The Apprehension of Plurality

Henry Flynt

(An instruction manual
for 1987 concept art)

I. Original Stroke-Numerals

Stroke-numerals were introduced in foundations of mathematics by the German mathematician David Hilbert early in the twentieth century. Instead of a given Arabic numeral such as '6', for example, one has the expression consisting of six concatenated occurrences of the stroke, e.g. 'IIIIII'.

To explain the use of stroke-numerals, and to provide a background for my innovations, some historical remarks about the philosophy of mathematics are necessary. Traditional mathematics had treated positive whole-number arithmetic as if the positive whole numbers (and geometrical figures also) were objective intangible beings. Plato is usually named as the originator of this view. Actually, there is a scholarly controversy over the degree to which Plato espoused the doctrine of Forms—over whether Aristotle's *Metaphysics* put words in Plato's mouth—but that is not important for my purposes. For an intimation of the objective intangible reality of mathematical objects in Plato's own words, see the remarks about "divine" geometric figures in Plato's "Philebus." Aristotle's *Metaphysics,* I.6, says that mathematical entities

> are intermediate, differing from things perceived in being eternal and unchanging, and differing from the Forms in that they exist in copies, whereas each Form is unique.

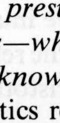
For early modern philosophers such as Hume and Mill, any such "Platonic" view was not credible and could not be defended seriously. Thus, attempts were made to explain number and arithmetic in ways which did not require a realm of objective intangible beings. In fact, Hume said that arithmetic consisted of tautologies; Mill that it consisted of truths of experience.

Following upon subsequent developments—the philosophical climate at the end of the nineteenth century, and specifically mathematical developments such as non-Euclidian geometry—Hilbert proposed that mathematics should be understood as a game played with meaningless marks. So, for example, arithmetic concerns nothing but formal terms—numerals—in a network of rules. Actually, what made arithmetic problematic for mathematicians was its infinitary character—as expressed, for example, by the principle of complete induction. Thus, the principal concern for Hilbert was that this formal game should not, as a result of being infinitary, allow the deduction of both a proposition and its negation, or of such a proposition as 0 = 1.

But at the same time (without delving into Hilbert's distinction between mathematics and metamathematics), the stroke-numerals replace the traditional answer to the question of what a number is. The stroke-numeral 'IIIIII' is a concrete semantics for the sign '6', and at the same time can serve as a sign in place of '6'. The problem of positive whole numbers as abstract beings is supposedly avoided by inventing e.g. a number-sign, a numeral, for six, which is identically a concrete semantics for six. Let me elaborate a little further. A string of six copies of a token having no internal structure is used as the numeral '6', the sign for six. Thus the numeral is itself a collection which supposedly demands a count of six, thereby showing its meaning. Hans Freudenthal calls this device an "ostensive numeral."

So traditionally, there is a question as to what domain of beings the propositions of arithmetic refer to, a question as to what the referents of number-words are. *Correlative to this, mathematicians' intentions require numerous presuppositions about content, and require extensive competancies—which the rationalizations for mathematics today are unable to acknowledge, much less to defend.*

For example, if mathematics rests on concrete signs, as Hilbert proposed, then, since concrete signs are objects of perception, the reliability of mathematics would depend on the reliability of perception. Given the script numeral



which is ambiguous between one and two, conventional mathematics would have to guarantee the exclusion of any such ambiguity as this. Yet foundations of mathematics excludes perception and the reliability of concrete signs as topics—much as Plato divorced mathematics from these topics. (Roughly, modern mathematicians would say that reliability of concrete signs does not interact with any advanced mathematical results. So this precondition can simply be transferred from the requisites of cognition in general. But it would not be sincere for Hilbert to give this answer. Moreover, my purpose is to investigate the possibility of reconstructing our intuitions of quantity beyond the limits of the present culture. In this connection, I need to activate the role of perception of signs.)

But the most characteristic repressed presuppositions of mathematics run in the opposite, supra-terrestrial direction. Mathematicians' intentions require a realm of abstract beings. Again, it is academically taboo today to expose such presuppositions.* But to recur to the purpose of this investigation, concept art is about reconstructing our intuitions of quantity beyond the limits of the present culture. This project demands an account of these repressed presuppositions. To compile such an account is a substantial task; I focus on it in a collateral manuscript entitled "The Repressed Content-Requirements of Mathematics." To uncover the repressed presuppositions, a combination of approaches is required.** I will not dwell further on the matter here-but a suitable sample of my results is the section "The Reality-Character of Pure Whole Numbers and Euclidian Figures" in "The Repressed Content-Requirements."

Returning to the original stroke-numerals, they were meant (among other things) to be part of an attempt to explain arithmetic without requiring numbers as abstract beings. They were meant as signs, for numbers, which are identically their own concrete semantics. Whether I think Hilbert succeeded in dispensing with abstract entities is not the point here. I am interested in how far the exercise of positing

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*Gödel and Quine admit the need to assume the non-spatial, abstract existence of classes. But they cannot elaborate this admission; they cannot provide a supporting metaphysics.

**One anthroplogist has written about "the locus of mathematical reality"—but, being an academic, he merely reproduces a stock answer outside his field (namely that the shape of mathematics is dictated by the physiology of the brain).

stroke-numerals as primitives can be elaborated. My notions of the original stroke-numerals are adapted from Hilbert, Weyl, Markov, Kneebone, and Freudenthal. For example, how does one test two stroke-numerals for equality? To give the answer that "you count the strokes, first in one numeral and then in the other," is not in the spirit of the exercise. For if that is the answer, then that means that you have a competency, "counting," which must remain a complete mystery to foundations of mathematics. What one wants to say, rather, is that you test equality of stroke-numerals by "cross-tallying": by e.g. deleting strokes alternately from the two numerals and finding if there is a remainder from one of the numerals. This is also the test of whether one numeral precedes the other. So, now, given an adult mastery of quality and abstraction, you can identify stroke-numerals without being able to "count."

In the same vein, you add two stroke-numerals by copying the second to the right of the first. You subtract a shorter numeral from a longer numeral by using the shorter numeral to tally deletion of strokes from the longer numeral. You multiply two stroke-numerals by copying the second as many times as there are strokes in the first: that is, by using the strokes of the first to tally the copying of the second numeral.

To say that all this is superfluous, because we already acquired these "skills" as a child, misses the point. The child does not face the question, posed in the Western tradition, of whether we can avoid positing whole numbers as abstract beings. To weaken the requirements of arithmetic to the point that somebody with an adult mastery of quality and abstraction can do feasible arithmetic "blindly"—i.e. without being able to "count," and without being able to see number-names ('five', 'seven', etc.) in concrete pluralities—is a notable exercise, one that correlates culturally with positivism and with the machine age.

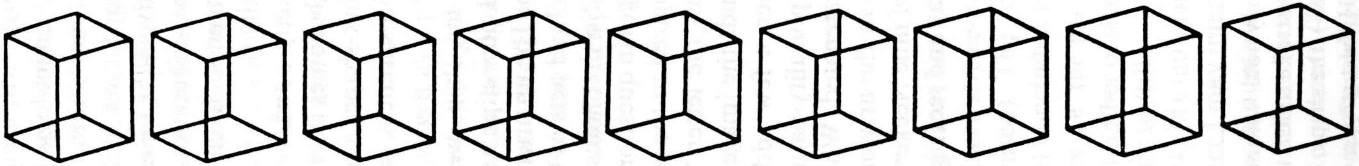
To reiterate, the stroke-numeral is meant to replace numbers as abstract beings by providing number-signs which are their own concrete semantics. Freudenthal said that we should communicate positive whole numbers to alien species by broadcasting stroke-numerals to them (in the form of time-series of beeps). Still, Freudenthal said that the aliens would have to resemble us psychologically to get the point. (*Lincos,* pp. 14-15.)

