every part of which can be divided into any multitude of parts whatsoever — or if we replace this by an equivalent definition in purely logical terms — we find it lends itself at once to mathematical demonstrations, and enables us to work with ease in topical geometry.

570. (3) Professor Royce wants to know* how I could, in a passage which he cites, attribute to Cantor the above opinion about infinitesimals. My intention in that passage was simply to acknowledge myself, in a general way, to be no more than a follower of Cantor in regard to infinity, not to make him responsible for any particular opinion of my own. However, Cantor proposed, if I remember rightly, so far to modify the kinetical theory of gases as to make the multitude of ordinary

* *Op. cit.*, p. 562n.

atoms equal to that of the integral numbers, and that of the atoms of ether equal to the multitude of possible collections of such numbers.* Now, since it is essential to that theory that encounters shall take place, and that promiscuously, it would seem to follow that each atom has, in the random distribution, certain next neighbors, so that if there are an infinite multitude in a finite space, the infinitesimals must be actual real distances, and not the mere mathematical conceptions, like $\sqrt{-1}$, which is all that I contend for.

* *Georg Cantor Gesammelte Abhandlung*, S. 275–6.

XIX

*NOMENCLATURE AND DIVISIONS OF DYADIC RELATIONS**

§1. NOMENCLATURE[E]

571. A *dyadic relation* is a character whose being consists in the logical possibility of a definite fact concerning an ordered pair, or *dyad*, of subjects; the first of these being termed the *relate*, the second the *correlate;* and the relation is said to *subsist* between the relate and correlate when the fact in whose possibility its being consists actually has place between these objects. The relation, by itself, is, therefore, an *ens rationis* and mere logical possibility; but its subsistence is of the nature of a fact. When the quality of the fact concerning two objects is considered, without reference to any distinction between these subjects other than that which this fact establishes, and therefore regardless of which of them is relate, which correlate, its possibility is termed by the author a *relationship*. (It is a useful distinction, but cannot be translated into every language.)

572. The broadest division of dyadic relations is into those which can only subsist between two subjects of different categories of being (as between an existing individual and a quality) and those which can subsist between two subjects of the same category. A relation of the former kind may advantageously be termed a *reference;* a relation of the latter kind, a *dyadic relation proper.*

573. A dyadic relation proper is either such as can only have place between two subjects of different universes of discourse (as the membership of a natural person in a corporation), or is such as can subsist between two objects of the same universe. A relation of the former description may be termed

* Printed separately in eight pages, circa 1903, apparently intended as the second part of *A Syllabus of Certain Topics of Logic*, published as a supplement to the Lowell Lectures of 1903. See vol. 1, bk. II, ch. 1, note.

a *referential relation;* a relation of the latter description, a *rerelation.*[1]

574. A rerelation may either be such as can only subsist between characters or between laws (such as the relation of "essentially depending upon"), or it may be such as can subsist between two existent individual objects. In the former case, it may be termed a *modal* relation (not a good term), in the latter case an *existential relation.* The author's writings on the logic of relations[2] were substantially restricted to existential relations; and the same restriction will be continued in the body of what here follows. A note at the end of this section will treat of modal relations.

575. The number of different species of existential relations for which technical designations are required is so great that it will be best to adopt names for them which shall, by their form, furnish technical definitions of them, in imitation of the nomenclature of chemistry. The following rules will here be used. Any name (for which in this statement of the rules of word-formation we may put x), having been adopted for all relations of a given description, the preposition *extra* (or *ex*, or *e*) will be prefixed to that name ("extra x") in order to form a name descriptive of any relation to which the primitive name does not apply; the preposition *contra* will be prefixed (forming "contra-x,") to make a name applicable only to such relations as consist precisely in the non-subsistence of corresponding relations to which the primitive name does apply; the preposition *juxta* will be prefixed so as to bear the sense of *contra-extra*, or (what is the same) *extra-contra;* the preposition *red* (or *re*)

[1] It is far better to invent a word for a purely technical conception than to use an expression liable to be corrupted by being employed by loose writers. I reduplicate the first syllable of relation to form this word, with little reference to the meaning of the syllable as a preposition. Still, relations of this kind are the only ones that might be asserted of the same relates transposed; and the reduplication of the preposition *re* connotes such transposition.

[2] I must, with pain and shame, confess that in my early days I showed myself so little alive to the decencies of science that I presumed to change the name of this branch of logic, a name established by its author and my master, Augustus De Morgan, to "the logic of relatives." I consider it my duty to say that this thoughtless act is a bitter reflection to me now, so that young writers may be warned not to prepare for themselves similar sources of unhappiness. I am the more sorry, because my designation has come into general use.

will be prefixed to form a name applicable to a relation if, and only if, the correlate of it stands to its relate in a relation to which the primitive name applies, so that, in other words, a "red-x" is a relation the converse of an x; the preposition *com* (or *con*, or *co*,) will be prefixed to form the general name of any relation which consists in its relate and correlate alike standing in one relation of the primitive kind to one and the same individual correlate; the preposition *ultra* will be prefixed to form a name applicable only to a relation which subsists between any given relate and correlate only in case the former stands in a relation of the primitive kind to some individual to which the latter does not stand in that same relation; the preposition *trans* will be used so as to be equivalent to *contra-red-ultra*, or (what is the same) *recontrultra*, so that A will be in a relation "*trans-x*" to B, if, and only if, there is an x-relation in which A stands to whatever individual there may be to which B stands in that very same relation; and the preposition *super* to form the name of a relation which is, at once, *ultra* and *trans*, in respect to the very same relation of the primitive kind. Any of these prepositions may be prefixed, in the same sense, occasionally (and where no misunderstanding could result) not only to names of classes of relations and their cognates, but also to relative terms. But it is chiefly the prepositions *com*, *ultra*, *trans* and *super*, that will be so used. For example, taking the relative term "loves," there will be little occasion to use the first four of the following expressions, especially, the first and third, which become almost meaningless, while the last four will often be convenient.

1. A *extra-loves* B; that is, stands in some other relation, whether loving besides or not;

2. A *contra-loves* B; that is, does not love B;

3. A *juxta-loves* B; that is, stands in some other relation than that of not loving, whether loving or not;

4. A *reloves* B; that is, is loved by;

5. A *coloves* B; that is, loves something loved by;

6. A *ultraloves* B; that is, loves something not loved by;

7. A *transloves* B; that is, loves whatever may be loved by;

8. A *superloves* B; that is, loves whatever may be loved by and something else.*

576. By a *seed* (*granum*) of an existential relation is to be understood an existing individual which not only stands in that relation to some correlate, but to which also some relate stands in that relation. By a *spike* of a relation is to be understood any collection of seeds of it of which it is both true that every one of them stands in that relation to some one of them; and it is also true that to every one seed of the spike some seed of the spike stands in that same relation.† Thus, two spikes of the same relation may have common seeds, or one may even be a part of another. A *simple spike* is a spike not containing any other spike as a part of it.

577. There are four general kinds of consideration on which divisions of existential dyadic relations may be based. The connections between the four systems of division so resulting have not been sufficiently studied to be treated here.

§2. FIRST SYSTEM OF DIVISIONS‡

578. An existential dyadic relation may be termed a *lation*, to express its possibly subsisting between two existing indi-

* I.e., 1. extraloves $a\bar{l}b$
2. contraloves $-(alb)$
3. juxtaloves $-(a\bar{l}b)$
4. reloves $a\breve{l}b$
5. coloves $(alc)(c\breve{l}b)$
6. ultraloves $(alc)(c\overset{*}{l}b)$
7. transloves $-[(a\bar{l}c)(c\breve{l}b)]$
8. superloves $\left\{-[(a\bar{l}c)(c\breve{l}b)][(ald)(d\overset{*}{l}b)]\right\}$

† See note to §4.

‡The following schedule may be of aid in this section:

1. $\Sigma_i\Sigma_j r_{ij}$ — lation
2. $\Sigma_i\Sigma_j \bar{r}_{ij}$ — contralation;
3. $\Pi_i\Pi_j \bar{r}_{ij}$ — extralation; $r=0$
4. $\Pi_i\Pi_j r_{ij}$ — juxtalation; $r=\infty$
5. $\Sigma_i\Pi_j r_{ij}$ — perlation
6. $\Sigma_i\Pi_j \bar{r}_{ij}$ — contraperlation
7. $\Pi_i\Sigma_j \bar{r}_{ij}$ — extraperlation
8. $\Pi_i\Sigma_j r_{ij}$ — juxtaperlation
9. $\Sigma_j\Pi_i r_{ij}$ — reperlation
10. $\Sigma_j\Pi_i \bar{r}_{ij}$ — contrareperlation
11. $\Pi_j\Sigma_i \bar{r}_{ij}$ — extrareperlation
12. $\Pi_j\Sigma_i r_{ij}$ — juxtareperlation
13. $\Sigma_i\Pi_j\Sigma_k r_{ik} \cdot r_{jk}$ — conlation
14. $\Sigma_i\Pi_j\Sigma_k r_{ik} \,.\, \bar{r}_{jk}$ — ultralation
15. $\Pi_i\Sigma_j\Pi_k - (\bar{r}_{ik} \,.\, r_{jk})$ — translation
16. $\Pi_i\Pi_j\Pi_k\Sigma_l - (\bar{r}_{ik} \,.\, r_{jk})(r_{il} \,.\, \bar{r}_{jl})$ — superlation
17. $\Sigma_i\Sigma_j r_{ij\infty}$ — essential perlation
18. $\Sigma_i\Sigma_j r_{\infty ij}$ — essential reperlation
19. $\Sigma_i\Sigma_j \bar{r}_{ij\infty}$ — contressentiperlation

$1=-3, 5=-7, 9=-11; 2=-4; 6=-8;\ 10=-12, 5 -\!\!< 12; 9 -\!\!< 8; 6 -\!\!< 11; 10 -\!\!< 7; 17.18 -\!\!< 3 \psi 4.$

viduals, and in opposition to an *extralation*, which is a fictive lation, looked upon as a lation, but which cannot (or, at least, does not) subsist between two existing individuals. In short, it is *incompossibility*.

579. A lation is either a *contralation* which does not necessarily subsist between the members of every dyad, or else it is *the juxtalation*, or *coëxistence*, which subsists between every such dyad.

580. Schröder, who always conscientiously follows the writer's terminology, except where he sees good and sufficient reasons for departing from it,[1] thinks that, in place of "coëxistence," the term "compossibility" should be used. This is a nice question. It is to be kept in view that an existential dyadic relation is to be regarded as a brute fact existing between two existing things, whatever may be thought about it. In saying this, we enunciate no metaphysical nor epistemological proposition, but simply say how the matter must be understood. Such a relation has no being at all unless the two things exist. It has not even that mode of being which consists in conceivability. That is, you cannot conceive that A strikes B, for example, while one of them is non-existent. But when we apply a general word to a class of such dyads, we are thinking of more than ever can exist. If the logician says "A strikes B," meaning (as is usual in logic, except when time is expressly under consideration) did strike, is striking, or wil strike B, he asserts only that that event either has happened or will, at some time in the endless future, have happened. But it is impossible, "in the nature of things," as we say, that is, is logically impossible, that all that ever will have happened should at any time actually have happened. Thus, the assertion transcends actual existence. This is because "strike" is a *general* word, and as such, relates to what is, or is not, *possible;* and because of the axiom that it is impossible that all that is possible should actually exist. Thus, in absolute strictness of

[1] Since this shows he felt the obligation, it is the more lamentable that his treatment of my notation showed no such scrupulosity. That notation had been most maturely considered and thoroughly put to the test; and his changes were, without exception, for the worse. I did not say this while he lived, out of regard for his feelings. His *additions* to the notation, to express what I had afforded no means of expressing, stand, of course, on a different footing; and I should be bound to follow him, here.

language, "striking" is not an existential relation, but only signifies a *class* of existential relations. But this having, once for all, been well-understood, it becomes permissible to speak of a general relation as an existential relation, meaning a general class of existential relations. We may, thus, even distinguish between what is *essentially* true of an existential relation and what is *accidentally*, but universally, true of it, although what in absolute strictness is an existential relation has no other mode of being than existence, and consequently has no essence. It is not possible, if it does not exist. Still, we have to bear in mind that though we allow ourselves to speak of general relations as existential, yet what we mean are classes of existential relations in the strictest sense; and therefore they have no possibility distinct from existence. Accordingly, *coexistence* is the right word, and compossibility is only a modal relation. It is quite another thing with the extralation, since this is *not* an existential relation, so that "incompossibility" seems the more appropriate word; but if I were to countenance any modification of my nomenclature for this pair of relations, it would be that of the substitution of "non-coexistence" for "incompossibility"; and yet, how could things not coexist? Non-coexistence is nonsense, while incompossibility is an intelligible modal relation borrowed to fill a blank in the scheme of divisions of lations.

581. A *contralation* (really, the only kind of lation there is) is to be termed a *perlation*, if, and only if, there is some individual that stands in this relation to every individual of the universe. A contralation is to be termed an *extraperlation*, if, and only if, there is nothing which stands in this relation to everything.

A contralation is to be termed a *contraperlation*, if, and only if, there is something which does not stand in this relation to anything. A contralation is to be termed a *juxtalation*,* if, and only if, everything stands to something or other in this relation.

A contralation is to be termed a *reperlation*, if, and only if, there is something to which everything stands in this relation. An *extrareperlation* is any contralation in which there is nothing to which something does not stand.†

* Juxta*per*lation.

† Better: ". . . there is nothing to which everything stands."

A *contrareperlation* is a contralation such that there is something to which nothing stands in it. If everything has something or other in a given relation to it, that relation (and none other than such) is a *juxtareperlation*.

Every perlation is necessarily a juxtareperlation; although not every juxtareperlation is a perlation; and similarly, every reperlation is a juxtaperlation.* So, every contraperlation is an extrareperlation; and every contrareperlation is an extraperlation.† The converse of none of these propositions is true. Consequently, the eight classes of contralations named divide all contralations into nine classes, as shown in this table:

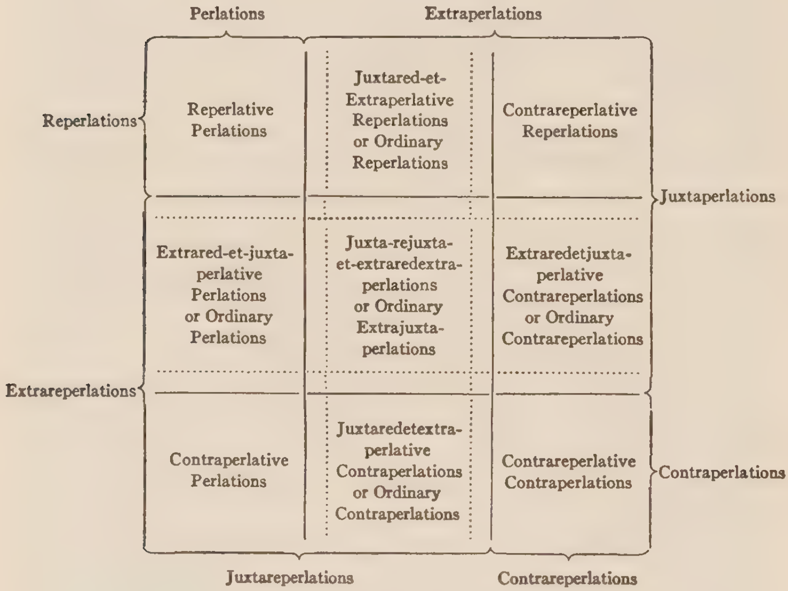


Figure 1 § (page 373)

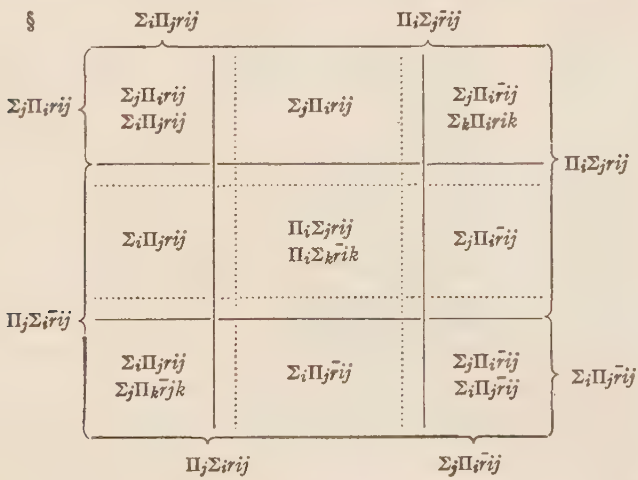
* I.e., $\Sigma_i \Pi_j r_{ij} \prec \Pi_j \Sigma_i r_{ij}$; $\Sigma_j \Pi_i r_{ij} \prec \Pi_i \Sigma_j r_{ij}$.

† $\Sigma_i \Pi_j \bar{r}_{ij} \prec \Pi_j \Sigma_i \bar{r}_{ij}$; $\Sigma_j \Pi_i \bar{r}_{ij} \prec \Pi_i \Sigma_j \bar{r}_{ij}$.

It is a little puzzling to find that a relation may be at once in a class and in the contra-class. Thus, taking the citizens of a town as a universe of discourse, if there is one of them who is helpful to himself and to all the others, helpfulness becomes for that universe, a *perlation*, and unhelpfulness a *contraperlation*. But if there is also in that town a citizen who is helpful to nobody, not even to himself, then helpfulness and unhelpfulness become alike contraperlative perlations. If, however, the first supposition remaining, there is no entirely unhelpful citizen, helpfulness will be a juxtaperlative perlation; and if there is a citizen who is helped by everybody, helpfulness and helpedness will be reperlative perlations.

582. The preposition *pene* may be prefixed to any one of these terms to show that in speaking of relations to everything or to something we mean everything *else* or something else, whether this limits or extends the class of relations. The dotted lines of the diagram show the effect of this prefix in slightly enlarging the border classes at the expense of the interior classes. If the character of a perlation is not due merely to the non-existence of anything that is not a correlate of the individual or individuals that make it a perlation, but is essential to the relation itself, regardless of any other general condition, then whatever is in this relation to anything *must be* in this relation to everything. A perlation of which this is true may be termed an *essential perlation*. In a similar sense we may

§



speak of an *essential reperlation*. Thus "loving something coexisting with" is an essential perlation, "coexisting with something that loves" is an essential reperlation. These are said to be formed from the *primitive* relation of loving. An essentially reperlative perlation must be the juxtalation unless it be the extralation, and so, no lation at all. Thus, either nothing is in the relation of "coexisting with a creator of something coexisting with" to anything, or else everything is in that relation to everything. Thus, an essentially reperlative perlation is not *essentially* either coexistence or non-coexistence, but is either an *accidental juxtalation* or an *accidental extralation*. In the algebra of dyadic relations such an expression is said to be *enveloped*. An essential perlation is essentially a contraperlation, and an essential reperlation is essentially a contrareperlation. [?] But a *contressentiperlation*, which consists in the absence of an essential perlation, is itself an essential perlation. Thus, "loving nobody that coexists with," is equivalent to "non-loving somebody coexisting with," although its relation to its primitive is different.

§3. SECOND SYSTEM OF DIVISIONS*

583. An existential relation may be termed a *suilation*, if, and only if, every individual of the universe stands in that relation to itself. An existential relation may be termed an

* The following schedule may be of aid in this section:

1. $\Pi_i r_{ii}$ — suilation
2. $\Pi_i \Pi_j \bar{r}_{ii} \bar{r}_{jj} r_{ij}$ — contrasuilation — alio-relative
3. $\Sigma_i \Sigma_j \bar{r}_{ii} \bar{r}_{jj} r_{ij}$ — extrasuilation
4. $\Sigma_i r_{ii}$ — juxtasuilation — self-relative

5. $\Pi_i \Pi_j r_{ij} \cdot i \neq j$ — ambilation
6. $\Pi_i \Pi_j r_{ii} r_{jj} \bar{r}_{ij}$ — contrambilation — concurrency
7. $\Sigma_i \Sigma_j \bar{r}_{ij} \cdot i \neq j$ — extrambilation
8. $\Sigma_i \Sigma_j r_{ij} \cdot i \neq j$ — juxtambilation — opponency

9. $\Sigma_i \Pi_j r_{ij} \cdot i \neq j$ — peneperlation
10. $\Sigma_j \Pi_i r_{ij} \cdot i \neq j$ — penereperlation
11. $\Sigma_i \Pi_j \bar{r}_{ij} \cdot i \neq j$ — penecontraperlation
12. $\Sigma_j \Pi_i \bar{r}_{ij} \cdot i \neq j$ — penecontrareperlation

$1 \prec 4$; $5 \prec 8$; $9 \prec 8$; $10 \prec 8$; $11 \prec 7$; $12 \prec 7$.

$1.5 = \infty$	$2.5 = \mathrm{N}.$
$1.6 = 1.$	$2.6 = \mathrm{O}.$

ambilation, if, and only if, every individual of the universe stands in that relation to every other.

584. Every suilation is a juxtasuilation and every ambilation a juxtambilation. Consequently, there will result nine classes of relations as shown in this table:

	Suilations	Extrasuilations		
Ambilations	Coëxistence	Ordinary Ambilations	Negation or, more accurately, Otherness	Juxtambilations
Extrambilations	Ordinary Suilations	Ordinary Extrambilative Extrasuilations	Extrambilative Contrasuilations	
	Identity	Extrasuilative Contrambilation	Incompossibility	Contrambilations
	Juxtasuilations		Contrasuilations	

Figure 2.*

*

	$\Pi_i r_{ii}$	$\Sigma_i\Sigma_j\bar{r}_{ii}\bar{r}_{jj}r_{ij}$		
$\Pi_i\Pi_j r_{ij} \cdot i \neq j$	∞	$\Pi_i\Pi_j r_{ij}$ $i \neq j$	N	$\Sigma_i\Sigma_j r_{ij} \cdot i \neq j$
$\Sigma_i\Sigma_j\bar{r}_{ij} \cdot i \neq j$	$\Pi_i r_{ii}$	$\Sigma_i\Sigma_j\Sigma_k\bar{r}_{ii}\bar{r}_{jj}$ $\bar{r}_{kk}r_{ij}r_{jk}$	$\Pi_i\Pi_j\bar{r}_{ii}$ $\bar{r}_{jj}r_{ij}$ $\Sigma_k\Sigma_l\bar{r}_{kl} \cdot k \neq l$	
	1	$\Sigma_i\Sigma_j\bar{r}_{ii}\bar{r}_{jj}r_{ij} \cdot$ $\Pi_k\Pi_l r_{kk}r_{ll}\bar{r}_{kl}$	0	$\Pi_i\Pi_j r_{ii}r_{jj}r_{ij}$
	$\Sigma_i r_{ii}$		$\Pi_i\Pi_j\bar{r}_{ii}\bar{r}_{jj}r_{ii}$	

585. The following terms were proposed by the author in 1870,* and since they have been generally accepted by writers on the subject, he is more bound to adhere to them than anybody else, although he does not now think they were very judiciously chosen. Namely, a juxtasuilation is termed a *self-relation*, a contrasuilation an *alio-relation;* a contrambilation is called a *concurrency*, a juxtambilation an *opponency.*

586. There is but one ambilative suilation. It is the juxtalation, or coëxistence. There is but one contrambilative suilation: it is the relation of individual *identity*, called *numerical identity* by the logicians. But the adjective seems needless. There is but one ambilative [contra] suilation: it is the relation of individual *otherness*, or *negation.* There is properly no contrambilative contrasuilation: it would be the absurd relation of incompossibility. These four relations are to be termed the *Four Cardinal Dyadic Relations of Second Intention.* It will be enough to call them the *cardinilations*, or *cardinal relations.*

587. Any peneperlation or penereperlation is a juxtambilation; any perlation or reperlation is, in addition, a juxtasuilation. Any penecontraperlation or penecontrareperlation is an extrambilation: any contraperlation or contrareperlation is, in addition, an extrasuilation. Every ambilation is a penereperlative penereperlation†: every contrambilation is a penecontrareperlative penecontraperlation. Every suilation is a juxtareperlative juxtaperlation: every contrasuilation is an extrareperlative extraperlation.‡

§4. THIRD SYSTEM OF DIVISIONS§

588. This system of divisions which depends upon the identity or otherness of relates and correlates not necessarily

* See 136.

† This should be: *peneperlation.*

‡ On one of his copies of this syllabus C. S. P. writes: "Here the fund for the printing gave out." The rest of this paper is from manuscript.

§ The following schedule may be of use in this section:

1. $\Pi_i\Pi_k\Sigma_j$ $\quad iRj \cdot jRk$ granilation
2. $\Sigma_i\Sigma_k\Pi_j - (iRj \cdot jRk)$ extragranilation
3. $\Pi_i\Pi_k\Sigma_j - (iRj \cdot jRk)$ contragranilation
4. $\Sigma_i\Sigma_k\Pi_j$ $\quad iRj \cdot jRk$ juxtagranilation
5. $\Pi_i\Pi_k\Pi_l\Pi_n\Sigma_j\Sigma_m\, iRj \cdot jRk \cdot lRm \cdot mRn \cdot \Pi_j\Sigma_m\, jRm \cdot \Pi_m\Sigma_j\, jRm$ spicalation
6. $\Pi_i\Sigma_j\Pi_k$ $\quad iRj \cdot jRk \multimap iRk$ transitive extraspicalation
7. $\Pi_i\Pi_j\Sigma_k$ $\quad iRj \cdot kRj \multimap iRk$ idempotent contraredultralation

(as in the Second System) relates and correlates of one another, is multiform and irregular, a condition, it can hardly be doubted, incident to its not having been sufficiently studied.

589. A relation having seeds may be termed a *granilation*. The term at present in use is a *repeating relation*, an *extragranilation* being called a non-repeating relation. Every juxtasuilation is a granilation, and every juxtagranilation is a suilation.

Extragranilations are of subsidiary importance (yet not of very little importance), since an essential extragranilation closely approximates to the nature of a reference.

590. A granilation may contain a spike,* or it may not. In the former case, it may be termed a *spicalation*, in the latter an *extraspicalation*. *Extressentispicalations*, or granilations not necessarily containing spikes, comprise the most important class of existential relations in logic. This is the class of *transitive relations* (the term was introduced by De Morgan).† A relation is *transitive* if, and only if, any individual object in that relation to a second which is in the same relation to a third is itself in that relation to this third. Thus, the relation of being as great as (in all such phrases "as" is, in logic, to be understood in the sense of "at least as") is essentially transitive. For if A is as great as B, and B is as great as C, then A is necessarily as great as C. The relation of "greater than" is also essentially transitive.

591. Imagine all the dyads (or ordered pairs) of individuals in the universe to be arrayed in a *matrix* (Cayley's term, though the application of the conception to the logic of relations was first made by the author) as follows:

A:A	A:B	A:C	A:D	A:E	A:F	etc.
B:A	B:B	B:C	B:D	B:E	B:F	etc.
C:A	C:B	C:C	C:D	C:E	C:F	etc.
D:A	D:B	D:C	D:D	D:E	D:F	etc.
E:A	E:B	E:C	E:D	E:E	E:F	etc.
F:A	F:B	F:C	F:D	F:E	F:F	etc.
etc.	etc.	etc.	etc.	etc.	etc.	etc.

Figure 3

* See 576.

† "On the Syllogism IV," *Transactions Cambridge Philosophical Society*, vol. 10, p. 346 (1859).

Let the horizontal rows be termed ranks and the vertical rows *columns* (Howard Staunton's terms?) Let the diagonal line containing the identical pairs A:A, B:B, etc. be termed the *principal*, *dexter*, or *leading diagonal* (all terms in common use.) Then a relation is transitive if, and only if, whatever pair of dyads be taken, of each of which the first individual is in that relation to the second, then in case the rank of either one of these dyads in the matrix crosses the column of the other on the dexter diagonal, the column of the former dyad crosses the rank of the latter in a dyad of which the first member is in the transitive relation to the second. Thus, suppose, to begin with: that the only individual in the relation whose transitivity is to be examined is B; and that it is in that relation to C alone. Then, it is an extragranilation; but it is, according to the definition, a transitive relation, since B does stand in that relation to whatever there may be that C is so related to so long as C is not so related to anything. But now suppose that, in addition, C is in the relation in question to F. Then, since the column of (B:C) meets the rank of (C:F) in (C:C) on the dexter diagonal, while the rank of (B:C) meets the column (C:F) in (B:F) the relation will not be transitive unless B stands in that relation to F. If, in addition, any other dyad in the same rank as (C:F) belongs to the relation, there will have to be a corresponding dyad of the relation in the rank of (B:C); and if there is any additional dyad in the column of (B:C), there will have to be a corresponding dyad of the relation in the column of (C:F).* Therefore, in examining the matrix of a relation in order to ascertain whether the latter is transitive or not, it will be sufficient to consider each pair in the dexter diagonal and examine whether for every dyad in its rank, there is a co-columnar dyad in the rank of every dyad in its (the dexter pair's) column, and conversely. Suppose, for example, we desire to represent a transitive relation in the seven-by-seven matrix of the first figure below with the condition that the squares with crosses shall be occupied by dyads of the relation.

Since the first, fourth, and fifth squares of the dexter diag-

* I.e., if (B:C) and (C:F) imply (B:F), then if (C:X) holds, (B:X) holds, X standing for any correlate; and if (Y:C) holds then (Y:F) holds, Y standing for any relate.

onals (or say, dexter squares) have blank ranks, they need no consideration; and the same is true of the second and sixth dexter squares, which have blank columns. It is thus only necessary to consider the third and seventh dexter squares.

\						
3	\	X	7	3		3
X		\	7	X		X
			\			
				\		
3		X	7	3	\	3
			X			\

Figure 4

The third will require dyads for the squares marked 3, *after which*, the seventh will require dyads in the squares marked 7; and, as it happens, these do not make any new dyads requisite, as they might have done. It will generally be necessary to reëxamine all the dexter squares until it is found that no new dyads are required.*

In this example, there is no need of filling any of the dexter squares; but any of them can be filled without affecting the transitivity of the relation; and in general no transitive extraspicalation is necessarily juxtasuilative; but it may be so in any way. That is, the modification of it by making it include any identical pairs will never destroy its transitivity.

592. On the other hand, a transitive spicalation must subsist between every possible dyad whose members are individuals of the spike. Thus, in order to render the relation whose matrix is shown in Figure 5 transitive, this relation having the two simple spikes A, B, and C, D, E, F, this matrix must be filled up as in Figure 6.

—	A:B	—	—	—	—
B:A	—	—	—	—	—
—	—	—	C:D	—	—
—	—	—	—	D:E	—
—	—	—	—	—	E:F
—	—	F:C	—	—	—

Figure 5

* I.e., BC, CA, CE, CG, FC, GD—<BA, BD, BE, BG, CD, FA, FD, FE, FG.

A:A	A:B	—	—	—	—
B:A	B:B	—	—	—	—
—	—	C:C	C:D	C:E	C:F
—	—	D:C	D:D	D:E	D:F
—	—	E:C	E:D	E:E	E:F
—	—	F:C	F:D	F:E	F:F

Figure 6

Such a relation, whose matrix is entirely composed of full squares having their dexter diagonals on the dexter diagonal of the whole matrix, may be termed a *simililation.* (The term formerly proposed by the author, "copulative relation,"* has not found much favor, and is too unsuitable.)

593. A transitive relation which is such that whatever individual stands in this relation to another stands in the same relation to some individual that stands in the same relation to that other is called an *idempotent* relation† (B. Peirce's Term,‡ which is generally received). An idempotent contrasuilation must subsist between individuals at least denumerable in multitude, and may be termed an *endlessly divisible idempotence.*

594. If a transitive relation is so connected with a relation, called a *primitive* of it, which may be a reference, that any individual stands in this transitive relation to an individual, if, and only if, the former individual stands in the primitive relation to whatever there may be to which the latter individual stands in that primitive relation, then, and only then, it is to be termed a *translation* of that primitive relation.§ Thus, taking "loving" as the primitive relation, the relation of "loving whatever may be loved by" is the translation of loving. Every translation is transitive suilation. Moreover, given any transitive relation, every relate of it stands to its correlate in a translation of that same relation. This is easily seen if we consider that, for example to say that "every lover of a servant of a mistress is a tormentor of that mistress" is the same as to say that "every lover is a tormentor of every mistress whom his beloved may be servant of"; and the same will evidently be

* See 136b.

† See 136b.

‡ See his *Linear Associative Algebras*, §25.

§ See editors' note to §2.

true in any analogous case. If, therefore (substituting "lover of" for "servant of" and for "tormentor of"), every lover of a lover of anybody is a lover of the last (which is as much as to say that loving is a transitive relation), it follows that every lover of anybody loves whatever may be loved by his beloved; that is, he stands in the translation of loving to that which he loves. But this does not imply that every transitive relation constitutes a translation. This is evidently untrue, since every translation is a suilation, but not every transitive relation has that character. Since, however, every translation is a transitive relation, it evidently follows that every transitive relation is precisely a compound of some relation and its translation; such as "at once loving and loving everything loved by." This was Schröder's most brilliant discovery, notwithstanding the simplicity of it, when once stated.

595. The *ultralation** of any primitive relation consists in any relate of it standing to its correlate in relation of standing in the primitive relation to something to which its correlate does not stand in that primitive relation. Thus, "loving something not loved by" is the ultralation of loving. The translation of any primitive is the contraredultralation of the same primitive. Thus, the redultralation of loving is "not loving something loved by" and the contraredultralation is "loving whatever may be loved by."

596. Almost every step in necessary reasoning depends on two premisses which come to the reasoner's knowledge from different quarters. He "puts two and two together," or, in Whewell's† terminology, he *colligates* the facts. But the two premisses may be (in fact they are, although this, as a mere psychological fact does not concern us), colligated, or *compounded* (De Morgan's‡ term) into a copulative proposition (the traditional term, *copulatum* being used by Aulus Gellius§ in the second century of our era, in this sense) from which as a single premiss the conclusion follows. This is to explain the frequent mention of *the premiss* of a necessary inference. The relation of a necessary conclusion to its premiss is an essential

* See editors' note to §2.

† *Novum Organon Renovatum*, bk. II, ch. IV.

‡ *Syllabus of Logic*, §§124, 144.

§ See Prantl, *Geschichte der Logik*, Bd. I, S. 521.

translation of "being true of." For that reason translations may, in all cases, give rise to necessary conclusions; and there is hardly any necessary inference that does not depend on a translation. Consequently, translations and ultralations are logically the most important of all dyadic relations.

597. The compound of the ultralation and translation of the same primitive is to be termed the *superlation* of that primitive. Thus, *superloving* is loving everything loved by and loving something not loved by. Essentially transitive relations (at least, all that are of importance) are either translations (and so, suilations) or are superlations (and so, contrasuilations).

598. A juxtasuilation is, strictly speaking, a spicalation having a spike consisting of a single seed. Such a spike may be called a unispike. A spike consisting of two individuals may be termed a bispike. A spike of which every individual stands in the relation, to which the spike belongs, to only one individual of the spike, and in the converse relation to only one individual of the spike, may be termed a cyclical spike.* Every *simple* spike, that is, every spike whose existence does not consist in the existence of other spikes, is a cyclical spike. It is true that a cyclical spike, and even a simple spike, may be composed of innumerable individuals; and in that case we are not compelled to regard it as running round into itself. Consider, for example, the relation "coming next after" in the both-ways endless series.

$$\ldots 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 10^{0}, 10^{1}, 10^{2}, 10^{3}, 10^{4}, 10^{5}, 10 \ldots$$

This, according to the definition, is a relation having a single simple spike. But since on one side, the values run toward zero, and on the other side toward infinity, it cannot well be considered as returning into itself. Yet it is a nice question whether we may not properly conceive such a series, if endless, as *reaching* the limit, which by definition it just fails to reach. For positive endlessness as much transcends the finite series as does the limit, and reaches that limit in becoming infinite.[1] The limiting term, however is identical with the next term. If that view of the matter be admissible (which is here left doubt-

* Cf. 136e.

[1] I am not here making infinity a limit but in making the state of exceeding the finite the limit of successive increments by unity.

ful), there is no spike, unless the series returns into itself. The question will be fully considered in the lectures.*

599. A spike is said *wholly to consist of simple spikes* only in case every dyad belonging to the relation forms a part of a simple spike belonging to the relation. Thus, the relation which subsists only between the following dyads, is composed wholly of simple spikes: (A:B), (B:C), (C:D), (D:A), (A:E), (E:F), (F:D). For every dyad either belongs to the simple spike (A:B), (B:C), (C:D), (D:A), or to the simple spike (D:A), (A:E), (E:F), (F:D), (D:A); and the circumstance that (D:A) belongs to both simple spikes does not conflict with this. But the spike (A:B), (A:D), (B:C), (C:A), (D:E), (E:F), (F:D) does not wholly consist of simple spikes, since (A:D) does not here belong to any simple spike.

600. A relation of which every correlate stands in this relation to every one of its relates, and consequently to every relate of which every one of its correlates stand in the same relation, is termed an *equiparance*† (the term has been used since Antonius Andreas about A.D. 1300‡). An *extrequiparance* is termed a *disquiparance*† (the term has been in common use since Franciscus Mayro§, about A. D. 1300). Equiparances and disquiparances are also termed *convertible* and *inconvertible* relations (De Morgan),¶ and quite commonly *reciprocating* and *non-reciprocating* relations. All the relates and correlates of an equiparance form a spike wholly consisting of spikes of two members and spikes of one member. (A spike of two members is either simple or is wholly composed of a simple spike of two members and, at most, two spikes of one member, every spike of one member being simple.)

§5. FOURTH SYSTEM OF DIVISIONS

601. This system depends upon the multitudes of relates and correlates. It has been treated with such elaboration by Mr. Kempe, whose "Memoir on the Theory of Mathematical Forms" may be found in the *Philosophical Transactions* for

* "The Lowell Lectures," 1903, for which this syllabus was intended as a supplement.

† See 136c.

‡ Cf. Prantl, *op. cit.*, Bd. III, S. 153, 282.

§ Cf. Prantl, *op. cit.*, Bd. III, S. 290.

¶ *Formal Logic*, p. 345.

1886, that only the most important points will here be noticed. Mr. Kempe's nomenclature is, in a few cases, unsuitable; but it is remarkable that the severest criticism has detected only one or two errors in the extensive work, and that that one, or may be two, attach to matters of minute detail toward the end of the memoir. It is one of the most solid treatises that have ever been written on the logic of relations — a work that goes toward the elevation of man.

602. In the first place, there are two cross dichotomies according as the relation has for its relates on the one hand, or for its correlates on the other, whatever there may be that possesses some general quality, or merely a singular object or singular objects. The names which the author proposed for these classes in 1870* were rightly rejected by Schröder, who names them by reference to the matrix.† The ranks and files of a matrix have always in German been called *Zeilen* and *Kolonnen* respectively; and Schröder, accordingly names a relative term applicable to a single relate an "Einzeiler," and one applicable to a single correlate an "Einkolonner." A compartment of the matrix is called in German "ein Auge";[1] and Schröder calls a relative term applicable only to a single dyad, whether like A:A or like A:B, an *Einauger*. The author's original term an *elementary relative* is sufficiently good. But the terms *one-rank* relation, *one-column* relation, *many-rank* relation, *many-column* relations can be used to advantage.

A one-rank relation may either be accidentally such (as it happens that there is but one "day-bringing" luminary), or essentially such (as "creator of"), and so of course, it is with one-column relations.

603. A general relation, and indeed, any many-ranked many-column relation, may be limited in respect to the multitude of relates *to each* correlate, and conversely. The sixth chapter of the third volume of Schröder's *Logik* contains a valuable and entirely original investigation of these. He describes these as relations with *empty* ranks (*Leerzeilen*) and *empty* columns, with *Singly occupied* (*Einbesetzten*) ranks or columns, with multioccupied (*Meerbesetzten*) ranks or columns, with

* Cf. 121ff.

† *Algebra der Logik*, 3.1, S. 11–12 and *passim*.

[1] But in the mathematical books, that which fills it is called an Element.

many-gapped (*Meerlückige-*) ranks or columns, *singly gapped* (*Einlück-*) ranks or columns, and with *full* (*Voll-*) ranks and columns.

These again may be accidental or essential. Thus, the relation of parentage has essentially two relates to each correlate, but an indefinite finite number of correlates to each relate.

604. Another pair of cross dichotomies depends upon whether the relates of each correlate, on the one hand, and the correlate of each relate, on the other, are, or are not, of a fixed number. Of particular consequence (meaning by "consequence" the leading to conclusions) are those relations for which both these numbers are fixed. Such a relation is termed a *correspondence* (a common term of mathematics). The number of relates to each correlate is usually prefixed to the designation, being followed by the number of correlates to each relate, with the proposition "to" intervening between the two numbers. Thus, we speak of a "two-to-three correspondence," meaning that each correlate has two relates and each relate three correlates.

605. There is an indefinite division of correspondences into the *regular* and the *irregular*. The regular are those in which there is some discernible rule as to what relates are joined to what correlates. Thus, the relations whose matrices are shown in Figures 7, 8 and 9 are all regular two-to-two correspondences; but their rules are different. It is certain, however, that no collection of individuals be it finite or infinite, can be arranged in succession without that succession's perfectly conforming to some kind of regularity. It is true that there is a difference between an *accidental* and an *essential* regularity. But the difference does not manifest itself in the existential

—	A:B	—	—	—	A:F	—	—
—	—	B:C	—	—	—	B:G	—
—	—	—	C:D	—	—	—	C:H
D:A	—	—	—	D:E	—	—	—
—	E:B	—	—	—	E:F	—	—
—	—	F:C	—	—	—	F:G	—
—	—	—	G:D	—	—	—	G:H
H:A	—	—	—	H:E	—	—	—

Figure 7

facts themselves. The problem of how an accidental regularity can be distinguished from an essential one is precisely the problem of inductive logic.

—	A:B	—	—	A:E	—	—	—
—	—	B:C	—	—	B:F	—	—
—	—	—	C:D	—	—	C:G	—
—	—	—	—	D:E	—	—	D:H
E:A	—	—	—	—	E:F	—	—
—	F:B	—	—	—	—	F:G	—
—	—	G:C	—	—	—	—	G:H
H:A	—	—	H:D	—	—	—	—

Figure 8

—	A:B	A:C	—	—	—	—	—
—	—	B:C	B:D	—	—	—	—
—	—	—	C:D	C:E	—	—	—
—	—	—	—	D:E	D:F	—	—
—	—	—	—	—	E:F	E:G	—
—	—	—	—	—	—	F:G	F:H
G:A	—	—	—	—	—	—	G:H
H:A	H:B	—	—	—	—	—	—

Figure 9

§6. NOTE ON THE NOMENCLATURE AND DIVISIONS OF MODAL DYADIC RELATIONS

606. A modal dyadic relation is either a relation between characters (including qualities and relations of individuals, of characters, and of concepts), or between symbols, or concepts.

607. Dyadic relations between characters mostly correspond to relations between the subjects of those characters or to relations between the symbols of them; and such need not be separately considered. There remain some relations between characters, especially between qualities, which do not seem to be derivative. Such are the relations of "being more intense than," of "being disparate to" (or in applicability to subjects of the same category, as multitude and intensity are disparate). But, so far as appears at present, no particular

logical interest attaches to such relations, and they will here be passed by.

608. Dyadic relations between symbols, or concepts, are matters of logic, so far as they are not derived from relations between the objects and the characters to which the symbols refer. Noting that we are limiting ourselves to modal *dyadic* relations, it may probably be said that those of them that are truly and fundamentally dyadic arise from corresponding relations between propositions. To exemplify what is meant, the dyadic relations of logical *breadth* and *depth*, often called denotation and connotation, have played a great part in logical discussions, but these take their origin in the triadic relation between a sign, its object, and its interpretant sign; and furthermore, the distinction appears as a dichotomy owing to the limitation of the field of thought, which forgets that concepts grow, and that there is thus a third respect in which they may differ, depending on the state of knowledge, or amount of information.* To give a good and complete account of the dyadic relations of concepts would be impossible without taking into account the triadic relations which, for the most part, underlie them; and indeed almost a complete treatise upon the first† of the three divisions of logic would be required.‡

* See 2.406ff.

† I.e., Speculative Grammar, for which see vol. 2, bk. II.

‡ The remainder of this syllabus is to be found in 2.233 seq., where triadic relations, and, in particular, the divisions of signs are treated at length.

XX

NOTES ON SYMBOLIC LOGIC AND MATHEMATICS[E]

§1. IMAGING*

609. A term proposed to translate *Abbildung* in its logical use. In order to apprehend this meaning, it is indispensable to be acquainted with the history of the meanings of *Abbildung*. This word was used in 1845 by Gauss† for what is called in English a map-projection, which is an incorrect term, since many such modes of representation are not geometrical rectilinear projections at all; and of those which Gauss had in view, but a single one is so. In mathematics *Abbildung* is translated *representation;* but this word is preëmpted in logic. Since *Bild* is always translated *image*, *imaging* will answer very well for *Abbildung*. If a map of the entire globe was made on a sufficiently large scale, and out of doors, the map itself would be shown upon the map; and upon that image would be seen the map of the map; and so on, indefinitely. If the map were to cover the entire globe, it would be an image of nothing but itself, where each point would be imaged by some other point, itself imaged by a third, etc. But a map of the heavens does not show the map itself at all. A Mercator's projection shows the entire globe (except the poles) over and over again in endlessly recurring strips.‡ Many maps, if they were completed, would show two or more different places on the earth at each point of the map (or at any rate on a part of it), like one map drawn upon another. Such is obviously the case with any rectilinear projection of the entire sphere, excepting only the stereographic. These two peculiarities may coexist in the same map.

610. Any mathematical function of one variable may be regarded as an image of its variable according to some mode

* *Dictionary of Philosophy and Psychology*, ed. by J. M. Baldwin, Macmillan & Co., N. Y., vol. 1, p. 518, by Peirce and H. B. Fine. 2d ed. 1911.

† "Untersuchung über Gegenstände der höhere Geodäsie; 2 Abhandlung;" *Abh.* d. Königl. G. d. W. zu Göttingen, 2[ter] u. 3[ter] Bd.

‡ See vol. 7.

of imaging. For the real and imaginary quantities correspond, one to one and continuously, to the assignable points on a sphere. Although mathematics is by far the swiftest of the sciences in its generalisations, it was not until 1879 that Dedekind (in the third edition of his recension of Lejeune-Dirichlet's *Zahlentheorie*, § 163, p. 470; but the writer has not examined the second edition) extended the conception to discrete systems in these words: "It very often happens in other sciences, as well as in mathematics, that there is a replacement of every element ω of a system Ω of elements or things by a corresponding element ω' [of a system Ω']. Such an act should be called a substitution. . . . But a still more convenient expression is found by regarding Ω' as the image of Ω, and ω' of ω, according to a certain mode of imaging." And he adds, in a footnote: "This power of the mind of comparing a thing ω with a thing ω', or of relating ω to ω', or of considering ω' to correspond to ω, is one without which no thought would be possible." (We do not translate the main clause.) This is an early and significant acknowledgment that the so-called "logic of relatives" — then deemed beneath the notice of logicians — is an integral part of logic. This remark remained unnoticed until, in 1895, Schröder devoted the crowning chapter of his great work (*Exakte Logik*, iii. 553–649) to its development. Schröder says that, in the broadest sense, any relative whatever may be considered as an imaging — "nämlich als eine eventuell bald 'undeutige,' bald 'eindeutige,' bald 'mehrdeutige' Zuordnung." He presumably means that the logical universe is thus imaged in itself. However, in a narrower sense, he says, a mode of imaging is restricted to a relative which fulfills one or other of the two conditions of being never *undeutig* or being never *mehrdeutig*. That is, the relation must belong to one or other of two classes, the one embracing such that every object has an image, and the other such that no object has more than one image. Schröder's definitions (however interesting his developments) break all analogy with the important property of the imaging of continua noticed above. If this is to be regarded as essential, an imaging must be defined as a generic relation between an object-class and an image-class, which generic relation consists of specific relations, in each of which one individual, and no more, of the image-class stands to each individual of the object-class, and in each of which every individual of the image-class stands to one individual and no more of the object-class. This is substantially a return to Dedekind's definition, which makes an *imaging* a synonym for a substitution.

§2. INDIVIDUAL*

611. (As a technical term of logic, *individuum* first appears in Boëthius,† in a translation from Victorinus, no doubt of ἄτομον, a word used by Plato (*Sophistes*, 229 D) for an indivisible species, and by Aristotle, often in the same sense, but occasionally for an individual. Of course the physical and mathematical senses of the word were earlier. Aristotle's usual term for individuals is τὰ καθ' ἕκαστα, Latin *singularia*, English *singulars*.) Used in logic in two closely connected senses. (1) According to the more formal of these an individual is an object (or term) not only actually determinate in respect to having or wanting each general character and not both having and wanting any, but is necessitated by its mode of being to be so determinate. See Particular (in logic).‡

612. This definition does not prevent two distinct individuals from being precisely similar, since they may be distinguished by their hecceities (or determinations not of a generalizable nature); so that Leibnitz' principle of indiscernibles is not involved in this definition. Although the principles of contradiction and excluded middle may be regarded as together constituting the definition of the relation expressed by "not," yet they also imply that whatever exists consists of individuals.§ This, however, does not seem to be an identical proposition or necessity of thought; for Kant's Law of Specification (*Krit. d. reinen Vernunft*, 1st ed., 656; 2d ed., 684; but it is requisite to read the whole section to understand his meaning), which has been widely accepted, treats logical quantity as a continuum in Kant's sense, i.e., that every part of which is composed of parts. Though this law is only regulative, it is supposed to be demanded by reason, and its wide acceptance

* *Dictionary of Philosophy and Psychology*, vol. 1, p. 537–38.

† See Prantl, *op. cit.*, Bd. I, S. 661, 684.

‡ Vol. 2, bk. II, ch. 4, §12.

§ Cf. 1.435.

as so demanded is a strong argument in favour of the conceivability of a world without individuals in the sense of the definition now considered. Besides, since it is not in the nature of concepts adequately to define individuals, it would seem that a world from which they were eliminated would only be the more intelligible. A new discussion of the matter, on a level with modern mathematical thought and with exact logic, is a desideratum. A highly important contribution is contained in Schröder's *Logik*, iii, Vorles. 10. What Scotus says (*Quaest. in Met.*, VII 9, xiii and xv) is worth consideration.

613. (2) Another definition which avoids the above difficulties is that an individual is something which reacts. That is to say, it does react against some things, and is of such a nature that it might react, or have reacted, against my will.

This is the stoical definition of a reality; but since the Stoics were individualistic nominalists, this rather favours the satisfactoriness of the definition than otherwise. It may be objected that it is unintelligible; but in the sense in which this is true, it is a merit, since an individual is unintelligible in that sense. It is a brute fact that the moon exists, and all explanations suppose the existence of that same matter. That existence is unintelligible in the sense in which the definition is so. That is to say, a reaction may be experienced, but it cannot be conceived in its character of a reaction; for that element evaporates from every general idea. According to this definition, that which alone immediately presents itself as an individual is a reaction against the will. But everything whose identity consists in a continuity of reactions will be a single logical individual. Thus any portion of space, so far as it can be regarded as reacting, is for logic a single individual; its spatial extension is no objection. With this definition there is no difficulty about the truth that whatever exists is individual, since existence (not reality) and individuality are essentially the same thing; and whatever fulfills the present definition equally fulfills the former definition by virtue of the principles of contradiction and excluded middle, regarded as mere definitions of the relation expressed by "not." As for the principle of indiscernibles, if two individual things are exactly alike in all other respects, they must, according to this definition, differ in their spatial relations, since space is nothing but the intuitional presentation of the conditions of reaction, or of some of them. But there will be no logical hindrance to two things being exactly alike in all other respects; and if they are never so, that is a physical law, not a necessity of logic. This second definition, therefore, seems to be the preferable one. Cf. Particular (in logic).*

§3. INVOLUTION†

614. A term of Symbolic Logic borrowed from algebra, where it means the raising of a base to a power. In logic it has two different senses. (1) Relative involution: let *lwm* denote any lover of a well-wisher of a man. That is, any individual *A* is denoted by *lwm*, provided there are in existence individuals *B* and *C* (who may be identical with each other or with *A*), such that *A* loves *B*, while *B* wishes well to *C*, and *C* is a man. Further, let $l^w m$ denote any individual *A*, if, and only if, there is in existence an individual *C*, who is a man, and who is such that taking any individual *B* whatever, if *B* is a well-wisher of *C*, then *A* is a lover of *B*. The operation of combining *l* and *w* in this statement is termed "progressive involution."‡ Again, let $l^w m$ denote any individual *A*, if, and only if, there is in existence an individual *B*, who is loved by *A*, and who is such that taking any individual *C* whatever, if *C* is wished well by *B*, then *C* is a man. The operation of combining *w* and *m* in this statement is termed "regressive involution."§ These designations were adopted because of the analogy of the general formulae to those of involution in the algebra of quantity.

These kinds of involution are not, at present, in use in symbolical logic; but they are, nevertheless, useful, especially in developing the conception of continuity. These two kinds of involution together constitute relative involution.

615. (2) Non-relative involution: consisting in the repeated introduction of the same premiss into a reasoning; as, for example, the half dozen simple premisses upon which the Theory of Numbers is based are introduced over and over again in the reasoning by which its myriad theorems are deduced. In exact logic the regular process of deduction begins

* Vol. 2, bk. II, ch. 4, §12.

† *Dictionary of Philosophy and Psychology*, vol. 1, p. 574.

‡ See 77–80.

§ See 113–118.

by non-relatively multiplying together all the premisses to make one conjunctive premiss, from which whatever can be deduced by using those premisses as often as they are introduced as factors, can be deduced by processes of "immediate inference" from that single conjunctive premiss. But the general character of the conclusion is found to depend greatly upon the number of times the same factor is multiplied in. From this circumstance the importance and the name of non-relative involution arise.

§4. LOGIC (EXACT)*

616. The doctrine that the theory of validity and strength of reasoning ought to be made one of the "exact sciences," that is, that generalisations from ordinary experience ought, at an early point in its exposition, to be stated in a form from which by mathematical, or expository, reasoning, the rest of the theory can be strictly deduced; together with the attempt to carry this doctrine into practice.

617. This method was pursued, in the past, by Pascal (1623–62), Nicolas Bernoulli (1687–1759), Euler (1708–83), Ploucquet (1716–90), Lambert (1728–77), La Place (1749–1827), De Morgan (1806–71), Boole (1815–64), and many others; and a few men in different countries continue the study of the problems opened by the last two named logicians as well as those of the proper foundations of the doctrine and of its application to inductive reasoning. The results of this method, thus far, have comprised the development of the theory of probabilities, the logic of relatives, advances in the theory of inductive reasoning (as it is claimed), the syllogism of transposed quantity, the theory of the Fermatian inference, considerable steps towards an analysis of the logic of continuity and towards a method of reasoning in topical geometry, contributions towards several branches of mathematics by applications of "exact" logic, the logical graphs called after Euler and other systems for representing in intuitional form the relations of premisses to conclusions, and other things of the same general nature.

618. There are those, not merely outside the ranks of exact logic, but even within it, who seem to suppose that the aim is

* *Dictionary of Philosophy and Psychology*, vol. 2, pp. 24–27.

to produce a calculus, or semi-mechanical method, for performing all reasoning, or all deductive inquiry; but there is no reason to suppose that such a project, which is much more consonant with the ideas of the opponents of exact logic than with those of its serious students, can ever be realised. The real aim is to find an indisputable theory of reasoning by the aid of mathematics. The first step in the order of logic towards this end (though not necessarily the first in the order of inquiry) is to formulate with mathematical precision, definiteness, and simplicity, the general facts of experience which logic has to take into account.

619. The employment of algebra in the investigation of logic is open to the danger of degenerating into idle trifling of too rudimentary a character to be of mathematical interest, and too superficial to be of logical interest. It is further open to the danger that the rules of the symbols employed may be mistaken for first principles of logic. An algebra which brings along with it hundreds of purely formal theorems of no logical import whatever must be admitted, even by the inventor of it, to be extremely defective in that respect, however convenient it may be for certain purposes. On the other hand, it is indisputable that algebra has an advantage over speech in forcing us to reason explicitly and definitely, if at all. In that way it may afford very considerable aid to analysis. It has been employed with great advantage in the analysis of mathematical reasonings.

Algebraic reasoning involves intuition just as much as, though more insidiously than, does geometrical reasoning; and for the investigation of logic it is questionable whether the method of graphs is not superior. Graphs cannot, it is true, readily be applied to cases of great complexity; but for that very reason they are less liable to serve the purposes of the logical trifler. In the opinion of some exact logicians, they lead more directly to the ultimate analysis of logical problems than any algebra yet devised. See Logical Diagram (or Graph).*

620. It is logical algebra, however, which has chiefly been pursued. De Morgan invented a system of symbols, which had the signal advantage of being entirely new and free from all associations, misleading or otherwise. Although he employed

* Vol. 4, bk.II, ch. 1., §1.

them for synthetical purposes almost exclusively, yet the great generality of some of the conceptions to which they led him is sufficient to show that they might have been applied with great advantage in analysis. Boole was led, no doubt from the consideration of the principles of the calculus of probabilities, to a wonderful application of ordinary algebra to the treatment of all deductive reasoning not turning upon any relations other than the logical relations between non-relative terms. By means of this simple calculus, he took some great steps towards the elucidation of probable reasoning; and had it not been that, in his pre-Darwinian day, the notion that certain subjects were profoundly mysterious, so that it was hopeless, if not impious, to seek to penetrate them, was still prevalent in Great Britain, his instrument and his intellectual force were adequate to carrying him further than he actually went. Most of the exact logicians of today are, from the nature of the case, followers of Boole. They have modified his algebra by disusing his addition, subtraction, and division, and by introducing a sign of logical aggregation. This was first done by Jevons; and he proposed ⸭ , a sign of division turned up, to signify this operation. Inasmuch as this might easily be read as three signs, it would, perhaps, be better to join the two dots by a light curve, thus ⸭. Some use the sign + for logical aggregation. The algebra of Boole has also been amplified so as to fit it for the logic of relatives. The system is, however, far from being perfect. See Relatives (logic of).*

621. Certain terms of exact logic may be defined as follows: —†

Copula is often defined as that which expresses the relation between the subject-term and the predicate-term of a proposition. But this is not sufficiently accurate for the purposes of exact logic. Passing over the objection that it applies only to categorical propositions, as if conditional and copulative propositions had no copula, contrary to logical tradition, it may be admitted that a copula often does fulfill the function mentioned; but it is only an accidental one, and its essential function is quite different. Thus, the proposition, "Some favoured patri-

* §8.

† A number of elementary definitions of such familiar terms as "aggregation," "absorption," "associative," "commutative," etc. have been omitted.

arch is translated" is essentially the same as "A translated favoured patriarch is"; and "Every mother is a lover of that of which she is a mother" is the same as "A mother of something not loved by her is not." In the second and fourth forms, the copula connects no terms; but if it is dropped, we have a mere term instead of a proposition. Thus the essential office of the copula is to express a relation of a general term or terms to the universe. The universe must be well known and mutually known to be known and agreed to exist, in some sense, between speaker and hearer, between the mind as appealing to its own further consideration and the mind as so appealed to, or there can be no communication, or "common ground," at all. The universe is, thus, not a mere concept, but is the most real of experiences. Hence, to put a concept into relation to it, and into the relation of describing it, is to use a most peculiar sort of sign or thought; for such a relation must, if it subsist, *exist* quite otherwise than a relation between mere concepts. This, then, is what the copula essentially does. This it may do in three ways: first, by a vague reference to the universe collectively; second, by a reference to all the individuals existent in the universe distributively; third, by a vague reference to an individual of the universe selectively. "It is broad daylight," I exclaim, as I awake. My universe is the momentary experience as a whole. It is that which I connect as object of the composite photograph of daylight produced in my mind by all my similar experiences. Secondly, "Every woman loves something" is a description of every existing individual in the universe. Every such individual is said to be coexistent only with what, so far as it is a woman at all, is sure to be a lover of some existing individual. Thirdly, "Some favoured patriarch is translated" means that a certain description applies to a select individual. A hypothetical proposition, whether it be conditional (of which the alternative, or disjunctive, proposition is a mere species, or *vice versa*, as we choose to take it) or copulative, is either general or *ut nunc*. A general conditional is precisely equivalent to a universal categorical. "If you really want to be good, you can be," means "Whatever determinate state of things may be admissibly supposed in which you want to be good is a state of things in which you can be good." The universe is that of determinate states of things that are admissible hypothetically. It is true that some logicians appear to dispute this; but it is manifestly indisputable. Those logicians belong to two classes: those who think that logic ought to take account of the difference between one kind of universe and another (in which case, several other *substantiæ* of propositions must be admitted); and those who hold that logic should distinguish between propositions which are necessarily true or false together, but which regard the fact from different aspects. The exact logician holds it to be, in itself, a defect in a logical system of expression, to afford different ways of expressing the same state of facts; although this defect may be less important than a definite advantage gained by it. The copulative proposition is in a similar way equivalent to a particular categorical. Thus, to say "The man might not be able voluntarily to act otherwise than physical causes make him act, whether he try or not," is the same as to say that there is a state of things hypothetically admissible in which a man tries to act one way and voluntarily acts another way in consequence of physical causes. As to hypotheticals *ut nunc*, they refer to no range of possibility, but simply to what is true, vaguely taken collectively.

622. Although it is thus plain that the action of the copula in relating the subject-term to the predicate-term is a secondary one, it is nevertheless necessary to distinguish between copulas which establish different relations between these terms. Whatever the relation is, it must remain the same in all propositional forms, because its nature is not expressed in the proposition, but is a matter of established convention. With that proviso, the copula may imply any relation whatsoever. So understood, it is the *abstract copula* of De Morgan (*Camb. Philos. Trans.*, x. 339). A *transitive* copula is one for which the mood Barbara is valid. Schröder has demonstrated the remarkable theorem that if we use IS in small capitals to represent any one such copula, of which "greater than" is an example, then there is some relative term *r*, such that the proposition "*S* IS *P*" is precisely equivalent to "*S* is *r* to *P* and is *r* to whatever *P* is *r* to." A *copula of correlative inclusion* is one for which both Barbara and the formula of identity hold good. Representing any one such copula by *is* in italics, there is a relative term *r*, such that the proposition "*S is P*" is pre-

cisely equivalent to "*S* is *r* to whatever *P* is *r* to." If the last proposition follows from the last but one, no matter what relative *r* may be, the copula is called the *copula of inclusion*, used by C. S. Peirce, Schröder, and others. De Morgan uses a copula defined as standing for any relation both transitive and convertible. The latter character consists in this, that whatever terms *I* and *J* may be, if we represent this copula by **is** in black-letter, then from "*I* **is** *J*" it follows that "*J* **is** *I*." From these two propositions, we conclude, by Barbara, that "*I* **is** *I*." Such copulas are, for example, "equal to," and "of the same colour as." For any such copula there will be some relative term *r*, such that the proposition "*S* **is** *P*" will be precisely equivalent to "*S* is *r* to everything, and only to everything, to which *P* is *r*." Such a copula may be called a copula of *correlative identity*. If the last proposition follows from the last but one, no matter what relative *r* may be, the copula is the *copula of identity* used by Thomson, Hamilton, Baynes, Jevons, and many others.

It has been demonstrated by Peirce that the copula of inclusion is logically simpler than that of identity.*

623. *Dialogism*. A form of reasoning in which from a single premiss a disjunctive, or alternative, proposition is concluded introducing an additional term; opposed to a syllogism, in which from a copulative proposition a proposition is inferred from which a term is eliminated.

Syllogism.

All men are animals, and all animals are mortal;
∴ All men are mortal.

Dialogism.

Some men are not mortal;
∴ Either some men are not animals, or some animals are not mortal.

624. *Dimension*. An element or respect of extension of a logical universe of such a nature that the same term which is individual in one such element of extension is not so in another. Thus, we may consider different persons as individual in one respect, while they may be divisible in respect to time, and in

* See 47n.

respect to different admissible hypothetical states of things, etc. This is to be widely distinguished from different universes, as, for example, of things and of characters, where any given individual belonging to one cannot belong to another. The conception of a multidimensional logical universe is one of the fecund conceptions which exact logic owes to O. H. Mitchell.* Schröder, in his then second volume, where he is far below himself in many respects, pronounces this conception "untenable." But a doctrine which has, as a matter of fact, been held by Mitchell, Peirce, and others, on apparently cogent grounds, without meeting any attempt at refutation in about twenty years, may be regarded as being, for the present, at any rate, tenable enough to be held.

625. *Dyadic relation.* A fact relating to two individuals. Thus, the fact that *A* is similar to *B*, and the fact that *A* is a lover of *B*, and the fact that *A* and *B* are both men, are dyadic relations; while the fact that *A* gives *B* to *C* is a triadic relation. Every relation of one order of relativity may be regarded as a relative of another order of relativity if desired. Thus, *man* may be regarded as *man coexistent with*, and so as a relative expressing a dyadic relation, although for most purposes it will be regarded as a monad or non-relative term.

§5. MULTITUDE (IN MATHEMATICS)†

626. That relative character of a collection which makes it greater than some collections and less than others. A collection, say that of the *A*'s, is *greater than* another, say that of the *B*'s, if, and only if, it is impossible that there should be any relation *r*, such that every *A* stands in the relation *r* to a *B* to which no other *A* is in the relation *r*.

627. The precise analysis of the notion is due to G. Cantor, whose definition is, however, a little different in its mode of expression, since it is more abstract. He defines the character in these words: "By Mächtigkeit or cardinal number of a collection (Menge) *M*, we mean the universal concept, which by the help of our active faculty of thought results from the collection *M* by abstraction from the characters of the different

* *Johns Hopkins Studies in Logic*, p. 87ff.

† *Dictionary of Philosophy and Psychology*, vol. 2, p. 117–18, by Peirce and H. B. Fine.

members (Elemente) of that collection and from the order in which they are given (Gegebensein)."*

628. A cardinal number, though confounded with multitude by Cantor, is in fact one of a series of vocables the prime purpose of which, quite unlike any other words, is to serve as an instrument in the performance of the experiment of counting; these numbers being pronounced in their order from the beginning, one as each member of the collection is disposed of in the operation of counting. If the operation comes to an end by the exhaustion of the collection, the last cardinal number pronounced is applied adjectivally to the collection, and expresses its multitude, by virtue of the theorem that a collection the counting of which comes to an end, always comes to an end with the pronunciation of the same cardinal number.

629. If the cardinal numbers are considered abstractedly from their use in counting, simply in themselves, as objects of mathematical reasoning, stripped of all accidents not pertinent to such study, they become indistinguishable from the similarly treated ordinal numbers, and are then usually called *ordinal numbers* by the mathematico-logicians. There is small objection to this; yet it is to be remarked that they are ordinal in different senses in grammar and in the logic of mathematics. For in grammar they are called ordinal as being adapted to express the ordinal places of other things in the series to which those things belong; while in the logic of mathematics the only relevant sense in which they are ordinal is as being defined by a serial order within their own system. The definition of this order is not difficult; but the syntax of ordinary language does not lend itself to the clear expression of such relations in the manner in which they ought to be expressed in order to bring out their logical character. It must, therefore, be here passed by. In fact, none of the doctrines of logic can be satisfactorily expressed under the limitations here imposed, however simple they may be. The doctrine of ordinal numbers is by Dedekind (*Was sind und was sollen die Zahlen?*) made to precede that of the cardinal numbers; and this is logically preferable, if hardly so imperative as Schröder considers it.

630. The doctrine of the so-called ordinal numbers is a doctrine of pure mathematics; the doctrine of cardinal num-

* See *Georg Cantor Gesammelte Abhandlung* S. 282, Berlin, (1932).

bers, or, rather, of multitude, is a doctrine of mathematics applied to logic. The smallest multitude is most conveniently considered to be *zero;* but this is a question of definition. A *finite* collection is one of which the syllogism of transposed quantity holds good. Of finite collections, it is true that the whole is greater than any part. It is singular that this is often taken as the type of an axiom, although it has from early times been a matter of familiar knowledge that it is not true of infinite collections. Every addition of one increases a finite multitude. An infinite collection cannot be separated into a lesser collection of parts all smaller than itself.

631. The multitude of all the different finite multitudes is the smallest infinite multitude. It is called the *denumeral* multitude. (Cantor uses a word equivalent to *denumerable;* but the other form has the advantage of being differentiated from words like *enumerable*, *abnumerable*, which denote classes of multitudes, not, like *denumeral*, a single multitude.) Following upon this is a denumeral series of multitudes called by C. S. Peirce the *first*, *second*, etc. *abnumerable* multitudes. Each is the multitude of possible collections formed from the members of a collection of the next preceding multitude. They seem to be the same multitudes that are denoted by Cantor as *Alephs*. The first of them is the multitude of different limits of possible convergent series of rational fractions, and therefore of all the quantities with which mathematical analysis can deal under the limitations of the doctrine of limits. (The imaginaries do not increase the multitude.) What comes after these is still a matter of dispute, and is perhaps of inferior interest. The transition to continuity is, however, a matter of supreme importance for the theory of scientific method; nor is it a very complicated matter; but it cannot be stated under the limitations of expression here imposed upon us.

§6. POSTULATE*

632. (1) The earliest definition we have of postulate, which was a technical term of Greek geometers, is by Aristotle.† The passage has an appearance of incoherence; it is, however, plain that Aristotle makes a distinction between *hypotheses*

* *Dictionary of Philosophy and Psychology*, vol. 2, pp. 315–16.

† See *Analytica Posteriora*, 76b, 26–31.

and *postulates* which Euclid does not draw, and which is irrelevant. Omitting the distinction, the two have this in common — that they are propositions not necessarily true which are assumed as the bases of deductions.

If we turn to the first book of Euclid's *Elements*, we observe, in the first place, that he calls axioms by the name of common notions, a deliberate choice by him, for Aristotle, before his day, had called them axioms, though Aristotle usually calls them τὰ κοινά, nearly Euclid's name. These matters of common knowledge, according to Euclid's enumeration of them, are not specially geometrical, except that magnitudes superposable are equal (see the *Cent. Dict.*, "Axiom"). On the other hand, the "postulates" of Euclid are all geometrical. They are as follows (according to the best MS. and all the evidence): —

(*a*) Between any two points a straight line can be drawn.

(*b*) Any terminated straight line can be prolonged at either end indefinitely.

(*c*) About any point in any plane as centre a circle may be described with any radius.

(*d*) All right angles are equal.

(*e*) If two straight lines in a plane are cut by a third, making the sum of the internal angles on one side less than two right angles, those two straight lines will meet if sufficiently produced.

(*f*) Two straight lines cannot enclose a space in a plane.

633. (2) Since Wolff it has been very common among Germans, and among English writers who follow them, to define a postulate as an indemonstrable practical proposition. That is to say, it is an indemonstrable *particular* proposition, asserting that some general description of an object *exists* (in the only sense in which pure geometrical forms can be said to exist), in contradistinction to *axioms*, which were supposed to be indemonstrable theoretical (*i.e.* universal) propositions, asserting that some general description of an object has no existence as a geometrical form.

It is certainly desirable to have two terms bearing these meanings; but it was an utter misunderstanding to suppose that such were the proper meanings either of the word *axiom* or of the word *postulate*. The manner in which this misunderstanding came about is somewhat instructive. An axiom was a perfectly indubitable statement *about things*, in contradistinction to a definition, which cannot be called in question. On the contrary, a postulate was an indemonstrable proposition, not indubitable. There was some question whether certain postulates might not be considered to be axiomatic. When that was done, all the remaining postulates were particular propositions; namely, the first three of Euclid's list. This view was aided by the illogical notion that definitions could be considered as among the foundations of geometrical truth. Some writers went so far as to say that definitions were, or ought to be, the sole foundation of geometry — an extreme nominalistic position. But if definitions are allowed to take such a position, one postulate, at most, suffices, without any axiom; and all the rest of geometry can be thrown into a single definition. Namely, it is only necessary to postulate, say, that a point is possible, and to define a point in such a way as to make it cover the whole of geometry. This was not seen; and the practice of throwing geometrical truth over into definitions so far prevailed as to aid in restricting postulates to particular propositions. That such assumptions of possibility had a markedly different logical function from assumptions of impossibility was sufficiently clear to Wolff and the earlier writers whom he followed to cause him to put forth his definitions of *axiom* and *postulate;* and they recommended themselves all the more, because the postulates had become so familiar that it was no longer recognized that they were open to doubt.

634. (3) Kant calls his principles of modality "postulates of empirical thought" in the sense of judgments which are objectively analytical but subjectively synthetical. In fact, the principles as stated by him are not synthetical in any sense whatever, but are mere definitions.

§7. PRESUPPOSITION*

635. Presupposition is either a conjecture or what is better called in English a *Postulate.* (q. v.)

As a philosophical term it translates the German *Voraussetzung*, and is presumably preferred to "postulate" by Germans and others imperfectly acquainted with the English language, because they suppose that postulate in English has the same

* *Dictionary of Philosophy and Psychology*, vol. 2, p. 338.

meaning as *Postulat* in German, which is not true; for the English retains the old meaning, while the German has generally adopted the conception of Wolff. If postulate does not exactly translate German *Voraussetzung*, it comes, at any rate, quite as near to doing so as presupposition; a good translation would be "assumption."

§8. RELATIVES*

636. If from any proposition having more than one subject (used to include "objects") we strike out the indices of the subjects, as in "——— praises ——— to ———," "——— dat in matrimonium ———," what remains and requires at least two insertions of subject-nouns to make a proposition is a "relative term," or "*relative rhema,*" called briefly a "relative." The relative may be converted into a complete assertion by filling up the blanks with proper names or abstract nouns; this serves as a criterion.

But in such a relative there must be such an idea of the difference between the subjects to be applied that "dat in matrimonium" shall be different from "datur in matrimonium." In order to free ourselves from the accidents of speech, we might represent the sentence by the following diagram:

Cinna — | dat in matrimonium | — Cossutiam
Caesari

or, as follows:

d_{ijk} (Cinna $=i$, Cossutia $=j$, Caesar $=k$).

Then the relative will appear as

— | dat in matrimonium | —

or as: d_{ijk}.

But in either case, in order to explain what is meant, it will be necessary to explain how those three tails, or the three letters

* *Ibid.*, vol. 2, pp. 447–50.

i, j, k, differ. The order shows which of three indices is given, which giver, which recipient.

637. Relatives may be more or less general like other terms, that is, one relative may be predicable of members of a set of which another is not, while the latter is predicable only of members of sets of which the former is predicable. By a set is meant an ordered system, so that *ABC* and *BCA*, though the same collection, are different sets. As any general term is predicable of any one of an aggregate of individuals, so a relative is predicable of any one of an aggregate of sets; and each such set may be regarded as an individual relative. By a system is meant an individual of which if anything is true, the truth of it consists in certain things being true of certain other individuals, called its *members*, regardless of the system. A *system* is either a *sorite*, *heap*, or *mere collection*, or it is a *set*. A *sorite* is a system of which, if anything is true, its truth consists of the truth of one predicate for any one of the members. A *set* is a system of which the truth of anything consists in the truth of different predicates. Of course the idea of relation is involved in the idea of a system. As it is very important for the understanding of relations that the conception of a system should be perfectly clear, let us consider the latter a moment in its simplest form, that of a *sorite* or mere collection. *ABC* is a sorite. Thus, it is true of it that it contains the three first letters of the alphabet, and the truth of that consists in *A*, *B*, and *C* being each one of the first three letters of the alphabet. It is true that it contains nothing but the first letters of the alphabet, because it is true of *A*, *B*, *C* severally that each is nothing but one of the first three letters of the alphabet. *AB* is a different sorite, because something is true of it which is not true of *ABC*. *A* may be regarded as a sorite provided we mean not *A* in its first intention and being, but a something whose being consists in *A*'s being. The collection *A* is not the letter *A*, but it contains *A* and nothing else. If it be said that there is no such thing, the reply is that every collection, every system may be said to be an *ens rationis*. To this point we shall return. Even Nothing may be said to be a collection. For when we say that Nothing is less than 1, we do not mean that a self-subsisting individual is so, but that an *ens rationis* whose mode of being consists in the absence of everything is less than 1.

The sorite *ABC* is other than *AB*Γ. But should I say that *ABC* contains two of the letters of Caesar's first name, and subsequently learn that that was a mistake, the real name being Gaius, that would not make *ABC* a different sorite.

638. That in the reality which corresponds to a proposition with a relative predicate is called the *fundamentum relationis*. A *relationship* is a system of such fundamenta. *Relation* is the relative character, conceived as belonging in different ways to the different relates, and (owing to the somewhat undue prominence given by familiar languages to one of these) especially to the relate which is denoted by the noun which is the subject nominative.

639. Relatives and relations are said to differ in their *orders*, according to the numbers of their relates. *Dyadic* or *dual* relations, or relatives of two relates, of which the second is called the *correlate*, differ somewhat widely from *plural*, or *polyadic*, relations. *Triadic* relations have all the principal characters of *tetradic* and higher relations. In fact, a compound of two triadic relatives may be a tetradic relative; as "praiser of —— to a maligner of —— to ——."

640. Relatives may be compounded in all the ways in which other terms can be compounded as well as in other ways closely related to those. Thus, *A* may be said to be at once a lover and a servant of *B*, and it may be said that there is something, *X*, such that *A* is a lover of *X*, while *X* is a servant of *B*; so that *A* is a lover of a servant of *B*. This mode of composition is called *relative multiplication*. So, not only may it be said that *A* is either a lover or a servant of *B* (not excluding both), but also that whatever *X* may be, either *A* is a lover of *X* or *X* is a servant of *B*; that is, *A* is a lover of everything there is besides servants of *B*. (This wording, by Schröder, slightly violates English idiom, but is valuable as showing the analogy to aggregation.) This mode of composition is called *relative addition*. So, again, it may not only be said that *A* is if a lover then a servant of *B*, but also that whatever *X* may be, if *A* is a lover of *X*, then *X* is a servant of *B*; that is, *A* is a lover only of servants of *B*. This is called *relative regressive involution*. Or it may be said that whatever *X* may be, *A* is a lover of *X*, if *X* is a servant of *B*, or *A* is a lover of whatever is a servant of *B*. This is called *relative progressive involution*. Polyadic relatives

are capable of other modes of composition. Thus, it may be said that anything whatever, X, being taken, something Y exists, such that A praises X to Y while X maligns Y to B; that is, A praises everybody to somebody maligned by him to B. Or we can say that there is something, Y, such that, whatever X may be, A praises X to Y while X maligns Y to B; or, A praises everybody to somebody whom everybody maligns to B.

641. Deductive logic can really not be understood without the study of the logic of relatives, which corrects innumerable serious errors into which not merely logicians, but people who never opened a logic-book, fall from confining their attention to non-relative logic. One such error is that demonstrative reasoning is something altogether unlike observation. But the intricate forms of inference of relative logic call for such studied scrutiny of the representations of the facts, which representations are of an *iconic* kind, in that they represent relations in the fact by analogous relations in the representation, that we cannot fail to remark that it is by *observation* of diagrams that the reasoning proceeds in such cases. We successively simplify them and are always able to remark that such observation is required, and that it is even thus, and not otherwise, that the conclusion of a simple syllogism is seen to follow from its premisses. Again, non-relative logic has given logicians the idea that deductive inference was a following out of a rigid rule, so that machines have been constructed to draw conclusions. But this conception is not borne out by relative logic. People commonly talk of *the* conclusion from a pair of premisses, as if there were but one inference to be drawn. But relative logic shows that from any proposition whatever, without a second, an endless series of necessary consequences can be deduced; and it very frequently happens that a number of distinct lines of inference may be taken, none leading into another. That this must be the case is indeed evident without going into the logic of relatives, from the vast multitude of theorems deducible from the few incomplex premisses of the theory of numbers. But ordinary logic has nothing but barren sorites to explain how this can be. Since Kant, especially, it has been customary to say that deduction only elicits what was implicitly thought in the premisses; and the famous distinction of analytical and

synthetical judgments is based upon that notion. But the logic of relatives shows that this is not the case in any other sense than one which reduces it to an empty form of words. Matter entirely foreign to the premisses may appear in the conclusion. Moreover, so far is it from being true, as Kant would have it, that all reasoning is reasoning in *Barbara*, that that inference itself is discovered by the microscope of relatives to be resolvable into more than half a dozen distinct steps. In minor points the doctrines of ordinary logic are so constantly modified or reversed that it is no exaggeration to say that deductive logic is completely metamorphosed by the study of relatives.

642. One branch of deductive logic, of which from the nature of things ordinary logic could give no satisfactory account, relates to the vitally important matter of abstraction. Indeed, the student of ordinary logic naturally regards abstraction, or the passage from "the rose smells sweet" to "the rose has perfume," to be a quasi-grammatical matter, calling for little or no notice from the logician. The fact is, however, that almost every great step in mathematical reasoning derives its importance from the fact that it involves an abstraction. For by means of abstraction the transitory elements of thought, the ἔπεα πτερόεντα, are made substantive elements, as James terms them, ἔπεα ἀπτερόεντα.* It thus becomes possible to study their relations and to apply to these relations discoveries already made respecting analogous relations. In this way, for example, operations become themselves the subjects of operations.

To take a most elementary example — from the idea of a particle moving, we pass to the idea of a particle describing a line. This line is then thought as moving, and so as generating a surface; and so the relations of surfaces become the subject of thought. An abstraction is an *ens rationis* whose being consists in the truth of an ordinary predication. A *collection*, or *system*, is an abstraction or abstract *ens;* and thus the whole doctrine of number is founded on the operation of abstraction. If we conceive an object to be a collective whole, but to be so in such a way that it has no part which is not itself a collective whole in the same way, then, if the collection is of the nature of a sorite, it is a *general*, whose parts are distinguished merely as

* *Principles of Psychology*, I, 243.

having additional characters; but if the collection is a *set*, whose members have other relations to one another, it is a *continuum*. The logic of continua is the most important branch of the logic of relatives, and mathematics, especially geometrical topic, or topical geometry, has its developement retarded from the lack of a developed logic of continua.

643. *Literature:* relatives have, since Aristotle, been a recognized topic of logic. The first germ of the modern doctrine appears in a somewhat trivial remark of *Robert Leslie Ellis. De Morgan* did the first systematic work in his fourth memoir on the syllogism in 1860 (*Cambridge Philosophical Transactions*, x. 331–358); he here sketched out the theory of dyadic relations. C. S. Peirce, in 1870,† extended Boole's algebra so as to apply to them, and after many attempts produced a good general algebra of logic, together with another algebra specially adapted to dyadic relations (*Studies in Logic*, by members of the Johns Hopkins University, 1883, Note B, 187–203).‡ Schröder developed the last in a systematic manner (which brought out its glaring defect of involving hundreds of merely formal theorems without any significance, and some of them quite difficult) in the third volume of his *Exakte Logik* (1895). Schröder's work contains much else of great value. . . .

§9. TRANSPOSITION§

644. Transposition consists in transferring a term from the subject to the predicate, or the reverse, with no change in the character of the connection; as, *No artists who are bankers are clever, No artists are clever bankers, No bankers are clever artists, None are at once artists and bankers and clever;* or as *All but a is b, All but b is a.* Any proposition may be "transformed" into other exactly equivalent forms: *e.g.* the transformation may consist in the change from one sort of connection to another (change of copula, in the extended meaning of that term), as — to take a compound proposition as an example — *It never rains but it pours = always either it pours or it does not rain*, but this is not transposition.

† See Paper No. III.

‡ See Paper No. XII.

§ *Dictionary of Philosophy and Psychology*, vol. 2, p. 713, by Christine Ladd-Franklin and Peirce.

645. Certain copulas permit transposition simply, with no variation in the quality of the term transposed (as in the instances just given); but with the non-symmetrical copulas there must be a change from positive to negative or the reverse (and, if the proposition is complex, from the conjunctive to the alternative combination and the reverse), if the change can be made at all: He who is an astronomer *and* un-devout is mad = Any astronomer is mad *or* devout = All are mad *or* devout *or* not astronomers. When both the whole subject and the whole predicate is transposed the change is commonly called contraposition if the copula is non-symmetrical (*All a is b* = *All non-b is non-a; None but a is b* = *None but non-b is non-a*), but simple conversion if it is symmetrical (*No a is b* = *No b is a*, *Some a is b* = *Some b is a*). The usual discussion in the logics of the doctrine of the equivalence of propositions is greatly simplified by taking this more general view of the subject.

APPENDIX

ON NONIONS[E]*

646. Readers of Professor Sylvester's communication entitled *Erratum* in the last number of these Circulars have perhaps inferred that my conduct in the matter there referred to had been in fault. Professor Sylvester's *Erratum* relates to his "Word upon Nonions," printed in the *Johns Hopkins University Circulars No.* 17, *p.* 242. In that article appears this sentence: "These forms [*i.e.* a certain group of nine Forms belonging to the algebra of Nonions] can be derived from an algebra given by Mr. Charles S. Peirce, (*Logic of Relatives*, 1870)." The object of Professor Sylvester's "*Erratum*" would seem to be to say that this sentence was inserted by me in his proof-sheet without his knowledge or authority on the occasion of the proof being submitted to me to supply a reference, and to repudiate the sentence, because he "knows nothing whatever" of the fact stated. But I think this view of Professor Sylvester's meaning is refuted by simply citing the following testimony of Professor Sylvester himself, printed in the *Johns Hopkins University Circulars*, *No.* 15, *p.* 203.

"Mr. Sylvester mentioned . . . that . . . he had come upon a system of Nonions, the exact analogues of the Hamiltonian Quaternions . . . Mr. Charles S. Peirce, it should be stated, had to the certain knowledge of Mr. Sylvester arrived at the same result many years ago in connection with his theory of the *logic of relatives;* but whether the result had been published by Mr. Peirce, he was unable to say."

This being so, I think that on the occasion of Professor Sylvester's publishing these forms I was entitled to some mention, if I had already published them, and *a fortiori* if I had not. When the proof-sheet was put into my hands, the request made to me, by an oral message, was not simply to supply a reference but to correct a statement relating to my work in the body of the text. And I had no reason to suppose that having thus submitted his text to me, Professor Sylvester would omit

* *Johns Hopkins University Circulars*, No. 22, pp. 86–88, April, 1883, entitled "A Communication from Mr. Peirce."

to look at his proof-sheet after it left my hands to see whether or not he approved of such alteration as I might have proposed. At any rate, when from these causes Professor Sylvester's "Word upon Nonions" had been published with the above statement concerning me, would it have been too much to expect that he should take the trouble to refer to my memoir in order to see whether the statement was not after all true, before publicly protesting against it?

647. I will now explain what the system of Nonions consists in and how I have been concerned with it.

The calculus of Quaternions, one of the greatest of all mathematical discoveries, is a certain system of algebra applied to geometry. A quaternion is a four-dimensional quantity; that is to say, its value cannot be precisely expressed without the use of a set of four numbers. It is much as if a geographical position should be expressed by a single algebraical letter; the value of this letter could only be defined by the use of two numbers, say the Latitude and Longitude. There are various ways in which a quaternion may be conceived to be measured and various different sets of four numbers by which its value may be defined. Of all these modes, Hamilton, the author of the algebra, selected one as the standard. Namely, he conceived the general quaternion q to be put into the form

$$q=xi+yj+zk+w,$$

where x, y, z, w, are four ordinary numbers, while i, j, k, are peculiar units, subject to singular laws of multiplication. For $ij=-ji$, the order of the factors being material, as shown in this multiplication table, where the first factor is entered at the side, the second at the top, and the product is found in the body of the table.

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

As long as x, y, z, and w in Hamilton's standard tetranomial form are confined to being *real* numbers, as he usually supposed them to be, no simpler or more advantageous form of conceiving the measurement of a quaternion can be found. But my father, Benjamin Peirce,* made the profound, original, and pregnant discovery that when x, y, z, w are permitted to be imaginaries, there is another very different and preferable system of measurement of a quaternion. Namely, he showed that the general quaternion, q, can be put into the form

$$q=xi+yj+zk+wl,$$

where x, y, z, w, are real or imaginary numbers, while i, j, k, l are peculiar units whose multiplication obeys this table.

	i	j	k	l
i	i	j	0	0
j	0	0	i	j
k	k	l	0	0
l	0	0	k	l

A quaternion does not cease to be a quaternion by being measured upon one system rather than another. Any quantity belonging to the algebra is a quaternion; the algebra itself is "quaternions." The usual formulæ of the calculus have no reference to any tetranomial form, and such a form might even be dispensed with altogether.

While my father was making his investigations in multiple algebra I was, in my humble way, studying the logic of relatives and an algebraic notation for it; and in the ninth volume of the *Memoirs of the American Academy of Arts and Sciences*,† appeared my first paper upon the subject. In this memoir, I was led, from logical considerations that are patent to those who read it, to endeavor to put the general expression of any

* In his *Linear Associative Algebras*, 1870.

† Paper No. III.

linear associative algebra into a certain form; namely as a linear expression in certain units which I wrote thus:

$$\begin{array}{lll} (u_1\!:\!u_1) & (u_1\!:\!u_2) & (u_1\!:\!u_3), \text{ etc.,} \\ (u_2\!:\!u_1) & (u_2\!:\!u_2) & (u_2\!:\!u_3), \text{ etc.,} \\ (u_3\!:\!u_1) & (u_3\!:\!u_2) & (u_3\!:\!u_3), \text{ etc.,} \\ \text{etc.} & \text{etc.} & \text{etc.} \end{array}$$

These forms, in their multiplication, follow these rules:

$$(u_a\!:\!u_b)\ (u_b\!:\!u_c) = (u_a\!:\!u_c) \qquad (u_a\!:\!u_b)\ (u_c\!:\!u_d) = 0.$$

I said, "I can assert, upon reasonable inductive evidence, that all such algebras can be interpreted on the principles of the present notation in the same way,"* and consequently can be put into this form. I afterwards published a proof of this.† I added that this amounted to saying that "all such algebras are complications and modifications of the . . . Hamilton's quaternions."‡ What I meant by this appears plainly in the memoir. It is that any algebra that can be put into the form proposed by me is thereby referred to an algebra of a certain class (afterwards named *quadrates* by Professor Clifford) which present so close an analogy with quaternions that they may all be considered as mere complications of that algebra. Of these algebras, I gave as an example, the multiplication table of that one which Professor Clifford afterward named *nonions*.[1] This is the passage:§

It will be seen that the system of nonions is not a group but an algebra; that just as the word "quaternion" is not restricted to the three perpendicular vectors and unity, so a nonion is any quantity of this nine-fold algebra.

So much was published by me in 1870; and it then occurred either to my father or to me (probably in conversing together) that since this algebra was thus shown (through his form of quaternions) to be the strict analogue of quaternions, there ought to be a form of it analogous to Hamilton's standard tetranomial form of quaternions. That form, either he or I

* See 130.

† See 150–51.

‡ See 130.

[1] It would have been more accurately analogical, perhaps, to call it *novenions*.

§ 127, beginning with "for example," was here reproduced.

certainly found. I cannot remember, after so many years, which first looked for it; whichever did must have found it at once. I cannot tell what his method of search would have been, but I can show what my own must have been. The following course of reasoning was so obtrusive that I could not have missed it.

Hamilton's form of quaternions presents a group of four square-roots of unity. Are there, then, in nonions, nine independent cube-roots of unity, forming a group? Now, unity upon my system of notation was written thus:

$$(u_1 : u_1) + (u_2 : u_2) + (u_3 : u_3).$$

Two independent cube-roots of this suggest themselves at once; they are

$$(u_1 : u_2) + (u_2 : u_3) + (u_3 : u_1)$$
$$(u_3 : u_2) + (u_2 : u_1) + (u_1 : u_3).$$

In fact these are hinted at in my memoir, p. 53.[?] Then, it must have immediately occurred to me, from the most familiar properties of the imaginary roots of unity, that instead of the coefficients

$$1, \quad 1, \quad 1,$$

I might substitute

$$1, \quad g, \quad g^2,$$

or

$$1, \quad g^2, \quad g,$$

where g is an imaginary cube-root of unity. The nine cube-roots of unity thus obtained are obviously independent and obviously form a group. Thus the problem is solved by a method applicable to any other quadrate.

648. My father, with his strong partiality for my performances, talked a good deal about the algebra of nonions in general and these forms in particular; and they became rather widely known as mine. Yet it is clear that the only real merit in the discovery lay in my father's transformation of quaternions. In 1875, when I was in Germany, my father wrote to me that he was going to print a miscellaneous paper on multiple algebra and he wished to have it accompanied by a paper by me, giving an account of what I had found out. I wrote such a paper, and

sent it to him; but somehow all but the first few pages of the manuscript were lost, a circumstance I never discovered till I saw the part that had reached him (and which he took for the whole) in print. I did not afterward publish the matter, because I did not attach much importance to it, and because I thought that too much had been made, already, of the very simple things I had done.

I here close the narrative. The priority of publication of the particular group referred to belongs to Professor Sylvester. But most readers will agree that he could not have desired to print it without making any allusion to my work, and that to say the group could be derived from my algebra was not too much.

INDEX OF PROPER NAMES*

*The numbers refer to paragraphs, not to pages

INDEX OF SUBJECTS*

*The numbers refer to paragraphs, not to pages.

VOLUME IV

THE SIMPLEST MATHEMATICS

Errata

4.*Preface* n* *for* 6–10 *read* 6–11
4.19 (p. 16, line 16) *for* ⋎ *read* –<
4.20n†, line 12 for *A*, *AA*; *A*, *AA* read *AA*, *A*; *AA*, *A*
p. 26, page heading *for* 43.7] *read* 4.37]
4.81n† *for* C. *read* Cf.
4.85n* *delete* see vol. 9, letters to Judge Russell
4.96, line 2 *for* intransitive *read* transitive
*4.214, line 3 for 2^{2^n} read n
4.214, line 8 *for the second* denumberable *read* primipostnumeral
*4.223, line 7 *for* three *read* two
4.230n† for *Isogogen sine* v *voces* read *Isagoge sive quinque voces*
4.264, line 4 for last $\bar{\bar{x}}$ read $\bar{\bar{\bar{x}}}$
4.264n‖ for $x\colon \vee :^{x}$ read $\bar{x} \vee x$
4.265n‡ (p. 217, note line 1) *add . at end of line*
4.295, line 19 *delete* and
4.345 (p. 285, line 17) *for* 1 *read* |
4.357, Fig. 17 should read 
4.357 (p. 307, line 16) *for* make *read* mark
4.457, Fig. 105 figure should be turned around
4.513n* *for* to be published in vol. 7 *read* "A quincuncial Projection of the Sphere" (1879)
4.536n* (p. 423) *for* vol. 9 *read* 8.327–379
4.569n† (p. 458) *for* vol. 9 *read* 8.380f
4.668n* (p. 566) delete note
4.677 (p. 576, line 7) *for* possible *read* possible [to say]
p. 581: Index s.n. Abbott *for* 150 *read* 50
for Abbott *read* Abbot
p. 582: Index s.n. Sylvester *for* 304 *read* 305
p. 584: Index s.n. Achilles, The *add* 677
p. 597: Index s.v. Reasoning, corollarial *insert* 613

* This emendation departs from Peirce's own text, but it seems required for the reasoning.

INTRODUCTION

Peirce's punctuation and spelling have, wherever possible, been retained. Titles supplied by the editors for papers previously published are marked with an ^{E}, while Peirce's titles for unpublished papers are marked with a ^{P}. Peirce's titles for previously published papers and the editors' titles for unpublished papers are not marked. Remarks and additions by the editors are enclosed in light-face square brackets. The editors' footnotes are indicated by various typographical signs, while Peirce's are indicated by numbers. Paragraphs are numbered consecutively throughout the volume. At the top of each page the numbers signify the volume and the first paragraph of that page. All references in the indices are to the numbers of the paragraphs.

The department and the editors desire to express their gratitude to Dr. Henry S. Leonard and Miss Isabel Stearns for their assistance with the proofs, references and editorial footnotes.

HARVARD UNIVERSITY
September, 1933

EDITORIAL NOTE

In addition to his published papers on exact logic and the foundations of mathematics (see vol. III), Peirce wrote a great number of others which he seems never to have given to publishers, although many of them are advances on his published work. A selected number of these papers, arranged in chronological order, form the first book of the present volume.

The logical graphs, which are presented in the second book, are, in Peirce's opinion, his greatest contribution to logic. They are considered by him to provide a more detailed and satisfactory analysis of the structure of logical argumentation than is permitted by previous symbolisms or the traditional Eulerian diagrams. As he treats them they are pertinent to discussions on the nature of mathematics, multiple-valued logics and modality, as well as to his theory of signs (see vol. II, bk. ii) and his pragmaticism (see vol. V, bk. ii). The papers in this book, particularly those which formulate the rules for the manipulation of the graphs, often overlap, since almost all his writings on this subject were different attempts to express the same truths. The repetitions could only have been avoided either by not publishing previously published papers (in violation of the plan of the edition), or by the omission of manuscripts upon which he bestowed great effort, or by the breaking of the thread of his discourse — none of which seemed desirable alternatives.

Peirce's last projected series of papers, serving to illustrate fundamental theorems in logic and mathematics, forms the third book of the volume. The frontispiece reproduces a page from the last article of this series, the last perhaps of his original papers, written when Peirce was about seventy years old.

Papers I, III, IV, VII, VIII of book i and the whole of book ii are recommended to students of exact logic; while papers IV–X of book i and those of book iii will be most useful to students of the foundations of mathematics. The general reader will perhaps find papers II and VII of book i, and chapters 4–6 of book ii to be the clearest and most interesting.

Bryn Mawr College
September, 1933

CONTENTS

CONTENTS

Book I

LOGIC AND MATHEMATICS

(UNPUBLISHED PAPERS)

PREFACE*

1. . . . Now what was the question of realism and nominalism?† I see no objection to defining it as the question of which is the best, the laws or the facts under those laws. It is true that it was not stated in this way. As stated, the question was whether *universals*, such as the Horse, the Ass, the Zebra, and so forth, were *in re* or *in rerum natura*. But that there is no great merit in this formulation of the question is shown by two facts; first, that many different answers were given to it, instead of merely yes and no, and second, that all the disputants divided the question into various parts. It was therefore a broad question and it is proper to look beyond the letter into the spirit of it. Most of those scholastics whose works are occasionally read today were matter-of-fact dualists; and when they used the phrase *in re* or *in rerum natura* in formulating the question, they took for granted something in regard to which other disputants, however confusedly, were at odds with them. For some of them regarded the universals as more real than the individuals. Therefore, the reality, or as I would say in order to avoid any begging of the question, the value or worth, not merely of the universals, but also that of the individuals was a part of the broad question. Finally, it was always agreed that there were other sorts of universals besides *genera* and *species*, and in using the word law, or regularity, we bring into prominence the kind of universals to which modern science pays most attention. Roughly speaking, the nominalists conceived the *general* element of cognition to be merely a convenience for understanding this and that fact and to amount to nothing except for cognition, while the realists, still more roughly speaking, looked upon the general, not only as the end and aim of knowledge, but also as the most important element of being. Such was and is the question. It is as pressing today

* 1–5 are from the second lecture on "Detached Ideas on Vitally Important Topics" of 1898. For the first lecture see vol. 1, bk. III, ch. 5. 6–10 are from "Phaneroscopy, $\phi\alpha\nu$ 1906."

† Cf. 1.15ff.

as ever it was, Ernst Mach,* for example, holding that generality is a mere device for economising labor while Hegeler,† though he extols Mach to the skies, thinks he has said that man is immortal when he has only said that his influence survives him.

According to the nominalistic view, the only value which an idea has is to represent the fact, and therefore the only respect in which a system of ideas has more value than the sum of the values of the ideas of which it is composed is that it is compendious; while, according to the realistic view, this is more or less incorrect depending upon how far the realism be pushed.

Dr. [F. E.] Abbot in his *Scientific Theism* [1885] has so clearly and with such admirable simplicity shown that modern science is realistic that it is perhaps injudicious for me to attempt to add anything upon the subject. Yet I shall try to put it into such a light that it may reflect some rays upon the worth or worthlessness of detached ideas. But I warn you that I shall not argue the question, but only indicate what my line of reasoning would be were I to enter upon it in detail.

The burden of proof is undoubtedly upon the realists, because the nominalistic hypothesis is the simpler. Dr. Carus‡ professes himself a realist and yet accuses me of inconsistency in admitting Ockham's razor although I am a realist, thus, implying that he himself does not accept it.§ But this brocard, *Entia non sunt multiplicanda praeter necessitatem*, that is, a hypothesis ought not to introduce complications not requisite to explain the facts, this is not distinctively nominalistic; it is the very roadbed of science. Science ought to try the simplest hypothesis first, with little regard to its probability or improbability, although regard ought to be paid to its consonance with other hypotheses, already accepted. This, like all the logical propositions I shall enunciate, is not a mere private impression of mine: it is a mathematically necessary deduction from unim-

* E.g., in "The Economical Nature of Physical Inquiry," *Popular Scientific Lectures*, Chicago (1894).

† E. C. Hegeler (1835–1910), founder of the *Open Court Monthly* and *The Monist.*

‡ Paul Carus (1852–1919), editor of the *Open Court Monthly* and *The Monist.*

§ See "The Founder of Tychism, his Methods and Criticisms," I, 4, *The Monist*, vol. III (1893).

peachable generalizations of universally admitted facts of observation. The generalizations are themselves allowed by all the world; but still they have been submitted to the minutest criticism before being employed as premisses. It appears therefore that in scientific method the nominalists are entirely right. Everybody ought to be a nominalist at first, and to continue in that opinion until he is driven out of it by the *force majeure* of irreconcilable facts. Still he ought to be all the time on the lookout for these facts, considering how many other powerful minds have found themselves compelled to come over to realism.

Nor has the wealth of thought that has been expended upon the defenses of nominalism especially by four great English philosophers who have engineered the works, I mean Ockham, Hobbes, Berkeley, and James Mill, by any means been wasted. It has on the contrary been most precious for the clear comprehension of logic and of metaphysics. But as for the average nominalist whom you meet in the streets, he reminds me of the blind spot on the retina, so wonderfully does he unconsciously smooth over his field of vision and omit facts that stare him in the face, while seeing all round them without perceiving any gap in his view of the world. That any man not demented should be a realist is something he cannot conceive.

My plan for defeating nominalism is not simple nor direct; but it seems to me sure to be decisive, and to afford no difficulties except the mathematical toil that it requires. For as soon as you have once mounted the vantage-ground of the logic of relatives, which is related to ordinary logic precisely as the geometry of three dimensions is to the geometry of points on a line, as soon as you have scaled this height, I say, you find that you command the whole citadel of nominalism, which must thereupon fall almost without another blow.

I am going to describe in general terms what this logic of relatives is, so far as it bears upon this great controversy. And in doing so I can at the same time, without lengthening the lecture by more than three or four minutes, make my account of this generalized logic illustrate some of [the] relative advantages and disadvantages of detached ideas and of systematic thought, by simply forming it into a narrative of how I myself became acquainted with that logic.

2. I came to the study of philosophy not for its teaching about God, Freedom, and Immortality, but intensely curious about Cosmology and Psychology. In the early sixties I was a passionate devotee of Kant, at least as regarded the Transcendental Analytic in the *Critic of the Pure Reason.* I believed more implicitly in the two tables of the Functions of Judgment and the Categories than if they had been brought down from Sinai. Hegel, so far as I knew him through a book by Vera* repelled me. Now Kant points out certain relations between the categories. I detected others; but these others, if they had any orderly relation to a system of conceptions, at all, belonged to a larger system than that of Kant's list. Here there was a problem to which I devoted three hours a day for two years, rising from it, at length, with the demonstrative certitude that there was something wrong about Kant's formal logic. Accordingly, I read every book I could lay hands upon on logic, and of course Kant's essay on the *falsche Spitzfindigkeit der vier syllogistischen Figuren;*† and here I detected a fallacy similar to that of the phlogistic chemists. For Kant argues that the fact that all syllogisms can be reduced to Barbara shows that they involve no logical principle that Barbara does not involve. A chemist might as well argue, that because water boiled with zinc dust evolves hydrogen, and the hydrogen does not come from the zinc, therefore water is a mere form of hydrogen. In short, Kant omits to inquire whether the very reasoning by which he reduces the indirect moods to Barbara may not itself introduce an additional logical principle. Pursuing this suggestion, I found that that was in truth the case, and I succeeded [in 1866] in demonstrating that the second and third figures each involved a special additional logical principle, both of which enter into the fourth figure.‡ Namely, the additional principle of the second figure is that by which we pass from judging that among dumb brutes no animal with a hand can be found, to judging that among animals with hands no dumb brute can be found; and the additional principle of the third figure is that by which we pass from judging that among human beings there are females to that of judging that among female

* Augusto Vera, *Introduction à la philosophie de Hegel*, Paris (1855).

† *Die falsche Spitzfindigkeit der vier syllogistischen Figuren erwiesen* (1762).

‡ See 2.485ff; 2.801ff.

animals there are human beings. Although I do not stop to give the proof, I assert that it is rigidly demonstrated that these are distinct principles of logic. Thus to find that the passage from one way of viewing a fact to another way of viewing the same fact should be a logical principle was naturally food for reflection. I remarked that while the circumstances under which propositions of the form No A is B and No B is A are true are identical, yet the circumstances under which such a pair of propositions indefinitely approximate to being true do not by any means indefinitely approximate toward being identical. For instance, the probability that a man taken at random will be a poet as great as Dante may be indefinitely near to zero; but it does not follow that the probability that a poet as great as Dante will be a man approximates to zero, at all. This reflection led me to inquire whether there might not be forms of probable reasoning analogous to the second and third figures of syllogism which were widely different from one another and from the first figure. Here, Aristotle's account of induction aided me; for Aristotle* makes induction to be a probable syllogism in the third figure.

3. I found that there was also a mode of probable reasoning in the second figure essentially different both from induction and from probable deduction.† This was plainly what is called reasoning from consequent to antecedent, and in many books is called adopting a hypothesis for the sake of the explanation it affords of known facts. It would be tedious to show how this discovery led to the thorough refutation of the third and most important of Kant's triads, and the confirmation of the doctrine that for the purposes of ordinary syllogism categorical propositions and conditional propositions, which Kant and his ignorant adherents call hypotheticals, are all one.‡ This led me to see that the relation between subject and predicate, or antecedent and consequent, is essentially the same as that between premiss and conclusion.§ It was interesting to see how the combined result of all these improvements and some others to which I have not alluded was decidedly to consolidate

* Cf., e.g., *Prior Analytic* II, 23.

† See 2.509; 2.706f.

‡ See, e.g., 2.345ff, 2.710.

§ See, e.g., 3.175.

that systematic or synthetic unity in the system of formal logic which occupied so large a place in Kant's thought. But though there was more unity than in Kant's system, still, as the subject stood, there was not as much as might be desired. Why should there be three principles of reasoning, and what have they to do with one another? This question, which was connected with other parts of my schedule of philosophical inquiry that need not be detailed, now came to the front. Even without Kant's categories, the recurrence of triads in logic was quite marked, and must be the croppings out of some fundamental conceptions. I now undertook to ascertain what the conceptions were. This search resulted in what I call my categories. I then* named them Quality, Relation, and Representation. But I was not then aware that undecomposable relations may necessarily require more subjects than two; for this reason *Reaction* is a better term. Moreover, I did not then know enough about language to see that to attempt to make the word *representation* serve for an idea so much more general than any it habitually carried, was injudicious. The word *mediation* would be better. Quality, reaction, and mediation will do. But for scientific terms, Firstness, Secondness, and Thirdness, are to be preferred as being entirely new words without any false associations whatever. How the conceptions are *named* makes, however, little difference. I will endeavor to convey to you some idea of the conceptions themselves.† It is to be remembered that they are excessively general ideas, so very uncommonly general that it is far from easy to get any but a vague apprehension of their meaning. . . .

4. [With regard to] my logical studies in 1867, various facts proved to me beyond a doubt that my scheme of formal logic was still incomplete. For one thing, I found it quite impossible to represent in syllogisms any course of reasoning in geometry, or even any reasoning in algebra, except in Boole's logical algebra. Moreover, I had found that Boole's algebra required enlargement to enable it to represent the ordinary syllogisms of the third figure; and though I had invented such an enlargement, it was evidently of a makeshift character, and there must be some other method springing out of the

* In 1867, see 1.551ff.

† See vol. 1, bk. III, for an extended discussion of these categories.

idea of the algebra itself. Besides, Boole's algebra suggested strongly its own imperfection. Putting these ideas together I discovered the logic of relatives.* I was not the first discoverer; but I thought I was, and had complemented Boole's algebra so far as to render it adequate to all reasoning about dyadic relations, before Professor De Morgan sent me his epoch-making memoir† in which he attacked the logic of relatives by another method in harmony with his own logical system. But the immense superiority of the Boolian method was apparent enough, and I shall never forget all there was of manliness and pathos in De Morgan's face when I pointed it out to him in 1870. I wondered whether when I was in my last days some young man would come and point out to me how much of my work must be superseded, and whether I should be able to take it with the same genuine candor. . . .‡

5. The great difference between the logic of relatives and ordinary logic is that the former regards the form of relation in all its generality and in its different possible species while the latter is tied down to the matter of the single special relation of similarity. The result is that every doctrine and conception of logic is wonderfully generalized, enriched, beautified, and completed in the logic of relatives.

Thus, the ordinary logic has a great deal to say about *genera* and *species*, or in our nineteenth century dialect, about *classes*. Now, a *class* is a set of objects comprising all that stand to one another in a special relation of similarity. But where ordinary logic talks of classes the logic of relatives talks of *systems*. A *system* is a set of objects comprising all that stand to one another in a group of connected relations. Induction according to ordinary logic rises from the contemplation of a sample of a class to that of the whole class; but according to the logic of relatives it rises from the contemplation of a fragment of a system to the envisagement of the complete system.

6. It is requisite that the reader should fully understand the relation of thought in itself to *thinking*, on the one hand,

* The steps of this development are clearly manifest in the early papers of vol. 3.

† "On the Syllogism IV, and on the Logic of Relations," *Cambridge Philosophical Transactions*, vol. 10, pp. 331–58.

‡ Peirce here gives a number of elementary graphs to illustrate the logic of relatives. The papers in book II of this volume cover the same ground.

and to graphs, on the other hand. Those relations being once magisterially grasped, it will be seen that the graphs break to pieces all the really serious barriers, not only to the logical analysis of thought, but also to the digestion of a different lesson, by rendering literally visible before one's very eyes the operation of thinking *in actu*. In order that the fact should come to light that the method of graphs really accomplishes this marvelous result, it is first of all needful, or at least highly desirable, that the reader should have thoroughly assimilated, in all its parts, the truth that thinking always proceeds in the form of a dialogue — a dialogue between different phases of the *ego* — so that, being dialogical, it is essentially composed of signs, as its matter, in the sense in which a game of chess has the chessmen for its matter. Not that the particular signs employed *are* themselves the thought! Oh, no; no whit more than the skins of an onion are the onion. (About as much so, however.) One selfsame thought may be carried upon the vehicle of English, German, Greek, or Gaelic; in diagrams, or in equations, or in graphs: all these are but so many skins of the onion, its inessential accidents. Yet that the thought should have *some* possible expression for some possible interpreter, is the very being of its being. . . .

7. How many writers of our generation (if I must call names, in order to direct the reader to further acquaintance with a generally described character — let it be in this case the distinguished Husserl*), after underscored protestations that their discourse shall be of logic exclusively and not by any means of psychology (almost all logicians protest that on file), forthwith become intent upon those elements of the process of thinking which seem to be special to a mind like that of the human race, *as we find it*, to too great neglect of those elements which must belong, as much to any one as to any other mode of embodying the same thought. It is one of the chief advantages of Existential Graphs, as a guide to Pragmaticism, that it holds up thought to our contemplation with the wrong side out, as it were; showing its construction in the barest and plainest manner, so that it [does not] seduce us into the bye-path of the distinctively English logicians (whether in that branch of it where the way is strewn, often in the most valuably suggestive works, such as Venn's *Empirical Logic*, with puerilities

*See, e.g., his *Logische Untersuchung*, Teil I, Kap. 3 (1900).

about words — and often not merely strewn with them but buried so deep in them, as by a great snowstorm, as to obstruct the reader's passage and render it fatiguing in the extreme, while the books of lesser inquirers, say Carveth Read,* Horace William Brindley Joseph,† and the last edition (greatly inferior to the first) of John Neville Keynes' *Formal Logic*, offer little reward for the labour of listening to their irrelevant baby-talk; or whether in the other branch of the same path where, as in the two Logics of Miss Constance Jones,‡ it seems to be forgotten that Latin Grammar does not furnish the only type even of Sud Germanic construction, which is itself a peculiarly specialized form of expression opposed in various particulars to the common ways of thinking of the great majority of mankind).

8. Nor does it lead us into the divarications of those who know no other logic than a "Natural History" of thought. As to this remark, I pray you, that "Natural History" is the term applied to the descriptive sciences of nature, that is to say, to sciences which describe different kinds of objects and classify them as well as they can while they still remain ignorant of their essences and of the ultimate agencies of their production, and which seek to explain the properties of those kinds by means of laws which another branch of science called "Natural Philosophy" has established. Thus a logic which is a natural history merely, has done no more than observe that certain conditions have been found attached to sound thought, but has no means of ascertaining whether the attachment be accidental or essential; and quite ignoring the circumstance that the very essence of thought lies open to our study; which study alone it is that men have always called "logic," or "dialectic."

Accordingly, when I say that Existential Graphs put before us moving pictures of thought, I mean of thought in its essence free from physiological and other accidents. . . .

9. The highest kind of symbol is one which signifies a growth, or self-development, of thought, and it is of that alone that a moving representation is possible; and accordingly, the central problem of logic is to say whether one given thought is truly, *i.e.*, is adapted to be, a development of a given other or

* *Logic, Deductive and Inductive* (1898).

† An Introduction to Logic (1906).

‡ *Elements of Logic as a Science of Propositions* (1890); *An Introduction to General Logic* (1892).

not. In other words, it is the critic of arguments. Accordingly, in my early papers I limited logic to the study of this problem. But since then, I have formed the opinion that the proper sphere of any science in a given stage of development of science is the study of such questions as one social group of men can properly devote their lives to answering*; and it seems to me that in the present state of our knowledge of signs, the whole doctrine of the classification of signs and of what is essential to a given kind of sign, must be studied by one group of investigators. Therefore, I extend logic to embrace all the necessary principles of semeiotic, and I recognize a logic of icons, and a logic of indices, as well as a logic of symbols; and in this last I recognize three divisions: *Stecheotic* (or stoicheiology), which I formerly called Speculative Grammar; *Critic*, which I formerly called Logic; and *Methodeutic*, which I formerly called Speculative Rhetoric.†

10. A fallacy is, for me, a supposititious thinking, a thinking that parades as a self-development of thought but is in fact begotten by some other sire than reason; and this has substantially been the usual view of modern logicians. For reasoning ceases to be Reason when it is no longer reasonable: thinking ceases to be Thought when true thought disowns it. A self-development of Thought takes the course that thinking will take that is sufficiently deliberate, and is not truly a self-development if it slips from being the thought of one object-thought to being the thought of another object-thought. It is, in the geological sense, a "fault" — an inconformability in the strata of thinking. The discussion of it does not appertain to pure logic, but to the application of logic to psychology. I only notice it here, as throwing a light upon what I *do not* mean by "Thought."

11. I trust by this time, Reader, that you are conscious of having some idea, which perhaps is not so dim as it seems to you to be, of what I mean by calling Existential Graphs a moving-picture of Thought. Please note that I have not called it a *perfect* picture. I am aware that it is not so: indeed, that is quite obvious. But I hold that it is considerably more nearly perfect than it seems to be at first glance, and quite sufficiently so to be called a portraiture of Thought.

* Cf. 1.236.

† See 2.93, 2.229.

THE SIMPLEST MATHEMATICS

I

*A BOOLIAN ALGEBRA WITH ONE CONSTANT**

12. Every logical notation hitherto proposed has an unnecessary number of signs. It is by means of this excess that the calculus is rendered easy to use and that a symmetrical development of the subject is rendered possible; at the same time, the number of primary formulæ is thus greatly multiplied, those signifying facts of logic being very few in comparison with those which merely define the notation. I have thought that it might be curious to see the notation in which the number of signs should be reduced to a minimum; and with this view I have constructed the following. The apparatus of the Boolian calculus consists of the signs, $=, >$ (not used by Boole, but necessary to express particular propositions) $+, -, \times, 1, 0$. In place of these seven signs, I propose to use a single one.

13. I begin with the description of the notation for conditional or "secondary" propositions. The different letters signify propositions. Any one proposition written down by itself is considered to be asserted. Thus,

$$A$$

means that the proposition A is true. Two propositions written in a pair are considered to be both denied. Thus,

$$A\ B$$

means that the propositions A and B are both false; and

$$A\ A$$

means that A is false. We may have pairs of pairs of propositions and higher complications. In this case we shall make use of commas, semicolons, colons, periods, and parentheses, just

* Untitled paper c. 1880. Compare H. M. Sheffer's: "A Set of Five Independent Postulates for Boolean Algebras, with application to logical constants," *Transactions, American Mathematical Society*, vol. 14, pp. 481–88 (1913), of which this is a striking anticipation. See also 264f., where the same idea is developed from a different angle.

as [in] chemical notation, to separate pairs which are themselves paired. These punctuation marks can no more count for distinct signs of algebra, than the parentheses of the ordinary notation.

14. To express the proposition: "If *S* then *P*," first write

A

for this proposition. But the proposition is that a certain conceivable state of things is absent from the universe of possibility. Hence instead of *A* we write

B B

Then *B* expresses the possibility of *S* being true and *P* false. Since, therefore, *SS* denies *S*, it follows that (*SS*, *P*) expresses *B*. Hence we write

SS, *P*; *SS*, *P*.*

15. Required to express the two premisses, "If *S* then *M*" and "if *M* then *P*." Let

A

be the two premisses. Let *B* be the denial of the first and *C* that of the second; then in place of *A* we write

B C

But we have just seen that *B* is (*SS*,*M*) and that *C* is (*MM*, *P*); accordingly we write

SS, *M*; *MM*, *P*.

16. All the formulæ of the calculus may be obtained by development or elimination. The development or elimination having reference say to the letter *X*, two processes are required which may be called the erasure of the *Xs* and the erasure of the double *Xs*. The erasure of the *Xs* is performed as follows:

17. Erase all the *Xs* and fill up each blank with whatever it is paired with. But where there is a double *X* this cannot be done; in this case erase the whole pair of which the double *X* forms a part, and fill up the space with whatever it is paired with. Go on following these rules.

A pair of which both members are erased is to be considered as doubly erased. A pair of which either member is doubly

* This can be symbolized as: *S*/*S*†*P*‡*S*/*S*†*P*; where the stroke is the sign of the logical multiplication of the contradictories of the constituents, and the number of cross bars indicates the inverse order of dissolution. Thus, (1)–(*S*/*S*†*P*); (2)–(–(*S*/*S*)–*P*); (3)–(*S*.–*P*).

erased is to be considered as only singly erased, without regard to the condition of the other member. Whatever is singly erased is to be replaced by the repetition of what it is paired with.

To erase the double *Xs*, repeat every *X* and then erase the *Xs*.*

18. If φ be any expression, $\frac{\varphi}{x}$ what it becomes after erasure of the *Xs*, and $\frac{\varphi}{xx}$ what it becomes after erasure of the double *Xs*, then

$$\varphi = \frac{\varphi}{x},\ x;\ \frac{\varphi}{xx},\ xx.$$

If φ be asserted, then

$$\frac{\varphi}{x}\frac{\varphi}{xx},\ \frac{\varphi}{x}\ \frac{\varphi}{xx}$$

may be asserted.

19. The following are examples. Required to develop *X* in terms of *X*. Erasing the *Xs* the whole becomes erased, and

$$\frac{\varphi}{x}x = xx.$$

* These rules may be reformulated as follows:

A. 1. For every single x (i.e., for every x that is not paired with an x) substitute its pair; e.g., $x\dagger a/a$ becomes $a/a\dagger a/a$.

2. In the case of double x consider the whole of which the double x is a part.

a. If that whole is itself paired, substitute its pair for it; e.g., $x/x\dagger a\ddagger a$ becomes a/a.

b. If that whole is not paired, substitute the other member of the whole for each of the xs;

e.g., $x/x\dagger a$ becomes $a/a\dagger a$

$x/x\dagger a/a$ becomes $a/a\dagger a/a\ddagger a/a$

3. Repeat operations as often as necessary.

4. Equate result to $\frac{\varphi}{x}$.

B. 1. Negate all xs; i.e., for every double x employ but one; for every one employ a double.

2. Perform steps 1–3 of *A*.

3. Equate to $\frac{\varphi}{xx}$.

Erasing the double Xs, the whole becomes doubly erased and $\frac{\varphi}{xx}\,xx$ is erased. If φ, then

$$\frac{\varphi}{x}\,x,\ \frac{\varphi}{xx}\,xx = xx,\ xx.$$

So that

$$X = xx,\ xx.$$

Required to eliminate X from $(xx,\ x;\ a)$.

$$\frac{\varphi}{x} = 00,\ 0;\ a = aa$$

$$\frac{\varphi}{xx} = 00,\ 00;\ 00 : a = aa$$

The following are additional illustrations to those in the text.

Let $\varphi = a/a\dagger x$ which
by $A1$ becomes $a/a\dagger a/a$, or a
so that $\frac{\varphi}{x} = a$
by $B1$ we get $a/a\dagger x/x$ which by
$A2b$ becomes $a/a\ddagger a/a\dagger a/a$, or $a/a\dagger a$
so that $\frac{\varphi}{xx} = a/a\dagger a$
and $\varphi \curlyvee a$, x being eliminated.

Let $\varphi = x/x\dagger a$ which
by $A2b$ becomes $a/a\dagger a$
so that $\frac{\varphi}{x} = a/a\dagger a$
by $B1$ we get $x/x\dagger x/x\ddagger a$ from which
by $A2b$ we get a/a
so that $\frac{\varphi}{xx} = a/a$
and $\varphi \mathrel{-\!\!<} a/a$, x being eliminated.

Let $\varphi = a\ddagger a/a\dagger x$ which
by $A1$ becomes $a\ddagger a/a\dagger a/a$, or a/a
so that $\frac{\varphi}{x} = a/a$
by $B1$ we get $a\ddagger a/a\dagger x/x$
and by $A2a$ we get a/a
so that $\frac{\varphi}{xx} = a/a$
and $\varphi = a/a$, x being eliminated

$$\therefore \varphi = {}^* \frac{\varphi}{x} \frac{\varphi}{xx}, \frac{\varphi}{x} \frac{\varphi}{xx} = aa, aa; aa, aa = aa.$$

Required to eliminate X from (xa, a).

$$\frac{\varphi}{x} = 0a, a = aa, a$$

$$\frac{\varphi}{xx} = 00, a; a = aa$$

$$\therefore \varphi = {}^* aa, a; aa : aa, a; aa = aa, aa = a. \dagger$$

Required to develop $(ax; b, xx : ab)$ according to X.

$$\frac{\varphi}{x} = a0; b, 00: ab = aa, aa; ab = a, ab = a\ddagger$$

$$\frac{\varphi}{xx} = a00; b, 0: ab = bb, bb; ab = b, ab = b\ddagger$$

$$\varphi = \frac{\varphi}{x} x, \frac{\varphi}{xx} xx = ax; b, xx\ddagger$$

Required to eliminate M from $(SS, M; MM, P)$.

$$\frac{\varphi}{M} = SS, 0; 00, P = SS, SS; SS, SS = SS$$

$$\frac{\varphi}{MM} = SS, 00; 0, P = P, P; P, P = P$$

$$\therefore SS, M; MM, P = {}^* SS, P; SS, P$$

which is the syllogistic conclusion.

We may now take an example in categoricals. Given the premisses "There is something besides Ss and Ms," and "There is nothing besides Ms and Ps," to find the conclusion. As the combined premisses state the existence of a non-S non-M and the non-existence of an MP,§ they are expressed by

$$SM, SM; MP.$$

* This should be a sign of implication and not of equivalence.

† But $aa, a; aa: aa, a; aa = aa$, which is the correct answer.

‡ This seems to be in error: $a\dagger a/b$ is not equal to a; nor is $b\dagger a/b$ equal to b. Peirce's conclusion should be: $a, ab; x: b, ab; xx$, or $a, bb; x: b, aa; xx$.

§ This should be: non-M non-P.

To eliminate M, we have

$$\frac{\varphi}{M} = S0, S0; 0P = SS, SS; PP = S, PP$$

$$\frac{\varphi}{MM} = S,\ 00;\ S,\ 00{:}\ 00, P = \text{erased}$$

$$\therefore \frac{\varphi}{M}\ \frac{\varphi}{MM}, \frac{\varphi}{M}\ \frac{\varphi}{MM} = {}^{*}S, PP;\ 0{:}\ S, PP;\ 0$$

$$= {}^{*}S, PP;\ S, PP{:}\ S, PP;\ S, PP = S, PP.$$

The conclusion therefore is that there is something which is not an S but is a P.

20. Of course, it is not maintained that this notation is convenient; but only that it shows for the first time the possibility of writing both universal and particular propositions with but one copula which serves at the same time as the only sign for compounding terms and which renders special signs for negation, for "what is" and for "nothing" unnecessary. It is true, that a 0 has been used, but it has only been used as the sign of an erasure.†

* This should be an entailment.

† With this notation only a single undefined or "primitive" idea and the principle of substitution are necessary in order to construct the propositions and to define all the signs used in a Boolian Algebra. The following is an indication of how this can be done:

Primitive Idea.	No. 1. $A\ B$.
Substituting $\frac{A}{B}$ in No. 1.	2. $A\ A \underset{\text{def}}{=} -A$.
Substituting $\frac{B}{A}$ in No. 1.	3. $B\ B \underset{\text{def}}{=} -B$.
Substituting $\frac{2}{A}, \frac{3}{B}$ in No. 1.	4. $AA;\ BB \underset{\text{def}}{=} A \times B$ (logical product).
Substituting $\frac{1}{A}, \frac{1}{B}$ in No. 1	5. $AB;\ AB \underset{\text{def}}{=} A + B$ (logical sum).
Substituting $\frac{2}{A}$ in No. 5.	6. $AA, B;\ AA, B \underset{\text{def}}{=} A \prec B$.
Substituting $\frac{3}{B}$ in No. 5.	7. $A, BB;\ A, BB \underset{\text{def}}{=} B \prec A$.
Substituting $\frac{A}{B}$ in No. 6.	8. $A, AA;\ A, AA \underset{\text{def}}{=} 1$.
Substituting $\frac{2}{B}$ in No. 1.	9. $A, AA \underset{\text{def}}{=} 0$.
Substituting $\frac{6}{A}, \frac{7}{B}$ in No. 4.	10. $A \prec B \times B \prec A \underset{\text{def}}{=} A = B$.

II

THE ESSENCE OF REASONING[P]*

§1. SOME HISTORICAL NOTES

21. . . . Logic having been written first in Greek had to be turned into Latin; and this was done for the most part by imitating the formation of each technical term. Thus, the Greek *hypothesis*, ὑπόθεσις was compounded of ὑπό, *under* and τίθεναι, *to put*. The preposition ὑπό was equivalent to the Latin *sub*—which is from the same root (being altered from *sup*), and τίθεναι was translated by *ponere*. Hence resulted *suppositio*. It is a very curious fact, by the way, that in this process it was always necessary to change the root. For, whether it be that there is something analogous to Grimm's law applying to *meanings*, as that applies to *sounds*, certain it is that the roots bear so uniformly different meanings that a different one must always be taken. Thus, the root of τίθεναι is the same as that of the Latin *facere*, so that *hypothetical* is the equivalent of sufficient, which widely diverges in meaning. *Ponere* is *po-sinere*, of which the root may be *sa*, to sow, to strew.

22. The earliest Latin work in which we find logical words so transferred from the Greek is supposed to be a treatise on Rhetoric (*Ad Herennium*) usually printed with the works of Cicero, but supposed to be written by one Cornificius, a little older than Cicero. Cicero himself made a number of words on that plan which are now very common, such as *quantity* and *quality*.†

23. Apuleius, the author early in the second century of our era of the celebrated novel of the *Golden Ass*, wrote a treatise on logic which has somehow come to be arranged as the third book of his work *De dogmate Platonis*. The termi-

* Chapter 6 of the "Grand Logic" of 1893. 53–79 are from an alternative draft.

† Cf. Prantl, *Geschichte der Logik im Abenlande*, Bd. I, S. 581.

nology of this treatise we may be pretty sure [Apuleius] did not invent, though it differs considerably from that of any other book which either preceded or for centuries followed it. That terminology has overridden other rival systems of translating the Greek words and has become largely ours. If the reader asks me what the quality was which lent it this staying power, he will be surprised at the answer. Namely, [Apuleius] had one of the most artificial, word-playing, fantastically and elaborately nonsensical styles that the Indo-European literatures can show. It sedulously cultivates every quality which writers upon style admonish us to avoid.

24. . . . Towards the end of the fifth century there was one Martianus Minneus Felix Capella, who wrote a work entitled the *Nuptials of Philology and of Mercury*. This Martianus Capella thought that beneath the stars there was nothing so beautiful nor so worthy of emulation as the style of Apuleius. He did his very best to outdo him; and in studying him became embued with his phraseology. Now in the book the Seven Liberal Arts are invited to the celebration of the Nuptials aforesaid, and each one entertains the company with the greatest good taste by talking shop for all she is worth. The consequence is that the book contains seven short treatises upon these disciplines, of which logic is one. Now the masters of the cathedral schools which at the fall of the Western Empire had to take the place of the old Roman schools found that in an age when one copy of one book and that not too large a one, was all that one school could commonly afford, the work of Capella was well adapted to their purpose. And thus it happened that for some centuries that was the only secular book that ordinary clerks had ever laid eyes upon. Thus, its borrowed terminology became traditional.

25. Anicius Manlius Severinus Boëtius (more commonly called Boëthius) was the author of a book which, whatever its merits and faults, was sincere and has in fact excited a degree of admiration such as has fallen to few works. He is most respectable as a thinker, a logician of positive strength, a man of great learning, a most estimable and sympathetic character, and the courageous supporter of calamities that touch every heart. . . .

26. Petrus Hispanus was a noble Portuguese who, having

taken degrees in all the faculties in Paris, returned to Lisbon and was appointed head of that school which ultimately developed into the University of Coimbra. Subsequently, he was head physician to Pope Gregory X, who created him Cardinal; and he was crowned Pope, September 20, 1276.* He began his pontificate with promise of grandeur; but a part of his palace fell upon him and he died in consequence of his injuries on May 16, 1277.* This man, who had he survived would surely have been reckoned among the world's great men, was according to the tradition, the author of the *Summulæ logicales*, the regular textbook in logic almost to the very end of scholasticism. There are, it is true not very many printed editions subsequent to 1520; but over fifty editions having by that time been printed upon substantial linen paper, copies could always be procured in plenty. Manuscript copies were also current long after printing came in.

There is a Greek text of the book; it has been printed with the name of Michael Psellus attached to it. That name was a common one in Constantinople. Even if any MS. carries it, which has been denied, it does not prove that any particular Michael Psellus was the author; and the language, which is intermediate between Greek and a kind of Romaic, absolutely negatives the idea of its being written by any Michael Psellus known. It is full of Latinisms, and of reminiscences of Latin authors. The Latin text on the other hand bears on its first page conclusive evidence that the author did not know Greek. Namely, we there read: "Dicitur enim *dyalectica* a *dya*, quod est duo, et *logos*, quod est fermo vel *lexis* quod est ratio." Nevertheless, some writers, especially Prantl,† have believed the Greek text to be the original. Charles Thurot has written ably on the other side. When the reader comes across anything about "Byzantine" logic, what is meant is that this book is supposed to be the relic of a development of logic in Constantinople, which in my opinion is an unfounded fancy of Prantl's taken up by many writers without sufficient examination, and solely because Prantl has looked into more logical books of the middle ages than anybody else. I am very grateful to him for what he has read and published in a most convenient form;

* These dates do not coincide exactly with those generally accepted today.

† *Op. cit.*, Bd. II, S.266.

but I find myself compelled to dissent from his judgment very many times. A more slap-dash historian it would be impossible to conceive.

27. There is a synchronism between the different periods of medieval architecture, and the different periods of logic. The great dispute between the Nominalists and Realists took place while men were building the round-arched churches, and the elaboration finally attained corresponds to the intricate character of the opinions of the later disputants in that controversy. From that style of architecture we pass to the early pointed architecture with only plate-tracery. The simplicity of it is perfectly paralleled by the simplicity of the early logics of the thirteenth century. Among these simple writings, I reckon the commentaries of Averroes and of Albertus Magnus. I would add to them the writings of the great psychologist, St. Thomas Aquinas. For Thomistic Logic, I refer to Aquinas,* to Lambertus de Monte† whose work was approved by the Doctors of Cologne, to the highly esteemed Logic of the Doctors of Coimbra,‡ and to the modern manual of [Antoine] Bensa.§

28. During the period of the Decorated Gothic, we have the writings of Duns Scotus, one of the greatest metaphysicians of all time, whose ideas are well worth careful study, and are remarkable for their subtilty, and their profound consideration of all aspects of the questions [of philosophy]. The logical upshot of the doctrine of Scotus is that real problems cannot be solved by metaphysics, but must be decided according to the evidence. As he was a theologian, that evidence was, for him, the dicta of the church. But the same system in the hands of a scientific man will lead to his insisting upon submitting everything to the test of observation. Especially, will he insist upon doing so as against so-called "experientialists," who, though they talk about experience as their guide, really reach the most important conclusions without any careful examination of experience. Whether their conclusions happen to be right or wrong, the Scotist will protest against the manner

* Various *Opuscula*.

† *Copulata pulcerrima in novam logicam Aristotelis*, (1493).

‡ *Commentariorum Collegii Conimbricensis in universam dialecticam Aristotilis Stagiritæ partes duæ*, Venice, (1616).

§ *Manuel de logique*, Paris, (1855).

in which they are taken up. Scotus added a great deal to the language of logic. Of his invention is the word *reality*. For Scotistic logic I refer to Scotus,* Sirectus,† and Tartaretus.‡

29. Scotus died in 1308. After him William of Ockham, who died in 1347, took up once more the nominalistic opinion and this gained ground more and more. Logic now took on a very elaborate, but fanciful and in great measure senseless development; and finally became so big and so useless, that men must have dropped it, even if a new awakening of thought had not occurred. This was during the flamboyant period of architecture in France, the perpendicular in England. The Occamists made important additions to the terminology. For Occamistic Logic, I refer to Ockham's own elaborate treatise,§ to the *Summulæ* of the Doctors of Mayence, to the commentaries of Bricot,¶ etc.

30. The new awakening consisted in the conviction that the classical authors had not been sufficiently studied. At the same time the reformation of the churches came. Logic once more became simple, and this time took on a rhetorical character. Ramus (Pierre de la Ramée),|| Ludovicus Vives,° Laurentius Valla,** were the names of logicians who contributed a few things, but on the whole, rather important things to the tradition of logic.

31. Upon the heels of that movement came another, which has not yet expended itself, nor even quite completed its conquest of minds. It arose from the conviction that man had everything to learn from observation. The first great investigators in this line were Copernicus, Tycho Brahe, Kepler, Galileo, Harvey, and Gilbert. None of them seemed to have any interest at all in the general theory—and that for a simple reason; namely, they knew no way of inquiry but the way of experiment; and their lives were so many experiments in regard to the efficacity of the method of experimentation. The first

* Various *Quæstiones*.

† *Formalitates de mente Scoti* (1501).

‡ See Prantl, *op. cit.*, Bd. IV, S.204ff.

§ *Summa totius logicæ* (1488).

¶ *Textus totius logices* (1492).

|| *Dialecticæ partitiones* (1543).

° *de Causis corruptarium artium*, bk. III, Antwerp (1531).

** *Dialecticæ disputationes*, (1541).

great writer on the theory of Induction, Francis Bacon, was no scientific man. He had no turn that way, though he wished to have, and though he came to his death by a foolish experiment; and his judgments of scientific men were uniformly mistaken. The details of his theory were equally at fault; yet as long as he remains upon the ground of generalities, his ponderous charges are excellent. His *Novum Organum*, like several other great works of this period upon method, is marked by complete contempt for the Aristotelian analysis of reasoning, which nevertheless has kept the field, and, on the whole, held its ground. Still, Bacon made some distinct contributions to the traditional stock of logical ideas. . . .

32. The works we are now coming to are of less *historical* interest, precisely because they have to be taken seriously. Truly to paint the ground where we ourselves are standing is an impossible problem in historical perspective. . . .

33. The nominalistic wing of the Lockian party, much influenced by Hobbes and Ockham, made a philosophical development, chiefly psychological, but also logical. Among their names are Hartley, Berkeley, Hume, James Mill, John Stuart Mill, Bain. Bentham's *Logic* I must confess I do not remember to have seen. That of Mill, which appeared in 1843, contributed some phrases, which many persons adhere to passionately without reference to their meaning, sometimes seeming to attach no meaning to them, except the general one of a party-reveil. The present writer cares nothing about social matters, and knows not what such things mean. He examines logical questions as such and question by question. He perceives that many adherents of John Stuart Mill seem to be in a passion about something. But until they can calm themselves sufficiently, any scientific discussion of the questions, which perhaps they care little about, anyway, is impossible.

34. All the Occamistic school, from the Venerable Inceptor down, have been more or less politicians. John Stuart Mill was hypochondrically scrupulous. Nevertheless, *every* man of action is, must be, and ought to be, cunning, worldly, and dishonest, or what seems so to a man of pure science. When such men dispute, the dispute has some other object than the ascertainment of scientific truth. Men accomplish, roughly speaking, what they desire. Government may be ever so much more

important than science; but only those men can advance science who desire simply to find out how things really are, without *arrière-pensée.*

35. Occamism is governed by a very judicious maxim of logic, called *Ockham's razor.* It runs thus: *Entia non sunt multiplicanda præter necessitatem*, that is, "Try the theory of fewest elements first; and only complicate it as such complication proves indispensible for the ascertainment of truth." It may seem, at the outset, that the more complicated theory is the more probable. Nevertheless, it is highly desirable to stop and carefully to examine the simpler theory, and not contenting oneself with concluding that it will not do, to note precisely what the nature of its shortcomings are. Realism can never establish itself except upon the basis of an ungrudging acceptance of that truth. The Occamists have followed out this rule in the most interesting manner, and have contributed much to human knowledge. Reasons will be given* for thinking that their simple theory will not answer; yet this in no wise detracts from their scientific merits, since the only satisfactory way of ascertaining the insufficiency of the theory was to push the application of it, just as they have done. But because the abandonment of the theory would imply the modification of their politics, they employ every means in their power to discredit and personally hamper those who reject it and to prevent the publication and circulation of works in which it is impartially examined. That is not the conduct of philosophers, however wise it may be from the point of view of statesmanship. . . .

36. As a logician [Leibniz] was a nominalist and leaned to the opinion of Raymond Lully, an absurdity here passed over as not worth mention. This very nominalism led Leibniz to an extraordinary metaphysical theory, his Monadology, of much interest. In regard to human knowledge, he put forth many ideas which had great influence, all of them rooted in nominalism, yet at the same time departing widely from the Occamistic spirit. Such were his tests of universality and necessity; and such was his *principle of sufficient reason*, which he regarded as one of the fundamental principles of logic. This principle is

* See 36, 68ff, and 1.26, 1.170, 1.422, 2.149. There are many pertinent discussions in vol. 1, bk. III and vols. 5 and 6.

that whatever exists has a *reason* for existing, not a blind cause, but a reason. A reason is something essentially general, so that this seems to confer reality upon generals. Yet if realism be accepted, there is no need of any principle of sufficient reason. In that case, existing things do not need supporting reasons; for they *are* reasons, themselves. A great deal of the Leibnizian philosophy consists of attempts to annul the effect of nominalistic hypotheses. . . .

37. Immanuel Kant, who made a revolution in philosophy by his *Critic of the Pure Reason*, 1781, had great power as a logician. He unfortunately had the opinion that the traditional logic was perfect and that there was no room for any further development of it.* That opinion did not prevent his introducing a number of ideas which have indirectly more than directly affected the traditional logic.

The merits of German philosophers since Kant as logicians have in the opinion of the present writer been small, while their errors and vagaries have been incessant.† At any rate, they have had little or no effect upon the ordinary logic. . . .

§2. THE PROPOSITION

38. Very little of the traditional logic relates to the subject of the present section. St. Thomas Aquinas‡ divides the operations of the Understanding in reference to the logical character of their products into

Simple Apprehension,
Judgment, and
Ratiocination, or Reasoning.

Prantl declares§ the commentary on the *Perihermeneias* in which this occurs not to be the work of Aquinas. But he does not explain how it could possibly happen that all the other books of the commentary should be genuine, as he admits they are, and this spurious. From the manner in which such books are written it is utterly inadmissible to suppose Aquinas passed over this book without comment. Such conduct would have

* See the Preface to the second edition, B VIII.

† See 2.152ff.

‡ *Summa totius logicæ Aristotelis* (*Opusculum* 48).

§ *Geschichte der Logik*, Bd. III, S.108.

excited a riot the noise of which would have reached our ears. If, then, the existing commentary is spurious, how could the genuine one have been lost? Thomas Aquinas was already an object of worship living. There was no school which adhered so religiously to the tenets of their master. Prantl himself complains that there is absolutely nothing in the works of Lambertus de Monte, and other Thomists except what St. Thomas had said. How could, then, all those schools be deceived into rejecting one of the works of their holy master, and taking in its place a writing that was not his? How is it that men of such learning as the doctors of Coimbra should get no wind of the substitution? Even Duns Scotus, writing directly after Aquinas, uses in his questions expressions which he probably derived from the book which Prantl suspects. Prantl gives no reason whatever for his rejection. He seems to think his judgment will be so commended by the comparison of it with manuscripts in other cases so entirely that he is placed quite above the necessity of giving reasons for his opinions. Similar ideas are apt to get possession of Germans.

39. *Simple Apprehension* produces *concepts* expressed by *names* or *terms*, "man," a state, suspended existence, the character of eating canned vacuum.

Judgment produces judgments, which are true or false, and are expressed by sentences, or *propositions*, as "Man is mortal," "some men may be insane."

Ratiocination or *reasoning* produces *inferences* or *reasonings*, which are expressed by *argumentations*, as, "I think, therefore I must exist," "Enoch, being a man, must have died; and since the Bible says he did not die, not everything in the Bible can be true."

40. A term names something but asserts nothing; a proposition asserts. Propositions differ in *modality*, which is the degree of positiveness of their assertion, as in *maybe*, *is*, *must be*. In another respect propositions are said to be *assertory*, *problematic*, and *apodictic*. The old statement* was that propositions were either *modal* or *de inesse*, *i.e.*, *assertoric*. They may also be *probable* assertions; they may further be *approximate* and *probable* assertions, as "about 51 per cent of the births in any one year will be male." Propositions are divided

* See Petrus Hispanus, *Summulæ logicales*, Tractatus I, cap. 39 (1597).

into the *Categorical* and the *Hypothetical.* "Propositionum alia categorica alia hypothetica,"* says the *Summulæ.* A *categorical* proposition is one whose immediate parts are terms; or as the *Summulæ* of the Mayence doctors say,† "cathegorica est illa quæ habet subiectum et prædicatum tanguam partes principales sui." A *hypothetical* proposition, better called by the Stoics‡ a *composite* proposition, is one which is composed of other propositions: "Propositio hypothetica est illa quæ habet duas propositiones cathegoricas tanquam partes principales sui."§ The old, and less incorrect doctrine about compound propositions was that they were of three kinds, conditional, copulative, and disjunctive.¶ A conditional proposition is one whose *members* are joined by an *if*, or its equivalent: "Conditionalis est illa in qua coniunguntur duæ cathegoricæ per hanc coniunctionem, si."|| That is, what is asserted is that in case one proposition, called the *antecedent*, is true, another proposition, called the consequent, is true. But how it may be in the opposite case in which the antecedent is not true is not stated. A *copulative* proposition is one in which the truth of every one of several propositions is affirmed. A *disjunctive* proposition is one in which the truth of *some* one of several propositions is affirmed. This enumeration is faulty because the *conditional* and *disjunctive* do not differ from one another in the same way in which both differ from the *copulative* proposition. For the conditional merely (or, at least, principally) asserts that unless one proposition is true another is true, that is, either the contrary of the former is true or the latter is true; and the disjunctive implies no more than that if the contradictions of all the alternatives but one be true, that one is true. Hence, either these two classes should be joined together, or we ought to include three other kinds of compound propositions, one which declares the repugnancy of two or more given propositions so that all cannot be true, one which declares the independence of one proposition of others

* *Ibid.*, cap. 12.

