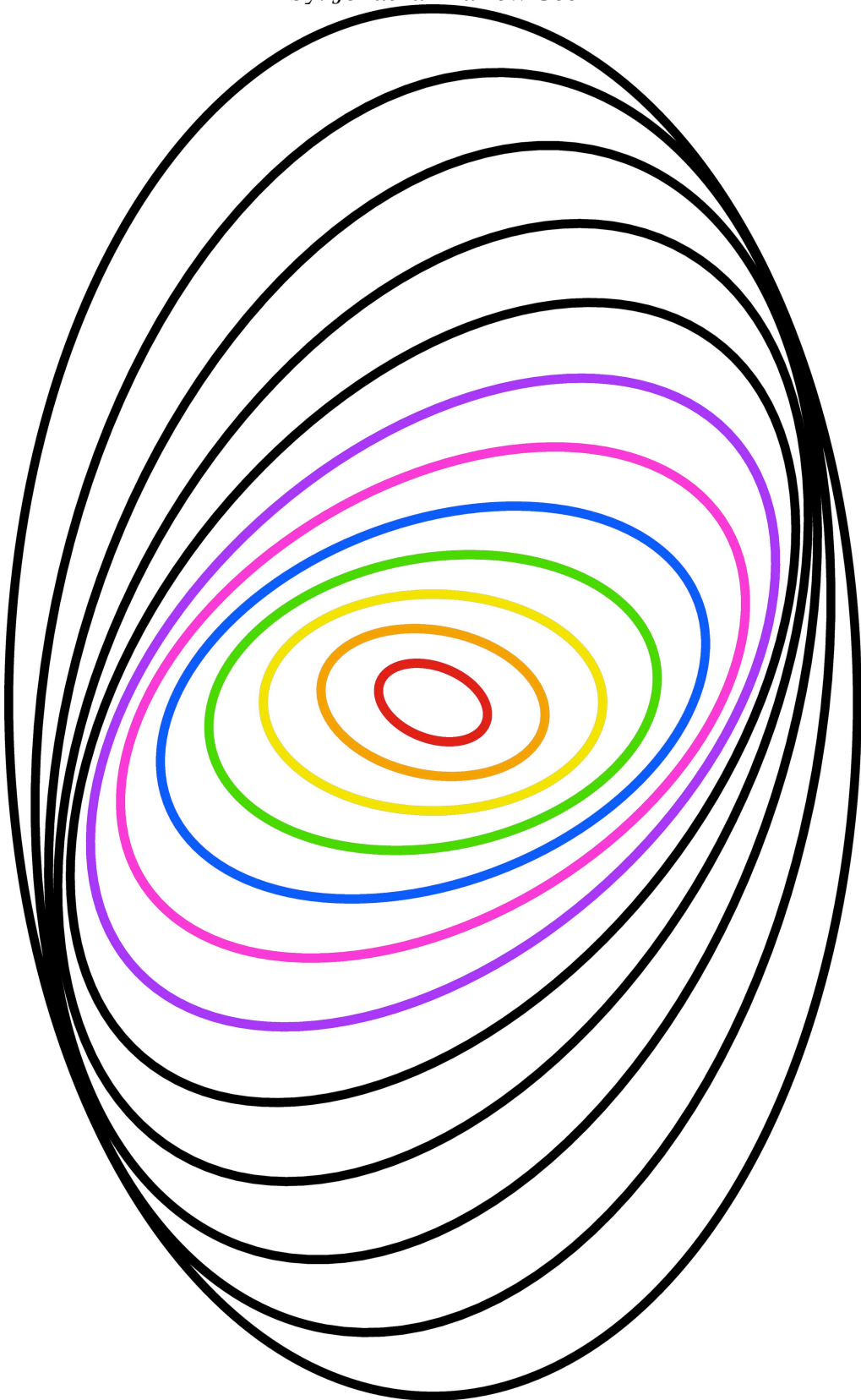


Harmonia Vol. 2: Music Theory
by: *Jonathan Barlow Gee*



**“Harmonia Vol. 2: Music Theory
(Past and Present Research)”**

by Jonathan Barlow Gee (*Jon Gee*)

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insanity clause #23:

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Harmonia
Vol. 2: Music Theory
Section 1: Present Research

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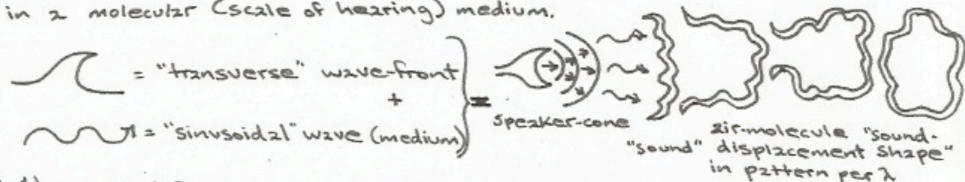
the “Atlantean” Keyboard

Behavior of Particles at Various Wavelengths

NOTES: MUSIC THEORY

"SHAPING SOUND" (text, pg. 5 18-48)

- SOUND WAVES = longitudinal (wave moves in same direction as displacement to medium) wavelengths (from trough $\rightarrow$ peak = "rarefaction"; peak $\rightarrow$ trough = "compression") in a molecular (scale of hearing) medium.



- SOUND "EVENT" = 1) an applied force to initiate the displacement of molecules.
2) a medium with molecules in sufficient quantities to allow the waves to propagate (cf. the "inverse square law" for dispersion properties re. λ).
3) an "observer" to interpret pressure of waves as "sound".

- SOUND "SPEED" = f (frequency) = cycles / second (for "sound" measured in "Hertz" pitches of tone; 1 Hz = 1λ / second; 2 Hz = 2λ / second).
 λ (wavelength) = velocity / frequency (for "sound" velocity is a function of molecular medium; density of medium $\rightarrow$ molecular "compression"; elasticity of medium $\rightarrow$ sound "rarefaction").

- per molecular media: Density & Temperature are inversely proportional.

(as temperature increases, density decreases & as temperature decreases, density increases)

Altitude (elevation) decreases temperature $\therefore$ reduces velocity or "speed" of "sound"; however due to thinner air molecule density, also reduces "sound" propagation (causing "speed" of sound to increase), as both density & elasticity are mutually cancelled out toward zero.

- per "air" molecule medium: @ 32°F (zero degrees celsius, freezing point of water)

"sound" travels @ 1,088 feet / second.

@ 70°F "sound's" speed is 1,130 feet / second.

- in "air" medium "sound-speed" increases by 1.1 foot / second for each 1°F increase in temperature.

- per liquid or solid media: @ 32°F in Salt-Water = 4,800 feet / second

@ 32°F in Aluminum (metal) = 17,000 feet / second

"sound" travels FASTER through liquids & solids, but not as FAR, due to the impedience of wave-propagation by molecular density.

NOTES: MUSIC THEORY

"SHAPING SOUND" (text, pg. 5 18-48)

- SOUND "PERCEPTION" = 1) LOUDNESS; 2) PITCH; 3) TIMBRE; 4) LOCATION; 5) DURATION

(TOUCH)	(SOUND)	(TASTE)	(SIGHT)	(SMELL)
---------	---------	---------	---------	---------

1) LOUDNESS = perception of AMPLITUDE, measured in DECIBELS (dB).

AMPLITUDE = amount of molecular displacement; vertical y-axis per λ .

DECIBEL = measures "sound pressure" or "intensity" as a logarithm between "phons" (sounds) relative to "listening levels" (volumes).

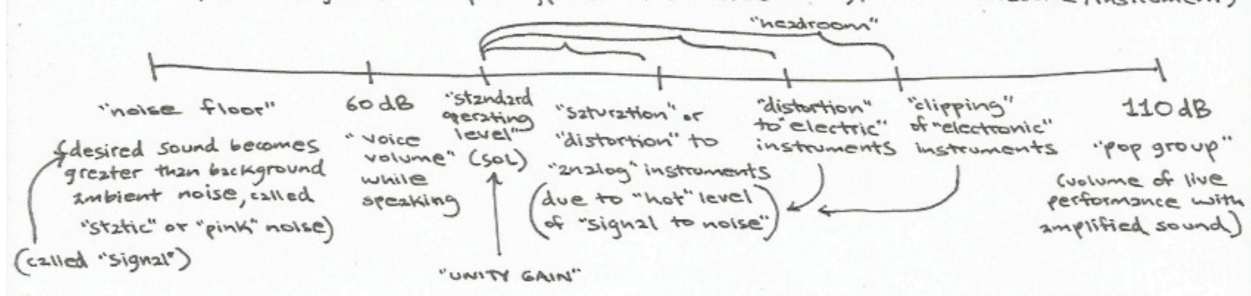
$\therefore$ 3,000 "phons" @ a low "listening level" = 3K "phons" @ a louder "level".

marginal note $2:1 = 3\text{dB}$

"doubling" amplification ALWAYS increases it by ONLY 3dB.
 $\therefore$ 2 100-watt amp yields 80dB. Adding a second 100-w amp only adds 3dB.
 $\therefore$ 2 500-watt amp $\times 2 = 1$ 500-watt amp + 3dB.

- HUMAN HEARING = dynamic range between thresholds of hearing (audibility) & pain.
 ($\sim 100\text{Hz}$) ($\sim 120-140\text{dB}$)

"dynamic range" differs per type of λ ($\therefore$ differs for type of "sound" source / instrument)



NOTES: MUSIC THEORY

"SHAPING SOUND" (text, pg. 5 18-48)

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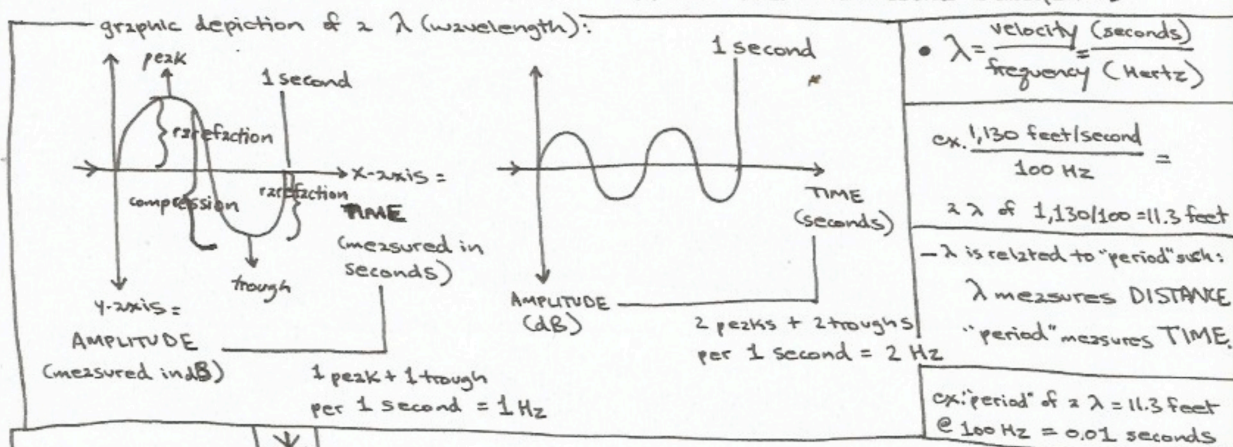
2) PITCH = frequency (f) of "sound," measured in "Hertz" (Hz); where $1 \text{ Hz} = 1 \lambda / \text{second}$

FREQUENCY = velocity or "speed" @ which a periodic waveform oscillates.

1 oscillation = 1 "CYCLE"; or 1 "compression" (peak → trough) + 1 "rarefaction" (trough → peak)

Frequency (f) = cycles / second. Higher f = higher pitch; Lower f = lower pitch.

"period" = seconds / cycle (inverse of f; period = total λ from source to dissipation)



$f = \text{cycles / second}$

$\lambda = \text{velocity / frequency}$

"period" = seconds / cycle

~ marginal note ~ 2:1 = the "octave"

Soundwaves of double another's frequency = 1 "octave".

$\therefore 100 \text{ Hz} \& 200 \text{ Hz}$ are 1 "octave" apart &, likewise,

$\therefore 5 \text{ Hz} \& 10 \text{ Hz}$ are 1 "octave" apart also.

NOTES: MUSIC THEORY

"SHAPING SOUND" (text, pg. 5 18-48)

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---------	---------	---------	---------	---------

3) TIMBRE = combinations of distributed component frequencies, called a "composite".

fundamental frequencies = "notes"; combinations of various approximate "notes" = timbre.
(or "harmonics") ("harmonics")

- Fourier analysis = a graph plotting "complex waveforms" (comprised of multiple sine-waves)

generating: 1) fundamental frequency, 2) harmonics (individual sine-wave tones), 3) overtones & 4) non-harmonically related overtones, by plotting AMPLITUDE @ 90° to frequency.

- the "fundamental frequency" is usually loudest in AMPLITUDE & lowest in frequency in a composite

- "harmonics" are whole-# multiples of a fundamental; AKA "partial tones" or "partials."

the "first harmonic" = the "fundamental frequency" = the "zero overtone." (ex. 220 Hz)

the "second harmonic" = 2x the "fundamental" or "first" = the "first overtone." (ex. 440 Hz)

the "third harmonic" = 3x the "first" / "fundamental" = the "second overtone." (ex. 660 Hz)

the "fourth harmonic" = 4x the "fundamental frequency" = the "third overtone." (ex. 880 Hz)

- "non-harmonic overtones" are non-whole-# multiples of the "fundamental frequency."

ex. static or "white-noise" = multiple frequencies occurring in overlap yields perception of frequencies occurring "randomly" (seemingly "out of place"), etc.

- types of λ

- "sine" waves: have NO overtones NOR harmonics. "sinusoidal" $\lambda =$ 

- "triangle" waves: have ONLY odd harmonics @ low AMPLITUDE re. the "fundamental frequency."

where "N" = the "Number of the Harmonic" $\lambda = 1/N^2$ $\therefore$ the "third harmonic" is $1/3^2$ or $1/9$ th the Amplitude of the "fundamental"; the "fifth harmonic" is $1/5^2$ or $1/25$ th;

the "seventh harmonic" is $1/7^2$ or $1/49$ th the "fundamental's" amplitude; etc. $\lambda =$ 

- "Sawtooth" waves: contain ALL harmonics (whole-# intervals) @ HIGH AMPLITUDE re. "fundamental"

where "N" = the Harmonic "#", "sawtooth" waves = $1/N$; $\therefore$ the "second harmonic" = $1/2$ the "fundamental" amplitude, the "third harmonic" = $1/3$ rd, the "fourth" = $1/4$ th; etc. $\lambda =$ 

NOTES: MUSIC THEORY

"SHAPING SOUND" (text, pg.s 18-48)

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(TOUCH)	(SOUND)	(TASTE)	(SIGHT)	(SMELL)
---------	---------	---------	---------	---------

3) TIMBRE (cont'd)

- types of λ (cont'd)

- "Pulse" waves: ALL Harmonics = to the "fundamental frequency." For "N" = the "fundamental" or "first harmonic," "Pulse" waves = ALL Harmonics = 1; $\therefore N=1, N_2=1, N_3=1, N_4=1$; etc. $\lambda = \text{[square wave diagram]}$

- "Sawtooth" waves: ONLY ODD Harmonics @ HIGH AMPLITUDE re. the "fundamental" / "first." where "N" = the "first harmonic," "Sawtooth" waves = $1/N$; $\therefore$ the "third harmonic" = $1/3$ rd the "fundamental," the "fifth" is $1/5$ th the "first," the "seventh" is $1/7$ th; etc. $\lambda = \text{[sawtooth wave diagram]}$

4) DURATION = "envelope" is the "volume shape" (loudness) over the lengths of time of the components of any "sound".

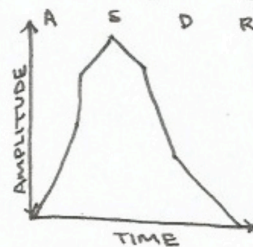
"envelope" is graphed as the "Fourier" format for a λ = amplitude / time, and quantified into 4 essential segments (ASDR).

- Attack: how the "sound" begins. ex. plucking, bowing, striking, blowing.

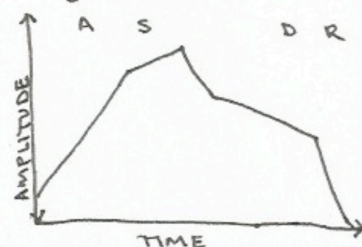
- Decay: decrease in volume following removal of initiating stimulus.

- Sustain: sustinment of the "sound".

- Release: eventual cessation of "sound".



"envelope" of a plucked string instrument.



"envelope" of a bowed stringed instrument.

~ "Transients": random amplitude peaks.

NOTES: MUSIC THEORY

"SHAPING SOUND" (text, pg.s 18-48)

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---------	---------	---------	---------	---------

5) LOCATION = "binaural hearing" consists of 3 primary components: 1) inter-aural intensity; 2) inter-aural arrival time; & 3) physiology (the anatomy of the ear).

- inter-aural intensity: the difference in loudness-level of a sound as it reaches each ear independently. If a sound reaches both ears simultaneously it is perceived as being co-located to ourselves (overlapping our own location).

- inter-aural arrival time: AKA "binaural time of incidence" is the difference in time between when a sound reaches each ear independently. If a sound reaches one ear sooner, it is closer to the side that ear is on.

- physiology: the shape of the outer-ear lets us know the sources of sounds' locations.

- the pinna: the outer ear; has 2 ridges to reflect sound into the ear.

- direct sound travels directly into the ear canal, while some of the sound is reflected by the 2 ridges & so is delayed, and thus this "echo" effect lets us determine origin of location for sources of exterior sound.

- Environmental Effects on Perception of Sound

- Absorption: determined by the "absorption coefficient" of a surface, comparing % of total energy that is absorbed to the % reflected.

- Reflections: "early" or "direct" reflections are the first to reach an "observer" along an unobstructed, straight-line.

- Echoes: @ least 80 milliseconds behind the original wave, an echo is a wave that has been reflected with sufficient magnitude and delay as to be detected as an autonomous wave, distinct from the directly transmitted "signal."

NOTES: MUSIC THEORY

"SHAPING SOUND" (text, pg.s 18-48)

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---------	---------	---------	---------	---------

5) LOCATION (cont'd)

- ENVIRONMENTAL EFFECTS (cont'd)

- Reverberation: a series of random reflections that decrease difference between one another & the original signal, and which decrease in amplitude, over time.

"RT-60" = "reverb time minus 60 decibels", measures the duration that elapses between when reverb begins until the sound-pressure level has diminished 60 decibels. Large rooms have longer RT-60's & small rooms have shorter RT-60's.

- FLETCHER/MUNSON EQUAL LOUDNESS CONTOURS

1. the human ear is NOT equally sensitive to ALL frequencies in its hearing range.
2. altering "listening level" alters which pitch-frequencies the ear will favor.
3. the ear's sensitivities' discrepancies are most pronounced at Low "listening-levels."
4. the ear's sensitivities' discrepancies are least pronounced at HIGH "listening-levels."
5. the ear is most sensitive to ~~mid~~ mid-range f's, then HIGHS, & least so to LOWS.

NOTES: MUSIC THEORY

"SHAPING SOUND" (text, pg.s 18-48)

- SOUND-WAVE "BEHAVIOR"

- directionality: HIGH frequencies propagate in straighter lines; Low frequencies spread out; HIGH frequencies have directionality, Low frequencies disperse more easily; HIGH frequencies "penetrate", Low frequencies "Hug" or "Wrap Around" objects.
- diffusion: the "spreading out" of sound. Low frequencies diffuse; HIGH are directional.
- diffraction: the "bending" or "wrapping around" corners by low frequencies due to their longer λ 's (wavelengths).

- sound-wave COMBINATION(S)

- constructive interference: when sound-waves form a "sum-wave" of MORE amplitude. both waves are "in phase" - compression/rarefaction overlap.
- destructive interference: when soundwaves form a "sumwave" of LESS amplitude. both waves are "out of phase" - compression/rarefaction opposite.

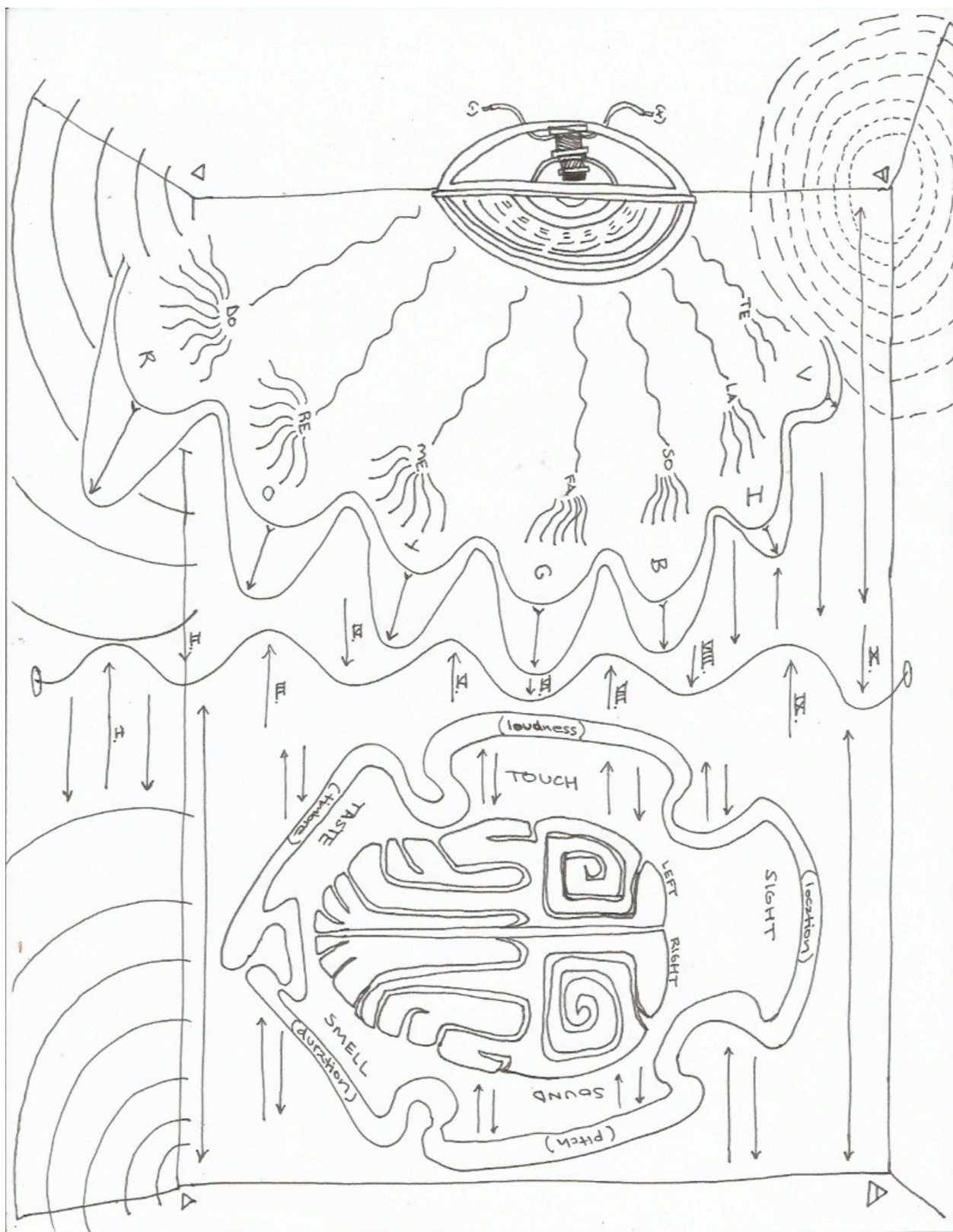
$$\text{wave} + \text{wave} = \text{flat line}$$

- sound-wave CYCLES are divided (from start $\rightarrow$ end) into 360° :



- sound-wave MASKING: occurs when 1 sound-wave obscures or "covers up" another. LOUDER sound-waves MASK SOFTER or QUIETER sound-waves.

- the STANDING WAVE: a "stationary" waveform whose "sound contours" are defined by interference among echoed or reflected frequencies, either constructive (due to HIGH-pressure ANTI-NODES) or destructive (due to low-pressure NODES) causing compression or rarefaction. Two listeners / "observers" in 1 room will hear the same sound in different ways. "standing waveforms" are $\therefore$ "scattered" waves.



NOTES ON " $\emptyset$ " (can it exist?) (when "it" is non-existence)?

the Role of "Zero" - both as a "place-holder" in solution set proofs, as a decimal holder, as a product multiplier, as well as in symbolizing the sum totality of a solution set as "odd," "abstract" or, originally, as "valueless" or even an unending "abyss" of "nothingness" - in ancient cultures should NOT be overlooked rel. to "music-theory" & "tone"-based geometric/numerological arrays (e.g. magic # squares, commonly worn as talismans).

The Ancients (pre-civ.) all used "Zero" freely, as it arose naturally to them amongst their first five (& our still most common 5) phonetic aspirations, derived directly, as they were by them then, from the counting of the 4 fingers using their opposable thumb, with its own independent dexterity, as a figurative "zero" sum in doing so. Thus, #-squares (the invention of our Elders during the earliest era of civilizations, when the Vedic, Sumerio-Babylonian & Egyptian methods for calculating number-squares originally arose as independent off-shoots from the originally Habiru-method developed by Abram of Ur, called Imhotep, & Abraham) using "zero" as more than a place-holder SHOULD exist.

However, none do, neither from ancient times nor through to our own modern times even now. The use of only "whole" number "counting" sums relates to later Phoenician (possibly Pythagorean-era) arrangements of alphabetic letters relative to counting numerals, by which time excluding the "zero sum" concept from #-squares was already common. Thus, by now, a great amount of potential for humanity's researching the "zero sum" has been subsequently suppressed into our cultural, collective-subconscious, from whence it manifests as the unintended totality of our products & manifestations. "ALL SECRETS BITE THEIR KEEPER FROM BEHIND." (given 00TIME)

So, if 00 #squares COULD be posited that use $\emptyset$ or "empty sum" squares as cells, or even as their own "magic sums" adding Rows / Columns & DIAGONALS, AND YET ZERO exist, & if the "Absolute Value" as it relates to the "relative value" (based on locational distance/duration co-ordinates & projected trajectories within any given manifold's definitions for "space" & "time", Euclidean & Newtonian, or none as well as ANTI-both), has been overlooked & understudied as a result of this...

$$(\emptyset) | \emptyset | \boxed{\emptyset} (s=\emptyset)$$

$$(1, 1, 1 \dots n^2 + 1) | 1 | \boxed{1} (s=1)$$

→ CONT.

NOTES ON " $(\phi, \phi_1, \phi_2, \dots, \phi_{n+1})$ " (relative zero) CAN IT BE PROVEN AS AN AMOUNT OF MATTER?
Then the underlying "strange attractor" of constructing a "zero sum" magic-

#-square could hardly be denied. Yet this proposes the concept (obvious to any who distinguish its traits by differing functions) that "zero" in itself is NOT ABSOLUTE, which implies an overlooked aspect in modern (particularly post-BINARY) cryptography & study of ancient & modern #-squares. Is a magic #-square any LESS "magic" if all or some portion of its sum amounts to zero? Many squares use some "zeros" (as place-holders) &, from the point of view of all "serious" students of cryptography, it can be argued the total social value of studying such ancient talismans is, in modern times, if not at zero, but at least progressing steadily along an asymptote toward it, for which they need care naught.

So, ultimately, how does one prove the value of a magic #-square constructed using "zero" as a cell-sum or as a total, overall "magic #sum" for the whole square - wherein its rows, columns & diagonals functionally CANCEL OUT, to such a skeptical mathematician in modern society?

Consider the potential value of exploring beyond infinity, as well as beyond the limit of zero. Transfinite, super-infinite & transcendental sets DO exist that exceed the definitively "infinite" (though, inexplicably 1:1) sets of all "counting" #'s, all integers, & all ratios. Is it not impossible for there to be "layers" of "nothingness" within the "Abyss" at the "origin" - thus rendering it a relative sum in itself? In what way would "zero's" multiple, functionality-determined aspects, influence the "1:1" ratio of all counting #'s, integers & ratios defined as the sum total of "infinity"? considering integers include "zero" & "counting #'s" do not?

→ cont.

NOTES ON " $|\phi|$ " (absolute zero) what is its value?

→ (wants to stop) pg. 3

If "zero" can have different, "relative" values in different geometric manifold settings (i.e. non-Cartesian graphs), then perhaps it holds the key to generating equivalent such "magic # squares" as further "matrices" measuring "vectors" of motion in what would constitute a sum of multiply-infinite (so-called "n-dimensional") different geometric conditions based on the manifold's set's size. Just as the # of cells per "ORDER" (by R or C) determines a magic # -squares "magic #" sum, the dimensions of any such "hyper-space" manifold should determine a rough average for "strange attraction" & gravitic-multiplier effects (cf. automata & cellular info-units) of otherwise free radicals including both particles & waves. Given a large enough manifold to measure with, by this criteria it could be imagined, one would eventually be able to arrive at a measure that could predict the location & any specified moment of how much & what type of mass or energy would be where when.