When Hilbert first announced stroke-numerals, certain difficulties were pointed out immediately. It is not feasible to write the stroke-numerals for very large integers. (And yet, if it is feasible to write the stroke-numeral for the integer n, then there is no apparent reason why it would not also be feasible to write the stroke-numeral for n+1. So stroke-numerals are closed under succession, and yet are contained in a finite segment of the classical natural number series.) Moreover, large feasible stroke-numerals, such as that for 10,001, are not surveyable.

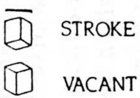
But this is not a study of metamathematical stroke-numerals. And I do not wish to go into Hilbert's question of the consistency of arithmetic as an infinitary game here; "The Repressed Content-Requirements" will have more to say on the consistency question. The purpose of this manual, and of the artworks which it accompanies, is to establish apprehensions of plurality beyond the limits of traditional civilizations (beyond the limits of Freudenthal's "us"). Moreover, these apprehensions of plurality are meant to violate the repressed presuppositions of mathematics. I refer back to original stroke-numerals because certain devices which I will use in assembling my novelties cannot be supposed to be intuitively comprehensible—certainly not to the traditionally-indoctrinated reader—and will more likely be understood if I mention that they are adaptations of features of original stroke-numerals. Let me mention one point right away. In our culture, we usually see numerals as positional notations—e.g. 111 is decimal $1 \times 10^2 + 1 \times 10^1 + 1$ or binary $1 \times 2^2 + 1 \times 2^1 + 1$. But stroke-numerals are not a positional notation (except trivially for base 1). Likewise, my novelties will not be positional notations; I will even nullify the reference to base 1. (Only much later in my investigations, when broad scope becomes important, will I use positional notation.) So the foregoing introduction to stroke-numerals has only the purpose of motivating my novelties. And references to the academic canon are given only for completeness. They cannot be norms for what I am "permitted" to posit.

II. Simple Necker-Cube Numerals

In my stroke-numerals, the printed figure, instead of being a stroke, is a Necker cube. (Refer to the attached reproduction, "Stroke-Numeral.") A Necker cube is a two-dimensional representation of a cubical frame, formed without foreshortening so that its perspective is perceptually equivocal or multistable. The Necker cube can be seen as flat, as slanting down from a central facet like a gem, etc.; but for the moment I am exclusively concerned with the two easiest variants in which it is seen as an ordinary cube, either projecting up toward the front or down toward the front.



STROKE-NUMERAL



Since I will use perceptually multistable figures as notations, I need a terminology for distinctions which do not arise relative to conventional notation. I call the ink-shape on paper a *figure.* I call the stable apparition which one sees in a moment—which has imputed perspective—the *image.** As you gaze at the figure, the image changes from one orientation to the other, according to intricate subjective circumstances. It changes spontaneously; also, you can change it voluntarily.

Strictly—and very importantly—it is the image which in this context becomes the notation. Thus, I will work with notations which are not ink-shapes and are not on a page. They arise as active interactions of awareness with an "external" or "material" print-shape or object.

So far, then, we have images—partly subjective, pseudo-solid shapes. I now stipulate an alphabetic role for the two orientations in question. The up orientation is a *stroke;* the down orientation is called *"vacant,"* and acts as the proofreaders' symbol ⁐ , meaning "close up space." (So that "vacant" is not "even" an alphabetic space.) Now the two images in question are *signs.* The transition from image to sign can be analogized to the stipulation that circles of a certain size are (occurances of) the letter "o."** I may say that one sees the image; one apprehends the image as sign.

When a few additional explanations are made, then the signs become plurality-names or "numerals." First, figures, Necker cubes, are concatenated. When this is done, a *display* results. So the stroke-numeral in the artwork, as an assembly of marks on a surface, is a display of nine Necker cubes. An *image-row* occurs when one looks at the display and sees nine subjectively oriented cubes, for just so long as

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*I may note, without wanting to be precious, that a bar does not count as a Hilbert stroke unless it is vertical relative to its reader.

**And—the shape, bar, positioned vertically relative to its reader, is the symbol, Hilbert stroke.

the apparition is stable (no cube reverses orientation). I chose nine Necker cubes as an extreme limit of what one can apprehend in a fixed field of vision. (So one must view the painting from several meters away, at least.) The reader is encouraged to make shorter displays for practice. Incidentally, if one printed a stroke-numeral so long that one could only apprehend it serially, by shifting one's visual field, it would be doubtful that it was well-defined. (Or it would incorporate a feature which I do not provide for.) The universe of pluralities which can be represented by these stroke-numerals is "small." My first goal is to establish "subjectified" stroke-numerals at all. They don't need to be large.

The concatenated signs which you apprehend in a moment of looking at the display are now apprehended or judged as a plurality-name, a *numeral.* At the level where you apprehend signs (which, remember, are alphabetized, partly subjective images, not figures), the apparition is disambiguated. Thus I can explain this step of judging the signs as plurality-names by using fixed notation. For nine Necker cubes with the assigned syntactical role, you might apprehend such permutations of signs as

a) I⁐⁐II⁐⁐⁐I

b) I⁐⁐⁐⁐⁐III

c) IIII⁐⁐⁐⁐⁐

d) IIII⁐⁐⁐⁐I

e) ⁐⁐⁐⁐⁐⁐⁐⁐⁐

My Necker-cube stroke-numerals are something new; but (a)-(e) are not—they are just a redundant version of Hilbert stroke-numerals (which nullifies the base 1 reference as I promised). The "close up space" signs function as stated; and the numeral concluded from the expression corresponds to the number of strokes; i.e. the net result is the Hilbert stroke-numeral having the presented number of strokes. So (a) and (b) and (c) all amount to IIII. (d) amounts to IIIII.

As for (e), it has the alphabetic role of a blank. My initial interpretation of this blank is "no numeral present." Later I may interpret the blank as "zero," so that every possibility will be a numeral. Let me explain further. Even when I will interpret the blank as "zero." it will not come about from having nine zeros mapped to one zero (like a sum of zeros). (e) has nine occurrences of "close up space," making a blank.

There is always only one way of getting "blank." (A two-place display allows two ways of getting "one" and one way of getting "two"; etc.) The notation is not positional. It is immaterial whether one "focuses" starting at the left or at the right.

Relative to the heuristic numerals (a)-(e), you may judge the intended numerals by counting strokes, using your naive competency in counting. (It is also possible to use such numerals as (a)-(e) "blindly" as explained earlier. This might mean that there would be no recognition of particular numbers as gestalts; identity of numbers would be handled entirely by cross-tallying.) The Necker-cube numerals, however, pertain to a realm which is in flux because it is coupled to subjectivity. My numerals provide plurality-names and models of that realm. Thus, the issue of what you do when you conclude a numeral from a sign in perception is not simple. *We have to consider different hermeneutics for the numerals—and the ramifications of those hermeneutics.* Here we begin to get a perspective of the mutability which my devices render manageable.

For one thing, given a (stable) image-row, and thus a sign-row, you can indeed use your naive arithmetical competency to count strokes, and so conclude the appropriate numeral. This is *bicultural hermeneutic,* because you are using the old numbers to read a new notation for which they were not intended. We use the same traditional counting, of course, to speak of the number of figures in a display.