† Tractatus I, pars. 3.

‡ Cf. Prantl, *op. cit.*, I, S. 446, 447.

§ Petrus Hispanus, *op. cit.*, cap. 29.

¶ But see below and 2.271, 2.345f.

|| Petrus Hispanus, *op. cit.*, cap. 30.

so that it can be false although they are all true, and one which declares that there is a possibility that all of certain propositions are false.*

41. The subject of a categorical proposition is that concerning which something is said, the predicate is that which is said of it. Most of the medieval logics teach that subject and predicate are the principal parts of the categorical proposition but that there is also a *Copula* which joins them together. . . . The Mayence doctors were quoted on this head, because Petrus Hispanus† makes the Subject, Predicate, and Copula to be all principal parts — one of the numerous evidences that the text is not a translation from the Greek, a language in which the copula may be dispensed with. Aristotle, however, in his treatise upon forms of propositions, the *De interpretatione*,‡ analyzes the categorical proposition into the *noun*, or *nominative*, and the *verb*.

42. Categorical propositions are said to be divided according to their *Quantity*, into the universal, the particular, the indefinite, and the singular. A *universal* proposition was said to be a proposition whose subject is a common term determined by a universal sign. A common term was defined as one which is adapted to being predicated of several things (*aptus natus prædicari de pluribus*§). The universal signs are *every*, *no*, *any*, etc. A *particular* proposition was said to be a proposition whose subject is a common term determined by a particular sign. The particular signs are, *some*, etc. An *indefinite* proposition was said¶ to be one in which the subject is a common term without any sign, "ut homo currit." That unfortunate "indefinite" man has been running on now for so many centuries, it is fair he should have a rest and that we should revert to Aristotle's example, "Man is just."|| A *singular* proposition was said to be one in which the subject is a *singular* term. A *singular* term was defined as "qui aptus natus est prædicari de uno solo,"** that is, it is a proper noun. Kant and other modern

* Cf. 2. 345f.

† *Op. cit.*, cap. 13.

‡ Ch. 1–5, passim.

§ Petrus Hispanus, *op. cit.*, cap. 14.

¶ *Ibid.*, cap. 15.

|| *De interpretatione*, ch. 10.

** Petrus Hispanus, *op. cit.*, cap. 16.

logicians very rightly drop the indefinite propositions which merely arise from the imperfect expression of what is meant. Singular propositions are for the purposes of formal logic equivalent to universal ones.

43. Propositions were further distinguished into propositions *per se* and propositions *per accidens*. But this was a complicated doctrine, which Kant very conveniently replaced by the distinction between *analytic*, or *explicatory*, and *synthetic*, or *ampliative*, propositions. Namely, the question is what we are talking about. If we are saying that some imaginable kind of thing does or does not occur in the real world, or even in any well-established world of fiction (as when we ask whether Hamlet was mad or not), then the proposition is *synthetic*. But when we are merely saying that such and such a verbal combination does or does not represent anything that can find a place in any self-consistent supposition, then, we are either talking nonsense, as when we say, "A woolly horse would be a horse," or else, we are, as Kant says,* expressing a result of inward experimentation and observation, as when I say, "Probability essentially involves the supposition that certain general conditions are fulfilled many times and that in the long run a specific circumstance accompanies them in some definite proportion of the occurrences." If such a proposition is true and we substitute for the subject what that subject means, the proposition is reduced to an *identical* proposition, or in Kantian terminology an *empty* form of judgment. But the real sense of it lies in its being only just now seen that such *is* the meaning of the subject, that subject having hitherto been obscurely apprehended.

44. Categorical propositions are further divided into *affirmative* and *negative* propositions. A *negative* is one which has the particle of exclusion, *not*, or *other than* attached to the copula. There is a confusing distinction between a *negative* proposition and an *infinite*, that is, an *indefinite* one. The former is like *homo non est equis*, the latter like *homo est non equis*. That is the negative does not imply the existence of the subject, while the affirmative does imply this. But this arrangement, as will be shown in another chapter,† greatly complicates the descrip-

* See his *Kritik der Reinen Vernunft*, A7, B11; B418.

† See 552n, 2.376, 2.453, 3.532.

tion of correct reasonings. For analytical propositions, though affirmative, cannot, as analytical, assert the real existence of anything.*

45. Ratiocination is defined by St. Thomas† as the operation by which reason proceeds from the known to the unknown. Inferences are of two kinds: the necessary and the probable. There are in either case (such is the traditional opinion which will be modified in this work‡) certain propositions called *premisses* laid down and granted; and these render another proposition, called the *conclusion* either necessary or probable, as the case may be. The conclusion is sometimes said to be *collected* from the premisses. It is also said to *follow* from them. The proposition that from such premisses such a conclusion *follows*, that is, is rendered necessary or probable, is called the *logical rule*, *dictum*, *law*, or *principle.* A necessary inference from a single premiss is called an *immediate inference*, from two premisses a *syllogism*, from more than two a *sorites*. The massing of a number of premisses into one conjunctive proposition, which, in general consonance with the doctrine of immediate inference, might be considered as the inference of the conjunctive proposition from its members, though it is *not* so conceived traditionally, is conveniently called by Whewell§ a *colligation.* It is plain that colligation is half the battle in ratiocination.¶

It may be mentioned that Scotus (Duns, of course, for Scotus Erigena was not a scholastic) and the later scholastics usually dealt, not with the Syllogism, but with an inferential form called a consequence. The consequence has only one expressed premiss, called the *antecedent;* its conclusion is called the *consequent;* and the proposition which asserts that in case the antecedent be true, the consequent is true, is called the *consequence.* . . .

46. Logic ought, for the realization of its germinal idea, to be *l'art de penser. L'art de penser!* What a sublime conception. A school to which an age can turn and here learn the most efficient method of solving its theoretical problems! Such is the idea of logic; but it manifestly asks that the

* See 2.456f, 3.178–9.

† *Summa totius logicæ Aristotelis* (*Opusculum* 48).

‡ See vol. 2, bk. III.

§ *Novum Organon Renovatum* II, iv.

¶ See 2.442f, 2.451, 2.469n.

logician should be head and shoulders above his age. That is not at all impossible. There are such men by the dozen in every age. Unfortunately, that is not enough. The man must not only live in realms of thought far removed from that of his fellow-citizens, and really *be* vastly their intellectual superior, but he must also be recognized as such; and that is a combination of events which hardly ever has happened. Aristotle, alone, by the extraordinary chance of adding to his vast powers, inherited wealth, and the close friendship of two kings the most powerful in the world, and both of them, men of gigantic intellect, came near to that ideal. That logic should really teach an age to think must be confessed impracticable. Let it aspire in each age to register the highest method of thinking to which that age actually attains, and it will be doing all that can be expected. This calls for the best minds. But in few ages has even this been done. The logicians instead of generally riding on the crest of the thought-wave, have, three-quarters waterlogged, drifted wherever the motion of thought was least. . . .

§3 THE NATURE OF INFERENCE

47. We now come to the proper subject of this chapter. What is the nature of inference? What says the traditional syllogism? That an inference consists of a colligation of propositions which if true render certain or probable another collected proposition. If, to get to the bottom of the matter, we ask what is the nature of a proposition, traditional logic tells us, that it consists of terms — two terms, usually connected together by another kind of sign, a copula.

This is tolerably explicit, and, so far, good.

48. The next question, in order, which we put to the traditional logic, is, how do you know that all that is true? to divide the question, tell us, first, how you know that that analysis of the nature of assertion is correct.

To this, the traditional logic has not one traditional word to say. It is perfectly plain, however, that the reason it thinks so, is that that seems a satisfactory analysis of a sentence. So it is of the majority of sentences in the Greek, Latin, English, German, French, Italian, Spanish, languages — in short, in the Indo-European languages; and European grammarians, true children of Procrustes, manage to exhibit sentences in other languages forced into the same formula.[1] But outside of that family of languages which bears somewhat the same relation to language in general as the phanerogams do to all plants, or the vertebrates to all animals — while there are of course proper names — it seems to me that *general terms*, in the logical sense, do not exist.* That the analysis of the proposition into subject and predicate represents tolerably the way *we*, Arians, think, I grant; but I deny that it is the *only* way to think. It is not even the clearest way nor the most effective way.

49. There appear to be very many languages in which the copula is quite needless. In the Old Egyptian language, which seems to come within earshot of the origin of speech, the most explicit expression of the copula is by means of a word, really the relative pronoun, *which*. Now to one who regards a sentence from the Indo-European point of view, it is a puzzle how "which" can possibly serve the purpose in place of "is." Yet nothing is more natural.† The fact that hieroglyphics came so easy to the Egyptians shows how their thought is pictorial. . . . [*e.g.*] "Aahmes what we write of is a soldier *which* what we write of is overthrown," means "Aahmes the soldier *is* overthrown." Are you on the whole quite sure that this is not the most effective way of analyzing the meaning of a proposition?

50.‡ Take, now, the other part of the question, namely, supposing the nature of assertion to be understood, what is the relation of inference to assertion, according to the traditional logic? Here we find a marked difference between the view taken down to A. D. 1300 or 1325 and the view which then

[1] I once had the privilege in the Levant of passing some weeks in the companionship of E. H. Palmer, and had a hundred convincing evidences of the high respect which was paid by Arabians to his wonderful mastery of their language, which much surpassed that of any native Sheikh we met. It gave me great pleasure after his death to find a super-learned Regius Professor find fault with Palmer's Arabic grammar because it followed the system which seemed right to those whose vernacular Arabic was, instead of "following the Greek and Latin methods."

* See 56, 2.341, 3.459.

† Cf. 2.354.

‡ Cf. 1.15ff.

gradually gained ground and became universal considerably before A. D. 1600, and remained so until long after A. D. 1800. After 250 years of contest in which it was always gaining ground, it remained for 250 years more in unchallenged possession of the field. The opinion referred to is nominalism. Ockham revived it. By the time the universities were reformed in the sixteenth century, it had gained a complete victory. Descartes, Leibniz, Locke, Hume, and Kant, the great landmarks of philosophical history, were all pronounced nominalists. Hegel first advocated realism; and Hegel unfortunately was about at the average degree of German correctness in logic. The author of the present treatise is a Scotistic realist. He entirely approved the brief statement of Dr. F. E. Abbott in his *Scientific Theism* that Realism is implied in modern science. In calling himself a Scotist, the writer does not mean that he is going back to the general views of 600 years back; he merely means that the point of metaphysics upon which Scotus chiefly insisted and which has since passed out of mind, is a very important point, inseparably bound up with the *most* important point to be insisted upon today. The author might with more reason, call himself a Hegelian; but that would be to appear to place himself among a known band of thinkers to which he does not in fact at all belong, although he is strongly drawn to them.

51. How, then, does Kant regard the *apodictic* inference? He holds that the conclusion is thought in the premisses although indistinctly. That that is Kant's view could be shown in a few words. But let us rather listen to his general tone in talking of reasoning. In the *Critic of The Pure Reason*, Transcendental Dialectic, Introduction, Section II, Subsection B, [A303, B359] he speaks of the logical employment of the Reason, as follows:

"A distinction is usual between things known immediately and things merely inferred. That in a figure bounded by three straight lines, there are three angles is known immediately; that the sum of these angles equals two right angles is a thing inferred. [When Kant wrote this no step in the modern revival of graphical geometry had been made. That three rays in a plane have three intersections, which, without any two rays coinciding, may reduce to one, is a *theorem* of graphics. But

Kant confounds this proposition with another, namely, that if three lines, straight or not, enclose a space on a surface, those three lines must have at least three intersections. This is a corollary from the *Census theorem* of topology. That the sum of the three angles of the triangle equals two right angles, depends, as Lambert had clearly explained, before Kant wrote, upon a particular system of measurement which, however much it may be recommended by what we observe in nature, is not the only admissible system of measurement. Thus, what Kant says is immediately known, is fairly demonstrable; but what he says is demonstrable, is not so. This is not merely true in this case, but would be true of any example which Kant would feel to be a good one. It casts suspicion, at once, upon what he has to say, which has been the result of his generalizations of such examples]. Having an incessant need of inferring we become so accustomed to it, that at last the distinction spoken of escapes us. Even so called deceptions of the senses, where evidently it is the *inferences* that are at fault, we take for immediate perceptions. In every inference, there is one initial proposition, another, the consequent, which is drawn from it, and finally there is the consequence, or proposition according to which the truth of the consequent invariably accompanies the truth of the antecedent. [This is the doctrine of *consequentia* which is so extensively employed by philosophers of the fourteenth and fifteenth centuries.] If the concluded judgment is so contained in the initial judgment, that it can be derived without the intervention of any third idea, the consequence is called *immediate*. [This well-known term Kant would find in Wolff.] I would rather term it an Understanding-consequence. [This Kant seems to think an original idea, but that such a consequence was not an argument was the established doctrine.] But in case, besides the knowledge assigned as reason, still another judgment be needful, in order to draw the conclusion, the inference is called a Reason-inference. In the proposition "All men are mortal" is contained the propositions, "Some men are mortal," "Some mortals are men," "No immortal is a man." These, therefore, follow *immediately* from that. On the other hand, the proposition "All *savans* are mortal" is not contained in our assumed judgment (which does not contain the notion of *savan*), so that this proposition cannot be deduced from that other without a mediating judgment." [This is a slipshod analysis. Kant, out of his well-founded contempt for the scholastic method of trying to answer real questions by drawing distinctions, was led virtually to put the stamp of his condemnation upon all accurate thought. "Subtleties," he often says, "may sharpen the wits, but they are of no use at all."[1] That was a very unfortunate opinion, which encouraged the down-at-the-heels, slouchy sort of logic to which Germans were prone enough and which has disgraced that country. To return to the present case, why does Kant consider only one kind of enthymeme and not another? Suppose the consequence to be the following — which represents an argument actually used by Kant against Boscovich —

All particles are bodies;
Ergo, All particles are extended.

Will Kant tell us there is any idea contained in this consequent not contained in its antecedent? Not so: he himself says,* "I need not go beyond the notion connected with the noun *body* to find that *extension* belongs to it." Will he, then, say that the consequence is no argument? It is put forward as such by himself; and such a doctrine would be a novelty in the traditional logic, with which he professes himself eminently satisfied, which were it involved in his doctrine, he certainly ought to have called attention to. But this example shows that in Kant's opinion the conclusion of a complete and perfect argumentation is implicitly contained in its premisses.]

52. "With the explanation of synthetical Knowledge," says Kant [Analytic of Principles, Chapter 2, Section 2, [A154, B193] of the highest principles of all synthetical judgments],"general logic has absolutely nothing to do." The reason is obvious. Reasoning, according to the doctrine of that work, is regulated entirely by the principle of contradiction, which is the principle of analytical thought. The one law of demonstrative reasoning is that nothing must be said in the conclusion which is not implied in the premisses, that is, nothing must be said in the conclusion, not actually thought

[1] See, for instance, Kant's *Werke*, Ed. Rosenkrantz u. Schubert, III, 181.

† *Kritik der Reinen Vernunft*, A 7, B 11.

in the premisses, though not so clearly and consciously.[1] The proposition that that is *actually thought*, though somewhat unconsciously, which is *implicitly contained* in what is thought, is absurd enough; but it is a *psychological* absurdity which may perhaps be passed over in logic. If that be true, nobody can tell by the most attentive introspection, what he thinks. For it will not be maintained that by carefully considering the few and simple premisses of the theory of numbers — by just contemplating these propositions ever so nicely — one could even discover the truth of Fermat's theorems. It would be impossible to adduce a single instance of the discovery of anything deserving the name of a mathematical theorem by any such means. Every mathematical discoverer knows very well that that is no way to succeed. If the implied proposition be thought, it is thought in some cryptic sense, and it in no wise tells us how it is that inference is performed, to say that in such sense the conclusion is thought as soon as the premisses are given. The distinction between analytical and synthetical judgments represents this conception of reasoning. The distinction may approximate to a just and valuable distinction; but it cannot be accepted as accurately defined. . . .

53.* A belief is a habit; but it is a habit of which we are conscious. The actual calling to mind of the substance of a belief, not as personal to ourselves, but as holding good, or true, is a *judgment*. An inference is a passage from one belief to another; but not every such passage is an inference. If noticing my ink is bluish, I cast my eye out of the window and my mind being awakened to color remark particularly a poppy, that is no inference. Or if without casting my eye out of the window, I call to mind the green tinge of Niagara or the blue of the Rhone, that is no inference. In inference one belief not only follows *after* another, but follows *from* it.

54. What does that mean? The proper method of finding the answer to this question is to compare pairs of beliefs which differ as little as possible except in that in one pair one belief follows *from* the other and in the other pair only follows *after*

[1] See *Prolegomena* 2. "Analytische Urtheile sagen im Prädicate nichts, als das, was im Begriffe des Subjects schon *wirklich*, obgleich nicht so Klar und mit gleichem Bewusstseyn *gedacht wird*."

* Cf. 3.160.

it; and then note what practical difference, or difference that might become practical, there is between those two pairs. . . .

55. I think the upshot of reflection will be this. If a belief is produced for the first time directly after a judgment or colligation of judgments and is suggested by them, then that belief must be considered as the result of and as following from those judgments. The idea which is the matter of the belief is suggested by the idea in those judgments according to some habit of association, and the peculiar character of believing the idea really *is* so, is derived from the same element in the judgments. Thus, inference has at least two elements: the one is the suggestion of one idea by another according to the law of association, while the other is the carrying forward of the *asserting* element of judgment, the holding for true, from the first judgment to the second. That these two things suffice [to] constitute inference I do not say. . . .*

56.† Let us now inquire in what the assertory element of a judgment consists. What is there in an assertion which makes it more than a mere complication of ideas? What is the difference between throwing out the word *speaking monkey*, and averring that *monkeys speak*, and inquiring *whether monkeys speak or not?* This is a difficult question.

In the first place, it is to be remarked that the first expression signifies nothing. The grammarians call it an "incomplete speech." But, in fact, it is no speech at all. As well call the termination *ability* — or *ationally* an incomplete speech. It is also to be remarked that the number of languages in which such an expression is possible is very small. In most languages that have nouns and adjectives, the participial adjective follows the noun and when left without other words the combination would mean *the monkey is speaking*.

In such languages you can't say "speaking monkey," and surely it is no defect in them; for after it is said, it is pure nonsense. . . . There are more than a dozen different families of languages, differing radically in their manner of thinking; and I believe it is fair to say that among these the Indo-European is only one in which words which are distinctively common nouns are numerous. And since a noun or combination of nouns

* Cf. vol. 2, bk. II, ch. 6, §2.

† Cf. vol. 2, bk. II, ch. 4, §5.

by itself says nothing, I do not know why the logician should be required to take account of it at all. Even in Indo-European speech the linguists tell us that the roots are all verbs. It seems that, speaking broadly, ordinary words in the bulk of languages are assertory. They assert as soon as they are in any way attached to any object. If you write GLASS upon a case, you will be understood to mean that the case contains glass. It seems certainly the truest statement for most languages to say that a *symbol* is a conventional sign which being attached to an object signifies that that object has certain characters. But a symbol, in itself, is a mere dream; it does not show what it is talking about. It needs to be connected with its object. For that purpose, an *index* is indispensable. No other kind of sign will answer the purpose. That a word cannot in strictness of speech be an index is evident from this, that a word is general — it occurs often, and every time it occurs, it is the same word, and if it has any meaning *as a word*, it has the same meaning every time it occurs; while an index is essentially an affair of here and now, its office being to bring the thought to a particular experience, or series of experiences connected by dynamical relations. A *meaning* is the associations of a word with images, its dream exciting power. An index has nothing to do with meanings; it has to bring the hearer to share the experience of the speaker by *showing* what he is talking about. The words *this* and *that* are indicative words. They apply to different things every time they occur.

It is the connection of an indicative word to a symbolic word which makes an assertion.

57.* The distinction between an assertion and an interrogatory sentence is of secondary importance. An assertion has its *modality*, or measure of assurance, and a question generally involves as part of it an assertion of emphatically low modality. In addition to that, it is intended to stimulate the hearer to make an answer. This is a rhetorical function which needs no special grammatical form. If in wandering about the country, I wish to inquire the way to town, I can perfectly do so by assertion, without drawing upon the interrogative form of syntax. Thus I may say, "This road leads, perhaps, to the city. I wish to know what you think about it." The most

* Cf. 3.515–6.

suitable way of expressing a question would, from a logical point of view, seem to be by an interjection: "This road leads, perhaps, to the city, eh?"

58. An index, then, is quite essential to a speech and a symbol equally so. We find in grammatical forms of syntax, a part of the sentence particularly appropriate to the index, another particularly appropriate to the symbol. The former is the *grammatical subject*, the latter the *grammatical predicate*. In the logical analysis of the sentence, we disregard the forms and consider the sense. Isolating the indices as well as we can, of which there will generally be a number, we term them the *logical subjects*, though more or less of the symbolic element will adhere to them unless we make our analysis more recondite than it is commonly worth while to do; while the purely symbolic parts, or the parts whose indicative character needs no particular notice, will be called the *logical predicate*. As the analysis may be more or less perfect — and perfect analyses are very complicated — different lines of demarcation will be possible between the two logical members.* In the sentence "John marries the mother of Thomas," John and Thomas are the logical subjects, marries-the-mother-of- is the logical predicate. . . .

59. In making general assertions it is not possible directly to indicate anything but the real world, or whatever world discourse may refer to. But it is necessary to give a general direction as to the manner in which an object intended may be found. Especially it is necessary to be able to say that any object whatever will answer the purpose, in which case the subject is said to be *universal*, and to be able to say that a suitable object occurs, in which case the subject is said to be *particular*.

60. If there are several subjects, some universal and some particular, it makes a difference in what order the selections of a universal and of a particular subject are made. For example, the four following statements are different:

1. Take any two things, A and C; then a thing, B, can be so chosen that if A and C are men, B is a man praised by A to C.

2. Take anything, A; then a thing, B, can be so chosen,

* See 2.358.

that whatever third thing, C, be taken, if A and C are men, B is a man praised by A to C.

3. Take anything, C; then a thing, B, can be so chosen, that whatever third thing, A, be taken, if A and C are men, B is a man praised by A to C.

4. A thing, B, can be so chosen that whatever things A and C may be, if A and C are men, B is a man praised by A to C.

We should usually express these as follows:

1. Every man praises some man or other to each man.
2. Every man praises some man to all men.
3. To every man some man is praised by all men.
4. There is a man whom all men praise to all men.*

61. . . . When we busy ourselves to find the answer to a question, we are going upon the hope that there is an answer, which can be called *the* answer, that is, the final answer. It may be there is none. If any profound and learned member of the German Shakespearian Society were to start the inquiry how long since Polonius had had his hair cut at the time of his death, perhaps the only reply that could be made would be that Polonius was nothing but a creature of Shakespeare's brain, and that Shakespeare never thought of the point raised. Now it is certainly conceivable that this world which we call the real world is not perfectly real but that there are things similarly indeterminate. We cannot be sure that it is not so. In reference, however, to the particular question which at any time we have in hand, we hope there is an answer, or something pretty close to an answer, which sufficient inquiry will compel us to accept.

62. Suppose our opinion with reference to a given question to be quite settled, so that inquiry, no matter how far pushed, has no surprises for us on this point. Then we may be said to have attained *perfect knowledge* about that question. True, it is conceivable that somebody else should attain to a like "perfect knowledge," which should conflict with ours. He might

* The difference between these four expressions is represented symbolically by a difference in the order of the quantifiers:

1. $\Pi a \Pi c \Sigma b$
2. $\Pi a \Sigma b \Pi c$
3. $\Pi c \Sigma b \Pi a$
4. $\Sigma b \Pi a \Pi c$.

know something to be white, which we should know was black. This is *conceivable;* but it is not possible, considering the social nature of man, if we two are ever to compare notes; and if we never do compare notes, and no third party talks with both and makes the comparison, it is difficult to see what meaning there is in saying we disagree. When we come to study the principle of continuity* we shall gain a more ontological conception of knowledge and of reality; but even that will not shake the definition we now give.

63. Perhaps we may already have attained to perfect knowledge about a number of questions; but we cannot have an unshakable opinion that we have attained such perfect knowledge about any given question. That would be not only perfectly to know, but perfectly to know that we do perfectly know, which is what is called *sure knowledge*. No doubt, many people opine that they surely know certain things; but after they have read this book, I hope many of them will be led to see that that opinion is not unshakable. At any rate, as they are, after all, in some measure reasonable beings, no matter how pig-headed they might be (I am only saying that pig-headed people exist, not that they are very frequently met with among my opponents), after a time, if they live long enough, reason must get the better of obstinate adherence to their opinion, and they must come to see that sure knowledge is impossible.

64. Nevertheless, in every state of intellectual development and of information, there are things that seem to us sure, because no little ingenuity and reflection is needed to see how anything can be false which all our previous experience seems to support; so that even though we tell ourselves we are *not* sure, we cannot clearly see *how* we fail of being so. Practically, therefore, life is not long enough for a given individual to rake up doubts about everything; and so, however strenuously he may hold to the doctrine of catalepsy, he will practically treat one proposition and another as certain. This is a state of *practically perfect belief*.

65. We have now to define the five words *necessary*, *unnecessary*, *possible*, *impossible*, and *contingent*. But first let me say that I use the word *information* to mean a state of knowl-

* See 121ff, 219ff, 639ff, and 6.112ff, 6.179ff.

edge, which may range from total ignorance of everything except the meanings of words up to omniscience; and by *informational* I mean relative to such a state of knowledge. Thus, by "informationally possible," I mean possible so far as we, or the persons considered, know. Then, the *informationally possible* is that which in a given information is not perfectly known not to be true. The *informationally necessary* is that which is perfectly known to be true. The *informationally contingent*, which in the given information remains uncertain, that is, at once possible and unnecessary.

66. The information considered may be our actual information. In that case, we may speak of what is possible, necessary, or contingent, *for the present*. Or it may be some hypothetical state of knowledge. Imagining ourselves to be thoroughly acquainted with all the laws of nature and their consequences, but to be ignorant of all particular facts, what we should then not know not to be true is said to be *physically possible;* and the phrase *physically necessary* has an analogous meaning. If we imagine ourselves to know what the resources of men are, but not what their dispositions and desires are, what we do not know will not be done is said to be *practically possible;* and the phrase *practically necessary* bears an analogous signification. Thus, the possible varies its meaning continually. We speak of things *mathematically* and *metaphysically possible*, meaning states of things which the most perfect mathematician or metaphysician does not *qua* mathematician or metaphysician know not to be true.

67. There are two meanings of the words *possible* and *necessary* which are of special interest to the logician more than to other men. These refer to the states of information in which we are supposed to know *nothing*, except the meanings of words, and their consequences, and in which we are supposed to know *everything*. These I term *essential* and *substantial possibility*, respectively; and of course necessity has similar varieties. That is *essentially* or *logically possible* which a person who knows no facts, though perfectly *au fait* at reasoning and well-acquainted with the words involved, is unable to pronounce untrue. The *essentially* or *logically necessary* is that which such a person knows is true. For instance, he would not know whether there was or was not such an animal as a *basilisk*, or

whether there are any such things as serpents, cocks, and eggs; but he would know that every basilisk there may be has been hatched by a serpent from a cock's egg. That is essentially necessary; because that is what the word *basilisk* means. On the other hand, the substantially possible refers to the information of a person who knows everything now existing, whether particular fact or law, together with all their consequences. This does not go so far as the omniscience of God; for those who admit Free-Will suppose that God has a direct intuitive knowledge of future events even though there be nothing in the present to determine them. That is to say, they suppose that a man is perfectly free to do or not do a given act; and yet that God already knows whether he will or will not do it. This seems to most persons flatly self-contradictory; and so it is, if we conceive God's knowledge to be among the things which exist at the present time. But it is a degraded conception to conceive God as subject to Time, which is rather one of His creatures. Literal fore-knowledge is certainly contradictory to literal freedom. But if we say that though God knows (using the word *knows* in a trans-temporal sense) he never did know, does not know, and never will know, then his knowledge in no wise interferes with freedom. The terms, *substantial necessity* and *substantial possibility*, however, refer to supposed information of the present in the present, including among the objects known all existing laws as well as special facts. In this sense, everything in the present which is possible is also necessary, and there is no present contingent. But we may suppose there are "future contingents." Many men are so cocksure that necessity governs everything that they deny that there is anything substantially contingent. But it will be shown in the course of this treatise that they are unwarrantably confident, that wanting omniscience we ought to presume there may be things substantially contingent, and further that there is overwhelming evidence that such things are. . . .*

68. To conclude from the above definitions that there is nothing analogous to possibility and necessity in the real world, but that these modes appertain only to the particular limited information which we possess, would be even less defensible than to draw precisely the opposite conclusion from the same

* See 6.35ff.

premisses. It is a style of reasoning most absurd. Unfortunately, it is so common, that the moment a writer sets down these definitions nine out of ten critics will set him down as a nominalist. The question of realism and nominalism, which means the question how far real facts are analogous to logical relations, and why, is a very serious one, which has to be carefully and deliberately studied, and not decided offhand, and not decided on the ground that one or another answer to it is "inconceivable." Nothing is "inconceivable" to a man who sets seriously about the conceiving of it.* There are those who believe in their own existence, because its opposite is inconceivable; yet the most balsamic of all the sweets of sweet philosophy is the lesson that personal existence is an illusion and a practical joke. Those that have loved themselves and not their neighbors will find themselves April fools when the great April opens the truth that neither selves nor neighborselves were anything more than vicinities; while the love they would not entertain was the essence of every scent.†

69. A leading principle of inference which can lead from a true premiss to a false conclusion is insofar bad; but insofar as it can only lead either from a false premiss or to a true conclusion, it is satisfactory; and whether it leads from false to false, from true to true, or from false to true, it is equally satisfactory. The first part of this theorem, that an inference from true to false is bad, [follows] from the essential characteristic of truth, which is its finality. For truth being our end and being able to endure, it can only be a false maxim which represents it as destroying itself. Indeed, I do not see how anybody can fail to admit that (other things being equal) it is a fault in a mode of inference that it can lead from truth to falsity. But it is by no means as evident that an inference from false to false is as satisfactory as an inference from true to true; still less, that such a one is as satisfactory as an inference from false to true. The Hegelian logicians seem to rate only that reasoning A1 which setting out from falsity leads to truth. But men of laboratories consider those truths as small that only an inward necessity compels. It is the great compulsion of the Experience of nature which they worship. On the other hand, the

* Cf. 2.29.

† Cf. 6.355ff.

men of seminaries sneer at nature; the great truths for them are the inward ones. Their god is enthroned in the depths of the soul. How shall we decide the question? Let us rationally inquire into it, subordinating personal prepossessions in view of the fact that whichever way these prepossessions incline, we can but admit that wiser men than we, more sober-minded men than we, and humbler searchers after truth, do today embrace the opinion the opposite of our own. How, then, shall we decide the question? Yes, how to decide questions is precisely the question to be decided. One thing the laboratory-philosophers ought to grant: that when a question can be satisfactorily decided in a few moments by calculation, it would be foolish to spend much time in trying to answer it by experiment. Nevertheless, this is just what they are doing every day. The wisest-looking man I ever saw, with a vast domelike cranium and a weightiness of discourse that left Solon in the distance, once spent a month or more in dropping a stick on the floor and seeing how often it would fall on a crack; because that ratio of frequency afforded a means of ascertaining the value of π, though not near so close as it could be calculated in five minutes; and what he did it for was never made clear. Perhaps it was only for relaxation; though some people might have found reading Goldsmith or Voltaire fully as lively an occupation. If it were not for the example of this distinguished LL.D., I should have ventured to say that nothing is more foolish than carrying a question into a laboratory until reflection has done all that it can do towards clearing it up — at least, all that it can do for the time being. Of course, for a seminary-philosopher, to send a question to the laboratory is to have done with it, to which he naturally has a reluctance; while the laboratory-philosopher is impatient to get a whack at it.

70. Suppose that, at any rate, we try applying this maxim of methodology to the question now in hand. Then the first thing that has to be remarked is that every inference proceeds according to a general rule — and *that*, a comprehended rule — so that in the very act of drawing it the reasoner thinks of there being other similar inferences to be drawn. For unless the premiss determines the conclusion according to a rule, there is no intelligible meaning in saying that it determines it

at all; unless, indeed, we are prepared to say that the conclusion feels compelled but knows not how; and if it knows not how, how can it know it was the premiss which compelled it? But a conclusion is not only determined by the premiss, but rationally determined, and that implies that in drawing said conclusion we feel we are following a rule and a comprehensible rule. . . .

71. Descartes marks the period when Philosophy put off childish things and began to be a conceited young man. By the time the young man has grown to be an old man, he will have learned that traditions are precious treasures, while iconoclastic inventions are always cheap and often nasty. He will learn that when one's opinion is beseiged and one is pushed by questions from one reason to another behind it, there is nothing illogical in saying at last, "Well, this is what we have always thought; this has been assumed for thousands of years without inconvenience." The childishness only comes in when tradition, instead of being respected, is treated as something infallible before which the reason of man is to prostrate itself, and which it is shocking to deny. In 1637, Descartes (aged 41) published his first work on philosophy, the *Discours de la méthode pour bien conduise sa raison et chercher la verité dans les sciences.* In the fourth part of this dissertation, after insisting upon the doubtfulness of everything, even the simplest propositions of mathematics, in a strain quite familiar to readers of the present work, he goes on to say how at one time "je me résolus de feindre que toutes les choses qui m'étoient jamais entrées en esprit n'étoient non plus vraies que les illusions de mes songes." Thereupon follows the grand passage: "Mais aussitôt après je pris garde que, pendant que je voulais ainsi penser que tout étoit faux, il falloit nécessairement que moi qui le pensois susse quelque chose; et, remarquant que cette vérité: *je pense, donc je suis*, étoit si ferme et si assurée que toutes les plus extravagantes suppositions des sceptiques n'étoient pas capable de l'ébranler, je jugeai que je pouvois la recevoir sans scrupule pour le premier principe de la philosophie que je cherchois."

Descartes thought this "très-clair"; but it is a fundamental mistake to suppose that an idea which stands isolated can be otherwise than perfectly blind. He professes to doubt the testimony of his memory; and in that case all that is left is a vague indescribable idea. There is no warrant for putting it into the first person singular. "I think" begs the question. "There is an idea: therefore, I am," it may be contended represents a compulsion of thought; but it is not a rational compulsion. There is nothing clear in it. Here is a man who utterly disbelieves and almost denies the dicta of memory. He notices an idea, and then he thinks he exists. The *ego* of which he thinks is nothing but a holder together of ideas. But if memory lies there may be only one idea. If that one idea suggests a holder-together of ideas, *how* it can do so is a mystery. To make the reflection that many of the things which appear certain to us are probably false, and that there is not one which may not be among the errors, is very sensible. But to make believe one does not believe anything is an idle and self-deceptive pretence. Of the things which seem to us clearly true, probably the majority are approximations to the truth. We never can attain absolute certainty; but such clearness and evidence as a truth can acquire will consist in its appearing to form an integral unbroken part of the great body of truth. If we could reduce ourselves to a single belief, or to only two or three, those few would not appear reasonable or clear.

72. Now, then, how is truth to be inferred from falsehood? First, it may happen accidentally, from the falsehood that Alexander the Great was the great-grandson of Benjamin Franklin it may be inferred there lived a great-grandson of Benjamin Franklin named Alexander, which happens to be true. It cannot be considered as a merit of a rule that its results accidentally have any character; for an accidental result *ex vi termini* is not determined by the rule. Secondly, truth may follow from falsehood because no lie is altogether false. Every precept of inference which does not lead from truth to falsity, must sometimes lead from falsity to truth. For let A be a true premiss and B a conclusion from it according to such a precept. Then B must be true. But if we add to A something false, B will follow from it just the same. A mode of inference may accordingly infer a larger proportion of true conclusions from false premisses than another simply by inferring less. But concluding falsehood from falsehood is by no means useless, provided it follows a precept which cannot conclude falsehood

from truth. For it hastens the detection and rejection of the falsity. Consequently of two modes of inference neither ever leading from truth to falsity, one of which infers something false from a false premiss from which the other infers something true, the former is rather to be preferred because it infers more. Suppose for instance it is false that the sides of a triangle measure 4 inches, 5 inches, and 6 inches, then the rule of inference which deduces for the area $\frac{15}{4}\sqrt{7}$ square inches is certainly superior to a rule of inference which only concludes that the area is finite. Thirdly, truth may follow from falsehood because that falsehood is impossible and refutes itself. But in this way, only what is logically necessary can be inferred, that is only what a person ought to know independently of any particular premisses. As this is a mode of inference which infers less than any other, its value is the least that any mode can have which never leads from truth to falsity. Many persons will be inclined to dispute this, and will point to the utility of the *reductio ad absurdum* in geometry. But the *reductio ad absurdum* is not a method of inferring truth from falsity; it is only a form of statement of an inference from truth to truth. . . .*

Now it may be that everything is so bound up with everything else that to understand perfectly any single fact, as it really is, would involve a knowledge of all facts. But this is not admitting that from any proposition, understood as it is understood, and not as the reality it represents ought to be understood, much can be inferred; far less that valuable truth can be deduced from falsehood.

It thus appears that the inference of truth from falsity is never so valuable as when it is accidental, in which case its value is precisely the same as that of an inference from false to false.

73. The inference from true to true has precisely the same value as that from false to false. For to infer B from A involves inferring the falsity of A from the falsity of B. The two inferences are inseparable; when either is made the other is made. Now if either of these is an inference from truth to truth, the other is an inference from falsity to falsity; and conversely, if

* Cf. 2.612.

either is an inference from false to false, the other is an inference from true to true. Accordingly it is impossible to set different values upon the two modes of inference.

74. Leading principles are of two classes: those whose pretension it is to lead always to the truth unless from the false, and never astray; and those which only profess to lead toward the truth in the long run. This distinction separates two great branches of reasoning, the one bringing to light the dark things of the hidden recesses of the soul, the other those hidden in nature. We may, for the present, call them Imaginative and Experiential reasoning; or reasoning by diagrams and reasoning by experiments.*

75. . . . The necessity for a sign directly monstrative of the connection of premiss and conclusion is susceptible of proof. That proof is as follows. When we contemplate the premiss, we mentally perceive that that being true the conclusion is true. I say we *perceive* it, because clear knowledge follows contemplation without any intermediate process. Since the conclusion becomes certain, there is some state at which it becomes directly certain. Now this no symbol can show; for a symbol is an indirect sign depending on the association of ideas. Hence, a sign directly exhibiting the mode of relation is required. This promised proof presents this difficulty: namely, it requires the reader actually to *think* in order to see the force of it. That is to say, he must represent the state of things considered in a direct imaginative way.

76. A large part of logic will consist in the study of the different monstrative signs, or icons, serviceable in reasoning. Suppose we reason

Enoch was a man,
Then, Enoch must have died.

Let this reasoning be called in question, and the reasoner searches his mind to discover the leading principle which actuated it. He finds this in the truth (as he assumes it to be) that

Every man dies.

He now repeats his reasoning, joining this proposition to the premiss previously assumed, to make the compound premiss,

Enoch was a man, and every man dies.

* Cf. 2.96.

This may be otherwise stated thus:

> If we are talking of Enoch, what we are talking of is a man; and if we are talking of a man, what we are talking of dies.

The conclusion is

> If we are talking of Enoch, what we are talking of dies.

Or we may state it thus:

> From being Enoch follows being a man, and from being a man follows being subject to death;
>
> Hence, from being Enoch follows being subject to death.

If this reasoning is called in question, the reasoner searches his mind for the leading principle and may state it thus:

> If one truth, A, makes another truth, B, certain, and if this truth, B, makes a third truth, C, certain; then, the truth, A, makes the truth, C, certain.

This is the logical principle called the *Nota notæ*, because one statement of it is, *nota notæ est nota rei ipsius*.*

Now shall the reader add this as a premiss to the compound premiss already adopted? He gains nothing by doing so. For he cannot reason at all without a monstrative sign of illation; and this sign is not really monstrative unless it makes clear the proposition here proposed to be abstractly stated. Nor could any use of that statement be made without using the truth which it expresses.

That if the fact A is certain evidence of the fact B and the fact B is certain evidence of the fact C, then the fact A is certain evidence of the fact C, appears to us perfectly clear. That appearance of evidence may be an argument that the proposition is probably about true; for our instincts are generally pretty well adapted to their ends. But its appearing clear will

* Cf. 2.590.

not prevent our reflecting that things that seem evident are often found to be mistakes, so that it may be the proposition is not true.

77. Now although the reader does not really doubt that the proposition is true, it may be instructive to feign such a doubt, and see what the nature of the source of knowledge is.

A common form of the maxim is this: The word *mortal* is applicable to everything to which the word *man* is applicable, and the word *man* is applicable to everything to which the word *Enoch* is applicable. Hence, the word *mortal* is applicable to everything to which the word *Enoch* is applicable. This mode of representing the matter is embodied in a maxim called the *Dictum de omni*:* if A is in any relation to all to which B is in the same relation, and if B is in this relation to all to which C is in this relation, then A is in this relation to all to which C is in this relation; that is, if the things to which A is applicable are wholly included among the things to which B is applicable, and the things to which B is applicable are wholly included among the things to which C is applicable, then the things to which A is applicable are wholly included among the things to which C is applicable.

Here we have a mental diagram representing receptacles or spaces successively included in one another; and the question of the truth of the maxim may be divided into two parts:

First: Is the maxim certainly true of the mental diagram; and if so how do we know it?

Second: Does the mental diagram represent the relations of truths of nature to one another, in fact?

As to the first question, there would seem to be no reason to doubt that we know it is true of our mental diagram, just as we know of *our idea* of numbers that 2 and 3 make 5. And no line can be drawn between this case and knowing that $\sqrt{2} = 1.414213562373095$ except that the latter is more complicated. It would thus appear that our certainty about the mental diagram is merely due to our having gone over it many times and being confident we could not be all wrong about a matter so simple. Still, as it is easy to make a mistake in calculating the $\sqrt{2}$, and that mistake may be repeated, it is barely *possible* that any conclusion reached in the same way is wrong.

* See 2.591; also H. W. B. Joseph, *An Introduction to Logic*, pp. 296n (1925).

Besides, how do I know I am not crazy and am not uttering the greatest absurdity when I enunciate the *Nota notæ?* Of course, it is not rational for a man to assume that he is utterly irrational. A man cannot be speaking the truth in saying that everything he says is false. For this very thing is one of the things he says; and if this be false then in what it says of itself it is true, and therefore false.* But this remark does not clear up the matter; and we shall leave the problem for the present, to return to it later.†

As to the second question, it is important to remark that the *Nota notæ* does not declare that there is any infallible mark of anything, or any rule without exceptions. If, as we have seen, the *Nota notæ* itself is not absolutely certain, nothing else ought to be so regarded. We cannot go so far as to declare that absolutely no rule is without exceptions; for this declaration is itself a rule. Nor can we say that no rule but this is without exceptions. For this rule either has exceptions or it has not. If it has exceptions and every other rule has exceptions, it has no exceptions. But if it has no exceptions, then in accordance with its declaration it has exceptions. We are thus obliged to admit that there are rules without exceptions, or at least that the denial of it has no sense.[1] But we ought not to suppose that we can identify any general proposition as being certainly or even probably without exceptions. The case is like the following. We say $\frac{1}{2}$ of $\frac{1}{3}$ is $\frac{1}{6}$. Now we do not really think we can divide anything into precisely equal parts; but we think that, barring the possibility that we have made a mistake in doing the sum, which is excessively improbable, the nearer we can come to $\frac{1}{2}$ of $\frac{1}{3}$ of anything, that is, to the ideal state of things in our imagination, the nearer we shall come to $\frac{1}{6}$ . . .

78. In like manner, it may be nothing in the world precisely conforms to rigidity of our idea of something steady enough to

* Cf. 2.618, 5.340.

† See 2.186ff.

[1] If the first thing Adam said was, "No man has ever begun to say anything not false," is it necessary to suppose Pre-adamites? If no man had ever begun to say anything at all, Adam was clearly right, except perhaps in regard to his own remark; and if that was false, it was not false in what it said of itself, and would therefore have to be false of something some Pre-adamite had said. How is this? Even if Adam did not say this before anything else, he might have done so.

be represented by a sign. The reader has had several examples of *insolubilia*,* as they are called by logicians, that is, cases in which every attempt to reason lands us in absurdity. Here are two more examples.

In order to prove black is white, you have only to say, "Either what I am saying is false or black is white." Is that proposition false? It cannot be so; for it only says that one or other of two things is true; and if either is true the proposition is true. It cannot, therefore, be false; for that is one of the alternatives that it leaves open. The proposition is true, then. Consequently, one of the alternatives is true. But not the first; therefore the second. Hence, black is white.

A man invented an ink containing Vanadium the like of which had never been made before. He was just about to try it for the first time, when a friend asked, "Has anything ever been written in Vanadium ink before?" "No." "Will you please write what I tell you for the first handsel of it?" "Yes." "Very well, here is a folded paper marked 'Exhibit A.' Write: What is written in exhibit A is true." He did so. "Now," said the friend, "do you know you have lied to me?" "Oh, but I only wrote that to please you. I did not mean to say it was true." "Very well; suppose it false. Then, what exhibit A says is false. Now read Exhibit A. It reads: 'Something written in Vanadium ink is false.' If that is false, what you have written must be true." "Good! So much the better!" "Not so fast, if you please. What you have written is, of course, true; and consequently exhibit A is true; and consequently something written in Vanadium ink is false. Now it is not what you have just written, for that is true; and therefore you must have lied when you told me nothing had ever been written with that ink before."

It may be that all the propositions in the world would, if subjected to a dialectical examination, prove thus elusive. But that does not affect the truth of the *Nota notæ*, which only says that so far as things conform to our idea of successive inclusion, so far (unless we have blundered almost inconceivably) the *Nota notæ* holds. . . .

79. The logician does not assert anything, as the geometrician does; but there are certain assumed truths which he

* See 2.618; also vol. 5, bk. II, No. 3, §2.

hopes for, relies upon, banks upon, in a way quite foreign to the arithmetician. Logic teaches us to expect some residue of dreaminess in the world, and even self-contradictions; but we do not expect to be brought face to face with any such phenomenon, and at any rate are forced to run the risk of it. The assumptions of logic differ from those of geometry, not merely in not being assertorically held, but also in being much less definite.

III

*SECOND INTENTIONAL LOGIC**P

80. Second intentional, or, as I also call it, Objective Logic, is much the larger part of formal logic. It is also the more beautiful and the interesting subject; and in serious significance it is superior in a far higher ratio. But it is highly abstract, remote from the bread and butter of all parties, and to yield to the temptation of going into it would be to forget

> That not to know at large of things remote
> From use, obscure and suttle, but to know
> That which before us lies in daily life,
> Is the prime Wisdom, what is more is fume,
> Or emptiness, or fond impertinence,
> And renders us in things that most concerne
> Unpractis'd, unprepar'd, and still to seek.

81. Second intentional logic treats at length of the properties of logical conceptions. First, come such simple relations as O, ∞, 1, T.† There is also an extensive doctrine concerning *q*, the relation of inherence. Kant, in his celebrated Appendix to the Transcendental Dialectic,‡ has set forth three sporadic propositions of this sort, whose significance can hardly be seen away from their crowd. Besides, it is more satisfactory to see these things set forth in a purely logical way and deduced mathematically, than to have them treated at their first presentation as regulative principles. As a part of this general doctrine of inherence, there is a special doctrine of the properties of relations. Of course, all logical treatises consider these things; but they do not consider them in a formal way, nor at all in the manner in which they are turned out by the machinery of this calculus. One of the questions which pertain to this branch of logic is that of the classification of relations. There are also some special relations of logical origin which have to be

* Ch. 14 of the "Grand Logic," 1893. Cf. 3.398ff. See also 126.

† O represents impossibility; ∞, coexistence; 1, identity and T, otherness. C. 3.339.

‡ *Kritik der reinen Vernunft* A651ff, B679ff.

considered, among which is that of *correspondence*, which has been studied by mathematicians without much logical analysis.

82. A number of interesting features of the logical calculus itself emerge in the application of it to the second intentions. One of these, for example, is that the subjacent letters I call indices do not essentially differ from any other letters. Thus we may define identity as follows:

$$\Pi_i \Pi_j \Pi_K \Pi_x \left\{ 1_K = \bar{r}_{Kij} \dotplus x_i \dotplus \bar{x}_j \right\}$$

That is, to say that anything whatever, K, is identity is to say that if any two things i and j are in the relation, K, the i to the j, then any proposition whatever, x, is true of i, or else that proposition is false of j. The point calling for notice is that x is put into the logisterium, although it is one of the principal letters of the Boolian.*

83. Another thing is that the forms of logisteria, themselves, become subjects of study, and certain general propositions with regard to them are expressed as if they were shops or trees; and yet these very propositions can be made use of in the calculus.

Among the forms of logisteria which require attentive study and which are found to possess interesting properties are particularly those which are infinite series, though the very purpose of the lectical symbols is to embrace infinite series. But we find that we have to resort to logisteria of logisteria, and to their logisteria again, and so on indefinitely, and that the distinctive characters of different such infinite series of logisteria have to be discriminated.

84. Another point of logical interest is that when our discourse relates to a universe of possibility which virtually embraces all logical possibility, everything is true [from] which no false consequence can possibly follow; and the only way of investigating certain propositions is by proving that they cannot give rise to any contradiction; and this being proved, they are proved true.

For example, it is required to prove that

$$\Pi_a \Sigma_K \Pi_b \ g_{Ka}(\bar{g}_{Kb} \dotplus 1_{ab})$$

* The "logisterium" and the "Boolian" are Peirce's respective names for the quantifying and quantified parts of the proposition. Cf. 346 and 3.500.

That is, that every object, a, whether individual or general, has a quality, K, which is a quality of no object, b, unless b is identical with a. Now this can only lead to an absurd result if K be eliminated. But without "logical involution," or the compounding of the premiss with itself, K can only be eliminated in two ways; first, by eliminating g_{Ka} and $\bar{g}_{Kb}$ independently, and second, by identifying b with a. In the first way, we can only get

$$\Pi_a\Pi_b \ (\bar{1}_{ab} \dotplus 1_{ab})$$

which is true; and in the second way we can only get

$$\Pi_a 1_{aa}$$

which is equally true. We next proceed, then, to square the proposition in question, and so get

$$\Pi_a \Sigma_K \Pi_b \Pi_c \ \ g_{Ka} \cdot (\bar{g}_{Kb} \dotplus 1_{ab}) \cdot (\bar{g}_{Kc} \dotplus 1_{ac}).$$

Unless we identified K in the factors, there could be no aid in eliminating K. But this identification forces us to identify a which is to the left of it in the logisterium. But it is easy to show that from the square so written K cannot be eliminated in any new way, nor from any higher power. Therefore, the proposition can lead to no absurdity, and is true.

Suppose, however, we were to subject the following (which seems hardly distinguishable from it to a loose thinker) to the same test

$$\Sigma_K \Pi_a \Pi_b \ \ g_{Ka} \cdot (\bar{g}_{Kb} \dotplus 1_{ab})$$

Squaring this, we have

$$\Sigma_K \Pi_a \Pi_b \Pi_c \Pi_d \ \ g_{Ka} \cdot g_{Kc} \cdot (\bar{g}_{Kb} \dotplus 1_{ab}) \cdot (\bar{g}_{Kd} \dotplus 1_{cd}).$$

Now identifying d with a and c with b, we get

$$\Pi_a \Pi_b 1_{ab}$$

which is as much as to say that everything is identical with everything. So that in a universe of more than one object this proposition is false.

IV

*THE LOGIC OF QUANTITY**P

§1. ARITHMETICAL PROPOSITIONS

85. Kant, in the introduction to his *Critic of the Pure Reason*†, started an extremely important question about the logic of mathematics. He begins by drawing a famous distinction, as follows:

"In judgments wherein the relation of a subject to a predicate is thought . . . this relation may be of two kinds. Either the predicate, *B*, belongs to the subject, *A*, as something covertly contained in *A* as a concept; or *B* is external to *A*, though connected with it. In the former case, I term the judgment analytical; in the latter synthetical. Analytical judgments, then, are those in which the connection of the predicate with the subject is thought to consist in identity, while those in which this connection is thought without identity, are to be called synthetical judgments. The former may also be called explicative, the latter ampliative judgments, since those by their predicates add nothing to the concept of the subject, which is only divided by analysis into partial concepts that were already thought in it though confusedly; while these add to the concept of the subject a predicate not thought in it at all, and not to be extracted from it by any analysis. For instance, if I say all bodies are extended, this is an analytical judgment. For I need not go out of the conception I attach to the word *body*, to find extension joined to it; it is enough to analyze my meaning, *i.e.*, merely to become aware of the various things I always think in it, to find that predicate among them. On the other hand, if I say, all bodies are heavy, that predicate is quite another matter from anything I think in the mere concept of a body in general."

* Ch. 17 of the "Grand Logic," 1893. Peirce said that this was the strongest paper he ever wrote; see vol. 9, letters to Judge Russell.

† A 7; B 10, 11.

Like much of Kant's thought this is acute and rests on a solid basis, too; and yet is seriously inaccurate. The first criticism to be made upon it is, that it confuses together a question of psychology with a question of logic, and that most disadvantageously; for on the question of psychology, there is hardly any room for anybody to maintain Kant right. Kant reasons as if, in our thoughts, we made logical definitions of things we reason about! How grotesquely this misrepresents the facts, is shown by this, that there are thousands of people who, believing in the atoms of Boscovich, do not hold bodies to occupy any space. Yet it never occurred to them, or to anybody, that they did not believe in corporeal substance. It is only the scientific man, and the logician who makes definitions, or cares for them.

86. At the same time, the unscientific, as well as the scientific, frequently have occasion to ask whether something is consistent with their own or somebody's meaning; and that sort of question they themselves widely separate from a question of how experience, past or possible, is qualified. The Aristotelian [logicians] — and, in fact, all men who ever have thought — have made that distinction. It is embodied in the conjugations of some barbarous languages. What was peculiar to Kant — it came from his thin study of syllogistic figure — was his way of putting the distinction, when he says we necessarily think the explicatory proposition although confusedly, whenever we think its subject. This is monstrous! The question whether a given thing is consistent with a hypothesis, is the question of whether they are logically compossible or not. I can easily throw all the axioms of number, which are neither numerous nor complicated, into the antecedent of a proposition — or into its *subject*, if that be insisted upon — so that the question of whether every number is the sum of three cubes, is simply a question of whether that is *involved* in the conception of the subject and nothing more. But to say that because the answer is *involved* in the conception of the subject, it is confusedly *thought* in it, is a great error. To be *involved*, is a phrase to which nobody before Kant ever gave such a psychological meaning. Everything is involved which can be evolved. But how does this evolution of necessary consequences take place? We can answer for ourselves after having worked a while in the

logic of relatives. It is not by a simple mental stare, or strain of mental vision. It is by manipulating on paper, or in the fancy, formulæ or other diagrams — experimenting on them, *experiencing* the thing. Such experience alone *evolves* the reason hidden within us and as utterly hidden as gold ten feet below ground — and this experience only differs from what usually carries that name in that it brings out the reason hidden within and not the reason of Nature, as do the chemist's or physicist's experiments.

87. There is an immense distinction between the Inward and the Outward truth. I know them alike by experimentation only. But the distinction lies in this, that I can glut myself with experiments in the one case, while I find it most troublesome to obtain any that are satisfactory in the other. Over the Inward, I have considerable control, over the Outward very little. It is a question of degree only. Phenomena that inward force puts together appear *similar;* phenomena that outward force puts together appear *contiguous.* We can try experiments establishing similarity so easily, that it seems as if we could see through and through that; while contiguity strikes us as a marvel. The young chemist precipitates Prussian blue from two nearly colorless fluids a hundred times over without ceasing to marvel at it. Yet he finds no marvel in the fact that any one precipitate when compared in color with the other seems similar every time. It is quite as much a mystery, in truth, and you can no more get at the heart of it, than you can get at the heart of an onion.

But nothing could be more extravagant than to jump to the conclusion that because the distinction between the Inward and the Outward is merely one of how much, therefore it is unimportant; for the distinction between the unimportant and the important is itself purely one of little and much. Now, the difference between the Inward and the Outward worlds is certainly very, very great, with a remarkable absence of intermediate phenomena.

88. The first question, then, to ask concerning arithmetical and geometrical propositions is, whether they are logically necessary and merely relate to hypotheses, or whether they are logically contingent and relate to experiential fact.

Beginning with the propositions of arithmetic, we have seen already* that arithmetical propositions may be syllogistic conclusions from ordinary particular propositions. From

$$A\ \overline{\overline{B}}$$
$$\text{and } \overline{A}\ \overline{\overline{B}},\dagger$$

taken together, or

Some *A* is *B*,
Some not-*A* is *B*,

it follows that there are at least two *B*'s. This inference is strictly logical, depending on the principle of contradiction, that is, on the non-identity of *A* and not-*A*. By the same principle, from

Some *A* is *B*,
Some not-*A* is *B*,
Any *B* is *C*,
Some not-*B* is *C*,

taken together it follows that there are at least three *C*'s.

89. Hamilton admits‡ that the arithmetical proposition, "Some *B* is not some-*B*," is so urgently called for in logic, that a special propositional form must be made for it. So, if a distributive meaning be given to "every," Every *A* is every *A*, implies that there is but one *A*, at most. This is what this proposition must mean, if it is to be the precise contradiction of the other. If a proposition is infra-logical in form, its denial must be admitted to be so.

90. It clearly belongs to logic to evolve the consequences of its own forms. Hence, the whole of the theory of numbers belongs to logic; or rather, it would do so, were it not, as pure mathematics, *prelogical*, that is, even more abstract than logic.§

91. These considerations are sufficient of themselves to refute Kant's doctrine that the propositions of arithmetic are "synthetical." As for the argument of J. S. Mill,¶ or what is usually attributed to him, for what this elusive writer really meant, if he precisely meant anything, about any difficult

* I.e., in a previous paper which is not being published. Cf. 2.607, 2.526.

† I.e., $-(A \prec \overline{B})$ and $-(\overline{A} \prec \overline{B})$.

‡ Cf. *Lectures on Logic*, App. V (d); (1860). See 2.532f.

§ Cf. 2.191.

¶ *System of Logic*, bk. II, ch. 6, §3.

point, it is utterly impossible to determine — I mean the argument that because we can conceive of a world in which when two things were put together, a third should spring up, therefore arithmetical propositions are experiential, this argument proves too much. For, in the existing world, this often happens; and the fact that nobody dreams of its constituting any infringement of the truths of arithmetic shows that arithmetical propositions are not understood in any experiential sense.

But Mill is wrong in supposing that those who maintain that arithmetical propositions are logically necessary, are therein *ipso facto* saying that they are verbal in their nature. This is only the same old idea that Barbara in all its simplicity represents all there is to necessary reasoning, utterly overlooking the construction of a diagram, the mental experimentation, and the surprising novelty of many deductive discoveries.

If Mill wishes me to admit that *experience* is the only source of any kind of knowledge, I grant it at once, provided only that by experience he means *personal history*, life. But if he wants me to admit that inner experience is nothing, and that nothing of moment is found out by diagrams, he asks what cannot be granted.

92. The very word *a priori* involves the mistaken notion that the operations of demonstrative reasoning are nothing but applications of plain rules to plain cases. The really unobjectionable word is *innate;* for that may be innate which is very abstruse, and which we can only find out with extreme difficulty. All those Cartesians who advocated innate ideas took this ground; and only Locke failed to see that learning something from experience, and having been fully aware of it since birth, did not exhaust all possibilities.

Kant declares that the question of his great work is "How are synthetical judgments *a priori* possible?" By *a priori* he means universal; by synthetical, experiential (*i.e.*, relating to experience, not necessarily derived wholly from experience). The true question for him should have been, "How are universal propositions relating to experience to be justified?" But let me not be understood to speak with anything less than profound and almost unparalleled admiration for that wonderful achievement, that indispensible stepping-stone of philosophy.

93. To return to number, there are various ways in which arithmetic may be conceived to connect itself with and spring out of logic. Besides the path of spurious propositions [as indicated in 88], there is another which I pursued on an early paper* in which I defined the arithmetical operations in terms of those of the reformed Boolian calculus. In a later paper,† I considered quantity from the point of view of the logic of relatives.

I shall in the present chapter endeavor, as much as I can, to avoid tedious questions of detail and seek to make clear some of the main points of the logic of mathematics.

§2. TRANSITIVE AND COMPARATIVE RELATIONS

94. I have certainly written to little purpose, and so has Dr. Schröder, if we have not succeeded in making readers perceive the pervasive working of balance and symmetry in every part of logic. Now, we have seen the ubiquitous logical agency of the form

$$l\check{l}.$$‡

We say that this is due to the formula $l\check{l} -\!\!< \mathrm{T}$,§ which is balanced by 1¶$-\!\!< l \dagger \check{l}$.|| But, be it observed, that this is a kind of balance which throws all the active work upon the shoulders of the former principle, and allows the latter to moulder in innocuous desuetude. Yet really, the form

$$l \dagger \check{l}$$

is all-important, inasmuch as it is the basis of all quantitative thought. For the relation expressed by it is a transitive relation. By a transitive relation, we mean a relation like that of the copula. If *A* be so related to *B*, and *B* be so related to *C*,

* Vol. 3, No. II.

† Vol. 3, No. VII.

‡ This can be read as: "lover of what is not loved by."

§ T means "other than."

¶ 1 means "identical with."

|| As † is the mark of relative addition, $l \dagger \check{l}$ can be read as: "lover of everything loved by." See 3.332n.

then A is so related to C. In other words, if t is a transitive relation,

$$tt -\!\!< t.$$

Now, this is the case with $l \dagger \breve{l}$. For that

$$(l \dagger \breve{l})(l \dagger \breve{l}) -\!\!< l \dagger \breve{l}$$

is obvious. Though the reader sees how, I will, in consideration of the importance of the matter, set down the steps:

$$\begin{aligned} &(l \dagger \breve{l})(l \dagger \breve{l}) \\ -\!\!<\ & l \dagger \breve{l}(l \dagger \breve{l}) \\ -\!\!<\ & l \dagger \breve{l}l \dagger \breve{l} \\ -\!\!<\ & l \dagger \mathrm{T} \dagger \breve{l} \\ -\!\!<\ & l \dagger \breve{l}. \end{aligned}$$

This is not only a transitive relation, but it is one which includes identity under it. That is,

$$1 -\!\!< l \dagger \breve{l}.$$

But it is further demonstrable that *every* transitive relation which includes identity under it is of the form

$$l \dagger \breve{l}.$$

For let t be such a relation that

$$1 -\!\!< t.$$

Multiplying by $\breve{t}$, we get,

$$\breve{t} -\!\!< 1\breve{t} -\!\!< t\breve{t} -\!\!< \mathrm{T}.$$

Hence,*

$$t \dagger \breve{t} -\!\!< t \dagger \mathrm{T} -\!\!< t.$$

On the other hand,

$$\begin{aligned} & t -\!\!< t1 \\ \therefore\ & t -\!\!< t(t \dagger \breve{t}) \\ \therefore\ & t -\!\!< tt \dagger \breve{t}. \end{aligned}$$

But because t is transitive,

$$\begin{aligned} & tt -\!\!< t \\ \therefore\ & t -\!\!< t \dagger \breve{t}. \end{aligned}$$

* By transposition.

Having just found

$$t \dagger \check{i} \prec t,$$

we can write

$$t = t \dagger \check{i},$$

so that t may be expressed in the form $l \dagger \check{l}$, Q. E. D.

95. I am now going to allow myself to be led aside out of the main channel of thought upon this subject merely to show how little interest there is in transitive relationship apart from the logical form $(l \dagger \check{l})$.

Let us use the zodiacal sign of Leo to signify a transitive relation, such that not everything is in that relation to itself. The inference holds,

$$x \mathfrak{L} y, \; y \mathfrak{L} z,$$
$$\therefore x \mathfrak{L} z.$$

Let L be an individual that is not in this relation to itself, which we may write,

$$L \overline{\mathfrak{L}} L.$$

Then, the (equivalent) inferences hold

$$L \mathfrak{L} x \qquad x \mathfrak{L} L$$
$$\therefore x \overline{\mathfrak{L}} L \qquad \therefore L \overline{\mathfrak{L}} x.$$

We may, therefore, divide all other individuals into three classes; *first*, K, J, etc. such that

$$K \mathfrak{L} L$$
$$J \mathfrak{L} L, \text{ etc.;}$$

second, M, N, etc., such that

$$L \mathfrak{L} M$$
$$L \mathfrak{L} N, \text{ etc.;}$$

and *third*, Γ, Δ, etc., such that

$$\Gamma \overline{\mathfrak{L}} L \qquad L \overline{\mathfrak{L}} \Gamma$$
$$\Delta \overline{\mathfrak{L}} L \qquad L \overline{\mathfrak{L}} \Delta, \text{ etc.}$$

Taking any one of the first class, K, and any one of the second, M, we have

$$M \overline{\mathfrak{L}} K.$$

Let G be a letter of the first class which is a non-Leo of itself; then

$$G \overline{\Omega} G,$$

and the first class may be subdivided into three with reference to G, just as all were divided relatively to L. So, if R be a letter of the second class which is non-Leo of itself, or

$$R \overline{\Omega} R.$$

We can then divide all possible individuals other than G, L, and R into ten classes, *viz.*:

First, Those which, as B, C, etc. are Leos of G; as

$$B \Omega G;$$

Second, Those which, as Γ, are neither Leos of nor Leo'd by G, but are Leos of L; as

$$\Gamma \overline{\Omega} G, \; G \overline{\Omega} \Gamma, \; \Gamma \Omega L;$$

Third, Those which, as H, K, etc., are Leo'd by G and are Leos of L; as

$$G \Omega K, \; K \Omega L;$$

Fourth, Those which, as Δ, are not Leo'd by G, nor are Leos of L, but are Leos of R; as

$$G \overline{\Omega} \Delta, \; \Delta \overline{\Omega} L, \; \Delta \Omega R;$$

Fifth, Those which, as Θ, are Leo'd by G, are neither Leos of nor Leo'd by L, and are Leos of R; as

$$G \Omega \Theta, \; L \overline{\Omega} \Theta, \; \Theta \overline{\Omega} L, \; \Theta \Omega R;$$

Sixth, Those which, as P, Q, etc. are Leo'd by L and are Leos of R; as

$$L \Omega P, \; P \Omega R;$$

Seventh, Those which as Ξ are not Leo'd by G nor are Leos of R; as

$$G \overline{\Omega} \Xi, \; \Xi \overline{\Omega} R;$$

Eighth, Those which as Π are Leo'd by G, are not Leo'd by L, and are not Leos of R;

$$G \Omega \Pi, \; L \overline{\Omega} \Pi, \; \Pi \overline{\Omega} R;$$

Ninth, Those as Σ which are Leo'd by L but are neither Leos of nor Leo'd by R;

$$L \Omega \Sigma, \; R \overline{\Omega} \Sigma, \; \Sigma \overline{\Omega} R;$$

Tenth, Those as X, Y, etc. which are Leo'd by R; as

$$R \Omega X.$$

96. The above gives some idea what the further doctrine of intransitive relations not including identity would be like. It is evidently more interesting to consider further the study of relatives of the form $(l \dagger \breve{\bar{l}})$ and others allied to them.

The converse of $l \dagger \breve{\bar{l}}$ is $\bar{l} \dagger \breve{l}$,* which is, of course, also transitive. The negative is $\bar{l}\breve{l}$,† which is not transitive, but which has the property,

$$\bar{l}\breve{l} \prec \bar{l}\breve{l} \dagger \bar{l}\breve{l}.$$

For $\bar{l}\breve{l} = \bar{l}1\breve{l} \prec \bar{l}(\breve{l} \dagger \bar{l})\breve{l} \prec \bar{l}\breve{l} \dagger \bar{l}\breve{l}$.

This is a property allied to transitiveness.

If *A* is a lover of something not loved by *B*, which is, in its turn, a lover of something not loved by *C*, the conclusion is, that *A* is a lover of something different from something not loved by *C*. That is,

$$(l\breve{\bar{l}})(l\breve{\bar{l}}) \prec l \,\mathrm{T}\, \breve{l}.$$

The natural current of thought next carries us to the hypothesis that the relation expressed by l be such that

$$l\breve{\bar{l}} \prec l \dagger \breve{\bar{l}}.$$

In this case, $l\breve{\bar{l}}$ is a transitive relation. For

$$l\breve{\bar{l}}l\breve{\bar{l}} \prec l\breve{\bar{l}}(l \dagger \breve{\bar{l}}) \prec l\breve{\bar{l}}.$$

Such a relation may well be termed a *comparative* relation. If Samson can lift something Ajax cannot lift, then Samson can lift everything Ajax can lift. Such a relation underlies all measurement; and the propriety of the designation I propose will be allowed.

With this conception, quantitative science begins. Note well how it has been suggested to us.

A trial of strength must begin by young Ajax, the challenger, doing various things which he "stumps" the champion, Samson, to imitate. If Samson cannot perform all of Ajax's feats, that settles it. But if it seems that Samson can do all that Ajax can do then he will, in his turn, do something which he proposes that Ajax shall imitate. If Ajax cannot do all that Samson can do, that again settles it. But if it seems that each can repeat all the performances of the other, we conclude that they are *equally* strong. Thus equality is a complex relation.‡

* This can be read as: "non-lover of all not loved by."

† This can be read as: "non-lover of what is loved by."

‡ Cf. 3.47n, 3.173n.

In a universe of *quantities* of one dimension (where are only *quantities*, not *quanta*) things equal are identical; so that, not only,

$$l\breve{\bar{l}} -\!\!< \mathrm{T},$$

which is always true and

$$l\breve{\bar{l}}l\breve{\bar{l}} -\!\!< \mathrm{T},$$

which is true for all comparative relations, but also

$$\mathrm{T} -\!\!< l\breve{\bar{l}} \pmb{+} \bar{l}\breve{l}.*$$

That is, if A and B are not identical, either A can do something that B cannot, or B can do something that A cannot.

We have thus analyzed the conception of quantity; and we see that nothing but logical conceptions enter into its constitution. The idea of being *able*, especially in the broad sense in which one quantity is said to be *able* to do something another is *unable* to do, is only a modification of the idea of possibility, the precise explanation of which is given in second intentional logic.

97. Although I cannot, in this work, carry the student deeply into second intentional logic, yet it will be indispensible to look upon quantity somewhat in that way; for quantitative thought, like the traditional "chimæra bombinans in vacuo," feeds upon second intentions.

That l is a comparative relation in a universe of quantity, may be expressed by the formula,

$$0 \dagger (1 \pmb{+} l\breve{\bar{l}} \pmb{+} \bar{l}\breve{l}) \dagger 0*$$

But the same thing may be expressed in another way, by throwing the relation l among the indices. Thus, let us use the three symbols, u, v, w.

w_{ij} means that i is an individual relation of which j is the general character,

u_{ij} means that i is an individual relation of which j is the first relate,

v_{ij} means that i is an individual relation of which j is the correlate.

* $\pmb{+}$ represents logical addition. 0 represents "inconsistent with." Cf. 3.348.

Then, we shall have

$$l_{ij} = \Sigma_k u_{ki} v_{kj} w_{kl},$$

that is, there is an individual relation of which i is the relate, j the correlate, and l the general character.

That being premissed, the proposition that l is a *comparative* relation may be written,

$$\Pi_h \Pi_k \Sigma_i \Sigma_j \Sigma_p \Pi_q \Pi_r \Sigma_s \quad 1_{hk} \Psi u_{ph} \cdot v_{pi} \cdot w_{pl} \cdot (\bar{u}_{qk} \Psi \bar{v}_{qi} \Psi \bar{w}_{ql}) \\ \Psi (\bar{u}_{rh} \Psi \bar{v}_{rj} \Psi \bar{w}_{rl}) \cdot u_{sk} \cdot v_{sj} \cdot w_{sl};$$

that is, of any two different objects, h and k, one or other is l to something to which the other is not l.

More simply, if

$$r_{ijk}$$

means that i is a relation in which j stands to k; we may write,

$$\Pi_h \Pi_k \Sigma_i \Sigma_j \quad 1_{hk} \Psi r_{lhi} \bar{r}_{lki} \Psi r_{lkj} \bar{r}_{lhj}$$

to express that l is a comparative relation.

98. We are next naturally led to remark that it is a very important thing to say of a class of objects, say the A's, that there is some *one* relation such that, of any two A's not identical, one stands in that relation to the other, while the second does not stand in that relation to the first. This we write

$$\Sigma_l \Pi_m \Pi_n \quad \bar{a}_m \Psi \bar{a}_n \Psi 1_{mn} \Psi r_{lmn} \bar{r}_{lnm} \Psi r_{lnm} \bar{r}_{lmn}$$

But this becomes particularly important, in case the relation l is a relation of comparison. If that be the case, we multiply in the above definition of such a relation.

99. The question arises, is it possible there should be a class which does not possess the property just defined? It is a difficult question, to which a good logician will be reluctant to give a negative answer. In order to answer it, we must have some way of constructing an icon of a class, in general. Now, a class may be said to comprise all of which something is true. Shall we say of the different individuals composing it that they are distinguished by having some of them qualities which the others do not possess? It seems far from evident that this is so; although, no doubt, after two instances have presented themselves, it is possible in the circumstances of the presentation to find distinguishing qualities. But supposing this difficulty surmounted, of two individuals of a class,

each may have qualities the other wants. If, then, we seek to establish an order of precedence among the things, such as a relationship of comparison supposes, we must first establish some order of precedence among the qualities. We are thus brought back to the question with which we set out, whether among a collection of objects an order of precedence *can* always be establishable. It thus becomes clear that no contradiction can emerge from the hypothesis of a class among the members of which no thoroughgoing order of precedence can be established, and to which all quantitative conceptions are quite inapplicable. About such classes we can reason, but we cannot reason quantitatively.

§3. ENUMERABLE COLLECTIONS

100. But supposing we have to do with a class of things throughout which a relationship of comparison *can* be established, the next question that balance and symmetry suggest is, whether, as we have

$$l\breve{l}l\breve{l} -\!\!< l\breve{l},$$

we have also

$$l\breve{l} -\!\!< l\breve{l}l\breve{l}.$$

It is clear enough, that cases can be imagined in which this shall not be true. Classes of a mixed character exist, too, where this holds in certain parts, but not in others. Such mixed cases are not, however, of much interest. The interesting cases are those in which

$$l\breve{l} -\!\!< l\breve{l}l\breve{l}$$

invariably holds,* and those in which whatever is $l\breve{l}$ to anything is $l\breve{l}$ to something to which it is *not* $l\breve{l}l\breve{l}$. To speak in more familiar terms, whatever is greater than anything may or may not always be *next* greater than something, that is, may or may not be always greater than something else, greater than that thing.

101. But a further distinction immediately arises, according as, on the one hand, one or other of these propositions is true for *every* comparative relation, or on the other hand for some comparative relations the one proposition is true and for others the other.

* Cf. 121.

This trichotomy constitutes the most important distinction between classes in respect to their *multitude.*

Let us first consider a class in which, no matter what comparative relation may be signified by $l\check{l}$,

$$l\check{l} \infty \multimap\!\!< [l\check{l} \cdot (\bar{l} \dagger \breve{l} \dagger \bar{l} \dagger \breve{l})] \infty .^{*}$$

That is, whatever is greater than anything is next greater than something, in *every sense* of being greater, that is, for *every* comparative relation, $l\check{l}$, for which

$$0 \dagger (1 \boldsymbol{+} l\check{l} \boldsymbol{+} \bar{l}\breve{l}) \dagger 0.$$

102. We have now pushed our way far enough into the theory of quantity and its complications of logic to meet with theorems. Such is the following: any class of which the conditions enunciated holds has a *maximum* and a *minimum* individual for each comparative relation, that is, one which is not $l\check{l}$ to any member of the class, and one which is not $\bar{l}\breve{l}$ to any member. I will first show that there is a maximum. For this purpose, assume a_0 to be any member of the class of A's, and consider the relation

$$(l\check{l} \boldsymbol{+} \bar{l}\breve{l})a_0\breve{a}_0 \boldsymbol{+} l\check{l}a_0\breve{a}_0\bar{l}\breve{l} \cdot l\check{l} \boldsymbol{+} \bar{l}\breve{l}a_0\breve{a}_0\bar{l}\breve{l} \boldsymbol{+} \bar{l}\breve{l}a_0\breve{a}_0 l\check{l} \cdot \bar{l}\breve{l}.$$

This is an aggregate of four relations, *viz.:*

First, That having for its relate any A, superior ($l\check{l}$) or inferior ($\bar{l}\breve{l}$) to a_0, and for its correlate a_0,

Second, That having for both relate and correlate A's superior to a_0, the relate being superior to the correlate,

Third, That having for relate any A inferior to a_0, and for correlate any A superior to a_0,

Fourth, That having for both relate and correlate A's inferior to a_0, the relate being inferior to the correlate.

This, I say, will be a quantitative relation. That is, it will be transitive, included under its own negative converse, and including negation under the aggregate of itself and its converse. That it is transitive, that is, that, if X is in this relation to Y, and Y to Z, then X is in this relation to Z, is plain; for, if Y is superior to a_0, then Z must either be a_0 (in which case, X, whatever A it may be, is in this relation to Z) or

* ∞ represents "coexistent with."

must be superior to a_0 but inferior to Y. Then, if X is superior to a_0, it is superior to Y, and consequently also to Z, and is in this relation to Z. But if X is inferior to a_0, it is in this relation to Y, which is superior. X can in no case be a_0. If Y is inferior to a_0, Z is either a_0 (when as before X will be in this relation to it) or superior to a_0, or inferior to a_0, but superior to Y. X will be inferior to Y, and thus will be in this relation to Z. This shows that the relation is transitive. That it is included under its negative converse, that is, inconsistent with its converse, is plain; for if U could be in this relation to V and in the converse relation, too, that is, V in this relation to U, then, since the relation is transitive, U would be in this relation to itself, which, it is easy to see, the definition excludes. That of any pair of different A's, one is in this relation to the other, is easily seen by running over the definition.

We will call this relation "second-superior." Now, I say, if the class of A's has no maximum for the relation $l\breve{l}$, then that A which is next inferior to a_0 is not next second-superior to any A. Will it be objected that I have not proved that there is an A next inferior to a_0? It is easy to supply the defect. For by hypothesis whatever A is superior is next superior to some A for every comparative relation. Now, we have only to substitute $\breve{l}$ for l and *vice versa*, and next inferior becomes next superior. Therefore, to say that for *every* comparative relation, whatever has a superior has a next superior, is the same as to say that for every comparative relation, whatever has an inferior has a next inferior. The A next inferior to a_0 is second-superior only to a_0, to A's superior to a_0 and to A's inferior to a_0 but superior to itself. Because it is *next* inferior, of the last there are none. That of which it is next superior is therefore superior to a_0, and any other A superior to it is second-superior to it and second-inferior to the next inferior to a_0. Thus, that to which the next inferior to a_0 is next second-superior, is the superior of all other A's superior to a_0; that is, it is the *maximum*. The proof that there will be a minimum is altogether similar.

Hence, any class of things in which whatever is *anywise* superior to another of the class is *next* superior to some one can be enumerated. For in enumeration, the objects of a class are singly *told*, and "told later than" evidently satisfies the three conditions of a quantitative relative. If there is a maximum, the telling comes to an end, the class is told out, it is enumerated. For that reason, it is convenient to term a class every member of which anywise superior to another is next superior to some, an *enumerable* collection.

103.* About an enumerable collection certain forms of reasoning hold which, though they had been used more or less since man began to be a reasoning animal, were first signalized in a logical work by De Morgan in 1847,[1] and constitute one of his claims to be considered the greatest of all formal logicians. In his *Formal Logic* he gives eight forms which in the Appendix to his fourth Memoir on the Syllogism are increased to 64. The eight are as follows:

For every Z there is an X that is Y,
Some Z is not Y;
∴ Some X is not Z.

For every Z there is an X not Y,
Some Z is Y;
∴ Some X is not Z.

For every non-Z there is an X that is Y,
There is something besides Y's and Z's;
∴ Some X is Z.

For every non-Z there is an X not Y,
Some Y is not Z;
∴ Some X is Z.

For every Z there is a Y not X,
Some Z is not Y;
∴ There is something besides X's and Z's.

For every Z there is something neither X nor Y,
Some Y is Z;
∴ There is something besides X's and Z's.

* Cf. 3.288; 3.402.

[1] *Formal Logic: or the Calculus of Inference, Necessary and Probable*, pp. 166 et seq. Also: *Cambridge Philosophical Transactions*, X. 355. (Here more thoroughly treated.)

For every non-Z there is a Y not X,
There is something besides Y's and Z's;
∴ Some Z is not X.

For every non-Z there is something neither X nor Y,
Some Y is not Z;
∴ Some Z is not X.

We might also have such a reasoning as this:

For every Z there is an X that is Y,
For every X not Z there is an X not Y;
∴ Every Z is X.

This is not one of De Morgan's Forms. He gives, however such as these:

For every Z an X is Y,
Every Y is Z;
∴ Every Z is X.

For every not Z is an X not Y,
For every X is a Y not Z;
∴ Every Z is X.

De Morgan termed these "syllogisms of transposed quantity," because they transfer the lexis from one term to another. His point of view was this: Take *Baroko*,

Any M is P,
Some S is not P;
∴ Some S is not M.

The converse,

Some M is not S,

does not follow; but if there are as many M's as there are S's, then this *does* follow. "For if M's, as many as there are S's, be among the P's, and some of the S's be not among the P's, though all the rest were, there would not be enough to match all the M's, or some M's are not S's."

The rank and file of old-fashioned logicians were not pleased with the syllogisms of transposed quantity. They belonged to that class of minds who decry originality, who dread novelty, who hate discoveries, and who will go to some trouble to inflict any personal injury on those who perpetrate them, provided they can inflict it without serious injury to themselves. They circulated an unfounded innuendo that De Morgan was a drunkard; their spitefulness was only bounded by their prudence. The idea of so lifeless a subject as formal logic — too abstract to be philosophical — exciting such passions is laughable. Yet such was the fact.

But they could not find anything better to say against those syllogisms of transposed quantity than that they were "extra-logical." If it had only occurred to them that they were not sound reasoning, that is, not universally valid, they would have seized upon that defect with glee. Nor, singularly enough, does De Morgan himself seem to have remarked the circumstance; although it ought to have been evident from the line of thought which led him to those forms. By the logic of relatives, we at once find that the statement that such a syllogism is necessary implies that a certain collection is enumerable.

The following is the first form De Morgan adduced:

> Some *X* is *Y*,
> For every *X* there is something neither *Y* nor *Z*;
> Hence, something is neither *X* nor *Z*.

Let us put for *X* "odd number," for *Y* "prime," for *Z* "either an even number or not a number," so that its negative is "a number not even." Then, the conclusion is false though the premisses are true. Thus:

> Some odd numbers are prime,
> Every odd number has for its square a number not even nor prime;
> Hence, some number not even is not odd.

104. Let us enclose the description of a class in square brackets to denote the number of individuals in it. Then the premisses of the above may be arithmetically stated thus:

$$[x \cdot y] > 0$$
$$[x] \leqq [\bar{y} \cdot \bar{z}]$$

Developing the last, we get

$$[x\cdot y\cdot z]+[x\cdot y\cdot \bar{z}]+[x\cdot \bar{y}\cdot z]+[x\cdot \bar{y}\cdot \bar{z}]\leqq[x\cdot \bar{y}\cdot \bar{z}]+[\bar{x}\cdot \bar{y}\cdot \bar{z}]$$

Cancelling $[x\cdot \bar{y}\cdot \bar{z}]$, we have

$$[x\cdot y\cdot z]+[x\cdot y\cdot \bar{z}]+[x\cdot \bar{y}\cdot z]\leqq[\bar{x}\cdot \bar{y}\cdot \bar{z}]$$

Developing the first premiss

$$[x\cdot y\cdot z]+[x\cdot y\cdot \bar{z}]>0.$$

It thus follows, not only that

$$[\bar{x}\cdot \bar{z}]>0$$

but even that, throwing aside *Y*'s, there are more non-*X*'s not *Z*'s than there are *X*'s that are *Z*'s. But the fallacy lies in assuming the Simple Simon proposition that *every part is less than its whole*. That is, because the odd squares are no fewer than all the odd numbers, we quietly reason as if they were more than a part of odd numbers; so that after taking away alike from odd squares and from odd numbers the odd primes, we should necessarily have as many odd squares left over as we have odd numbers. Of collections not enumerable it is not generally true that the part is less than the whole. Every integer has a square; and thus there are as many squares as there are integers; although the squares form but a part of all the integers.

Take this example:

Every woman marries a man,
For every man there is a woman;
∴ Every man is married to a woman.

The necessity of this plainly arises from the fact that after every woman has got a husband, the collection of men is *exhausted*. To say this, is to imply that for every quantitative relation it would have a *maximum*, that is, a *last reached*, in any order of running it through. . . .

105. The commonest sort of paralogism by far among thoughtful persons consists in reasoning as if collections were enumerable which are, in fact, inenumerable. How often do we hear one, speaking of objects in a linear series, say there must be a first or must be a last! Logic lends no color to such ideas; but, on the contrary, shows them to be pure assumptions.

In metaphysics, particularly, it is frequently argued that something is analyzable into a series — by pure abstract reasoning — and then because "there must be a first" some consequence truly startling follows. Years of experience bring us to expect, as a matter of course, some fallacy, big or little, in every demonstration which seems to advance knowledge very much.

106. De Morgan's syllogism of transposed quantity does not seem very clearly or accurately to set forth precisely what the nature of the reasoning specially applicable to enumerable collections is. What does precisely describe it is, that in whatever order you pass, one by one, through the collection, you come to a last unit. But let us logically analyze this. Here there is a relation of the later-taken to the earlier-taken. The earlier is taken at a time at which the later is not taken. If l signify "taken at a time," then "taken at a time at which was not taken" is written

$$l\check{\bar{l}}.$$

We see at once that this running through the collection is only a specialized way of saying that we have to do with a quantitative relation, a way of expressing it which brings in the irrelevant idea of *time*. The better statement is that, in reference to every quantitative relation, the enumerable collection has a last. From this it quite obviously follows that there is also a first, and that every superior of anything is next superior of something, and so also with the inferior. Another form of definition of the enumerable collection is, to say that it is a collection any part of which is less than the whole. That is, given a class, the b's, such that

$$\infty \prec \check{\bar{b}} \dagger a,$$

or every b is a, while

$$\infty \prec \breve{a}\bar{b},$$

or some a is not b; if, further, k is such a relative that

$$\check{k}bk \prec 1,$$

then

$$\infty \prec \breve{a}(\check{\bar{k}} \dagger \bar{b}).$$

This is a good statement of the kind of inference peculiar to enumerable classes. It has three premisses, involving two class terms and a relative term; and it reposes directly upon the axiom that an enumerable part is less than the whole. One of the three premisses is implicitly assumed but not stated by De Morgan; the relative is his "for every," and one of his three terms is superfluous. Thus he puts the argument about the checks into form as follows:*

"For every memorandum of a purchase a countercheque is a transaction involving the drawing of a cheque,

"Some purchases are not transactions involving the drawing of checks;

"Therefore, some countercheques are not memoranda of purchases."

But I should put it in form as follows:

Some payment for purchase was not a check,
Every payment by purchase is told off against a check;
No two payments by purchase are told against the same check;

∴ Some checks are not payments by purchase.

That is, we prove the checks are not a part of the payments by purchase, because they are not less than the payments by purchase; and it is assumed they were enumerable. If they ran on endlessly, each payment by purchase might be told off against a check of a subsequent day, the purchases increasing in number day by day.

§4. LINEAR SEQUENCES

107. The second, or middling, grade of multitude is that of collections which have different attributes for different quantitative collections; namely, for some such relations, every member of the class superior to another member is *next* superior to some member, definitely designatable, while for other quantitative relations it is not so. I undertake to show that there is always some quantitative relation for which (1) the class has a minimum but no maximum, (2) for which every member of the class that is superior to another is *next* superior to some

* Cf. 3.402n.

other, (3) and for which the partial class consisting of any two members of the class we are speaking of, together with all that are superior to one of these two members but inferior to the other, is *enumerable*. Let us begin by thinking of a member of this class, say a_x. Then, considering a quantitative relation in which every a superior to an a is next superior to an a, let us think of that to which a_x is next superior. Then, think of that to which the last is next superior. Then consider a partial class to all of which a_x is superior, and the next inferior of each member of which is also included under it, so that either there is a *minimum*, which is not superior to any member of the class, or else, if a_y is any member of the class to which a_x is superior, and the a's at once inferior to a_x and superior to a_y are enumerable, it follows that a_y is a member of the partial class. For if not, of all the a's superior to a_y and inferior to a_x, a part belongs to the partial class, and this part of an enumerable collection, being itself (as such) enumerable, must have a *minimum*. But by the definition of the partial class, whatever is next inferior to any member of it also belongs to it. To this partial class, then, belongs every a inferior to a_x, so long as between it and a_x, the collection of a's is enumerable. We do not know that there is an a next superior to a_x. But we define a second partial class as containing the a next superior to a_x, if there be any, and as containing nothing else, except that it contains the a next superior to any a that it contains. Then, it will either contain all the a's superior to a_x up to some *maximum*, which need not be the maximum of all the a's, but which has no a next superior to it, or, in the absence of such a maximum, it will contain all the a's up to and beyond any a superior to a_x, but such that the a's inferior to it and superior to a_x form an enumerable collection. The proof of this (so plain that it hardly needs statement) is as follows: if this be not the case let a_z be an a superior to a_x and such that the a's inferior to a_z but superior to a_x form an enumerable multitude. Then, those of those which belong to the second partial class, being part of an enumerable collection. are themselves enumerable. Hence, they have a maximum, contrary to the hypothesis. Taking the first and second partial classes together, I propose to call such a series of a's a linear sequence. I will repeat its characteristics:

1. It contains a_x.

2. It contains every a inferior to a_x and identical with or superior to a_y, no matter what a_y may be, so long as it is inferior to a_x, and so long as the a's superior to a_y and inferior to a_x form an enumerable collection.

3. It contains every a superior to a_x and identical with or inferior to a_z, no matter what a_z may be, so long as it is superior to a_x, and so long as the a's inferior to a_z and superior to a_x form an enumerable collection.

4. It has no minimum unless that *minimum* be the minimum of all the a's.

5. It has no maximum unless that maximum be an a which has no a next superior to it.

6. Unless the linear series happens to have a minimum and a maximum, it is itself an *inenumerable* collection of a's.

108. Having formed this linear sequence, if there be any a's not included in it, let a'_x be one of them. We then proceed to form a second linear sequence in which a'_x takes the place of a_x. If, after that, there still be a's not included in the linear sequences already formed we proceed to form a *next succeeding* linear sequence, and so on indefinitely. The multitude of linear sequences may be inenumerable; but as necessary consequences of the rule for the formation of these sequences, the following propositions hold.

First, The linear sequences are formed successively. If we take

$$t$$

to signify "already formed at a time —", then

$$\bar{t}\breve{t} \prec \bar{t} \dagger \breve{t}$$

that is "what is not yet formed at some time at which X was already formed," is included under "formed only at times at which X was already formed." Moreover

$$0 \dagger (1 + \bar{t}\breve{t} + t\breve{\bar{t}}) \dagger 0.$$

That is, of two linear sequences not the same one was formed earlier than the other.

Second, None of the linear sequences was $t\breve{\bar{t}}$ to the first.

Third, Every sequence $\bar{\imath}\breve{\imath}$, that is formed subsequent to another, was formed on the occasion of its having been found that some a's were left over not included in the previous sequences, and thus was not $\bar{\imath}\breve{\imath}\bar{\imath}\breve{\imath}$, or was formed *next* subsequent to some other.

Fourth, All the sequences formed previous to a given sequence, have a first and last, and are an enumerable collection.

109. It is well to remark, as a matter of language, that whenever a quantitative relation is applied to a class which has for that relation a minimum, and every member of the class superior to another in respect to this relation is next superior to some other, and the partial class consisting of all inferior to any given member is enumerable, then we can conveniently speak of the relationship as constituting an arithmoidal order in which the individuals are taken, the next superior being said to come next after, etc.

110. I propose now to show that all the a's can be embraced in such a serial order. But for this purpose, I must first establish a preliminary arithmoidal order in each of the linear sequences. The first sequence may have both a minimum and a maximum, and if so, it is enumerable. If it has a minimum but no maximum, the arithmoidal order will already exist, the minimum being the first. In every sequence which has a maximum, the arithmoidal order is established by simply considering the converse of the quantitative relation for which that maximum is a maximum. It remains only to establish an arithmoidal order in each of the sequences which has neither minimum nor maximum. This we do by taking arbitrarily any a of the sequence as first of the series, and for the one next after it, the a to which it is next superior, and thereafter the following rule is to be used; next after any a, as a_w, which is inferior to the a next preceding it, which we may call a_v, is to be taken the a next superior to a_v; but next after any a, as a_u, which is superior to the a next preceding it, which we may call u_w, is to be taken the a next inferior to u_w. The demonstration that this reduces the sequence to such a series is so easy that it may be omitted. We then take all the a's together in the following order: first, we take the whole of the first sequence if it is enumerable; next, we take the first a of the first inenu-

merable series; next, after any *a* of any inenumerable series we take the first *a* not already taken of the next inenumerable series, unless the last taken were the first of its series, when next after it we take the first not already taken of the first inenumerable series. The demonstration that this has the desired effect is sufficiently easy to be left to the student.

Such an arithmoidal series is just like the series of positive whole numbers. I call it with reference to its grade of multitude, *dinumerable*. That is, it corresponds one to one to the numbers, yet the count of it cannot be completed. To such a series applies the kind of reasoning called by me the *Fermatian inference*. This consists in proving a proposition to be true of such a series, because otherwise it must be false of an enumerable collection, such falsity, by reasoning on the principle of the part being less than the whole, being shown to be impossible. Fermat himself called it indefinite descent. He states his "manière de demonstrer," which he calls "une route tout à fait singulière" as consisting in showing that if the proposition to be proved were false of any number, it would be false of some smaller number. This statement shows a good comprehension of its nature.*

111. If we take all the whole numbers and write opposite to each the same figures in inverse order with a decimal point before them — as, opposite 1894, for example, .4981 — and then arrange the numbers in the order of these decimal fractions, we shall have established a quantitative relation according to which no number is next superior or next inferior to any other. A story is told of a bar of tin which being sent into Russia in the depths of winter, arrived in good order only that every atom had broken away from every other. If this tale did not serve to put money in somebody's pocket, it at least affords a pretty simile of the condition of the numbers when looked upon from that below-zero point of view. To bind them together after they are in their new order would require a multitude of new units inexpressibly more numerous relative to these numbers than the totality of them is to one.

112. If from the entire series of integer numbers, ranged in regular order, we imagine none, certain ones, or all to be omitted, we have what we may call a broken series, and the

* Cf. 3.286–7.

multitude of the entire collection of all such broken series possible is so great that they cannot be arranged by means of any quantitative relation so that whatever one is superior to another is next superior to some one. The proof which I offer of this is at bottom not mine. It seems to me sound, and if so is wonderful. In order to show that those broken series cannot themselves be arranged in an arithmoidal order, let us first arrange them in any quantitative order. This is easy, for each number may represent a place of decimals in the binary system of numeration, 1 the $\frac{1}{2}$ place, 2 the $\frac{1}{2^2}$ place, 3 the $\frac{1}{2^3}$ place, and n the $\frac{1}{2^n}$ place. The absence of a number being represented by zero, and its presence by 1, each series is represented by a binarial fraction, and these may be arranged in the order of their values. (Of course, the series could not be represented by integer numbers by reflection from the binarial point, because so they would all be infinite.) Now the question is whether the series of whole numbers can in any way whatever be made to correspond to these fractions. And in order that the matter may appear in a clearer light let us suppose that parallel to the series of infinite binarial fractions is ranged the entire series of values of rational fractions between 0 and 1, expressed in the same notation and set down in the same order. Let us first take a mode of correspondence which obviously will not fulfill the purpose, but which will serve to show the difference between such a series as that of the rational fractions and the series with which we are dealing. Suppose that the numbers correspond to the fractions in the order of their simplicity. Thus our first two fractions (I won't take the trouble to write them in binary notation) are:

$\frac{1}{3}$ .3333333333 $\frac{1}{2}$ 0.5000000000;

between these the first two are

$\frac{2}{5}$ .4000000000 $\frac{3}{7}$ 0.4285742857;

between these the first two are

$\frac{7}{17}$ .4117647059 $\frac{5}{12}$ 0.4166666667;

between these the first two are

$\frac{12}{29}$ .4137931034 $\qquad$ $\frac{17}{41}$ 0.4146341463;

between these the first two are

$\frac{41}{99}$ .4141414141 $\qquad$ $\frac{29}{70}$ 0.4142857143;

between these the first two are

$\frac{70}{169}$ .4142011834 $\qquad$ $\frac{99}{239}$ 0.4142259414.

All these are found in both parallel rows; but they are converging toward

$$0.41421356$$

or $\sqrt{2}-1$ which is not a rational fraction.

Now the fact that in this case the numbers happened to be rational fractions had nothing to do with the result. It is plain that in every case, when between two values we insert two, and between those two, two more again, and so on indefinitely, there remains a limit which is never reached; and the multitude which includes all such limits cannot be made to correspond, one to one, to any dinumerable collection.

§5. THE METHOD OF LIMITS

113. Let us settle the terminology as shown in this diagram

Enumerable $\qquad$ Inenumerable

Dinumerable

Numerable $\qquad$ Innumerable

The relation of the Innumerable to the Dinumerable is analogous to that of the Dinumerable to the Enumerable. Dinumerable is the multitude of enumerable numbers; innumerable is the multitude of dinumerable series. The dinumerable follows after the enumerable; but so closely after that as soon as you have passed all that is enumerable you have passed the dinumerable; so that we rightly reason such-and-such must be the character of the dinumerable, for if not there must be an enumerable which wants this character. In like manner the innumerable lies beyond the dinumerable; it is its limit; but it lies so closely beyond that we rightly reason, such-and-such must be the character of the innumerable; for if not, there must be a gap between this character and that of the dinumerable.

All reasoning about the Innumerable derives its force from the conception of a *limit*. We therefore have to study this conception. But two or three prolegomena are called for.

114. The idea that there can be any vigorous and productive thought upon any great subject without reasoning like that of the differential calculus is a futile and pernicious idea. Some newspapers maintain that all doctrines involving such reasoning ought to be struck out of political economy because that science is of no service unless everybody, or the great majority of voters, individually comprehend it and assent to its reasonings. I do not observe that it is a fact that voters are such asses as to insist upon thinking they personally comprehend the effects of tariff-laws, etc. But whether they be so or not, it is certain that the ratio of the circumference to the diameter is 3.14159265358979323846264338327950288 41971694 . . . whether the reasoning that proves it is hard or easy. That I feel sure of, although I personally have not verified the above figures; and if I had, I should not feel perceptibly more sure of the matter than I am. Certainly, if on attempting to verify them I got a slightly different result, I should feel pretty sure it was I who had committed an error. But whether people be wise or foolish, it remains that there is no possible way of establishing the true doctrines of political economy except by reasonings about *limits*, that is, reasoning essentially the same as that of the differential calculus. (I do not know why I should hesitate to say that the journal which I have particularly in mind is the New York *Evening Post*, incontestably one of the very best newspapers in the world, and especially remarkable for the sagacity of its judgments upon all questions of public policy.)

115. The reasoning of Ricardo about rent is this.* When competition is unrestrained by combination, producers will carry production to the limit at which it ceases to be profitable. Thus, a man will put fertilizers on his land, until the point is reached where, were he to add the least bit more, his little increased production would no more than just pay the increased expense. Every piece of land will be treated in this way, and every grade of land will be used down to the limit of the land upon which the product can just barely pay.

The whole reasoning of political economy proceeds in this fashion. If we put an import duty upon any article, the price to the consumer cannot be raised by the full amount of the tax. For the price before the imposition was such as to sell a certain amount. Now, if the price is raised, less can be sold. If less can be sold, less will be produced. But production will only be diminished by the producer getting a less price; and it is this less price *plus* the duty that the consumer pays. Of course, we must understand by the *duty*, not merely what goes to the government, but what has to be paid in consequence to brokers, bankers, and increased expenses of all kinds caused by the change of the law. Looking at the matter from this point of view (and abstracting from other considerations) the best articles on which to levy duties are those upon the production of which our demand is so influential that a small decrease in the demand will cause a relatively large fall in the price.[1]

116. As another preliminary to the analysis of the conception of Limit, I now pass to a widely different topic. The student has not failed to remark how much I have insisted upon balance and symmetry in logic. It is a great point in the art of reasoning; although I do not know that one could say that logic requires it. As long ago as 1867 I spoke of a trivium of formal sciences of symbols in general. "The first," I said, "would treat of the formal conditions of symbols having meaning, and this might be called formal grammar; the second,

* See *On the Principles of Political Economy and Taxation*, ch. II.

[1] Some remarks of mine to this effect were characterized by the *Evening Post* as "too much like the differential calculus." No doubt the reasoning was too sound for the convenience of those who maintain the consumer pays the whole duty.

logic, would treat of the formal conditions of the truth of symbols; and the third would treat of the formal conditions of the force of symbols, or their power of appealing to a mind, and this might be called formal rhetoric."[1] It would be a mistake, in my opinion, to hold the last to be a matter of psychology. That which needs no further premisses for its support than the universal data of experience that we cannot suppose a man not to know and yet to be making inquiries, that I do not think can advantageously be thrown in with observational science. Each of these kinds of science is strong where the other is weak; and hence it is well to discriminate between them. Now, the *Grundsatz* of Formal Rhetoric is that an idea should be presented in a unitary, comprehensive, systematic shape. Hence it is that many a diagram which is intricate and incomprehensible by reason of the multitude of its lines is instantly rendered clear and simple by the addition of more lines, these additional lines being such as to show that those that were there first were merely parts of a unitary system. The mathematician knows this well. We have seen what endless difficulties there are with "some's" and "all's". The mathematician almost altogether frees himself from "some's"; for wherever something outlying and exceptional occurs, he enlarges his system so as to make it regular. I repeat that this is the prime principle of the rhetoric of self-communing. Nobody who neglects it can attain any great success in thinking.

117. The innumerable appears in two different shapes. In the first place, if we append to the entire series of finite integral numbers, which is a dinumerable collection, all the infinite numbers, we obtain an innumerable collection. Or, if we take the series of rational fractions, also a dinumerable collection, and add to them the limits of all infinite series of such fractions, then again we obtain an innumerable collection. In this latter case, each instance taken from the innumerable collection is a limit which may be passed through. This latter is a more balanced conception than the other; but the mathematician reduces the other to it by conceiving, that in the former case also, after passing through infinity as a mere point, we pass into a new region — a new world. We pass off, for example,

[1] *Proceedings of the American Academy of Arts and Sciences*, VII, 295. [1.559.]

in a straight line parallel to the earth's axis northwardly: after passing through infinity we pass into an imaginary region from which after an infinite passage we re-emerge into our space at the extreme south. Or, it may be that this imaginary world reduces to nothing and that the points at infinity north and south coincide. This is the way the mathematician supplements facts in the interest of formal rhetoric. Of course, in doing so he has to take care not to misrepresent the real world; but his ideal addition to it may have any properties that simplicity dictates. This is an immense engine of thought in mathematics. It affords a little difficulty to the mind at first presentation; but that passes away very soon, and then it is found to be greatly in the interest of comprehensibility. Every mathematician will tell you this; if you are not already aware of it. But even among mathematicians there is a trace of that human weakness, the stupidity of adhering to what ought to be obsolete; and consequently the idea that infinity is something to pass through has not been everywhere carried out.

118. In many mathematical treatises the limit is defined as a point that can "never" be reached. This is a violation not merely of formal rhetoric but of formal grammar. True, in the world of real experience, "never" has at least an approximate meaning. But in the Platonic world of pure forms with which mathematics is always dealing, "never" can only mean "not consistently with —." To say that a point can never be reached is to say that it cannot be reached consistently with—, and has no meaning until the blank is filled up. And thereupon, the mathematical and balanced conception must be that the point is instantly passed through. The metaphysicians have in this instance been clearer than the mathematicians — and that upon a point of mathematics; for they have always declared that a limit was inconceivable without a region beyond it.

I understand that Jordan has rewritten the first volume of his *Cours d'analyse*. I have not seen this new edition (for all my life my studies have been cruelly hampered by my inability to procure necessary books), but I can guess to some extent what the character of it will be; and it no doubt contains much, most pertinent to the subject now under our attention. It was, I presume, this work which suggested to Klein a remark which he makes in his Evanston Colloquium, to the effect that there is a distinction between the *naïve* and the *refined* geometrical intuition. "In imagining a line," he says, "we do not picture to ourselves length without breadth, but a *strip* of a certain width. Now such a strip has, of course, always a tangent; *i.e.*, we can always imagine a straight strip having a small portion (element) in common with the curved strip."* The psychological remark seems to me incorrect. I, for my individual part, imagine a curve (even of an odd degree, which I convert into an even degree by doubling it, or by crossing it by a line) as the boundary between two regions pink and bluish grey; and I do not think I imagine the line as a strip. But it is of little consequence what individual ways of imagining may be. Klein's *naïve* view has a real importance far greater than his adjective imports, at which I have hinted in the Century Dictionary, under *Limits*, *Doctrine of*, where I say that this doctrine "should be understood to rest upon the general principle that every proposition must be interpreted as referring to a possible experience."[1] What I mean is that absolute exactitude cannot be revealed by experience, and therefore every boundary of a figure which is to represent a possible experience ought to be blurred. If this is the case, it is both needless and useless to talk of infinitesimals. Still thought of this inexact kind (I mean upon these essentially

* *The Evanston Colloquium; Lectures on Mathematics*, Lecture vi, p. 42, (1894).

[1] *Method or doctrine of limits:* The doctrine that we cannot reason about infinite and infinitesimal quantities and that phrases in mathematics containing these and cognate words are not to be understood literally, but are to be interpreted as meaning that the functions spoken of behave in certain ways when their variables are indefinitely increased or diminished, and that the fundamental formulæ of the differential calculus should be based on the conception of a limit. The first of these positions is not now tenable; the hypothesis of infinite and infinitesimal quantities is consistent and can be reasoned about mathematically, but the doctrine of limits should be understood to rest upon the general principle that every proposition must be interpreted as referring to a possible experience. The problems to which this method is applied belong to three types: the summation of series, the problem of differentials and the problem of quadratics. It is the same as Newton's method of prime and ultimate ratios. Its rival is the method of infinitesimals which is almost excluded from the textbooks at present. — *Century Dictionary and Cyclopædia*, p. 3458. Cf. 125ff, and 3.563ff.

inexact premisses) will be found *much more intricate and difficult* than the exact doctrine.

119. To define a limit, mathematicians usually write

$$x_n,$$

where x_1, x_2, x_3, etc. are supposed to successively approximate toward a value. Then they say that if after, perhaps, some scattering values, the successive x_n's at length come nearer and nearer to a constant which they indefinitely approach but *never* reach, that quantity is the limit. By saying they never reach it, they mean that as the $_n$ of x_n passes through infinity, x_n passes through the limit. This $n=\infty$ of course marks the point at which the collection which n measures becomes dinumerable. At that point x_n ceases to vary with n; else the value would be indeterminate.

120. I insert here a few remarks. The dinumerable is to the innumerable as logarithmic infinity is to ordinary infinity. The analogy may be traced in two ways; first the number of numbers expressible by n decimal points is, of course, b^n where b is the base of the system of numeration; but the innumerable is the number of numbers expressible by dinumerable decimal points. In the second place, the innumerable is not only dinumerably more than the dinumerable but is innumerably more.

§6. THE CONTINUUM

121. It may be asked whether there be not a higher degree of multitude than that of the points upon a line. At first sight, the points on a surface seem to be more; but they are not so. For points on surfaces can be discriminated by two coördinates with values running to a dinumerable multitude of decimal places. Now two such numbers or any enumerable multitude of them can be expressed by one series of numbers. Thus to express two, write a number such that the succession of figures in the odd places of decimals gives one coördinate, and those in the even places, the other. Thus,

$u=32.174118529821685238548599709435$ .... will mean
$x=3.141592653589793$ . . .
$y=2.718281828459045$ . . .

This method would break down if the number of dimensions were dinumerable; but even then another method could be found. But if the number of dimensions were innumerable, it is difficult to say without more study than I have given, how to proceed. The idea of space with innumerable dimensions does not, at first blush, at least, strike one as presenting great difficulty.

But if b^n when n is dinumerable gives a new grade of multitude, we might expect that when n was innumerable, a still higher grade would be given.

Yet, on the other hand, looking at the matter from the point of view of the original definitions given above, the three classes of multitude seem to form a closed system. Still, nothing in those definitions prevents there being many grades of multiplicity in the third class. I leave the question open, while inclining to the belief that there are such grades.* Cantor's theory of manifolds appears to me to present certain difficulties; but I think they may be removed.

Let us now consider what is meant by saying that a line, for example, is continuous. The multitude of points, or limiting values of approximations upon it, is of course innumerable. But that does not make it continuous. Kant† defined its continuity as consisting in this, that between any two points upon it there are points. This is true, but manifestly insufficient, since it holds of the series of rational fractions, the multitude of which is only dinumerable. Indeed, Kant's definition applies if from such a series any two, together with all that are intermediate, be cut away; although in that case a finite gap is made. I have termed the property of infinite intermediety, or divisibility, the *Kanticity* of a series. It is *one* of the defining characters of a continuum. We had better define it in terms of the algebra of relatives. Be it remembered that continuity is not an affair of multiplicity simply (though nothing but an innumerable multitude can be continuous) but is an affair of arrangement also. We are therefore to say not merely that there *can* be a quantitative relation but that there is such, with reference to which the collection is continuous.

* Cf. 214ff, 3.549n.

† See *Kritik der reinen Vernunft*, A169, 659; B211, 687.

Let $l\breve{\bar{l}}$ denote this relation. Then, as quantitative, this has, as we have seen,* these properties:

$$l\breve{\bar{l}} -\!\!< l \dagger \breve{\bar{l}},$$

and

$$0 \dagger (1 + l\breve{\bar{l}} + \bar{l}\breve{l}) \dagger 0.$$

Then the property of Kanticity consists in this:

$$l\breve{\bar{l}} -\!\!< l\breve{\bar{l}}l\breve{\bar{l}}.$$

122. To complete the definition of a continuum, the a's, we require the following property. Namely, if there be a class of b's included among the a's but all inferior to a certain a, that is, if

$$b -\!\!< a,$$
$$1 -\!\!< \breve{a}(l\breve{\bar{l}} \dagger \bar{b});$$

and if further there be for each b another next superior to it; that is,

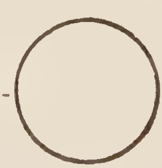
$$1 -\!\!< \breve{\bar{b}}\{l\breve{\bar{l}} \cdot [\bar{l} \dagger \breve{l} \dagger (\bar{b} + \bar{l} \dagger \breve{l})]\} \dagger b,$$

then there is an *a next* superior to all the b's. That is,

$$1 -\!\!< \breve{a}\{(l\breve{\bar{l}} \dagger \bar{b}) \cdot [\bar{l} \dagger \breve{l} \dagger (\bar{l} \dagger \breve{l})b]\}.$$

I call this the *Aristotelicity* of the series, because Aristotle seems to have had it obscurely in mind in his definition of a continuum as that whose parts have a common limit.†

123. If we consider a line (which, for rhetoric's sake, we will consider as returning into itself, though if it did not, it would give no difficulty further than an intolerably tedious complexity) it consists in a connection of points, such that by virtue of it, if any two points, A and Ω, be taken on that line, the points are divided into two parts, say the a_0 and the a_∞, such that a certain continuous quantitative relation, say l_0, subsists between all the a_0's having A for minimum and Ω for maximum, and another continuous quantitative relation, say l_∞, subsists between all the a_∞'s having the same maximum and minimum. The student is invited to state this



* In 96 and 97.

† *Metaphysica*, 1069a, 5.

in algebraic form using Π's, Σ's, and indices. He begins, for example

$$\Sigma_i \Pi_j \Pi_k \; \bar{a}_j \dotplus \bar{a}_k \dotplus q_{ij} \cdot \bar{q}_{ik} \dotplus \bar{q}_{ij} \cdot q_{ik} \dotplus l_{0jk} \dotplus l_{0kj} \dotplus l_{\infty jk} \dotplus l_{\infty kj}.$$

To this I wish to add something, which seems to require a preliminary remark. There are certain quantitative relations between the points such that if one of them were to govern an arrangement of the points in space, it would derange their connection in a line, in this sense, that it would cause some four points which are connected in the cyclical order *PQRS* (=*PSRQ*) to be brought in to one of the two orders *PQSR* (=*PRSQ*) or *PSQR* (=*PRQS*). We will call such a relation, for short, incompatible. Of course, there is nothing to prevent its existing; only the points cannot be arranged according to this order and remain in their order in the line. I now say that by no compatible continuous quantitative order can we pass from any a_0 to any a_∞ without passing through A or Ω. The student will do well to express this in terms of the algebra. Of course this statement requires modification in case the line forks. But for the purposes of logic it does not seem necessary to examine such details.

124. Pass we now to the study of the Surface. It is here that the mathematical conception of a "spread" — as Clifford calls it in insular but expressive language — at length displays itself. For an example of a surface, think of something irregularly round — multiple-connected surfaces complicate the subject, without advantage. They are easily taken into account later and the modifications they require made. A surface contains an innumerable multitude of lines. Let one of these — a complete oval — be marked upon it. Then the connection of points is such that this line separates those that do not lie upon it into two classes, such that it is impossible to pass from any point of one class to any point of the other by a *compatible* continuous quantitative order without passing through some point of the line. This is, however, perhaps not quite clear. Let us endeavor to make a better statement. Upon the oval take two points A and Ω. Then by virtue of the connection of points on the surface there is an innumerable series of continuous quantitative orders, each beginning at A and ending at Ω. Two of these, signified by the relative terms l_0 and l_∞,

follow the two parts of the oval. These orders are such that no two of them embrace the same point, except the initial and final points which are common to them all. And all other lines (or compatible continuous quantitative orders), twice cutting the oval, cut these different lines in the same order, say $l_0 \ldots l_n \ldots l_\infty \ldots l_n \ldots l_0 \ldots$ Every point on the surface lies on one of these lines.

§7. THE IMMEDIATE NEIGHBORHOOD

125. I wish to remark that it is a serious fault in the ordinary treatment of the fundamentals of geometry that attention is not paid to the distinction between the two sides of points on a line, lines on a surface, and surfaces in space. This is why certain theorems indubitably true are so difficult of formal proof. It is that a part of the fundamental properties of space have no expression among the Postulates of Geometry.

I think that I have thus described the nature of the connection of points upon the surface — and nothing need be added.

But there are three very important ideas I have left undefined. I mean those of the *simplest line* (straight line on a plane, great circle on a sphere, perhaps the geodetic line on other surfaces), the *immediate neighborhood*, and *measurement.* I have also imaginary quantity still to consider.

The easiest of these ideas seems to be that of the *immediate neighborhood.* It supposes that we recognize that every region stands in relation to a certain scale of quantity. We do not yet assign its quantity but we are able to say whether it is connected with an enumerable, a dinumerable, or an innumerable multitude. Two regions which are connected with quantities of the same class are said to be *about alike.* Suppose, then, we have a region about like the whole surface, or about like some region which we take as a standard. Suppose a thunderbolt rends this into two parts about alike, a crack separating them. Suppose a second thunderbolt similarly rends both parts; and each successive thunderbolt rends all the parts the last left into two new parts about alike. Suppose these thunderbolts to follow at the completion of each rational fraction of a minute. Then, at the end of the minute, the region will be rent into innumerable parts about alike. These parts are neighborhoods or infinitesimals.

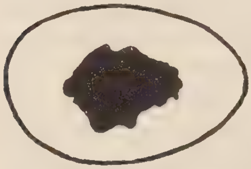
126. It will at once be objected that there is no reason to suppose that this operation would leave any parts at all, or if it did there is no reason to suppose they would be surfaces rather than angels, or oranges, or precessions of the equinoxes; for the only reason for thinking they remained of the same genus is that no one thunderbolt would change them. Reasoning from that premiss, however, would be a Fermatian inference, and would, as such, only hold good for the dinumerable.

But the reply is that there is no need of calling in the Fermatian inference. The minute of thunderbolts does not differ from any other minute, as far as the character of the surface goes. The parts have been moved a little, but all their mutual relations are undisturbed. Even if the operation broke it up into single points, which is an unfounded proposition, still all the cracks that have been made in no wise alter the relations of the points to one another. The space the region occupies, though interfiltrated through with another space, remains the same, and the relations of its parts the same. If this conception is too difficult, imagine that the thunderbolts do not rend the regions, but only cause a mind to imagine them rent.

It would, however, be quite out of order to consider the question of whether these parts are single points or what their composition may be, until it first be fully admitted that the logical division of the region into innumerable parts is logically possible. But there is no room for dispute here. It has been irrefragably demonstrated that the points of a line, and *a fortiori* of a surface, are innumerable. Now, as no two coincide, there is nothing in logic to prevent their being drawn asunder. My definition of a continuum only prescribes that, after every innumerable series of points, there shall be a *next* following point, and does not forbid this to follow at the interval of a mile. That, therefore, certainly permits cracks everywhere; for there is no ordinal place in the series where such a limit point is not inserted. But if anybody thinks my definition is in error here, still it will not be maintained that that definition *involves any contradiction*. Hence, there is no contradiction in the separation into parts, even if I am wrong in saying that it involves no breach of continuity. There is no contradiction involved in breaking the region anywhere. But perhaps it may be said, the contradiction lies, not in breaking it anywhere, nor in breaking it into as many parts as it has points, but that the idea of an innumerable multitude involves a contradiction. That it does not can be formally demonstrated by second-intentional logic; but that part of this book having been excised, it is necessary to find other arguments. There is no difficulty about the existence of π, and therefore none in the existence of incommensurable limits. There is no more difficulty about the existence of any one number not accurately expressible in a finite number of decimals than in any other. Therefore, there is no logical contradiction in supposing all numbers to which decimals can indefinitely approximate to exist, *i.e.*, as all the objects of mathematics exist, as abstract hypotheses. Besides, that the innumerable multitudes are logically possible is shown by the fact that many propositions (namely all that are true of the dinumerable but not of the numerable) cannot be demonstrated in a way which will stand logical examination unless it be expressly introduced as a premiss that a given multitude is numerable. Now a logically necessary proposition is of no avail as a premiss. On the whole then, there is nothing in logic to prevent a region from being divided into innumerable parts about alike.

Now I say that each of those parts contains innumerable points. For if that were not the case each of these parts could be so arranged that every point had another next after it; and, since a continuum has no molecular constitution, the divisions could everywhere be made between points having other points next them; and so, after rearranging the parts (no matter how the continuity might be broken up) all the points would have points next after them. But this is contrary to the fact that the points are innumerable. Besides, going back to the unanalyzed idea of continuity, it is evident that in a continuum the points are so connected that every part, irrespective of its magnitude, contains innumerable points. It may be objected that the single points are parts. But that is not properly true. The single points are parts of the collection; but they cannot be broken off by a division of parts unless they are on the outer boundary of a region, or unless they are not continuous with the rest. They can be extracted from the middle; but doing this breaks the continuity. Thus the incommensurable numbers taken by themselves do not form a continuum.

127. A drop of ink has fallen upon the paper and I have walled it round. Now every point of the area within the walls is either black or white; and no point is both black and white.



That is plain. The black is, however, all in one spot or blot; it is within bounds. There is a line of demarcation between the black and the white. Now I ask about the points of this line, are they black or white? Why one more than the other? Are they (A) both black and white or (B) neither black nor white? Why A more than B, or B more than A? It is certainly true,

First, that every point of the area is either black or white,

Second, that no point is both black and white,

Third, that the points of the boundary are no more white than black, and no more black than white.

The logical conclusion from these three propositions is that the points of the boundary do not exist. That is, they do not exist in such a sense as to have entirely determinate characters attributed to them for such reasons as have operated to produce the above premisses. This leaves us to reflect that it is only as they are connected together into a continuous surface that the points are colored; taken singly, they have no color, and are neither black nor white, none of them. Let us then try putting "neighboring part" for point. Every part of the surface is either black or white. No part is both black and white. The parts on the boundary are no more white than black, and no more black than white. The conclusion is that the parts near the boundary are half black and half white. This, however (owing to the curvature of the boundary), is not exactly true unless we mean the parts in the *immediate neighborhood* of the boundary. These are the parts we have described. They are the parts which must be considered if we attempt to state the properties *at precise points* of a surface, these points being considered, as they must be, in their connection of continuity.

One begins to see that the phrase "immediate neighborhood," which at first blush strikes one as almost a contradiction in terms, is, after all, a very happy one.

What is the velocity of a particle *at* any instant? I answer it is the ratio of space traversed to time of traversing, in the *moment*, or time in the immediate neighborhood, of that *instant*, or point of time. Some logicians object to this. They say that the velocity means nothing but the limiting value of the ratio of the space to the time when the time is indefinitely diminished. But they say they use the expression "immediate neighborhood" to mean nothing more than that, as a convenience of language. Sometimes we meet with an assertion difficult to refute because it involves several difficult logical blunders. The position just stated is an example of this. People who talk in this way do not see that what they say is a justification of the idea of a part such as the whole contains an innumerable multitude of. I do not yet say "immeasurably small," because we have not yet studied the conception of measurement. These people do not seem to have analyzed the conception of a "meaning,"* which is, in its primary acceptation, the translation of a sign into another system of signs, and which, in the acceptation here applicable, is a second assertion from which all that follows from the first assertion equally follows, and *vice versa*. It is true that, when we find with reference to a continuous motion that something would be true at the limit of a dinumerable series, it follows this is true for the part about the point considered. . . . This is as much as to say that the one assertion "means" the other. But do these people mean to say that when I think of a particle as having a velocity, I can only think, or that it is convenient to think, simply that at different times it is stationed at different points? Do they mean to say I have no direct, clear icon of a movement? If so, they are shutting their eyes to the plain truth. Remember it is by icons only that we really reason, and abstract statements are valueless in reasoning except so far as they aid us to construct diagrams. The sectaries of the opinion I am combating seem, on the contrary, to suppose that reasoning is performed with abstract "judgments," and that an icon is of use only as enabling me to frame abstract statements as premisses.

The idea of "immediate neighborhood" is an exceedingly easy one, into which everybody is continually slipping, though he fancies it unjustifiable. Klein says of his "refined intuition"

* Cf. 1.339, 1.343f, 2.293.

that, strictly speaking, it is not an intuition. But, speaking as strictly as that, there is no such psychological phenomenon as an intuition.* The strip, which he says makes the curve in the *naïve* intuition, makes two parallel curves with a region between. But the simple idea is that of a blurred outline, to which we all, wise and simple, append the mental note that its breadth is such that an innumerable number would be contained in any surface.

Those who, finding themselves betrayed into the use of the expression "immediate neighborhood" or something equivalent, seek to justify it by the *exigencies of speech*, are mistaken. It is not English grammar which forces these words upon them, but it is the very grammar of thought — formal grammar — which forces the idea upon them. The idea of supposing that they can think about motion without an image of something moving!

We must return to this subject after having considered the nature of measurement.

§8. LINEAR SURFACES

128. Euclid† defines a straight line as a line which lies evenly between its points. This is the real Greek acuteness; it is as much as to say that if a straight line be moved, its new position intersects its old one in one point at most. This is substantially the idea of all modern geometry. Legendre,‡ it is true, defined the straight line as the shortest distance between two points, as it most indubitably is. Nor do I think that it would be fair to object that this definition is *metrical*, that is, supposes a definition of measurement. For all kinds of measurement known make the straight line the shortest (or the longest, sometimes, if there be a longest) distance, if there be a shortest distance. But a more serious objection to Legendre's definition is that, if that be adopted, its property of two straight lines not intersecting in two places follows as a consequence; while, if Euclid's definition be adopted, there must be a separate postulate to the effect that there is a shortest distance. Thus, Euclid's definition involves a more thor-

* See 5.213ff.

† Bk. I, def. 4; but cf. Heath, *The Thirteen Books of Euclid's Elements*, pp. 165–169, (1926).

‡ *Elements de géometrie*, livre premier, def. 3.

ough analysis of the properties of space. Legendre conceived the other way, which wraps up as much as possible in one formula, to be the best. It certainly is not so for the purposes of logic.

When instead of a plane we consider a roundish surface, it is difficult to say what sort of an oval best corresponds to a straight line. Most writers have assumed that it is the geodetic line which is the shortest (or longest) distance between its points. But they seem to have forgotten that a geodetic line on almost any surface but a perfect sphere generally intersects itself a dinumerable multitude of times. The discussion of this question would involve very difficult mathematics, quite out of place in this work.

We must, therefore, confine ourselves to the plane. Now it is evident that the definition we have adopted supposes straight lines to move about in the plane without ceasing to be straight. Hitherto, all the properties of the connection of points are such as might hold if the plane were a fluid; for though discontinuous fluid motion is conceivable, it has no place in the usual conceptions of the student of hydrokinetics. But now we propose that the straight line should move about as if it were a rigid bar. However, it is not necessary to broach the theory of elasticity, a doctrine of Satanic perplexity. We may call a straight line the path of a ray of light, or the shadow of a dark point cast from a luminous point. That is rather a pretty idea. Or going down to the roots of physics, we may define the straight line as the path of a particle, not deflected by any force. This is, so far as we can see, the origin of the importance of the straight line in the physical world. But, then, at present it is doubtful whether we are concerned at all with the physical world. We would like, if we could, to find some logical property of the straight line distinguishing it from other curves. I fear, however, there is none, if we are to leave its shortness out of account. We can perfectly well conceive of a cubic curve, such as is shown in the figure, moving about with modifications of its shape, so as in any position to cut any other position once and once only (in real space).

A mathematician will easily write down the conditions for this. Namely, the equation of the serpentine is

$$y = \frac{1}{x + \frac{1}{x}},$$

and that of the different cubics is

$$\frac{x}{a} + \frac{\left\{ y - \frac{1}{x + \frac{1}{x}} \right\}}{b} = 1.$$

There is nothing in the plane geometry of the straight line which is not equally true, *mutatis mutandis*, of such a system of cubics.

But the intersectional properties of straight lines in a plane are not exhausted in saying that any two straight lines intersect once and once only.

129. Let us resort to our algebra of relatives. Denote unlimited straight lines by lower case italic letters. Let capitals denote points. Let Greek minuscules denote certain marks of lines. All these letters are treated as indices; but they will be written on the line.

Let aB (or any similar pair of letters) mean that the line a is considered as having the point B, which lies on it. If the point B is not on the line a, then $\overline{aB}$; but even if B be on a, it does not necessarily follow that B is regarded as belonging to the line a, and if not, again $\overline{aB}$. A point may belong to two lines, at once.

Let αb (or any similar pair of letters) signify that the line b has the mark α, the nature of which will appear in the sequel.

Let αB, etc., signify that the point B belongs to some line that has the mark α.

Let us now endeavor to sum up in a series of propositions the fundamental truths about the intersections of lines.

First proposition.

$$\Pi A \Pi B \Sigma c \quad cA \cdot cB$$

that is, any two points *may be* regarded as belonging to one straight line.

Second proposition.

$$\Pi\alpha\Pi\beta\Sigma c \;\; \alpha c \cdot \beta c,$$

that is, given any two marks, an unlimited straight line having them both may be drawn.

Third proposition.

$$\Pi a \Pi b \Pi C \Pi D \;\; \overline{aC} \dagger \overline{aD} \dagger \overline{bC} \dagger \overline{bD} \dagger 1ab \dagger 1CD,$$

that is, if two points are regarded as belonging to two lines, either the two points or the two lines coincide.

Fourth proposition.

$$\Pi\alpha\Pi\beta\Pi c\Pi d \;\; \overline{\alpha c} \dagger \overline{\alpha d} \dagger \overline{\beta c} \dagger \overline{\beta d} \dagger 1\alpha\beta \dagger 1cd,$$

that is, if two marks belong to two lines, either the two marks are coextensive or the two lines coincide.

Fifth proposition.

$$\Pi A \Pi b \Sigma\gamma \;\; \gamma b \cdot \gamma A,$$

that is, given any line, any point may be regarded as belonging to a line having a mark in common with the given line.

Sixth proposition.

$$\Pi\alpha\Pi b \Sigma C \;\; \alpha b \cdot \alpha C,$$

that is, given any mark and any line, it is always possible to find a point which may be regarded as belonging to the given line and to some line having the given mark.

Seventh proposition.

$$\Pi\alpha\Pi\beta\Pi c\Pi D \;\; \overline{\alpha c} \dagger \overline{\alpha D} \dagger \overline{\beta c} \dagger \overline{\beta D} \dagger cD \dagger 1\alpha\beta,$$

that is, if two marks belong to a given line and to lines to which a given point is regarded as belonging, that point must be regarded as belonging to that line, unless the two marks are coextensive.

Eighth proposition.

$$\Pi a \Pi b \Pi C \Pi D \;\; \overline{aC} \dagger \overline{aD} \dagger \overline{bC} \dagger \overline{bD} \dagger ab \dagger 1CD,$$

that is, if two points are regarded as belonging to a given line and to lines having a given mark, that line has that mark unless the two points coincide.

Ninth proposition.

$$\Pi\alpha\Pi b\Pi c\Pi d\Pi E\ \overline{\alpha b} + \overline{\alpha c} + \overline{\alpha d} + \overline{bE} + \overline{cE} + \overline{dE} + 1bc + 1bd + 1cd,$$

that is, any three lines either have no common point or no common mark.

Tenth proposition.

$$\Pi b\Pi c\Pi d\Sigma\alpha\Sigma E \quad \alpha b\cdot \alpha c\cdot \alpha d + bE\cdot cE\cdot dE$$ *[1]

130. . . . The student may object, at first blush, that the marks indicated by Greek letters *have no meaning.* This is a great mistake; they have precisely the meaning that is pertinent; but it is true they have no meaning in the sense of anything which particularly strikes ordinary attention. Reflect upon this. What people call an "interpretation" is a thing totally irrelevant, except that it may show by an example that no slip of logic has been committed.

131. At this point, I should like to give some account of Schubert's *calculus of enumerative geometry*†, which is the most extensive application of the Boolian algebra which has ever been made, and is of manifestly high utility. But I do not feel that I could possibly condense the elementary explanations or clarify them more than Schubert has himself done in his book. He has by no means exhausted the powers of his method. There is plenty of room for new researches; but his work will stand as the classical treatise upon geometry as viewed from the standpoint of arithmetic for an indefinite future.

§9. THE LOGICAL AND THE QUANTITATIVE ALGEBRA

132. Cauchy‡ first gave the correct logic of imaginaries, and very instructive it is. But the majority of writers of text-

* Any three lines have a common mark or a common point.

[1] Is this right? I have been working right on end for over twelve hours and with slight interruption for over 20 hours and I can hardly tell what I write. — A marginal note, addressed apparently to Judge Russell, to whom the "Grand Logic" was submitted for criticism.

† *Kalkül der Abzählenden Geometrie* Hermann Schubert, Leipzig, (1879).

‡ See *e.g.*, his *Nouveaux Exercises d'Analyse et de Physique mathématique*, t. III et IV (1840–1847).

books, who reason by the rule of thumb, do not understand it to this day. The square of the imaginary unit, i, is -1, and therefore it may be allowable to speak of i and $-i$ as being two square roots of -1. But to speak of them as *the* two square roots of -1 will not do. The algebraist sets out with a single continuous quantitative relation. But when he comes to quadratics he finds himself confronted with impossible problems. He says: "I want a square root of negative unity. Now there is no such thing in the universe: clearly, then, I must import it from abroad." Let us see how one would go about to prove there is no square root of negative unity. He would reason indirectly: that is the mathematician's recipe for everything. He would say let i be this square root if there be one. Then, whether its sign be $+$ or $-$, its square will have a positive sign, contrary to the hypothesis. Then the whole impossibility depends upon this, that every quantity is supposed to be positive or negative. Suppose we make i neither positive nor negative. "But there is no such thing," some rule-of-thumb man says. Really? In that respect it is just like all the other objects the mathematician deals with. They are one and all mere figments of the brain.* "But to say that a quantity is neither positive nor negative *means nothing*," objects the thumbist. I reply, the meaning of a sign is the sign it has to be translated into. Now in mathematics, which is merely tracing out the consequences of hypotheses, to say a thing has no meaning is to say it is not included in our hypothesis. In that case, all we have to do is to enlarge the hypothesis and put it in. That is your course when you have a concrete hypothesis. That was our conduct when we called a debt, negative property. But, at present, we are dealing with algebra in the abstract. The only hypothesis we make is that our letters obey the laws of algebra. If there is one of those laws which requires a quantity to be either positive or negative, find out which it is and delete it. If you have a system of laws which is self-consistent, it will not be less so when one of them is wiped out. But let us see what the laws of algebra are and how they are affected toward a quantity whose square is negative. We have,

* Cf. 2.191–2, 2.305, 2.778, 3.426.

(1) If $x=y$, then x may anywhere be substituted for y.
(2) $x+y=y+x$.
(3) $x+(y+z)=(x+y)+z$.
(4) $xy=yx$.
(5) $x(yz)=(xy)z$.
(6) $(x+y)z=xz+yz$.
(7) $x+0=x$.
(8) $x1=x$.
(9) $x+\infty=\infty$.
(10) If $x+y=x+z$, either $y=z$ or $x=\infty$.
(11) If $xy=xz$, either $y=z$, or $x=0$, or $x=\infty$.
(12) If $x>y$, not $y>x$.
(13) If $x>y$, then there is a quantity a such that $a>0$ [and] $a+y=x$.
(14) If $x>y$, then $x+z>y+z$.
(15) If $x>0$ and $y>0$, then $xy>0$.
(16) Either $x>y$, or $x=y$, or $y>x$.
(17) $1>0$.

It is plain that from these equations it is impossible to prove that $x^2<0$ is not true except by the aid of one of the last six formulæ, and further that it will be requisite to consider the factors of x^2. Now (15) is the only one of the last six formulæ, directly containing a product. This gives $x^2>0$ provided $x>0$. Also, if $0>x$, let $x+\xi=0$, by (13), where $\xi>0$.

Then, by (6), $$x\xi+\xi^2=0\xi$$

But by (7),
$$y+0=y,$$
and by (6),
$$(y+0)\xi=y\xi+0\xi=y\xi,$$
and by (7),
$$y\xi+0\xi=y\xi+0,$$
and by (10),
$$\text{either } 0\xi=0 \text{ or } y\xi=\infty \text{ whatever } y \text{ may be.}$$

But the last alternative is absurd; for then by (9),
$$y0=y(x+\xi)=yx+y\xi=yx+\infty=\infty.$$

But if $y=1$ by (8),
$$y0=10=0.$$

Hence we should have

$$0=\infty$$

whence by (7) and (9),

$$z=0=\infty$$

whatever z may be. Hence by (1),

$$\text{If } u>v \quad v>u$$

Hence by (12), in no case is $u>v$. But this contradicts (17).

We have, then,

$$0\xi=0.$$

Hence by (2) and (7),

$$x\xi+\xi^2=0,$$

But by (15),

$$\xi^2>0.$$

Hence by (14),

$$x\xi+\xi^2>x\xi+0.$$

Hence by (7),

$$x\xi+\xi^2>x\xi$$

or,

$$0>x\xi.$$

But by (6) and (4),

$$x(x+\xi)=x^2+x\xi=x0=0x=0.$$

Hence,

$$x^2+x\xi>x\xi.$$

Now since $0>x\xi$ by (13) there is a quantity a such that

$$a>0, \quad a+x\xi=0$$

Hence, by (14),

$$x^2+x\xi+a>x\xi+a$$

Hence, finally, by (3),

$$x^2+0>0$$

or by (7),

$$x^2>0.$$

Hence, by (16), in every case

$$x^2>0 \quad \text{or} \quad x=0.$$

But it is plain that without (16) this conclusion could not be drawn, since no other of the formulæ (12)–(17) have anything to say about quantities neither greater, less, nor equal to one another.

It thus appears that we have only to strike out (16), and the quantity i such that $i^2=-1$, becomes perfectly possible, and perfectly *conceivable*, in the only clear sense of that word, namely, that we can write down

$$i^2+1=0$$

without conflict with any formula. If we define $-x$ by the formula

$$x+(-x)=0,$$

then, necessarily, if $i^2=-1$, we have also $(-i)^2=-1$. Ordinary algebra *assumes* there is no other quantity except these two whose square equals -1. Thus, if the algebraist finds $x^2=y^2$, he at once writes $x=\pm y$. This is because he chooses to exclude all other square roots of -1. I will return to this point shortly.*

133. Men are anxious to learn what the square root of negative unity *means*. It just means

$$i^2+1=0;$$

precisely as -1 means

$$1+(-1)=0.$$

The algebraic system of symbols is a *calculus;* that is to say, it is a language to *reason in*. Consequently, while it is perfectly proper to define a *debt* as negative property, to explain what a negative quantity is, by saying that it is what debt is to property, is to put the cart before the horse and to explain the more intelligible by the less intelligible. To say that algebra means anything else than just its own forms is to mistake an *application* of algebra for the *meaning* of it.[1] But to this statement a proviso should be attached. If an application of algebra consists in another system of diagrams having properties analogous to those of the sixteen fundamental formulæ, or to the greater number of them, and if that other system of diagrams is a good one to reason in, and may advantageously be taken as an adjunct of the algebraic system in reasoning, then

* See 138.

[1] See my "Description of a Notation for the Logic of Relatives." Also, my brochure entitled "Brief Description of the Algebra of Relatives." [Vol. 3, Nos. III and IX.]

such system of diagrams should be regarded as more than a subject for the *application* of logic, and though it is too much to say it is *the* meaning of the algebra, it may be conceived as a secondary, or junior-partner meaning. Such junior interpretations are especially, the *logical* and the *geometrical.*

134. Logical algebra ought to be entirely self-developed. Quantitative algebra, on the contrary, ought to be developed as a special case of logical algebra. I do not mean that elementary teaching should set it on that basis; but that should be recognized as the fundamental philosophy of it. The seminary logicians have often seemed to think that those who study logic algebraically entertain the opinion that logic is a branch of the science of quantity. Even if they did, the error would be a trifling one; since it would be an isolated opinion, having no influence upon the main results of their studies, which are purely formal. But with the possible exception of Boole himself, whose philosophical views have not been lauded by any of his followers, none of the algebraic logicians do hold any such opinion. For my part, I consider that the business of drawing demonstrative conclusions from assumed premisses, in cases so difficult as to call for the services of a specialist, is the sole business of the mathematician. Whether this makes mathematics a branch of logic, or whether it cuts off this business from logic, is a mere question of the classification of the sciences. I adopt the latter alternative, making the business of logic to be analysis and theory of reasoning, but not the practice of it. To show how reasoning about quantity may be facilitated by considering logical interpretations, I may instance the Enumerate Geometry of Schubert,* which works by means of the logical calculus, and Mr. MacColl's† method of transposing the limits of multiple integrals, which is done by the Boolian algebra. Dr. Fabian Franklin has effected some difficult algebraical demonstrations by considering quantities as expressive of probabilities. I myself made two additions to the theory of multiple algebra by considering it as expressive of the logic of relatives.‡

* *Kalkül der Abzählenden Geometrie*, Leipzig, (1879).

† "The Calculus of Equivalent Statements and Integration Limits," *Proceedings*, London Mathematical Society, vol. ix, pp. 9–20 (1877–8).

‡ See 3.150f, and 3.324f.

135. The idea of multiplication has been widely generalized by mathematicians in the interest of the science of quantity itself. In quaternions, and more generally in all linear associative algebra, which is the same as the theory of matrices, it is not commutative. The general idea which is found in all of these is that the product of two units is the pair of units regarded as a new unit. Now there are two senses in which a "pair" may be understood, according as BA is, or is not, regarded as the same as AB. Ordinary arithmetic makes them the same. Hence, 2×3 or the pairs consisting of one unit of a set of 2, say, I, J, and another unit of a set of 3, say X, Y, Z, the pairs IX, IY, IZ, JX, JY, JZ, are the same as the pairs formed by taking a unit of the set of 3 first, followed by a unit of the set of 2. So when we say that the area of a rectangle is equal to its length multiplied by its breadth, we mean that the area consists of all the units derived from coupling a unit of length with a unit of breadth. But in the multiplication of matrices, each unit in the Pth row and Qth column, which I write $P{:}Q$, of the multiplier coupled with a unit in the Qth row and Rth column, or $Q{:}R$ gives

$$(P{:}Q)(Q{:}R)=P{:}R$$

or a unit of the Pth row and Rth column of the multiplicand. If their order be reversed,

$$(Q{:}R)(P{:}Q)=0,$$

unless it happens that $R=P$.

136. In my earlier papers on the logic of relatives I made an application of the sign of involution* which, I am persuaded, is less special than it seems at first sight to be. Namely, I there wrote

$$l^s$$

for the lover of *every* servant, while ls was the lover of *some* servant.

$$l^{(sm)}=(l^s)^m,$$

or the lover of everything that is servant to a man stands to every man in the relation of lover of every servant of his.

$$l^{w \Psi m}=l^w \cdot l^m$$

* 3.77ff.

or the lover of everything that is either woman or man is the same as the lover of every woman and, at the same time, lover of every man.

$$(l \cdot b)^m = l^m \cdot b^m$$

or that which is to every man at once lover and benefactor is the same as a lover of every man who is benefactor of every man.

$$(e +\!\!, c)^f = e^f +\!\!, [f] e^{f-1^*} \cdot c^{1^*} +\!\!, \frac{[f] \cdot ([f]-1)}{2} \cdot e^{f-2^*} \cdot c^{2^*} +\!\!, \text{etc.}$$

that is to say, those things each of which is to every Frenchman either emperor or conqueror consist first of the emperors of all Frenchmen; second, of a number of classes equal to the number of Frenchmen, each class consisting of all emperors of all Frenchmen but one who are at the same time conquerors of that one; third, of a number of classes equal to half the product of the number of Frenchmen by one less than the number of Frenchmen, each class consisting of every individual which is emperor of all Frenchmen but two and conqueror of those two; etc.

This makes

$$l^m = lm_1 \cdot lm_2 \cdot lm_3 \cdot lm_4 \cdot \text{etc.}$$

Of course, the ordinary idea which makes of involution the iteration of an operation, is a special case under this.

Thus, quantitative algebra is only a special development of logical algebra. On the other hand, it is equally true that the Boolian algebra is nothing but the mathematics of numerical congruences having 2 for their modulus.

