

DOUGLASS B. MORRIS, *Some models of ZF + $\neg$ AC.*

Let ZF be Zermelo-Fraenkel set theory with axiom of foundation, GCH the generalized continuum hypothesis, AC the axiom of choice, and ZFC ZF + AC. Using Cohen's method of forcing the following results are proved. In each, M is a countable model of ZFC and N is a Cohen extension of M preserving cardinals and cofinalities. A different construction is used for each result. (Some related results are also proved.) The results concern large linearly ordered sets with small proper initial segments.

Let $U(X)$ be " $\exists \langle T, < \rangle (\bar{T} > \bar{X}$ and $<$ linearly orders T such that every proper initial segment has cardinality $< \bar{X}$)." Let $U'(X)$ be the same but with every subset of T required to have cardinality commensurable with $\bar{X}$.

For any $\aleph_\alpha$ in M there is an $N \models \text{ZF} + U(\aleph_\alpha)$. If $\aleph_\alpha$ is regular, $N \models$ dependent choices.

Suppose $M \models \text{GCH}$. For any class J of alephs in M satisfying certain conditions there is an $N \models \text{ZF} + \forall \aleph(U(\aleph) \text{ iff } \aleph \in J)$. J may be the class of uncountable alephs. Many members of J may be made to satisfy U' .

There is an $N \models \text{ZF} + \exists \langle X, < \rangle (X \notin W \text{ and } < \text{ linearly orders } X \text{ with every proper initial segment countable})$, where W is the class defined by Keisler, whose members are formed from well-ordered sets by transfinitely iterating the taking of well-ordered unions.

EDUARD WETTE, *On new paradoxes in formalized mathematics.*

I have pointed out repeatedly since 1967 that it is possible to refute the semantical explanation of Gödel's "incompleteness" theorem (1931). Indeed, there will be no defense against a finite derivation of 'consistency'-propositions which should be underivable from indirect and reflecting considerations.

I refer to my abstract in Tagungsbericht 10/68 (pp. 7-8) of Mathematisches Forschungsinstitut Oberwolfach, the last sentence (6.) of which was concerned with contradictions in number theory.

1. Let $\mathcal{B}$ be the following formal system: intuitionistic predicate calculus, $A(a) \rightarrow A'(a) \Rightarrow A(0) \rightarrow A(b)$, $V \rightarrow 0 \leq a$, $a \leq b \leftrightarrow 'a \leq 'b$, $'a \leq 0 \rightarrow \Delta$, $V \rightarrow +a0a$, $+a'b'c \leftrightarrow +abc$, $V \rightarrow \cdot a00$, $\cdot a'bc \leftrightarrow V_x \cdot abz \wedge +zac$. How to justify informally the *theorem* $\vdash_{\mathcal{B}} \Delta$ and its *lemma* $\vdash_{\mathcal{B}} \text{cons}_{\mathcal{B}} (\Leftrightarrow \text{ded}_{\mathcal{B}} (d, \ulcorner \Delta \urcorner) \rightarrow \Delta)$?

In contrast to Lorenzen's first book (1955, XXIII 99 (7); 1969)—cf. p. 23, lines 7 and 8—we can exhibit a calculus $\mathcal{R}$ which codifies each transformation term T [for derivations D according to a system K of rules R] as required to eliminate the applications of a meta^k-rule ${}^kR(A)$ that enters by way of the "operative" interpretation of a $\mathcal{B}$ -deduced formula $A(*, \dots, *)$; and then $\mathcal{R}, \mathcal{B}$ turn out to be "too" complete—rather than "in" complete—, since the arithmetizations of $\mathcal{R}$ as well as of semiotic inductions for the T, D, K in $\mathcal{R}$, and of the proof that the specified ${}^kR(A)$ are "relatively admissible" with respect to a reflection-free calculus

$$\begin{aligned} K_0(: \Rightarrow \leq 00, 0u \Rightarrow \leq 0'u, \leq vw \Rightarrow \leq 'v'w; \\ \Rightarrow > '00, > u0 \Rightarrow > 'u0, > vw \Rightarrow > 'v'w) \end{aligned}$$

—see l.c. §§2-9, 13—have finite expressions within $\mathcal{B}$.

(The entire inconsistency-derivation should be controllable by a computer.)

2. $\mathcal{R}$ enumerates (i) a formal-operative translation ${}^kR(A)$ for each $\mathcal{B}$ -deduced A , (ii) the derivations D according to each K which consists of the rules R corresponding to the endformulae of a $\mathcal{B}$ -deduction and of its subdeductions (as per (i)), (iii) for each K (as per (ii)) a term T which transforms each K -derivation into a derivation according to K^- [without the last rule of K] under preservation of the (variable-free) result, (iv) the defining equalities between an implicit derivation TD and the explicitly transformed derivation D_1 .

$\mathcal{R}$ introduces a reading term e^{K_1} for D that singles out the result of D together with an identification of the system $*$ of the rules as applied in D , and with an examination whether the result of D is built up from the prime objects of K_1 only.