(This prescription of a hermeneutic is not entirely straightforward. The competency called counting is required in traditional mathematics. But such counting is already paradoxical "phenomenologically." I explain this in the section called "Phenomenology of Counting" in "The Repressed Content-Requirements." As for the Necker-cube numerals, the elements counted are not intended in a way which supports the being of numbers as eternally self-identical. So the Necker-cube numerals might resonate with the phenomenological paradoxes of ordinary counting. The meaning of ordinary numbering, invoked in this context, might begin to dissolve. But I mention this only to hint at later elaborations. At this stage, it is proper to recall one's inculcated school-counting; and to suppose that e.g. the number of figures in a display is fixed in the ordinary way.)

Then, there is the *ostensive hermeneutic.* Recall that I explained Hilbert stroke-numerals as signs which identically provide a concrete semantics for themselves; and as an attempt to do arithmetic without assuming that one already possesses arithmetic in the form of competency in counting, or of seeing number-names in pluralities. My intention was to prepare the reader for features to be explained now. On the other hand, at present we drop the notion of handling identity of numerals by cross-tallying.* For the ostensive hermeneutic, it is crucial that the display is short enough to be apprehended in a fixed field of vision.

With respect to short Hilbert numerals, I ask that when you see e.g.

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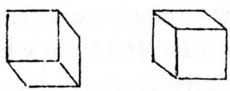
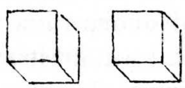
marked on a wall, you grasp it as a sign for a definite plurality, without mediation—without translating to the word "two." A similar intention is involved in recognizing

as a definite plurality, as a gestalt, without translating to "five."

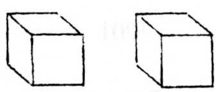
Now I ask you to apply this sort of hermeneutic to Necker-cube stroke-numerals. I ask you to grasp the sign-row as a numeral, as a gestalt. (Without using ordinary counting to call off the strokes.) For a two-place display, you are to take such images as



and



as plurality-names without translating into English words. (Similarly for



in the case where I choose to read "blank" as "zero.") Perhaps it is necessary to spend considerable time with this new symbolism before

*Because this notion corresponds to a situation in which we are unable to appraise image-rows as numerals, as gestalts.

recognition is achieved. Again, I encourage the reader to make short displays for practice. I have set a display of nine figures as the upper limit for which it might be possible to learn to grasp every sign-row as a numeral, as a gestalt.

The circumstance that the apprehended numeral may be different the next moment is not a mistake; the apprehended numeral is supposed to be in flux. So when you see image-rows, you take them as identical signs/semantics for the appearing pluralities.

But who wants such numerals—where are there any phenomena for them to count? For one thing, they count the very image-rows which constitute them. The realm of these image-rows is a realm of subjective flux: its plurality is authentically represented by my numerals, and cannot be authentically represented by traditional arithmetic.

A further remark which may be helpful is that here numerals arise only visually. So far, my numerals have no phonic or audio equivalent. (Whereas Freudenthal in effect posited an audio version of Hilbert numerals, using beeps.)

To repeat, by the "ostensive hermeneutic" I mean grasping the sign-row, without mediation, as a numeral. But there is, as well, the point that the Necker-cube numerals are *ostensive numerals.* That is, the (momentary) numeral for six would in fact be an image-row with just six occurrences of the image "upward cube." (Compare e.g. IIIↄↄIIↄI) The numeral is a collection in which only the "copies" of "upward cube" contribute positively, so to speak; and these copies demand a count of six (bicuturally). This feature needs to be clear, because later I will introduce numerals for which it does not hold.

Let me add another proviso concerning the ostensive hermeneutic which will be important later. I will illustrate the feature in question with an example which, however, is only an analogy. Referring to Arabic decimal-positional numerals, you can appraise the number-name of

1001

(comma omitted) immediately. But consider

786493015201483492147

Here you cannot appraise the number-name without mediation. That is, if you are asked to read the number aloud, you don't know whether to begin with "seven" or "seventy-eight" or "seven hundred eighty-six."

Lacking commas, you have to group this expression from the right, in triples, to find what to call it. An act of analysis is required.

In the case of Necker-cube numerals and the ostensive hermeneutic, I don't want you to see traditional number-names in the pluralities. However, I ask you to grasp a sign-row as a numeral, as a gestalt. I now add that the gestalt appraisal is definitive. I rule out appraising image-rows analytically (by procedures analogous to mentally grouping an Arabic number in triples). (I established a display of nine figures as the upper limit to support this.)

The need for this proviso will be obscure now. It prepares for a later device in which, even for short displays, gestalt appraisal and appraisal by analysis give different answers, either of which could be made binding.

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The bicultural hermeneutic is applied, in effect, in my uninterpreted calculus "Derivation," which serves as a simplified analogue of my early concept art piece "Illusions." (Refer to the reproductions on the next four pages.) Strictly, though, "Derivation" does not concern a Necker-cube stroke-numeral. The individual figures are not Necker cubes, but "Wedberg cubes," formed with some foreshortening to make one of the two orientations more likely to be seen than the other. What is of interest is not apprehension of image-rows as numerals, but rather appraisal of lengths of the image-rows via ordinary counting. As for the lessons of this piece, a few simple observations are made in the piece's instructions. But to pursue the topic of concept art as uninterpreted calculi, and derive substantial lessons from it, will require an entire further study—taking off from earlier writings on post-formalism and uncanny calculi, and from my current writings collateral to this essay.

1987 Concept Art — Henry Flynt

"DERIVATION" (August 1987 corrected version)

Purpose: To provide a simplified analogue of my 1961 concept art piece "Illusions" which is discrete and non-"warping."* Thereby certain features of "Illusions" become more clearly discernible.

Given a perceptually multistable figure, the "Wedberg cube," which can be seen in two orientations: as a cube; as a prism (trapezohedron.)
Call what is seen at an instant an *image*.
Nine figures are concatenated to form the *display*.

An *element* is an image of the display for as long as that image remains constant (Thus, elements include: the image from the first instant of a viewing until the image first changes; an image for the duration between two changes; the image from the last change you see in a viewing until the end of the viewing.)

The *length* of an element equals the number of prisms seen. Lengths from 0 through nine are possible. Two different elements can have the same length. Length of element X is written *l*(X).

Elements are seen in *temporal order* in the lived time of the spectator. I refer to this order by words with prefix 'T'. T-first; T-next; etc.

Element Y *succeeds* element X if and only if
i) *l*(X) = *l*(Y), and Y is T-next after X of all elements with this length; *or*
ii) Y is the T-earliest element you ever see with length *l*(X) + 1.
Note that (ii) permits Y to be T-earlier than X: the relationship is rather artificial.

The *initial element* A is the T-first element. (*l*(A) may be greater than 0; but it is likely to be 0 because the figure is biased.)

The *conclusion* C is the T-earliest element of length 9 (exclusive of A in the unlikely case in which *l*(A) = 9).

A derivation is a series of elements in lived time which contains A and C and in which every element but A succeeds some other element.

Discussion
To believe that you have seen a derivation, you need to keep track that you see each possible length, and to force yourself to see lengths which do not occur spontaneously.

You may know that you have seen a derivation, without being able to identify in memory the particular successions.