137. The geometrical interpretation affords great aid in reasoning, because man has, so to speak, a natural genius for geometry. Thus we see easily enough, algebraically, that

$$x+yi=\sqrt{x^2+y^2}\left\{\sqrt{\frac{\frac{x}{y}}{\frac{x}{y}+\frac{y}{x}}}+\sqrt{\frac{\frac{y}{x}}{\frac{x}{y}+\frac{y}{x}}}\,.\,i\right\}$$

and further that

$$(x+yi)(u+vi)=\sqrt{(x^2+y^2)(u^2+v^2)}\left\{\sqrt{\frac{\frac{x}{y}\frac{u}{v}-\frac{y}{x}\frac{v}{u}}{\left(\frac{x}{y}+\frac{y}{x}\right)\left(\frac{u}{v}+\frac{v}{u}\right)}}+\sqrt{\frac{\frac{x}{y}\frac{v}{u}+\frac{y}{x}\frac{u}{v}}{\left(\frac{x}{y}+\frac{y}{x}\right)\left(\frac{u}{v}+\frac{v}{u}\right)}}.i\right\}.$$

But that which is by no means obvious algebraically, but becomes obvious geometrically, is that when we plot x and y as abscissa and ordinate of rectangular coördinates and u, v as other values of the same coördinates, and the product in the same way, the angle from the axis of abscissas of the product is equal to the sum of those of the two factors. This once found out, in the geometrical way, is easily put into algebraical form. Geometry here renders a precisely similar service to that which the theory of probabilities often lends. There are several instances in which mechanical instincts have been valuable in the same way. A choice collection of such lemmas would be interesting.

§10. THE ALGEBRA OF REAL QUATERNIONS

138. I now turn back to square roots of negative unity, not supposing multiplication to be commutative. That is, we do not generally have $xy=yx$. Suppose we have two quantities i and j, such that

$$i^2+1=0$$
$$j^2+1=0$$

Then it is plain that

$$(iji)(iji)=(ij)ii(ji)=-(ij)(ji)=-i.jj.i=ii=-1,$$

so that iji and jij are also square roots of negative unity.

Five cases may be studied:

First,	$iji=i$
Second,	$iji=-i$
Third,	$iji=j$
Fourth,	$iji=-j$
Fifth,	$iji=k$ (a third unit).

First Case. $iji = i$. Then,

$$i.iji = -ji = ii$$
$$-j = i.$$

Second Case. $iji = -i$. Then,

$$i.iji = -ji = -ii$$
$$j = i.$$

Third Case. $iji = j$. Then,

$$ijij = jj = -1$$
$$jiji = jj = -1$$

and ij and ji are also square roots of negative unity.

$$iji.i = -ij = ji.$$

But,

$$(ij)i = j$$
$$i(ij) = -j,$$

equations that do not hold for $ij = i$ nor for $ij = j$.

Nor can we put

$$ij = \sin\ \theta.\ i + \cos\ \theta.j$$

For then,

$$i.ij = -\sin\theta + \cos\theta.ij = -\sin\theta + \cos\theta\ \sin\theta.i + \cos^2\theta.j$$

and we have

$$\cos\theta = \sqrt{-1}.$$

Let us then write

$$ij = k$$
$$ji = -k$$

Then,

$$ki = iji = j$$
$$ik = iij = -j$$
$$kj = ijj = -i$$
$$jk = -jji = i$$

This is the algebra of quaternions.*

* See 3.130f.

Fourth Case. $iji = -j$. Then,

$$iji.i = -ij = -ji$$

or

$$ji = ij.$$

Hence, since

$$i^2 = j^2$$

$$(i-j)(i+j) = 0$$

$$j = \pm i.$$

Fifth Case. $iji = k$. The multiplication cannot be commutative. We may then have four infinite series of units

$i_1 = i$	$j_1 = j$	$k^1 = ij$	$l^1 = ji$
$i_2 = iji$	$j_2 = jij$	$k^2 = ijij$	$l^2 = jiji$
$i_3 = ijiji$	$j_3 = jijij$	$k^3 = ijijij$	$l^3 = jijiji$
$i_4 = ijijiji$	$j_4 = jijijij$	$k^4 = ijijijij$	$l^4 = jijijiji$
etc.	etc.	etc.	etc.

Here

$$i_n = il^{n-1} = k^{n-1}i$$

$$j_n = jk^{n-1} = l^{n-1}j$$

$$i_m j_n = k^{m+n-1}$$

$$j_m i_n = l^{m+n-1}$$

It is possible to suppose these all different.

If, on the other hand, any two are equal, there are but a finite number of different units. For example, if

$$ijijiji = jij$$

then,

$$ijiji = jijij$$

And all forms of more than five letters are equal to forms of quite as few as five letters. Thus,

$$ijijij = jijij.j = -jiji.$$

139.* But the moment we suppose the number of linearly independent letters is finite we can reason as follows. Taking any expression, A, some power of it is a linear function of inferior powers. Hence, there is some equation

$$\Sigma_m(a_m A^m) + a_0 = 0.$$

* Cf. 3.301ff.

By the theory of equations, this is resoluble into quadratic factors. One of these, then, must equal zero. Let it be

$$(A-s)^2+t^2=0.$$

Then,

$$\left(\frac{A-s}{t}\right)^2=-1$$

or, every expression, upon subtraction of a real number from it, can be converted, in one way only, into a square root, of a negative number. Let us call such a square root, the *vector* of the first expression, and the real number subtracted, the *scalar* of it.

Let $v^2=-1$, $j^2=-1$, and $ij=s+v$,

where s is scalar, and v vector. Then it is impossible to find three real numbers, a, b, c, such that

$$v=a+bi+cj$$

For assume this equation. Then, since

$$ij.j=-i,$$

$$-i=sj+vj=-c+(s+a)j+bij$$
$$=bs-c+ab+b^2i+(s+a+bc)j.$$

Whence,

$$b^2=-1,$$

and b could not be real.

Moreover, we shall have

$$ji=r(s-v),$$

where v is a real number. For write

$$ji=s'+v',$$

where s' is the scalar, and v' the vector, of ji. Let us write, too,

$$vv'=s''+v'',$$

where s'' and v'' are again the scalar and vector of vv'. Then,

$$ij.ji=(s+v)(s'+v')=ss'+sv'+s'v+s''+v''.$$

But

$$ij.ji=1.$$

Hence,

$$v''=1-ss'-s''-sv'-s'v.$$

But it has just been proved that the vector of the product of two vectors is linearly independent of these vectors and of unity. Hence

$$v''=0.$$

That is,

$$sv'=1-ss'-s''-s'v.$$

But it has just been shown that a quantity can be separated into a scalar and a vector part in only one way. Hence

$$sv'=-s'v$$
$$s''=1-ss'.$$

The former equation makes

$$ji=\frac{s'}{s}(s-v).$$

Let us next consider such an expression as

$$ai+bj=S+V$$

where S and V are the scalar and vector of the first member. Squaring the vector, we get

$$V^2=-N=(ai+bj-S)^2=-a^2-b^2+S^2+abs+abs'-2aSi-2bSj$$
$$+ab\left(1-\frac{s'}{s}\right)v$$

or

$$ab\left(1-\frac{s'}{s}\right)v=-N+a^2+b^2-S^2-abs-abs'+2aSi+2bSj.$$

But since v is the vector of ij it must, as we have seen, be linearly independent of unity, i, and j. Hence the first member must vanish. But if $v=0$, $ij=s$, whence $-i=sj$, contrary to hypothesis. Hence,

$$1-\frac{s'}{s}=0$$

or

$$s'=s.$$

Whence,

$$s''=1-s^2.$$

But s'' being the negative of the square of v is positive. Hence,

$$s^2\leqq 1.$$

We know that $ai+bj$ cannot be a scalar; for then a quantity could in two ways be resolved into a scalar and vector part. Now

$$(ai+bj)^2 = -a^2-b^2+2abs.$$

This must therefore be negative. For $(ai+bj) = S+V$, and V does not vanish. Hence

$$(ai+bj)^2 = S^2+V^2+2SV;$$

and since, by comparison, it appears the vector part $2SV$ vanishes, it follows that $S=0$, and the sum, or linear function, of two vectors is a vector.

The same thing is evident because $s^2 \leqq 1$; whence

$$-a^2-b^2+2abs = -p(a+b)^2-(1-p)(a-b)^2,$$

where $p = \dfrac{1+s}{2}$ and $1-p = \dfrac{1-s}{2}$, both of which are positive, or zero.

Let us then assume a vector j, such that

$$j_1 = \frac{si+j}{\sqrt{1-s^2}}$$

$$j_1^2 = \frac{-s^2-1+2s^2}{1-s^2} = -1$$

$$ij_1 = \frac{-s+s+v}{\sqrt{1-s^2}} = \frac{v}{\sqrt{1-s^2}} \qquad j_1 i = -\frac{v}{\sqrt{1-s^2}}$$

$$j_1 \frac{v}{\sqrt{1-s^2}} = j_1 i j_1 = -j_1^2 i = i \qquad \frac{v}{\sqrt{1-s^2}} j_1 = ij_1 j_1 = -i$$

$$i \frac{v}{\sqrt{1-s^2}} = iij_1 = -j_1 \qquad \frac{v}{\sqrt{1-s^2}} i = -j_1 ii = j_1$$

$$\left(\frac{v}{\sqrt{1-s^2}}\right)^2 = -ij_1 j_1 i = -1.$$

Writing j for j_1 and k for $\dfrac{v}{\sqrt{1-s^2}}$ and the above formulæ define the algebra of real *quaternions*.

140.* I will now prove that it is not possible to add to this a fourth linearly independent vector. For suppose [l] to be such a unit vector. Write

$$jl = S' + V',$$
$$li = S'' + V''.$$

Substitute for l,

$$l_1 = S''i + S'j + l$$

Then,

$$\begin{aligned} jl_1 &= -S''k - S' + S' + V' = -S''k + V',\\ l_1j &= \ \ S''k - S' + S' - V' = \ \ S''k - V',\\ l_1i &= -S'' - S'k + S'' + V'' = -S''k + V'',\\ il_1 &= -S'' + S'k + S'' - V'' = \ \ S''k - V''.\end{aligned}$$

Let us further assume

$$kl_1 = S''' + V'''$$

Whence,

$$l_1k = S''' - V'''.$$

But

$$l_1j = -jl_1 \quad \text{and } l_1i = -il_1.$$

Hence,

$$kl = i.jl_1 = -i.l_1j = -il_1.j = l_1i.j = l_1.ij = lk.$$

So,

$$kl_1 = l_1k$$

or,

$$kl_1 - l_1k = 0$$

But we have seen that

$$kl_1 - l_1k = S''' + V''' - S''' + V''' = 2V'''.$$

Hence $V''' = 0$. Then these vectors are not linearly independent, and a fourth unit vector is impossible.

But this proof does not apply when the multitude of linearly independent expressions is endless; such algebras are nonlinear.

We thus see that even when we annul the commutative law of multiplication, there are but three linear algebras, real single algebra, ordinary imaginary algebra, and the algebra of real quaternions which obey all the other algebraic laws. The law which so limits the number is:

If $xy = xz$, then $y = z$, unless $x = 0$ or $x = \infty$.

* Cf. 3.305.

141. In all other algebras this law fails and with it goes all semblance of importance for the inverse operation of division.[1] The algebra of logic illustrates the vanity of that device for solving equations, which must on the contrary usually be solved by producing special known quantities by direct operations.

§11. MEASUREMENT

142. It was necessary to say something about imaginaries before coming to the subject of measurement since the modern theory of measurement (due to the researches of Cayley, Clifford, Klein, etc.) depends essentially upon imaginaries.

Let us first consider measurement in one dimension. There is a certain absurdity in talking about measurement in one dimension. This is seen in the instance of time. Suppose we only knew the flow of our inward sensations, but nothing spread into two dimensions, how could one interval of time be compared with another? Certainly, their contents might be so alike that we should judge them equivalent. But that is not shoving a scale along. It does not enable us to compare intervals unless they happen to have similar contents. However, it is convenient to put that consideration aside, and to begin (with Klein) at unidimensional measurement.

We are to measure, then, along a line. We will, for formal rhetoric's sake, conceive that line as returning into itself. We will, first, in order that we may apply numerical algebra, give a preliminary numbering to all the points of that line, so that every point has a number and but one number, and every real number, positive or negative, rational or surd, has a point and but one point, and so that the succession of any four numbers is the same as the succession on the line of the four corresponding points. Now, we must make a scale to shift along that line.

[1] Professor Sylvester investigates the "general case" of multiple algebra. This is like enunciating the great truth that every human being who ever lived has been caught up into heaven (excepting only those who were neither Enoch nor Elijah). Only it is more extreme. For as Sylvester allows imaginary coefficients, his "general case" of multiple algebra has not one single multiple, algebra or group under it. It is pure moonshine. Professor Sylvester ventilates his scorn for my father's work; but if he had studied it, he would have escaped the absurdity into which he falls. [See Sylvester's "Lectures on the Principles of Universal Algebra," *American Journal of Mathematics*, vol. 6, pp. 270–286, (1884).]

We must imagine that we have a movable line which lies everywhere in coincidence with the fixed line, and which can be shifted. In the shifting, parts of it may become expanded or contracted, for we cannot tell whether they do or not unless we had some third standard to shift along to tell us; and then the same question would arise. But the continuity and succession of points shall not be broken in the shifting; and moreover, when the movable line has any one point brought back to coincidence with a former position, all the points shall be brought back. Now imagine all this extended to the imaginary numbers. Then, it is shown in the mathematical theory of functions, that if x be the number against which any point of the movable line falls in any one position and y be the number the same point falls against in any other position, it follows, because for each value of x there is just one value of y and for each value of y just one value of x, that x and y are connected by an equation linear in each, that is, an equation of the form

$$xy+Ax+By+C=0.$$

This gives

$$y=-\frac{Ax+C}{x+B}.$$

Now this is a function which forms the subject of some very beautiful and simple algebraical studies.[1] It is convenient to put

$$A=B-\alpha-\beta$$
$$C=B^2-(\alpha+\beta)B+\alpha\beta.$$

Then

$$y=\frac{(\alpha+\beta-B)x-B^2+(\alpha+\beta)B-\alpha\beta}{x+B}$$
$$=\frac{(\alpha+\beta)x+(\alpha+\beta)B-\alpha\beta}{x+B}-B$$
$$=\frac{(\alpha^2-\beta^2)(x+B)-\alpha\beta(\alpha-\beta)}{(x+B)(\alpha-\beta)}-B$$
$$=\frac{(x+B-\beta)\alpha^2-(x+B-\alpha)\beta^2}{(x+B-\beta)\alpha^1-(x+B-\alpha)\beta^1}-B$$

[1] See Boole, *Calculus of Finite Differences*, 2d ed., ch. xv. Clebsch, *Geometrie*, Klein, *Das Ikosaeder*, Forsyth, *Theory of Functions*, ch. xxii, "Automorphic Functions."

But

$$x = \frac{(x+B-\beta)\alpha^1 - (x+B-\alpha)\beta^1}{(x+B-\beta)\alpha^\circ - (x+B-\alpha)\beta^\circ} - B.$$

So that the effect of the shifting has been to raise the exponents of α and β by 1.

It is easily proved that the same operation, performed any number t times, gives

$$\frac{(x+B-\beta)\alpha^{t+1} - (x+B-\alpha)\beta^{t+1}}{(x+B-\beta)\alpha^t - (x+B-\alpha)\beta^t} - B.$$

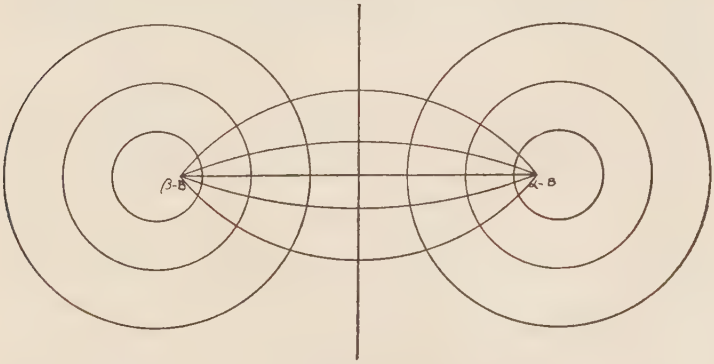
If α has a modulus greater than that of β, it is easily seen that when t becomes a very large positive number, the first terms of numerator and denominator will become indefinitely greater than the second terms and the value will indefinitely approximate to

$$\alpha - B.$$

But when t is a very large negative quantity, the reverse will occur, and the value will approximate toward

$$\beta - B.$$

143. If we look at the field of imaginary quantity, what we shall see is shown in the diagram.



Here we have a stereographic projection of the globe. At the south pole is $\beta - B$; at the north pole, $\alpha - B$. The parallels are not at equal intervals of latitude but are crowded together infinitely about the pole. Now an increase of t by unity carries a point of the scale along a meridian from one parallel to the one next nearer the north pole. But an addition to t of an *imaginary* quantity carries the point of the scale round along a parallel.

If the real line of the scale lies along a meridian all real shiftings of it will crowd its parts toward the north or the south pole; and the distance of either pole, as measured by the multitude of shiftings required to reach it, is *infinite.*[1] The scale is *limited*, but *immeasurable.*

But if the real line of the scale lies along a parallel, real shiftings, that is shiftings from real points to real points, will carry it round, so that a finite number of shiftings will restore it to its first position. Such is the scale of rotatory displacement. It is *unlimited*, but finite, or *measurable.* A scale of measurement, in the sense here defined, cannot be both limited and finite. We seem to have such a scale in the measurement of probabilities. But it is not so. Absolute certainty, or probability 0 or probability 1 are unattainable; and therefore, the numbers attached to probabilities do not constitute any proper scale of measurement, *which can be shifted along.* But it is possible to construct a true scale for the measurement of belief.[2] It was a part of the definition of a scale that in all its shiftings it should cover the whole of the line measured. ("For every point of the line a number of the scale in every position.") Hence the shifting can never be arrested by abuttal against a limit. If there is a limit, it must be at an immeasurable distance.

144. But there is a special case of measurement, very different from the one considered. Namely, it may happen that the nature of the shifting is such that [given] the equation

$$xy+Ax+By+C=0,$$

where A, B, C, may have any values, real or imaginary, we have

$$C=\frac{1}{4}(A+B)^2.$$

[1] The word *infinite* does not, as its etymology would suggest, mean *unlimited;* for we do not call the surface of a pea *infinite.* It means immeasurably great.

[2] See *Wenzel Šimerka*, "Die Kraft der Ueberzeugung," *Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften,* Wien; Philosophisch-Historische Classe (CIV, 2 Heft, S. 511–571).

Substituting this in the expressions for A and C in terms of α, β, and B, we get

$$\frac{1}{4}(\alpha+\beta)^2=\alpha\beta$$

or

$$\alpha=\beta.$$

This necessitates an altogether different treatment. In this case, we have

$$y=-\frac{Ax+\frac{1}{4}(A+B)^2}{x+B}$$

$$2y=-(A+B)+\frac{(B-A)(2x+A+B)}{(B-A)+(2x+A+B)}$$

$$2x=-(A+B)+\frac{(B-A)(2x+A+B)}{(B-A)+0(2x+A+B)}.$$

And t shifts give

$$-(A+B)+\frac{(B-A)(2x+A+B)}{(B-A)+t(2x+A+B)}.$$

This gives for $t=\pm\infty$, $-(A+B)$. The scale is in this case then unlimited and immeasurable. This is the manner in which the Euclidean geometry virtually conceives lengths to be measured; but whether this method accords precisely with measurement by a rigid bar is a question to be decided experimentally, or irrationally, or not at all.

145. The fixed limits of measurement are very appropriately termed by mathematicians the Absolute.[1] It is clear that even when measurement is not practical, even when we can hardly see how it ever can become so, the very idea of measuring a quantity, considerably illuminates our ideas about it. Naturally, the first question to be asked about a continuous quantity is whether the two points of its absolute coincide; if not, a second less important, but still significant question is whether they are in the real line of the scale or not. These

[1] Cayley gave the name to the "circular points." ["On Evolutes and Parallel Curves," *Quarterly Journal of Pure and Applied Mathematics*, pp. 183–200, vol. XI (1871).] It seems to indicate that even at that early day, he had some insight into the philosophy of the subject. Yet, had he seen more clearly, he would have made the double line at infinity a part of the absolute.

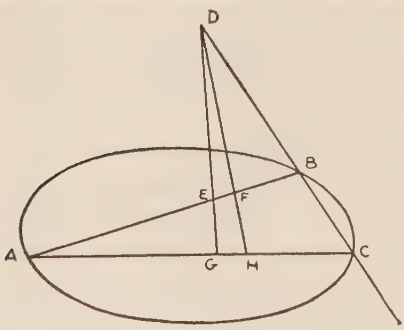
are ultimately questions of fact which have to be decided by experimental indications; but the answers to them will have great bearing on philosophical and especially cosmogonical problems.

146. The mathematician does not by any means pretend that the above reasoning flawlessly establishes the absolute in every case. It is evident that it involves a premiss in regard to the imaginary points which only indirectly relates to anything in visible geometry, and which, of course, may be supposed not true. Nevertheless, the doctrine of the mathematical absolute holds with little doubt for all cases of measurement, because the assumptions virtually made will hardly ever fail.

147. When we pass to measurement in several dimensions, there seems to be little difference between one number of dimensions and another; and therefore we may as well limit ourselves to studying measurement on a plane, the only spatial spread for which our intuition is altogether effortless.

Radiating from each point of the plane is a continuity of lines. Each of these has upon it its two absolute points (possibly imaginary, and even possibly coincident); and assuming these to be continuous, they form a curve which, being cut in two points only by any one line, is of the second order. That is, it is a conic section, though it may be an imaginary or even degenerate one.

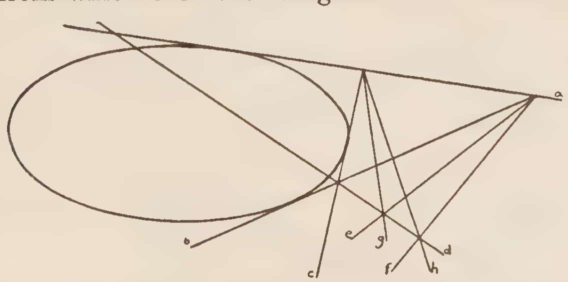
Now as the foot has different lengths in different countries, so the ratios of units of lengths along different lines in the plane is somewhat arbitrary. But the measurement is so made that first, every point infinitely distant from another along a straight line is also infinitely distant along any broken line; and second, if two straight lines intersect at a point, A, on the absolute conic and respectively cut it again at B and C; and from D, any point collinear with B and C, two straight lines be drawn, the segments of the first two lines, EF and GH, which these cut off, are equal. I omit the geometrical



proof that this involves no inconsistency. This proposition enables us to compare any two lengths.

148. We now have to consider angular magnitude. In the space of experience, the evidence is strong that, when we turn around and different landscapes pass panorama-wise before our vision we come round to the same direction, and not merely to a new world much like the old one. In fact, I know of no other theory for which the evidence is so strong as it is for this. But it is quite conceivable that this should not be so; there might be a world in which we never could get turned round but should always be turning to new objects. But certain conveniences result from assuming for the measurement of the angles between lines the same absolute conic which is assumed as the absolute of linear measure. Thus, it is assumed that two straight lines meeting at infinity have no inclination to one another, just as it is assumed that in a direction such that the opposite infinities should coincide, all other points would have no distance from one another. The latter is another way of saying that if a point is at an infinite distance from another point on a straight line, it is so on a broken line. The other assertion is that if an infinite turning is requisite to reach a line from one centre, it is equally so if you attempt to reach it by turning successively about different centres. The analogue of the proposition for which the last figure was drawn is as follows:

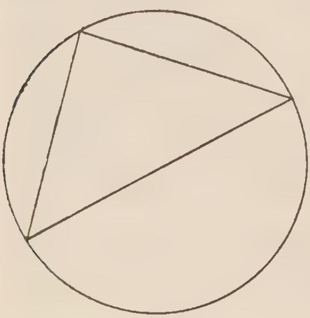
Upon a line, *a*, tangent to the absolute let two points be taken from which the other tangents to the absolute are *b*



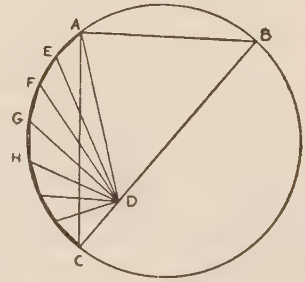
and *c*. Through the intersection of *b* and *c* draw any line, *d*, then any two lines *e* and *f*, meeting at the intersection of *a* and *b*, make the same angle with one another as two other lines having the same intersections with *d*, and cutting one another at the intersection of a and c. This enables us to compare all angles.

149. Suppose a man to be standing upon an infinitely extended plane free from all obstructions. Would he see something like a horizon line, separating earth from sky, being the foreshortened parts of the plane at an infinite distance? If space is infinite, he would. Now suppose he sets up a plane of glass and traces upon it the projection of that horizon, from his eye as a centre. Would that projection be a straight line? Euclid virtually says, "Yes." Modern geometers say it is a question to be decided experimentally. As a logician, I say that no matter how near straight the line may seem, the presumption is that sufficiently accurate observation would show it was a conic section. We shall see the reason for this, when we come to study probable inference.

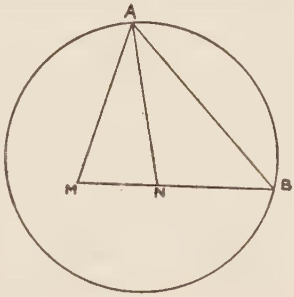
Let us suppose, then, that the horizon is not a straight line upon a level with the eye, but is a small circle below that level.



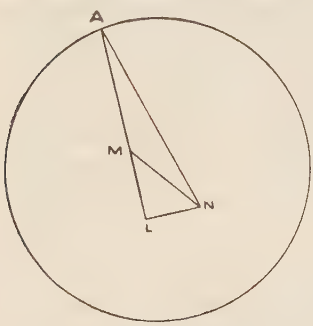
If then two straight lines meet at infinity, their other ends must be infinitely distant; but the angle between them is null. Hence, there may be a triangle having all its angles null, and all its sides infinite. Let us assume (what might, however, be proved) that two triangles, having all the sides and angles of the one respectively and in their order equal to those of the other, are of equal area. Then all triangles having the sum of their angles *null* are of equal area. Call this T. Then the area of an ordinary polygon of V vertices all on the absolute is $(V-2)T$. The area of the absolute is therefore infinite.



If a triangle has two angles null and the third $\frac{1}{N}$ part of 180°, what is its area? Let ABD be the triangle, AB being on the absolute. Continue BD the absolute at C. Let ADE, EDF, etc., be $N-1$ triangles having their angles at D all equal to ADB. Then these N triangles are all equal, because their sides and angles are equal. They make a polygon of $N+1$ vertices on the absolute, the area of which is $(N-1)T$. Hence, the area of each triangle is $\left(1-\frac{1}{N}\right)T$.

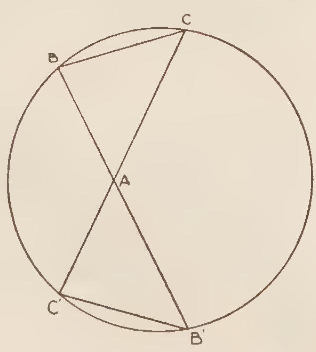


What is the area of a triangle having one angle zero and the others $m\times180°$ and $n\times180°$? Let AMN be such a triangle; extend MN on the side of N to the absolute at B. Then the area of ABM is $(1-m)T$ and that of ABN is nT. Hence the area of AMN, which is their difference, is $(1-m-n)T$.



What is the area of a triangle having its three angles equal to $l\times180°$, $m\times180°$, and $n\times180°$? Let LMN be such a triangle. Produce LM on the side of M to the absolute at A and join AN. Then if the angle MNA is put equal to $x.180°$, the area of ALN is $(1-l-n-x)T$, while that of AMN is $(m-x)T$. Hence, that of LMN, which is their difference, is $(1-l-m-n)T$.

Thus, the area of a triangle is proportional to the amount by which the sum of its angles falls short of two right angles. Of course, this does not forbid that amount being infinitely small for all triangles whose sides are finite.



150. The above reasoning may appear to be fallacious because it forgets that subtraction is not applicable to infinites. But it does not fall into that

error. I may remark, however, that subtraction is applicable to infinites, in case their transformations are so limited that $x+y$ cannot equal $x+z$ unless $y=z$. For instance, we have considered the triangle ABC having two vertices on the absolute. This triangle is finite. But we might perfectly well reason about the infinite sector ABC provided this sector be not allowed to vary so as to change the area of the triangle, and provided, further, that we always add to each sector, BAC, its equal vertical sector $B'AC'$.

Looking at a triangle from this point of view, we see that the sum of the six sectors (two for each angle) is twice three times the triangle *plus* all the rest of the plane, or twice the area of the triangle plus the whole plane. The whole plane is four right angled sectors. But we have thus reckoned together with the sectors of the angles of the triangle their equal vertical sectors. Dividing by two, we find the sum of the angular sectors of a triangle is two right angles *plus* the area of the triangle.

Now since the sector is proportional to its angle, and since further, for the largest possible sector the angle is zero, it follows that the sector is equal to the negative of the angle, whence we find

$$\text{area } \Delta = 2\perp - \Sigma \text{ Angles.}$$

151. . . . Such are the ideas which the mathematician is using every day. They are as logically unimpeachable as any in the world; but people, who are not sure of their logic, or who, like many men who pride themselves on their soundness of reason, are totally destitute of it, and who substitute for reasoning an associational rule of thumb, are naturally afraid of ideas that are unfamiliar, and which might lead them they know not whither.

As compared with imaginaries, with the absolute, and with other conceptions with which the mathematician works fearlessly — because good logicians, the Cauchys and the like,[1] have led the way — as compared with these, the idea of an infinitesimal is exceedingly natural and facile. Yet men are

[1] Cauchy is often called a bad logician because he has made many logical blunders. It is true he did so; but nevertheless he knew how to reason better than others, who grope and never make a misstep because they only shuffle along.

afraid of infinitesimals, and resort to the cumbrous method of limits. This timidity is a psychological phenomenon which history explains. But I will not occupy space with that here.*

It was Fermat, a wonderful logical and still more wonderful mathematical genius, whose light was almost extinguished by the bread-and-butter difficulties which the secret plotting of worldlings forced upon him, who first taught men the method of reasoning which lies at the bottom of all modern science and modern wealth, the method of the differential calculus.† He gave a variety of instructive examples, did this lawyer, this "conseiller de minimis," as the jealous Descartes was base enough to call him, joining himself to the "born missionaries" who were determined to "head off" this hope of mankind. But the first and simplest of them is the solution of the problem to divide a number, a, into two parts so that their product shall be a maximum. Let the parts be x and $a-x$. Let e be a quantity such that $a+e$ is "adequal" or πάρισος, say perequal to a. Then, the product being a maximum is at the point when increase of x ceases to cause it to increase. Hence Fermat writes

$$x(a-x)=(x+e)\ (a-x-e)$$

which gives

$$0=-xe+e(a-x-e)$$

or

$$0=e(a-2x-e)$$

Fermat now divides both sides by e (which assumes e is not zero). Whence

$$0=a-2x-e.$$

But $a-2x-e$ is "adequal" to $a-2x$; and the e may consequently be "elided." Thus we get

$$0=a-2x,$$

or

$$x=\frac{1}{2}a.$$

The peculiar properties of e, which we now call, after Leibniz, the infinitesimal, are:

First, that if $pe=qe$, then $p=q$, contrary to the property of zero; while

* See 118 and 3.563f.

† See *Oeuvres*, t. 1, p. 133f. Paris, (1891).

Second, that, under certain circumstances, we treat e as if zero, writing

$$p+e=p.$$

Of course, we cannot adopt the last equation without reservation. For it would follow that

$$e=0,$$

whence, since

$$4\times 0=5\times 0,$$
$$4e=5e,$$

and then by the first property,

$$4=5.$$

The method of indivisibles[1] had recognized that infinitely large numbers may have definite ratios, so that division is applicable to them.

152. The simplest way of defending the algebraical device is to say that e represents a quantity immeasurably small, that is, so small that the Fermatian inference does not hold from these quantities to any that are assignable. That no contradiction is involved in this has been shown in the former part of this chapter.* In the sense of measurement, then, $p+e=p$, while from a formally logical point of view, it is assumed that $e>0$. This is the most natural way, a perfectly logical way, and the way the most consonant with modern mathematics.

It is also possible to conceive the reasoning to represent the following. (The problem is the same as above.) Let x be the unknown. Then, since $x(a-x)$ is a maximum,

$$x(a-x)>(x+e)(a-x-e)$$

for all neighboring values of e. That is

$$0>e(a-2x-e).$$

Then the sign of $a-2x-e$ is opposite to that of e no matter what the value of e. It follows that $2x$ differs from a by less than any assignable quantity.

[1] Newcomb errs in saying (Johnson's *Cyclopædia*, 1894, IV, 567) this method is "medieval," and his description of it is not very characteristic. He is also wrong (Funk's *Dictionary*, *indivisible*) in calling it an application of the method of limits.

* See 125ff.

The great body of modern mathematicians repudiate infinitesimals in the above literal sense, because it is not clear that such quantities are possible, or because they cannot entirely satisfy themselves with that mode of reasoning. They therefore adopt the method of limits, which is a method of establishing the fundamental principles of the differential calculus. I have nothing against it, except its timidity or inability to see the logic of the simpler way. Let x be a variable quantity which takes an unlimited series of values $x_1, x_2, \ldots x_n$, so that n will be a variable upon which x_n depends. If, then, there be a quantity c such that

$$x_\infty = c,$$

that is, as the mathematicians prefer to say, in order to avoid speaking of infinity, if for every positive quantity e sufficiently small, there be a positive quantity ν such that for all values of n greater than ν

$$\text{Modulus } (x_n - c) < e$$

then c is said to be the *limit* of x.

Upon this definition is raised quite an imposing theory about limits which I can only regard with admiration, when it is erected with modern accuracy. Only, I wish to point out that the need for such a definition is not limited to its application to $n = \infty$, nor because infinity presents peculiar difficulties. It is only because ∞ is not an assignable number with which we can perform arithmetical processes. Let the function $x_n = n^2$, then the same difficulty arises when $n = \Pi$, and the same definition of a limit is called for.

The differential calculus deals with continuity, and in some shape or other, it is necessary to define continuity. I accept the above definition, with unimportant modifications, as a good definition of continuity. From it, as it appears to me, the idea of infinitesimals follows as a consequence; but, if not, no matter — so long as the algebraic expression of the infinitesimal be accepted, which is really the essential point. Infinitesimals may exist and be highly important for philosophy, as I believe they are. But I quite admit that as far as the calculus goes, we only want them to reason with, and if they be admitted into our reasoning apparatus (which is the algebra) that is all we need care for.

V

*A THEORY ABOUT QUANTITY**P*

§1. THE CARDINAL NUMERALS

153. Quantity presents certain metaphysical difficulties, to appreciate which it would be as hopeless a task to bring this generation as to bring it to a sense of sin. Medieval doctors apprehended such points of logic far more clearly. Why a mass distant one yard from a pound of matter should gravitate toward that pound by precisely that fraction of an inch per second that it does, neither more nor less — how it is possible that the exact value of this quantity ever should be explained and brought under the manifest governance of that unyielding and universal law which is supposed to regulate all facts, how this can be when the general properties of an inch are nowise different from a mile, is a question which those philosophers who oppose my tychism† would find a puzzling one, could they once be brought to understand what this question is. My present purpose, however, is not to discuss this problem in its entirety, but merely to follow out, in a rambling spirit, a pretty little opening of thought suggested by an objection to a part of my solution of that problem.

154. That part of my solution is that Quantity is merely the mathematician's idealization of meaningless vocables invented for the experimental testing of orders of sequence. In our experience we often have occasion to remark that something is true of two or more things — say, for example, that one thing eats another, or that one day is pleasanter than another. A fact true of several subjects is called a "relation" between them. A fact true of a pair of subjects (as in the examples just suggested) is a "dyadic relation." Of some kinds of dyadic relation we find that if one thing, *A*, be so related to a second, *B*, while *B* is so related to a third thing, *C*, then

* From "Recreations in Reasoning," c. 1897.

† See vol. 6, bk. I, ch. 2.

A is always related in that same way to *C*. When this is so, the relation is said to be "transitive." We also call it a "succession" or "sequence." Now I hold that numbers are a mere series of vocables serving no other purpose than that of expressing such transitive relations, or, at least, no other purpose except one whose accomplishment is necessarily involved in that. I admit that our senses may inform us, not merely that *A* is heavier than *B*, but that *A* is a great deal more heavier than *B*, than *C* is heavier than *D*; and undoubtedly numbers do serve to express this verdict of sense with greater precision than sense can render it. But this, as I think, is a use of numbers which necessarily results from their primary use; the judgment, that one thing is *much* heavier than another, being a mere complexus of judgments each that one thing is heavier by a unit than another.

155. When a number is mentioned, I grant that the idea of a succession, or transitive relation, is conveyed to the mind; and insofar the number is not a meaningless vocable. But then, so is this same idea suggested by the children's gibberish

"Eeny, meeny, miney, mo,"

Yet all the world calls these meaningless words, and rightly so. Some persons would even deny to them the title of "words," thinking, perhaps, that every word properly means something. That, however, is going too far. For not only "this" and "that," but all proper names, including such words as "yard" and "metre" (which are strictly the names of individual prototype standards), and even "I" and "you," together with various other words, are equally devoid of what Stuart Mill* calls "connotation." Mr. Charles Leland informs us that "eeny, meeny," etc. are gipsy numerals.† They are certainly employed in counting nearly as the cardinal numbers are employed. The only essential difference is that the children count on to the end of the series of vocables round and round the ring of objects counted; while the process of counting a collection is brought to an end exclusively by the exhaus-

* See *A System of Logic*, bk. 1, ch. ii, §5.

† The editors have been unable to find this statement in the writings of Mr. Leland. But see his *Gypsy Sorcery and Fortune Telling*, p. 210, London (1891).

tion of the collection, to which thereafter the last numeral word used is applied as an adjective. This adjective thus expresses nothing more than the relation of the collection to the series of vocables.

156. Still, there is a real fact of great importance about the collection itself which is at once deducible from that relation; namely, that the collection cannot be in a one-to-one correspondence with any collection to which is applicable an adjective derived from a subsequent vocable, but only to a part of it; nor can any collection to which is applicable an adjective derived from a preceding collection be in a one-to-one correspondence with this collection, but only with a part of it; while, on the other hand, this collection is in one-to-one correspondence with every collection to which the same numeral adjective is applicable. This, however, is not essentially implied as a part of the significance of the adjective. On the contrary, it is only shown by means of a theorem, called "The Fundamental Theorem of Arithmetic,"* that this is an attribute of the collections themselves and not an accident of the particular way in which they have been counted. Nevertheless, this is a complete justification for the statement that quantity — in this case, multitude, or collectional quantity — is an attribute of the collections themselves. I do not think of denying this; nor do I mean that any kind of quantity is merely subjective. I am simply not using the word quantity in that acception. I am not speaking of physical, but of mathematical, quantity.

157. Were I to undertake to establish the correctness of my statement that the cardinal numerals are without meaning, I should unavoidably be led into a disquisition upon the nature of language quite astray from my present purpose. I will only hint at what my defence of the statement would be by saying that, according to my view, there are three categories of being; ideas of feelings, acts of reaction, and habits.† Habits are either habits about ideas of feelings or habits about acts of reaction. The ensemble of all habits about ideas of feeling constitutes one great habit which is a World; and the ensemble of all habits about acts of reaction constitutes a second

* See 163, 187n.

† See vol. 1, bk. III.

great habit, which is another World. The former is the Inner World, the world of Plato's forms. The other is the Outer World, or universe of existence. The mind of man is adapted to the reality of being. Accordingly, there are two modes of association of ideas: inner association, based on the habits of the inner world, and outer association, based on the habits of the universe.* The former is commonly called association by resemblance; but in my opinion, it is not the resemblance which causes the association, but the association which constitutes the resemblance. An idea of a feeling is such as it is within itself, without any elements or relations. One shade of red does not in itself resemble another shade of red. Indeed, when we speak of a shade of red, it is already not the idea of the feeling of which we are speaking but of a cluster of such ideas. It is their clustering together in the Inner World that constitutes what we apprehend and name as their resemblance. Our minds, being considerably adapted to the inner world, the ideas of feelings attract one another in our minds, and, in the course of our experience of the inner world, develop general concepts. What we call sensible qualities are such clusters. Associations of our thoughts based on the habits of acts of reaction are called associations by contiguity, an expression with which I will not quarrel, since nothing can be contiguous but acts of reaction. For to be contiguous means to be near in space at one time; and nothing can crowd a place for itself but an act of reaction. The mind, by its instinctive adaptation to the Outer World, represents things as being in space, which is its intuitive representation of the clustering of reactions. What we call a Thing is a cluster or habit of reactions, or, to use a more familiar phrase, is a centre of forces.† In consequence, of this double mode of association of ideas, when man comes to form a language, he makes words of two classes, words which denominate things, which things he identifies by the clustering of their reactions, and such words are proper names, and words which signify, or *mean*, qualities, which are composite photographs of ideas of feelings, and such words are verbs or portions of verbs, such as are adjectives, common nouns, etc.

* Cf. 1.383 and vol. 8.

† Cf. 1.436.

158. Thus, the cardinal numerals in being called meaningless are only assigned to one of the two main divisions of words. But within this great class the cardinal numerals possess the unique distinction of being mere instruments of experimentation. "This" and "that" are words designed to stimulate the person addressed to perform an act of observation; and many other words have that character; but these words afford no particular help in making the observation. At any rate, any such use is quite secondary. But the sole uses of the cardinal numbers are, first, to count with them, and second, to state the results of such counts.

159. Of course, it is impossible to count anything but clusters of acts, *i.e.*, events and things (including persons); for nothing but reaction-acts are individual and discrete. To attempt, for example, to count all possible shades of red would be futile. True, we count the notes of the gamut; but they are not all possible pitches, but are merely those that are customarily used in music, that is, are but habits of action. But the system of numerals having been developed during the formative period of language, are taken up by the mathematician, who, generalizing upon them, creates for himself an ideal system after the following precepts.

§2. PRECEPTS FOR THE CONSTRUCTION OF THE SYSTEM OF ABSTRACT NUMBERS[P]

160. First, There is a relation, *G*, such that to every *number*, *i.e.*, to every object of the system, a different number is *G* and is *G* to that number alone; and we may say that a number to which another is *G* is "*G*'d" by that other;

Second, There is a number, called zero, 0, which is *G* to no cardinal number;

Third, The system contains no object that it is not necessitated to contain by the first two precepts. That is to say, a given description of number only exists provided the first two precepts require the existence of a number which may be of that description.

161. This system is a cluster of ideas of individual things; but it is not a cluster of real things. It thus belongs to the world of ideas, or Inner World. Nor does the mathematician,

though he "creates the idea *for himself*," create it absolutely. Whatever it may contain of [that which is] impertinent [to Mathematics] is soilure from [elsewhere]. The idea in its purity is an eternal being of the Inner World.

162. This idea of discrete quantity having an absolute minimum subsequently suggests the ideas of other systems, all of which are characterized by the prominence of transitive relations. These mathematical ideas, being then applied in physics to such phenomena as present analogous relations, form the basis of systems of measurement. Throughout them all, succession is the prominent relation; and all measurement is affected by two operations. The first is the experiment of superposition, the result of which is that we say of two objects, A and B, A is (or is not) in the transitive relation, t, to B, and B is (or is not) in the relation, t, to A; while the second operation is the experiment of counting. The question "How much is A?" only calls for the statement, A has the understood transitive relation to such things, and such things have this relation to A.

§3. APPLICATION OF THE THEORY TO ARITHMETIC[P]

163. According to the theory partially stated above, pure arithmetic has nothing to do with the so-called Fundamental Theorem of Arithmetic.* For that theorem is that a finite collection counts up to the same number in whatever order the individuals of it are counted. But pure arithmetic considers only the numbers themselves and not the application of them to counting.

164. In order to illustrate the theory, I will show how the leading elementary propositions of pure arithmetic are deduced, and how it is subsequently applied to counting collections.

Corollary 1. No number is G of more than one number. For every number necessitated by the first precept is G to a single number, and the only number necessitated by the second precept, by itself, is G to no number. Hence, by the third precept, there is no number that is G to two numbers.

Corollary 2. No number is G'd by two numbers. For were

* See 187n.

there a number to which two numbers were G, one of the latter could be destroyed without any violation of the first two precepts, since the destruction would leave no number without a G which before had one, nor would it destroy 0, since that is not G. Hence, by the third precept, there is no number which is G to a number to which another number is G.

Corollary 3. No number is G to itself. For every number necessitated by the first precept is G to a different number, and to that alone; and the only number necessitated by the second precept, by itself, is G to no number.

Corollary 4. Every number except zero is G of a number. For every number necessitated by the first precept is so, and the only number directly necessitated by the second is zero.

Corollary 5. There is no class of numbers every one of which is G of a number of that class. For were there such a class, it could be entirely destroyed without conflict with precepts 1 and 2. For such destruction could only conflict with the first precept if it destroyed the number that was G to a number without destroying the latter. But no number of such a class could be G of any number out of the class by the first corollary. Nor could zero, the only number required to exist by the second precept alone, belong to this class, since zero is G to no number. Therefore, there would be no conflict with the first two precepts, and by the third precept such a class does not exist.

165. The truly fundamental theorem of pure arithmetic is not the proposition usually so called, but is the Fermatian principle, which is as follows:

Theorem I. The Fermatian Principle: *Whatever character belongs to zero and also belongs to every number that is* G *of a number to which it belongs, belongs to all numbers.*

Proof. For were there any numbers which did not possess that character, their destruction could not conflict with the first precept, since by hypothesis no number without that character is G to a number with it. Nor would their destruction conflict with the second precept directly, since by hypothesis zero is not one of the numbers which would be destroyed. Hence, by the third precept, there are no numbers without the character.

166. *Definition* 1. Any number, M, is, or is not, said to

be *greater than*, a number, *N*, and *N* to be, or not to be, *less than M*, according to [whether] the following conditions are, or are not, fulfilled:

First, Every number *G* to another is greater than that other;

Second, Every number greater than a second, itself greater than a third, is greater than that third;

Third, No number is greater than another unless the above two conditions necessitate its being so.

Theorem II. Every cardinal number except zero is greater than zero.*

Theorem III. No cardinal number *L* is greater than any number, *M*, unless *L* is *G* to a cardinal number, *N*, which either is greater than [or equal to] *M*.

Corollary 1. By the same reasoning (substituting everywhere less for "greater" and *G'd by* for "*G* of") no number *M* is *less* than any number *L* unless *L* be *G* to *M* or be greater than the number that is *G* to *M*.

Corollary 2. Hence, by the first and second conditions of the definition, if a cardinal number, *L* is greater than a cardinal number, *M*, then the number that is *G* to *L* is greater than the number that is *G* to *M*.

Corollary 3. Zero is greater than no number.

Corollary 4. Every number greater than a number is *G* of some number.

Theorem IV. No cardinal number is both greater and less than the same cardinal number.

Corollary 1. No number is either greater or less than itself.

Corollary 2. No cardinal number, *M*, is greater than a cardinal number, *N*, and less than *GN*.

Theorem V. Of any two different cardinal numbers, one is greater than the other.

Corollary. If the cardinal number, *GL*, that is *G* to *L*, be greater than the cardinal number, *GM*, that is *G* to *M*, by Theorem IV it cannot be less. Hence by Corollary 2 from Theorem III, *L* cannot be less than *M*. But by the first corollary from Theorem IV, *GL* is not *GM*, and therefore *L* is not *M*. Hence *L* is greater than *M* if *GL* is greater than *GM*.

167. *Theorem VI.* (*Modified Fermatian Principle.*) *If a character, α, be such that, taking any two cardinal numbers,* A

* The proofs of the less important theorems have been omitted by the editors.

and Z, *either* α *does not belong to both* A *and* Z, *or no cardinal number is greater than* A *and less than* Z, *or* SOME *cardinal number greater than* A *and less than* Z *has the character*, α, *then*, α *is also such that, taking any two cardinal numbers*, B *and* Y, *either* α *does not belong both to* B *and* Y, *or no cardinal number is greater than* B *and less than* Y, *or* EVERY *cardinal number greater than* B *and less than* Y *has the character*, α.

Proof. For were there an exception, there would be, at least, one cardinal number of a class we may call the n's fulfilling the conditions that every n is greater than B and less than Y and no number at once greater than an n and less than an n possess α. Then, by Corollary 4 of Theorem III, every n, and also every number greater than every n, would be G to some cardinal number; and by Corollary 5 from the general precepts there would be some n G to a cardinal number, not an n, which we may call M, and there would be some number greater than every n which would be G to some n which we may call N. But then GN and M would possess α, and if any n's existed, they would be greater than M and less than GN and yet there would be *no* cardinal number greater than M and less than GN having α. Hence it is absurd to suppose any exception.

168. *Definition* 2. A *sum* of a cardinal number, M, *added to* a cardinal number, N, is a cardinal number which fulfills the following conditions:

First, A sum of zero added to zero is zero;

Second, A sum of zero added to a cardinal number which is G to any cardinal number, N, is a number which is G to a sum of zero added to N;

Third, A sum of a cardinal number that is G to any cardinal number, M, added to any cardinal number, N, is a cardinal number which is G to a cardinal number which is a sum of M added to N;

Fourth, No cardinal number is a sum of a cardinal number added to a cardinal number unless it is necessitated to be so by the above conditions.

Theorem VII. There is one cardinal number, and but one, which is a sum of a cardinal number, M, *added to a cardinal number*, N.

Corollary 1. Whatever cardinal numbers M and N may be, $M+N>0$ unless $M=N=0$.

Corollary 2. Whatever cardinal numbers M and N may be $M+N>N$ unless $M=0$, and $M+N>M$ unless $N=0$.

Corollary 3. Whatever cardinal numbers M and N may be, $M+GN=G(M+N)$.

Corollary 4. Whatever cardinal number N may be, $0+N=N$.

Corollary 5. Whatever cardinal number N may be, $N+0=N$.

Corollary 6. Whatever cardinal numbers M and N may be, $M+N=N+M$.

Corollary 7. Whatever cardinal numbers L, M, and N may be, $L+(M+N)=(L+M)+N$.

Theorem VIII. The sum of a greater cardinal number, L, *added to any cardinal number,* N, *is greater than the sum of a lesser cardinal number,* M, *added to the same cardinal number,* N.

Corollary 1. Whatever cardinal numbers L, M, and N may be if $L>M$, then $N+L>N+M$.

Corollary 2. Whatever cardinal numbers A, B, C, D may be, if $A>C$ and $B>D$ then $A+B>C+D$.

Corollary 3. Whatever cardinal numbers L, M, and N may be unless $L=M$, $L+N$ or $N+L$ is not $M+N$ or $N+M$.

Corollary 4. If $L+N>M+N$ then $L>M$.

Corollary 5. If $A+B>C+D$, either A or B is greater than C and than D or else either C or D is less than A and than B.

Theorem IX. Whatever cardinal numbers L *and* M *may be, there is one and only one cardinal number,* N, *such that either* N+M=L *or* N+L=M.

Definition 3. The *difference* between two cardinal numbers, L and M, is such a number, N, that either $N+M=L$ or $N+L=M$. It is said to be the *remainder* after *subtracting* the smaller as *subtrahend*, from the larger, as *minuend*. It is best denoted by the "minus sign" written after the larger of L and M and before the smaller.

Corollary 1. $L-M$ is no cardinal number unless $L>M$.

Corollary 2. If $L>M$, then $(L-M)+M=L$.

Definition 4. The *product* of a cardinal number, M, *multiplied into* a cardinal number, N, or the product of the *multiplicand*, N, *multiplied by* the *multiplier*, M, is a cardinal number, written $M\times N$, or $M.N$, or MN, subject to the following conditions:

First, zero is a product of any cardinal number multiplied by 0;

Second, a product of a cardinal number, N, multiplied by the cardinal number, GM, that is G to any cardinal number, M, is the sum, $N+M.N$, of N added to the product of M multiplied into N;

Third, no cardinal number is a product of cardinal numbers unless necessitated to be so by the foregoing conditions.

A product is said to be a *multiple* of its multiplicand.

Theorem X. There is one cardinal number and but one which is M.N *a product of one cardinal number,* M, *multiplied into a given cardinal number,* N.

Corollary 1. The product of any cardinal number, N, multiplied into zero is zero.

Corollary 2. Whatever cardinal numbers M and N may be $M\times N>0$ unless $M=0$ or $N=0$.

Corollary 3. The product of any cardinal number, N, multiplied by the cardinal number that is G to zero (which is called 1, one) is N.

Corollary 4. The product of any cardinal number, N, multiplied into $G0$, or 1, is N.

Corollary 5. Whatever cardinal numbers M and N may be, $M\times GN=M+M.N$.

Corollary 6. Whatever cardinal numbers M and N may be, $M\times N>M$ unless $M=0$ or $N=0$ or $N=G0$, and $M\times N>N$ unless $N=0$ or $M=0$ or $M=G0$.

Corollary 7. Whatever cardinal numbers L, M, and N may be, $L.(M+N)=L.M+L.N$.

Corollary 8. Whatever cardinal numbers L, M, N may be $L.(M.N)=(L.M).N$.

Corollary 9. Whatever cardinal numbers M and N may be $M.N=N.M$.

Theorem XI. Of two products of the same multiplicand not zero, that by the greater multiplier is the greater.

Corollary 1. If $L>M$, $N.L>N.M$ unless $N=0$.

Corollary 2. If $A>C$ and $B>D$, $A\times B>C\times D$ in all cases.

Corollary 3. Unless $L=M$, $L\times N$ is not $M\times N$, unless $N=0$.

Corollary 4. If $L\times N>M\times N$, then $L>M$ in all cases.

Corollary 5. If $A\times B>C\times D$,either A or B is greater than C and than D, or C or D is less than A and than B.

Corollary 6. If either B or C is greater than A or than D, then $B.C>A.D$, unless $A+D>B+C$.

Definition 5. A *divisor* of a cardinal number, N, is a cardinal number which multiplied by a cardinal number gives N as product. The number, N, is said to be *exactly divisible* by its divisor.

Abbreviations. We may write $N \equiv N'$ (mod M) where M is any cardinal number, not zero, to express that N and N' are cardinal numbers leaving the same remainder after division by M. We may denote the remainder and quotient of N divided by M by $R_m N$ and $Q_m N$, respectively. Then $N = R_m N + (Q_m N) \cdot M$.

We may denote $GG0$ by Q.

Scholium. The number Q is logically and mathematically peculiar. In old arithmetics multiplication and division by Q are considered as peculiar operations, *Duplation* and *Mediation*. We have need of an arithmetic of two, even in reasonings which do not concern quantity in the ordinary sense.

Theorem XII. Every cardinal number, N, *has with reference to every cardinal number,* M, *except zero, a remainder,* $\mathrm{R_m N}$, *and a quotient,* $\mathrm{Q_m N}$; *and only one number is remainder or quotient.*

Corollary 1. If the cardinal number, N, is less than the modulus, M, its remainder, $R_m N = N$.

Corollary 2. The remainder of the sum of two numbers, N and N', is the remainder of the sum of their remainders.

Corollary 3. The remainder of the product of two numbers, N and N', is the remainder of the product of the remainders.

Corollary 4. The quotient of the sum of two numbers is the sum of the quotient of the sum of the remainders added to the sum of the quotients of the numbers.

Corollary 5. The quotient of the product of two numbers, N and N', is the sum of the product of N by $Q_m N'$, the quotient of the other, added to the quotient of the product of N by $R_m N'$, the remainder of the other. Or $Q_m(N.N') = N.Q_m N' + Q_m(N.R_m N')$.

Corollary 6. Given any cardinal number, N, and any modular number, M, there is a multiple of M greater than N. For $(GQ_m N).M$ is such a multiple.

Definition 6. The *powers* of any cardinal number, B, called the *base* of the powers, are a class of cardinal numbers, each having a cardinal number, E, connected with it, called its exponent; and the power is written, B^E, and powers and exponents are defined by the following conditions:

First, $G0$ is a power of B whose exponent is zero;

Second, The product of the power B^E, of B with exponent E, multiplied by B is B^{GE}, a power of B with exponent GE;

Third, No cardinal number is a power of a cardinal number, unless necessitated to be so by the foregoing conditions.

Corollary. $B^{G0} = B \times B^0 = B \times G0 = B$.

Theorem XIII. Given any two cardinal numbers, B *and* E, *there is one, and but one cardinal number which is a power,* B^E, *of* B *with exponent,* E.

Corollary 1. Hence, $0^0 = G0$ and is not indeterminate. In this respect, the definition here assumed differs from the usual one, which substitutes for the first condition $B^1 = B$ and adds the condition that $B^E = P$ if $B^{GE} = B.P$. But practically the present definition is just as useful, if not more so, than the usual one.

Theorem XIV. A given exponent of two powers with the same exponent greater than zero, that with the greater base is the greater, and two powers of the same base greater than G0, *that with the greater exponent is the greater.*

Definition 7. An *even number* is a cardinal number whose remainder, relative to $GG0$ as modulus, is *zero*.

An odd number is a cardinal number whose remainder, relative to $GG0$ as modulus, is $G0$.

Corollary 1. Every cardinal number, N, is even or odd; and if N be even, GN is odd and *vice versa.*

Corollary 2. The double of a cardinal number, N, is $N+N$, the sum of N added to itself.

Corollary 3. If a number is even, it has a cardinal number that is half of it, but if it is odd, it has not.

Corollary 4. If the difference of two cardinal numbers, M and N, is even, those two numbers have a cardinal number as their arithmetical mean, and the difference between this mean and either M or N is half the difference between M and N.

169. *Theorem XV.* (Binary form of the Fermatian Principle.) *If any character belongs to every power of* $GG0$ *and also to the mean of any two numbers having a mean, if it belongs to the numbers themselves, then it belongs to every cardinal number except* 0.

VI

*MULTITUDE AND NUMBER**[P]

§1. THE ENUMERABLE

170. Let us consider the relation of a constituent unit to the collective whole of which it forms a part. Suppose A to be such a unit and B to be such a whole. Then in order to avoid the circumlocution of saying that A is a constituent unit of B as the collective whole of which it is a unit, I shall simply say A is a unit of B, and shall write "A is a *u* of B"; or I may reverse the order in which A and B are mentioned by writing "B is *u*'d by A."

The only logical peculiarities of this relation are as follows:

First, Whatever is *u* of anything is *u*'d by itself and by nothing else. Hence, if anything is *u*'d by anything not itself, it is not itself *u* of anything; and consequently nothing that is *u*'d by anything but itself is *u*'d by itself.

Second, Whatever is not *u*'d by anything does not exist.

171. By a *collection*, I mean anything which is *u*'d by whatever has a certain quality, or general description, and by nothing else. That is, if C is a collection, there is some quality, α, such that taking anything whatever, say x, either x possesses the quality of α and is a unit of C, or else it neither possesses the quality α nor is a unit of C. On the other hand, if C is not a collection, no matter what quality or general description, β may be taken, there is either something possessing the quality β without being a unit of C, or there is some unit of C which does not possess the quality, β.

It will be perceived, therefore, that there is a collection corresponding to every common noun or general description. Corresponding to the common noun "man" there is a collection of men; and corresponding to the common noun "fairy" there is a collection of fairies. It is true that this last collec-

* 1897. See 217 where the present paper is spoken of as if it were a lecture.

tion does not exist, or as we say, the total number of fairies is zero. But though it does not exist, that does not prevent it from being of the nature of a collection, any more than the non-existence of fairies deprives them of their distinguishing characteristics. . . .

172. Whether the constituent individuals or units of a collection have each of them a distinct identity of its own or not, depends upon the nature of the universe of discourse. If the universe of discourse is a matter of objective and completed experience, since experience is the aggregate of mental effect which the course of life has forced upon a man, by a brute bearing down of any will to resist it, each such act of brute force is destitute of anything reasonable (and therefore of the element of generality, or continuity, for continuity and generality are the same thing), and consequently the units will be individually distinct. It is such collections that I desire first to call your attention. I put aside then, for the present, such collections as the drops of water in the sea; and assume that the units are of such a kind that they may be absolutely distinguished from one another. Then, I say, as long as the discourse relates to a common objective and completed experience, those units *retain* each its distinct identity. If you and I talk of the great tragedians who have acted in New York within the last ten years, a definite list can be drawn up of them, and each of them has his or her proper name. But suppose we open the question of how far the general influences of the theatrical world at present favor the development of female stars rather than of male stars. In order to discuss that, we have to go beyond our *completed* experience, which may have been determined by accidental circumstances, and have to consider the possible or probable stars of the immediate future. We can no longer assign proper names to each. The individual actors to which our discourse now relates become largely merged into general varieties; and their separate identities are partially lost. Again, statisticians can tell us pretty accurately how many people in the city of New York will commit suicide in the year after next. None of these persons have at present any idea of doing such a thing, and it is very doubtful whether it can properly be said to be determinate now who they will be, although their number is approximately fixed. There is an

approach to a want of distinct identity in the individuals of the collection of persons who are to commit suicide in the year 1899. When we say that of all possible throws of a pair of dice one thirty-sixth part will show sixes, the collection of possible throws which have not been made is a collection of which the individual units have no distinct identity. It is impossible so to designate a single one of those possible throws that have not been thrown that the designation shall be applicable to only one definite possible throw; and this impossibility does not spring from any incapacity of ours, but from the fact that in their own nature those throws are not individually distinct. The possible is necessarily general; and no amount of general specification can reduce a general class of possibilities to an individual case. It is only actuality, the force of existence, which bursts the fluidity of the general and produces a discrete unit. Since Kant it has been a very wide-spread idea that it is time and space which introduce continuity into nature. But this is an *anacoluthon*. Time and space are continuous because they embody conditions of possibility, and the possible is general, and continuity and generality are two names for the same absence of distinction of individuals.

When the universe of discourse relates to a common experience, but this experience is of something imaginary, as when we discuss the world of Shakespeare's creation in the play of Hamlet, we find individual distinction existing so far as the work of imagination has carried it, while beyond that point there is vagueness and generality. So, in the discussion of the consequences of a mathematical hypothesis, as long as we keep to what is distinctly posited and its positive implications, we find discrete elements, but when we pass to mere possibilities, the individuals merge together. This remark will be fully illustrated in the sequel.

173. A *part* of a collection called its *whole* is a collection such that whatever is *u* of the part is *u* of the whole, but something that is *u* of the whole is not *u* of the part.

174. It is convenient to use this locution; namely, instead of saying A is in the relation, *r*, to B, we may say A is an *r* to B, or of B; or, if we wish to reverse the order of mentioning A and B, we may say B is *r*'d by A.

If a relation, *r*, is such that nothing is *r* to two different things, and nothing is r'd by two different things, so that some things in the universe are perhaps r to nothing while all the rest are r, each to its own distinct correlate, and there are some things perhaps to which nothing is r, but all the rest have each a single thing that is r to it, then I call r a *one-to-one relation*. If there be a one-to-one relation, r, such that every unit of one collection is r to a unit of a second collection, while every unit of the second collection is r'd by a unit of the first collection, those two collections are commonly said to be in a one-to-one correspondence with one another.*. . .

175. I shall use the word *multitude* to denote that character of a collection by virtue of which it is greater than some collections and less than others, provided the collection is *discrete*, that is, provided the constituent units of the collection are or may be distinct. But when the units lose their individual identity because the collection exceeds every positive existence of the universe, the word multitude ceases to be applicable. I will take the word *multiplicity* to mean the greatness of any collection discrete or continuous.

176. We have to note the precise meaning of saying that a relation of a given description exists. A relation of the kind here considered has been called an *ens rationis;* but it cannot be said that because nobody has ever constructed it — perhaps never will — it exists any the less on that account. Its existence consists in the fact that, if it were constructed, it would involve no contradiction. An easy dilemma will show that to suppose three things to be in one-to-one correspondence with individuals of a pair involves contradiction. But it is much more difficult to prove that a given hypothesis *involves no contradiction*. In mathematics, such propositions are usually replaced by so-called "problems." That is to say, a construction shows *how* the thing in question can take place. When we know how it can take place, we know, of course, that it is possible. Cases are rare in mathematics in which anything is shown to be possible without its being shown how. But when we come to philosophical questions, such a construction is generally practically beyond our powers; and it becomes necessary to examine the principles of logic in order to discover a general method of proving that a given hypothesis

* Cf. 3.537.

involves no contradiction. Without a thorough mastery of the principles of logic such a search must be fruitless.

Mathematics never has hypotheses forced upon it that are perplexing from [their] seemingly irresoluble mistiness — which is the aspect of such a question of philosophical possibility, at first sight. Mathematics does not *need* to take up any hypothesis that is not crystal-clear. Unfortunately, philosophy cannot choose its first principles at will, but has to accept them as they are.

177. For example, the relations of equality and excess of multitude having been defined after Cantor, philosophy can not avoid the question which instantly springs up: must every two collections be either equal or the one greater than the other, or can they be so multitudinous that the units of neither can be in one-to-one relation to units of the other?

To say that the collection of M's and the collection of N's are equal is to say:

There is a one-to-one relation, *c*, such that every M is *c* to an N; and there is a one-to-one relation, *d*, such that every N is *d* to an M.

To say that the collection of M's is less than the collection of N's is to say:

There is a one-to-one relation, *c*, such that every M is *c* to an N; but whatever one-to-one relation *d* may be, some N is not *d* to any M.

To say the collection of M's is greater than the collection of N's is to say:

Whatever one-to-one relation *c* may be, some M is not *c* to any N; but there is a one-to-one relation *d* such that every N is *d* to an M.

Now, formal logic suggests the fourth relation:

Whatever one-to-one relation *c* may be, some M is not *c* to any N, and whatever one-to-one relation *d* may be, some N is not *d* to any M.

Or this last may be stated more simply thus:

Whatever one-to-one relation *c* may be, some M is not *c* to any N and some N is not *c*'d by any M.

How shall we proceed in order to find out whether this last relation is a possible one, or not? . . .

178. In the first place, it must not be supposed that even if a collection is so great that the constituent units lose their individual identity, a one-to-one relation necessarily becomes impossible. If such a relation implied an actual operation performed, it would indeed be impossible, I suppose. But this is not the case. As the collection enlarges and the individual distinctions are little by little merged, it also passes out of the realm of brute force into the realm of ideas which is governed by rules. This sounds vague, because until I have shown you how to develop the idea of such a collection, I can offer you no example. But it is not necessary actually to construct the correspondence. It suffices to suppose that a certain number of units of the two collections having been brought into such a relation (and, in fact, they always are in such relations), then the general rules of the genesis of the two collections necessitate the falling of all the other individuals into their places in the correspondence. All this will become quite clear in the sequel.

179. That difficulty, then, having been removed, we have two collections, the M's and the N's; and the question is whether there is, no matter what these collections may be, always either some one-to-one relation, *c*, such that any M is *c* to an N or else some one-to-one relation, *d*, such that every N is *d*'d by an M. To begin with, there are vast multitudes of relations such that taking any one of them, *r*, every M is *r* to an N and every N is *r*'d by an M. For example, the relations of coexistence, maker of, non-husband of, etc. In general, each M can have any set of N's whatever as its correlates, except that there must be *one* of the M's that shall have among its correlates all those N's that are not *r*'d by any other M. And all those sets of N's for each M can be combined in any way whatever. In order to make our ideas more clear, let us for the moment suppose that the M's are equal to the finite number, μ, and the N's are equal to the finite number, ν. Then, for each M except one there are $2^\nu - 1$ different sets of N's, any one of which can be its correlates. Hence, there are $(2^\nu - 1)^{\mu-1}$ different forms of the relation *r*, without taking account of the variety of different sets of correlates which the remaining M may have. Suppose we had a diagram of each of those relations, each diagram showing the collection of M's above and

the collection of N's below, with lines drawn from each M to all the N's of which it was *r*. Each of that stupendous multitude of relations may be modified so as to reduce it [to] what we may call a one-to-*x* relation, by running through the N's and cutting away the connection of each N with every M but one; and each of the *r* relations could be thus cut down in a vast multitude of different ways. Call any such resulting relation, *s*. Then, every N would be *s*'d by a single M. Each one of the *r* relations could also be so modified as to reduce it to what we may call an *x*-to-one relation, by running through the M's and cutting off the connection of each M with every N but one. Call such a resulting relation, *t*. Then, every M would be *t* to a single N. Suppose we had a collection of diagrams showing all the ways in which every *r* relation could thus be reduced to an *s* relation or a *t* relation, that is, be reduced to a one-to-*x* relation or to an *x*-to-one relation. The question is, could the multitudes of M and N, be such that there would not be a single one-to-one relation among all those one-to-*x* relations [which each M has to an N] and *x*-to-one relations [which each N has to an M]? If among the diagrams of the one-to-*x* relations there were not one where the one-to-*x* relation was a one-to-one relation, it would be because in each case there was some M which was *s* [*i.e.*, one-to-*x*] to two or more N's. If, then, there were any of these diagrams in which some M was not *s* to any N, those diagrams could be thrown out of consideration, because there was no necessity for a pluralism of lines to one M, as long as there were M's to which no line ran; and since there was no necessity for it, there is no need of modifying those diagrams so as to take away plural lines from some of the M's so as to give lines to all the M's, because, *since there is a diagram for every possible modification changing an r* [*x-x*] *relation to an s* [*one-x*] *relation*, there must already be a diagram remedying this fault. There must, therefore, be among the diagrams, some diagrams in which every M is *s* to an N — unless indeed there is a diagram where the *s* is a one-to-one relation. Taking, then, any diagram in which every M is *s* to an N, all it is necessary to do is to erase all the lines but one which go to each M, and the relation so resulting, which we may call *u*, is such that every M is *u* to an N, no N is *u*'d by two M's (for no N is *s*'d by two M's and the erasures cannot increase the relates of any N), and no M is *u* of two N's. In other words, *u* is a one-to-one relation, and every M is *u* of an N. Q.E.D.

Is this demonstration sound? It may be doubted; at any rate I can show you how by a very small modification it would certainly become unsound; and thus direct your attention to the point which requires scrutiny. If, instead of casting aside those diagrams of *s* relations which showed some M's that are not *s* to any N, I had proposed to cure them by changing the course of lines from M's having two or more lines to M's having none, until there were either no M's left without any lines or no M's left with pluralities of lines, I should have fallen into a gross *petitio principii*. For I should be assuming that, of those two classes of M's, the whole of one (whichever it might be) could be put into a one-to-one relation with the whole or a part of the other; and whether or not this is always possible is the very question at issue.

But the true argument is this: Nothing can force *all* of the *s* diagrams to show pluralities of lines to M's except the fact that *some* of them show lines to all the M's. For since all possibilities are represented in the diagrams, if all the diagrams show pluralities of lines to M's, there must be a logical necessity for this, so that the conditions would be contradicted if it were not so. Now the only logical necessity there can be in making some lines terminate at M's, that already have lines, is that there are no M's that have not already lines. Hence, in some cases, at least, all the M's must have lines.

The gist of this argument is that it considers in what way contradiction can arise, and thus shows that the only circumstance which could render the one-to-one correspondence impossible in one way, necessarily renders it possible in another way.

180. I will now prove two general theorems of great importance. The first is, that the collection of possible sets of units (including the set that includes no units at all) which can be taken from discrete collections is always greater than the collection of units.* . . .

The other theorem, which gives great importance to the first, is that if a collection is not too great to be discrete, that is, to have all its units individually distinct, neither is the col-

* This proof is being omitted, having been given in 3.548. Cf. also 204.

lection of sets of units that can be generally formed from that collection too great to be discrete.

For we may suppose the units of the smaller collection to be independent characters, and the larger collection to consist of individuals possessing the different possible combinations of those characters. Then, any two units of the larger collection will be distinguished by the different combinations of characters they possess, and being so distinguished from one another they must be distinct individuals.

On those two theorems, I build the whole doctrine of collections.

181. I will now run over the different grades of multitude of discrete collections, and point out the most remarkable properties of those multitudes.

The lowest grade of multitude is that of a collection which does not exist, or the multitude of *none*. A collection of this multitude has obvious logical peculiarities. Namely, nothing asserted of it can be false. For of it alone contradictory assertions are true. It is a collection and it is not a collection. Given the premisses that all the X's are black and that all the X's are pure white, what is the conclusion? Simply that the multitude of the X's is zero.

The least difference by which one multitude can exceed another is by a single unit. But I do not say that the multitude next greater than a given multitude always exceeds it by a single unit.

The multitude of ways of distributing nothing into two abodes is *one*. This is the next grade of multitude. This again has certain logical peculiarities. Namely, in order to prove that every individual of it possesses one character, it suffices to prove that every individual of it does not possess the negative of that character.

The multitude of ways of distributing a single individual into two houses is *two*. This is the next grade of multitude. This again has certain logical peculiarities which have been noted in Schröder's *Logik*.

The multitude of combinations of two things is four, which is not the next grade of multitude. The multitude of combinations of four things is 16. The multitude of combinations of 16 things is 65,536. The multitude of combinations of 65,536 things is large. It is written by 20,036 followed by 19,725 other figures. The multitude of combinations of that many things is a number to write which would require over 600,000 thousand trimillibicentioctagentiseptillions of figures on the so-called English system of numeration. What the number itself would be called it would need a multimillionaire to say. But I suppose the word trimillillillion might mean a million to the trimillillionth power; and a trimillillion would be a million to the three thousandth power. But the multitude considered is far greater than a trimillillillion. It is safe to say that it far exceeds the number of chemical atoms in the gallactic cluster. Yet this is one of the early terms of a series which is confined entirely to finite collections and never reaches the really interesting division of multitudes, which comprises these that are infinite.

182. The finite collections, however, or, as I prefer to call them, the *enumerable* collections, have several interesting properties. The first thing to be considered is, how shall an enumerable multitude be defined? If we say that it is a multitude which can be reached by starting at 0, the lowest grade of multitude, and successively increasing it by one, we shall express the right idea. The difficulty is that this is not a clear and distinct statement. As long as we discuss the subject in ordinary language, the defect of distinctness is not felt. But it is one of the advantages of the algebra which is now used by all exact logicians, that such a statement cannot be expressed in that logical algebra until we have carefully thought out what it really means. An enumerable multitude is said to be one which can be constructed from zero by "successive" additions of unity. What does "successive," here, mean? . Does it allow us to make *innumerable* additions of unity? If so, we certainly should get beyond the enumerable multitudes. But if we say that by "successive" additions we mean an enumerable multitude of additions, we fall into a *circulus in definiendo*. A little reflection will show that what we do mean is, that the enumerable multitudes are those multitudes which are necessarily reached, provided we start at zero, and provided that, any given multitude being reached, we go on to reach another multitude next greater than that. The only fault of this statement is, that it is logically inelegant.

It sounds as if there were some special significance in the "reaching," which by the principles of logic there cannot be. For the enumerable multitudes are defined as those which are *necessarily* so reached. Now the kind of necessity to which this "necessarily" plainly refers is logical necessity. But the perfect logical necessity of a result never depends upon the material character of the predicate. If it is necessary for one predicate, it is equally so for any other. Accordingly, what is meant is that the enumerable multitudes are those multitudes every one of which possess any character whatsoever which is, in the first place, possessed by zero and, in the second place, if it is possessed by any multitude, M, whatsoever, is likewise possessed by the multitude next greater than M. We, thus, find that the definition of enumerable multitude is of this nature, that it asserts that that famous mode of reasoning which was invented by Fermat* applies to the succession of those multitudes. The enumerable multitudes are defined by a logical property of the whole collection of those multitudes.

183. Since the whole collection of enumerable multitudes has this logical property it follows *a fortiori* that every single enumerable multitude has the same property.

184. But it further follows from the same definition that every single enumerable collection has a further logical property.

This property is, that if an enumerable collection be counted, the counting process eventually comes to an end by the exhaustion of the collection. This property follows from the other, in this sense, that it is true of the *zero* collection, and if it be true of any collection whatever, it is equally true of every collection that is greater than that by one individual. Hence, it is true of all enumerable collections, by Fermatian reasoning.

185. You may ask why I should call this a *logical* property. It does not at first sight appear to be of that nature. But that is because it is not distinctly expressed. In place of "coming next after in the count," we may substitute any relation, *r*, such that not more than one individual (at least of the collection in question) is an *r* to any one. Then, the property is that

* See *Oeuvres de Fermat*, t. III, pp. 431–436; Paris (1891–94). Cf. 110, 165.

if the M's form an enumerable collection, then and only then, if every M is *r* to an M [say, L], then every M is *r*'d by an M [say, N]. For example, in a count no M is immediately preceded by more than one M, hence it cannot be that every M immediately precedes an M (so that the collection is never exhausted) unless every M is immediately preceded by an M (in which case, the count would have no beginning). Because this is a logical necessity, the property is a logical property and is the foundation of that mode of inference for which De Morgan first gave the logical rules, under the name of the *syllogism of transposed quantity*. He, however, overlooked the fact that this mode of reasoning is only valid of enumerable collections.*. . .

186. A remarkable and important property of enumerable collections is, that every finite part is less than a whole. If the finite part is measured, the multitude of units it contains is enumerable; and if it is incommensurable with the unit, the unit can be changed so as to make the finite part commensurable. Thus, to say that a finite part is less than its whole is the same as to say that an enumerable collection which is part of another is less than that other. There are two cases: first, when the whole is enumerable; and second, when the whole is inenumerable. Let us consider the first case. Let the M's be contained among the N's (which form an enumerable collection). Suppose however that the collection of M's is not less than that of the N's. Then, by the definition of equality, there is such a one-to-one relation *d*, that every N is *d*'d by an M. Then, since this M is an N, every N is *d*'d by an N. But *d* being a one-to-one relation, there are not two N's that are *d*'d by the same N. Hence, by the syllogism of transposed quantity, every N is *d* of an N. But the N's are, by their equality to the M's, *d*'d by nothing but M's. Hence, every N is an M. That is, we have shown that if the N's form an enumerable collection, the only collection at once contained in that collection and equal to that collection is the collection itself, and is not a part of the collection. That is, no part of an enumerable collection is equal to the collection. But the relation of inclusion is a one-to-one relation of every unit of the part to a unit of the whole. Hence, the part cannot be greater than the whole, and must be less than the whole.

* Cf. 183 and 106.

We now take up the second case. But we can go further, and show that every inenumerable collection is greater than any enumerable collection. It is to be shown that it is absurd to suppose that every unit of an inenumerable collection, the N's, is in a one-to-one relation, *c*, to a unit of any one enumerable collection, the M's. Let *r* be such a one-to-one relation that every M except one is *r* to an M. Then, by the syllogism of transposed quantity, every M except just one is *r*'d by an M. (For if every M were *r* to an M, every M would be *r*'d by an M; and since *r* is a one-to-one [relation], if there is a single one of the connections or relations between pairs of individuals, which is excluded from *r*, it leaves just one M not *r* to an M and just one M not *r*'d by an M.) This is so whatever one-to-one relation *r* may be. Hence, were every N *c* to an M, it would follow that every N but one would be *c* to an M that was *r* to an M that was *c*'d by an N; and this compound relation of being '*c* to an *r* of something *c*'d by' would be a one-to-one relation, being compounded of one-to-one relations. And invariably, whatever one-to-one relation *r* might be, one N would be the last in a count of the N's which should proceed from each N, say N_i, to that N, say N_j, such that N_i was *c* of that M that was *r* of that M which was *c*'d by N_j. In every such mode of counting, I say, some N would be the last N completing the count. And the M's being equal to the N's, and the one collection tied to the other by the relative, for every possible order of counting of the N's there would be some *r* relation among the M's; and thus in every possible counting of the N's there would be a last N, contrary to the hypothesis that the N's form an inenumerable collection. Thus, it is shown to be impossible that an inenumerable collection should be no greater than an enumerable collection, and the demonstration that a finite part is less than its whole is complete.

Now it is singular that every time Euclid reasons that a part is less than its whole, he falls into some fallacy, even though the part he is speaking of be finite.[1] I can only account

[1] Euclid has been so over-admired by men who were far from seeing all the depth of thought in the first book of the *Elements*, that it is hard to speak of him as he deserves without risk of being understood to admire what is not admirable. Undoubtedly, too, some of the merits of the *Elements* were not

for it by supposing that owing to the falsity of his axiom, he learned to think that very wonderful things could be proved by its aid, things that he would know could never be proved by any other axiom; for when a man appeals to an axiom he is pretty sure to be reasoning fallaciously. And thus he was prevented from suspecting and thoroughly criticizing those places in his reasoning. . . .

187. It is a curious illustration of how even that part of mankind who reason for themselves more than any others — I mean the mathematicians — yet how even they follow phrases and forget their meanings, that while everybody is in the habit of calling the proposition that a part is less than its whole an axiom, yet when this proposition is stated in another form of words — for the transformation amounts to little more — we always speak of it as the *fundamental theorem of arithmetic*. The statement is that if in counting a collection with the cardinal numerals the count of a collection comes to a stop from the exhaustion of the individuals it always comes to a stop at the same numeral. I say that this amounts pretty much to saying that an enumerable part can not equal its whole. For to say that the same collection can in one order of counting count 16 and in another order of counting count 15 would be the same as to say that the first 16 numerals could (through the identity of the objects counted) be put into a one-to-one correspondence with the first 15 numerals; and this,

original with Euclid. It is only the first book in which he has elaborated the logic as far as he was able. One of the remarkable merits of it is that Euclid had evidently gone far toward an understanding of the non-Euclidean geometry, and must undoubtedly be classed among the non-Euclideans. One evidence of that is that he puts his famous postulate about parallels into the form in which it most obtrusively displays its hypothetic character. He ranks it, too, as a postulate, that is as a dubitable proposition not demonstrated. Then, too, he arranges his theorems with those which hold for all systems of measurement first. But the greatest blunder of Euclid is in setting it down as an axiom that a part is less than its whole. That this is not true in regard to inenumerable parts can be shown by a simple example. The collection of all the even numbers is only a part of the collection of all the whole numbers; for only every other number is even. But if we imagine the whole numbers written in a row, and under each imagine its double written, there will be a distinct and separate even number written under every whole number. That is to say, the even numbers and all the whole numbers will be in one-to-one correspondence with one another, so that, by the definition of equality, the two collections are equal, although one is but a part of the other.

by the definition of equality, would be to say that the collection of the first 15 numerals was equal to the collection of the first 16 numerals, although the former collection is an enumerable part of the latter.

It is generally understood to be very difficult to demonstrate this theorem logically, and so it is somewhat so if the principles of logic are not attended to. At any rate several of the proposed demonstrations egregiously beg the question.[1]

§2. THE DENUMERABLE

188. But I have lingered too long among enumerable multitudes. Let us go on to inquire what is the smallest possible multitude which is inenumerable?

Take the collection of M's. If this collection be such that taking any one-to-one relation *r* whatever, if every M is *r* to an M it necessarily follows that every M is *r*'d by an M, the collection of M's thereby fulfills the definition of an enumerable collection. We can substitute a phrase for the letter *r* in this statement and say that to call the collection of M's enumerable is the same as to assert that if every M, in any order of

[1] I shall use the language of the logic of relatives [to prove the "fundamental proposition of arithmetic"]. Namely, supposing λ signifies a class of ordered pairs of which PQ is one (QP may, or may not, belong to the class), then I shall say that P is a λ of Q and that Q is λ'd by P.

I next define a *finite* class. Suppose a lot of things, say the A's, is such that whatever class of ordered pairs λ may signify, the following conclusion shall hold. Namely, if every A is a λ of an A, and if no A is λ'd by more than one A, then every A is λ'd by an A. If that necessarily follows, I term the collection of A's a *finite* class.

I now proceed to prove the difficult part of the proposition, namely, that every collection of things the count of which can be completed by counting them in a suitable order of succession is finite. For suppose there be a collection of which this is not true, and call it the A's. Then, by the definition of a finite class, there must be some relative, or class of ordered pairs, λ, such that while every A is a λ of an A, and no two A's λ of the same A, there is some A not λ'd by any A. Then, I say that if this A, not λ'd by any A, be removed from the class of A's, the same thing will remain true. Namely, first, every A is λ of an A, for so it was before the removal, and no A λ'd by an A has been removed; second, no two A's are λ of the same A, for the removal could not increase the number fulfilling any positive condition; and third, there is still an A not λ'd by any A, namely, that A which was λ'd by the removed A, and by no other A. Now, the class of A's is said to have been counted, and by the definition of counting, some number must have been called out in counting the A that was arrangement, is immediately succeeded by another M, and *that* an M which does not so immediately succeed any other of the M's, then every M immediately succeeds another M, and there is some ring arrangement without any first. To say that if there be no last there can be no first, is to say the collection spoken of is enumerable.

To deny that the M's are enumerable is, then, as much as to assert that there is a possible arrangement in which each M is immediately followed by another M which so follows no third M, and yet there is an absolutely first M which does not follow any M. If now we deny that the collection of M's is enumerable but, at the same time, restrict it to including no individual that need not be included to make the collection inenumerable, we shall plainly have a collection of the lowest order of multitude which any inenumerable collection can have. Such a collection I call *denumerable*. To say, then, that the collection of M's is denumerable, is the same as to assert that it contains nothing except one particular object and except what is implied in the fact that there is a one-to-one relation r such that every M is r to an M. This is a logical character;

afterward removed. Let every number higher than that be lowered by unity, and a count of the class, after A is removed, results.

It follows, then, that if there be a collection not finite the count of which can, by a suitable arrangement, be terminated by any number N, then the same is true of some collection the count of which can be terminated by a lower number. This implies there is no lowest number; but by definition of number, there is a lowest number, namely, *one*. Thus, the hypothesis that a class whose count in any order can be completed is not finite is reduced to absurdity.

Now, suppose a finite class to be counted twice. By the definition of a finite class, each count must stop. For make λ mean "next followed in the counting by" and the definition states that if the counting does not stop, then there is no A at which it begins, which is contrary to the definition of counting. If the two counts do not stop at the same number, call that the superior which stops at the higher number.

Let the cardinal numbers used in this "superior" count be called the S's. Let a number of this count be said to be "successor" of the number which in the inferior count was called out against the same thing. Then, every S is successor of an S, but no two S's are successors of the same S, (since, by the definition of counting, no number was used twice in the inferior count). Consequently, the number of S's being finite by the definition of a finite class, every S is succeeded by an S, or, in other words, every S, including the greatest, was used in the inferior count. Hence, the two counts end with the same number.— From "The Critic of Arguments," III (1892).

for it is the same as to say that the syllogism of transposed quantity does not hold good of it but that the Fermatian inference does. That is, if the collection of M's is denumerable, every character which is true of a certain M, say M_0, and is also true of every M which is in a certain one-to-one relation to an M of which it is true, is necessarily true of every M of the collection.

For example, the entire collection of whole numbers forms a denumerable collection. For zero is a whole number, which is not greater by one than any number, there is a number greater by one than any given whole number, and there is no number or numbers which could be struck out of the collection and still leave it true that zero belonged to the collection and that there was a number of the collection greater by one than each number of the collection.

189. I have already shown by the example of the even numbers that a part of a denumerable collection may be equal to the whole collection. I will now prove that all denumerable collections are equal. For suppose that the M's and the N's are two denumerable collections. Then, a certain M can be found which we may call M_0, such that taking a certain one-to-one relation, *r*, every M except M_0 is *r* to an M, and there is an *r* to every M; and in like manner there is a one-to-one relation, *s*, such that every N except one, N_0, is *s* to an N, and every N is *s*'d by an N. Then, I say, that the relation, *c*, can be so defined that every M is *c* to an N, and every N is *c*'d by an M. For let M_0 be *c* to N_0 and to nothing else; and let N_0 be *c*'d by nothing but M_0, and if anything, X, is *c* to anything, Y, let the *r* to X (and it alone) be *c* to the *s* of Y and to nothing else. Then, evidently *c* is a one-to-one relation. But every M is *c* to an N, because M_0 is *c* to an N (namely to N_0) and if any M is *c* to an N, then the *r* of that M is *c* to an N (for it is, by the definition of *c*, *c* to the *s* of the N to which the former M is *c*). And in like manner every N is *c*'d by an M, because N_0 is *c*'d by an M (namely by M_0), and if any N is *c*'d by an M, then the *s* of that N is *c*'d by an M (for it is, by the definition of *c*, *c*'d by the *r* of the M by which the former N is *c*'d). Q. E. D.

Accordingly, there is but a single grade of denumerable multitude. So it is to be noted as a defect in my nomenclature, which I unfortunately did not remark when I first published it,* that *enumerable* and *denumerable*, which sound so much alike, denote, the one a whole category of grades of multitude and the other a simple grade like, *zero*, or *twenty-three*.

190. It will be convenient to make here a few remarks about arithmetical operations upon multitude. Please observe that I have not said one word as yet about number, and I do not propose even to explain at all what numbers are until I have fully considered the subject of multitude, which is a radically different thing. Arithmetical operations can be performed upon both multitudes and upon numbers, just as they can be performed upon the terms of logic, the vectors of quaternions, the operations of the calculus of functions, and other subjects. What I ask you at this moment to consider is, not at all the addition and multiplication of numbers, for you do not know what I mean by numbers — it is safe to say so, since the word bears so many different meanings — but the addition and multiplication of multitudes.

Addition in general differs from aggregation inasmuch as a unit is increased by being *added* to itself but not by being aggregated to itself. When mutually exclusive terms are aggregated, that is the same as the addition of them. Addition might, therefore, be defined as the aggregation of the positings of terms. Two positings of the same term being different positings, their aggregate is different from a single positing of the term. The sum of two multitudes is the multitude of the aggregate of two mutually exclusive collections of those multitudes. The *aggregate* of a collection of collections of units may be defined as that collection of units, every unit of which is a unit of one of those collections, and which has every unit of any of those collections among its units.

191. It is easily proved that the sum of an enumerable collection of enumerable multitudes is an enumerable multitude.† . . .

192. The sum of an enumerable multitude and the denumerable multitude is denumerable. The proof is excessively simple; for we have only to count the enumerable collection in

* See 3.546.

† Peirce's proof of this runs into several pages. As it is not an original proof, it has been omitted.

linear series, first. The count of that has to end; and then the denumerable series may follow in its primal order.

193. That the denumerable multitude added to itself gives itself is made plain by zigzagging through two denumerable series. But this comes more properly under the head of multiplication of multitudes, which I propose to consider.

Mathematicians seem to be satisfied so far to generalize the conception of multiplication as to make it the application of one operation to the result of another. But the conception may be still further generalized, and in being further generalized it returns more closely to its primitive type. The more general conception of multiplication to which I allude is expressed in the following definition: *Multiplication* is the pairing of every unit of one quantity with every unit of another quantity so as to make a new unit. Since there are two acceptions of the term *pair* — the ordered acception, according to which AB and BA are different pairs, and the unordered acception — there are two varieties of multiplication, the non-commutative and the commutative. Multiplication may further be distinguished into the *free* and the *dominated*. In free multiplication the idea of pairing remains in all its purity and generality. In dominated multiplication, the product of two units is that which results from the special mode of pairing which is of preëminent importance with reference to the particular kind of units that are paired. Thus, in reference to length and breadth the pairing of their units in units of area is preëminently important; in reference to an operator and its operant the pairing of their units in units of the result is preëminently important; in logic, in reference to two general terms, the pairing of their units in identical units which reunite their essential characters is preëminently important, etc. In the multiplication of multitudes we have one of the very rare instances of free multiplication. The *product* of a collection of multitudes called its *factors* may be defined as the multitude of possible sets of units any one of which could be formed out of units taken one from each of a collection of mutually exclusive collections of units having severally the multitudes of the factors. For example, to multiply 2 and 3, we take a collection of two objects, as A and B, and a distinct collection of three objects, as X, Y, and Z, and form the pairs AX, AY, AZ, BX, BY, BZ, which are all the sets that can be formed each from one unit of each collection. Then, since the multitude of these pairs is 6, the product of the multitudes, 2 and 3, is the multitude of 6.

194. The same general idea affords us a definition of involution. *Involution* is the formation of a new quantity a *power* from two quantities, a *base*, and an *exponent*, each unit of the power resulting from the attachment of all the units of the exponent each to some one unit of the base, without reference to how many units of the exponent are attached to any one unit of the base. Thus, 3 to the 2 power is the multitude of different ways in which both of two units, A and B, can be joined each to some one of three objects, X, Y, and Z. . . .

195. The product of two multitudes, μ and ν, is equal to the multitude of units in μ mutually exclusive collections each of ν units. For since there is one unit and but one for each of the ν units of each of the μ collections, these units are in one-to-one correspondence with the possible descriptions of single units each of which pairs a unit of a multitude of ν with a unit of a multitude of μ; and the multitude of such pairs is the product of μ and ν.

The μ power of ν is equal to the product of μ mutually exclusive collections each of ν units.

The product of two enumerable multitudes is an enumerable multitude.

The product of an enumerable multitude and the denumerable multitude is the denumerable multitude.

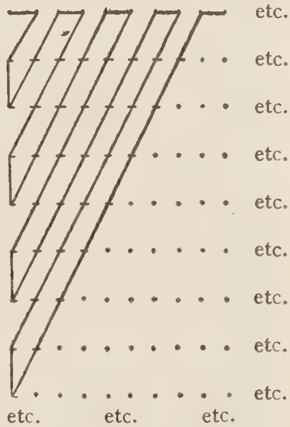
An enumerable power of the denumerable multitude is the denumerable multitude.

196. That the second power of the denumerable multitude is the denumerable multitude is easily seen by aggregating a denumerable series of collections, each a denumerable series of units.

Now we can start at the corner and proceeding from each unit we reach to a single next one and can reach any unit whatever in time without completing the proceeding. Hence, the whole forms one denumerable series. This proof is substantially that of Cantor.* The proposition being proved for two factors instantly extends itself to any enumerable multitude of factors. Of course, there is not the slightest difficulty in express-

* *Georg Cantor Gesammelte Abhandlung*, S. 294–5 (1932).

ing this idea so as to construct the most rigidly formal demonstration. Let $\aleph$ denote the denumerable multitude. Then, I am to show that $\aleph^2 = \aleph$. Let the M's be a denumerable collection. That is, suppose



First: a certain object M_0, is an M;

Second: there is a certain non-identical one-to-one relation, *r*, such that every M is *r*'d by an M;

Third: whatever is not necessitated to be an M by the above statements is not an M.

Let A and B constitute a collection of two objects not M's. Let us define the relation ς as follows:

First: the pair of attachments of A to M_0 and B to M_0 is ς'd by nothing;

Second: every pair of attachments of A to an M which we may call M_i other than M_0 and of B to an M, which we may call M_j, is ς to the pair of attachments of A to that M which is *r*'d by M_i, and of B to that M which is *r* of M_j;

Third: every pair of attachments of A to M_0 and of B to an M, which we may call M_k, is ς to the pair of attachments of A* to that M which is *r* of M_k, and of B† to M_0;

Fourth: if one thing is not necessitated by the above rules to be ς to another, it is not ς to that other.

It is evident, then, that ς is a one-to-one relation; and it is evident that every pair of attachments of A to any M, say M_x, and of B to any M, say M_y, is ς of another such pair of attachments, that one such pair of attachments is ς'd by nothing, and that nothing is a pair of such attachments that is not necessitated to exist by the fact that everything is ς of something. Hence, the multitude of those pairs of attachments is denumerable; and that is the same as to say that the second power of the denumerable multitude is the denumerable multitude.

* B?

† A?

197. Dr. George Cantor* first substantially showed that between the units of any denumerable collection certain remarkable relations exist, which I call indefinitely divident relations. Namely, let the M's be any denumerable collection, and let f be any relation indefinitely divident of the collection of the M's. Then, no M is f to itself, but of any two different M's one is f to the other; and if an M is f to another it is f to every M that is f'd by that other; and if an M is f'd by another it is f'd by every M that is f to that other. And now comes the remarkable feature: If one M is f to another, it is f to an M that is not f'd by that other; whence, necessarily if one M is f'd by another, it is f'd by an M that is not f to that other.

There are vast multitudes of such indefinitely divident relations. I will instance a single one. If we take the whole series of vulgar fractions, those of the same denominator being taken immediately following one another in the increasing order of the numerators and those of different denominators in the increasing order of the denominators,

$$\frac{1}{2} \quad \frac{1}{3}\,\frac{2}{3} \quad \frac{1}{4}\,\frac{2}{4}\,\frac{3}{4} \quad \frac{1}{5}\,\frac{2}{5}\,\frac{3}{5}\,\frac{4}{5} \quad \frac{1}{6}\,\frac{2}{6}\,\frac{3}{6}\,\frac{4}{6}\,\frac{5}{6} \quad \text{etc.,}$$

these evidently form a denumerable collection, for they form the aggregate of a denumerable collection of enumerable collections of units. If from this collection we omit those fractions which are equal to other fractions of lower denominations we plainly have still a denumerable collection. Now for the first of these denumerable collections, that of all the vulgar fractions, an indefinitely divident relation is that of being "greater than or equal to but of higher terms than". For the second of those denumerable collections, that of all the rational quantities greater than 0 and less than 1, an indefinitely divident relation is that of being "greater than." . . .

Numbers in themselves cannot possibly signify any magnitude other than the magnitudes of collections, or multitude; but what they principally represent is place in a serial order. Numbers do not contain the idea of the equality of parts and consequently a fraction cannot in itself signify anything involv-

* *Ibid.*, S. 296ff.

ing equality of parts. They merely express the ordinal place in such a uniformly condensed series. . . .*

198. A striking difference between enumerable and denumerable collections is this, that no arrangement of an enumerable collection has any different properties from any other arrangement; for the units are or may be in all respects precisely alike, that is, have the same *general* characters, although they differ individually, each having its proper designation. But it is not so with regard to denumerable collections. Every such collection has a *primal* arrangement, according to its *generating relation.* There is one unit, at least, which arbitrarily belongs to the collection just as every unit of an enumerable collection belongs to that collection. But after that one unit, or some enumerable collection of units, has been arbitrarily posited as belonging to the collection, the rest belong to it by virtue of the general rule that there is in the primal arrangement one unit of the collection next after each unit of the collection. Those last units cannot be all individually designated, although any one of them may be individually designated. Nor is this merely owing to an incapacity on our part. On the contrary, it is logically impossible that they should be so designated. For were they so designated there would be no contradiction in supposing a list of them all to be made. That list would be complete, for that is the meaning of *all.* There would therefore be a last name on the list. But that is directly contrary to the definition of the denumerable multitude.

The same truth may be stated thus: It is impossible that all the units of a denumerable collection should have the same general properties. For the existence of the primal arrangement is essential to it, being involved in the very definition of the denumerable collection as that of smallest multitude greater than every enumerable collection. Now, this primal arrangement is an arrangement according to a general rule, and its statement constitutes, therefore, general differences between the units of the denumerable collection.

On the other hand any unit whatever of a denumerable collection may be individually designated, as well as all those

* The editors omitted 16 manuscript pages of proof showing that there are a vast multitude of indefinitely divident relations between the units of any denumerable collection.

which precede it in the primal arrangement. And these can be all exactly alike in their general qualities. Yet there must always be a latter part of the collection which is not individually designated but is only generally described. In this part we recognize an element of ideal being as opposed to the brute and surd existence of the individual.

The denumerable collection of whole numbers, for example, constitutes a discrete series, in the sense that there is not one which may not be distinguished completely and individually from its neighbors.

But we cannot with any clearness of thought carry these reflections further until we are in possession of an instance of a greater collection.

199. The arrangement of a denumerable collection according to an indefinitely divident relation like the rational numbers — or to take a simpler instance, like the fractions which can be written in the binary system of arithmetical notation with enumerable series of figures — is a very recondite arrangement, not at all naturally suggested by the primal arrangement. This is shown by the fact that the world had to wait for George Cantor to inform it that the collection of rational fractions was a collection precisely like that of the whole numbers.*

This remark will be found important in the sequel.

§3. THE PRIMIPOSTNUMERAL

200. So much, for the present, for the denumerable multitude. Let us now inquire, what is the smallest multitude which exceeds the denumerable multitude? An enumerable or denumerable multitude is a multitude such that whatever in *any* arrangement of an enumerable collection, in the *primal* arrangement of a denumerable collection, is true of the first unit, and is further true of any unit which comes next after any unit of which it is true, is true of all and every unit of the collection. . . . It has not yet been proved that there is any such minimum multitude among those which exceed the denumerable; but it is convenient to say that in fact there is. I have hitherto named this multitude, which was first clearly described

* *Ibid.*, S. 304.

by Cantor,* the *first abnumeral* multitude.† But I find that a name in one word is wanted. So I will hereafter name it the *primipostnumeral* multitude.

201. Suppose it to be true of a collection that in whatever way its units be arranged in a horizontal line with one unit to the extreme left, and a unit next to the right of each unit, there is something which is true of the first unit and which if true of any unit is always true of the next unit to the right, which nevertheless is not true of all the units; and suppose furthermore that the collection is no greater than it need be to bring about that state of things. Then, that collection is by definition a primipostnumeral collection. Or by the aid of the logic of relatives, we may state the matter as follows:

1. Let there be an existent collection, R;
2. Let R include no unit which is not necessitated by that condition;
3. Let *r* be a one-to-one relation between units;
4. Let there be a collection, the Q's, such that no Q is R;
5. Let there be a Q that is *r* to each unit of the collections of the Q's and R's;
6. Let the collection of the Q's include nothing not necessitated by the foregoing conditions;
7. Let *h* be a one-to-one relation of a unit to a collection;
8. Let there be a collection, P, such that no P is a Q or R;
9. Let there be a P which is *h* to every (denumerable) collection of Q's.
10. Let there be no P which is not necessitated in order to fulfill the foregoing conditions.

Then, the collection of P's is a primipostnumeral collection.

It would be easy to make this statement more symmetrical in appearance; but I prefer to make it perspicuous. Thus, we might make *r* a relation between a unit and an enumerable collection; and we might make the P's include an *h* for every denumerable collection of P's, Q's and R's, etc. The word "denumerable" in the ninth condition is added merely for the sake of perspicuity.

The second, sixth, and tenth conditions are not very clear. The meaning is that the multitude is no larger than need be.

* *Ibid.*, S. 288; 325ff.

† Cf. 117.

202. The definitions of a primipostnumeral collection just given suppose it to be constructed from a denumerable collection. But if we attempt to form a primipostnumeral collection from a denumerable collection in its primal arrangement we shall fail ignominiously.

Let us, for example, imagine a series of dots representing, the first [dot] the position of the tortoise when Achilles began to run after him, and each successive dot the position of the tortoise at the instant when Achilles reached the position represented by the preceding dot. If there are no more dots than are necessary to fulfill this condition, the collection of dots is denumerable. If we add a dot to represent the position of the tortoise at the moment when Achilles catches up with him, the Fermatian inference seems at first sight not to hold good. For the first dot represents a position of the tortoise *before* Achilles had caught up with him, and if any dot represents the position of the tortoise before Achilles caught up with him, so likewise does the dot which immediately succeeds it. The Fermatian inference then would seem to be that every dot represents a position of the tortoise before Achilles had caught up with him. Yet this is not true of the last dot which represents the position of the tortoise at the moment when Achilles caught up with him. Yet but one dot has been added to the denumerable collection, and of course, it remains denumerable. The only reason that the inference does not hold is that the dots are no longer in their primal arrangement. Put the last dot at the beginning, so as to preserve the primal arrangement, and any Fermatian inference whose premisses were true would hold good. The point I wish to make is that the denumerable collection in its primal order leads to no way of constructing or of conceiving of a primipostnumeral collection. Of course, we can say, "Let there be a dot for each denumerable collection of the tortoise-places;" but we might as well omit the tortoise-places and say, "Let there be a primipostnumeral collection of dots." The primal arrangement of the denumerable collection affords no definite places nor approximations to the places for the primipostnumeral collection.

203. The reason is that the latter part of the denumerable collection, which is its denumerable point, is all concentrated towards one point, whether that point be a metrically ordinary

point or a point at infinity. This fault is remedied in the indefinitely divident arrangement. Here the denumerable part of the collection is spread over a line.

In this case, if we imagine all those subdivisions to be performed which are implied by saying that the intervals resulting from each set of subdivisions are all subdivided in the next following set of subdivisions, the multitude of subdivisions is $2^{\aleph}$ where $\aleph$ is the denumerable multitude; and this is no mere algebraical form without meaning. It has a perfectly exact meaning which I explained in speaking of the effects of addition, multiplication, and involution upon multitudes.

Moreover, you will remember that I distinctly and fully proved* that the multitude of possible sets of units each of which can be formed from the units of a collection always exceeds the multitude of that collection, provided it be a discrete collection.

204. Do you not think it possible that the stellar universe extends throughout space? If so, the whole collection of worlds is at least denumerable. At any rate, it is perfectly possible that the whole collection of intelligent beings who live, have lived, or will live anywhere is at least equal to the collection of whole numbers. It is conceivable that they are all immortal and that each one should be given each hour throughout eternity the name of one of them and he should assign that person in wish to heaven or to hell, so that in the course of eternity he would wish every one of them to heaven or to hell. Could they by all making different wishes wish among them for every possible distribution of themselves to heaven or to hell? If not, the multitude of such possible distributions is greater than the denumerable multitude. But they plainly could not wish for all possible such distributions. For if they did, some one would necessarily be perfectly satisfied with every possible distribution. But one possible distribution would consist in sending each person to the place he did not wish himself to go; and that would satisfy nobody. It was Cantor who first proved that the surd quantities form a collection exceeding the collection of rational quantities.† But his method was only applicable to that particular case. My method is

* See 180n.

† *Ibid.*, S. 278–80; 288.

applicable to any discrete multitude whatever and shows that $2^{\mu} > \mu$ in every case in which μ is a discrete multitude.

205. I will give a few more examples of primipostnumeral collections. The collection of quantities between zero and unity, to the exact discrimination of which decimals can indefinitely approximate but never attain, is evidently $10^{\aleph}$, which of course equals $2^{\aleph}$. For $16^{\aleph} = (2^4)^{\aleph} = 2^{(4\aleph)} = 2^{\aleph}$.

The collection of all possible limits of convergent series, whose successive approximations are vulgar fractions, although it does not, according to any obvious rule of one-to-one correspondence, give a limit for every possible denumerable collection of vulgar fractions, does nevertheless in an obvious way correspond each limit to a denumerable collection of vulgar fractions, and to so large a part of the whole that it is *primipostnumeral*, as Cantor has strictly proved.*

206. Just as there is a primal arrangement of every denumerable collection, according to a generating relation, so there is a primal arrangement of every primipostnumeral collection, according to a generating arrangement. This primal arrangement of the primipostnumeral collection springs from a highly recondite arrangement of the denumerable collection. Namely, we must arrange the denumerable collection in an indefinitely divident order, and then the units, which are implied in saying that the denumerable succession of subdivisions have been completed constitute the primipostnumeral collection. But when I say that the primipostnumeral collection *springs from an arrangement* of the denumerable collection, I do not mean that it is formed from the denumerable collection itself; for that would not be true. On the contrary, the primipostnumeral collection can only be constructed by a method which *skips* the denumerable collection altogether. In order to show what I mean I will state the definition of a primipostnumeral collection in terms of relations. There are two or three trifling explanations to be made here. First an aggregate of collections is a collection of the units of those collections. It is also an aggregate of the collections, which are called its aggregants. Just as to say that Alexander cuts some knot implies that a knot exists, although to say Alexander cuts every knot, *i.e.*, whatever knot there may be, does not imply the existence of

* *Ibid.*, S. 288.

any knot, the latter by its generality referring to an ideal being, not to a brute individual existence, so to say that a collection has a certain collection as its aggregant implies the existence of the latter collection and therefore that it contains at least one unit. I must also explain that whenever I say either one thing or another is true I never thereby mean to exclude both.

207. I will now describe a certain collection A, whose units I will call the *P*'s [Π's?].

First, The Π's can be arranged in linear order. That is, there is a relation, *p*, such that taking as you will any Π's, individually designable as Π_1, Π_2, and Π_3, either Π_3 is not *p* to Π_2 or Π_2 is not *p* to Π_1 or (if Π_3 is *p* to Π_2 and Π_2 is *p* to Π_1), Π_3 is *p* to Π_1;

Second, The line of arrangement of the Π's can be taken so as not to branch. That is, taking as you will Π's, individually designable as Π_4 and Π_5, either Π_4 is *p* to Π_5 or Π_5 is *p* to Π_4; (of course this permits both to be true, but that I proceed to forbid).

Third, The line of arrangement of the Π's can further be so taken as not to return into itself, circularly. That is, taking as you will any Π, individually designable as Π_6, Π_6 is not *p* to Π_6;

Fourth, There are certain parts of A called "packs" of Π, which are mutually exclusive. That is, taking any pack whatever and any unit of that pack, that unit is a Π; and taking as you will any packs individually designable as P_7* and P_8, and any Π's individually designable as Π_7 and Π_8, either P_7 is identical with P_8 or Π_7 is not a unit of P_7, or Π_8 is not a unit of P_8, or else Π_7 is not identical with Π_8;

Fifth, The packs can be arranged in linear order. That is, there is a relation, *s*, such that taking as you will any *P*'s, individually designable as P_1, P_2, and P_3, either P_3 is not *s* to P_2, or P_2 is not *s* to P_1, or P_3 is *s* to P_1;

Sixth, The line of arrangement of the packs can be taken so as not to branch. That is, taking as you will any *P*'s, individually designable as P_4 and P_5, either P_4 is *s* to P_5 or P_5 is *s* to P_4;

Seventh, The line of arrangement of the packs can be further

* Peirce used a square instead of a *P*.

taken so as not to return into itself. That is, taking as you will any pack individually designable as P_6, P_6 is not s to P_6;

Eighth, The arrangement of the packs can further be such that each pack is immediately succeeded by a next following pack. That is, taking as you will any pack individually designable as P_9, a pack individually designable as P_{10} can be found such that P_{10} is s to P_9; and such that taking thereafter as you will any pack individually designable as P_{11}, either P_{11} is not p to P_9, or P_{11} is not p'd by P_{10};

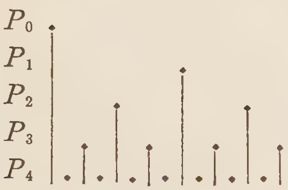
Ninth, Such a succession of packs is not a mere idea, but actually exists if the collection A exists. That is, a certain collection, P_0, is such a pack;

Tenth, Each pack contains a unit which, in the linear order of the Π's, comes next after each unit of any of those packs which precede this pack in the linear order of the packs. That is, taking as you will any packs, individually designable as P_{12} and P_{13}, and any unit, individually designable as Π_{12}, a unit, individually designable as Π_{13}, can be thereafter found such that, taking as you will any pack individually designable as P_{14} and any unit individually designable as Π_{14}, either P_{13} is not s to P_{12}, or Π_{12} is not a unit of P_{12}, or Π_{13} is p to Π_{12}; and either P_{13} is not s to P_{14} or Π_{14} is not a unit of P_{14}, or Π_{12} is p to Π_{14}, or Π_{13} is not p to Π_{14};

Eleventh, No varieties of descriptions of Π's exist than those which are necessitated by the foregoing conditions;

Twelfth, No varieties of descriptions of packs exist than those which are necessitated by the foregoing conditions.

This collection of Π's is primipostnumeral; and you will see what I mean by saying that the construction skips the denumerable multitude, if you consider how many Π's are contained in each pack. The pack P_0 is obliged by the ninth condition to exist, so that it must contain at least one Π. But nothing obliges it to contain a Π which is other than any Π which it contains; and therefore the twelfth condition forbids it to contain [more than] one Π. It consists, therefore, of a single Π. If we arrange the Π's in a horizontal row so that p shall be equivalent to being "further to the right than," then that P which is s to P_0, but is not s to any other pack, which pack we may call P_1, must contain one Π to the right of the Π of [P_0]. It need contain no other, and therefore cannot contain any other.



P_2 contains a Π immediately to the right of that of the P_0 and another to the right of that of P_1, and after this each pack contains double the units of the preceding. Thus, P_{n+1} contains 2^n units. As long as n is enumerable, this is enumerable. But as soon as n becomes denumerable, it skips the denumerable multitude and becomes primipostnumeral.

208. In order to prove that any proposition is generally true of every member of a denumerable collection, it is always necessary — unless it be some proposition not peculiar to such a collection — to consider the collection either in its primal arrangement, or in reference to some relation by which the collection is generable, and then reason as follows, where r is the generating relation, and M_0 is that M which is not r to any M:

M_0 is X,
If any M is X then the r of M is X;
∴ Every M is X.

Without this Fermatian syllogism no progress would ever have been made in the mathematical doctrine of whole numbers; and though by the exercise of ingenuity we may seem to dispense with this syllogism in some cases, yet either it lurks beneath the method used, or else by a generalization the proposition is reduced to a case of a proposition not confined to the denumerable multitude.

209. In like manner, in order to prove that anything is true of a primipostnumeral collection, unless it is more generally true, we must consider that collection in its primal arrangement or with reference to a relation equivalent to that of its primal arrangement. The special mode of reasoning will be as follows:

Π_0, the unit of P_0, is X,

If every Π of any pack is X, then every Π of the pack which is *s* of that pack is X;

Hence, every Π is X.

This may be called the primipostnumeral syllogism.

210. Every mathematician knows that the doctrine of real quantities is in an exceedingly backward condition. It cannot be doubted by any exact logician that the reason of this is the neglect of the primipostnumeral syllogism without which it is as impossible to develop the doctrine of real quantities, as it would be to develop the theory of numbers without Fermatian reasoning.

I do not mean to say that the primipostnumeral syllogism is altogether unknown in mathematics; for the reasoning of Ricardo* in his theory of rent, reasoning which is of fundamental importance in political economy, as well as much of the elementary reasoning of the differential calculus, is of that nature. But these are only exceptions which prove the rule; for they strongly illustrate the weakness of grasp, the want of freedom and dexterity with which the mathematicians handle this tool which they seem to find so awkward that they can only employ it in a few of its manifold applications.

211. In the denumerable multitude we noticed the first beginnings of the phenomenon of the fusion of the units. All the units of the first part of the primal order of a denumerable multitude can be individually designated as far as we please, but those in the latter part cannot. In the primipostnumeral multitude the same phenomenon is much more marked. It is impossible to designate individually all the units in any part of a primipostnumeral multitude. Any one unit may be completely separated from all the others without the slightest disturbance of the arrangement.

Thus, we may imagine points measured off from 0 as origin

toward A to represent the real quantities from zero toward $\sqrt{2}$. Let A be the point which according to this measurement would represent $\sqrt{2}$. But we may modify the rule of one-to-one

* See *On the Principles of Political Economy and Taxation*, ch. II. Cf. 115.

correspondence between quantities and points, so that, for all values less than $\sqrt{2}$, the points to the left of A represent those values, while another point an inch or two to the right shall represent $\sqrt{2}$, and all quantities greater than $\sqrt{2}$ shall be represented by points as many inches or parts of an inch to the right of a third point, C, several inches to the right of B, as there are units and parts of units in the excess of those quantities over $\sqrt{2}$. This mode of representation is just as perfect as the usual unbroken correspondence. It represents all the relations of the quantities with absolute fidelity and does not disturb their arrangement in the least.

It is, therefore, perfectly possible to set off any one unit of a primipostnumeral collection by itself, and equally possible so to set off any enumerable multitude of such units. Nor are there any singular units of the collection which resist such separation.

I will give another illustration. It is perfectly easy to exactly describe many surd quantities simply by stating what their expressions in the Arabic system of notation would be. This may sound very false; but it is so, nevertheless. For instance, that quantity, which is expressed by a decimal point followed by a denumerable series of figures, of which every one which stands in a place appropriated to $\left(\frac{1}{10}\right)^n$ where n is *prime* shall be a figure 1, while every one which stands in a place whose logarithm n is composite shall be a *cipher*, is, we know, an irrational quantity. Now, I do not think there can be much doubt that, however recondite and complicated the descriptions may be, *every* surd quantity is capable in some such way of having its expression in decimals exactly described.

Thus every unit of a primipostnumeral collection admits of being individually designated and exactly described in such terms as to distinguish it from every other unit of the collection. Thus, notwithstanding a certain incipient cohesiveness between its units, it is a discrete collection, still. . . .

212. It is one of the effects of the deplorable neglect by mathematicians of the properties of primipostnumeral collections that we are in complete ignorance of an arrangement of such a collection, which should be related to its primal arrangement in any manner analogous to the relation of the arrange-

ment [of] the primal arrangement of the denumerable collection to that indefinitely divident arrangement, which leads to a clear conception of the next grade of multitude.

I have had but little time to consider this problem; but I can produce an arrangement which will be of some service. Suppose that instead of proceeding, as in the usual generation of the primipostnumeral multitude, to go through a denumerable series of operations each consisting in interpolating a unit between every pair of successive units, we go through a denumerable multitude of operations each consisting in replacing every pair by an image of the whole collection. For example, using the binary system of arithmetical notation, suppose we begin with a collection of two objects, zero and one-half.

.0 .1

Each operation may consist of replacing each number by a sub-collection of [all the] numbers, each consisting of two parts, the first part being the figures of the number replaced, the second [being the figures of one of] the numbers composing the whole collection. Thus, the result of the first operation will be

.00 .01 .10 .11

The result of the second operation will be

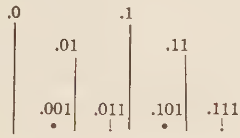
.0000	.0001	.0010	.0011	.0100	.0101	.0110
.0111	.1000	.1001	.1010	.1011	.1100	.1101
.1110	.1111					

The next result would be 256 numbers, the next 65,536 numbers, the next 4,294,867,296. The result of a denumerable succession of such operations will evidently be to give all the real quantities between zero and one, which is a primipostnumeral collection.

§4. THE SECUNDOPOSTNUMERAL AND LARGER COLLECTIONS

213. Although I have not touched upon half the questions of interest concerning the primipostnumeral multitude, I must hurry on to inquire, what is the least multitude greater than the primipostnumeral multitude? Time forbids my going through a fundamentally methodical discussion of this problem. But the speediest route to a correct solution of a difficult

logical crux lies almost always through that paradox or sophism which depends upon that crux. Let us recur then for a moment to the indefinitely divident arrangement of a primipostnumeral collection. It will be convenient to use the binary system of arithmetical notation. We begin with .0 as our Π_0. P_1 consists



of a fraction equal to that but carried into the first place of secundals and of corresponding units which differ only in having a 1 in the first place of secundals. P_2 consists of fractions equal to those but carried into the second place of secundals, together with fractions differing from them only in having a 1 in the second place of secundals. And so on. Now if we use all the enumerable places of secundals, but stop before we reach any denumerable place, we shall have, among all the packs, all the fractions whose denominators are powers of 2 with enumerable exponents, and therefore we shall plainly have only a denumerable collection. But if n is the number of packs up to a given pack, then the number of fractions will be $2^n - 1$. When we have used all the enumerable places of secundals and no others, how many packs have we used? Plainly a denumerable collection, since the multitude of enumerable whole numbers is denumerable. It would appear, then, that $2^n - 1$, when n is denumerable, is denumerable. But on the contrary, if we consider only that pack which fills every enumerable place of secundals, since it contains the expression in secundals of every real quantity between 0 and 1, it alone is a primipostnumeral collection. Moreover, the number of Π's in P_n is 2^n, and since n is denumerable for this collection, it follows that 2^n is primipostnumeral. And it is impossible that the subtraction of one unit should reduce a primipostnumeral collection to a denumerable collection. Again, every pack contains a multitude of individuals only 1 more than that of all the packs that precede it in the order of the packs. How then can the former be primipostnumeral while the latter is denumerable?

The explanation of this sophism is that it confounds two categories of characters of collections, their *multitudes* and their *arithms*. The arithm of a multitude is the multitude of multitudes less than that multitude. Thus, the arithm of 2 is 2; for the multitudes less than 2 are 0 and 1. By *number* in one of its senses, that in which I endeavor to restrict it in exact discussions, is meant an enumerable arithm. Thus, the arithm or number of any enumerable multitude is that multitude. The arithm of the denumerable multitude, also, is that multitude. But the arithm of the primipostnumeral multitude is the denumerable multitude. The maximum multitude of an increasing endless series that converges to a limit is the arithm of that limit, in this sense, that by the limit of an increasing endless series is meant the smallest multitude greater than all the terms of the series. If there is no such smallest multitude the series is not convergent. If, then, by the maximum multitude of an increasing series we mean the multitude of *all* the multitudes which would converge increasingly to the given limit, this maximum multitude is plainly the arithm of the series. Thus, the series of whole numbers is an increasing endless series. Its limit is the denumerable multitude. The arithm of this multitude is the maximum multitude of the series. If in 2^n we substitute the different whole numbers for n, we get an increasing endless series whose limit is the primipostnumeral multitude. Its arithm, which is the maximum multitude of the series, is denumerable only. It is strictly true that the multitude of pack P_n, in the example to which the sophism relates, is 2^n. But it is not strictly true that the multitude of Π's in all preceding packs is 2^n-1. It happens to be so when n is a *number*, that is, is enumerable. But strictly it is the multitude *next smaller* than the multitude of 2^n. If the latter is the primipostnumeral multitude, the former can be nothing but the denumerable multitude. This is what we find to be the case, as it must be; and there is nothing paradoxical in it, when rightly understood. There is no value of n for which 2^n is denumerable.

The limit of 2^n is primipostnumeral. The denumerable is skipped. But were we to reach the denumerable as we may, if we erroneously assume the sum of 2^n is 2^n-1, when we double that on the principle that $2^n=2\times(2^n-1)$, we, of course, only have the denumerable as the result.

214. Let us now consider 2^{2^n}. Since 2^n can never be denumerable, but skips at once from the enumerable to the primipostnumeral, when 2^{2^n} is denumerable, it follows that 2^{2^n} can never be denumerable nor primipostnumeral. For there is no value which 2^n could have to make 2^{2^n} denumerable; and in order that 2^{2^n} should be primipostnumeral, 2^n would have to be denumerable, which is impossible. Thus, 2^{2^n} skips the denumerable and the denumerable multitudes. But if we use square brackets to denote the arithm, so that $[2]=2$, $[3]=3$, $[\infty]=\infty$, etc., then since $[2^\infty]$ is denumerable, $2^{[2^\infty]}$ is primipostnumeral.

215. When we start with .0 and .1, and repeating these varieties in the next figures, get .00 .01 .10 .11, and then repeating these varieties in the next figures, get .0000 .0001 etc., and then repeating these varieties in the next figures, get .00000000 .00000001 etc., if we say that, when this operation is carried out until the number of figures is denumerable, we get a primipostnumeral collection, we are assuming what is not true, that by continually doubling an enumerable multitude we shall ever get to a denumerable multitude. That is not true. In that process the denumerable multitude is skipped. We are assuming that because [the] multitude of all the arithmetical places which we pass by is denumerable, when the operation has been performed a denumerable multitude of times, therefore the multitude reached is denumerable. That is, we are confusing 2^∞ with $[2^\infty]$.

The function 2^{2^x} is no doubt the simplest one which skips the denumerable and primipostnumeral multitudes. Therefore the multitude of this when x is denumerable is, no doubt, the smallest multitude greater than the primipostnumeral multitude. It is the *secundopostnumeral* multitude.

216. Although there can remain no doubt whatever to an exact logician of the existence in the world of mathematical ideas, of the secundopostnumeral multitude, yet I have been unable, as yet, to form any very intuitionally conception* of the construction of such a collection. But I must confess I have not bestowed very much thought upon this matter. I give a few constructions which have occurred to me.

Imagine points on a line to be in one-to-one correspondence

* I.e. . . . any intuitional concept.

with all the different real quantities between 0 and 1. Imagine the line to be repeated over and over again in each repetition having a different set of those points marked. Then the entire collection of repetitions is a *secundopostnumeral* collection.

Imagine a denumerable row of things, which we may call the B's. Let every set of B's possess some character, which we may call its crane* different from the crane of any other set. Imagine a collection of houses which we may call the beths* such that each house contains an object corresponding to each crane-character, and according as that object does or does not possess that character, the beth is said to possess or want that character. Then, the different possible varieties of beths, due to their possessing or not possessing the different cranes, form a secundopostnumeral collection.

According to the hypothesis of Euclidean projective geometry there is a plane at infinity. That plane we virtually see when we look up at the blue spread of the sky. A straight line at infinity, although it is straight and looks straight, is called a great semi-circle of the heavens. At two opposite points of the horizon we look at the same point of the plane at infinity. Of course, we cannot look both ways at once. We measure distances on an ordinary straight line by metres and centimetres. We measure distance on a straight line in the sky by degrees and minutes. The entire circuit of the straight line is 180 degrees, and the circuits of all straight lines are equal. But in metres the measure is infinite. If by a projection we make a position of a straight line in the sky correspond to a straight line near at hand, we perhaps make a degree correspond to a metre, although in reality a metre is to a degree in the proportion, 180 degrees to infinity. Imagine that, upon a straight filament in the sky, points are marked off metrically corresponding to all the real quantities. Then let that filament be brought down to earth. If one of those real quantities' points is at any near point, there will not be another at any finite number of kilometres from it. For were there two, when it was in the sky they would have been closer together than any finite fraction of a second of arc. If, however, when you had pulled the filament down from the sky you were to find that each of those things you took for points was really a doubly refracting

* The drawings have been omitted.

crystal and that these acted quite independently of one another, so that when you looked through two others you saw four images, when you looked through three you saw eight images, and so on, then if you were to look along the filament through all the crystals, one for each real quantity, the collectum of images you would see would be a secundopostnumeral collection.

217. In like manner, there will be a *tertiopostnumeral* multitude. $2^{2^{2^{\aleph}}}$, a *quartopostnumeral* multitude $2^{2^{2^{2^{\aleph}}}}$ and so on *ad infinitum.*

All of these will be discrete multitudes although the phenomenon of the incipient cohesion of units becomes more and more marked from one to another.

These multitudes bear no analogy to the orders of infinity of the calculus; for $\infty^1 \times \infty^1 = \infty^2$. But any of these multiplied by itself gives itself. I had intended to explain these infinites of the calculus. But I find I cannot cram so much into a single lecture.

218. I now inquire, is there any multitude larger than all of these? That there is a multitude greater than any of them is very evident. For every postnumeral multitude has a next greater multitude. Now suppose collections one of each postnumeral multitude, or indeed any denumerable collection of postnumeral multitudes, all unequal. As all of these are possible their aggregate is *ipso facto* possible. For aggregation is an existential relation, and the aggregate exists (in the only kind of existence we are talking of, existence in the world of noncontradictory ideas) by the very fact that its aggregant parts exist. But this aggregate is no longer a discrete multitude, for the formula $2^n > n$ which I have proved holds for all discrete collections cannot hold for this. In fact writing Exp. n for 2^n, (Exp.)$\aleph^{\aleph}$ is evidently so great that this formula ceases to hold and it represents a collection no longer discrete.

§5. CONTINUA

219. Since then there is a multiplicity or multiplicities greater than any discrete multitude, we have to examine continuous multiplicities. Considered as a mere multitude, we might be tempted to say that continuous multiplicities are

incapable of discrimination. For the nature of the differences between them does not depend upon what multitudes enter into the denumerable series of discrete multitudes out of which the continuous multiplicity may be compounded; but it depends on the manner in which they are connected. This connection does not spring from the nature of the individual units, but constitutes the mode of existence of the whole.

The explanation of the paradoxes which arise when you undertake to consider a line or a surface as a collection of points is that, although it is true that a line is nothing but a collection of points of a particular mode of multiplicity, yet in it the individual identities of the units are completely merged, so that not a single one of them can be identified, even approximately, unless it happen to be a topically singular point, that is, either an extremity or a point of branching, in which case there is a defect of continuity at that point. This remark requires explanation, owing to the narrowness of the common ways of conceiving of geometry. Briefly to explain myself, then, geometry or rather mathematical geometry, which deals with pure hypotheses, and unlike physical geometry, does not investigate the properties of objectively valid space — mathematical geometry, I say, consists of three branches; Topics (commonly called Topology), Graphics (or pure projective geometry), and Metrics. But metrics ought not to be regarded as pure geometry. It is the doctrine of the properties of such bodies as have a certain hypothetical property called absolute rigidity, and all such bodies are found to slide upon a certain individual surface called the Absolute. This Absolute, because it possesses individual existence, may properly be called a thing. Metrics, then, is not pure geometry; but is the study of the graphical properties of a certain hypothetical thing. But neither ought graphics to be considered as pure geometry. It is the doctrine of a certain family of surfaces called the *planes*. But when we ask what surfaces these planes are, we find that no other purely geometrical description can be given of them than that there is a threefold continuum of them and that every three of them have one point and one only in common. But innumerable families of surfaces can be conceived of which that is true. For imagine space to be filled with a fluid and that all the planes, or a sufficient collection of them, are marked by dark films in that fluid. Suppose the fluid to be slightly viscous, so that the different parts of it cannot break away from one another. Then give that fluid any motion. The result will be that those films will be distorted into a vast variety of shapes of all degrees of complexity, and yet any three of them will continue to possess a particle in common. The family of surfaces they then occupy will have every purely geometrical property of the family of planes; and yet they will be planes no longer. The distinguishing character of a plane is that if any particle lying in it be luminous and any filament lying in it be opaque, the shadow of that filament from that luminous particle lies wholly in the plane. Hence it is that unlimited straight lines are called *rays*. Graphics then is not pure geometry but is geometrical perspective. If, however, any geometer replies that the family of planes ought not to be limited to optical planes, but ought to be considered as any tridimensional continuum of surfaces, any three of which have just one point in common, then my rejoinder is that if we are to allow the planes to undergo any sort of distortion so long as the connections of the different planes of the family are preserved, then the whole doctrine of graphics is manifestly nothing but a branch of topics. For this is just what topics is. It is the study of the continuous connections and defects of continuity of loci which are free to be distorted in any way so long as the integrity of the connections and separations of all their parts is maintained. All strictly pure geometry, therefore, is topics. I now proceed to explain my remark that in a continuous *locus* no point has any individual identity, unless it be a topically singular point, that is, an isolating point, or either the extremity of a line, or a point from which three or more branches of a line, or two or more sheets of a surface extend. Consider for example an oval line, and let that oval line be broken so as to make a line with two extremities. It may be said that when this happens a point of the oval bursts into two. But I say that there is no particular point of the yet unbroken oval which can be identified, even approximately, with the point which bursts. For to say that the different points of an oval move round the oval, without ever moving out of it, is a form of words entirely destitute of meaning. The points are but places; and the oval and all its parts subsist unchanged whether we regard the points as standing still or running round. In like manner, when we say the oval bursts, we introduce time with a second dimension. Considering the time, the place of the oval is a two dimensional place. This is cylindrical at the bursting and is a ribbon afterward. If one of the dimensions has a different quality from the other, the couple, consisting of a point and instant on the two dimensional continuum where the bursting takes place, has an individual identity. But it cannot be identified with any particular line in the cylindrical part of the two dimensional, even approximately. That line has no individuality.

220. If instead of an oval *place*, we consider an oval *thing*, say a filament, then it certainly means something to say that the parts revolve round the oval. For any one particle might be marked black and so be seen to move. And even if it were not actually marked, it would have an individuality which would make it capable of being marked. So that the filament would have a definite velocity of rotation whether it could be seen to move or not. But the reply to this is, that the marking of a single particle would be a discontinuous marking; and if the particles possess all their own individual identities, that is to suppose a discontinuity of existence everywhere, notwithstanding the continuity of place. But I go further. If those particles possess each its individual existence there is a discrete collection of them, and this collection must possess a definite multitude. Now this multitude cannot equal the multiplicity of the aggregate of all possible discrete multitudes; because it is a discrete multitude, and as such it is smaller than another possible multitude. Hence, it is not equal to the multitude of points of the oval. For that is equal to the aggregate of all possible discrete multitudes, since the line, by hypothesis, affords room for any collection of discrete points however great. Hence, if particles of the filament are distributed equally along the line of the oval, there must be, in every sensible part, continuous collections of points, that is, *lines*, that are unoccupied by particles. These lines may be far less than any *assignable* magnitudes, that is, far less than any parts into which the system of real quantities enables us to divide the line. But there is no contradiction whatever in-

volved in that. It thus appears that true continuity is logically absolutely repugnant to the individual designation or even approximate individual designation of its units, except at points where the character of the continuity is itself not continuous.

221. In view of what has been said, it is not surprising that those arithmetical operations of addition and multiplication, which seemed to have lost their significance forever, now reappear in reference to continua. It is not that the points, as points, can be one more or less; but if there are defects of continuity, those discontinuities can have perfect individual identity and so be added and multiplied.

222. In regard to lines, there are two kinds of defects of continuity. The first is, that two or more particles moving in a line-figure may be unable to coalesce. The possible number of such non-coalescible particles may be called the *chorisis* of a figure. Any kind of a geometrical figure has chorisis whether it be a point-figure, a line-figure, a surface-figure, or what. Thus the chorisis of three [not overlapping] ovals is three. The chorisis may be any discrete multitude.

223. The other defect of continuity that can affect a line-figure is that there may be a collection of points upon it from which a particle can move in more or fewer ways than from the generality of points of the figure. These *topically singular points*, as I call them, are of two kinds: those away from which a particle can move on the line in less than two ways and those from which a particle can move in the line in more than three ways. Of the first kind are, first, isolated points,* or *topical acnodes*, and extremities. Those, from which a particle can move in more than two ways, are points of branching, or *topical nodes*. The negative of what Listing calls the *Census number* of a line is, if we give a further extension to his definition, that which I would call the total *singularity* of the line; namely, it is half the sum of the excesses over two of the number of ways in which a particle could leave the different singular points of the line. No line can have a fractional total singularity.

224. In regard to surfaces, the chorisis is very simple and calls for no particular attention.

The theory of the singular places of surfaces is somewhat

* Such as the centre of a circle.

complicated. The singular places may be points, and those are either isolated points or points where two or more sheets are tacked together. Or the singular places may be isolated lines, and those are either totally isolated, or they may cut the surface. Such lines can have singularities like lines generally. Or the singular places may be lines which are either bounding edges or lines of splitting of the surface, or they may be in some parts edges and in other parts lines of splitting. They have singular points at which the line need not branch. All that is necessary is that the identities of the sheets that join there should change. If such a line has an extremity or point of odd branches, an even number of the sheets which come together there must change.

225. In addition to that, surfaces are another kind of defect of continuity, which Listing calls their *cyclosis*. That is, there is room upon them for oval filaments which cannot shrink to nothing by any movement in the surface. The number of operations each of a kind calculated to destroy a simple cyclosis which have to be [employed] in order to destroy the cyclosis of a surface is the number of the cyclosis. A puncture of a surface which does not change it from a closed surface to an open surface increases the cyclosis by one. A cut from edge to edge which does not increase the chorisis diminishes the cyclosis by one.

The cyclosis of a spherical surface is 0; that of an unlimited plane is 1; that of an anchor-ring is 2, that of a plane with a fornix (or bridge from one part to another) is 3; that of an anchor-ring with a fornix is 4, etc.

Euler's theorem* concerning polyhedra is an example of the additive arithmetic of continua.

226. The multiplicity of points upon a surface must be admitted, as it seems to me, to be the square of that of the points of a line, and so with higher dimensions. The multitude of dimensions may be of any discrete multitude.

* See his "Elementa doctrinæ solidarum" *Novi Commentarii Petropolitanæ*, T. IV, p. 119, (1752–3).

VII

*THE SIMPLEST MATHEMATICS**P

§1. THE ESSENCE OF MATHEMATICS

227. In this chapter, I propose to consider certain extremely simple branches of mathematics which, owing to their utility in logic, have to be treated in considerable detail, although to the mathematician they are hardly worth consideration. In Chapter 4,† I shall take up those branches of mathematics upon which the interest of mathematicians is centred, but shall do no more than make a rapid examination of their logical procedure. In Chapter 5,† I shall treat formal logic by the aid of mathematics. There can really be little logical matter in these chapters; but they seem to me to be quite indispensable preliminaries to the study of logic.

228. It does not seem to me that mathematics depends in any way upon logic. It reasons, of course. But if the mathematician ever hesitates or errs in his reasoning, logic cannot come to his aid. He would be far more liable to commit similar as well as other errors there. On the contrary, I am persuaded that logic cannot possibly attain the solution of its problems without great use of mathematics. Indeed all formal logic is merely mathematics applied to logic.‡

229. It was Benjamin Peirce,§ whose son I boast myself, that in 1870 first defined mathematics as "the science which draws necessary conclusions."¶ This was a hard saying at the time; but today, students of the philosophy of mathematics generally acknowledge its substantial correctness.

230. The common definition, among such people as ordi-

* Chapter 3 of the "Minute Logic," dated January-February, 1902. For the previous chapters see vol. 2, bk. I, ch. 1 and 2, and vol. 1, bk. II, ch. 2.

† These chapters were not written. See 1.584n.

‡ Cf. 1.247.

§ See 2.9n.

¶ "Linear Associative Algebra" (1870), sec. 1; see *American Journal of Mathematics*, vol. 4 (1881).

nary schoolmasters, still is that mathematics is the science of quantity. As this is inevitably understood in English, it seems to be a misunderstanding of a definition which may be very old,[1] the original meaning being that mathematics is the science of *quantities*, that is, forms possessing quantity. We perceive that Euclid was aware that a large branch of geometry had nothing to do with measurement (unless as an aid in demonstrating); and, therefore, a Greek geometer of his age (early in the third century B.C.) or later could not define mathematics as the science of that which the abstract noun quantity expresses. A line, however, was classed as a quantity, or *quantum*, by Aristotle* and his followers; so that even perspective (which deals wholly with intersections and projections, not at all with lengths) could be said to be a science of quantities, "quantity" being taken in the concrete sense. That this was what was originally meant by the definition "Mathematics is the science of quantity," is sufficiently shown by the circumstance that those writers who first enunciate it, about A.D. 500, that is Ammonius Hermiæ† and Boëthius,‡ make astronomy and music branches of mathematics; and it is confirmed by the reasons they give for doing so.[2] Even Philo of Alexandria (100 B.C.), who defines mathematics as the science of ideas furnished by sensation and reflection in respect to their necessary consequences, since he includes under mathematics, besides its more essential parts, the theory of numbers and geometry, also the practical arithmetic of the Greeks, geodesy, mechanics, optics (or projective geometry), music, and astronomy, must be said to take the word 'mathematics' in a different sense from ours. That Aristotle did not regard mathematics as the science of quantity, in the modern abstract sense, is evidenced in various ways. The subjects of mathematics are, according to him, the how much and the

* *Metaphysica*, 1020a, 14–20.

† In *Porphyrii Isogogen sine* v *voces*, p. 5v., 1.11 *et seq.*

‡ *de institutione Arithmetica*, L. I, c. 1.

[1] From what is said by Proclus Diadochus, A.D. 485 [*Commentarii in Primum Euclidis Elementorum Librum*, Prologi pars prior, c. 12], it would seem that the Pythagoreans understood mathematics to be the answer to the two questions "how many?" and "how much?"

[2] I regret I have not noted the passage of Ammonius to which I refer. It is probably one of the excerpts given by Brandis. My MS. note states that he gives reasons showing this to be his meaning.

continuous. (See *Metaph.* K iii 1061 a 33). He referred the continuous to his category of *quantum;* and therefore he did make *quantum*, in a broad sense, the one object of mathematics.

231. Plato, in the Sixth book of the *Republic*,[1] holds that the essential characteristic of mathematics lies in the peculiar kind and degree of its abstraction, greater than that of physics but less than that of what we now call philosophy; and Aristotle* follows his master in this definition. It has ever since been the habit of metaphysicians to extol their own reasonings and conclusions as vastly more abstract and scientific than those of mathematics. It certainly would seem that problems about God, Freedom, and Immortality are more exalted than, for example, the question how many hours, minutes, and seconds would elapse before two couriers travelling under assumed conditions will come together; although I do not know that this has been proved. But that the methods of thought of the metaphysicians are, as a matter of historical fact, in any aspect, not far inferior to those of mathematics is simply an infatuation. One singular consequence of the notion which prevailed during the greater part of the history of philosophy, that metaphysical reasoning ought to be similar to that of mathematics, only more so, has been that sundry mathematicians have thought themselves, as mathematicians, qualified to discuss philosophy; and no worse metaphysics than theirs is to be found.

232. Kant† regarded mathematical propositions as synthetical judgments *a priori;* wherein there is this much truth, that they are not, for the most part, what he called analytical judgments; that is, the predicate is not, in the sense he intended, contained in the definition of the subject. But if the propositions of arithmetic, for example, are true cognitions, or even forms of cognition, this circumstance is quite aside from their mathematical truth. For all modern mathematicians agree with Plato and Aristotle that mathematics deals exclusively with hypothetical states of things, and asserts no matter of fact whatever; and further, that it is thus alone that the

[1] 510C to the end; but in the *Laws* his notion is improved.

* See *Metaphysica*, 1025^b1–1026^a33; 1060^b31–1061^b34.

† *Kritik der reinen Vernunft* Einleitung, B, §V.

necessity of its conclusions is to be explained.[1] This is the true essence of mathematics; and my father's definition is in so far correct that it is impossible to reason necessarily concerning anything else than a pure hypothesis. Of course, I do not mean that if such pure hypothesis happened to be true of an actual state of things, the reasoning would thereby cease to be necessary. Only, it never would be known apodictically to be true of an actual state of things. Suppose a state of things of a perfectly definite, general description. That is, there must be no room for doubt as to whether anything, itself determinate, would or would not come under that description. And suppose, further, that this description refers to nothing occult—nothing that cannot be summoned up fully into the imagination. Assume, then, a range of possibilities equally definite and equally subject to the imagination; so that, so far as the given description of the supposed state of things is general, the different ways in which it might be made determinate could never introduce doubtful or occult features. The assumption, for example, must not refer to any matter of fact. For questions of fact are not within the purview of the imagination. Nor must it be such that, for example, it could lead us to ask whether the vowel *OO* can be imagined to be sounded on as high a pitch as the vowel *EE*. Perhaps it would have to be restricted to pure spatial, temporal, and logical relations. Be that as it may, the question whether in such a state of things, a certain other similarly definite state of things, equally a matter of the imagination, could or could not, in the assumed range of possibility, ever occur, would be one in reference to which one of the two answers, *Yes* and *No*, would be true, but never both. But all pertinent facts would be within the beck and call of the imagination; and consequently nothing but the operation of thought would be necessary to render the true answer. Nor, supposing the answer to cover the whole range of possibility assumed, could this be rendered otherwise than by reasoning that would be apodictic, general, and exact. No knowledge of what actually is, no *positive* knowledge, as we say, could result. On the other hand, to assert that any source of information that is restricted to actual facts could afford us

[1] A view which J. S. Mill (*Logic* II, V, §2) rather comically calls "the important doctrine of Dugald Stewart."

a necessary knowledge, that is, knowledge relating to a whole general range of possibility, would be a flat contradiction in terms.