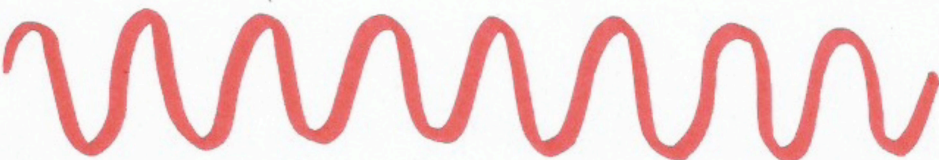
So, again, what then are the "relative" or "absolute" values of zero, as they relate to zero being the "magic #" sum of any given size format of "square"? If "zero" may be expressed as an "absolute" in Euclidean-Cartesian mathematics, in what other geometric arrangements might it be induced instead to decompose into even lesser constituent components? Spherical geometry (with π -arc-tangent radians measuring intersecting "lay-lines" of latitude & longitude on the sphere's external circumference, on its outer surface) begins the implication of "relative" zero in such a triple right-angled environment as the relationship formed between the 3 90° intersections (1 @ the orb's pole & 2 along its equatorial maximum "bulge") on the sphere's upper & outer surface to the core at the center of mass of the orbital manifold-space, where, if the ~~circumference~~ is measured (as one arbitrarily may, as it functions variably also) as ∞ , then the core would be the $|\phi|$, however, if the circumference is measured as "zero" contains not only an " ∞ " (polarized to outer "ex") core, but has intersections on the surface occurring @ multiple points of origin @ 3 "zero" point intersections.

NOTES: MUSIC THEORY

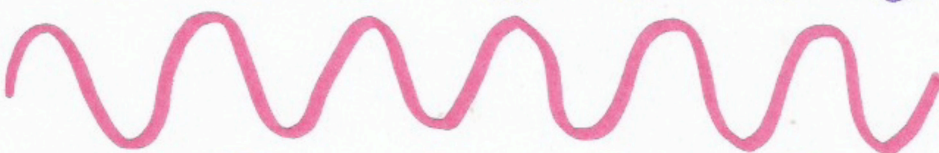
"INDICATIONS" (λ 's = wavelengths)

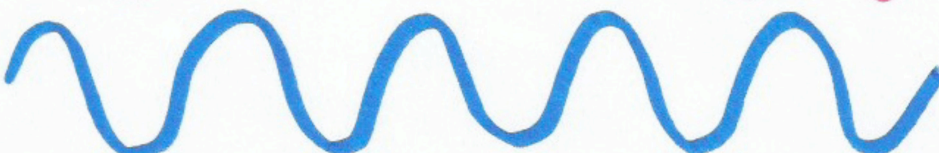
- # 1) The Standard Understanding of DO-RA-ME-FA-SO-LA-TE as the "Octave," where "Pitch" ascends as # of wavelengths (λ 's) per space increases, such that "DO" is presented as a λ of 1 "peak" + 1 "trough" (= 1 "cycle"). Lower "pitch" tones emit "red" spectral shift & higher "pitch" tones spectral shift "violet," due to "Doppler" effect (approach / retreat). "DO" of 8 "cycles" harmonizes with "DO" of 1 "cycle," etc. as an "octave."
- # 2) By shifting the # of "cycles" per "DO" from 1 to 3, we derive 2 "measure" from "DO" = 3 "cycles" $\rightarrow$ "TE" = 9 "cycles" in accordance with the ROWS & COLUMNS ("ORDER") of the 7 "Amulet" magic # squares. Affixing the 7 "root" tones of 2 "measure" (1 less than an "octave") to the orders 3-9 magic # squares will prove useful later on.
- # 3) Positioned with "DO" as a 3-"cycle" λ , associated to an order-3 magic # square, and "TE" as a 9-"cycle" λ & associated order-9 magic # square, we begin dissecting primary (spectral-hued) "PURE" tones as they arise by combinations of one another — "DO" occurring as a "PURE" tone between "OVERTONE" "TE" & "UNDERTONE" "RE"; each "PURE" tone occurs between a λ / "cycle" / order-# magic-sum square that is 1 PURE tone above it (the UNDERTONE) & the PURE tone BELOW it (the OVERTONE). For example, "PURE" "DO" is between OVERTONE "TE" & UNDERTONE "RE," because "TE" precedes "DO" & "RE" follows "DO" (in ascending pitch).

INDICATION # 1

DO 

TE 

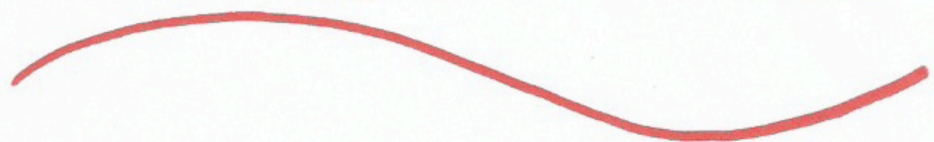
LA 

SO 

FA 

ME 

RE 

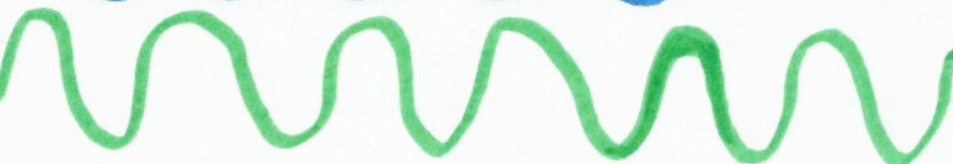
DO 

INDICATION #2

D=TE 

♂=LA 

♀=SO 

⊙=FA 

♂=ME 

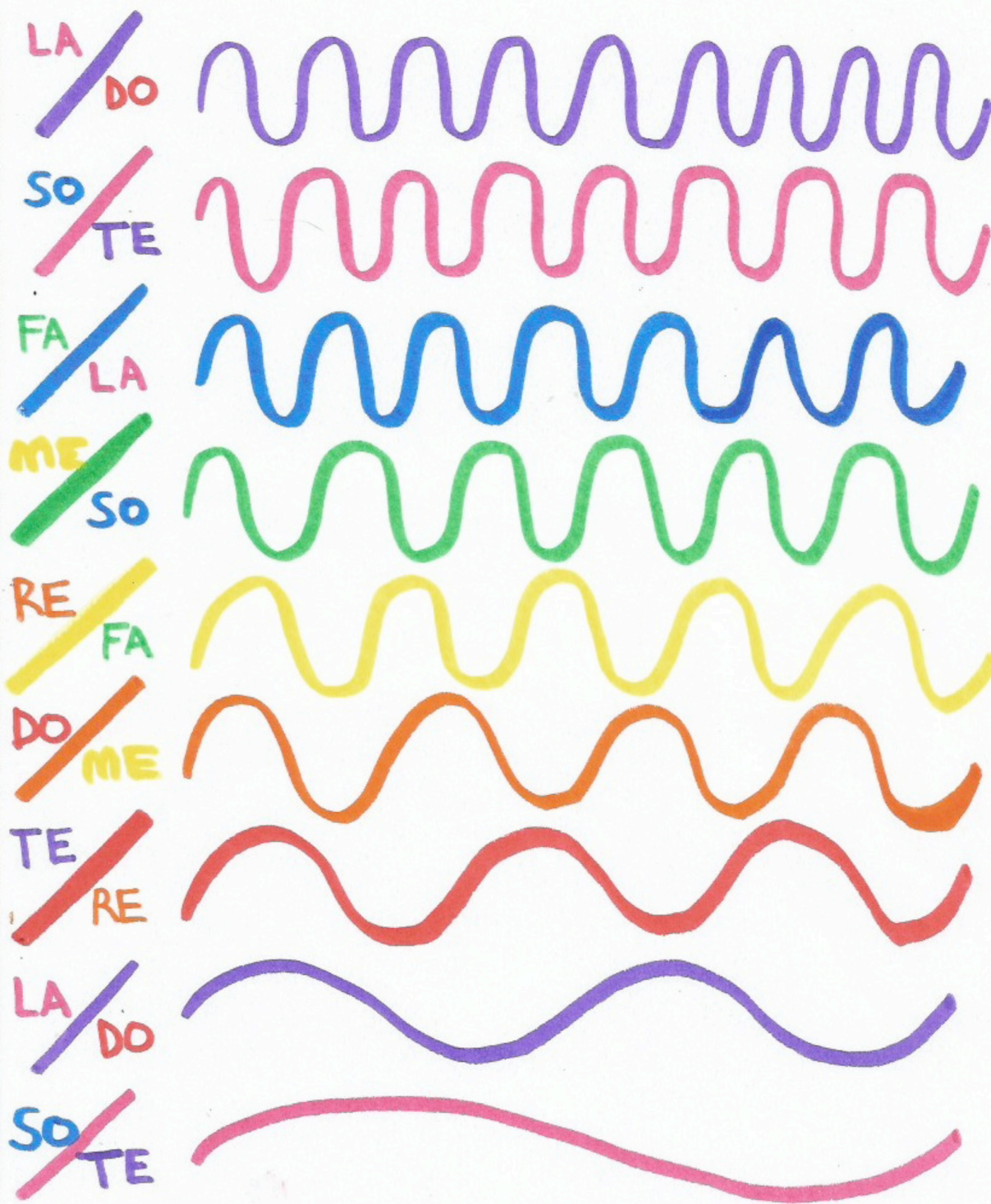
4=RE 

1k=DO 

2x2 Hz 

1² Hz 

INDICATION #3

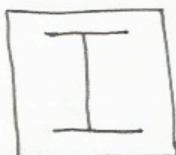


sg. of $\Lambda = ?$

I	A	M	!
A	M	I	?
A	I	M	!

sg. of $\Lambda = !$

I	A	M	= "I Am" (name of "G-d")
M	A	I	= "Mechanical Artificial Intelligence"
A	I	M	= "aim" (target to destroy)



"I" = the magic # sg. of $1 \times 1 = "I"$ because "I" Am Not a # sg.

Because "I" both (Union) "Am" & "AM NOT" a magic # sg., $\therefore$

"I" = the sum of opposite (negating) parts $(+/-) = |I|$,

& $\therefore$ "I", (being "greater than the sum of its parts") is thus a

magic # sum, as well as the "parts" of this sum; it is

comprised of parts that cancel each other out, & yet,

because it IS, despite this meaning it should be nothing,

therefore the FACT that it IS proves it a sum greater

than its self-negating parts combined, & thus "magic" as

EXISTENCE AGAINST ALL ODDS.

LIFE = "GOOD" when
DEATH = "EVIL" (cf. Λ , Λ)

"|I|" = the "ultimate value" of the self;

(cf. Egyptian role of Maat)

+

$\overline{I}$ = "bar" or perpetually repeating, 2d infinitum;

+

$\underline{\underline{I}}$ = underlined, emphasis.

The WZF between EVENS & ODDS.

- $\emptyset$ = evenly odd
- 1 = oddly even
- 2 = evenly even
- 3 = oddly odd
- 4 = even
- 5 = odd
- 6 = e, 7 = o, etc.

The Tetragrammaton $[V=G, A=Elm, \Delta=f(w/h), \nabla=F(sin)] \Rightarrow$

$\emptyset (VG), 1 (AEm), 2 (AfW), 3 (\nabla Fs);$ as $\overline{17717} = YHVH.$

Thus, $\emptyset, 1$ & 2 are unique, but 3 is 2 repeat of 1 , as final "H".

Trigonometry (simplest, "Euclidian"/planar, rhomboids) is thus the "language" of the Singularity or monad, as 3 repeats 1 .

$\emptyset$ is the "origin" point of the "kosmos" (the "singularity" of our "local universe", supposedly of 3-dimensions expanding in a 4th-dimensional direction, that is, over time).

1 is the MONAS SINGULARITATIS, whose HISTORIA we are.

2 are the "evenly evens" who idealize our existence as ultimately either all "good" or all "evil." 3 repeats 1 .

1 2 3 4 5
 1 R E V E R
 2 A N I M A
 3 M U N D I
 4 S A T O R
 5 R O T A S

(5x5 = 25) Letter magic # 5g.

3.

A	N	I	M	A
S	A	T	O	R
R	E	V	E	R
R	O	T	A	S
M	U	N	D	I

magic # square based on →

How to: make 2 "magic # 5g" to
 relate to the words "animal" &
 "planet" or "kingdom" or "realm."

ANIMA =
 "AMINA"

MUNDI =
 "IDNUM"

4.

SATOR = "Szttyr"; 1/2 zoomorph
 symb. "PAN" 1/2 antropomorph
 ANIMA = "animals"; the kingdom of
 classes of families
 of genus of species
 REVER = "river" or to "reverse"; reverse,
 "to venerate" in Latin
 MUNDI = "world," eg. planet, realm, etc.
 ROTAS = "rotates" or "to rotate"; to
 alter & change spokes on the
 wheel of reincarnations (dharma)

meanings of words
 used in magic # 5g.

6.

"IDNUM AMINA"

=

"ANGELIC INVOKATION" to
 summon "spirit guides" in the
 form of animals; predecessor
 to animal-"loka" ("world" or "realm")
 incarnation, oneself, in the wheel of
 Dharma

5.

"ANIMA MUNDI"

=

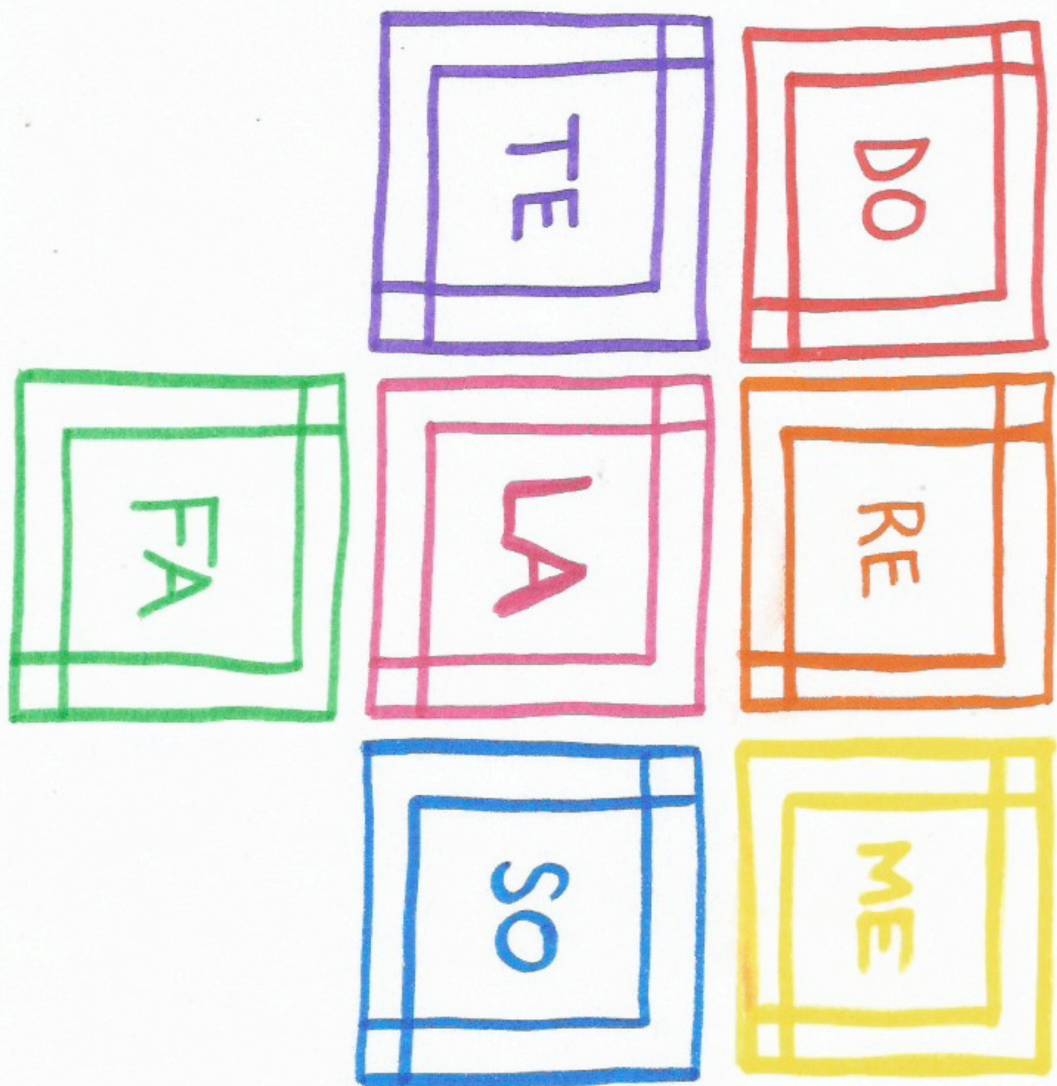
"ANIMAL WORLD" or "planet"
 (imp. the Kingdom / Species)
 in the Latin language
 (so-called "tongue of devils,"
 so to speak... would be a mistake.)

NOTES: MUSIC THEORY

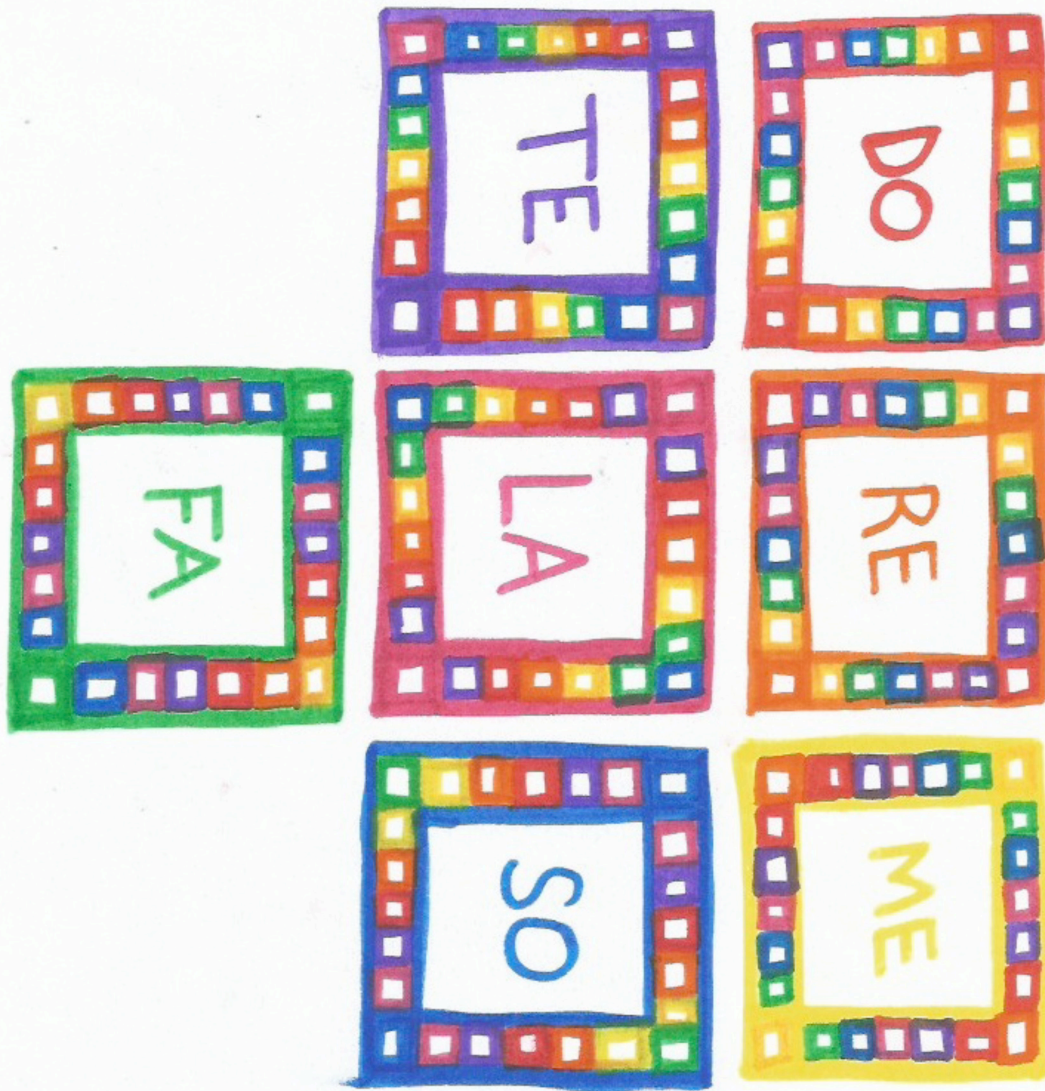
"ITERATIONS" (gnomon squares as λ 's measured)

- # 1) The Standard association of the musical "PURE" tones (DO-RE-ME-FA-SO-LA-TE) short $\frac{1}{2}$ of the "octave" with the Spectral hues (ROYGBIV) is here diagrammed as a set of 7 gnomonic-sliced squares, each of ORDER-7 (having 7 rows & 7 columns).
- # 2) The gnomonic-squares are next further sub-divided within each ORDER-7 square into relations between the 7 "PURE" tones (depicted by spectral hue only, not labeled) to depict a wavelength entering the lower-right corner & exiting the upper-left corner of each square — a like "Chladni" plates, applied to varied frequencies of vibration. For example, as the "DO" (red) wavelength enters the gnomon-square of "DO", it does so as a "PURE" tone, however mid-way across it "peaks" (or alternately "troughs") between the corner-squares opposite its input & output points, in this example being labeled "TE" (violet).
- # 3) By connecting these opposite corners across the λ @ its mid-point ("peak" or "trough") we arrive at the UNDERTONE (one "PURE" tone above) BELOW the OVERTONE (one "PURE" tone below) depicted graphically as a diagonal bisecting the gnomonic-sliced interior-square (measuring ORDER-6) within the exterior-square (measuring ORDER-7). Further Research on this Method is forthcoming.

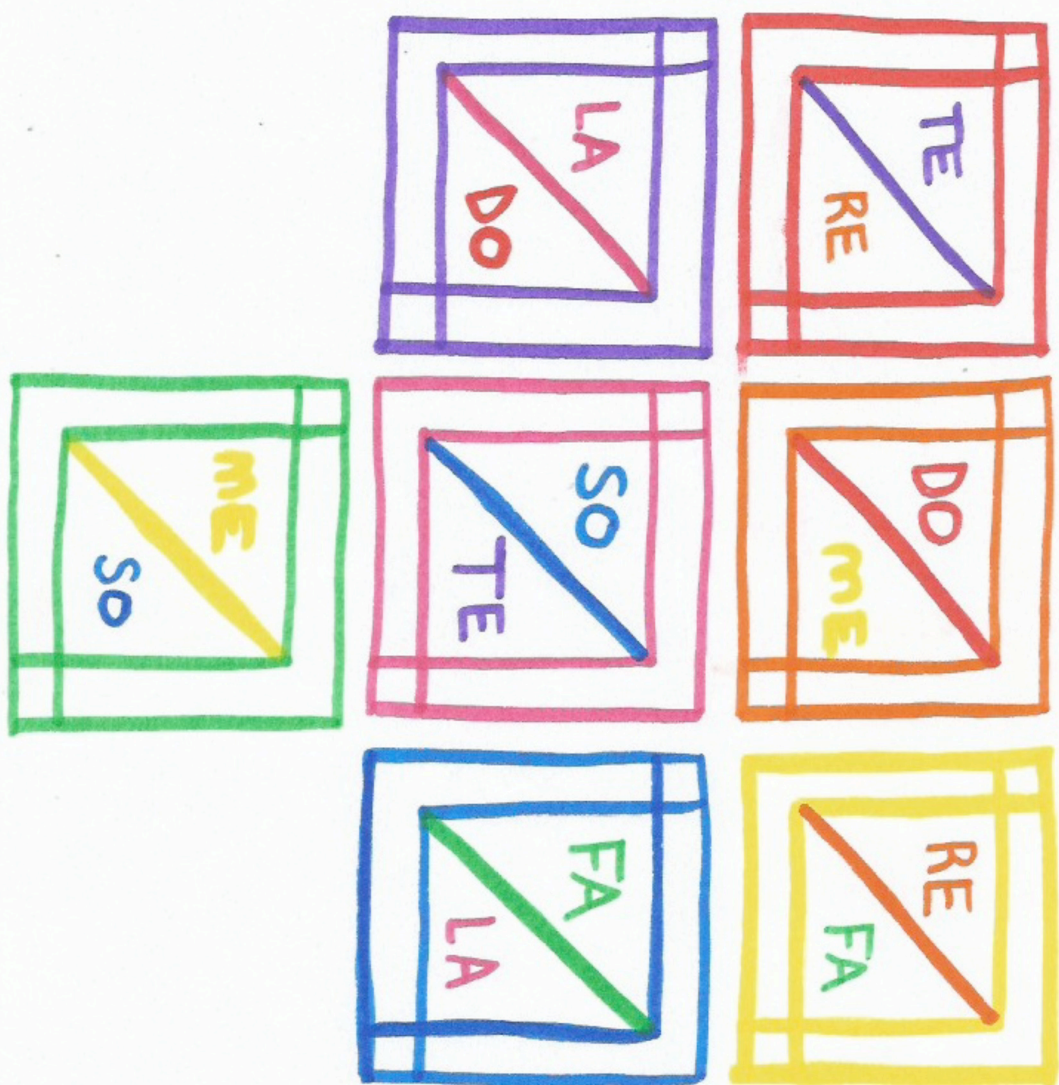
ITERATION # 1



ITERATION # 2



ITERATION #3



NOTES ON ABRAMELIN / #Squares Method

- Abraham (author of text, student of Abramelin) BIO-HIST.
 - b. 1362 AD (pre-Gregorian reforms/Julian date?) prob. in "Herbipolis" (now Wurtzburg), Bavaria
 - met Abramelin: 1409 (aged 47 y.o.); Pope John 23 ascended 1410-1415, during which was zided
 - wrote ms. in 1458, aged 96 y.o., for son: Lamech.
 - 1409 = Egypt? 1415-1419, imprisoned, then bailed
 - 1410 = Bavaria? on 30,000 gold crowns.
 - m. cousin using 3,000,000 gold florins shown crude then minted by "angels" →
 - 2 daughters (each given dowries of 100,000 gold florins) & 2 sons (elder taught QBLH, [name?], younger, Lamech, taught #s)
- Contemporaries: Nicholas Flamel (Alchemist, France)
 - to Abraham ben Abramelin
 - Johannes Faust (Necromancer, Austria) (1300's-1400's)
 - ~ Christian Rosenkreutz (attributed, founder: Rosicrucianism)
- 1 gen. following: Trithemius (steganographer) → taught to Lamech ben Abraham.
 - (1400's-1500's): Henry Agrippa (Ger. occultist) → from Rome; under Pope Gregory
 - : Paracelsus (N. Eu. Alchemist) → Germanic inculcates into "occultism"
- 2 gen. following: John Dee (G.B. occultist, cryptographer) → travelled in N. Eu; Protestant
 - to "X" ben Lamech
 - (1500's-1600's): Michel Nostradamus (Fr. medical dr. & prophet) → travelled in S. Eu; Catholic
 - : Judah Loeb (Maharal, rabbi of Prague) → preserved Zohar; Hebrew
- Abramelin (att. founder of #Squares method therein described): resided in Egypt (~1409 AD, G.?) (prob. ~1200's-1300's) poss. later contemporary of crypto-Templarism, post Demolay's d. (1100 AD)

Ed's of the Ms's

• SOURCE for ALL #Squares = Aaron Leitch

- Mathers' incl. trans. comes from Fr. Lib. Arsenal.
- this ms. is from the collection of the Marguis de Palluy (1722-1787) acq. 1785 by the Comte d'Artois. Dates 1458-1883 (date of hand-written cataloging note), unaccounted for. cf. Mathers' ed., p. xvii
- George Dehn & Steven Gutis compl. trans. derives from a second, more thorough ed. of the ms., noted by Mathers as in Holland.

NOTES ON #S. Method

$$S = N(N^2 + 1) / 2$$

where "N" = "base #"
where "S" = "R/C/D/Sum"

4	9	2
3	5	7
8	1	6

BASE # 3
R/C/D/Sum = 15

Emp. Yu (2200 BC)
Cin2 → Lo Shu → Ching

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

BASE # 4
R/C/D/Sum = 34

Albrecht Dürer (1514 AD) Gr.