"Derivation" is not isomorphic to "Illusions" for a number of reasons. "Illusions" doesn't require you to see individually every possible ratio between the T-first ratio and unity. "Illusions" allows an element to succeed itself. The version of "Derivation" presented here is a compromise between mimicking "Illusions" and avoiding a trivial or cluttered structure. Any change such as allowing elements to succeed themselves would require several definitions to be modified accordingly.

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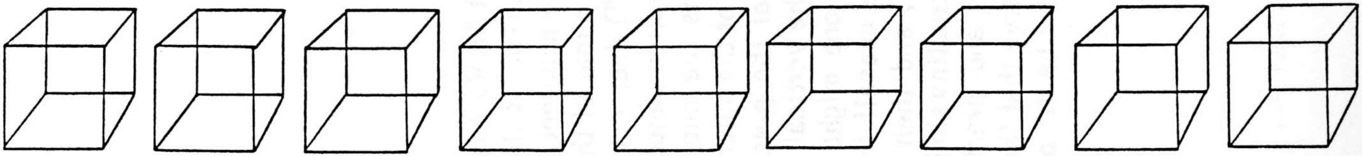
*In "Illusions," psychic coercion, which may be called "false seeing" or "warping," is recommended to make yourself see the ration as unity. In "Derivation," this warping is not necessary; all that may be needed is that you see certain lengths willfully.



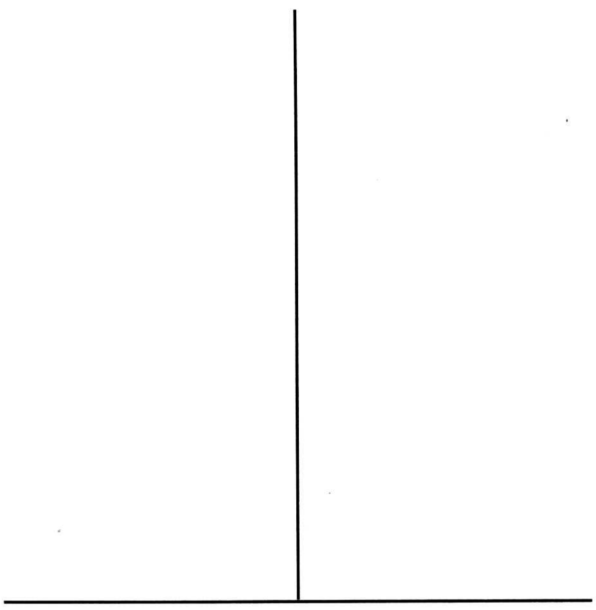
"DERIVATION"

THE DISPLAY

Concept Art Version of Mathematics System 3/26/61(6/19/61)

An "element" is the facing page (with the figure on it) so long as the apparent, perceived, ratio of the length of the vertical line to that of the horizontal line (the element's "associated ratio") does not change.

A "selection sequence" is a sequence of elements of which the first is the one having the greatest associated ratio, and each of the others has the associated ratio next smaller than that of the preceding one. (To decrease the ratio, come to see the vertical line as shorter, relative to the horizontal line, one might try measuring the lines with a ruler to convince oneself that the vertical one is not longer than the other, and then trying to see the lines as equal in length; constructing similar figures with a variety of real (measured) ratios and practicing judging these ratios; and so forth.) [Observe that the order of elements in a selection sequence may not be the order in which one sees them.]



An elaboration of "Stroke-Numeral" should be mentioned here, the piece called "an Impossible Constancy." (Refer to the facing page.) As written, this piece presupposes the bicultural hermeneutic, and that is probably the way it should be formulated. The point of this piece, paradoxically, is that one seeks to annul the flux designed into the apprehended numeral. Viewing of the Necker-cube numeral is placed in the context of a lived experience which is interconfirmationally weak: namely, memory of past moments within a dream (a single dream). Presumably, appraisals of the numeral at different times could come out the same because evidence to the contrary does not survive. So inconstancy passes as constancy. Either hermeneutic can be employed; but when I explained the hermetic hermeneutic, I encouraged you to follow the flux. Here you wouldn't do that—you wouldn't stare at the display over a retentional interval.

As for the concept of *equality* with regard to Necker-cube numerals, what can be said about it at this point? We have equality of numbers of figures in displays, by ordinary counting. We have two hermeneutics for identifying an apprehended numeral. In the course of expounding them, I expounded equivalence of different permutations of "stroke" and "vacant." Nevertheless, given that, for example, a display of two figures can momentarily count the numeral apprehended from a display of three figures,* we are in unexplored territory. Cross-tallying, suitable for judging equality of Hilbert numerals, seems maladapted to Necker-cube numerals; in fact, I dismissed it when introducing the ostensive hermeneutic.

If the "impossible constancy" from the paragraph before last were manageable, then one might consider restricting the ultimate definition of equality to impossible constancies. That is, with respect to a single display, if one wanted to investigate the intention of constancy (self-equivalence of the apprehended numeral), one might start with the impossible constancy. Appraisals of a given display become constant (the numeral becomes self-equivalent) in the dream. Then two *displays* which are *copies* might become constantly equivalent to each other, in the dream.

Such is a possibility. To elaborate the basics and give an incisive notion of equality is really an open problem, though. Other avenues might require additional devices such as the use of figures with distinctions of appearance.

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*that it is not assured that copies of a numeral will be apprehended or appraised correlatively

1987 Concept Art — Henry Flynt

Necker-Cube Stroke-Numeral: AN IMPOSSIBLE CONSTANCY

The purpose of this treatment is to say how a Necker-cube stroke numeral may be judged (from the standpoint of private subjectivity) to have the same value at different times; even though the conventional belief-system says that the value is likely to change frequently.

This is accomplished by selecting a juncture in an available mode of illusion, namely dreaming, which annuls any distinction between an objective circumstance, and the circumstance which exists according to your subjective judgment. In the first instance, I don't ask you to change your epistemology. Instead, to repeat, I select an available juncture in lived experience at which the conventional epistomology gets collapsed.

You have to occupy yourself with the stroke-numeral to the point that you induce yourself to dream about it.

When, in apprehending a stroke-numeral, you "judge" the value of the numeral, the number, this refers to the image you see and to the number-word which you may conclude from the image.

Suppose that in a single dreamed episode, you judge the value of the numeral at two different moments. Suppose that at the second moment, you do not register any discrepancy between the value at the second moment and what the value was at the first moment. Then you are permitted to disregard fallibility of memory, and to conclude that the values were the same at both moments: because if your memory has changed the past, it has done so tracelessly. A tracelessly-altered past may be accepted as the genuine past.

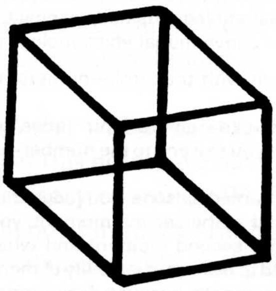
Refinements. The foregoing dream-construct may be "lifted" to waking experience, as per the lengthy explanations in "An Epistemic Calculus." Now you are asked to alter your epistemology, selectively to suspend a norm of realism.

Now that we are concerned with waking experience, a supporting refinement is possible. Suppose I make an expectation (which may be unverbalized) that the value of the numeral at a future moment will be the same that it is now. This expectation cannot be proved false, if: the undetermined time-reference "future moment" is applied only at those later moments when the value is the same as at the moment the expectation was made. (Any later moment when the value is not the same is set aside as not pertinent, or forgotten at still later moments when the value is the same.)