233. Mathematics is the study of what is true of hypothetical states of things. That is its essence and definition. Everything in it, therefore, beyond the first precepts for the construction of the hypotheses, has to be of the nature of apodictic inference. No doubt, we may reason imperfectly and jump at a conclusion; still, the conclusion so guessed at is, after all, that in a certain supposed state of things something would necessarily be true. Conversely, too, every apodictic inference is, strictly speaking, mathematics. But mathematics, as a serious science, has, over and above its essential character of being hypothetical, an accidental characteristic peculiarity — a *proprium*, as the Aristotelians used to say — which is of the greatest logical interest. Namely, while all the "philosophers" follow Aristotle in holding no demonstration to be thoroughly satisfactory except what they call a "direct" demonstration, or a "demonstration why" — by which they mean a demonstration which employs only general concepts and concludes nothing but what would be an item of a definition if all its terms were themselves distinctly defined — the mathematicians, on the contrary, entertain a contempt for that style of reasoning, and glory in what the philosophers stigmatize as "mere" indirect demonstrations, or "demonstrations that." Those propositions which can be deduced from others by reasoning of the kind that the philosophers extol are set down by mathematicians as "corollaries." That is to say, they are like those geometrical truths which Euclid did not deem worthy of particular mention, and which his editors inserted with a garland, or corolla, against each in the margin, implying perhaps that it was to them that such honor as might attach to these insignificant remarks was due. In the theorems, or at least in all the major theorems, a different kind of reasoning is demanded. Here, it will not do to confine oneself to general terms. It is necessary to set down, or to imagine, some individual and definite schema, or diagram — in geometry, a figure composed of lines with letters attached; in algebra an array of letters of which some are repeated. This schema is constructed so as to conform to a hypothesis set forth in general terms in the thesis

of the theorem. Pains are taken so to construct it that there would be something closely similar in every possible state of things to which the hypothetical description in the thesis would be applicable, and furthermore to construct it so that it shall have no other characters which could influence the reasoning. How it can be that, although the reasoning is based upon the study of an individual schema, it is nevertheless necessary, that is, applicable, to all possible cases, is one of the questions we shall have to consider. Just now, I wish to point out that after the schema has been constructed according to the precept virtually contained in the thesis, the assertion of the theorem is not evidently true, even for the individual schema; nor will any amount of hard thinking of the philosophers' corollarial kind ever render it evident. Thinking in general terms is not enough. It is necessary that something should be DONE. In geometry, subsidiary lines are drawn. In algebra permissible transformations are made. Thereupon, the faculty of observation is called into play. Some relation between the parts of the schema is remarked. But would this relation subsist in every possible case? Mere corollarial reasoning will sometimes assure us of this. But, generally speaking, it may be necessary to draw distinct schemata to represent alternative possibilities. Theorematic reasoning invariably depends upon experimentation with individual schemata. We shall find that, in the last analysis, the same thing is true of the corollarial reasoning, too; even the Aristotelian "demonstration why." Only in this case, the very words serve as schemata. Accordingly, we may say that corollarial, or "philosophical" reasoning is reasoning with words; while theorematic, or mathematical reasoning proper, is reasoning with specially constructed schemata.

234. Another characteristic of mathematical thought is the extraordinary use it makes of abstractions. Abstractions have been a favorite butt of ridicule in modern times. Now it is very easy to laugh at the old physician who is represented as answering the question, why opium puts people to sleep, by saying that it is because it has a dormative virtue. It is an answer that no doubt carries vagueness to its last extreme. Yet, invented as the story was to show how little meaning there might be in an abstraction, nevertheless the physician's answer does contain a truth that modern philosophy has gen-

erally denied: it does assert that there really is in opium *something* which explains its always putting people to sleep. This has, I say, been denied by modern philosophers generally. Not, of course, explicitly; but when they say that the different events of people going to sleep after taking opium have really nothing in common, but only that the mind classes them together — and this is what they virtually do say in denying the reality of generals — they do implicitly deny that there is any true explanation of opium's generally putting people to sleep.

235. Look through the modern logical treatises, and you will find that they almost all fall into one or other of two errors, as I hold them to be; that of setting aside the doctrine of abstraction (in the sense in which an abstract noun marks an abstraction) as a grammatical topic with which the logician need not particularly concern himself; and that of confounding abstraction, in this sense, with that operation of the mind by which we pay attention to one feature of a percept to the disregard of others. The two things are entirely disconnected. The most ordinary fact of perception, such as "it is light," involves *precisive* abstraction, or *prescission*.* But *hypostatic* abstraction, the abstraction which transforms "it is light" into "there is light here," which is the sense which I shall commonly attach to the word abstraction (since *prescission* will do for precisive abstraction) is a very special mode of thought. It consists in taking a feature of a percept or percepts (after it has already been prescinded from the other elements of the percept), so as to take propositional form in a judgment (indeed, it may operate upon any judgment whatsoever), and in conceiving this fact to consist in the relation between the subject of that judgment and another subject, which has a mode of being that merely consists in the truth of propositions of which the corresponding concrete term is the predicate. Thus, we transform the proposition, "honey is sweet," into "honey possesses sweetness." "Sweetness" might be called a fictitious thing, in one sense. But since the mode of being attributed to it *consists* in no more than the fact that some things are sweet, and it is not pretended, or imagined, that it has any other mode of being, there is, after all, no fiction. The only profession made is that

* Cf. 1.549n; 2.428.

we consider the fact of honey being sweet under the form of a relation; and so we really can. I have selected sweetness as an instance of one of the least useful of abstractions. Yet even this is convenient. It facilitates such thoughts as that the sweetness of honey is particularly cloying; that the sweetness of honey is something like the sweetness of a honeymoon; etc. Abstractions are particularly congenial to mathematics. Everyday life first, for example, found the need of that class of abstractions which we call *collections*. Instead of saying that some human beings are males and all the rest females, it was found convenient to say that *mankind* consists of the male *part* and the female *part*. The same thought makes classes of collections, such as pairs, leashes, quatrains, hands, weeks, dozens, baker's dozens, sonnets, scores, quires, hundreds, long hundreds, gross, reams, thousands, myriads, lacs, millions, milliards, milliasses, etc. These have suggested a great branch of mathematics.[1] Again, a point moves: it is by abstraction that the geometer says that it "describes a line." This line, though an abstraction, itself moves; and this is regarded as generating a surface; and so on. So likewise, when the analyst treats operations as themselves subjects of operations, a method whose utility will not be denied, this is another instance of abstraction. Maxwell's notion of a tension exercised upon lines of electrical force, transverse to them, is somewhat similar. These examples exhibit the great rolling billows of abstraction in the ocean of mathematical thought; but when we come to a minute examination of it, we shall find, in every department, incessant ripples of the same form of thought, of which the examples I have mentioned give no hint.

236. Another characteristic of mathematical thought is that it can have no success where it cannot generalize. One cannot, for example, deny that chess is mathematics, after a fashion; but, owing to the exceptions which everywhere confront the mathematician in this field — such as the limits of the board; the single steps of king, knight, and pawn; the finite

[1] Of course, the moment a collection is recognized as an abstraction we have to admit that even a percept is an abstraction or represents an abstraction, if matter has parts. It therefore becomes difficult to maintain that all abstractions are fictions.

number of squares; the peculiar mode of capture by pawns; the queening of pawns; castling — there results a mathematics whose wings are effectually clipped, and which can only run along the ground. Hence it is that a mathematician often finds what a chess-player might call a gambit to his advantage; exchanging a smaller problem that involves exceptions for a larger one free from them. Thus, rather than suppose that parallel lines, unlike all other pairs of straight lines in a plane, never meet, he supposes that they intersect at infinity. Rather than suppose that some equations have roots while others have not, he supplements real quantity by the infinitely greater realm of imaginary quantity. He tells us with ease how many inflexions a plane curve of any description has; but if we ask how many of these are real, and how many merely fictional, he is unable to say. He is perplexed by three-dimensional space, because not all pairs of straight lines intersect, and finds it to his advantage to use quaternions which represent a sort of four-fold continuum, in order to avoid the exception. It is because exceptions so hamper the mathematician that almost all the relations with which he chooses to deal are of the nature of correspondences; that is to say, such relations that for every relate there is the same number of correlates, and for every correlate the same number of relates.

237. Among the minor, yet striking characteristics of mathematics, may be mentioned the fleshless and skeletal build of its propositions; the peculiar difficulty, complication, and stress of its reasonings; the perfect exactitude of its results; their broad universality; their practical infallibility. It is easy to speak with precision upon a general theme. Only, one must commonly surrender all ambition to be certain. It is equally easy to be certain. One has only to be sufficiently vague. It is not so difficult to be pretty precise and fairly certain at once about a very narrow subject. But to reunite, like mathematics, perfect exactitude and practical infallibility with unrestricted universality, is remarkable. But it is not hard to see that all these characters of mathematics are inevitable consequences of its being the study of hypothetical truth.

238. It is difficult to decide between the two definitions of

mathematics; the one by its method, that of drawing necessary conclusions; the other by its aim and subject matter, as the study of hypothetical states of things. The former makes or seems to make the deduction of the consequences of hypotheses the sole business of the mathematician as such. But it cannot be denied that immense genius has been exercised in the mere framing of such general hypotheses as the field of imaginary quantity and the allied idea of Riemann's surface, in imagining non-Euclidian measurement, ideal numbers, the perfect liquid. Even the framing of the particular hypotheses of special problems almost always calls for good judgment and knowledge, and sometimes for great intellectual power, as in the case of Boole's logical algebra. Shall we exclude this work from the domain of mathematics? Perhaps the answer should be that, in the first place, whatever exercise of intellect may be called for in applying mathematics to a question not propounded in mathematical form [it] is certainly not pure mathematical thought; and in the second place, that the mere creation of a hypothesis may be a grand work of poietic* genius, but cannot be said to be scientific, inasmuch as that which it produces is neither true nor false, and therefore is not knowledge. This reply suggests the further remark that if mathematics is the study of purely imaginary states of things, poets must be great mathematicians, especially that class of poets who write novels of intricate and enigmatical plots. Even the reply, which is obvious, that by *studying* imaginary states of things we mean *studying* what is true of them, perhaps does not fully meet the objection. The article *Mathematics* in the ninth edition of the *Encyclopaedia Britannica*† makes mathematics consist in the study of a particular sort of hypotheses, namely, those that are exact, etc., as there set forth at some length. The article is well worthy of consideration.

239. The philosophical mathematician, Dr. Richard Dedekind,‡ holds mathematics to be a branch of logic. This would not result from my father's definition, which runs, not that mathematics is the science of *drawing* necessary conclusions — which would be deductive logic — but that it is the science

* From ποιέω.

† By George Chrystal.

‡ *Was sind und was sollen die Zahlen; Vorwort;* (1888.)

which *draws* necessary conclusions. It is evident, and I know as a fact, that he had this distinction in view. At the time when he thought out this definition, he, a mathematician, and I, a logician, held daily discussions about a large subject which interested us both; and he was struck, as I was, with the contrary nature of his interest and mine in the same propositions. The logician does not care particularly about this or that hypothesis or its consequences, except so far as these things may throw a light upon the nature of reasoning. The mathematician is intensely interested in efficient methods of reasoning, with a view to their possible extension to new problems; but he does not, *quâ* mathematician, trouble himself minutely to dissect those parts of this method whose correctness is a matter of course. The different aspects which the algebra of logic will assume for the two men is instructive in this respect. The mathematician asks what value this algebra has as a calculus. Can it be applied to unravelling a complicated question? Will it, at one stroke, produce a remote consequence? The logician does not wish the algebra to have that character. On the contrary, the greater number of distinct logical steps, into which the algebra breaks up an inference, will for him constitute a superiority of it over another which moves more swiftly to its conclusions. He demands that the algebra shall analyze a reasoning into its last elementary steps. Thus, that which is a merit in a logical algebra for one of these students is a demerit in the eyes of the other. The one studies the science of drawing conclusions, the other the science which draws necessary conclusions.

240. But, indeed, the difference between the two sciences is far more than that between two points of view. Mathematics is purely hypothetical: it produces nothing but conditional propositions. Logic, on the contrary, is categorical in its assertions. True, it is not merely, or even mainly, a mere discovery of what really is, like metaphysics. It is a normative science. It thus has a strongly mathematical character, at least in its methodeutic division; for here it analyzes the problem of how, with given means, a required end is to be pursued. This is, at most, to say that it has to call in the aid of mathematics; that it has a mathematical branch. But so much may be said of every science. There is a mathematical

logic, just as there is a mathematical optics and a mathematical economics. Mathematical logic is formal logic. Formal logic, however developed, is mathematics. Formal logic, however, is by no means the whole of logic, or even its principal part. It is hardly to be reckoned as a part of logic proper. Logic has to define its aim; and in doing so is even more dependent upon ethics,* or the philosophy of aims, by far, than it is, in the methodeutic branch, upon mathematics. We shall soon come to understand how a student of ethics might well be tempted to make his science a branch of logic; as, indeed, it pretty nearly was in the mind of Socrates. But this would be no truer a view than the other. Logic depends upon mathematics; still more intimately upon ethics; but its proper concern is with truths beyond the purview of either.

241. There are two characters of mathematics which have not yet been mentioned, because they are not exclusive characteristics of it. One of these, which need not detain us, is that mathematics is distinguished from all other sciences† except only ethics, in standing in no need of ethics. Every other science, even logic — logic, especially — is in its early stages in danger of evaporating into airy nothingness, degenerating, as the Germans say, into an anachrioid [?] film, spun from the stuff that dreams are made of. There is no such danger for pure mathematics; for that is precisely what mathematics ought to be.

242. The other character — and of particular interest it is to us just now — is that mathematics, along with ethics and logic alone of the sciences, has no need of any appeal to logic. No doubt, some reader may exclaim in dissent to this, on first hearing it said. Mathematics, they may say, is preëminently a science of reasoning. So it is; preëminently a science that reasons. But just as it is not necessary, in order to talk, to understand the theory of the formation of vowel sounds, so it is not necessary, in order to reason, to be in possession of the theory of reasoning. Otherwise, plainly, the science of logic could never be developed. The contrary objection would have more excuse, that no science stands in need of logic, since our natural power of reason is enough. Make of logic what the

* Cf. 1.577.

† But cf. 1.611.

majority of treatises in the past have made of it, and a very common class of English and French books still make of it — that is to say, mainly formal logic, and that formal logic represented as an art of reasoning — and in my opinion this objection is more than sound, for such logic is a great hindrance to right reasoning. It would, however, be aside from our present purpose to examine this objection minutely. I will content myself with saying that undoubtedly our natural power of reasoning is enough, in the same sense that it is enough, in order to obtain a wireless transatlantic telegraph, that men should be born. That is to say, it is bound to come sooner or later. But that does not make research into the nature of electricity needless for gaining such a telegraph. So likewise if the study of electricity had been pursued resolutely, even if no special attention had ever been paid to mathematics, the requisite mathematical ideas would surely have been evolved. Faraday, indeed, did evolve them without any acquaintance with mathematics. Still it would be far more economical to postpone electrical researches, to study mathematics by itself, and then to apply it to electricity, which was Maxwell's way. In this same manner, the various logical difficulties which arise in the course of every science except mathematics, ethics, and logic, will, no doubt, get worked out after a time, even though no special study of logic be made. But it would be far more economical to make first a systematic study of logic. If anybody should ask what are these logical difficulties which arise in all the sciences, he must have read the history of science very irreflectively. What was the famous controversy concerning the measure of force but a logical difficulty? What was the controversy between the uniformitarians and the catastrophists but a question of whether or not a given conclusion followed from acknowledged premisses? This will fully appear in the course of our studies in the present work.*

243. But it may be asked whether mathematics, ethics, and logic have not encountered similar difficulties. Are the doctrines of logic at all settled? Is the history of ethics anything but a history of controversy? Have no logical errors been committed by mathematicians? To that I reply, first, as to logic, that not only have the rank and file of writers on the

* This point is not discussed in the "Minute Logic." But see 1.104f and vol. 6, bk. I.

subject been, as an eminent psychiatrist, Maudsley, declares, men of arrested brain-development, and not only have they generally lacked the most essential qualification for the study, namely mathematical training, but the main reason why logic is unsettled is that thirteen different opinions are current as to the true aim of the science.* Now this is not a logical difficulty but an ethical difficulty; for ethics is the science of aims. Secondly, it is true that pure ethics has been, and always must be, a theatre of discussion, for the reason that its study consists in the gradual development of a distinct recognition of a satisfactory aim. It is a science of subtleties, no doubt; but it is not logic, but the development of the ideal, which really creates and resolves the problems of ethics. Thirdly, in mathematics errors of reasoning have occurred, nay, have passed unchallenged for thousands of years. This, however, was simply because they escaped notice. Never, in the whole history of the science, has a question whether a given conclusion followed *mathematically* from given premisses, when once started, failed to receive a speedy and unanimous reply. Very few have been even the apparent exceptions; and those few have been due to the fact that it is only within the last half century that mathematicians have come to have a perfectly clear recognition of what is mathematical soil and what foreign to mathematics. Perhaps the nearest approximation to an exception was the dispute about the use of divergent series. Here neither party was in possession of sufficient pure mathematical reasons covering the whole ground; and such reasons as they had were not only of an extra-mathematical kind, but were used to support more or less vague positions. It appeared then, as we all know now, that divergent series are of the utmost utility.[1]

* See vol. 2, ch. 1, §3.

[1] It would not be fair, however, to suppose that every reader will know this. Of course, there are many series so extravagantly divergent that no use at all can be made of them. But even when a series is divergent from the very start, some use might commonly be made of it, if the same information could not otherwise be obtained more easily. The reason is — or rather, one reason is — that most series, even when divergent, approximate at last somewhat to geometrical series, at least, for a considerable succession of terms. The series $\log(1+x) = x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+$, etc., is one that would not be judiciously employed in order to find the natural logarithm of 3, which is 1.0986, its successive terms being $2-2+8/3-4+32/5-32/3+$, etc. Still, employing the common device

Struck by this circumstance, and making an inference, of which it is sufficient to say that it was not mathematical, many of the old mathematicians pushed the use of divergent series beyond all reason. This was a case of mathematicians disputing about the validity of a kind of inference that is not mathematical. No doubt, a sound logic (such as has not hitherto been developed) would have shown clearly that that non-mathematical inference was not a sound one. But this is, I believe, the only instance in which any large party in the mathematical world ever proposed to rely, in mathematics, upon unmathematical reasoning. My proposition is that true mathematical reasoning is so much more evident than it is possible to render any doctrine of logic proper — without just such reasoning — that an appeal in mathematics to logic could only embroil a situation. On the contrary, such difficulties as may arise concerning necessary reasoning have to be solved by the logician by reducing them to questions of mathematics. Upon those mathematical dicta, as we shall come clearly to see, the logician has ultimately to repose.

244. So a double motive induces me to devote some preliminary chapters to mathematics. For, in the first place, in studying the theory of reasoning, we are concerned to acquaint ourselves with the methods of that prior science of which acts of reasoning form the staple. In the second place, logic, like any other science, has its mathematical department, and of that, a large portion, at any rate, may with entire convenience be studied as soon as we take up the study of logic, without any propedeutic. That portion is what goes by the name of Formal Logic.[1] It so happens that the special kind of mathematics needed for formal logic, which, therefore, we need to study in detail, as we need not study other branches of mathe-

of substituting for the last two terms that are to be used, say M and N, the expression $M/(1-N/M)$, the succession of the first six values is 0.667, 1.143, 1.067, 1.128, 1.067, which do show some approximation to the value. The mean of the last two, which any professional computer would use (supposing him to use this series, at all) would be 1.098, which is not very wrong. Of course, the computer would practically use the series $\log 3 = 1 + 1/12 + 1/80 + 1/448 +$, etc., of which the terms written give the correct value to four places, if they are properly used.

[1] "Formal Logic" is also used, by Germans chiefly, to mean that sect of Logic which makes Formal Logic pretty much the whole of Logic.

matics, is so excessively simple as neither to have much mathematical interest, nor to display the peculiarities of mathematical reasoning. I shall, therefore, devote the present chapter — a very dull one, I am sorry to say, it must be — to this kind of mathematics. Chapter 4 will treat of the more truly mathematical mathematics; and Chapter 5 will apply the results of the present chapter to the study of Formal Logic.*

§2. DIVISION OF PURE MATHEMATICS[P]

245. We have to make a rapid survey of pure mathematics, in so far as it interests us as students of logic. Each branch of mathematics will have to be reconnoitered and its methods examined. Those parts of the calculus of which use has to be made in the study of reasoning must receive a fuller treatment. Finally, having so collected some information about mathematics, we may venture upon some useful generalizations concerning the nature of mathematical thought. But this plan calls for a preliminary dissection of mathematics into its several branches.

246. Each branch of mathematics sets out from a general hypothesis of its own. I mean by its general hypothesis the substance of its postulates and axioms, and even of its definitions, should they be contaminated with any substance, instead of being the pure verbiage they ought to be. We have to make choice, then, between a division of mathematics according to the matter of its hypotheses, or according to the forms of the schemata of which it avails itself. These latter are either geometrical or algebraical. Geometrical schemata are linear figures with letters attached; the perfect imaginability, on the one hand, and the extreme familiarity, on the other hand, of spatial relations are taken advantage of, to enable us to see what will necessarily be true under supposed conditions. The algebraical schemata are arrays of characters, sometimes in series, sometimes in blocks, with which are associated certain rules of permissible transformation. With these rules the algebraist has perfectly to familiarize himself. By virtue of these rules, become habits of association, when one array has

* See 227n.

been written or assumed to be permissibly scriptible, the mathematician just as directly perceives that another array is permissibly scriptible, as he perceives that a person talking in a certain tone is angry, or [is] using certain words in such and such a sense.

247. The primary division of mathematics into algebra and geometry is the usual one. But, in all departments, it appears both *a priori* and *a posteriori*, that divisions according to differences of purpose should be given a higher rank than divisions according to different methods of attaining that purpose.* The division of pure mathematics into algebra and geometry was first adopted before the modern conception of pure mathematics had been distinctly prescinded, and when geometry and algebra seemed to deal with different subjects. It remains, a vestige of that old unclearness and a witness that not even mathematicians are able entirely to shake off the sequelæ of exploded ideas. For now that everybody knows that any mathematical subject, from the theory of numbers to topical geometry, may be treated either algebraically or geometrically, one cannot fail to see that so to divide mathematics is to make twice over the division according to fundamental hypotheses, to which one must come, at last. This duplication is worse than useless, since the geometrical and algebraical methods are by many writers continually mixed. No such inconvenience attends the other plan of classification; for two sets of fundamental hypotheses could not, properly speaking, be mixed without self-contradiction.

248. Let us, then, divide mathematics according to the nature of its general hypotheses, taking for the ground of primary division the multitude of units, or elements, that are supposed; and for the ground of subdivision that mode of relationship between the elements upon which the hypotheses focus the attention.

249. From a logician's point of view this plan of classification would seem to call for a preliminary analysis of what is meant by multitude. But to execute this analysis satisfactorily, considerable studies of logic would be indispensable preliminaries. Besides, it is not at all in the spirit of mathematics to analyze the ideas with which it works farther than

* Cf. 1.205f.

is needful for using them in deducing consequences, nor sooner than that need comes to be felt. It is true that we, as students of logic, are not bound to embrace the mathematical ways of thought as far as that, but the other circumstance, that it is, at the present stage of our studies, impossible to make the analysis, must be conclusive.

§3. THE SIMPLEST BRANCH OF MATHEMATICS[P]

250. Were nothing at all supposed, mathematics would have no ground at all to go upon. Were the hypothesis merely that there was nothing but one unit, there would not be a possibility of a question, since only one answer would be possible. Consequently, the simplest possible hypothesis is that there are two objects, which we may denote by **v** and **f**. Then the first kind of problem of this algebra will be, given certain data concerning an unknown object, x, required to know whether it is **v** or **f**. Or similar problems may arise concerning several unknowns, x, y, etc. Or when the last problem cannot be resolved, we may ask whether, supposing x to be **v**, will y be **v** or **f**? And similarly, supposing x to be **f**. Again, given certain data concerning x, we may ask, what else needs to be known in order to compel x to be **v** or to be **f**. Or again, given certain information about x, y, and z, what relations between x and z remain unchanged whether y be **v** or **f**?

251. Let us call **v** and **f** the two possible *values*, one of which must be attached to any unknown. For the form of reasoning will be the same whether we talk of identity or attachment. The attachment may be of any kind so long as each unknown must be, or be attached to, **v** or **f**, but cannot be or be attached to, both **v** and **f**. This idea of a system of values is one of the most fundamental abstractions of the algebraic method of mathematics. An object of the universe, whose *value* is generally unknown, though it may in special cases be known — that is to say, an object which, to phrase the matter differently, *is* one of the values, though perhaps we do not know which — is called, when we speak of it as "*having*" a value, a *quantity*. For example, suppose the problem

under consideration be to determine, upon a certain hypothesis, the numerical definition of the instant, or, as we may say, to determine the exact *date*, at which two couriers will meet. This date is some one of the series of numbers each of which is expressible, at least to any predesignate degree of approximation, in our usual method of numeral notation. That series of numbers will be the system of values; and the number we want *is* one of them. But we find it convenient to use a different phrase, and to say that the date *is* defined to be the date at which the couriers meet, that *this* fixes its *identity*, and that what we seek to know is what value becomes *attached* to it in consequence of the conditions the problem supposes. It will be convenient to conceive of this statement as a "mere" variation of phraseology, although, as we shall learn, the word "mere" in such cases is often inappropriate, since great mathematical results are attainable by such means. Dichotomic algebra can be applied wherever there are just two possible alternatives. Thus, we might call the **v** the truth, and **f** falsity. Then, in regard to a given proposition we may seek to know whether it is true or false; that is, whether it is or is not a partial description of the real universe, or say, whether what it means is identical with the existent truth or identical with nothing. Looking at the matter in a different way, or phrasing it differently, we say that a proposition has one or other of two values, being either true and good for something, or false and good for nothing. The point of view of mathematics is the point of view which looks upon those two points of view as no more than different phrases for the same fact.

252. There is another little group of algebraical words which must now be defined in the imperfect way in which they can be defined for dichotomic mathematics. In the first place, there are the pair of terms, *constant*, or *constant quantity*, and *variable*, or *variable quantity*. These words were introduced by the Marquis d'Hôpital* in 1696. Suppose two couriers to set out, at the same instant, from two points 12 miles apart and to travel toward one another, the one at the rate of 7 miles an hour, the other at the rate of 8 miles an hour: when will they meet? They evidently approach one another at the rate

* *Analyse des infiniment petits pour l'intelligence des lignes courbes*, §1, Def. I.

of 7 *plus* 8, or 15 miles an hour; and they will reduce the distance of 12 miles to nothing in 12/15 of an hour, or 12 times 4, or 48, minutes. But suppose we find the distance was wrongly given; that it is 12 1/2 miles. Then, the *date*, or numerical designation of the instant of meeting, becomes different. But if we choose to say that the *quantity* sought is defined as the time of meeting, and that it remains the same quantity, having the same definition, but that its value only is altered, then that quantity is said to be *variable*. A quantity is said to be *variable* when we propose to consider it as taking different values in different states of things; or, to phrase the matter differently, when we consider a group of questions together, as one general question, the single questions having different values for their answers. The most usual case is where we suppose the quantity to take all possible values under different circumstances. A quantity is called *constant* when the hypothesis includes no states of things in which its value changes. The difference between an *unknown* quantity and a variable quantity is trifling. The unknown quantity is variable at first; but special hypotheses being adopted, it is restricted to certain values, perhaps to a single value.

253. The word *function* (a sort of semi-synonym of "operation") was first used in something like its present mathematical sense in 1692, by a writer who was doubtless Leibniz.* It soon came into use with the circle of analysts of whom Leibniz was the centre. But the first attempt at a definition of it was by John Bernouilli,† in 1718. There has since been much discussion as to what precise meaning can most advantageously be applied to it; but the most general definition, that of Dirichlet,‡ is confined to a system of numerical values. Since I wish to apply the word to all sorts of algebra, I shall, under these circumstances, take the liberty of generalizing the meaning in the manner which seems to me to be called for. I shall say then, that, given two ordered sets of the same number of quantities, $x_1, x_2, x_3, \ldots x_n$, and $y_1, y_2, y_3 \ldots y_n$, any quantity, say x_2, of the one set is the *same function* of the other quanti-

* See his *Mathematische Schriften*, h. von C. I. Gerhardt, Bd. I, S. 268; (1858).

† *Histoire de l'Academie Royale des Sciences*, pp. 100–139; (1718). Reprinted in his *Opera Omnia*, t. II, p. 241; (1742).

‡ *Werke*, Bd. I, S. 133–160; (1889).

ties of that same set, which are called its *arguments*, that the corresponding quantity, y_2, according to the order of arrangement of the other set, is of the remaining quantities of that set, if and only if every set of values which either set of quantities, in their order, can take, can likewise be taken by the other set. Thus, to say that a quantity is a *given function* of certain quantities as *arguments* is simply to say that its value stands in a given relation to theirs; or that a given proposition is true of the whole set of values in their order. To say simply that one quantity is *some* function of certain others is to say nothing; since of every set of values something is true. But this no more renders *function* a useless word than the fact, that it means nothing to say of a set of things that there is some relation between them, renders *relation* a useless word.

I may mention that the old and usual expression is "a function of variables"; but the word *argument* here is not unusual and is more explicit. The function is also called the *dependent variable;* the arguments, the *independent variables*. Of course, any one of the whole set of quantities composed of the function and its arguments is just as dependent as any other. It is a mere way of referring to them. The function is often conceived, very conveniently, as resulting from an *operation* performed on the arguments, which are then called *operands*. The idea is that the definition of the *same function* implies a rule which permits such sets of values as may conform to its conditions and excludes others; and the *operation* is the operation of actually applying this rule, when the values of all the quantities but one are given, in order to ascertain what the value of the remaining quantity can be.

254. Among functions, or operations, there is one extensive class which is of particular importance. I call it the class of *correspondential* functions, or operations. Namely, if all the variables but one, independent and dependent, have a set of values assigned to them, then, if the relation between them is a *correspondence*, the number of different values which the remaining variable can have, is *generally* the same, whatever the particular set of assigned values may be; although this number is *not* necessarily the same when different quantities are thus left over to the last. I say *generally* the same, because

there may be peculiar isolated exceptions, though this limitation can have no significance in dichotomic mathematics. A function which is in correspondence with its arguments may be called a correspondential function. It may be remarked that it is not the habit of mathematicians, in general statements, to pay attention to isolated exceptions; and when a mathematician uses the phrase "in general" he means to be understood as not considering possible peculiar cases. Thus, I have known a great mathematician to enunciate a proposition concerning multiple algebra to be true "in general" when the state of the case was that there were just two instances of its being true against an infinity of instances of its being false.

255. A function which has but one value for any one set of values of the arguments is called *monotropic*. A function which, when all the arguments except a certain one take any fixed values, always changes its value with a change of that one, may be called *distinctive* for that argument.

256. If the relation between a function and its arguments is such that one of the latter may take any value for every set of the values of the others without altering the function, the function may be said to be *invariable* with that argument. If the function can take any value, whatever values be assigned to the arguments, it may be said to be independent of the arguments. In either of these cases, the function may be called a *degenerate* function.

257. With this lexical preface, we come down to our dichotomic mathematics, which I shall treat algebraically. The first thing to be done is to fix upon a sign to show that any quantity, say x, has the value **v**, and upon another to signify that it has the value **f**. The simplest suggestion is that universally used since man began to keep accounts; namely, to appropriate a place in which we are to write whatever is **v**, say the upper of two lines, the lower of which is appropriated to quantities whose value is **f**. That is, we open one account for **v**, and another for **f**. In doing this, we put **v** and **f** in a radically different category from the other letters, very much as two opposite qualities, say *good* and *bad*, are attributed to concrete objects. I do not mean that there is any other analogy than that the values, **v** and **f**, are made to be of a different nature

from the *quantities*, x, y, z, etc. One or other of the values, but not both, is connected, in some definite sense, and it matters not what the sense may be, so long as it is definite, with each quantity. But here an important remark obtrudes. Non-connection in any definite way is only another equally definite mode of connection; especially in a strictly dichotomic state of things. If, for example, every man either does good and eschews evil, or does evil and eschews good, then the former is thereby connected with evil by eschewing it, as he is connected with good in the mode of connection called doing it. Note how the perfect balance of our initial dichotomy generates new dichotomies: first, two categories, those of value and of quantity; then, two modes of connection between a value and a quantity.

258. Let us modify our mode of signifying the attachment of a quantity to a value, so as to show its contrary attachment to the opposite value. For this purpose,

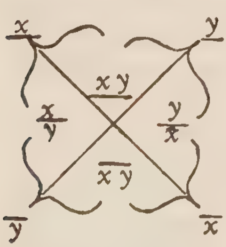
from a centre, O, let us draw a horizontal arm to the right, which we will call the **v**-radius, and another to the left, which we will call the **f**-radius. Now, then, any quantity x put in the upper or **v** account, will be so situated that a right-handed, or clock-wise, revolution around O will bring it first to the **v**-radius; as it will bring a quantity, y, in the **f** account, to the **f**-radius; while a left-handed, or counter-clock-wise, turn around O will carry the quantities each to the other radius. This diagram suggests another way of signifying the value of a quantity. Let a heavy line, representing the horizontal bar of the diagram, be drawn under the sign of a quantity, thus, $\underline{x}$, to signify that its value is **v**; and the same bar be drawn above it, thus, $\bar{y}$, to signify that its value is **f**.

259. It may be mentioned that this mode of indicating the value by a bar has a historical appropriateness. For although the two values **f** and **v** are, at present, merely distinguished, without any definite difference between them being admitted — and mathematically they do stand upon a precise par, and will continue to do so — yet when dichotomic algebra comes to be applied to logic, it will be found necessary

to call one of them *verity* and the other *falsity;* and the letters **v** and **f** were chosen with a view to that. We shall find it impossible later to prevent this affecting our purest practicable mathematics, in some measure. Now it has been the practice, from antiquity, to draw a heavy line under that whose truth it was desired to emphasize. On the other hand, the *obelus*, or spit, is already mentioned by Lucian, in the second century A. D., as the sign of denial; and that is why it is frequently even now used in several European countries to denote an *n*, for *non*, or the other nasal letter *m*.

The Greek word ὀβελός means a spit, (for example, πεμπωβελός is a five-pronged fork) so that the original notion was that that which is beneath it was transfixed; just as it used to be usual to nail false coins to the counter.

260. There is a small theorem about multitude that it will be convenient to have stated, and the reader will do well to fix it in his memory correctly, with the "each" number as exponent. If each of a set of *m* objects be connected with some one of a set of *n* objects, the possible modes of connection of the sets will number n^m. Now an assertion concerning the value of a quantity either admits as possible or else excludes each of the values **v** and **f**. Thus, **v** and **f** form the set of *m* objects each connected with one only of *n* objects, *admission* and *exclusion*. Hence there are, n^m, or 2^2, or 4, different possible assertions concerning the value of any quantity, x. Namely, one assertion will simply be a form of assertion without meaning, since it admits either value. It is represented by the letter, x. Another assertion will violate the hypothesis of dichotomies by excluding both values. It may be represented by $\bar{\underline{x}}$.



Of the remaining two, one will admit **v** and exclude **f**, namely, $\underline{x}$; the other will admit **f** and exclude **v**, namely $\bar{x}$.

261. Now, let us consider assertions concerning the values of two quantities, x and y. Here there are two quantities, each of which has one only of two values; so that there are 2^2, or 4, possible states of things, as shown in this diagram.

Above the line, slanting upward to the right, are placed the cases in which x is **v**; below it, those in which x is **f**. Above the line but slanting downward to the right, are placed the cases in which y is **v**; below it, those in which y is **f**. Now in each possible assertion each of these states of things is either admitted or excluded; but not both. Thus, m will be 2^2, while n will be 2; and there will be n^m, or 2^4, or 16, possible assertions. They may be represented by drawing the lines of the diagram between x and y and closing over the compartments for the excluded sets of values. . . .*

262. Of three quantities, there are 2^3, or 8, possible sets of values, and consequently 2^8, or 256, different forms of propositions. Of these, there are only 38 which can fairly be said to be expressible by the signs [used in a logic of two quantities]. It is true that a majority of the others might be expressed by two or more propositions. But we have not, as yet, expressly adopted any sign for the operation of compounding propositions. Besides, a good many propositions concerning three quantities cannot be expressed even so. Such, for example, is the statement which admits the following sets of values:

x	y	z
v	**v**	**v**
v	**f**	**f**
f	**v**	**f**
f	**f**	**v**

Moreover, if we were to introduce signs for expressing [each of] these, of which we should need 8, even allowing the composition of assertions, still 16 more would be needed to express all propositions concerning 4 quantities, 32 for 5, and so on, *ad infinitum*.

263. The remedy for this state of things lies in simply giving the values **v** and **f** to propositions; that is, in admitting them to the universe of quantities. Here I will make an observation, by the way. Although formal logic is nothing but mathematics applied to logic, yet not a few of those who

* At this point Peirce introduces sixteen novel signs — one for each of the possible dyadic connections of P, $\bar{\text{P}}$, Q and $\bar{\text{Q}}$. As he below abandons these signs for the conventional dot (to represent logical multiplication) and for a sign of logical disjunction, his sixteen signs are not being reproduced. Wherever it was necessary to differentiate the sixteen cases a more conventional symbolism was substituted by the editors.

have cultivated it have had distinctly unmathematical minds. Indeed, in man's first steps in mathematics, he always draws back from mathematical conceptions. To first make **v** represent, let us say, Julius Caesar, and **f**, Pompey, since they may represent any subjects that are individual and definite, and thereupon further to propose to make every proposition either **v** or **f**, shocks the lower order of formal logicians. Such a mind will say, "If we have to distinguish propositions into two categories, let us denote their values by accented, or otherwise modified, letters, say **v**′ and **f**′, and not call them Caesar and Pompey, which is absurd." But I reply that that sort of stickling for usage bars the progress of mathematical thought; that the very fact that it is absurd that a proposition should be Caesar or Pompey proves that there will be no inconvenience, not in calling propositions what you mean by Caesar and Pompey, which, as *you say*, nobody *could* mean to do, but in generalizing the conception of Caesar, so as to make it include those propositions which are destined to triumph over the others. To protest against this, is virtually to protest against generalization; and to protest against generalization is to protest against thought; and to protest against thought is a pretty kind of logic. But still the unmathematical mind will ask, why not, however, adopt the **v**′ and **f**′; for he cannot conquer his shrinking from any generalization that can be evaded. It is the spirit of conservatism, the shrinking from the *outré*, which is commendable in its proper place; only it is unmathematical: instead of shrinking from generalizations, the part of the mathematician is to go for them eagerly. However, it would not even answer the purpose to distinguish **v**′ and **f**′ from **v** and **f**,[1] for the reason that there would be equal reason for distinguishing propositions about quantities being **v** or being **f** from propositions about quantities being **v**′ or being **f**′; so that we should require a **v**″ and an **f**″, and so on, *ad infinitum*. Now this would hamper us, because we should find we had occasion to form many a proposition about two propositions, as to whether one of the two was **v**″ or **f**″, for example, and at the same time whether the other were, say, $\mathbf{v}^{\text{IV}}$ or $\mathbf{f}^{\text{IV}}$, etc. We should, therefore, require still other **v**'s and **f**'s all to no mathematical purpose whatsoever; but, on

[1] Nor a relative from a non-relative universe.

the contrary, interfering fatally with a very different diversification of **v**'s and **f**'s which, we shall find, really will be needed.

264. If we assign the values **v** and **f** to propositions, we must either say that $\underline{x}$ has the same value as x, in which case $\bar{x}$ will have the contrary value, and $x \equiv \underline{x} \equiv \underline{\underline{x}} \equiv \underline{\underline{\underline{x}}} \equiv$ etc., while $-(x \equiv \bar{x}), -(\bar{x} \equiv \bar{\bar{x}})$, so that, $x \equiv \bar{\bar{x}} \equiv \bar{\bar{\bar{\bar{x}}}} \equiv$ etc., $\bar{x} \equiv \bar{\bar{\bar{x}}} \equiv$ etc., or else we must say that $\bar{x}$ has the same value as x, in the which case, $\underline{x}$ will have the contrary value, so that we shall have $x \equiv \bar{x} \equiv \bar{\bar{x}} \equiv$ etc. But $x \equiv \underline{\underline{x}} \equiv \underline{\underline{\underline{\underline{x}}}} \equiv$ etc., $\underline{x} \equiv \underline{\underline{\underline{x}}} \equiv$ etc. and $-(x \equiv \underline{x})$, $-(\underline{x} \equiv \underline{\underline{x}})$, etc. A choice has to be made; and there is no reason for one choice rather than the other, except that I have selected the letters **v** and **f**, and the other signs, so as to make the former choice accord with usual conventions about signs.

Adopting that former convention, we shall make the value of $\underline{x}$ the same as that of x. Where it becomes necessary, as it sometimes will, to distinguish them, we may either use the bar, or *vinculum*, below the line, or we may make use of the admirable invention of Albert Girard, who, in 1629, introduced the practice of enclosing an expression in parentheses to show that it was to be understood as signifying a quantity.* For example, $x \curlywedge y$† signifies that x is **f** and y is **f**. Then $(x \curlywedge y) \curlywedge z$, or $\underline{x \curlywedge y} \curlywedge z$,‡ will signify that z is **f**, but that the statement that x and y are both **f** is itself **f**, that is, is *false*. Hence, the value of $x \curlywedge x$ is the same as that of $\bar{x}$; and the value of $\underline{x \curlywedge x} \curlywedge x$§ is **f**, because it is necessarily false; while the value of $\underline{x \curlywedge y} \curlywedge \underline{x \curlywedge y}$¶ is only **f** in case $x \curlywedge y$ is **v**; and $(\underline{x \curlywedge x} \curlywedge x) \curlywedge (x \curlywedge \underline{x \curlywedge x})$|| is necessarily true, so that its value is **v**.

With these two signs, the vinculum (with its equivalents, parentheses, brackets, braces, etc.) and the sign $\curlywedge$, which I will call the *ampheck* (from ἀμφηκής, cutting both ways), all assertions as to the values of quantities can be expressed.

* But cf. F. Cajori, *A History of Mathematics*, p. 158.

† This is another anticipation of the Shefferian stroke-function; cf. 12ff.

‡ I.e., $(x \vee y).\bar{z}$.

§ I.e., $x.\bar{x}$.

¶ I.e., $x \vee y$.

|| I.e., $x :\vee: x$.

Thus,

$$x \text{ is } \underline{x \curlywedge x} \curlywedge \underline{x \curlywedge x}$$

$$\bar{x} \text{ is } x \curlywedge x$$

$$x : \vee : \bar{x} \text{ is } (\underline{x \curlywedge x} \curlywedge x) \curlywedge (x \curlywedge \underline{x \curlywedge x})$$

$$x \cdot \bar{x} \text{ is } \underline{x \curlywedge x} \curlywedge x$$

$$-(x\bar{x}y\bar{y}) \text{ is } \{\underline{x \curlywedge y} \curlywedge (\underline{x \curlywedge y} \curlywedge \underline{x \curlywedge y})\} \curlywedge \{(\underline{x \curlywedge y} \curlywedge \underline{x \curlywedge y}) \curlywedge \underline{x \curlywedge y}\}$$

$$x\bar{x}y\bar{y} \text{ is } \underline{x \curlywedge y} \curlywedge (\underline{x \curlywedge y} \curlywedge \underline{x \curlywedge y})$$

$$x \equiv y \text{ is } (x \curlywedge \underline{y \curlywedge y}) \curlywedge (\underline{x \curlywedge x} \curlywedge y)$$

$$-(x \equiv y) \text{ is } \underline{x \curlywedge y} \curlywedge (\underline{x \curlywedge x} \curlywedge \underline{y \curlywedge y})$$

$$x \vee y \text{ is } \underline{x \curlywedge y} \curlywedge \underline{x \curlywedge y}$$

$$\bar{x} \vee \bar{y} \text{ is } (\underline{x \curlywedge x} \curlywedge \underline{y \curlywedge y}) \curlywedge (\underline{x \curlywedge x} \curlywedge \underline{y \curlywedge y}) \quad [\text{or } x \curlywedge y]$$

$$\bar{x} \vee y \text{ is } (\underline{x \curlywedge y} \curlywedge y) \curlywedge (y \curlywedge \underline{x \curlywedge y}) \quad [\text{or } (y \curlywedge \underline{x \curlywedge x}) \curlywedge (y \curlywedge \underline{x \curlywedge x})]$$

$$x \cdot y \text{ is } \underline{x \curlywedge x} \curlywedge \underline{y \curlywedge y}$$

$$\bar{x} \cdot y \text{ is } x \curlywedge \underline{x \curlywedge y} \quad \text{or} \quad x \curlywedge \underline{y \curlywedge y}$$

$$\left[\begin{array}{l} \bar{x} \cdot \bar{y} \text{ is } x \curlywedge y \\ x \vee \bar{y} \text{ is } (x \curlywedge \underline{y \curlywedge y}) \curlywedge (x \curlywedge \underline{y \curlywedge y}) \\ x \cdot \bar{y} \text{ is } \underline{x \curlywedge x} \curlywedge y \\ y \text{ is } \underline{y \curlywedge y} \curlywedge \underline{y \curlywedge y} \\ \bar{y} \text{ is } y \curlywedge y \end{array}\right]^*$$

265. It is equally possible to express all propositions concerning more than two quantities. Thus, the one between three noticed above† is $\{x \curlywedge [\underline{y \curlywedge z} \curlywedge (\underline{y \curlywedge y} \curlywedge \underline{z \curlywedge z})]\} \curlywedge \{\underline{x \curlywedge x} \curlywedge [(y \curlywedge \underline{z \curlywedge z}) \curlywedge (\underline{y \curlywedge y} \curlywedge z)]\}$. That we can equally express every proposition by means of the vinculum [and] one $\curlywedge$ is sufficiently shown by the fact that $x \curlywedge y$ can be so expressed.‡ It is

$$(\underline{x \curlywedge x} \curlywedge \underline{y \curlywedge y}) \curlywedge (\underline{x \curlywedge x} \curlywedge \underline{y \curlywedge y}) \ldots$$

* Peirce omitted the consideration of these five cases, though he did provide for them in the elided passages.

† In 262.

‡ I.e., a dichotomic algebra or logic can be developed through the use of but one logical constant; in this case, through the use of a single symbol representing the *disjunction* of the negatives of the symbolized constituents. In the first paper of this volume and in 264, Peirce used, as the one constant, a symbol

266. In order that a sign, say O, should be associative, it is requisite either that whatever quantity x may be, $xOx=x$, or else, that whatever quantities x and y may be, $xOy=yOx$ and either $x=y$, or $xOx=x$, or $yOy=y$. This may be otherwise stated as follows:

First, Suppose $\mathbf{v}O\mathbf{v}=\mathbf{v}$ and $\mathbf{f}O\mathbf{f}=\mathbf{f}$. Then I will show that the operation is associative. For if not, it would be possible to give such values to p, q, r, that $-\{pO(qOr)\}\equiv(pOq)Or$. But of these three values, p, q, r, some two must be equal. But all three cannot be equal, since then, because of $\mathbf{v}O\mathbf{v}=\mathbf{v}$ and $\mathbf{f}O\mathbf{f}=\mathbf{f}$, the inequality would not hold. Suppose then first that $p\equiv r$, $\bar{p}\equiv q$. If then $pOq\equiv qOp$, substituting p for r, $pO(qOp)\equiv pO(pOq)\equiv(pOq)Op\equiv(pOq)Or$, contrary to the hypothesis. Suppose, secondly, then, that $q\equiv r$, $\bar{q}\equiv p$. Then, substituting q for r, $pO(qOq)\equiv pOq$; and since this is unequal to $(pOq)Oq$, it follows that $-(p\equiv pOq)$. But in that case, there being only two different values possible, $pOq\equiv q$, and $pO(qOq)\equiv pOq\equiv q$ while $(pOq)Oq\equiv qOq\equiv q$, contrary to the hypothesis. The third supposition, that $p\equiv q$ would evidently lead to an absurdity analogous to the last; so that in no way can the associativeness fail in this case.

Second, Suppose $\mathbf{v}O\mathbf{v}=\mathbf{f}$ and $\mathbf{f}O\mathbf{f}=\mathbf{v}$. Then I will show that the operation is not associative. For on the one hand,

$$(\mathbf{v}O\mathbf{v})O\mathbf{f}\equiv\mathbf{f}O\mathbf{f}\equiv\mathbf{v};$$

while, on the other hand, whether $\mathbf{v}O\mathbf{f}=\mathbf{v}$, so that

$$\mathbf{v}O(\mathbf{v}O\mathbf{f})\equiv\mathbf{v}O\mathbf{v}\equiv\mathbf{f},$$

or whether $\mathbf{v}O\mathbf{f}=\mathbf{f}$, so that

$$\mathbf{v}O(\mathbf{v}O\mathbf{f})\equiv\mathbf{v}O\mathbf{f}\equiv\mathbf{f},$$

in either case, the associative rule is broken.

representing the *conjunction* of the negatives of the symbolized constituents He now shows that either one of these logical constants can be defined in terms of the other, thus:

$$\text{As } x\curlyvee y=\bar{\bar{x}}\cdot\bar{\bar{y}}$$
$$\text{As } x\curlywedge y=\bar{\bar{x}}\vee\bar{\bar{y}}$$
$$\text{And as } \bar{\bar{x}}\cdot\bar{\bar{y}}=-[-(\bar{\bar{x}}\vee\bar{\bar{x}})\vee-(\bar{\bar{y}}\vee\bar{\bar{y}})]$$
$$\text{It must be true that } x\curlyvee y=(\underline{x\curlywedge x}\curlywedge\underline{y\curlywedge y})\curlywedge(\underline{x\curlywedge x}\curlywedge\underline{y\curlywedge y}).$$
$$\text{And as } \bar{\bar{x}}\vee\bar{\bar{y}}=-[-(\bar{\bar{x}}\cdot\bar{\bar{x}})\cdot-(\bar{\bar{y}}\cdot\bar{\bar{y}})]$$
$$\text{It must be true that } x\curlywedge y=(\underline{x\curlyvee x}\curlyvee\underline{y\curlyvee y})\curlyvee(\underline{x\curlyvee x}\curlyvee\underline{y\curlyvee y})]$$

Third, Suppose **v**O**v** = **f**O**f** and **v**O**f** = **f**O**v**. Then I will show that the operation is associative. For otherwise it would be possible to give such values to p, q, r, that

$$-\{pO(qOr)\} \equiv (pOq)\, Or.$$

But since **v**O**f** = **f**O**v**, it follows that the second side of the inequality would be equal to

$$rO(qOp)$$

so that the inequality requires $\bar{p} \equiv r$. But then also $-(qOp) \equiv qOr$ and consequently, the two assumed equations are inconsistent with the inequality, and the operation must be associative.

Fourth, Suppose **v**O**v** ≡ **f**O**f**, but −(**v**O**f**) ≡ **f**O**v**. Then I will show that the operation is not associative. For either

(**v**O**v**)O**v** ≡ **f**O**v**, while **v**O(**v**O**v**) ≡ **v**O**f**
or (**f**O**f**)O**f** ≡ **v**O**f**, while **f**O(**f**O**f**) ≡ **f**O**v**;

and in either case since −(**v**O**f**) ≡ **f**O**v**, the rule of association is violated. The four propositions thus proved, when taken together, are equivalent to the proposition [at the top of page 217]. Of these four, the first shows that ∨, ·, x,* y* are associative; the third that :∨:−, ≡, −≡, −(:∨:−), are so. The second shows that λ, $\bar{x}$,* $\bar{y}$,* ⫛, are non-associative; the fourth that —<, ∨−, −(∨−), −(—<) are so.

267. Another important property of some signs is that a quantity over a vinculum can be interchanged with one beyond the vinculum, so that—

either	$xO(yOz)$	and	$yO(xOz)$
or	$(xOy)Oz$	and	$(xOz)Oy$

will have the same value. In order that the former formula should hold, it is necessary and sufficient that if xOy, whatever x and y may be, always has the contrary value to that of y when x and y have contrary values, then it should have the

* By the sign x is meant any sign such that $xOy \equiv x$; by y is meant any sign such that $xOy \equiv y$; by $\bar{x}$ is meant any sign such that $xOy \equiv \bar{x}$; and by $\bar{y}$ is meant any sign such that $xOy \equiv \bar{y}$.

contrary value to that of y when x and y have the same value; and conversely, if it has the contrary value to that of y when x and y have the same value, it should also have the contrary value to that of y when x and y have different values. The same rule holds in regard to the second formula interchanging x and y in the statement.

The proposition may be otherwise stated as follows. Let P be the proposition that either $\mathbf{v}O\mathbf{v}=\mathbf{v}$ or $\mathbf{f}O\mathbf{f}=\mathbf{f}$ and Q the proposition that either $\mathbf{f}O\mathbf{v}=\mathbf{v}$ or $\mathbf{v}O\mathbf{f}=\mathbf{f}$. Then the [first] formula holds if P and Q are both true or both false; but fails if either is true while the other is false.

First, Suppose P and Q both false; so that

$$\mathbf{v}O\mathbf{v}=\mathbf{f},\quad \mathbf{f}O\mathbf{f}=\mathbf{v},\quad \mathbf{f}O\mathbf{v}=\mathbf{f},\quad \mathbf{v}O\mathbf{f}=\mathbf{v},$$

Then,

$$\mathbf{f}O(\mathbf{v}O\mathbf{v})=\mathbf{f}O\mathbf{f}=\mathbf{v},\quad \mathbf{v}O(\mathbf{f}O\mathbf{v})=\mathbf{v}O\mathbf{f}=\mathbf{v}$$
$$\mathbf{f}O(\mathbf{v}O\mathbf{f})=\mathbf{f}O\mathbf{v}=\mathbf{f},\quad \mathbf{v}O(\mathbf{f}O\mathbf{f})=\mathbf{v}O\mathbf{v}=\mathbf{f},$$

and the two formulæ hold.

Second, Suppose the first formula fails. That is,

$$-\{\mathbf{f}O(\mathbf{v}O\mathbf{v})\equiv\mathbf{v}O(\mathbf{f}O\mathbf{v})\}.$$

Now if $\mathbf{v}O\mathbf{v}=\mathbf{f}O\mathbf{v}=\mathbf{v}$, evidently every expression ending in $\mathbf{v}$ would be equal to $\mathbf{v}$, contrary to the hypothesis of the inequality just written. Hence, either $\mathbf{v}O\mathbf{v}=\mathbf{f}$ or $\mathbf{f}O\mathbf{v}=\mathbf{f}$.

If $\mathbf{v}O\mathbf{v}=\mathbf{f}$, but $\mathbf{f}O\mathbf{v}=\mathbf{v}$, the second side of the inequality is $\mathbf{v}O\mathbf{v}=\mathbf{f}$; so that the first side must be $\mathbf{f}O\mathbf{f}=\mathbf{v}$, and P is false.

If $\mathbf{f}O\mathbf{v}=\mathbf{f}$, but $\mathbf{v}O\mathbf{v}=\mathbf{v}$, the first side of the inequality is $\mathbf{f}O\mathbf{v}=\mathbf{f}$; so that the second side must be $\mathbf{v}O\mathbf{f}=\mathbf{v}$, and Q is false.

If $\mathbf{v}O\mathbf{v}=\mathbf{f}$ and $\mathbf{f}O\mathbf{v}=\mathbf{f}$, the inequality becomes $-(\mathbf{f}O\mathbf{f})=\mathbf{v}O\mathbf{f}$. Hence, either $\mathbf{f}O\mathbf{f}=\mathbf{v}$, when P is false or $\mathbf{v}O\mathbf{f}=\mathbf{v}$, when Q is false; and in any case either P or Q is false.

To suppose that the second of the two formulæ fails, that is, that

$$-\{\mathbf{v}O(\mathbf{f}O\mathbf{f})\equiv\mathbf{f}O(\mathbf{v}O\mathbf{f})\}$$

is merely changing $\mathbf{f}$ to $\mathbf{v}$ and $\mathbf{v}$ to $\mathbf{f}$ throughout the supposition just examined. Consequently, the result must be obtained by making the same interchange in the result of that supposition. But such interchange will neither change the falsity of P nor that of Q. Consequently, whichever formula fails either P or

Q is false; although, by the first supposition, if both are false both formulæ hold. It remains then to examine the cases in which of P and Q one is true and the other false.

Third, Suppose P to be false, and one or other formula to fail. If it be the first that fails, or

$$-\{\mathbf{f}O(\mathbf{v}O\mathbf{v}) \equiv \mathbf{v}O(\mathbf{f}O\mathbf{v})\},$$

since the first member is $\mathbf{f}O(\mathbf{v}O\mathbf{v}) = \mathbf{f}O\mathbf{f} = \mathbf{v}$, the second member must be $\mathbf{f}$. This will be the case, if, and only if, either $\mathbf{f}O\mathbf{v} = \mathbf{v}$ or $\mathbf{v}O\mathbf{f} = \mathbf{f}$; that is, if and only if Q is true. If, however, it be the second formula that fails, or

$$-\{\mathbf{v}O(\mathbf{f}O\mathbf{f}) \equiv \mathbf{f}O(\mathbf{v}O\mathbf{f})\},$$

again, by interchange of $\mathbf{f}$ and $\mathbf{v}$, Q must be true.

Fourth, Suppose Q to be false (that is, $\mathbf{f}O\mathbf{v} = \mathbf{f}$ and $\mathbf{v}O\mathbf{f} = \mathbf{v}$) and one or other formula to fail. If it be the first, that is, if

$$-\{\mathbf{f}O(\mathbf{v}O\mathbf{v}) \equiv \mathbf{v}O(\mathbf{f}O\mathbf{v})\},$$

since the second member is $\mathbf{v}O(\mathbf{f}O\mathbf{v}) = \mathbf{v}O\mathbf{f} = \mathbf{v}$, the first member must equal $\mathbf{f}$ and either $\mathbf{v}O\mathbf{v} = \mathbf{v}$ or $\mathbf{f}O\mathbf{f} = \mathbf{f}$, that is, P will be true. The same results, by interchange of $\mathbf{f}$ and $\mathbf{v}$, if the second formula fails.

It follows, then, that if P and Q are both true or both false the formulæ hold, while [if] P and Q differ in respect to truth both formulæ fail, and must be replaced by inequalities.

It will be remarked that one or other of the two formulæ

$$xO(yOz) \equiv yO(xOz)$$
$$(xOy)Oz \equiv (xOz)Oy$$

is true of 14 out of the 16 signs, or all but ʎ and ɹ.

Another very commonly true pair of formulæ are

$$(xOy)Ox \equiv (yOy)Ox$$
$$xO(yOx) \equiv xO(yOy).$$

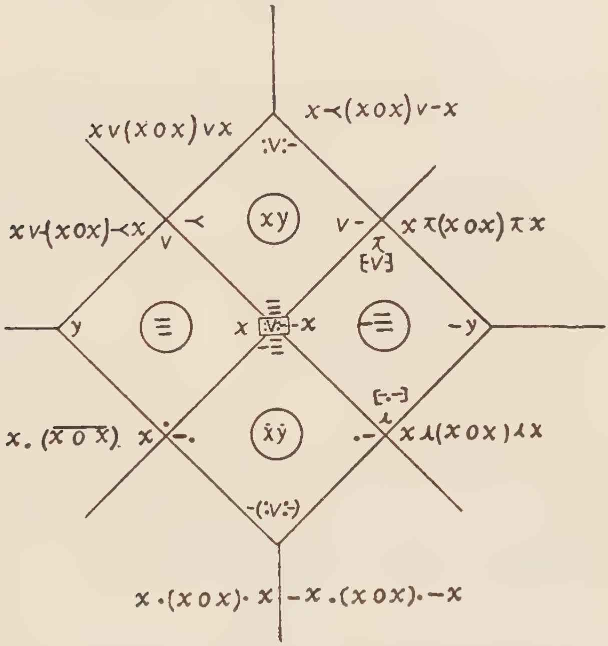
One formula or the other holds for all the signs except $\equiv$ and $-\equiv$; and both hold except for these and x, y, $\bar{x}$, $\bar{y}$.

Somewhat similar to the above are the formulæ

$$(xOy)Oy \equiv (yOx)Ox$$
$$xO(xOy) \equiv yO(yOx).$$

The first formula holds for :∨:–, ∨, —<, ·,–·, –(:∨:–) and the second for :∨:–, ∨,∨–, ·, ·–, –(:∨:–).

268. The following table is a key to all the propositions which can be written with one vinculum, at most, and are either necessarily true or necessarily false, whatever be the value of the single letter they contain.



269. A proposition of the form xOx is necessarily true, if O be replaced by any one of the signs in the four corners of the upper quadrant; that is, by :∨:–, —<, ∨–, ≡, which are, therefore, *affirmative* signs, according to De Morgan's* definition of that word. A proposition of the same form is necessarily false, if O be replaced by any of the former signs in the corners of the lower quadrant; that is, by –≡, –·, ·–, –(:∨:–), which are accordingly *negative* signs. The others are *neutral*.

Above the horizontal line, there are, outside the large square, four compartments each of which heads a pair of quadrants making half the large square. In each of these compartments is written a formula of the form

$$x \diamondsuit (xOx) \heartsuit x.$$

* *Formal Logic*, p. 56; *Syllabus*, §19.

If we strike out the heart and last letter, and replace the diamond by the sign found in its place in any one of those compartments, and replace the *O* by any one of the eight signs in the half-square that compartment heads, we shall get a proposition necessarily true. Such, for example, are

$$x \multimap\!\!< (x \equiv x)$$

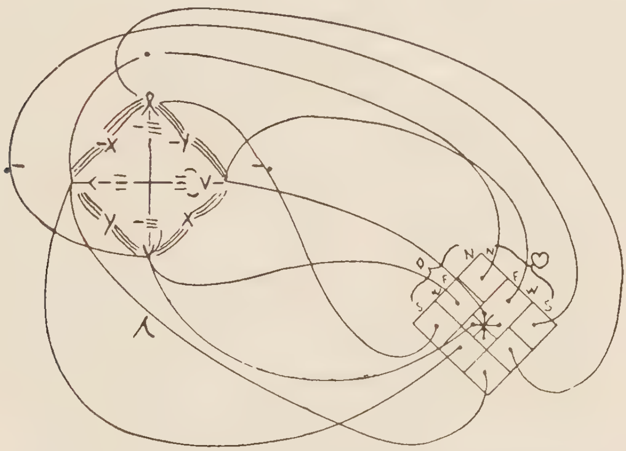
and $$x \vee (x ⅄ x).$$

There will be a similar result if we strike off the first letter and use only the heart; instead of striking off the heart and last letter and using only the diamond. On the other hand, if we enter in either of these ways one of the compartments below the horizontal line, we get a proposition necessarily false. Thus $x.\ (\overline{x\,.\,x})$ is necessarily false. If we replace the diamond by the left-hand, or only, sign [*i.e.* by ≡] in an oval, and the *O* by any of the signs in the corners of the same square [*i.e.* by ∨, *y*, *x*, .] we have a proposition necessarily true; and so if we replace the heart by the right-hand, or only, letter in an oval [*i.e.* by −≡]. If instead of replacing the *O* by one of the four signs in the corners of the same square, we take one of those in the opposite square, we get a proposition necessarily false. If the diamond or heart is replaced by :∨:−, which is in the little square in the middle, and the *O* by any of the 16 signs in the large square, the proposition will be necessarily true; but if −(:∨:−) replaces the diamond or heart, the proposition will necessarily be false.

270. Of propositions necessarily true of the form

$$(x \diamond x)\ O\ (x \heartsuit x)$$

there are just eleven hundred. But in 256 of these *O* is :∨:−, while ◇ and ♡ can be any signs whatever. The remaining 844 are exhibited in the following diagram, not very elegantly, it must be confessed. The sign in the place of the diamond is first to be sought in the first diagram; and its quadrant there is to be denoted by the corresponding cardinal point upon an ordinary map. That is to say, N is either :∨:−, —<, ∨−, ≡; E is either ⅄, $\bar{x}$, $\bar{y}$, or $\curlywedge$; W is either ∨, $\mathfrak{x}$, *y* or . ; S is either −≡, −.,.− or −(:∨:−). We do the same with the sign in place of the heart. We enter the square on the right hand side below, with the letter for the diamond on the left, and that for the heart at the right. At the intersection of the two rows will be found a spot, from which a line leads to three signs in the left-hand part of the diagram, or to seven signs in case neither



letter is E or W. Any one of these signs being taken as *O*, the proposition will be necessarily true.

271. Rather than inflict upon the reader more of these inconsequential technicalities, I will skip much that systematic thoroughness would require, and will at once notice some propositions necessarily true of the forms:

(α)	(xRy) ♡ [(ySz) ◇ (x*O*z)]	[(xRy) ◇ (ySz)] ♡ (x*O*z)	(ζ)
(β)	(xRy) ♡ [(x*O*z) ◇ (ySz)]	[(xRy) ◇ (x*O*z)] ♡ (ySz)	(η)
(γ)	(ySz) ♡ [(xRy) ◇ (x*O*z)]	[(ySz) ◇ (xRy)] ♡ (x*O*z)	(θ)
(δ)	(ySz) ♡ [(x*O*z) ◇ (xRy)]	[(ySz) ◇ (x*O*z)] ♡ (xRy)	(ι)
(ϵ)	(x*O*z) ♡ [(xRy) ◇ (ySz)]	[(x*O*z) ◇ (xRy)] ♡ (ySz)	(κ)
(F)	(x*O*z) ♡ [(ySz) ◇ (xRz)]	[(x*O*z) ◇ (ySz)] ♡ (xRy)	(λ)

In all cases, R is on the left margin, S on the right margin, and *O* in the body of the table. If these parts of the margins are used which intersect in ɹ, then ◇ is to be taken as ɹ and ♡, in the six left-hand formulæ ($\alpha-$F), as ∨ −, but in the others ($\zeta-\lambda$) as −<; or both ◇ and ♡ may be taken as ∨. If those parts of the margin are used which intersect in −•, in formulæ α, β, ϵ, F, ♡ is ∨ −; in θ, ζ, ι, λ, it is −<. In α, F, θ,

ι, ◇ is .–; in β, ϵ, ζ, λ, it is – . . In γ, δ, η, κ, ♡ is ⅄ while ◇ is ⅄. But, on the other hand, we may use a different interpretation, and in α, β, ϵ, F, ζ. θ, ι, λ, make ♡ to be ∨; while in α, F, θ, ι, we put ─< for ◇, and in β, ϵ, ζ, λ, we put ∨ – for ◇. Then, in γ, δ, we can put ─<, and in η and κ, can put ∨ –, for ♡; putting ∨ in all four for ◇. If the parts of the margin intersecting in .– are used, then in one system of interpretation, in γ, δ, ϵ, F, we may put ∨ – for ♡, and in ζ, η, θ, k, may put ─< for ♡; while in γ, ϵ, ζ, η, we put .–, and in δ, F, θ, κ, we put – . for ◇. Then in α, β, ι, λ, we shall put ⅄ for ♡ and ⅄ for ◇. In another system of interpretation, we may in γ, δ, ϵ, F, ζ, η, θ, κ, put ∨ for ♡, while in γ, ϵ, ζ, η, we put ─< and in δ, F, θ, κ, we put ∨ –, for ◇. In α and β, we shall put ─<, in ι and λ, we shall put ∨ –, for ♡, and in all four shall put ∨ for ◇. If the parts of the margin intersecting in . are used, then, in the first system, in α, β, γ, δ, we put ─<, while in η, ι, κ, λ, we put ∨ –, for ♡, and in α, γ, η, ι, we put ─<, in β, δ, κ, λ, we put ∨ –, for ◇. In the other system in α, β, γ, δ, η, ι, κ, λ, we put ⅄ for ♡, in α, γ, η, ι, we put .–, in β, δ, κ, λ, we put – ., for ◇. In ϵ, F, ζ, θ, we may put ∨ for ♡ and ⅄ for ◇; or else in ϵ, F, we may put ∨ –, in ζ, θ we may put – ∨ for ♡, in all four . for ◇.

The 24376 formulæ which this table yields are all of this class that it seems worthwhile to give.

272. In order to form a proposition necessarily true of the form $(x \diamond y)\ O\ (x \heartsuit y)$, it is only necessary to rewrite it, replacing ◇ and ♡ by the signs they stand for in their most iconic forms, but with :∨:– in place of O. This middle sign is now to be modified as follows:

If there is a quadrant open in both right- and left-hand signs, the top quadrant of the middle one must be left open;

If there is a quadrant open in the left-hand sign alone, the left-hand quadrant of the middle sign must be left open;

If there is a quadrant open in the right-hand sign alone, the right-hand quadrant of the middle sign must remain open;

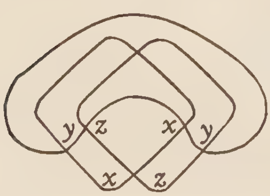
If there is a quadrant open neither in the right- nor in the left-hand sign, the bottom quadrant of the middle sign must be open.

Any quadrant which is not compelled to be open by this rule, may be closed.

273. The rule may be used inversely, of course. Its principal results are shown in the following table.*

-(:v:-)	λ	·	.–	–.	-y	-x	≡-	≡	x	y	v-	-v	λ	v	:v:-	
‾.	y	y	y	y	y	y	y	y	y	y	y	y	y	y	·	:v:-
-x	≡-	-v	-v	-v	v	v	-v	v	-v	-v	v	v	v	≡	x	v
-x	-v	≡-	-v	-v	-v	-v	-v	v	v	v	v	v	≡	v	x	λ
-x	-v	-v	≡-	-v	v	-v	v	-v	v	-v	v	≡	v	v	x	-v
-x	-v	-v	-v	≡-	-v	v	v	-v	-v	v	≡	v	v	v	x	v-
-x	λ	-v	λ	-v	≡-					≡	v	v-	v	v-	x	y
-x	λ	-v	-v	λ		≡-			≡		v-	v	v	v-	x	x
-x	-v	-v	λ	λ			≡-	≡			v-	v-	v	v	x	≡
-x	λ	λ	-v	-v			≡	≡-			v	v	v-	v-	x	≡-
-x	-v	λ	λ	-v		≡			≡-		v	v-	v-	v	x	-x
-x	-v	λ	-v	λ	≡					≡-	v-	v	v-	v	x	-y
-x	λ	λ	λ	≡	λ	v-	v-	λ	λ	v-	≡-	v-	v-	v-	x	–.
-x	λ	λ	≡	λ	v-	λ	v-	λ	v-	λ	v-	≡-	v-	v-	x	.–
-x	λ	≡	λ	λ	λ	λ	λ	v-	v-	v-	v-	v-	≡-	v-	x	·
-x	≡	λ	λ	λ	v-	v-	λ	v-	λ	λ	v-	v-	v-	≡-	x	λ
λ	-y	-y	-y	-y	-y	-y	-y	-y	-y	-y	-y	-y	-y	-y	.‾	-(:v:-)

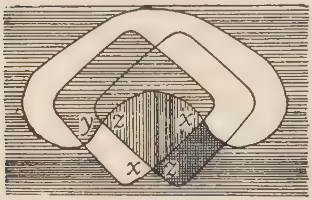
274. From two propositions, one relating to the values of x and y, the other to those of y and z, some proposition concerning the values of x and z can invariably be deduced; although it may be the absurd one, $-(x:\vee:-z)$, which is often valuable. In order to ascertain what this is, the value concerning y and z should be written to the left of that concerning x and y, and then lines should be drawn as in this figure.



Whatever quadrants are closed should be shaded, in the one case by vertical shading and in the other by horizontal shading; and this shading should extend over that line which does not pass

* This is a clearer (and more accurate) representation, by reason of a reduction in the number of signs used, of the table Peirce gives. It says that if the symbols on the top of the columns be used to connect x and y, and the symbols at the end of the rows be used to connect x and y, then the first pair is connected necessarily with the second pair by the symbol at the intersection.

through the sign concerned. We then read off the conclusion below, noting well that each quadrant consists of two compartments, since the line that does not pass through the sign concerned is disregarded; and unless *both* are shaded, the quadrant is not closed. For example, given that $y \vee -z$ and $x \equiv -y$, our diagram becomes as here shown.



It will be seen that although three quadrants appear, at first glance, to be cut off from the lower sign, yet two of them are not really cut off, since the more remote parts of them are unshaded. The conclusion, therefore, is, that *x*⅄*z*.

Such diagrams go by the name of Euler's diagrams,* although they are said by Hamilton† and by Drobisch‡ to be far older than Euler. The rudimentary idea of them is very likely ancient. But the plan of shading them is due to Mr. Venn.§ Further on [in book II], I shall show how they may be rendered even more efficient.

So far we have refrained from making use of the obelus, a practice in the style of De Morgan. We shall now see what instant and complete simplification results from the use of it. In the first place, all the signs then become expressible by means of any one of the eight.¶

∨   ·

∨ —   — ∨   · —   — ·

— ∨ —   — · —

275. Dichotomic mathematics, in itself considered, is a trivial thing. Early students of it — in the days of Boole, and later, I mean — may be excused for fancying it could turn out important; I myself long entertained that chimerical dream.

* See bk. II.

† *Lectures on Logic*, Lecture XIV.

‡ *Neue Darstellung der Logik*, §84.

§ See *Symbolic Logic*, p. 122n.

¶ Obviously, if all the signs can be expressed by either ɹ or ⅄, they can be expressed by any of the other six, for these other six differ from ɹ and ⅄ only through the addition of signs of denial. Thus, $x \,.\, \bar{y}$ is $\bar{x}$ɹy; $\bar{x} \vee y$ is x⅄$\bar{y}$, etc. As $\bar{x}$ and $\bar{y}$ are themselves expressible by ɹ and ⅄, the six signs may be viewed as abbreviations of some combination of either ɹ or ⅄.

The real importance of it lies in the fact that it is a most important aid to the clear understanding of speculative grammar, and even of critical logic; and in the circumstance that, for logical reasons, every mathematical doctrine involves dichotomic mathematics. Where, for example, would be the algebra without a sign of equality? Yet that sign is a dichotomic sign. Were dichotomic algebra only to be used in the study of logic, the simplification of the apparatus would be a secondary consideration; although even then it would be dreary waste of time always to be going back to first principles. But when we reflect that every algebra must involve a dichotomic algebra, we see that such simplification is a serious desideratum. It will be best, therefore, to retain at least two signs of the relations between the values of two quantities. We want these to be as free from necessary formulæ, which will be rules to be borne in mind, as possible; and what rules there are should be as simple as possible. It is, therefore, best to select signs with which it is impossible to construct a formula which is either necessarily true or necessarily false. Moreover, associative signs simplify rules greatly. These two conditions are connected. That is, a sign which satisfies the first necessarily satisfies the second; and the only two signs which do this are $\vee$ and $\cdot$.

276. Suppose, then, we proceed to build the algebra upon these, together with the obelus. We shall then have no need of ⅄ and ⅄̣, since

$$x ⅄ y \text{ is } \bar{x} \vee \bar{y} \text{ and } x ⅄̣ y \text{ is } \bar{x} \cdot \bar{y}.$$

But the very frequently occurring relations $x \equiv y$ and $-(x \equiv y)$ are still only capable of being expressed in a too complicated way

$$x \equiv y \text{ is } \underline{x \cdot y} \vee \underline{\bar{x} \cdot \bar{y}} \text{ or } \underline{x \vee \bar{y}} \cdot \underline{\bar{x} \vee y}$$

and

$$-(x \equiv y) \text{ is } \underline{\bar{x} \cdot y} \vee \underline{x \cdot \bar{y}} \text{ or } \underline{x \vee y} \cdot \underline{\bar{x} \vee \bar{y}}$$

It will be best to retain one or both of them. Moreover, for logical reasons, the sign $-\!\!<$ is of the greatest importance. From a logical point of view, we may say, as we shall see, that it is the most important of any; and even from an algebraical point of view, if we retain $\vee$ and $\cdot$ as our usual signs, the most important formula of the algebra, apart from those of the commutative and associative principles, requires this sign for its convenient and clear expression. Namely, this formula is

$$(\underline{x \vee y} \cdot z) \mathrel{-\!\!<} (x \vee \underline{y \cdot z})$$
$$(x \cdot \underline{y \vee z}) \mathrel{-\!\!<} (\underline{x \cdot y} \vee z)$$

expressing a sort of associativeness involving the two signs $\vee$ and $\cdot$ . . .

277. Adopting these signs, I proceed to state, as definitions, those fundamental conventions from which all the properties of the signs, and all the necessary rules and formulæ of the algebra can be deduced. In doing so, I shall, after Schröder, imitate a practice introduced into geometry, I believe, by Gergonne (about 1820), of writing reciprocally related propositions in parallel columns. The perfect correspondence between aggregation and composition was vaguely asserted by De Morgan, but was first definitely applied to dichotomic algebra and demonstrated, by me.*

278. *Definition of the Quantities of a Dichotomic Algebra.*

If x is any quantity of this algebra, then

No predicate and its negative can both be true of x. That is, x is *definite.*	Every predicate or its negative must be true of x. That is, x is *individual.*

Every proposition concerning quantities of the algebra which is such that it must be true or false but cannot be both is itself a quantity of the algebra; and no other propositions except those which are primarily quantities of the algebra, and those which relate to the values of quantities of the algebra, are quantities of the algebra.

Definitions of **v** *and* **f**

No quantity of this algebra has, at once, the two values **v** and **f**.	Every quantity of this algebra has either the value, **v**, or the value, **f**.
Every true proposition which is a quantity of this algebra has the value **v**.	Every false proposition which is a quantity of this algebra has the value, **f**.

* See 3.4 and 3.199f.

279. *Definitions of the Vinculum and Obelus*

Every quantity written upon a sheet of assertion, is either written

with a *vinculum* extending under the whole expression, and is thereby asserted to have the value **v**.	with an *obelus* extending over the whole of it, and is thereby asserted to have the value **f**.

But such assertions may be in what is equivalent to *oratio obliqua*, and are not necessarily *direct*.

280. *Definitions of Composition and Aggregation.*

If x and y are any quantities of this algebra (whether the same or different)

If $x \cdot y = \mathbf{v}$, then $x = \mathbf{v}$;	If $x \dotplus y$* $= \mathbf{f}$, then $x = \mathbf{f}$;
If $x \cdot y = \mathbf{v}$, then $y = \mathbf{v}$;	If $x \dotplus y = \mathbf{f}$, then $y = \mathbf{f}$;
If $x = \mathbf{v}$, then if $y = \mathbf{v}$,	If $x = \mathbf{f}$, then if $y = \mathbf{f}$,
so likewise $x \cdot y = \mathbf{v}$.	so likewise $x \dotplus y = \mathbf{f}$.

Substantially these definitions of composition and aggregation were given by me in 1880 (*Am. Jour. of Math. III, 33*).† They have the effect of reducing all legitimate transformations to successive legitimate insertions and omissions. The demonstration of this is very easy; but I think it will be more accurately appreciated if I postpone giving it until I have first shown what transformations are, by the above definitions, legitimated through insertions and omissions.

281. I. *Any proposition in this algebra, being written on a sheet of assertions, can, without loss of truth, receive the insertion of a vinculum below it.*

Demonstration. For, according to the definition of quantities, any proposition of this algebra is a quantity; and by the definition of vinculum and obelus, if written on a sheet of assertions, it must receive either a vinculum or an obelus, either of which being added to it, the result is an assertion concerning the value of the quantity. But this assertion is, by

* $\dotplus$ is Peirce's alternative symbol for logical addition, which was just now symbolized by $\vee$.

† In 3.199.

the definition of quantities, itself a quantity. And since this assertion asserts itself to be true, by the definition of **v** and **f**, it assigns to itself the value, **v**. But, by the definition of a vinculum, the assertion that a quantity has the value **v** is expressed by writing it upon a sheet of assertions with a vinculum below it; and therefore the insertion of the vinculum is legitimated by the assertion of the proposition. For example, if x appears upon the field of assertions we may write $\underline{x}$, or if $\bar{y}$ appears, we may write $\underline{\bar{y}}$. For a transformation is legitimate if it proceeds in accordance with a rule which can never transform a true proposition into a false one.

282. II. *A quantity, written on a sheet of assertions with a vinculum under it may, without loss of truth, be transformed by the omission of the vinculum.*

Demonstration. Let x be any quantity which is written on the sheet of assertion with a vinculum under it, $\underline{x}$. We may confine ourselves to the consideration of the case of the assertion x being true; for if it be not true, it certainly cannot sustain a loss of truth, since, by the definition of quantities, an assertion which has any other than the two grades of truth does not enter into the algebra. Let us first suppose that x is a proposition. Now every asserted proposition virtually asserts its own truth. That is to say, it asserts a fact, which being assumed real, whoever perceives that the proposition asserts that fact, has a perception which can be formulated by saying that what the proposition asserts is true. Therefore, if the vinculum is removed, nothing false is asserted by x, assuming $\underline{x}$ to be true. The only difference is that $\underline{x}$ directly asserts what x virtually asserts, and that x implies what $\underline{x}$ directly asserts. If, on the other hand, x is not an assertion, still its being written upon a sheet of assertions would make it assert itself to be an assertion; and whatever asserts itself to be something asserts something. But, being a quantity of this algebra, unless it is itself primarily an assertion, which would be contrary to our present hypothesis, the only assertion it can be, by the definition of quantities, is that a quantity of the algebra has some value. But the only quantity of the algebra to which x could refer would be itself. It must, therefore, assert that it has itself either the value **v** or the value **f**. But, by I, it must assert something which implies the truth of $\underline{x}$.

Hence, x must assert that its value is **v**. But this is no more than is asserted by $\underline{x}$; and therefore no falsity can be introduced by omitting the vinculum.

283. III. *A quantity, written on a sheet of assertion, may, without loss of truth, be transformed by the insertion of two obeli over it.*

Demonstration. Let x represent what is written. Being written upon the sheet of assertions, it is an assertion. It, therefore, virtually, at least, consists in the affirmation that some proposition pertinent to the algebra is true, that is, has the value **v**. Then, since every such proposition is **v** or **f**, but not both, it virtually implies that $\bar{x}$ has the value **f**. But $\bar{\bar{x}}$ merely asserts that $\bar{x}$ has the value **f**. Hence $\bar{\bar{x}}$ asserts no more than is virtually implied in x, and there will be no loss of truth in inserting the two obeli.

284. IV. *If a quantity upon a sheet of assertions has two obeli over it, it may be transformed, without loss of truth, by the omission of the two obeli.*

Demonstration. Let $\bar{\bar{x}}$ represent what is written. This asserts that $\bar{x}$ has the value **f**. That is to say, it asserts that the assertion of $\bar{x}$ is false. But, then, since x must either have the value **v**, when x is true, or the value **f**, when $\bar{x}$ is true, and since the latter is not the case, it follows that x is true. That is, by II, x can be written on the sheet of assertions. Or, in other words, the two obeli can be omitted, without loss of truth.

285. V. *If a quantity, were it written by itself on a sheet of assertions, could be transformed, without loss of truth, into another quantity, then were the latter written under an obelus on the field of assertions, it could be transformed without loss of truth into the former under the obelus.*

Demonstration. Let x represent the first quantity, y the second, so that it is supposed that if x were written on the sheet of assertions, it could in every case be transformed into y without loss of truth. Suppose, now, that it were not true that $\bar{y}$ could in every case be transformed into $\bar{x}$ without loss of truth. Then, there must be some possible case in which $\bar{y}$ would be true and $\bar{x}$ not true. To say that $\bar{x}$ is not true, is expressed by writing $\bar{\bar{x}}$; so that while $\bar{y}$ was true, it would be possible for $\bar{\bar{x}}$ to be true, and hence, by IV, for x to be true.

But $\bar{y}$ is the expression of the assertion that y is **f**, or y is false. Hence, it would be possible for x to be true but y false; when it would not be true that x would in all cases be transformed into y without loss of truth, which is contrary to the hypothesis. Hence, if x could be transformable into y, without loss of truth, it would be absurd to suppose $\bar{y}$ not transformable into $\bar{x}$ without loss of truth.

286. VI. *If a quantity, were it written by itself on a sheet of assertions, could be transformed into another with an obelus over it, then the latter, were it written by itself, without the obelus, could be transformed into the former covered with an obelus.*

Demonstration. Let x be the former, y the latter quantity. Then it is assumed that x, were it asserted, could be transformed into $\bar{y}$ without loss of truth. It follows, then, from V, that $\bar{\bar{y}}$ could be transformed into $\bar{x}$ without loss of truth. But by III, y, were it written on a sheet of assertions, could be transformed into $\bar{\bar{y}}$ without loss of truth. Therefore, y could be, by the two steps, transformed into $\bar{x}$ without loss of truth.

287. VII. *If a quantity, were it written with an obelus over it on a sheet of assertions, could be transformed, obelus and all, into another quantity, without loss of truth, then the latter, were it written with an obelus over it, could be transformed, obelus and all, into the former without its obelus.*

Demonstration. Let x be the former, y the latter quantity. Then we assume that $\bar{x}$ could, in all cases, be transformed into y, without loss of truth; and I am to prove that $\bar{y}$ could, in all cases, be transformed into x without loss of truth. Since $\bar{x}$ could be transformed into y, it follows from V that $\bar{y}$ could be transformed into $\bar{\bar{x}}$. But by IV, $\bar{\bar{x}}$ could then be transformed into x; so that $\bar{y}$ would be transformable into x.

288. VIII. *If a quantity, were it written on a sheet of assertions under an obelus, could be transformed, without loss of truth, into another quantity under the same obelus, then, were the latter written without the obelus, it could, without loss of truth, be transformed into the former (without the obelus).*

Demonstration. Let x represent the former, y the latter quantity. Then, we assume that $\bar{x}$ could be transformed into $\bar{y}$, without loss of truth; and I am to show that, under that assumption, y could be transformed into x, without loss of truth. By VI, y could be transformed into x, and by IV, $\bar{\bar{x}}$

could be transformed into x. Hence, y could be transformed, in two steps, into x.

289. IX. *Any composite which can be written could, were it written on a sheet of assertions, be transformed, without loss of truth, by the omission of any component with its compositor.*

Demonstration. A composite can only be written by writing one component after another, singly. It will therefore be sufficient to show that, if the proposition is true for any given composite, it will be true for every composite resulting from the compounding with that composite of an additional component; provided, that I further show the proposition to be true for every composite with the writing of which the writing of any composite begins. For having proved *that*, I shall have shown that the proposition is true for every composite which can result from the successive affixation of components. For if, notwithstanding this reasoning, a composite could be written of which the proposition were not true, let

$$a \cdot b \cdot c \cdot d \ldots . \cdot k \cdot l \cdot m \cdot n$$

represent such a composite. Now if this composite can be written, it manifestly can be written by first writing a, then affixing $\cdot b$, then $\cdot c$ and so on, that is, by a series of changes each of which consists merely in affixing a new component with its compositor.[1] But if the proposition is true for the first compound, $a \cdot b$, so resulting, and none of the steps of the series of operations renders it false for the result of that step, then it never can have been rendered false at all, and must be true for the composite $a \cdot b \cdot c \cdot d \cdots \cdot k \cdot l \cdot m \cdot n$. Now, if the proposition is false, there must be some single composite for which it is false. It cannot then be false, if I prove that no affixation of a single component can render it false, and if I further prove that it is true at the beginning of writing any composite whatsoever. This general method of proof was invented by the great mathematician Pierre de Fermat* (1601–1665).

Let then $a \cdot b \cdot c \cdots \cdot k \cdot l \cdot m$ represent any composite whatever which can be written and of which it is true either that

[1] It will be remarked that the most logical treatment of associativeness and commutativeness is here considered to be that of ignoring them altogether.

* *Oeuvres de Fermat*, t. III, pp. 431–436, Paris, (1894).

this composite is false, or that the composite which results from it by erasing any component with an adjacent compositor is true. Then I say that, no matter what quantity n may be, it is true of the composite $a \cdot b \cdot c \cdots \cdot k \cdot l \cdot m \cdot n$ that either it is false, or every composite which results from erasing from this composite any component with an adjacent compositor is true.

First, then, suppose $a \cdot b \cdot c \cdots \cdot k \cdot l \cdot m$ to be false. Then, $a \cdot b \cdot c \cdots \cdot k \cdot l \cdot m \cdot n$ must be false. For otherwise, the latter would be true, and consequently have the value **v**. But in this case, by the first clause of the definition of composition $a \cdot b \cdot c \cdots \cdot k \cdot l \cdot m$ would be equal to **v**, and therefore true and not false, contrary to the hypothesis.

Secondly, consider the alternative, namely that $a \cdot b \cdot c \cdots \cdot k \cdot l \cdot m$, as well as every composite resulting from the omission from $a \cdot b \cdot c \cdots \cdot k \cdot l \cdot m$ of a component with its adjacent compositor, is true. Then every such composite will have the value **v**; and if n likewise has the value **v**, by the third clause of the definition of composition, every composite resulting from $a \cdot b \cdot c \cdots \cdot k \cdot l \cdot m \cdot n$ by the omission of one component will have the value **v**, and consequently, will be true. But if n has not the value **v**, then, by the second clause of the definition of composition, the composite $a \cdot b \cdot c \cdots \cdot k \cdot l \cdot m \cdot n \cdot$ will not be **v**, and will therefore not be true, but false.

The simplest form of composite, with which the writing of any other must begin, is that in which there are but two components. By the first and second clauses of the definition of composition, if this is true, that is, equal to **v**, then each component is equal to **v**, that is, is true. Therefore, in this case either component can be erased without loss of truth; and the demonstration is thus completed.

290. X. *Every quantity written on a sheet of assertions may be transformed without loss of truth, by the insertion of an aggregator with any quantity whatever as aggregant; and if any aggregate which can be written out is written on a sheet of assertions, any additional aggregant, with its aggregator, may be anywhere inserted, without loss of truth.*

Demonstration. The demonstration is altogether similar to that of IX. Namely, suppose that $a \dotplus b \dotplus c \dotplus \cdots \dotplus k \dotplus l \dotplus m$ be an aggregate which can be written out, of which this proposition is true; that is to say, that if it be written on a sheet of assertions, that assertion is true, if it be possible to find an aggregant which, being omitted with its adjacent aggregator, the quantity remaining could be written on the sheet of assertions with truth. Then, I say, that the same thing is true, no matter what quantity n may be, of the aggregate $a+b+c+ \cdots +k+l+m+n$; and moreover I aver that the proposition holds of every aggregate with which the writing of an aggregate begins, namely of every aggregate of two aggregants only.

For assume that the proposition is true of $a+b+c+ \cdots +k+l+m$; that is to say, that either this is true or the quantity which results from the omission from it of any aggregant with an adjacent aggregator is false. Then, I say, of the aggregate $a+b+c+ \cdots +k+l+m+n$, that it either is true, or else that no matter what single aggregant of it with an adjacent aggregator be omitted, the resulting quantity, being written on a sheet of assertions, makes a false assertion. Consider, in the first place, the case in which $a+b+c+ \cdots +k+l+m$ is true, that is equals **v**. Then it does not equal **f** and consequently, by the first clause of the definition of aggregation, $a+b+c+ \cdots k+l+m+n$ cannot equal **f**, but must equal **v**, that is, must be true.

Consider next the other alternative, that both $a+b+c+ \cdots +k+l+m$ and also every quantity resulting from the omission from it of an aggregant and an adjacent aggregator are false. Then, these are all equal to **f** and if, besides, n is equal to **f**, by the third clause of the definition of aggregation, every quantity which results from the omission from $a+b+c+ \cdots k+l+m+n$ is also equal to **f**, is therefore false. If, however, n is not equal to **f**, then by the second clause of the definition of aggregation $a+b+c+ \cdots k+l+m+n$ is not equal to f, and must be equal to **v** and hence, when written on a sheet of assertions, must be true.

But if an aggregate has two aggregants only, then, by the first and second clauses of the definition of aggregation, if either of those aggregants is true, and so equal to **v**, and not to **f**, then the aggregate cannot have the value **f**, but must have the value **v**, and when written on a sheet of assertions must constitute a true assertion.

Corollary. It follows that every composite is true or equal to **v** only in case all its components are so; being false, or equal to **f**, if any one of its components is so; while an aggregate is true, or equal to **v**, if any one of its aggregants is so, and is only false, or equal to **f**, if all its aggregants are so. But this applies only to composites and aggregants which can be written out.

291. XI. *If every one of the components of a composite might, with truth, be written upon a sheet of assertions, the composite itself can be so written, if it can be written out.*

Demonstration. I shall use the Fermatian method. Assume that it is true of the composite (which can be written out) $a \cdot b \cdot c \cdots \cdot k \cdot l \cdot m$. Then it is also true of the composite $a \cdot b \cdot c \cdots \cdot k \cdot l \cdot m \cdot n$, by the third clause of the definition of composition. It is true of every composite of two components by the same clause. Hence it is always true.

292. XII. *If an aggregate might, with truth, be written out upon a sheet of assertions, then some one of its aggregants might be so written.*

Demonstration. Let $a \dotplus b \dotplus c \dotplus \cdots \dotplus k \dotplus l \dotplus m$ be an aggregate of which it is true either that it cannot be written with truth upon a sheet of assertions or that some aggregant of it might be so written (although we may not know which one). Then the same will be true of the aggregate $a \dotplus b \dotplus c \dotplus \cdots \dotplus k \dotplus l \dotplus m \dotplus n$. For consider, first, the alternative that $a \dotplus b \dotplus c \dotplus \cdots \dotplus k \dotplus l \dotplus m$ cannot be asserted with truth. Then its value is not **v** but **f**. If then n is likewise **f**, it follows from the third clause of the definition of aggregation that $a \dotplus b \dotplus c \dotplus \cdots \dotplus k \dotplus l \dotplus m \dotplus n$ is **f** and so could not be asserted with truth. But if n is not **f**, then this is an aggregant of $a \dotplus b \dotplus c \dotplus \cdots \dotplus k \dotplus l \dotplus m \dotplus n$ which can be asserted with truth. The other alternative is that some aggregant of $a \dotplus b \dotplus c \dotplus \cdots \dotplus k \dotplus l \dotplus m$ can be asserted with truth. But that aggregant will equally be an aggregant of $a \dotplus b \dotplus c \dotplus \cdots k \dotplus l \dotplus m \dotplus n$. Finally, if there are but two aggregants, the proposition is true by the third clause of the definition of aggregation.

293. XIII. *As long as the only signs used are those described, any quantity which, if written alone on a sheet of assertions, could be transformed into a certain other, without loss of truth, can be so transformed wherever it occurs as a part of*

a directly asserted quantity, so long as it is not under an obelus.

Demonstration. Let x be a quantity which, if asserted to have the value **v**, could be replaced by y without loss of truth.

Then, in the first place, if x is a component of a directly asserted composite, it can be replaced by y without loss of truth. For if any of the other components is equal to **f**, and so is false, by IX, the composite is not true, and is therefore not capable of sustaining a loss of truth. But if every other component is equal to **v**, and this component is also **v**, then by XI, the composite is likewise **v**; while if this component is **f**, by X, the composite is **f**. Thus, the value of the whole is the same as that of this component x; and if this component is transformed into y, which is necessarily **v** if x is **v**, the whole composite remains **v**.

Next, suppose that that x is an aggregant of the asserted quantity. Then, if any of the other aggregants is **v**, the aggregate will be **v**, whatever transformation x may undergo. But if all the others are **f**, and x is **f**, the aggregate, will, by XII, be **f**, while if x is **v**, by X, the aggregate will be **v**. Thus the value of the whole aggregate will be the same as that of x; and if x is replaced by y, which is necessarily **v** if x is **v**, the value of the aggregate will remain **v**.

Thus the transformations of an aggregant, as well as those of a component, follow the same rules as those of an entire asserted quantity; and consequently, if x be an aggregant of a component, or a component of an aggregant, or be in any other relation to the asserted quantity describable by the alternate use of component and aggregate, as it must be if we use no other signs than those described, and if x is not under an obelus, then it is subject to the same rules of transformation as if it were asserted alone.

294. XIV. *As long as the only signs used are those described, any quantity into which another could be transformed without loss of truth, if the latter were asserted alone, can, if it occurs anywhere under a single obelus in a directly asserted quantity, be there transformed into that latter.*

Demonstration. Let x and y represent the two quantities, so that in the assertion that x has the value **v**, x could be replaced by y, without loss of truth. Then, I say that if, in a directly asserted quantity, y occurs anywhere under a single obelus, it can be transformed into x without loss of truth. For let Y be the entire expression which is under the same obelus as y; and let X be what Y would become if y were replaced by x. Then, by XIII, if X were asserted, it could be transformed into Y. And consequently, by V, $\overline{\mathrm{Y}}$, if it were asserted alone could be transformed into $\overline{\mathrm{X}}$. Hence, by XIII, situated as $\overline{\mathrm{Y}}$ is, not under any obelus, it can be transformed into $\overline{\mathrm{X}}$. But this transformation consists only in replacing y by x under a single obelus.

295. *Definition.* Let us say that two operations are *internal negatives* of one another, if and only if, either gives a result of contrary value to the result of the other when whatever quantities it operates upon are of contrary value to those operated upon by that other; and let us write an obelus over a sign of the combination of two quantities to signify the internal negative of that sign.

Then we shall have the following pairs of internal negatives

:∨:– and –(:∨:–)
ψ or ∨ and ·
λ and ʎ
—< and –·
∨– and ·–
≡ and – (≡)
x and $\bar{x}$
y and $\bar{y}$

The obelus and vinculum will each be its own internal negative.

Since a quantity operates upon nothing, its internal and negative will be its own negative; that is x and $\bar{x}$, **v** and **f** will be internal negatives of each other.

296. XV. *Every expression of a quantity has the contrary value to the expression which results from the substitution in it of the internal negative of each quantity and operation.*

Demonstration. For, in the first place, this is true regarding single letters. For let x represent any single letter. Then, since x operates upon nothing, $\bar{x}$ gives the contrary value to x when it operates upon whatever x operates upon. In the second place, it is true of expressions operated upon by the vinculum and obelus. For since x has the contrary value to $\bar{x}$, the vinculum is its own internal negative; and since $\bar{x}$ has the contrary value to $\bar{\bar{x}}$, the obelus is its own internal negative. In the next place, the proposition is true of any operator upon two operands; for, by the definition, xOy has the contrary value to that of $\bar{x}\bar{O}\bar{y}$. Hence, if y is $u \Diamond v$, $xO(u \Diamond v)$ has the contrary value to $\bar{x}\bar{O}(\bar{u}\bar{\Diamond}\bar{v})$. And so if $y = \bar{w}$, $xO\bar{w}$ has the contrary value to $\bar{x}\bar{O}w$. And by Fermatian reasoning, it is evident that this will be so in every case.

297. XVI. *Every necessarily true proposition concerning the values, or relations between the values, of quantities of dichotomic algebra will remain necessarily true after it has been modified by everywhere interchanging* **f** *and* **v**, *and internal negatives of operators upon two letters, and when categorical affirmation and denial, protasis and apodosis of the same conditional sentence, copulation and disjunction are likewise interchanged.*

Demonstration. I have to prove that if a certain form of algebraic expression is necessarily true, then if that form is modified by the interchange of **f** and **v**, of · and ✢, of ≡ and −(≡), of —< and —·, etc., it becomes necessarily false; and further it being necessarily true, that from the truth of one expression the truth of another follows, it will be equally true that from the truth of the latter, modified as above, the truth of the former, modified in the same way, equally follows by necessity. Finally, I have to show that to say that if any quantity, A, is **v**, then both the quantities B and C are **v**, is the same as to say that if either $\bar{\text{B}}'$ or $\bar{\text{C}}'$ is **v**, then $\bar{\text{A}}'$ is **v**; where, $\bar{\text{A}}'$, $\bar{\text{B}}'$, $\bar{\text{C}}'$ are the above-described modifications of A, B, C.

I first remark that to call a proposition concerning the values, or relations between the values, of quantities of dichotomic algebra *necessarily* true is to say that it is true whatever be the values of the quantities to which it relates. If, then, a proposition is necessarily true, it remains so when for all the single ordinary letters (not **v** and **f**) that are mentioned in its statement, the negatives of these letters are substituted. But if this substitution be made, and the internal negative of each combination be then substituted for the combinations themselves, the result will be the same modification which is described in the enunciation of this theorem. For the single letters will be restored to their original conditions with respect to having obeli over them, but **f** and **v** will be interchanged; the vincula and obeli will remain unchanged; and the signs of operations upon two quantities will be changed into their internal negatives. The quantities described by the combinations so modified will thus have values contrary to the values of the corresponding unmodified combinations, after the obeli are applied to the single letters. Consequently, if we make the proper changes in what is said of them, the initial proposition which was necessarily true will remain necessarily true of these so modified combinations. In particular, if the original proposition represents the assertion that a combination, C, is necessarily true, or has the value **v**; and if C′ is the modification produced by putting obeli over the single letters of C, then that C′ is necessarily true, or has the value **v**, will be equally true; and consequently, that $\bar{\text{C}}'$ is necessarily false will be equally true; where $\bar{\text{C}}'$ will be the modification of C described in the enunciation. So if the necessarily true proposition is that if a certain combination, A, is true, then a certain combination, B, is true, it remains equally necessary that if A′ is true B′ is true; and consequently, equally true that if $\bar{\text{B}}'$ is true (and therefore B′ is not true) then $\bar{\text{A}}'$ will be true (*i.e.*, A′ will not be true). So, if the original proposition be that if A is true, then both B and C are true, and [if] this be necessarily true, it is equally necessary that if A′ is true, then both B′ and C′ are true, and consequently, [it] is equally necessary that if either $\bar{\text{B}}'$ or $\bar{\text{C}}'$ be true (so that either B′ or C′ is false), then $\bar{\text{A}}'$ is true (or, A′ is false). The necessary truth of the proposition is thus made plain.

298. XVII. *If a quantity have the value* **v**, *then this quantity may be inserted as a component of the whole or any part of an asserted quantity, and any component of a part (or the whole) of an asserted quantity, can be repeated, without loss of truth, as a component of this part or of any part of it.*

Demonstration. By the first clause of the definition of composition if *a* ·**v** has the value **v**, so has *a*; and by the third clause of the same definition, if *a* has the value **v**, so has *a* ·**v**. Thus *a* and *a* ·**v** have in all cases the same value, and the substitution of the latter for the former in any part of an expression must be without effect upon the value of it, since the value of an expression of a given form depends exclusively on the values of its parts.

Consequently, if a has the value **v**, its introduction anywhere as a component cannot change the value of the expression into which it is introduced; but if a has the value **f**, and is a component, the composite has in any case the value **f**, whatever be done to the other components. Thus, its introduction, as a component of a part of the composite of which it is a component, can never alter the value.

Corollary. It follows, by XVI, that a quantity having the value **f** may be introduced as an aggregant into any part of an assertion; and further, that any quantity, repeated as an aggregate of any part of any part of an assertion of which it is an aggregant, may be omitted in that inner repetition.

Corollary. We may place here the propositions that composition and aggregation are commutative and associative.

Given $a \cdot b$, we can, by insertion, write $b \cdot a \cdot b$, for if $a \cdot b$ is **v**, so is b, and if b and $a \cdot b$ are **v**, so is $b \cdot a \cdot b$. But then, by omission, we get $b \cdot a$. For if $b \cdot a \cdot b$ is **v**, so is $a \cdot b$, and if $a \cdot b$ is **v**, so is a. Also, if $b \cdot a \cdot b$ is **v**, so is b; and if both b and a are **v**, so is $b \cdot a$.

Again, given $(a \cdot b) \cdot c$, we get by insertion $(a \cdot b \cdot c) \cdot c$. For if $(a \cdot b) \cdot c$ is **v**, so is c; and if c is **v**, $b \cdot c$ is **v**, if b is **v**; and if $a \cdot b \cdot c$ is **v** if $a \cdot b$ is **v**, and if $(a \cdot b) \cdot c$ is **v**, so is $(a \cdot b \cdot c) \cdot c$. But then by omission we get $a \cdot b \cdot c$.

Since composition has these properties, so, by XVI has aggregation.

299. XVIII. *The negative of a component may be introduced into any part of its composite as an aggregant, without loss of truth.*

Demonstration. For if a has the value **v**, its negative is **f**, and as an aggregant can affect the value of nothing. But if a has the value **f**, being a component, the composite is **f**, no matter how the other components be transformed.

Corollary. It follows that if the negative of an aggregant occurs in any part of the aggregate as a component, it may be omitted.

300. XIX. *A quantity which is a component of every aggregant of an aggregate may be introduced as a component of*

the whole, without altering the value; and it may then be omitted from the aggregants, without altering the value.

Demonstration. I have to prove that $a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc. has the same value as $(a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc.$)\cdot x$, and also the same value as $(a +\!\!, b +\!\!, c +\!\!,$ etc.$)\cdot x$.

First, suppose that the value of $a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc. is **v**. Then, by the third clause of the definition of composition, the value of $(a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc.$)\cdot(a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc.$)$ will also be **v**.

But, then, omitting components, $(a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc.$)$ $\cdot(x +\!\!, x +\!\!, x +\!\!,$ etc.$)$ will also be **v**. But, then, $(a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc.$)\cdot x$ will be **v**. For if $(a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc.$)\cdot(a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc.$)$ is **v**, then by the second clause of the definition of composition, $a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc. is **v**; and then by the third clause of the definition of aggregation, either $a\cdot x$ or $b\cdot x$ or $c\cdot x$ etc. is **v**; and then, by the second clause of the definition of composition, x is **v**; and then, since by the first clause of the definition of composition, $(a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc.$)\cdot(a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc.$)$ is **v**, $a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc. is **v**, it follows from the third clause of the same definition that $(a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc.$)\cdot x$ is **v**. And further, if $(a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc.$)\cdot x$ is **v**, it follows, by omission of components that $(a +\!\!, b +\!\!, c +\!\!,$ etc.$)\cdot x$ is **v**.

Secondly, suppose that $(a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc.$)$ is **f**. Then, by the first clause of the definition of composition, $(a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc.$)\cdot x$ is **f**. And further, $a\cdot x$ and $b\cdot x$ and $c\cdot x$ etc. have by the first and second clauses of the definition of aggregation, all severally the value **f**. Then, if x is **v**, by the third clause of the definition of composition, a, b, c, etc. have all severally the value **f**, and by the third clause of the definition of aggregation, $a +\!\!, b +\!\!, c +\!\!,$ etc. has the value **f**; and by the first clause of the definition of composition $(a +\!\!, b +\!\!, c +\!\!,$ etc.$)\cdot x$ has the value **f**. Suppose, on the other hand, that x has the value **f**. Then, by the second clause of the definition of composition $(a +\!\!, b +\!\!, c +\!\!,$ etc.$)\cdot x$ has the value **f**.

Thus, in every case, $a\cdot x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc. and $(a\,x +\!\!, b\cdot x +\!\!, c\cdot x +\!\!,$ etc.$)\cdot x$ and $(a +\!\!, b +\!\!, c +\!\!,$ etc.$)\cdot x$ have the same value.

Corollary. It follows that

$$
\begin{aligned}
&(a +\!\!, x)\cdot(b +\!\!, x)\cdot(c +\!\!, x) \text{ etc.}\\
&= (a +\!\!, x)\cdot(b +\!\!, x)\cdot(c +\!\!, x)\cdot \text{ etc. } +\!\!, x\\
&= (a +\!\!, b +\!\!, c +\!\!, \text{ etc.})\cdot x.
\end{aligned}
$$

When we come to the algebra of logic, there will be a highly important remark to make concerning this theorem.

301. The above are, I believe, all the theorems of dichotomic algebra with which it is worthwhile to trouble the reader. There are, however, a few problems to be considered. Of these, I shall give those methods of solution which seem to me to be upon the whole the most useful, taking into consideration something besides their brevity in very complicated cases — making this, indeed, decidedly a secondary consideration, in view of the excessive rarity of cases in which the reader will ever have occasion to apply the algebra to complicated problems, and in view of the very moderate degree of mathematical ingenuity requisite to clearing away the complexity even from these few. What seems to me desirable is that the procedure should have that kind of simplicity which makes it easy to remember or to reconstruct if its details are forgotten.

The first problem is to put an expression into such a form that a certain letter appears only as an aggregant of a component, or as the negative of an aggregant of a component. I first gave a general solution of this problem which I do not think can be improved upon.* Let x be the letter in question; and let the expression be Fx. Then,

$$\mathrm{F}x = (\mathrm{Ff} +\!\!\!\prec x)\cdot(\mathrm{Fv} +\!\!\!\prec \bar{x}).$$

I must say that there was little originality in this solution, since it was but the reciprocal of a proposition of Boole's. If it is desired to separate *two* letters in this way, we have

$$\mathrm{F}(x, y) = (\mathrm{Ff, f} +\!\!\!\prec x +\!\!\!\prec y)\cdot(\mathrm{Ff, v} +\!\!\!\prec x +\!\!\!\prec \bar{y})$$
$$(\mathrm{Fv, f} +\!\!\!\prec \bar{x} +\!\!\!\prec y)\cdot(\mathrm{Fv, v} +\!\!\!\prec \bar{x} +\!\!\!\prec \bar{y})$$

The procedure for a greater number of letters will be similar. But it is never really necessary to separate more than one.

The proof is simple enough. If Fx has the value v, either x has the value v, when Fx becomes Fv or $\bar{x}$ has the value v, so that, in any case, $\mathrm{Fv} +\!\!\!\prec \bar{x}$ has the value v; and further, either $\bar{x}$ has the value v, when Fx becomes Ff or x has the value v, so that, in any case, $\mathrm{Ff} +\!\!\!\prec x$ has the value v. Consequently, if Fx has the value v, the composite $(\mathrm{Fv} +\!\!\!\prec \bar{x})\cdot(\mathrm{Ff} +\!\!\!\prec x)$ has the

* See 3.9, (18′).

same value. But if Fx has the value f, either x has the value v, when Fx becomes Fv, and $\mathrm{Fv} +\!\!, \bar{x}$ has the value f, or x has the value f, when Fx becomes Ff, and $\mathrm{Ff} +\!\!, x$ has the value f. Thus, if Fx has the value f, one or other of the components of $(\mathrm{Fv} +\!\!, x)\cdot(\mathrm{Ff} +\!\!, x)$ has the value f, and again $(\mathrm{Fv} +\!\!, \bar{x})\cdot(\mathrm{Ff} +\!\!, x)$ has the same value as Fx.

Boole's reciprocal problem is to put an expression into a form in which a given letter or its negative appears only as a component of an aggregant. The solution is

$$\mathrm{F}x = (\mathrm{Fv})\cdot x +\!\!, (\mathrm{Ff})\cdot\bar{x}.$$

We now have to take a step which I first took in my memoir dated, 1870 Jan. 26.[1] Let us start from the sixteen assertions concerning the values of two dichotomic quantities, and take the signs of the operations they involve in a new sense, which may be distinguished by a dot over these signs, regarding these operations themselves as quantities, and at the same time operators upon the second of the two quantities, thereby producing the first. Then, the four $\dot{:}$, $\cdot\dot{-}$, $\dot{-}\cdot$ $\dot{\curlywedge}$, will by aggregation give all the rest, except $-(:\vee:-)$. Thus,

$\dot{\equiv}$ will be $\dot{:} +\!\!, \dot{\curlywedge}$ since $x \equiv y = (x\cdot y) +\!\!, (x \curlywedge y)$

$-(\dot{\equiv})$ will be $\cdot\dot{-} +\!\!, \dot{-}\cdot$ since $-(x \equiv y) = (x\cdot\bar{y}) +\!\!, (\bar{x}\cdot y)$

$\dot{y}$ will be $\dot{:} +\!\!, \dot{-}\cdot$ since $y = (x\cdot y) +\!\!, (\bar{x}\cdot y)$

$\dot{x}$ will be $\dot{:} +\!\!, \cdot\dot{-}$

$\dot{\bar{y}}$ will be $\cdot\dot{-} +\!\!, \dot{\curlywedge}$

$\dot{\bar{x}}$ will be $\dot{-}\cdot +\!\!, \dot{\curlywedge}$

$\dot{\vee}$ will be $\dot{:} +\!\!, \cdot\dot{-} +\!\!, \dot{-}\cdot$

$\dot{\curlywedge}$ will be $\dot{\curlywedge} +\!\!, \cdot\dot{-} +\!\!, \dot{-}\cdot$

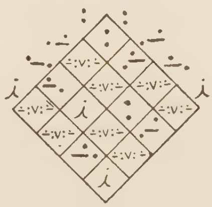
$-\!\!\dot{<}$ will be $\dot{:} +\!\!, \dot{\curlywedge} +\!\!, \dot{-}\cdot$

$\dot{\vee}-$ will be $\dot{:} +\!\!, \dot{\curlywedge} +\!\!, \cdot\dot{-}$

$(:\vee:-)$ will be $\dot{:} +\!\!, \cdot\dot{-} +\!\!, \dot{-}\cdot +\!\!, \dot{\curlywedge}$

[1] *Memoirs of the American Academy of Arts and Sciences, IX*, pp. 317–378 [Vol. 3, No. 3]; also with a separate title page and paging of its own, the title being *Description of a Notation for the Logic of Relatives, resulting from an Amplification of the Conceptions of Boole's Calculus of Logic.* My calling De Morgan's *logic of relations* by a slightly different name, for no better reason than that all logic treats of relations, was a youthful piece of bad manners of which I am now heartily ashamed. My work was due, of course, to the combined study of Boole's *Laws of Thought*, 1854, De Morgan "On the Syllogism and the Logic of Relatives" (*Cambridge Philosophical Transactions*, X, 1860,

302. Now mathematicians have long ago agreed upon generalizing the meaning of the word 'multiplication' so as to make it signify the operation of applying, as multiplier, an operation to the result of another operation, which last operation is regarded as the multiplicand. I identify this operational multiplication, which is commonly called *functional* multiplication, with my *relative* multiplication, which is the operation of so combining a relative term — such as '*lover of*' — as multiplier, with a correlate as multiplicand, so as to yield, as product, the relate which is in the relation signified by the relative term to the object indicated as correlate. A mathematical operator is nothing but a mathematical relative term. Then, the relative, or functional, multiplication table of :, ·−̇, −̇·, ⅄̇ will be as here shown.



303. My father afterward, but independently, obtained two multiplication tables of similar form in his *Linear Associative Algebra* (1st Ed., pp. 30, 59; 2nd Ed. [*American Journal of Mathematics*, 1881], edited by me, pp. 111, 132), which I showed* was due to their being, in essence, the same thing. He further discovered that by means of ordinary imaginary algebra, quaternions, or rather Hamilton's biquaternions, can be put into this form.† Namely, if we put

$$:+\dot{⅄} = 1$$
$$(:-\dot{⅄})\sqrt{-1} = i$$
$$(\cdot\dot{-}+\dot{-}\cdot)\sqrt{-1} = j$$
$$(\cdot\dot{-}-\dot{-}\cdot) = k$$

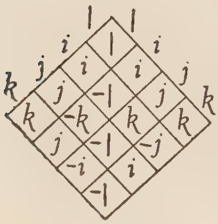
we get the multiplication table of quarternions, which is as here shown.

April 23). I interested my father in the subject, and his *Linear Associative Algebra* was issued to his friends before the printing of my memoir was complete. We were, therefore, working simultaneously upon closely related subjects, and continually discussing them together; and consequently, it is impossible to say precisely what was due to each. Of course, in mathematics, he was my master, and vastly my superior in genius; so that, in case of doubt, it is safer to attribute any mathematical step to him.

* See 3.126f.

† See 3.647.

304. This is only interesting here as showing how dichotomic mathematics may influence higher orders of mathematics. But quaternions proper does not admit of imaginary coefficients; and biquaternions is essentially a different thing. Quaternions is a quaternary, or tetrachotomic, algebra of which [I gave] the proper representation in February, 1882.*



305. If the operations are to be considered as quantities, what we have considered, and what are, the quantities of dichotomic algebra, sink to the rank of *umbræ*, or ingredients of quantities. It was Leibniz† who in 1693, April 28, first introduced into algebra this conception of ingredients of quantities; and unfortunately, he neglected to provide a suitable name for them. It so happened that it was not until three half centuries later, when they had been used by a hundred writers, at the very least, and had figured in familiar textbooks, that the seething brain of Sylvester allowed him to claim them as his own invention, and to bestow upon them the name of *umbræ*.‡ The name could not well be more inappropriate; for whoever heard of shadows conspiring to create a substance? Besides, there is nothing to prevent umbræ being identified with ordinary quantities — or rather all ordinary quantities being identified with some umbræ: such a step is, often, a most useful mathematical generalization. An *umbra*, or better, an *ingredient* of a quantity, is a logical symbol, a set of which systematically, and from a logical point of view, describes a quantity, without any necessary reference to its value. For example, if the velocities of two couriers are denoted respectively by u_1 and u_2, then u, 1, and 2, are umbræ, or ingredients, of the quantities u_1, and u_2. The first example given by Leibniz was of three general simultaneous linear equations between two quantities, which he wrote thus:

$$10+11x+12y=0$$
$$20+21x+22y=0$$
$$30+31x+32y=0$$

* See vol. 3, No. X.

† *Mathematische Schriften*, 2 Abt., 3 Bd., S. 160–3.

‡ *Collected Mathematical Papers*, vol. I, pp. 241–50.

Here the numbers have nothing to do with the values of the coefficients, which may be anything. The first figure shows simply what equation is referred to and the second what term of that equation. My particular umbral notion of 1870* for relative terms, which has been generally approved, was A:B, where A and B are individual objects, and A:B is that operation upon B which produces A, but operating upon any other individual than B produces **f**, even if that other individual have the same value as B. Nevertheless, since values, in all sorts of algebra, are singular and definite objects, to which the principles of contradiction and excluded middle apply, there is nothing to prevent our taking the colon in a special sense, so that A:B shall operate, not upon B logically considered, but upon the value of B, giving the value of A. It is in this sense that we may write

$$: = \mathbf{v}:\mathbf{v} \qquad \dot{\curlywedge} = \mathbf{f}:\mathbf{f}$$

$$\cdot\dot{-} = \mathbf{v}:\mathbf{f} \qquad \dot{-}\cdot = \mathbf{f}:\mathbf{v}. \ldots$$

306. If we identify **v** with $\dot{x}$, and **f** with $\dot{\bar{x}}$, the distinction between relative multiplication and composition will disappear in regard to the ordinary quantities of dichotomic algebra. For just as

$$\begin{array}{lll} \mathbf{v}\cdot\mathbf{v}=\mathbf{v} & \text{so} & \dot{x}:\dot{x}=\dot{x}\dagger \\ \mathbf{v}\cdot\mathbf{f}=\mathbf{f} & & \dot{x}:\dot{\bar{x}}=\dot{\bar{x}} \\ \mathbf{f}\cdot\mathbf{v}=\mathbf{f} & & \dot{\bar{x}}:\dot{x}=\dot{\bar{x}} \\ \mathbf{f}\cdot\mathbf{f}=\mathbf{f} & & \dot{\bar{x}}:\dot{\bar{x}}=\dot{\bar{x}} \end{array}$$

But in regard to these new quantities, the distinction will be maintained. For example,

$$\dot{-}\!< :^{\dagger} \dot{\vee}- = \dot{\curlywedge}. \quad (\dot{-}\!<\cdot\,\dot{\vee}-) = (\dot{\equiv}) = (\dot{\vee}-\cdot\ \dot{-}\!<). \quad \dot{\vee}- :^{\dagger} -\!< = \cdot.$$

I shall not further enlarge upon this matter at this point, although the conception mentioned opens a wide field; because it cannot be set in its proper light without overstepping the limits of dichotomic mathematics.

* See 3.123.

† : here represents relative multiplication.

§4. TRICHOTOMIC MATHEMATICS[P]

307. We have already, along one line, traversed the marches between dichotomic and trichotomic mathematics; for the general idea of operational multiplication is as purely triadic as it could well be, involving no ideas but those of the triad, operator, operand, and result. Relative multiplication, however, involves a marked dichotomic element since (A:B):(C:D) is one of the two, **f** or A:D, according as (B:C) is one of the two **f** or **v**.

308. Trichotomic mathematics is not quite so fundamentally important as the dichotomic branch; but the need of a study of it is much greater, its applications being most vital and its difficulties greater than the dichotomic. Nevertheless, it has received hardly any direct attention. The permutations of three letters have, of course, been noticed, along with other permutations. The theory of the cubic equation is fully made out; along with those of plane and twisted cubic curves. There is also an algebra of novenions. In addition, considerable studies have been made in a particular province of trichotomic mathematics by logicians, without their recognizing the triadic character of the subject.

A trichotomic mathematics entirely free from any dichotomic element appears to be impossible. For how is the mathematician to take a step without recognizing the duality of truth and falsehood? Hegel and others have dreamed of such a thing; but it cannot be. Trichotomic mathematics will therefore be a 2 x 3 affair, at simplest.

309. I will begin this topic by a glance at some of the logico-mathematical generalities, without being too scrupulous about excluding higher numbers than three.

The most fundamental fact about the number three is its generative potency.* This is a great philosophical truth having its origin and rationale in mathematics. It will be convenient to begin with a little *a priori* chemistry.† An atom of helion, neon, argon, xenon, crypton, appears to be a medad (if I may be allowed to form a patronymic from μηδέν). Argon gives us, with its zero valency, the one single type

A.

* Cf. 1.347.

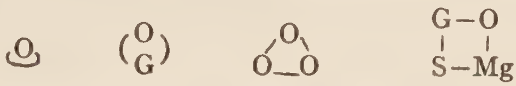
† Cf. 1.289f, 3.421, 3.469, 5.469.

Supposing H, L, Na, Ag, etc. and F, Cl, Br, I to have strictly unit valency (which appears not to be true; at least, not for the halogens), then they afford only the two types

H – H   H – F,

if these can be called two.

Assuming G (glucinum), etc. with O, S, etc., to have valency 2 (certainly not true), they might give an endless series of saturated rings, by themselves.

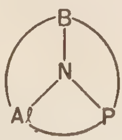


and so on, *ad infinitum*. With the monads, these dyads would give terminated lines.

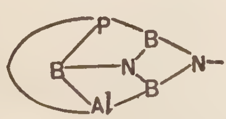
H–O–H   H–O–O–H   Cl–O–O–O–H.

and so on, *ad infinitum*. But they can give no other types than single rings and terminated lines.

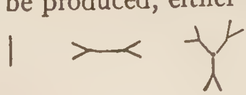
Triads, on the other hand, will give every possible variety of type. Thus, we may imagine the atom of argon to be really formed of four triads, thus

A = 

We may imagine the monadic atom to be composed of seven triads; thus;

H = 

A dyad will be obtained by breaking any bond of A; while higher valencies may be produced, either simply



or in an intricate manner.

One atom forms one type without a ring  , one with a one-atom ring  : two in all.

Two atoms form, in one piece, one acyclic type  , one with one protocycle  , one diprotocyclic type  , one monodeuterocyclic , one dideuterocyclic type  : five in all.

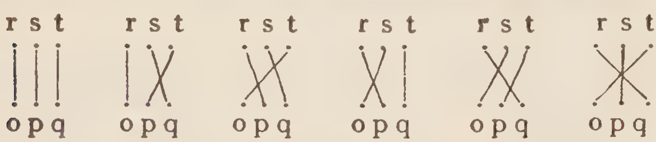
Three atoms form one acyclic type , one monoprotocyclic  , one diprotocyclic  , one monodeuterocyclic , one protodeuterocyclic  , one tritocyclic  one deuterotritocyclic : seven in all.

Four atoms form two acyclic types   , two monoprotocyclic  , two diprotocyclic   one triprotocyclic  , two monodeuterocyclic  , one dideuterocyclic , two monodeuteromonoprotocyclic  , one monodeuterodiprotocyclic  , one monotritocyclic  , one ditritocyclic  (where there is, of course, also a four-atom ring), one tritritocyclic  (so I name it, although there are four three-atom rings and three four-atom rings), one prototritocyclic  , one deuterotritocyclic  , one protodeuterotritocyclic  ; one tettarto-cyclic  , one monodeutero-tettartocyclic  , one dideuterotettartocyclic , one tritotettartocyclic  : twenty-three in all, if I have repeated none. With larger numbers of atoms the types multiply astonishingly.

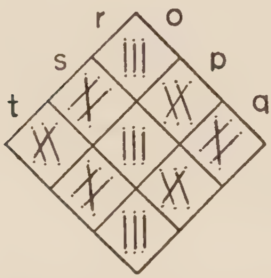
310. It would scarcely be an exaggeration to say that the whole of mathematics is enwrapped in these trichotomic graphs; and they will be found extremely pertinent to logic. So prolific is the triad in forms that one may easily conceive that all the variety and multiplicity of the universe springs from it, though each of the thousand corpuscles of which an atom of hydrogen consists be as multiple as all the telescopic heavens, and though all our heavens be but such a corpuscle which goes with a thousand others to make an atom of hydrogen of a single molecule of a single cell of a being gazing through a telescope at a heaven as stupendous to him as ours to us. All that springs from the

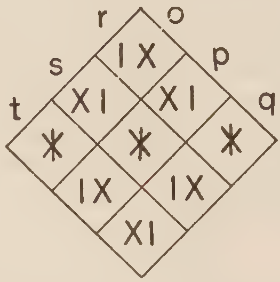
— an emblem of fertility in comparison with which the holy phallus of religion's youth is a poor stick indeed.

311. Let us now glance at the permutations of three things. To say that there are six permutations of three things is the same as to say the two sets of three things may correspond, one to one in six ways. The ways are here shown



No one of these has any properties different from those of any other. They are like two ideal rain drops, distinct but not different. Leibniz's "principle of indiscernibles" is all nonsense. No doubt, all things differ; but there is no logical necessity for it. To say there is, is a logical error going to the root of metaphysics; but it was an odd hodge-podge, Leibniz's metaphysics, containing a little to suit every taste. These arrangements are just like so many dots, as long as they are considered in themselves. There is nothing that is true of one that is not equally true of any other — so long as in the proposition no other is definitely mentioned. But when we come to speak of them in pairs, we find that *pairs* of permutations differ greatly. To show this let us make a table like that which we formed in dichotomic algebra. On one side we enter the table with *r*, *s* or *t*, on the other with [*o*, *p*, or *q*], and at the intersections of the rows we find the figure of the permutation in which the two correspond. In order to avoid putting two symbols in one square I repeat the table. We have then





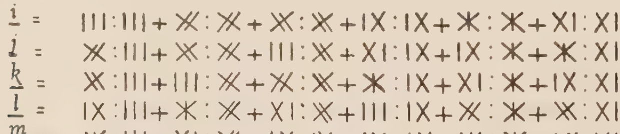
It will be seen that from this point of view, that of their relations to one another, the permutations separate themselves into two sets. In any one set, there are no two permutations which make the same letter correspond to the same letter; while of pairs of permutations of opposite sets, each agrees in respect to the correspondence of one letter.

Since there are six permutations, there could be 2^6, or sixty-four different assertions which might be made concerning an unknown two, as to what ones of the six they were. But from these sixty-four, it will be interesting to select a set of six, such that any one, being relatively multiplied by any other, the product is one of the six, and such that the product is different from what it would be if either multiplier or multiplicand were alone different. Not that this is to be true of all quantities of the algebra, but only of the six single letters. For the larger condition would render the problem impossible. Let the six letters conforming to this condition be i, j, k, l, m, n. Since every product of these letters is to be one of the letters, and since i^2, ji, ki, li, mi, ni are all different, it follows that some one of them must be equal to i. This one may or may not be i. Suppose, first, that it is not so; but that, say, $ji=i$. Then $j^2i=j(ji)=ji=i$. Thus j^2 is some letter which multiplied into i gives i. But since this is true of j, it cannot, by our hypothesis, be true of any other letter, and it must be that $j^2=j$. Thus, if i^2 is not i, there is some other letter whose square is the letter itself. We may assume, then,

i to be that letter; and we shall have $i^2=i$. Then what will any other letter, say j, give when multiplied by i? Since $i=i^2$, we have $ij=i^2j=i(ij)$. But according to our hypothesis, if j were not equal to ij, as the product, then ij could not be equal to $i(ij)$, as it is. Hence $ij=j$, and similarly $ik=k$, $il=l$, $im=m$, $in=n$; and by like reasoning $ji=j$, $ki=k$, $li=l$, $mi=m$, $ni=n$. It is plain, then, that no other quantity can have its square equal to itself. For, if $j=j$ while $ij=j$, we should have by our hypothesis $i=j$.

312. Since the product of two letters is a letter in every case, and since the number of letters is finite, it follows that some power of a letter is equal to some other power of the same letter. But suppose that $j^p=j^{p+q}=j^p{\cdot}j^q$. But j^p is equal to a letter and therefore, as we have seen, $j^{p\cdot i}=j^p$. Thus, $j^p{\cdot}i=j^p{\cdot}j^q$, where j^q is likewise equal to a letter. But it is assumed that multiplication is invertible for the letters; so that $j^q=i$. That is, some power of each letter is i. Moreover, j^{q-1} will also be equal to a letter; and $j{\cdot}j^{q-1}=j^{q-1}{\cdot}j=i$. There is, therefore, for each letter some letter which multiplied by or into it gives i as the product; and since multiplication is invertible for the letters, there will be no other letter that multiplied either by or into that letter will give i as the product. Consequently, if the product of two letters is i, each of them is equal to some power of the other. If, then, two letters, say j and k, are not powers of one another, their product cannot be a power of either; for were $j{\cdot}k=k^q$, since multiplication is invertible for the letters, we should have $j=k^{p-1}$, that is, j would be a power of k. If, then, there are two series of letters, the letters of each series, powers of one another, but no two letters of the one and the other powers of one another, then there must be a third series of which this is true; but any of these series may consist of a single square root of i. Of the five letters j, k, l, m, n, not more than three can be square roots of i. In the first place, all five of the letters j, k, l, m, n cannot be square roots of i. For if $j^2=k^2=i$ then jk must be a different letter from either j or k. Call it l. Since then $jk=l$, we have $lk=jk^2=ji=j$ and $jl=j^2k=ik=k$. Moreover, if $l^2=i$, that is, if $jkjk=i$, multiplying into k we have $kj=ikj=j^2kj=j^2kji=j^2kjk^2=j(jk{\cdot}jk)k=jl^2k=jik=jk$, so that $kj=jk=l$ and $lj=kj^2=ki=k$, $kl=k^2j=ij=j$. We thus

find the products of j into i, j, k, l to be j, i, l, k, leaving nothing for jm to be except n. For, by hypothesis, it must be some letter different from the product of j into any of the other letters. But in like manner, the products of k into i, j, k, l would be k, l, i, j respectively; and the products of l into i, j, k, l, would be l, k, j, i, respectively; so that in each case we could only have $jm = n$, $km = n$, $lm = n$, violating the hypothesis that multiplication is invertible for the letters. That just four of the five letters, j, k, l, m, n, cannot be square roots of i is quite obvious. For what root of i could the fifth one be? It could not be a fourth root of i, since then its cube must be a separate letter; it could not be any other root of i, since then its square must be a separate letter; while all the other letters would be preoccupied. Indeed, for the same reason no root of i higher than the cube root (except, of course, a sixth root) can exist among the five letters. No more can two independent cube roots of i; although this is perhaps a trifle less obvious. Thus, the only possible cases are, first, where one of the letters is a sixth root of i, of which there are one hundred twenty varieties, ὁι πολλοὶ, which vie with one another in their utter want of interest. The only remaining possible case is where there are three square roots of i and a cube root of i with its square, to which of course the same description applies. I here give the multiplication table.



If these signs were to be used in any general application, some of them would require modification to prevent their being mistaken for two or three characters. It will be remarked that i, j, k have each permutation in its own class indicated by the even or odd number of crossings in its character; while l, m, n reverse the class of each permutation.

313. In regard to the multiplication table, it is, in the first place, noticeable that if we take six new quantities, as follows:

$$i'' = \frac{(i\text{-}j)\text{-}(k\text{-}i)}{3} \qquad l'' = \frac{(l\text{-}m)\text{-}(n\text{-}l)}{3}$$

$$j'' = \frac{(j\text{-}k)\text{-}(l\text{-}j)}{3} \qquad m'' = \frac{(m\text{-}n)\text{-}(l\text{-}m)}{3}$$

$$k'' = \frac{(k\text{-}i)\text{-}(j\text{-}k)}{3} \qquad n'' = \frac{(n\text{-}l)\text{-}(m\text{-}n)}{3}$$

we have

$$i''\text{-}j'' = i\text{-}j \qquad j''\text{-}k'' = j\text{-}k \qquad k''\text{-}i'' = k\text{-}i$$

$$l''\text{-}m'' = l\text{-}m \qquad m''\text{-}n'' = m\text{-}n \qquad n''\text{-}l'' = n\text{-}l;$$

whence it follows that all formulæ concerning the unaccented differences have reciprocal formulæ exactly like them concerning accented differences. It is true that the reverse is not always true.

For example,

$$i''+j''+k''=o \qquad l''+m''+n''=o;$$

while, without the accents, these equations would not hold good. But this very fact will lead us, in due time, to an analogy, not only very pretty, but of the highest importance for logic.

It is to be added that the multiplication table of i'', j'', k'', l'', m'', n'' is precisely the same as that of i, j, k, l, m, n, except that accents are everywhere added. But it is seriously important, both for the sake of the purely ideal beauty of this algebra, and still more in view of its great application to logic, that we should not allow its triadic purity to be violated. Now the reciprocal relation between the unaccented and the accented letters, were it allowed to stand alone, would be a purely dichotomic one. In order to avoid this, we ought to introduce the following singly accented letters:

$$i' = \frac{j\text{-}k}{\sqrt{-3}} \qquad i'' = \frac{j'\text{-}k'}{\sqrt{-3}} \qquad l' = \frac{m\text{-}n}{\sqrt{-3}} \qquad l'' = \frac{m'\text{-}n'}{\sqrt{-3}}$$

$$j' = \frac{k\text{-}i}{\sqrt{-3}} \qquad j'' = \frac{k'\text{-}i'}{\sqrt{-3}} \qquad m' = \frac{n\text{-}l}{\sqrt{-3}} \qquad m'' = \frac{n'\text{-}l'}{\sqrt{-3}}$$

$$k' = \frac{i\text{-}j}{\sqrt{-3}} \qquad k'' = \frac{i'\text{-}j'}{\sqrt{-3}} \qquad n' = \frac{l\text{-}m}{\sqrt{-3}} \qquad n'' = \frac{l'\text{-}m'}{\sqrt{-3}}$$

314. . . . In pure algebra, the symbols have no other meaning than that which the formulæ impose upon them. In other words, they signify any relations which follow the same laws. Anything more definite detracts needlessly and injuriously from the generality and utility of the algebra. It is that high principle which we all learned at a tender age that one cannot eat his cake and have it too; one cannot devote a thing to a particular use without making it less available for other applications. The logicians call it the principle of the inverse proportionality of comprehension and extension. Yet in this particular instance, we can adapt our doctrine better to thoroughgoing trichotomy by derogating a little from the dignified meaninglessness of pure algebra. In multiple algebra, it is generally assumed that the coefficients can be any numbers. My father, for example, even allowed them to be imaginary; though I cannot approve of that. But for the purposes of trichotomic mathematics, it should be recognized that each quantity has one of three values. Call them, for the moment, O°, 120°, 240° — regarding 360° as the same as O°. Or one might call them night, morning, and afternoon. Let us denote the three values by o (for ὄρθρος), δ (for δείλη), ν (for νύξ). Then, we must adopt the addition table:

$o+o=o$	$o+\delta=\delta$	$o+\nu=\nu$
$\delta+o=\delta$	$\delta+\delta=\nu$	$\delta+\nu=o$
$\nu+o=\nu$	$\nu+\delta=o$	$\nu+\nu=\delta$

For multiplication [a] table of these numbers will be obtained by assuming $\delta\delta=\delta$. Then $\delta\nu=\delta(\delta+\delta)=\delta\delta+\delta\delta=\delta+\delta=\nu$. $\delta o=\delta(\nu+\delta)=\delta\nu+\delta\delta=\nu+\delta=o$. For the sake of showing the consistency of the formulæ, we may complete the cycle

$$\delta\delta=\delta(\nu+\nu)=\delta\nu+\delta\nu=\nu+\nu=\delta;$$
$$\nu\nu=\nu(\delta+\delta)=\nu\delta+\nu\delta=\nu+\nu=\delta, \quad \text{(just as } - - = +\text{)}$$
$$\nu o=\nu(\nu+\delta)=\nu\nu+\nu\delta=\delta+\nu=o$$
$$oo=(\nu+\delta)o=\nu o+\delta o=o+o=o$$

Instead of assuming the distributive principle in its entirety we might have evolved the multiplication table from these three equations

$$oo=o \qquad x(y+\delta)=xy+x \qquad (x+\delta)y=xy+y.$$

In the same spirit, the addition table might have been derived from the equations,

$$o+\delta=\delta \quad \delta+\delta=\nu \quad \nu+\delta=o \quad (x+\delta)+y=x+(y+\delta)=(x+y)+\delta$$

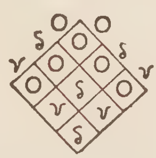
Thus,

$$o+\nu=o+(\delta+\delta)=(o+\delta)+\delta=\delta+\delta=\nu$$
$$o+o=o+(\nu+\delta)=(o+\nu)+\delta=\nu+\delta=o$$
$$\nu+\nu=\nu+(\delta+\delta)=(\nu+\delta)+\delta=o+\delta=\delta.$$

315. As for involution, until somebody can give me some good reason for attaching a given definite interpretation to $(120°)^{(120°)}$, I may be excused from attempting to introduce it into this algebra. I apprehend that, multiple algebra being essentially linear, there is no demand for any involution of its units.

316. It will be seen that, if we are to accept the premisses upon which the addition-table and multiplication-table are based, we cannot avoid giving peculiar properties to each of the three values δ, ν, o, and that the connection of them with some such sensuous images as day, night, and dawn is by no means an idle fancy. Let us put these tables into form. I add the subtraction-table

ADDITION TABLE



MULTIPLICATION TABLE

SUBTRACTION TABLE

We see that the multiplication-table recognizes a characteristic property in each member of the triad o, δ, ν. Multiplication by δ effects nothing. Multiplication by ν may have peculiar effects, but it is undone by a second multiplication by ν. Multiplication by O can never be undone, nor the same effect be otherwise produced.

317. We thus see that it is impossible to deal with a triad without being forced to recognize a triad of which one member is positive but ineffective, another is the opponent

of that, a third, intermediate between these two, is all-potent. The ideas of our three categories could not be better stated in so few words. A man must be wedded to a system of metaphysics not to see the philosophical importance of the fact that these ideas thus insist upon intruding where we have done our best to bolt and bar the doors against them, by assuming the members of the triad to be more alike than three rain-drops.

318. What is experience? It is the resultant ideas that have been forced upon us. We find we cannot summon up what images we like. Try to banish an idea and it only comes home with greater violence later. Hence, we find the only wisdom is to accept, at once, the ideas that sooner or later we must accept; and we even go to work solicitous to find out what are the ideas which are going ultimately to be forced upon us. Three such ideas are the three categories; and it will be wise to pitch overboard promptly the metaphysics which preaches against them. To recognize the triad is a step out of the bounds of mere dualism; but to attempt [to deny] independent being to the dyad and monad, Hegel-wise, is only another one-sidedness.

319. We are not bound at all times to introduce the triad: it is not needed on every occasion. But we should be prepared to introduce it whenever it is needed. We must not absolutely restrict ourselves to the notion that two triads can at one time correspond to one another in only one way, so that a given member of the one must be down-right, absolutely connected with a given member of the other, or else be down-right, absolutely, disconnected from it — two alternatives differing as day from night — as *δείλη* from *νύξ*. We must be prepared, if occasion be, to admit a possible intermediate dawn. For to say that two things are disconnected is but to say that they are connected in a way different from the way under contemplation. For everything is in some relation to each other thing. It is connected with it by otherness, for example. We should, therefore, be prepared to say that two atoms, one of each triad, have either a positive connection, such as is under the illumination of thought at the time, or a dark other mode of connection, or a vague glimmering intermediate form of connection. Nor are we to

rest there, as a finality. We must not restrict ourselves to saying absolutely that between a pair there either down-right is a given kind of connection, or down-right is *not* that kind. We must be prepared to say, if need be, that the pair has a δ connection with a given mode of connection, or an opposite ν-connection with it, or a neutral o-connection with it. We can push this sort of thing as far as may be — indefinitely. Still, however far we carry it, ultimately there will always be a dichotomic alternative between the truth and falsity of what is said. Why it should be so, we shall see in the proper place and time for such an inquiry. At present, it is pertinent to note that the fact that it is so is forced upon our attention in pure mathematics.

320. Our algebra of i, j, k, l, m, n, supposes a triad to be in one-to-one correspondence with a second triad. We may conveniently identify that second triad with o, δ, ν; since these three values, by hypothesis, are any three things we please. As to the mode of connection of the first triad with o, δ, ν, we ought, in order to make full use of the algebra, to have a state of things which is, or is analogous to, a state in which each of the three things composing the triad is in two distinct modes connected with one of the three values, o, δ, ν; or with a value of whatever system of values may be appropriate. To indicate this we may use a sign like this

$$\begin{matrix} \delta & \delta & o \\ \cdot & \cdot & \cdot \\ \nu & \delta & \nu \end{matrix}$$

The three dots represent the triad. The letters above show what values are attached to the members of the triad in the first mode; the letters below show what values are so attached in the second mode. In order to translate this into the algebra, the values of the upper line, in their order, are to be taken as the coefficients of i, j, k respectively; those in the lower line as the coefficients of l, m, n respectively. Thus the sign written would be

$$\delta i + \delta j + ok + \nu l + \delta m [+ \nu n.]$$

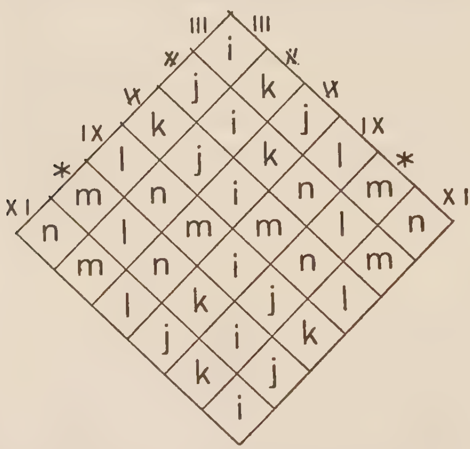
We may commonly write O for o, 1 for δ, -1 for ν. It often happens in the application of algebra that the absolute values of the coefficients are without significance, their ratios being alone important. If, in addition, the second mode of connection of the members of the triad with the values is identical with the first mode, we shall have

$$i+j+k=o \qquad l+m+n=o$$

just as $$i'+j'+k'=o \qquad l'+m'+n'=o.$$

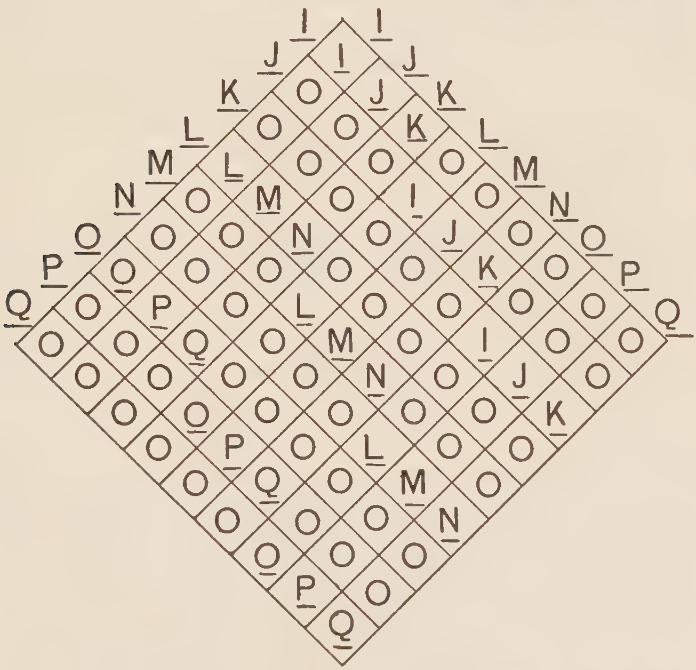
Indeed it was the circumstance, that with the unaccented letters these equations do not generally hold good, which led me to the above remarks, which give an interpretation to a sum of letters of the algebra.