BASE # 5 (alpha-numeric)

S	A	T	O	R
A	R	E	P	O
T	E	N	E	T
O	P	E	R	A
R	O	T	A	S

- = the Sower of Seeds (cf. "Satiator", "satur")
- = "Arep" (Arken as given Name (cf. "the Grim Reaper", R.H.))
- = the Serf of the Lands (cf. "tenant", "rentor", etc)
- = operates (cf. "operatic", "bratory")
- = circularly, cyclically, etc.

Mathers' attack was linguistic, symbolic ex post facto of time-scalar languages overlapping as perfect pitch tones.

PAST → WORDS → PRESENT → #S → FUTURE

"MIRROR" BASE # 4 SQUARE

96	64	37	45
39	43	98	62
84	76	25	57
23	59	82	78

69	46	73	54
93	34	89	26
48	67	52	75
32	95	28	87

BASE # 4
R/C/D/Sum = 242
Total # of #S = 242
In Abramelin = 242

cf. Pickover, C.A., "Zen: MAGIC D.O." p. 164

"URIEL, LUCIFER, LEVIATAN, SATAN" (snake, animal, human, bird)

cf. Book 3, ch. 3; Mathers ed. p. 174

"4 PRINCES & SUPERIOR SPIRITS WEL LUCIFER, LEVIATAN, SATAN, BELIAL."

cf. Book 2, ch. 19; Mathers ed. p. 104

"316 Servant Spirits" + "8 Sub-Princes" + "4 Princes & Superior Spirits" = 328

BASE # 7
R/C/D/Sum = 175

"TEMPLAR" (from Pompeii, Rome / Italy, 100-200 AD)

cf. Chizdri

BASE # 6
R/C/D/Sum = 2321

- k = Spirit / demon
- Agel / Zazel (45) / (45)
- 4 = Yaphiel / Hasmael (136) / (136)
- 8 = Grathiel (325) / (325) Barsabel
- 0 = Nakiel / Sorath (111) / (666)
- 9 = Hagiel / Kedemel (19) / (175)
- 6 = Bene / Raphael / Haxia / Seraphim (1252) / (2080)
- D = Malka / Haxmodai (369) / Betharshim (3321)

R	Sh	B
G	H	Z
Ch	A	V

 $k=3^2$

D	YD	YSh	A
L	Z	V	YB
H	YA	Y	Ch
YV	B	G	YB

 $4=4^2$

YAKD	Z	K	G
D	YB	KHCh	YV
YZ	H	YG	KA
Y	YCh	A	YDKB
KGV	YL	B	YH

 $5^2=5^2$

V	SB	G	SDSh	A
Z	YAKZ	KCh	Ch	S
YL	YD	VYH	KG	KB
YCh	K	KB	KAYZ	YG
KH	KL	Y	L	KVYB
SV	H	SG	D	BSA

 $6^2=6^2$

KBMZYV	MA	Y	Sh	D
H	KGMCh	YZ	MB	YAKL
S	V	KD	MLYCh	SVYB
YG	SA	Z	KH	MGYLSZ
SCh	YD	BA	KV	MDK
KASL	Ch	SG	B	KZMH
MVYH	M	L	SD	GKCh

 $7^2=7^2$

Ch	NGNLH	D	Th	ThGA
MLYH	YD	NB	NGYA	YNV
MA	KG	KB	MD	MHYL
SB	SD	SH	KL	KChSChSLCh
M	KV	KZ	SZ	SV
YZ	MZ	NV	K	KAMG
L	NH	MD	YB	YGNA
Th	D	B	G	Th

 $8^2=8^2$

SZ	Ch	KLU	KATH	BYGND	H
V	SCh	UL	S	WAKB	Th
MZ	Z	SL	P	SA	VBKGNHYH
YV	MCh	Ch	M	PASB	ThDKD
NZ	YZ	ML	L	MAUG	SGTh
KV	NCh	YCh	N	A	MBUD
Th	KZ	NLY	NA	B	MGU
SV	Th	Ch	Th	YANB	G
UZ	KCh	Th	L	KTh	YBNG

 $D=9^2$



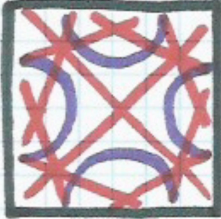
$k=9$



$4=16$



$\sigma^2=25$



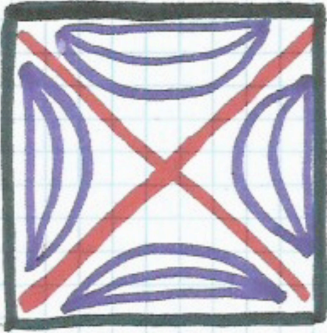
$\odot=36$



$\sigma^2=49$



$\sigma^2=64$



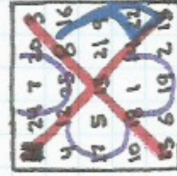
$\sigma^2=81$



$k=15$



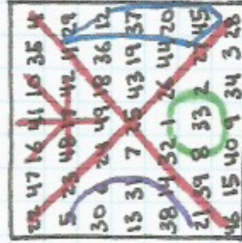
$4=34$



$\sigma^2=65$



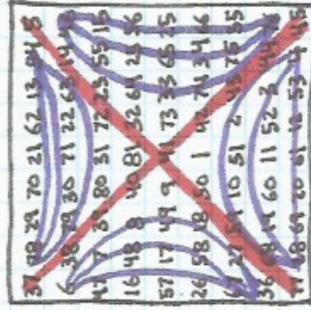
$\odot=111$



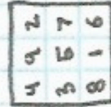
$\sigma^2=175$



$\sigma^2=260$



$\sigma^2=369$



$q=15$



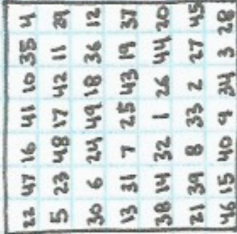
$16=34$



$25=65$



$36=111$



$49=175$



$64=260$



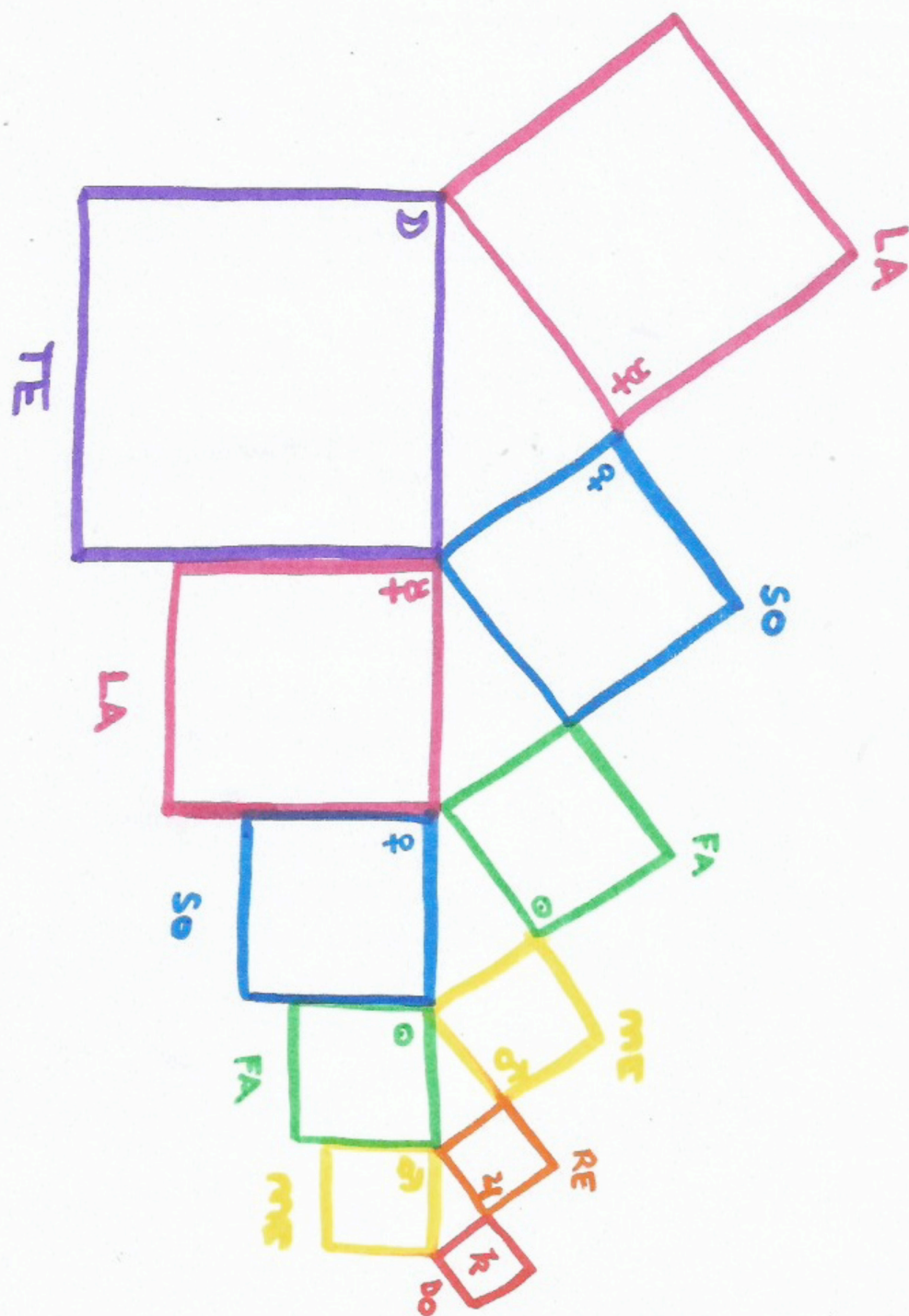
$81=369$

NOTES : MUSIC THEORY

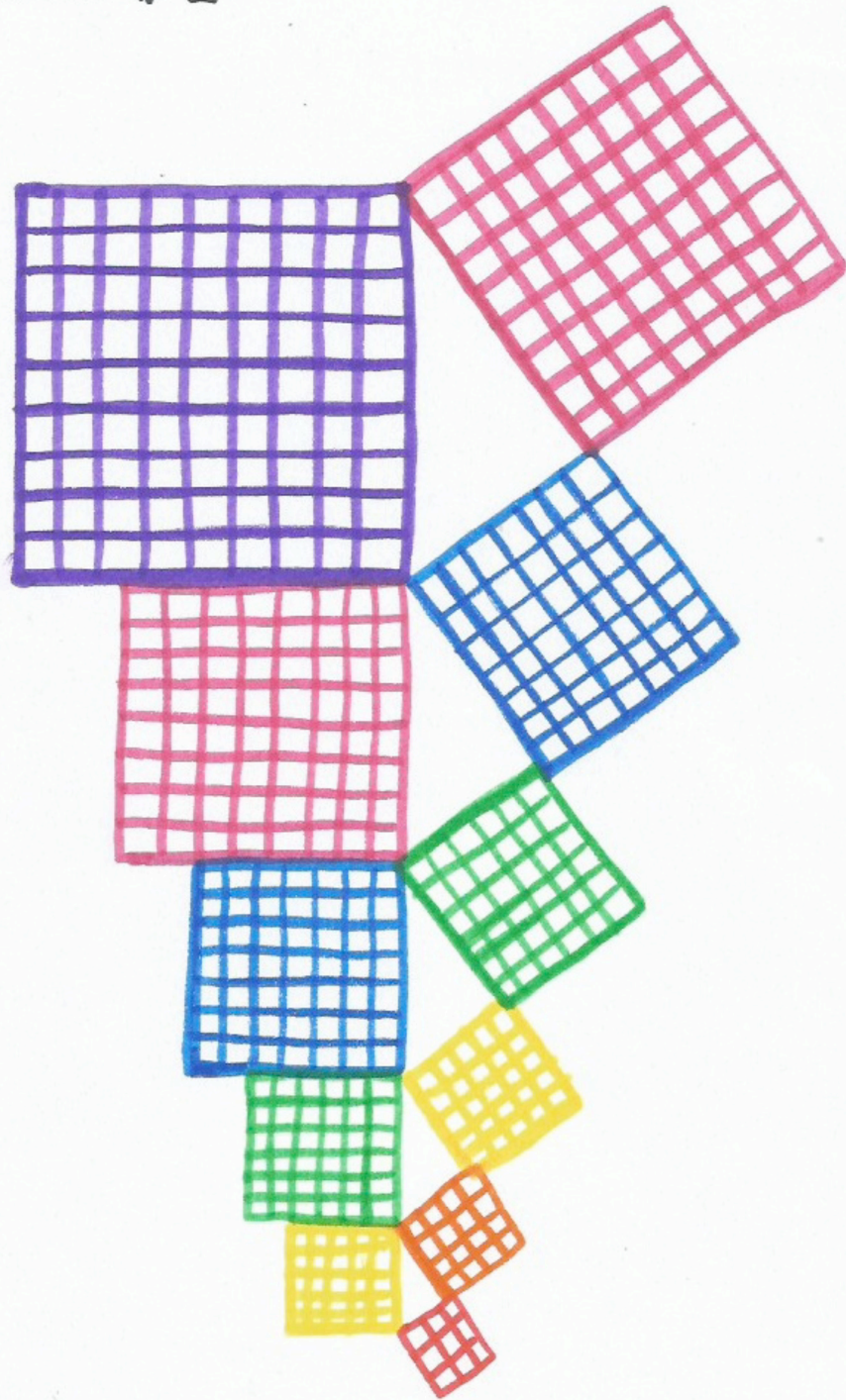
"PROPAGATIONS" (λ 's 25 POINCARÉ SECTIONS & CHLADNI PLATE SURFACES)

- #1) DO-RE-ME-FA-SO-LA-TE (the usual "OCTAVE" minus-one) are correlated here, again, to the divided Prismatic spectrum (ROYGBIV) by ascending orders of pure-tone pitch (Hz frequencies 3-9) as well as to the ORDER magic-# squares by scale of $3 \times 3 \rightarrow 9 \times 9$, arranged in ascending size-scale according to a forced Pythagorean triangle relationship (occurring 5 times per 7 square/tones). The five triangles comprise the 5 "mixed" pitch or " $\frac{1}{2}$ note" tones on a scale.
- #2) "DO", indicating the LOWEST pitch frequency with the LONGEST wavelength (λ), is thus depicted as a RED square of 3 Rows & 3 columns - an "ORDER 3" # square. So is "RE", the next pure tone of 4 Hz (cycles per second), an orange square of ORDER 4, & so on up to "TE", a purple square of 9^2 smaller, "unit" squares.
- #3) The Chaldean-era originating Izmens of the "planetary Kamez" - derived by numerical relations between the Rows' & columns' # - sums when each ORDER # square is factored to produce a "magic # - sum" & this derived into its additive components arrayed in a matrix across the grid - depicted per each color (ROYGBIV) for the 7 "planetary Kamez" ($k, 4, \sigma, \odot, \otimes, \frac{\sigma}{\tau}, D$). These ancient Izmens may relate to the Chladni plate patterns for Hz frequencies 3-9 respectively.

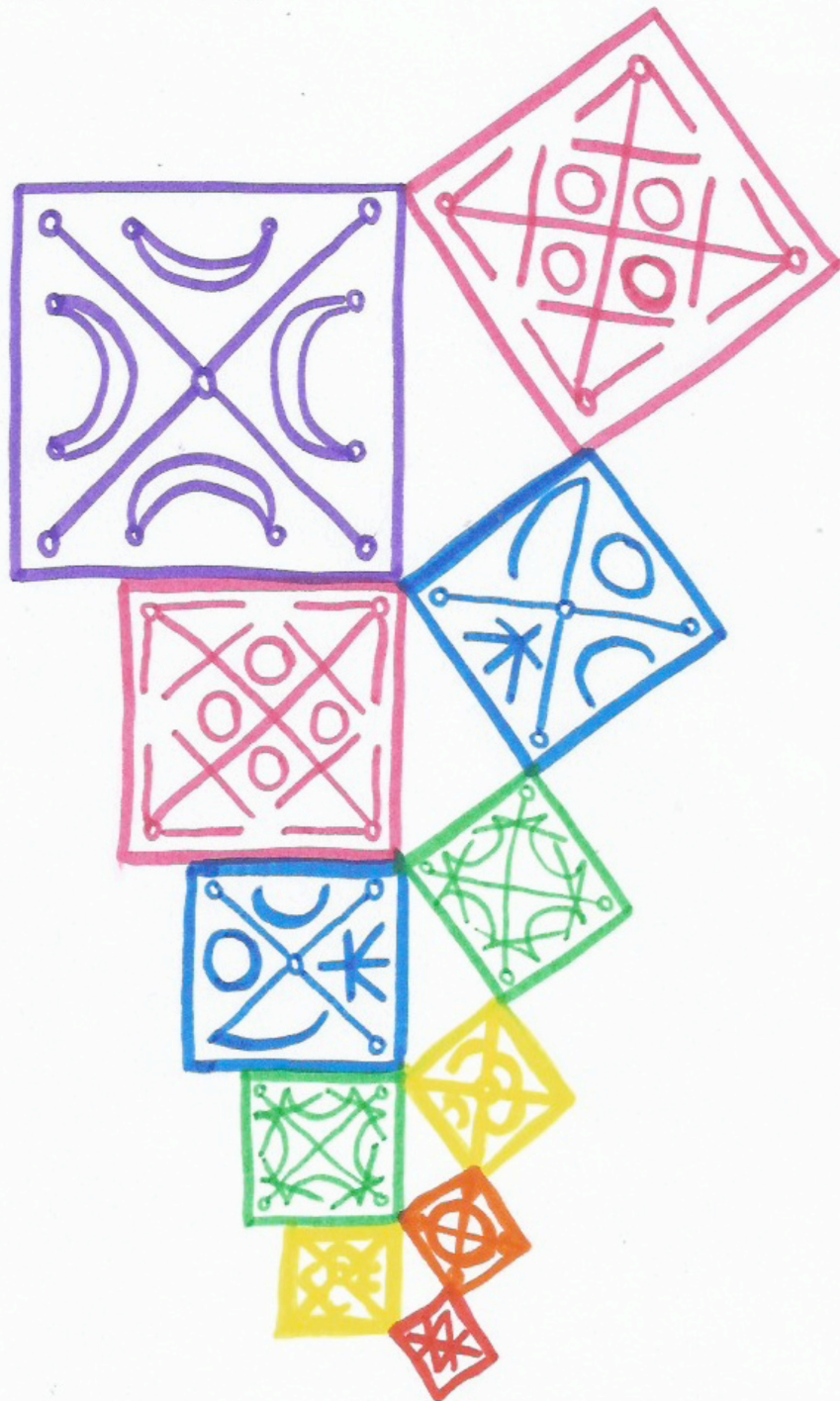
PROPAGATION # 1



PROPAGATION # 2



PROPAGATION #3

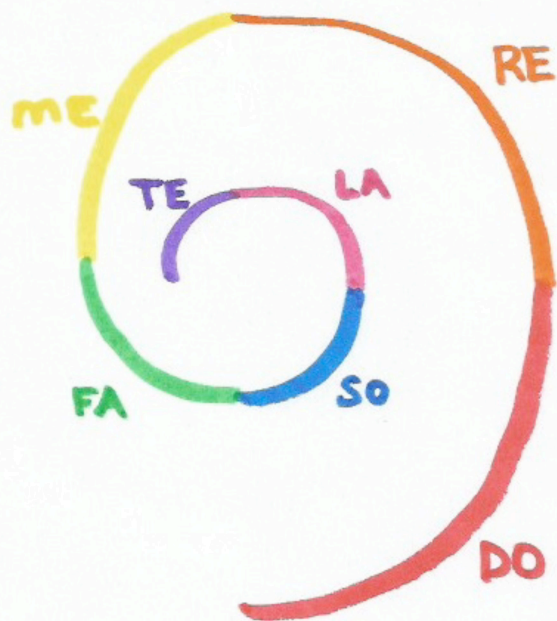


NOTES: MUSIC THEORY

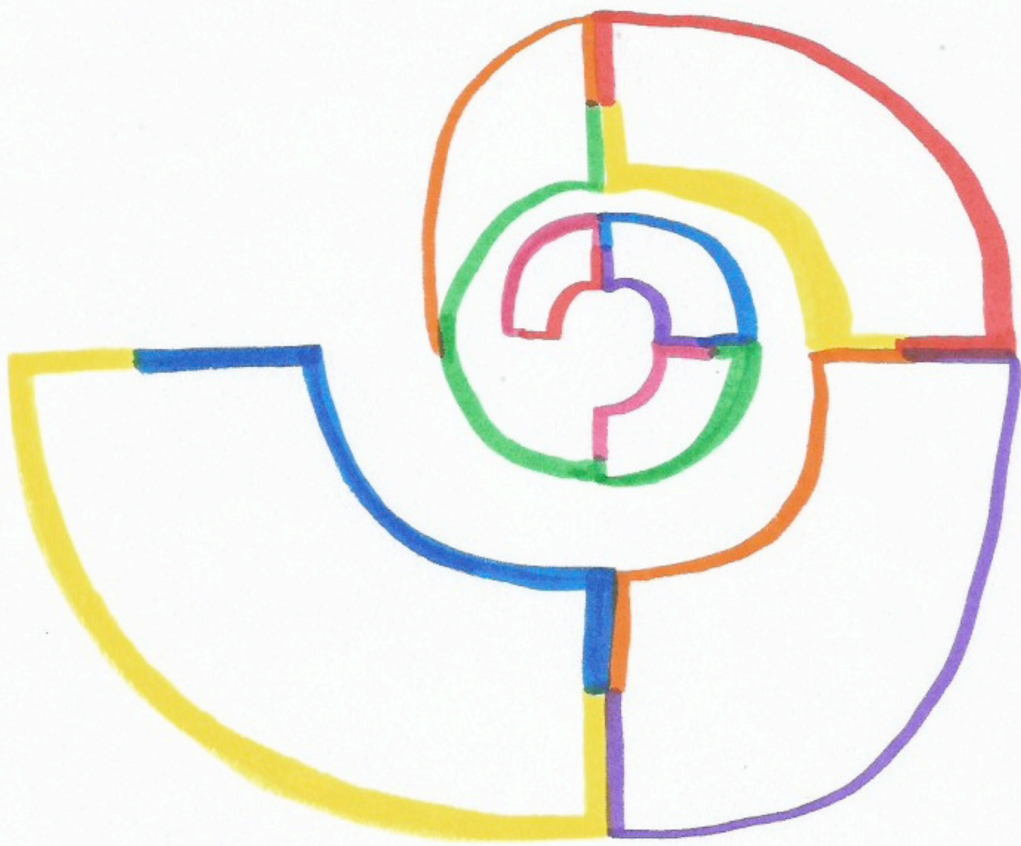
"SPANS" (the "Staggered" comm2 measuring λ 's)

- # 1) Lower "Pitch" Tones are comprised of longer λ 's (wavelengths) and spectral-shifted toward the "hot" end of the spectrum due to the "Doppler" effect (compression while approaching, expansion while retreating). Higher "pitch" tones are shorter λ 's and of the "cool" end of the spectrum of hues. THEREFORE, "DO" (red) is a "longer" λ than "TE" (violet), which is very short by comparison. This relationship is here depicted as a "staggered" comm2, showing a spiral divided into an "octave", however with 1 missing PURE tone between the LOWER 4 pitches & the UPPER 3 pitches.
- # 2) The same "staggered" comm2, showing, instead of WHOLE "pure" tones, the combinations of OVERTONES (outer spiral edge) over UNDERTONES (inner spiral edge) that effectively combine to form the "PURE" tones.
- # 3) Both the "OVERTONES/UNDERTONES" & "PURE" tones "Staggered" comm2s combined into one chart.