After introduction of formal variables $X, \dots$ for D the above mentioned relative admissibilities can be ascertained in the form

$$\vdash_{\mathcal{R}} e^{K_0}_{K_0 R_1 \dots R_n} X \equiv e^{K_0}_{K_0} T_1 \dots T_n X.$$

The deciphered duplicate for $\mathcal{S}$ -consistency is

$$\vdash_{\mathcal{S}} e^{K_0 R_1 \dots R_n} X \neq > 00.$$

In case of a ^{K+1}R the term $^{K}T_0$ that eliminates the conclusion $^{K}R_0$ of ^{K+1}R is expressed uniformly in terms $^{K}T_1, \dots$ as eliminating already the premises $^{K}R_1, \dots$ of ^{K+1}R .

(In order to parry criticisms as to negligent "inversions" we use Łukasiewicz's notation, and, for the sake of uniqueness, I provide formalized instructions for each application of a rule even within the official (1-dimensional) spelling of the D's.)

PAUL BENIOFF, *A theorem on undefinability.*

Let T be ZF set theory and $L(T)$ the language of T with appropriate conservative extensions to include the probability theoretic concepts used below. Let Ω denote the set of all infinite sequences of natural numbers, P be a probability measure defined on the usual minimal σ -field of subsets of Ω , and φ and θ be sequence variables.

Define $E(T, P, \varphi)$ by

$$\forall q [q \in L(T) \text{ and } q \text{ has exactly one free sequence variable and at most one probability measure variable and } P[\theta \mid q_P(\theta)] = 1 \Rightarrow q_P(\varphi)]$$

where q ranges over all formulas of $L(T)$ and $q_P(\theta)$ is the expression obtained from q by replacing any free probability measure variable (if any) with P .

$E(T, P, \varphi)$ has many uses in mathematical logic and physics. Among them are:

- (1) It is the essential part of the presently given definitions of randomness [1], [2].
- (2) It is the essential part of the proposed definition of agreement, with respect to T , between a physical theory and experiment for a large class of physical theories [2].
- (3) It gives a limitation on which sequences in Ω are definable in L [2].

The latter is given by the following theorem:

For any φ and P such that P is nontrivial, $E(T, P, \varphi)$ implies that φ is not definable in T .

This theorem will be proven and discussed. In particular, it and the existence theorem, $\exists \varphi [E(T, P, \varphi)]$, give stringent limitations on the addition of names of sequences in Ω to $L(T)$.

REFERENCES

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- [2] P. A. BENIOFF, *Journal of mathematical physics*, vol. II (1970), p. 2553, vol. 12 (1971), p. 360.

CLEMENT F. KENT, *Provability as an order map of the Lindenbaum algebra*, I. Preliminary Report.

Throughout this abstract, $\vdash$ denotes provability, and $\text{Prov}(\)$ represents the provability predicate of a formal theory, U , of arithmetic, or containing arithmetic. The symbol, $\ulcorner A \urcorner$, denotes the Gödel number of the formula A , while $\overline{A}$, denotes the canonical primitive recursive function which, for each substitution of numerals for the free variables in A , calculates the Gödel number of the substituted form. The following "derivability conditions" (see Hilbert and Bernays, *Grundlagen der Mathematik*, II) are assumed. (i) $\vdash A \rightarrow B \Rightarrow \vdash \text{Prov } \overline{A} \rightarrow \text{Prov } \overline{B}$. (ii) $\vdash A \Leftrightarrow \vdash \text{Prov } \ulcorner A \urcorner$. (iii) $\vdash \text{Prov } \overline{A} \wedge \text{Prov } \overline{A} \rightarrow B \rightarrow \text{Prov } \overline{B}$. (iv) $\vdash \text{Prov } \overline{A} \wedge \overline{B} \leftrightarrow \text{Prov } \overline{A} \wedge \text{Prov } \overline{B}$. (v) $\vdash \text{Prov } \ulcorner A \urcorner \rightarrow \text{Prov } \overline{A}$. (vi) $\vdash A \rightarrow \text{Prov } \overline{A}$, for any formula A whose prenex normal form has only existential quantifiers, (Σ_1^0 formula).

From (i) and (iv) it follows that Prov is an order preserving map of the Lindenbaum Algebra (the Boolean Algebra of classes of formulas induced by the pre-order relation $A \leq B \Leftrightarrow \vdash A \rightarrow B$) into the sublattice of classes containing Σ_1^0 formulas, denoted $\hat{\Sigma}_1^0$. By Löb's theorem this JOURNAL, vol. 20 (1955), $\vdash \text{Prov } \overline{A} \rightarrow A \Leftrightarrow \vdash \overline{A}$. Thus, the map is never decreasing. From (vi) it follows that it is nondecreasing on $\hat{\Sigma}_1^0$, and increasing everywhere on $\hat{\Sigma}_1^0$, except on the unit. Basic question: Does the Prov mapping increase any element not in $\hat{\Sigma}_1^0$? For reference, let C denote the assertion: If $\vdash A \rightarrow \text{Prov } \ulcorner A \urcorner$, then A belongs to a member of $\hat{\Sigma}_1^0$.