321. I have known such astounding blunders and oversights to be committed by the most powerful and exact intellects, that I have come to the conclusion that it is folly to attempt to set limits which human stupidity cannot overpass. Otherwise, I should venture to say that no intelligent reader of this chapter could, while wide awake, for two minutes harbor the notion that a transposition, or passage from one



permutation to another — the mathematicians call it a substitution — could possibly be the interpretation of a sum of transpositions; for the very idea of a multiple algebra is that a polynomial in the unit letters cannot be expressed as a numerical multiple of a single unit. A transposition produces a permutation in which all the members of the set are connected with different members of another set. In a sum of transpositions connections between members of two sets are created in which two or more of one set may be connected with one of the other. Precisely what such connections may be is sufficiently shown above. Another illustration is afforded by the diagram on page 260.

This is a sort of division table. The operand being entered on the right and the product on the left, the one operation effecting this transmutation is found at the intersection of the rows. We see that any permutation can be transmuted into any other, but by only one letter; and since there are only six letters in all while there are nine pairs composed each of a



member of each of two triads, no combination of letters can express every possible change of connections of the members of two triads (but only changes of permutations and their sums, as above explained). When it is requisite to be able to do that, as we shall find that for some logical problems it is, we must resort to another algebra published by me in 1870* and for which I prefer the name *novenions*, rather than 'nonions,'

* See 3.127.

the designation proposed by Clifford and employed by many mathematicians. Let u_1, u_2, u_3, be three "umbræ," or ingredients of quantities. Then take

$$\begin{array}{lll} i=u_1:u_1 & l=u_2:u_1 & o=u_3:u_1 \\ j=u_1:u_2 & m=u_2:u_2 & p=u_3:u_2 \\ k=u_1:u_3 & n=u_2:u_3 & g=u_3:u_3 \end{array}$$

and the multiplication table is as shown on page 261.

322. Sometime after my first publication, either my father or I myself (under the instigation of my father's ideas) transformed[1] this algebra by means of the following equations, where, as above, ρ is an imaginary cube root of unity:

$$\begin{aligned}
\mathrm{I} &= (u_1:u_1)+\quad (u_2:u_2)+\quad (u_3:u_3) \\
\mathrm{J} &= (u_1:u_1)+\rho\ (u_2:u_2)+\rho^2(u_3:u_3) \\
\mathrm{K} &= (u_1:u_1)+\rho^2(u_2:u_2)+\rho\ (u_3:u_3) \\
\mathrm{L} &= (u_1:u_2)+\quad (u_2:u_3)+\quad (u_3:u_1) \\
\mathrm{M} &= (u_1:u_2)+\rho\ (u_2:u_3)+\rho^2(u_3:u_1) \\
\mathrm{N} &= (u_1:u_2)+\rho^2(u_2:u_3)+\rho\ (u_3:u_1) \\
\mathrm{O} &= (u_1:u_3)+\quad (u_3:u_2)+\quad (u_2:u_1) \\
\mathrm{P} &= (u_1:u_3)+\rho\ (u_3:u_2)+\rho^2(u_2:u_1) \\
\mathrm{Q} &= (u_1:u_3)+\rho^2(u_3:u_2)+\rho\ (u_2:u_1)
\end{aligned}$$

323. Other points concerning trichotomic mathematics are more of logical than of mathematical interest, and are so woven with logic in my mind that I will not attempt to set them forth from a purely mathematical point of view. Here, then, I conclude what I have to say of these very simple branches of mathematics which lie at the root of formal logic. Those of which the interest is more purely mathematical must be treated in a very different manner in the next chapter.*

[1] This transformation somehow escaped publication at the time; I think probably because I was abroad, so that my father and I could not consult, and he thought it had been discovered by me, and I by him. It certainly was an obvious transformation of my algebra in view of certain ideas of his. It thus happened that Sylvester first published it long after, saying in his first mention of it, "To my certain knowledge this result was obtained by Mr. C. S. Peirce many years ago." [See 3.646f.]

* That chapter does not seem to have been written. See 227n.

VIII

*NOTES ON THE LIST OF POSTULATES OF DR. HUNTINGTON'S SECTION 2**P

324. Dr. Huntington's Postulates at the head of section 2 of his paper† seem to me to have so great and so permanent an interest, that I am prompted to append to them some remarks with a view to establishing some other points of view which seem to me to be worth examining. My remarks inevitably present more or less opposition to Dr. Huntington's ideas; but that opposition is insignificant. For I certainly hold it to be undeniable that Dr. Huntington has successfully achieved his purpose; and that that purpose was a most important one for the study of logic. But in philosophy, as every student of it well understands, discoveries are worked out by reflection upon matters of common observation, instead of resulting from new opportunities for observation, as in all the positive sciences, or from pure thought concerning creations of the learned, as in mathematics, so that an idea is worthy of publication until one finds that it has not occurred to some profound student of the subject; and this one knows only by its opposition to something he has said.

325. By a 'postulate,' Dr. Huntington seems to understand any one of a body of propositions such that nothing can be deduced from one that could equally be deduced from another, while, from them all, every proposition of a given branch of mathematics might be deduced. The utility of such a body of premisses for the logical analysis of the branch of mathematics in question is beyond dispute. But I think we ought to distinguish between *postulates* and *definitions*. As for *axioms*, or propositions already well-known to the student who takes up the branch of mathematics in question, the ancients themselves admitted that they might be omitted without detriment to the course of deduction of the

* c. 1904.

† *Transactions*, American Mathematical Society, vol. 5, pp. 288–309; (1904).

theorems. Indeed, an axiom could only be a maxim of logical nature. A postulate is a statement that might be questioned or denied without absurdity. A definition, or rather, one of the pair, or larger number, of propositions that constitute a definition, cannot be questioned, because it merely states the logical relation of a conception thereby introduced to conceptions already in use. It is quite true, on the one hand, that a postulate, after all, is merely a part of the definition of the underlying conception of the branch of mathematics to which it refers, (Euclid's celebrated fifth postulate, for example, being merely a part of the definition of Euclidean space); while on the other hand, a definition is a statement of positive fact about the use of the word defined, and may thus be regarded as a sort of postulate. But to argue from these truths that there is no important difference between a postulate and a definition would be to fall into a fallacy of a very common kind, that of denying all important difference between two things because they are in an important respect alike, or of denying all important likeness between two things because they are in an important respect unlike. There is a vast difference between the logical relations to a branch of mathematics of propositions defining its very purpose in defining its fundamental hypothesis, and those of propositions that merely define conceptions which it is convenient or even which it is necessary to introduce in order to develop that branch.

326. I shall consider only the postulates of Dr. Huntington's section 2. This section refers to a special form of the Boolian algebra of logic. The algebra which Boole himself used was simply ordinary numerical algebra as applied to a collection of quantities each of which was assumed to be subject to the quadratic equation $x(1-x)=0$, and Boole showed how this hypothesis could be applied to the solution of many logical problems. For him, therefore, addition and multiplication were nothing but the numerical operations, greatly restricted in their application. An essential, not to say the vital, element of Boole's method lay in its applicability to the calculation of probabilities and statistical relations. But this feature disappeared in the algebra as it was modified by all his followers except Mr. Venn, a circumstance that gives a special value to Venn's *Symbolic Logic*, a work that has many

other merits. The rest of us assigned to the terms and operations purely logical meanings which we thought had sufficient analogy with the numerical conceptions to receive either exactly the same symbols, or, in my own case, to receive symbols closely resembling the numerical symbols. We* made three changes which affected the working of the method. These were as follows:†

First, we introduced what we inappropriately called 'logical addition,' writing something exactly or nearly like $x \,+\!\!,\, y$ for what Boole wrote as $x+y+xy$.

Second, we introduced the copula of inclusion, writing

$$x -\!\!\!< y$$

where Boole would write

$$x=xy.$$

Third, we introduced the negative of this copula; writing

$$x \,\overline{-\!\!\!<}\, y$$

for what Boole had no correct way of writing except "$x=xy$ is false."

327. Now the Boolian algebra to which Dr. Huntington's section 2 relates is Boole's as modified in the first two of these ways. It is to be added, moreover, that Schröder, with the majority of the Boolians, abandoned Boole's conception that every logical term has one or other of two values. For my part, I have always retained that conception, as far as non-relative terms go, which correspond to the quantities of ordinary algebra. But I introduced relative terms which correspond to what Sylvester called the *umbræ* of quantities (the conception is due to Leibniz), and employed various signs of operation upon these *umbræ*.‡ At the same time I showed that a non-relative term can be considered as a relative, that, in another sense, a relative term may be considered as non-relative, and that the non-relative operations equally apply to relatives.§ I regard a logical term as an indefinite proposition

* More accurately, Peirce.

† See 3.3, 3.47 and 3.165. There are a considerable number of other improvements as is evident from the papers in volume 3.

‡ See 305.

§ See 3.73, 3.331f.

or blank form of proposition; *man*, for example, meaning "x is a man." Now every proposition has one or other of two values; the lesser, that of being false; the greater, that of being true. A term, or rheme, is like $\frac{0}{0}$, in itself indeterminate in value, yet having one or the other of two values in each particular case. Thus, when in the ordinary Boolian algebra we write $m \prec l$, meaning "every man is a liar," according to me this means "if x (which is any individual object you may choose) is a man, then x is a liar," m signifying that x is a man, and l, signifying that x is a liar. Schröder, on the other hand, would say that m 'denotes' the entire collection of men (though I do not know what definite idea can be attached to the word 'denotes'), that l 'denotes' the entire collection of liars, and that the formula states that the former collection is included in the latter. Now it is certain that Dr. Huntington does not embrace my conception, since he would have greatly simplified his list of postulates if he had done so; but it is not clear that he unequivocally rejects this conception.

328. In my opinion, in algebra generally, the distinction between a quantity and an operation is a subsidiary one, and that we ought to allow operations to be operated upon as much as quantities; and this, perhaps, will be commonly conceded. If so, in the algebra of logic, the signs of the so-called logical addition and multiplication are substantially nothing but relative terms of a special description. But I would allow the spirit of unification to have the further effect that no radical separation be recognized between signs of operation and copulas, such as = and $\prec$. Thus, in any paper of 1880,* I have such expressions as $(a \prec b) \prec (c \prec d)$ which is equivalent to

$$(\bar{a} + b) \prec (\bar{c} + d)$$

or to

$$a.\bar{b} + \bar{c} + d.$$

Dr. Huntington evidently does not contemplate any such handling of the algebra of logic.

329. Dr. Huntington's purpose is, while considering the Boolian algebra in a purely morphic light, without regard to

* Vol. 3, No. VI.

its interpretation, to draw up a list of independent propositions which shall justify every illative transformation of the algebra in one way, and in one way only. Of his postulates, which are ten in number, the first three give a morphic definition of the copula ⧀ which is the purely morphic generalization of the copula of inclusion, these three propositions also showing that the algebra regards all entities as interchangeable which a direct application of this copula would not distinguish. The next two postulates define the two terms which I regard as the two values. The next two amount (according to my well-known definitions of the logical aggregate and compound)* to asserting that any terms may be logically added and multiplied. The eighth and ninth morphically define the sign of negation. The tenth is the morphic generalization of the logical postulate that it is possible to express a proposition that is false.

330. Perhaps this arrangement has not been the subject of much consideration. In defining any branch of mathematics, I should begin with stating the multitude of values it admits; and therefore in this case I should put postulate No. 10 first. Dr. Huntington seems to have left it to the last because he considered it "trivial." But it is certainly not trivial in the sense of being inessential. If it were it should be omitted altogether. Indeed in the sense in which it is trivial, this is the character of the whole algebra. In the sense in which the algebra is of interest and significance, its having just two values is its most significant and characteristic feature. If he had begun with the statements

There are at least two expressions not interconvertible, Z and I.

Every expression, a, is so related to Z, that a ⧀ Z may be written alone.

Every expression, a, is so related to I, that I ⧀ a may be written alone.

Every expression, which is such that, taking any expression that is a part of it, the whole could permissibly be written alone whether Z or I were substituted for this part, may be written alone.†

* Cf. 3.199.

† The manuscript ends here.

IX

*ORDINALS**

331. ... I pay full homage to Cantor. He is indisputably the *Hauptförderer* of the mathematico-logical doctrine of numbers. As for Dedekind, his little book *Was sind und was sollen die Zahlen?* is most ingenious and excellent. But it proves no difficult theorem that I had not proved or published years before, and my paper† had been sent to him. His definition‡ of an infinite collection is precisely my previous definition of a finite collection reversed.§ His introduction of Gauss's concept of the *Abbild*,¶ which has been spoken of as something quite great, might have been borrowed from my paper, though I made no fuss about it. Since my priority about the distinction of the finite and the infinite has been pointed out in Germany, in a prominent way,|| Dedekind has said that he had the idea some years earlier. He seems to think this an important circumstance. I may mention that my habit has always been to record ideas that seemed to me valuable in a certain large blank book with the dates at which I set them down, almost always not until I had had the ideas long enough to be quite convinced of their value. This idea about finite and infinite collections was thought worthy of record.** But I do not see that it has any interest for anybody but myself; and from Dedekind's conduct, I infer he would prefer I should not give it.

332. An extremely difficult question about whole numbers

* c. 1905; 331–334 are from a proposed lecture to the Academy of American Arts and Sciences; the remainder of the paper is from "Topics," a revised version of the latter part of that lecture.

† Paper No. VII of vol. 3.

‡ *Op. cit.*, §64.

§ Cf. 3.281f., 3.564.

¶ *Op. cit.*, §21.

|| See Ernst Schröder's article, "Ueber zwei Definitionen der Endlichkeit u. G. Cantor'sche Sätze," *Nova Acta*, Bd. 71, S. 301 (1898).

**No such record has been found.

is as to which are the more fundamental, ordinal numbers or the cardinal numbers considered as expressing the multitudes of collections. The solution of this problem is contained in the following six propositions, which are all capable of proof.*

First. The general idea of *plurality* is involved in the fundamental concept of *Thirdness*, a concept without which there can be no suggestion of such a thing as *logic*, or such a character as *truth*. *Plurality*, therefore, is an idea much more fundamental than that of the ordinal place of a member of a linear series.

Second. The conception that there is *a* transitive relation of greater and less among multitudes is logically prior to the conception of ordinal place in a linear series. But *that* relation of greater and less is by no means Bolzano's relation† which is at the foundation of the doctrine of multitude. Nothing more in the way of a conception of greater or less collections is involved in the concept of ordinal place than the idea that the greater collection can result from the incorporation into the lesser collection of another collection (using this word in a sense in which a collection may have but a single member).

Third. Cantor‡ represents the two ways in which a unit may be added to an endless series, namely by incorporation into the series, or by immediately following the endless series, as differing only in respect to the order of performance of the addition. But this is incorrect. The original concept of greater involved in the general concept of ordinal place is that of incorporation into a series. A contradiction is involved in speaking of a unit being incorporated into an endless series after all the members of the series, as well as in speaking of an endless series being incorporated into a finite series. The concept of a unit coming immediately after the endless series is a different concept.

Fourth. Both the concept of incorporation into a series and of attachment immediately after an endless series are involved in the complete conception of ordinal quantity; but they do not suffice to make it up. They do not even make up the full concept of what Cantor calls a well-ordered series,§

* Cf. 337 and 657ff.

† *Paradoxien des Unendlichen*, §19; *Wissenschaftslehre*, §84f.

‡ *Georg Cantor Gesammelte Abhandlungen*, S. 302.

§ *Ibid.*, S. 312f.

but which I propose to call a Cantorian series, in order to pay due honor to the completer of the doctrine of ordinal quantity, by attaching his name to this invaluable concept. This concept does not of itself make up the concept of ordinal quantity, but it is its most important ingredient.

Fifth. Bolzano's concept* of being multitudinally *greater than* is in no way involved in the concept of ordinal quantity.

Sixth. The concept of multitudinal quantity does not involve the concept of ordinal quantity as a system, nor even that of an ordinal quantity; but it does involve every ingredient of the concept of ordinal quantity except the *subjectal abstraction* of it. The logical term subjectal abstraction here requires a word of explanation; for there are few treatises on logic which notice subjectal abstraction under any name, except so far as to confuse it with precisive abstraction which is an entirely different logical function.† When we say that the Columbia library building is *large*, this remark is a result of precisive abstraction by which the man who makes the remark leaves out of account all the other features of his image of the building, and takes [to represent the size] the word "large" which is entirely unlike that image — and when I say the word is unlike the image, I mean that the general signification of the word is utterly disparate from the image, which involves no predicates at all. Such is *precisive abstraction*. But now if this man goes on to remark that the largeness of the building is very impressive, he converts the applicability of that predicate from being a way of thinking about the building to being itself a subject of thought, and that operation is *subjectal abstraction*. Subjectal abstraction is one of the most constantly employed tools of the mathematician. In thinking of the system of multitudinal quantity, we do not need to think about ordinal quantities, but we do need to attribute, to the objects we are thinking about, ordinal places in a series. The very system of multitudinal quantities themselves consists in their being ordinally arranged.

333. It is indispensable to my argument about continuity that I should, at this point, give a formal definition of the system of finite numbers, regarded as ordinals. I will give two

* *Op. cit.*

† See 1.549n., 2.428.

such definitions. The first is new, and is, I think, the best definition yet given of the finite ordinals, although because of its novelty I find the second handier, which was substantially given by me in 1883.* I may mention that in a paper of mine published in 1867,† I attempted a definition of the cardinal numbers considered as multitudinal. But although I received some complimentary letters about that paper at the time, it is now utterly unintelligible to me, and is, I trust, by far the worst I ever published. Nevertheless, it is founded upon an interesting idea, worthy of a better development; and the curious contrast between all the operatives of arithmetic when viewed multitudinally and when viewed ordinally is also worth showing.

In both these definitions, for the sake of simplicity, I speak of 'ordinals' meaning *finite ordinals*, or places in a simple endless series. Also when I speak of a definition of ordinals what I mean is a definition of the system of relations between ordinals. . . .

334. If α denote a character, then what I should mean by calling it an *ordinary* character would be that it would be absurd to say of two imaginary objects, M and N, that M possesses α while N does not possess it, but that in respect to all other characters — or in respect to all other characters of a given line of characters — M and N do not differ. For ordinary characters so blend into one another that no one can be singled out. It is not in their nature. By a *singular* character on the other hand, I mean a character which differs decidedly from every other however nearly like it. A *branch* character is a singular character which belongs to a certain collection of characters which I call a *system of branch characters*, and the characteristic property of such a system is as follows:

Let α and β be any two different characters of the system whatsoever, then of the three propositions:

P possesses both α and β
Q possesses α but not β
R possesses β but not α

* 1881?; see 3.260ff.
† 3.43–44.

one or other is contrary to the nature of things, even of imaginary things, while two of the three propositions (as the proposition "S possesses neither α nor β") are possible as far as the nature of the characters go. Understand then, that if of a system of singular characters it is possible to find among them two such (say *beauty* and *virtue*) that all that triad of propositions might be true of different imaginary objects, that system of characters is *not* a system of branch characters. Again if a system of singular characters be such that it is possible to find any two of them (like being a prime number and exactly dividing the number next greater than the continued product of all smaller integers) so related that two of those three forms of proposition would be contrary to the nature of the characters, *that* system is not a system of branch characters. Once more if a system of singular characters be such that it is possible to find among them any two of such a nature that both could not be absent from any object, *that* system is not a system of branch characters. Finally, if it be possible to find in a system a single ordinary character, it is not a system of branch characters. But if none of these four things be possible it *is* a system of branch characters.

The reason I call these branch characters is that a collection of characters each of which consists in being on some one branch of a tree forms for small objects like insects, as subjects, a system of branch characters, for if the two branches are separate nothing can be on both, but a thing may be on either and not on the other, or it may be on the trunk and so not on either. The only case in which a thing can be on two branches at once is when one branch is itself on the other, so that to be on the former is *ipso facto* to be on the latter. But then it cannot be on the former and not on the latter, though it may be on the latter and not on the former.

335. *First Definition of Ordinals*

Clause 1. Of any two non-identical ordinals, there is a branch character of a certain system of branch characters (here to be designated simply as 'the system') which one of them possesses while the other lacks it.

Clause 2. The branch characters of the system are singular.

Clause 3. It would be logically possible for an object susceptible of any branch character of the system, and actually having one of them, to change to one other, and to change from every character, to which it had been changed, to one branch character of the system to which it had never been changed, without ever being restored to its first character.

Clause 4. On the other hand, this would be logically impossible if the changes were restricted to those branch-characters of the system that are possessed by any one ordinal.

Clause 5. (From clauses 3 and 4 it follows that every branch character of the system has at least one other immediately dependent upon it; that is, dependent upon it, but not dependent on a third that is dependent upon it), but no branch character of the system has more than one immediately dependent upon it.

Clause 6. Every combination of possession and non-possession of branch characters of the system which is logically consistent with clauses 1 to 5 inclusive and with the definitions of branch characters, dependence, etc., is actually realized in some ordinal; (and that ordinal which possesses none of the branch characters of the system is called zero).

336. The statement of the second definition will be facilitated by the explanations of three peculiar locutions. In order to abbreviate oft-recurring phrases like 'A stands in the relation *r* to B,' we may say indifferently either 'A is *r* to B' or 'B is *r*'d by A.' We may say that a relation, *r*, is an *appurtenance*, if, and only if, it is out of the nature of things for anything to be *r* to two different correlates. We may call a relation, *r*, a *comparative fulfillment*, if, and only if, to say that A is *r* to B is the same as to say that A is not *r*'d by anything that *r*'s B, or nothing is *r* to A that is not *r* to B. Thus, to say that 'John' is as short a word as 'Jack' is the same as to say that nothing is as short as 'John' that is not as short as 'Jack.'

Second Definition of Ordinals

Clause I. The most fundamental relation of ordinals, N, ('next after'), is an appurtenance.

Clause II. Every ordinal is N'd by an ordinal.

Clause III. There is an ordinal, zero, or O, that is not N to any ordinal.

Clause IV. The relation g (as high as) is a comparative fulfillment.

Clause V. Whatever ordinal is N of an ordinal is g of that ordinal.

Clause VI. Whatever ordinals p and q may be, either the facts that certain ordinals are N to certain ordinals, taken in conjunction with the preceding clauses of this definition, logically necessitate p's being g to q, or else p is not g to q.

Clause VII. Whatever ordinals p and q may be, either p is g to q or q is g to p.

If we translate either definition into the terms of the other all its clauses may be deduced from those of the other. Thus, the branch characters of the system are the characters of being N to an ordinal that is g to a, where a is any one ordinal, constant for any one branch character but varying with the branch character. On the other hand, to say that n is g to m is the same as to say that n possesses every branch character of m, while to say that n is N to m is to say that n possesses a branch character, say ν, not possessed by m, while whatever branch character that is not ν and is not possessed by m, ξ may be, n does not possess ξ. The demonstration of each definition by the other will be found an instructive exercise, but it need not be worked out for our purposes. A person who wishes to try it should begin by proving by the second definition that no ordinal is N to itself (for in Clause II it is not said that every ordinal is N'd by *another*), and that no two different ordinals are N to the same ordinal (for this is not implied in any single clause of the definition).

337. All that it is necessary to insist upon here is that the only thing that whole numbers can express is the relative place of objects in a simple, discrete, linear series; and whole numbers are applicable to enumerable multitudes and enumerable collections, only because it happens that those multitudes have each its place in a simple, discrete, linear series. It is true that Dr. Georg Cantor, the great founder and *Hauptförderer* of the logico-mathematical doctrine of numbers, begins his exposition with what he calls "cardinal numbers,"* but which ought properly to be called *multitudes*. For cardinal numbers proper are nothing but the vocables of a certain

* *Op. cit.*, S. 282.

series of vocables that are used in the operation of ascertaining the multitude of a collection, by counting, and thence are applied as appellatives of collections to signify their multitudes. Multitude itself, however, belongs to various different collections in various different grades, where cardinal number has no application, at all. Cantor,* however, has partially shown, what is entirely true, that the whole doctrine of multitude can be developed without any reference to ordinal numbers. But in treating of ordinals we are obliged to say, *in substance*, what their multitude is. Thus, when we look at the matter from a certain point of view, it seems that the doctrine of multitude is more fundamental than that of ordinals, and that all whole numbers really express multitudes. But this is a logical fallacy. That the concepts of *multitude* and of *ordinal place in a simple, discrete, linear series* are very intimately connected is true. The latter involves the consideration of facts constituting the applicability of definite conceptions of multitude; but it does not involve these conceptions themselves. Multitude, on the other hand, is nothing but the place of a series in one or the other of two simple, discrete, linear series, and it is impossible to define it at all without the use of the ordinal conception itself.

That positive whole numbers can express nothing but places in a linear series is proved by the fact that from either of the definitions above of ordinals, neither of which involves any concept not involved in the concept of such a series, any property of whole numbers can be deduced. If the statement of the property involves the triadic relation of being 'sum of' or being 'product of' of course this relation must first be defined. In case the first definition is used, N being defined in terms of branch characters as above, and in case the second definition is used, without that definition of N, the definition of *sum* is as follows:

Any ordinal, *s*, is a *sum* of an ordinal *m* as summand, and an ordinal *n* as addend, if, and only if, either *m* is N to an ordinal, *l*, and *s* is N to an ordinal that is a sum of *l* and *n*, or *m* is not N to any ordinal, and *s* is *n*.

A product may be defined thus: An ordinal, *p*, is a *product* of an ordinal as *multiplier* and of an ordinal as *multiplicand*,

* *Ibid.*, S. 284ff.

if, and only if, either the multiplier, m, is N to an ordinal, l, and p is the sum of the multiplicand, n, and of an ordinal that is a product of l and n, or else m is not N to any ordinal, and p is m.

338. Many a person who will readily admit that whole numbers can express nothing but places in a linear series is inclined to insist that with fractions it is otherwise, fractions essentially involving the idea of equality of measure among the parts of a whole. Indeed, more than one highly esteemed writer might be named who has emphasized this as an essential characteristic of fractions, and in support of the assertion has averred that fractions cannot be added or subtracted until they have been reduced to a common denominator, and indeed that until that is done one cannot always tell which of two fractions is the larger. It thus becomes necessary to enter upon a proof that what is true of whole numbers is equally true of fractions; namely, that they can express nothing but relative places in a linear series; and this shall be done by defining first the system of rationals, or rational quantities, and then the system of fractional expressions, without any reference to measure, purely in terms of the relations of linear series, and in showing that from these definitions all properties of rationals and of fractions can be logically deduced.

339. In order to do this it becomes convenient, and indeed little short of indispensable, to make use of the secundal system of numerical notation, which may be familiarly described as a system exactly like the Arabic notation, except that Two is taken as the base of numeration instead of Ten. It may be formally defined as follows:

1. There is a collection of objects called *secundal places* (not *places of secundals*, which are used only in fractional expressions), and this one collection of places is the same for all secundal numerical expressions.

2. Every secundal place is in a certain relation to an ordinal called *being designated by* that ordinal, and is designated by no other ordinal.

3. Every ordinal designates a secundal place, and designates no other secundal place.

4. Every secundal integer expression denotes an ordinal and denotes no other ordinal.

5. Every ordinal is denoted by a secundal integer expression and is denoted by no other secundal integer expression.

6. Every secundal integer expression is a system of perceptual objects, called *figures*, each having perceptual characters in itself, and each having a perceptual relation to each secundal place.

7. In every secundal integer expression each figure has a perceptual relation to a secundal place, called *being in* that place, and is in no other place.

8. In every secundal integer expression, every secundal place has a figure in it and has no other figure in it.

9. In every secundal integer expression, the figure in each secundal place is perceptually distinguishable as having a certain character, called being a *unit*, or as not having that character, when it is called a *blank*. . . .

10. In every secundal integer expression, there is a secundal place, such that in every secundal place that is *g* to it there is a blank;

11. Every secundal expression denoting an ordinal *n* that is N to an ordinal, *m*, is such that:

Firstly, some figure of it is a blank.

Secondly, in that secundal place, to which every place that is *g* contains a blank in the secundal expression for *m*, in the secundal expression for *n* there is a unit.

Thirdly, in every secundal place to which that place is N*g* there is a blank in the expression for *n* instead of the unit there in the expression for *m*.

Fourthly, in every secundal place that is N*g* to that place the figure in the expression for *n* has the same character as the figure in the same place in the expression for *m*.

It is usual to write those blanks to whose places the place of some unit is N*g* as O, leaving those of which this is not true unwritten. The figure in the O place is best drawn with heavy lines.

340. The system of rationals, or positive rational quantities, may now be defined as follows:

1. Every rational is in a certain triadic relation to an ordinal called its *antecedent*, and to an ordinal called its *consequent*, namely, the relation of being the *ratio* (for the sake of brevity, I omit the qualification *in lowest terms*) *of* the ante-

cedent *to* the consequent; but is the ratio (in lowest terms, always) of no antecedent to any other consequent, nor of any other antecedent to any consequent. Nor is any other rational a ratio of the same antecedent to the same consequent. But not every two ordinals are the antecedent and consequent of a rational (in lowest terms).

2. Every ordinal is a rational, being the ratio of itself as antecedent, to NO (the ordinal that is N to zero, which in the secundal notation is written 1).

3. The relation g may be taken in a generalized sense, so as to be applicable to all rationals. Writing g' for this more general relation, every ordinal that is g to an ordinal, is also g' to the same ordinal, and if any rational be g' to an ordinal that is g to an ordinal, the first rational is g to the latter ordinal, and if an ordinal be g to an ordinal that is g' to a rational, the former ordinal is g' to the last rational.

4. If the converse of the negative of g' be called γ (being greater than) every rational stands to any non-identical rational either in the relation of being γ to it or in that of being γ'd by it.

5. There is a peculiar relation, to be here called *being indicated by* or *having as indicator*, in which every rational stands to a secundal integer expression, and to nothing else.

6. Every secundal integer expression is indicator of a rational, but of no other rational.

7. If from the indicator of any rational, on the one hand, the figure in the zero secundal place be struck off and the result be called the near subindicator, and on the other hand all those figures be struck off which are in places g'd by the place of that figure unlike the figure in the zero place whose place is g'd by all the figures unlike the figure in the zero place and call the result the far subindicator, then unless the far subindicator contains no unit, the irrational of the first indicator has for its antecedent the sum of the antecedents of the rationals indicated by the near and far subindicators and for its consequent the sum of the consequents of the same rationals; but if the far subindicator contains no unit, the antecedent of the first rational is N to the antecedent of the rational indicated by the near subindicator, and its consequent is the ordinal that is N to O.

0 $\frac{0}{1}$ $\frac{1}{0}$

1 $\frac{1}{1}$

10 $\frac{1}{10}$ 11 $\frac{10}{1}$

100 $\frac{1}{11}$ 101 $\frac{10}{11}$ 110 $\frac{11}{10}$ 111 $\frac{11}{1}$

1000 $\frac{1}{100}$ 1001 $\frac{10}{101}$ 1010 $\frac{11}{101}$ 1011 $\frac{11}{100}$ 1100 $\frac{100}{11}$ 1101 $\frac{101}{11}$ 1110 $\frac{101}{10}$ 1111 $\frac{100}{1}$

10000 $\frac{1}{101}$ 10001 $\frac{10}{111}$ 10010 $\frac{11}{1000}$ 10011 $\frac{11}{111}$ 10100 $\frac{100}{111}$ 10101 $\frac{101}{1000}$ 10110 $\frac{101}{111}$ 10111 $\frac{100}{101}$ 11000 $\frac{101}{100}$ 11001 $\frac{111}{101}$ 11010 $\frac{1000}{101}$ 11011 $\frac{111}{100}$ 11100 $\frac{111}{11}$ 11101 $\frac{1000}{11}$ 11110 $\frac{111}{10}$ 11111 $\frac{101}{1}$

100	100	100	100	100	100	100	100	101	101	101	101	101	101	101	101	110	110	110	110	110	110	110	110	111	111	111	111	111	111	111	111
000	001	010	011	100	101	110	111	000	001	010	011	100	101	110	111	000	001	010	011	100	101	110	111	000	001	010	011	100	101	110	111
$\frac{1}{110}$	$\frac{10}{1100}$	$\frac{11}{1011}$	$\frac{11}{1010}$	$\frac{100}{1011}$	$\frac{101}{1101}$	$\frac{101}{1100}$	$\frac{100}{1001}$	$\frac{101}{1001}$	$\frac{111}{1100}$	$\frac{1001}{1111}$*	$\frac{111}{1011}$	$\frac{111}{1010}$	$\frac{1000}{1011}$	$\frac{111}{1001}$	$\frac{101}{110}$	$\frac{110}{101}$	$\frac{1001}{111}$	$\frac{1011}{1000}$	$\frac{1010}{111}$	$\frac{1011}{111}$	$\frac{1101}{1000}$†	$\frac{1100}{111}$	$\frac{1001}{101}$	$\frac{1001}{100}$	$\frac{1100}{101}$	$\frac{1101}{1001}$‡	$\frac{1011}{100}$	$\frac{1010}{11}$	$\frac{1011}{11}$	$\frac{1001}{10}$§	$\frac{1010}{1}$¶

* $\frac{1000}{1001}$? † $\frac{1001}{1000}$? ‡ $\frac{1101}{101}$? § $\frac{1100}{10}$? ¶ $\frac{110}{1}$?

8. The rational whose indicator is zero is zero.

It now becomes easy to arrange the rationals in order of their magnitudes. The table on preceeding page shows a few of them, [with] the rationals under the indicators.

I will now give the antecedents, or numerators, only for the right-hand-half of the table in Arabic figures.

1

2

3 3

4 5 5 4

5 7 8 7 7 8 7 5

6 9 11 10 11 13 12 9 9 12 13 11 10 11 9 6

7 11 14 13 15 18 17 10 14 19 21 18 17 19 16 11 11 16 19 17 18 21 19 14 13 17 18 15 13 14 11 7

The objection that there is no way of finding out which of two fractions is the greater without reducing them to a common denominator is an amusing self-betrayal; for all good arithmeticians proceed by setting down the two fractions side by side and then subtracting numerator from numerator and denominator from denominator, putting down the remainders, as a new fraction on the side of the fraction with larger terms, and so continuing until one has $\frac{0}{1}$ at one end of the row and $\frac{1}{0}$ on the other. For $\frac{1}{0} > \frac{0}{1}$. This can always be done if a little ingenuity varies the process a little. Thus, which is the greater $\frac{487}{830}$ or $\frac{301}{513}$?

Proceed thus

$$\frac{487}{830}\ \frac{301}{513}$$

$$\frac{186}{317}\ \frac{487}{830}\ \frac{301}{513}$$

$$\frac{1}{0}\ \frac{1}{1}\ \frac{2}{3}\ \frac{3}{5}\ \frac{10}{17}\ \frac{27}{46}\ \frac{71}{121}\ \frac{186}{317}\ \frac{487}{830}\ \frac{301}{513}\ \frac{115}{196}\ \frac{44}{75}\ \frac{17}{29}\ \frac{7}{12}\ \frac{4}{7}\ \frac{1}{2}\ \frac{0}{1}$$

This series everywhere increases in the same direction. Hence $\frac{487}{830} > \frac{301}{513}$. Fractions can be added on the same principle.

X

*ANALYSIS OF SOME DEMONSTRATIONS CONCERNING DEFINITE POSITIVE INTEGERS**P

341. Let the Universe of l. c. italics be the aggregate of all definite Integers not negative. Let the Universe of Greek Minuscules be the aggregate of possible characters of such Integers. Let $q_{\alpha u}$ mean, as in 3.398, that the Integer, u, has the character, α.

Hypotheses

(1) $\Pi_\alpha \Sigma_\beta \Pi_u \ (q_{\alpha u} \Psi q_{\beta u}) \cdot (\bar{q}_{\alpha u} \Psi \bar{q}_{\beta u})$ *i.e.*, every character has a negative.

(2) $\Pi_\alpha \Pi_\beta \Sigma_\gamma \Pi_u (\bar{q}_{\alpha u} \Psi \bar{q}_{\beta u} \Psi q_{\gamma u}) \cdot (q_{\alpha u} \cdot q_{\beta u} \Psi \bar{q}_{\gamma u})$ *i.e.*, of every two possibilities there is a compounded possibility. Instead of introducing an unanalyzed relation of 'as small as,' let us, at first, conceive a character of characters, consisting in each of certain characters belonging to every integer lower than an integer to which it belongs.

(3) $\Pi_u \Sigma_\alpha \ q_{\alpha u} \cdot s_\alpha$,† every Integer has a character common to all lower Integers. (A formal proposition.)

(4) $\Pi_\alpha \Sigma_u \ \bar{s}_\alpha \Psi \bar{q}_{\alpha u}$ *i.e.*, there is no highest Integer.

(5) $\Sigma_u \Pi_\alpha \Pi_v \ q_{\alpha u} \Psi \bar{s}_\alpha \Psi \bar{q}_{\alpha v}$ *i.e.*, there is a lowest Integer (*i.e.*, Zero).

(6) $\Pi_\alpha \Pi_\beta \Pi_u \Pi_v \ \bar{s}_\alpha \Psi \bar{s}_\beta \Psi \bar{q}_{\alpha u} \Psi q_{\beta u} \Psi q_{\alpha v} \Psi \bar{q}_{\beta v}$ *i.e.*, an Integer having any s-character that another has not has every one that other has.

(7) $\Pi_u \Pi_v \Sigma_\alpha \Pi_\beta \ s_\alpha \cdot (q_{\alpha u} \Psi q_{\alpha v}) \cdot (\bar{q}_{\alpha u} \Psi \bar{q}_{\alpha v}) \Psi q_{\beta u} \cdot q_{\beta v} \Psi \bar{q}_{\beta u} \cdot \bar{q}_{\beta v}$ *i.e.*, unless one of any two integers has an s-character which the other has not, they are alike in *all* characters, and therefore, being definite, are identical.

(8) $\Pi_u \Sigma_\alpha \Sigma_v \Pi_w \Pi_\beta \Pi_\gamma \ q_{\alpha u} \cdot s_\alpha \cdot \bar{q}_{\alpha v} \cdot (\bar{s}_\beta \Psi \bar{q}_{\beta u} \Psi q_{\beta w} \Psi \bar{s}_\gamma \Psi \bar{q}_{\gamma w} \Psi q_{\gamma v})$ *i.e.*, every integer has another next higher. This is a consequence of the following with (4):

* 1905.

†S_α means that every unit less than u has the character α.

(9) $\Pi_\alpha\Pi_u\Sigma_v\Pi_w\Pi_\beta\, \bar{q}_{\alpha u} \boldsymbol{+} q_{\alpha v}\cdot(\bar{q}_{\alpha w} \boldsymbol{+} \bar{s}_\beta \boldsymbol{+} \bar{q}_{\beta w} \boldsymbol{+} q_{\beta v})$ *i.e.*, every class of integers has a lowest member.

For formula (10) see below [343].

Addition

Let $(i+j)_k$ mean that the integer *k* can result from adding the integer *i* to the integer *j*. This can be negatived by an obelus over it like any other expression.

Addition is definable by the following six formulæ:

(11) $\Pi_i\Pi_j\Sigma_k(i+j)_k$

(12) $\Pi_k\Pi_j\Sigma_i\Sigma_\alpha\, s_\alpha\cdot q_{\alpha j}\cdot \bar{q}_{\alpha k} \boldsymbol{+} (i+j)_k$

(13) $\Pi_i\Pi_k\Sigma_j\Sigma_\alpha\, s_\alpha\cdot q_{\alpha i}\cdot \bar{q}_{\alpha k} \boldsymbol{+} (i+j)_k.$

(14) $\Pi_i\Pi_j\Pi_k\Pi_u\Pi_v\Pi_w\Sigma_\alpha\Pi_\beta\overline{(i+j)_k} \boldsymbol{+} \overline{(u+v)_w}$
$\boldsymbol{+} s_\alpha\cdot(q_{\alpha u}\cdot\bar{q}_{\alpha i} \boldsymbol{+} q_{\alpha v}\cdot\bar{q}_{\alpha j}) \boldsymbol{+} \bar{s}_\beta \boldsymbol{+} \bar{q}_{\beta w} \boldsymbol{+} q_{\beta k}$

(15) $\Pi_i\Pi_j\Pi_k\Pi_u\Pi_v\Pi_w\Sigma_\alpha\Pi_\beta\overline{(i+j)_k} \boldsymbol{+} \overline{(u+v)_w}$
$\boldsymbol{+} s_\alpha\cdot(q_{\alpha v}\cdot\bar{q}_{\alpha j} \boldsymbol{+} q_{\alpha w}\cdot\bar{q}_{\alpha k}) \boldsymbol{+} \bar{s}_\beta \boldsymbol{+} \bar{q}_{\beta u} \boldsymbol{+} q_{\beta i}$

(16) $\Pi_i\Pi_j\Pi_k\Pi_u\Pi_v\Pi_w\Sigma_\alpha\Pi_\beta\overline{(i+j)_k} \boldsymbol{+} \overline{(u+v)_w}$
$\boldsymbol{+} s_\alpha\cdot(q_{\alpha u}\cdot\bar{q}_{\alpha i} \boldsymbol{+} q_{\alpha w}\cdot\bar{q}_{\alpha k}) \boldsymbol{+} \bar{s}_\beta \boldsymbol{+} \bar{q}_{\beta v} \boldsymbol{+} q_{\beta j}$

342. It would be illuminating to exhibit the above fifteen propositions scribed in existential graphs;* but it would be aside from my present purpose. I proceed to indicate sketchily in what manner the leading theorems concerning the addition of positive integers can be deduced from the fifteen propositions by means of the rules given in 3.396. (Though those rules might now be amended much, so as to render them more efficient.) If (14) be iterated, it becomes

$$\Pi_i\Pi_j\Pi_k\Pi_u\Pi_v\Pi_w\Sigma_\alpha\Pi_{i'}\Pi_{j'}\Pi_{k'}\Pi_{u'}\Pi_{v'}\Pi_{w'}\Sigma_{\alpha'}\Pi_\beta\Pi_{\beta'}$$
$$\{\overline{(i+j)_k} \boldsymbol{+} \overline{(u+v)_w} \boldsymbol{+} s_\alpha\cdot(q_{\alpha u}\cdot\bar{q}_{\alpha i} \boldsymbol{+} q_{\alpha v}\cdot\bar{q}_{\alpha j}) \boldsymbol{+} \bar{s}_\beta \boldsymbol{+} \bar{q}_{\beta w} \boldsymbol{+} q_{\beta k}\}$$
$$\cdot\{\overline{(i'+j')_{k'}} \boldsymbol{+} \overline{(u'+v')_{w'}} \boldsymbol{+} s_{\alpha'}\cdot(q_{\alpha' u'}\cdot\bar{q}_{\alpha' i'} \boldsymbol{+} q_{\alpha' v'}\cdot\bar{q}_{\alpha' j'})$$
$$\boldsymbol{+} \bar{s}_{\beta'} \boldsymbol{+} \bar{q}_{\beta' w'} \boldsymbol{+} q_{\beta' k'}\}.$$

Next (I go into detail with this first example farther than I shall with others), we may, by the fifth rule, identify u, i′, and u′, with i; v, j′, and v′ with j; k′ with w; w′ with k; and β′ with β (for though the rule as given in the memoir is the

*See book II for a detailed analysis of these graphs.

right one, theoretically, yet in practice the operation of this and part of the sixth can generally be reduced with convenience to the identification of the index of any Π with any index to the left of it in the quantifier). We, at the same time, apply Rule 6 somewhat, remembering that $q_{\alpha i}\cdot\bar{q}_{\alpha i} \prec 0$ etc. and applying the principle $(A \dotplus B)\cdot(A \dotplus C) = A \dotplus B\cdot C$, and then applying Rule 7 we get

$$\Pi_i\Pi_j\Pi_k\Pi_w\Pi_\beta \overline{(i+j)_k} \dotplus \overline{(i+j)_w} \dotplus \bar{s}_\beta \dotplus (\bar{q}_{\beta w} \dotplus q_{\beta k})\cdot(\bar{q}_{\beta k} \dotplus q_{\beta w}) \text{ or}$$

$$\Pi_i\Pi_j\Pi_k\Pi_w\Pi_\beta \overline{(i+j)_k} \dotplus \overline{(i+j)_w} \dotplus \bar{s}_\beta \dotplus \bar{q}_{\beta w}\cdot\bar{q}_{\beta k} \dotplus q_{\beta w}\cdot q_{\beta k}.$$

Let us now compound this with (7) in which, to avoid confusion, we may write m for u, n for v, υ for α, and ϕ for β. We thus get

$$\Pi_m\Pi_n\Sigma_\upsilon\Pi_\phi\Pi_i\Pi_j\Pi_k\Pi_w\Pi_\beta \{ \overline{(i+j)_k} \dotplus \overline{(i+j)_w} \dotplus \bar{s}_\beta \dotplus \bar{q}_{\beta w}\cdot\bar{q}_{\beta k} \dotplus q_{\beta w}\cdot q_{\beta k} \}$$
$$\{ s_\upsilon\cdot(q_{\upsilon m} \dotplus q_{\upsilon n})\cdot(\bar{q}_{\upsilon m} \dotplus \bar{q}_{\upsilon n}) \dotplus q_{\phi m}\cdot q_{\phi n} \dotplus \bar{q}_{\phi m}\cdot\bar{q}_{\phi n} \}.$$

Now identifying β with υ, w with m, k with u, the formula with an obvious reduction of the Boolian, becomes

(17) $\Pi_m\Pi_n\Pi_\phi\Pi_i\Pi_j \overline{(i+j)_n} \dotplus \overline{(i+j)_m} \dotplus q_{\phi m}\cdot q_{\phi n} \dotplus \bar{q}_{\phi m}\cdot\bar{q}_{\phi n}$ *i.e.*, if $(i+j)_m$ and $(i+j)_n$, then $m=n$; or the sum of two definite positive integers has but a single value.

Without writing down the formulæ, a little close attention will enable one to convince himself that (15) and (16), treated almost exactly as (14) has been above, show that if

(18) $(i+j)_k$ and $(u+j)_k$ then $u=i$ and that

(19) if $(i+j)_k$ and $(i+v)_k$ then $j=v$.

343. *Abbreviations.* Having thus illustrated how the notation works, it will be well to introduce some abbreviations.

First, although obviously indefinite individuals may be alike in respect to every character, yet different in their (real or pretended) brute existence, such as the different parts of space and the different vertices of the regular dodecahedron of pure mathematics, still since the Universe of l. c. italics is confined to definite integers, we may, by introducing I_{ij} to mean that i and j are the same individual, write the following principle:

(10) $\Pi_u\Pi_v\Sigma_\alpha\, s_\alpha\cdot(q_{\alpha u} \dotplus q_{\alpha v})\cdot(\bar{q}_{\alpha u} \dotplus \bar{q}_{\alpha v}) \dotplus \mathrm{I}_{uv}.$

Of course, the negative of I_{ij} will be T_{ij}.

344. One may entertain the theory that all vagueness is due to a defect of cogitation or cognition. It is a natural kind of nominalism the justice of which it would be remote from the purpose of this analysis to consider. The vagueness of characters is of different kinds. The quality of redness and the quality of blueness differ without differing in any essential character which one has but the other lacks. The otherness of them is as irrational as the qualities themselves, if not more so. It appears to consist in a mutual war between them, in our taste. But the characters of integers are not of this irrational kind. In another regard, however, they are vague. Thus we say that the two characters of 4, of being the sum of 2 and 2, and of being the product of 2 and 2, are different characters, so that we cannot, in imitation of (10), write

$$\Pi_{\alpha}\Pi_{\beta}\Sigma_{n}(q_{\alpha n} \Psi q_{\beta n}) \cdot (\bar{q}_{\alpha n} \Psi \bar{q}_{\beta n}) \Psi \mathsf{I}_{\alpha\beta}$$

This is [because] we do not think out the meaning of $2+2$ and 2×2 to the very bottom. In this respect, the objects we denote by Greek minuscules are not generally definite.

345. The character, I, which I introduced in 1882, when I was teaching logic in the Johns Hopkins University, was in my mind one of a class of notations which I left unmentioned in order that some one of my pupils might have the pleasure of finding it out for himself; but as nobody has, so far as I have noticed, in the three-fourths of a generation that has elapsed, I will give some illustrations of the class:

I_{ij} means j is a member of the *singlet*, i.

2_{ij} means j is a member of the *doublet*, i, or unordered pair, or *couple*.

3_{ij} means j is a member of the *triplet*, i, or unordered trio, or *leash*.

4_{ij} means j is a member of the *quadruplet*, i, or unordered collection of 4.

9_{ij} means j is a member of the *nonuplet*, i, or unordered collection of 9.

x_{ij} means j is a member of the *decuplet*, i, or unordered collection of 10.

Ordered collections I call, *medads* (0), *monads*, *dyads*, *triads*, etc. Indeterminate as to being ordered are *binion* (or pair), *trine*, *quaternion*, *quine*, *senion*, *septene*, *octone*, *novene*, *dene* (or *denion*), etc.

By an ordered collection, I mean one of which each member has a peculiar relation to the whole; as for example, if one is definitely the first, another definitely the second, a definite one the third, etc., or if there is any other formal relation by which each is different from all the others. There are also *diduct* collections which are formally divided into subcollections and it may be in more than one way, whether inadequately, adequately, or superfluously. By *adequately*, I mean just sufficiently to make the collection an ordered one.

With this notation (7) can be expressed as follows, using Hebrew letters to denote definite collections:

$$\Pi_{\aleph}\Sigma_i\Sigma_j\Sigma_\alpha \quad 2_{\aleph i}\cdot 2_{\aleph j}\cdot s_\alpha\cdot q_{\alpha i}\cdot \bar{q}_{\alpha j}$$

The utility of the symbols 1, 2, 3, etc. is increased by employing them as follows: $\mathsf{1}_i$, $\mathsf{1}_{ij}$, $\mathsf{1}_{ijk}$, etc. means that the indices denote the same existing individual.

$2_{\aleph i}$, $2_{\aleph ij}$, $2_{\aleph ijk}$, etc. mean that the individuals denoted by the indices belong to the doublet $\aleph$.

2_{ijk}, 2_{ijkl}, etc. mean that all the individuals denoted by the indices are members of one doublet.

$(2\cdot\mathsf{T})_{ij}$, $(2\cdot\mathsf{T})_{ijk}$, etc., mean that the individuals denoted by the indices belong to one doublet but are not all one individual.

$3_{\aleph i}$, $3_{\aleph ij}$, $3_{\aleph ijk}$, $3_{\aleph ijkl}$ mean that i, j, etc. all belong to the triplet $\aleph$.

3_{ijkl}, 3_{ijklm}, etc. mean that i, j, etc. all belong to one triplet.

$(3\cdot\mathsf{T})_{ijkl}$ means that i, j, k, l all belong to one triplet but are not all identical.

$(3\cdot\bar{2})_{ijkl}$ means that i, j, k, l are three different existing individuals.

$(32)_{ijklmnpq}$ (where note the absence of a dot — *not* 3·2, but 32) means that the individuals indicated are all members of a triplet of doublets.

$(3 \maltese 2)_{ijklmno}$ means that every individual denoted by an index is either a member of a triplet or of a doublet.

I would use a special form of parenthesis (I will not recommend any particular form as more appropriate than another) which I would use in the following way:

$\Pi_i[\overline{\odot}\prec l]_i$ means any object which is sun is, *as such*, the member of a singlet, *i.e.*, $\Pi_i\Pi_j\overline{\odot}_i\prec\overline{\odot}_j\prec l_{ij}$.

If $\int_{+}$ means is a satellite of Jupiter, then

$\Pi_i[\overline{\int_{+}}\prec 5]_i$ means that whatever is a satellite of Jupiter is, *as such*, a member of a quintuplet, *i.e.*,

$$\Sigma_i\Sigma_j\Sigma_k\Sigma_l\Sigma_m\Pi_n \quad \int_{+i}\cdot\int_{+j}\cdot\int_{+k}\cdot\int_{+l}\cdot\int_{+m}\cdot T_{ij}\cdot T_{ik}\cdot T_{il}\cdot T_{im} \cdot T_{jk}\cdot T_{jl}\cdot T_{jm}\cdot T_{kl}\cdot T_{km}\cdot T_{lm}\cdot(\overline{\int_{+n}}\prec l_{in} \prec l_{jn}\prec l_{kn}\prec l_{ln}\prec l_{mn}).$$

The saving here is enormous.

Intimately connected with these abbreviations are others, some of which I have mentioned elsewhere. The rules of their application would form an elaborate logical doctrine, which I have not time to develop, because I am working at more fundamental parts of logic. Whoever undertakes it in the light of what I have said here and elsewhere will have other symbols forced upon his attention.

I pass to another and very simple abbreviation, which consists in using the symbol σ so that σ_{ij} shall mean that j is at least as low an integer as i. That is,

$$(20) \quad \sigma_{ij}=\Pi_\alpha\, \bar{s}_\alpha\prec\bar{q}_{\alpha i}\prec q_{\alpha j} \qquad \bar{\sigma}_{ij}=\Sigma_\alpha\, s_\alpha\cdot q_{\alpha i}\,\bar{q}_{\alpha j}$$

It immediately follows that

$$(21) \qquad \Pi_i\sigma_{ii} \text{ and}$$

$$(22) \qquad \Pi_i\Pi_j\Pi_k\ \bar{\sigma}_{ij}\prec\bar{\sigma}_{jk}\prec\sigma_{ik}$$

From (20) and (10) it follows that

$$(23) \qquad \Pi_i\Pi_j\Pi_k\ \sigma_{ij}\prec\sigma_{jk}\prec\bar{\sigma}_{ik}$$

Hypothesis (5) represents that there is an integer as low as any and by (10) this is lower than any other. We may therefore give it the proper name, o, which will possess the singularity of being definable. Thus

$$(24) \qquad \Pi_i\, l_{oi}\prec\bar{\sigma}_{oi} \qquad \Pi_i\,\sigma_{io}$$

a definition which is also singular in being in a single proposition. But this is owing to (10).

The long formula (8) requires abbreviation; and we may write

$$(25)\quad H_{uv}=\Sigma_{\alpha}\Pi_{\beta}\Pi_{\gamma}\Pi_{w}\ s_{\alpha}\cdot q_{\alpha u}\cdot \bar{q}_{\alpha v}\cdot(\bar{s}_{\beta}\pitchfork\bar{q}_{\beta u}\pitchfork q_{\beta w}\pitchfork\bar{s}_{\gamma}\pitchfork q_{\beta v}\pitchfork\bar{q}_{\beta w})$$
$$=\Pi_{w}\bar{\sigma}_{uv}\cdot(\sigma_{uw}\pitchfork\sigma_{wv})$$

We may further take the index 1 as such a proper name that

(26) H_{o_1}. I will also write r_{Hi} for $\Sigma_j\, r_j\cdot H_{ij}$ and $(H_i+H_j)_k$ for $\Sigma_u H_{iu}\cdot\Sigma_v H_{wv}\cdot(u+v)_k$

Formulæ (14)–(16) may be put in the form

$$(14)\quad \Pi_i\Pi_j\Pi_k\Pi_u\Pi_v\Pi_w\quad \bar{\sigma}_{ui}\pitchfork\bar{\sigma}_{vj}\pitchfork\overline{(i+j)}_k\pitchfork\overline{(u+v)}_w\pitchfork\sigma_{wk}$$
$$(15)\quad \Pi_i\Pi_j\Pi_k\Pi_u\Pi_v\Pi_w\quad \bar{\sigma}_{wk}\pitchfork\bar{\sigma}_{vj}\pitchfork\overline{(i+j)}_k\pitchfork\overline{(u+v)}_w\pitchfork\sigma_{ui}$$
$$(16)\quad \Pi_i\Pi_j\Pi_k\Pi_u\Pi_v\Pi_w\quad \bar{\sigma}_{ui}\pitchfork\bar{\sigma}_{wk}\pitchfork\overline{(i+j)}_k\pitchfork\overline{(u+v)}_w\pitchfork\sigma_{vj}$$

Putting, in (14), o for i, j, and w, it becomes

$$\Pi_k\Pi_u\Pi_v(o+o)_k\pitchfork(u+v)_o\pitchfork\bar{\sigma}_{uo}\pitchfork\bar{\sigma}_{vo}\pitchfork\sigma_{ok}$$

Multiplying this by the third power of (24), *i.e.*, (24)·(24)·(24), we get

$$(27)\qquad (o+o)_o$$

(9) may be put in the form

$$(9')\qquad \Pi_{\alpha}\Pi_u\Sigma_v\Pi_w\quad \bar{q}_{\alpha u}\pitchfork q_{\alpha v}\cdot(\bar{q}_{\alpha w}\pitchfork\sigma_{wv})$$

Putting for q_{α} the expression $\bar{\sigma}_u$, this becomes

$$(28)\qquad \Pi_u\Sigma_v\Pi_w\quad \bar{\sigma}_{uv}\cdot(\sigma_{uw}\pitchfork\sigma_{wv})$$

346. I will now return to addition. I will remark, by the way (for I do not make this paper at all systematic), that Schröder's notation $\sum\limits_{u}$ and $\prod\limits_{v}$ and the like, which is his chief modification of my two logical algebras (which, by the way, can perfectly well be mixed), made long after my second intentional section of the paper, No. XIII in vol. 3, has several advantages over mine, both theoretical and practical, and ought to be employed freely. But it fails to do what my invention was made in order to do, namely, to enable us to perform the operation of hypostatic abstraction, and freely make use of *entia rationis*. But that is neither here nor there.

I will start with (14) in its last form and will trace out the steps of the algebraical transformation in closer detail than I purpose generally to do in this paper. For the inference I am coming to employs the *Rule of Diduction*, or diversification, which I fully treated of in a paper I drew up in Grammercy Park in 1885*, but the dignity of science does not permit me to go begging to have its results printed. This paper I am writing will probably never be seen by other eyes than those that see it written; but I record this for my own gratification. The rule is that after any quantifier of the *Peircean* (whether it be Π or Σ) can be inserted a Σ with a new index, into which the preceding index can be transmuted in any of the places where it occurs, remaining untransmuted in the other places. Thus $\Pi_i l_{ii}$, everybody loves himself, can be changed to $\Pi_i \Sigma_j l_{ij}$, everybody loves somebody. Identifying v with j, in (14) we get

$$\Pi_i \Pi_j \Pi_k \Pi_w \Pi_u \overline{(i+j)}_k \dotplus \sigma_{wk} \dotplus \overline{(u+j)}_w \dotplus \bar{\sigma}_{ui}$$

Let us insert the aggregant $\bar{q}_{\beta u}$:

$$\Pi_i \Pi_j \Pi_k \Pi_w \Pi_\beta \Pi_u \overline{(i+j)}_k \dotplus \sigma_{wk} \dotplus \bar{q}_{\beta u} \dotplus \overline{(u+j)}_w \dotplus \bar{\sigma}_{ui}$$

The insertion of this aggregant authorizes the insertion of its negative as component of another aggregant.

$$\Pi_i \Pi_j \Pi_k \Pi_w \Pi_\beta \Pi_u \ \overline{(i+j)}_k \dotplus \sigma_{wk} \dotplus \bar{q}_{\beta u} \dotplus \overline{(u+j)}_w \dotplus q_{\beta u} \cdot \bar{\sigma}_{ui}.$$

Let the index u now be diduced, becoming x in the last aggregant:

$$\Pi_i \Pi_j \Pi_k \Pi_w \Pi_\beta \Pi_u \Sigma_x \overline{(i+j)}_k \dotplus \sigma_{wk} \dotplus \bar{q}_{\beta u} \dotplus \overline{(u+j)}_w \dotplus q_{\beta x} \cdot \bar{\sigma}_{xi}$$

Let $\bar{q}_{\alpha w}$ be inserted as an aggregant:

$$\Pi_i \Pi_j \Pi_k \Pi_\alpha \Pi_w \Pi_\beta \Pi_u \Sigma_x \ \overline{(i+j)}_k \dotplus \bar{q}_{\alpha w} \dotplus \sigma_{wk} \dotplus \bar{q}_{\beta u} \dotplus \overline{(u+j)}_w \dotplus q_{\beta x} \cdot \bar{\sigma}_{xi}$$

The insertion of this aggregant authorizes the insertion of its negative as a component of another aggregant.

$$\Pi_i \Pi_j \Pi_k \Pi_\alpha \Pi_w \Pi_\beta \Pi_u \Sigma_x \ \overline{(i+j)}_k \dotplus \bar{q}_{\alpha w} \dotplus \sigma_{wk} \dotplus q_{\alpha w} \cdot \{ \bar{q}_{\beta u} \dotplus \overline{(u+j)}_w \} \dotplus q_{\beta x} \cdot \bar{\sigma}_{xi}$$

* The Note to Paper XIII, 3.403A ff.?

We now diduce w, transmuting it in one term into z, and thus obtain finally,

(29)

$$\Pi_i\Pi_j\Pi_k\Pi_\alpha\Pi_w\Sigma_z\Pi_\beta\Pi_u\Sigma_x\ \overline{(i+j)}_k \maltese \bar{q}_{\alpha w} \maltese \sigma_{wk} \maltese q_{\alpha z}\cdot\left\{\bar{q}_{\beta u} \maltese \overline{(u+j)}_z\right\} \maltese q_{\beta x}\cdot\bar{\sigma}_{xi}$$

Here we have a nice little theoremidion, obvious though not self-evident. Namely, if any three positive integers, i, j, k, are such that k can result from adding i to j, then, selecting any class of integers we please, and speaking of the character of being an integer of this class as "the character α" either all integers of this class are as large as or larger than any integer k that can result from adding i to j, or else (if that is not the case) there is an integer of this class, z, if we take any second class of integers whatever (inclusion in which shall be called the character β) no integer u of this second class can on being added to j give the integer z, unless there be an integer x of the second class which is smaller than i. The form of statement is too strictly logical and formal for an ordinary mind readily to grasp it; but let us dilute it with a little verbiage, as follows. Suppose k is a positive integer which can result as the sum of j, as augend, and i, as addend. We select a first class of positive integers, say for example the cubes above 0 and 1. Now it may be that k does not exceed any of these. As to that case we say nothing. But should there be one or more of the first class that exceed k, then it may be that one of them is such that it cannot result from adding any positive integer to *j* as augend, because it may be less than j. It would have been better if, instead of writing $\Sigma_z\Pi_\beta\Pi_u$ in the *Peircean*, I had written $\Pi_\beta\Pi_u\Sigma_z$, for it is always allowable to carry Σ's to the right. Then the second class being selected first, it might happen that there was an integer of the first class that could not result from adding any integer of the second class to j. . . .

BOOK II

EXISTENTIAL GRAPHS[P]

MY CHEF D'OEUVRE

CHAPTER 1

EULER'S DIAGRAMS[E]

§1. LOGICAL DIAGRAM*

347. A diagram composed of dots, lines, etc., in which logical relations are signified by such spatial relations that the necessary consequences of these logical relations are at the same time signified, or can, at least, be made evident by transforming the diagram in certain ways which conventional 'rules' permit.

348. In order to form a system of graphs which shall represent ordinary syllogisms, it is only necessary to find spatial relations analogous to the relations expressed by the copula of inclusion and its negative and to the relation of negation. Now all the formal properties of the copula of inclusion are involved in the principle of identity and the *dictum de omni*. That is, if *r* is the relation of the subject of a universal affirmative to its predicate, then, whatever terms *X*, *Y*, *Z* may be,

> Every *X* is *r* to an *X*; and if every *X* is *r* to a *Y*, and every *Y* is *r* to a *Z*, every *X* is *r* to a *Z*.

Now, it is easily proved by the logic of relatives, that to say that a relation *r* is subject to these two rules, implies neither more nor less than to say that there is a relation *l*, such that, whatever individuals *A* and *B* may be,

> If nothing is in the relation *l* to *A* without being also in the same relation *l* to *B*, then *A* is in the relation *r* to *B*; and conversely, that,
> If *A* is *r* to *B*, there is nothing that is *l* to *A* except what is *l* to *B*.

349. Consequently, in order to construct such a system of graphs, we must find some spatial relation by which it shall

* *Baldwin's Dictionary of Philosophyand Psychology*, vol. 2, p. 28, (2d edition, 1911). The Macmillan Co., New York.

appear plain to the eye whether or not there is anything that is in that relation to one thing without being in that relation to the other. The popular Euler's diagrams fulfill one-half of this condition well by representing A as an oval inside the oval B. Then, l is the relation of being included within; and it is plain that nothing can be inside of A without being inside B. The relation of the copula is thus represented by the spatial relation of 'enclosing only what is enclosed by'. In order to represent the negation of the copula of inclusion (which, unlike that copula, asserts the existence of its subject), a dot may be drawn to represent some existing individual. In this case the subject and predicate ovals must be drawn to intersect each other, in order to avoid asserting too much. If an oval already exists cutting the space in which the dot is to be placed, the latter should bę put on the line of that oval, to show that it is doubtful on which side it belongs; or, if an oval is to be drawn through the space where a dot is, it should be drawn through the dot; and it should further be remembered that if two dots lie on the boundaries of one compartment, there is nothing to prevent their being identical. The relation of negation here appears as 'entirely outside of'. For a later practical improvement see Venn, *Symbolic Logic*, chapter xi.

§2. OF EULER'S DIAGRAMS*

350. In the second volume of the great Leonard Euler's *Lettres à une Princesse d'Allegmagne*, which appeared in 1772 (four years after the first volume), the nature of the syllogism is illustrated by means of circles, in substantially the following manner. Let the syllogism whose cogency is to be exhibited be the following:

All men are passionate,
All saints are men;
Therefore, All saints are passionate.

Imagine the entire collection of saints and nothing else to be enclosed in the imaginary circle, S, of Fig. 1; imagine the entire collection of men and nothing else to be enclosed in the imaginary circle, M — which will therefore enclose whatever

* From "Graphs," c. 1903.

is enclosed in the circle S, since all saints are men. Imagine the entire collection of passionate beings and nothing else to be enclosed in the circle, P, which will thus enclose whatever is enclosed in the circle M, since all men are passionate. We see, then, that whatever is enclosed in the circle, S, is enclosed in the circle, P; that is, that all saints are passionate.

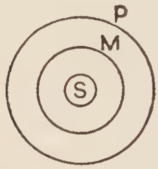


Fig. 1

Fig. 2

It will be remarked that the way in which any facts of enclosure relating to the circle M can inform us about any relation of enclosure between two other circles, is by the facts about M being that one of the other circles is on the inside of M and the other on the outside of it; so that, so far as this mode of representation exhibits the true nature of the syllogism, there ought to be just two kinds of syllogisms, one corresponding to Fig. 1, and the other to Fig. 2, which latter figure illustrates the following syllogism:

No man is perfect,
But any saint is a man;
Hence, no saint is perfect.

351. . . . We may now define the system of Euler's Diagrams, as he left it, in the following rules:

First, every area of the diagram represents the entire collection, or aggregate, of possibilities of a certain description.

Second, if a circle is drawn within an area, the part of the area within the circle represents the entire aggregate of those possibilities represented by the area to which a certain description applies, while the area outside the circle represents the entire aggregate of those possibilities represented by the area to which that description does not apply; and in general the area common to two areas represents the entire aggregate of possibilities which are at once represented by each of those two areas.

352. From these two principles it follows that to draw two circles or other areas, A and B, so that they have no common area, is to represent that there is no possibility which is at once of the description represented by A and of the description represented by B; *and this is the only way in which a Euler's diagram can represent a state of things to be a fact.* It is essential that this should be understood. Thus, in Fig. 3, let the entire area of the sheet represent all men now living. Let the circle G enclose all Greeks, and the circle C all courageous men. Then the four parts into which the two circles divide the whole sheet represent respectively whatever, i, Courageous Greeks; ii, Greeks not Courageous; iii, Courageous men not Greeks; iv, Men, neither Courageous nor Greeks, there may be among men now living. These are represented

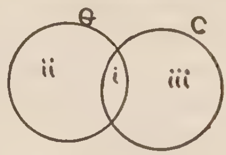


Fig. 3

merely as possible classes, without any assertion. Fig. 4 represents that all men consist of whatever Greeks not Courageous, Courageous Men not Greeks and Men neither Greeks nor Courageous there may be, and thus asserts that no Greek is Courageous; while Fig. 5 represents that men consist of whatever Courageous Greeks, Courageous Men not Greeks, and Men neither Courageous nor Greek there may be, and thus asserts that whatever Greek men there may be now living are courageous.

353. The history of this System of Graphs has been discussed by Mr. Venn;* and though he does not consider every

Fig. 4

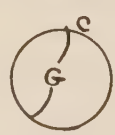


Fig. 5

historical question upon which one might desire to be informed, nothing additional will here be brought forward. The results

* *Symbolic Logic*, ch. 20, II, 2d ed., London, (1894).

are briefly these: Eight years before Euler's publication appeared, the *Neues Organon* of John Henry Lambert (Alsatian by birth, French by descent, but German by residence and by the honor and support that country rendered him) in which* the author made the same use of the stretches of parallel lines essentially as Euler did of the areas of circles, with an additional feature of dotted lines and extensions of lines. Lambert, however, does not seem to aim at any mathematical accuracy of thought in using his lines. He certainly does not attain it; nor could he do so, as long as he failed to perceive that the only purpose such diagrams could subserve is that of representing the necessity with which the conclusion follows from the premisses of a necessary reasoning, and that that necessity is not a compulsion in thinking (although there is such a compulsion) but is a relation between the *facts* represented in the premisses and the facts represented in the conclusion. The failure to comprehend the true nature of logical necessity and the confounding of it with a psychological compulsion is common to German logicians generally,† excepting only the Herbartians. Thus, he represents 'Some A is B' by this diagram.

B   ———
A . . . .

Fig. 6

Thus, a distinction is represented between 'Some A is B,' and 'Some B is A,' although they express the same fact. No doubt, there is a way of regarding such a fact, or supposed fact, as that 'Some Germans are given to subjective ways of thinking' which renders that a more natural mode of expression than 'Some men given to subjective ways of thinking [are] Germans.' But it is one thing to admit that this is so and quite another to admit that the sentence "expresses" that way of thinking, rather than the fact itself. The sentence is an assertion; and an assertion is of a fact and not of a way of thinking the fact. When a writer makes an assertion, his principal purpose is to induce the reader to believe in the reality of the fact asserted. He has the subsidiary design of causing the reader to follow along his line of thinking. . . . Throughout

* Bd. I, s. 111ff.

† Cf. 2.152ff.

Lambert's whole treatment of syllogistic, the way of thinking is made the principal thing. Under these circumstances, it was impossible for him to have a clear conception of the proper nature of a system of syllogistic graphs.

My reason for insisting at such length upon this point is that it is a passage of Lambert's *Architektonik** which is the principal authority for one of the main points of the current account of the history of the Eulerian diagrams. The usual assertion is that the voluminous pedagogist Christian Weise, the author of two works on logic (*Doctrina Logica*, 1690, and *Nucleus Logicæ*, 1691), who died 1708 October 21, made use of the system of diagrams in question. Nobody has ever examined Weise's own editions to see whether they bear out the assertion.† But it is said that one Johann Christian Lange in a book by him (*von ihm verfassten*) and entitled *Nucleus Logicæ Weisianæ*, published in 1712, tells how Weise so taught logic. Nobody, however, except Hamilton,‡ claims to have seen even this book. They appeal to a vague account of its contents in Lambert's *Architektonik*. Now this book of Lambert's preceded Euler's publication by a year; and in view of Lambert's crude notions of what such diagrams ought to be, and in view of his not apparently being greatly struck by what would, to his mathematical mind in its benighted condition concerning syllogistic, have been a great light, the passage of Lambert is rather against the claims made for Weise than in their favor. It is curious that even Ueberweg§ talks of Lange as the *author* of the publication of 1712. But to anybody familiar with such literature the title proclaims it to be a work by Weise probably with a running commentary or copious notes by Lange. The passage in Lambert's *Architektonik* was first brought to light by Drobisch in 1851 in the second edition of his *Neue Darstellung der Logik*.¶ But Hamilton in his fourteenth lecture on logic, publicly delivered in 1837–8 and regularly afterwards till his death, says|| "I find

* *Anlage zur Architektonik*, i. 28.

† Cf. Venn, *Symbolic Logic*, 2 ed., p. 509.

‡ *Lectures on Metaphysics and Logic*, III, 256 (1874).

§ *System of Logic*, p. 302 (1871).

¶ Th. 1, Abs. 1, §38, n. 2.

|| *Op. cit.*, p. 256.

it [*i.e.*, the mode of sensualizing by circles the abstractions of logic] in the *Nucleus Logicæ Weisianæ*, which appeared in 1712; but this was a posthumous publication, and the author, Christian Weise, who was Rector of Zittan, died in 1708." Hamilton was mistaken in supposing that the book had not appeared before; for it was published originally in 1691. What Hamilton here attributes to Weise falls very far short indeed of the system of Eulerian diagrams. It is true that Hamilton appears to confound the two; but no careful student of this strikingly unmathematical scholar will attribute any importance to such a unification. In this very same passage, he attributes to Alstedius, in 1614, the use of Lambert's linear graphs, which his editors are compelled to admit is a gross exaggeration.* How utterly unfounded it was is shown by Venn.† When we think of the great reputation of Weise in his own day, it is almost incredible that so striking an idea as that of Euler's diagrams should have been developed by so prominent a man without attracting universal attention. Until further evidence is adduced his claims to their authorship must be pronounced quite unsupported. But Friedrich Albert Lange in his remarkable *Logische Studien* (p. 10) says that substantially the Eulerian method is mentioned by the celebrated Juan Luis Vives‡ early in the sixteenth century, and in an offhand manner (*die schlichte Art ihrer Einführung*) that would seem to indicate that it was traditional in the schools. Venn§ copies the passage and diagram, which shows the cardinal idea of the Eulerian diagrams, that the middle term is like a boundary separating the two regions in which the other two terms respectively lie — and this much probably was traditional — but gives no hint of any development of this idea into a sort of calculus, such as Euler's system is. Of this, the principal achievement, Euler is the author. After Euler several attempts were made to improve the system; but all of them were blunders until Venn's publications in 1880. Venn made a distinct improvement, and I shall endeavor to contribute others; but before giving an account

* *Ibid.*, Note β.

† *Op. cit.*, pp. 507–8.

‡ *Opera*, i, 607.

§ *Op. cit.*, p. 507.

of them, it will be requisite to study critically Euler's original proposal.

354. What is it, then, that these diagrams are supposed to accomplish? Is it to prove the validity of the syllogistic formula? That sounds rather ridiculous — as if anything could be more evident than a syllogism — yet that is not far from the opinion of Friedrich Albert Lange, a thinker of no ordinary force. Suppose we ask ourselves *why* it is that, if a circle P wholly encloses a circle M which itself wholly encloses a circle S, the circle P necessarily wholly encloses the circle S. In order to express the answer, it will be well to avail ourselves of a phraseology proper to the logic of relatives. I use the words *relation* and *relative* in a somewhat narrow sense, which I begin by explaining. Take, then, any assertion[1] whatever about a number of designate individuals. These individuals may be persons, material objects, actions, collections of things, possible courses of events, qualities, abstractions of any kind, and, in short, of any one nature or of any several natures whatsoever; only each of them must be well-known and rated by a proper name, and each must belong to some universe, or total aggregate of things of the same wide class, and the assertion must be such that if any one of the individuals did not really occur in its universe, independently of whether you, I, or any collection of men or other cognoscitive beings should opine that it did or that it did not, then that assertion would be false. For example, if in the assertion that Mrs. Harris was unbeknown to Betsy Prig except by hearsay, "unbeknown" be understood in such a sense that the non-existence of Mrs. Harris would render it true, then not only does this assertion not fulfill the condition, but — still taking "unbeknown" in the same sense — no more would the assertion that Sairey Gamp was unbeknown to Betsy Prig except by hearsay; while if "unbeknown" be taken in such a sense that the first assertion is rendered false by the non-existence of Mrs. Harris, then, although that assertion would not fulfill the condition because Mrs. Harris did not belong to the universe of characters in *Martin Chuzzlewit*, yet, taking the word in this sense, the assertion, "Sairey Gamp is unbeknown to

[1] Two different sentences having the same meaning precisely are expressions of the same assertion.

Betsy Prig except by hearsay," will perfectly fulfill the condition; and neither its falsity nor the fictitiousness of the universe to which Sairey Gamp and Betsy Prig belong are any objections. . . .

An assertion fulfilling the condition having been obtained, let a number of the proper designations of individual subjects be omitted, so that the assertion becomes a mere blank form for an assertion which can be reconverted into an assertion by filling all the blanks with proper names. I term such a blank form a *rheme*. If the number of blanks it contains is zero, it may nevertheless be regarded as a rheme, and under this aspect, I term it a *medad*. A medad is, therefore, merely an assertion regarded in a certain way, namely as subject to the inquiry, How many blanks has it? If the number of blanks is one, I term the rheme a *monad*. If the number of blanks exceeds one, I term it a *Relative Rheme*. If the number of blanks is two, I term the rheme a *Dyad*, or *Dyadic Relative*. If the number of blanks exceeds two, I term it a *Polyad*, or *Plural Relative*, etc. A *Relation* is a substance whose being and identity precisely consist in this; its being, in the possibility of a fact which could be precisely asserted by filling the blanks of a corresponding relative rheme with proper names; its identity, in its being in all cases so expressible by the same relative rheme.* It must be confessed that it would have been better if a modifying adjective had been attached to the words *relative* and *relation* to form the technical terms to designate what have just been defined as a relative rheme and a relation. But now that these terms have been established by me, my convictions of the ethics of terminology† forbid me to attempt to alter the meanings attached to them. I use the word "signify" in such a sense that I say that a relative rheme *signifies* its corresponding relation. In the technical language of the logic of relatives, letters of the alphabet are employed as pronouns to denote relatives, just as, in ordinary and especially in legal language, they are often used as relative pronouns. The ancient grammarians defined a pronoun as a word used to replace a noun, a most preposterous attempt at analysis. It would have been far nearer the truth

* Cf. 3.466; 3.571.

† See vol. 2, bk. II, ch. 1.

to describe a common noun as a word used in place of a pronoun.* In the middle ages, Duns Scotus and others brought a correcter definition into vogue; but the humanists of the reformation stickled for the ancient definition and that of the scholastics was quite forgotten. . . . A relative pronoun designates a subject by indicating, through its position and agreement, a noun that designates that subject. This nearly corresponds to the use of letters in the Catechism, "What is your Name? *Answer* N., or M." and the priest in dipping the child in the water so "discretely and warily," is represented as saying "N. I baptize thee in the name" etc. The point is that in neither case is it meant that the letter is pronounced, but this letter designates the person through indicating by its position that it is to be replaced by the Christian name. . . . So in logic, *Barbara* is described as the syllogistic form

Any M is P,
Any S is M;
∴ Any S is P.

What is meant is that the letters S, M, and P, in this formula, may be replaced by any terms whatever; only each letter must everywhere in the formula be replaced by the same term. In the logic of relatives, the letters r, s, ρ, σ, etc., are frequently employed as substitutes for dyadic relatives, so that "A is r of B" and "B is r'd by A" stand for different expressions of the same fact, analogous to "A is lover of B" and "B is loved by A."

355. With this explanation of terms, we can intelligibly answer the question, "*Why* does a circle, P, that wholly encloses a circle M, itself wholly enclosing a circle, S, likewise necessarily enclose the circle, S?" Namely, understanding by this question, "What is the peculiarity of the relation of wholly enclosing which renders this necessary?" we answer, "It is because the relation of wholly enclosing is such that there is a dyadic relative, r, such that to say that any place, X, wholly encloses a place, Y, is equivalent to saying that X is at once r of Y and is r of everything that is r'd by Y." To show that this is the explanation, we must prove two propositions: firstly, that there is a dyadic relative, r, such that, on the

* Cf. 2.287n.

one hand, if the place X wholly encloses the place Y, the place X is *r* of Y and is *r* of everything *r*'d by Y, while on the other hand, if X does not enclose Y, either X is not *r* of Y or else there is something *r*'d by Y that is not *r*'d by X; and secondly, that from that first proposition it necessarily results that if any circle, P, wholly encloses a circle, M, itself wholly enclosing a circle, S, then P wholly encloses the circle, S. Before proving the first of these propositions, it is to be remarked that whether we affirm that the place X wholly encloses the place Y, [or whether] we say that the place, X, does not wholly enclose the place, Y, we are to be understood as recognizing X and Y as definite places in space, so that if either of them is not of that nature, both the one assertion and the other are false. Now in order to prove the first proposition, it will suffice to make *r* signify the relation of *not* being quite at a distance from (with intervening place), so that the first clause of the first proposition will be that if the place, X, wholly encloses the place, Y, then X is not altogether at a distance from Y, nor is it altogether at a distance from any place from which Y is not altogether at a distance. That, if X wholly encloses Y it is not altogether at a distance from Y, is self-evident. Moreover, if we consider any place Z, from which Y is not altogether removed, there must be some point of Z from which Y is not altogether removed, and from this point, X will not be altogether removed. Hence it is evident that X is not altogether removed from any place from which Y is not altogether removed; and the first clause of the proposition is found to be true. The other clause of the proposition is that, if X does not wholly enclose Y, then either X is not *r* of Y or else there is something *r*'d by Y that is not *r*'d by X; that is, if X does not wholly enclose Y, either X is altogether remote from Y or else there is some place not altogether distant from Y from which X is entirely remote. This is plainly true, since if X does not wholly enclose Y, there is some point of Y which lies quite outside of X; and such a point will be a place from which Y is not remote but from which X is remote. Thus the first proposition is true. It remains then to be shown that, from this peculiar form of the relation of total inclusion, it follows that a circle, P, wholly enclosing a circle, M, itself wholly enclosing a circle, S, likewise wholly encloses the circle,

S. . . . Now it is clear that as long as P is *r* of whatever is *r*'d by M, if S is *r*'d by M, so long will P be *r* of S; while as long as P is *r* of whatever is *r*'d by M, and whatever is *r*'d by S is *r*'d by M, P is *r* of whatever is *r*'d by S. Thus if P is *r* of whatever is *r*'d by M and if both S and whatever is *r*'d by S are *r*'d by M, P is *r* both of S and of whatever is *r*'d by S; *quod erat demonstrandum.* Thus the reason, that the geometrically wholly enclosed by the wholly enclosed is itself wholly enclosed, is shown. But this is the very same reason substantially that Aristotle* gives for the validity of the syllogism in *Barbara.*

Any M is P,
Any S is M;
∴ Any S is P.

For Aristotle's doctrine is that this depends on the essential nature of being *dictum de omni*, or universally predicated. This essential nature he says is, that to say that X is predicated of the whole of Y, is to say that X is predicated of Y and of whatever Y is predicated of.

That is, the relation of universal predication is also of the form, "At once *r* of and *r* of whatever is *r*'d by." He might have avoided the apparent *circulus in definiendo* by stating the matter thus: X is predicated of all Y if and only if X is not foreign to Y nor to any term to which Y is not foreign. Thus, as far as logical dependence goes, the validity of the syllogism and the property of the Eulerian diagram depend upon a common principle. They are analogous phenomena neither of which is, properly speaking, the cause or principle of the other. Lange† is of opinion that all reasoning proceeds by the observation of imaginary Euler's diagrams or of something closely similar; and I,‡ for my part, share his opinion so far as to admit that an imaginary observation is the most essential part of reasoning. But the psychological process is not the matter in question. This brings us back to the inquiry, What purpose are the diagrams fitted to subserve?

* *Prior Analytics*, I, 1, 24b, 28; see also vol. 2, bk. III, ch. 4, §14 and Joseph's *An Introduction to Logic*, p. 296n and p. 308n, 2d. edition, revised (1916).

† *Op. cit.*, S. 9ff.

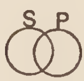
‡ Cf. 2.77 and 2.444.

They may help to analyze reasonings, and this either in a practical way by aiding a person in rendering his ideas clear, or theoretically. In either regard it is desirable that they should be adequate to represent the gist of every kind of deductive reasoning.

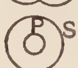
356. As Euler left the system, it had the following faults:

First, two circles cannot be each inside the other; so that while, as Mrs. Franklin has shown (*Johns Hopkins Studies in Logic*, p. 64), there are fifteen or sixteen different ways in which two terms may be related in reference to the possibility or impossibility of their different combinations, Euler's original diagrams show but eight of these, as follows:

Fig.

(7) 

Entire Ignorance

(8) 

Any P is S

(9) 

No S is P

(10) 

There is no P

(11) 

Any S is P

(12) 

S and P identical

Fig.

(13) 

There is no S

(14) There is neither S nor P

The states of possibility not represented are as follows:

Everything is either S or P; [S ∨ P]
Everything is S; [S]
No S is P, but everything but S is P; [S ≡ −P]
Everything is S and nothing is P; [S. −P]
Everything is P; [P]
Everything is both S and P; [S.P]
Nothing is S but everything is P; [−S.P]
The Universe is absurd and impossible; [P· −P]

Second, in regard to every combination of terms (that is, in regard to each of the possible parts of the universe, when we are in complete ignorance), the system is limited to expressing its non-existence or to not expressing whether it exists or not. *It cannot affirm the existence* of any description of an object. But a categorical, though possibly partial, description of the universe in its relation to two terms can, in reference to each of the four possible parts into which those two terms can divide the universe of possibility, either affirm its existence, or deny its existence, or say nothing. Therefore, excluding the absurd assertion that nothing exists, there are 3^4-1, or eighty, possible categorical descriptions of the universe, of which this system can express but one tenth part.

Third, the system affords no means of expressing a knowledge that one or another of several alternative states of things occurs. Of the sixteen possible dichotomic states of things with reference to two terms, a state of knowledge may either exclude or admit each, though it cannot exclude all. There are therefore $2^{16}-1$, or 65535, possible states of dichotomous information about two terms of which the system permits the expression of only eight, or one out of every 8192.

Fourth, the system affords no means of expressing any other than dichotomous, or qualitative, information. It cannot express enumerations, statistical facts, measurements, or probabilities. In short, it affords no room for the introduction of quantitative premisses into its reasonings.

Fifth, the system affords no means of exhibiting reasoning, the gist of which is of a relational or abstractional kind. It does not extend to the logic of relatives.

357. Some of these imperfections are, however, easily removed. This first of them was done away with by an im-

Fig. 15

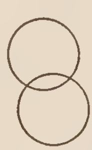


Fig. 16

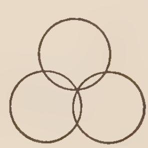


Fig. 17

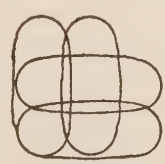


Fig. 18

provement introduced by Mr. Venn in 1880. Namely, Mr. Venn in his *Symbolic Logic* (I use the first edition of 1881) recommends drawing the diagrams so as always to exhibit all the possible parts into which terms, to the number employed, would, in the absence of all information, divide the universe. That done, if information is received that certain of these parts do not exist, the corresponding regions of the diagrams are shaded. Thus the areas representing the terms may be arranged in one of the following ways according as they are one, two, three, or four in number. With more than four terms the system becomes cumbrous; yet, by having on hand lithographed blank forms showing the four-term figure on a large scale, all the compartments containing repetitions of one figure, whether that for one term, for two terms, for three or for four, and considering corresponding regions of all sixteen of the large compartments to represent together the extension of one term, it is possible without much inconvenience to increase the number of terms to eight. Beyond eight terms, the best way will simply be to make a list of the regions, numbered in the dichotomous system of arithmetical notation, one numerical place being appropriated to each term.

Instead of shading excluded regions we may simply make them with the character 0, for zero.

358. The unmodified Eulerian system gives two syllogistic diagrams as shown above, Figs. 1 and 2. These with the modi-

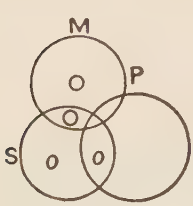


Fig. 19

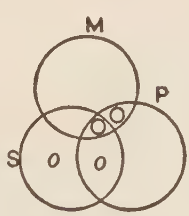


Fig. 20

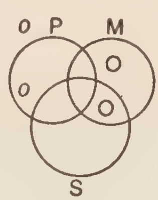


Fig. 21*

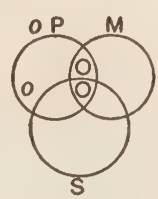


Fig. 22†

fication are shown in Figs. 19 and 20. The exclusions by different premisses are marked differently. Venn's modification furnishes two new syllogistic diagrams shown in Figs. 21 and 22.

359. The second imperfection of the system is also very readily remedied; and the remedy almost inevitably suggests a partial remedy for the third imperfection. Namely, why not draw the character X in any compartment in order to signify

* *I.e.*, All men are passionate and all non-saints are men.

† *I.e.*, No men are passionate and all non-saints are men.

that something of the corresponding description occurs in the universe? We shall thus get these three forms of propositions:

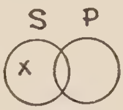


Fig. 23

Some S is not P.

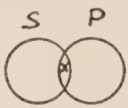


Fig. 24

Some S is P.

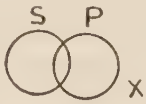


Fig. 25

There is something beside S and P.

The precise denial of each of these is produced by substituting 0 for X. But when a third term is present some further rule has to be determined. How shall we mark the following diagram in order to express "Some S is not P"? The proposi-

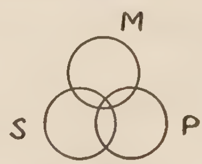


Fig. 26

tion will here take the form, Either some S that is M but not P exists or some S that is neither M nor P exists. One suggestion would be that a cross be made on the circumference of M. But this would only provide for a special class of disjunctions. The question would then become, How shall we express, "Either something that is at once S and P exists or something that is neither S nor P exists"? Since we have drawn *zeros* at once in two compartments to signify the non-existence of either of two classes of objects, if we are to adhere to the principle that precise denial is produced by substituting crosses for zeros and conversely, it would follow that two crosses in two compartments would signify that something exists either of the one or of the other class. But this decision would render it impossible to give any systematic interpretation to a cross in one compartment and a zero in another.

Suppose, then, that signs in different compartments, if disconnected, are to be taken conjunctively, and, if connected, disjunctively, or *vice versa*. Then precise denial will be effected by reversing the characters of the signs and of their relations as to connection or disconnection. There are perhaps no very compulsive reasons for adopting one interpretation of the connection of signs rather than the other. But it would seem strange if the insertion of a new and disconnected sign should cause a diagram to assert less; while the modification of an existing sign, by attaching to it a line of connection terminating in a new sign, might well enough diminish the assertion. It seems also quite natural that to mark the same compartment independently with contradictory signs, as in Fig. 27, should be absurd, while that if the two opposite signs are connected, as in Fig. 28, they should simply annul one another and be equivalent to no sign at all.

Fig. 27


Fig. 28

Moreover, a cross on a boundary line may very naturally be understood to be equivalent to two connected crosses on the two sides of the boundary. Another consideration, perhaps more decisive, is that we shall necessarily regard the connected assertions as being put together directly, while the detached connexi of assertions are afterward compounded. It is therefore a question between using copulations of disjunctions [or] disjunctions of copulations. The former is the more convenient. . . .

360. Let this rule then be adopted:

> Connected assertions are made alternatively, but disconnected ones independently, *i.e.*, copulatively.

361. As a consequence of this rule and of the introduction of the cross, the permissible transformations of diagrams, which transformations of course signify inferences, become so various that it is time to draw up a code of Rules for them. "Rules" is here used in the sense in which we speak of the

"rules" of algebra; that is, as a permission under strictly defined conditions.[1]

362. *Rules of Transformation of Eulerian Diagrams*

Rule 1. Any entire sign of assertion (*i.e.*, a cross, zero, or connected body of crosses and zeros) can be erased.

Rule 2. Any sign of assertion can receive any accretion. Thus Fig. 29 may be transformed into Fig. 30.*

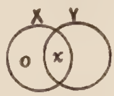


Fig. 29

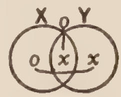


Fig. 30

Rule 3. Any assertion which could permissively be written, if there were no other assertion, can be written at any time, detachedly.

Rule 4. In the same compartment repetitions of the same sign, whether mutually attached or detached, are equivalent to one writing of it. Two different signs in the same compartment, if attached to one another are equivalent to no sign at all, and may be erased or inserted. But if they are detached from one another they constitute an absurdity. All the foregoing supposes the signs to be unconnected with any in other compartments. If two contrary signs are written in the same compartment, the one being attached to certain others, P,

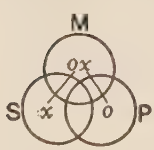


Fig. 31

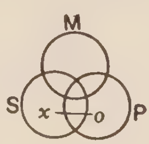


Fig. 32

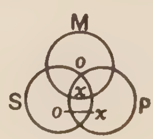


Fig. 33

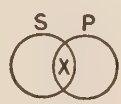


Fig. 34

[1] This curious use of the word *Rule* is doubtless derived from the use of the word in Vulgar Arithmetic, where it signifies a method of computation adapted to a particular class of problems; as the Rule of Three, the Rule of Alligation, the Rule of False, the Rule of Fellowship, the Rule of Tare and Tret, the Rule of Coss. Here the Rule is a body of directions for performing an operation successfully. But when we speak of the Rule of Transposition, the directions are so simple, that the Rule becomes principally a permission.

* *I.e.*, "All x is y and Some x is y" can be transformed to "Either All x is y or Some $\bar{x}$ is y, and Some x is y or All $\bar{x}$ is y."

and the other to certain others, Q, it is permitted to attach P to Q and to erase the two contrary signs.

Thus, Fig. 31 can be transformed into Fig. 32.*

Rule 5. Any Area-boundary, representing a term, can be erased, provided that, if, in doing so, two compartments are thrown together containing independent zeros, those zeros be connected, while if there be a zero on one side of the boundary to be erased which is thrown into a compartment containing no independent zero, the zero and its whole connex be erased.

Thus, Fig. 33 can be transformed into Fig. 34.†

Rule 6. Any new Term-boundary can be inserted; and if it cuts every compartment already present, any interpretation desired may be assigned to it. Only, where the new boundary passes through a compartment containing a cross, the new boundary must pass through the cross, or what is the same thing, a second cross connected with that already there must be drawn and the new boundary must pass between

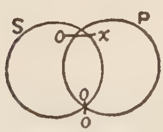


Fig. 35

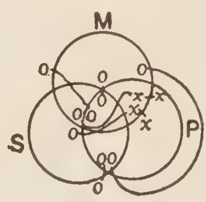


Fig. 36

them, regardless of what else is connected with the cross. If the new boundary passes through a compartment containing a zero, it will be permissible to insert a detached duplicate of the whole connex of that zero, so that one zero shall be on one side and the other on the other side of the new boundary.

Thus, Fig. 35 can be converted into Fig. 36.‡

These six rules have been written down entirely without

* *I.e.*, "Either Some $S\bar{P}$ is $\bar{M}$ or All M is $S \vee P$, and Some $M\bar{P}$ is $\bar{S}$ or All P is $M \vee S$" is transformable into "Either Some $S\bar{P}$ is $\bar{M}$ or All P is $M \vee S$."

† *I.e.*, "Either All S is $P \vee M$ or Some $P\bar{M}$ is $\bar{S}$, and Some SM is P and All M is $S \vee P$" is transformable into "Some S is P."

‡ *I.e.*, "Either All S is P or Some P is $\bar{S}$, and either No S is P or No $\bar{S}$ is $\bar{P}$" is transformable into "$\bar{M}\bar{S}\bar{P}=0$ or $MSP=0$; and $MSP=0$ or $M\bar{S}\bar{P}=0$; and $M\bar{S}\bar{P}=0$ or $\bar{M}SP=0$; and $\bar{M}SP=0$ or $\bar{M}\bar{S}\bar{P}=0$; and $MS\bar{P}=0$ or Some $\bar{S}M$ is P or Some $\overline{SM}$ is P; and $\bar{M}S\bar{P}=0$, or Some $\bar{S}M$ is P or Some $\bar{S}\bar{M}$ is P."

preconsideration; and it is probable that they might be simplified, and not unlikely that some have been overlooked.

363. As thus improved, Euler's diagrams are capable of giving an instructive development of the particular syllogism. The premisses of *Darii* are as follows:

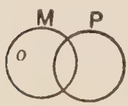


Fig. 37

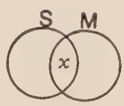


Fig. 38

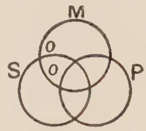


Fig. 39

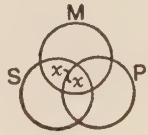


Fig. 40

Fig. 37: Any M is P. S being inserted this gives, by Rule 6, Fig. 39. Fig. 38: Some S is M, P being inserted, this becomes, by Rule 6, Fig. 40. Uniting Figs. 39 and 40 by Rule 3, we get Fig. 41, and by Rule 4, Fig. 42. Now erasing M by Rule 5,

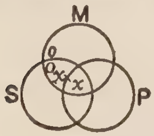


Fig. 41

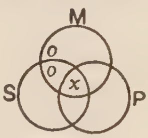


Fig. 42

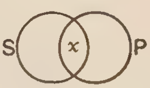


Fig. 43

we get Fig. 43. *Baroko*, *Bokardo*, and *Frisesomorum* proceed in the same way. The premisses of the last are as follows:

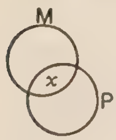


Fig. 44

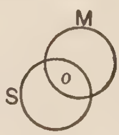


Fig. 45

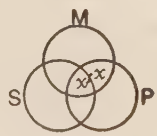


Fig. 46

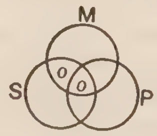


Fig. 47

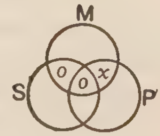


Fig. 48

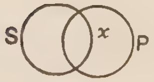


Fig. 49

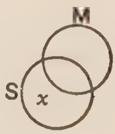


Fig. 50

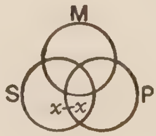


Fig. 51

Fig. 44: Some M is P, which, by Rule 6, gives Fig. 46. Fig. 45: No S is M, which, by Rule 6, may give Fig. 47.

Combining these by Rule 3, Rule 4 gives Fig. 48 and Rule 5, Fig. 49. Let us now make the second premiss particular, as well as the first. We thus have Fig. 50 in place of Fig. 45; and on inserting P, we have Fig. 51 in place of Fig. 47. Uniting Figs. 46 and 51 we get Fig. 52. We now introduce two new and *undescribed* terms, as in Fig. 53, and on erasing M, we get Fig. 54 of which the interpretation is "Some S is

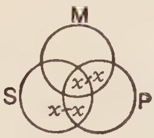


Fig. 52

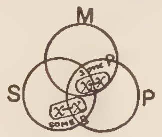


Fig. 53

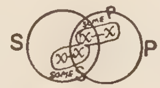


Fig. 54

not some P." The objection may be raised that this method of dealing with the spurious* syllogism does not seem to follow from general principles, as a matter of course. In view of that objection we may put a single cross on the boundary instead of two connected crosses. The reasoning then proceeds, by uniting Figs. 44 and 50, as shown in Figs. 55 and 56. A portion of the boundary of M is retained in Fig. 56 to show that on whichever sides of the boundaries the two crosses may belong, they can in no case fall within the same region. Let it be noted, by the way, as a suggestive circumstance, that the portion of the boundary of M now remaining is simply a sign of negation.

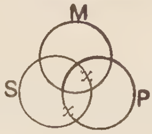


Fig. 55

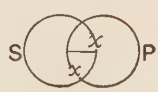


Fig. 56

364. This proposition "Some S is not some P" is called by Mr. B. I. Gilman, in a paper which constitutes a distinct step in logical research, but which is buried in the Johns Hopkins Bulletins,† a proposition "particular in the second degree." An ordinary particular proposition asserts the existence of at least one individual of a given description. A proposition particular in the second degree asserts the existence of at least two individuals. It is an inference from two particular propositions each of which affirms the existence of one

* See 2.526n., 2.607 for the meaning of this term.

† *Johns Hopkins University Circular*, August, 1882.

of the two individuals. We should therefore expect that, from a particular proposition of the second degree combined with one of the first degree, the inference should affirm the existence of three objects. Let us try the experiment. Fig. 57 shows that the conclusions from the two premisses

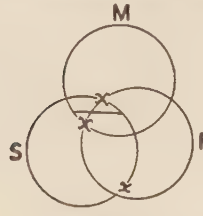


Fig. 57

Some S is not some M, and
Some P is not M,

is "Some S is other than something other than some P." But the S and the P in question are represented by the two lower crosses in the figure; and since these border upon the same compartment they may refer to the same individual. But if in addition to two ordinary particular premisses we take a universal premiss we can get a conclusion affirming the existence of three individuals. Take for instance the premisses

Some S is not M
Some M is P
No N is P
Some M is N

These premisses are combined in Fig. 58; and it will be seen that the three connexes of crosses must be all different individuals; so that the conclusion is "Some S is other than and other than something other than some P." This line of study is far from being a trivial matter, however it may appear to superficial thinkers. But it does not enter into the purpose of the present paper to pursue it further.

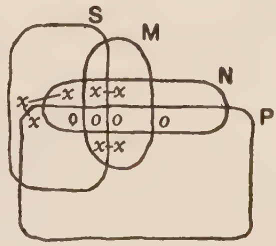


Fig. 58

365. In remedying the second imperfection we have gone far to remove the third and have even done something toward a treatment of the fourth. Let us consider a moment how far it can now be said that the method is inadequate to dealing with disjunctions. If by a disjunctive proposition we mean the sort of propositions usually given in the books as examples of this form, there never was any difficulty at all in dealing

with them by Euler's diagrams in their original form. But such a proposition as "Every A is either B or C" which merely declares the non-existence of an A that is at once not B and not C, is not properly a disjunctive proposition. It is only disjunctions of conjunctions that cause some inconvenience; such as "Either some A is B while everything is either A or B, or else All A is B while some B is not A." Even here there is no serious difficulty. Fig. 59 expresses this proposition. It is merely that there is a greater complexity in the expression than is essential to the meaning. There is, however, a very easy and very useful way of avoiding this. It is to draw an Euler's Diagram of Euler's Diagrams each surrounded by a circle to represent its Universe of Hypothesis. There will be no need of connecting lines in the enclosing diagram, it being understood that its compartments contain the several possible cases. Thus, Fig. 60 expresses the same proposition as Fig. 59.

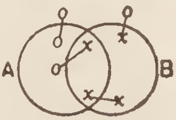


Fig. 59

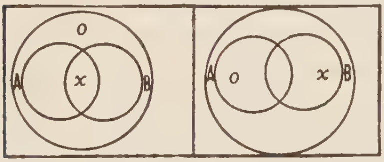


Fig. 60

366. Let us now consider the fourth imperfection. We are already in condition to express minimal multitudes. Thus Fig. 61 expresses that there are at least four A's. The precise

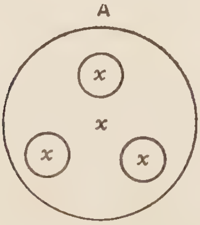


Fig. 61

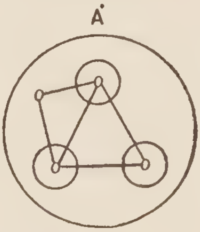


Fig. 62

denial of a minimal proposition will be a maximal proposition; and consequently, Fig. 62 must express that there are not as many as four A's. It is necessary here that the whole area of A should be covered by the parts.

This mode of expression becoming impracticable, except for very small numbers, it naturally occurs to us to write a number in a compartment to express the precise multitude of individuals it contains. By extending this to algebraic expressions, not merely ratios but all sorts of numerical relations can be expressed.

367. The fifth fault of the system is by far the worst; and if there is any cure for it, not the smallest hopeful indication of its possibility appears at present.

368. Let us now endeavor to seize upon the spirit and characteristic of this system of graphs, and to estimate its value. Its beauty — a violent inappropriate word, yet apparently the best there is to express the satisfactoriness of it upon mere contemplation — and its other merits, which are fairly considerable, spring from its being veridically iconic, naturally analogous to the thing represented, and not a creation of conventions. It represents logic because it is governed by the same law. It works the syllogism as the planet integrates the equation of Laplace, or as the motion of the air about a pendulum solves a mathematical problem in ideal hydrodynamics. Still more closely, it resembles the application of geometry to algebra. By this I mean what is commonly called the application of algebra to geometry, but surely quite preposterously and contrarily to the spirit of the study. I hope no set argument is needed to defend this statement. The habitual neglect by students of analytical geometry of the *real* properties of loci, of which very little is known, and their almost exclusive interest in the *imaginary* properties, which are non-geometrical, sufficiently show that it is geometry that is the means, algebra the end. Geometry is not a perfect fit to algebra, in some respects falling short, in others over-running; elliptic in the absence of the imaginary, hyperbolic in presenting a continuity to which analytic quantity can hardly be said to make any approach. Yet even its partial analogy has been so helpful to modern algebra (and it was not less so to the older doctrine) that the phrase "it has been the making of it" is not too strong. For no doubt it was geometry that suggested the importance of the linear transformation, that of invariance, and in short almost all the profounder conceptions. The analogy of the doctrine of the Eulerian diagrams to non-relative logic is *proportionately* fully as great; although, owing to the greater simplicity of the subject and to its having fewer char-

acters in all, the absolute number and weight of the points of resemblance are necessarily less. Such mathematics, as there may be connected with non-relative logic, we should have a right to expect would be much illuminated by the Eulerian Diagrams. Only this [is] mathematics of the most rudimentary conceivable kind; and hardly stands in need of any particular illumination. The different branches of pure mathematics are distinguished by their different systems of quantity; that is, of systems of points, units, or elements. In algebra, these points are so distributed over a surface that, in whatever manner any one is related to a single other exclusively, in that same manner is this other related to a third, and so on, *ad infinitum;* and moreover this infinite series may tend toward a definite limit, which limit is, in every case, included in the system. This is the most highly organized system of quantity that mathematicians have ever succeeded in definitely conceiving. On the other hand, the very simplest and most rudimentary of all conceivable systems of quantity is that one which distinguishes only two values. This [is] the system of evaluation which ethics applies to actions in dividing them into the right and the wrong, and which non-relative logic applies to assertions in dividing them into the true and the false. The mathematics of such a system — dichotomous mathematics — amounts to very little. Those who seek to make a calculus of the algebra of logic struggle vainly after mathematical interest by complicating their problems. They do not succeed: mere complication has not even a mathematical interest.

369. Dichotomous mathematics does not amount to much, but it does amount to something. For example, the subject of higher particular propositions, in consequence of not being perfectly familiar, will call for considerable reflection to understand in its entirety and in its connections. Complicated questions of non-relative deductive reasoning are rare, it is true; still, they do occur, and if they are garbed in strange disguises, will now and then make the quickest minds hesitate or blunder. Euler's diagrams are the best aids in such cases, being natural, little subject to mistake, and every way satisfactory. It is true that there is a certain difficulty in applying them to problems involving many terms; but it is an easy art to learn to break such problems up into manageable fragments.

The improved Boolian algebra has some advantages for those who are expert in its use, and who do not allow their instrument to rust from want of use. But the diagrams are always ready . . . .*

370. Any broad mathematical hypothesis, like that of a system of values, will attract three classes of students by three different interests that attach to it. The first is the special interest in the circumstance that that hypothesis necessarily involves certain relations among the things supposed, over and above those that were supposed in the definition of it. This is the mathematical interest proper. The second is the methodeutic interest in the devices which have to be employed to bring those new relations to light. This is a matter of supreme interest to the mathematician and of considerable, though subordinate, interest to the logician. The third is the analytical interest in the essential elements of the hypothesis and of the deductive processes of the second study, in their intellectual pedigrees and in their conceptual affiliations with ideas met with elsewhere. This is the logical interest, *par excellence.* In the case of non-relative deductive logic, that is, the doctrine of the relations of truth and falsity between combinations of non-relative terms, the methodeutic interest is slight owing to the extreme simplicity of the methods. The logical interest, on the other hand, limited as the subject is when relative terms are excluded, is very considerable, not to say great. In the inquiries which it prompts, it is the simplest cases which will chiefly attract attention, and therefore the circumstance, that the system of Eulerian diagrams becomes too cumbrous and laborious in complicated problems, is no objection to it. While the student cannot be counselled to confine himself to any single method of representation, the system of Eulerian diagrams is probably the best of any single one for the purely non-relative analysis of thought. Thus, it at once directs attention to the circumstance that the syllogism may be considered as a special case of the inference from Fig. 63 to Fig. 64, where the blots may either be zero or crosses or one a zero and the other a cross. Another example of the analytical interest of the system lies in the higher particular propositions,

* Peirce here illustrates the method by solving a complicated problem through the use of one hundred and thirty-five circles.

where we see an evolution of the conception of multitude. Multitude, or maniness, is a property of *collections*. Now a collection is an *ens rationis*, or abstraction; and abstraction appears as the highest product of the development of the logic

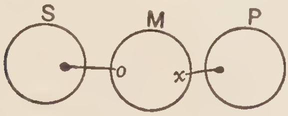


Fig. 63

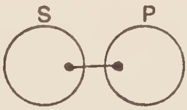


Fig. 64

of relatives. The student is thus directed to the deeply interesting and important problem of just how it is that the conception of multitude merges in the Eulerian diagrams.

371. The value of the system is thus considerable. Its fatal defect seems to be that it has no vital power of growth beyond the point to which it has here been carried. But this seeming may perhaps only be the reflection of the present student's own stupidity.

CHAPTER 2

*SYMBOLIC LOGIC**

372. If symbolic logic be defined as logic — for the present only deductive logic — treated by means of a special system of symbols, either devised for the purpose or extended to logical from other uses, it will be convenient not to confine the symbols used to algebraic symbols, but to include some graphical symbols as well.

373. The first requisite to understanding this matter is to recognize the purpose of a system of logical symbols. That purpose and end is simply and solely the investigation of the theory of logic, and not at all the construction of a calculus to aid the drawing of inferences. These two purposes are incompatible, for the reason that the system devised for the investigation of logic should be as analytical as possible, breaking up inferences into the greatest possible number of steps, and exhibiting them under the most general categories possible; while a calculus would aim, on the contrary, to reduce the number of processes as much as possible, and to specialize the symbols so as to adapt them to special kinds of inference. It should be recognized as a defect of a system intended for logical study that it has two ways of expressing the same fact, or any superfluity of symbols, although it would not be a serious fault for a calculus to have two ways of expressing a fact.

374. There must be operations of transformation. In that way alone can the symbol be shown determining its interpretant. In order that these operations should be as analytically represented as possible, each elementary operation should be either an insertion or an omission. Operations of commutation, like $xy \therefore yx$, may be dispensed with by not recognizing any order of arrangement as significant. Associative trans-

* Peirce's contribution to an article of that title in *Baldwin's Dictionary of Philosophy and Psychology*, vol. 2, pp. 645–50; 393 is by Peirce and Mrs. C. L. Franklin.

formations, like $(xy)\,z \therefore x\,(yz)$, which is a species of commutation, will be dispensed with in the same way; that is, by recognizing an equiparant* as what it is, a symbol of an unordered set.

375. It will be necessary to recognize two different operations, because of the difference between the relation of a symbol to its object and to its interpretant. Illative transformation (the only transformation, relating solely to truth, that a system of symbols can undergo) is the passage from a symbol to an interpretant, generally a partial interpretant. But it is necessary that the interpretant shall be recognized without the actual transformation. Otherwise the symbol is imperfect. There must, therefore, be a sign to signify that an illative transformation would be possible. That is to say, we must not only be able to express "*A* therefore *B*," but "If *A* then *B*." The symbol must, besides, separately indicate its object. This object must be indicated by a sign, and the relation of this to the significant element of the symbol is that both are signs of the same object. This is an equiparant, or commutative relation. It is therefore necessary to have an operation combining two symbols as referring to the same object. This, like the other operation, must have its actual and its potential state. The former makes the symbol a proposition "*A* is *B*;" that is, "Something *A* stands for, *B* stands for." The latter expresses that such a proposition *might* be expressed, "This stands for something which *A* stands for and *B* stands for." These relations might be expressed in roundabout ways; but two operations would always be necessary. In Jevons's modification† of Boole's algebra the two operations are aggregation and composition. Then, using non-relative terms, "nothing" is defined as that term which aggregated with any term gives that term, while "what is" is that term which compounded with any term gives that term. But here we are already using a third operation; that is, we are using the relation of equivalence; and this is a composite relation. And when we draw an inference, which we cannot avoid, since it is the end and aim of logic, we use still another. It is true that if our purpose were to make a calculus, the two operations, aggregation and

* See 3.136c.

† *Pure Logic*, chs. 6 and 15; (1864).

composition, would go admirably together. Symmetry in a calculus is a great point, and always involves superfluity, as in homogeneous coördinates and in quaternions. Superfluities which bring symmetry are immense economies in a calculus. But for purposes of analysis they are great evils.

376. A proposition *de inesse* relates to a single state of the universe, like the present instant. Such a proposition is altogether true or altogether false. But it is a question whether it is not better to suppose a general universe, and to allow an ordinary proposition to mean that it is sometimes or possibly true. Writing down a proposition under certain circumstances asserts it. Let these circumstances be represented in our system of symbols by writing the proposition on a certain sheet. If, then, we write two propositions on this same sheet, we can hardly resist understanding that both are asserted. This, then, will be the mode of representing that there is something which the one and the other represent — not necessarily the same quasi-instantaneous state of the universe, but the same universe. If writing A asserts that A may be true, and writing B that B may be true, then writing both together will assert that A may be true and that B may be true.

377. By a *rule* of a system of symbols is meant a permission under certain circumstances to make a certain transformation; and we are to recognize no transformations as elementary except writing down and erasing. From the conventions just adopted, it follows, as *Rule* 1, that *anything written down may be erased, provided the erasure does not visibly affect what else there may be which is written along with it.*

378. Let us suppose that two facts are so related that asserting the one gives us the right to assert the other, because if the former is true, the latter must be true. If A having been written, we can add B, we may then, by our first rule, erase A; and consequently A may be transformed into B by two steps. We shall need to express the fact that writing A gives us a right, under all circumstances, to add B. Since this is not a reciprocal relation, A and B must be written differently; and since neither is positively asserted, neither must be written so that the other could be erased without affecting it. We need some place on our sheet upon which we can write a proposition without asserting it. The present writer's habit is

to cut it off from the main sheet by enclosing it within an oval line; but in order to facilitate the printing, we will here enclose it in square brackets. In order, then, to express "If A can under any circumstances whatever be true, B can under some circumstances be true," we must certainly enclose A in square brackets. But what are we to do with B? We are not to assert positively that B can be true; yet it is to be more than hypothetically set forth, as A is. It must certainly, in some fashion, be enclosed within the brackets; for were it detached from the brackets, the brackets with their enclosed A could, by Rule 1, be erased; while in fact the dependence upon A cannot be omitted without danger of falsity. It is to be remarked that, in case we can assert that "If A can be true, B can be true," then, *a fortiori*, we can assert that "If both A and C can be true, B can be true," no matter what proposition C may be. Consequently, we have, as Rule 2, that, *within brackets already written, anything whatever can be inserted.* But the fact that "If A can be true, B can be true" does *not* generally justify the assertion "If A can be true, both B and D are true"; yet our second rule would imply that, unless the B were cut off, in some way, from the main field within the brackets. We will therefore enclose B in parentheses, and express the fact that "If A can be true, B can be true" by

$$[A(B)] \text{ or } [(B)A] \text{ or } \begin{bmatrix} A \\ (B) \end{bmatrix}, \text{ etc.}$$

The arrangement is without significance. The fact that "If A can be true, both B and D can be true," or $[A(BD)]$, justifies the assertion that "If A is true B is true," or $[A(B)]$. Hence the permission of Rule 1 may be enlarged, and we may assert that anything unenclosed or enclosed both in brackets and parentheses can be erased if it is separate from everything else. Let us now ask what $[A]$ means. Rule 2 gives it a meaning; for by this rule $[A]$ implies $[A(X)]$, whatever proposition X may be. That is to say, that $[A]$ can be true implies that "If A can under any circumstances be true, then anything you like, X, may be true." But we may like to make X express an absurdity. This, then, is a *reductio ad absurdum* of A; so that $[A]$ implies, for one thing, that A cannot under any circumstances be true. The question is, Does it express anything further? According to this, $[A(B)]$ expresses that $A(B)$ is impossible. But what is this? It is that A can be true while something expressed by (B) can be true. Now, what can it be that renders the fact that "If A can ever be true, B can sometimes be true" incompatible with A's being able to be true? Evidently the falsity of B under all circumstances. Thus, just as $[A]$ implies that A can never be true, so (B) implies that B can never be true. But further, to say that $[A(B)]$, or "If A is ever true, B is sometimes true," is to say no more than that it is impossible that A is ever true, B being never true. Hence, the square brackets and the parentheses precisely deny what they enclose. A logical principle can be deduced from this: namely, if $[A]$ is true $[A(X)]$ is true. That is, if A is never true, then we have a right to assert that "If A is ever true, X is sometimes true," no matter what proposition X may be. Square brackets and parentheses, then, have the same meaning. Braces may be used for the same purpose.

379. Moreover, since two negatives make an affirmative, we have, as *Rule* 3, that *anything can have double enclosures added or taken away, provided there be nothing within one enclosure but outside the other.* Thus, if B can be true, so that B is written, Rule 3 permits us to write $[(B)]$, and then Rule 2 permits us to write $[X(B)]$. That is, if B is sometimes true, then "If X is ever true, B is sometimes true." Let us make the apodosis of a conditional proposition itself a conditional proposition. That is, in $\mathbf{(}C\{D\}\mathbf{)}$ let us put for D the proposition $[A(B)]$. We thus have $\mathbf{(}C\{[A(B)]\}\mathbf{)}$. But, by Rule 3, this is the same as $\mathbf{(}CA(B)\mathbf{)}$.

380. All our transformations are analysed into insertions and omissions. That is, if from A follows B, we can transform A into AB and then omit the B. Now, by Rule 1, from AB follows A. Treating this in the same way, we first insert the conclusion and say that from AB follows ABA. We thus get as *Rule* 4 that *any detached portion of a proposition can be iterated.*

381. It is now time to reform Rule 2 so as to state in general terms the effect of enclosures upon permissions to transform. It is plain that if we have written $[A(B)]C$, we can write $[A(BC)]C$, although the latter gives us no right to the former. In place, then, of Rule 2 we have:

Rule 2 (amended). *Whatever transformation can be performed on a whole proposition can be performed upon any detached part of it under additional enclosures even in number, and the reverse transformation can be performed under additional enclosures odd in number.*

But this rule does not permit every transformation which can be performed on a detached part of a proposition to be performed upon the same expression otherwise situated.

382. Rule 4 permits, by virtue of Rule 2 (amended), all iteration under additional enclosures and erasure of a term inside enclosures if it is iterated outside some of them.

383. We can now exhibit the *modi tollens et ponens*. Suppose, for example, we have these premisses: "If A is ever true, B is sometimes true," and "B is never true." Writing them, we have $[A(B)](B)$. By Rule 4, from (B) we might proceed to $(B)(B)$. Hence, by Rule 2 (amended), from $[A(B)](B)$ we can proceed to $[A](B)$, and by Rule 1 to $[A]$. That is, "A is never true." Suppose, on the other hand, our premisses are $[A(B)]$ and A. As before, we get $[(B)]A$, and by Rule 3, BA, and by Rule 1, B. That is, from the premisses of the *modus ponens* we get the conclusion. Let us take as premisses "If A is ever true, B is sometimes true," and "If B is ever true, C is sometimes true." That is, $(A\{B\})[B(C)]$. Then iterating $[B(C)]$ within two enclosures, we get $(A\{B[B(C)]\})[B(C)]$, or, by Rule 1, $(A\{B[B(C)]\})$. But we have just seen that $B[B(C)]$ can be transformed to C. Performing this under two enclosures, we get $(A\{C\})$, which is the conclusion, "If A is ever true, C is sometimes true." Let us now formally deduce the principle of contradiction $[A(A)]$. Start from any premiss X. By Rule 3 we can insert $[(X)]$, so that we have $X[(X)]$. By insertion under odd enclosures we have $X[A(X)]$. By iteration under additional enclosures we get $X[A(AX)]$; by erasures under even enclosures $[A(A)]$.

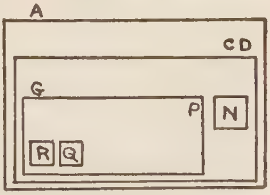


Fig. 65

384. In complicated cases the multitude of enclosures become unmanageable. But by using ruled paper and drawing lines for the enclosures, composed of vertical and horizontal lines, always writing what is more en-

closed lower than what is less enclosed, and what is evenly enclosed, on the left-hand part of the sheet, and what is oddly enclosed, on the right-hand part, this difficulty is greatly reduced. The diagram on page 325 illustrates the general style of arrangement recommended.

385. It is now time to make an addition to our system of symbols. Namely, *AB* signifies that *A* is at some quasi-instant true, and that *B* is at some quasi-instant true. But we wish to be able to assert that *A* and *B* are true at the same quasi-instant. We should always study to make our representations iconoidal; and a very iconoidal way of representing that there is one quasi-instant at which both *A* and *B* are true will be to connect them with a heavy line drawn in any shape, thus:

A—*B* or ┌*A*
└*B*

If this line be broken, thus *A*— —*B*, the identity ceases to be asserted. We have evidently:

Rule 5. *A line of identity may be broken where unenclosed.* —*A* will mean "At some quasi-instant *A* is true." It is equivalent to *A* simply. But —(*A*) will differ from (—*A*) or (*A*) in merely asserting that at some quasi-instant *A* is not true, instead of asserting, with the latter forms, that at no quasi-instant is *A* true. Our quasi-instants may be individual things. In that case —*A* will mean "Something is *A*"; —(*A*), "Something is not *A*"; [—(*A*)], "Everything is *A*"; (—*A*), "Nothing is *A*." So *A*—*B* will express "Some *A* is *B*"; (*A*—*B*), "No *A* is *B*"; *A*—(*B*), "Some *A* is not *B*"; [*A*—(*B*)], "Whatever *A* there may be is *B*"; (*A*)(*B*), "There is something besides *A* and *B*";* [(*A*)(*B*)], "Everything is either *A* or *B*."

386. The rule of iteration must now be amended as follows:

Rule 4 (amended). *Anything can be iterated under the same enclosures or under additional ones, its identical connections remaining identical.*

Thus, [*A*—(*B*)] can be transformed to [*A*(*A*—*B*)]. By the same rule *A*—(—*B*), *i.e.*, "Something is *A* and nothing is *B*," by iteration of the line of identity, can be transformed to

* Better : Something is $\bar{A}$ and $\bar{B}$.

A(–B), *i.e.*, "Some *A* is not coexistent with anything that is *B*," whence, by Rules 5 and 2 (amended), it can be further transformed to A(–B), *i.e.*, "Some *A* is not *B*."

387. But it must be most carefully observed that two unenclosed parts cannot be illatively united by a line of identity. The enclosure of such a line is that of its least enclosed part. We can now exhibit any ordinary syllogism. Thus, the premisses of *Baroko*, "Any *M* is *P*" and "Some *S* is not *P*,"

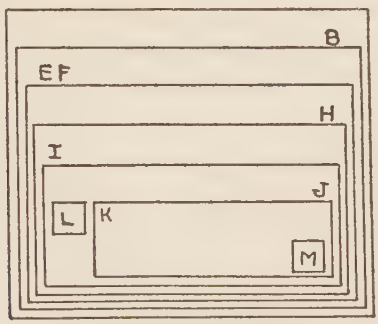


Fig. 66

may be written {M[P]}S(P). Then, as just seen, we can write {M[P]}S(P). Then, by iteration, {M[P(P)]}S(P). Breaking the line under even enclosures, we get {[P(P)]M}S(P). But we have already shown that [P(P)] can be written unenclosed. Hence it can be struck out under one enclosure; and the unenclosed (*P*) can be erased. Thus we get {*M*}*S*, or "Some *S* is not *M*." The great number of steps into which syllogism is thus analysed shows the perfection of the method for purposes of analysis.

388. In taking account of relations, it is necessary to distinguish between the different sides of the letters. Thus let *l* be taken in such a sense that *X*—*l*—*Y* means "*X* loves *Y*." Then *X* –*l*– *Y* will mean "*Y* loves *X*." Then, if *m*— means "Something is a man," and —*w* means "Something is a woman," *m*—*l*—*w* will mean "Some man loves some woman";

m [(—l—) —w] will mean "Some man loves all women"; [(m—l—)w] will mean "Every woman is loved by some man," etc.

389. Since enclosures signify negation, by enclosing a part of the line of identity, the relation of otherness is represented. Thus, A (|) B will assert "Some A is not some B." Given the premisses "Some A is B" and "Some C is not B," they can be written A—BC (B). By Rule 3, this can be written A {[B]} C(B). By iteration, this gives A {[B(B)]} C(B). The lines of identity are to be conceived as passing through the space between the braces outside of the brackets. By breaking the lines under even enclosures, we get A {[B (B)]} C(B). As we have already seen, oddly enclosed [B(B)] can be erased. This, with erasure of the detached (B), gives A { } C. Joining the lines under odd enclosures, we get A { | } C, or "Some A is not some C."

390. For all considerable steps in ratiocination, the reasoner has to treat qualities, or collections, (they only differ grammatically), and especially relations, or systems, as objects of relation about which propositions are asserted and inferences drawn. It is, therefore, necessary to make a special study of the logical relatives "—— is a member of the collection ——," and "——is in the relation —— to ——." The key to all that amounts to much in symbolical logic lies in the symbolization of these relations. But we cannot enter into this extensive subject in this article.

391. The system, of which the slightest possible sketch has been given, is not so iconoidal as the so-called Euler's diagrams; but it is by far the best *general* system which has yet been devised. The present writer has had it under examination for five years with continually increasing satisfaction. However, it is proper to notice some other systems that are now in use. Two systems which are merely extensions of Boole's algebra of logic may be mentioned. One of these is called by no more proper designation than the "general algebra of logic."* The other is called "Peirce's algebra of

* See 3.351f, 3.499f.

dyadic relatives."* In the former there are two operations — aggregation, which Jevons† (to whom its use in algebra is due) signifies by a sign of division turned on its side, thus ·|·. (I prefer to join the two dots, in order to avoid mistaking the single character for three); and composition, which is best signified by a somewhat heavy dot, •.

Thus, if *A* and *B* are propositions, *A*·|·*B* is the proposition which is true if *A* is true, is true if *B* is true, but is such that if *A* is false and *B* is false, it is false. *A*·*B* is the proposition which is true if *A* is true and *B* is true, but is false if *A* is false and false if *B* is false. Considered from an algebraical point of view, which is the point of view of this system, these expressions *A*·|·*B* and *A*·*B* are *mean functions;* for a mean function is defined as such a symmetrical function of several variables, that when the variables have the same value, it takes that same value. It is, therefore, wrong to consider them as addition and multiplication, unless it be that *truth* and *falsity*, the two possible states of a proposition, are considered as logarithmic infinity and zero. It is therefore well to let o represent a false proposition and ∞ (meaning logarithmic infinity, so that $+\infty$ and $-\infty$ are different) a true proposition. A heavy line, called an "obelus," over an expression negatives it.

The letters *i*, *j*, *k*, etc., written below the line after letters signifying predicates, denote individuals, or supposed individuals, of which the predicates are true. Thus, l_{ij} may mean that *i* loves *j*. To the left of the expression a series of letters Π and Σ are written, each with a special one of the individuals *i*, *j*, *k* attached to it in order to show in what order these individuals are to be selected, and how. Σ_i will mean that *i* is to be a suitably chosen individual, Π_j that *j* is any individual, no matter what. Thus,

$$\Sigma_i \Pi_j l_{ji}$$

means that there is an individual *i* such that every individual *j* loves *i*; and

$$\Pi_j \Sigma_i l_{ji}$$

will mean that taking any individual *j*, no matter what, there is some individual *i*, whom *j* loves. This is the whole of this

* See 3.330ff, 3.492ff.

† See his *Substitution of Similars;* §41 (1869); *Pure Logic,* p. 111 (1890).

system, which has considerable power. This use of Σ and Π was probably first introduced by O. C. Mitchell in his epoch-making paper in *Studies in Logic*,* by members of the Johns Hopkins University.

392. In Peirce's algebra of dyadic relatives the signs of aggregation and composition are used; but it is not usual to attach indices. In place of them two relative operations are used. Let *l* be "lover of," *s* "servant of." Then *ls*, called the relative product of *s* by *l*, denotes "lover of some servant of"; and *l*†*s*, called the relative sum of *l* to *s*, denotes "lover of whatever there may be besides servants of." In ms. the tail of the cross will naturally be curved. The sign I is used to mean "numerically identical with," and T to mean "other than." Schröder, who has written an admirable treatise on this system (though his characters are very objectionable, and should not be used†), has considerably increased its power by various devices, and especially by writing, for example, Π_u before an expression containing *u* to signify that *u* may be any relative whatever, or Σ_u to signify that it is a possible relative. In this way he introduces an abstraction or term of second intention.

393. Peano has made considerable use of a system of logical symbolization of his own. Mrs. Ladd-Franklin‡ advocates eight copula-signs to begin with, in order to exhibit the equal claim to consideration of the eight propositional forms. Of these she chooses "No *a* is *b*" and "Some *a* is *b*" ($a\overline{\vee}b$ and $a\vee b$) as most desirable for the elements of an algorithmic scheme; they are both symmetrical and natural. She thinks that a symbolic logic which takes "All *a* is *b*" (Boole, Schröder) as its basis is cumbrous; for every statement of a theorem, there is a corresponding statement necessary in terms of its contrapositive. This, she says, is the source of the parallel columns of theorems in Schröder's *Logik*; a single set of theorems is all-sufficient if a symmetrical pair of copulas is chosen. Some logicians (as C. S. P.) think the objections to Mrs. Ladd-Franklin's system outweigh its advantages. Other systems, as that of Wundt, §show a complete misunderstanding of the problem.

* p. 79.

† See 3.510.

‡ *Johns Hopkins Studies in Logic*, p. 25ff.

§ *Logik*, (1880, 1883).

CHAPTER 3

*EXISTENTIAL GRAPHS**

A. The Conventions

§1. ALPHA PART

394. *Convention No. Zero.* Any feature of these diagrams that is not expressly or by previous conventions of languages required by the conventions to have a given character may be varied at will. This "convention" is numbered zero, because it is understood in all agreements.

395. *Convention No. I.* These Conventions are supposed to be mutual understandings between two persons: a *Graphist*, who expresses propositions according to the system of expression called that of *Existential Graphs*, and an *Interpreter*, who interprets those propositions and accepts them without dispute.

A *graph* is the propositional expression in the System of Existential Graphs of any possible state of the universe. It is a Symbol,† and, as such, general, and is accordingly to be distinguished from a *graph-replica*.[1] A graph remains such though not actually asserted. An expression, according to the conventions of this system, of an impossible state of things (conflicting with what is taken for granted at the outset or has been asserted by the graphist) is not a graph, but is termed *The pseudograph*, all such expressions being equivalent in their absurdity.

396. It is agreed that a certain sheet, or blackboard, shall, under the name of *The Sheet of Assertion*, be considered as

* *A Syllabus of Certain Topics of Logic*, pp. 15–23. Alfred Mudge & Son, Boston (1903). Continuing 2.226.

† Most of the terms such as "symbol," "replica," "rheme," "legisign" used in this paper are defined in vol. 2, bk. II, ch. 2.

[1] "I abandon this inappropriate term, *replica*, Mr. Kempe having already ('Memoir on the Theory of Mathematical Form' [*Philosophical Transactions, Royal Society* (1886)], §170) given it another meaning. I now call it an instance."— marginal note, c. 1910.

representing the universe of discourse, and as asserting whatever is taken for granted between the graphist and the interpreter to be true of that universe. The sheet of assertion is, therefore, a graph. Certain parts of the sheet, which may be severed from the rest, will not be regarded as any part of it.

397. The graphist may place replicas of graphs upon the sheet of assertion; but this act, called *scribing* a graph on the sheet of assertion, shall be understood to constitute the assertion of the truth of the graph scribed. (Since by 395 the conventions are only "supposed to be" agreed to, the assertions are mere pretence in studying logic. Still they may be regarded as actual assertions concerning a fictitious universe.) "Assertion" is not defined; but it is supposed to be permitted to scribe some graphs and not others.

Corollary. Not only is the sheet itself a graph, but so likewise is the sheet together with the graph scribed upon it. But if the sheet be blank, this *blank*, whose existence consists in the absence of any scribed graph, is itself a graph.

398. *Convention No. II.* A graph-replica on the sheet of assertion having no scribed connection with any other graph-replica that may be scribed on the sheet shall, as long as it is on the sheet of assertion in any way, make the same assertion, regardless of what other replicas may be upon the sheet.

The graph which consists of all the graphs on the sheet of assertion, or which consists of all that are on any one area severed from the sheet, shall be termed the *entire* graph of the sheet of assertion or of that area, as the case may be. Any part of the entire graph which is itself a graph shall be termed a *partial* graph of the sheet or of the area on which it is.

Corollaries. Two graphs scribed on the sheet are, both of them, asserted, and any entire graph implies the truth of all its partial graphs. Every blank part of the sheet is a partial graph.

399. *Convention No. III.* By a *Cut* shall be understood to mean a self-returning linear separation (naturally represented by a fine-drawn or peculiarly colored line) which severs all that it encloses from the sheet of assertion on which it stands itself, or from any other area on which it stands itself. The whole space within the cut (but not comprising the cut itself) shall be termed the *area* of the cut. Though the area

of the cut is no part of the sheet of assertion, yet the cut together with its area and all that is on it, conceived as so severed from the sheet, shall, under the name of the *enclosure* of the cut, be considered as on the sheet of assertion or as on such other area as the cut may stand upon. Two cuts cannot intersect one another, but a cut may exist on any area whatever. Any graph which is unenclosed or is enclosed within an even number of cuts shall be said to be *evenly enclosed;* and any graph which is within an odd number of cuts shall be said to be *oddly enclosed.* A cut is not a graph; but an enclosure is a graph. The sheet or other area on which a cut stands shall be called the *place* of the cut.

400. A pair of cuts, one within the other but not within any other cut that that other is not within, shall be called a *scroll.* The outer cut of the pair shall be called the *outloop,* the inner cut the *inloop,* of the scroll. The area of the inloop shall be termed the *inner close* of the scroll; the area of the outloop, *excluding the enclosure of the inloop* (and not merely its area), shall be termed the *outer close* of the scroll.

401. The enclosure of a scroll (that is, the enclosure of the outer cut of the pair) shall be understood to be a graph having such a meaning that if it were to stand on the sheet of assertion, it would assert *de inesse* that if the entire graph in its outer close is true, then the entire graph in its inner close is true. No graph can be scribed across a cut, in any way; although an enclosure is a graph.

(A conditional proposition *de inesse* considers only the existing state of things, and is, therefore, false only in case the consequent is false while the antecedent is true. If the antecedent is false, or if the consequent is true, the conditional *de inesse* is true.)

402. The filling up of any entire area with whatever writing material (ink, chalk, etc.) may be used shall be termed *obliterating* that area, and shall be understood to be an expression of the pseudograph on that area.

Corollary. Since an obliterated area may be made indefinitely small, a single cut will have the effect of denying the entire graph in its area. For to say that if a given proposition is true, everything is true, is equivalent to denying that proposition.

§2. BETA PART

403. *Convention No. IV.* The expression of a rheme in the system of existential graphs, as simple, that is without any expression, according to these conventions, of the analysis of its signification, and such as to occupy a superficial portion of the sheet or of any area shall be termed a *spot*. The word "spot" is to be used in the sense of a *replica;* and when it is desired to speak of the symbol of which it is the replica, this shall be termed a *spot-graph*. On the periphery of every spot, a certain place shall be appropriated to each blank of the rheme; and such a place shall be called a *hook* of the spot. No spot can be scribed except wholly in some area.

404. A heavy *dot* scribed at the hook of a spot shall be understood as filling the corresponding blank of the rheme of the spot with an indefinite sign of an individual, so that when there is a dot attached to every hook, the result shall be a proposition which is particular in respect to every subject.

405. *Convention No. V.* Every heavily marked point, whether isolated, the extremity of a heavy line, or at a furcation of a heavy line, shall denote a single individual, without in itself indicating what individual it is.

406. A heavily marked line without any sort of interruption (though its extremity may coincide with a point otherwise marked) shall, under the name of *a line of identity*, be a graph, subject to all the conventions relating to graphs, and asserting precisely the identity of the individuals denoted by its extremities.

Corollaries. It follows that no line of identity can cross a cut.

Also, a point upon which three lines of identity abut is a graph expressing the relation of *teridentity*.

407. A heavily marked point may be on a cut; and such a point shall be interpreted as lying in the place of the cut and at the same time as denoting an individual identical with the individual denoted by the extremity of a line of identity on the area of the cut and abutting upon the marked point on the cut. Thus, in

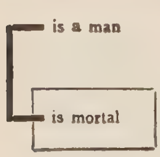


Fig. 67

Fig. 67, if we refer to the individual denoted by the point where the two lines meet on the cut, as X, the assertion is, "Some individual, X, of the universe is a man, and nothing is at once mortal and identical with X"; *i.e.*, some man is not mortal. So in Fig. 68, if X and Y are the individuals denoted by the points on the [inner] cut, the interpretation is,

"If X is the sun and Y is the sun, X and Y are identical."

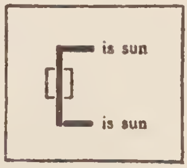


Fig. 68

A collection composed of any line of identity together with all others that are connected with it directly or through still others is termed a *ligature*. Thus ligatures often cross cuts, and, in that case, are not graphs.

408. *Convention No. VI*. A symbol for a single individual, which individual is more than once referred to, but is not identified as the object of a proper name, shall be termed a *Selective*. The capital letters may be used as selectives, and may be made to abut upon the hooks of spots. Any ligature may be replaced by replicas of one selective placed at every hook and also in the outermost area that it enters. In the interpretation, it is necessary to refer to the outermost replica of each selective first, and generally to proceed in the interpretation from the outside to the inside of all cuts.

§3. GAMMA PART

409. *Convention No. VII*. The following spot-symbols shall be used, as if they were ordinary spot-symbols, except for special rules applicable to them: (Selectives are placed against the hooks in order to render the meanings of the new spot-symbols clearer).

A*q*, A is a monadic character;

A*r*, A is a dyadic relation;

A*s*, A is a triadic relation;

$X \wedge_0$, X is a proposition or fact;

$X \wedge_1 Y$, Y possesses the character X;

$X \wedge_2 {}^{Y}_{Z}$, Y stands in the dyadic relation X to Z;

$X \overset{Y}{\underset{W}{\wedge_3}} Z$, Y stands in the triadic relation X to Z for W.

410. *Convention No. VIII.* A cut with many little interruptions* aggregating about half its length shall cause its enclosure to be a graph, expressing that the entire graph on its area is logically contingent (non-necessary).

411. *Convention No. IX.* By a *rim* shall be understood an oval line making it, with its contents, the expression either of a rheme or a proper name of an *ens rationis.* Such a rim may be drawn as a line of peculiar texture, or a gummed label with a colored border may be attached to the sheet. A dotted rim containing a graph, some part of which is itself enclosed by a similar inner dotted oval and with heavy dotted lines proceeding from marked points of this graph to hooks on the rim, shall be a spot expressing that the individuals denoted by lines of identity attached to the hooks (or the single such individual) have the character, constituted by the truth of the graph, to be possessed by the individuals denoted by those points of it to which the heavy dotted lines are attached, in so far as they are connected with the partial graph within the inner oval.

412. A rim represented by a wavy line containing a graph, of which some marked points are connected by wavy lines with hooks on the rim, shall be a spot expressing that the individuals denoted by lines of identity abutting on these hooks form a collection of sets, of which collection each set has its members characterized in the manner in which those individuals must be which are denoted by the points of attachment of the interior graph, when that graph is true.

413. A rim shown as a saw line denotes an individual collection of individual single objects or sets of objects, the members of the collection being all those in existence, which are such individuals as the truth of the graph within makes those to be that are denoted by points of attachment of that graph to saw lines passing to hooks of the rim.

* *I.e.*, a broken cut.

B. Rules of Transformation

Pure Mathematical Definition of Existential Graphs, Regardless of Their Interpretation

§1. ALPHA PART

414. 1. The *System of Existential Graphs* is a certain class of diagrams upon which it is permitted to operate certain transformations.

2. There is required a certain surface upon which it is practicable to scribe the diagrams and from which they can be erased in whole or in part.

3. The whole of this surface except certain parts which may be severed from it by "cuts" is termed the *sheet of assertion*.

4. A *graph* is a legisign (*i.e.*, a sign which is of the nature of a general type) which is one of a certain class of signs used in this system. A *graph-replica* is any individual instance of a graph. The sheet of assertion itself is a graph-replica; and so is any part of it, being called the *blank*. Other graph-replicas can be scribed on the sheet of assertion, and when this is done the graphs of which those graph-replicas are instances is said to be "scribed on the sheet of assertion"; and when a graph-replica is erased, the graph is said to be erased. Two graphs scribed on the sheet of assertion constitute one graph of which they are said to be *partial graphs*. All that is at any time scribed on the sheet of assertion is called the *entire scribed graph*.

5. A *cut* is a self-returning finely drawn line. A cut is not a graph-replica. A cut drawn upon the sheet of assertion severs the surface it encloses, called the *area* of the cut, from the sheet of assertion; so that the area of a cut is no part of the sheet of assertion. A cut drawn upon the sheet of assertion together with its area and whatever is scribed upon that area constitutes a graph-replica scribed upon the sheet of assertion, and is called the *enclosure* of the cut. Whatever graph might, if permitted, be scribed upon the sheet of assertion might (if permitted) be scribed upon the area of any cut. Two graphs scribed at once on such area constitute a graph, as they would on the sheet of assertion. A cut can (if permitted) be drawn upon the area of any cut, and will sever the surface which it

encloses from the area of the cut, while the enclosure of such inner cut will be a graph-replica scribed on the area of the outer cut. The sheet of assertion is also an area. Any blank part of any area is a graph-replica. Two cuts one of which has the enclosure of the other on its area and has nothing else there constitute a *double cut.*

6. No graph or cut can be placed partly on one area and partly on another.*

7. No transformation of any graph-replica is permitted unless it is justified by the following code of Permissions.

Code of Permissions

415. *Permission No.* 1. *In each special problem such graphs may be scribed on the sheet of assertion as the conditions of the special problem may warrant.*

Permission No. 2. *Any graph on the sheet of assertion may be erased, except an enclosure with its area entirely blank.*

Permission No. 3. *Whatever graph it is permitted to scribe on the sheet of assertion, it is permitted to scribe on any unoccupied part of the sheet of assertion, regardless of what is already on the sheet of assertion.*

Permission No. 4. *Any graph which is scribed on the inner area of a double cut on the sheet of assertion may be scribed on the sheet of assertion.*

Permission No. 5. *A double cut may be drawn on the sheet of assertion; and any graph that is scribed on the sheet of assertion may be scribed on the inner area of any double cut on the sheet of assertion.*

Permission No. 6. *The reverse of any transformation that would be permissible on the sheet of assertion is permissible on the area of any cut that is upon the sheet of assertion.*

Permission No. 7. *Whenever we are permitted to scribe any graph we like upon the sheet of assertion, we are authorized to declare that the conditions of the special problem are absurd.*

§2. BETA PART

416. 8. The beta part adds to the alpha part certain signs to which new permissions are attached, while retaining all the alpha signs with the permissions attaching to them.

* But see 579.

9. The *line of identity* is a Graph any replica of which, also called a line of identity, is a heavy line with two ends and without other topical singularity (such as a point of branching or a node), not in contact with any other sign except at its extremities. Otherwise, its shape and length are matters of indifference. All lines of identity are replicas of the same graph.

10. A *spot* is a graph any replica of which occupies a simple bounded portion of a surface, which portion has qualities distinguishing it from the replica of any other spot; and upon the boundary of the surface occupied by the spot are certain points, called the *hooks* of the spot, to each of which, if permitted, one extremity of one line of identity can be attached. Two lines of identity cannot be attached to the same hook; nor can both ends of the same line.