SPAN #1

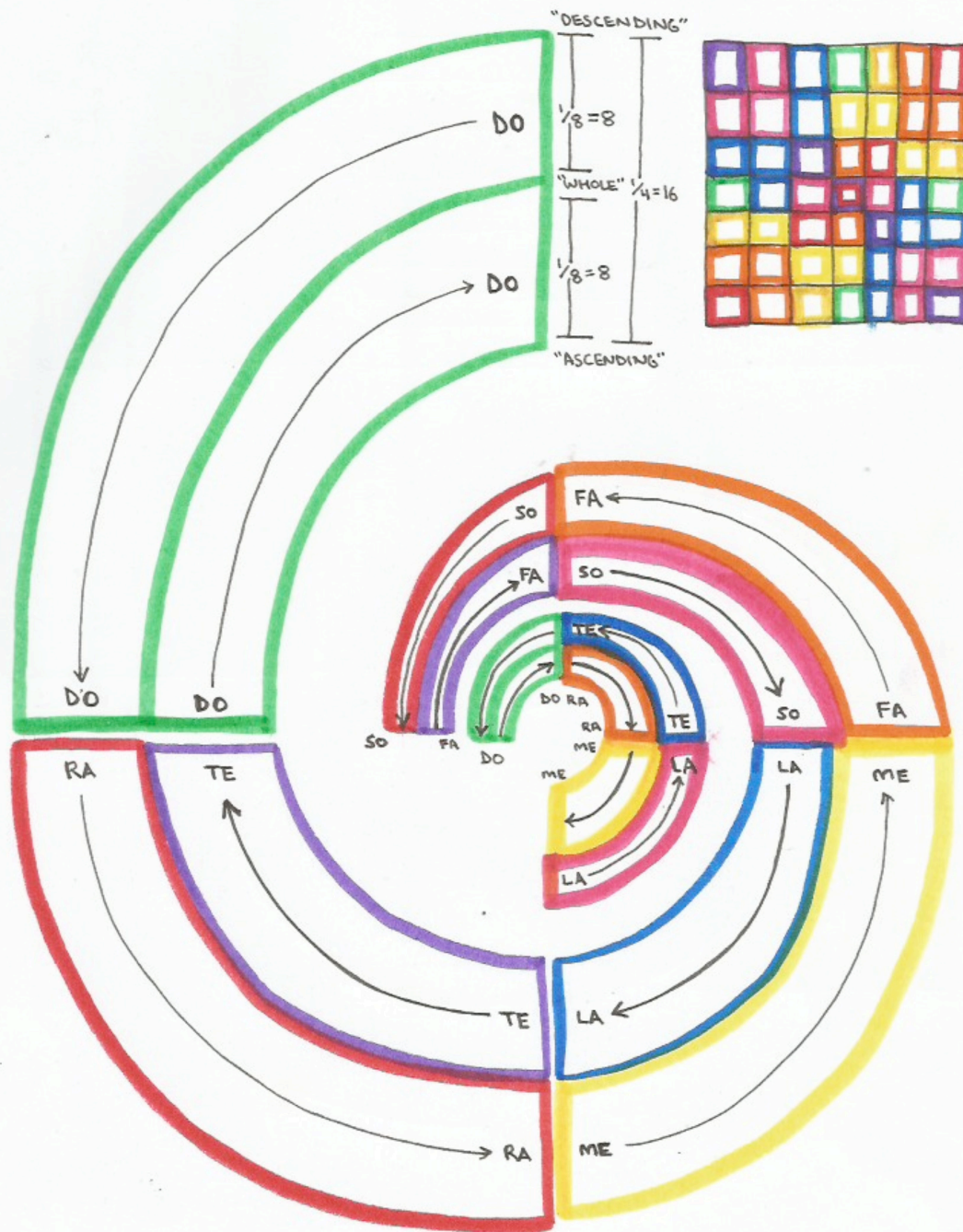


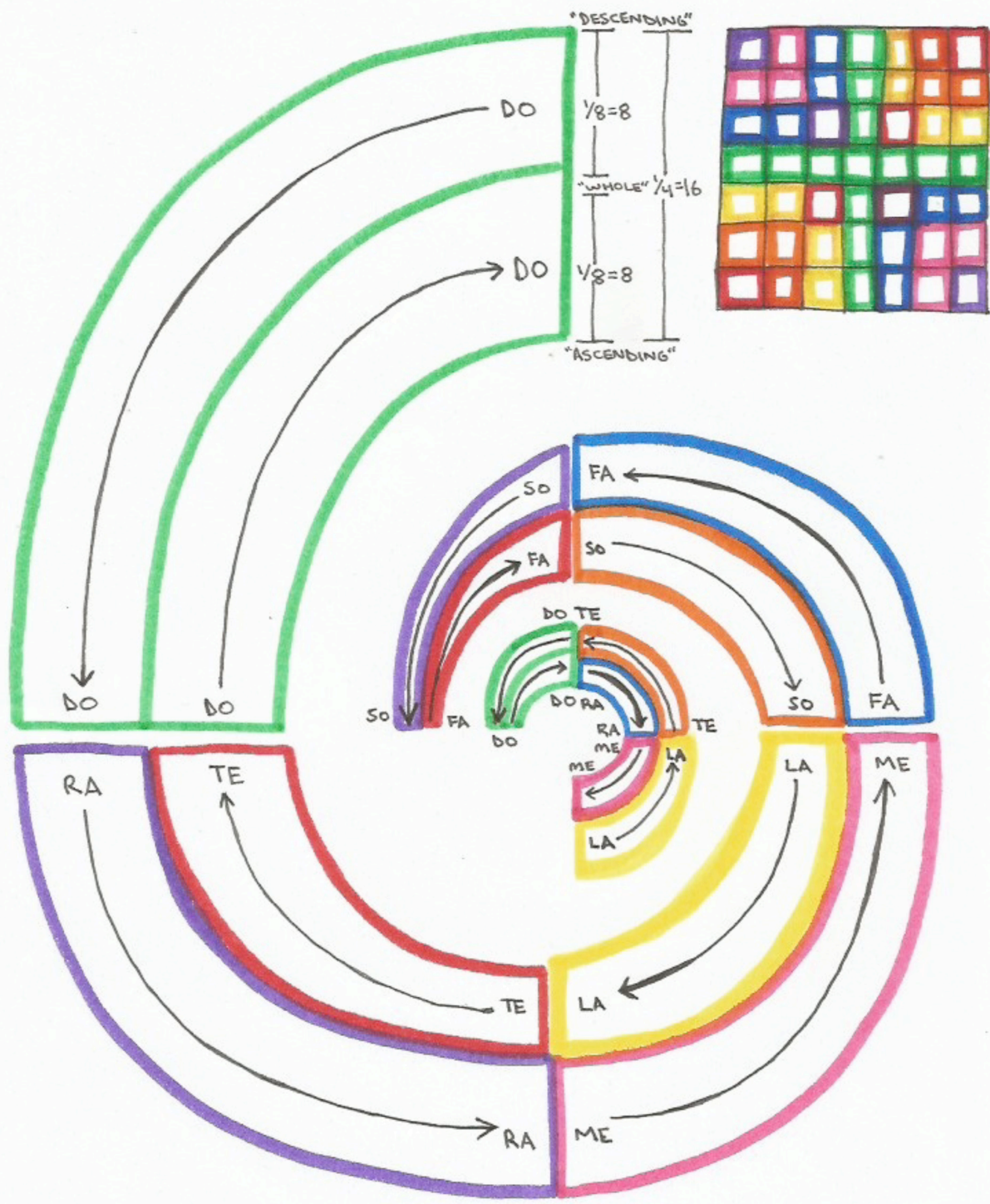
SPAN # 2

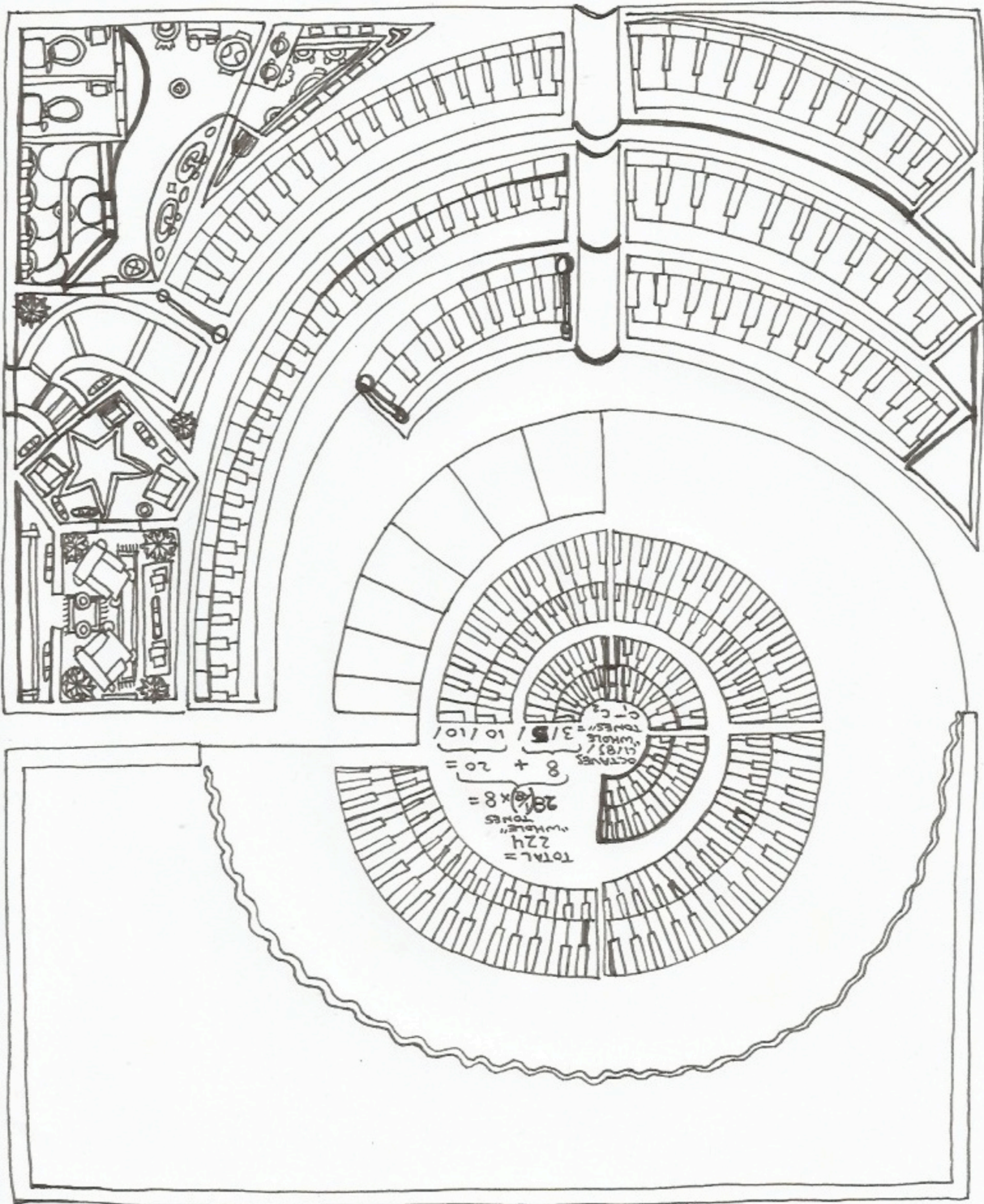


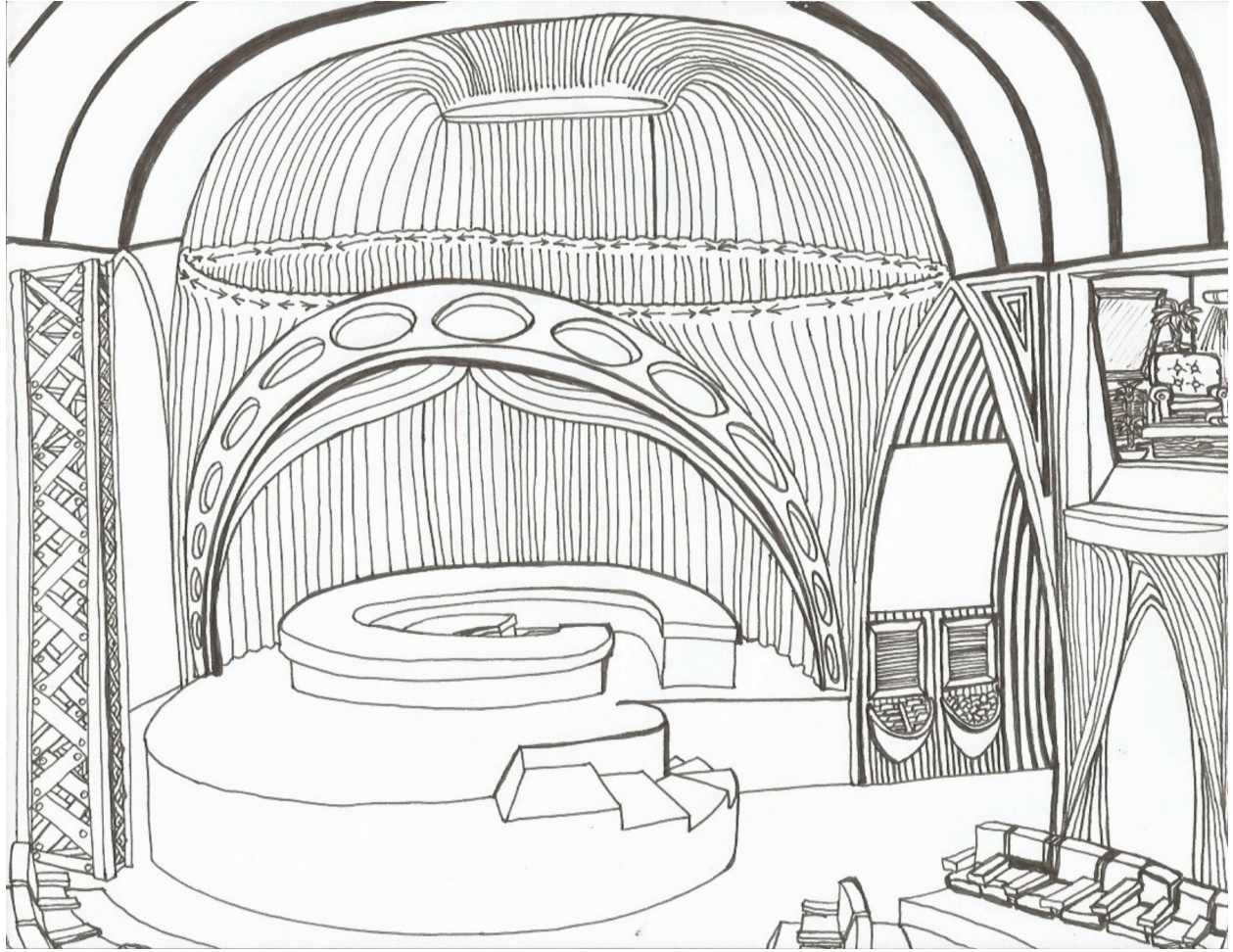
SPAN #3

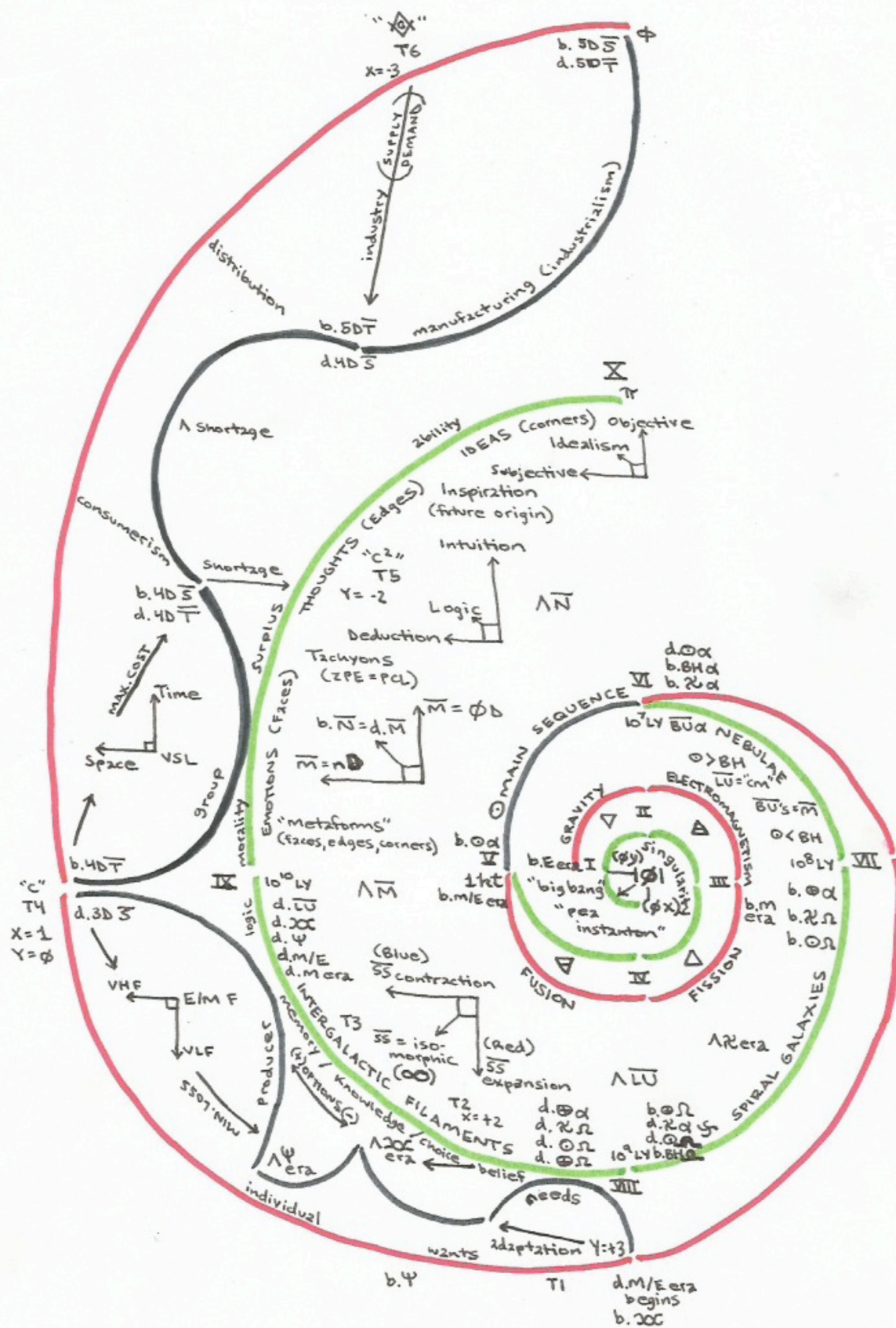


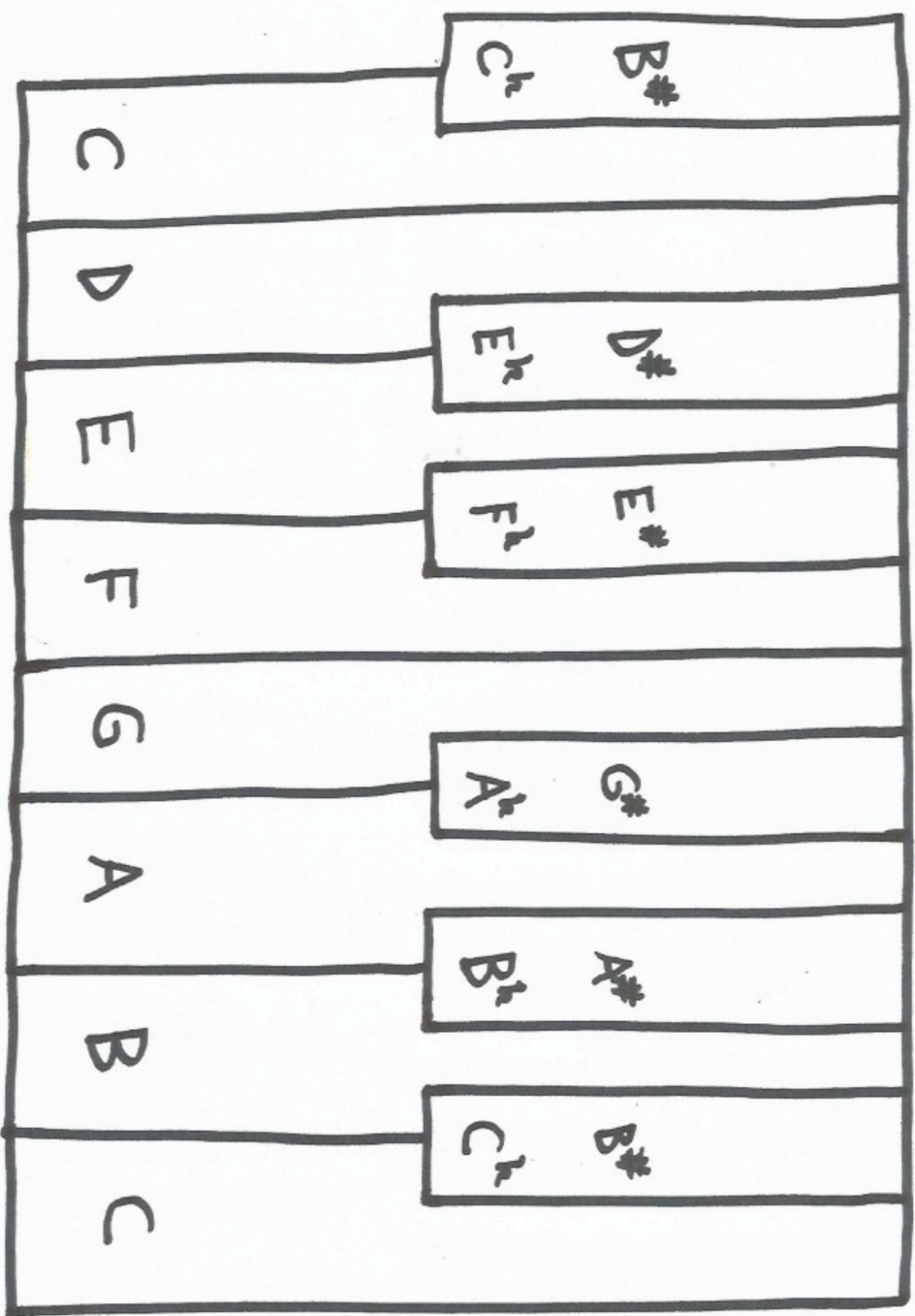




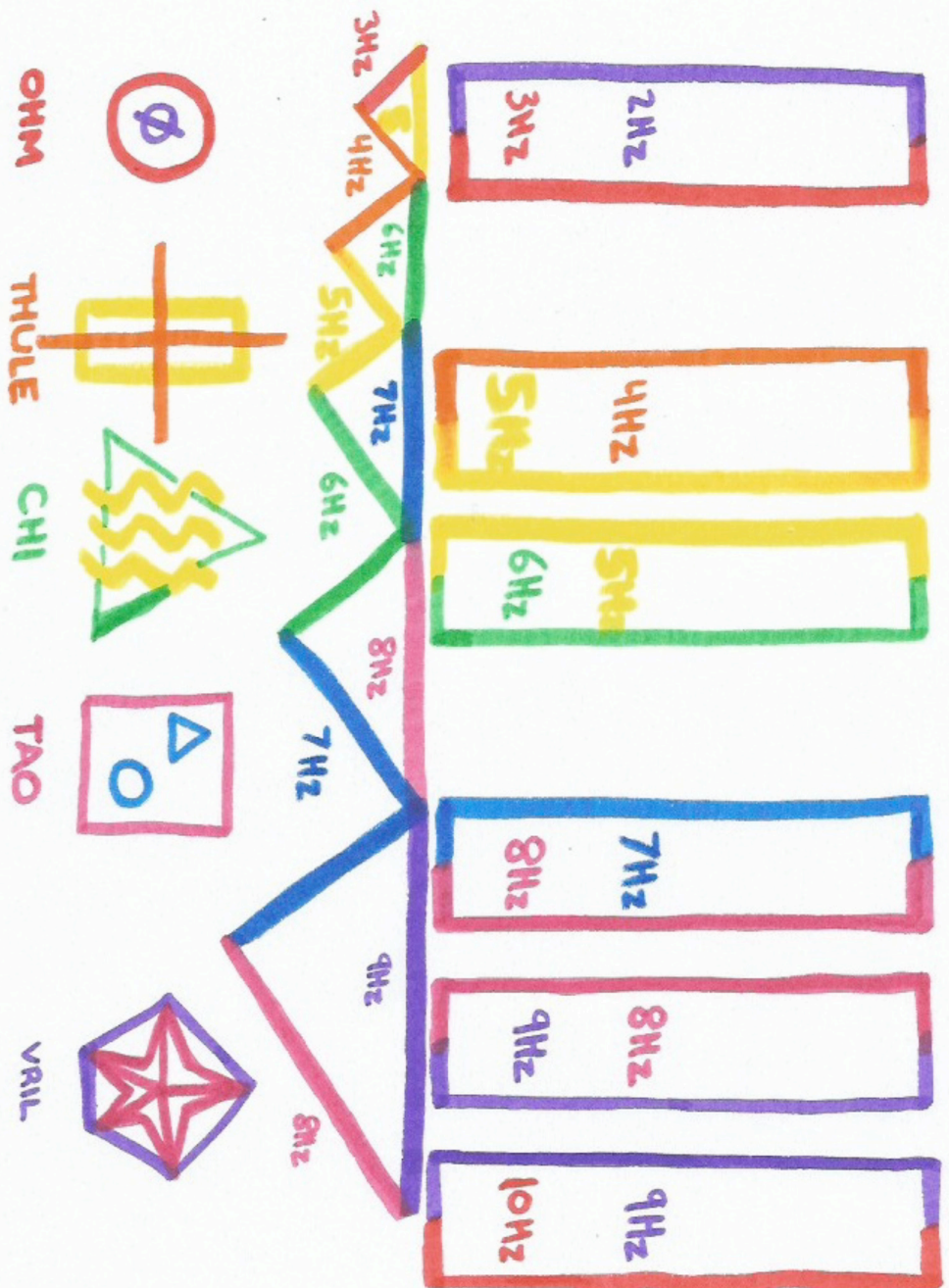


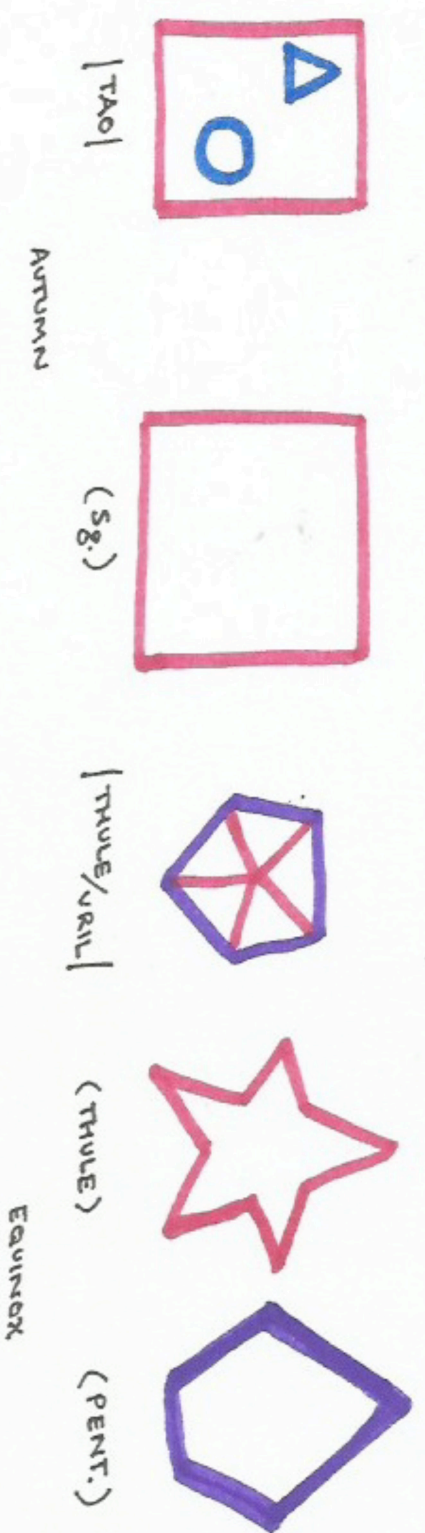
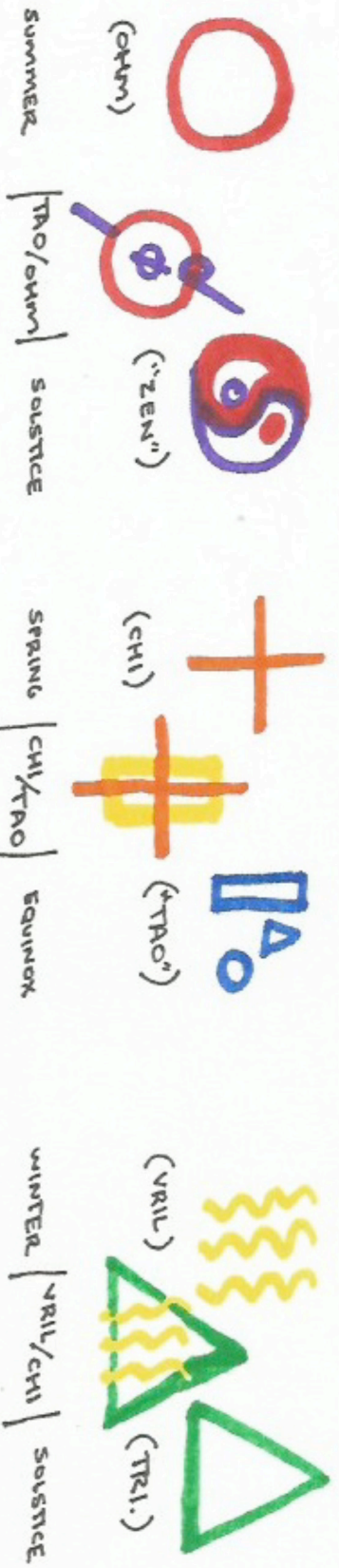






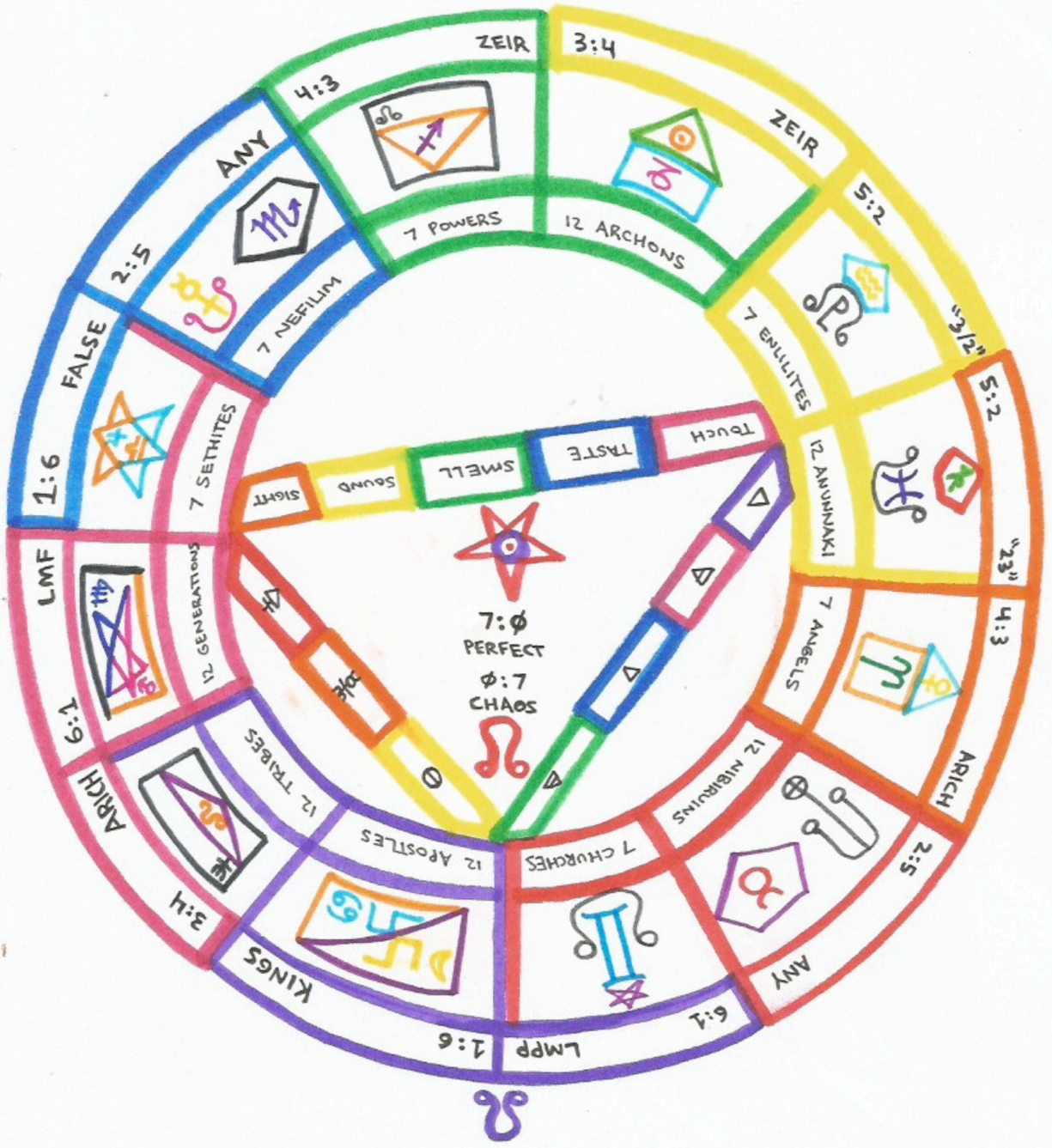
$\text{DO} = \text{r}$	r / u
$\text{RE} = \text{v}$	v / q
$\text{ME} = \text{q}$	q / o
$\text{FA} = \text{o}$	
$\text{SO} = \text{o} +$	$\text{o} + / \text{o} +$
$\text{LA} = \text{o} +$	$\text{o} + / \text{u}$
$\text{TE} = \text{u}$	u / r
$\text{DO} = \text{r}$	