As a postscript, there is another respect in which testing a fact requires trust in a comparable fact. Suppose I make a verbalized expectation that the value of the numeral in the future will be the same as at present. Then to test this expectation in the future depends on my memory of my verbalization. My expectation cannot be belied unless I have a sound memory that the number I verbalized in my expectation is different from the number I conclude from the image now.

III. Inconsistently-Valued Numerals

As the "Wedberg cube" illustrates, a cubical frame can be formed in different ways, altering the likelihood that one or another image is seen. With respect to the initial uses of the Necker-cube stroke-numeral a figure is wanted which lends itself to the image of a cube projecting up, or of a cube projecting down, with an approximately equal likelihood for the two images—and which makes other images unlikely. Now let a Necker cube be drawn large, with heavy line-segments, with all segments equally long, with rhomboid front and back faces; and display it below eye level.



As you look for the up and down orientations, there should be moments when paradoxically you see the figure taking on both of these mutually-exclusive orientations at once—yielding an apparition which is a logical/geometric impossibility. The sense-content in this case is dizzying.

That we have perceptions of the logically impossible when we suffer illusions has been mentioned by academic authors. (Negative afterimages of motion—the waterfall illusion.) Evidently, though, these phenomena are so distasteful to sciences which are still firmly Aristotelian that the relations of perception, habituation, language, and logic manifested in these phenomena have never been assessed academically. For me to treat the paradoxical image thoroughly here would be too much of a digression from our subject, the apprehension of plurality. However, a sketchy treatment of the features of the impossible image is necessary here.

To begin with, the paradoxical image of the Necker cube is not the same phenomenon as the "impossible figures" shown in visual perception textbooks. The latter figures employ "puns" in perspective coding such that parts of a figure are unambiguous, but the entire figure cannot be grasped as a gestalt coherently. Then, the paradoxical Necker-cube image is not an inconsistently oriented *object* (as the reader may have noted). It is an *apparitional depiction* of an inconsistently oriented object. But this is itself remarkable. For since a dually-oriented cube (in Euclidean 3-space) is self-contradictory by geometric standards, a picture of it amounts to a non-vacuous semantics for an inconsistency. Another way of saying the same thing is that the paradoxically-oriented image is real *as an apparition.*

If one is serious about wanting a "logic of contradictions"—a logic which admits inconsistencies, without a void semantics and without entailing everything—then one will not attempt to get it by a contorted weakening of received academic logic. One will start from a concrete phenomenon which demands a logic of contradictions for its authentic representation—and will let the contours of the phenomenon shape the logic.

In this connection, the paradoxically-oriented Necker-cube image provides a lesson which I must explain here. Consider states or properties which are mutually exclusive, such as "married" and "bachelor." Their conjunction—in English, the compound noun "married bachelor"—is inconsistent.* On the other hand, the joint denial "unmarried nonbachelor" is perfectly consistent and is satisfied by nonpersons: a table is an unmarried nonbachelor. "Married" and "bachelor" are mutually exclusive, but not exhaustive, properties. Only when the domain of possibility, or intensional domain, is *restricted* to persons, so "married" and "bachelor" become exhaustive properties.** Then, by classical logic, "married bachelor" and "unmarried nonbachelor" both have the same semantics: they are both inconsistent, and thus vacuous, and thus indistinguishable. For exhaustive opposites, joint affirmation and joint denial are identically vacuous.

But the paradoxically-oriented Necker-cube image provides a concrete phenomenon which combines mutually exclusive states—as an apparition. We can ascertain whether a concrete case behaves as the tenets of logic prescribe. As I have said, various images can be seen in a Necker cube, including a flat image. Thus, the "up" and "down" cubes

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*If I must show that it is academically permitted to posit notions such as these, then let me mention that Jan Mycielski calls "triangular circle" inconsistent in *The Journal of Symbolic logic,* Vol. 46, p. 625.

**I invoke this device so that I may proceed to the main point quickly. If it is felt to be too artificial, perhaps it can be eliminated later.

are analogous to "married" and "bachelor" in that they are not exhaustive of a domain unless the domain is produced by restriction. Then "neither up nor down" is made inconsistent. (It is very helpful if you haven't learned to see any stable images other than "up" and "down.") The great lesson here is that given "both up and down" and "neither up nor down" as inconsistent, their concrete reference is quite different. To see a cube which manifests both orientations at the same time is one paradoxical condition, which we know how to realize. To see a cube which has no orientation (absence of "stroke" and absence of "vacant" both) would be a different paradoxical condition, which we do not know how to realize and which may not be realizable from the Necker-cube figure. I don't claim that this is fully worked out; but it intimates a violation of classical logic so important that I had to mention it. *When concept art reaches the level of reconstructing our inferential intuitions as well as our quantitative intuitions, such anomalies as these will surely be important.*

Referring back to the Necker cube of page 210, let us now intend it as a stroke-numeral (display of one figure). Let me modify the previous assignments and stipulate that "blank" means "zero," rather than "no numeral present." (It is more convenient if every sign yields a numeral.) When you see the paradoxical image, you are genuinely seeing "a" numeral which is the simultaneous presence of two mutually exclusive numerals "one" and "zero"—because it is the simultaneous presence of images which are mutually exclusive geometrically.***

It's not the same thing as

—or as an alternative,

—because these are merely ambiguous scripts. In the Necker-cube case, two determinate images which by logic preclude each other are present at once; and as these images are different numerals, we have a genuine

*For brevity, I may compress the three levels image, sign, numeral in exposition.

inconsistently-valued numeral.

This situation changes features of the Necker-cube numerals in important ways, however. Lessons from above become crucial. We transfer the ostensive hermeneutic to the new situation, and find an inconsistent-valued numeral. But this is no longer an *ostensive numeral.* We have a name which is one and zero simultaneously, but this is because of the impossible shape (orientation) of the notation-token. What we do not have is a collection of images of a single kind (the stroke) which paradoxically requires a count of one and a count of zero. "Stroke" is positively present, while "vacant" is positively present in the same place. We will find that a display with two figures can be inconsistent as zero and two; but it is not an ostensive numeral, because the number of strokes present is two uniquely.* Here the numerals are not identically their semantics: for the anomaly is not an anomaly of counting. The ambiguous script numeral is a proper analogy in this respect. To give an anomaly of counting which serves as a concrete semantics for the inconsistently-valued numerals, I will turn to an entirely different modality.

From work with the paradoxical image, we learn that the Necker cube allows some apprehensions which are not as common as others—but which can be fostered by the way the figure is made and by indicating what is to be seen. These rare apprehensions then become intersubjectively determinate. If one observes Necker-cube displays for a long time, one may well observe subtle, transient effects. For example, you might see the "up" and "down" orientations at the same time, but see one as dominating the other. In fact, there are too many such effects and their interpersonal replicability is dubious. If we accepted such effects as determining numerals, the interpersonal replicability of the symbols would be eroded. Also the concrete definiteness of my anomalous, paradoxical effects would be eroded. So I must stipulate that every subtle transient effect which I do not acknowledge explicitly is not definitive, and is unwanted, when the display is intended as a symbolism.