11. Any indefinitely small dot may be a spot replica called a *spot of ter-identity*, and three lines of identity may be attached to such a spot. Two lines of identity, one outside a cut and the other on the area of the same cut, may have each an extremity at the same point on the cut. The totality of all the lines of identity that join one another is termed a *ligature*. A ligature is not generally a graph, since it may be part in one area and part in another. It is said to lie within any cut which it is wholly within.

417. 12. The following are the additional permissions attaching to the beta part.

Code of Permissions — Continued

Permission No. 8. All the above permissions apply to all spots and to the line of identity, as Graphs; and Permission No. 2 is to be understood as permitting the erasure of any portion of a line of identity on the sheet of assertion, so as to break it into two. Permission No. 3 is to be understood as permitting the extension of a line of identity on the sheet of assertion to any unoccupied part of the sheet of assertion. Permission No. 3 must not be understood [*as stating that*] *that because it is permitted to scribe a graph without certain ligatures therefore it is permissible to scribe it with them, or the reverse.*

Permission No. 9. It is permitted to scribe an unattached line of identity on the sheet of assertion, and to join such unat-

tached lines in any number by spots of ter-identity. This is to be understood as permitting a line of identity, whether within or without a cut, to be extended to the cut, although such extremity is to be understood to be on both sides of the cut. But this does not permit a line of identity within a cut that is on the sheet of assertion to be retracted from the cut, in case it extends to the cut.

Permission No. 10. *If two spots are within a cut (whether on its area or not), and are not joined by any ligature within that cut, then a ligature joining them outside the cut is of no effect and may be made or broken. But this does not apply if the spots are joined by other hooks within the cut.**

Permission No. 11. *Permissions Nos.* 4 *and* 5 *do not cease to apply because of ligatures passing from without the outer of two cuts to within the inner one, so long as there is nothing else in the annular area.*†

* But see 580.

† For the code of permissions for the Gamma part, which was not discussed in this printed pamphlet, see below, 470–1, and chapters 5 and 7.

CHAPTER 4

*ON EXISTENTIAL GRAPHS, EULER'S DIAGRAMS, AND LOGICAL ALGEBRA**P

§INTRODUCTION

418. A *diagram* is a representamen† which is predominantly an icon of relations and is aided to be so by conventions. Indices are also more or less used. It should be carried out upon a perfectly consistent system of representation, founded upon a simple and easily intelligible basic idea.

419. A *graph* is a superficial diagram composed of the sheet upon which it is written or drawn, of spots or their equivalents, of lines of connection, and (if need be) of enclosures. The type, which it is supposed more or less to resemble, is the structural formula of the chemist.

420. A *logical graph* is a graph representing logical relations iconically, so as to be an aid to logical analysis.

421. An *existential graph* is a logical graph governed by a system of representation founded upon the idea that the sheet upon which it is written, as well as every portion of that sheet, represents one recognized universe, real or fictive, and that every graph drawn on that sheet, and not cut off from the main body of it by an enclosure, represents some fact existing in that universe, and represents it independently of the representation of another such fact by any other graph written upon another part of the sheet, these graphs, however, forming one composite graph.

422. No other system of existential graphs than that herein set forth having hitherto been proposed, this one will need, for the present, no more distinctive designation. Should such designation hereafter become desirable, I desire that this system should be called the Existential System of 1897, in which year I wrote an account of it and offered it for publica-

* From "Logical Tracts, No. 2," c. 1903. "Logical Tracts, No. 1" is largely a repetition of the papers on signs in vol. 2, bk. II, ch. 2.

† Most of the terms such as "representamen," "icons," "indices," etc. are defined in vol. 2, bk. II, ch. 2.

tion to the Editor of *The Monist,* who declined it on the ground that it might later be improved upon. No changes have been found desirable since that date, although it has been under continual examination; but the exposition has been rendered more formal.

423. The following exposition of this system will be arranged as follows:

Part I will explain the expression of ordinary forms of language in graphs and the interpretation of the latter into the former in three sections, as follows:

A will state all the fundamental conventions of the system, separating those which are essentially different, showing the need which each is designed to meet together with the reasons for meeting it by the particular convention chosen, so far as these can be given at this stage of the development. A complete discussion will be given in an Appendix* to this part. To aid the understanding of all this, various logical analyses will be interspersed where they become pertinent.

B will enunciate other rules of interpretation whose validity will be demonstrated from the fundamental conventions as premisses. This section will also introduce certain modifications of some of the signs established in *A*, the modified signs being convenient, although good reasons forbid their being considered fundamental.

C will redescribe the system in a compact form, which, on account of its uniting into one many rules that had, in the first instance, to be considered separately, is more easily grasped and retained in the mind.

Part II will develop formal "rules," or permissions, by which one graph may be transformed into another without danger of passing from truth to falsity and without recurring to any interpretation of the graphs; such transformations being of the nature of immediate inferences. The part will be divided into sections corresponding to those of Part I.

A will prove the basic rules of transformation directly from the fundamental conventions of *A* of Part I.

B will deduce further rules of transformation from those of *A*, without further recourse to the principles of transformation.

C will restate the rules in more compact form.

Part III will show how the system may be made useful.*

* This does not seem to have been written.

PART I. PRINCIPLES OF INTERPRETATION[P]

A. Fundamental Conventions[P]

§1. OF CONVENTIONS NOS. 1 AND 2*[P]

424. In order to understand why this system of expression has the construction it has, it is indispensable to grasp the precise purpose of it, and not to confuse this with four other purposes, to wit:

First, although the study of it and practice with it will be highly useful in helping to train the mind to accurate thinking, still that consideration has not had any influence in determining the characters of the signs employed; and an exposition of it, which should have that aim, ought to be based upon psychological researches of which it is impossible here to take account.

Second, this system is not intended to serve as a universal language for mathematicians or other reasoners, like that of Peano.

Third, this system is not intended as a calculus, or apparatus by which conclusions can be reached and problems solved with greater facility than by more familiar systems of expression. Although some writers† have studied the logical algebras invented by me with that end apparently in view, in my own opinion their structure, as well as that of the present system, is quite antagonistic to much utility of that sort. The principal desideratum in a calculus is that it should be able to pass with security at one bound over a series of difficult inferential steps. What these abbreviated inferences may best be, will depend upon the special nature of the subject under discussion. But in my algebras and graphs, far from anything of that sort being attempted, the whole effort has been to dissect the operations of inference into as many distinct steps as possible.

Fourth, although there is a certain fascination about these graphs, and the way they work is pretty enough, yet the system is not intended for a plaything, as logical algebra has sometimes been made, but has a very serious purpose which I proceed to explain.

* These conventions, together with No. 3, define the Alpha Part of Graphs.

† *E.g.*, Schröder; see 3.510ff.

425. Admirable as the work of research of the special sciences — physical and psychical — is, as a whole, the reasoning [employed in them] is of an elementary kind except when it is mathematical, and it is not infrequently loose. The philosophical sciences are greatly inferior to the special sciences in their reasoning. Mathematicians alone reason with great subtlety and great precision. But hitherto nobody has succeeded in giving a thoroughly satisfactory logical analysis of the reasoning of mathematics. That is to say, although every step of the reasoning is evidently such that the collective premisses cannot be true and yet the conclusion false, and although for each such step, A, we are able to draw up a self-evident general rule that from a premiss of such and such a form such and such a form of conclusion will necessarily follow, this rule covering the particular inferential step, A, yet nobody has drawn up a complete list of such rules covering all mathematical inferences. It is true that mathematics has its calculus which solves problems by rules which are fully proved; but, in the first place, for some branches of the calculus those proofs have not been reduced to self-evident rules, and in the second place, it is only routine work which can be done by simply following the rules of the calculus, and every considerable step in mathematics is performed in other ways.

426. If we consult the ordinary treatises on logic for an account of necessary reasoning, all the help that they afford is the rules of syllogism. They pretend that ordinary syllogism explains the reasoning of mathematics; and books have professed to exhibit considerable parts of the reasoning of the first book of Euclid's *Elements* stated in the form of syllogisms. But if this statement is examined, it will be found that it represents transformations of statements to be made that are not reduced to strict syllogistic form; and on examination it will be found that it is precisely in these transformations that the whole gist of the reasoning lies. The nearest approach to a logical analysis of mathematical reasoning that has ever been made was Schröder's statement, with improvements, in a logical algebra of my invention, of Dedekind's reasoning (itself in a sort of logical form) concerning the foundations of arithmetic.* But though this relates only to an exceptionally

* *Vorlesungen über die Algebra der Logik*, Bd. 3, §23 and §31, (1895).

simple kind of mathematics, my opinion — quite against my natural leanings toward my own creation — is that the soul of the reasoning has even here not been caught in the logical net.

427. No other book has, during the nineteenth century, been deeply studied by so large a proportion of the strong intellects of the civilized world as Kant's *Critic of the Pure Reason;* and the reason has undoubtedly been that they have all been greatly struck by Kant's logical power. Yet Kant, for all this unquestionable power, had paid so little attention to logic that he makes it manifest that he supposed that ordinary syllogism explains mathematical reasoning, and indeed [in] the simplest mood of syllogism, *Barbara.* Now, at the very utmost, from n propositions only $\frac{1}{4}n^2$ conclusions can be drawn by *Barbara.* In the thirteen books of Euclid's Elements there [are] 14 premisses (5 postulates and 9 axioms) excluding the definitions, which are merely verbal. Therefore, even if these premisses were related to one another in the most favorable way, which is far from being the case, there could only be 49 conclusions from them. But Euclid draws over ten times that number (465 propositions, 27 corollaries, and 17 lemmas) besides which his editors have inserted hundreds of corollaries. There are 48 propositions in the first book. Moreover, in *Barbara* or any sorites, or complexus of such syllogisms, to introduce the same premiss twice is idle. But throughout mathematics the same premisses are used over and over again. Moreover a person of fairly good mind and some logical training will instantly see the syllogistic conclusions from any number of premisses. But this is far from being true of mathematical inferences.

428. There is reason to believe that a thorough understanding of the nature of mathematical reasoning would lead to great improvements in mathematics. For when a new discovery is made in mathematics, the demonstration first found is almost always replaced later by another much simpler. Now it may be expected that, if the reasoning were thoroughly understood, the unnecessary complications of the first proof would be eliminable at once. Indeed, one might expect that the shortest route would be taken at the outset. Then again, consider the state of topical geometry, or geometrical topics,

otherwise called topology. Here is a branch of geometry which not only leaves out of consideration the proportions of the different dimensions of figures and the magnitudes of angles (as does also graphics, or projective geometry — perspective, etc.) but also leaves out of account the straightness or mode of curvature of lines and the flatness or mode of bending of surfaces, and confines itself entirely to the connexions of the parts of figures (distinguishing, for example, a ring from a ball). Ordinary metric geometry equally depends on the connections of parts; but it depends on much besides. It, therefore, is a far more complicated subject, and can hardly fail to be of its own nature much the more difficult. And yet geometrical topics stands idle with problems to all appearance very simple staring it unsolved in the face, merely because mathematicians have not found out how to reason about it. Now a thorough understanding of mathematical reasoning must be a long stride toward enabling us to find a method of reasoning about this subject as well, very likely, as about other subjects that are not even recognized to be mathematical.

429. This, then, is the purpose for which my logical algebras were designed but which, in my opinion, they do not sufficiently fulfill. The present system of existential graphs is far more perfect in that respect, and has already taught me much about mathematical reasoning. Whether or not it will explain all mathematical inferences is not yet known.

Our purpose, then, is to study the workings of necessary inference. What we want, in order to do this, is a method of representing diagrammatically any possible set of premisses, this diagram to be such that we can observe the transformation of these premisses into the conclusion by a series of steps each of the utmost possible simplicity.

430. What we have to do, therefore, is to form a perfectly consistent method of expressing any assertion diagrammatically. The diagram must then evidently be something that we can see and contemplate. Now what we see appears spread out as upon a sheet. Consequently our diagram must be drawn upon a sheet. We must appropriate a sheet to the purpose, and the diagram drawn or written on the sheet is to express an assertion. We can, then, approximately call this sheet our *sheet of assertion*. The *entire graph*, or all that is drawn on the

sheet, is to express a proposition, which the act of writing is to assert.

431. But what are our assertions to be about? The answer must be that they are to be about an arbitrarily hypothetical universe, a creation of a mind. For it is *necessary* reasoning alone that we intend to study; and the necessity of such reasoning consists in this, that not only does the conclusion happen to be true of a pre-determinate universe, but *will* be true, so long as the premisses are true, howsoever the universe may subsequently turn out to be determined. Thus, conformity to an *existing*, that is, entirely determinate, universe does not make necessity, which consists in what always *will be*, that is, what is determinately true of a universe not yet entirely determinate. Physical necessity consists in the fact that whatever may happen will conform to a law of nature; and logical necessity, which is what we have here to deal with, consists of something being determinately true of a universe not entirely determinate as to what is true, and thus not *existent*. In order to fix our ideas, we may imagine that there are two persons, one of whom, called the *grapheus*, creates the universe by the continuous development of his idea of it, every interval of time during the process adding some *fact* to the universe, that is, affording justification for some assertion, although, the process being continuous, these facts are not distinct from one another in their mode of being, as the propositions, which state some of them, are. As fast as this process in the mind of the grapheus takes place, that which is thought acquires *being*, that is, *perfect definiteness*, in the sense that the effect of what, is thought in any lapse of time, however short, is definitive and irrevocable; but it is not until the whole operation of creation is complete that the universe acquires *existence*, that is, *entire determinateness*, in the sense that nothing remains undecided. The other of the two persons concerned, called the *graphist*, is occupied during the process of creation in making *successive* modifications (*i.e.*, not by a continuous process, since each modification, unless it be final, has another that follows *next* after it), of the entire graph. Remembering that the entire graph is *whatever* is, at any time, expressed in this system on the sheet of assertion, we may note that before anything has been drawn on the sheet, the *blank* is, by that definition, a

graph. It may be considered as the expression of whatever must be well-understood between the graphist and the interpreter of the graph before the latter can understand what to expect of the graph. There must be an interpreter, since the graph, like every sign founded on convention, only has the sort of being that it has if it is interpreted; for a conventional sign is neither a mass of ink on a piece of paper or any other individual existence, nor is it an image present to consciousness, but is a special habit or rule of interpretation and consists precisely in the fact that certain sorts of ink spots — which I call its *replicas* — will have certain effects on the conduct, mental and bodily, of the interpreter. So, then, the blank of the blank sheet may be considered as expressing that the universe, in process of creation by the grapheus, is perfectly definite and entirely determinate, etc. Hence, even the first writing of a graph on the sheet is a modification of the graph already written. The business of the graphist is supposed to come to an end before the work of creation is accomplished. He is supposed to be a mind-reader to such an extent that he knows some (perhaps all) the creative work of the grapheus so far as it has gone, but not what is to come. What he intends the graph to express concerns the universe as it will be when it comes to exist. If he risks an assertion for which he has no warrant in what the grapheus has yet thought, it may or may not prove true.

432. The above considerations constitute a sufficient reason for adopting the following convention, which is hereby adopted:

Convention No. 1. A certain sheet, called the **sheet of assertion**, *is appropriated to the drawing upon it of such graphs that whatever may be at any time drawn upon it, called* **the entire graph**, *shall be regarded as expressing an assertion by an imaginary person, called the* **graphist**, *concerning a universe, perfectly definite and entirely determinate, but the arbitrary creation of an imaginary mind, called* **the grapheus**.

433. The convention which has next to be considered is the most arbitrary of all. It is, nevertheless, founded on two good reasons. A diagram ought to be as iconic as possible; that is, it should represent relations by visible relations anal-

ogous to them. Now suppose the graphist finds himself authorized to write each of two entire graphs. Say, for example, that he can draw:

The pulp of some oranges is red;

and that he is equally authorized to draw:

To express oneself naturally is the last perfection of a writer's art.

Each proposition is true independently of the other, and either may therefore be expressed on the sheet of assertion. If both are written on different parts of the sheet of assertion, the independent presence on the sheet of the two expressions is analogous to the independent truth of the two propositions that they would, when written separately, assert. It would, therefore, be a highly iconic mode of representation to understand,

The pulp of some oranges is red.
To express oneself naturally is the last perfection
of a writer's art.

where both are written on different parts of the sheet, as the assertion of both propositions.

434. It is a subsidiary recommendation of a mode of diagrammatization, but one which ought to be accorded some weight, that it is one that the nature and habits of our minds will cause us at once to understand, without our being put to the trouble of remembering a rule that has no relation to our natural and habitual ways of expression. Certainly, no convention of representation could possess this merit in a higher degree than the plan of writing both of two assertions in order to express the truth of both. It is so very natural, that all who have ever used letters or almost any method of graphic communication have resorted to it. It seems almost unavoidable, although in my first invented system of graphs, which I call *entitative graphs*,* propositions written on the sheet together were not understood to be independently asserted but to be *alternatively* asserted. The consequence was that a blank sheet instead of expressing only what was taken

* See 3.468ff.

for granted had to be interpreted as an absurdity. One system seems to be about as good as the other, except that unnaturalness and aniconicity haunt every part of the system of entitative graphs, which is a curious example of how late a development simplicity is. These two reasons will suffice to make every reader very willing to accede to the following convention, which is hereby adopted.

Convention No. 2. Graphs on different parts of the sheet, called **partial graphs**, *shall independently assert what they would severally assert, were each the entire graph.*

§2. OF CONVENTION NO. 3[P]

435. If a system of expression is to be adequate to the analysis of all necessary consequences,[1] it is requisite that it should be able to express that an expressed consequent, C, follows necessarily from an expressed antecedent, A. The conventions hitherto adopted do not enable us to express this. In order to form a new and reasonable convention for this purpose we must get a perfectly distinct idea of what it means to say that a consequent follows from an antecedent. It means that in adding to an assertion of the antecedent an assertion of the consequent we shall be proceeding upon a general principle whose application will never convert a true assertion into a false one. This, of course, means that so it will be in the universe of which alone we are speaking. But when we talk logic — and people occasionally insert logical remarks into ordinary discourse — our universe is that universe which embraces all others, namely *The Truth*, so that, in such a case, we mean that in no universe whatever will the addition of the assertion of the consequent to the assertion of the antecedent be a conversion of a true proposition into a false one. But before we can express any proposition referring to a general principle, or, as we say, to a "range of possibility," we must first find means to express the simplest kind of conditional proposition, the *conditional de inesse*, in which "If A is true, C is true" means only that, principle or no prin-

[1] In the language of logic "consequence" does not mean that which follows, which is called the *consequent*, but means the fact that a consequent follows from an antecedent.

ciple, the addition to an assertion of A of an assertion of C will not be a conversion of a true assertion into a false one. That is, it asserts that the graph of Fig. 69, anywhere on the sheet of assertion, might be transformed into the graph of Fig. 70 without passing from truth to falsity.

a

Fig. 69

a c

Fig. 70

This conditional *de inesse* has to be expressed as a graph in such a way as distinctly to express in our system both *a* and *c*, and to exhibit their relation to one another. To assert the graph thus expressing the conditional *de inesse*, it must be drawn upon the sheet of assertion, and in this graph the expressions of *a* and of *c* must appear; and yet neither *a* nor *c* must be drawn upon the sheet of assertion. How is this to be managed? Let us draw a closed line which we may call a *sep* (*sæpes*, a fence), which shall cut off its contents from the sheet of assertion. Let this sep together with all that is within it, *considered as a whole*, be called an *enclosure*, this close, being written on the sheet of assertion, shall assert the conditional *de inesse;* but that which it encloses, considered separately from the sep, shall not be considered as on the sheet of assertion. Then, obviously, the antecedent and consequent must be in separate compartments of the close. In order to make the representation of the relation between them iconic, we must ask ourselves what spatial relation is analogous to their relation. Now if it be true that "If *a* is true, *b* is true" and "If *b* is true, *c* is true," then it is true that "If *a* is true, *c* is true." This is analogous to the geometrical relation of inclusion. So naturally striking is the analogy as to be (I believe) used in all languages to express the logical relation; and even the modern mind, so dull about metaphors, employs this one frequently. It is reasonable, therefore, that one of the two compartments should be placed within the other. But which shall be made the inner one? Shall we express the conditional *de inesse* by Fig. 71 or by Fig. 72? In order to decide which is the more appropriate mode of representation, one should observe that the consequent of a conditional proposition asserts what is true, not throughout the whole universe of possibilities considered, but in a subordinate universe marked off by the antecedent. This is not a fanciful notion, but a truth. Now in Fig. 72, the consequent appears in a special part of the sheet representing the universe, the space between the two lines containing the definition of the sub-

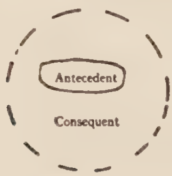


Fig. 71*

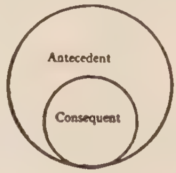


Fig. 72

universe. There is no such expressiveness in Fig. 71 — or, if there be, it is only of a superficial and fanciful sort. Moreover, the necessity of using two kinds of enclosing lines — a necessity which, we shall find, does not exist in Fig. 72 — is a defect of Fig. 71; and when we come to consider the question of convenience, the superiority of Fig. 72 will appear still more strongly. This, then, will be the method for us to adopt.

436. The two seps of Fig. 72, taken together, form a curve which I shall call a *scroll*. The node is of no particular significance. The scroll may equally well be drawn as in Fig. 73. The only essential feature is that there should be two seps, of which the inner, however drawn, may be called the *inloop*. The node merely serves to aid the mind in the interpretation, and will be used only when it can have this effect. The two compartments will be called the *inner*, or *second*, *close*, and the *outer close*, the latter excluding the former. The outer close considered as containing the inloop will be called the *close*.

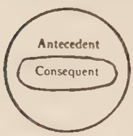


Fig. 73

437. *Convention No. 3. An* **enclosure** *shall be a graph consisting of a* **scroll** *with its contents.*

The **scroll** *shall be a real curve of two closed branches, the one within the other, called* **seps**, *and the inner specifically called the* **loop**; *and these branches may or may not be joined at a node.*

The contents of the scroll shall consist of whatever is in the area enclosed by the outer sep, this area being called the **close**

* See 515f. on the broken cut.

and consisting of the **inner**, *or* **second**, **close**, *which is the area enclosed by the loop, and the* **outer**, *or* **first close**, *which is the area outside the loop but inside the outer sep.*

When an enclosure is written on the sheet of assertion, although it is asserted as a whole, its contents shall be cut off from the sheet, and shall not be asserted in the assertion of the whole. But the enclosure shall assert **de inesse** *that if every graph in the outer close be true, then every graph in the inner close is true.*

§3. OF CONVENTIONS NOS. 4 TO 9*[P]

438. Let a heavy dot or dash be used in place of a noun which has been erased from a proposition. A blank form of proposition produced by such erasures as can be filled, each with a proper name, to make a proposition again, is called a *rhema*, or, relatively to the proposition of which it is conceived to be a part, the *predicate* of that proposition. The following are examples of rhemata:

——— is good
every man is the son of ———
——— loves ———
God gives ——— to ———

Every proposition has one predicate and one only. But what that predicate is considered to be depends upon how we choose to analyze it. Thus, the proposition

God gives some good to every man

may be considered as having for its predicate either of the following rhemata:

——— gives ——— to ———
——— gives some good to ———
——— gives ——— to every man
God gives ——— to ———
God gives some good to ———
God gives ——— to every man
——— gives some good to every man
God gives some good to every man.

* The conventions Nos. 4 to 12 define the Beta Part of Graphs.

In the last case the entire proposition is considered as predicate. A rhema which has one blank is called a *monad*; a rhema of two blanks, a *dyad*; a rhema of three blanks, a *triad*; etc. A rhema with no blank is called a *medad*, and is a complete proposition. A rhema of more than two blanks is a *polyad*. A rhema of more than one blank is a *relative*. Every proposition has an *ultimate predicate*, produced by putting a blank in every place where a blank can be placed, without substituting for some word its definition. Were this done we should call it a *different proposition*, as a matter of nomenclature. If on the other hand, we transmute the proposition without making any difference as to what it leaves unanalyzed, we say the *expression* only is different, as, if we say,

Some good is bestowed by God on every man.

Each part of a proposition which might be replaced by a proper name, and still leave the proposition a proposition is a *subject* of the proposition.[1] It is, however, the rhema which we have just now to attend to.

[1] This, it will be remarked, makes what modern grammars call the direct and indirect objects, as well as much else, to be *subjects;* and some persons will consider this to be a bad abuse of the word *subject*. Come, let us have this out. I grant you that in polite literature usage is, not only almost, but altogether, the *arbitrium et jus et norma loquendi*. And if I am asked *whose* usage, I reply, that of the public whom you are addressing. If, with Vaugelas [*Remarques sur la langue française*], you are addressing the court, then the usage of the court. If you are lecturing the riffraff of a great city, then their usage. If anybody were to dispute this and ask me to prove it, I should reply that whatever ultimate purpose the polite *littérateur* may have, it is indispensable to that purpose that he should make the reading of what he writes agreeable; and in order that it may be agreeable, it is necessary that it should be easily understood by those who are addressed. But with logical writings it is different. If there be any sciences which can flourish without any words having any exact meanings, logic is not one of them. It cannot pursue its truths without a terminology of which every word shall have a single exact definition. To a great extent it already possesses such a terminology, notwithstanding the frequent abuse of its terms. But where this terminology is unsettled, to follow usage would simply be to prolong the confusion. There are conflicting individual predilections which must be made to give way; and there is only one thing to which they will consent to give way. It is some rational principle; which, stated generally, will recommend itself to all. Where are we to seek such a principle? In experience. He must profit by the experience of those sciences which have had the greatest difficulties with their terminology, and which have successfully surmounted those difficulties. Wherever this has been accomplished, it has been

439. A rhema is, of course, not a proposition. Supposing, however, that it be written on the sheet of assertion, so that we have to adopt a meaning for it as a proposition, what can it most reasonably be taken to mean? Take, for example, Fig. 74. Shall this, since it represents the universe, be taken to mean that "Something in the universe is beautiful," or

by adopting a rational general principle; and that principle has always been essentially the same. Any taxonomic zoölogist or botanist will tell you what it is. *He who introduces a conception into the science shall have the right and the duty of assigning to it a suitable technical expression; and whoever thereafter uses that expression*, technically *in any other sense commits a grave misdemeanor*, since he thereby inflicts an injury upon the science. [Cf. 2.219–26.]

Now let us apply this rule to the word *subject*. This was made a term of logic about A.D. 500 with this definition: "Subjectum est dequo dicitur id quod praedicatur" (*Boethii Opera*, Eds. of 1546 and 1570, p. 823, in *Comm. in Ciceronis Topica*, lib. v.) Now unless we were prepared to say that for different languages there are different doctrines of logic (which would be contrary to the essence of logic, as all will admit) we cannot, in this definition, take the preposition *de* in so narrow a sense as to exclude the grammatical accusative, dative, genitive and ablative of the verb. For dispersed through all the families of speech there are a dozen languages which either habitually or frequently express a proposition completely without putting any noun in the nominative. Among the European languages, Gaelic is an example, in which the principal subject is most commonly put in the genitive. But the logical fact is simply that it frequently makes a difference in the sense of a proposition which of the different nouns, naming objects to which the verb refers, is considered to be immediately attached to the verb, which to the combination of these two, and so on. Thus, in the sentence, "Some angel gives every man some gift," the verb "gives" is directly applied to "some gift," making "gives-a-gift"; then this action of gift-giving is applied to "every man"; finally the compound "gives-gift to every man" is applied to a certain angel; while in the sentence "A certain gift (perhaps, speech) is given to every man by some angel or other" the verb "is given by" is applied directly to "some angel," making "is angel-given to," which is applied to "every man," and then "is angel-given to every man" is applied to a certain gift. One sentence represents one angel as distributing gifts to all men, the other represents one gift as bestowed by one or another angel on each man. Thus, the subject-nominative is ordinarily of all the subjects the one of which the verb is least directly said. I quite admit that I use the word *subject* as Boëthius never contemplated its being used; but it would be destructive to science to say that a term must be applied to nothing that its originator did not contemplate its being applied to. It is the definition only that holds.

As a term of grammar, the word *subject* did not come into use until late in the eighteenth century. It would be somewhat impertinent, therefore, for grammarians to claim that, to their usage, the millennial usage of those from whom they borrowed the term, must bow.

that "Anything in the universe is beautiful," or that "The universe, as a whole, is beautiful"? The last interpretation may be rejected at once for the reason that we are generally unable to assert anything of the universe not reducible to one of the other forms except what is well-understood between graphist and interpreter. We have, therefore, to choose between interpreting Fig. 74 to mean "Something is beautiful"

——————is beautiful

Fig. 74

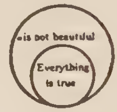


Fig. 75

and to mean "Anything is beautiful." Each asserts the rhema of an individual; but the former leaves that individual to be designated by the grapheus, while the latter allows the rhema [interpreter!] to fill the blank with any proper name he likes. If Fig. 74 be taken to mean "Something is beautiful," then Fig. 75 will mean "Everything is beautiful"; while if Fig. 74 be taken to mean "Everything is beautiful," then Fig. 75 will mean "Something is beautiful." In either case, therefore, both propositions will be expressible, and the main question is, which gives the most appropriate expressions? The question of convenience is subordinate, as a general rule; but in this case the difference is so vast in this respect as to give this consideration more than its usual importance.

440. In order to decide the question of appropriateness, we must ask which form of proposition, the universal or the particular, "Whatever salamander there may be lives in fire," or "Some existing salamander lives in fire," is more of the nature of a conditional proposition; for plainly, these two propositions differ in form from "Everything is beautiful" and "Something is beautiful" respectively, only in their being limited to a subsidiary universe of salamanders. Now to say "Any salamander lives in fire" is merely to say "If anything, X, is a salamander, X lives in fire." It differs from a conditional, if at all, only in the identification of X which it involves. On the other hand, there is nothing at all conditional in saying "There is a salamander, and it lives in fire."

Thus the interpretation of Fig. 74 to mean "Something is beautiful" is decidedly the more appropriate; and since reasonable arrangements generally prove to be the most convenient in the end, we shall not be surprised when we come to find, as we shall, the same interpretation to be incomparably the superior in that respect also.

441. *Convention No. 4. In this system, the unanalyzed expression of a rhema shall be called a* **spot**. *A distinct place on its periphery shall be appropriated to each blank, which place shall be called a* **hook**. *A spot with a dot at each hook shall be a graph expressing the proposition which results from filling every blank of the rhema with a separate sign of an indesignate individual existing in the universe and belonging to some determinate category, usually that of "things."*

442. In many reasonings it becomes necessary to write a copulative proposition in which two members relate to the same individual so as to distinguish these members. Thus we have to write such a proposition as,

A is greater than something that is greater than B,

so as to exhibit the two partial graphs of Fig. 76.

A is greater than ——
—— is greater than B

Fig. 76

The proposition we wish to express adds to those of Fig. 76 the assertion of the identity of the two "somethings." But this addition cannot be effected as in Fig. 77.

A is greater than ——
—— is greater than B
—— is greater than ——

Fig. 77

For the "somethings," being indesignate, cannot be described in general terms. It is necessary that the signs of them should be connected in fact. No way of doing this can be more perfectly iconic than that exemplified in Fig. 78.

A is greater than ┐
└is greater than B.

Fig. 78

Any sign of such identification of individuals may be called a *connexus*, and the particular sign here used, which we shall do well to adopt, may be called a *line of identity*.

443. *Convention No.* 5. *Two coincident points, not more, shall denote the same individual.*

444. *Convention No.* 6. *A heavy line, called a* **line of identity**, *shall be a graph asserting the numerical identity of the individuals denoted by its two extremities.*

445. The next convention to be laid down is so perfectly natural that the reader may well have a difficulty in perceiving that a separate convention is required for it. Namely, we may make a line of identity branch to express the identity of three individuals. Thus, Fig. 79 will express that some black bird is thievish. No doubt, it would have been easy to draw up Convention No. 4 in such a form as to cover this procedure. But it is not our object in this section to find ingenious modes of statement which, being borne in mind, may serve as rules for as many different acts as possible. On the contrary, what we are here concerned to do is to distinguish all proceedings that are essentially different. Now it is plain that no number of mere bi-terminal bonds, each terminal occupying a spot's hook, can ever assert the identity of three things, although

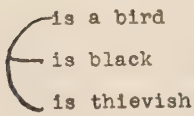


Fig. 79

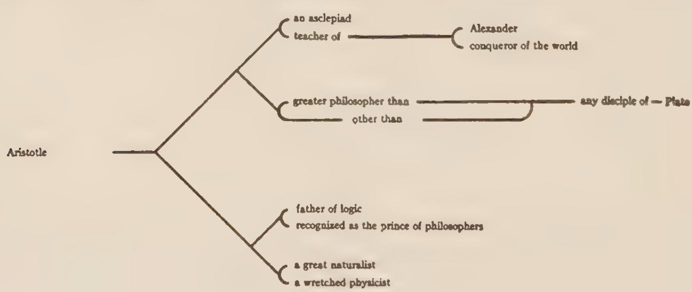


Fig. 80

when we once have a three-way branch, any higher number of terminals can be produced from it, as in Fig. 80.

446. We ought to, and must, then, make a distinct convention to cover this procedure, as follows:

Convention No. 7. *A branching line of identity shall express a triad rhema signifying the identity of the three individuals, whose designations are represented as filling the blanks of the rhema by coincidence with the three terminals of the line.*

447. Remark how peculiar a sign the line of identity is. A sign, or, to use a more general and more definite term, a *representamen*, is of one or other of three kinds:* it is either an *icon*, an *index*, or a *symbol*. An icon is a representamen of what it represents and for the mind that interprets it as such, by virtue of its being an immediate image, that is to say by virtue of characters which belong to it in itself as a sensible object, and which it would possess just the same were there no object in nature that it resembled, and though it never were interpreted as a sign. It is of the nature of an appearance, and as such, strictly speaking, exists only in consciousness, although for convenience in ordinary parlance and when extreme precision is not called for, we extend the term *icon* to the outward objects which excite in consciousness the image itself. A geometrical diagram is a good example of an icon. A pure icon can convey no positive or factual information; for it affords no assurance that there is any such thing in nature. But it is of the utmost value for enabling its interpreter to study what would be the character of such an object in case any such did exist. Geometry sufficiently illustrates that. Of a completely opposite nature is the kind of representamen termed an *index*. This is a real thing or fact which is a sign of its object by virtue of being connected with it as a matter of fact and by also forcibly intruding upon the mind, quite regardless of its being interpreted as a sign. It may simply serve to identify its object and assure us of its existence and presence. But very often the nature of the factual connexion of the index with its object is such as to excite in consciousness an image of some features of the object, and in that way affords evidence from which positive assurance as to truth of fact may be drawn. A photograph, for example, not only excites an image, has an appearance, but, owing to its optical connexion with the object, is evidence that that appearance corresponds to a reality. A *symbol* is a representamen

* Cf. vol. 2, bk. II, ch. 2, §5.

whose special significance or fitness to represent just what it does represent lies in nothing but the very fact of there being a habit, disposition, or other effective general rule that it will be so interpreted. Take, for example, the word "*man*." These three letters are not in the least like a man; nor is the sound with which they are associated. Neither is the word existentially connected with any man as an index. It cannot be so, since the word is not an existence at all. The word does not consist of three films of ink. If the word "man" occurs hundreds of times in a book of which myriads of copies are printed, all those millions of triplets of patches of ink are embodiments of one and the same word. I call each of those embodiments a *replica* of the symbol. This shows that the word is not a thing. What is its nature? It consists in the really working general rule that three such patches seen by a person who knows English will effect his conduct and thoughts according to a rule. Thus the mode of being of the symbol is different from that of the icon and from that of the index. An icon has such being as belongs to past experience. It exists only as an image in the mind. An index has the being of present experience. The being of a symbol consists in the real fact that something surely will be experienced if certain conditions be satisfied. Namely, it will influence the thought and conduct of its interpreter. Every word is a symbol. Every sentence is a symbol. Every book is a symbol. Every representamen depending upon conventions is a symbol. Just as a photograph is an index having an icon incorporated into it, that is, excited in the mind by its force, so a symbol may have an icon or an index incorporated into it, that is, the active law that it is may require its interpretation to involve the calling up of an image, or a composite photograph of many images of past experiences, as ordinary common nouns and verbs do; or it may require its interpretation to refer to the actual surrounding circumstances of the occasion of its embodiment, like such words as *that*, *this*, *I*, *you*, *which*, *here*, *now*, *yonder*, etc. Or it may be pure symbol, neither *iconic* nor *indicative*, like the words *and*, *or*, *of*, etc.

448. The value of an icon consists in its exhibiting the features of a state of things regarded as if it were purely imaginary. The value of an index is that it assures us of posi-

tive fact. The value of a symbol is that it serves to make thought and conduct rational and enables us to predict the future. It is frequently desirable that a representamen should exercise one of those three functions to the exclusion of the other two, or two of them to the exclusion of the third; but the most perfect of signs are those in which the iconic, indicative, and symbolic characters are blended as equally as possible. Of this sort of signs the line of identity is an interesting example. As a conventional sign, it is a symbol; and the symbolic character, when present in a sign, is of its nature predominant over the others. The line of identity is not, however, arbitrarily conventional nor purely conventional. Consider any portion of it taken arbitrarily (with certain possible exceptions shortly to be considered) and it is an ordinary graph for which Fig. 81 might perfectly well be substituted. But when we consider the

—— is identical with ——

Fig. 81

connexion of this portion with a next adjacent portion, although the two together make up the same graph, yet the identification of the something, to which the hook of the one refers, with the something, to which the hook of the other refers, is beyond the power of any graph to effect, since a graph, as a symbol, is of the nature of a *law*, and is therefore general, while here there must be an identification of individuals. This identification is effected not by the pure symbol, but by its *replica* which is a thing. The termination of one portion and the beginning of the next portion denote the same individual by virtue of a factual connexion, and that the closest possible; for both are points, and they are one and the same point. In this respect, therefore, the line of identity is of the nature of an index. To be sure, this does not affect the ordinary parts of a line of identity, but so soon as it is even *conceived*, [it is conceived] as composed of two portions, and it is only the factual junction of the replicas of these portions that makes them refer to the same individual. The line of identity is, moreover, in the highest degree iconic. For it appears as nothing but a continuum of dots, and the fact of the identity of a thing, seen under two aspects, consists merely in the continuity of being in passing from one apparition to another. Thus uniting, as the line of identity does, the natures of symbol, index, and icon, it is fitted for playing an extraordinary part in this system of representation.

449. There is no difficulty in interpreting the line of identity until it crosses a sep. To interpret it in that case, two new conventions will be required.

How shall we express the proposition "Every salamander lives in fire," or "If it be true that something is a salamander then it will always be true that *that something* lives in fire"? If we omit the assertion of the identity of the somethings, the expression is obviously given in Fig. 82. To that, we wish to add the expression of individual identity. We ought to use our line of identity for that. Then, we must draw Fig. 83. It would be unreasonable, after having adopted the line of identity as our instrument for the expression of individual identity, to hesitate to employ it in this case. Yet to regularize such a mode of expression two new conventions are required. For, in the first place, we have not hitherto had any such sign as a line of identity crossing a sep. This part of the line of identity is not a graph; for a graph must be either outside or inside of each sep.* In order, therefore, to legitimate our interpretation of Fig. 83, we must agree that a line of identity crossing a sep simply asserts the identity of the individual denoted by its outer part and the individual denoted by its inner part. But this agreement does not of itself necessitate our interpretation of Fig. 83; since this might be understood to mean, "There is *something* which, if it be a salamander, lives in fire," instead of meaning, "If there be *anything* that is a salamander, *it* lives in fire." But although the last interpretation but one would involve itself in no positive contradiction, it would annul the convention that a line of identity crossing a sep still asserts the identity of its extremities—not, indeed, by conflict with that convention, but by rendering it nugatory. What

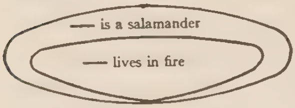


Fig. 82

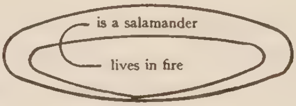


Fig. 83

*See 579.

does it mean to assert *de inesse* that there is something, which if it be a salamander, lives in fire? It asserts, no doubt, that there is something. Now suppose that anything lives in fire. Then of that it will be true *de inesse* that if it be a salamander, it lives in fire; so that the proposition will then be true. Suppose that there is anything that is not a salamander. Then, of that it will be true *de inesse* that if it be a salamander, it lives in fire; and again the proposition will be true. It is only false in case whatever there may be is a salamander while nothing lives in fire. Consequently, Fig. 83 would be precisely equivalent to Fig. 84, and there would be no need of any line of identity's crossing a sep. It would then be impossible to express a universal categorical analytically except by resorting to an unanalytic expression of such a proposition or something substantially equivalent to that.[1]

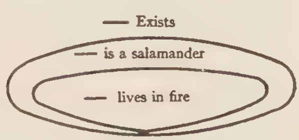


Fig. 84

Two conventions, then, are necessary. In stating them, it will be well to avoid the idea of a graph's being cut through by a sep, and confine ourselves to the effects of joining dots on the sep to dots outside and inside of it.

450. *Convention No. 8. Points on a sep shall be considered to lie outside the close of the sep so that the junction of such a point with any other point outside the sep by a line of identity shall be interpreted as it would be if the point on the sep were outside and away from the sep.*

451. *Convention No. 9. The junction by a line of identity of a point on a sep to a point within the close of the sep shall assert of such individual as is denoted by the point on the sep, according to the position of that point by Convention No. 8, a hypothetical conditional identity, according to the conventions applicable to graphs situated as is the portion of that line that is in the close of the sep.*

452. It will be well to illustrate these conventions by some examples. Fig. 85 asserts that if it be true that something is good, then this assertion is false. That is, the assertion is that nothing is good. But in Fig. 86, the terminal of the line of identity on the outer sep asserts that something, X, exists,

[1] This will be proved in a later note. [This was not done, but see 472f., 561n.]

and it is only of this existing individual, X, that it is asserted that if *that* is good the assertion is false. It therefore means

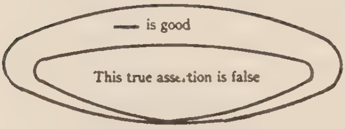


Fig. 85

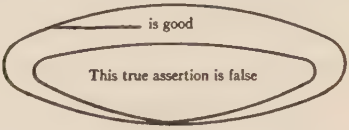


Fig. 86

"Something is not good." On Fig. 87 and Fig. 88 the points on the seps are marked with letters, for convenience of reference. Fig. 87 asserts that something, A, is a woman; and that if there is an individual, X, that is a catholic, and an individual, Y, that is identical with A, then X adores Y; that is, some woman is adored by all catholics, if there are any. Fig. 88 asserts that if there be an individual, X, and if X is a catholic, then X adores somebody that is a woman. That is, whatever

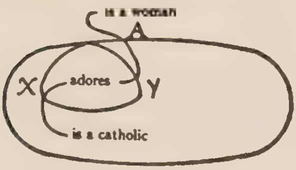


Fig. 87

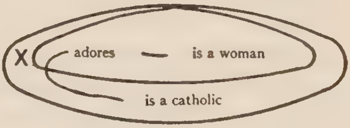


Fig. 88

catholic there may be adores some woman or other. This does not positively assert that any woman exists, but only that if there is a catholic, then there is a woman whom he adores.

453. A triad rhema gives twenty-six affirmative forms of simple general propositions, as follows:

Nos.

Fig. 89. ——blames⌋ to—— Somebody blames somebody to somebody 1

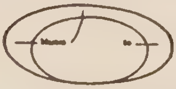
Fig. 90.  Everybody blames everybody to everybody 1

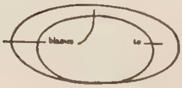
Fig. 91.  Somebody blames everybody to everybody 3 such

Fig. 92.  Everybody blames everybody to somebody or other 3 such

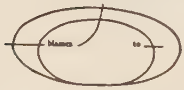
Fig. 93.  Somebody blames somebody to everybody 3 such

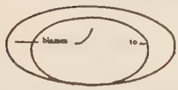
Fig. 94.  Everybody blames somebody to somebody 3 such

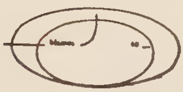
Fig. 95.  Somebody blames everybody to somebody or other 6 such

Fig. 96.  Everybody to somebody or other blames all 6 such

Total 26

For a tetrad there are 150 such forms; for a pentad 1082; for a hexad 9366; etc.

B. Derived Principles of Interpretation[P]

§1. OF THE PSEUDOGRAPH AND CONNECTED SIGNS[P]

454. It is, as will soon appear, sometimes desirable to express a proposition either absurd, contrary to the understanding between the graphist and the interpreter, or at any rate well-known to be false. From any such proposition, as antecedent, any proposition whatever follows as a consequent *de inesse*. Hence, every such proposition may be regarded as implying that everything is true; and consequently all such propositions are equivalent. The expression of such a proposition may very well fill the entire close in which it is, since nothing can be added to what it already implies. Hence we may adopt the following secondary convention.

Convention No. 10. *The* **pseudograph**, *or expression in this system of a proposition implying that every proposition is true, may be drawn as a black spot entirely filling the close in which it is.*

455. Since the size of signs has no significance, the blackened close may be drawn invisibly small. Thus Fig. 97 [may be

Fig. 97

Fig. 98

Fig. 99

Fig. 100

Fig. 101

scribed] as in Fig. 98, or even as in Fig. 99, Fig. 100, or lastly as in Fig. 101.*

456. *Interpretational Corollary* 1. *A scroll with its contents having the pseudograph in the inner close is equivalent to the precise denial of the contents of the outer close.*

For the assertion, as in Fig. 97, that *de inesse* if *a* is true everything is true, is equivalent to the assertion that *a* is not true, since if the conditional proposition *de inesse* be true *a* cannot be true, and if *a* is not true the conditional proposition *de inesse*, having *a* for its antecedent, is true. Hence the one is always true or false with the other, and they are equivalent.

This corollary affords additional justification for writing Fig. 97 as in Fig. 101, since the effect of the loop enclosing the pseudograph is to make a precise denial of the absurd proposition; and to deny the absurd is equivalent to asserting nothing.

457. *Interpretational Corollary* 2. *A disjunctive proposition may be expressed by placing its members in as many inloops of one sep. But this will not exclude the simultaneous truth of several members or of all.*

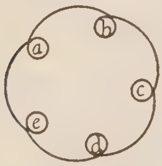


Fig. 102

Thus, Fig. 102 will express that either *a* or *b* or *c* or *d* or *e* is true. For it will deny the simultaneous denial of all.

458. *Interpretational Corollary* 3. *A graph may be interpreted by copulations and disjunctions. Namely, if a graph within an odd number of seps be said to be* **oddly enclosed**, *and a graph within no sep or an even number of seps be said to be*

*Cf. 564n.

evenly enclosed, *then spots in the same compartment are copulated when evenly enclosed, and disjunctively combined when oddly enclosed; and any line of identity whose outermost part is evenly enclosed refers to* **something**, *and any one whose outermost part is oddly enclosed refers to* **anything** *there may be. And the interpretation must begin outside of all seps and proceed inward. And spots evenly enclosed are to be taken affirmatively; those oddly enclosed negatively.*

For example, Fig. 83 may be read, Anything whatever is either not a salamander or lives in fire. Fig. 87 may be read, Something, A, is a woman, and whatever X may be, either X is not a catholic or X adores A. Fig. 88 may be read, Whatever X may be, either X is not a catholic or there is something Y, such that X adores Y and Y is a woman. Fig. 96 may be read, Whatever A may be, there is something C, such that whatever B may be, A blames B to C. Fig. 103 may be read, Whatever X and Y may be, either X is not a saint or Y is not a saint or X loves Y; that is, Every saint there may be loves every saint. So Fig. 104 may be read, Whatever X and Y may be, either X is not best or Y is not best or X is identical with Y; that is, there are not two bests. Fig. 105 may be read, Whatever X and Y may be, either X does not love Y or Y does not love X; that is, no two love each other. Fig. 106 may be read, Whatever X and Y may be either X does not love Y or there is something L and X is not L but Y loves L; that is, nobody loves anybody who does not love somebody else.

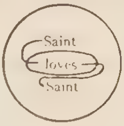


Fig. 103

Fig. 104

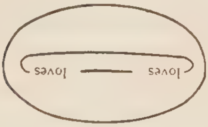


Fig. 105

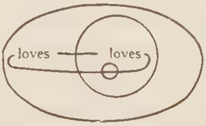


Fig. 106

459. *Interpretational Corollary* 4. *A sep which is vacant, except for a line of identity traversing it, expresses with its contents the non-identity of the extremities of that line.*

§2. SELECTIVES AND PROPER NAMES[P]

460. It is sometimes impossible upon an ordinary surface to draw a graph so that lines of identity will not cross one another. If, for example, we express that x is a value that can result from raising z to the power whose exponent is y, by means of Fig. 107, and express that u is a value that can result from multiplying w by v, by Fig. 108, then in order to express that

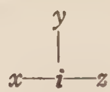


Fig. 107

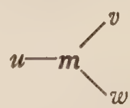


Fig. 108

whatever values x, y, and z may be, there is a value resulting from raising x to a power whose exponent is a value of the product of z by y which same value is also one of the values resulting from raising to the power z a value resulting from raising x to the power y (this being one of the propositions expressed by the equation $x^{(yz)} = (x^y)^z$), we may draw Fig. 109; but there is an unavoidable intersection of two lines of identity. In such a case, and indeed in any case in which the lines of identity become too intricate to be perspicuous, it is advantageous to replace some of them by signs of a sort that in this system are called *selectives*. A selective is very much of the same nature as a proper name; for it denotes an individual and its outermost occurrence denotes a wholly indesignate individual of a certain category (generally a thing) existing in the universe, just as a proper name, on the first occasion of hearing it, conveys no more. But, just as on any subsequent hearing of a proper name, the hearer identifies it with that individual concerning which he has some information, so all occurrences of the selective other than the outermost must be understood to denote that identical individual. If, however, the outermost occurrence of any given selective is oddly enclosed, then, on that first occurrence the selective will refer to any individual whom the interpreter may choose, and in all other occurrences to the same individual. If there be no one outermost occur-

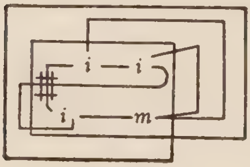


Fig. 109

rence, then any one of those that are outermost may be considered as the outermost. The later capital letters are used for selectives. For example, Fig. 109 is otherwise expressed in Figs. 110 and 111.

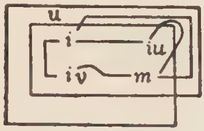


Fig. 110

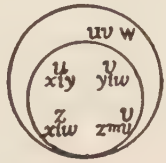


Fig. 111

Fig. 111 may be read, "Either no value is designated as U, or no value is designated as V, or no value is designated as W, or else a value designated as Y results from raising W to the V power, and a value designated as Z results from multiplying U by V, and a value designated as X results from raising Y to the U power, while this same value X results from raising W to the Z power."

461. *Convention No.* 11. *The capital letters of the alphabet shall be used to denote single individuals of a well-understood category, the individual existing in the universe, the early letters preferably as proper names of well-known individuals, the later letters, called* **selectives**, *each on its first occurrence, as the name of an* **individual** (*that is, an object existing in the universe in a well-understood category; that is, having such a mode of being as to be determinate in reference to every character as wholly possessing it or else wholly wanting it*), *but an individual that is* **indesignate** (*that is, which the interpreter receives no warrant for identifying*); *while in every occurrence after the first, it shall denote that same individual. Of two occurrences of the same selective, either one may be interpreted as the earlier, if and only if, enclosed by no sep that does not enclose the other. A selective at its first occurrence shall be asserted in the mode proper to the compartment in which it occurs. If it be on that occurrence evenly enclosed, it is only affirmed to exist under the same conditions under which any graph in the same close is asserted; and it is then asserted, under those conditions, to be the subject filling the rhema-blank corresponding to any hook against which it may be placed. If, however, at its first occurrence, it be oddly enclosed, then, in the disjunctive mode of interpretation, it will be denied, subject to the*

conditions proper to the close in which it occurs, so that its existence being disjunctively denied, a non-existence will be affirmed, and as a subject, it will be **universal** (*that is, freed from the condition of wholly possessing or wholly wanting each character*) *and at the same time* **designate** (*that is, the interpreter will be warranted in identifying it with whatever the context may allow*), *and it will be, subject to the conditions of the close, disjunctively denied to be the subject filling the rhema-blank of the hook against which it may be placed. In all subsequent occurrences it shall denote the individual with which the interpreter may, on its first occurrence, have identified it, and otherwise will be interpreted as on its first occurrence.*

Resort must be had to the examples to trace out the sense of this long abstract statement; and the line of identity will aid in explaining the equivalent selectives. Fig. 112 may be read

X is good

Fig. 112

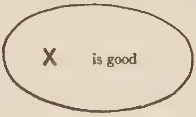


Fig. 113

there exists something that may be called X and it is good. Fig. 113, the precise denial of Fig. 112, may be read "Either there is not anything to be called X or whatever there may be is not good," or "Anything you may choose to call X is not good," or "all things are non-good." "Anything" is not an individual subject, since the two propositions, "Anything is good" and "Anything is bad," do not exhaust the possibilities. Both may be false.

462. *Convention No.* 12. *The use of selectives may be avoided, where it is desired to do so, by drawing parallels on both sides of the lines of identity where they appear to cross.**

§3. OF ABSTRACTION AND *ENTIA RATIONIS*†[P]

463. The term *abstraction* bears two utterly different meanings in philosophy. In one sense it is applied to a psy-

* To illustrate this, two complicated graphs are given. They are not reproduced because the ambiguity in Peirce's explanations makes them unilluminating.

†This section deals in part with the Gamma Part of Graphs; see particularly 470–471. Cf. also 516ff.

chological act by which, for example, on seeing a theatre, one is led to call up images of other theatres which blend into a sort of composite in which the special features of each are obliterated. Such obliteration is called precisive abstraction. We shall have nothing to do with abstraction in that sense. But when that fabled old doctor, being asked why opium put people to sleep, answered that it was because opium has a dormative virtue, he performed this act of immediate inference:

Opium causes people to sleep;
Hence, Opium possesses a power of causing sleep.

The peculiarity of such inference is that the conclusion relates to something — in this case, a power — that the premiss says nothing about; and yet the conclusion is necessary. *Abstraction*, in the sense in which it will here be used, is a necessary inference whose conclusion refers to a subject not referred to by the premiss; or it may be used to denote the characteristic of such inference. But how can it be that a conclusion should necessarily follow from a premiss which does not assert the existence of that whose existence is affirmed by it, the conclusion itself? The reply must be that the new individual spoken of is an *ens rationis;* that is, its being consists in some other fact. Whether or not an *ens rationis* can *exist* or be *real*, is a question not to be answered until existence and reality have been very distinctly defined. But it may be noticed at once, that to deny every mode of being to anything whose being consists in some other fact would be to deny every mode of being to tables and chairs, since the being of a table depends on the being of the atoms of which it is composed, and not *vice versa*.

464. Every symbol is an *ens rationis*, because it consists in a habit, in a regularity; now every regularity consists in the future conditional occurrence of facts not themselves that regularity. Many important truths are expressed by propositions which relate directly to symbols or to ideal objects of symbols, not to realities. If we say that two walls collide, we express a real relation between them, meaning by a *real relation* one which involves the existence of its correlates. If we say that a ball is red, we express a positive quality of feeling really connected with the ball. But if we say that the ball is not

blue, we simply express — as far as the direct expression goes — a relation of inapplicability between the predicate blue, and the ball or the sign of it. So it is with every negation. Now it has already been shown that every universal proposition involves a negation, at least when it is expressed as an existential graph. On the other hand, almost every graph expressing a proposition not universal has a line of identity. But identity, though expressed by the line as a dyadic relation, is not a relation between two things, but between two representamens of the same thing.

465. Every rhema whose blanks may be filled by signs of ordinary individuals, but which signifies only what is true of symbols of those individuals, without any reference to qualities of sense, is termed a *rhema of second intention*. For *second intention* is thought about thought as symbol. Second intentions and certain *entia rationis* demand the special attention of the logician. Avicenna defined logic as the science of second intentions, and was followed in this view by some of the most acute logicians, such as Raymund Lully, Duns Scotus, Walter Burleigh, and Armandus de Bello Visu; while the celebrated Durandus à Sancto Porciano, followed by Gratiadeus Esculanus, made it relate exclusively to *entia rationis*, and quite rightly.

466. *Interpretational Corollary* 5. A blank, considered as a medad, expresses what is well-understood between graphist and interpreter to be true; considered as a monad, it expresses "—— exists" or "—— is true"; considered as a dyad, it expresses "—— coexists with ——" or "and."

467. *Interpretational Corollary* 6. An empty sep with its surrounding blank, as in Fig. 114, is the pseudograph. Whether


Fig. 114


Fig. 115


Fig. 116

it be taken as medad, monad, or dyad, for which purpose it will be written as in Figs. 115, 116, it is the denial of the blank.

Fig. 117

468. *Interpretational Corollary* 7. A line of identity traversing a sep will signify non-

identity. Thus Fig. 117 will express that there are at least two men.

469. *Interpretational Corollary* 8. A branching of a line of identity enclosed in a sep, as in Fig. 118, will express that three individuals are not all identical.

Fig. 118

We now come to another kind of graphs which may go under the general head of second intentional graphs.*

470. *Convention No.* 13. *The letters*, ρ_0, ρ_1, ρ_2, ρ_3, etc., *each with a number of hooks greater by one than the subscript number, may be taken as rhemata, signifying that the individuals joined to the hooks, other than the one vertically above the* ρ, *taken in their order clockwise, are capable of being asserted of the rhema indicated by the line of identity joined vertically to the* ρ.

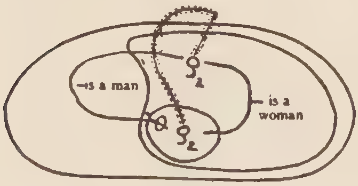


Fig. 119

Thus, Fig. 119 expresses that there is a relation in which every man stands to some woman to whom no other man stands in the same relation; that is, there is a woman corresponding to every man or, in other words, there are at least as many women as men. The dotted lines, between which, in Fig. 119, the line of identity denoting the *ens rationis* is placed, are by no means necessary.

471. *Convention No.* 14. *The line of identity representing an* **ens rationis** *may be placed between two rows of dots, or it may be drawn in ink of another colour, and any graph, which is to be spoken of as a thing, may be enclosed in a dotted oval with a dotted line attached to it. Other* **entia rationis** *may be treated in the same way, the patterns of the dotting being varied for those of different category.*

The graph of Fig. 120 is an example. It may be read, as follows: "Euclid† enunciates it as a postulate that if two straight lines are cut by a third straight line so that those angles the two make with the third, these angles lying between the first two lines (τὰς ἐντὸς γωνίας) and on the same side of

* *I.e.*, the Gamma Part of Graphs.

† Bk. I, Postulate 5.

the third, are less than two right angles, then that those two lines shall meet on that same side; and in this enunciation, by a side, *μέρη* of the third line must be understood part of a plane that contains that third line, which part is bounded by that line and by the infinitely distant parts of the plane." . . .

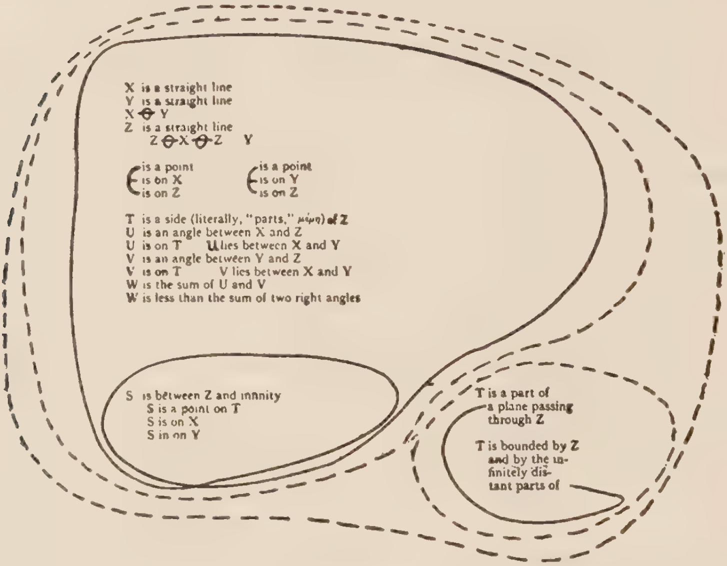


Fig. 120

C. *Recapitulation*[P]

472. The principles of interpretation may now be restated more concisely and more comprehensibly. In this *resumé,* it will be assumed that selectives, which should be regarded as a mere abbreviating device, and which constitute a serious exception to the general idea of the system, are not used. A person, learning to use the system and not yet thoroughly expert in it, might be led to doubt whether every proposition is capable of being expressed without selectives. For a line of identity cannot identify two individuals within enclosures outside of one another without passing out of both enclosures, while a selective is not subject to that restriction. It can be shown, however, that this restriction is of no importance nor even helps to render thought clear. Suppose then that two desig-

nations of individuals are to be identified, each being within a separate nest of seps, and the two nests being within a common nest of outer seps. The question is whether this identification can always be properly effected by a line of identity that passes out of the two separate nests of seps, and if desired, still farther out. The answer is plain enough when we consider that, having to say something of individuals, some to be named by the grapheus, others by the graphist, we can perfectly well postpone what we have to say until all these individuals are indicated; that is to say, the order in which they are to be specified by one and the other party. But if this be done, these individuals will first appear, even if selectives are used, in one nest of seps entirely outside of all the spots; and then these selectives can be replaced by lines of identity.

473. The respect in which selectives violate the general idea of the system is this; the outermost occurrence of each selective has a different significative force from every other occurrence — a grave fault, if it be avoidable, in any system of regular and exact representation. The consequence is that the meaning of a partial graph containing a selective depends upon whether or not there be another part, which may be written on a remote part of the sheet in which the same selective occurs farther out. But the idea of this system is that assertions written upon different parts of the sheet should be independent of one another, if, and only if, they have no common part. When lines of identity are used to the exclusion of selectives, no such inconvenience can occur, because each line of one partial graph will retain precisely the same significative force, no matter what part outside of it be removed (though if a line be broken, the identity of the individuals denoted by its two parts will no longer be affirmed); and even if everything outside a sep be removed (the sep being unbreakable by any removal of a partial graph, or part which written alone would express a proposition) still there remains a point on the sep which retains the same force as if the line had been broken quite outside and away from the sep.

474. Rejecting the selectives, then, the principles of interpretation reduce themselves to simple form, as follows:

1. The writing of a proposition on the sheet of assertion unenclosed is to be understood as asserting that proposition;

and that, independently of any other proposition on the sheet, except so far as the two may have some part or point in common.

2. A "spot," or unanalyzed expression of a rhema, upon this system, has upon its periphery a place called a "hook" appropriated to every blank of the rhema; and whenever it is written a heavily marked point occupies each hook. Now every heavily marked point, whether isolated or forming a part of a heavy line, denotes an indesignate individual, and being unenclosed affirms the existence of some such individual; and if it occupy a hook of a spot it is the corresponding subject of the rhema signified by the spot. A heavy line is to be understood as asserting, when unenclosed, that all its points denote the same individual, so that any portion of it may be regarded as a spot.

3. A sep, or lightly drawn oval, when unenclosed is with its contents (the whole being called an *enclosure*) a graph, entire or partial, which precisely denies the proposition which the entire graph within it would, if unenclosed, affirm. Since, therefore, an entire graph, by the above principles, copulatively asserts all the partial graphs of which it is composed, and takes every indesignate individual, denoted by a heavily marked point that may be a part of it, in the sense of "something," it follows that an unenclosed enclosure disjunctively denies all the partial graphs which compose the contents of its sep, and takes every heavily marked point included therein in the sense of "anything" whatever. Consequently, if an enclosure is oddly enclosed, its evenly enclosed contents are copulatively affirmed; while if it be evenly enclosed, its oddly enclosed contents are disjunctively denied.

4. A heavily marked point upon a sep, or line of enclosure, is to be regarded as no more enclosed than any point just outside of and away from the sep, and is to be interpreted accordingly. But the effect of joining a heavily marked point within a sep to such a point upon the sep itself by means of a heavy line is to limit the disjunctive denial of existence (which is the effect of the sep upon the point within it) to the individual denoted by the point upon the sep. No heavy line is to be regarded as cutting a sep; nor can any graph be partly within a sep and partly outside of it; although the entire

enclosure (which is not inside the sep) may be part of a graph outside of the sep.*

5. A dotted oval is sometimes used to show that that which is within it is to be regarded as an *ens rationis*.

Part II. The Principles of Illative Transformation[P]

A. Basic Principles[P]

§1. SOME AND ANY

475. The first part of this tract was a grammar of this language of graphs. But one has not mastered a language as long as one has to think about it in another language. One must learn to think *in* it *about* facts. The present part is designed to show how to reason *in* this language without translating it into another, the language of our ordinary thought. This reasoning, however, depends on certain first principles, for the justification of which we have to make a last appeal to instinctive thought.

476. The purpose of reasoning is to proceed from the recognition of the truth we already know to the knowledge of novel truth. This we may do by instinct or by a habit of which we are hardly conscious. But the operation is not worthy to be called reasoning unless it be deliberate, critical, self-controlled. In such genuine reasoning we are always conscious of proceeding according to a general rule which we approve. It may not be precisely formulated, but still we do think that all reasoning of that perhaps rather vaguely characterized kind will be safe. This is a doctrine of logic. We never can really reason without entertaining a logical theory. That is called our *logica utens*.†

477. The purpose of logic is attained by any single passage from a premiss to a conclusion, as long as it does not at once happen that the premiss is true while the conclusion is false. But reasoning proceeds upon a rule, and an inference is not *necessary*, unless the rule be such that in every case the fact stated in the premiss and the fact stated in the conclusion are so related that either the premiss will be false or the conclu-

* But see 579.

† Cf. 2.186ff.

sion will be true. (Or *both*, of course. "*Either* A *or* B" does not properly exclude "*both* A *and* B.") Even then, the reasoning may not be *logical*, because the rule may involve matter of fact, so that the reasoner cannot have sufficient ground to be absolutely certain that it will not sometimes fail. The inference is only logical if the reasoner can be *mathematically certain* of the excellence of his rule of reasoning; and in the case of necessary reasoning he must be mathematically certain that in every state of things whatsoever, whether now or a million years hence, whether here or in the farthest fixed star, such a premiss and such a conclusion will never be, the former true and the latter false. It would be far beyond the scope of this tract to enter upon any thorough discussion of how this can be. Yet there are some questions which concern us here — as, for example, how far the system of rules of this section is eternal verity, and how far it merely characterizes the special language of existential graphs — and yet trench closely upon the deeper philosophy of logic; so that a few remarks meant to illuminate those pertinent questions and to show how they are connected with the philosophy of logic seem to be quite in order.

478. Mathematical certainty is not absolute certainty. For the greatest mathematicians sometimes blunder, and therefore it is *possible* — barely possible — that all have blundered every time they added two and two. Bearing in mind that fact, and bearing in mind the fact that mathematics deals with imaginary states of things upon which experiments can be enormously multiplied at very small cost, we see that it is not impossible that inductive processes should afford the basis of mathematical certainty; and any mathematician can find much in the history of his own thought, and in the public history of mathematics to show that, as a matter of fact, inductive reasoning is considerably employed in making sure of the first mathematical premisses. Still, a doubt will arise as to whether this is anything more than a psychological need, whether the reasoning really rests upon induction at all. A geometer, for example, may ask himself whether two straight lines can enclose an area of their plane. When this question is first put, it is put in reference to a concrete image of a plane; and, at first, some experiments will be tried in the imagination.

Some minds will be satisfied with that degree of certainty: more critical intellects will not. They will reflect that a closed area is an area shut off from other parts of the plane by a boundary all round it. Such a thinker will no longer think of a closed area by a composite photograph of triangles, quadrilaterals, circles, etc. He will think of a predictive rule — a thought of what experience one would intend to produce who should intend to establish a closed area.

479. That step of thought, which consists in interpreting an image by a symbol, is one of which logic neither need nor can give any account, since it is subconscious, uncontrollable, and not subject to criticism. Whatever account there is to be given of it is the psychologist's affair. But it is evident that the image must be connected in some way with a symbol if any proposition is to be true of it. The very truth of things must be in some measure representative.

480. If we admit that propositions express the very reality, it is not surprising that the study of the nature of propositions should enable us to pass from the knowledge of one fact to the knowledge of another.[1]

481. We frame a system of expressing propositions — a written language — having a syntax to which there are abso-

[1] Some reader may think that I am expending energy in trying to explain what needs no explanation. He may argue that the mathematician reasons about a diagram in which there appears to be nothing at all corresponding to the structure of the proposition — no predicate and subjects. Nor does the mathematician's premiss or conclusion at all pretend to represent the diagram in that respect. It may seem to this reader satisfactory to say that the conclusion follows from the premiss, because the premiss is only applicable to states of things to which the conclusion is applicable. If he thinks that satisfactory, the purpose of this tract does not compel me to dispute it. It is only to defend myself against the charge of giving a needless and doubtful explanation that I point out that it is precisely this relation of applicability that requires to be explained. How comes it that the conclusion is applicable whenever the premiss is applicable? I suppose the answer will be that its only meaning is a part of what the premiss means. The "meaning" of a proposition is what it is intended to convey. But when a mathematician lays down the premisses of the theory of numbers, it cannot be said that he then intends to convey all the propositions of that theory, of which the great majority will occasion him much surprise when he comes to learn them. If to avoid this objection a distinction be drawn between what is explicitly intended and what is implicitly intended, I submit that this manifestly makes a vicious circle; for what can it be *implicitly* to intend anything, except to intend whatever may be a necessary consequence of what is explicitly intended?

lutely no exceptions. We then satisfy ourselves that whenever a proposition having a certain syntactical form is true, another proposition definitely related to it — so that the relation can be defined in terms of the appearance of the two propositions on paper — will necessarily also be true. We draw up our code of basic rules of such illative transformations, none of these rules being a necessary consequence of others. We then proceed to express in our language the premisses of long and difficult mathematical demonstrations and try whether our rules will bring out their conclusions. If, in any case, not, and yet the demonstration appears sound, we have a lesson in logic to learn. Some basic rule has been omitted, or else our system of expression is insufficient. But after our system and its rules are perfected, we shall find that such analyses of demonstrations teach us much about those reasonings. They will show that certain hypotheses are superfluous, that others have been virtually taken for granted without being expressly laid down; and they will show that special branches of mathematics are characterized by appropriate modes of reasoning, the knowledge of which will be useful in advancing them. We may now lay all that aside, and begin again, constructing an entirely different system of expression, developing it from an entirely different initial idea, and having perfected it, as we perfected the former system, we shall analyze the same mathematical demonstrations. The results of the two methods will agree as to what is and what is not a necessary consequence. But a consequence that either method will represent as an immediate application of a basic rule, and therefore as simple, the other will be pretty sure to analyze into a series of steps. If it be not so, in regard to some inference the one method will be merely a disguise of the other. To say that one thing is simpler than another is an incomplete proposition, like saying that one ball is to the right of another. It is necessary to specify what point of view is assumed, in order to render the sentence true or false.

482. This remark has its application to the business now in hand, which is to translate the effect of each simple illative transformation of an existential graph into the language of ordinary thought and thus show that it represents a necessary consequence. For it will be found that it is not the operations

which are simplest in this system that are simplest from the point of view of ordinary thought; so that it will be found that the simplest way to establish by ordinary thought the correctness of our basic rules will be to begin by proving the legitimacy of certain operations that are less simple from the point of view of the existential graphs.

483. The first proposition for assent to which I shall appeal to ordinary reason is this; when a proposition contains a number of *any*s and *some*s, or their equivalents, it is a delicate matter to alter the form of statement while preserving the exact meaning. Every *some*, as we have seen,* means that under stated conditions, an individual could be specified of which that which is predicated of the *some* is [true], while every *any* means that what is predicated is true of no matter what [specified] individual; and the specifications of individuals must be made in a certain order, or the meaning of the proposition will be changed. Consider, for example, the following proposition: "A certain bookseller only quotes a line of poetry in case it was written by some blind authoress, and he either is trying to sell any books she may have written to the person to whom he quotes the line or else intends to reprint some book of hers." Here the existence of a bookseller is categorically affirmed; but the existence of a blind authoress is only affirmed conditionally on that bookseller's quoting a line of poetry. As for any book by her, none such is positively said to exist, unless the bookseller is not endeavoring to sell all the books there may be by her to the person to whom he quotes the line.

484. Now the point to which I demand the assent of reason is that all those individuals, whose selection is so referred to, might be named to begin with, thus: "There is a certain individual, A, and no matter what Z and Y may be, an individual, B, can be found such that whatever X may be, there is something C, and A is a bookseller and if he quotes Z to Y, and if Z is a line of poetry and Y is a person, then B is a blind poetess who has written Z, and either X is not a book published by B or A tries to sell X to Y or else C is a book published by B and A intends to reprint C." This is the precise equivalent of the original proposition, and any proposition involving *some*s and *any*s, or their equivalents, might equally be expressed by

* In 439–40.

first thus defining exactly what these *somes* and *anys* mean, and then going on to predicate concerning them whatever is to be predicated. This is so evident that any proof of it would only confuse the mind; and anybody who could follow the proof will easily see how the proof could be constructed. But after the *somes* and *anys* have thus been replaced by letters, denoting each one individual, the subsequent statement concerns merely a set of designate individuals.

§2. RULES FOR DINECTED GRAPHS

485. In order, then, to make evident to ordinary reason what are the simple illative transformations of graphs, I propose to imagine the lines of identity to be all replaced by selectives, whose first occurrences are entirely outside the substance of the graph in a nest of seps, where each selective occurs once only and with nothing but existence predicated of it (affirmatively or negatively according as it is evenly or oddly enclosed). I will then show that upon such a graph certain transformations are permissible, and then will suppose the selectives to be replaced by lines of identity again. We shall thus have established the permissibility of certain transformations without the intervention of selectives.

486. There will therefore be two branches to our inquiry. First, what transformations may be made in the inner part of the graph where all the selectives have proper names, and secondly what transformations may be made in the outer part where each selective occurs but once. It will be found that the second inquiry almost answers itself after the first has been investigated, and further, that the first class of transformations are precisely the same as if all the first occurrences of selectives were erased and the others were regarded as proper names. We therefore begin by inquiring what transformations are permissible in a graph which has no connexi at all, neither lines of identity nor selectives.

487. First of all, let us inquire what are those modes of illative transformation by each of which any graph whatever, standing alone on the sheet of assertion, may be transformed, and, at the same time, what are those modes of illative transformation from each of which any graph whatever, standing alone on the sheet of assertion, might result. Let us confine

ourselves, in the first instance, to transformations not only involving no connexi, but also involving no *entia rationis* nor seps. Let us suppose a graph, say that of Fig. 121, to be alone upon the sheet of assertion. In what ways can it be illatively transformed without using connexi nor seps nor other *entia rationis?* In the first place, it may be erased; for the result of erasure, asserting nothing at all, can assert nothing false. In the second place, it can be iterated, as in Fig. 122; for the result of the iteration asserts nothing not asserted already. In the third place, any graph, well-understood (before the original graph was drawn) to be true, can be inserted, as in Fig. 123. Evidently, these are the only modes of transformation that conform to the assumed conditions. Next, let us inquire in what manner any graph, say that of Fig. 124, can result. It cannot, unless of a special nature, result from insertion, since the blank is true and the graph may be false; but it can result by any omission, say of *y* from the graph of Fig. 125, whether *y* be true or false, or whatever its relation to *z*, since the result asserts nothing not asserted in the graph from which it results.

a

Fig. 121

aa

Fig. 122

The universe
is here
a

Fig. 123

z

Fig. 124

y z

Fig. 125

488. We may now employ the following:

Conditional Principle No. 1. If any graph, *a*, were it written alone on the sheet of assertion, would be illatively transformable into another graph, *z*, then if the former graph, *a*, is a partial graph of an entire graph involving no connexus or sep, and written on the sheet of assertion, *a* may still be illatively transformed in the same way.

For let *a* be a partial graph of which the other part is *m*, in Fig. 126. Then, both *a* and *m* will be asserted. But since *a* would be illatively transformable into *z* if it were the entire graph, it follows that if *a* is true *z* is true. Hence, the result of the transformation asserts only *m* which is already asserted, and *z* which is true if *a*, which is already asserted, is true.

a m

Fig. 126

z m

Fig. 127

489. By means of this principle we can evidently deduce the following:

Categorical Basic Rules for the Illative Transformation of Graphs dinectively built up from partial graphs not separated by seps.

1. Any partial graph may be erased.
2. Any partial graph may be iterated.
3. Any graph well-understood to be true may be inserted.

It is furthermore clear that no transformation of such graphs is *logical*, that is, results from the mere form of the graph, that is not justified by these rules. For a transformation not justified by these rules must insert something not in the premiss and not well-understood to be true. But under those circumstances, it may be false, as far as appears from the form.

490. Let us now consider graphs having no connexi or *entia rationis* other than seps. Here we shall have the following

Conditional Principle No. 2. If a graph, *a*, were it written alone on the sheet of assertion, would be illatively transformable into a sep containing nothing but a graph, *z*, then in case nothing is on the sheet of assertion except this latter graph, *z*, this will be illatively transformable into a sep containing nothing but *a*.

For to say that Fig. 123 [?121] is illatively transformable into Fig. 128, is to say that if *a* is true, then if *z* were true, anything you like would be true; while to say that Fig. 124 is illatively transformable into Fig. 129 is to say that if *z* is true, then if *a* were true, anything you like would be true. But each of these amounts to saying that if *a* and *z* were both true anything you like would be true. Therefore, if either [transformation] is true so is the other.

491. *Conditional Principle No.* 3. If a sep containing nothing but a graph, *a*, would, were it written alone on the sheet of assertion, be illatively transformable into a graph, *z*, then if a sep, containing nothing but the latter graph, *z*, were written alone on the sheet of assertion, [this would] be illatively transformable into the graph, *a*.

Fig. 128

Fig. 129

For to say that Fig. 129 is illatively transformable into Fig. 124 is to say that by virtue of the forms of *a* and *z*, if *a* is false, *z* is true; in other words, by virtue of their forms, either *a* or *z* is true. But this is precisely the meaning of saying that Fig. 128 is illatively transformable into Fig. 123 [?121].

492. By means of these principles we can deduce the following:

Basic Categorical Rules for the Illative Transformation of Graphs dinectively built up from Partial Graphs and from Graphs separated by seps.

Rule 1. Within an even finite number (including none) of seps, any graph may be erased; within an odd number any graph may be inserted.

Rule 2. Any graph may be iterated within the same or additional seps, or if iterated, a replica may be erased, if the erasure leaves another outside the same or additional seps.

Rule 3. Any graph well-understood to be true (and therefore an enclosure having a pseudograph within an odd number of its seps) may be inserted outside all seps.

Rule 4. Two seps, the one enclosing the other but nothing outside that other, can be removed.

493. These rules have now to be demonstrated. The former set of rules, already demonstrated, apply to every graph on the sheet of assertion composed of dinected partial graphs not enclosed; for the reasoning of the demonstrations so apply. It is now necessary to demonstrate, from Conditional Principle No. 2, the following *Principle of Contraposition:* If any graph, say that of Fig. 123 [?121], is illatively transformable into another graph, say that of Fig. 124, then an enclosure consisting of a sep containing nothing but the latter graph, as in Fig. 130, is illatively transformable into

Fig. 130

Fig. 131

Fig. 132

Fig. 133

Fig. 134

an enclosure consisting of a sep containing nothing but the first graph, as in Fig. 131. In order to prove this principle, we must first prove that any graph on the sheet of assertion is illatively transformable by having two seps drawn round it, the one containing nothing but the other with its contents. For let *z* be the original graph. Then, it has to be shown that Fig. 124 is transformable into Fig. 132. Now Fig. 130 on the sheet of assertion is illatively transformable into itself since any graph is illatively transformable into any graph that by virtue of its form cannot be false unless the original graph be false, and Fig. 130 cannot be false unless Fig. 130 is false. But from this it follows, by Conditional Principle No. 2, that Fig. 124 is illatively transformable into Fig. 132. Q. E. D. The principle of contraposition, which can now be proved without further difficulty, is that if any graph, *a*, (Fig. 123 [?121]) is illatively transformable into any graph, *z*, (Fig. 124) then an enclosure (Fig. 130) consisting of a sep enclosing nothing but the latter graph, *z*, is transformable into an enclosure (Fig. 131) consisting of a sep containing nothing but the first graph, *a*. If *a* is transformable into *z*, then, by the rule just proved, it is transformable into Fig. 132, consisting of *z* doubly enclosed with nothing between the seps. But if Fig. 123 [?121] is illatively transformable into Fig. 132, then, by Conditional Principle No. 2, Fig. 130 is illatively transformable into Fig. 131, Q. E. D.

494. Supposing, now, that Rule 1 holds good for any insertion or omission within not more than any finite number, N, of seps, it will also hold good for every insertion or omission within not more than N+1 seps. For in any graph on the sheet of insertions of which a partial graph is an enclosure consisting of a sep containing only a graph, *z*, involving a nest of N seps, let the partial graph outside this enclosure be *m*, so that Fig. 133 is the entire graph. Then application of the rule within the N+1 seps will transform *z* into another graph, say *a*, so that Fig. 134 will be the result. Then *a*, were it written on the sheet of assertion unenclosed and alone, would be illatively transformable into *z*, since the rule is supposed to be valid for an insertion or omission within N seps. Hence, by the principle of contraposition, Fig. 130 will be transformable into Fig. 131, and by Conditional Principle No. 1, Fig. 133 will be transformable into Fig. 134. It is therefore proved that if Rule 1 is valid within any number of seps up to any finite number, it is valid for the next larger whole number of seps. But by Rule 1 of the former set of rules, it is valid for N=0, and hence it follows that it is valid within seps whose number can be reached from 0 by successive additions of unity; that is, for any finite number. Rule 1 is, therefore, valid as stated. It will be remarked that the partial graphs may have any multitude whatsoever; but the seps of a nest are restricted to

a finite multitude, so far as this rule is concerned. A graph with an endless nest of seps is essentially of doubtful meaning, except in special cases. Thus Fig. 135, supposed to continue the alternation endlessly, evidently merely asserts the truth of *a*.* But if instead of *ba*, *b* were everywhere to stand alone, the graph would certainly assert either *a* or *b* to be true and would certainly be true if *a* were true, but whether it would be true or false in case *b* were true and not *a* is essentially doubtful.

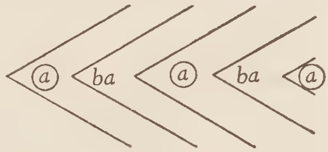


Fig. 135

495. Rule 2 is so obviously demonstrable in the same way that it will be sufficient to remark that unenclosed iterations of unenclosed graphs are justified by Rule 2 of the former set of rules. Then, since Fig. 136 is illatively transformable into

am n

Fig. 136

am an

Fig. 137

Fig. 138

Fig. 139

Fig. 140

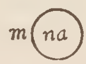


Fig. 141

Fig. 137, it follows from the principle of contraposition that Fig. 138 is illatively transformable into Fig. 139. Or we may reason that to say that Fig. 137 follows from Fig. 136 is to say that, *am* being true, *an* follows from *n*; while to say that Fig. 139 follows from Fig. 138, is to say that, *am* being true, as before, if from *an* anything you like follows, then from *n* anything you like follows. In the same way Fig. 140 is transformable into Fig. 141.

496. The transformations the reverse of these, that is of Fig. 137 into Fig. 136, of Fig. 139 into Fig. 138, and of Fig. 141 into Fig. 140 are permitted by Rule 1. Then by the same Fermatian reasoning by which Rule 1 was demonstrated, we easily show that a graph can anywhere be illatively inserted or omitted, if there is another occurrence of the same graph in the same compartment or farther out by one sep. For if Fig. 138 is transformable into Fig. 139, then by the principle of contraposition, Fig. 142 is transformable into Fig. 143, and by Conditional Principle No. 1, Fig. 144 is transformable in

* Cf. 2.356.

Fig. 145. Having thus proved that iterations and deiterations are always permissible in the same compartment as the leading replica or in a compartment within one additional sep, we have no difficulty in extending this to any finite interval.

Fig. 142

Fig. 143

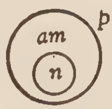


Fig. 144

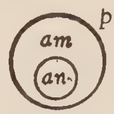


Fig. 145

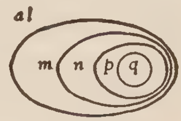


Fig. 146

Thus, Fig. 146 is transformable into Fig. 147, this into Fig. 148, this successively into Figs. 149 to 153. Thus, the second rule is fully demonstrable.

Rule 3 is self-evident.

497. We have thus far had no occasion to appeal to Conditional Principle No. 3; but it is indispensable for the proof of Rule 4. We have to show that if any graph, which [we] may denote by z is surrounded by two seps with nothing be-

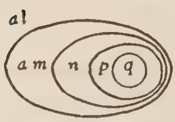


Fig. 147

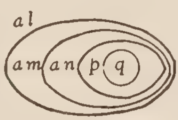


Fig. 148

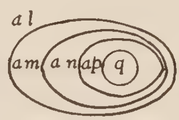


Fig. 149

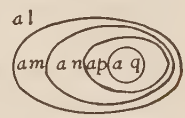


Fig. 150

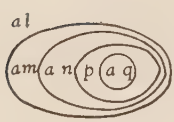


Fig. 151

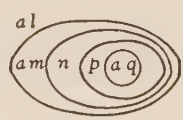


Fig. 152

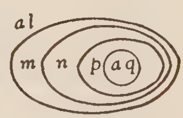


Fig. 153

tween as in Fig. 132, then the two seps may be illatively removed as in Fig. 124. Now if the graph, z, occurred within one sep, as in Fig. 130, this, as we have seen, would be transformed into itself. Hence, by Conditional Principle No. 3, Fig. 132, can be illatively transformed into Fig. 124. Q. E. D.

498. The list of rules given for dinected graphs is complete. This is susceptible of proof; but the proof belongs in the next section of this chapter, where I may perhaps insert it. It is not interesting.

B. Rules for Lines of Identity

499. We now pass to the consideration of graphs connected by lines of identity. A small addition to our nomenclature is required here. Namely, we have seen that a line of identity is a partial graph; and as a graph it cannot cross a sep. Let us, then, call a series of lines of identity abutting upon one another at seps, a *ligature*; and we may extend the meaning of the word so that even a single line of identity shall be called a ligature. A ligature composed of more than one line of identity may be distinguished as a *compound ligature*. A compound ligature is not a graph, because by a graph we mean something which, written or drawn alone on the sheet of assertion, would, according to this system, assert something. Now a compound ligature could not be written alone on the sheet of assertion, since it is only by means of the intercepting sep, which is no part of it, that it is rendered compound. The different spots, as well as the different hooks, upon which a ligature abuts, may be said to be *ligated* by that ligature; and two replicas of the same graph are said to have the same *ligations* only when all the corresponding hooks of the two are ligated to one another. When a ligature cuts a sep, the part of the ligature outside the sep may be said to be *extended* to the point of intersection on the sep, while the part of the ligature inside may be said to be *joined* to that point.

500. It has already* been pointed out that the mass of ink on the sheet by means of which a graph is said to be "scribed" is not, strictly speaking, a symbol, but only a replica of a symbol of the nature of an index. Let it not be forgotten that the significative value of a symbol consists in a regularity of association, so that the identity of the symbol lies in this regularity, while the significative force of an index consists in an existential fact which connects it with its object, so that the identity of the index consists in an existential fact or thing. When symbols, such as words, are used to construct an assertion, this assertion relates to something real. It must not only *profess* to do so, but must really *do* so; otherwise, it could not be true; and still less, false. Let a witness take oath, with every legal formality, that John Doe has committed

* In 447.

murder, and still he has made no assertion unless the name John Doe denotes some existing person. But in order that the name should do this, something more than an association of ideas is requisite. For the person is not a conception but an existent thing. The name, or rather, *occurrences* of the name, must be existentially connected with the existent person. Therefore, no assertion can be constructed out of pure symbols alone. Indeed, the pure symbols are immutable, and it is not *them* that are joined together by the syntax of the sentence, but occurrences of them — *replicas* of them. My aim is to use the term "graph" for a graph-symbol, although I dare say I sometimes lapse into using it for a graph-replica. To say that a graph is scribed is accurate, because "to scribe" means *to make a graphical replica of.* By "a line of identity," on the other hand, it is more convenient to mean a replica of the linear graph of identity. For here the indexical character is more positive; and besides, one seldom has occasion to speak of the graph. But the only difference between a line of identity and an ordinary dyadic spot is that the latter has its hooks marked at points that are deemed appropriate without our being under any factual compulsion to mark them at all, while a simple line such as is naturally employed for a line of identity must, from the nature of things, have extremities which are at once parts of it and of whatever it abuts upon. This difference does not prevent the rules of the last list from holding good of such lines. The only occasion for any additional rule is to meet that situation, in which no other graph-replica than a line of identity can ever be placed, that of having a hook upon a sep.

501. As to this, it is to be remarked that an enclosure — that is, a sep with its contents — is a graph; and those points on its periphery, that are marked by the abuttal upon them of lines of identity, are simply the hooks of the graph. But the sep is outside its own close. Therefore an unmarked point upon it is just like any other vacant place outside the sep. But if a line inside the sep is prolonged to the sep, at the instant of arriving at the sep, its extremity suddenly becomes identified — as a matter of fact, and there as a matter of signification — with a point outside the sep; and thus the prolongation suddenly assumes an entirely different character

from an ordinary, insignificant prolongation. This gives us the following:

Conditional Principle No. 4. Only the connexions and continuity of lines of identity are significant, not their shape or size. The connexion or disconnexion of a line of identity outside a sep with a marked or an unmarked point on the sep follows the same rules as its connexion or disconnexion with any other marked or unmarked point outside the sep, but the junction or disjunction of a line of identity inside the sep with a point upon the sep always follows the same rules as its connexion or disconnexion with a marked point inside the sep.

In consequence of this principle, although the categorical rules hitherto given remain unchanged in their application to lines of identity, yet they require some modifications in their application to ligatures.

502. In order to see that the principle is correct, first consider Fig. 154. Now the rule of erasure of an unenclosed graph certainly allows the transformation of this into Fig. 155, which must therefore be interpreted to mean "Something is not ugly," and must not be confounded with Fig. 156, "Nothing is ugly." But Fig. 156 is transformable into Fig. 157; that is, the line of identity with a loose end can be carried to any vacant place within the sep. If, therefore, Fig. 155 were to be treated as if the end of the line were loose, it could be illatively transformed into Fig. 156. But the line can no more be separated from the point of the sep than it could from any marked point within the sep — any more, for example, than

Fig. 154

Fig. 155

Fig. 156

Fig. 157

Fig. 158, "Nothing good is ugly" could be transformed into Fig. 159, "Either nothing is ugly or nothing is good." So Fig. 160 can, by the rule of insertion within odd seps, be transformed to Fig. 161, and must be interpreted, like that, "Everything acts on everything," and not, as in Fig. 162, "Everything acts on something or other." But if the vacant point on the sep could be treated like an ordinary point, Fig. 162

could be illatively transformed into Fig. 160, which the interpretation forbids. Although in this argument special graphs

Fig. 158

Fig. 159

Fig. 160

Fig. 161

Fig. 162

are used, it is evident that the argument would be just the same whatever others were used, and the proof is just as conclusive as if we had talked of "any graph whatever, x," etc., as well as being clearer. The principle of contraposition renders

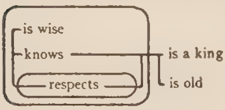


Fig. 163

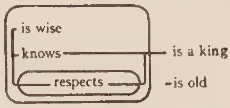


Fig. 164

Fig. 165

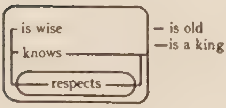


Fig. 166

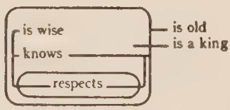


Fig. 167

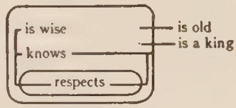


Fig. 168

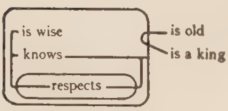


Fig. 169

Fig. 170

Fig. 171

Fig. 172

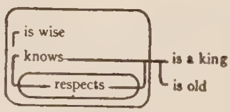


Fig. 173

Fig. 174

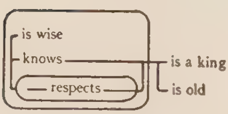


Fig. 175

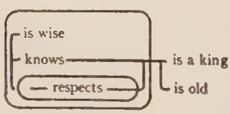


Fig. 176

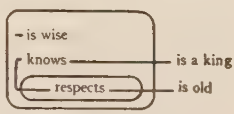


Fig. 177

it evident that the same thing would hold for any finite nests of seps.

503. On the other hand, it is easy to show that the illative connexion or disconnexion of a line exterior to the sep with a point on the sep follows precisely the same rules as if the point were outside of and away from the sep.

Figs. 163–177 furnish grounds for the demonstration of this. Fig. 163 asserts that there is an old king whom every wise person that knows him respects. The connexion of "is old" with "is king" can be illatively severed by the rule of erasure, as in Fig. 164; so that the old person shall not be asserted to be identical with the king whom all wise people that know him respect; and once severed the connexion cannot be illatively restored. So it is precisely if the line of identity outside the outer sep is cut at the sep, as in Fig. 165, which asserts that somebody is respected by whatever wise person there may be that knows him, and asserts that there is an old king, but fails to assert that the old king is that respected person. Here, as before, the line can be illatively severed but cannot be illatively restored. It is evident that this is not because of the special significance of the "spots" or unanalyzed rhemata, but that it would be the same in all cases in which a line of identity should terminate at a point on a sep where a line inside that sep should also terminate. Fig. 166 shows both lines broken, so that this might equally and for the same reason result from the illative transformation of Fig. 164 or of Fig. 165. The lines, being broken as in Fig. 166, can be distorted in any way and their extremities can be carried to any otherwise vacant places outside the outer sep, and afterwards can be brought back to their present places. In this respect, a vacant point on a sep is just like any other vacant point outside the close of the sep. If the line of identity attached to "is old" be carried to the sep, as in Fig. 167, certainly no addition is thereby made to the assertion. Once the ligature is carried as far as the sep, the rule of insertion within an odd number of seps permits it to be carried still further, as is done in Fig. 167, with the ligature attached to "is a king." This whole graph may be interpreted, "Something is old and something is a king." But this last does not exist unless something is respected by whatever that is wise there may be that knows it. The graph of Fig. 167 can be illatively retransformed into Fig. 166, by first severing the ligature attached to "is a king" outside the sep by the rule of erasure, when the part of the ligature inside may be erased by the rule of deiteration, and finally the part outside the close of the sep may be erased by the rule of erasure. On the other hand the ligatures attached in Fig. 167 to "is old" and "is a king" might, after Fig. 167 had been converted in Fig. 168, be illatively joined inside the sep by the rule of insertion, as in Fig. 169, which asserts that there is something old and there is a king; and if there is an old king something is respected by whatever wise thing there may be that knows it. This is not illatively retransformable in Fig. 168. It thus abundantly shows that an unenclosed line can be extended to a point on an unenclosed sep under the same conditions as to any other unenclosed point. For there is evidently nothing peculiar about the characters of being old and of being a king which render them different in this respect from graphs in general. Let us now see how it is in regard to singly enclosed lines in their relations to points on seps in the same close. If in Fig. 163 we sever the ligature denoting the object accusative of "respects," just outside the inner sep, as in Fig. 170, the interpretation becomes, "There is an old king, and whoever that is wise there may be who knows him, respects everybody." This is illatively transformable into Fig. 163 by the rule of insertion under odd enclosures, just as if the marked point on the sep were a hook of any spot. We may, of course, by the rule of erasure within even seps, cut away the ligature from the sep internally, getting Fig. 171, "There is an old king, whom anybody that knows respects somebody or other." The point on the sep being now unmarked, it makes no difference whether the outside ligature is extended to it, as in Fig. 172, or not. It is the same if the ligature denoting the subject nominative of "respects" be broken outside the inner sep, as in Fig. 174. Whether this be done, or whether the line of identity joining "is wise" to "knows" be cut, as in Fig. 173, in either case we get a graph illatively transformable into Fig. 163, but not derivable from Fig. 163 by any illative transformation. If, however, the line of identity within the inner sep be retracted from the sep, as in Figs. 175 and 176, it makes no difference whether the line outside the sep be extended to the unmarked point on the sep or not. One cannot even say that one form of interpretation better fits the one figure and another the other: they are absolutely equivalent. Thus, the unmarked point on the oddly enclosed sep is just like any other unmarked point exterior to the close of the sep as far as its relations with exterior lines of identity are concerned.

504. The principle of contraposition extends this Conditional Principle No. 4 to all seps, within any finite number of seps.

By means of this principle the rules of illative transformation hitherto given will easily be extended so as to apply to graphs with ligatures attached to them, and the one rule which it is necessary to add to the list will also be readily deduced. In the following statement, each rule will first be enunciated in an exact and compendious form and then, if necessary, two remarks will be added, under the headings of "Note A" and "Note B." Note A will state more explicitly how the rule applies to a line of identity; while Note B will call attention to a transformation which might, without particular care, be supposed to be permitted by the rule but which is really not permitted.

C. Basic Categorical Rules for the Illative Transformation of All Graphs[P]

505. *Rule* 1. Called *The rule of Erasure and of Insertion.* In even seps, any graph-replica can be erased; in odd seps any graph-replica can be inserted.

Note A. By even seps is meant any finite even number of seps, including none; by odd seps is meant any odd number of seps.

This rule permits any ligature, where evenly enclosed, to be severed, and any two ligatures, oddly enclosed in the same seps, to be joined. It permits a branch with a loose end to be added to or retracted from any line of identity.

It permits any ligature, where evenly enclosed, to be severed from the inside of the sep immediately enclosing that evenly enclosed portion of it, and to be extended to a vacant point of any sep in the same enclosure. It permits any ligature to be joined to the inside of the sep immediately enclosing that oddly enclosed portion of it, and to be retracted from the outside of any sep in the same enclosure on which the ligature has an extremity.

Note B. In the erasure of a graph by this rule, all its ligatures must be cut. The rule does not permit a sep to be so inserted as to intersect any ligature, nor does it permit any erasure to accompany an insertion.

It does not permit the insertion of a sep within even seps.

506. *Rule* 2. Called *The Rule of Iteration and Deiteration.* Anywhere within all the seps that enclose a replica of a graph, that graph may be iterated with identical ligations, or being iterated, may be deiterated.

Note A. The operation of iteration consists in the insertion of a new replica of a graph of which there is already a replica, the new replica having each hook ligated to every hook of a graph-replica to which the corresponding hook of the old replica is ligated, and the right to iterate includes the right to draw a new branch to each ligature of the original replica inwards to the new replica. The operation of deiteration consists in erasing a replica which might have illatively resulted from an operation of iteration, and of retracting outwards the ligatures left loose by such erasure until they are within the same seps as the corresponding ligature of the replica of which the erased replica might have been the iteration.

The rule permits any loose end of a ligature to be extended inwards through a sep or seps or to be retracted outwards through a sep or seps. It permits any cyclical part of a ligature to be cut at its innermost part, or a cycle to be formed by joining, by inward extensions, the two loose ends that are the innermost parts of a ligature.

If any hook of the original replica of the iterated graph is ligated to no other hook of any graph-replica, the same should be the case with the new replica.

Note B. This rule does not confer a right to ligate any hook to another nor to deligate any hook from another unless the same hooks, or corresponding hooks of other replicas of the same graphs (these replicas being outside every sep that the hooks ligated or deligated are outside), be ligated otherwise, and outside of every sep that the new ligations or deligations are outside of.

This rule does not confer the right to extend any ligature outwardly from within any sep, nor to retract any ligature inwardly from without any sep.

507. *Rule* 3. Called *The Rule of Assertion.* Any graph well-understood to be true may be scribed unenclosed.

Note A. This rule is to be understood as permitting the explicit assertion of three classes of propositions; first, those that are involved in the conventions of this system of existential graphs; secondly, any propositions known to be true but which may not have been thought of as pertinent when the graph was first scribed or as pertinent in the way in which it is now seen to be pertinent (that is to say, premisses may be added if they are acknowledged to be true); thirdly, any propositions which the scription of the graph renders true or shows to be true. Thus, having graphically asserted that it snows, we may insert a graph asserting "that it snows is asserted" or "it is possible to assert that it snows without asserting that it is winter."

508. *Rule* 4. Called *The Rule of Biclosure.* Two seps, one within the other, with nothing between them whose significance is affected by seps, may be withdrawn from about the graph they doubly enclose.

Note A. The significance of a ligature is not affected by a sep except at its outermost part, or if it passes through the close of the sep; and therefore ligatures passing from outside the outer sep to inside the inner one will not prevent the withdrawal of the double sep; and such ligatures will remain unaffected by the withdrawal.

Note B. A ligature passing twice through the outer sep without passing through the inner one, or passing from within the inner one into the intermediate space and stopping there, will be equivalent to a graph and will preclude the withdrawal.

509. *Rule* 5. Called *The Rule of Deformation.* All parts of the graph may be deformed in any way, the connexions of parts remaining unaltered; and the extension of a line of identity outside a sep to an otherwise vacant point on that sep is not to be considered to be a connexion.

CHAPTER 5

*THE GAMMA PART OF EXISTENTIAL GRAPHS**

510. The alpha part of graphs . . . is able to represent no reasonings except those which turn upon the logical relations of general terms.

511. The beta part . . . is able to handle with facility and dispatch reasonings of a very intricate kind, and propositions which ordinary language can only express by means of long and confusing circumlocutions. A person who has learned to think in beta graphs has ideas of the utmost clearness and precision which it is practically impossible to communicate to the mind of a person who has not that advantage. Its reasonings generally turn upon the properties of the relations of individual objects to one another.

But it is able to do nothing at all with many ideas which we are all perfectly familiar with. Generally speaking it is unable to reason about abstractions. It cannot reason for example about qualities nor about relations as subjects to be reasoned about. It cannot reason about ideas. It is to supply that defect that the gamma part of the subject has been invented. But this gamma part is still in its infancy. It will be many years before my successors will be able to bring it to the perfection to which the alpha and beta parts have been brought. For logical investigation is very slow, involving as it does the taking up of a confused mass of ordinary ideas, embracing we know not what and going through with a great quantity of analyses and generalizations and experiments before one can so much as get a new branch fairly inaugurated. . . .

512. The gamma part of graphs, in its present condition, is characterized by a great wealth of new signs; but it has no sign of an essentially different kind from those of the alpha and beta part. The alpha part has three distinct kinds of signs, the *graphs*, the *sheet of assertion*, and the *cuts*. The beta part adds two quite different kinds of signs, *spots*, or

* From "Lowell Lectures of 1903." Lecture IV.

lexeis, and *ligatures* with *selectives*. It is true that a line of identity is a graph; but the terminal of such a line, especially a terminal on a cut where two lines of identity have a common point, is radically different. So far, all the gamma signs that have presented themselves, are of those same kinds. If anybody in my lifetime shall discover any radically disparate kind of sign, peculiar to the gamma part of the system, I shall hail him as a new Columbus. He must be a mind of vast power. But in the gamma part of the subject all the old kinds of signs take new forms. . . . Thus in place of a sheet of assertion, we have a book of separate sheets, tacked together at points, if not otherwise connected. For our alpha sheet, as a whole, represents simply a universe of existent individuals, and the different parts of the sheet represent facts or true assertions made concerning that universe. At the cuts we pass into other areas, areas of conceived propositions which are not realized. In these areas there may be cuts where we pass into worlds which, in the imaginary worlds of the outer cuts, are themselves represented to be imaginary and false, but which may, for all that, be true, and therefore continuous with the sheet of assertion itself, although this is uncertain. You may regard the ordinary blank sheet of assertion as a film upon which there is, as it were, an undeveloped photograph of the facts in the universe. I do not mean a literal picture, because its elements are propositions, and the meaning of a proposition is abstract and altogether of a different nature from a picture. But I ask you to imagine all the true propositions to have been formulated; and since facts blend into one another, it can only be in a continuum that we can conceive this to be done. This continuum must clearly have more dimensions than a surface or even than a solid; and we will suppose it to be plastic, so that it can be deformed in all sorts of ways without the continuity and connection of parts being ever ruptured. Of this continuum the blank sheet of assertion may be imagined to be a photograph. When we find out that a proposition is true, we can place it wherever we please on the sheet, because we can imagine the original continuum, which is plastic, to be so deformed as to bring any number of propositions to any places on the sheet we may choose.

513. So far I have called the sheet a photograph, so as

not to overwhelm you with all the difficulties of the conception at once. But let us rather call it a map — a map of such a photograph if you like. A map of the simplest kind represents all the points of one surface by corresponding points of another surface in such a manner as to preserve the continuity unbroken, however great may be the distortion. A Mercator's chart, however, represents all the surface of the earth by a strip, infinitely long, both north and south poles being at infinite distances, so that places near the poles are magnified so as to be many times larger than the real surfaces of the earth that they represent, while in longitude the whole equator measures only two or three feet; and you might continue the chart so as to represent the earth over and over again in as many such strips as you pleased. Other kinds of map, such as my Quincuncial Projection which is drawn in the fourth volume of the *American Journal of Mathematics*,* show the whole earth over and over again in checkers, and there is no arrangement you can think of in which the different representations of the same place might not appear on a perfectly correct map. This accounts for our being able to scribe the same graph as many times as we please on any vacant places we like. Now each of the areas of any cut corresponds exactly to some locus of the sheet of assertion where there is mapped, though undeveloped, the real state of things which the graph of that area denies. In fact it is represented by that line of the sheet of assertion which the cut itself marks.

514. By taking time enough I could develop this idea much further, and render it clearer; but it would not be worth while, for I only mention it to prepare you for the idea of quite different kinds of sheets in the gamma part of the system. These sheets represent altogether different universes with which our discourse has to do. In the Johns Hopkins *Studies in Logic*† I printed a note of several pages on the universe of qualities —*marks*, as I then called them. But I failed to see that I was then wandering quite beyond the bounds of the logic of relations proper. For the relations of which the so-called "logic of relatives" treats are *existential* relations, which the nonexistence of either relate or correlate reduces to nullity. Now,

* Vol. 2, pp. 394–6 (1879); to be published in vol. 7.

† See 2.517ff.

qualities are not, properly speaking, individuals. All the qualities you actually have ever thought of might, no doubt, be counted, since you have only been alive for a certain number of hundredths of seconds, and it requires more than a hundredth of a second actually to have any thought. But all the qualities, any one of which you readily can think of, are certainly innumerable; and all that might be thought of exceed, I am convinced, all multitude whatsoever. For they are mere logical possibilities, and possibilities are general, and no multitude can exhaust the narrowest kind of a general. Nevertheless, within limitations, which include most ordinary purposes, qualities may be treated as individuals. At any rate, however, they form an entirely different universe of existence. It is a universe of logical possibility. As we have seen, although the universe of existential fact can only be conceived as mapped upon a surface by each point of the surface representing a vast expanse of fact, yet we can conceive the facts [as] sufficiently separated upon the map for all our purposes; and in the same sense the entire universe of logical possibilities might be conceived to be mapped upon a surface. Nevertheless, in order to represent to our minds the relation between the universe of possibilities and the universe of actual existent facts, if we are going to think of the latter as a surface, we must think of the former as three-dimensional space in which any surface would represent all the facts that might exist in one existential universe. In endeavoring to begin the construction of the gamma part of the system of existential graphs, what I had to do was to select, from the enormous mass of ideas thus suggested, a small number convenient to work with. It did not seem to be convenient to use more than one actual sheet at one time; but it seemed that various different kinds of cuts would be wanted.

515. I will begin with one of the *gamma cuts*. I call it the *broken cut*. I scribe it thus

Fig. 178

This does not assert that it does not rain. It only asserts that the alpha and beta rules do not compel me to admit that it rains, or what comes to the same thing, a person altogether ignorant, except that he was well versed in logic so far as it embodied in the alpha and beta parts of existential graphs, would not know that it rained.*

516. The rules of this cut are very similar to those of the alpha cut.

Rule 1. *In a broken cut already on the sheet of assertion any graph may be inserted.*

Rule 2. *An evenly enclosed alpha cut may be half erased so as to convert it into a broken cut, and an oddly enclosed broken cut may be filled up to make an alpha cut. Whether the enclosures are by alpha or broken cuts is indifferent.*

Consequently

Fig. 179

will mean that the graph *g* is beta-necessarily true.† By Rule 2, this is converted into

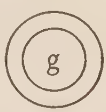


Fig. 180

which is equivalent to g

Fig. 181

the simple assertion of *g*. By the same rule Fig. 180 is transformable into

Fig. 182

* *I.e.*, It is possible that it does not rain.

† *I.e.*, It is false that *g* is possibly false.

which means that the beta rules do not make *g* false.* That is *g* is beta-possible.†

So if we start from

Fig. 183

which denies the last figure and thus asserts that it is beta-impossible that *g* should be true,‡ Rule 2 gives

Fig. 184

equivalent to

Fig. 185

the simple denial of *g*.§

And from this we get again

Fig. 186¶

517. It must be remembered that possibility and necessity are relative to the state of information.

Of a certain graph *g* let us suppose that I am in such a state of information that it *may be true* and *may be false;* that is I can scribe on the sheet of assertion Figs. 182 and 186. Now I learn that it is true. This gives me a right to scribe on the sheet Figs. 182, 186 and 181. But now relative to this new state of information, Fig. 186 ceases to be true; and therefore relatively to the new state of information we can scribe Fig. 179.||

* *I.e.*, It is possibly false that *g* is false; or it is possible that *g* is true.

† The passage from Figs. 179 to 181, from 181 to 182, and from 179 to 182 represent C. I. Lewis' subsequent "strict implications," 4.1, 4.12, 4.13 respectively. See his *Survey of Symbolic Logic*, p. 306–7. The broken cut represents Lewis' −∼−.

‡ *I.e.*, *g* is impossible.

§ This transition is Lewis' 1.7, *ibid.*, p. 295.

¶ *I.e.*, *g* is not necessary, or it is possible that *g* is false. 183–6 is the transformation 179–182 with $\bar{g}$ substituted for *g*.

|| *I.e.*, if *g* be possibly true and false, and also true, it is necessarily true.

518. You thus perceive that we should fall into inextricable confusion in dealing with the broken cut if we did not attach to it a sign to distinguish the particular state of information to which it refers. And a similar sign has then to be attached to the simple *g*, which refers to the state of information at the time of learning that graph to be true. I use for this purpose cross marks below, thus:

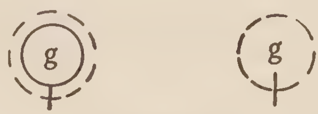


Fig. 187

These selectives are very peculiar in that they refer to states of information as if they were individual objects. They have, besides, the additional peculiarity of having a definite order of succession, and we have the rule that from Fig. 188 we can infer Fig. 189.*

Fig. 188

Fig. 189

These signs are of great use in cleaning up the confused doctrine of *modal propositions* as well as the subject of logical breadth and depth.

519. There is not much utility in a *double broken cut*. Yet it may be worth notice that Fig. 181 and

Fig. 190

can neither of them be inferred from the other. The outer of the two broken cuts is not only relative to a state of information but to a state of reflection. The graph [190] asserts that it is possible that the truth of the graph *g* is necessary.

* *I.e.*, with respect to the given state of information, if *g* is true, it is necessarily true.

It is only because I have not sufficiently reflected upon the subject that I can have any doubt of whether it is so or not.

520. It becomes evident, in this way, that a modal proposition is a simple assertion, not about the universe of things, but about the universe of facts that one is in a state of information sufficient to know. [Fig. 186] without any selective, merely asserts that there is a possible state of information in which the knower is not in a condition to know that the graph *g* is true, while Fig. 179 asserts that there is no such possible state of information. Suppose, however, we wish to assert that there is a conceivable state of information of which it would not be true that, in that state, the knower would not be in condition to know that *g* is true. We shall naturally express this by Fig. 191. But this is to say that there is a conceivable state of information in which the knower would know that *g* is true. [This is expressed by] Fig. 188.

Fig. 191

521. Now suppose we wish to assert that there is a conceivable state of information in which the knower would know *g* to be true and yet would not know another graph *h* to be true. We shall naturally express this by Fig. 192.

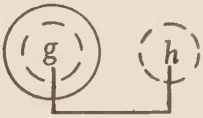


Fig. 192

Here we have a new kind of ligature, which will follow all the rules of ligatures. We have here a most important addition to the system of graphs. There will be some peculiar and interesting little rules, owing to the fact that what one knows, one has the means of knowing that one knows — which is sometimes incorrectly stated in the form that whatever one knows, one knows that one knows, which is manifestly false. For if it were the same to say "A whale is not a fish" and "I know that a whale is not a fish," the precise denials of the two would be the same. Yet one is "A whale is a fish" and the other is "I do not know that a whale is not a fish."

522. The truth is that it is necessary to have a graph to

signify that one state of information follows after another. If we scribe, A—▬—B to express that the state of information B follows after the state of information A, we shall have

Fig. 193

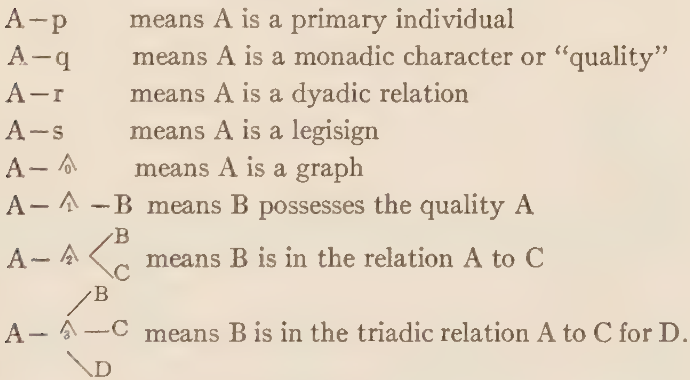
523. It is clear, however, that the matter must not be allowed to rest here. For it would be a strangely, and almost an ironically, imperfect kind of logic which should recognize only *ignorance* and should ignore *error*. Yet in order to recognize *error* in our system of graphs, we shall be obliged still further to introduce the idea of time, which will bring still greater difficulties. *Time* has usually been considered by logicians to be what is called "extra-logical" matter. I have never shared this opinion.* But I have thought that logic had not reached that state of development at which the introduction of temporal modifications of its forms would not result in great confusion; and I am much of that way of thinking yet. The idea of time really is involved in the very idea of an argument. But the gravest complications of logic would be involved, [if we took] account of time [so as] to distinguish between what *one knows* and what one *has sufficient reason to be entirely confident of*. The only difference, that there seems to be room for between these two, is that what one *knows*, one always will have *reason to be confident of*, while what one now has ample reason to be entirely confident of, one may conceivably in the future, in consequence of a new light, find reason to doubt and ultimately to deny. Whether it is really possible for this to occur, whether we can be said truly to have sufficient reason for entire confidence unless it is manifestly impossible that we should have any such new light in the future, is not the question. Be that as it may, it still remains *conceivable* that there should be that difference, and therefore there is a difference in the *meanings* of the two phrases. I confess that my studies heretofore have [not] progressed so far that I am able to say precisely what modification of our logical forms will be required when we come to take account, as some day we must, of all

* Cf. 3.446.

the effects of the possibilities of error, as we can now take account, in the doctrine of modals, of the possibilities of ignorance. Nor do I believe that the time has yet come when it would be profitable to introduce such complications. But I can see that, when that time does come, our logical forms will become very much more metamorphosed, by introducing that consideration, than they are in modal logic, where we take account of the possibility of ignorance as compared with the simple logic of propositions *de inesse* (as non-modal propositions, in which the ideas of possibility and necessity are not introduced, are called) . . .

524. I introduce certain spots which I term Potentials. They are shown on this diagram:

The Potentials



525. It is obvious that the lines of identity on the left-hand side of the potentials are quite peculiar, since the characters they denote are not, properly speaking, individuals. For that reason and others, to the left of the potentials I use selectives not ligatures.

526. As an example of the use of the potentials, we may take this graph, which expresses a theorem of great importance: The proposition is that for every quality Q whatsoever, there is a dyadic relation, R, such that, taking any two different individuals both possessing this quality, Q, either the first stands in the relation R to some thing to which the second does not stand in that relation, while there is nothing to which the second stands in that relation without the first standing in

the same relation to it; or else it is just the other way, namely that the second stands in the relation, R, to which the first does not stand in that relation, while there is nothing to which the first stands in that relation, R, without the second also standing in the same relation to it. The proof of this, which is a little too intricate to be followed in an oral statement

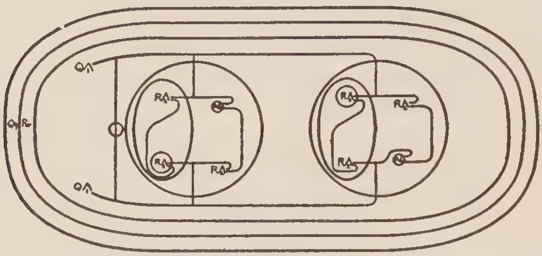


Fig. 194

(although in another lecture* I shall substantially prove it) depends upon the fact that a relation is in itself a mere logical possibility.

527. I will now pass to another quite indispensable department of the gamma graphs. Namely, it is necessary that we should be able to reason in graphs about graphs. The reason is that a reasoning about graphs will necessarily consist in showing that something is true of every possible graph of a certain general description. But we cannot scribe every possible graph of any general description, and therefore if we are to reason in graphs we must have a graph which is a general description of the kind of graph to which the reasoning is to relate.

528. For the alpha graphs, it is easy to see what is wanted. Let A, the old Greek form of the letter A, denote the sheet of assertion. Let $-\gamma$ be "is a graph." Let Y—)—X mean that X is scribed or placed on Y. Let W–k–Z mean that Z is the area of the cut W. Let U– {V} mean that U is a graph, precisely expressing V. It is necessary to place V in the *saw-rim*, as I call the line about it, because in thus speaking of a

* That lecture is not being published.

sign *materialiter*, as they said in the middle ages, we require that it should have a hook that it has not got. For example

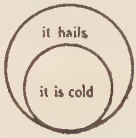


Fig. 195

asserts, of course, that if it hails, it is cold *de inesse.*

Now a graph asserting that this graph is scribed on the sheet of assertion, will be

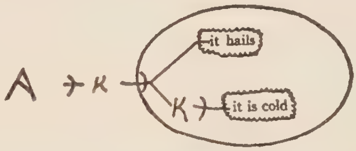
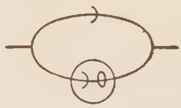


Fig. 196

This graph only asserts what the other does assert. It does not say what the other does not assert. But there would be no difficulty in expressing that. We have only to place instead

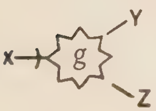
of  , wherever it occurs, 

529. We come now to the graphical expressions of beta graphs. Here we require the following symbols,

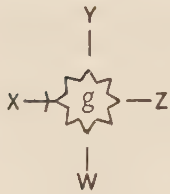
Gamma Expressions of Beta Graphs

x  y means Y is a ligature whose outermost part is on X.

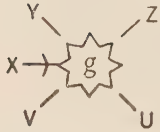
means g is expressed by a monad spot on X whose hook is joined to the ligature Y on X.



means g is expressed by a dyad graph on X whose first and second hooks respectively are joined on X to the ligatures Y and Z.



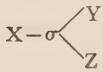
means g is expressed by a triad graph on X whose first, second, and third hooks are joined on X to the ligatures Y, Z, W, respectively.



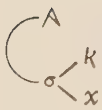
means g is expressed by a tetrad spot on X whose first to fourth hooks are joined to Y, Z, U, V, respectively.*

* The following was found on a separate sheet, apparently for use in a similar lecture:

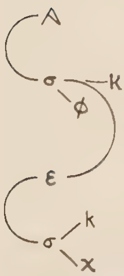
X—A means "X is the sheet of assertion"
X—ε—Y means "X is the area of the enclosure Y"
X—ϕ means "X is a permission"
X—χ means "X is a fact"
X—μ means "X is a blank"
X—k means "X is an enclosure"



means "X carries Y as its entire graph insofar as it is of the nature of Z to make it do so," that is to say, for example

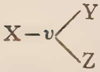


means "An enclosure is the entire graph on the sheet of assertion as a fact."



means "It is permitted to place on the sheet of assertion, as the entire graph, an enclosure on whose area an enclosure is placed as a fact."

X—δ—Y means "The graph-replica denoted by X contains as a part of it, the replica Y."



means "X is a line of identity having its terminals at Y and Z."

X—ζ—Y means "X is a replica of the same graph of which Y is a replica, or is equivalent to Y."

X—γ means "X is a graph-replica."

CHAPTER 6

*PROLEGOMENA TO AN APOLOGY FOR PRAGMATICISM**

§1. SIGNS[E]†

530. Come on, my Reader, and let us construct a diagram to illustrate the general course of thought; I mean a System of diagrammatization by means of which any course of thought can be represented with exactitude.

"But why do that, when the thought itself is present to us?" Such, substantially, has been the interrogative objection raised by more than one or two superior intelligences, among whom I single out an eminent and glorious General.

Recluse that I am, I was not ready with the counter-question, which should have run, "General, you make use of maps during a campaign, I believe. But why should you do so, when the country they represent is right there?" Thereupon, had he replied that he found details in the maps that were so far from being "right there," that they were within the enemy's lines, I ought to have pressed the question, "Am I right, then, in understanding that, if you were thoroughly and perfectly familiar with the country, as, for example, if it lay just about the scenes of your childhood, no map of it would then be of the smallest use to you in laying out your detailed plans?" To that he could only have rejoined, "No, I do not say that, since I might probably desire the maps to stick pins into, so as to mark each anticipated day's change in the situations of the two armies." To that again, my sur-rejoinder should have been, "Well, General, that precisely corresponds to the advantages of a diagram of the course of a discussion. Indeed, just there, where you have so clearly pointed it out, lies the advan-

* *The Monist*, pp. 492–546, vol. 16 (1906), with some minor corrections as listed in vol. 17, p. 160. The two preceding articles of this series are in vol. 5, bk. II, Nos. 6 and 7.

† A detailed study of signs is to be found in vol. 2, bk. II.

tage of diagrams in general. Namely, if I may try to state the matter after you, one can make exact experiments upon uniform diagrams; and when one does so, one must keep a bright lookout for unintended and unexpected changes thereby brought about in the relations of different significant parts of the diagram to one another. Such operations upon diagrams, whether external or imaginary, take the place of the experiments upon real things that one performs in chemical and physical research. Chemists have ere now, I need not say, described experimentation as the putting of questions to Nature. Just so, experiments upon diagrams are questions put to the Nature of the relations concerned." The General would here, may be, have suggested (if I may emulate illustrious warriors in reviewing my encounters in afterthought), that there is a good deal of difference between experiments like the chemist's, which are trials made upon the very substance whose behavior is in question, and experiments made upon diagrams, these latter having no physical connection with the things they represent. The proper response to that, and the only proper one, making a point that a novice in logic would be apt to miss, would be this: "You are entirely right in saying that the chemist experiments upon the very object of investigation, albeit, after the experiment is made, the particular sample he operated upon could very well be thrown away, as having no further interest. For it was not the particular sample that the chemist was investigating; it was the molecular *structure*. Now he was long ago in possession of overwhelming proof that all samples of the same molecular structure react chemically in exactly the same way; so that one sample is all one with another. But the object of the chemist's research, that upon which he experiments, and to which the question he puts to Nature relates, is the Molecular Structure, which in all his samples has as complete an identity as it is in the nature of Molecular Structure ever to possess. Accordingly, he does, as you say, experiment upon the Very Object under investigation. But if you stop a moment to consider it, you will acknowledge, I think, that you slipped in implying that it is otherwise with experiments made upon diagrams. For what is there the Object of Investigation? It is the *form of a relation*. Now this Form of Relation is the very form of the relation between the two corresponding parts of the diagram. For example, let f_1 and f_2 be the two distances of the two foci of a lens from the lens. Then,

$$\frac{1}{f_1}+\frac{1}{f_2}=\frac{1}{f_0}.$$

This equation is a diagram of the form of the relation between the two focal distances and the principal focal distance; and the conventions of algebra (and all diagrams, nay all pictures, depend upon conventions) in conjunction with the writing of the equation, establish a relation between the very *letters* f_1, f_2, f_0 regardless of their significance, the form of which relation is the *Very Same* as the form of the relation between the three focal distances that these letters denote. This is a truth quite beyond dispute. Thus, this algebraic Diagram presents to our observation the very, identical object of mathematical research, that is, the Form of the harmonic mean, which the equation aids one to study. (But do not let me be understood as saying that a Form possesses, itself, Identity in the strict sense; that is, what the logicians, translating ἀριθμῷ, call 'numerical identity.')"

531. Not only is it true that by experimentation upon some diagram an experimental proof can be obtained of every necessary conclusion from any given Copulate of Premisses, but, what is more, no "necessary" conclusion is any more apodictic than inductive reasoning becomes from the moment when experimentation can be multiplied *ad libitum* at no more cost than a summons before the imagination. I might furnish a regular proof of this, and am dissuaded from doing so now and here only by the exigency of space, the ineluctable length of the requisite explanations, and particularly by the present disposition of logicians to accept as sufficient F. A. Lange's persuasive and brilliant, albeit defective and in parts even erroneous, apology for it.* Under these circumstances, I will content myself with a rapid sketch of my proof. First, an analysis of the essence of a sign, (stretching that word to its widest limits, as *anything which, being determined by an object, determines an interpretation to determination, through it, by the same object*), leads to a proof that every sign is determined by

* In his *Logische Studien* (1877).

its object, either first, by partaking in the characters of the object, when I call the sign an *Icon;* secondly, by being really and in its individual existence connected with the individual object, when I call the sign an *Index*; thirdly, by more or less approximate certainty that it will be interpreted as denoting the object, in consequence of a habit (which term I use as including a natural disposition), when I call the sign a *Symbol.*[1] I next examine into the different efficiencies and inefficiencies of these three kinds of signs in aiding the ascertainment of truth. A Symbol incorporates a habit, and is indispensable to the application of any *intellectual* habit, *at least.* Moreover, Symbols afford the means of thinking about thoughts in ways in which we could not otherwise think of them. They enable us, for example, to create Abstractions, without which we should lack a great engine of discovery. These enable us to count; they teach us that collections are individuals (individual =individual object), and in many respects they are the very warp of reason. But since symbols rest exclusively on habits already definitely formed but not furnishing any observation even of themselves, and since knowledge is habit, they do not enable us to add to our knowledge even so much as a necessary consequent, unless by means of a definite preformed habit. Indices, on the other hand, furnish positive assurance of the reality and the nearness of their Objects. But with the assurance there goes no insight into the nature of those Objects. The same Perceptible may, however, function doubly as a Sign. That footprint that Robinson Crusoe found in the sand, and which has been stamped in the granite of fame, was an Index to him that some creature was on his island, and at the same time, as a Symbol, called up the idea of a man. Each Icon partakes of some more or less overt character of its Object. They, one and all, partake of the most overt character of all lies and deceptions — their Overtness. Yet they have more to do with the living character of truth than have either Symbols or Indices. The Icon does not stand unequivocally for this or that existing thing, as the Index does. Its Object may be a pure fiction, as to its existence. Much less is its Object

[1] In the original publication of this division, in 1867 [1.558] the term "representamen" was employed in the sense of a sign in general, while "sign" was taken as a synonym of *Index,* and an *Icon* was termed a "likeness."

necessarily a thing of a sort habitually met with. But there is one assurance that the Icon does afford in the highest degree. Namely, that which is displayed before the mind's gaze — the Form of the Icon, which is also its object — must be *logically possible*. This division of Signs is only one of ten different divisions of Signs which I have found it necessary more especially to study.* I do not say that they are all satisfactorily definite in my mind. They seem to be all trichotomies, which form an attribute to the essentially triadic nature of a Sign. I mean because three things are concerned in the functioning of a Sign; the Sign itself, its Object, and its Interpretant. I cannot discuss all these divisions in this article; and it can well be believed that the whole nature of reasoning cannot be fully exposed from the consideration of one point of view among ten. That which we can learn from this division is of what sort a Sign must be to represent the sort of Object that reasoning is concerned with. Now reasoning has to make its conclusion manifest. Therefore, it must be chiefly concerned with forms, which are the chief objects of rational insight. Accordingly, Icons are specially requisite for reasoning. A Diagram is mainly an Icon, and an Icon of intelligible relations. It is true that what must be is not to be learned by simple inspection of anything. But when we talk of deductive reasoning being necessary, we do not mean, of course, that it is infallible. But precisely what we do mean is that the conclusion follows from the form of the relations set forth in the premiss. Now since a diagram, though it will ordinarily have Symbolide Features, as well as features approaching the nature of Indices, is nevertheless in the main an Icon of the forms of relations in the constitution of its Object, the appropriateness of it for the representation of necessary inference is easily seen.

§2. COLLECTIONS[E]

532. But since you may, perhaps, be puzzled to understand how an Icon can exhibit a necessity — a Must-be — I will here give, as an example of its doing so, my proof† that the single members of no collection or plural, are as many as

* See 536n and 2.235n.

† Cf. 3.547ff.

are the collections it includes, each reckoned as a single object, or, in other words, that there can be no relation in which every collection composed of members of a given collection should (taken collectively as a single object) stand to some member of the latter collection to which no other such included collection so stands. This is another expression of the following proposition, namely: that, taking any collection or plural, whatsoever, be it finite or infinite, and calling this the *given collection;* and considering all the collections, or plurals, each of which is composed of some of the individual members of the given collection (but including along with these *Nothing* which is to be here regarded as a collection having no members at all; and also including the single members of the given collection, conceived as so many collections each of a single member), and calling these the *involved collections;* the proposition is that there is no possible relation in which each involved collection (considered as a single object), stands to a member of the given collection, without any other of the involved collections standing in the same relation to that same member of the given collection. This purely symbolic statement can be rendered much more perspicuous by the introduction of Indices, as follows. The proposition is that no matter what collection C may be, and no matter what relation R may be, there must be some collection, c', composed exclusively of members of C, which does not stand in the relation R to any member, k, of C, unless some other collection, c'', likewise composed of members of C, stands in the same relation R to the same k. The theorem is important in the doctrine of multitude, since it is the same as to say that any collection, no matter how great, is less multitudinous than the collection of possible collections composed exclusively of members of it; although formerly this was assumed to be false of some infinite collections. The demonstration begins by insisting that, if the proposition be false, there must be some definite relation of which it is false. Assume, then, that the letter R is an index of any one such relation you please. Next divide the members of C into four classes as follows:

Class I is to consist of all those members of C (if there be any such) to each of which no collection of members of C stands in the relation R.

Class II is to consist of all those members of C to each of which one and only one collection of members of C stands in the relation *R*; and this class has two sub-classes, as follows:

Sub-Class 1 is to consist of whatever members of Class II there may be, each of which is contained in that one collection of members of C that is in the relation *R* to it.

Sub-Class 2 is to consist of whatever members of Class II there may be, none of which is contained in that one collection of members of C that is in the relation *R* to it.

Class III is to consist of all those members of C, if there be any such, to each of which more than one collection of members of C are in the relation *R*.

This division is complete; but everybody would consider the easy diagrammatical proof that it is so as needless to the point of nonsense, implicitly relying on a Symbol in his memory which assures him that every Division of such construction is complete.

I ought already to have mentioned that, throughout the enunciation and demonstration of the proposition to be proved, the term "collection included in the given collection" is to be taken in a peculiar sense to be presently defined. It follows that there is one "possible collection" that is included in every other, that is, which excludes whatever any other excludes. Namely, this is the "possible collection" which includes only the Sphinxes, which is the same that includes only the Basilisks, and is identical with the "possible collection" of all the Centaurs, the unique and ubiquitous collection called "Nothing," which has no member at all. If you object to this use of the term "collection," you will please substitute for it, throughout the enunciation and the demonstration, any other designation of the same object. I prefix the adjective "possible," though I must confess it does not express my meaning, merely to indicate that I extend the term "collection" to Nothing, which, of course, has no existence. Were the suggested objection to be persisted in by those *soi-disant* reasoners who refuse to think at all about the object of this or that description, on the ground that it is "inconceivable," I should not stop to ask them how they could say that, when that involves thinking of it in the very same breath, but should simply say that for them it would be necessary to except collections consisting of single individuals. Some of these mighty intellects refuse to allow the use of any name to denote single individuals, and also plural collections along with them; and for them the proposition ceases to be true of pairs. If they would not allow pairs to be denoted by any term that included all higher collections, the proposition would cease to be true of triplets and so on. In short, by restricting the meaning of "possible collection," the proposition may be rendered false of *small* collections. No general formal restriction can render it false of *greater* collections.

I shall now assume that you will permit me to use the term "possible collection" according to the following definition. A "possible collection" is an *ens rationis* of such a nature that the definite plural of any noun, or possible noun of definite signification, (as "the A's," "the B's," etc.) denotes one, and only one, "possible collection" in any one perfectly definite state of the universe; and there is a certain relation between some "possible collections," expressed by saying that one "possible collection" *includes* another (or the same) "possible collection," and if, and only if, of two nouns one is universally and affirmatively predicable of the other in any one perfectly definite state of the universe, then the "possible collection" denoted by the definite plural of the former *includes* whatever "possible collection" is *included* by the "possible collection" denoted by the definite plural of the latter, and of any two different "possible collections," one or other must *include* something not *included* by the other.

A diagram of the definition of "possible collection" being compared with a diagram embracing whatever members of subclasses 1 and 2 that it may, excluding all the rest, will now assure us that any such aggregate is a possible collection of members of the class C, no matter what individuals of Classes I and III be included or excluded in the aggregate along with those members of Class II, if any there be in the aggregate.

We shall select, then, a single possible collection of members of C to which we give the proper name, *c*, and this possible collection shall be one which contains no individual of Subclass 1, but contains whatever individual there may be of Subclass 2. We then ask whether or not it is true that *c* stands in the relation *R* to a member of C to which no other possible collection of members of C stands in the same relation; or, to put this question into a more convenient shape, we ask, Is there any member of the Class C to which *c* and no other possible collection of members of C stands in the relation *R*? If there be such a member or members of C, let us give one of them the proper name T. Then T must belong to one of our four divisions of this class. That is,

either T belongs to Class I (but that cannot be, since by the definition of Class I, to no member of this class is any possible collection of members of C in the relation *R*);
or T belongs to Subclass 1 (but that cannot be, since by the definition of that subclass, every member of it is a member of the only possible collection of members of C that is *R* to it, which possible collection cannot be *c*, because *c* is only known to us by a description which forbids its containing any member of Subclass 1. Now it is *c*, and *c* only, that is in the relation *R* to T);
or T belongs to Subclass 2 (but that cannot be, since by the definition of that subclass, no member of it is a member of the only possible collection of members of C that is *R* to it, which possible collection cannot be *c*, because the description by which alone *c* can be recognized makes it contain every member of Subclass 2. Now it is *c* only that is in the relation *R* to T);
or T belongs to Class III (but this cannot be, since to every member of that class, by the definition of it, more than one collection of members of C stand in the relation *R*, while to T only one collection, namely, *c*, stands in that relation).

Thus, T belongs to none of the classes of members of C, and consequently is not a member of C. Consequently, there is no such member of C; that is, no member of C to which *c*, and no other possible collection of members of C, stands in the relation *R*. But *c* is the proper name we were at liberty to give to whatever possible collection of members of C we pleased.

Hence, there is no possible collection of members of C that stands in the relation *R* to a member of the class C to which no other possible collection of members of C stands in this relation *R*. But *R* is the name of *any* relation we please, and C is any class we please. It is, therefore, proved that no matter what class be chosen, or what relation be chosen, there will be some possible collection of members of that class (in the sense in which Nothing is such a collection) which does not stand in that relation to any member of that class to which no other such possible collection stands in the same relation.

§3. GRAPHS AND SIGNS[E]

533. When I was a boy, my logical bent caused me to take pleasure in tracing out upon a map of an imaginary labyrinth one path after another in hopes of finding my way to a central compartment. The operation we have just gone through is essentially of the same sort, and if we are to recognize the one as essentially performed by experimentation upon a diagram, so must we recognize that the other is performed. The demonstration just traced out brings home to us very strongly, also, the convenience of so constructing our diagram as to afford a clear view of the mode of connection of its parts, and of its composition at each stage of our operations upon it. Such convenience is obtained in the diagrams of algebra. In logic, however, the desirability of convenience in threading our way through complications is much less than in mathematics, while there is another desideratum which the mathematician as such does not feel. The mathematician wants to reach the conclusion, and his interest in the process is merely as a means to reach similar conclusions. The logician does not care what the result may be; his desire is to understand the nature of the process by which it is reached. The mathematician seeks the speediest and most abridged of secure methods; the logician wishes to make each smallest step of the process stand out distinctly, so that its nature may be understood. He wants his diagram to be, above all, as analytical as possible.

534. In view of this, I beg leave, Reader, as an Introduction to my defence of pragmatism, to bring before you a very simple system of diagrammatization of propositions which I

term the System of Existential Graphs. For, by means of this, I shall be able almost immediately to deduce some important truths of logic, little understood hitherto, and closely connected with the truth of pragmaticism;[1] while discussions of other points of logical doctrine, which concern pragmaticism but are not directly settled by this system, are nevertheless much facilitated by reference to it.

535. By a *graph* (a word overworked of late years), I, for my part, following my friends Clifford* and Sylvester,† the introducers of the term, understand in general a diagram composed principally of spots and of lines connecting certain of the spots. But I trust it will be pardoned to me that, when I am discussing Existential Graphs, without having the least business with other Graphs, I often omit the differentiating adjective and refer to an Existential Graph as a Graph simply. But you will ask, and I am plainly bound to say, precisely what kind of a Sign an Existential Graph, or as I abbreviate that phrase here, a *Graph* is. In order to answer this I must make reference to two different ways of dividing all Signs. It is no slight task, when one sets out from none too clear a notion of what a Sign is — and you will, I am sure, Reader, have noticed

[1] You apprehend in what way the system of Existential Graphs is to furnish a test of the truth or falsity of Pragmaticism. Namely, a sufficient study of the Graphs should show what nature is truly common to all significations of concepts; whereupon a comparison will show whether that nature be or be not the very ilk that Pragmaticism (by the definition of it) avers that it is. It is true that the two terms of this comparison, while in substance identical, yet might make their appearance under such different garbs that the student might fail to recognize their identity. At any rate, the possibility of such a result has to be taken into account; and therewith it must be acknowledged that, on its negative side, the argument may not turn out to be sufficient. For example, *quâ* Graph, a concept might be regarded as the passive object of a geometrical *intuitus*, although Pragmaticism certainly makes the essence of every concept to be exhibited in an influence on possible conduct; and a student might fail to perceive that these two aspects of the concept are quite compatible.

But, on the other hand, should the theory of Pragmaticism be erroneous, the student would only have to compare concept after concept, each one, first, in the light of Existential Graphs, and then as Pragmaticism would interpret it, and it could not but be that before long he would come upon a concept whose analyses from these two widely separated points of view unmistakably conflicted. . . . — from Phaneroscopy *φαν*"; one of a number of fragmentary manuscripts designed to follow the present article. See 540n; 553n and 1.306n.

* "Remarks on the Chemico-Algebraic Theory," *Mathematical Papers*, No. 28.

† "Chemistry and Algebra," *Mathematical Papers*, vol. III, no. 14.

that my definition of a Sign is not convincingly distinct — to establish a single vividly distinct division of all Signs. The one division which I have already given has cost more labor than I should care to confess. But I certainly could not tell you what sort of a Sign an Existential Graph is, without reference to two other divisions of Signs. It is true that one of these involves none but the most superficial considerations, while the other, though a hundredfold more difficult, resting as it must for a clear comprehension of it upon the profoundest secrets of the structure of Signs, yet happens to be extremely familiar to every student of logic. But I must remember, Reader, that your conceptions may penetrate far deeper than mine; and it is to be devoutly hoped they may. Consequently, I ought to give such hints as I conveniently can, of my notions of the structure of Signs, even if they are not strictly needed to express my notions of Existential Graphs.

536. I have already noted that a Sign has an Object and an Interpretant, the latter being that which the Sign produces in the Quasi-mind that is the Interpreter by determining the latter to a feeling, to an exertion, or to a Sign, which determination is the Interpretant. But it remains to point out that there are usually two Objects, and more than two Interpretants. Namely, we have to distinguish the Immediate Object, which is the Object as the Sign itself represents it, and whose Being is thus dependent upon the Representation of it in the Sign, from the Dynamical Object, which is the Reality which by some means contrives to determine the Sign to its Representation. In regard to the Interpretant we have equally to distinguish, in the first place, the Immediate Interpretant, which is the interpretant as it is revealed in the right understanding of the Sign itself, and is ordinarily called the *meaning* of the sign; while in the second place, we have to take note of the Dynamical Interpretant which is the actual effect which the Sign, as a Sign, really determines. Finally there is what I provisionally term the Final Interpretant, which refers to the manner in which the Sign tends to represent itself to be related to its Object. I confess that my own conception of this third interpretant is not yet quite free from mist.* Of the ten divisions of signs which have seemed to me to call for my special

* Cf. 5.475ff.

study, six turn on the characters of an Interpretant and three on the characters of the Object.* Thus the division into Icons, Indices, and Symbols depends upon the different possible relations of a Sign to its Dynamical Object.† Only one division is concerned with the nature of the Sign itself, and this I now proceed to state.

537. A common mode of estimating the amount of matter in a MS. or printed book is to count the number of words.[1] There will ordinarily be about twenty *the*'s on a page, and of course they count as twenty words. In another sense of the word "word," however, there is but one word "the" in the English language; and it is impossible that this word should lie visibly on a page or be heard in any voice, for the reason that it is not a Single thing or Single event. It does not exist; it only determines things that do exist. Such a definitely significant Form, I propose to term a *Type*.‡ A Single event which happens once and whose identity is limited to that one happening or a Single object or thing which is in some single place at any one instant of time, such event or thing being significant only as occurring just when and where it does, such as this or that word on a single line of a single page of a single copy of a book, I will venture to call a *Token*.‡ An indefinite significant character such as a tone of voice can neither be called a Type nor a Token. I propose to call such a Sign a *Tone*.‡ In order that a Type may be used, it has to be embodied in a Token which shall be a sign of the Type, and thereby of the object the Type signifies. I propose to call such a Token of a Type an *Instance* of the Type. Thus,

* Signs can be classified on the basis of the characters which (1) they, (2) their immediate and (3) their dynamical objects, and their (4) immediate, (5) dynamical and (6) final interpretants possess, as well as on the basis of the nature of relations which (7) the dynamical objects and the (8) dynamical and (9) final interpretants have to the sign and which the (10) final interpretant has to the object. These ten divisions provide thirty designations for signs (each division being trichotomized by the categories, First, Second and Third). When properly arranged, they are easily shown to yield but sixty-six classes of possible signs. The principle determining that conclusion is stated in the introduction to vol. 2 and in 2.235n. See also the letters to Lady Welby, vol. 9.

† (7) of the previous footnote. Cf. 2.243; 2.247.

[1] Dr. Edward Eggleston originated the method.

‡ The type, token and tone are the legisigns, sinsigns and qualisigns discussed in 2.243f and form division (1) in the note to 536.

there may be twenty Instances of the Type "the" on a page. The term (Existential) *Graph* will be taken in the sense of a Type; and the act of embodying it in a *Graph-Instance* will be termed *scribing* the Graph (not the Instance), whether the Instance be written, drawn, or incised. A mere blank place is a Graph-Instance, and the Blank *per se* is a Graph; but I shall ask you to assume that it has the peculiarity that it cannot be abolished from any Area on which it is scribed, as long as that Area exists.