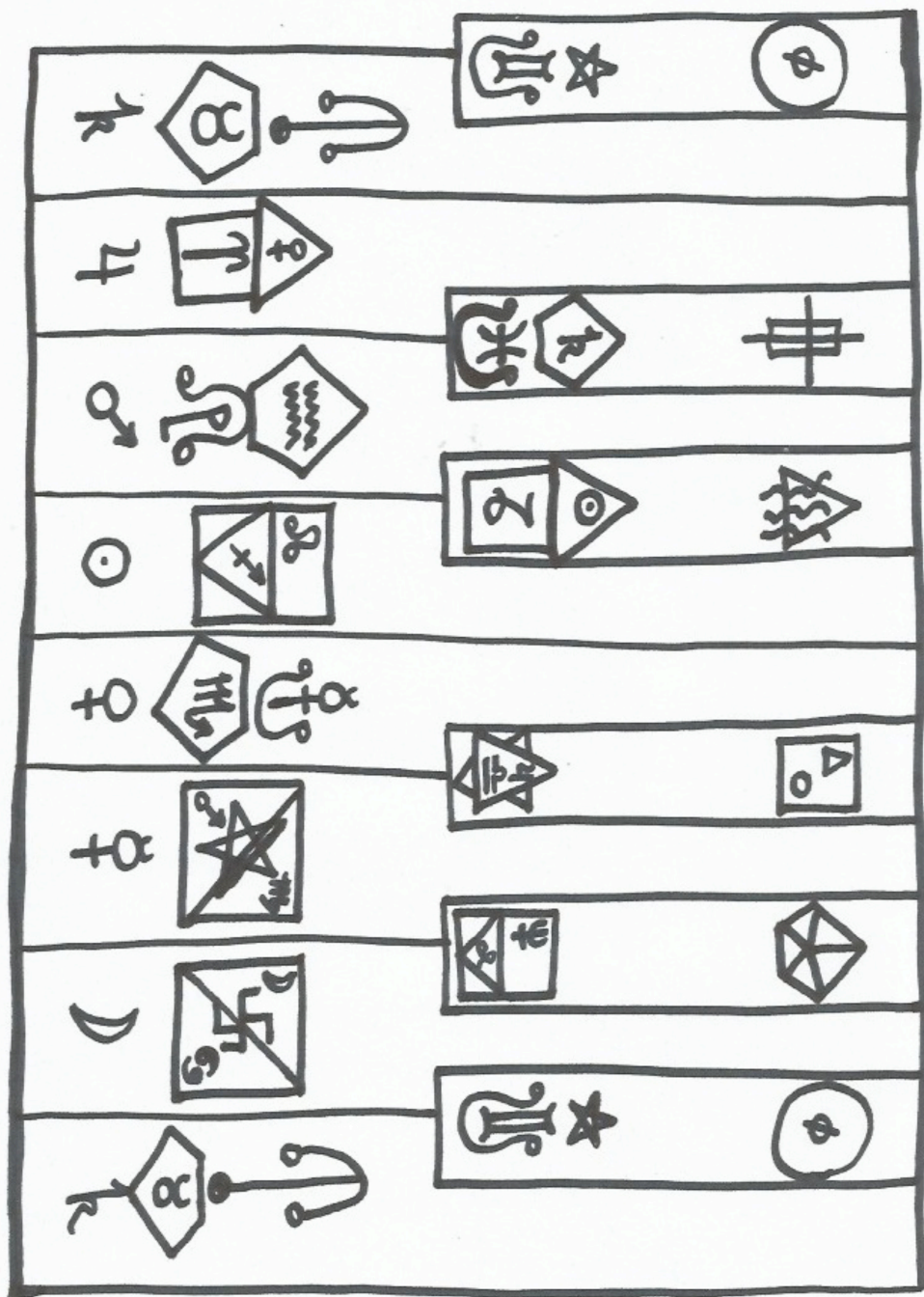


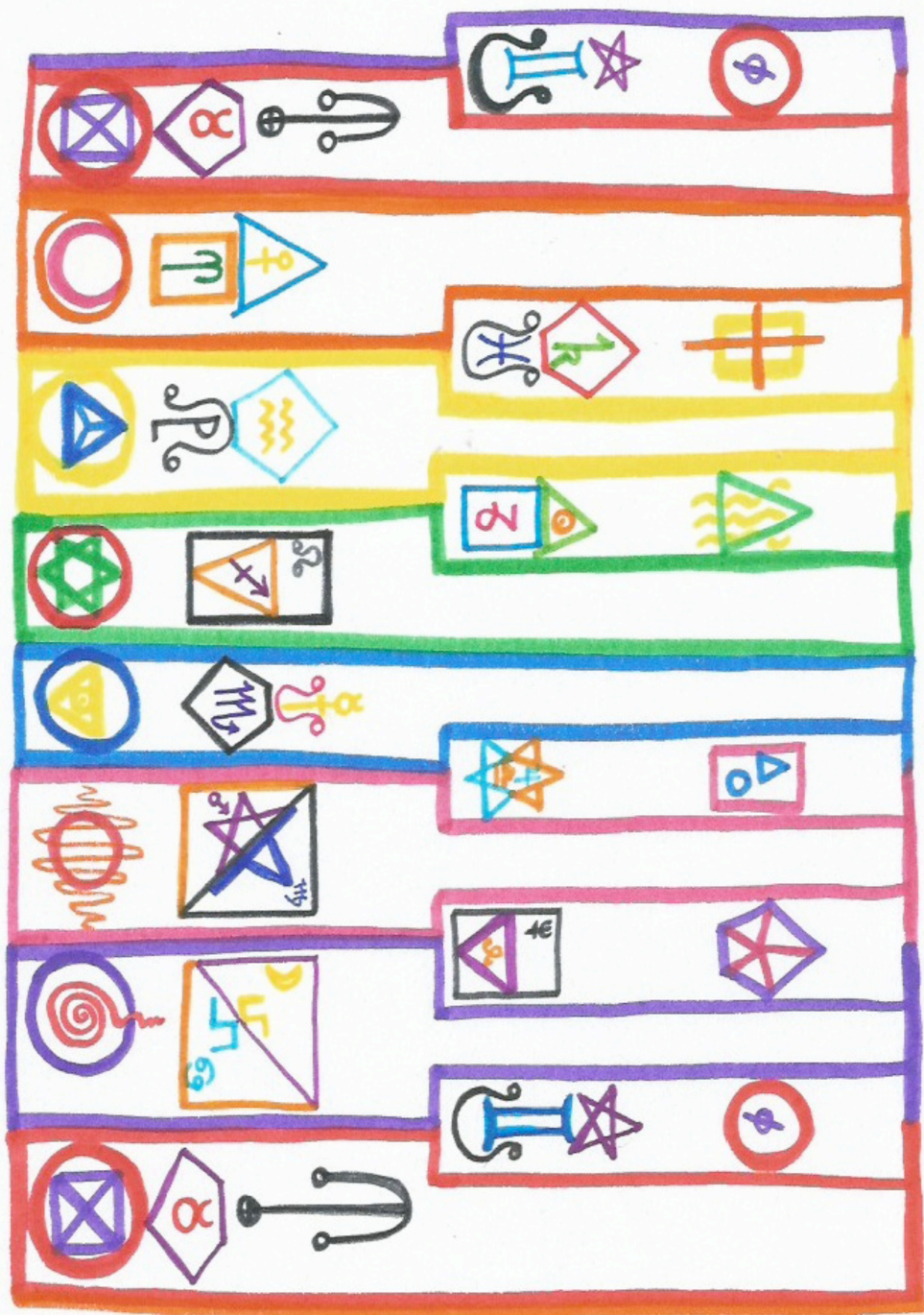


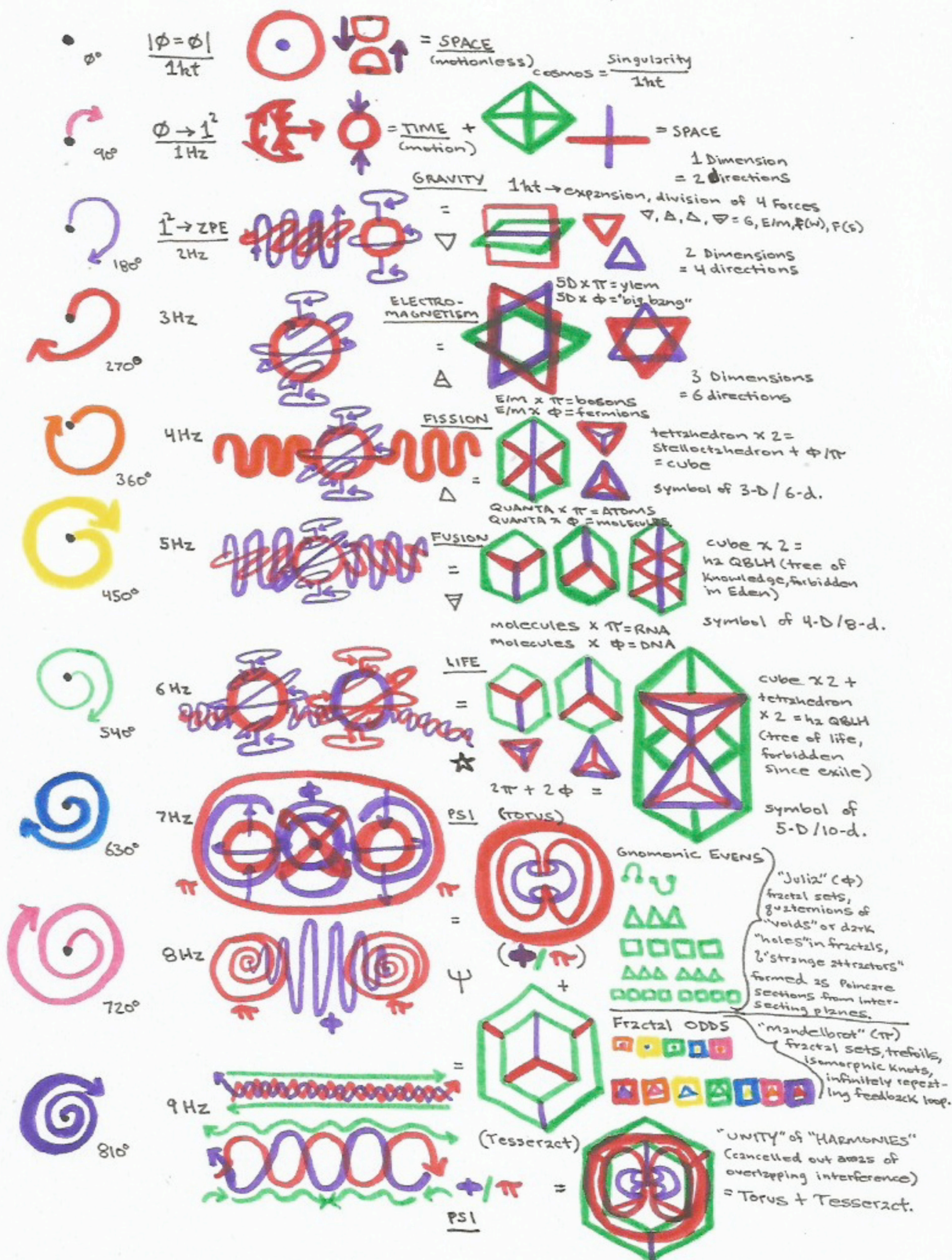
I		OHM	$ \emptyset = (\emptyset, \emptyset, \emptyset, \dots) = 0$	♀
II		TAO/OHM	$1/\phi = (1, \phi), (\phi, 1)$	▽
III		"ZEN"	$ 1 = I$	▽▽
IV		CHI	$(2) = E$	♀
V		CHI/TAO	$(2) = EA$	△
VI		"TAO"	$(2) = A$	△△
VII		VRIL	$(3) = A$	♂
VIII		VRIL/CHI	$ 3 = AE$	△
IX		TRI.	$(3) = E$	△△
X		TAO	$ 4 = U$	▽
XI		Sq.	$(4) = Y$	▽▽
XII		THULE/VRIL	$ 4 , 5 = U$	✕
XIII		THULE	$ 5 = U$	1 _k
XIV		PENT.	$(5) = Y$	2











Harmonia
Vol. 2: Music Theory
Section 2: Past Research

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Pythagorean Tuning

Superparticular ratio

In mathematics, a **superparticular ratio**, also called a **superparticular number** or **epimoric ratio**, is a ratio of the form

$$\frac{n+1}{n} = 1 + \frac{1}{n}.$$

Thus:

A superparticular number is when a great number contains a lesser number, to which it is compared, and at the same time one part of it. For example, when 3 and 2 are compared, they contain 2, plus the 3 has another 1, which is half of two. When 3 and 4 are compared, they each contain a 3, and the 4 has another 1, which is a third apart of 3. Again, when 5, and 4 are compared, they contain the number 4, and the 5 has another 1, which is the fourth part of the number 4, etc.

—Throop (2006).

Superparticular ratios were written about by Nicomachus in his treatise "Introduction to Arithmetic". Although these numbers have applications in modern pure mathematics, the areas of study that most frequently refer to the superparticular ratios by this name are music theory and the history of mathematics.

Mathematical properties

As Euler observed, the superparticular numbers (including also the multiply superparticular ratios, numbers formed by adding an integer other than one to a unit fraction) are exactly the rational numbers whose continued fraction terminates after two terms. The numbers whose continued fraction terminates in one term are the integers, while the remaining numbers, with three or more terms in their continued fractions, are superpartient.

The Wallis product

$$\prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right) = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots = \frac{4}{3} \cdot \frac{16}{15} \cdot \frac{36}{35} \cdots = 2 \cdot \frac{8}{9} \cdot \frac{24}{25} \cdot \frac{48}{49} \cdots = \frac{\pi}{2}$$

represents the irrational number π in several ways as a product of superparticular ratios and their inverses. It is also possible to convert the Leibniz formula for π into an Euler product of superparticular ratios in which each term has a prime number as its numerator and the nearest multiple of four as its denominator:

$$\pi/4 = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdots$$

In graph theory, superparticular numbers (or rather, their reciprocals, $1/2$, $2/3$, $3/4$, etc.) arise via the Erdős-Stone theorem as the possible values of the upper density of an infinite graph.

Other applications

In the study of harmony, many musical intervals can be expressed as a superparticular ratio. Indeed, whether a ratio was superparticular was the most important criterion in Ptolemy's formulation of musical harmony. In this application, Størmer's theorem can be used to list all possible superparticular numbers for a given limit; that is, all ratios of this type in which both the numerator and denominator are smooth numbers.

These ratios are also important in visual harmony. Aspect ratios of $4:3$ and $3:2$ are common in digital photography, and aspect ratios of $7:6$ and $5:4$ are used in medium format and large format photography respectively.

Ratio names and related intervals

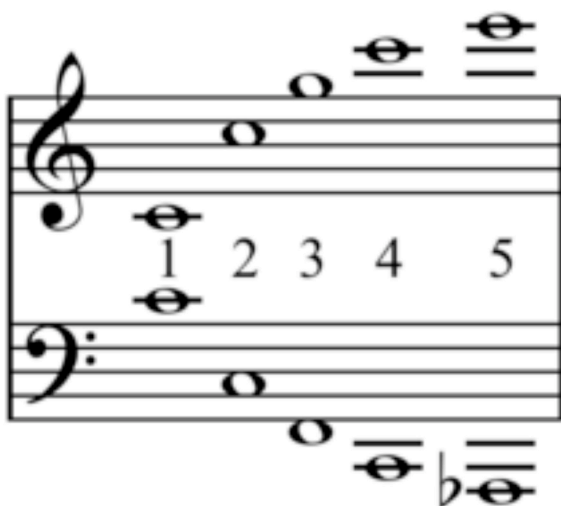
Many individual superparticular ratios have their own names, either in historical mathematics or in music theory. These include the following:

Examples

Ratio	Name	Related musical interval
2:1	duplex	octave
3:2	sesquialterum	perfect fifth
4:3	sesquitercium	perfect fourth
5:4	sesquiquartum	major third
6:5	sesquiquintum	minor third
7:6		septimal minor third
9:8	sesquioctavum	major second
10:9	sesquinona	minor tone
16:15		just diatonic semitone
25:24		just chromatic semitone
81:80		syntonic comma
126:125		septimal semicomma
4375:4374		ragisma

The root of some of these terms comes from Latin *sesqui*- "one and a half" (from *semis* "a half" + *-que* "and") describing the ratio 3:2.

Otonality and Utonality



5-limit Otonality and Utonality: overtone and "undertone" series, partials 1-5 numbered. Play Otonality, Utonality, major chord on C, and minor chord on F.

Otonality and **Utonality** are terms introduced by Harry Partch to describe chords whose pitch classes are the harmonics or subharmonics of a given fixed tone (identity). For example: 1/1, 2/1, 3/1,... or 1/1, 1/2, 1/3,....

Definition

G (=1/1), A(=9/8), $\flat$ H(=5/4), $\sharp$ C(=11/8), D(=3/2), $\flat$ F(=7/4)

Otonality on G = lower line of the tonality diamond bottom left to top right.

G (=1/1), F(=16/9), $\flat$ E(=8/5), $\flat$ D(16/11), C(=4/3), $\flat$ A(=8/7)

Utonality under G = lower line of the tonality diamond bottom right to top left.

An Otonality is a collection of pitches which can be expressed in ratios, expressing their relationship to the fixed tone, that have equal denominators. For example, 1/1, 5/4, and 3/2 (just major chord) form an Otonality because they can be written as 4/4, 5/4, 6/4. Every Otonality is therefore composed of members of a harmonic series. Similarly, the ratios of an Utonality share the same numerator. 7/4, 7/5, 7/6, and 1/1 (7/7) form an Utonality. Every Utonality is therefore composed of members of a subharmonic series.

An Otonality corresponds to an arithmetic series of frequencies, or lengths of a vibrating string. Brass instruments naturally produce Otonalities, and indeed Otonalities are inherent in the harmonics of a single fundamental tone. Tuvan khoomei singers produce Otonalities with their vocal tracts.

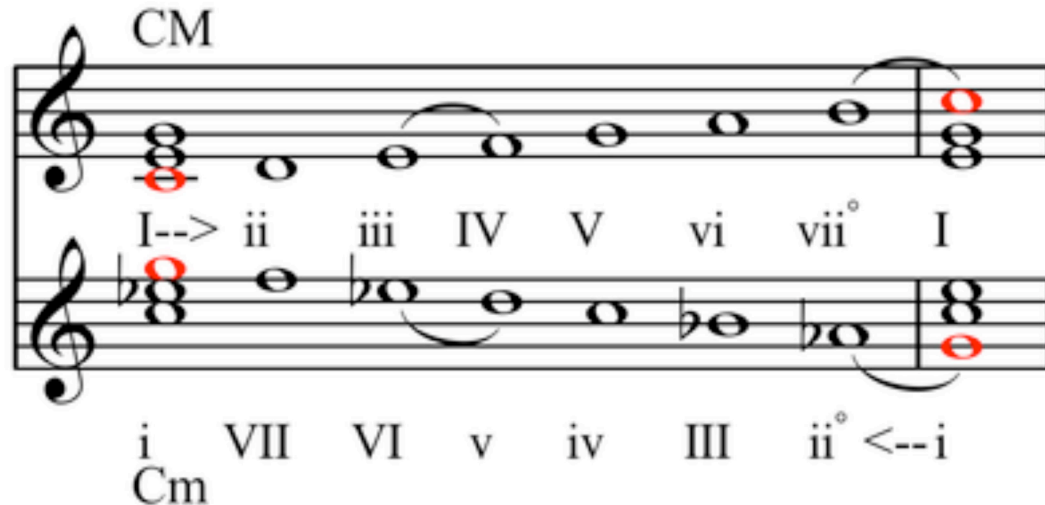
Utonality is the opposite, corresponding to a subharmonic series of frequencies, or an arithmetic series of wavelengths (the inverse of frequency). The *arithmetical proportion* "may be considered as a demonstration of Utonality ('minor tonality')." [1]

Relationship to standard Western music theory

Partch said that his 1931 coinage of "Otonality" and "Utonality" was, "hastened," by having read Henry Cowell's discussion of undertones in *New Musical Resources* (1930).

The 5-limit Otonality is simply a just major chord, and the 5-limit Utonality is a just minor chord. Thus Otonality and Utonality can be viewed as extensions of major and

minor tonality respectively. However, whereas standard music theory views a minor chord as being built up from the root with a minor third and a perfect fifth, an Utonality is viewed as descending from what's normally considered the "fifth" of the chord, so the correspondence is not perfect. This corresponds with the dualistic theory of Hugo Riemann:



Minor as upside down major.

In the era of meantone temperament, augmented sixth chords of the kind known as the German sixth (or the English sixth, depending on how it resolves) were close in tuning and sound to the 7-limit Otonality, called the tetrad. This chord might be, for example, A $\flat$ -C-E $\flat$ -G $\flat$ [F $\sharp$]. Standing alone, it has something of the sound of a dominant seventh, but considerably less dissonant. It has also been suggested that the Tristan chord, for example, F-B-D $\sharp$ -G $\sharp$ can be considered a Utonality, or 7-limit utonal tetrad, which it closely approximates if the tuning is meantone, though presumably less well in the tuning of a Wagnerian orchestra.

Criticism

Though Partch presents Otonality and Utonality as being equal and symmetric concepts, when played on most physical instruments an Otonality sounds much more consonant than a similar Utonality, due to the presence of the missing fundamental phenomenon. In an Otonality, all of the notes are elements of the same harmonic series, so they tend to partially activate the presence of a "virtual" fundamental as though they were harmonics of a single complex pitch. Utonal chords, while containing the same dyads as Otonal chords, do not tend to activate this phenomenon as strongly.

Combination tone



Sum and difference tones of A220 and unison, just perfect fifth, and octave. Hz marked.

A **combination tone**, also called a **sum tone** or a **difference tone** (also occasionally resultant tone), can be any of at least three similar psychoacoustic phenomena. When two tones are played simultaneously, a listener can sometimes perceive an additional tone whose frequency is a sum or difference of the two frequencies. The discovery of some of these phenomena is credited to the violinist Giuseppe Tartini, and so the tones are also called **Tartini tones**.

Explanation

One way a difference tone can be heard is when two tones with fairly complete sets of harmonics make a just fifth. This can be explained as an example of the missing fundamental phenomenon. If f is the missing fundamental frequency, then $2f$ would be the frequency of the lower tone, and its harmonics would be $2f$, $6f$, $8f$, etc. Since a fifth corresponds to a frequency ratio of $2:3$, the higher tone and its harmonics would then be $3f$, $6f$, $9f$, etc. When both tones are sounded, there are components with frequencies of $2f$, $3f$, $4f$, $6f$, $8f$, $9f$, etc. The missing fundamental is heard because so many of these components refer to it.

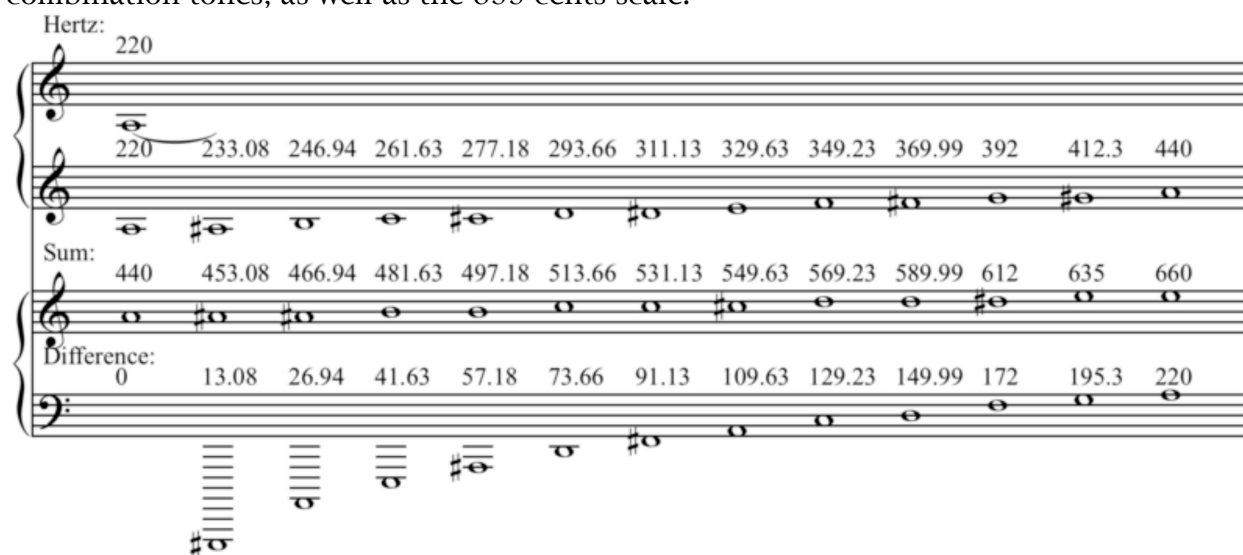
The specific phenomenon that Tartini discovered was physical. Sum and difference tones are thought to be caused sometimes by the non-linearity of the inner ear. This causes intermodulation distortion of the various frequencies which enter the ear. They are combined linearly, generating relatively faint components with frequencies equal to the sums and differences of whole multiples of the original frequencies. Any components which are heard are usually lower, with the most commonly heard frequency being just the difference tone, $f_2 - f_1$, though this may be a consequence of the other phenomena. Although much less common, the following frequencies may also be heard:

$$2f_1 - f_2, 3f_1 - 2f_2, \dots, f_1 - k(f_2 - f_1)$$

For a time it was thought that the inner ear was solely responsible whenever a sum or difference tone was heard. However, experiments show evidence that even when using headphones providing a single pure tone to each ear separately, listeners may still hear a difference tone. Since the peculiar, non-linear physics of the ear doesn't

come into play in this case, it is thought that this must be a separate, neural phenomenon. Compare binaural beats.

Heinz Bohlen proposed what is now known as the Bohlen-Pierce scale on the basis of combination tones, as well as the 833 cents scale.

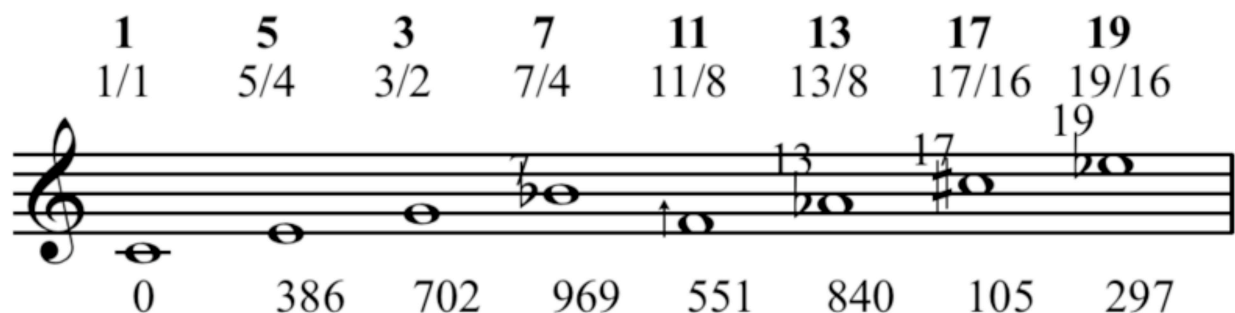


Resultant tone

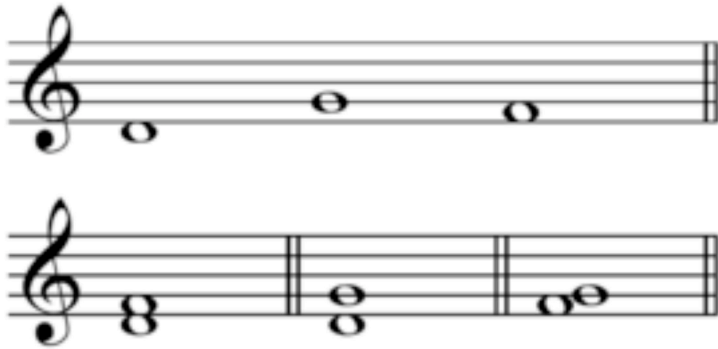
In pipe organs, a **resultant tone** is the sound of a combination of organ pipes that allows the listener to perceive a lower pitch.