Let me continue the explanation, for the inconsistently-valued

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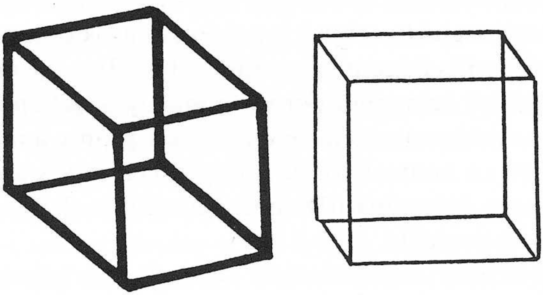
*Referring to my "person-world analysis" and to the dichotomy of Paradigm 1 and Paradigm 2 expounded in "Personhood III," this token which is two mutually exclusive numerals because its shape is inconsistent is outside that dichotomy: because established signs acquire a complication which is more or less self-explanatory, but the meanings do not follow suit.

numerals, for displays of more than one figure. When the display consists of two Necker cubes, and the paradoxical images are admitted, what are the variations? In the first place, one figure might be seen (in a moment) as a paradoxical image and the other as a unary image. Actually, if it is important to obtain this variant, we can compel it, by drawing one of the cubes in a way which hampers the double image. (Thin lines, square front and back faces, the four side segments much shorter than the front and back segments.) Then we stipulate that the differently-formed cubes continue to have the same assigned interpretation.



Reading the two-figure display, then, the paradoxical and unary images concatenate so that the resulting numeral is in one case one and two at the same time; and in the other case zero and one at the same time. Of course, it is only in a moment that either of these two cases will be realized. At other moments, one may have only unary images, so that the numeral is noncontradictorily zero, one, or two as the case may be. (If it is important to know that we can obtain a numeral which is both one and two at the same time without using dissimilar figures, then, of course, we can use a single figure and redefine the signs as "one" and "two.")

Now let us consider a display of two copies of the cube which lends itself to the paradoxical image. Suppose that two paradoxical images are seen; what is the numeral? Here is where I need the proviso which I introduced earlier. Every sign-row is capable of being grasped as a numeral, as a gestalt; and the appraisal of image-rows as numerals, analytically, is ruled out. Let me explain how this proviso applies when two paradoxical images are seen.

Indeed, let me begin with the case of a pair of ambiguous script-numerals:

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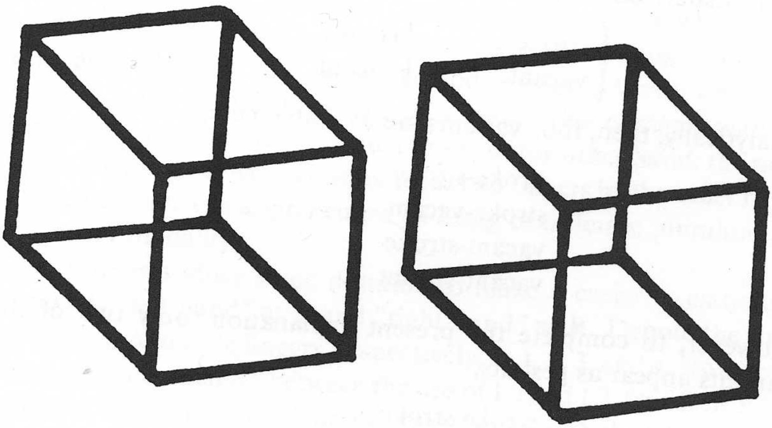


When these numerals are formed as exact copies, and I appraise the expression as a numeral, as a gestalt, then I see 11 or I see 22. ("Concatenating in parallel") I do not see 21 or 12—although these variants are possible to an analytical appraisal of the expression. In the gestalt, it is unlikely to intend the left and right figures differently. This case is helpful heuristically, because it provides a situation in which the perceptual modification is only a matter of emphasis (as opposed to imputation of depth). To this degree, the juncture at issue is externalized; and it is easier to argue a particular outcome. On the other hand, the mechanics differ essentially in the script case and the Necker-cube case.

In the Necker-cube case, one sees both the left and the right image determinately both ways at once. This case may be represented as

$$\begin{cases}\text{stroke}\\\text{vacant}\end{cases} \qquad \begin{cases}\text{stroke}\\\text{vacant}\end{cases}$$

Analytically, then, four variants are available here,

stroke-stroke
stroke-vacant
vacant-stroke
vacant-vacant

However, to complete the present explanation, only two of these variants appear as gestalts,

stroke-stroke
vacant-vacant

I chose to rule out the three-valued numeral which would be obtained by analytically inventorying the permutations of the signs afforded in the perception. The two-valued numeral arising when the sign-row is grasped as a gestalt is definitive.

Let me summarize informally what I have established. Relative to a two-figure display with paradoxical images admitted, we have a numeral which is inconsistenly two and zero. We can also have a numeral which is inconsistently one and zero, and a numeral which is inconsistently two and one. (In fact, these variants occur in several ways.) But we don't have a numeral which is inconsistently zero, one, and two—even though such a variant is available in an analytical appraisal—because such a numeral does not appear, in perception, as a gestalt.

Academic logic would never imagine that there is a situation which demands just this configuration as its representation. Certain definite positive inconsistencies are available in perception. Other definite positive inconsistencies, very near to them, are not available. Once again, *if one wants a vital "logic of contradictions," one has to develop it as a representation of concrete phenomena; not as an unmotivated contortion of received academic logics.*

* * *

But what is the use of inconsistently-valued numerals? I shall now provide the promised concrete semantics for them. This semantics utilizes another experience of a logical impossibility in perception. This time the sensory modality is touch; and the experienced contradiction is one of enumeration. Aristotle's illusion is well known in which a rod, placed between the tips of crossed fingers, is felt as two rods. (Actually, the greater oddity is that when the rod is held between uncrossed fingers, it is felt as one even though it makes two contacts with the hand.) I now replace the rod with a finger of the other hand: the same finger is felt as one finger in one hand, as two fingers by the other hand. So the same entity is apprehended as being of different pluralities, in one sensory modality.

Let me introduce some notation to make it easier to elaborate. Abbreviate "left-hand" as L and "right-hand" as R. Denote the first, middle, ring, and little fingers, respectively, as 1, 2, 3, and 4. Now cross L2 and L3, and touch R3 between the tips of L2 and L3. One feels R3 as one finger in the right hand, and as two fingers with the left hand. As apparition, R3 gets a count of both one and two, apprehended in the same sensory modality at the same time. Here is a phenomenon authentically signified by a Necker-cube numeral which is both "1" and "2."

The crossed-finger device is obviously unwieldy. The possibilities can, however, be enlarged somewhat, to make a further useful point. For example, touch L1 and R3, while touching crossed L2 and L3 with R4. Here we have a plurality, concatenated from one unary and one paradoxical constituent, which numbers two and three at the same time.

Then, we may cross L1 and L2 and touch R3, while crossing L3 and L4 and touching R4. Now we have a plurality which is two and four at the same time. In terms of perceptual structure, it is analogous to the numeral concatenated from two paradoxical images. As gestalt, we concatenate in parallel. In the case of the fingers, we do not find a plurality of three unless we appraise the perception analytically (blocking concatenation in parallel).

If one wants the inconsistently-valued numerals to be ostensive numerals, then one can use finger-apparitions to constitute stroke-numerals. Referring back to the first example, if we specify that the stroke(s) is your R3-perception, or the apparition R3, then we obtain a stroke which is single and double at the same time. Now the inconsistently-valued numeral is identically its semantics: it authentically names the token-plurality which constitutes it.

I choose not to rely heavily on this device because it is so unwieldy. The visual device is superior in that considerably longer constellations are in the grasp of one person. Of course, if one chose to define fingers as the tokens of ordinary counting, one might keep track of numbers larger than ten by calling upon more than one person. The analogous device could be posited with respect to the inconsistently-valued numbers; but then postulates about intersubjectivity would have to be stated formally. I do not wish to pursue this approach.