538. A familiar logical triplet is Term, Proposition, Argument.* In order to make this a division of all signs, the first two members have to be much widened. By a *Seme*,† I shall mean anything which serves for any purpose as a substitute for an object of which it is, in some sense, a representative or Sign. The logical Term, which is a class-name, is a Seme. Thus, the term "The mortality of man" is a Seme. By a *Pheme*‡ I mean a Sign which is equivalent to a grammatical sentence, whether it be Interrogative, Imperative, or Assertory. In any case, such a Sign is intended to have some sort of compulsive effect on the Interpreter of it. As the third member of the triplet, I sometimes use the word *Delome* (pronounce deeloam, from δήλωμα), though *Argument* would answer well enough. It is a Sign which has the Form of tending to act upon the Interpreter through his own self-control, representing a process of change in thoughts or signs, as if to induce this change in the Interpreter.

A Graph is a Pheme, and in my use hitherto, at least, a Proposition. An Argument is represented by a series of Graphs.

§4. UNIVERSES AND PREDICAMENTS[E]

539. The Immediate Object of all knowledge and all thought is, in the last analysis, the Percept. This doctrine in no wise conflicts with Pragmaticism, which holds that the Immediate Interpretant of all thought proper is Conduct. Nothing is more indispensable to a sound epistemology than a

* These are defined in terms of the relation of the final interpretant to the sign. They constitute division (9) in the note to 536. Cf. 2.250f.

† Or rheme. But cf. 560.

‡ Or dicisign.

crystal-clear discrimination between the Object and the Interpretant of knowledge; very much as nothing is more indispensable to sound notions of geography than a crystal-clear discrimination between north latitude and south latitude; and the one discrimination is not more rudimentary than the other. That we are conscious of our Percepts is a theory that seems to me to be beyond dispute; but it is not a fact of Immediate Perception. A fact of Immediate Perception is not a Percept, nor any part of a Percept; a Percept is a Seme, while a fact of Immediate Perception or rather the Perceptual Judgment of which such fact is the Immediate Interpretant, is a Pheme that is the direct Dynamical Interpretant of the Percept, and of which the Percept is the Dynamical Object, and is with some considerable difficulty (as the history of psychology shows), distinguished from the Immediate Object, though the distinction is highly significant.* But not to interrupt our train of thought, let us go on to note that while the Immediate Object of a Percept is excessively vague, yet natural thought makes up for that lack (as it almost amounts to), as follows. A late Dynamical Interpretant of the whole complex of Percepts is the Seme of a Perceptual Universe that is represented in instinctive thought as determining the original Immediate Object of every Percept.† Of course, I must be understood as talking not psychology, but the logic of mental operations. Subsequent Interpretants furnish new Semes of Universes resulting from various adjunctions to the Perceptual Universe. They are, however, all of them, Interpretants of Percepts.

Finally, and in particular, we get a Seme of that highest of all Universes which is regarded as the Object of every true Proposition, and which, if we name it [at] all, we call by the somewhat misleading title of "The Truth."

540. That said, let us go back and ask this question: How is it that the Percept, which is a Seme, has for its direct Dynamical Interpretant the Perceptual Judgment, which is a

* *I.e.*, The perceptual judgment is a proposition of existence determined by the percept, which it interprets. See 541, 5.115ff and 5.151ff.

† *I.e.*, A complex of percepts yields a picture of a perceptual universe. Without reflection, that universe is taken to be the cause of such objects as are represented in a percept. Though each percept is vague, as it is recognized that its object is the result of the action of the universe on the perceiver, it is so far clear.

Pheme? For that is not the usual way with Semes, certainly. All the examples that happen to occur to me at this moment of such action of Semes are instances of Percepts, though doubtless there are others. Since not all Percepts act with equal energy in this way, the instances may be none the less instructive for being Percepts. However, Reader, I beg you will think this matter out for yourself, and then you can see — I wish I could — whether your independently formed opinion does not fall in with mine. My opinion is that a pure perceptual Icon — and many really *great* psychologists have evidently thought that Perception is a passing of images before the mind's eye, much as if one were walking through a picture gallery — could not have a Pheme for its direct Dynamical Interpretant. I desire, for more than one reason, to tell you *why* I think so, although that you should today appreciate my reasons seems to be out of the question. Still, I wish you to understand me so far as to know that, mistaken though I be, I am not so sunk in intellectual night as to be dealing lightly with philosophic Truth when I aver that weighty reasons have moved me to the adoption of my opinion; and I am also anxious that it should be understood that those reasons have not been psychological at all, but are purely logical. My reason, then, briefly stated and abridged, is that it would be *illogical* for a pure Icon to have a Pheme for its Interpretant, and I hold it to be impossible for thought not subject to self-control, as a Perceptual Judgment manifestly is not, to be illogical. I dare say this reason may excite your derision or disgust, or both; and if it does, I think none the worse of your intelligence. You probably opine, in the first place, that there is no meaning in saying that thought which draws no Conclusion is illogical, and that, at any rate, there is no standard by which I can judge whether such thought is logical or not; and in the second place, you probably think that, if self-control has any essential and important relation to logic, which I guess you either deny or strongly doubt, it can only be that it is that which makes thought *logical*, or else which establishes the distinction between the logical and the illogical, and that in any event it has to be such as it is, and would be logical, or illogical, or both, or neither, whatever course it should take. But though an Interpretant is not necessarily a Conclusion, yet a Conclusion is necessarily an Interpretant. So that if an Interpretant is not subject to the rules of Conclusions there is nothing monstrous in my thinking it is subject to some generalization of such rules. For any evolution of thought, whether it leads to a Conclusion or not, there is a certain normal course, which is to be determined by considerations not in the least psychological, and which I wish to expound in my next article;* and while I entirely agree, in opposition to distinguished logicians, that normality can be no criterion for what I call rationalistic reasoning, such as alone is admissible in science, yet it is precisely the criterion of instinctive or common-sense reasoning, which, within its own field, is much more trustworthy than rationalistic reasoning. In my opinion, it is self-control which makes any other than the normal course of thought possible, just as nothing else makes any other than the normal course of action possible; and just as it is precisely that that gives room for an ought-to-be of conduct, I mean Morality, so it equally gives room for an ought-to-be of thought, which is Right Reason; and where there is no self-control, nothing but the normal is possible. If your reflections have led you to a different conclusion from mine, I can still hope that when you come to read my next article, in which I shall endeavor to show what the forms of thought are, in general and in some detail, you may yet find that I have not missed the truth.

541. But supposing that I am right, as I probably shall be in the opinions of *some* readers, how then is the Perceptual Judgment to be explained? In reply, I note that a Percept cannot be dismissed at will, even from memory. Much less can a person prevent himself from perceiving that which, as we say, stares him in the face. Moreover, the evidence is overwhelming that the perceiver is aware of this compulsion upon him; and if I cannot say for certain how this knowledge comes to him, it is not that I cannot conceive how it could come to him, but that, there being several ways in which this might happen, it is difficult to say which of those ways actually is followed. But that discussion belongs to psychology; and I will not enter upon it. Suffice it to say that the perceiver

* This is the last published article of the present series. A number of incompleted papers, intended as the next article, have been found and published in part. See e.g., 1.305n, 1.306n, 534n, 553n, 561n, 564n, 5.549f.

is aware of being compelled to perceive what he perceives. Now existence means precisely the exercise of compulsion. Consequently, whatever feature of the percept is brought into relief by some association and thus attains a logical position like that of the observational premiss of an explaining Abduction,[1] the attribution of Existence to it in the Perceptual Judgment is virtually and in an extended sense, a logical Abductive Inference nearly approximating to necessary inference. But my next paper will throw a flood of light upon the logical affiliation of the Proposition, and the Pheme generally, to coercion.

542. That conception of Aristotle which is embodied for us in the cognate origin of the terms *actuality* and *activity* is one of the most deeply illuminating products of Greek thinking. Activity implies a generalization of *effort*; and effort is a two-sided idea, effort and resistance being inseparable, and therefore the idea of Actuality has also a dyadic form.

543. No cognition and no Sign is absolutely precise, not even a Percept; and indefiniteness is of two kinds, indefiniteness as to what is the Object of the Sign, and indefiniteness as to its Interpretant, or indefiniteness in Breadth and in Depth.* Indefiniteness in Breadth may be either Implicit or Explicit. What this means is best conveyed in an example. The word *donation* is indefinite as to who makes the gift, what he gives, and to whom he gives it. But it calls no attention, itself, to this indefiniteness. The word *gives* refers to the same sort of fact, but its meaning is such that that meaning is felt to be incomplete unless those items are, at least formally, specified; as they are in "Somebody gives something to some person (real or artificial)." An ordinary Proposition† ingeniously contrives to convey novel information through Signs whose significance depends entirely on the interpreter's familiarity with them; and this it does by means of a "Predicate," *i.e.*, a term explicitly indefinite in breadth, and defining its breadth by means of "Subjects," or terms whose breadths are somewhat

[1] Abduction, in the sense I give the word, is any reasoning of a large class of which the provisional adoption of an explanatory hypothesis is the type. But it includes processes of thought which lead only to the suggestion of questions to be considered, and includes much besides.

* Cf. 2.407ff.

† Cf. vol. 2, bk. II, ch. 4.

definite, but whose informative depth (*i.e.*, all the depth except an essential superficies) is indefinite, while conversely the depth of the Subjects is in a measure defined by the Predicate. A Predicate is either non-relative, or *a monad*, that is, is explicitly indefinite in one extensive respect, as is "black"; or it is a dyadic relative, or dyad, such as "kills," or it is a polyadic relative, such as "gives." These things must be diagrammatized in our system.

Something more needs to be added under the same head. You will observe that under the term "Subject" I include, not only the subject nominative, but also what the grammarians call the direct and the indirect object, together, in some cases, with nouns governed by prepositions. Yet there is a sense in which we can continue to say that a Proposition has but one Subject, for example, in the proposition, "Napoleon ceded Louisiana to the United States," we may regard as the Subject the ordered triplet, "Napoleon — Louisiana — the United States," and as the Predicate, "has for its first member, the agent, or party of the first part, for its second member the object, and for its third member the party of the second part of one and the same act of cession." The view that there are three subjects is, however, preferable for most purposes, in view of its being so much more analytical, as will soon appear.

544. All general, or definable, Words, whether in the sense of Types or of Tokens, are certainly Symbols. That is to say, they denote the objects that they do by virtue only of there being a habit that associates their signification with them. As to Proper Names, there might perhaps be a difference of opinion, especially if the Tokens are meant. But they should probably be regarded as Indices, since the actual connection (as we listen to talk), of Instances of the same typical words with the same Objects, alone causes them to be interpreted as denoting those Objects. Excepting, if necessary, propositions in which all the subjects are such signs as these, no proposition can be expressed without the use of Indices.[1] If, for example, a man remarks, "Why, it is raining!" it is only by some such *circumstances* as that he is now standing here looking out at a window as he speaks, which would serve as an Index (not,

[1] Strictly pure Symbols can signify only things familiar, and those only in so far as they are familiar.

however, as a Symbol) that he is speaking of this place at this time, whereby we can be assured that he cannot be speaking of the weather on the satellite of Procyon, fifty centuries ago. Nor are Symbols and Indices together generally enough. The arrangement of the words in the sentence, for instance, must serve as *Icons*, in order that the sentence may be understood. The chief need for the Icons is in order to show the Forms of the synthesis of the elements of thought. For in precision of speech, Icons can represent nothing but Forms and Feelings. That is why Diagrams are indispensable in all Mathematics, from Vulgar Arithmetic up, and in Logic are almost so. For Reasoning, nay, Logic generally, hinges entirely on Forms. You, Reader, will not need to be told that a regularly stated Syllogism is a Diagram; and if you take at random a half dozen out of the hundred odd logicians who plume themselves upon not belonging to the sect of Formal Logic, and if from this latter sect you take another half dozen at random, you will find that in proportion as the former avoid diagrams, they utilize the syntactical Form of their sentences. No pure Icons represent anything but Forms; no pure Forms are represented by anything but Icons. As for Indices, their utility especially shines where other Signs fail. Extreme precision being desired in the description of a red color, should I call it vermillion, I may be criticized on the ground that vermillion differently prepared has quite different hues, and thus I may be driven to the use of the color-wheel, when I shall have to Indicate four disks individually, or I may say in what proportions light of a given wave-length is to be mixed with white light to produce the color I mean. The wave-length being stated in fractions of a micron, or millionth of a meter, is referred through an Index to two lines on an individual bar in the Pavillon de Breteuil, at a given temperature and under a pressure measured against gravity at a certain station and (strictly) at a given date, while the mixture with white, after white has been fixed by an Index of an individual light, will require at least one new Index. But of superior importance in Logic is the use of Indices to denote Categories and Universes,[1] which are

[1] I use the term *Universe* in a sense which excludes many of the so-called "universes of discourse" of which Boole [*An Investigation of the Laws of Thought*, etc., pp. 42, 167], DeMorgan [*Cambridge Philosophical Transactions*, VIII, 380. *Formal*

classes that, being enormously large, very promiscuous, and known but in small part, cannot be satisfactorily defined, and therefore can only be denoted by Indices. Such, to give but a single instance, is the collection of all things in the Physical Universe. If anybody, your little son for example, who is such an assiduous researcher, always asking, What is the Truth (Τί ἐστιν ἀλήθεια); but like "jesting Pilate," will not always stay for an answer, should ask you what the Universe of things physical is, you may, if convenient, take him to the Rigi-Kulm, and about sunset, point out all that is to be seen of Mountains, Forests, Lakes, Castles, Towns, and then, as the stars come out, all there is to be seen in the heavens, and all that though not seen, is reasonably conjectured to be there; and then tell him, "Imagine that what is to be seen in a city back yard to grow to all you can see here, and then let this grow in the same proportion as many times as there are trees in sight from here, and what you would finally have would be harder to find in the Universe than the finest needle in America's yearly crop of hay." But such methods are perfectly futile: Universes cannot be described.

545. Oh, I overhear what you are saying, O Reader: that a Universe and a Category are not at all the same thing; a Universe being a receptacle or class of Subjects, and a Category being a mode of Predication, or class of Predicates. I never said they were the same thing; but whether you describe the two correctly is a question for careful study.

546. Let us begin with the question of Universes. It is rather a question of an advisable point of view than of the truth of a doctrine. A logical universe is, no doubt, a collection of *logical* subjects, but not necessarily of meta-physical Subjects, or "substances"; for it may be composed of characters, of elementary facts, etc. See my definition in Baldwin's Dictionary.* Let us first try whether we may not assume that there is but one kind of Subjects which are either existing things or else quite fictitious. Let it be asserted that there is some married woman who will commit suicide in case her husband fails in business. Surely that is a very different proposi-

Logic, pp. 37–8] and many subsequent logicians speak, but which, being perfectly definable, would in the present system be denoted by the aid of a graph.

* 2.536.

tion from the assertion that some married woman will commit suicide if all married men fail in business. Yet if nothing is real but existing things, then, since in the former proposition nothing whatever is said as to what the lady will or will not do if her husband does *not* fail in business, and since of a given married couple this can only be false if the fact is contrary to the assertion, it follows it can only be false if the husband *does* fail in business and if the wife then fails to commit suicide. But the proposition only says that there is *some* married couple of which the wife is of that temper. Consequently, there are only two ways in which the proposition can be false, namely, first, by there not being any married couple, and secondly, by *every* married man failing in business while *no* married woman commits suicide. Consequently, all that is required to make the proposition true is that there should either be some married man who does not fail in business, or else some married woman who commits suicide. That is, the proposition amounts merely to asserting that there is a married woman who will commit suicide if *every* married man fails in business. The equivalence of these two propositions is the absurd result of admitting no reality but existence. If, however, we suppose that to say that a woman will suicide if her husband fails, means that every *possible* course of events would either be one in which the husband would not fail or one in which the wife would commit suicide, then, to make that false it will not be requisite for the husband actually to fail, but it will suffice that there are *possible* circumstances under which he would fail, while yet his wife would not commit suicide. Now you will observe that there is a great difference between the two following propositions:

First, There is some *one* married woman who under all possible conditions would commit suicide or else her husband would not have failed.

Second, Under all possible circumstances there is some married woman *or other* who would commit suicide, or else her husband would not have failed.

The former of these is what is really meant by saying that there is some married woman who would commit suicide if her husband were to fail, while the latter is what the denial of any possible circumstances except those that really take place logically leads to [our] interpreting (or virtually interpreting), the Proposition as asserting.

547. In other places,* I have given many other reasons for my firm belief that there are real possibilities. I also think, however, that, in addition to actuality and possibility, a *third* mode of reality must be recognized in that which, as the gipsy fortune-tellers express it, is "sure to come true," or, as we may say, is *destined*,[1] although I do not mean to assert that this is affirmation rather than the negation of this Mode of Reality. I do not see by what confusion of thought anybody can persuade himself that he does not believe that tomorrow is destined to come. The point is that it is today really true that to-morrow the sun will rise; or that, even if it does not, the clocks or *something*, will go on. For if it be not real it can only be fiction: a Proposition is either True or False. But we are too apt to confound destiny with the impossibility of the opposite. I see no impossibility in the sudden stoppage of everything. In order to show the difference, I remind you that "impossibility" is that which, for example, describes the mode of falsity of the idea that there should be a collection of objects so multitudinous that there would not be characters enough in the universe of characters to distinguish all those things from one another. Is there anything of that sort about the stoppage of all motion? There is, perhaps, a *law of nature* against it; but that is all. However, I will postpone the consideration of that point. Let us, at least, *provide* for such a mode of being in our system of diagrammatization, since it *may* turn out to be needed and, as I think, surely will.

548. I will proceed to explain why, although I am not prepared to deny that every proposition can be represented, and that I must say, for the most part very conveniently, under your view that the Universes are receptacles of the Subjects alone, I, nevertheless, cannot deem that mode of analyzing propositions to be satisfactory.

* *E.g.*, in 1.422. See also 580.

[1] I take it that anything may fairly be said to be *destined* which is sure to come about although there is no necessitating reason for it. Thus, a pair of dice, thrown often enough, will be sure to turn up sixes some time, although there is no necessity that they should. The probability that they will is 1: that is all. *Fate* is that special kind of *destiny* by which events are supposed to be brought about *under definite circumstances* which involve no necessitating cause for those occurrences.

And to begin with, I trust you will all agree with me that no analysis, whether in logic, in chemistry, or in any other science, is satisfactory, unless it be thorough, that is, unless it separates the compound into components each entirely homogeneous in itself, and therefore free from the smallest admixture of any of the others. It follows that in the Proposition, "Some Jew is shrewd," the Predicate is "Jew-that-is-shrewd," and the Subject is *Something*, while in the proposition "Every Christian is meek," the Predicate is "Either not Christian or else meek," while the Subject is *Anything;* unless, indeed, we find reason to prefer to say that this Proposition means, "It is false to say that a person is Christian of whom it is false to say that he is meek." In this last mode of analysis, when a Singular Subject is not in question (which case will be examined later), the only Subject is *Something*. Either of these two modes of analysis [differentiates] quite [clearly] the Subject from any Predicative ingredients; and at first sight, either seems quite favorable to the view that it is only the Subjects which belong to the Universes. Let us, however, consider the following two forms of propositions:

A* Any adept alchemist could produce a philosopher's stone of some kind or other,

B There is one kind of philosopher's stone that any adept alchemist could produce.

We can express these in the principle that the Universes are receptacles of Subjects as follows:

A[1] The Interpreter having selected any individual he likes, and called it A, an object B can be found, such that, Either A would not be an adept alchemist, or B would be a philosopher's stone of some kind, and A could produce B.

B[1] Something, B, might be found, such that, no matter what the Interpreter might select and call A, B would be a philosopher's stone of some kind, while either A would not be an adept alchemist, or else A could produce B.

* The numeration has been changed to avoid ambiguity. Originally A, A[1] and 1 were all numbered 1; B, B[1] and 2 were all numbered 2, and not differentiated in the text.

In these forms there are two Universes, the one of individuals selected at pleasure by the interpreter of the proposition, the other of suitable objects.

I will now express the same two propositions on the principle that each Universe consists, not of Subjects, but the one of True assertions, the other of False, but each to the effect that there is something of a given description.

1. This is false: That something, P, is an adept alchemist, and that this is false, that while something, S, is a philosopher's stone of some kind, P could produce S.
2. This is true: That something, S, is a philosopher's stone of some kind; and this is false, that something, P, is an adept alchemist while this is false, that P could produce S.

Here, the whole proposition is mostly made up of the truth or falsity of assertions that a thing of this or that description exists, the only conjunction being "and." That this method is highly analytic is manifest. Now since our whole intention is to produce a method for the perfect analysis of propositions, the superiority of this method over the other for our purpose is undeniable. Moreover, in order to illustrate how that other might lead to false logic, I will tack the predicate of B^1, in its objectionable form, upon the subject of A^1 in the same form, and *vice versa*. I shall thus obtain two propositions which that method represents as being as simple as are Nos. 1 and 2. We shall see whether they are so. Here they are:*

3. The Interpreter having designated any object to be called A, an object B may be found such that

   B is a philosopher's stone of some kind, while either A is not an adept alchemist or else A could produce B.
4. Something, B, may be found, such that, no matter what the interpreter may select, and call A,

   Either A would not be an adept alchemist, or B would be a philosopher's stone of some kind, and A could produce B.

Proposition 3 may be expressed in ordinary language thus: There is a kind of philosopher's stone, and if there be any adept alchemist, he could produce a philosopher's stone of some

* 3, 5 and 7 were all numbered 3; and 4, 6 and 8 were all numbered 4 in the original and not distinguished in the text.

kind. That is, No. 3 differs from A, A^1 and 1 only in adding that there is a kind of philosopher's stone. It differs from B, B^1 and 2 in not saying that any two adepts could produce the same kind of stone (nor that any adept could produce any existing kind); while B, B^1 and 2 assert that some kind is both existent and could be made by every adept.

Proposition 4, in ordinary language, is: If there be (or were) an adept alchemist, there is (or would be) a kind of philosopher's stone that any adept could produce. This asserts the substance of B, B^1 and 2, but only conditionally upon the existence of an adept; but it asserts, what A, A^1 and 1 do not, that all adepts could produce some one kind of stone, and this is precisely the difference between No. 4 and A^1.

To me it seems plain that the propositions 3 and 4 are both less simple than No. 1 and less simple than No. 2, each adding some thing to one of the pair first given and asserting the other conditionally. Yet the method of treating the Universes as receptacles for the metaphysical Subjects only, involves as a consequence the representation of 3 and 4 as quite on a par with 1 and 2.

It remains to show that the other method does not carry this error with it. [If] it is the states of things affirmed or denied that are contained in the universes, then the propositions [3 and 4] become as follows:

5. This is true: that there is a philosopher's stone of some kind, S, and that it is false that there is an adept, A, and that it is false that A could produce a philosopher's stone of some kind, S′. (Where it is neither asserted nor denied that S and S′ are the same, thus distinguishing this from 2.)
6. This is false: That there is an adept, A, and that this is false: That there is a stone of a kind, S, and this is false: That there is an adept, A′, and that this is false: That A′ could produce a stone of the kind S. (Where again it is neither asserted nor denied that A and A′ are identical, but the point is that this proposition holds even if they are not identical, thus distinguishing this from 1.)

These forms exhibit the greater complexity of Propositions 3 and 4, by showing that they really relate to *three* individuals each; that is to say, 3 to two possible different kinds of stone, as well as to an adept; and 4 to two possible different adepts, and to a kind of stone. Indeed, the two forms 3 and 4* are absolutely identical in meaning with the following different forms on the same theory. Now it is, to say the least, a serious fault in a method of analysis that it can yield two analyses so different of one and the same compound.

7. An object, B, can be found, such that whatever object the interpreter may select and call A, an object, B′, can thereupon be found such that B is an existing kind of philosopher's stone, and either A would not be an adept or else B′ is a kind of philosopher's stone such as A could produce.
8. Whatever individual the Interpreter may choose to call A, an object, B, may be found, such that whatever individual the Interpreter may choose to call A′, Either A is not an adept or B is an existing kind of philosopher's stone, and either A′ is not an adept or else A′ could produce a stone of the kind B.

But while my forms are perfectly analytic, the need of diagrams to exhibit their meaning to the eye (better than merely giving a separate line to every proposition said to be false) is painfully obtrusive.[1]

549. I will now say a few words about what you have called Categories, but for which I prefer the designation Predicaments, and which you have explained as predicates of predicates. That wonderful operation of hypostatic abstraction by which we seem to create *entia rationis* that are, nevertheless, sometimes real, furnishes us the means of turning predicates from being signs that we think or think *through*, into being subjects thought of. We thus think of the thought-sign itself, making it the object of another thought-sign. Thereupon, we can repeat the operation of hypostatic abstraction, and from these second intentions derive third intentions. Does this series proceed endlessly? I think not. What then are the characters

* Originally ". . . forms of statement of 3 and 4 on the other theory of the universes . . ."; a locution necessary so long as 3 and 5, and 4 and 6 were not distinguished.

[1] In correcting the proofs, a good while after the above was written, I am obliged to confess that in some places the reasoning is erroneous; and a much simpler argument would have supported the same conclusion more justly; though some weight ought to be accorded to my argument here, on the whole.

of its different members? My thoughts on this subject are not yet harvested. I will only say that the subject concerns Logic, but that the divisions so obtained must not be confounded with the different Modes of Being:* Actuality, Possibility, Destiny (or Freedom from Destiny). On the contrary, the succession of Predicates of Predicates is different in the different Modes of Being. Meantime, it will be proper that in our system of diagrammatization we should provide for the division, whenever needed, of each of our three Universes of modes of reality into *Realms* for the different Predicaments.

550. All the various meanings of the word "Mind," Logical, Metaphysical, and Psychological, are apt to be confounded more or less, partly because considerable logical acumen is required to distinguish some of them, and because of the lack of any machinery to support the thought in doing so, partly because they are so many, and partly because (owing to these causes), they are all called by one word, "Mind." In one of the narrowest and most concrete of its logical meanings, a Mind is that Seme of The Truth, whose determinations become Immediate Interpretants of all other Signs whose Dynamical Interpretants are dynamically connected.† In our Diagram the same thing which represents The Truth must be regarded as in another way representing the Mind, and indeed, as being the Quasi-mind of all the Signs represented on the Diagram. For any set of Signs which are so connected that a complex of two of them can have one interpretant, must be Determinations of one Sign which is a *Quasi-mind*.

551. Thought is not necessarily connected with a brain. It appears in the work of bees, of crystals, and throughout the purely physical world; and one can no more deny that it is really there, than that the colors, the shapes, etc., of objects are really there. Consistently adhere to that unwarrantable denial, and you will be driven to some form of idealistic nominalism akin to Fichte's. Not only is thought in the organic world, but it develops there. But as there cannot be a General without Instances embodying it, so there cannot be thought

* Usually called categories by Peirce. See vol. 1, bk. III.

† *I.e.*, Mind is a propositional function of the widest possible universe, such that its values are the meanings of all signs whose actual effects are in effective interconnection.

without Signs. We must here give "Sign" a very wide sense, no doubt, but not too wide a sense to come within our definition. Admitting that connected Signs must have a Quasi-mind, it may further be declared that there can be no isolated sign. Moreover, signs require at least two Quasi-minds; a *Quasi-utterer* and a *Quasi-interpreter;* and although these two are at one (*i.e.*, *are* one mind) in the sign itself, they must nevertheless be distinct. In the Sign they are, so to say, *welded*. Accordingly, it is not merely a fact of human Psychology, but a necessity of Logic, that every logical evolution of thought should be dialogic. You may say that all this is loose talk; and I admit that, as it stands, it has a large infusion of arbitrariness. It might be filled out with argument so as to remove the greater part of this fault; but in the first place, such an expansion would require a volume — and an uninviting one; and in the second place, what I have been saying is only to be applied to a slight determination of our system of diagrammatization, which it will only slightly affect; so that, should it be incorrect, the utmost *certain* effect will be a danger that our system *may* not represent every variety of non-human thought.

§5. TINCTURED EXISTENTIAL GRAPHS[E]

552. There now seems to remain no reason why we should not proceed forthwith to formulate and agree upon

THE CONVENTIONS

Determining the Forms and Interpretations of Existential Graphs

Convention the First: Of the Agency of the Scripture. We are to imagine that two parties[1] collaborate in composing a Pheme, and in operating upon this so as to develop a Delome. (Provision shall be made in these Conventions for expressing every kind of Pheme as a Graph;[2] and it is certain that the Method could be applied to aid the development and analysis of any kind of purposive thought. But hitherto no Graphs have been

[1] They may be two bodies of persons, two persons, or two mental attitudes or states of one person.

[2] A *Graph* has already been defined in 535 *et seq.*

studied but such as are Propositions; so that, in the resulting uncertainty as to what modifications of the Conventions might be required for other applications, they have mostly been here stated as if they were only applicable to the expression of Phemes and the working out of necessary conclusions.)

The two collaborating parties shall be called the *Graphist* and the *Interpreter*. The Graphist shall responsibly scribe each original Graph and each addition to it, with the proper indications of the Modality to be attached to it, the *relative Quality*[1] of its position, and every particular of its dependence on

[1] The traditional and ancient use of the term propositional *Quality* makes it an affair of the mode of expression solely. For "Socrates is mortal" and "Socrates is immortal" are equally Affirmative; "Socrates is not mortal" and "Socrates is not immortal" are equally Negative, provided "is not" translates *non est*. If, however, "is not" is in Latin *est non*, with no difference of meaning, the proposition is infinitated. Without anything but the merest verbiage to support the supposition that there is any corresponding distinction between different meanings of propositions, Kant insisted on raising the difference of expression to the dignity of a category. In [3.532; but cf. 5.450] I gave some reason for considering a relative proposition to be affirmative or negative according as it does or does not unconditionally assert the existence of an indefinite subject. Although at the time of writing that, nine and a half years ago, I was constrained against my inclinations, to make that statement, yet I never heartily embraced that view, and dismissed it from my mind, until after I had drawn up the present statement of the Conventions of Existential Graphs, I found, quite to my surprise, that I had herein taken substantially the same view. That is to say, although I herein speak only of "relative" quality, calling the assertion of any proposition the Affirmation of it, and regarding the denial of it as an assertion *concerning* that proposition as subject, namely, that it is false; which is my distinction of Quality Relative to the proposition either itself Affirmed, or of which the falsity is affirmed, if the Relative Quality of it is Negative, yet since every Graph in itself either recognizes the existence of a familiar Singular subject or asserts something of an indefinite subject asserted to exist in some Universe, it follows that every relatively Affirmative Graph unconditionally asserts or recognizes the occurrence of some description of object in some Universe; while no relatively Negative Graph does this. The logic of a Limited Universe of Marks [2.519ff.] suggests a different view of Quality, but careful analysis shows that it is in no fundamental conflict with the above.

A question not altogether foreign to the subject of Quality is whether Quality and Modality are of the same general nature. In selecting a mode of representing Modality, which I have not done without much experimentation, I have finally resorted to one which commits itself as little as possible to any particular theory of the nature of Modality, although there are undeniable objections to such a course. If any particular analysis of Modality had appeared to me to be quite evident, I should have endeavored to exhibit it unequivocally. Meantime, my opinion is that the Universe is a Subject of every Proposition and that any and connections with other graphs. The Interpreter is to make such erasures and insertions of the Graph delivered to him by the Graphist as may accord with the "*General Permissions*" deducible from the Conventions and with his own purposes.

553. *Convention the Second; Of the Matter of the Scripture, and the Modality*[1] *of the Phemes expressed.* The matter which the Graph-instances are to determine, and which thereby becomes the *Quasi-mind* in which the Graphist and Interpreter are at one, being a Seme of *The Truth*,[2] that is, of the widest

Modality shown by its indefiniteness to be Affirmative, such as Possibility and Intention, is a special determination of the Universe of The Truth. Something of this sort is seen in Negation. For if we say of a Man that he is not sinless, we represent the sinless as having a place only in an ideal universe which, or the part of which that contains the imagined sinless being, we then positively sever from the identity of the man in question.

[1] I may as well, at once, acknowledge that, in Existential Graphs, the representation of Modality (possibility, necessity, etc.) lacks almost entirely that pictorial, or Iconic, character which is so striking in the representation in the same system of every feature of propositions *de inesse*. Perhaps it is in the nature of things that it should be so in such wise that for Modality to be iconically represented in that same "pictorial" way in which the other features are represented would constitute a falsity in the representation. If so, it is a perfect vindication of the system, upon whose accusers, I suppose, the burden of proof lies. Still, I confess I suspect there is in the heraldic representation of modality as set forth [below] a defect capable of being remedied. If it be not so, if the lack of "pictorialness" in the representation of modality cannot be remedied, it is because modality has, in truth, the nature which I opined it has (which opinion I expressed toward the end of the footnote [to 552], and if that be the case, Modality is not, properly speaking, conceivable at all, but the difference, for example, between possibility and actuality is only recognizable much in the same way as we recognize the difference between a dream and waking experience, supposing the dream to be ever so detailed, reasonable, and thoroughly consistent with itself and with all the rest of the dreamer's experience. Namely, it still would not be so "vivid" as waking experience. . . — from "Phaneroscopy, φαν," c. 1906; part of the manuscript used in 534n.

[2] It was the genius of my gifted student, Dr. O. H. Mitchell, in [*Studies in Logic*, ed. by C. S. Peirce, p. 73ff] that first opened our eyes to the identity of the subject of all assertions, although in another sense one assertion may have several individual subjects, which may even belong to what Mitchell called (quite justifiably, notwithstanding a certain condemnatory remark, as superficial as it was supercilious), different dimensions of the logical Universe. The entire Phemic Sheet and indeed the whole Leaf [see 555] is an image of the universal field of interconnected Thought (for, of course, all thoughts are interconnected). The field of Thought, in its turn, is in every thought, confessed to be a sign of that great external power, that Universe, the Truth. We all agree that we refer

Universe of Reality, and at the same time, a Pheme of all that is tacitly taken for granted between the Graphist and Interpreter, from the outset of their discussion, shall be a sheet, called the *Phemic Sheet*, upon which signs can be scribed and from which any that are already scribed in any manner (even though they be incised) *can* be erased. But certain parts of other sheets not having the significance of the Phemic

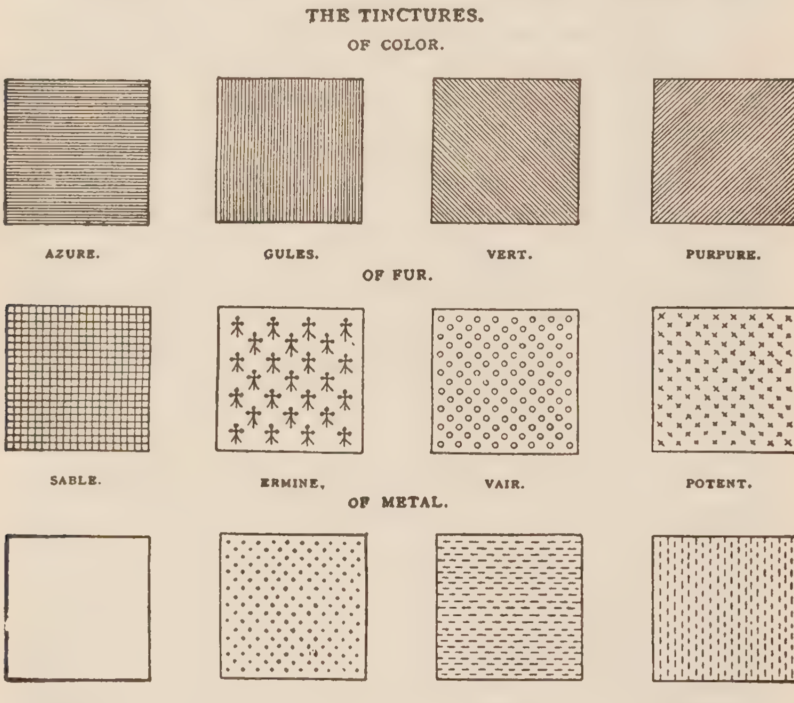


Fig. 197

to the same real thing when we speak of the truth, whether we think aright of it, or not. But we have no cognition of its essence that can, in strictness, be called a *concept* of it: we only have a direct perception of having the matter of our Thought forced upon it from outside our own control. It is thus, neither by immediate feeling, as we gaze at a red color, that we mean what we mean by the Truth; for Feeling tells of nothing but itself. Nor is it by the persuasion of reason, since reason always refers to two other things than itself. But it is by what I call a *dyadic consciousness.*— from "The Bedrock beneath Pragmaticism," c. 1906, one of a number of fragmentary manuscripts designed to follow the present article.

sheet, but on which Graphs can be scribed and erased, shall be sometimes inserted in the Phemic sheet and exposed to view, as the Third Convention shall show. Every part of the exposed surface shall be tinctured in one or another of twelve tinctures. These are divided into three *classes* of four tinctures each, the class-characters being called *Modes of Tincture*, or severally, Color, Fur, and Metal. The tinctures of Color are Azure, Gules, Vert, and Purpure. Those of Fur are Sable, Ermine, Vair, and Potent. Those of Metal are Argent, Or, Fer, and Plomb. The Tinctures will in practice be represented as in Fig. 197.[1] The whole of any *continuous* part of the exposed surface in one tincture shall be termed a *Province*. The border of the sheet has one tincture all round; and we may imagine that it was chosen from among twelve, in agreement between the Graphist and the Interpreter at the outset. The province of the border may be called the *March*. Provinces adjacent to the March are to be regarded as overlying it; Provinces adjacent to those Provinces, but not to the March, are to be regarded as overlying the provinces adjacent to the March, and so on. We are to imagine that the Graphist always finds provinces where he needs them.

554. When any representation of a state of things consisting in the applicability of a given description to an individual or limited set of individuals otherwise indesignate is scribed, the Mode of Tincture of the province on which it is scribed shows whether the Mode of Being which is to be affirmatively or negatively attributed to the state of things described is to be that of Possibility, when Color will be used; or that of Intention, indicated by Fur; or that of Actuality shown by Metal. Special understandings may determine special tinctures to refer to special varieties of the three genera of Modality. Finally, the Mode of Tincture of the March may determine whether the Entire Graph is to be understood as Interrogative, Imperative, or Indicative.

[1] It is chiefly for the sake of these convenient and familiar modes of representation of Petrosancta, that a modification of heraldic tinctures has been adopted. Vair and Potent here receive less decorative and pictorial Symbols. Fer and Plomb are selected to fill out the quaternion of metals on account of their monosyllabic names.

555. *Convention the Third: Of Areas enclosed within, but severed from, the Phemic Sheet.* The Phemic Sheet is to be imagined as lying on the smoother of the two surfaces or sides of a *Leaf*, this side being called the *recto*, and to consist of so much of this side as is continuous with the March. Other parts of the *recto* may be *exposed* to view. Every Graph-instance on the Phemic Sheet is posited unconditionally (unless, according to an agreement between Graphist and Interpreter, the Tincture of its own Province or of the March should indicate a condition) and every Graph-instance on the *recto* is posited affirmatively and, in so far as it is indeterminate, indefinitely.

556. Should the Graphist desire to negative a Graph, he must scribe it on the *verso*, and then, before delivery to the Interpreter, must make an incision, called a *Cut*, through the Sheet all the way round the Graph-instance to be denied, and must then turn over the excised piece, so as to *expose* its rougher surface carrying the negatived Graph-instance. This reversal of the piece is to be conceived to be an inseparable part of the operation of making a *Cut*.[1] But if the Graph to be negatived includes a Cut, the twice negatived Graph within that Cut must be scribed on the *recto*, and so forth. The part of the exposed surface that is continuous with the part just outside the Cut is called the *Place of the Cut*. A Cut is neither a Graph nor a Graph-instance; but the Cut, together with all that it encloses, exposed is termed an *Enclosure*, and is conceived to be an Instance of a Graph *scribed* on the Place of the Cut, which is also termed the *Place of the Enclosure*. The surface within the Cut, continuous with the parts just within it, is termed the *Area* of the Cut and of the Enclosure; and the part of the *recto* continuous with the March (*i.e.*, the Phemic Sheet), is likewise termed an *Area*, namely the Area of the Border. The Copulate of all that is scribed on any one Area, including the Graphs of which the Enclosures whose Place is this Area are Instances, is called the *Entire Graph* of that Area; and any part of the Entire Graph, whether graphically connected with or disconnected from the other parts, provided

[1] I am tempted to say that it is the reversal alone that effects the denial, the Cut merely cutting off the Graph within from assertion concerning the Universe to which the Phemic Sheet refers. But that is not the only possible view, and it would be rash to adopt it definitely, as yet.

it might be the Entire Graph of the Sheet, is termed a Partial Graph of the Area.

557. There may be any number of Cuts, one within another, the Area of one being the Place of the next, and since the Area of each is on the side of the leaf opposite to its Place, it follows that *recto* Areas may be *exposed* which are not parts of the Phemic Sheet. Every Graph-instance on a *recto* Area is affirmatively posited, but is posited conditionally upon whatever may be signified by the Graph on the Place of the Cut of which this Area is the Area. (It follows that Graphs on Areas of different Enclosures on a *verso* Place are only alternately affirmed, and that while only the Entire Graph of the Area of an Enclosure on a *recto* Place is denied, but not its different Partial Graphs, except alternatively, the Entire Graphs of Areas of different Enclosures on one *recto* Place are copulatively denied.)

558. Every Graph-instance must lie upon one Area,[1] although an Enclosure may be a part of it. Graph-instances on different Areas are not to be considered as, nor by any permissible latitude of speech to be called, Parts of one Graph-instance, nor Instances of Parts of one Graph; for it is only Graph-instances on one Area that are called Parts of one Graph-instance, and that only of a Graph-instance on that same Area; for though the Entire Graph on the Area of an enclosure is termed the *Graph of the Enclosure*, it is no Part of the Enclosure and is connected with it only through a denial.

559. *Convention the Fourth: Concerning Signs of Individuals and of Individual Identity.* A single dot, not too minute, or single congeries of contiguous pretty large dots, whether in the form of a line or surface, when placed on any exposed Area, will refer to a single member of the Universe to which the Tincture of that Area refers, but will not thereby be made to refer determinately to any one. But do not forget that

[1] For, of course, the Graph-instance must be on one sheet; and if part were on the *recto*, and part on the *verso*, it would not be on one continuous sheet. On the other hand, a Graph-instance can perfectly well extend from one Province to another, and even from one *Realm* (or space having one Mode of Tincture) to another. Thus, the Spot, "—is in the relation—to—," may, if the relation is that of an existent object to its purpose, have the first Peg on Metal, the second on Color, and the third on Fur. Cf. 579.

separate dots, or separate aggregates of dots, will not necessarily denote different Objects.

560. By a *rheme*, or *predicate*, will here be meant a blank form of proposition which might have resulted by striking out certain parts of a proposition, and leaving a *blank* in the place of each, the parts stricken out being such that if each blank were filled with a proper name, a proposition (however nonsensical) would thereby be recomposed. An ordinary predicate of which no analysis is intended to be represented will usually be *written* in abbreviated form, but having a particular point on the periphery of the written form appropriated to each of the blanks that might be filled with a proper name. Such written form with the appropriated points shall be termed a *Spot;* and each appropriated point of its periphery shall be called a *Peg* of the Spot. If a heavy dot is placed at each Peg, the Spot will become a Graph expressing a proposition in which every blank is filled by a word (or concept) denoting an indefinite individual object, "something."

561. A heavy line shall be considered as a continuum of contiguous dots; and since contiguous dots denote a single individual, such a line without any point of branching will signify the identity of the individuals denoted by its extremities, and the type of such unbranching line shall be the Graph of Identity, any instance of which (on one area, as every Graph-instance must be) shall be called a *Line of Identity.* The type of a three-way point of such a line (Fig. 198) shall be the *Graph of Teridentity;* and it shall be considered as composed of three contiguous Pegs of a Spot of Identity. An extremity

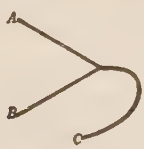


Fig. 198

Fig. 199

of a Line of Identity not abutting upon another such Line in another area shall be called a *Loose End.* A heavy line, whether confined to one area or not (and therefore not generally being a Graph-instance) of which two extremities abut upon pegs of spots shall be called a *Ligature.* Two lines cannot abut upon the same peg other than a point of teridentity. (The purpose of this rule is to force the recognition of the demonstrable logical truth that the concept of teridentity is not mere identity. It is identity *and* identity, but this "and" is a distinct concept, and is precisely that of teridentity.) A Ligature crossing a Cut is to be interpreted as unchanged in meaning by erasing the part that crosses to the Cut and attaching to the two Loose Ends so produced two Instances of a Proper Name nowhere else used; such a Proper name (for which a capital letter will serve) being termed a *Selective.*[1] In the interpretation of Selectives it is often necessary to observe the rule which holds throughout the System, that the Interpretation of Existential Graphs must be *endoporeutic*, that is, the application of a Graph on the Area of a Cut will depend on the predetermination of the application of that which is on the Place of the Cut.

In order to avoid the intersection of Lines of Identity, either a Selective may be employed, or a *Bridge*, which is imagined to be a bit of paper ribbon, but will in practice be pictured as in Fig. 199.

[1] The essential error, τὸ πρῶτον ψεῦδος, of the Selectives, and their inevitable error, τὸ πρωτὸν ψεῦδος, lies in their putting forth, in a system which aims at giving, in its visible forms, a diagram of the logical structure of assertions, as a representation, for example, of the assertion that Tully and Cicero are the same man, a type of image which does not differ in form from the assertion that Julius Cæsar and Louis Seize were both men:

(is Tully (is S	(is Julius Cæsar (is a man
(is S (is Cicero	(is a man (is Louis Seize

. . . [The] purpose of the System of Existential Graphs, as it is stated in the Prolegomena [533], [is] to afford a method (1) as *simple* as possible (that is to say, with as small a number of arbitrary conventions as possible), for representing propositions (2) as *iconically*, or diagrammatically and (3) as *analytically* as possible. (The reason for embracing this purpose was developed through the first dozen pages of this paper.) These three essential aims of the system are, every one of them, missed by Selectives. The first, that of the utmost attainable simplicity, is so, since a selective cannot be used without being attached to a Ligature, and Ligatures *without Selectives* will express all that Selectives with Ligatures express. The second aim, to make the representations as iconical as possible, is likewise missed; since Ligatures are far more iconic than Selectives. For the comparison of the above figures shows that a Selective can only serve

562. *Convention the Fifth: Of the Connections of Graph-Instances.* Two partial Graph-Instances are said to be *individually and directly connected*, if, and only if, in the Entire Graph, one individual is, either unconditionally or under some condition, and whether affirmatively or negatively, made a Subject of both. Two Graph-Instances connected by a liga-

its purpose through a special habit of interpretation that is otherwise needless in the system, and that makes the Selective a Symbol and not an Icon; while a Ligature expresses the same thing as a necessary consequence regarding each sizeable dot as an Icon of what we call an "individual object"; and it must be such an Icon if we are to regard an invisible mathematical point as an Icon of the strict individual, absolutely determinate in all respects, which imagination cannot realize. Meantime, the fact that a special convention (a clause of the fourth) is required to distinguish a Selective from an ordinary univalent Spot constitutes a second infraction of the purpose of simplicity. The third item of the idea of the System, that of being as analytical as possible, is infringed by Selectives in no less than three ways. This, at least, is the case if it be true, as I shall endeavour further on to convince the reader that it is, that Concepts are capable of being compounded only in a way differing but in one doubtful particular from that in which the so-called "substances"—i.e., species — of Organic Chemistry are compounded, according to the established theory of that science. (That respect is that the different bonds and *pegs* of the Spots of Graphs are different, while those of chemical atoms are believed to be all alike. But on the one hand, it may possibly be that a more nearly ultimate analysis of Concepts would show, as Kempe's "A Memoir on the theory of Mathematical Form" [*Philosophical Transactions* of the Royal Society, v. 177, pp. 1–70, 1886] seems to think, that the pegs of simple concepts are all alike. On the other hand, the carbon-atom seems to be the only one for the entire similarity of whose bonds there is much positive evidence. In the case of nitrogen, for example, two of the five *valencies* seem to be of such different quality from the others as to suggest that the individual *bonds* may likewise be different; and if there were such difference between different bonds of atoms generally, obvious probable causes would prevent our discovering [them] in the present state of chemistry. Looking at the question from the point of view of Thomson's corpuscles, it seems very unlikely that the looser electrons all fulfill precisely the same function in all cases.) For if this be true, the fact that two or more given concepts can be put together to produce one concept, without either of those that are so put together being separated into parts, is *conclusive proof* that the concept so produced is a compound of those that were put together. The principle, no doubt, requires to be proved. For it might easily be thought that the concept of a scalar as well as that of a vector (in quaternions) can equally result from putting together the concepts of a tensor and a versor in different ways, while at the same time the concept of a tensor and that of a versor can, in their turn, result from putting together those of a scalar and of a vector in different ways; so that no one of the four concepts is more or less composite than any of the others.
Were such a view borne out by exact analysis, as it certainly is not, a radical

ture are explicitly and definitely individually and directly connected. Two Graph-Instances in the same Province are thereby explicitly, although indefinitely, individually and directly connected, since both, or one and the negative of the other, or the negative of both, are asserted to be true or false together, that is, under the same circumstances, although these circumstances are not formally defined, but are left to be interpreted according to the nature of the case. Two Graph-Instances not in the same Province, though on the same Mode of Tincture, are

disparateness between the composition of concepts and that of chemical species would be revealed. But this could scarcely fail to entail such a serious revolution in accepted doctrines of logic as it would be unwarrantable *gratuitously* to suppose that further investigation will bring about. It will be found that the available evidence is decidedly that Concepts can only be combined through definite "pegs." The first respect in which Selectives are not as analytical as they might be, and therefore ought to be, is in representing identity. The identity of the two S's above is only symbolically expressed. . . . Iconically, they appear to be merely coexistent; but by the special convention they are interpreted as identical, though identity is not a matter of interpretation — that is of logical depth — but is an assertion of unity of Object, that is, is an assertion regarding logical breadth. The two S's are instances of one symbol, and that of so peculiar a kind that they are interpreted as *signifying*, and not merely *denoting*, one individual. There is here no analysis of identity. The suggestion, at least, is, quite decidedly, that identity is a simple relation. But the line of identity which may be substituted for the selectives very explicitly represents Identity to belong to the genus Continuity and to the species Linear Continuity. But of what variety of Linear Continuity is the heavy line more especially the Icon in the System of Existential Graphs? In order to ascertain this, let us contrast the Iconicity of the line with that of the surface of the Phemic Sheet. The continuity of this surface being two-dimensional, and so polyadic, should represent an external continuity, and especially, a continuity of experiential appearance. Moreover, the Phemic Sheet iconizes the Universe of Discourse, since it more immediately represents a field of Thought, or Mental Experience, which is itself directed to the Universe of Discourse, and considered as a sign, denotes that Universe. Moreover, it [is because it must be understood] *as* being directed to that Universe, that it is iconized by the Phemic Sheet. So, on the principle that logicians call "the *Nota notae*" that the sign of anything, X, is itself a sign of the very same X, the Phemic Sheet, in representing the field of attention, represents the general object of that attention, the Universe of Discourse. This being the case, the continuity of the Phemic Sheet in those places, where, nothing being scribed, no *particular* attention is paid, is the most appropriate Icon possible of the continuity of the Universe of Discourse — where it only receives *general* attention as that Universe — that is to say of the continuity in experiential appearance of the Universe, relatively to any objects represented as belonging to it.— From "The Bedrock beneath Pragmaticism" (2) 1906; one of a number of fragmentary manuscripts designed to follow the present article.

only in so far connected that both are in the same Universe. Two Graph-Instances in different Modes of Tincture are only in so far connected that both, or one and the negative of the other, or the negative of both, are posited as appertaining to the Truth. They cannot be said to have any individual and direct connection. Two Graph-Instances that are not individually connected within the innermost Cut which contains them both cannot be so connected at all; and every ligature connecting them is meaningless and may be made or broken.

563. Relations which do not imply the occurrence in their several universes of all their correlates must not be expressed by Spots or single Graphs,[1] but all such relations can be expressed in the System.

564. I will now proceed to give a few examples of Existential Graphs in order to illustrate the method of interpretation, and also the *Permissions of Illative Transformation* of them.

If you carefully examine the above conventions, you will find that they are simply the development, and excepting in their insignificant details, the inevitable result of the development of the one convention that if any Graph, A, asserts one state of things to be real and if another graph, B, asserts the same of another state of things, then AB, which results from setting both A and B upon the sheet, shall assert that both states of things are real. This was not the case with my first system of Graphs, described in Vol. VII of *The Monist*,* which I now call *Entitative Graphs*. But I was forced to this principle by a series of considerations which ultimately arrayed themselves into an exact logical deduction of all the features of Existential Graphs which do not involve the Tinctures. I have no room for this here; but I state some of the points arrived at somewhat in the order in which they first presented themselves.

In the first place, the most perfectly analytical system of representing propositions must enable us to separate illative transformations into indecomposable parts. Hence, an illative transformation from any proposition, A, to any other, B, must

[1] It is permissible to have such spots as "possesses the character," "is in the real relation to," but it is not permissible to have such a spot as "can prevent the existence of."

* Vol. 3, No. XVI, §4.

in such a system consist in first transforming A into AB, followed by the transformation of AB into B. For an omission and an insertion appear to be indecomposable transformations and the only indecomposable transformations. That is, if A can be transformed by insertion into AB, and AB by omission in B, the transformation of A into B can be decomposed into an insertion and an omission. Accordingly, since logic has primarily in view argument, and since the conclusiveness of an argument can never be weakened by adding to the premisses nor by subtracting from the conclusion, I thought I ought to take the general form of argument as the basal form of composition of signs in my diagrammatization; and this necessarily took the form of a "scroll," that is (see Figs. 200, 201, 202) a curved line without contrary flexure and returning

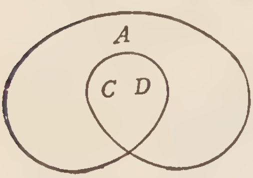


Fig. 200

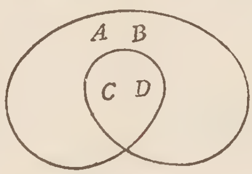


Fig. 201

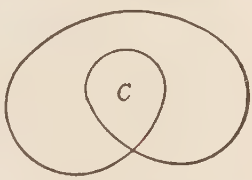


Fig. 202

into itself after once crossing itself, and thus forming an outer and an inner "close." I shall call the outer boundary the *Wall;* and the inner, the *Fence.* In the outer I scribed the Antecedent, in the inner the Consequent, of a Conditional Proposition *de inesse.* The scroll was not taken for this purpose at hap-hazard, but was the result of experiments and reasonings by which I was brought to see that it afforded the most faithful Diagram of such a Proposition. This form once obtained, the logically inevitable development brought me speedily to the System of Existential Graphs. Namely, the idea of the scroll was that Fig. 200, for example, should assert that if A be true (under the actual circumstances), then C and D are both true. This justifies Fig. 201, that if both A and B are true, then both C and D are true, no matter what B may assert, any insertion being permitted in the outer close, and any omission from the inner close. By applying the former clause of this rule to Fig. 202, we see that this scroll with the outer close void, justifies the assertion that if no matter what be true, C is in any case true; so that the two walls of the scroll, when nothing is between them, fall together, collapse, disappear, and leave only the contents of the inner close standing, asserted, in the open field. Supposing, then, that the contents of the inner scroll had been CD, these would have been left standing, both asserted; and we thus return to the principle that writing assertions together on the open sheet asserts them all. Now, Reader, if you will just take pencil and paper and scribe the scroll expressing that if A be true, then it is true that if B be true C and D are true, and compare this with Fig. 201, which amounts to the same thing in meaning, you will see that scroll walls with a void between them collapse even when they belong to different scrolls; and you will

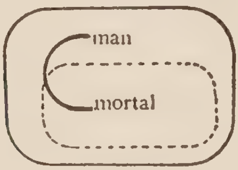


Fig. 203

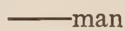


Fig. 204

further see that a scroll is really nothing but one oval within another. Since a Conditional *de inesse* (unlike other conditionals) only asserts that either the antecedent is false or the consequent is true, it all but follows that if the latter alternative be suppressed by scribing nothing but the antecedent, which may be any proposition, in an oval, that antecedent is thereby denied.[1] The use of a heavy line as a juncture signifying identity is inevitable; and since Fig. 203 must mean that if anything is a man, it is mortal, it will follow that Fig. 204 must mean "Something is a man."

[1] I can make this blackened Inner Close as small as I please, at least, so long as I can still see it there, whether with my outer eye or in my mind's eye. Can I not make it quite invisibly small, even to my mind's eye? "No," you will say, "for then it would not be scribed at all." You are right. Yet since confession will be good for my soul, and since it will be well for you to learn how like walking on smooth ice this business of reasoning about logic is — so much so that I have often remarked that nobody commits what is called a "logical fallacy," or hardly ever does so, except logicians; and they are slumping into such stuff continually — it is my duty to [point out] this error of assuming that, because the blackened Inner Close can be made indefinitely small, therefore it can be struck out entirely like an infinitesimal. That led me to say that a Cut around a Graph-

565. The first permission of illative transformation is now evident as follows:

First Permission, called "The Rule of Deletion and Insertion." Any Graph-Instance can be deleted from any recto *Area (including the severing of any Line of Identity), and any Graph-Instance can be inserted on any* verso *Area (including as a Graph-Instance the juncture of any two Lines of Identity or Points of Teridentity).*

566. The justice of the following will be seen instantly by students of any form of Logical Algebra, and with very little difficulty by others:

Second Permission, called "The Rule of Iteration and Deiteration." Any Graph scribed on any Area may be Iterated in or (if already Iterated) may be Deiterated by a deletion from that Area or from any other Area included within that. This involves the Permission to distort a line of Identity, at will.

instance has the effect of denying it. I retract: it only does so if the Cut enclosed also [has] a blot, however small, to represent iconically, the blackened Inner Close. I was partly misled by the fact that in the Conditional *de inesse* the Cut may be considered as denying the contents of its Area. That is true, so long as the entire Scroll is on the Place. But that does not prove that a single Cut, without an Inner Close, has this effect. On the contrary, a single Cut, enclosing only A and a blank, merely says: "If A," or "If A, then" and there stops. If what? you ask. It does not say. "Then *something* follows," perhaps; but there is no assertion at all. This can be proved, too. For if we scribe on the Phemic Sheet the Graph expressing "If A is true, Something is true," we shall have a Scroll with A alone in the Outer Close, and with nothing but a Blank in the Inner Close. Now this Blank is an Iterate of the Blank-instance that is always present on the Phemic Sheet; and this may, according to the rule, be deiterated by removing the Blank in the inner close. This *will* do, what the blot would not; namely, it *will* cause the collapse of the Inner Close, and thus leaves A in a single cut. We thus see that a Graph, A, enclosed in a single Cut that contains nothing else but a Blank has no signification that is not implied in the proposition, "If A is true, Something is true." When I was in the twenties and had not yet come to the full consciousness of my own gigantic powers of logical blundering, with what scorn I used to think of Hegel's confusion of Being with Blank Nothing, simply because it had the form of a predicate without its matter! Yet here am I after devoting a greater number of years to the study of exact logic than the probable number of hours that Hegel ever gave to this subject, repeating that very identical fallacy! Be sure, Reader, that I would have concealed the mistake from you (for vanity's sake, if for no better reason), if it had not been "up to" me, in a way I could not evade, to expose it.— From "Copy T," c. 1906; one of a number of fragmentary manuscripts designed to follow the present article.

To *iterate* a Graph means to scribe it again, while joining by Ligatures every Peg of the new Instance to the corresponding Peg of the Original Instance. To *deiterate* a Graph is to erase a second Instance of it, of which each Peg is joined by a Ligature to a first Instance of it. One Area is said to be *included within* another if, and only if, it either is that Area or else is the Area of a Cut whose Place is an Area which, according to this definition, must be regarded as *included within* that other. By this Permission, Fig. 205 may be transformed into Fig. 206, and thence, by Permission No. 1, into Fig. 207.

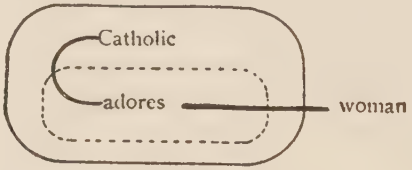


Fig. 205

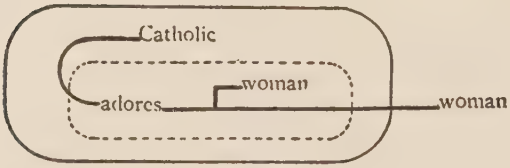


Fig. 206

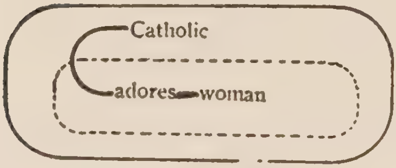


Fig. 207

567. We now come to the Third Permission, which I shall state in a form which is valid, sufficient for its purpose, and convenient in practice, but which cannot be assumed as an undeduced Permission, for the reason that it allows us to regard the Inner close, after the Scroll is removed, as being a part of the Area on which the Scroll lies. Now this is not strictly either an Insertion or a Deletion; and a perfectly analytical System of Permissions should permit only the indecomposable operations of Insertion and Deletion of Graphs that are simple

in expression. The more scientific way would be to substitute for the Second and Third Permissions the following Permission:

If an Area, ϒ, *and an Area,* Ω, *be related in any of these four ways,* viz., (1) *If* ϒ *and* Ω *are the same Area;* (2) *If* Ω *is the Area of an Enclosure whose Place is* ϒ; (3) *if* Ω *is the Area of an Enclosure whose Place is the Area of a second Enclosure whose Place is* ϒ; *or* (4) *if* Ω *is the Place of an Enclosure whose Area is vacant except that it is the Place of an Enclosure whose Area is* ϒ, *and except that it may contain ligatures, identifying Pegs in* Ω *with Pegs in* ϒ; *then, if* Ω *be a* **recto** *area, any simple Graph already scribed upon* ϒ *may be iterated upon* Ω; *while if* Ω *be a* **verso** *Area, any simple Graph already scribed upon* ϒ *and iterated upon* Ω *may be deiterated by being deleted or abolished from* Ω.

These two Rules (of Deletion and Insertion, and of Iteration and Deiteration) are substantially all the undeduced Permissions needed; the others being either Consequences or Explanations of these. Only, in order that this may be true, it is necessary to assume that all indemonstrable implications of the Blank have from the beginning been scribed upon distant parts of the Phemic Sheet, upon any part of which they may, therefore, be iterated at will. I will give no list of these implications, since it could serve no other purpose than that of warning beginners that necessary propositions not included therein were deducible from the other permissions. I will simply notice two principles the neglect of which might lead to difficulties. One of these is that it is physically impossible to delete or otherwise get rid of a Blank in any Area that contains a Blank, whether alone or along with other Graph-Instances. We may, however, assume that there is one Graph, and only one, an Instance of which entirely fills up an Area, without any Blank. The other principle is that, since a Dot merely asserts that some individual object exists, and is thus one of the implications of the Blank, it may be inserted in any Area; and since the Dot will signify the same thing whatever its size, it may be regarded as an Enclosure whose Area is filled with an Instance of that sole Graph that excludes the Blank. The Dot, then, denies that Graph, which may, therefore, be understood as the absurd Graph, and its signification may be formulated as "Whatever you please is true." The absurd Graph may also take the form of an Enclosure with its Area entirely Blank, or enclosing only some Instance of a Graph implied in the Blank. These two principles will enable the Graphist to thread his way through some Transformations which might otherwise appear paradoxical and absurd.

Third Permission, called "The Rule of the Double Cut." Two Cuts one within another, with nothing between them, unless it be Ligatures passing from outside the outer Cut to inside the inner one, may be made or abolished on any Area.

568. Let us now consider the Interpretation of such Ligatures. For that purpose, I first note that the Entire Graph of any *recto* Area is a wholly particular and affirmative Proposition or Copulation of such Propositions. By "wholly particular," I mean, having for every Subject an indesignate individual. The Entire Graph of any *verso* Area is a wholly universal negative proposition or a disjunction of such propositions.

The first time one hears a Proper Name pronounced, it is but a name, predicated, as one usually gathers, of an existent, or at least historically existent, individual object, of which, or of whom, one almost always gathers some additional information. The next time one hears the name, it is by so much the more definite; and almost every time one hears the name, one gains in familiarity with the object. A Selective is a Proper Name met with by the Interpreter for the first time. But it always occurs twice, and usually on different areas. Now the Interpretation, by Convention No. 3, is to be Endoporeutic, so that it is the outermost occurrence of the Name that is the earliest.

569. Let us now analyze the interpretation of a Ligature passing through a Cut. Take, for example, the Graph of Fig. 208. The partial Graph on the Place of the Cut asserts that there exists an individual denoted by the extremity of the line of identity on the Cut, which is a millionaire. Call that individual C. Then, since contiguous dots denote the same individual objects, the extremity of the line of identity on the Area of the cut is also C, and the Partial Graph on that Area, asserts that, let

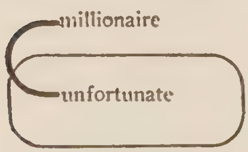


Fig. 208

the Interpreter choose whatever individual he will, that individual is either not C, or else is not unfortunate. Thus, the Entire Graph asserts that there exists a millionaire who is not unfortunate. Furthermore, the Enclosure lying in the same Argent Province as the "millionaire," it is asserted that this individual's being a millionaire *is connected* with his not being unfortunate. This example shows that the Graphist is permitted to extend any Line of Identity on a recto Area so as to carry an end of it to any Cut in that area. Let us next interpret Fig. 209. It obviously asserts that there exists a Turk who is at once the husband of an Individual denoted by a point on the Cut, which individual we may name U, and is the husband of an Individual, whom we may name V, denoted by another point on the Cut. And the Graph on the Area of the cut, declares that whatever Individual the Interpreter may select either is not, and cannot be, U, or is not and cannot be V. Thus, the Entire Graph asserts that there is an existent Turk who is husband of two existent persons; and the "husband," the "Turk" and the enclosure, all being in the same Argent province, although the *Area* of the Enclosure is on color, and thus denies the *possibility* of the identity of U and V, all four predications are true *together*, that is, are true under the same circumstances, which circumstances should be defined by a special convention when anything may turn upon what they are. For the sake of illustrating this, I shall now scribe Fig. 210 all in one province. This may be read, "There is some married woman who will commit suicide in case her husband fails in business." This evidently goes far beyond saying that if every married man fails in business some married woman will commit suicide. Yet note that since the Graph is on Metal it asserts a conditional proposition *de inesse* and only means that there is a married woman whose husband does not fail or else she com-

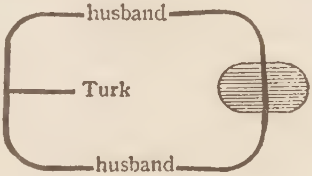


Fig. 209

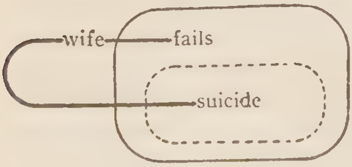


Fig. 210

mits suicide. That, at least, is all it will seem to mean if we fail to take account of the fact, that being all in one Province, it is said that her suicide is *connected* with his failure. Neglecting that, the proposition only denies that every married man fails, while no married woman commits suicide.* The logical principle is that to say that there is some one individual of which one or other of two predicates is true is no more than to say that there either is some individual of which one is true or else there is some individual of which the other is true. Or, to state the matter as an illative permission of the System of Existential Graphs,

Fourth Permission. If the smallest Cut which wholly contains a Ligature connecting two Graphs in different Provinces has its Area on the side of the Leaf opposite to that of the Area of the smallest Cut that contains those two Graphs, then such Ligature may be made or broken at pleasure, as far as these two Graphs are concerned.†

570. Another somewhat curious problem concerning ligatures is to say by what principle it is true, as it evidently is true, that the passage of ligatures from without the outer of two Cuts to within the inner of them will not prevent the two from collapsing in case there is no other Graph-Instance between them. A little study suffices to show that this may depend upon the ligatures' being replaceable by Selectives where they cross the Cuts, and that a Selective is always, at its first occurrence, a new predicate. For it is a principle of Logic that in introducing a new predicate one has a right to assert what one likes concerning it, without any restriction, as long as one implies no assertion concerning anything else. I will leave it to you, Reader, to find out how this principle accounts for the collapse of the two Cuts. Another solution of this problem, not depending on the superfluous device of Selectives, is afforded by the second enunciation of the Rule of Iteration and Deiteration; since this permits the Graph of the Inner Close to be at once iterated on the Phemic Sheet. One may choose between these two methods of solution.

571. The System of Existential Graphs which I have now

* Cf. 580.

† In a letter to F. A. Woods, in 1913, Peirce expressed scepticism as to the universal validity of this permission; see vol. 9. See also 580.

sufficiently described — or, at any rate, have described as well as I know how, leaving the further perfection of it to others — greatly facilitates the solution of problems of Logic, as will be seen in the sequel, not by any mysterious properties, but simply by substituting for the symbols in which such problems present themselves, concrete visual figures concerning which we have merely to say whether or not they admit

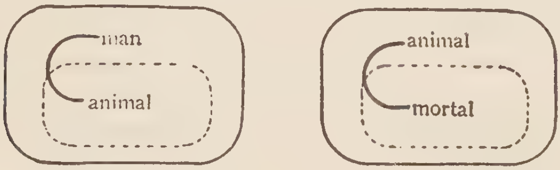


Fig. 211

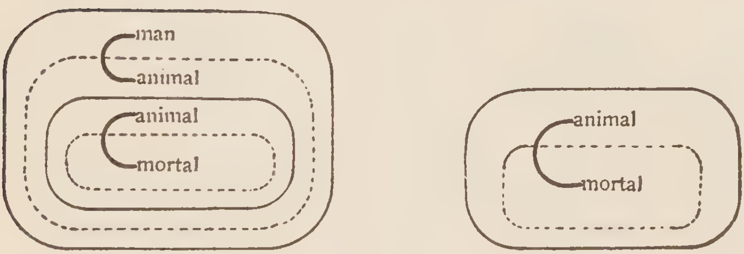


Fig. 212

certain describable relations of their parts. Diagrammatic reasoning is the only really fertile reasoning. If logicians would only embrace this method, we should no longer see attempts to base their science on the fragile foundations of metaphysics or a psychology not based on logical theory; and there would soon be such an advance in logic that every science would feel the benefit of it.

This System may, of course, be applied to the analysis of reasonings. Thus, to separate the syllogistic illation, "Any man would be an animal, and any animal would be mortal; therefore, any man would be mortal," the Premisses are first scribed as in Fig. 211. Then by the rule of Iteration, a first illative transformation gives Fig. 212. Next, by the permission to erase from a recto Area, a second step gives Fig. 213.

Then, by the permission to deform a line of Identity on a recto Area, a third step gives Fig. 214. Next, by the permission to insert in a verso Area, a fourth step gives Fig. 215. Next, by Deiteration, a fifth step gives Fig. 216. Next, by

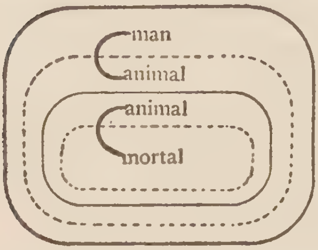


Fig. 213

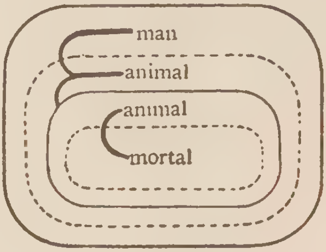


Fig. 214

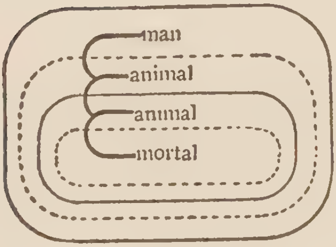


Fig. 215

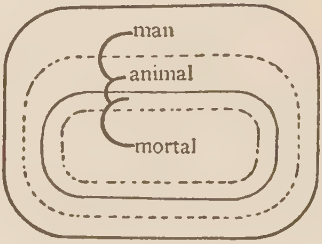


Fig. 216

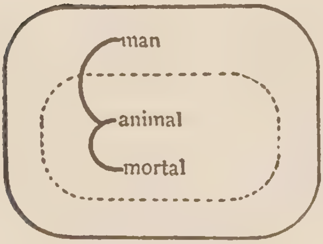


Fig. 217

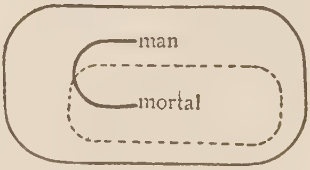


Fig. 218

the collapse of two Cuts, a sixth step gives Fig. 217; and finally, by omission from a recto Area, a seventh step gives the conclusion Fig. 218. The analysis might have been carried a little further, by means of the Rule of Iteration and Deiteration, so as to increase the number of distinct inferential steps to nine, showing how complex a process the drawing of a syllogistic conclusion really is. On the other hand, it need scarcely be said that there are a number of *deduced* liberties of transformation, by which even much more complicated inferences than a syllogism can be performed at a stroke. For that sort of problem, however, which consists in drawing a conclusion or assuring oneself of its correctness, this System is not particularly adapted. Its true utility is in the assistance it renders — the support to the mind, by furnishing concrete diagrams upon which to experiment — in the solution of the most difficult problems of logical theory.

572. I mentioned on an early page of this paper that this System leads to a different conception of the Proposition and Argument from the traditional view that a Proposition is composed of Names, and that an Argument is composed of Propositions. It is a matter of insignificant detail whether the term Argument be taken in the sense of the Middle Term, in that of the Copulate of Premisses, in that of the setting forth of Premisses and Conclusion, or in that of the representation that the real facts which the premisses assert (together, it may be, with the mode in which those facts have come to light) logically signify the truth of the Conclusion. In any case, when an Argument is brought before us, there is brought to our notice (what appears so clearly in the Illative Transformations of Graphs) a process whereby the Premisses bring forth the Conclusion, not informing the Interpreter of its Truth, but appealing to him to assent thereto. This Process of Transformation, which is evidently the kernel of the matter, is no more built out of Propositions than a motion is built out of positions. The logical relation of the Conclusion to the Premisses might be asserted; but that would not be an Argument, which is essentially intended to be understood as representing what it represents only in virtue of the logical habit which would bring any logical Interpreter to assent to it. We may express this by saying that the Final (or quasi-intended) Interpretant of an Argument represents it as representing its Object after the manner of a Symbol. In an analogous way the relation of Predicate to Subject which is *stated* in a Proposition might be merely described in a Term. But

the essence of the Proposition is that it intends, as it were, to be regarded as in an existential relation to its Object, as an Index is, so that its assertion shall be regarded as evidence of the fact. It appears to me that an assertion and a command do not differ essentially in the nature of their Final Interpretants as in their Immediate, and so far as they are effective, in their Dynamical Interpretants; but that is of secondary interest. The Name, or any Seme, is merely a substitute for its Object in one or another capacity in which respect it is all one with the Object. Its Final Interpretant thus represents it as representing its Object after the manner of an Icon, by mere agreement in idea. It thus appears that the difference between the Term, the Proposition, and the Argument, is by no means a difference of complexity, and does not so much consist in structure as in the services they are severally intended to perform.

For that reason, the ways in which Terms and Arguments can be compounded cannot differ greatly from the ways in which Propositions can be compounded. A mystery, or paradox, has always overhung the question of the Composition of Concepts. Namely, if two concepts, A and B, are to be compounded, their composition would seem to be necessarily a third ingredient, Concept C, and the same difficulty will arise as to the Composition of A and C. But the Method of Existential Graphs solves this riddle instantly by showing that, as far as propositions go, and it must evidently be the same with Terms and Arguments, there is but one general way in which their Composition can possibly take place; namely, each component must be indeterminate in some respect or another; and in their composition each determines the other. On the *recto* this is obvious: "Some man is rich" is composed of "Something is a man" and "something is rich," and the two somethings merely explain each other's vagueness in a measure. Two simultaneous independent assertions are still connected in the same manner; for each is in itself vague as to the Universe or the "Province" in which its truth lies, and the two somewhat define each other in this respect. The composition of a Conditional Proposition is to be explained in the same way. The Antecedent is a Sign which is Indefinite as to its Interpretant; the Consequent is a Sign which is Indefinite as to its Object. They supply each the other's lack. Of course, the explanation of the structure of the Conditional gives the explanation of negation; for the negative is simply that from whose Truth it would be true to say that anything you please would follow *de inesse.*

In my next paper, the utility of this diagrammatization of thought in the discussion of the truth of Pragmaticism shall be made to appear.*

* See 540n.

CHAPTER 7

*AN IMPROVEMENT ON THE GAMMA GRAPHS**[E]

573. In working with Existential Graphs, we use, or at any rate imagine that we use, a sheet of paper of different tints on its two sides. Let us say that the side we call the *recto* is cream white while the *verso* is usually of somewhat bluish grey, but may be of yellow or of a rose tint or of green. The *recto* is appropriated to the representation of existential, or actual, facts, or what we choose to make believe are such. The *verso* is appropriated to the representation of possibilities of different kinds according to its tint, but usually to that of subjective possibilities, or subjectively possible truths. The special kind of possibility here called subjective is that which consists in ignorance. If we do not know that there are not inhabitants of Mars, it is subjectively possible that there are such beings. . . .

574. The *verso* is usually appropriated to imparting information about *subjective possibilities* or what may be true for aught we know. To scribe a graph is to impart an item of information; and this item of information does one of two things. It either adds to what we know to exist or it cuts off something from our list of subjective possibilities. Hence, it must be that a graph scribed on the *verso* is thereby denied.

575. Now the denial of a subjective possibility usually, if not always, involves the assertion of a truth of existence; and consequently what is put upon the *verso* must usually have a definite connection with a place on the *recto*.

576. In my former exposition of Existential Graphs, I said that there must be a department of the System which I called the Gamma part into which I was as yet able to gain mere glimpses, sufficient only to show me its reality, and to rouse my intense curiosity, without giving me any real insight into it. The conception of the System which I have just set forth

* From "For the National Academy of Science, 1906 April Meeting in Washington."

is a very recent discovery. I have not had time as yet to trace out all its consequences. But it is already plain that, in at least three places, it lifts the veil from the Gamma part of the system.

577. The new discovery which sheds such a light is simply that, as the main part of the sheet represents existence or actuality, so the area within a cut, that is, the *verso* of the sheet, represents a kind of possibility.

578. From thence I immediately infer several things that I did not understand before, as follows:

First, the cut may be imagined to extend down to one or another depth into the paper, so that the overturning of the piece cut out may expose one stratum or another, these being distinguished by their tints; the different tints representing different kinds of possibility.

This improvement gives substantially, as far as I can see, nearly the whole of that Gamma part which I have been endeavoring to discern.

579. Second, In a certain partly printed but unpublished "Syllabus of Logic," which contains the only formal or full description of Existential Graphs that I have ever undertaken to give, I laid it down, as a rule, that no graph could be partly in one area and partly in another;* and this I said simply because I could attach no interpretation to a graph which should cross a cut. As soon, however, as I discovered that the *verso* of the sheet represents a universe of possibility, I saw clearly that such a graph was not only interpretable, but that it fills the great lacuna in all my previous developments of the logic of relatives. For although I have always recognized that a possibility may be *real*, that it is sheer insanity to deny the reality of the possibility of my raising my arm, even if, when the time comes, I do *not* raise it; and although, in all my attempts to classify relations, I have invariably recognized, as one great class of relations, the class of *references*, as I have called them, where one correlate is an existent, and another is a mere possibility; yet whenever I have undertaken to develop the logic of relations, I have always left these references out of account, notwithstanding their manifest importance, simply because the algebras or other forms of

* 414 (6).

diagrammatization which I employed did not seem to afford me any means of representing them.* I need hardly say that the moment I discovered in the *verso* of the sheet of Existential Graphs a representation of a universe of possibility, I perceived that a *reference* would be represented by a graph which should cross a cut, thus subduing a vast field of thought to the governance and control of exact logic.

580. Third, My previous account of Existential Graphs

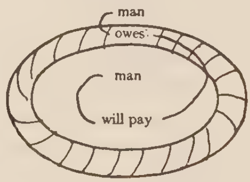


Fig. 219†

was marred by a certain rule which, from the point of view from which I thought the system ought to be regarded, seemed quite out of place and inacceptable, and yet which I found myself unable to dispute.‡ I will just illustrate this matter by an example. Suppose we wish to assert that there is a man every dollar of whose indebtedness will be paid by some man

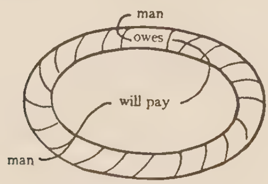


Fig. 220

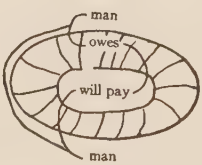


Fig. 221

or other, perhaps one dollar being paid by one man and another by another man, or perhaps all paid by the same man. We do not wish to say how that will be. Here will be our graph, Fig. 219. But if we wish to assert that one man will pay the whole, without saying in what relation the payer stands to the debtor, here will be our graph, Fig. 220. Now

* See e.g. 3.572.

† The shaded portions represent the verso.

‡ See 569.

suppose we wish to add that this man who will pay all those debts is the very same man who owes them. Then we insert two graphs of teridentity and a line of identity as in Fig. 221. The difference between the graph with and without this added line is obvious, and is perfectly represented in all my systems. But here it will be observed that the graph "*owes*" and the graph "*pays*" are *not only* united on the *left* by a line *outside* the smallest area that contains them both, but likewise on the *right*, by a line *inside* that smallest common area. Now let us consider a case in which this *inner* connection is lacking. Let us assert that there is a man A and a man B, who may or may not be the same man, and if A becomes bankrupt then B will suicide. Then, if we add that A and B *are* the same man, by drawing a line outside the smallest common area of the

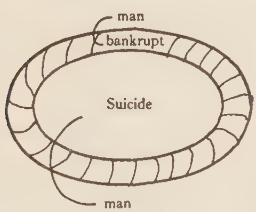


Fig. 222

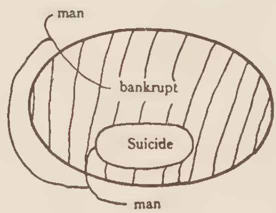


Fig. 223

graphs joined, which are here bankrupt and suicide, the strange rule to which I refer is that such outer line, because there is no connecting line within the smallest common area, is null and void, that is, it does not affect the interpretation in the least. . . . The proposition that there is a man who if *he* goes bankrupt will commit suicide is false only in case, taking any man you please, he *will* go bankrupt, and *will not* suicide. That is, it is falsified only if every man goes bankrupt without suiciding. But this is the same as the state of things under which the other proposition is false; namely, that every man goes broke while no man suicides. This reasoning is irrefragable as long as a mere possibility is treated as an absolute nullity. Some years ago,* however, when in consequence of an invitation to deliver a course of lectures in Harvard Uni-

* 1903, see vol. 5, bk. I.

versity upon Pragmatism, I was led to revise that doctrine, in which I had already found difficulties, I soon discovered, upon a critical analysis, that it was absolutely necessary to insist upon and bring to the front, the truth that a mere possibility may be quite real. That admitted, it can no longer be granted that every conditional proposition whose antecedent does not happen to be realized is true, and the whole reasoning just given breaks down.

581. I often think that we logicians are the most obtuse of men, and the most devoid of common sense. As soon as I saw that this strange rule, so foreign to the general idea of the System of Existential Graphs, could by no means be deduced from the other rules nor from the general idea of the system, but has to be accepted, if at all, as an arbitrary first principle — I ought to have asked myself, and should have asked myself if I had not been afflicted with the logician's *bêtise*, What compels the adoption of this rule? The answer to that must have been that the *interpretation* requires it; and the inference of common sense from that answer would have been that the interpretation was too narrow. Yet I did not think of that until my operose method like that of a hydrographic surveyor sounding out a harbour, suddenly brought me up to the important truth that the *verso* of the sheet of Existential Graphs represents a universe of possibilities. This, taken in connection with other premisses, led me back to the same conclusion to which my studies of Pragmatism had already brought me, the reality of some possibilities. This is a striking proof of the superiority of the System of Existential Graphs to either of my algebras of logic.* For in both of them the incongruity of this strange rule is completely hidden behind the superfluous machinery which is introduced in order to give an appearance of symmetry to logical law, and in order to facilitate the working of these algebras considered *as reasoning machines*. I cannot let this remark pass without protesting, however, that in the construction of no algebra was the idea of making a calculus which would turn out conclusions by a regular routine other than a very secondary purpose. . . .†

582. The sheet of the graphs in all its states collectively,

* See e.g. 3.332ff, 3.492ff; and 3.351ff, 3.499f.

† See 3.485, 3.618.

together with the laws of its transformations, corresponds to and represents the *Mind* in its relation to its thoughts, considered as signs. That thoughts *are* signs has been more especially urged by nominalistic logicians; but the realists are, for the most part, content to let the proposition stand unchallenged, even when they have not decidedly affirmed its truth. The scribed graphs are *determinations* of the sheet, just as thoughts are *determinations* of the mind; and the mind itself is a comprehensive thought just as the sheet considered in all its actual transformation-states and transformations, taken collectively, is a graph-instance and taken in all its permissible transformations is a graph. Thus the system of existential graphs is a rough and generalized diagram of the Mind, and it gives a better idea of what the mind is, from the point of view of logic, than could be conveyed by any abstract account of it.

583. The System of Existential Graphs recognizes but one mode of combination of ideas, that by which two indefinite propositions define, or rather partially define, each other on the *recto* and by which two general propositions mutually limit each other upon the *verso;* or, in a unitary formula, by which two indeterminate propositions mutually determine each other in a measure. I say in a measure, for it is impossible that any sign whether mental or external should be perfectly determinate. If it were possible such sign must remain absolutely unconnected with any other. It would quite obviously be such a sign of its entire universe, as Leibniz and others have described the omniscience of God to be, an intuitive representation amounting to an indecomposable feeling of the whole in all its details, from which those details would not be separable. For no reasoning, and consequently no abstraction, could connect itself with such a sign. This consideration, which is obviously correct, is a strong argument to show that what the system of existential graphs represents to be true of propositions and which must be true of *them*, since every proposition can be analytically expressed in existential graphs, equally holds good of concepts that are *not* propositional; and this argument is supported by the evident truth that no sign of a thing or kind of thing — the ideas of signs to which concepts belong — can arise except in a proposition; and no logical operation upon a proposition can result in anything but a

proposition; so that non-propositional signs can only exist as constituents of propositions. But it is not true, as ordinarily represented, that a proposition can be built up of non-propositional signs. The truth is that concepts are nothing but indefinite problematic judgments. The concept of *man* necessarily involves the thought of the possible being of a man; and thus it is precisely the judgment, "There may be a man." Since no perfectly determinate proposition is possible, there is one more reform that needs to be made in the system of existential graphs. Namely, the line of identity must be totally abolished, or *rather* must be understood quite differently. We must hereafter understand it to be *potentially* the graph of *teridentity* by which means there always will virtually be at least one loose end in every graph. In fact, it will not be truly a graph of *teridentity* but a graph of indefinitely multiple identity.

584. We here reach a point at which novel considerations about the constitution of knowledge and therefore of the constitution of nature burst in upon the mind with cataclysmal multitude and resistlessness. It is that synthesis of tychism and of pragmatism for which I long ago proposed the name, Synechism,* to which one thus returns; but this time with stronger reasons than ever before. But I cannot, consistently with my own convictions, ask the Academy to listen to a discourse upon Metaphysics.

* See 6.102ff, 6.169ff.

Book III

THE AMAZING MAZES

CHAPTER 1

*THE FIRST CURIOSITY**[E]

"Mazes intricate.
Eccentric, interwov'd, yet regular
Then most, when most irregular they seem."

Milton's Description of the Mystical Angelic Dance.

§1. STATEMENT OF THE FIRST CURIOSITY[E]

585. About 1860 I cooked up a *mélange* of effects of most of the elementary principles of cyclic arithmetic; and ever since, at the end of some evening's card-play, I have occasionally exhibited it in the form of a "trick" (though there is really no trick about the phenomenon) with the uniform result of interesting and surprising all the company, albeit their mathematical powers have ranged from a bare sufficiency for an altruistic tolerance of cards up to those of some of the mightiest mathematicians of the age, who assuredly with a little reflection could have unraveled the marvel.

586. The following shall describe what I do; but you, Reader, must do it too, if you are to appreciate the curiosity of the effect. So be good enough as to take two packets of playing-cards, the one consisting of a complete red suit and the other of a black suit without the king, the cards of each being arranged in regular order in the packet, so that the face-value of every card is equal to its ordinal number in the packet.

N.B. *Throughout all my descriptions of manipulations of cards, it is to be understood, once for all, that the observance of the following* STANDING RULES *is taken for granted in all cases where the contrary is not expressly directed:* Firstly, *that a pack or packet of cards* held in the hand *is, unless otherwise directed, to be held with backs up* (*though not, of course, while they are in process of arrangement or rearrangement*), *while a pile of cards* FORMED *on the table* (*in contra-distinction to a pile placed, ready formed, on the table, as well as to rows of single cards spread upon*

* *The Monist*, pp. 227–241, vol. 18, April, 1908. The original title was "Some Amazing Mazes."

the table) is always to be formed with the faces displayed, and left so until they are gathered up. Secondly, *that, whether a packet in the hand or a pile on the table be referred to, by the "ordinal, or serial, number" of a single card or of a larger division of the whole is meant its number, counting in the order of succession in the packet or pile, from the card or other part at the* BACK *of the packet or at the* BOTTOM *of the pile as* "Number 1," *to the card or other part at the* FACE *of the packet or the* TOP *of the pile; the ordinal or serial number of this last being equal to the cardinal number of cards (or larger divisions* COUNTED) *in the whole packet or pile; and the few exceptions to this rule will be noted as they occur;* Thirdly, *that by the "face-value" is meant the number of pips on a plain card, the ace counting as one; while, of the picture-cards, the knave, for which* J *will usually be written, will count as eleven, the queen, or* Q, *as twelve, and the king,* K, *either as thirteen or as the zero of the next suit;* and Fourthly, *that when a number of piles that have been formed upon the table by dealing out the cards, are to be gathered up, the uniform manner of doing so is to be as follows: The first pile to be taken (which pile this is to be will appear in due time) is to be grasped as a whole and placed* (faces up) *upon the pile that is to be taken next. Then those two piles are to be grasped as a whole, and placed* (faces up) *upon the pile that is next to be taken; and so on, until all the piles have been gathered up; when, in accordance with the first Standing Rule, the whole packet is to be turned back up.* And note, by the way, that in consequence of the manner in which the piles are gathered, each, after the first, being placed at the back of those already taken, while in observance of the second Standing Rule, we always count places in a packet from the back of it, it follows that the last pile taken will be the first in the regathered packet, while the first taken will become the last, and all the others in the same complementary way, the ordinal numbers of their gathering and those of their places in the regathered packet adding up to one more than the total number of piles.

587. Of course, while the red packet and the black packet are getting arranged so that the face-value of each card shall also be its ordinal, or serial, number in the packet, the cards must needs be held faces up. But as soon as they have been arranged, the packet of thirteen cards is to be laid on the

table, *back up*. You then deal — for, let me repeat it, Reader, by the inexorable laws of psychology, if you do not actually take cards (and the United States Playing-Card Company's "Fauntleroy" playing-cards are the most suitable, although any that run smoothly will do), and actually go through the processes, the whole description can mean nothing to you; — *you* deal, then, the twelve black cards, one by one, into two piles, the first card being turned to form the bottom of the first pile, the second that of the second pile (on the right hand of the first pile), the third card going on the first pile again, the fourth on the second, and every following card being placed immediately upon the card whose ordinal, or serial, number in the packet before the deal was two lower than the former's ordinal, or serial, number then was. *The last card, however, is to be exceptionally treated.* Instead of being placed on the top of the second pile according to the rule just given, it is to be placed on the table, face up, and apart from the other cards, to make the bottom card of an isolated pile, to be called the "*discard pile*"; while, in place of it, the first card of the pile of cards of the red suit, which card will, of course, be the ace, is to be placed face up on the top of the second of the two piles formed by the dealing, where that discarded card would naturally have gone. Now you gather up these two piles by grasping the first, or left-hand pile, placing it, face up, upon the second, or right-hand, pile, and taking up the two together; and you then at once turn the packet back up in compliance with the first standing rule. This whole operation of *firstly*, dealing out into two piles the packet that was at first entirely composed of black cards; but *secondly*, placing the last card, face up, on the discard pile, and *thirdly*, substituting for it the card then at the top of the pile of red cards, by placing this latter, face up, upon the top of the second pile of the deal, and then, *fourthly*, putting the left-hand, or first, pile of the deal, face up, upon the second, and having taken up the whole packet, turning it with its back up — this whole quadripartite operation, I say, is to be performed, in all, twelve times in succession. My statement that in this operation the last card is treated *exceptionally* was quite correct, since its treatment made an exception to the rule of placing each card on the card that before the deal came two places in advance of it in the packet. Had I said it was treated *irregularly*, I should have written very carelessly, since it is just one of those cases in which a violation of a regularity of a low order establishes a regularity of a much higher order (if John Milton knew the meaning of the word "regular") — a pronouncement which must be left for the issue of the performance to ratify; and you shall see, Reader, that the event will ratify it with striking emphasis. Already, we begin to see some regularity in the process, since each of the twelve cards placed on the discard-pile in the twelve performances of the quadripartite operation is seen to belong to the black suit; so that the packet held in the hand and dealt out, from being originally entirely black, has now become entirely red. Having placed the red king upon the face of this packet, you now lay down the latter in order to have your hands free to manipulate the discard-pile. Holding this discard pile as the first standing rule directs, you take the cards singly from the top and range them, one by one, from left to right, in a row upon the table, with their backs up. The length of the table from left to right ought to be double that of the row; and this is one of the reasons for preferring cards of a small size. To guard against any mistake, you may take a peek at the seventh card, to make sure that it is the ace, as it should be. The row being formed, I remark to the company, as you should do in substance, that I reserve the right to move as many of these black cards as I please, at any and all times, from one end of the row to the other; but that beyond doing that, I renounce all right to disarrange those cards. Then, taking up the red cards, and holding the packet with its back up, I (and so must you) request any person to cut it. When he does so, you place the cards he leaves in your hand at the back of the partial packet he removes. This is my proceeding, and must be yours. You then ask some person to say into how many piles (less than thirteen) the red cards shall be dealt. When he has prescribed the number of piles, you are to hold the packet of red cards back up, and deal cards one by one from the back of it, placing each card on the table face up, and each to the right of the last card dealt. When you have dealt out enough to form the bottom cards of piles to the number commanded, you return to the extreme left-hand pile, *which you are to imagine as lying next to, and to the right of, the extreme right-hand pile* — as in fact it would come next in clockwise order, if the row were bent down at the ends in the manner shown in Fig. 224, where the piles (here supposed to be eight in all) are numbered in the order in which their bottom cards are laid down. Indeed, when more than seven piles are ordered, it is not a bad plan actually to arrange them so. So, counting the piles round and round, whether you place them in a circle or not, you place each card on the pile that comes clockwise next after, or to the right of the pile upon which the card next before it was placed (regulating your imagination as above stated), and so you continue until you have dealt out the whole packet of thirteen cards. You now proceed to gather up the piles according to the Fourth Standing Rule.

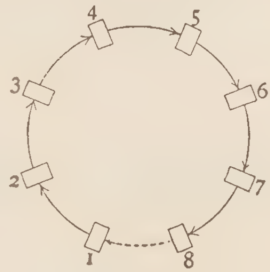


Fig. 224

588. That rule, however, does not determine the order of succession in which the piles are to be taken up. I will now give the rule for this. It applies to the dealing of any prime number of cards, or of any number of cards one less than a prime number, into any number of piles less than that prime number. It happens that that form of statement of this rule which is decidedly the most convenient when the number of piles does not exceed seven, as well as when the whole number of cards differs by less than three from some multiple of the number of piles, becomes quite confusing in other cases. A slight modification of it which I will give as a second form of the rule, sometimes greatly mitigates the inconvenience; and it will be well to acquaint yourself with it. But for the most part, when the first form threatens to be confusing, it will be best to resort to that form of the rule which I describe as the third.

For the purpose of this "first curiosity" (indeed, in every case where a prime number of real cards are dealt out), it matters not what pile you take up first. But in certain cases we shall have occasion to deal out into piles a number of cards, such as 52, which is one less than a prime number. In such case, it will be necessary to add *an imaginary card* to the pack, since a real card would interfere with certain operations. Now imaginary cards, if allowed to get mixed in with real ones, are liable to get lost. Consequently, in such cases, we have to keep the imaginary card constantly at the face of the pack by taking up first the pile on which it is imagined to fall, that is, the pile next to the right of the one on which the last real card falls. I now proceed to state, in its three forms, the rule for determining what pile is to be taken up next after any given pile that has just been taken. It is assumed that the whole pack of cards dealt consists of a prime number of cards; but, of these cards, the last may be an imaginary one, provided the pile on which it is imagined regularly to fall be taken up first.

First Form of the Rule. Count from the place of the extreme right-hand pile, as zero, either way round, clockwise or counter-clockwise — preferably in the shortest way — to the place of the pile on which the last card, real or imaginary, fell. Then, counting the original places of piles, whether the piles themselves still remain in those places or have already been picked up, from the place of the pile last taken, in the same direction, up to the same number, you will reach the place of the next pile to be taken.

Fig. 225

Example. If 13 cards are dealt into five piles, the thirteenth card will fall on the second pile from the extreme right-hand pile going round counter-clockwise. Supposing, then, that the first pile taken is the right-handmost but one, they are all to be taken in the order marked in Fig. 225.

Second Form. Proceed as in the first form of the rule until you have repassed the place of the first pile taken. You will then always find that the place of the last pile taken is nearer to that of some pile, P, previously taken, than it is to the place of that taken immediately before it. Then, the next pile to be taken will be in the same relation of places to the pile taken next after the pile P.

Example. Let 13 cards be dealt into 9 piles. Then the last card will fall on the pile removed 4 places clockwise from the extreme right-hand pile. Then, when you have removed four piles according to the first form of the rule, you will at once perceive, as shown in Fig. 226 (where it is assumed that

Fig. 226

the extreme left-hand pile was the one to be taken up first), that for the rest of the regathering, you have simply to take the pile that stands immediately to the left of the place of the last previous removal but one.

Third Form. In this form of the rule vacant places are not counted, but only the remaining piles, which is sometimes much less confusing. It is requisite, however, carefully to note the place of the pile first taken. You begin as in the first form of the rule; but every time you pass over the place whence the first pile was removed, you diminish the number of your count by one, beginning with the count then in progress; and you adhere to this number until you pass the same place again, and consequently again diminish the number of your count, which will thus ultimately be reduced to one, when you will take every pile you come to.

Example. Let a pack of 52 cards be dealt into 22 piles. The first pile taken up must be the one upon which the imaginary fifty-third card falls. It is assumed that, before the deal the cards were arranged in suits in the order ♢ ♠ ♡ ♣ and in each suit in the order of their face-values. Then the different columns of Fig. 227 show the cards at the tops of the different piles while the different horizontal rows show what piles remain, just before you come to count the left-handmost of the remaining piles, as your countings successively pass through the whole row of piles. The gap between the columns just after the place where the imaginary card is supposed to have fallen, contains the direction thereafter to diminish by one the number of piles you count. Beneath the designations of the top cards are small type numbers which are the numbers in your different countings through the row of piles; and the last number in each count is followed by a note of admiration that is to be understood as a command to gather up that pile. Beneath it is a heavy faced number, which is the ordinal number of that removal.

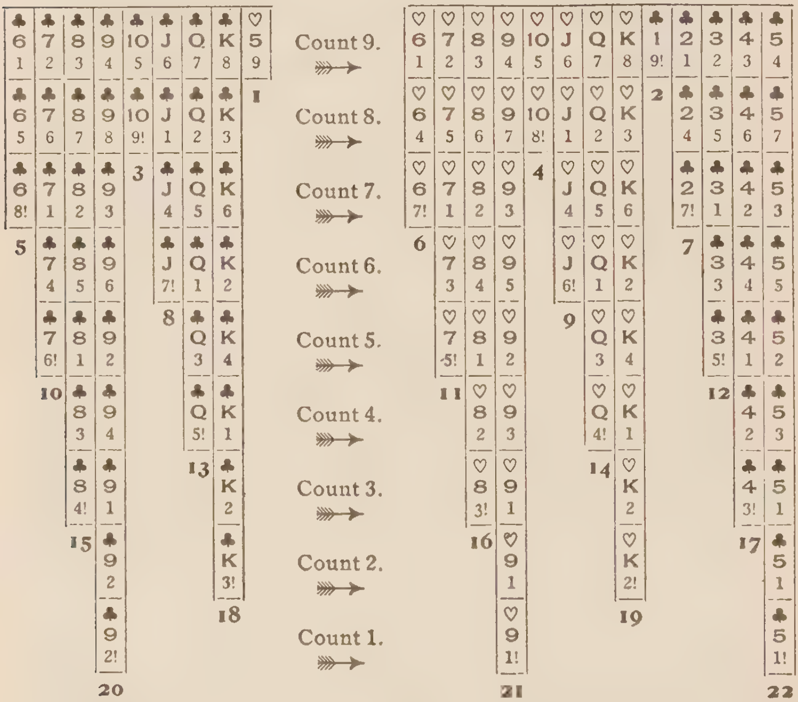


Fig. 227

589. I hate to bore readers who are capable of exact thought with redundancies; but others often deploy such brilliant talents in not understanding the plainest statements that have no familiar jingle, that I must beg my more active-minded readers to have patience under the infliction while I exhibit in Fig. 228 the orders in which 5, 8, 9, 10, and 11 piles formed by dealing 13 cards are to be taken up.

590. When the red cards have thus been regathered, you again hold out the packet to somebody to cut, and again request somebody to say into how many piles they shall be dealt

"in order that the mixing may be as thorough as it may." You follow his directions, and regather the piles according to the same rule as before. If your company is not too intelligent, you might venture to ask somebody, before you regather the piles, to say what pile you shall take up first; but this will be presuming a good deal upon the stupidity of the company; for an inference might be drawn which would go far toward destroying the surprise of the result. Nothing absolutely prevents the cards from being cut and dealt any number of times.

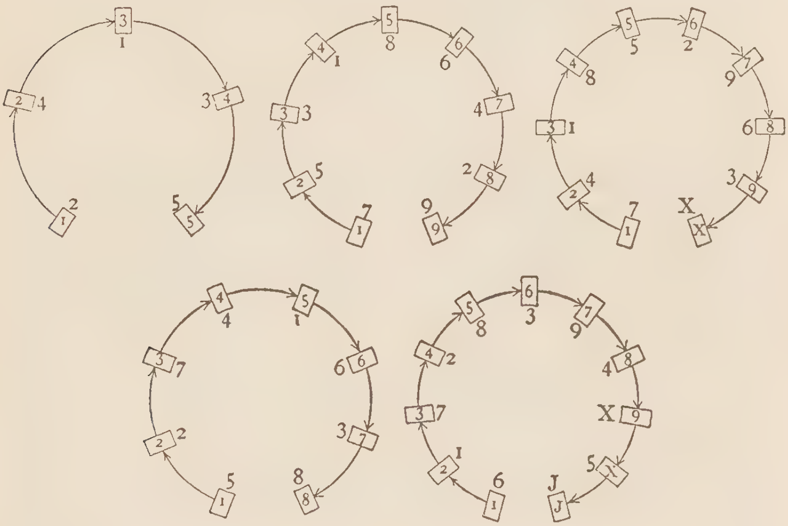


Fig. 228

591. When the number of piles for the last dealing has been given out, you will have to ascertain what transposition of the black cards is required. There are three alternative ways of doing this, which I proceed to describe. The best way is to multiply together the numbers of piles of the different dealings of the red cards, subtracting from each product the highest multiple of 13, if there be any, that is less than that product. The result is the cyclical product. By "the different dealings," you here naturally understand those that have taken place since the last shifting of the black row. If a wrong shift has been made, the simplest way to correct it, after new cuttings and dealings, is to resort to a peep at the black ace,

and to determining where it ought to be in the third way explained below.

Thus, if the red cards have been dealt into 5 piles and into 3 piles, since 3 times 5 make 15, and $15-13=2$, the cyclical product is 2. You now proceed to ascertain how many times 1 has to be cyclically doubled to make that cyclical product. But if 6 doublings do not give it — which six doublings will give

1 doubling, twice 1 are 2,
2 doublings, twice 2 are 4,
3 doublings, twice 4 are 8,
4 doublings, twice 8 less 13 make 3,
5 doublings, twice 3 are 6,
6 doublings, twice 6 are Q, —

I say if none of the first six doublings gives the cyclical product of the numbers of piles in the dealings, you resort to successive cyclical halvings of 1. The cyclical half of an even number is the simple half; but to get the cyclical half of an odd number, add 7 to half of one less than that number. Thus,

The cyclical half of 1 is $(0\div 2)+7=7$;
The cyclical half of 7 is $(6\div 2)+7=X$;
The cyclical half of X is 5;
The cyclical half of 5 is $(4\div 2)+7=9$;
The cyclical half of 9 is $(8\div 2)+7=J$;
The cyclical half of J is $(X\div 2)+7=Q$.

If the cyclical product of the numbers of piles in the dealings is one of the first six results of doubling one, you will have (when the time comes) to bring one card from the right-hand end of the row of black cards to the left-hand end for each such doubling. Thus, if the red cards have twice been deal into 4 piles, four cards must be brought from the right end to the left end of the row of black cards. For $4\times 4-13=3$ and $1\times 2^4-13=3$. But if that cyclical product is one of the first six results of successive cyclical halvings of one, one card must be carried from the left to the right end of the row of black cards for every halving. Thus, if the red cards have been dealt into 6 and into 8 piles, 4 black cards must be carried from the left-hand end of the row to the right-hand end of the row. $6\times8-3\times13=9$ and it takes 4 cyclical halvings to give 9. If the product of the numbers of piles in the dealings is one more than a multiple of 13, the row of black cards is to remain unshifted.

The second way of determining how the black cards are to be transposed is simply, during the last of the dealings, to note what card is laid upon the king. The face-value of this card is the ordinal, or serial place in the row, counting from the left-hand extremity of it, which the ace must be brought to occupy. Now if you remember, as you always ought to do, where the ace is in the row, you will know how many cards to carry from one end to the other so as to bring the ace into that place. But if in the last dealing the king happens to fall at the top of one of the piles, two lines of conduct are open to you. One would be, in regathering the piles, by a pretended awkwardness in taking up the pile that is to be taken next before the one that the king heads, at first to leave its bottom card on the table, so as to get a glimpse of it before you take it up, as you would regularly have done at first; and if the king should happen to be the last card dealt, the card at the back of the packet would be the one for you to get sight of, by a similar imitation blunder. In either case, the card you so aim to get sight of would show the right place for the ace in the row. But if you doubt your ability to be gracefully awkward, it always remains open to you to ask to have the red packet cut again and a number of piles for a new deal to be ordered.

The third way of determining the proper transposition of the black cards is a slight modification of the second. It consists in looking at the card whose back is against the face of the king, when you come to cut the red packet so as to bring the king to the face. (Any practical psychologist, such as a prestidigitator must be, can, with the utmost ease, look for the card he wants to see, and can inspect it without detection.)

But whichever of these methods you employ, you should not touch the row of black cards until the red cards having been regathered after the last dealing, you have said something like this: "Now I think that all these dealings and cuttings and exchanges of the last cards have sufficiently mixed up the red cards to give a certain interest to the fact that I am going to show you; namely, that this row of black cards forms an index showing where any red card you would like to see is to be found in the red pack. But since there is no black king in the row, of course the place of the red king cannot be indicated; and for that reason, I shall just cut the pack of red cards so as to bring the king to the face of it, and so render any searching for that card needless." You then cut the red cards. That speech is quite important as restraining the minds of the company from reflecting upon the relation between the effect of your cutting and that of theirs. Without much pause you go on to say that you shall leave the row of black cards just as they are, simply putting so many of them from one end of the row to the other. You now ask some one, "Now, what red card would you like to find?" On his naming the face-value of a card, you begin at the left-hand end of the row of black cards and count them aloud and deliberately, pointing to each one as you count it, until you come to the ordinal number which equals the face-value of the red card called for; and in case that card is the knave or queen, you call "knave" instead of "eleven" on pointing at the eleventh card, and "queen" on pointing at the last card. When you come to call the number that equals that of the red card called for, you turn the card you are pointing at face up. Suppose it is the six, for example. Then you say, naming the card called for, that that card will be the sixth; or if the card turned up was the knave, you say that the card called for will be "in the knave-place," and so in other cases. You then take up the red packet, and counting them out, aloud and deliberately, from one hand to the other, and from the back toward the face of the packet, when you come to the number that equals the face-value of the black card turned, you turn over this card as soon as you have counted it, and lo! it will be the card called for.

592. The company never fail to desire to see the thing done again; and on their expressing this wish, after impressing on your memory the present place of the black ace, you have only to hold out the red cards to be cut again, and you again go through the rest of the performance, now abbreviating it by having the cards dealt only once. The third time you do it, since you will now have given them the enjoyment of their

little astonishment, there will no longer be any reason for not asking somebody to say what pile you shall take up first, although that will soon lead to their seeing that all the cuttings are entirely nugatory. Still they will not thoroughly understand the phenomenon.

593. If you wish for an explanation of it, the wish shows that you are not thoroughly grounded in cyclic arithmetic, and that you consequently still have before you the delight of assimilating the first three *Abschnitte* (for that matter the first hundred pages would suffice to reveal the foundations of the present mystery; but I confess I do not particularly admire the first *Abschnitt*) of Dedekind's lucid and elegant redaction of the unerring Lejeune-Dirichlet's "*Vorlesungen über Zahlentheorie.*" But, perhaps, on another occasion* I will myself give a little essay on the subject, "adapted to the meanest capacity," as some of the books of my boyhood used, not too respectfully, to express it.

§2. EXPLANATION OF CURIOSITY THE FIRST†

594. You remember that at the end of my description of the card "trick" that made my first curiosity, I half promised to give, some time, an explanation of its rationale. This half promise I proceed to half redeem.

Suppose a prime number, P, of cards to be dealt into S (for *strues*) piles, where $S<P$. (Were $S=P$, it would be impossible to regather the cards, according to the rule given in the description of the "trick.") Then, in each pile, every card that lies directly on another occupied, before the deal, the ordinal, or serial, place in the packet whose number was S more than that of the other; and using Q to denote the integral part of the quotient of the division of P by S, so that $P-QS$ is positive, while $P-(Q+1)S$ is negative (for P being prime, neither can be zero), and assuming that the piles lie in a horizontal row, and that each card is dealt out upon the pile that is next on the right of the pile on which the last preceding card was dealt, it follows that the left-hand piles, to the number of $P-QS$ of them, contain each $Q+1$ cards, while the $(Q+1)S-P$

* See the next section.

† *The Monist*, pp. 416–64, vol. 18, July, 1908.

piles to the right contain each only Q cards. It is plain, then, that, in each pile, every card above the bottom one is the one that before the dealing stood S places further from the back of the packet than did the card upon which it is placed in dealing. But in what ordinal place in the packet before the dealing did that card stand which after the regathering of the piles comes next in order after the card which just before the regathering of the piles lay at the top of any pile whose ordinal place in the row of piles, counting from the left, may be called the *s*th? In order to answer this question, we have first to consider that the effect of Standing Rule No. IV is that the pile that comes next after any given pile in the order of the regathered packet, counting, as we always do, from back to face, is the pile which was taken up *next before* that given pile; and of course it is the bottom card of that pile to which our question refers. Now the rule of regathering is that, after taking up any pile we next take up, either the pile that lies $P-QS$ places to the right of it, or else that which lies $(Q+1)S-P$ places to the left of it. In other words, the pile that is taken up *next before* any pile, numbered s from the left of the row, is either the pile numbered $s+QS-P$ (and so lies toward the *left* of pile s) or else is the pile numbered $s+(Q+1)S-P$ (and so lies toward the *right* of pile s). But if pile number s were one of those which contain $Q+1$ cards each, since these are the first $P-QS$ piles, we should have $s \leqq P-QS$, and the pile taken next before it, if it were to the left of it, would be numbered less than or equal to zero; and there is no such pile. Consequently in that case, that pile taken up next before pile s will be to the right of the pile numbered s, and its number will be $s+(Q+1)S-P$, which will also have been the number of its bottom card in the packet before the dealing; while, since the bottom card of pile number s was card number s before the dealing, and since this pile contains Q other cards, each originally having occupied a place S further on than the one next below it in the pile, it follows that its top card was, before the dealing, the card whose ordinal number was $s+QS$. Thus, while every other card of any of the first $P-QS$ piles is followed after the regathering by a card whose original place was numbered S more than its own, the top card of such a pile will then be followed by a card whose original place was

S more than its own, *counting round a cycle of* P *cards.* In a similar way, if pile number s contains only Q cards, it is one of the last $(Q+1)S-P$ piles. Then it cannot be that the pile taken up, according to the rule, next before it lay to the right of it; for in that case the number of this previously taken pile would exceed S. It must therefore be pile number $s+QS-P$; and this will be the original number of its bottom card, while the original number of the top card of pile number s (since this contains only Q cards), will be $s+(Q-1)S$. Hence, as before, the top card will be followed after the regathering by a card whose original place would be S greater than its own, but for the subtraction of P in counting round a cycle of P numbers. This rule then holds for all the cards.

It follows that if, after the regathering, the last card, that at the face of the pack or in the P place, is the one whose original place may be called the Πth, then any other card, as that whose place after the gathering is the lth, was originally in the $\Pi+lS-mP$, where mP is the largest multiple of P that is less than $\Pi+lS$. If, however, after the regathering, the pack be cut so as to bring the card which was originally the Pth, or last, that is, which was at the face of the pack, back to that same situation, then, since the original places increase by S (round and round a cycle of P places) every time the regathered places increase by 1, it follows that the original place of the card that is first subsequently to that cutting will have been S, that of the second, 2S, etc.; and in general, that of the lth will have been $lS-mP$. If the cards had originally been arranged in the order of their face-values, the face-value of the card in the lth place after the cut will be $lS-mP$, which we may briefly express by saying that the dealing into S piles with the subsequent cutting that brings the face card back to its place, "cyclically multiplies the face-value of each card by S," the cycle being P. If after dealing into S piles, another dealing is made into T piles, and another into U piles, etc., after which a cut brings the face card back to its place, the face-value of every card will be cyclically multiplied by $S\times T\times U\times$ etc. Moreover, if cuttings were made before each of the dealings, since each cutting only cyclically adds the same number to the place of every card, the cards will still follow after one another according to the same rule; so that the final cutting that restores the face card to its place, annuls the effect of all those previous cuttings.

595. My hints as to the rationale of the exceptional treatment of the last card in twelve initial deals, and as to the extraordinary relation which results between the orders of succession of the black and of the red cards must be prefaced by some observations on the effects of reiterated dealings into a constant number of piles. What I shall say will apply to a pack of any prime number of cards greater than two; but to convey more definite ideas I shall refer particularly to a suit of 13 cards, each at the outset having its ordinal number in the packet equal to its face-value. The effect of one cyclic multiplication of the face-values by 2, brought about by dealing the suit into 2 piles, regathering, and cutting, if need be, so as to restore the king to the face of the packet, will be to shift all the cards except the king in one circuit. That is, the order before and after the cyclic multiplication being as here shown.

Before the cyclic doubling of
the face-values.......1, 2, 3, 4, 5, 6, 7, 8, 9, X, J, Q, K,
After the same ....2, 4, 6, 8, X, Q, 1, 3, 5, 7, 9, J, K,

the 2 takes the place of the 1, the 4 that of the 2, the 8 that of the 4, the 3 that of the 8, the 6 that of the 3, the Q that of the 6, the J that of the Q, the 9 that of the J, the 5 that of the 9, the X that of the 5, the 7 that of the X, and the 1 that of the 7; so that the values are shifted as shown by the arrows on the circumference of the circle of Fig. 229. If 7, instead of 2, be the number of piles into which the thirteen cards are dealt there will be a similar shift round the same circuit, but in the direction opposite to the pointings of the arrows; and if the cards are dealt into 6 or into 11 piles, there will be a shift in a similar single circuit along the sides of the inscribed stellated polygon. But if the 13 cards are dealt into a number of piles other than 2, 6, 7, or 11, the single circuit will break into 2, 3, 4, or 6 separate circuits of shifting. Thus, if the dealing be into 4 or into 10 piles, there will be two such circuits, each along the sides of a hexagon whose vertices are at alternate points along the circumference of the circle in the same figure (or, what comes to the same thing, at alternate vertices, along the periphery of the stellated polygon).

Dealing into 4 piles makes one round from 1 to 4, from 4 to 3, from 3 to Q, from Q to 9, from 9 to X, and from X back to 1; while another round is from 2 to 8, from 8 to 6, from 6 to J, from J to 5, from 5 to 7, and from 7 back to 2. Dealing into 5 or into 8 piles will make three circuits each from one vertex to the next one of 3 squares inscribed in the circle. Dealing into 3 or into 9 piles will give 4 circuits round three inscribed equilateral triangles. Finally, dealing into 12 piles, with regathering, etc., according to rule, simply reverses the

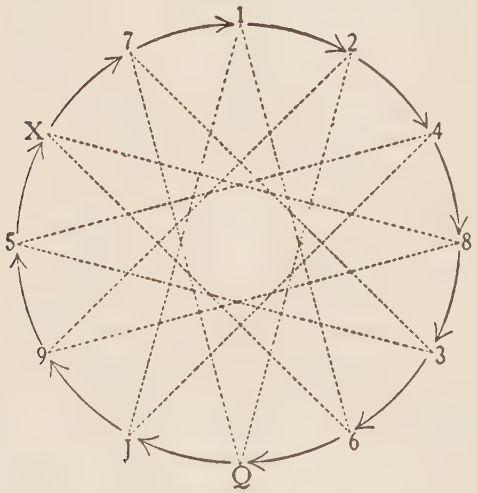


Fig. 229

order so that the ace and queen, the 2 and knave, the 3 and ten, etc., change places.

It has already been made evident that if any prime number, P, of cards, each inscribed with a number, so that, when operations begin this number shall be equal to the ordinal place of the card in the pack, be dealt into any lesser number, S, of piles, and these be regathered, etc., according to rule, the effect is cyclically to multiply by S the number inscribed on any card which is identified solely by its resulting ordinal place, that is, to multiply in counting the numbers round and round a cycle of P numbers — or, to state it otherwise, the ordinary product has the highest lesser multiple of P subtracted from it, though this seems to me to be a needlessly complicated form of conceiving the cyclical product. In counting round and round the number of numbers in the cycle, the so-called "modulus of the cycle" is the same as zero; so that the product of its multiplication by S is zero; or, regarding the matter in the other way, SP diminished by the largest lesser multiple of P gives P. Consequently, the face card will not change its face-value. Let the dealing, etc., be reiterated until it has been performed δ times. The effect will be to multiply the face-values (of cards identified only by their final ordinal places) by S^δ. Since this is the same multiplier for all the cards, it follows that when δ attains such a value that the card in any one place, with the exception of the face card of the pack, *which alone retains an unchanging value*, recovers its original value, every one of the $P-1$ cards of (apparently) changing values equally recovers its original value; and if the values do not shift round a single circuit of $P-1$ cards, all the circuits must be equal; for otherwise the single number S^δ would not fix the values of all the cards. And since zero, or P, is the only number that remains unchanged by a multiplication where the multiplier is not unity (and S is always cyclically greater, that is, more advanced clockwise, than 1 and less than P), it follows that the moduli of the shifts must all be the same divisor of $P-1$, and consequently $P-1$ deals, whatever be the constant number of piles, must restore the original order. The pure arithmetical statement of this result is that S^{P-1}, whenever P is a prime number and S not a multiple of it, must exceed by one some multiple of P. This proposition goes by the name of its discoverer, perhaps the most penetrating mind in the history of mathematics, being known as "Fermat's theorem"; although from our present point of view, it may seem too obvious to be entitled to rank as a "theorem." The books give half a dozen demonstrations of it. It lies at the root of cyclic arithmetic.

596. Fermat said he possessed a demonstration of his theorem; and there is every reason for believing him; but he did not publish any proof.* About 1750, the mathematician König asserted that he held an autograph manuscript of Leibniz containing a proof of the proposition, but it has never been

* See *Oeuvres*, t. II, p. 209; Paris, (1894).

published, so far as I know.* Euler,† at any rate, first published a proof of it; and Lambert‡ gave a similar one in 1769. Subsequently Euler§ gave a proof less encumbered with irrelevant considerations; and this second proof is substantially the same as that in Gauss's celebrated *Disquisitiones Arithmeticæ* of 1801, §49. Several other simple proofs have since been given; but none, I think, better than that derived from the consideration of repeated deals.

597. But what concerns the curious phenomenon of my little "trick" is not so much Fermat's theorem as it is the more comprehensive fact that, whatever odd prime number, P, the number of cards in the pack may be, there is some number, S, such that in repeated deals into that number of piles, all the numbers less than P shift round a single circuit. I hope and trust, Reader, that you will not take my word for this. If fifty years spent chiefly with books makes my counsel about reading of any value, I would submit for your approbation the following maxims:

I. There are more books that are really worth reading than you will ever be able to read. Confine yourself, therefore, to books worth reading and re-reading; and as far as you can, own the good books that are valuable to you.

II. Always read every book critically. A book may have three kinds of value. First, it may enrich your ideas with the mere possibilities, the mere ideas, that it suggests. Secondly, it may inform you of facts. Thirdly, it may submit, for your approbation, lines of thought and evidences of the reasonable connection of possibilities and facts. Consider carefully the attractiveness of the ideas, the credibility of the assertions, and the strengths of the arguments, and set down your well-matured objections in the margins of your own books.

III. Moreover, procure, in lots of twenty thousand or more, slips of stiff paper of the size of postcards, made up into pads of fifty or so. Have a pad always about you, and note upon one of them anything worthy of note, the subject being stated

* See *Leibnizens Mathematische Schriften*, ed. C. 1. Gerhardt, VII, S. 180.

† *Commentarii Academiæ Petropolitanæ*, t. VIII, pp. 141–6.

‡ *Nova Acta Eruditorum*, p. 109 (1769).

§ *Op. cit.*, t. I, p. 25 (1747 et 1748); t. VII, p. 70 (1758 et 1759).

at the top and reference being made below to available books or to your own note books. If your mind is active, a day will seldom pass when you do not find a dozen items worth such recording; and at the end of twenty years, the slips having been classified and arranged and rearranged, from time to time, you will find yourself in possession of an encyclopaedia adapted to your own special wants. It is especially the small points that are thus to be noted; for the large ideas you will carry in your head.

598. If you are the sort of person to whom anything like this recommends itself, you will want to know what evidence there is of the truth of what I assert, that there is some number of piles into which any prime number of cards must be dealt out one less than that prime number of times before they return to their original order.

If these maxims meet your approval, and you read this screed at all, you will certainly desire to see my proposition proved. At any rate, I shall assume that such is your desire. Very well; proofs can be found in all the books on the subject from the date of Gauss's immortal work down. But all those proofs appear to me to be needlessly involved, and I shall endeavor to proceed in a more straightforward way, which "mehr rechnend zu Werke geht." Indeed, I think I shall render the matter more comprehensible by first examining a few special cases. But at the outset let us state distinctly what it is that is to be proved. It is that if P is any prime number greater than 2, then there must be some number of piles, S, into which a pack of P cards must be dealt (and regathered and cut, according to the rule) $P-1$ times in order to bring them all round to their original places again. The reason I limit the proposition to primes will presently appear: the reason I limit the primes to those that are greater than 2 is that two cards cannot, in accordance with the rule, be dealt, etc., into more than one pile (if you call that dealing); and of course this does not alter the arrangement; and since there is no number of piles less than one, the theorem, in this case, reduces itself to an identical proposition; while if 1 be considered to be a prime number, the proposition is falsified since there is no number of piles into which one card can be dealt and regathered according to the rule, which requires S to be less than P.

Let our first example be that of P = 17. Then P − 1 = 16; and unless there be a single circuit of 16 face-values, which my whole present object is to show that there must be, all the circuits must either be one or more sets of 8 circuits of 2 values each, or sets of 4 circuits of 4 values each, or sets of 2 circuits of 8 values each; unless, indeed, we count in, as we ought to do, the case of 16 circuits of 1 value each. This last means that each of the 16 cards retains its face-value after a single deal. It is obtrusively obvious that this can only be when S = 1. But since in these hints toward a demonstration of the proposition the particular values of S do not concern us, and had better be dismissed from our minds, we will denote this value of S by S^{xvi}, meaning that it is a value that gives 16 circuits. We will not ask what is the number of piles into which 2 dealings will restore the face-value of every card; or, in other words, will give 8 circuits of 2 values each. Letting x denote that unknown quantity, the number of piles, or the cyclic multiplier, the equation to determine it is $x^2 = 1$. To many readers two values satisfying this equation will be apparent. But I do not care what they are, further than that the value $x = 1$ obviously satisfies the equation $x^2 = 1$. I do care, however, to show that there can be but two solutions of the equation $x^2 = 1$. For suppose that $x_1^2 = 1$ and $x_2^2 = 1$. Then $[x_1^2] - x_2^2 = (x_1 + x_2) \cdot (x_1 - x_2) = 0$ or equals mP. Now if a multiple of a prime number be separated into two or more factors, one of these, at least, must itself be a multiple of that prime, just as in the algebra of real and of imaginary quantities and in quaternions, if the product of several quantities be zero, one or other of those factors must be zero; and just as in logic, if an assertion consisting of a number of asserted items be false, one or more of these items must be false. In addition, every summand has its own independent effect; but every unit of a product is compounded of units of all the several factors. This is the formal, or purely intellectual, principle at the root of all the reasons for making the number of cards dealt, especially in reiterated dealings, to be a prime. It follows, then, that there are but two numbers of piles, dealings into each of which will restore the original arrangement after 2 deals; and one of these is $x = 1$; for evidently (bear this in mind), if $x^a = 1$, then also $x^{(ab)} = (x^a)^b = 1$. There is then but one number of piles, dealings into which shift the values of the cards in eight, and only eight, circuits; and this number we will denote by S^{viii}. Then, reserving x to denote any root of the equation $x^2 = 1$, and taking ξ to denote that one of the two roots that is not 1, we will take y to denote any number of piles, after dealing into which 4 times, the resulting arrangement of the values will be the original arrangement. That is to say, y will be any root of the cyclic equation $y^4 = 1$. But $x^4 = (x^2)^2 = 1^2 = 1$; so that any value of x is a value of y. Let η denote any value of y that is not a value of x; and let us suppose that there are two values of η, which we may denote by S^{iv} and S^{xii}. It will be easy to show that there is no third value of η. For $(\eta^2)^2 = 1$, where η^2 fulfills the definition of x and is thus either 1 or ξ. But the roots of the equation $\eta^2 = 1$ fulfill the definition of x, whose values are excluded from the definition of η. Hence we can only have $\eta^2 = \xi$; and that this has but two roots is proved by the same argument as was used above. Namely, η_1 and η_2 being any two of these, $(\eta_1^2 - \eta_2^2) = (\eta_1 + \eta_2)\cdot(\eta_1 - \eta_2) = 0$, so that unless η_1 and η_2 are equal, and $\eta_1 - \eta_2 = 0_1$, then $\eta_1 + \eta_2 = 0$, or η_1 and η_2 are negatives of each other. Now no more than 2 quantities can be each the negative of each of the others. We now pass to the consideration of those numbers of piles into which eight successive dealings result in the original arrangement. Denoting by z any such number, it is defined by the equation $z^8 = 1$. But every value of y (of which we have seen that there cannot be more than 4), satisfies this equation, since $y^8 = (y^4)^2 = 1^2 = 1$. Let ζ denote any value of z which is not a value of y. We may suppose that there are two of these for each of the two values of η, which we will designate as S^{ii}, S^{vi}, S^{x}, S^{xiv}. I need not assert that there are so many; but my argument requires me to prove that there are no more. The equation $(z^2)^4 = z^8 = 1$ shows that z^2 fulfills the definition of y and can therefore have no more than the four values, 1, ξ, and the two values of η. Now if $z^2 = 1$, z can, as we have seen in the case of x, have no other values than $z = 1$ and $z = \xi$, both of which are values of y.

If $z^2 = \xi$, as we have seen in regard to y, z can have no other values than the two values of η, which are again values of y. Now let us suppose that z has four values, S^{ii}, S^{vi}, S^{x}, and

S^{xiv}, that are not values of y; and let us define ζ as any value of z that is not a y. The proof that there can be no more than four ζ's is so exactly like the foregoing as to be hardly worth giving. I will relegate it to a paragraph of its own that shall be both eusceptic and euskiptatic — "what horrors!" I hear from the mouths of those moderns who abominate all manufactures of Hellenic raw materials, like "skip" and "skimp."

We have seen that either $z^2=1$, or $z^2=\xi$, or $z^2=\eta$; and also that, in the first case, either $z=1$ or $z=\xi$, both of which are values of y; and that, in the second case, z has one or other of the two values of η. Accordingly, it only remains that $\zeta^2=\eta$. There are but two values of η and if ζ_1 and ζ_2 are two different values of ζ whose squares are the same value of η, $\zeta_1^2-\zeta_2^2 =(\zeta_1+\zeta_2)\cdot(\zeta_1-\zeta_2)=0$. Hence, since $\zeta_1-\zeta_2$ is not zero, it follows that every value of ζ differs from every other value derived from the same η only by being the negative of it. Now no number has two different negatives; and therefore there can be no more than two ζs to every η; and there being no more than two ηs, there can be no more than four ζs.

Now this is the summary of the whole argument: the 17 cards of the pack being consecutively inscribed with numbers from the back to the face of the pack, each number of piles into which they are dealt etc. according to the rule acts as a cyclic multiplier of the face-value of every card. Every such multiplier leaves $0(=17)$ unchanged, and shifts the other 16 face-values in a number of circuits having the same number of values in each. The possible consequences, excluding the case of a single circuit of 16 values, are the following:

16 circuits of 1 value each can result from but	1 multiplier at the utmost
8 circuits of 2 values each can result but from	1 other multiplier
4 circuits of 4 values each can result but from	2 other multipliers
2 circuits of 8 values each can result but from	4 other multipliers
In all, the number of multipliers that give more than 1 circuit (of all 16 values) is.......	8 at most
But there are in all........................	16 multipliers
Hence, the number of multipliers that shift the values in 1 circuit of 16 values is.........	8, at least.

In point of fact, it is precisely 8.

599. Let us now consider a pack of 31 cards. Here, the zero card not changing its value, there are 30 values which are shifted in one of these ways:

In 30 circuits of 1 value each;
In 15 circuits of 2 values each;
In 10 circuits of 3 values each;
In 6 circuits of 5 values each;
In 5 circuits of 6 values each;
In 3 circuits of 10 values each;
In 2 circuits of 15 values each;
In 1 circuit of 30 values.

I propose to show as before that if we exclude the last case, the others do not account for the effects of so many as 30 different multipliers. In the first place, as in the last example, but one multiplier will give circuits of one value each; and but one other multiple will give circuits of only two values each. We may call the former S^{xxx} and the latter S^{xv}.

The problems of 10 circuits of 3 values each and of 6 circuits of 5 values each can be treated by exactly the same method, 3 and 5 being prime numbers. I shall exhibit in full the solution of the more complicated of the two, leaving the other to the reader.

I propose, then, to show that there are at most but 5 different values which satisfy an equation of the form $s^5=1$. The general idea of my proof will be to assume that there are 5 different values (for it is indifferent to my purpose whether there be so many or not) and then to show that there is such an equation between these five, that given any four, there is but one value that the fifth can have; that being as much as to say that there are not more than five such values in all. This assumes that every one of the five values differs from every one of the other four; making ten premisses of this kind that have to be introduced. Now to introduce a premiss into a reasoning, is to make some inference which would not necessarily follow if that premiss were not true. Assuming, then, that $s^5=1$, $t^5=1$, $u^5=1$, $v^5=1$, $w^5=1$, are the five assumed equations, I note that the division by one divisor of both sides of an equation necessarily yields equal quotients only if the divisor is known not to be zero. Hence if I divide my equations by $s-t$, by $s-u$, by $s-v$, by $s-w$, by $t-u$, by $t-v$, by $t-w$, by $u-v$, [by] $u-w$, and by $v-w$, I shall certainly introduce the ten premisses that all the five values are different; and with a little ingenuity — a *very* little, as it turns out — I ought to reach my legitimate conclusion.

I will begin then by subtracting $t^5=1$ from $s^5=1$, giving $s^5-t^5=0$; and dividing this by $s-t$, and using $\cdot|\cdot$ as the logical sign of disjunction, that is, to mean "or else," I get

(1) $s^4+s^3t+s^2t^2+st^3+t^4=0 \quad \cdot|\cdot \; s=t.$

By analogy, I can equally write

$$s^4+s^3u+s^2u^2+su^3+u^4=0 \quad \cdot|\cdot \; s=u.$$

Subtracting the latter of these from the former, I get

$$s^3(t-u)+s^2(t^2-u^2)+s(t^3-u^3)+t^4-u^4=0 \quad \cdot|\cdot \; s=t \cdot|\cdot \; s=u.$$

And dividing this by $t-u$, I obtain

(2) $s^3+s^2(t+u)+s(t^2+tu+u^2)+t^3+t^2u+tu^2+u^3=0 \quad \cdot|\cdot \; s=t \cdot|\cdot \; s=u \cdot|\cdot \; t=u.$

By analogy, I can equally write

$$s^3+s^2(t+v)+s(t^2+tv+v^2)+t^3+t^2v+tv^2+v^3=0 \quad \cdot|\cdot \; s=t \cdot|\cdot \; s=v \cdot|\cdot \; t=v.$$

Subtracting the last equation from the last but one, I get

$$(s^2+st+t^2)(u-v)+(s+t)(u^2-v^2)+u^3-v^3=0 \quad \cdot|\cdot \; s=t \cdot|\cdot \; s=u \cdot|\cdot \; s=v \cdot|\cdot \; t=u \cdot|\cdot \; t=v.$$

And dividing by $u-v$, I have

(3) $s^2+st+t^2+(s+t)(u+v)+u^2+uv+v^2=0 \quad \cdot|\cdot \; s=t \cdot|\cdot \; s=u \cdot|\cdot \; s=v \cdot|\cdot \; t=u \cdot|\cdot \; t=v \cdot|\cdot \; u=v.$

By analogy, I can equally write

$$s^2+st+t^2+(s+t)(u+w)+u^2+uw+w^2=0 \quad \cdot|\cdot \; s=t \cdot|\cdot \; s=u \cdot|\cdot \; s=w \cdot|\cdot \; t=u \cdot|\cdot \; t=w \cdot|\cdot \; u=w.$$

Subtracting the last from the last but one, and dividing by $v-w$, I get

(4) $s+t+u+v+w=0 \quad \cdot|\cdot s=t \cdot|\cdot \; s=u \cdot|\cdot \; s=v \cdot|\cdot \; s=w \cdot|\cdot \; t=u \cdot|\cdot \; t=v \cdot|\cdot \; t=w \cdot|\cdot \; u=v \cdot|\cdot \; u=w \cdot|\cdot \; v=w.$

This shows at once that there cannot be more than 5 different numbers, which, counting round any prime cycle, all have their fifth powers equal to 1. By a similar process, as you can almost see without slate and pencil, from $x^3=1$, $y^3=1$, $z^3=1$ one can deduce $x+y+z=0 \cdot|\cdot\ x=y \cdot|\cdot\ x=z \cdot|\cdot\ y=z$. The existence of these 5 and these 3 numbers must, for the present, be regarded as problematic, except that we cannot shut our eyes to the fact that 1 is one of the members of each set; as indeed $1^\delta=1$, whatever the exponent may be.

I have numbered some of the equations obtained in the proof that there are no more than 5 fifth roots of unity. You will observe that (1) equates to zero the sum of all possible terms of the fourth degree formed by two roots; that (2) equates to zero the sum of all possible terms of the third degree formed by three roots; that (3) equates to zero the sum of all possible terms of the second degree formed from four roots; and that (4) equates to zero the sum of all possible terms of the first degree formed by all five roots. Now it is plain that if we assume that there are n unequal nth roots of unity, then by subtracting $x_2^n=1$ from $x_1^n=1$, and dividing by x_1-x_2, we shall equate to zero the sum of all possible terms of the $(n-1)$th degree in x_1 and x_2. And if we have proved, in regard to any m of the roots, that (all being unequal) the sum of all possible terms of the $(n-m+1)$th degree in these roots is equal to zero; then by taking two such equations of the $(n-m+1)$th degree in $m-1$ roots common to the two, with one root in each equation not entering into the others; by subtracting one of these equations from the other, and then dividing by the difference between the two roots which enter each into but one of these equations, we shall get an equation of the $(n-m)$th degree in $m+1$ roots. For $x^n-y^n=(x-y)\cdot \Sigma_i{}_0^{n-1} x^i y^{n-i-1}$. Accordingly, by repetitions of this process, we shall ultimately find that the sum of the n roots, if there be so many, is 0. This proves that there can be no more than n unequal nth roots of unity in cyclic arithmetic any more than in unlimited real or imaginary arithmetic.

600. But if the root of unity be of an order not prime but composite, so that it is the root of an equation of the form $x^{pq}=1$, it is evident that it is satisfied by every root of $y^p=1$

and by every root of $y^q=1$; since every power of 1 is 1. Accordingly, exclusive of roots of a lower order, the number of roots of unity of order n, that is, the number of roots of $x^n=1$, additional to those that are roots of unity of lower order, cannot be greater than the number of numbers not greater than n and prime to it. A number is said to be prime to a number when they have no other common divisor than 1. I shall write the expression of two or more numbers separated by heavy vertical lines to denote the greatest common divisor of those numbers. Thus, I shall write $12 \mid 18 = 6$. This vertical line may be considered as a reminiscence of the line that separates numbers in the usual algorithm of the greatest common divisor. A prime number is a number prime to every other number. Consequently, 1 is a prime number. It is the only prime number that is prime to itself; for $p \mid p = p$. The number of numbers not exceeding a number, n, but prime to it is now called the *totient* of n. In the books of the first four-fifths of the nineteenth century, the totient of n was denoted by $\phi(n)$; but since the invention of the word *totient** about 1880, Tn has become the preferable notation. $\mathrm{T}1=1$; but if p be a prime not prime to itself, $\mathrm{T}p=p-1$. It is quite obvious that the totient of any number, n, whose prime factors, not prime to themselves, are p', p'', p''', etc., is obtained by subtracting from n the p'th part of it, and then successively from each remainder the p''th, etc., part of it, but not using any prime factor twice. Thus $\mathrm{T}4=2$ (for $4 \mid 1 = 1$ and $4 \mid 3 = 1$; but $4 \mid 2 = 2$ and $4 \mid 4 = 4$); $\mathrm{T}6=2$ (for $6-\frac{1}{2}\cdot 6=3$ and $3-\frac{1}{3}\cdot 3=2$); $\mathrm{T}8=4$ (for $8-\frac{1}{2}\cdot 8=4$), $\mathrm{T}9=6$, $\mathrm{T}10=4$, etc. If $m \mid n=1$, then $\mathrm{T}mn=(\mathrm{T}m)(\mathrm{T}n)$. On the other hand, if p is a prime and m any exponent, $\mathrm{T}p^n=(p-1)p^{n-1}$. A "perfect number" is defined as one which is equal to the sum of its "aliquot parts," that is, of all its divisors except itself; but, in a more philosophical sense, *every* number is a perfect number. That is to say, it is equal to the sum of the totients of *all* its divisors; a proposition which is perfectly obvious if regarded from the proper point of view. However, since this proposition has some relevancy to the proposition I am endeavoring to prove; namely, that there is some number of piles, dealing into which shifts all the face-values of the cards along a single cycle, I

* By Sylvester; see his *Mathematical Papers*, IV, 102.

will repeat a pretty demonstration of the former proposition that I find in the books. Having selected any number, m, rule a sheet of paper into columns, a column for each divisor of m; and write these divisors, in increasing order from left to right each at the top of its column as its principal heading. Just beneath this, write in parentheses, as a subsidiary heading to the column, the complementary divisor, *i.e.*, the divisor whose product into the principal heading is the number m; and draw a line under this subsidiary heading. Now, to fill up the columns, run over all the numbers in regular succession, from 1 up to m inclusive, writing each in one column, and in one only; namely in that column which is furthest to the right of all the columns of whose principal headings the number to be written is a multiple. Here, for example, is the table for $m = 20$:

1 (20)	2 (10)	4 (5)	5 (4)	10 (2)	20 (1)
1	2				
3		4	5		
	6				
7		8			
9				10	
11		12			
13	14		15		
		16			
17	18				
19					20

Fig. 230

By this means it is obvious that each column will receive all those multiples of the principal heading whose quotients by that heading are prime to the subsidiary heading, and will receive no other numbers. Thus, every column will contain just one number for each number prime to the subsidiary heading but not greater than it; (since no number is entered which exceeds the product of the two headings). In other words, the number of numbers in each column equals the totient of the subsidiary heading; and since the subsidiary headings are all the divisors, and the total number of numbers entered is m, the sum of the totients of all the divisors of m is m, whatever number m may be. It will be convenient to have a name for this principle; and since, as I remarked, it renders every number a perfect number in a perfected sense of that term, or say a *perfecti perfect* number, I will refer to it as the *rule of perfection.*

According to this, although $x^6=1$ may have 6 roots, yet since x^2, x^3, and x^6 are also roots, by the rule of perfection there can be but $T6=T2 \cdot T3=1 \cdot 2=2$ numbers of piles into which dealing must be made 6 times successively in order to restore the original arrangement; and similarly for the other divisors. So then the number of ways of dealing (*i.e.*, number of piles into which the cards can be dealt, etc.) which will restore 31 cards to their original order in less than 30 deals cannot exceed $T1+T2+T3+T5+T6+T10+T15$. There are, however, in all, 30 ways of dealing; and by the rule of perfection $30=T1+T2+T3+T5+T6+T10+T15+T30$. Hence, there must be $T30=T2 \cdot T3 \cdot T5=1 \cdot 2 \cdot 4=8$ ways of dealing which shift the 30 values in a single circuit. And so with any other prime number than 31. This argument is so near a perfect demonstration that there always must be such ways of dealing that I may leave its perfectionment to the reader.