This is done by having two pipes, one pipe of the note being played, and another harmonically related, typically at its fifth, being sounded at the same time. The result is a pitch at a common subharmonic of the pitches played (one octave below the first pitch when the second is the fifth, 3:2, two octaves below when the second is the major third, 5:4). This effect is useful especially in the lowest ranks of the pipe organ where cost or space could prohibit having a rank of such low pitch. For example, a 64' pipe would be costly and take up at least 32' of space (if capped) for each pipe. Using a resultant tone for such low pitches would eliminate the cost and space factor, but would not sound as full as a true 64' pipe.

This effect is most often used in the lowest octave of the organ only.



Interval



Melodic and harmonic intervals.

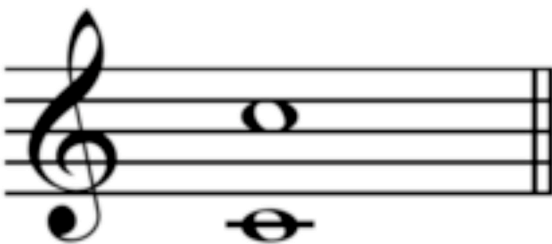
In music theory, an **interval** is the difference between two pitches.[1] An interval may be described as **horizontal**, **linear**, or **melodic** if it refers to successively sounding tones, such as two adjacent pitches in a melody, and **vertical** or **harmonic** if it pertains to simultaneously sounding tones, such as in a chord.[2][3]

In Western music, intervals are most commonly differences between notes of a diatonic scale. The smallest of these intervals is a semitone. Intervals smaller than a semitone are called microtones. They can be formed using the notes of various kinds of non-diatonic scales. Some of the very smallest ones are called commas, and describe small discrepancies, observed in some tuning systems, between enharmonically equivalent notes such as C# and D♭. Intervals can be arbitrarily small, and even imperceptible to the human ear.

In physical terms, an interval is the ratio between two sonic frequencies. For example, any two notes an octave apart have a frequency ratio of 2:1. This means that successive increments of pitch by the same interval result in an exponential increase of frequency, even though the human ear perceives this as a linear increase in pitch. For this reason, intervals are often measured in cents, a unit derived from the logarithm of the frequency ratio.

In Western music theory, the most common naming scheme for intervals describes two properties of the interval: the quality (perfect, major, minor, augmented, diminished) and number (unison, second, third, etc.). Examples include the minor third or perfect fifth. These names describe not only the difference in semitones between the upper and lower notes, but also how the interval is spelled. The importance of spelling stems from the historical practice of differentiating the frequency ratios of enharmonic intervals such as G-G# and G-A♭.

Size



Example: Perfect octave on C in equal temperament and just intonation: $2/1 = 1200$ cents.

The size of an interval (also known as its width or height) can be represented using two alternative and equivalently valid methods, each appropriate to a different context: frequency ratios or cents.

Frequency ratios

The size of an interval between two notes may be measured by the ratio of their frequencies. When a musical instrument is tuned using a just intonation tuning system, the size of the main intervals can be expressed by small-integer ratios, such as 1:1 (unison), 2:1 (octave), 3:2 (perfect fifth), 4:3 (perfect fourth), 5:4 (major third), 6:5 (minor third). Intervals with small-integer ratios are often called *just intervals*, or *pure intervals*. To most people, just intervals sound consonant, that is, pleasant and well tuned.

Most commonly, however, musical instruments are nowadays tuned using a different tuning system, called 12-tone equal temperament, in which the main intervals are typically perceived as consonant, but none is justly tuned and as consonant as a just interval, except for the unison (1:1) and octave (2:1). As a consequence, the size of most equal-tempered intervals cannot be expressed by small-integer ratios, although it is very close to the size of the corresponding just intervals. For instance, an equal-tempered fifth has a frequency ratio of $2^{7/12}$, approximately equal to 1.498:1, or 2.997:2 (very close to 3:2). For a comparison between the size of intervals in different tuning systems, see section Size in different tuning systems.

Cents

The standard system for comparing interval sizes is with cents. The cent is a logarithmic unit of measurement. If frequency is expressed in a logarithmic scale, and along that scale the distance between a given frequency and its double (also called octave) is divided into 1200 equal parts, each of these parts is one cent. In twelve-tone equal temperament (12-TET), a tuning system in which all semitones have the same size, the size of one semitone is exactly 100 cents. Hence, in 12-TET the cent can be also defined as one hundredth of a semitone.

Mathematically, the size in cents of the interval from frequency f_1 to frequency f_2 is

$$n = 1200 \cdot \log_2 \left(\frac{f_2}{f_1} \right).$$

Main intervals

The table shows the most widely used conventional names for the intervals between the notes of a chromatic scale. A perfect unison (also known as perfect prime) is an interval formed by two identical notes. Its size is zero cents. A semitone is any interval between two adjacent notes in a chromatic scale, a whole tone is an interval spanning two semitones (for example, a major second), and a tritone is an interval spanning three tones, or six semitones (for example, an augmented fourth). Rarely, the term ditone is also used to indicate an interval spanning two whole tones (for example, a major third), or more strictly as a synonym of major third.

Intervals with different names may span the same number of semitones, and may even have the same width. For instance, the interval from D to F $\sharp$ is a major third, while that from D to G $\flat$ is a diminished fourth. However, they both span 4 semitones. If the instrument is tuned so that the 12 notes of the chromatic scale are equally spaced (as in equal temperament), these intervals will also have the same width. Namely, all semitones will have a width of 100 cents, and all intervals spanning 4 semitones will be 400 cents wide.

The names listed here cannot be determined by counting semitones alone. The rules to determine them are explained below.



Fifth from C to G in the A $\flat$ major scale.

The number of an interval is the number of staff positions it encompasses.[*citation needed*] Both lines and spaces (see figure) are counted, including the positions of both notes forming the interval. For instance, the interval C-G is a fifth (denoted **P5**) because the notes from C to G occupy five consecutive staff positions, including the positions of C and G. The table and the figure above show intervals with numbers ranging from 1 (e.g., **P1**) to 8 (e.g., **P8**). Intervals with larger numbers are called compound intervals.

There is a one-to-one correspondence between staff positions and diatonic-scale degrees (the notes of a diatonic scale).[9] This means that interval numbers can be also determined by counting diatonic scale degrees, rather than staff positions, provided that the two notes which form the interval are drawn from a diatonic scale. Namely, C-G is a fifth because in any diatonic scale that contains C and G, the sequence from C to G includes five notes. For instance, in the A $\flat$ -major diatonic scale, the five notes are C-D $\flat$ -E $\flat$ -F-G (see figure). This is not true for all kinds of scales. For instance, in a chromatic scale, the notes from C to G are eight (C-C $\sharp$ -D-D $\sharp$ -E-F-F $\sharp$ -G). This is the reason interval numbers are also called *diatonic numbers*, and this convention is called *diatonic numbering*.

If one takes away any accidentals from the notes which form an interval, by definition the notes do not change their staff positions. As a consequence, any interval has the same interval number as the corresponding natural interval, formed by the same notes without accidentals. For instance, the intervals C-G $\sharp$ (spanning 8 semitones) and C $\sharp$ -G (spanning 6 semitones) are fifths, like the corresponding natural interval C-G (7 semitones).

Interval numbers do not represent exactly interval widths. For instance, the interval C-D is a second, but D is only one staff position, or diatonic-scale degree, above C. Similarly, C-E is a third, but E is only two staff positions above C, and so on. As a consequence, joining two intervals always yields an interval number one less than their sum. For instance, the intervals C-E and E-G are thirds, but joined together they form a fifth (C-G), not a sixth. Similarly, a stack of three thirds, such as C-E, E-G, and G-B, is a seventh (C-B), not a ninth.

The rule to determine the diatonic number of a compound interval (an interval larger than one octave), based on the diatonic numbers of the simple intervals from which it is built is explained in a separate section.

Quality

	C and ...	D and ...	E and ...	F and ...	G and ...	A and ...	B and ...
C	P1						
D	M2	P1					
E	M3	M2	P1				
F	P4	m3	m2	P1			
G	P5	P4	m3	M2	P1		
A	M6	P5	P4	M3	M2	P1	
B	M7	M6	P5	A4 (TT)	M3	M2	P1
c	P8	m7	m6	P5	P4	m3	m2
d		P8	m7	M6	P5	P4	m3
e			P8	M7	M6	P5	P4
f				P8	m7	m6	d5 (TT)
g					P8	m7	m6
a						P8	m7
b							P8

Intervals formed by the notes of a C major diatonic scale.

The name of any interval is further qualified using the terms **perfect (P)**, major (**M**), minor (**m**), augmented (**A**), and diminished (**d**). This is called its *interval quality*. It is possible to have doubly diminished and doubly augmented intervals, but these are quite rare, as they occur only in chromatic contexts. The quality of a compound interval is the quality of the simple interval on which it is based.

Perfect



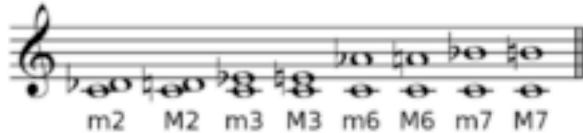
Perfect intervals on C.

Perfect intervals are so-called because they were traditionally considered perfectly consonant, although in Western classical music the perfect fourth was sometimes regarded as a less than perfect consonance, when its function was contrapuntal. Conversely, minor, major, augmented or diminished intervals are typically considered to be less consonant, and were traditionally classified as mediocre consonances, imperfect consonances, or dissonances.

Within a diatonic scale all unisons (**P1**) and octaves (**P8**) are perfect. Most fourths and fifths are also perfect (**P4** and **P5**), with five and seven semitones respectively. There's one occurrence of a fourth and a fifth which are not perfect, as they both span six semitones: an augmented fourth (**A4**), and its inversion, a diminished fifth (**d5**). For instance, in a C-major scale, the **A4** is between F and B, and the **d5** is between B and F (see table).

By definition, the inversion of a perfect interval is also perfect. Since the inversion does not change the pitch of the two notes, it hardly affects their level of consonance (matching of their harmonics). Conversely, other kinds of intervals have the opposite quality with respect to their inversion. The inversion of a major interval is a minor interval, the inversion of an augmented interval is a diminished interval.

Major/minor

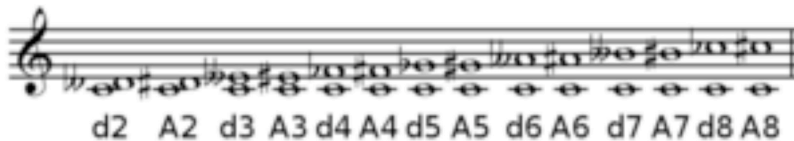


Major and minor intervals on C.

As shown in the table, a diatonic scale[9] defines seven intervals for each interval number, each starting from a different note (seven unisons, seven seconds, etc.). The intervals formed by the notes of a diatonic scale are called diatonic. Except for unisons and octaves, the diatonic intervals with a given interval number always occur in two sizes, which differ by one semitone. For example, six of the fifths span seven semitones. The other one spans six semitones. Four of the thirds span three semitones, the others four. If one of the two versions is a perfect interval, the other is called either diminished (i.e. narrowed by one semitone) or augmented (i.e. widened by one semitone). Otherwise, the larger version is called major, the smaller one minor. For instance, since a 7-semitone fifth is a perfect interval (**P5**), the 6-semitone fifth is called "diminished fifth" (**d5**). Conversely, since neither kind of third is perfect, the larger one is called "major third" (**M3**), the smaller one "minor third" (**m3**).

Within a diatonic scale, unisons and octaves are always qualified as perfect, fourths as either perfect or augmented, fifths as perfect or diminished, and all the other intervals (seconds, thirds, sixths, sevenths) as major or minor.

Augmented/diminished



Augmented and diminished intervals on C.

Augmented and diminished intervals are so called because they exceed or fall short of either a perfect interval, or a major/minor pair by one semitone, while having the same interval number (i.e., encompassing the same number of staff positions). For instance, an augmented third such as C-E# spans five semitones, exceeding a major third (C-E) by one semitone, while a diminished third such as C#-E♭ spans two semitones, falling short of a minor third (C-E♭) by one semitone.

Except for the above-mentioned augmented fourth (**A4**) and diminished fifth (**d5**), augmented and diminished intervals do not appear in diatonic scales(see table).

Example

Neither the number, nor the quality of an interval can be determined by counting semitones alone. As explained above, the number of staff positions must be taken into account as well.

For example, as shown in the table below, there are four semitones between A♭ and B#, between A and C#, between A and D♭, and between A# and E, but

- A♭-B# is a second, as it encompasses two staff positions (A, B), and it is doubly augmented, as it exceeds a major second (such as A-B) by two semitones.
- A-C# is a third, as it encompasses three staff positions (A, B, C), and it is major, as it spans 4 semitones.
- A-D♭ is a fourth, as it encompasses four staff positions (A, B, C, D), and it is diminished, as it falls short of a perfect fourth (such as A-D) by one semitone.
- A#-E is a fifth, as it encompasses five staff positions (A, B, C, D, E), and it

is triply diminished, as it falls short of a perfect fifth (such as A-E) by three semitones.

Number of semitones	Interval name	Staff positions				
		1	2	3	4	5
4	doubly <u>augmented second</u>	A $\flat$	B $\sharp$			
4	<u>major third</u>	A		C $\sharp$		
4	<u>diminished fourth</u>	A			D $\flat$	
4	triply <u>diminished fifth</u>	A $\sharp$				E $\flat\flat$

Shorthand notation

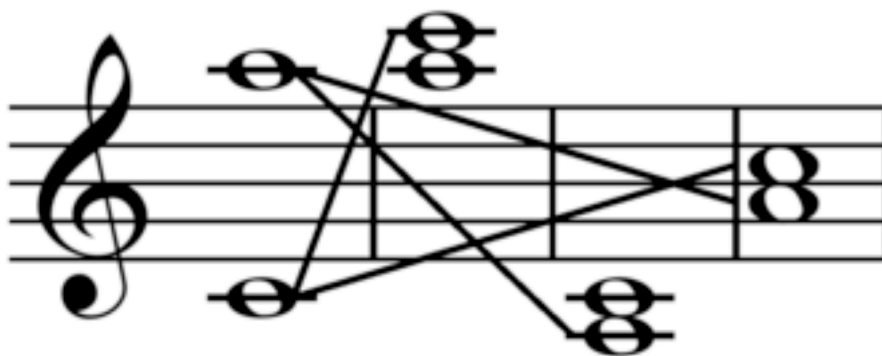
Intervals are often abbreviated with a **P** for perfect, **m** for minor, **M** for major, **d** for diminished, **A** for augmented, followed by the interval number. The indication M and P are often omitted. The octave is P8, and a unison is usually referred to simply as "a unison" but can be labeled P1. The tritone, an augmented fourth or diminished fifth is often **TT**. The interval qualities may be also abbreviated with **perf**, **min**, **maj**, **dim**, **aug**. Examples:

- m2 (or min2): minor second,
- M3 (or maj3): major third,
- A4 (or aug4): augmented fourth,
- d5 (or dim5): diminished fifth,
- P5 (or perf5): perfect fifth.

Inversion



Interval inversions



Major 13th (compound Major 6th) inverts to a minor 3rd by moving the bottom note

up two octaves, the top note down two octaves, or both notes one octave

A simple interval (i.e., an interval smaller than or equal to an octave) may be inverted by raising the lower pitch an octave, or lowering the upper pitch an octave. For example, the fourth from a lower C to a higher F may be inverted to make a fifth, from a lower F to a higher C.

There are two rules to determine the number and quality of the inversion of any simple interval:

1. The interval number and the number of its inversion always add up to nine ($4 + 5 = 9$, in the example just given).

2. The inversion of a major interval is a minor interval, and vice versa; the inversion of a perfect interval is also perfect; the inversion of an augmented interval is a diminished interval, and vice versa; the inversion of a doubly augmented interval is a doubly diminished interval, and vice versa.

For example, the interval from C to the E[?] above it is a minor third. By the two rules just given, the interval from E[?] to the C above it must be a major sixth.

Since compound intervals are larger than an octave, "the inversion of any compound interval is always the same as the inversion of the simple interval from which it is compounded."

For intervals identified by their ratio, the inversion is determined by reversing the ratio and multiplying by 2. For example, the inversion of a 5:4 ratio is an 8:5 ratio.

For intervals identified by an integer number of semitones, the inversion is obtained by subtracting that number from 12.

Since an interval class is the lower number selected among the interval integer and its inversion, interval classes cannot be inverted.

Classification

Intervals can be described, classified, or compared with each other according to various criteria.



Melodic and harmonic intervals.

Melodic and harmonic

An interval can be described as

- Vertical or harmonic if the two notes sound simultaneously
- Horizontal, linear, or melodic if they sound successively.

Diatonic and chromatic

In general,

- A *diatonic interval* is an interval formed by two notes of a diatonic scale.
- A *chromatic interval* is a non-diatonic interval formed by two notes of a chromatic scale.



Ascending and descending chromatic scale on C.

The table above depicts the 56 diatonic intervals formed by the notes of the C major scale (a diatonic scale). Notice that these intervals, as well as any other diatonic

interval, can be also formed by the notes of a chromatic scale.

The distinction between diatonic and chromatic intervals is controversial, as it is based on the definition of diatonic scale, which is variable in the literature. For example, the interval B-E $\flat$ (a diminished fourth, occurring in the harmonic C-minor scale) is considered diatonic if the harmonic minor scales are considered diatonic as well. Otherwise, it is considered chromatic.

By a commonly used definition of diatonic scale (which excludes the harmonic minor and melodic minor scales), all perfect, major and minor intervals are diatonic. Conversely, no augmented or diminished interval is diatonic, except for the augmented fourth and diminished fifth.



The A $\flat$ -major scale.

The distinction between diatonic and chromatic intervals may be also sensitive to context. The above-mentioned 56 intervals formed by the C-major scale are sometimes called *diatonic to C major*. All other intervals are called *chromatic to C major*. For instance, the perfect fifth A $\flat$ -E $\flat$ is chromatic to C major, because A $\flat$ and E $\flat$ are not contained in the C major scale. However, it is diatonic to others, such as the A $\flat$ major scale.

Consonant and dissonant

Consonance and dissonance are relative terms that refer to the stability, or state of repose, of particular musical effects. Dissonant intervals are those that cause tension, and desire to be *resolved* to consonant intervals.

These terms are relative to the usage of different compositional styles.

- In the Middle Ages, only the unison, octave, perfect fourth, and perfect fifth were considered consonant harmonically.

- In 15th- and 16th-century usage, perfect fifths and octaves, and major and minor thirds and sixths were considered harmonically consonant, and all other intervals dissonant, including the perfect fourth, which by 1473 was described (by Johannes Tinctoris) as dissonant, except between the upper parts of a vertical sonority—for example, with a supporting third below ("6-3 chords"). In the common practice period, it makes more sense to speak of consonant and dissonant chords, and certain intervals previously thought to be dissonant (such as minor sevenths) became acceptable in certain contexts. However, 16th-century practice continued to be taught to beginning musicians throughout this period.

- Hermann von Helmholtz (1821–1894) defined a harmonically consonant interval as one in which the two pitches have an upper partial (an overtone) in common (specifically *excluding* the seventh harmonic). This essentially defines all seconds and sevenths as dissonant, and the above thirds, fourths, fifths, and sixths as consonant.

- Pythagoras defined a hierarchy of consonance based on how small the numbers are that express the ratio. 20th-century composer and theorist Paul Hindemith's system has a hierarchy with the same results as Pythagoras's, but defined by fiat rather than by interval ratios, to better accommodate equal temperament, all of whose intervals (except the octave) would be dissonant using acoustical methods.

- David Cope (1997) suggests the concept of **interval strength**, in which an interval's strength, consonance, or stability is determined by its approximation to a lower and stronger, or higher and weaker, position in the harmonic series. See also: Lipps-Meyer law.

- #Interval root

All of the above analyses refer to vertical (simultaneous) intervals.

Simple and compound



Simple and compound major third.

A simple interval is an interval spanning at most one octave. Intervals spanning more than one octave are called compound intervals.

In general, a compound interval may be defined by a sequence or "stack" of two or more simple intervals of any kind. For instance, a major tenth (two staff positions above one octave), also called *compound major third*, spans one octave plus one major third. A major seventeenth (two staff positions above two octaves) is another example of compound major third, and can be built either by adding up two octaves and one major third, or four perfect fifths.

Any compound interval can be always decomposed into one or more octaves plus one simple interval. For instance, a seventeenth can be always decomposed into two octaves and one major third, and this is the reason why it is called a compound major third, even when it is built using four fifths.

The diatonic number DN_c of a compound interval formed from n simple intervals with diatonic numbers $DN_1, DN_2, \dots, DN_n$, is determined by:

$$DN_c = 1 + (DN_1 - 1) + (DN_2 - 1) + \dots + (DN_n - 1),$$

which can also be written as:

$$DN_c = DN_1 + DN_2 + \dots + DN_n - (n - 1),$$

The quality of a compound interval is determined by the quality of the simple interval on which it is based. For instance, a compound major third is a major tenth ($1+(8-1)+(3-1) = 10$), or a major seventeenth ($1+(8-1)+(8-1)+(3-1) = 17$), and a compound perfect fifth is a perfect twelfth ($1+(8-1)+(5-1) = 12$) or a perfect nineteenth ($1+(8-1)+(8-1)+(5-1) = 19$). Notice that two octaves are a fifteenth, not a sixteenth ($1+(8-1)+(8-1) = 15$). Similarly, three octaves are a twenty-second ($1+3*(8-1) = 22$), and so on.

Intervals larger than a seventeenth seldom need to be spoken of, most often being referred to by their compound names, for example "two octaves plus a fifth" rather than "a 19th".

Steps and skips

Linear (melodic) intervals may be described as *steps* or *skips*. A *step*, or *conjunct motion*, is a linear interval between two consecutive notes of a scale. Any larger interval is called a *skip* (also called a *leap*), or *disjunct motion*. In the diatonic scale, a step is either a minor second (sometimes also called *half step*) or major second (sometimes also called *whole step*), with all intervals of a minor third or larger being skips.

For example, C to D (major second) is a step, whereas C to E (major third) is a skip.

More generally, a step is a smaller or narrower interval in a musical line, and a skip is a wider or larger interval, with the categorization of intervals into steps and skips is determined by the tuning system and the pitch space used.

Melodic motion in which the interval between any two consecutive pitches is no more than a step, or, less strictly, where skips are rare, is called *stepwise* or *conjunct* melodic motion, as opposed to *skipwise* or *disjunct* melodic motions, characterized by frequent skips.

Enharmonic intervals



Enharmonic tritones: A4 = d5 on C.

Two intervals are considered to be *enharmonic*, or *enharmonically equivalent*, if they both contain the same pitches spelled in different ways; that is, if the notes in the two intervals are themselves enharmonically equivalent. Enharmonic intervals span the same number of semitones.

For example, the four intervals listed in the table below are all enharmonically equivalent, because the notes F# and Gb indicate the same pitch, and the same is true for A# and Bb. All these intervals span four semitones.

Number of semitones	Interval name	Staff positions			
		1	2	3	4
4	major third	F#		A#	
4	major third		Gb		Bb
4	diminished fourth	F#			Bb
4	doubly augmented second		Gb	A#	

When played on a piano keyboard, these intervals are indistinguishable as they are all played with the same two keys, but in a musical context the diatonic function of the notes incorporated is very different.

Minute intervals



Pythagorean comma on C. The note depicted as lower on the staff (B#+++) is slightly higher in pitch (than C#).

There are also a number of minute intervals not found in the chromatic scale or labeled with a diatonic function, which have names of their own. They may be described as microtones, and some of them can be also classified as commas, as they describe small discrepancies, observed in some tuning systems, between enharmonically equivalent notes. In the following list, the interval sizes in cents are approximate.

- A *Pythagorean comma* is the difference between twelve justly tuned perfect fifths and seven octaves. It is expressed by the frequency ratio 531441:524288 (23.5 cents).
- A *syntonic comma* is the difference between four justly tuned perfect fifths and two octaves plus a major third. It is expressed by the ratio 81:80 (21.5 cents).
- A *septimal comma* is 64:63 (27.3 cents), and is the difference between the Pythagorean or 3-limit "7th" and the "harmonic 7th".
- A *diesis* is generally used to mean the difference between three justly tuned major thirds and one octave. It is expressed by the ratio 128:125 (41.1 cents). However, it has been used to mean other small intervals: see diesis for details.
- A *diaschisma* is the difference between three octaves and four justly tuned perfect fifths plus two justly tuned major thirds. It is expressed by the ratio 2048:2025 (19.6 cents).
- A *schisma* (also *skhisma*) is the difference between five octaves and eight justly tuned fifths plus one justly tuned major third. It is expressed by the ratio 32805:32768 (2.0 cents). It is also the difference between the Pythagorean and syntonic commas. (A schismic major third is a schisma different from a just major third, eight fifths down and five octaves up, F♭ in C.)
- A *kleisma* is the difference between six minor thirds and one *tritave* or *perfect twelfth* (an octave plus a perfect fifth), with a frequency ratio of 15625:15552 (8.1 cents).
- A *septimal kleisma* is six major thirds up, five fifths down and one octave up, with ratio 225:224 (7.7 cents).
- A *quarter tone* is half the width of a semitone, which is half the width of a whole tone. It is equal to exactly 50 cents.

Intervals in chords

Chords are sets of three or more notes. They are typically defined as the combination of intervals starting from a common note called the root of the chord. For instance a major triad is a chord containing three notes defined by the root and two intervals (major third and perfect fifth). Sometimes even a single interval (dyad) is considered to be a chord. Chords are classified based on the quality and number of the intervals which define them.

Chord qualities and interval qualities

The main chord qualities are: major, minor, augmented, diminished, half-diminished, and dominant. The symbols used for chord quality are similar to those used for interval quality (see above). In addition, **+** or **aug** is used for augmented, **°** or **dim** for diminished, **ø** for half diminished, and **dom** for dominant (the symbol - alone is not used for diminished).

Deducing component intervals from chord names and symbols

The main rules to decode chord *names or symbols* are summarized below.

1. For 3-note chords (triads), major or minor always refer to the interval of the third above the root note, while augmented and diminished always refer to the interval of the fifth above root. The same is true for the corresponding symbols (e.g., Cm means Cm3, and C+ means C+5). Thus, the terms third and fifth and the corresponding symbols 3 and 5 are typically omitted. This rule can be generalized to all kinds of chords,[21] provided the above mentioned qualities appear immediately after the root note, or at the beginning of the chord name or symbol. For instance, in the chord symbols Cm and Cm7, m refers to the interval m3, and 3 is omitted. When these qualities do not appear immediately after the root note, or at the beginning of the name or symbol, they should be considered interval qualities, rather than chord qualities. For instance, in Cm/M7 (minor major seventh chord), m is the chord quality and refers to the m3 interval, while M refers to the M7 interval. When the

number of an extra interval is specified immediately after chord quality, the quality of that interval may coincide with chord quality (e.g., CM7 = CM/M7). However, this is not always true (e.g., Cm6 = Cm/M6, C+7 = C+/m7, CM11 = CM/P11).[21] See main article for further details.

2. Without contrary information, a major third interval and a perfect fifth interval (major triad) are implied. For instance, a C chord is a C major triad, and the name C minor seventh (Cm7) implies a minor 3rd by rule 1, a perfect 5th by this rule, and a minor 7th by definition (see below). This rule has one exception (see next rule).

3. When the fifth interval is diminished, the third must be minor. This rule overrides rule 2. For instance, Cdim7 implies a diminished 5th by rule 1, a minor 3rd by this rule, and a diminished 7th by definition (see below).

4. Names and symbols which contain only a plain interval number (e.g., “Seventh chord”) or the chord root and a number (e.g., “C seventh”, or C7) are interpreted as follows:

- If the number is 2, 4, 6, etc., the chord is a major added tone chord (e.g., C6 = CM6 = Cadd6) and contains, together with the implied major triad, an extra major 2nd, perfect 4th, or major 6th (see names and symbols for added tone chords).
- If the number is 7, 9, 11, 13, etc., the chord is dominant (e.g., C7 = Cdom7) and contains, together with the implied major triad, one or more of the following extra intervals: minor 7th, major 9th, perfect 11th, and major 13th (see names and symbols for seventh and extended chords).
- If the number is 5, the chord (technically not a chord in the traditional sense, but a dyad) is a power chord. Only the root, a perfect fifth and usually an octave are played.

5.