It is worth mentioning that if you hold a rod vertically in the near center of your visual field, hold a mirror beyond it, and focus your gaze on the rod, then you will see the rod reflected double in the mirror. This is probably not an inconsistent perception, because the inconsistent counts don't apply to the same apparition. (But if we add Kant's postulate that a reflection exactly copies spacial relations among parts of the object, then the illusion does bring us close to inconsistency.) The illusion illustrates, though, that there is a rich domain of phenomena which support mutable and inconsistent enumeration.

IV. Magnitude Arithmatic

I will end this stage of the work with an entirely different approach to subjectively variable numerals and quantities. I use the horizontal-vertical illusion, the same that appeared in "Ilusions," to form numerals. The numeral called "one" is now the standard horizontal-vertical illusion with a measured ratio of one between the segments. The numeral called "two" becomes a horizontal-vertical figure such that the vertical has a measured ratio of two to the horizontal segment. Etc. If "zero" is wanted, it consists of the horizontal segment only.

The meaning of each numeral is defined as the apparent, perceived length-ratio of the vertical to the horizontal segment. Thus, for example, the meaning of the numeral called "one" admits subjective variation above the measured magnitude. For brevity, I call this approach magnitude arithmetic—although the important thing is how the magnitudes are realized.

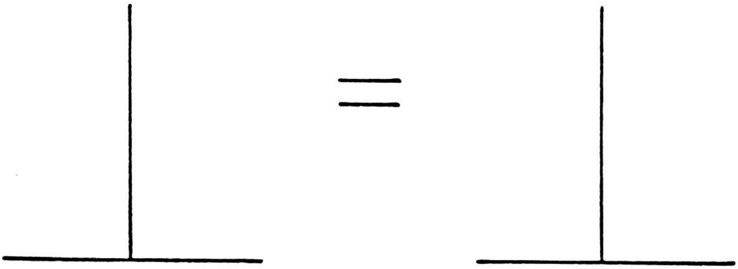
In all of the work with stroke-numerals, numbers were determinations of plurality. An ostensive numeral was a numeral formed from a quantity of simple tokens, which quantity was named by the expression. The issue in perception was the ability to make gestalt judgments of assemblies of copies of a simple token.

The magnitude numerals establish a different situation. Magnitude numerals pertain to quantity as magnitude. They relate to plurality only in the sense that in fact, measured vertical segments are integer multiples of a unit length; and e.g. the apprehended meaning of "two" will be a magnitude always between the apprehended meanings of "one" and "three"—etc.

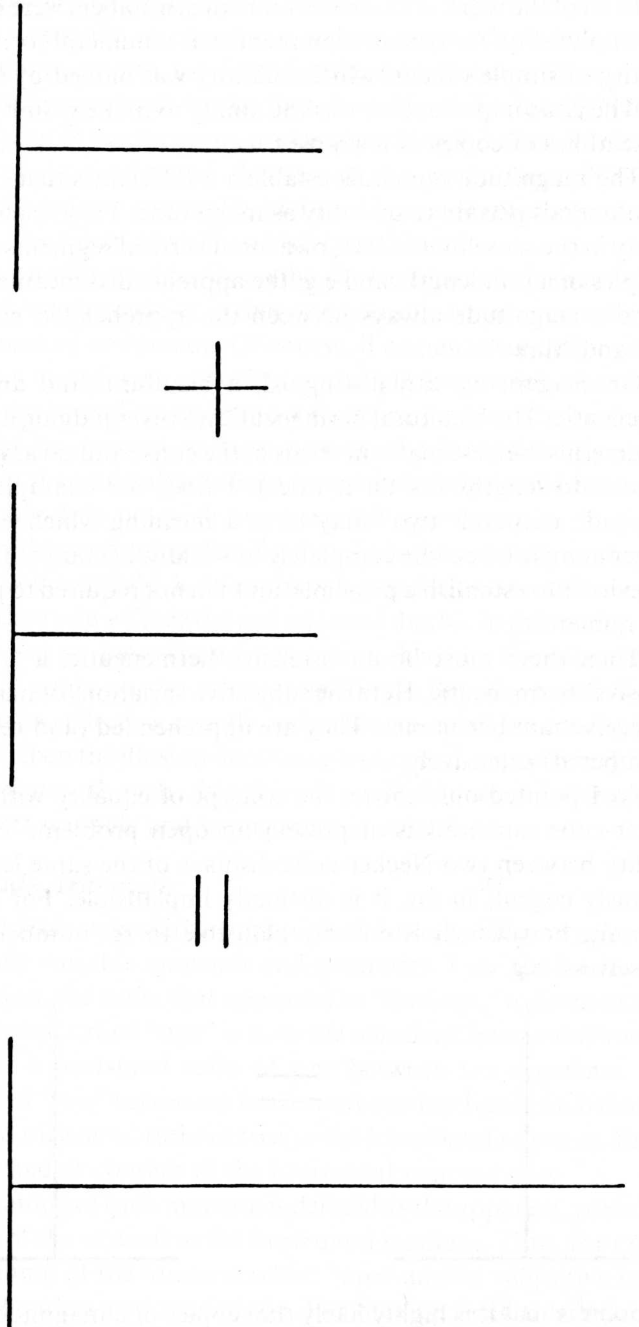
Once again we can distinguish a bicultural and an ostensive hermeneutic. The bicultural hermeneutic involves judging meanings of the numerals with estimates in terms of the conventional assignment of fractions to lengths (as on a ruler). I find, for example, that the magnitude numeral "two" may have a meaning which is almost 3. (Larger numerals become completely unwieldly, of course. The point of the device is to establish a principle, and I'm not required to provide for large numerals.)

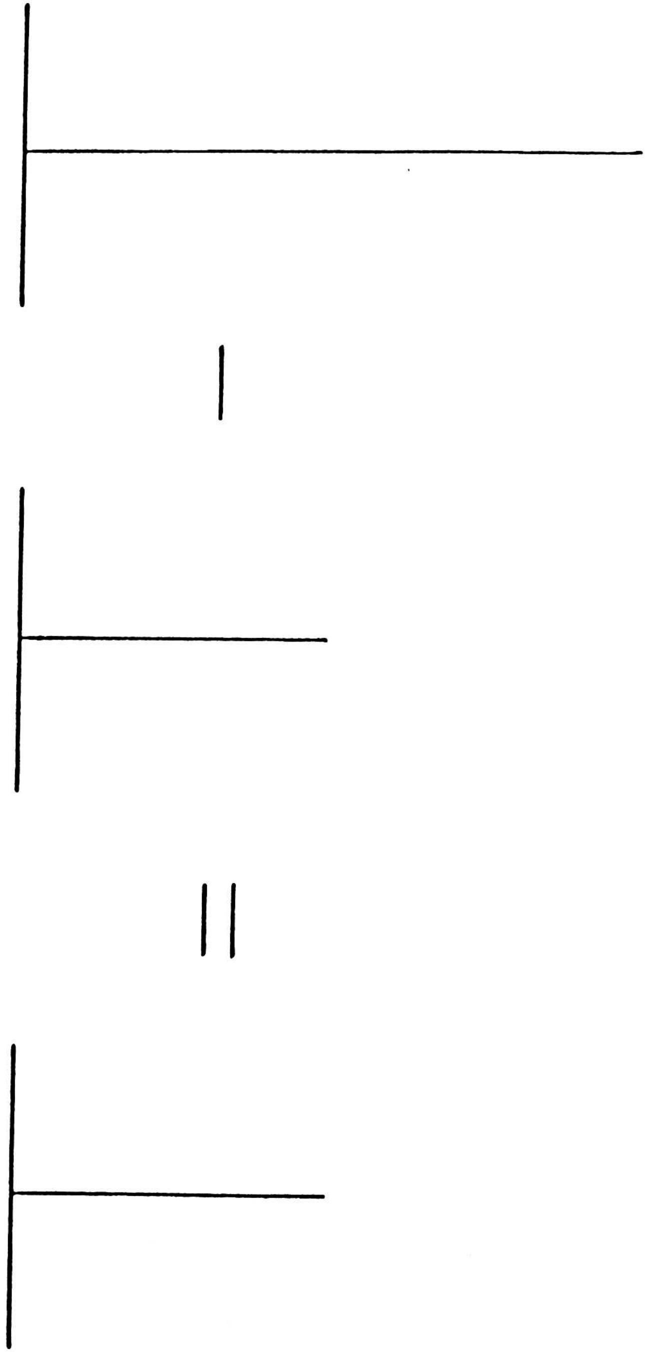
Then there must be an ostensive hermeneutic, a "magnitude-ostensive" hermeneutic. Here the subjective variations of magnitude do not receive number-names. They are apprehended (and retentionally remembered) ostensively.