601. I do not know of any general rule for ascertaining what the particular numbers of piles are into which the prime number p of cards must be dealt $p-1$ times in order to bring round the original arrangement again. It seems that there is a *Canon Arithmeticus* got out by Jacobi, which gives the numbers for the first 170 primes or so. It was published in the year of my birth;* so that it was clearly the purpose of the Eternal that I should have the advantage of it. But that purpose must have been frustrated; for I never saw the book. The *Tables Arithmétiques* of Hoüel (Gauthiers-Villars: 1866, 8vo, pp. 44) gives those numbers for all primes less than 200. From these tables it appears that for about five-eighths of the primes one such number is either 2 or $p-2$. Now as soon as one has been found, it is easy to find the rest which are all the powers of that one whose exponents are prime to $p-1$. In case $p-1$ has few prime factors, the numbers any one of which we seek must be nearly a third, perhaps nearly

* 1839.

or quite half of all the $p-1$ numbers: so that ere many trials have been made, one is likely to light upon one of them. Thus if $p=17$, try 2. Now $2^4=16=-1$; so this will not do. Nor will -2. Try 3. We have $3^2=9=-8$; $3^3=27=-7$, $3^4=81=-4$, $3^8=(3^4)^2=(-4)^2=16=-1$. Evidently 3 is one of the numbers and the others are $3^3=-7$, $3^5=-12=5$, $3^7=(3^3)(3^4)=(-7)\cdot(-4)=28=-6$, and the negatives of these. If the prime factors are many, a different procedure may be preferable. Take the case of $p=31$. Here $p-1=2\cdot 3\cdot 5$. Turning to that table of the first nine powers of the first hundred numbers which is given in so many editions of Vega, I find in the column of cubes, $5^3=125=4(31)+1$, and $6^3=216=7\cdot 31-1$ and in the column of fifth powers, I find $3^5=243=8(31)-5$. Consequently, $(3^5)^3=3^{15}=-1$. This renders it *likely* that 3 may be such a number as I seek. $3^2=9$, $3^3=-4$, $3^4=-12$, $3^5=-5$, $3^6=16=-15$, $3^{10}=-6$, $3^{12}=+8$, $3^{15}=(3^5)^3=-125=-1$. It is evident that 3 is one of the numbers. The other seven are $3^7=3^5\cdot 3^2=-45=-14$, $3^{11}=3\cdot 3^{10}=-18=13$, $3^{13}=3\cdot 3^{12}=24=-7$, $3^{17}=3^{15}\cdot 3^2=-9$; $3^{19}=3^{15}\cdot 3^4=+12$, $3^{23}=3^{19}\cdot 3^4=-144=+11$, $3^{29}=3^{17}\cdot 3^6=(-9)\cdot(-15)-135=+11$.

602. Since, then, whatever prime number not prime to itself p may be, there are always $T(p-1)$ numbers of which the lowest power equal to 1 (counting round the p cycle) is the $(p-1)$th and these powers run through all the values of the cycle excepting only $p=0$, it follows that these numbers may appropriately be called *basal* (or *primitive*) roots of the cycle; and that their exponents are true *cyclic logarithms* of all the numbers of the cycle except zero. But since, if b be such a basal root, its $(p-1)$th power, like that of any other number, equals 1 (counting round the p-cycle), it follows that these exponents run round a cycle smaller by one unit than that of their powers; or in other words, the *modulus* of the cycle of logarithms is $p-1$, while the modulus of the cycle of natural numbers is p.[1]

[1] This being the first [but see 595] occasion I have had in this essay to employ the word "modulus," I will take occasion to say that its general meaning is now well established. It means that signless quantity which measures the magnitude of a quantity and is a factor of it. So that if M and M′ are the moduli of two quantities, $M\mu$ and $M'\mu'$, their product is $MM' \cdot \mu\mu'$, where MM' is an ordinary product, but $\mu\mu'$ may be a peculiar function. Thus, the absolute

603. The cyclic logarithms form an entirely distinct number-system from that of the corresponding natural numbers. For the modulus of their cycle is composite instead of prime, a circumstance which essentially modifies some of the principles of arithmetic. For example, every natural number of a cycle of prime modulus gives an unequivocal quotient when divided by another. But some numbers in a cycle of composite modulus give two or more quotients when divided by certain others, while others are not divisible without remainders. The whole doctrine shall be set forth here. I will preface it with a statement of the essential differences between the system of all positive finite integers, the system of all real finite integers, and any cyclical system. I omit the Cantorian system, partly because the full explanation of it would be needed and would be long, and partly because there is a doubt whether it really possesses an important character which Cantor attributes to it.

value of -2, or 2, is its "modulus", as 3 is of -3; and $(-2) \cdot (-3) = +6$ where $2 \times 3 = 6$ by ordinary multiplication, but $(-1) \times (-1) = +1$ by an extension of ordinary multiplication. So the "modulus" of $A+Bi$, where $i^2 = -1$, is $\sqrt{A^2+B^2}$. The tensor of a quaternion and the determinant of a square matrix are other examples of moduli. The cardinal number of numbers in a cycle has no sign and may properly be called the modulus of the cycle. But I sometimes refer to it as "the cycle," for short. The present usage of mathematicians is to use, what seems to me a too involved way of conceiving of cyclic arithmetic which carries with it an irregular use of the word "modulus." Legendre [in his *Théorie des Nombres*] and the earlier writers on cyclic arithmetic conceived of its numbers as signifying the lengths of different steps along a cycle of objects, and thus spoke of 18 as being *equal* to 1 on a cycle of 17, just as we say that the 1st, 15th, 22d, and 29th days of August fall on *the same* day of the week, and just as we say that 270° of longitude west of any meridian and 90° east of it are *the very same* longitude. Gauss [in his *Disquisitiones Arithmeticæ*], however, introduced a different locution, involving quite another form of thought. Instead of saying that 18 *is*, or *equals*, 1 in counting round a cycle of modulus 17, he prefers to say that 18 and 1 belong to the same *class* of numbers *congruent to* one another for the *modulus* 17. Here the idea of a cycle appears to be rejected in favor of the idea that $(18-1)/17$ is a whole number.

Now I fully admit that the conception of an indefinitely advancing series is involved in that of a cycle, and further that non-cyclical numbers have to be used to some extent in cyclic arithmetic. But at the same time it seems to me that the theoric idea of a cycle ought to take the lead in this branch of mathematics. In particular, I cannot see why the term *cyclic logarithms* is not perfectly correct and far more expressive than Gauss's colorless name of "indices."

604. It is singular that though the systems to be defined possess, besides several independent common characters, others in respect to which they differ, yet *all* the properties of each system are necessary consequences of a single principle of immediate sequence. In stating this, I shall abbreviate a frequently recurring phrase of nine syllables by writing, "*m* is A of (or to) *n*," or even "*m* is A*n*," to mean that the member, *m*, of the system is in a certain relation of immediate antecedence to the member *n*. I shall express the same thing by writing "*n* is A'd by *m*." But when I call A an abbreviation, I do not mean to imply that the words "immediately antecedent" express its meaning in a satisfactory way. On the contrary, in part, they suggest something repugnant to its meaning, which must be gathered exclusively from the following definitions of the three kinds of systems:

605. A *cyclical system* of objects is such a collection of objects that, the expression "*m* is A to *n*" signifying some recognizable relation of *m* to *n*, every member of the system is A to some member or other, and whatever predicate, P, may be, if P is true of no member of the system without being true of some member of it that is A'd by that member, then P is true either of no member or of every member.

606. The system of all positive whole numbers is a single collection of numbers, the general essential character of which collection is that there is a recognizable relation signified by A, such that every positive integer is A to a positive integer, and there is one, and one only, initial positive integer, 0, (or, if this be excluded, then 1) such that, whatever predicate P may be, if P is true of no positive integer without being also true of some positive integer to which the former is A, then either this predicate is false of that initial positive integer or else is true of all positive integers.*

607. The system of all real integers is a collection of numbers of which the general essential character is that there is recognizable relation signified by one being A to another, such that every number of the system is both A to a number of the system, and is A'd by a number of the system, and whatever predicate P may be, if this be not true of any number, *n*, of the system without being both true of some number that is

* Cf. 110, 188, 337ff, 3.258; 3.562B.

A of *n*, and true also of some number that is A'd by *n*, then P is either false of every number of the system or is true of every number of the system.*

608. A *Cantorian* system is essentially a system of objects positively determined by every collection of objects of the system being A to some object of the system, and by a certain object, 0, being a member of the system; while it is negatively determined by the principle that, whatsoever predicate P may be, if P is not true of every member of any collection of the system without being also true of some member that is A'd by that collection, then either P is not true of the member, 0, or it is true of every member of the system.†

609. Now for several reasons, partly for the sake of the logical interest and instruction that will accrue I will proceed to show precisely *how* all the fundamental properties common to cyclical systems follow from my definition. In accordance with the usage of logicians and mathematicians, I shall call this "demonstrating" those properties. The reader must not fall into the error of supposing that, by this expression, I mean *rationally convincing* him that all cyclical systems have these properties; for I know well that he is perfectly cognizant of that already. All I am seeking to convince him of is, first, *that*, and second, *how*, their truth of all cyclical systems follows from my definition. But in the course of doing so, I shall endeavor to bring to his notice some things well worth knowing concerning necessary reasonings in general. Especially, I shall try to point out errors of logical doctrine which students of the subject who neglect the logic of relations are apt to fall into.

610. A brace of these errors, are, first, that nothing of importance can be deduced from a single premiss; and secondly, that from two premisses one sole complete conclusion can be drawn. Persons who hold the latter notion cannot have duly considered the paucity of the premisses of arithmetic and the immensity of higher arithmetic, otherwise called the "theory of numbers," itself. As to the former belief, aside from the consideration that whatever follows from two propositions equally follows from the one which results from their

* Cf. 110.

† Cf. 332, 675.

copulation, they will have occasion to change their opinion when they come to see what can be deduced from the definition of a cyclic system, which definition is not a copulative proposition.

611. That couple of logical heresies, being married together, legitimately generates a third more malignant than either; namely, that necessary reasoning takes a course from which it can no more deviate than a good machine can deviate from its proper way of action, and that its future work might conceivably be left to a machine — some Babbage's analytical engine or some logical machine (of which several have actually been constructed).* Even the logic of relations fails to eradicate that notion completely, although it does show that much unexpected truth may often be brought to light by the repeated reintroduction of a premiss already employed; and in fact, this proceeding is carried to great lengths in the development of any considerable branch of mathematics. Although, moreover, the logic of relations shows that the introduction of abstractions — which nominalists have taken such delight in ridiculing — is of the greatest service in necessary inference, and further shows that, apart from either of those manoeuvres — either the iteration of premisses or the introduction of abstractions — the situations in which the necessary reasoner finds several lines of reasoning open to him are frequent. Nevertheless, in spite of all this, the tendency of the logic of relations itself — the highest and most rational theory of necessary reasoning yet developed — is to insinuate the idea that in necessary reasoning one is always limited to a narrow choice between quasi-mechanical processes; so that little room is left for the exercise of invention. Even the great mathematician, Sylvester, perhaps the mind the most exuberant in original ideas of pure mathematics of any since Gauss, was infected with this error; and consequently, conscious of his own inventive power, was led to preface his "Outline Trace of the Theory of Reducible Cyclodes," with a footnote which seems to mean that mathematical conclusions are not always derived by an apodictic procedure of reason. If he meant that a man might, by a happy guess, light upon a truth which might have been made a mathematical conclusion, what he said was a

* See 2.56n.

truism. If he meant that the hint of the way of solving a mathematical problem might be derived from any sort of accidental experience, it was equally a matter of course. But the truth is that all genuine mathematical work, except the formation of the initial postulates (if this be regarded as mathematical work) is necessary reasoning. The mistake of Sylvester and of all who think that necessary reasoning leaves no room for originality — it is hardly credible however that there is anybody who does not know that mathematics calls for the profoundest invention, the most athletic imagination, and for a power of generalization in comparison to whose everyday performances the most vaunted performances of metaphysical, biological, and cosmological philosophers in this line seem simply puny — their error, the key of the paradox which they overlook, is that originality is not an attribute of the *matter* of life, present in the whole only so far as it is present in the smallest parts, but is an affair of *form*, of the way in which parts none of which possess it are joined together. Every action of Napoleon was such as a treatise on physiology ought to describe. He walked, ate, slept, worked in his study, rode his horse, talked to his fellows, just as every other man does. But he combined those elements into shapes that have not been matched in modern times. Those who dispute about Free-Will and Necessity commit a similar oversight. Notwithstanding my tychism, I do not believe there is enough of the ingredient of pure chance now left in the universe to account at all for the indisputable fact that mind acts upon matter.* I do not believe there is any amount of *immediate* action of that kind sufficient to show itself in any easily discerned way. But one endless series of mental events may be immediately followed by a beginningless series of physical transformations.† If, for example, all atoms are vortices in a fluid, and every fluid is composed of atoms, and these are vortices in an underlying fluid, we can imagine one way in which a beginningless series of transformations of energy[1]

* Cf. vol. 6, bk. I, chs. 9 and 10.

† See 628.

[1] You may well be puzzled, dear Reader, to iconize the consecution of a beginningless series upon an endless series. But you have only to imagine a dot to be placed upon the rim of a half-circle at each point whose angular distance from the beginning of the semicircumference has a positive or negative

might take place in a fraction of a second. Now whether this particular way of solving the paradox happens to be the actual way, or not, it suffices to show us that from the supposed fact that mind acts *immediately* only on mind, and matter *immediately* only on matter, it by no means follows that mind cannot act on matter, and matter on mind, without any *tertium quid*. At any rate, our power of self-control certainly does not reside in the smallest bits of our conduct, but is an effect of building up a character. All supremacy of mind is of the nature of Form.

612. The plan of a demonstration can obviously not spring up in the mind complete at the outset; since when the plan is perfected, the demonstration itself is so. The thought of the plan begins with an act of ἀγχίνοια[1] which, in consequence

whole number for its natural tangent. These dots will, then, occur at the following angular distances from the origin of measurement.

ANGULAR DISTANCE		TANGENT	ANGULAR DISTANCE		TANGENT	ANGULAR DISTANCE		
0°	00′	0	87°	24′	+22	93°	01′	−19
45	00	+ 1	87	31	+23	93	11	−18
63	26	+ 2	87	37	+24	93	21	−17
71	34	+ 3	87	43	+25	93	35	−16
75	58	+ 4				93	49	−15
78	41	+ 5	and so on endlessly. But			94	05	−14
80	32	+ 6	after all positive integer			94	24	−13
81	52	+ 7	values have been passed			94	46	−12
82	52	+ 8	through before 90°			95	12	−11
83	40	+ 9	(where there will not be			95	43	−10
84	17	+10	any dot), a beginningless			96	20	− 9
84	48	+11	series of dots will suc-			97	08	− 8
85	14	+12	ceed, for which the tan-			98	08	− 7
85	36	+13	gents are negative; and			99	28	− 6
85	55	+14	then			101	19	− 5
86	11	+15				104	02	− 4
86	25	+16	92°	17′	−25	108	26	− 3
86	38	+17	92	23	−24	116	34	− 2
86	49	+18	92	29	−23	135	09	− 1
86	59	+19	92	36	−22	180	00	0
87	08	+20	92	44	−21	225	00	+ 1
87	16	+21	92	52	−20	etc.		

[1] See *Charmides*, p. 160A, and the last chapter of the First *Posterior Analytics* [A: 34].

of pre-existent associations, brings out the idea of a possible object, this idea not being itself involved in the proposition to be proved. In this idea is discerned that the possibility of its object follows in some way from the condition, general subject, or antecedent of the proposition to be proved, while the known characters of the object of the new idea will, it is perceived, be at least adjuvant to the establishment of the predicate or consequent of that proposition.

613. I shall term the step of so introducing into a demonstration a new idea not explicitly or directly contained in the premisses of the reasoning or in the condition of the proposition which gets proved by the aid of this introduction, a *theō'ric* step. Two considerable advantages may be expected from such a step besides the demonstration of the proposition itself. In the first place, since it is a part of my definition that it really aids the demonstration, it follows that without some such step the demonstration could not have been effected, or at any rate only in some very peculiar way. Now to propositions which can only be proved by the aid of theoric steps (or which, at any rate, could *hardly* otherwise be proved), I propose to restrict the application of the hitherto vague word "*theorem*," calling all others, which are deducible from their premisses by the general principles of logic, by the name of *corollaries*.* A theorem, in this sense, once it is proved, almost invariably clears the way to the corollarial or easy theorematic proof of other propositions whose demonstrations had before been beyond the powers of the mathematicians. That is the first secondary advantage of a theoric step. The other such advantage is that when a theoric step has once been invented, it may be imitated, and its analogues applied in proving other propositions. This consideration suggests the propriety of distinguishing between varieties of theorems, although the distinctions cannot be sharply drawn. Moreover, a theorem may pass over into the class of corollaries, in consequence of an improvement in the system of logic. In that case, its new title may be appended to its old one, and it may be called a *theorem-corollary*. There are several such, pointed out by De Morgan, among the theorems of Euclid, to whom they were theorems and are reckoned as such, though to a modern exact logician

* Cf. 2.267.

they are only corollaries. If a proposition requires, indeed, for its demonstration, a theoric step, but only one of a familiar kind, that has become quite a matter of course, it may be called a *theoremation*.[1] If the needed theoric step is a novel one, the proposition which employs it most fully may be termed a *major theorem;* for even if it does not, as yet, appear particularly important, it is likely eventually to prove so. If the theoric invention is susceptible of wide application, it will be the basis of a mathematical method.

614. But mathematicians are rather seldom logicians or much interested in logic; for the two habits of mind are directly the reverse of each other;* and consequently a mathematician does not care to go to the trouble (which would often be very considerable) of ascertaining whether the theoric step he proposes to himself to take is absolutely indispensable or not, so long as he clearly perceives that it will be exceedingly convenient; and the consequence is that many demonstrations introduce theoric steps which relieve the mind and obviate confusing complications without being logically necessary. Such demonstrations prove corollaries more easily by treating them as if they were theorems. They may be called *theoric corollaries*, or if one is not sure that they are so, *theorically proved propositions*.

615. I wish a historical study were made of all the remarkable theoric steps and noticeable classes of theoric steps. I do not mean a mere narrative, but a critical examination of just what and of what mode the logical efficacy of the different steps has been. Then, upon this work as a foundation, should be erected a logical classification of theoric steps; and this should be crowned with a new methodeutic of necessary reasoning. My future years — whatever can have become of them, they do not seem so many now as they used, when, at De Morgan's *Open Sesame*, the Aladdin matmûrah of relative logic had been nearly opened to my mind's eye; but the remains of them shall, I hope, somehow contribute toward setting such an enterprise on foot. I shall not be so short-sighted as to expect any cut-and-dried rules nor yet any higher sort of con-

[1] *θεωρηματιον* is entered in L. & S. [Liddell & Scott, *Greek-English Lexicon*], with a reference to the Diatribes of Epictetus.

* Cf. 239f.

trivance, to supersede in the least that ἀγχίνοια — that penetrating glance at a problem that directs the mathematician to take his stand at the point from which it may be most advantageously viewed. But I do think that that faculty may be taught to nourish and strengthen itself, and to acquire a skill in fulfilling its office with less of random casting about than it as yet can.

616. Euclid always begins his presentation of a theorem by a statement of it in *general terms*, which is the form of statement most convenient for applying it. This was called the πρότασις, or *proposition*. To this he invariably appends, by a λέγω, "I say," a translation of it into *singular terms*, each general subject being replaced by a Greek letter that serves as the proper name for a single one of the objects denoted by that general subject. Yet the generality of the statement is not lost nor reduced, since the understanding is that the letter may be regarded as the name of any one of those objects that the student may select. This second statement was called the ἔκθεσις, or *exposition*. Euclid lived at a time when the surpassing importance of Aristotle's *Analytics* was not appreciated. The use, probably by Euclid himself, of the term πρότασις, which in Aristotle's writings means a premiss, to denote the conclusion to be proved, illustrates this, and confirms other reasons for thinking that Euclid was unacquainted with the doctrine of the *Analytics*. The invariable appending by Euclid of an ἔκθεσις to the πρότασις (except in a few cases in which the proposition is expressed in the ecthetic form alone) inclines me to think that it was, for him, a principle of logic that any general proposition can be so stated; and such a form of statement was always convenient in demonstration; sometimes, necessary. If this surmise be correct, Euclid probably looked upon the function of the ἔκθεσις as that of merely supplying a more convenient form for expressing no more than the πρότασις had already asserted. Yet inasmuch as the πρότασις does not mention those proper names consisting of single letters, the ἔκθεσις certainly does supply ideas that, however obvious they be, are not contained in the πρότασις; so that it must be regarded as taking a little theoric step. The principal theoric step of the demonstration is, however, taken in what immediately

follows; namely, in "preparation" for the demonstration, the παρασκευή, usually translated "the construction." The Greek word is applied to any thing got up with some elaboration with a view to its being used in any contemplated undertaking: a near equivalent to a frequent use of it is "apparatus." Euclid's παρασκευή consists of precise directions for drawing certain lines, rarely for spreading out surfaces; for though his work entitled "*Elements*," appears to have been intended as an introduction to theoretical mathematics in general (the art of computation being the *métier* — the 'mister, as Chaucer would say, of the Pythagoreans), yet Euclid always conceives arithmetical quantities — even when distinguishing between prime and composite integers — as being lengths of lines. It was his mania. Those lines which are drawn in the παρασκευή are not only all that are referred to in the condition of the proposition, but also all the additional lines which he is about to consider in order to facilitate the demonstration of which this παρασκευή is thus the soul, since in it the principal theoric step is taken. But the construction of these additional lines is introduced by γάρ, here meaning "for," and sometimes the text does not very sharply separate some parts of the παρασκευή from the next step, the ἀπόδειξις, or demonstration. This latter contains mere corollarial reasoning, though, in consequence of its silently assuming the truth of all that has been previously proved or postulated (which Mr. Gow, in his *Short History of Greek Mathematics*,* gives as the reason for Euclid's having called his work Στοίχεῖα; which seems to me very dubious), this corollarial reasoning will sometimes be a little puzzling to a student who has not so thoroughly assimilated what went before as to have the approximate proposition ready to his mind. After this, a sentence always using ἄρα, "hence," "*ergo*," repeats the πρότασις (not often the ἔκθεσις) so as to impress the proposition on the mind of the student, in its new light and new authority, expressed in the form most convenient in future applications of it. This is called συμπέρασμα, the "conclusion," which sounds highly Aristotelian. Yet the classical use of the verb to signify coming to a final conclusion, rendered this noun inevitable as soon as these neuter abstracts came into the frequent use that they

* But cf. the edition of 1884, p. 199.

had by Euclid's time. The conclusion always ends with the words ὑπερ ἔδει δεῖξαι, "which had to be shown," *quod erat demonstrandum,* for which Q. E. D. is now put.

I will take at random the twentieth proposition of the first book, to illustrate the matter. "In every triangle, any two sides, taken together are always greater than the third.

"For let ΑΒΓ be a triangle. I say that any two sides taken together are greater than the third; ΒΑ and ΑΓ than ΒΓ, ΑΒ and ΒΓ than ΑΓ, and ΒΓ and ΓΑ than ΑΒ.

"For extend ΒΑ to the point Δ, taking ΑΔ equal to ΓΑ [which he has shown in the second proposition always to be possible]; and join Δ to Γ by a straight line.

"Now since ΔΑ is equal to ΑΓ, the angle under ΑΔΓ is equal to that under ΑΓΔ [by the *pons asinorum*]. Hence, the angle under ΒΓΔ will be greater than that under ΑΔΓ. [This is a fallacy of a kind to which Euclid is subject from assuming that every figure drawn according to the παρασκευή will necessarily have its parts related in the same way, when it can only be otherwise if space is finite, which he has never formally adopted as a postulate. In the present case, if ΑΔ is more than half-way round space, the triangle ΑΓΔ will include the triangle ΑΒΓ within it; and then the angle ΒΓΔ will be less than the angle ΑΔΓ.] And since ΔΓΒ is a triangle having the angle under ΒΓΔ greater than that under ΒΔΓ, but the greater side subtends under the greater angle [which is the theorem that had just previously been demonstrated], therefore ΔΒ is greater than ΒΓ. But ΔΑ is equal to ΑΓ. Therefore, ΒΔ and ΑΓ are greater than ΒΓ. Similarly, we shall [*i.e.*, could] show that ΑΒ and ΒΓ are greater than ΓΑ, and ΒΓ and ΓΑ than ΑΒ.

"In every triangle, then, any two sides joined together are greater than the third, which is what had to be shown."

617. I will now return to the consideration of cyclical systems, and will begin by expressing my definition of such a system in those Existential Graphs which have been explained in *The Monist* [book II, ch. 6]. In reference to those graphs, it is to be borne in mind that they have not been contrived with a view to being used as a calculus, but on the contrary for a purpose opposed to that. Nevertheless, if anyone cares to amuse himself by drawing inferences by machinery, the

graphs can be put to this work, and will perform it with a facility about equal to that of my universal algebra of logic* and as much beyond that of my algebra of dyadic relatives,† of which the lamented Schroeder was so much enamoured.‡ The only other contrivances for the purpose appear to me to be of inferior value, unless it be considered worth while to bring a pasigraphy into use. Such ridiculously exaggerated claims have been made for Peano's system,§ though not, so far as I am aware, by its author, that I shall prefer to refrain from expressing my opinion of its value. I will only say that if a person chooses to use the graphs to work out difficult inferences with expedition, he must devote some hours daily for a

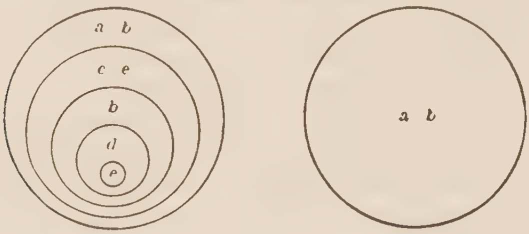


Fig. 231 Fig. 232

week or two to practice with it; and the most efficacious, instructive, and entertaining practice possible will be gained in working out his own method of using the graphs for his purpose. I will just give these little hints. Some slight shading with a blue pencil of the oddly enclosed areas will conduce to clearness. Abbreviate the parts of the graph that do not concern your work. Extend the rule of iteration and deiteration, by means of a few theorems which you will readily discover. Do not forget that useful iteration is almost always into an evenly enclosed area, while useful deiteration is, as usually, from an oddly enclosed area. Perform the iteration and the immediately following deiteration at one stroke, in your mind's eye. Do not forget that the ligatures may be considered as graph-instances scribed in the areas where their

* Cf. 3.499.

† Cf. 3.492ff.

‡ See his *Algebra u. Logik der Relative*, passim.

§ *E.g.*, by Russell in his *Principles of Mathematics*, p. 10.

least enclosed parts lie, and repeated at their attachments. Their intermediate parts may be disregarded. Reflect well on each of the four permissions* (especially that curious fourth one)† until you vividly comprehend the why and wherefore of each, and the bearings of each from every point of view that is habitual with you. Do not forget that an enclosure upon whose area there is a vacant cut can everywhere be inserted and erased, while an unenclosed vacant cut declares your initial assumption, first scribed, to have been absurd. You will thus, for example, be enabled to see at a glance that from Fig. 231 can be inferred Fig. 232. The cuts perform two functions; that of denial and that of determining the order of

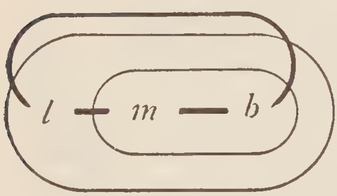


Fig. 233

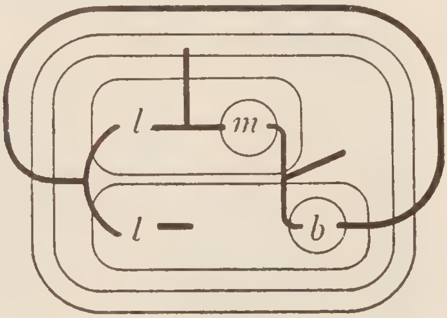


Fig. 234

selection of the individual objects denoted by the ligatures. If the outer cuts of any graph form a nest with no spot except in its innermost area, then all that part of the assertion that is therein expressed will need no nest of cuts, but only cuts outside of one another, none of them containing a cut with more than a single spot on it. It will seldom be advisable to apply this to a complicated case, owing to the great number of cuts required; but you should discover and stow away in some sentry-box of your mind whence the beck of any occasion may instantly summon it, the simple rule that expresses all possible complications of this principle. As an example of one of the simplest cases, Fig. 233 and Fig. 234 are seen precisely equivalent.

* See 565–69.
† See 569 and 580.

618. Owing to my Existential Graphs having been invented in January of 1897 and not published until October, 1906, it slipped my mind to remark when I finally did print a description of it, what any reader of the volume entitled *Studies in Logic by Members of the Johns Hopkins University* (Boston, 1883), might perceive, that in constructing it, I profited by entirely original ideas both of Mrs. and Mr. Fabian Franklin, as well as by a careful study of the remarkable work of O. H. Mitchell, whose early demise the world of exact logic has reason deeply to deplore.

619. My reason for expressing the definition of a cyclic system in Existential Graphs is that if one learns to think of relations in the forms of those graphs, one gets the most distinct and ecthetically as well as otherwise intellectually, iconic conception of them likely to suggest circumstances of theoric utility, that one can obtain in any way. The aid that the system of graphs thus affords to the process of logical analysis, by virtue of its own analytical purity, is surprisingly great, and reaches further than one would dream. Taught to boys and girls before grammar, to the point of thorough familiarization, it would aid them through all their lives. For there are few important questions that the analysis of ideas does not help to answer. The theoretical value of the graphs, too, depends on this.

620. Strictly speaking, the term "definition" has two senses — Firstly, this term is sometimes quite accurately applied to the composite of characters which are requisite and sufficient to express the signification of the "definitum," or predicate defined; but I will distinguish the definition in this sense by calling it the "definition-term." Secondly, the word definition is correctly applied to the double assertion that the definition-term's being true of any conceivable object would always be both requisite and sufficient to justify predicating the definitum of that object. I will distinguish the definition in this sense by calling it the "definition-assertion-pair." In the present case, as in most cases, it is needless and would be inconvenient to express the entire definition-assertion-pair with strict accuracy, since we only want the definition in order to prove certain existential facts of subjects of which we *assume* that the definitum, "cyclic-system," is predicable. We do not

care to *prove* that it is predicable, and therefore the assertion that the definitum is predicable of the definition-term is not relevant to our purpose. In the second place, we do not care to meddle with that universe of concepts with which the definition deals; and it would considerably complicate our premisses to no purpose to introduce it. We only care for the predication of the definition-term concerning the definitum so far as it can concern existential facts. All that we care to express in our graph is so much as may be required to deduce every existential fact implied in the existence of a cyclic system.

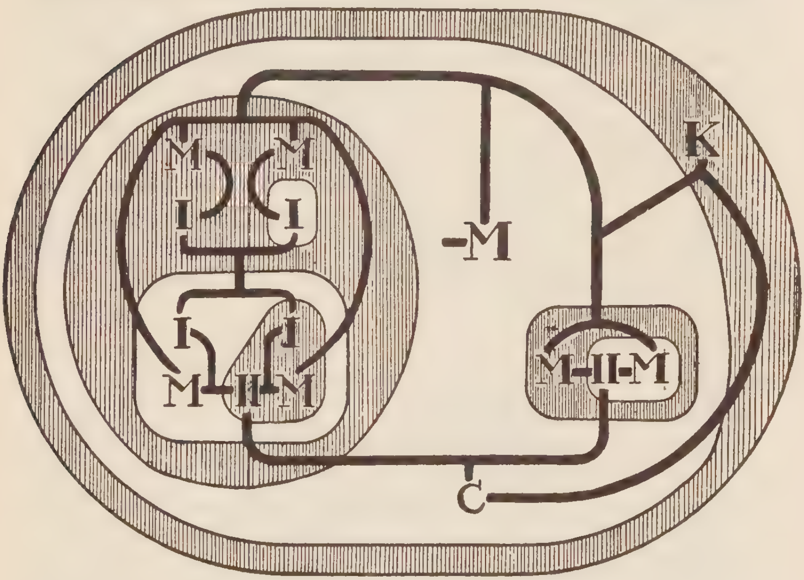


Fig. 235

621. A cyclic system is a system; and a system is a collection having a regular relation between its members. One member suffices to make a collection, and is requisite to the existence of the collection. The definition, so far as we need it, is then expressed in the graph of Fig. 235. Here K with a "peg" * at the side asserts that the object denoted by the peg is a cyclic system. The letter M with one peg at the top and another placed on either side without any distinction

* See 560.

of meaning, asserts that the object denoted by the side-peg is a *member* of the system denoted by the top-peg. The letter C, with a peg at the top and another at the side, asserts that the object denoted by the top-peg is a relation [*sic*] involved in that relation between all the members which constitutes the entire collection of them as the system that it is, and asserts that the object denoted by the side-peg is such a system. The Roman numerals each having one peg placed at the top or bottom of the numeral and a number of side-pegs equal to the value of the numeral, all these side-pegs being carefully distinguished, are used to express the truth of the proposition resulting from filling the blanks of the rheme denoted by the top or bottom peg, with indefinite signs of objects denoted by the side-pegs taken in their order, all the left-hand pegs being understood to precede all the right-hand pegs, and on each side a higher peg to precede a lower one. With this understanding, the graph of Fig. 235, where for the sake of perspicuity the oddly enclosed, or negating areas are shaded, may be translated into the language of speech in either of the two following equivalent forms (besides many others):

It is false that

there is a cyclic system while it is false that
this system has a member
and involves a relation ("being A to ," the bottom peg of II),
and that it is false that
the system has a member of which it is false that
it is in that relation, A, to a member of the system,
while it is false that
there is a definite predicate, P (the top or bottom peg of I), that is true of a member of
the system and is false of a member of the system,
and that it is false that
this predicate is true of a member of the system of which it is false that
it is A to a member of the system of which P is true.

This more analytic statement is equivalent to saying that every cyclic system (if there be any) has a member, and involves a relation called "being A to" (not the graph but perspicuity of speech requires it to be so named), such that every member of the system is A to a member of the system, and any definite predicate, P, whatsoever, that is at once true of one member of the system and untrue of another, is true of some member of the system that is not A to any member of which P is true.

622. To anybody who has no notion of logic this may seem a queer attempt to explain what is meant by a cyclic system; and it is true that it would be a needlessly involved *verbal* definition; a verbal definition being an explanation of the meaning of a word or phrase intended for a person to whose mind the idea expressed is perfectly distinct. But it is not intended to serve as a *verbal*, but as a *real* definition, that is, to explain to a person to whom the idea may be familiar enough, but who has never picked it to pieces and marked its structure, exactly how the idea is composed. As such, I believe it to be the simplest and most straightforward explanation possible. When you say that the days of the week "come round in a set of seven," you think of the week everything here expressed of K. I do not mean that all this is *actually* existent in your thought; for thinking no more needs the actual presence in the mind of what is thought than knowing the English language means that at every instant while one knows it the whole dictionary is actually present to his mind. Indeed, thinking, if possible, even *less* implies presence to the mind than knowing does; for it is tolerably certain that a mind to whom a word is present with a sense of familiarity knows that word; whereas a mind which being asked to *think* of anything, say a locomotive, simply calls up an image of a locomotive, has, in all probability, by bad training, pretty nearly lost the power of thinking; for really to think of the locomotive means to put oneself in readiness to attach to it any of its essential characters that there may be occasion to consider; and this must be done by general signs, not by an image of the object. But the truth of the matter will more fully be brought out as we proceed.

623. All that we require of the definition may be put into a simpler shape by omitting the letter M, since the interpreter of the graph must well understand that the whole talk of the graphist for the time being, so far as it refers to things and

not to the attributes or relations, has reference to the members of a cyclic system. We may consequently use the graph of Fig. 236 in place of Fig. 235.

It will be remarked that the graph of Fig. 236 is no more a definition of a cyclic system than it is of the relation of immediate antecedence; and this is as it should be; for plainly a system cannot be defined, without virtually defining the relation between its members that constitutes it a system.

Fig. 236

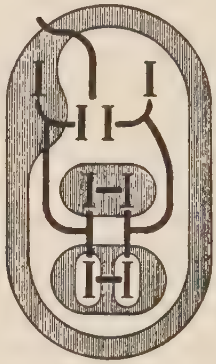


Fig. 237

624. I will now begin by drawing one of several corollaries that are right at my hand. I am always using the words *corollary* and *theorem* in the strict sense of the foregoing* definition. This corollary results from the logical principle that to every predicate there is a negative predicate which is true if the former is false, and is false if the former is true. This purely logical principle is expressed in the graph of Fig. 237. Obviously, if any predicate is both true of some member and false of some member of the system, the same will be the case with its negative. Consequently, by the definition, this negative will be true of some member without being true of any to which that member is A; or, in other words, the original predicate will be false of some member without being A to any member of which it is false. Thus, if any predicate is

* See 613.

neither true of all nor false of all the members of any cyclic system, but is true of some one and false of some other, there will be two different members of one of which it is true without being true of any to which that member is A, while of the other it is false without being false of any to which that member is A. Or, to put the corollary in a different light, taking any predicate, P, whatsoever, then, in case you can prove that there cannot be more than one exception to the rule that every member of the system resembles some one of those to which it is A in respect to the truth or falsity concerning it of P, then if P be true of one member, it is true of all, and if it be false of one, it is false of all.

625. I am now going to apply this proposition to a theoric proof of a proposition which is really only a corollary from the definition of a cyclic system. My motive for this departure from good method is that it will afford a good illustration of the advantage of making the selected predicate, P, as special and characteristic of the state of things you are reasoning about as possible. The proposition I am going to prove is, that in any cyclic system that contains more than one member no member will be A to itself. For this purpose I will consider any member of the system you please, and will give it the proper name, N. This ecthetic step is already theoric, but is a matter of course. Another theoric step, not a matter of course, shall consist in my selecting, as the predicate to be considered, "is N." Now if N is A to itself, every member of the system of which this predicate is true (which can be none other than N itself) will be A to a member of which the predicate is also true; and consequently, by the definition of a cyclic system this predicate cannot be true of one member and false of another. But if there be any other member of the system than N, it will be false of that one. Whence, if N were A to itself and were not the only member of the system, there would be no member of which it would be true that it was N. But by the definition, every cyclic system has some member, and N was chosen as such. So that it must be, either that the system has no other member, or that any member you please, and consequently *every one*, is non-A to itself.

Now what I wanted to point out was that if instead of "is N," I had selected, as my predicate to be considered, "is

A to itself," it would merely have followed that since any member that is A to itself is A to a member that is A to itself, by the general definition either every member of the system is A to itself or none is so.

I will now prove that this proposition, that no member of a cyclical system is A to itself unless it is the only member of the system, is not a theorem, in any strict sense, by proving it corollarially. For this purpose I first prove that no cyclical system, by virtue of the same relation A, involves another as a part, but not the whole of it. For suppose that certain members of a cyclical system form by themselves a cyclical system constituted by the same A-hood. Then, by the part of the definition of a cyclical system that has been expressed as graph in Fig. 235 and in Fig. 236, there is a member of this minor system; and every member of it is A to a member of the major system that is a member of the minor system. Hence, by that same partial definition, the predicate "is a member of the minor system" being true of one is true of all members of the major system. The minor system is, then, the whole of the major system. To go further, I must employ that assertion of the definitum "is a cyclic system" concerning the definition-term, which assertion has not been expressed as a graph, in order to prove, by its conformity with the definition that a single object, having a relation, identity, to itself, that relation conforming to the conditions of the constitutive relation of a cyclical system, must be admitted to be a cyclical system of a single member. If, therefore, one of the members of a cyclical system of more than one member were A to itself, it would be a cyclical system which was a part but not the whole of another cyclical system, which we have seen to be impossible.

626. I shall now employ the first corollary to prove that every member of a cyclical system is A'd by some member. For take any member you please of any such system you please; and I will assign to it the proper name N. If then, N is the only member of the system, by the definition N is A to itself. But if there be another member, it is one of which the predicate "is N" is not true, though there is some member, namely N, of which that predicate is true. Consequently, by that first corollary, there must be a member of which it is not

true that it is N which is A to nothing of which this is not true. But, by the definition, every member of a cyclic system is A to some member; and therefore that member which is not A to any member of which "is N" is not true, must be true of a member of which "is N" is true, which, by hypothesis, is only N itself; consequently any member of any cyclic system which one may choose to select is A'd by some member, and by another than itself, if there be another. Q. E. D.

627. Further investigation of the properties of cyclic systems will need a somewhat more recondite theoric step. Certainly, however, I must not convey the idea that I claim to be quite sure of this. As yet, I have not sufficiently studied the methodeutic of theorematic reasoning. I only have an indistinct apprehension of a principle which seems to me to prove what I say; and I must confess that of all logical habits that of confiding in deductions from vague conceptions is quite the most vicious, since it is just such reasonings that to the intellectual rabble are the most convincing; so that the conclusions get woven into the general common sense so closely, that it at length seems paradoxical and absurd to deny them, and men of "good sense" cling to them long after they have been clearly disproved. However, whether it be absolutely necessary or not, the only way I see, at present, of demonstrating the remaining properties of a cyclic system is to suppose a predicate to be formed by a process which will seem somewhat complicated. I shall not state what this predicate is, but only suppose it to be formed according to a rule; and even this rule will not be exactly stated but only a description of its provisions will be given. I shall suppose that one member of the system is selected by the rule as one of the class of subjects of which the predicate is true, and that the remaining members of this class shall be taken into it from among the members of the system *one by one*, according to the rule that when the member last taken in is not A to any member already taken in, one and one only of the members of the system not yet taken in to which that last adopted member is A is to be added to the class; and this new addition may, in the same way, require another. If the system were infinite (as we shall soon see that it cannot be), this might go on endlessly; and so far, we have not seen that this cannot happen. But as

soon as it happens that the member last admitted to the class is A to a member already admitted (and consequently that every member admitted to the class is A to an admitted member) the admissions to the class are to be brought to a stop. There are now two supposable cases to be provided for which we shall later find will never occur; but if we did not determine what was to be done if they should (this not being proved impossible) our first proof would involve a *petitio principii.* One is the case in which the finally adopted member is A to a member already having an A that had previously been admitted to the class. The other is the case in which the last (but not necessarily the final) adopted member is not only A'd by the *last previously* adopted member (for the sake of providing which with a member A'd by it, the very last was taken in) but is also A'd by an earlier adopted member. In the latter case, in which the member last adopted, which we may name V, is not only A'd by the last previous one, which we may name U, but is also A'd by a previously adopted member of the class which we may name K, we are to reject from the class all that were admitted after K to U inclusive; so that we revert to what would have been the case, as it might have been, if next after K we had admitted V, to which K is A. We should thus make the class smaller, which we shall soon see could not happen. In the other case, where the last adopted member, which we will name Z, is A to a previously adopted one, which we will name J, which was not the first member adopted into the class, but is A'd by another, which we will name I, we reject from the class both I and all that were adopted previously to I.

After these supposititious rejections, there is no object of which the predicate, "is a member of the class so formed," is true that is not A of any object of which the same predicate is true, and therefore, by the definition so often appealed to, this predicate cannot be both true of a member of the cyclic system and false of another such member. Now it plainly is true of some member, since the first object taken into it as well as every one subsequently taken into it were members of the cyclic system. Therefore, this predicate cannot be false of any member of the cyclic system. In other words, the class so formed includes all the members of the cyclic system. Consequently, there cannot have been any rejections.

Since there were no rejections, the first member adopted must remain a member of the class; and since we have seen in a former corollary that every member of a cyclic system is A'd by a member of the same system, this first adopted member must be A'd by some member of the system, that is, by some member of the class. But by the rule of formation of the class no member of it except the finally adopted one can be A to a previously adopted member. It follows that there must be a finally adopted one; and by the same rule no member of the class except the first was adopted without there being a *last previously adopted* member. It follows that the succession of adoptions cannot, at any part of it, have been endless. This is one of the most difficult theorems that I had to prove.

Moreover, every member of the class is by the mode of formation A to one, and only to one, member of the class; and consequently the same is true of all the members of every cyclic system.

Moreover, every member of the class except the first was only taken in so as to be A'd by the last, or, at any rate, by one member only; and the first adopted member as we have seen is A'd by the finally adopted member. It cannot be A'd by any other, since by the rule of formation, such another would thereby have become the finally adopted member. Hence, no member of a cyclic system is A'd (in the same sense) by any two members of the system; or no two members are A to the same member.

628. I have thus, by means of this θεωρία of the formation of a certain kind of class, succeeded in demonstrating, what one might well have doubted, that from the proposition expressed in Fig. 235 follows the double uniqueness of the cyclical relation of A-hood or immediate antecedence. This is the principal, as I think, of those properties that are common and peculiar to cyclical systems. The same theoric step, or a reduplication of it, will enable the reader to prove other properties, common but not peculiar to cyclic systems; and especially that a collection the count of whose members in one order comes to an end can never in any order involve an endless process, whether it comes to an end or does not. There is, by the way, an important logical interest in that mode of succession in which an endless succession, say, of odd

numbers, is followed by a beginningless diminishing succession of even numbers. For it shows that two classes of objects may have such a connection with a transitive relation, such as are those of causation, logical implication, etc., that any member of either class is *immediately* in this relation only to a member of the same class, while yet every member of one of the classes may be in this same relation to every member of the other class. Thus, it may be that thought only acts upon thought *immediately*, and matter *immediately* only upon matter; and yet it may be that thought acts on matter and matter upon thought, as in fact is plainly the case, somehow.

629. In this theoric step, it is noticeable that I have had to embody the idea of *antecedence* generally, in order to prove the properties of cyclical *immediate antecedence*. Any reasoner is always entitled to assume that the mind to which he makes appeal is familiar with the properties of antecedence in general; since if he were not so, he could not even understand what reasoning was at all about. For logical antecedence is an idea which no reasoner can unload or dispense with. It would have been easy to replace, in my demonstrations, all the "previously"s, etc., by relations of inference. I have not done so in order not to burden the reader's mind with needlessly intricate forms of thought.

630. A corollary from what has already been proved is that if we regard the definition of Fig. 236 as the definition of A-hood, or cyclical immediate antecedence, then A-hood is not a single relation but is any one of a class of relations which, if the collection of all the members of the system is not very small, is a large class. For taking any two members of the system, and naming them Y and ϒ, we may form such a relation, that of A′-hood, that whatever is neither Y nor ϒ, nor is A to Y nor to ϒ, is A′ to whatever is A'd by it, while whatever is A to Y is A′ to ϒ, whatever is A to ϒ is A′ to Y, whatever is A'd by Y is A″d by ϒ, and whatever is A'd by ϒ is A″d by Y; and then A′ will have the same general properties as A. Thus, if the number of members of a cyclic system is m, the number of relations of A-hood is $(m-1)!$ if m be seven, the number of A-relations is 720; etc.

631. There is no relation in a cyclic system exactly an-

swering to general antecedence in a denumeral[1] system.

632. As a finitude is a positive complication (as is shown by a form of inference being valid in a finite system that is not elsewhere valid) so in place of the relation of betweenness which in a linear system endless both ways, which, if those ways are not distinctively characterized, is triadic, we have in a cyclic system a tetradic relation expressible by α with four tails, so that Fig. 238, which means that an object which can, wherever it be in the cycle, pass from its position to that which is *next* to that position, being either A to it or A'd by it, will if at I be *opposite* to an object at J, relatively to any objects at U and at V. That is, such an object cannot move from I to J without passing through U and V. This implies that U is opposite to V relatively to I and J; that no other pair out of the four are opposite to each other relatively to the

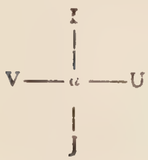


Fig. 238

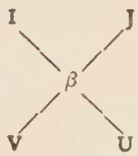


Fig. 239

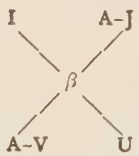


Fig. 240

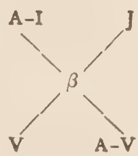


Fig. 241

other pair; and that that way of passing round the cycle in which U is reached next after I is the way in which J is reached next after U, V next after J, and I next after V; while that way in which V is reached next after I is the way in which J is reached next after V, U next after J, and I next after U. This supposes that I, J, U, and V are all different, as those that are opposite must be unless two that are adjacent are identical, in which case we may understand the relation as always being true and meaningless.

We may modify this relation, so as to render it exact, by defining Fig. 239 as true, if I and J are identical while U and V are also identical; or if I and U are identical while J and V are identical, and also if Fig. 240 or Fig. 241 is true; but as not true unless necessarily so according to these principles. This last clause, by the way, has a very important logical form; but I shall not stop to comment upon it.

[1] See Note at the end of the article [639ff].

It will be observed that if Fig. 239 is true, then one or other of the graphs Figs. 242 and 243 must be true. And if two α-relations hold, having three of their four correlates identical,

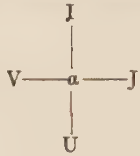


Fig. 242

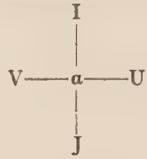


Fig. 243

and not the same pair being opposite in both, then two α-conclusions may be drawn in which the two correlates that only appeared once each in the premisses, appear together, and

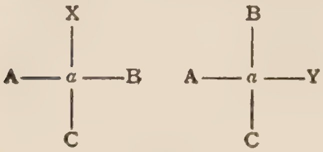


Fig. 244

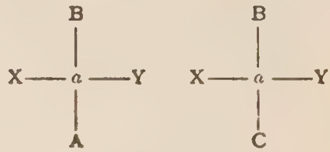


Fig. 245

opposite to one another. Thus, from Fig. 244 may be inferred Fig. 245. The β-relation lends itself to much further inferential

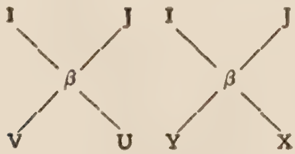


Fig. 246

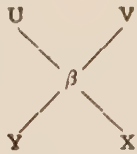


Fig. 247

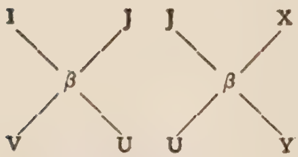


Fig. 248

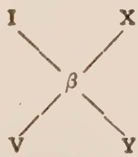


Fig. 249

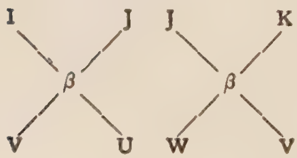


Fig. 250

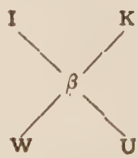


Fig. 251

procedure. In the first place in Fig. 239, the whole graph may be turned round on the paper so as to bring each correlate into the place of its opposite. It may also be turned through 180° round a vertical axis in the sheet. (It may consequently be turned 180° round a horizontal axis in the sheet.) Moreover, the two correlates on the left, I and V, may be interchanged.

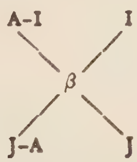


Fig. 252

(And so, consequently, may J and U.) Moreover, from Fig. 246 we can infer Fig. 247. (Whence it follows that from Fig. 248 we can infer Fig. 249.) Also, from Fig. 250 we can infer Fig. 251. Whence there follow very obviously several transformations. For example, Fig. 252 will be true; and if any three of the four graphs of Fig. 253 are true, so is the other

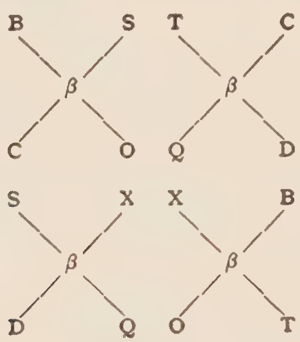


Fig. 253

one. It is obvious that the relation β involves cyclical addition-subtraction, by its definition.

633. Cyclic arithmetic involves no other *ordinal*, or climacote, numbers than cyclic ordinals. But if we define a *cardinal* number as an adjective essentially applicable, universally and exclusively, to a plural of a single multitude, then even the relations α and β may be said to depend upon the value of a cardinal number; namely, upon the modulus of the

cycle; and no cardinal number is cyclic. Dedekind and others* consider the pure abstract integers to be ordinal; and in my opinion they are not only right, but might extend the assertion to all real numbers.† (But what I mean by an ordinal number precisely must be explained further on.‡) Nevertheless, the operations of addition, multiplication, and involution can be more simply defined if they are regarded as applied to cardinals, that is to multitudes, than if they are regarded in their application to ordinals.

Thus, the sum of two multitudes, M and N, is simply the multitude of a collection composed of the mutually exclusive collections of the multitudes M and N. The ordinal definition, on the other hand, must be that $0+X=X$, whatever X may be, while (the ordinal next after Y)+X is the ordinal next after (Y+X). So the product of two multitudes M and N is simply the multitude of units each composed of a unit of a collection of multitude M and a unit of multitude N; while the ordinal definition must be that $0\times 0=0$ and that $X\times$(the ordinal next after Y) is $X+(X\cdot Y)$ and the ordinal next after $X\times Y$ is $(X\cdot Y)+Y$. So finally the multitude M raised to the power whose exponent is N, is the multitude of ways in which every member of a collection of multitude N can be related in

X	Y	Z	X	Y	Z	X	Y	Z
AB			A	B		A		B
B	A			AB			A	B
B		A		B	A			AB

Fig. 254

a given way, each to some single member or other of a collection of multitude M. Thus $3^2=9$ because the different configurations of Fig. 254 are nine in number; while $2^3=8$ because the different configurations of Fig. 255 are eight in

* *E.g.*, Schröder.

† Cf. 332ff, 659ff, 673ff.

‡ See 635.

number. But a definition of involution which shall be *purely*

A	B		A	B		A	B		A	B
XYZ			XY	Z		XZ	Y		X	YZ

A	B		A	B		A	B		A	B
YZ	X		Y	XZ		Z	XY			XYZ

Fig. 255

ordinal must be quite a complicated affair. We may say, for example, that $X^1 = X$ and $X^{1+Y} = X \cdot X^Y$.

634. In cyclic addition, that is, in the α and β relations, there is but a single cardinal number to be dealt with; and this is fully dealt with in counting round and round the single cycle. But in multiplication there is always another cycle, and thus another cardinal number to be considered, although the modulus of the second cycle is usually such that it is not brought to our attention. But suppose that in a cycle of 72 we multiply the successive integers from zero up by 54. The following will be the result:

$$0 \times 54 = 0 = 72$$
$$1 \times 54 = 54 = -18$$
$$2 \times 54 = 36$$
$$3 \times 54 = 18$$
$$4 \times 54 = 72 = 0$$

It will be seen that there is a cycle of modulus 4. Suppose that, instead of 54, we take 27 as the multiplicand. Then we shall have

$$0 \times 27 = 0 = 72$$
$$1 \times 27 = 27$$
$$2 \times 27 = 54 = -18$$
$$3 \times 27 = 9$$
$$4 \times 27 = 36$$
$$5 \times 27 = 63 = -9$$
$$6 \times 27 = 18$$
$$7 \times 27 = 45 = -27$$
$$8 \times 27 = 72 = 0$$

By halving the multiplicand we have doubled the modulus. Suppose, however, that, instead of $\frac{1}{2} \times 54$, we take $\frac{1}{3} \times 54 = 18$, as the multiplicand. Read the column of successive multiples of 54 upwards, and we shall see that the multiples of 18 have a cycle of modulus 4.

With 6 as the multiplicand we get a cycle of 12 for its multiples, the numbers being as follows:

$$6, 12, 18, 24, 30, 36, -30, -24, -18, -12, -6, 0$$

With 2×6 we get a cycle of $\frac{1}{2}\times12$, every other one. With 4×6 as multiplicand, we get a cycle of $\frac{1}{4}\times12=3$, with 8×12 as multiplicand; since 3 cannot be halved we still get 3. With $3\cdot6=18$ as multiplicand; we get a cycle of $\frac{1}{3}\times12$, or every third of the multiples of 6; but with $3\cdot18=54$ as modulus, since 4 is not divisible by 3, we still get a cycle of 4. With $6\cdot6=36$ as multiplicand, we get every sixth multiple of 6, or two in all, 0 and 36. With 5×6, 7×6, and 11×6 since 12 is not divisible by 5, 7, or 11, we still get a modulus of 12. With 30, the order is as follows:

$$0, 30, -12, 18, -24, 6, 36, -6, 24, -18, 12, -30, 0.$$

This principle is obvious: if the multiples of a number N form a cycle of modulus K, and p is a prime number, then the multiples of pN will form a cycle of K/p, provided K is divisible by p; but otherwise, the modulus will remain K. Suppose, then, that the cycle of multiples of 1, that is to say, the cycle of our entire system of numbers is $p^a\cdot q^b$, where p and q are primes, and a and b are any whole numbers. If, then, we multiply 1 by $r^c\cdot s^d\cdot t^e$, where r, s, t are other primes than p and q, the modulus of the cycle of multiples of $r^c\cdot s^d\cdot t^e$ will remain $p^a\cdot q^b$. But every time we multiply this by p we divide the modulus by p, until we have so multiplied it a times. On the other hand, if, instead of multiplying 1 by $r^c\cdot s^d\cdot t^e$, we multiply it by $p^a\cdot q^b$ to get a new multiplicand, the modulus of the cycle of multiples of $p^a\cdot q^b$ will be 1; that is, all multiples will be equal. It will follow by the distributive principle, that $p^a\cdot q^b$ added to any number leaves that number unchanged. That is to say, the modulus of a cycle is the *zero* of that cycle. But right here I must explain what I mean by an *ordinal number*.

635. Take any enumerable, or finite, collection of distinct objects. Let there be recognized one special relation in which

each of them stands to a single one of them, and no two to the same one, and such that any predicate whatsoever that is true of any one of them and is true of the one to which any one of which it is true stands in that relation, is true of all of them. This substantially defines that relation as the relation of "being A'd by." Thereby, that collection is recognized as forming a cyclical system of which those objects are members. But those objects will not in general be numbers of any kind. They may be days of the week or certain meridians of the Globe. But now consider a single "step," or substitution, by which the A of any member of the cyclic system is replaced by the member itself. From what member this step, or substitution, began remains indefinite. The "step" still leads to a single member, and the step is a single kind of step even if that member be any member you please, in which case it is not a single, *i.e.*, a singular, but the general member. I will condescend to meet the reader's probably indurated habit of crass nominalist thought by saying that, in the one case, it is a single member not definitely described, and in the other is a single member, left to him to choose; and there is no objection to this, if the member be supposed to be both existent and intelligible, both of which however it need not be. Give this kind of a step a proper name. Next consider in succession all the kinds of step each of which consists in first taking a step of the last previously considered kind and then substituting for the member which it puts in place of another, the member of which that member is A; so that the kinds of steps may be

From the A of a member to that member,
From the A of the A of a member to that member,
From the A of the A of the A of a member to that
member, etc., etc.

Now if each of these has a name, whether pronounced, scribed, or merely thought, those names will come round in a cycle of the same modulus as the original system. They will therefore form a cyclic system, but not a system of objects not essentially ordered, as the original system may have been. This system of names is a cyclic system of numbers. These are ordinal, or climacote, numbers. By ordinal numbers in general I mean names essentially denoting kinds of steps each from any member whatever of a system of objects to, at most, a single object of the system (*i.e.*, one or another object, depending on what object the step replaces by this other). Thus, as I use the term "ordinal number" I do not mean the absolute first, second, third, etc. member of a row of objects, but rather such as these: the same as the first after, the second after, the third before, etc. These numbers are certainly "ordinal" in the sense of expressing relative order; yet it might be better to avoid possible misunderstanding by calling them *metrical numbers*, or more specifically, *climacode* or *climacote numbers*.

636. In order to push further our study of this subject, let us suppose a pack of 72 cards, numbered in order upon their faces, to be dealt into two piles. We will not directly consider those serial face-values, but only their differences. The two piles cannot regularly be reunited, because the difference of successive face-values in each, comes round in a cycle in each pile, the bottom card of the one pile, 1, being 2 more than the top card 71 (counting round the cycle of modulus 72) and that of the other pile also coming round in a cycle. The difference between the face-values of any two cards in either pile is a multiple of 2, the multiplier being the difference of position in that pile. If now we desire so to re-deal the cards of the one pile and the other into any number n of piles, as to produce the same effect as if they had originally been dealt into $2n$ piles, we must first deal the first pile leaving room between every two of the new piles for the piles to be produced by dealing the second pile. If for the number, n, we take 8, we shall get sixteen piles, the first 8 of 5 cards each and the last 5 of 4; and now it is allowable and proper to place each of the first 8 piles on the pile 8 piles further advanced; or equally so to place each of the last 8 piles on the pile 8 piles further advanced, counting round and round the cycle of modulus 16. In either case the cards of each composite pile so formed will form a cycle, successive face-values increasing (round and round the cycle of 72) by 16. The rule for gathering the piles is just the same as that previously given, except that one must confine oneself to piles *of the same set*. For instance if 72 cards, numbered as just de-

scribed, get in any way dealt into 15 piles, the top cards of the piles will have these values:

61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 58, 59, 60

Now since 15 | 72 = 3 these are in 5 sets of 3 piles, thus

61,	64,	67,	70,	58,
62,	65,	68,	71,	59,
63,	66,	69,	72,	60.

We shall therefore put the pile headed by 72 on the pile headed by 69, because there is only one pile of the set to the right of the former, and these on the pile headed by 66, and these on that headed by 63, and finally all four on the one headed by 60. So we shall in the next set begin with the pile headed by 71, the last of the larger piles.

We shall thus get the whole pack divided into three portions, and there is absolutely no way of getting them back into a single pack except by *undealing* them, that is by cutting the cards one by one from the three portions in turn, round and round.

This general rule holds in all cases; as much when the entire number of cards is prime as when it is composite. For a prime number is one whose greatest common divisor with any smaller positive integer is 1, while, of course, like any other number, its greatest divisor common to itself is itself.

637. Having thus fully explained the dealing into any number of piles of any number of cards, prime or composite, I revert, after this almost interminable disquisition, to the subject of cyclic logarithms. I have confined, and shall continue to confine, my study of these to logarithms of numbers whose cycle has a prime modulus. Then, the modulus of the cycle of the logarithms being one less than that of the natural numbers cannot be prime. Still so long as it is a question of employing the logarithms merely to multiply two numbers, the logarithm of the product is simply the sum of the logarithms of multiplier and multiplicand; and in addition it makes no difference whether the modulus be prime or composite. But when it comes to raising numbers to powers or to extracting their roots, the divisors of the number one less than the modulus have to be considered. The modulus being prime,

the number one less must be divisible by 2. If 2 be the only prime factor, the modulus must be 3 or 5 or 17 or 65537 or much greater yet. As an example, let us take the modulus 17. Then the following two pairs of tables show the logarithms for the 8 different bases, 3, 5, 6, 7, 10, 11, 12, 14.

Nat. nos.	−16	−14	−8	−7	−4	−12	−2	−6	−1	−3	−9	−10	−13	−5	−15	−11	−16
	1	3	9	10	13	5	15	11	16	14	8	7	4	12	2	6	1
Logs.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	−16	−15	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0

Nat. nos.	−16	−15	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Logs.	0	14	1	12	5	15	11	10	2	3	7	13	4	9	6	8
	−16	−2	−15	−4	−11	−1	−5	−6	−14	−13	−9	−3	−12	−7	−10	−8

Nat. nos.	−16	−12	−9	−11	−4	−3	−15	−7	−1	−5	−8	−6	−13	−14	−2	−10
	1	5	8	6	13	14	2	10	16	12	9	11	4	3	15	7
Logs.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	−16	−15	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1

Nat. nos.	−16	−15	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Logs.	0	6	13	12	1	3	15	2	10	7	11	9	4	5	14	8
	−16	−10	−3	−4	−15	−13	−1	−14	−6	−9	−5	−7	−12	−11	−2	−8

Of course, none of the even numbers can be logarithms of a possible base of another system since with a modulus 16 no multiple of an even number can be 1, the logarithm of the base. On the other hand, every odd number is in every system of logarithms the logarithm of some base.

638. If, instead of 13 cards and 12, the "trick" be done with 17 and 16, say the first eight hearts *increasingly* and then the first eight diamonds *decreasingly*, with the joker or king of hearts to make up 17 and with the first eight spades to correspond with the hearts and the first eight clubs to correspond with the diamonds, laying down the black cards on the table, in *two* rows, one of eight from left to right, and the other below from right to left, after having dealt the black cards 16 times into three piles and every time exchanging the top card of the middle pile for the topmost red card, so as to bring the ace of spades into the right-handmost place of the

upper row, then having done the trick substantially as above described, there is a very pretty way in which you can ask into what *odd* number of piles the *black* cards shall be dealt and then dealing out the red cards, *minus* the extra one, 16 times exchanging a card each time for the three court cards and ten of each suit, so as to again render the black ones the index of the places of the red ones. But I leave it to the reader's ingenuity to find out exactly how this is to be done. *Beware of the moduli.*

There is much more to be said on this subject, but I leave it for the reader to investigate.

§3. A NOTE ON CONTINUITY*[E]

639. Denumeral is applied to a collection in one-to-one correspondence to a collection in which every member is immediately followed by a single other member, and in which but a single member does not, immediately or mediately, follow any other. A collection is in one-to-one correspondence to another, if, and only if, there is a relation, *r*, such that every member of the first collection is *r* to some member of the second to which no other member of the first is *r*, while to every member of the second some member of the first is *r*, without being *r* to any other member of the second. The positive integers form the most obviously denumeral system. So does the system of all real integers, which, by the way, does not pass through infinity, since infinity itself is not part of the system. So does a Cantorian collection in which the endless series of all positive integers is immediately followed by ω_1, and this by ω_1+1, this by ω_1+2, and so on endlessly, this endless series being immediately followed by $2\omega_1$. Upon this follow an endless series of endless series, all positive integer coefficients of ω_1 being exhausted, whereupon immediately follows ω_1^2, and in due course $x\omega_1^2+y\omega_1+z$, where x, y, z, are integers; and so on; in short, any system in which every member can be described so as to distinguish it from every other by a finite number of characters joined together in a finite number of ways, is a denumeral system. For writing

* This note was referred to in 631. Cf. also 121ff, 200ff, 219ff.

the positive whole numbers in any way, most systematically thus:

1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, etc.

it is plain that an infinite square matrix of pairs of such numbers can be arranged in one series, by proceeding along successive bevel lines thus: (1, 1); (1, 10); (10, 1): (1, 11); (10, 10); (11, 1): (1, 100); (10; 11); (11; 10); etc., and consequently whatever can be arranged in such a square can be arranged in one row. Thus an endless square of quaternions such as the following can be so arranged:

[(1, 1) (1, 1)]:[(1, 1) (1, 10)]; [(1, 1) (10, 1)]:[(1, 1) (1, 11)]; etc.
[(1, 10) (1, 1)]:[(1, 10) (1, 10)]; [(1, 10) (10, 1)]:[(1, 10) (1, 11)]; etc.
[(10, 1) (1, 1)]:[(10, 1) (1, 10)]; [(10, 1) (10, 1)]:[(10, 1) (1, 11)]; etc.
[(1, 11) (1, 1)]:[(1, 11) (1, 10)]; [(1, 11) (10, 1)]:[(1, 11) (1, 11)]; etc.

Consequently whatever can be arranged in a block of any finite number of dimensions can be arranged in a linear succession. Thus it becomes evident that any collection of objects, every one of which can be distinguished from all others by a finite collection of marks joined in a finite number of ways can be no greater than the denumeral multitude. (The bearing of this upon Cantor's ω^ω is not very clear to my mind.) But when we come to the collection of all irrational fractions, to exactly distinguish each of which from all others would require an endless series of decimal places, we reach a greater multitude, or grade of maniness, namely, the *first abnumerable multitude*. It is called "abnumerable," to mean that there is not only no way of counting the single members of such a collection so that, at last, every one will have been counted (in which case the multitude would be *enumerable*), but, further, there is no way of counting them so that every member will after a while get counted (which is the case with the single multitude called *denumeral*). It is called the *first* abnumerable multitude, because it is the smallest of an endless succession of abnumerable multitudes each smaller than the next. For whatever multitude of a collection of single members μ may denote, 2^μ, or the multitude of different collections, in such collection of multitude μ, is always greater than μ. The different members of an abnumerable collection are not capable of being distinguished, each one from all others, by any finite collection of marks or of finite sets of marks. But by the very definition of the first abnumerable multitude, as being the multitude of collections (or we might as well say of denumeral collections) that exist among the members of a denumeral collection, it follows that all the members of a first-abnumerable collection are capable of being ranged in a linear series, and of being so described that, of any two, we can tell which comes earlier in the series. For the two denumeral collections being each serially arranged, so that there is in each a first member and a singular next later member after each member, there will be a definite first member in respect to containing or not containing which the two collections differ, and we may adopt either the rule that the collection that contains, or the rule that the collection that does not contain, this member shall be earlier in the series of collections. Consequently a first-abnumerable collection is capable of having all its members arranged in a linear series. But if we define a *pure* abnumerable collection as a collection of all collections of members of a denumeral collection each of which includes a denumeral collection of those members and excludes a denumeral collection of them, then there will be no two among all such pure abnumerable collections of which one follows next after the other or of which one next precedes the other, according to that rule. For example, among all decimal fractions whose decimal expressions contain each an infinite number of 1s and an infinite number of 0s, but no other figures, it is evident that there will be no two between which others of the same sort are not intermediate in value. What number for instance is next greater or next less than one which has a 1 in every place whose ordinal number is prime and a zero in every place whose ordinal number is composite? .111010100010100010100010000001 etc. Evidently, there is none; and this being the case, it is evident that all members of a pure second-abnumerable collection, which both contains and excludes among its members first-abnumerable collections formed of the members of a pure first-abnumerable collection, cannot, in any *such* way, be in any linear series. Should further investigation prove that a second-abnumeral multitude can in *no way* be linearly arranged, my former opinion* that the common conception of a line im-

* See *e.g.* 3.567f.

plies that there is room upon it for any multitude of points whatsoever will need modification.

640. Certainly, I am obliged to confess that the ideas of common sense are not sufficiently distinct to render such an implication concerning the continuity of a line evident. But even should it be proved that no collection of higher multitude than the first abnumerable can be linearly arranged, this would be very far from establishing the idea of certain mathematico-logicians that a line consists of points.* The question is not a physical one: it is simply whether there can be a consistent conception of a more perfect continuity than the so-called "continuity" of the theory of functions (and of the differential calculus) which makes the continuum a first-abnumerable system of points. It will still remain true, after the supposed demonstration, that no collection of points, each distinct from every other, can make up a line, no matter what relation may subsist between them; and therefore whatever multitude of points be placed upon a line, they leave room for the same multitude that there was room for on the line before placing any points upon it. This would generally be the case if there were room only for the denumeral multitude of points upon the line. As long as there is certainly room for the first denumerable multitude, no denumeral collection can be so placed as to diminish the room, even if, as my opponents seem to think, the line is composed of actual determinate points. But in my view the unoccupied points of a line are mere possibilities of points, and as such are not subject to the law of contradiction, for what merely *can be* may also *not be*. And therefore there is no cutting down of the possibility *merely* by some possibility having been actualized. A man who can see does not become deprived of the power merely by the fact that he has seen.

641. The argument which seems to me to prove, not only that there is such a conception of continuity as I contend for, but that it is realized in the universe, is that if it were not so, nobody could have any memory. If time, as many have thought, consists of discrete instants, all but the feeling of the present instant would be utterly non-existent. But I have argued this elsewhere.† The idea of some psychologists of meet-

* *E.g.*, Russell, *Principles of Mathematics*, p. 437.

† See *e.g.*, 5.289.

ing the difficulties by means of the indefinite phenomenon of the span of consciousness betrays a complete misapprehension of the nature of those difficulties.

642. *Added*, 1908, May 26. In going over the proofs of this paper, written nearly a year ago, I can announce that I have, in the interval, taken a considerable stride toward the solution of the question of continuity, having at length clearly and minutely analyzed my own conception of a *perfect continuum* as well as that of an *imperfect continuum*, that is, a continuum having *topical singularities*, or places of lower dimensionality where it is interrupted or divides. These labors are worth recording in a separate paper, if I ever get leisure to write it.* Meantime, I will jot down, as well as I briefly can, one or two points. If in an otherwise unoccupied continuum a figure of lower dimensionality be constructed — such as an oval line on a spheroidal or anchor-ring surface — either that figure is a part of the continuum or it is not. If it is, it is a topical singularity, and according to my concept of continuity, is a breach of continuity. If it is not, it constitutes no objection to my view that all the parts of a perfect continuum have the same dimensionality as the whole. (Strictly, all the *material*, or *actual* parts, but I cannot now take the space that minute accuracy would require, which would be many pages.) That being the case, my notion of the essential character of a perfect continuum is the absolute generality with which two rules hold good, first, that every part has parts; and second, that every sufficiently small part has the same mode of immediate connection with others as every other has. This manifestly vague statement will more clearly convey my idea (though less distinctly) than the elaborate full explication of it could. In endeavoring to explicate "immediate connection," I seem driven to introduce the idea of time. Now if my definition of continuity involves the notion of immediate connection, and my definition of immediate connection involves the notion of time; and the notion of time involves that of continuity, I am falling into a *circulus in definiendo*. But on analyzing carefully the idea of Time, I find that to say it is continuous is just like saying that the atomic weight of oxygen is 16, meaning that that shall be the standard for all other

* That paper does not seem to have been written.

atomic weights. The one asserts no more of Time than the other asserts concerning the atomic weight of oxygen; that is, just nothing at all. If we are to suppose the idea of Time is wholly an affair of immediate consciousness, like the idea of royal purple, it cannot be analyzed and the whole inquiry comes to an end. If it can be analyzed, the way to go about the business is to trace out in imagination a course of observation and reflection that might cause the idea (or so much of it as is not mere feeling) to arise in a mind from which it was at first absent. It might arise in such a mind as a hypothesis to account for the seeming violations of the principle of contradiction in all alternating phenomena, the beats of the pulse, breathing, day and night. For though the *idea* would be absent from such a mind, that is not to suppose him blind to the *facts*. His hypothesis would be that we are, somehow, in a situation like that of sailing along a coast in the cabin of a steamboat in a dark night illumined by frequent flashes of lightning, and looking out of the windows. As long as we think the things we see are the same, they seem self-contradictory. But suppose them to be mere aspects, that is, relations to ourselves, and the phenomena are explained by supposing our standpoint to be different in the different flashes. Following out this idea, we soon see that it means nothing at all to say that time is unbroken. For if we all fall into a sleeping-beauty sleep, and *time itself stops during the interruption*, the instant of going to sleep is absolutely unseparated from the instant of waking; and the interruption is merely in our way of thinking, not in time itself. There are many other curious points in my new analysis. Thus, I show that my true continuum might have room only for a denumeral multitude of points, or it might have room for just any abnumeral multitude of which the units are in themselves capable of being put in a linear relationship, or there might be room for all multitudes, supposing no multitude is contrary to a linear arrangement.

CHAPTER 2

*A SECOND CURIOSITY**

643. A phenomenon easier to understand depends on the fact that, in counting round and round a cycle of 53 numbers, $\sqrt{-1}=\pm 30$. (For $30^2=900=17\cdot 53-1$.) This, likewise, may be exhibited in the form of a "trick." You begin with a pack of 52 playing cards arranged in regular order. For this purpose, it is necessary to assign ordinal numbers to the four suits. It seems appropriate to number the spade-suit as 1, because its ace carries the maker's trade-mark. I would number the heart-suit 2, because the pips are partially cleft in two; the club-suit 3, because a "club," as the French term *trèfle* reminds us, is a trefoil; and the diamond-suit as 4 or 0, because the pips are quadrilaterals, and counting round and round a cycle of 4, $4=0$. But it is convenient, in numbering the cards, to employ the system of arithmetical notation whose base is 13. It will follow that if the cards of each suit are to follow the order 1 2 3 4 5 6 7 8 9 X J Q K, the king of each suit must be numbered as if it were a zero-card of the following suit. The inconvenience of this is very trifling compared with the convenience of directly availing oneself of a regular system of notation; for the exhibitor of the "trick" will have many a "long multiplication" to perform in his head, as will shortly appear. Another slight inconvenience is that the cycle of numeration must be fifty-three, or 4♠, which, or its highest possible multiple, must be subtracted from every product that exceeds 4♠. It is to be remembered that ♢, ♠, ♡, ♣, are used as nothing but other shaped characters for 0, 1, 2, 3, respectively. Thirteen is the base of numeration, but fifty-three, or 4♠, is the cycle of numeration. I adopt ♢, rather than K, as the zero-sign in order to avoid denoting the king of diamonds by ♠ K, etc. In order to exhibit the trick in the highest style, the performer should have this multiplication table by heart in which I have been forced to put 10 in place

* *The Monist*, pp. 36–45, vol. 19, January 1909, Peirce's last published paper.

of x most incongruously simply because I am informed that the latter would transcend the resources of the printing office.

Yet I do it quite passably without possessing that accomplishment. In those squares of the multiplication-table where two lines are occupied, the upper gives the simple product in tridecimal notation, and the lower the remainder of this after subtracting the highest less multiple of fifty-three, *i.e.*, of 4♠.

◇ ♠	◇ ♡	◇ ♣	◇ 4	◇ 5	◇ 6	◇ 7	◇ 8	◇ 9	◇ 10	◇ J	◇ Q
◇ ♡	◇ 4	◇ 6	◇ 8	◇ 10	◇ Q	♠ ♠	♠ ♣	♠ 5	♠ 7	♠ 9	♠ J
◇ ♣	◇ 6	◇ 9	◇ Q	♠ ♡	♠ 5	♠ 8	♠ J	♡ ♠	♡ 4	♡ 7	♡ 10
◇ 4	◇ 8	◇ Q	♠ ♣	♠ 7	♠ J	♡ ♡	♡ 6	♡ 10	♣ ♠	♣ 5	♣ 9
◇ 5	◇ 10	♠ ♡	♠ 7	♠ Q	♡ 4	♡ 9	♣ ♠	♣ 6	♣ J	4 ♣ ◇ ♡	4 8 ◇ 7
◇ 6	◇ Q	♠ 5	♠ J	♡ 4	♡ 10	♣ 3	♣ 9	4 ♡ ◇ ♠	4 8 ◇ 7	5 ♠ ♠ ◇	5 7 ♠ 6
◇ 7	♠ ♠	♠ 8	♡ ♡	♡ 9	♣ ♣	♣ 10	4 4 ◇ ♣	4 J ◇ 10	5 5 ♠ 4	5 Q ♠ J	6 6 ♡ 5
◇ 8	♠ ♣	♠ J	♡ 6	♣ ♠	♣ 9	4 4 ◇ ♣	4 Q ◇ J	5 7 ♠ 6	6 ♡ ♡ ♠	6 10 ♡ 9	7 5 ♣ 4
◇ 9	♠ 5	♡ ♠	♡ 10	♣ 6	4 ♡ ◇ ♠	4 J ◇ 10	5 7 ♠ 6	6 ♣ ♡ ♡	6 Q ♡ J	7 8 ♣ 7	8 4 ◇ ♡
◇ 10	♠ 7	♡ 4	♣ ♠	♣ J	4 8 ◇ 7	5 5 ♠ 4	6 ♡ ♡ ♠	6 Q ♡ J	7 9 ♣ 8	8 6 ◇ 4	9 ♣ ♠ ♠
◇ J	♠ 9	♡ 7	♣ 5	4 ♣ ◇ ♡	5 ♠ ♠ ◇	5 Q ♠ J	6 10 ♡ 9	7 8 ♣ 7	8 6 ◇ 4	9 4 ♠ ♡	10 ♡ ♡ ◇
◇ Q	♠ J	♡ 10	♣ 9	4 8 ◇ 7	5 7 ♠ 6	6 6 ♡ 5	7 5 ♣ 4	8 4 ◇ ♡	9 ♣ ♠ ♠	10 ♡ ♡ ◇	J ♠ ♡ Q

Fig. 256

644. In order to exhibit the trick, while you are arranging the cards in regular order, you may tell some anecdote which involves some mention of the numbers 5 and 6. For instance, you may illustrate the natural inaptitude of the human animal for mathematics, by saying how all peoples use some multiple of 5 as the base of numeration, because they have 5 fingers on a hand, although any person with any turn for mathematics would see that it would be much simpler, in counting on the fingers, to use 6 as the base of numeration.

For having counted 5 on the fingers of one hand, one would simply fold a finger of the other hand for 6, and then make the first finger of the first hand to continue the count. The object of telling this anecdote would be to cause the numbers 5 and 6 to be uppermost in the minds of the company. But you must be very careful not at all to emphasize them; for if you do, you will cause their avoidance. The pack being arranged in regular sequence, you ask the company into how many piles you shall deal them, and if anybody says 5 or 6, deal into that number of piles. If they give some other number, manifest not the slightest shade of preference for one number of piles over another; but have the cards dealt again and again, until you can get for the last card either ♠ x, that is, the ten of the second suit (*i.e.*, suit number one; since the first suit is numbered ♢, or zero), or ♡4, the four of the third suit, or ♠6, or ♡8. If you cannot influence the company to give you any of the right numbers, after they have ordered several deals, you can say, "Now let me choose a couple of numbers," and by looking through the pack, you will probably find that one or other of those can be brought to the face of the pack in two or three deals. For every deal multiplies the ordinal place of each card by a certain number, counting round and round a cycle of 53. And this multiplier is that number which multiplied by the number of piles in the deal gives +1 or −1 in counting round and round the cycle of 53. For it makes no difference to which end of the pack the card is drawn. After each deal the piles are to be gathered up according to the same rule as in the first "trick," except that the first pile taken must not be the one on which the fifty-second card fell, but the one on which the fifty-third would have fallen if there had been 53 cards in the pile. The last deal having been made, you lay all the cards now, backs up, in 4 rows of 13 cards in each row, leaving small gaps between the third and fourth and sixth and seventh cards counting from each end, thus:

Fig. 257

The object of these gaps is to facilitate the counting of the places from each end, both by yourself and by the company of onlookers. If the first or last card is either ♠ x or ♡ 4, the first card of the pack will form the left-hand end of the top row, and each successive card will be next to the right of the previously laid card, until you come to the end of a row, when the next card will be the extreme left-hand card of the row next below that last formed. But if the first or last card is either ♠ 6 or ♡ 8, you begin at the top of the extreme right-hand column, and lay down the following three cards each under the last, the fifth card forming the head of the column next to the left, and so on, the cards being laid down in successive columns, passing downward in each column, and the successive columns toward the right being formed in regular order.

You now explain to the company, very fully and clearly, that the upper row consists of the places of the diamonds; and you count the places, pointing to each, thus "Ace of diamonds, two of diamonds, three; four, five, six; the seven, a little separated, the eight, nine, and ten, together; then a little gap, and the knave, queen, king of diamonds together. The next row is for the spades in the same regular order, from that end to this (you will not say "right" and "left," because the spectators will probably be at different sides of the table), next the hearts, and last the clubs. Please remember the order of the suits, diamond (you sweep your finger over the different rows successively), spades, hearts, and clubs. But (you continue), those are the places beginning at *that* (the upper left-hand) corner. In addition, every card has a *second* place, beginning at *this* opposite corner (the lower right-hand corner). The order is the same; only you count backwards, toward the right in each row; and the order of the suits is the same, diamonds, spades, hearts, clubs; only the places of the diamonds are in the bottom row, the places of the spades next above them, the places of the hearts next above them, and the clubs at the top. These are the regular places for the cards. But owing to their having been dealt out so many times, they are now, of course, all out of both their places." You now request one of the company (not the least intelligent of them) simply to turn over any card in its place. Suppose he turns up the fifth card in the third row. It will be either the ♡ 3 or ♠ J. Suppose it is the former. Then you say, "Since the three of hearts is in the place of the five of hearts, counting from *that* corner, it follows *of course*" (don't omit this phrase, nor emphasize it; but say it as if what follows were quite a syllogistically evident conclusion), "that the five of hearts will be in the place of the three of hearts counting from the opposite corner." Thereupon, you count "Spades, hearts: one, two, three," and turn up the card, which, sure enough, will be ♡ 5. "But," you continue, "counting from the first corner, the five of hearts is in the place of the knave of spades, and accordingly, the knave of spades will, of course, be in the place of the five of hearts, counting from the opposite corner." You count, first, to show that ♡ 5 is in the place of ♠ J, and then, always pointing as you count, and counting, first the rows, by giving successively the names of the suits, "diamonds, spades, hearts," and then the places in the row, "one, two, three, four, five," and turning up the card you find it to be, as predicted, the ♠ J. "Now," you continue, "the knave of spades is in the place of the nine of spades counting from the first corner, so that we shall necessarily find the nine of spades in the place of the knave of spades counting from the opposite corner." You count as before, and find your prediction verified. (I will here interrupt the description of the "trick" to remark that the number of different arrangements of the fifty-two cards all possessing this same property is thirty-eight thousand three hundred and eighty-two billions (or millions squared), three hundred and seventy-six thousand two hundred and sixty-six millions, two hundred and forty thousand, $= 6\times 10\times 14\times 18\times 22\times 26\times 30\times 34\times 38\times 42\times 46\times 50$, not counting a turning over of the block as altering the arrangement. But of these only one arrangement can be produced by dealing the cards according to our general rule. Either of the four *simplest* arrangements having the property in question will be obtained by first laying out the diamonds in a row so that the values of the cards increase regularly in passing along the row in either direction, then laying out the spades in a parallel row either above or below the diamonds, but leaving space for another row between the diamonds and spades, their values increasing in the counter-direction to the diamonds, then laying out the hearts in a parallel row close upon the other side of the diamonds, their values increasing in the same direction as the spades, and finally laying out the clubs between the diamond-row and the spade-row, their values increasing in the same direction as the former.

Not to let slip an opportunity for a logical remark, let me note that, *in itself considered*, *i.e.*, regardless of their sequence of values, any one arrangement of the cards is as *simple* as any other; just as any continuous line that returns into itself, without crossing or touching itself, or branching, is just as simple, *in itself*, as any other; and relatively to the sequence of values of the cards, only, the arrangement produced in "trick," in which the value of each card is i times the ordinal number of its place, where $i = \pm\sqrt{-1}$, is far simpler than the arrangement just described. But in calling the latter arrangement the "simpler," I use this word in the sense that is most important in logical methodeutic; namely, to mean more facile of human imagination. We form a detailed icon of it in our minds more readily.)

You now promptly turn down again the four cards that have been turned up (for some of the company may have the impression that the proceeding might continue indefinitely; and you do not wish to shatter their pleasing illusions), and ask how many piles they would like to have the cards dealt in next. If they mention 5 or 6, you say, "Well we will deal them into 5 and 6. Or shall we deal them into 4, 5, 6? Or into 2 and 7? Take your choice." Which ever they choose, you say, "Now in what order shall I make the dealings?" It makes no difference. But how the cards are to be taken up will be described below. After gathering the cards in the mode described in the next paragraph, deal them out, *without turning the cards up*. (I have never tried what I am now describing; but for fear of error, I shall do so before my article goes to press.) After that, you say, "Oh, I don't believe they are sufficiently shuffled. I will milk them." You proceed to do so. That is, holding the pack backs up, you take off the cards now at the top and bottom, and lay them backs up, the card from the bottom remaining at the bottom; and this you repeat 25 times more, thus exhausting the pack. Many persons insist that the proper way of milking the cards is to begin by putting the card that is at the back of the pack at its face; but when I speak of "milking," I mean this *not* to be done. Having milked the pack three times, you count off the four top cards (*i.e.*, the cards that are at the top as you hold the pack with the faces down) one by one from one hand to the other, putting each card above the last, so as to reverse their positions. You then count the next four into the same receiving hand, *under* the four just taken, so that their relative positions remain the same. The next four are to be counted, one by one, upon the first four, so that their relative positions are reversed, and the next four are to be counted into the receiving hand under those it already holds. So you proceed alternately counting four to the top and four to the bottom of those already in the receiving hand, until the pack is exhausted. You then say, "Now we will play a hand of whist." You allow somebody to cut the cards and deal the pack, as in whist, one by one into four "hands," or packets, turning up the last card for the trump. It will be found that you hold all the trumps, and each of the other players the whole of a plain suit.

645. I now go back to explain how the cards are to be taken up. If it is decided that the cards are to be dealt into 5 and into 6 piles (the order of the dealing always being immaterial), you take them up row by row, in consecutive order, from the upper left-hand to the lower right-hand corner. If they are to be dealt into 4, 5 and 6 piles, or into 2 and 7 piles, in any order, you take them up column by column, from the upper right-hand to the lower left-hand corner. The exact reversal of all the cards in the pack will make no difference in the final result. They may also be taken up in columns and dealt into piles whose product is 14 or 39 (as, for example, into 2 piles and 7 piles, or into 3 piles and 13 piles). They may be taken up in rows and dealt into any number of piles whose product is thirty, or, by the multiplication table is ♡4. The following are some of the sets of numbers whose products, counted round a cycle of 53, equal 30: 6·5; 17·8; 7·5·4·4; 9·7·3; 9·8·7·7; 9·6·6·5; 9·9·5·4; X·8·7; X·9·8·7·6; J·J·2; J·8·4·4; J·5·5·3; Q·X·X·4; Q·X·8·5; Q·7·7·6; K·K·3; ♠X·♠X·4 (decimally, 23·13·4); ♠6·♠4·6; ♠5·◇9·◇X.

The products of the following sets count round a cycle of 53 to −30=23; 4·♠6; 2·7·K; K·Q·X; 8·6·6; 9·8·4; X·X·5; Q·J·7; Q·Q·2; 5·5·5·4; 6·4·4·3; X·9·7·5; J·7·6·2; 11·7·4·3; 13·X·6·2; 13·8·5·3; 7·6·5·5·3; 7·7·7·5·4; 9·7·5·5·2; 11·6·5·4·3; 9·8·8·5·4·4; 8·8·7·7·4·4; 11·8·7·7·2·2; 12·11·9·8·7·6.

The products required to prepare the cards for being laid down column by column are ♠6, decimally expressed, 19; and ♡8, decimally expressed, 34.

The following are some of the sets of numbers whose continued products are 19: 9·8; Q·6; 5·5·5; 6·4·3; J·7·3; 13·6·5; 13·10·3; 8·7·6·4; 9·9·8·6; J·9·5·4; 11·10·9·2; 12·8·7·7; 13·10·8·7; 9·8·8·5·4; 10·7·7·6·5; 10·10·10·10·2; 12·7·7·5·5; 7·4·4·4·3; 13·7·4·4·4·4·4·4; 4·4·4·4·4·4·4·4·4·4·3. The following are sets of numbers whose continued product is 34: ♠4·2; ♠X·K; 29·3; 7·5·4; 9·3·♠♡; 9·9·5; X·7·2; J·8·4; Q·X·X; 17·11·5; 17·12·9; 19·13·4; 23·11·6; 23·13; 23·17·7; 41·3·2; 5·5·4·4·3; 9·7·7·6·3; 8·6·5·5; 9·9·7·7·2; 13·13·7·2; 17·12·9; 8·4·4·4·4; 2·2·2·2·2·2·2·2·2·2·2·11·10·7·5; 13·12·9; 23·13.

646. This "trick" may be varied in endless ways. For example, you may introduce the derangement that is the inverse of milking. That is, you may pass the cards, one by one, from one hand to the other, placing them alternately at the top and the bottom of the cards held by the receiving hand. Twelve such operations will bring the cards back to their original order. But a pack of 72 cards would be requisite to show all the curious effects of this mode of derangement.

CHAPTER 3

*ANOTHER CURIOSITY**

§1. COLLECTIONS AND MULTITUDES

647. A character which is not sometimes true and sometimes false of the same singular is a *kind*. A kind may not exist at all; or it may exist in but one sole singular, which the old logics used to say was the case with the kind called *sun*. Two kinds may, neither of them, exist except in singulars in which the other exists; and when this is the case, they are said to be *coextensive*. If two kinds, A and B, are so related that of whatever singulars A could possibly be true, B would necessarily also be true, then A is said to *involve* B.

This necessity may be of any of the modes of necessity. In particular, if A involves B because of the definitions, or very ideas, of the two kinds, A is said *essentially to involve*, or, in other words, to *imply* B. A kind all whose singulars seem, according to experience, normally to belong to other kinds not implied in the former kind, is called (especially if the other kinds are numerous) a *natural* kind.†

648. I consider a *kind* to be an *ens rationis*, although that may be open to dispute, at least as regards some kinds; but there can, I think, be no doubt that a *class* is an *ens rationis*. For a *class*, unlike a *kind*, is not a character, but is the totality of all those singulars that possess a definite existent character, which is the *essential character* of the class.‡ Should observation show that two classes having different essential characters embraced the very same singulars, then since it is the singulars, and not the kinds, that constitute the existence of the class, we should say that the two classes, though *entitatively*,

* From "Some Amazing Mazes, Fourth Curiosity," c. 1909. Neither the third nor the fourth papers of this series were previously published. The "Third Curiosity" contains little new. In the manuscript the present chapter follows shortly after 6.348.

† Cf. 1.203ff.

‡ Cf. 3.66.

that is, in their possibilities, they were diverse, were yet *existentially* one. Such, I think, is the modern notion of a class,* though I must confess that it appears to me to be rather hazy. The characters which go to define a class are not necessarily permanent characters of the singulars, as a kind is. On the contrary we speak with perfect propriety of the class of human males between the ages of fourteen and twenty-one, though there is evidently no such *kind*. In fluid with viscosity the belonging to a given vortex would be a kind; for once in that vortex, particles would for all eternity be in it. But the particles of a given wave, though not of one kind, would be of one class, which would endure as long as the wave endured; that is, forever. But the singulars would be continually passing in and passing out of that class, as those of the adolescent class do. Yet if the wave were to subside and cease to exist, that class would cease to be; or if the fecundity of a population were to be destroyed, after a few years the class of adolescents would cease to be a class. Such appears to be the notion of a class, whether it be consistent or not.

649. A *collection* or *plural* is different.† Here there always must have been some characters common and peculiar to all the singulars, however trifling and unnoticed they may have been. They have, for example, the common and peculiar character of having been chosen to go to the making up of the collection. The word *collection* does not imply that the singulars themselves are gathered together or are, in any way, externally affected. It is only the ideas of them that are grouped. Though we may strive to make our collection as promiscuous as possible, yet in spite of all our efforts, it always must embrace whatever there may be in the universe that has a certain character, and it will embrace nothing else. The essential characters of all possible collections are of one and the same type; each such character consists in all those characters that are common and peculiar to such objects as have the character of having been taken on some definite and determinate occasion to be included in one collection. Thus, if I on Monday consider the collection composed of Don Quixote's helmet, the procession of the equinoxes, Jean Dare's children,

* See *e.g.*, Russell, *Principles of Mathematics*, p. 68.

† Cf. 3.537n.

and the star Mira Cete at its maximum brilliancy, and you on Wednesday, without knowing of my collection determine to take the first, second, and fourth of those objects as a collection, or lot, then, because Jean Dare had no children, your collection and mine will be identically the same collection, having precisely the same essential character.

650. In like manner, two apparent and highly interesting ornithological collections, the one of whatever phoenixes there ever were or will be, the other of whatever cockatrixes there are at this moment, are one and the same collection, having one and the same essential character. It is that quite unique collection that goes by the name of *Nothing*. Some writers whose logical conceptions would seem to be in a state of disintegration have supposed the collection whose sole member is Gaius Julius Caesar to be identical with Gaius Julius Caesar himself — a strange confusion considering that the latter was a man of immense force of intellect who was brought into the world by a grossly unskillful operation of surgery, while the other is nothing but an *ens rationis* brought into being by the idea of that man being chosen without any surgery at all and utterly deprived of any force of intellect or life. So likewise that pair of objects which consists of Julius Caesar himself and the collection whose sole member is Julius Caesar is very different from the pair that consists of Julius Caesar himself and the collection whose sole member is the collection whose sole member is the collection whose sole member is Julius Caesar.

651. The conception of *multitude* which is now current among mathematico-logicians upon which I am unable to make any substantial improvement is due to a remarkable definition of the relation of equality of collections first put forward in the book *Paradoxes of the Infinite** of Bernardo Bolzano, a catholic priest at Buda-Pesth, and the author of a logic in four volumes. Since he was in the priesthood at the time he made this notable contribution to the clearness of human conceptions, it is needless to say that he was severely punished for an act so contrary to the sacerdotal functions. Without altering the main idea of Bolzano, I shall modify the definition as follows: Any collection or plural, say that of the

* *Paradoxien des Unendlichen*, §22, Leipzig (1851).

X's, is *more* or *greater* than any collection, say that of the Y's, if and only if, there is no relation *r* whatsoever, such that every X stands in the relation *r* to a Y to which no other X stands in this same relation, *r*.

652. I give this form of the definition because it is one that I have employed for a demonstration.* But there is another which is closer to the original idea of Bolzano, and which has the great merit of not masking the intrinsic absurdity of the whole idea. It is this. Let us call a *substitution*† a dyadic relation in which every singular in the universe stands to one and only one singular and in which to every singular in the universe stands one and only one singular. Then the X's are more multitudinous than the Y's if and only if, whatever substitution be considered, there is some X which is not in this relation to any Y, or, in other words, if and only if there is no substitution such that every X stands in this relation to a Y. As long as we deal with a universe the multitude of whose singulars is not abnumerable this definition involves no absurdity, so that it will answer very well for all enumerable multitudes. But if we are to consider all multitudes, it would be necessary that the universe should be a collection of units of a multitude such that it would be absurd to suppose a greater multitude. Now since I have proved in Vol. VII of the *Monist*‡ that there can be no maximum possible multitude, we have here an absurdity to begin with. Then when we come to speak of *every possible* substitution, this supposes a collection which, if M be the impossible maximum multitude, has a multitude equal to M! or greater than the impossible greatest multitude. Not that that adds anything to the absurdity, since that maximum multitude would, according to the definition, be greater than itself, and therefore it could not be identical with itself. There are still other points of view from which the arrant nonsense of it appears. For the only thing that exceeds the manifoldness of all collections is a continuum. Therefore to speak of every possible substitution is equivalent to speaking of the collection of *all possible curves* in which, regardless of continuity, there is but one value of F for each

* See 3.546.

† Cf. 321, 635, 3.232.

‡ See 3.547f.

value of x, which is absurd since it supposes a smallest possible distance between successive values of x, or, to put it better, supposes this continuum to be utterly discontinuous, without any continuity at any point.

653. The truth is that Bolzano's definition, if it is to be applied to *all* collections, must be replaced by one which does not introduce the idea of *all possible collections*, since that idea is intrinsically absurd. But if we confine ourselves to finite multitudes, or even to any fixed multitude, such as that of all possible irrational quantities, the absurdity disappears. I am not prepared to give any better definition of fewer and manier. I should like to have the leisure to work at the problem, for, since my paper in Vol. VII of the *Monist*, I have had ten good years of training in logic and am much stronger in it than I was then. However, I am to bethink me that in order to get time to make what work I have done generally useful before extreme old age overtakes me, I must leave new problems or difficulties to another generation, however much they may tempt me.

654. In my eagerness to express myself, I have permitted myself to talk of *multitude* without defining it. It is that respect in which discrete collections of singulars of which one is greater than the other disagree. It has two denumeral series of absolute grades, the one consisting of all *multitudes*, that is, of all absolute grades of multitudes such that the count of any collection of any such grade of multitude can be completed, which multitudes are distinguished by the cardinal numbers proper, that is, the finite cardinal numbers; these grades of enumerable multitude running from 0 up endlessly, are followed by another similar series of abnumerable multitudes, beginning with the multitude of abnumerability zero, which is the multitude of a simply endless succession of singulars; and each following multitude being the multitude of all the possible collections that can be formed of the singulars of a collection of the next lower multitude, so that this second and last series of multitudes forms another simply endless series.*

655. The lowest multitude is *None*. It increases by a step as a singular is affixed to any collection whose multitude it is, and this goes on endlessly, for all the finite multitudes, that

* Cf. 113, 218, 674 and 3.550.

is to say for all collections for which the following is necessarily true: "If every singular of the collection of Hottentots kills a singular of the same collection, and if no singular of that collection is killed by more than one singular of that collection, then every singular of that collection is killed by a singular of that collection."* Of course in place of "killing," any other dyadic relation may be substituted, and in place of the "Hottentots," any other plural, or collection, may be substituted.

656. That multitude which is greater than any such multitude but is not greater than any other multitude, is termed the *denumeral* multitude, which in the higher, or second, series of multitudes corresponds to *zero* in the lower, or first, series. After it follow one by one an endless series of abnumerable multitudes. Yet so far as I know (I am not acquainted with the work of Borel,† of which I have only quite vaguely heard), it has never been exactly proved that there are no multitudes between two successive abnumerable multitudes, nor, which is more important, that there is no multitude greater than all the abnumerable multitudes. Each abnumerable multitude after the denumeral multitude is the multitude of all possible collections whose singulars are members of a collection whose multitude is the next lower abnumerable multitude, the denumeral multitude being considered as the abnumerable multitude of grade Zero.

§2. CARDINAL AND ORDINAL NUMBERS

657. The *cardinal numbers*, strictly understood, are vocables or written signs, of which one is attached to each finite multitude. But Cantor uses the term cardinal number to mean any multitude whatsoever.‡ According to me, the proper extension of cardinal numbers consists in taking in the *arithms*, or indices, of abnumerable multitudes, which I have explained in Vol. VII of the *Monist*.§

* Cf. 3.288.

† *Leçons sur la théorie des Fonctions*, Paris, 1898. Borel does not prove the point here at issue.

‡ See Georg Cantor, *Gesammelte Abhandlung*, S. 282, Berlin (1932).

§ See 3.546f.

658. Let me now discuss after the fashion of a scholastic disputation the following

Question: *Whether the cardinal or the ordinal numbers are the pure and primitive mathematical numbers.*

It would seem that the cardinals are so: for

Firstly. All the writers of arithmetic books say so: Fibonacci,* or Leonardo of Pisa, the effective introducer into Europe of what we call the Arabic system of numerical notation in 1205, although Geber [Gerbert], who became Pope Sylvester II in 999, had brought the figures to Europe and had taught arithmetic in his school, and although I believe that passage of the Geometry of Boëthius† to be genuine which gave the forms of the characters representing the nine digits about A.D. 500; the Saxon Jordanus Nemorarius,‡ another early [thirteenth] century mathematician; the English Johannes Sacrobosco (Holliwood),§ Roger Bacon,¶ Adelard of Bath||; Thomas Bradwardine,° the Doctor Profundus of Merton College, made Archbishop of Canterbury in [1349], who anticipated and outstripped our most modern mathematicologicians, and gave the true analysis of continuity; the Cardinal of Cusa**; Prosdocimo de' Beldamandi,†† the Bamberg *Rechenbuch*,‡‡ the first printed arithmetic; Johannes Widmann§§ who first used the signs + and − about as we now do; the Arithmetic of Treviso of 1478¶¶; the *Arithmetica* of Borgi of 1484||||; Luca Paciuolo°°; *Le Triparty en la Science des Nombres**** which gives the words Byllion, etc., up to Nonyllion "et ainsi des aultres se plus oultre on voulait proceder," and who seemed first to have virtually used negative exponents;

* *Liber Abaci* (1202).

† *Ars Geometricæ*, Leipzig (1867).

‡ *Arithmetica demonstrata* (1496).

§ *Tractatus de Arte Numerandi*, Strasburg (1488).

¶ *Opus Majus*, Part 4.

|| *Regule Abaci, Bull. di Bibliographia*, T. XIV.

° *Arithmetica Speculativa*, Paris (1495).

** *Opuscula*, Strasburg (1490).

†† *Algorisimus*, Padua (1483).

‡‡ By U. Wagner (1482).

§§ *Betrede vnd hubsche Rechnung*, Pforzheim (1489).

¶¶ Anonymous.

|||| *Libro de Abacho de Arithmetica*, Venice (1484).

°° *Suma*, Venice (1494).

*** By Nicolas Chuquet (1484).

Oronce Fine*; Michael Stifel†; Cuthbert Tunstall, Bishop of London, *De arte Supputandi*,‡ which he wrote as a farewell to science on taking holy orders; Robert Recorde in his celebrated *Grounde of Artes*§; Masterson¶; Blundeville||; Hylles°; Oughtred**; Cocker†† who gave the name to the most famous of all English arithmetics, though it seems pretty clear that he did not write it and never saw it; Pliny Earl Chase‡‡ who wrote the best introduction to the art I ever saw; from which I learned to cipher as a boy; and though he wrote (probably under the influence of idiotic publishers) several very inferior arithmetics, I never saw but the one copy of his only excellent work, the one I studied in school at my father's dictation; but I still often refer to the arithmetic of Pliny Earl Chase and Horace Mann§§; all these make cardinals the fundamental numbers;

Secondly. The forms of the words in all languages show the cardinals to be the oldest; and since they thus appear to have been first conceived, that conception must be the simplest;

Thirdly. Any person whose head is not cracked by too much study of logic will say without hesitation that the cardinals are the original numbers. It is common-sense; and common-sense is the safest guide.

Fourthly. It is impossible to form a clear conception of multiplication without resort to cardinals; thus 3 times 5 is a collection of 3 members, each a collection of 5 units. No sense can be attached to the "third fifth," unless you really mean that you form three collections of five each. In like manner 2^3 is the number of different ways in which 3 objects can be distributed among 2 places, while 3^2 is the number of ways in which 2 objects can be distributed among 3 places. But there is no such clear conception of the involution of ordinals.

* *Protomathesis*, Paris (1530).

† *Arithmetica Integra*, Nürnberg (1544).

‡ Published in 1522.

§ Published in 1543.

¶ *Arithmeticke*, London (1592).

|| *Exercises*, London (1594).

° *The Arte of Vulgar Arithmeticke*, London (1600).

** *Clavis Mathematicæ*, London (1631).

†† *Arithmetick*, ed. by Hawkins (1678).

‡‡ *Elements of Arithmetic*, Philadelphia (1844).

§§ *Elements of Arithmetic*, Philadelphia (1851, 1855).

Fifthly. With ordinals alone there would be no fractions; except perhaps in Washington, where there is a "4 1-2th" Street!

659. On the other hand, it may be argued:

Firstly. What, after all, are the cardinal numbers? What do they signify? They signify the *grades* of multitude. Now a grade is a rank; it is an ordinal idea. The English word *grade* which came in with the nineteenth century, was evidently from Latin *gradus*, a stride, being the Latinized form of the old English word *gree*, which the Scotch still use in the sense of that which one strives to attain. It is the French *gré*. It is from an Aryan root found in "*greedy*." See Fick's list of roots in the International Dictionary, No. 49, [?√34]. There never was any idea of multitude attached to this root. Some think the principal idea is desire; others, that it is that of stepping out. It seems to me it is the idea of pushing on to the attainment of what one hankers after. Thus, cardinal numbers are nothing but a special class of ordinals. To say that a plural is five means that it is of the fifth grade of multitude. It would be the sixth, if we were to count *none*, or the foot of the staircase, as the first number; but we ought in consistency to call it the "*none-th*" number. The ordinal "*none-th*" is a desideration of *gree*, of thought that I have lately won. Just ponder the utility of that view, my candid reader. Now Number is *the* mathematical conception *par excellence;* and therefore the question is whether limiting the grades we refer to in mathematics to grades of multitude advances and aids mathematics to attain a higher grade of perfection or not. But this answers itself. All that is essential to the mathematics of numbers is *succession* and definite relations of succession, and that is just the idea that ordinal number developes.

Secondly. The essence of anything lies in what it is intended to do. Numbers are simply vocables used in counting. In order to subserve that purpose best, their sequence should stick in the memory, while the less signification they carry the better. The children are quite right in counting as they do:

Onery; uery; ickari; Ann;
Filason; folason; Nicholas Jan;
Queevy; quavy; English navy;
Stingalum; stangalum; Buck!

The fact that there are generally thirteen of these vocables suggests that they may have originated in counting out a panel in order to get a jury. These children's vocables are purely ordinal.

Thirdly. But the ultimate utility of counting is to aid reasoning. In order to do that, it must carry a *form* akin to that of reasoning. Now the inseparable form of reasoning is that of proceeding *from* a starting-point *through* something else, *to* a result. This is an ordinal, not a collective idea.

660. Now in answer to the above arguments on the side of the cardinal ideas.

As to the first argument [on the side of the cardinal ideas] the first reply is that all the authorities cited are worthless as to a question of logical analysis. The only opinions worth consideration are those of the modern mathematico-logicians, Georg Cantor, Richard Dedekind, Ernst Schröder, and their fellows; and of these Dedekind* emphatically and Schröder† probably are on the ordinal side; though Cantor‡ by basing ordinal upon the doctrine of cardinals has the appearance (perhaps it is a deceptive appearance), of taking the side of the cardinals.

But *secondly*, arguments from authority are of no authority in a question of logic.

661. To the second argument, likewise, two replies may be made. *Firstly*, it is almost always found that when a new idea is born into the living world of thought, it labors under all sorts of inconsequential and inconvenient adjuncts. A new machine, for example, is at first needlessly complicated, and has to be simplified later. We should therefore not expect to find that the earliest forms of numbers were the neatest and purest.

Yet, *secondly*, there can hardly be a doubt that the original numbers were meaningless vocables used for counting, such as children invent; and there is no reason to suppose that these were at first less purely ordinal than the children's are.

662. According to the principle of the third argument, which seems to be the widely disseminated tenet that the less

* *Was sind. u. was sollen die Zahlen*, §73, §161.

† *Lehrbuch der Arithmetik u. Algebra*, Leipzig (1873).

‡ *Op. cit.*, S. 284f.

thought a man has bestowed upon a question, the more valuable his opinion about it is likely to be, when it is applied to the question of how much that very third argument is worth, must result in according a perfectly crushing strength to my judgment, which is that it is beneath contempt.

663. The fourth argument, much the most respectable of the list, certainly shows that the device of considering numbers as multitudes gives very pretty demonstrations of the values of products and powers of whole numbers; but the first fault of the argument is that there are countless parallel instances of devices giving charmingly clear intuitions of mathematical truth, although nobody in his senses could say that the imported considerations were essentially involved in the subjects to which the theorems relate. Thus, a number of difficult evaluations of integrals can be obtained most delightfully by considering those integrals as the values of probabilities and then applying common sense, or some simple reasoning, to answering the question of probability. Yet who would say that the idea of probability was essentially involved in the idea of an abstract integral? The proper inference is the converse of that; I mean that the idea of the integral is essentially involved in the idea of the problem in probabilities. Just so, in the instances adduced; what they evidently prove is that the abstract ideas of multiplication and of involution are involved, the one in the more concrete idea of a collection whose units are collections, and the other in the concreter idea of the different ways of distributing the members of one collection into connection with the several units of another collection. I admit, with all my heart, the instructiveness of these remarks and to the fact that they shed a brilliant illumination upon the essential nature of the arithmetical and algebraical results. Indeed, they are so rich in their curiosity and their eye-opening virtues, that I will not spoil their effect by tagging any discussion of them upon this already exorbitant paper. I will only say that if on another occasion I ring up the curtain upon what they have to show, it will be seen that one of their first lessons is that numbers may stand for grades of any kind and not exclusively for grades of multitude. You will observe that, for example, in the iconization of involution, it was not members of a *multitude* that were put into the dif-

ferent parts of another *multitude*, but members of a *collection* which are attached to different singulars of a *collection*. Now while numbers *may* on occasion be, or represent, multitudes, they can never be collections, since collections are not *grades* of any kind, but are single things. It may be reckoned a second fault of that fourth argument that it quite overlooks the necessity of *proving* the exclusive limitation of numbers to a single variety of grades; and a third fault of it is that it baldly asserts, with not so much as an imitation-reason, that it is impossible to obtain a clear conception of multiplication without appeal to cardinals. That is a gage that I am obliged to take up. Let me first call attention to the fact that an object of pure mathematical thought does not possess this or that definite sensible quality, but is distinguished from other such objects by the form of relation involved in its structure. It must further be noticed that there are different kinds of multiplication, especially the "internal" and the "external";* and besides that, there are different allowable ways of using the term, so that what at one time would be called multiplication, at another time would not be multiplication. I have to define what could with propriety be called multiplication with the proper strictness and proper looseness. Above all, extreme care will be needed to avoid vicious circles and phrases that seem to have a meaning but really have none. For example, I shall have to mention addition in defining multiplication, and, consequently must begin by defining that. Now if I were to say that addition consists in simply putting two quantities together, that would sound as if it meant something; yet I do not clearly see what it would or well could mean; for if anybody were to ask me what kind of "putting together" I meant, why, what I should find myself meaning is simply the *adding* of them together. So since addition is of course adding, my statement might just as well be omitted, and no meaning would be lost with the omission.

664.† A *quantity* is in one sense or another an object of *almost* any category; but most appropriately the word is used to denote a dyadic relation, which is considered as having conceivable exact determinations differing from one another only

* See 107, 154, 3.242 and 3.331.

† Cf. 3.253ff.

in a linear respect, that is, so that there is a dyadic relation of "being *r* to" such that, of any two of the determinations in the same linear respect, one is *r* to whatever the other is *r* to, and is *r* to something the other is not *r* to; and to know all the possible determinations to which any determination was in that linear relation would be to know the determination exactly, the determinations being defined as such as to satisfy (especially so as just to satisfy) some general condition. If there is but a single linear respect such that, whatever two conceivable determinations of a quantity be taken, they can differ in that respect alone, it is called a *simple quantity;* but a quantity whose determinations can differ from others in different linear respects is called a *complex quantity.* The expressed determination of a quantity in all its linear respects of determination (especially if the expression be such that the determination is *exact*, that is, is a single one and is not any other), is called the *value* of the quantity.

665. The *ens rationis* whose complete being consists in the alternative possibility of all the conceivable values of a quantity under all conditions, is called the *scale of values* of the quantity. If the different values of the scale of conceivable values q q' etc., denoting the values, *consists of the discrepancies*, *according to a definite rule of comparison*, between the values of the scale of values Q, Q′ etc., so that for every value of the first scale, which is called the *relative scale*, and for any value Q′ that may be assumed on the second scale, called the *absolute scale*, there is some value Q on the absolute scale, whose discrepancy, according to the rule of comparison, from Q′ has the given value q of the relative scale, then if the rule of comparison is a convenient one, it will be possible by inserting, if necessary, *fictitious values* in the relative scale to have a value on the relative scale for the discrepancy of any value of the absolute scale from every other. There will, therefore, then be some value, qo, of the relative scale which shall represent the *nil* discrepancy of Q′ from itself; and consistency will require this to be, at the same time, the representative of the *nil* discrepancy of every value Q of the absolute scale from itself; and this value qo on the relative scale will be called and written, "zero," 0. Then every discrepancy $Q_a - Q_b$ of values on the absolute scale will be the same as discrepancy of

$q_a = Q_a - Q^1$ from $q_b = Q_b - Q^1$, where q_b will be any arbitrarily taken value on the relative scale and q_a will be a suitably chosen value of the same scale. Thus, the relative scale will itself fulfill the functions of the absolute scale, and may be identified with it.

666. Addition* is a "mathematical operation," *i.e.*, a certain triadic relation, of a suitable value s by any arbitrarily taken operand, called an *augend*. For such a scale of difference-values as that just described, if the discrepancy of q_s from q_a is represented by q_b, then the discrepancy of q_s from q_b will be q_a; and q_s will be the *sum* by q_a as *addend* upon q_b as *augend*. Or stating the matter otherwise,[1] $(q_x - q_y) + (q_y - q_z) = (q_x - q_z)$. In any case the rule for determining the sum from any given addend and augend, will be such that a given value as addend will produce a definite effect upon the sum depending exclusively upon its own value, regardless of what the augend may be; and the same value as augend will produce the very same definite effect upon the sum. From this follow all the properties of addition. In the case of the scale of values of the positive whole numbers, the ordinal rule will be that $0+0=0$ and that the value next following any given value will as addend or as augend give a sum of the value next following that given by that given value as addend or augend. That is to say, if we denote by Nx the positive whole number next following after x in the natural order of those numbers, and if $x+y=z$, then $Nx+y=Nz$ and $x+Ny=Nz$. Moreover, if $N0=1$, then, since $0+0=0$, we have $0+1=0+N0=N0=Z$ and so $1+0=1$, and it necessarily follows, as it is easy to see and to prove $Nx=x+1$.

667. I am now prepared to give a perfectly clear ordinal definition of multiplication. Only, I must warn you that mathematical clearness, as understood since Weierstrass, does not mean producing a sensuous impression of naturalness, but means logical clearness, clearness of *thought*. I will call attention to the circumstance that the idea of multiplication founded on multitude embraces only the multiplication of whole num-

* Cf. 190f., 3.262f., 3.562H.

[1] When I write a+b, I conceive a to be addend and b to be the augend, on the general principle of putting the operator before the operand, though addition is usually conceived to violate this rule.

bers. For notwithstanding the assertion of the fifth argument that fractions cannot be dealt with ordinally, which it is the principal purpose of the present paper to disprove, and I shall come to that disproof in a little while now, that fifth argument forgets that it ought, in order to have any force at all, to have shown that you *can* deal with fractions from the point of view of multitude. Now few things would seem more obvious than that there is no such thing as a fractional collection, and that that argument, as soon as I shall have shown how the rational fractions can be ordinally inserted in their places, will be turned in a convincing manner against the cardinal conception of pure mathematical number. And if anybody is doubtful whether I am right in saying that there is no fractional multitude, he ought to be convinced by the helplessness of the cardinal method when it attempts to represent the multiplication of fractions. It should also be remembered that mathematicians have a thoroughly well-grounded generalization of the conception of multiplication, in the multiplication of matrices, which is the same thing as the multiplication of quaternions and of other forms of multiple algebra. I shall embrace such multiplication in my first general ordinal conception of the operation, afterward showing both *how* and *why* its rules become specialized in the multiplication of numbers, whole or fractional, or even imaginary. I might show, were it not too far from my main theme, that the algebra of real quarternions is unique among all possible algebras in the closeness of its properties to numerical algebra, notwithstanding its non-commutative character.* A few writers have alluded to a so-called "symbolic" multiplication which is not even associative; but they have completely failed to show any advantage in regarding it as multiplication. I have myself made some studies along this line which have led me to the conviction that, except when such operations break up into sections which are associative, it is quite useless and idle to talk of it as multiplication. Accordingly, I shall pay no attention to it, although there would be no difficulty in treating it ordinally.

668. *Multiplication*† is another mathematical operation,

* Cf. 3.130, 3.327, 3.647.

† Cf. 193ff., 3.263f., 3.5621.

the triadic relation of a *product* "by" an operator called the *multiplier* "into" an operand called a *multiplicand.* The multiplicand is said to be multiplied *by* the multiplier, and the latter to be multiplied *into* the former. It is based, as is addition, upon a sort of discrepancy between numbers; upon a discrepancy, however, of quite a different kind, for it is, so to speak, double-ended, and has other remarkable peculiarities. It is not confined to a single line. Just as the basis of addition is that

$$(i-j)-(u-v)=(i-u)-(j-v)$$

and

$$x-x=0 \qquad x-0=x$$

so multiplication may be regarded as based upon the double discrepancy x/y and $y\backslash x$.* This, if my memory is right, was Grassmann's view.† But while, for addition, this point of view is in reason almost compulsory, because it alone accounts for the *zero* and determines all the properties of addition, for multiplication it has little to recommend it beyond the analogy of addition. The formulæ here, whose get-up is less smart because of the lack of the commutative principle, are

$$a/b\backslash a=b \qquad \text{and } a/(b\backslash a)=b$$
$$a/a=a\backslash a=1 \qquad 1\backslash a=a/1=a$$

But these formulæ by no means imply all the principles of division.

669. Any system of values which fully illustrates all the features of external multiplication of which I regard internal multiplication as a special case,[1] must have at least three

* $x/y=\dfrac{x}{y}$; $y\backslash x=y\cdot x$.

† Hermann Grassmann, *Die Ausdehnungslehre*, S. 11 (1878).

[1] I remark that in my memoir of 1870 on "The Logic of Relatives" [3.53f.], although I insisted with emphasis on there generally being these two kinds of multiplication, I made no reference to Grassmann nor designated them as "internal" and "external" which I am all but absolutely sure that I should have done had I been acquainted with either of Grassmann's volumes. [But cf. 3.152, 3.242n.] So I infer that the too exclusive admiration of Hamilton in our household prevented my acquaintance with that great system. The matter interests me as showing that a person who was studying algebra purely from the point of view of logic was quite independently led to the recognition of the presence of the two kinds of multiplication *in associative systems generally,* in spite of an undisputed admiration for Hamilton.

linear respects. I will take such a system for illustration. Let the three linear respects be

$$\begin{matrix} 0, aU, & bU, & cU, & \text{etc.} \\ 0, aV, & bV, & cV, & \text{etc.} \\ 0, aW, & bW, & cW, & \text{etc.} \end{matrix}$$

where a, b, c, etc. are any real numbers, 0 is ordinary zero, and U, V, W, are non-numerical, non-relative, units of different kinds, and linearly independent, so that the equation $xU+yV+zW=0$ is impossible, x, y, z, being numerical and not all 0.

670. Since multiplication is not generally commutative, there must be a progressive and a regressive division. For the present I will consider only the former of these. It will furnish a matrix of nine dyadic units; so that the general form of those dyadic quantities, so far as progressive division furnishes them will be

$$\begin{aligned} &a(U/U)+b(U/V)+c(U/W) \\ &+d(V/U)+e(V/V)+f(V/W) \\ &+g(W/U)+h(W/V)+i(W/W) \end{aligned}$$

where a, b, c, . . . i can take any real numerical values.

671. How multiplication follows this rule I proceed to state; and why it should do so will appear in the sequel.

$$\begin{matrix} (U/U)U=U, & (V/V)V=V, & (W/W)W=W \\ (W/U)U=W, & (U/V)V=U, & (V/W)W=V \\ (V/U)U=V, & (W/V)V=W, & (U/W)W=U. \end{matrix}$$

But all products in which the last term of the dyad is a different non-numerical unit from the multiplicand are equal to numerical *zero*. The numerical coefficient of a product is the product of the numerical coefficient of the factors.

Such products as

$$(U/V)(V/W)=(U/W)$$

obey the same principle, the multiplication and division being associative. That is to say,

$$(U/V)(V/W)=\{(U/V)V\}/W=U/W.$$

It will be seen that it necessarily follows that multiplication is associative. For

$$\{(U/V)(V/W)\}W = \{U/W\}W = U$$

and

$$(U/V)\{(V/W)W\} = (U/V)V = U$$

as before.

672. I may here note that in every system of values the sum of all the quotients of non-numerical units divided by themselves as divisors, in this system taken as our illustration, $(U/U)+(V/V)+(W/W)$ is numerical unity. The proof of this is perhaps the very simplest of all possible proofs, and certainly is as simple as any proof can be. It rests on the definition of a "system." Now everybody ought to have and must be expected to have a perfectly distinct notion of what he is talking about, at least, he must when he is undertaking to talk scientifically. But the truth of a proposition that follows from the definition of a notion that is perfectly distinct to a man must be seen with certainty by that man as soon as it is enunciated; for that is the definition of a "perfectly distinct" notion. Consequently, everybody who undertakes to discuss scientifically a *system* of values ought, and must be expected, to see, as soon as it is enunciated, the truth of any proposition that follows from the definition of a system of values, especially when that proof is excessively simple. Now it is part of the definition of a system of values that any collection of values, which does not contain *any* value that satisfies a system of equations in these values, although there is a value which would satisfy them if it were admitted into the system, is not a complete system of values. Consider then the system of equations which consists of all the equations of the form $xa=a$, where x is the unknown, and a takes in the different equations all the values of the system. There is a value which would satisfy all these equations were that value admitted to the system. Numerical unity is such a value, for $1a=a$, whatever value a may have. Consequently, given any complete system of values, it must either contain numerical unity as one of its values, or else it contains some other value which satisfies all three equations, so that, were numerical unity admitted into the system it would contain two different

values that satisfy the system of equations. Now if the latter alternative be the true one, let I and J be two different values that would belong to the system if numerical unity were admitted into it, each of these two being such that there would be no value in the system which would be altered by being multiplied by or into either I or J. Then the product IJ would be equal to I because J is the only other factor and would also be equal to J because I is the only other factor. Hence, since the same value is the value both of I and of J, I and J are the same value contrary to the hypothesis. Could there be a proof sounder or more needless than that? Suppose we were to admit that every collection of values, which contains, for every system of linear equations in the values of that collection which are capable of being satisfied linearly, some value that does satisfy them, is a complete system of values. Suppose, then, that the following were offered as a proof. Under that definition all possible values must constitute a complete system of values; for since it contains all possible values it must contain every possible value that satisfies any system of linear equations. But we have just seen that no complete system of values can contain two different values each of which is such that its product by or into any value whatever of the system gives that same value as the product. Therefore there is no other possible value than numerical unity of which this is true. Suppose, I say, that were offered as a proof, then I should like to put to you, reflective Reader, two questions. The first is this: What do you think of the value of that proposed proof, and precisely why? The second question is, What do you think I think of it? Do not think me impertinent; for I have two pertinent birds to hit with the two stones. One is to show that it is possible to have an extra clear ordinal conception of multiplication; and the other is to put myself in condition to please you better in some article on some subject.

673. There is one reply that might be made by upholders of the cardinal view of the pure mathematical conception of number that I think I ought briefly to notice. It might be said, "Well, granting for the sake of argument that even the cardinal view of number involves the ordinal idea, still it is equally true that one cannot count objects in a row without

regarding them as forming a collection, and thus the two views, if they be two, are quite on a par." I shall be able to make a very brief reply as a partial indemnity for the last affliction. It is not the question whether we are obliged when we think of number ordinally to think of some numbers as a collection; for of course we are so obliged. Nor is it the question whether in thinking of such collection we have to attend particularly to its multitude, although that is not so clear. The question is whether there is any possible theorem or reasoning of mathematics or any strictly numerical conception which calls for anything more with which to build it up than the ordinal idea supplies, and if there be, whether the idea of multitude supplies that further idea.

674. Now in my opinion the arguments already given are conclusive; but I announced my intention of discussing the question in the thorough manner of the best of the fourteenth century scholastic doctors. As yet their scheme of a disputation has not been filled out. It is now time that like them I should introduce such a filling out of the body of reasons as what has gone before suggests. Now what has gone before does suggest a new argument that seems to me decisive. The whole list of different multitudes is as follows. Beginning with the multitude of nothing, which is *none* (corresponding to the ordinal variously called "zero," "the origin," and "noneth," these three words marking three different aspects of that ordinal), we take the sole collection that has that multitude, which is Nothing, and successively add units to it as long as each added unit increases the multitude. This gives us the whole series of finite multitudes; each of which, of course, has its corresponding ordinal. We next take the multitude of all the positive integer numbers. It is a vulgar fallacy to reason that because any *collection* of that multitude, such as the collection of all the positive integers is *endless*, that is, has no last member, therefore there can be no corresponding ordinal number. But hold — I am wrong! It is *not* a *fallacy*, because it is not a reasoning at all. It is a jump of the rank of the Achilles-and-tortoise catch, or that of "It either rains or it doesn't rain; now it rains; *ergo* it doesn't rain." To say no corresponding ordinal can be supplied, when the multitudes themselves including the denumeral are in a linear series, is

absurd. In point of fact, there is a corresponding ordinal, namely, "infinitesimal," although this is ordinarily restricted to the *infinitieth* term of infinite converging series, or to the magnitude that such term would have, if there were any such, which there is not. Still there are many series of objects of thought which I do not halt when the finite numbers have been exhausted. Such, for example, is the series of values furnished by a converging series; such is the series which shows in perspective points at equal distances along a straight line, where the "vanishing-point" is the *infinitieth*, or infinitesimal, point; and such is the series of multitudes, which by no means ceases to increase after the denumeral multitude. Cantor denotes this ordinal by ω, and that notation is generally adopted. The next larger multitude than any infinite multitude, say that of the M's, if there be endlessly many M's, is the multitude of all possible different collections each consisting of M's; or, what is the same thing, the multitude of different ways in which all the M's might be distributed among different heads (for whether the number of heads were 2, 3, or any other number up to M would make no difference in the multitude). These were named by me (who had the right to name them, having been the first to define them as well as the first to prove that the multitude of ways of distributing the singulars of any collection under two heads is always greater than the multitude of those singulars themselves,* although it was soon found that they coincided with certain multitudes less clearly and less accurately defined by Cantor), the *abnumeral* or *abnumerable* multitudes. I prefer the latter term as corresponding to *enumerable* or finite multitudes and collections; while I prefer to speak of the *denumeral* multitude, giving the word a different termination because there is but one such multitude, while of the *abnumerable*, as well as of the *enumerable*, multitudes there is a denumeral collection. The multitude of ways of distributing the singulars of a denumeral collection is the *first abnumerable multitude.* It is the multitude of all irrational values (whether imaginaries be included or excluded). The multitude of ways of distributing a first-abnumerable collection is the *second-abnumerable* multitude. The multitude of ways of distributing a second abnumerable

* See 3.548f.

collection is the *third-abnumerable* multitude; and in general, the multitude of all possible collection is the (N+1)th-abnumerable multitude. There is no ω-abnumerable collection; which is a corollary drawn by me from my proof that 2^x is always greater than x. There is no multitude greater than the finitely-abnumerable multitudes. Consequently, the total multitude of possible multitudes is denumeral.* The *objects* of *any* abnumerable collection are in greater multitude than all multitudes. The reason, of course, is that the addition of a unit to an infinite collection never increases its multitudes. But that new unit will always carry a new ordinal number. If therefore, we extend the term "cardinal number" so as to make it apply to infinite collections, a multitude of ordinal numbers will be possible exceeding that of all possible cardinal numbers in any infinitely great ratio you please, without having begun to exhaust the ordinals in the least. The system of ordinals is thus infinitely more rich than the system of cardinals. In fact, those two denumeral series of ordinals which are alone required to count all the cardinals seem to the student of this branch of mathematical logic as most beggarly.

675. Dr. Georg Cantor, of Halle, undertook that research which I have mentioned as of the greatest urgency for logic, for metaphysics, and for cosmogony, that of ascertaining whether or not the singulars of every collection, however great, can be the subjects of a linear relation, and if not what is the greatest multitude of singulars that can be so arranged. To this end he introduced the immensely valuable concept of what he calls a "well-ordered" series, by which he means a linear series every portion of which has a first member.† He undertook to describe such a series, and name its members, the series containing more than any conceivable multitude of members. The momentous series so described ought to be called the "Cantorian Series," in everlasting memory of the man who so clearly perceived the supreme importance of the problem, and took so considerable a step, at least, toward its fulfillment. Students generally are either doubtful of his success or even deny it. For my part, I am not sure that I understand his papers. At any rate, I think that by deviating

* See 113, 218, 3.550.

† *Op. cit.*, S. 168, S. 312.

somewhat from his method, we may be able to attain clearer certitude, one way or the other. I prefer to construct a well-ordered series upon slightly different principles from these that Cantor has used. However, I cannot here go into details. The problem is to construct a well-ordered series which shall embrace as great a multitude of members as possible.

676. Let us make use of that system of numerical notation whose base is two. Let us take the different multitudes in succession and represent the different ordinals by the different ways of distributing objects of each multitude among two places. These two places may be called the affirmative and the negative places (for a non-existent object, or one not considered, is to be regarded as in the negative place), and we may represent each arrangement by marking objects in the affirmative place by 1's and those in the negative place by 0, or if an object is not considered it need not be marked. We will take the different secundal "places" of numerical notation from right to left to represent the different objects. Thus 1001 will represent that the first (or, as we had better call it, the zero object) is in the affirmative place, the second and third, or better, the first and second are in the negative place, and the fourth is in the affirmative place again, and so, this series of characters shall be, or represent, one of our ordinal numbers. We begin with the lowest multitude, which is that of the unique collections, Nothing. In how many ways can whatever members this collection contains (which is none, 0) be distributed among two places? The answer is $2^0=1$. Therefore we need one ordinal to describe it, and one object, which we will call the zero object, with which to construct that ordinal. There being no member of the collection of multitude zero, 0 will properly represent the sole arrangement. The next multitude is 1 and the number of ways of distributing one object among two places is $2^1=2$, one *additional* ordinal is required. The same object we used before will answer the purpose, being now put in the affirmative place; so that 1 represents the ordinal. The next multitude is 2 and the number of ways of distributing 2 objects among two places is $2^2=4$, *i.e.*, we need 4 minus 2 or 2 additional ordinals. For that purpose we take another object, represented by the secundal place next to the left of the one we have been using

and represent the two new arrangements by 10 and 11. In this way, all the finite ordinal numbers can be written down, in the sense in which it is true, in the mode of the possible, that of a subject of which anything can be predicated distributively the same thing can be predicated collectively; and these finite ordinals will be marked just as in the secundal system of arithmetic. In the following table, "distr." means modes of distributing under two heads.

Considering no object	0	The single mode of distributing nothing.
Now considering an object	1	The "distr." of 1 object when it is put under the affirmative head
Another object considered.	1 0 1 1	The "distr." of 2 objects when a certain one of them is put under the affirmative head.
A third object considered.	1 0 0 1 0 1 1 1 0 1 1 1 etc.	The "distr." of 3 objects of which a certain one is put under the affirmative head.

677. I guess that a good many people, among whom many mathematicians must be included, to judge by their often writing –, 1, 2, 3, ... ∞, . . . have a notion that nothing but a limitation attached to human powers prevents a finite collection receiving successive finite increments until it becomes denumeral; though I do not suppose that any modern mathematician would deliberately say that the positive integers strictly run up to the denumeral. It is not because of an human imperfection that we cannot add units to a collection until it becomes denumeral, but it is because the supposition involves a contradiction in itself, and therefore cannot be rendered definite in all respects. For the denumeral is and by definition that which *cannot* be reached by successive additions of unity. Nothing, however, prevents an endless series being followed by some definite unit as its *limit;* and this is what Cantor means, and expressly says he means, by his ω.* It is not *produced* by additions of unity but it is the first ordinal number after having passed through an endless series. There is no contradiction in the idea of passing through an endless series; for it is only endless in the sense of being incapable of production by successive additions of unity, just as Achilles

* *Op. cit.*, S. 324f.

can easily overtake the tortoise although he can never do so by repeatedly going *only part way* to where the tortoise will be the instant Achilles gets there. So we can and often do reach the ω term of a series, though not by merely passing through all previous terms. Yet while reaching the denumeral does not consist in passing from one number to the number exceeding that by 1, though this be done to any extent; nevertheless because the series of finite numbers is endless, it follows that to pass *all* finite numbers is to pass *beyond* them all, and in doing *that* to attain the denumeral. There are in Cantor's exposition of his ordinal numbers several points like that which will give the unmathematical student difficulty, not because he lacks intelligence, but because he thinks so exactly as to see the difficulties, while not being sufficiently acquainted with the subtleties of mathematics he is unable to solve them, while many mathematicians, especially of the pre-Weierstrassian school have their ideas hazy on these points, although they may be perfectly clear for all mathematical purposes. There is certainly no really sound objection to anything in Cantor's system of ordinals until the second abnumerable ordinals are reached; and even then in my opinion my modification of his law of progression removes any possible error that there may there be. But my article is already so long that I must cut that short. Suffice it to say that there is certainly a possible series of ordinals of the first abnumerable multitude, while the entire multitude of all possible multitudes is only denumeral. On that point there is no possible doubt for a competent judge. It follows that the cardinal numbers, even in the extended sense in which Cantor employs the term, to denote any multitudes whatever, cannot be so rich in relations and therefore must belong to a lower [order] than that of ordinals, which are merely exact grades, regardless of what sort of states they are grades of; and hence the restriction of number to cardinals involves a serious lopping off of the highest part of mathematics. Indeed it is not necessary to consider Cantor's ordinals to reach that conclusion, since the multitude of all possible irrational values, say between 0 and 1, is abnumerable and therefore can in no way be reduced to cardinals, of which the entire multitude is infinitely less.

To be sure, it might be said that the irrational numbers, even if they be not cardinals, are not ordinals, but are ratios, and involve the idea of equality of parts. But I propose to disprove that, by showing that all rational fractions are ordinal. It is well known that all fractions can be arranged in a well-ordered Cantorián series, and that in indefinitely many ways, but it may be said that when that is done, it is no longer possible (certainly far from evident) which of any two is the greater in value; for which purpose they must be reduced to a common denominator; and that the possibility of the reduction to a common denominator is not involved in the idea of the special Cantorian series. But I am going to show in two ways that such series are possible in which the relative magnitude of any two fractions is expressed in the series itself.

678. We have to distinguish the system of *rational fractions* themselves, which are merely expressions, denoting rational values, from the values themselves. Thus $\frac{5}{10}$ and $\frac{1}{2}$ are two different *fractions* denoting one and the same *rational value*. Now I am going to show how in the first place all positive numerical fractions can be arranged in a well-ordered Cantorian series, carrying between the members of each successive pair of fractions of the series either the sign $<$ to show that the succeeding fraction is the greater or the sign $=$ to show that the two fractions are equal. Afterward I shall exhibit a somewhat similar series whose terms are all the positive rational values expressed in their lowest terms.

679. For the present, I confine myself to the fractional expressions. The conception of the series will be built up in the following way. We are to suppose, in the first place, that all the positive fractions of denominator 1 (which fractions are all equal to their several numerators) to be ranged in the order of values of their numerators with the sign $<$ between every successive two, thus:

(First *state*): $$\frac{0}{1}<\frac{1}{1}<\frac{2}{1}<\frac{3}{1}< \quad \text{etc.}$$

We then go on to conceive that first all the fractions of denominator 2 are placed in the order of values of their numerators one in every "space" of that series; where by a "space," I mean an interval either between a fraction of that series and its following copula, <, or between a copula and its following fraction; so that the fractions of denominator two will be inserted in the spaces indicated by the "carets" of the following line:

$$\frac{0}{1}\wedge<\wedge\frac{1}{1}\ \wedge>\wedge\frac{2}{1}, \text{ etc.}$$

Each fraction when inserted will be accompanied by a copula either preceding or following it according as the space, before the insertion, was preceded or followed by a fraction.

For the sake of clearness, I will postpone saying what each copula is to be, but will only indicate it by a C. At first, then, we have the series in the form

(Second *state*):

$$\frac{0}{1}\wedge \mathrm{C} \wedge\frac{1}{1}\wedge \mathrm{C} \wedge\frac{2}{1}\wedge \mathrm{C} \wedge\frac{3}{1} \text{ etc.}$$

and after the insertion of the fractions of denominator 2 the series will become (without the carets), where I distinguish the *new* carets by italicizing the C's

(Third *state*):

$$\frac{0}{1}C\frac{0}{2}C\frac{1}{2}C\frac{1}{1}C\frac{2}{2}C\frac{3}{2}C\frac{2}{1}C\frac{4}{2}C\frac{5}{2}C\frac{3}{1}.$$

Now the general rule for the *carets* (which are only temporary scaffolding) is that after inserting the fractions of any denominator, N, where N is whatever whole number comes next in the order of magnitude of whole numbers, a caret is to be inserted at every N^{th} space from the beginning; so that in the state of the series just represented, the carets will appear as here shown:

(Fourth *state*):

$$\frac{0}{1}\mathrm{C}\wedge\frac{0}{2}\mathrm{C}\wedge\frac{1}{2}\mathrm{C}\wedge\frac{1}{1}\mathrm{C}\wedge\frac{2}{2}\mathrm{C}\wedge\frac{3}{2}\mathrm{C}\wedge\frac{2}{1}\mathrm{C}\wedge\frac{4}{2}\mathrm{C}\wedge\frac{5}{2}\mathrm{C}\wedge\frac{3}{1}\mathrm{C} \text{ etc.}$$

Then will be inserted the fractions of the next higher denominator each with its copula (which I will again italicize) thus:

(Fifth *state*):

$$\frac{0}{1}\,C\,\frac{0}{3}\,C\,\frac{0}{2}\,C\,\frac{1}{3}\,C\,\frac{1}{2}\,C\,\frac{2}{3}\,C\,\frac{1}{1}\,C\,\frac{3}{3}\,C\,\frac{2}{2}\,C\,\frac{4}{3}\,C\,\frac{3}{2}\,C\,\frac{5}{3}\,C \text{ etc.}$$

The carets will now be inserted as follows:

(Sixth *state*):

$$\frac{0}{1}\,C\,\frac{0}{3}\wedge C\,\frac{0}{2}\,C\wedge\,\frac{1}{3}\,C\wedge\,\frac{1}{2}\wedge C\,\frac{2}{3}\,C\wedge\,\frac{1}{1}\,C\wedge\,\frac{3}{3}\wedge C\,\frac{2}{2}\,C\wedge\,\frac{4}{3}\,C\,\frac{3}{2}\wedge C\,\frac{5}{3} \text{ etc.}$$

The carets are always forthwith replaced each by a fraction of the lowest denominator not used to replace any previous set of carets; and each fraction will be accompanied by a copula either before it, in case of a fraction immediately preceding the caret, or after it in case a fraction follows the caret.

It now only remains to state what the copulas are and the description of the series is complete. A newly inserted copula coming before a newly inserted fraction is always $=$, but a newly inserted copula which follows the fraction inserted with it in place of the same caret, is of the same kind as the old copula preceding that newly inserted fraction. In the only case not thus provided for, the copula is $<$.

680. Let the fractions be *defined* by the series thus formed, and not otherwise defined, and (the arithmetic of whole numbers being supposed) the *entire doctrine of fractions is contained in this series, or rather, in its governing definition, or rule of construction*. The very easy proof of this may be omitted.

681. I fear this series will not attract the attention it really merits; for it is dressed in a fantastic garb of artificiality that does not do it justice. It is like an honest, self-respecting actor, who, by some unfortunate mistake, happened to be arrested in his theatrical dress, should appear before the judge in the guise of Jeremy Diddler.

For my series of rational values, I hope better things. Here there will be no need of copulas, because all the terms are different values. We begin by writing

$$\frac{0}{1} \quad \frac{1}{0}$$

as the zero state of the series (although $\frac{1}{0}$ is not properly a rational value), and we go through the series from beginning to end, time and again ceaselessly; every time inserting between every two adjacent fractions a new fraction whose numerator shall be the sum of the numerators, and its denominator the sum of the denominators of the two fractions between which it is inserted. The result will be that every positive rational value will be inserted, once only, and expressed in its lowest terms.

Zero state:

$$\frac{0}{1}\quad\frac{1}{0}$$

First state:

$$\frac{0}{1}\quad\frac{1}{1}\quad\frac{1}{0}$$

Second state:

$$\frac{0}{1}\quad\frac{1}{2}\quad\frac{1}{1}\quad\frac{2}{1}\quad\frac{1}{0}$$

Third state:

$$\frac{0}{1}\quad\frac{1}{3}\quad\frac{1}{2}\quad\frac{2}{3}\quad\frac{1}{1}\quad\frac{3}{2}\quad\frac{2}{1}\quad\frac{3}{1}\quad\frac{1}{0}$$

Fourth state:

$$\frac{0}{1}\,\frac{1}{4}\,\frac{1}{3}\,\frac{2}{5}\,\frac{1}{2}\,\frac{3}{5}\,\frac{2}{3}\,\frac{3}{4}\,\frac{1}{1}\,\frac{4}{3}\,\frac{3}{2}\,\frac{5}{3}\,\frac{2}{1}\,\frac{5}{2}\,\frac{3}{1}\,\frac{4}{1}\,\frac{1}{0}$$

Fifth state:

$$\frac{0}{1}\,\frac{1}{5}\,\frac{1}{4}\,\frac{2}{7}\,\frac{1}{3}\,\frac{3}{8}\,\frac{2}{5}\,\frac{3}{7}\,\frac{1}{2}\,\frac{4}{7}\,\frac{3}{5}\,\frac{5}{8}\,\frac{2}{3}\,\frac{5}{7}\,\frac{3}{4}\,\frac{4}{5}\,\frac{1}{1}\quad \text{etc.}$$

This series has many curious properties, some of which are very easily proved. For example, the series of numerators in any state always begins with the series of the preceding state; thus:

01
011
01121
011213231
01121323143525341
011213231435253415473857275837451

All the properties of rational values and of their expressions in their lowest terms follow from the general fact that they are all contained in their order in the series constructed according to this rule.

This at once proves that the ideas of the rational values essentially involves no other relation than that of linear succession, and that the equality of parts is not presupposed. And since the irrational values are nothing but the limits of series of rational values, they also suppose nothing but the linear form of relation. It is because [of] this form of relation of rational consequence that numbers are of such stupendous importance in reasoning.

But the highest and last lesson which the numbers whisper in our ear is that of the supremacy of the forms of relation for which their tawdry outside is the mere shell of the casket.

INDEX OF PROPER NAMES*

*The numbers refer to paragraphs, not to pages

INDEX OF SUBJECTS*

*The numbers refer to paragraphs, not to pages

INDEX OF SUBJECTS

INDEX OF SUBJECTS