The table shows the intervals contained in some of the main chords (**component intervals**), and some of the symbols used to denote them. The interval qualities or numbers in **boldface** font can be deduced from chord name or symbol by applying rule 1. In symbol examples, C is used as chord root.

Main chords		Component intervals		
Name	Symbol examples	Third	Fifth	Seventh
Major triad	C	maj3	perf5	
	CM, or Cmaj	maj3	perf5	
Minor triad	Cm, or Cmin	min3	perf5	
Augmented triad	C+, or C ^{aug}	maj3	aug5	
Diminished triad	C°, or C ^{dim}	min3	dim5	
Dominant seventh chord	C7, or C ^{dom} 7	maj3	perf5	min7
Minor seventh chord	Cm7, or C ^{min} 7	min3	perf5	min7
Major seventh chord	CM7, or C ^{maj} 7	maj3	perf5	maj7
Augmented seventh chord	C+7, C ^{aug} 7, C7 ^{#5} , or C7 ^{aug5}	maj3	aug5	min7
Diminished seventh chord	C°7, or C ^{dim} 7	min3	dim5	dim7
Half-diminished seventh chord	C ^ø 7, Cm7 ^{b5} , or C ^{min} 7 ^{dim5}	min3	dim5	min7

Size of intervals used in different tuning systems

Number of semitones	Name	5-limit tuning (pitch ratio)	Comparison of interval width (in cents)			
			5-limit tuning	Pythagorean tuning	1/4-comma meantone	Equal temperament
0	Perfect unison	1:1	0	0	0	0
1	Minor second	16:15	112	90	117	100
2	Major second	9:8 10:9	204 182	204	193	200
3	Minor third	6:5 75:64 32:27	316 (wolf) 275 294	294 318	310 (wolf) 269	300
4	Major third	5:4 512:405 32:25 81:64	386 406 (wolf) 427 408	408 384	386 (wolf) 427	400
5	Perfect fourth	4:3 675:512 27:20	498 478 520	498 (wolf) 522	503 (wolf) 462	500
6	Augmented fourth Diminished fifth	45:32 64:45	590 610	612 588	579 621	600
7	Perfect fifth	3:2 40:27 1024:675	702 680 722	702 (wolf) 678	697 (wolf) 738	700
8	Minor sixth	8:5	814	792	814	800
9	Major sixth	5:3	884	906	890	900
10	Minor seventh	9:5 16:9	1018 996	996	1007	1000
11	Major seventh	15:8	1088	1110	1083	1100
12	Perfect octave	2:1	1200	1200	1200	1200

In this table, the interval widths used in four different tuning systems are compared. To facilitate comparison, just intervals as provided by 5-limit tuning (see symmetric scale n.1) are shown in **bold** font, and the values in cents are rounded to integers. Notice that in each of the non-equal tuning systems, by definition the width of *each* type of interval (including the semitone) changes depending on the note from which the interval starts. This is the price paid for seeking just intonation. However, for the sake of simplicity, for some types of interval the table shows only one value (the most often observed one).

In 1/4-comma meantone, by definition 11 perfect fifths have a size of approximately 697 cents ($700 - \sum$ cents, where $\sum \approx 3.42$ cents); since the average size of the 12 fifths must equal exactly 700 cents (as in equal temperament), the other one must have a size of about 738 cents ($700 + 11\sum$, the wolf fifth or diminished sixth); 8 major thirds

have size about 386 cents ($400-4\Sigma$), 4 have size about 427 cents ($400+8\Sigma$, actually diminished fourths), and their average size is 400 cents. In short, similar differences in width are observed for all interval types, except for unisons and octaves, and they are all multiples of Σ (the difference between the 1/4-comma meantone fifth and the average fifth). Note that 1/4-comma meantone was designed to produce just major thirds, but only 8 of them are just (5:4, about 386 cents).

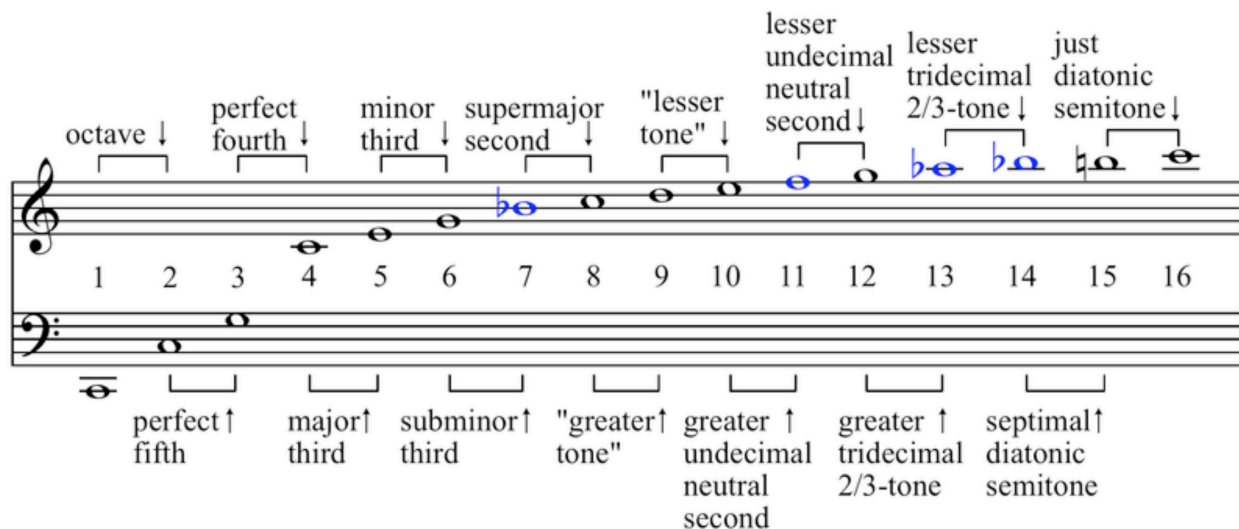
The Pythagorean tuning is characterized by smaller differences because they are multiples of a smaller Σ ($\Sigma \approx 1.96$ cents, the difference between the Pythagorean fifth and the average fifth). Notice that here the fifth is wider than 700 cents, while in most meantone temperaments, including 1/4-comma meantone, it is tempered to a size smaller than 700.

The 5-limit tuning system uses just tones and semitones as building blocks, rather than a stack of perfect fifths, and this leads to even more varied intervals throughout the scale (each kind of interval has three or four different sizes). Note that 5-limit tuning was designed to maximize the number of just intervals, but even in this system some intervals are not just (e.g., 3 fifths, 5 major thirds and 6 minor thirds are not just; also, 3 major and 3 minor thirds are wolf intervals).

The above mentioned symmetric scale 1, defined in the 5-limit tuning system, is not the only method to obtain just intonation. It is possible to construct juster intervals or just intervals closer to the equal-tempered equivalents, but most of the ones listed above have been used historically in equivalent contexts. In particular, the asymmetric version of the 5-limit tuning scale provides a juster value for the minor seventh (9:5, rather than 16:9). Moreover, the tritone (augmented fourth or diminished fifth), could have other just ratios; for instance, 7:5 (about 583 cents) or 17:12 (about 603 cents) are possible alternatives for the augmented fourth (the latter is fairly common, as it is closer to the equal-tempered value of 600 cents). The 7:4 interval (about 969 cents), also known as the harmonic seventh, has been a contentious issue throughout the history of music theory; it is 31 cents flatter than an equal-tempered minor seventh. Some assert the 7:4 is one of the blue notes used in jazz.

In the diatonic system, every interval has one or more *enharmonic equivalents*, such as augmented second for minor third.

Interval root



Intervals in the harmonic series.

Although intervals are usually designated in relation to their lower note, David Cope and Hindemith both suggest the concept of **interval root**. To determine an interval's root, one locates its nearest approximation in the harmonic series. The root

of a perfect fourth, then, is its *top* note because it is an octave of the fundamental in the hypothetical harmonic series. The bottom note of every odd diatonically numbered intervals are the roots, as are the tops of all even numbered intervals. The root of a collection of intervals or a chord is thus determined by the interval root of its strongest interval.

As to its usefulness, Cope provides the example of the final tonic chord of some popular music being traditionally analyzable as a "submediant six-five chord" (added sixth chords by popular terminology), or a first inversion seventh chord (possibly the dominant of the mediant V/iii). According the interval root of the strongest interval of the chord (in first inversion, CEGA), the perfect fifth (C-G), is the bottom C, the tonic.

Interval cycles

Interval cycles, "unfold [i.e., repeat] a single recurrent interval in a series that closes with a return to the initial pitch class", and are notated by George Perle using the letter "C", for cycle, with an interval-class integer to distinguish the interval. Thus the diminished-seventh chord would be C3 and the augmented triad would be C4. A superscript may be added to distinguish between transpositions, using 0-11 to indicate the lowest pitch class in the cycle.

Alternative interval naming conventions

As shown below, some of the above mentioned intervals have alternative names, and some of them take a specific alternative name in Pythagorean tuning, five-limit tuning, or meantone temperament tuning systems such as quarter-comma meantone. All the intervals with prefix *sesqui-* are justly tuned, and their frequency ratio, shown in the table, is a superparticular number (or epimoric ratio). The same is true for the octave.

Typically, a comma is a diminished second, but this is not always true (for more details, see Alternative definitions of comma). For instance, in Pythagorean tuning the diminished second is a descending interval (524288:531441, or about -23.5 cents), and the Pythagorean comma is its opposite (531441:524288, or about 23.5 cents). 5-limit tuning defines four kinds of comma, three of which meet the definition of diminished second, and hence are listed in the table below. The fourth one, called syntonic comma (81:80) can neither be regarded as a diminished second, nor as its opposite.

Number of semitones	Generic names			Specific names		
	Quality and number		Other naming convention	Pythagorean tuning	5-limit tuning	1/4-comma meantone
	Full	Short				
0	perfect unison or perfect prime	P1				
	diminished second	d2		descending Pythagorean comma (524288:531441)	lesser diesis (128:125) diaschisma (2048:2025) greater diesis (648:625)	
1	minor second	m2	semitone diatonic semitone, minor semitone	limma (256:243)		
	augmented unison or augmented prime	A1	half tone, half step chromatic semitone, major semitone	apotome (2187:2048)		
2	major second	M2	tone, whole tone , whole step	sesquioctavum (9:8)		
3	minor third	m3			sesquiquintum (6:5)	
4	major third	M3			sesquiquartum (5:4)	
5	perfect fourth	P4		sesquitertium (4:3)		
6	diminished fifth	d5	tritone ^[6]			
	augmented fourth	A4				
7	perfect fifth	P5		sesquialterum (3:2)		
12	perfect octave	P8		duplex (2:1)		

Additionally, some cultures around the world have their own names for intervals found in their music. For instance, 22 kinds of intervals, called shrutis, are canonically defined in Indian classical music.

Latin nomenclature

Up to the end of the 18th century, Latin was used as an official language throughout Europe for scientific and music textbooks. In music, many English terms are derived from Latin. For instance, semitone is from Latin *semitonus*.

The prefix semi- is typically used herein to mean "shorter", rather than "half". Namely, a semitonus, semiditonus, semidiatessaron, semidiapente, semihexachordum, semiheptachordum, or semidiapason, is shorter by one semitone than the corresponding whole interval. For instance, a semiditonus (3 semitones, or about 300 cents) is not half of a ditonus (4 semitones, or about 400 cents), but a ditonus shortened by one semitone. Moreover, in Pythagorean tuning (the most commonly used tuning system up to the 16th century), a semitritonus (d5) is smaller than a tritonus (A4) by one Pythagorean comma (about a quarter of a semitone).

Number of semitones	Quality and number	Short	Latin nomenclature
0	Perfect unison	P1	unisonus
1	Minor second	m2	semitonus
	Augmented unison	A1	unisonus superflua
2	Major second	M2	tonus
	Diminished third	d3	
3	Minor third	m3	semiditonus
	Augmented second	A2	tonus superflua
4	Major third	M3	ditonus
	Diminished fourth	d4	semidiatessaron
5	Perfect fourth	P4	diatessaron
	Augmented third	A3	ditonus superflua
6	Diminished fifth	d5	semidiapente, semitritonus
	Augmented fourth	A4	tritonus
7	Perfect fifth	P5	diapente
	Diminished sixth	d6	semihexachordum
8	Minor sixth	m6	hexachordum minus, semitonus maius cum diapente, tetratonus
	Augmented fifth	A5	diapente superflua
9	Major sixth	M6	hexachordum maius, tonus cum diapente
	Diminished seventh	d7	semiheptachordum
10	Minor seventh	m7	heptachordum minus, semiditonus cum diapente, pentatonus
	Augmented sixth	A6	hexachordum superflua
11	Major seventh	M7	heptachordum maius, ditonus cum diapente
	Diminished octave	d8	semidiapason
12	Perfect octave	P8	diapason
	Augmented seventh	A7	heptachordum superflua

Pitch-class intervals

In post-tonal or atonal theory, originally developed for equal-tempered European classical music written using the twelve-tone technique or serialism, integer notation is often used, most prominently in musical set theory. In this system, intervals are named according to the number of half steps, from 0 to 11, the largest interval class being 6.

In atonal or musical set theory, there are numerous types of intervals, the first being the ordered pitch interval, the distance between two pitches upward or downward. For instance, the interval from C upward to G is 7, and the interval from G downward

to C is 7. One can also measure the distance between two pitches without taking into account direction with the unordered pitch interval, somewhat similar to the interval of tonal theory.

The interval between pitch classes may be measured with ordered and unordered pitch-class intervals. The ordered one, also called directed interval, may be considered the measure upwards, which, since we are dealing with pitch classes, depends on whichever pitch is chosen as 0.

Generic and specific intervals

In diatonic set theory, specific and generic intervals are distinguished. Specific intervals are the interval class or number of semitones between scale steps or collection members, and generic intervals are the number of diatonic scale steps (or staff positions) between notes of a collection or scale.

Notice that staff positions, when used to determine the conventional interval number (second, third, fourth, etc.), are counted including the position of the lower note of the interval, while generic interval numbers are counted excluding that position. Thus, generic interval numbers are smaller by 1, with respect to the conventional interval numbers.

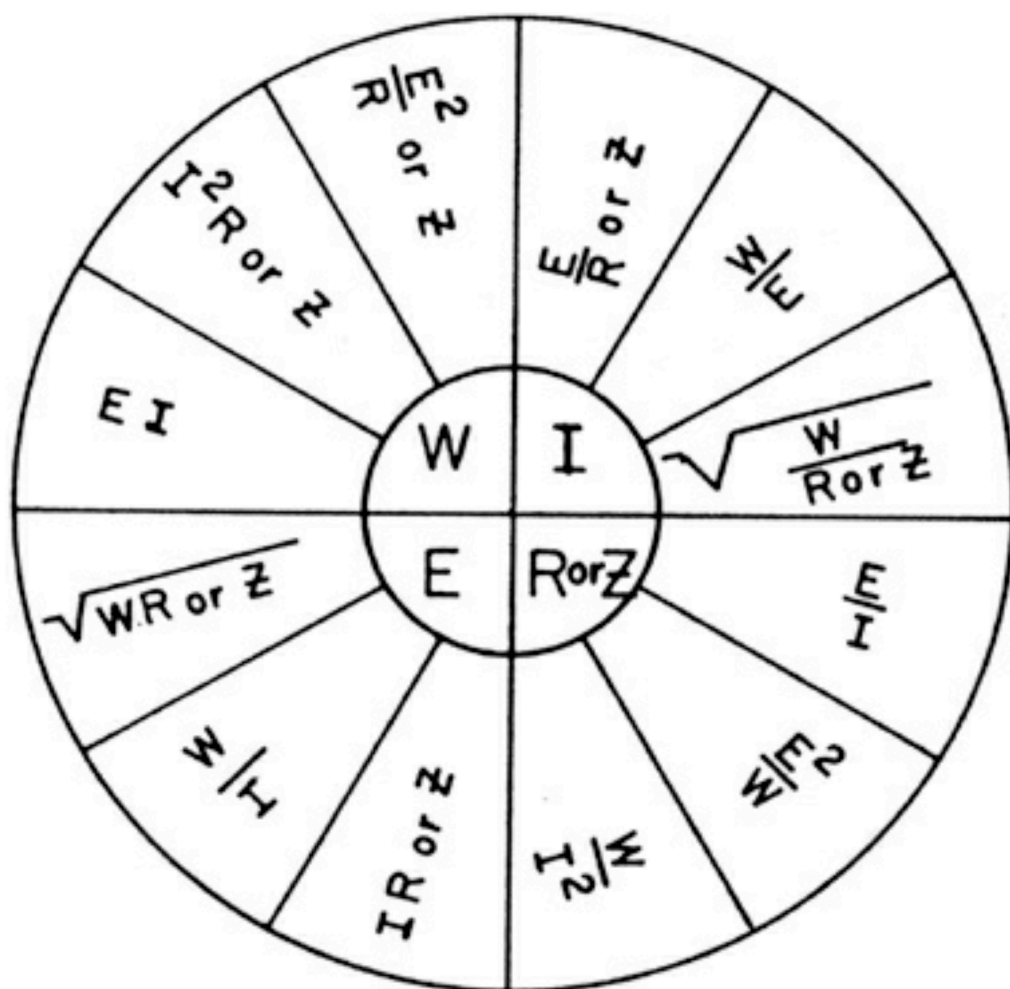
<u>Specific interval</u>			
Number of semitones	<u>Interval class</u>		
		<u>Generic interval</u>	Diatonic name
0	0	0	Perfect unison
1	1	1	Minor second
2	2	1	Major second
3	3	2	Minor third
4	4	2	Major third
5	5	3	Perfect fourth
6	6	3 4	Augmented fourth Diminished fifth
7	5	4	Perfect fifth
8	4	5	Minor sixth
9	3	5	Major sixth
10	2	6	Minor seventh
11	1	6	Major seventh
12	0	7	Perfect octave

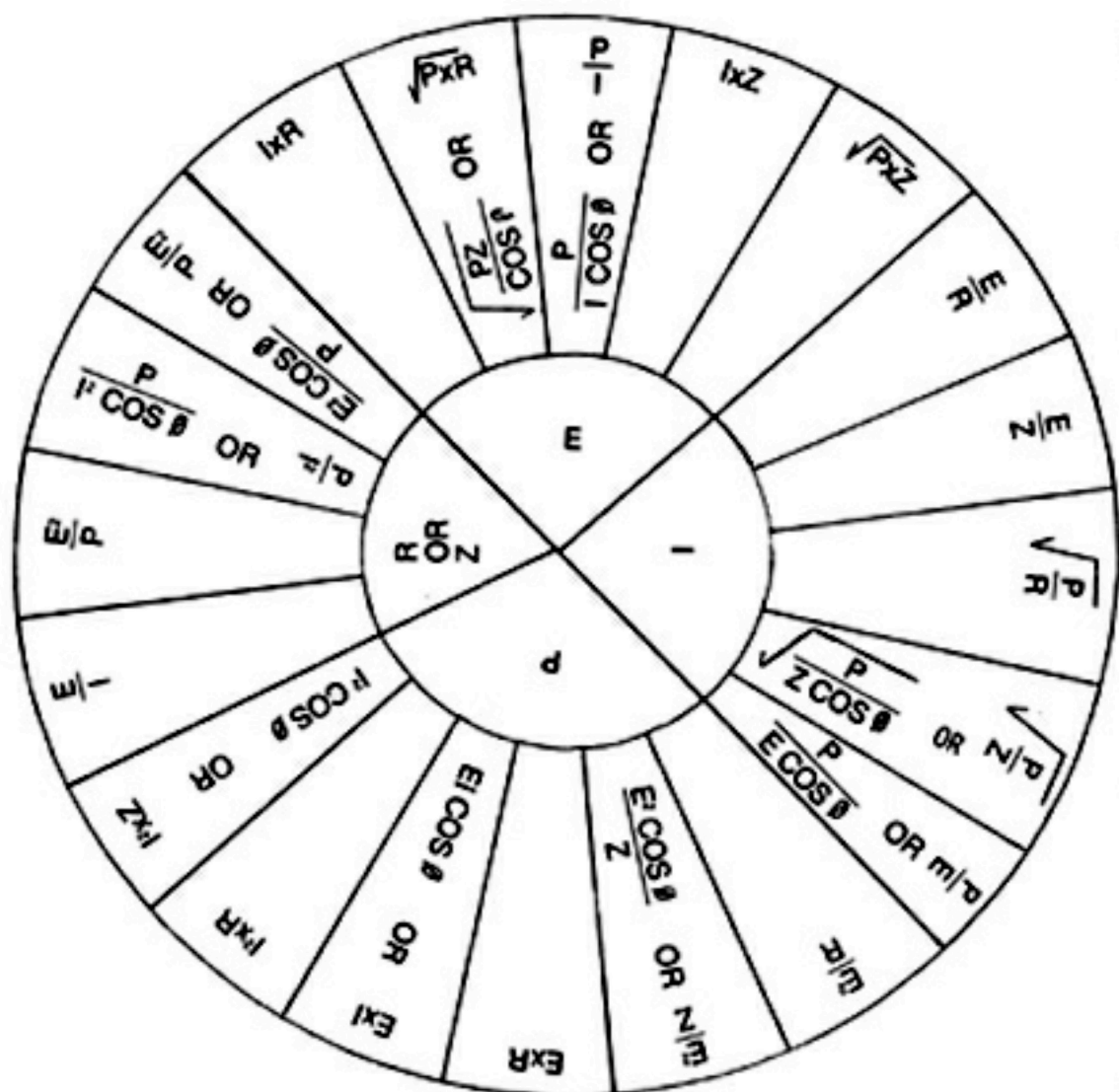
Generalizations and non-pitch uses

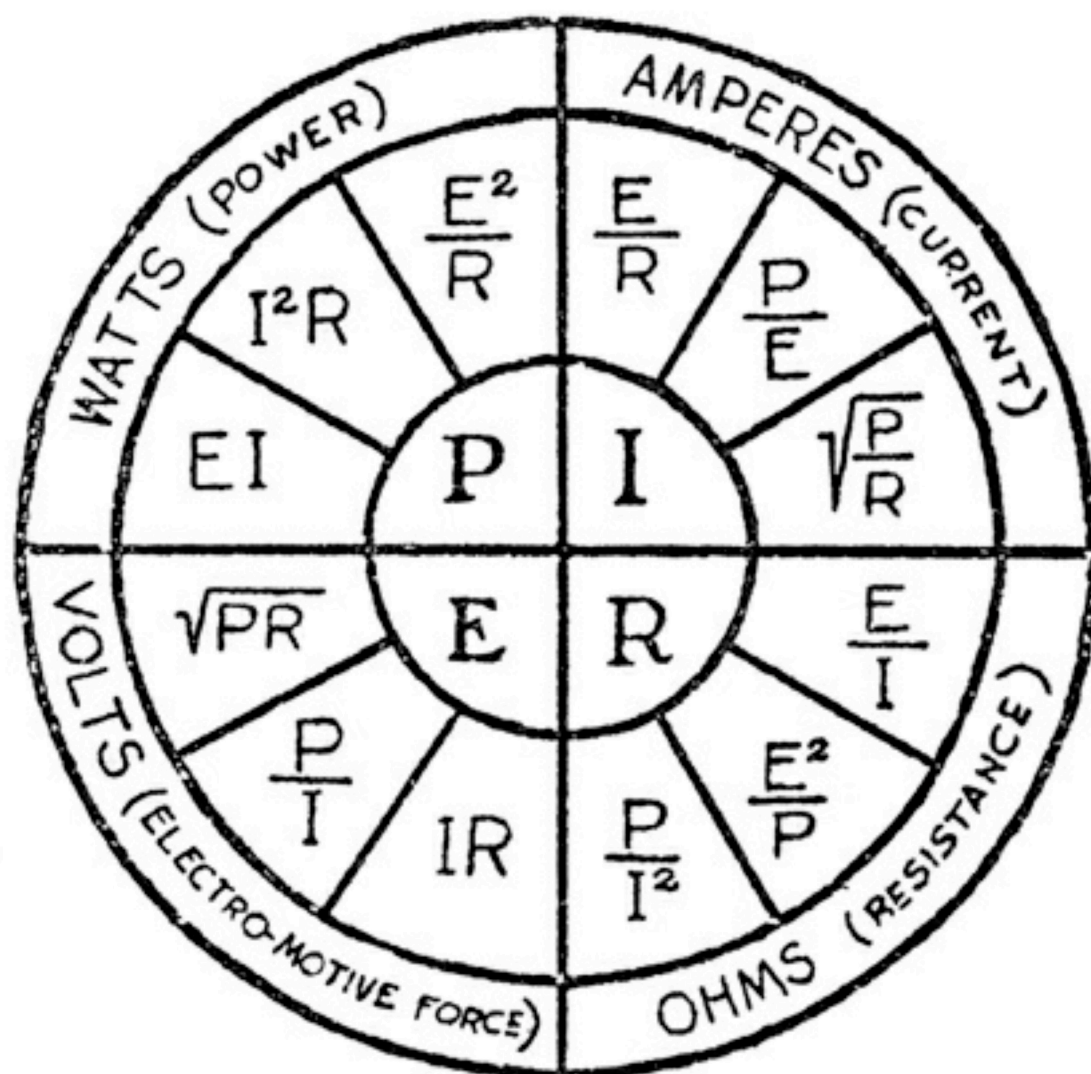


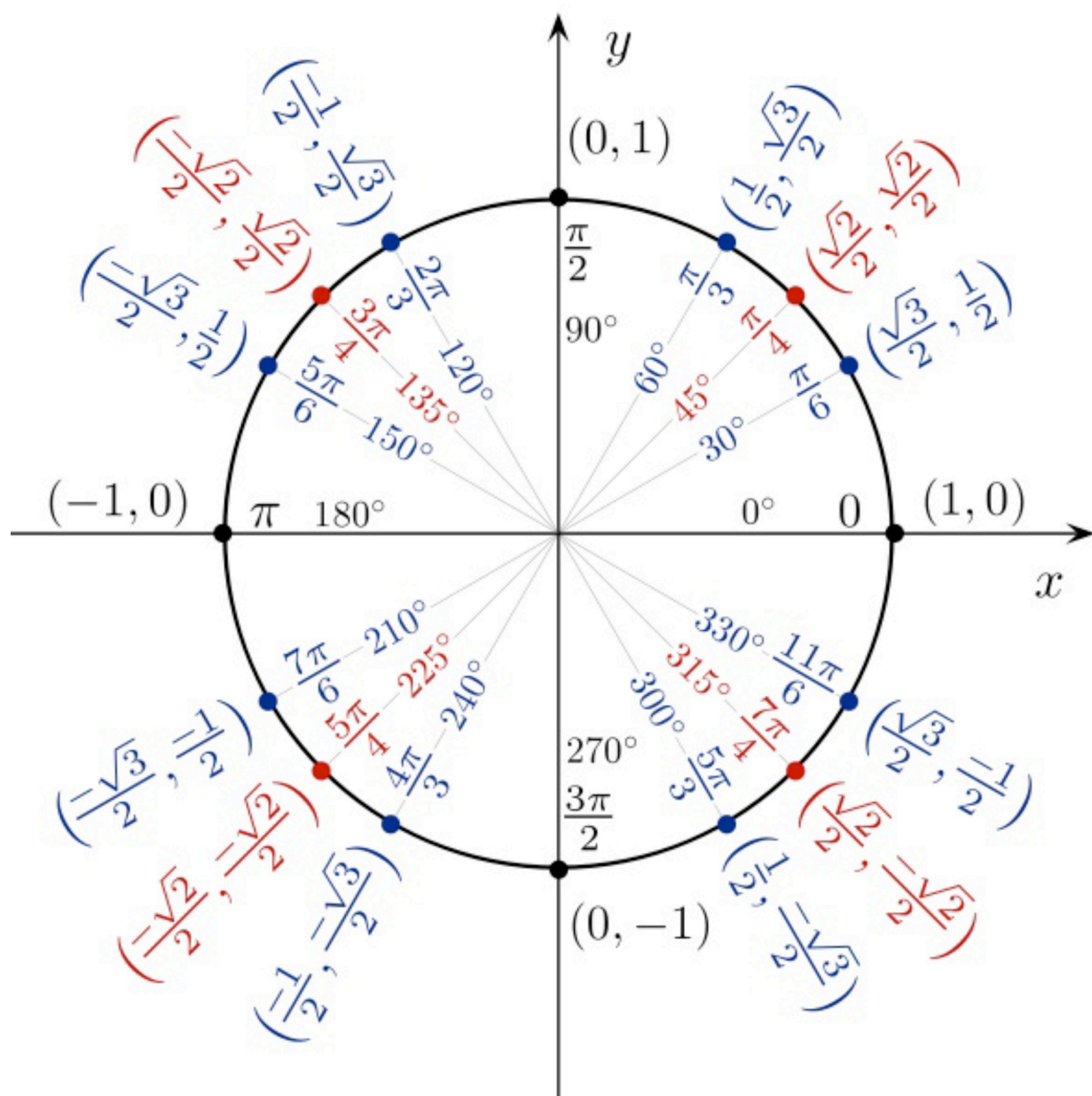
Division of the measure/chromatic scale, followed by pitch/time-point series.