As I pointed out, above, the concept of equality with regard to Necker-cube numerals is at present an open problem. To write an equality between two Necker-cube displays of the same length is not obviously cogent; in fat, it is distinctly implausible. For magnitude numerals, however, it is entirely plausible to set numbers equal to themselves—e.g.



The point is that it is highly likely that copies of a magnitude numeral will be apprehended or appraised correlatively. This was by no means guaranteed for copies of a Necker-cube numeral displayed in proximity.





Upon being convinced that these simplest of equations are meaningful, we may stipulate a simple addition, "one" plus "one" equals "two." (It was not possible to do anything this straightforward with Necker-cube numerals.) Continuing, we may write a subtraction with these numerals. There may now appear a complication in the rationale of combination of these quantities. The "two" in the subtraction may appear shorter than the "two" in the addition. A dependence of perceptions of these numbers on context may be involved.

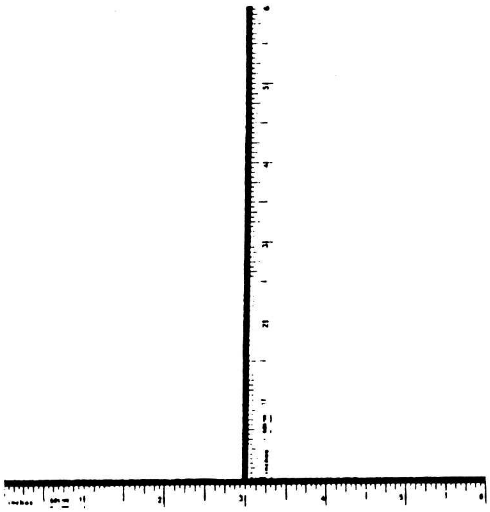
We find, further, that "readings" of these equations according to the bicutural hermeneutic yield propositions which are false when referred back to school-arithmetic—e.g. the addition might be read as

$$1^1/_5 + 1^1/_5 = 2^4/_5$$

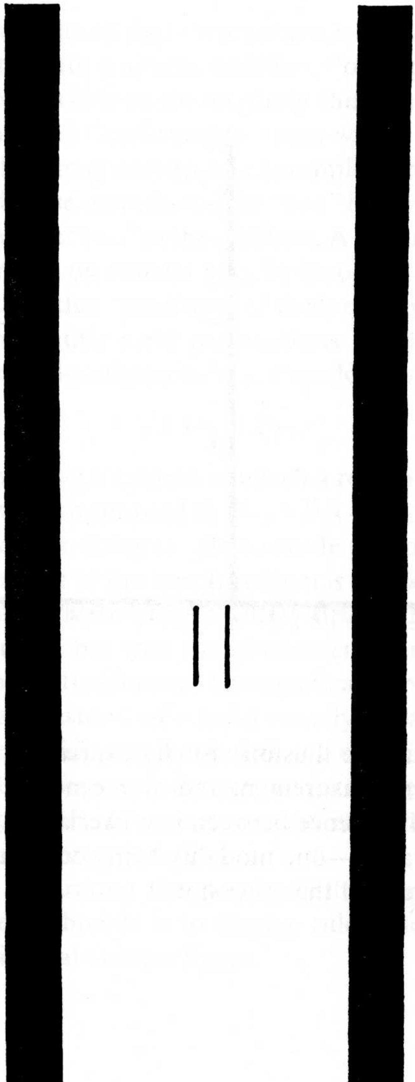
So the effect of inventing a context in which a relationship called "one plus one equals two" is appraised as $1^1/_5 + 1^1/_5 = 2^4/_5$ (where there is a palpable motivation for doing this) is to erode school-arithmetic.

Another approach to the same problem is to ask whether magnitude arithmetic authentically describes any palpable phenomenon. The answer is that it does, but that the phenomenon in question is the illusion, or rationale of the illusion. The significant phenomenon arises from having both a measured ratio and a visually-apparent ratio, which diverge. This is very different from claiming equations among non-integral magnitudes without any motivation for doing so. Indeed, given that the divergence is the phenomenon, the numerals are not really ostensive in a straightforward way.

One way of illustrating the power of the phenomenon which models magnitude arithmetic is to display ruler grids flush with the segments of a horizontal-vertical figure.



What we find is that the illusion visually captures the ruler grids: it withstands objective measurement and overcomes it. We have a non-trivial, systematic divergence between two overlapping modalities for appraising length-ratios—one modality being considered by this culture to be subjective, and the other not.



In "Derivation" I used multistable cube figures to give a simplified, discrete analogue of the potentially continuous "vocabulary" in "Illusions." I could try something similar for magnitude numerals. Take as the magnitude unit a black bar representing an objective unit of twenty 20ths, concatenated with a row of five Necker cubes. Each cube seen in the "up" orientation adds another 20th to the judged magnitude of the subjective unit, so that the unit's subjective magnitude can range to 1¼. When, however, we write the basic equality between units, it becomes clear that this device does not function as it is meant to. In particular, the claim of equality applied to the Necker-cube tails is not plausible, because it is not guaranteed that these tails will be apprehended or appraised correlatively. I have included this case as another illutration of the sort of inventiveness which this work requires; and also to illustrate how a device may be inadequate.

* * *

This completes the present stage of the work. Let me emphasize that this manual does little more than define certain devices developed in the summer of 1987. These devices can surely give rise to substantial lessons and substantial applications.

There is my pending project in *a priori* neurocybernetics. Given that mechanistic neurophysiology arrives at a mind-reading machine—called, in neurophysiological theory, an autocerebroscope—devise a text for the human subject such that reading it will place the machine in an impossible state (or short-circuit it). Such a problem is treated facetiously in Raymond Smullyan's *5000 B.C.;* and more seriously by Gordon G. Globus' "Mind, Structure, and Contradiction," in *Consciousness and the Brain,* ed. Gordon Globus *et al.* (New York, 1976), p. 283 in particular. But I imagine that my Necker-cube notations will be the key to the first profound, extra-cultural solution.

In any case, this essay is only the beginning of an enterprise which requires collateral studies and persistence far into the future to be fulfilled. (I may say that I first envisioned the possibility of the present results about twenty-five years ago.)

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