The term "interval" can also be generalized to other music elements besides pitch. David Lewin's *Generalized Musical Intervals and Transformations* uses interval as a generic measure of distance between time points, timbres, or more abstract musical phenomena.

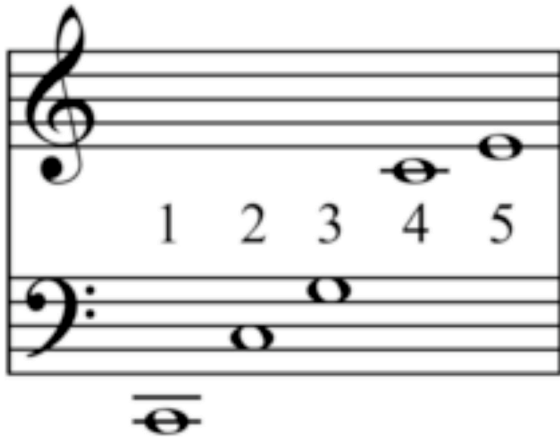








Just intonation



Harmonic series, partials 1-5 numbered

In music, **just intonation** (sometimes abbreviated as **JI**) or **pure intonation** is any musical tuning in which the frequencies of notes are related by ratios of small whole numbers. Any interval tuned in this way is called a **pure** or **just interval**. The two notes in any just interval are members of the same harmonic series. Frequency ratios involving large integers such as 1024:927 are not generally said to be justly tuned. "Just intonation is the tuning system of the later ancient Greek modes as codified by Ptolemy; it was the aesthetic ideal of the Renaissance theorists; and it is the tuning practice of a great many musical cultures worldwide, both ancient and modern."

Just intonation can be contrasted and compared with equal temperament, which dominates Western instruments of fixed pitch and default MIDI tuning. In equal temperament, all notes are defined as multiples of the same basic interval. Two notes separated by the same number of steps always have exactly the same frequency ratio. However, except for doubled frequencies (octaves), no other intervals are exact ratios of integers. Each just interval differs a different amount from its analogous, equally tempered interval.

Justly tuned intervals can be written as either ratios, with a colon (for example, 3:2), or as fractions, with a solidus ($3/2$). For example, two tones, one at 300 Hertz (cycles per second), and the other at 200 hertz are both multiples of 100 Hz and as such members of the harmonic series built on 100 Hz.

Examples

Just intonation An A-major scale, followed by three major triads, and then a progression of fifths in just intonation.

Equal temperament An A-major scale, followed by three major triads, and then a progression of fifths in equal temperament. By listening to the above file, and then listening to this one, one might be able to hear a slight buzzing in this file.

Equal temperament and just intonation compared A pair of major thirds, followed by a pair of full major chords. The first in each pair is in equal temperament; the second is in just intonation. Piano sound.

Equal temperament and just intonation compared with square waveform A pair of major chords. The first is in equal temperament; the second is in just intonation. The pair of chords is repeated with a transition from equal temperament to just temperament between the two chords. In the equal temperament chords a roughness or beating can be heard at about 4 Hz and about 0.8 Hz. In the just intonation triad

this roughness is absent. The square waveform makes the difference between equal and just temperaments more obvious.

History

Origins

Harmonic intervals come naturally to horns and vibrating strings.

Recorded history

Pythagorean tuning, perhaps the first tuning system to be theorized in the West,[2] is a system in which all tones can be found using powers of the ratio 3:2, an interval known as a perfect fifth. It is easier to think of this system as a cycle of fifths. Because a series of 12 fifths with ratio 3:2 does not reach the same pitch class it began with, this system uses a wolf fifth at the end of the cycle, to obtain its closure.

Quarter-comma meantone obtained a more consonant tuning of the major and minor thirds, but when limited to twelve keys (see split keys), the system does not close, leaving a very dissonant diminished sixth between the first and last tones of the cycle of fifths.

In Pythagorean tuning, the only highly consonant intervals were the perfect fifth and its inversion, the perfect fourth. The Pythagorean major third (81:64) and minor third (32:27) were dissonant, and this prevented musicians from using triads and chords, forcing them for centuries to write music with relatively simple texture. In late Middle Ages, musicians realized that by slightly tempering the pitch of some notes, the Pythagorean thirds could be made consonant. For instance, if one decreases by a syntonic comma (81:80) the frequency of E, C-E (a major third), and E-G (a minor third) become just. Namely, C-E is flattened to a justly intonated ratio of

$$(81:64) \times (80:81) = 5:4$$

and at the same time E-G is sharpened to the just ratio of

$$(32:27) \times (81:80) = 6:5$$

The drawback is that the fifths A-E and E-B, by flattening E, become almost as dissonant as the Pythagorean wolf fifth. But the fifth C-G stays consonant, since only E has been flattened (C-E * E-G = (5:4) * (6:5) = 3:2), and can be used together with C-E to produce a C-major triad (C-E-G).

By generalizing this simple rationale, Gioseffo Zarlino, in the late sixteenth century, created the first justly intonated 7-tone (diatonic) scale, which contained pure perfect fifths (3:2), pure major thirds, and pure minor thirds:

F -> A -> C -> E -> G -> B -> D

This is a sequence of just major thirds (M3, ratio 5:4) and just minor thirds (m3, ratio 6:5), starting from F:

F + M3 + m3 + M3 + m3 + M3 + m3

Since M3 + m3 = P5 (perfect fifth), i.e. (5:4) * (6:5) = 3:2, this is exactly equivalent to the diatonic scale obtained in 5-limit just intonation.

The Guqin has a musical scale based on harmonic overtone positions. The dots on its soundboard indicate the harmonic positions: 1/8, 1/6, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 5/6, 7/8.

Modern practice

The image displays two musical staves illustrating just intonation scales. The top staff is for C major (C utonal) and the bottom staff is for E-flat major (Eb-otonal). Both staves show the sequence of notes and their corresponding ratios.

Top Staff (C utonal):

- Notes: C, D, E, F, G, A, B, C
- Ratios: 1/1, 9/8, 5/4, 4/3, 3/2, 8/5, 16/13, 1/1
- Labels: (C utonal)

Bottom Staff (Eb-otonal):

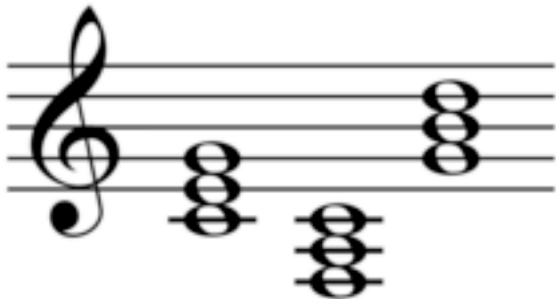
- Notes: Eb, F, G, Ab, Bb, C, D, Eb
- Ratios: 1/1, 16/11, 8/5, 16/13, 4/3, 3/2, 5/4, 13/8, 7/4, 11/8, 1/1
- Labels: (Eb-otonal)

Primary forms of the just tone row from Ben Johnston's *String Quartet No. 7*, mov. 2. Each permutation contains a just chromatic scale, however, transformations (transposition and inversion) produce pitches outside of the primary row form, as already occurs in the inversion of P0.

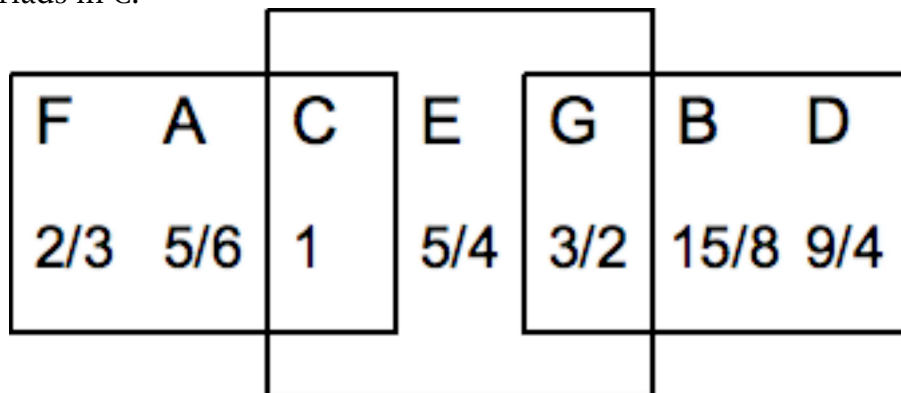
Today, despite the dominance of repertoire composed under equal-tempered systems and the prominence of the piano in musical training, musicians often approach just intonation either by accident or design because it is much easier to find (and hear) a point of stability than a point of calculated instability. A cappella groups that depend on close harmonies, such as barbershop quartets, usually use just intonation by design. Bagpipes, tuned correctly, also use just intonation. There are several conventionally used instruments which, while not associated specifically with just intonation, can handle it quite well, including the trombone and the violin family of instruments.

Diatonic scale

It is possible to tune the familiar diatonic scale or chromatic scale in just intonation in many ways, all of which make certain chords purely tuned and as consonant and stable as possible, and the other chords not accommodated and considerably less stable.



Primary triads in C.



Just tuned diatonic scale derivation.

The prominent notes of a given scale are tuned so that their frequencies form ratios of relatively small integers. For example, in the key of G major, the ratio of the frequencies of the notes G to D (a perfect fifth) is $\frac{3}{2}$, while that of G to C (a perfect fourth) is $\frac{4}{3}$. Three basic intervals can be used to construct any interval involving the prime numbers 2, 3, and 5 (known as *5-limit just intonation*):

- $16:15 = s$ (Semitone)
- $10:9 = t$ (Minor tone)
- $9:8 = T$ (Major tone)

which combine to form:

- 6:5 = Ts (minor third)
- 5:4 = Tt (major third)
- 4:3 = Tts (perfect fourth)
- 3:2 = TTts (perfect fifth)
- 2:1 = TTTtss (octave)

A just diatonic scale may be derived as follows. Suppose we insist that the chords F-A-C, C-E-G, and G-B-D be just major triads (then A-C-E and E-G-B are just minor triads, but D-F-A is not).

Then we obtain this scale (Ptolemy's intense diatonic scale):

Note	Name	C	D	E	F	G	A	B	C
	Ratio	1/1	9/8	5/4	4/3	3/2	5/3	15/8	2/1
	Natural	24	27	30	32	36	40	45	48
	Cents	0	204	386	498	702	884	1088	1200
Step	Name		T	t	s	T	t	T	s
	Ratio		9/8	10/9	16/15	9/8	10/9	9/8	16/15
	Cents		204	182	112	204	182	204	112

The major thirds are correct, and two minor thirds are right, but D-F is a 32:27 semiditone. Others approaches are possible (see Five-limit tuning), but it is impossible to get all six above-mentioned chords correct. Concerning triads, the triads on I, IV, and V are 4:5:6, the triad on ii is 27:32:40, the triads on iii and vi are 10:12:15, and the triad on vii is 45:54:64.

Twelve tone scale

There are several ways to create a just tuning of the twelve tone scale.

Pythagorean tuning

The oldest known form of tuning, Pythagorean tuning, can produce a twelve tone scale, but it does so by involving ratios of very large numbers, corresponding to natural harmonics very high in the harmonic series that do not occur widely in physical phenomena. This tuning uses ratios involving only powers of 3 and 2, creating a sequence of just fifths or fourths, as follows:

Note	G $\flat$	D $\flat$	A $\flat$	E $\flat$	B $\flat$	F	C	G	D	A	E	B	F $\sharp$
Ratio	1024:729	256:243	128:81	32:27	16:9	4:3	1:1	3:2	9:8	27:16	81:64	243:128	729:512
Cents	588	90	792	294	996	498	0	702	204	906	408	1110	612

The ratios are computed with respect to C (the *base note*). Starting from C, they are obtained by moving six steps to the left and six to the right. Each step consists of a multiplication of the previous pitch by 2/3 (descending fifth), 3/2 (ascending fifth), or their inversions (3/4 or 4/3).

Between the enharmonic notes at both ends of this sequence, is a difference in pitch of nearly 24 cents, known as the Pythagorean comma. To produce a twelve tone scale, one of them is arbitrarily discarded. The twelve remaining notes are repeated by increasing or decreasing their frequencies by a multiple of 2 (the size of one or more octaves) to build scales with multiple octaves (such as the keyboard of a piano). A drawback of Pythagorean tuning is that one of the twelve fifths in this scale is badly tuned and hence unusable (the wolf fifth, either F $\sharp$ -D $\flat$ if G $\flat$ is discarded, or B-G $\flat$ if F $\sharp$ is discarded). This twelve tone scale is fairly close to equal temperament, but it does not offer much advantage for tonal harmony because only the perfect intervals

(fourth, fifth, and octave) are simple enough to sound pure. Major thirds, for instance, receive the rather unstable interval of 81/64, sharp of the preferred 5/4 by an 81/80 ratio. The primary reason for its use is that it is extremely easy to tune, as its building block, the perfect fifth, is the simplest and consequently the most consonant interval after the octave and unison.

Pythagorean tuning may be regarded as a "3-limit" tuning system, because the ratios are obtained by using only powers of n , where n is at most 3.

Quarter-comma meantone

The quarter-comma meantone tuning system uses a similar sequence of fifths to produce a twelve tone scale. However, it flattens the fifths by about 5.38 cents with respect to their just intonation, in order to generate justly tuned major thirds (with interval ratio 5:4).

Although this tuning system is based on a just ratio (5:4), it cannot be considered a just intonation system, because most of its intervals are irrational numbers (i.e. they cannot be expressed as fractions of integers). For instance:

- the ratio of most semitones is $S = 8 : 5^{5/4}$,
- the ratio of most tones is $T = \sqrt{5} : 2$,
- the ratio of most fifths is $P = 5^{1/4}$.

Five-limit tuning

A twelve tone scale can also be created by compounding harmonics up to the fifth. Namely, by multiplying the frequency of a given reference note (the **base note**) by powers of 2, 3, or 5, or a combination of them. This method is called five-limit tuning.

To build such a twelve tone scale, we may start by constructing a table containing fifteen pitches:

Factor		1/9	1/3	1	3	9
5	note	D	A	E	B	F#
	ratio	10/9	5/3	5/4	15/8	45/32
	cents	182	884	386	1088	590
1	note	B $\flat$	F	C	G	D
	ratio	16/9	4/3	1	3/2	9/8
	cents	996	498	0	702	204
1/5	note	G $\flat$	D $\flat$	A $\flat$	E $\flat$	B $\flat$
	ratio	64/45	16/15	8/5	6/5	9/5
	cents	610	112	814	316	1018

The factors listed in the first row and column are powers of 3 and 5, respectively (e.g., $1/9 = 3^{-2}$). Colors indicate couples of enharmonic notes with almost identical pitch. The ratios are all expressed relative to C in the centre of this diagram (the base note for this scale). They are computed in two steps:

1. For each cell of the table, a *base ratio* is obtained by multiplying the corresponding factors. For instance, the base ratio for the lower-left cell is $1/9 \times 1/5 = 1/45$.

2. The base ratio is then multiplied by a negative or positive power of 2, as large as needed to bring it within the range of the octave starting from C (from 1/1 to 2/1). For instance, the base ratio for the lower left cell (1/45) is multiplied by 26, and the resulting ratio is 64/45, which is a number between 1/1 and 2/1.

Note that the powers of 2 used in the second step may be interpreted as ascending or descending octaves. For instance, multiplying the frequency of a note by 26 means increasing it by 6 octaves. Moreover, each row of the table may be considered to be a sequence of fifths (ascending to the right), and each column a sequence of major thirds (ascending upward). For instance, in the first row of the table, there is an ascending fifth from D and A, and another one (followed by a descending octave) from A to E. This suggests an alternative but equivalent method for computing the same ratios. For instance, one can obtain A, starting from C, by moving one cell to the left and one upward in the table, which means descending by a fifth and ascending by a major third:

$$\frac{2}{3} \cdot \frac{5}{4} = \frac{10}{12} = \frac{5}{6}.$$

Since this is below C, one needs to move up by an octave to end up within the desired range of ratios (from 1/1 to 2/1):

$$\frac{5}{6} \cdot \frac{2}{1} = \frac{10}{6} = \frac{5}{3}.$$

A 12 tone scale is obtained by removing one note for each couple of enharmonic notes. This can be done in at least three ways, which have in common the removal of G?, according to a convention which was valid even for C-based Pythagorean and 1/4-comma meantone scales. We show here only one of the possible strategies (the others are discussed in Five-limit tuning). It consists of discarding the first column of the table (labeled "1/9"). The resulting 12-tone scale is shown below:

Factor		1/9	1/3	1	3	9
5	note	D	A	E	B	F#
	ratio	10/9	5/3	5/4	15/8	45/32
	cents	182	884	386	1088	590
1	note	Bb	F	C	G	D
	ratio	16/9	4/3	1	3/2	9/8
	cents	996	498	0	702	204
1/5	note	Gb	Db	Ab	Eb	Bb
	ratio	64/45	16/15	8/5	6/5	9/5
	cents	610	112	814	316	1018

Extension of the twelve tone scale

The table above uses only low powers of 3 and 5 to build the base ratios. However, it can be easily extended by using higher positive and negative powers of the same numbers, such as $5^2 = 25$, $5^{-2} = 1/25$, $3^3 = 27$, or $3^{-3} = 1/27$. A scale with 25, 35 or even more pitches can be obtained by combining these base ratios (see Five-limit tuning

for further details).

Indian scales

In Indian music, the just diatonic scale described above is used, though there are different possibilities, for instance for the 6th pitch (*Dha*), and further modifications may be made to all pitches excepting *Sa* and *Pa*.

Note	Sa	Re	Ga	Ma	Pa	Dha	Ni	Sa
Ratio	1/1	9/8	5/4	4/3	3/2	5/3 or 27/16	15/8	2/1
Cents	0	204	386	498	702	884 or 906	1088	1200

Some accounts of Indian intonation system cite a given 22 Srutis. According to some musicians, one has a scale of a given 12 pitches and ten in addition (the tonic, *Shadja* (*Sa*), and the pure fifth, *Pancham* (*Pa*), are inviolate):

Note	C	D ^b	D ^b	D	D	E ^b	E ^b	E	E	F	F	F [#]
Ratio	$\frac{1}{1}$	$\frac{256}{243}$	$\frac{16}{15}$	$\frac{10}{9}$	$\frac{9}{8}$	$\frac{32}{27}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{27}{20}$	$\frac{45}{32}$
Cents	0	90	112	182	204	294	316	386	408	498	520	590

Note	F [#]	G	A ^b	A ^b	A	A	B ^b	B ^b	B	B	C
Ratio	$\frac{64}{45}$	$\frac{3}{2}$	$\frac{128}{81}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{27}{16}$	$\frac{16}{9}$	$\frac{9}{5}$	$\frac{15}{8}$	$\frac{243}{128}$	$\frac{2}{1}$
Cents	610	702	792	814	884	906	996	1018	1088	1110	1200

Where we have *two* ratios for a given letter name, we have a difference of 81:80 (or 22 cents), which is known as the syntonic comma. One can see the symmetry, looking at it from the tonic, then the octave.

(This is just one example of "explaining" a 22-Sruti scale of tones. There are many different explanations.)

Practical difficulties

Some fixed just intonation scales and systems, such as the diatonic scale above, produce wolf intervals. The above scale allows a minor tone to occur next to a semitone which produces the awkward ratio 32:27 for F:D, and still worse, a minor tone next to a fourth giving 40:27 for A:D. Moving D down to 10/9 alleviates these difficulties but creates new ones: G:D becomes 27:20, and B:G becomes 27:16.

One can have more frets on a guitar to handle both A's, 9/8 with respect to G and 10/9 with respect to G so that C:A can be played as 6:5 while D:A can still be played as 3:2. 9/8 and 10/9 are less than 1/53 octave apart, so mechanical and performance considerations have made this approach extremely rare. And the problem of how to tune chords such as C-E-G-A-D is left unresolved (for instance, A could be 4:3 below D (making it 9/8, if G is 1) or 4:3 above E (making it 10/9, if G is 1) but not both at the same time, so one of the fourths in the chord will have to be an out-of-tune wolf interval). However the frets may be removed entirely—this, unfortunately, makes in-tune fingering of many chords exceedingly difficult, due to the construction and mechanics of the human hand—and the tuning of most complex chords in just intonation is generally ambiguous.

Some composers deliberately use these wolf intervals and other dissonant intervals as a way to expand the tone color palette of a piece of music. For example, the extended piano pieces *The Well-Tuned Piano* by LaMonte Young, and *The Harp Of New Albion* by Terry Riley use a combination of very consonant and dissonant intervals for

musical effect. In "Revelation," Michael Harrison goes even farther, and uses the tempo of beat patterns produced by some dissonant intervals as an integral part of several movements.

For many instruments tuned in just intonation, one can not change keys without retuning the instrument. For instance, a piano tuned in just intonation intervals and a minimum of wolf intervals for the key of G, then only one other key (typically E-flat) can have the same intervals, and many of the keys have a very dissonant and unpleasant sound. This makes modulation within a piece, or playing a repertoire of pieces in different keys, impractical to impossible.

Synthesizers have proven a valuable tool for composers wanting to experiment with just intonation. Many commercial synthesizers provide the ability to use built-in just intonation scales or to program your own. Wendy Carlos used a system on her 1986 album *Beauty in the Beast*, where one electronic keyboard was used to play the notes, and another used to instantly set the root note to which all intervals were tuned, which allowed for modulation. On her 1987 lecture album *Secrets of Synthesis* there are audible examples of the difference in sound between equal temperament and just intonation.

Singing

The human voice is among the most pitch-flexible instruments in common use. Pitch can be varied with no restraints and adjusted in the midst of performance, without needing to retune. Although the explicit use of just intonation fell out of favour concurrently with the increasing use of instrumental accompaniment (with its attendant constraints on pitch), most a cappella ensembles naturally tend toward just intonation because of the comfort of its stability. Barbershop quartets are a good example of this.

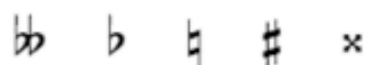
Western composers

Most composers don't specify how instruments are to be tuned, although historically most have assumed one tuning system which was common in their time; in the 20th century most composers assumed equal temperament would be used. However, a few have specified just intonation systems for some or all of their compositions, including John Luther Adams, Glenn Branca, Martin Bresnick, Wendy Carlos, Lawrence Chandler, Tony Conrad, Fabio Costa, Stuart Dempster, David B. Doty, Arnold Dreyblatt, Kyle Gann, Kraig Grady, Lou Harrison, Michael Harrison, Ben Johnston, Elodie Lauten, György Ligeti, Douglas Leedy, Pauline Oliveros, Harry Partch, Robert Rich, Terry Riley, Marc Sabat, Wolfgang von Schweinitz, Adam Silverman, James Tenney, Michael Waller, Daniel James Wolf, and La Monte Young. Eivind Groven's tuning system was schismatic temperament, which is capable of far closer approximations to just intonation consonances than 12-note equal temperament or even meantone temperament, but still alters the pure ratios of just intonation slightly in order to achieve a simpler and more flexible system than true just intonation.

Music written in just intonation is most often tonal but need not be; some music of Kraig Grady and Daniel James Wolf uses just intonation scales designed by Erv Wilson explicitly for a consonant form of atonality, and Ben Johnston's *Sonata for Microtonal Piano* (1964) uses serialism to achieve an atonal result. Composers often impose a limit on how complex the ratios used are: for example, a composer may write in "7-limit JI", meaning that no prime number larger than 7 features in the ratios they use. Under this scheme, the ratio 10/7, for example, would be permitted, but 11/7 would not be, as all non-prime numbers are octaves of, or mathematically and tonally related to, lower primes (example: 12 is a double octave of 3, while 9 is a square of 3). Yuri Landman derived a just intoned musical scale from an initially considered atonal prepared guitar playing technique based on adding a third bridge under the strings. When this bridge is positioned in the noded positions of the harmonic series the volume of the instrument increases and the overtone becomes clear and has a

consonant relation to the complementary opposed string part creating a harmonic multiphonic tone.

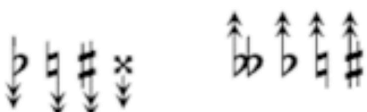
Staff notation



Pythagorean series of fifths – the open strings
(... c g d a e ...)



lowers / raises by a syntonic comma
 $81 : 80 = \text{circa } 21.5 \text{ cents}$



lowers / raises by two syntonic commas
circa 43 cents



lowers / raises by a septimal comma
 $64 : 63 = \text{circa } 27.3 \text{ cents}$



lowers / raises by two septimal commas
circa 54.5 cents



raises / lowers by an 11-limit undecimal quarter-tone
 $33 : 32 = \text{circa } 53.3 \text{ cents}$



lowers / raises by a 13-limit tridecimal third-tone
 $27 : 26 = \text{circa } 65.3 \text{ cents}$



lowers / raises by a 17-limit schisma
 $256 : 255 = \text{circa } 6.8 \text{ cents}$



raises / lowers by a 19-limit schisma
 $513 : 512 = \text{circa } 3.4 \text{ cents}$



raises / lowers by a 23-limit comma
 $736 : 729 = \text{circa } 16.5 \text{ cents}$

Ex. 1: Legend of the HE Accidentals



Pythagorean diatonic scale on C. Johnston's notation.



Just intonation diatonic scale on C. Johnston's notation (Pythagorean major scale in Helmholtz-Ellis notation).



Just intonation diatonic scale on C. Helmholtz-Ellis notation.



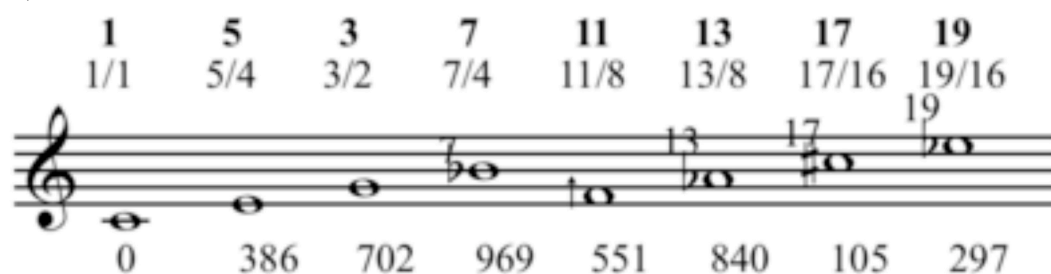
Just harmonic seventh chord on C. 7th: 968.826 cents, a septimal quarter tone lower than B $\flat$.

Originally a system of notation to describe scales was devised by Hauptmann and modified by Helmholtz (1877) in which Pythagorean notes are started with and subscript numbers are added indicating how many commas ($81/80$, syntonic comma) to lower by. For example the Pythagorean major third on C is C+E) while the just major third is C+E1). A similar system was devised by Carl Eitz and used in Barbour (1951) in which Pythagorean notes are started with and positive or negative superscript numbers are added indicating how many commas ($81/80$, syntonic comma) to adjust by. For example, the Pythagorean major third on C is C-E0 while the just major third is C-E $\wedge$ -1.

While these systems allow precise indication of intervals and pitches in print, more recently some composers have been developing notation methods for Just Intonation using the conventional five-line staff. James Tenney, amongst others, preferred to combine JI ratios with cents deviations from the equal tempered pitches, indicated in a legend or directly in the score, allowing performers to readily use electronic tuning devices if desired.[16] Beginning in the 1960s, Ben Johnston had proposed an alternative approach, redefining the understanding of conventional symbols (the seven "white" notes, the sharps and flats) and adding further accidentals, each designed to extend the notation into higher prime limits. Johnston's method is based on a diatonic C Major scale tuned in JI, in which the interval between D ($9/8$ above C) and A ($5/3$ above C) is one syntonic comma less than a Pythagorean perfect fifth $3:2$. To write a perfect fifth, Johnston introduces a pair of symbols representing this comma, + and ?. Thus, a series of perfect fifths beginning with F would proceed C G D A+ E+ B+. The three conventional white notes A E B are tuned as Ptolemaic major thirds ($5:4$) above F C G respectively. Johnston introduces new symbols for the septimal (67 & 97), undecimal (up & down), tridecimal (13 & 31), and further prime

extensions to create an accidental based exact JI notation for what he has named "Extended Just Intonation". For example, the Pythagorean major third on C is C-E+ while the just major third is C-E♭.

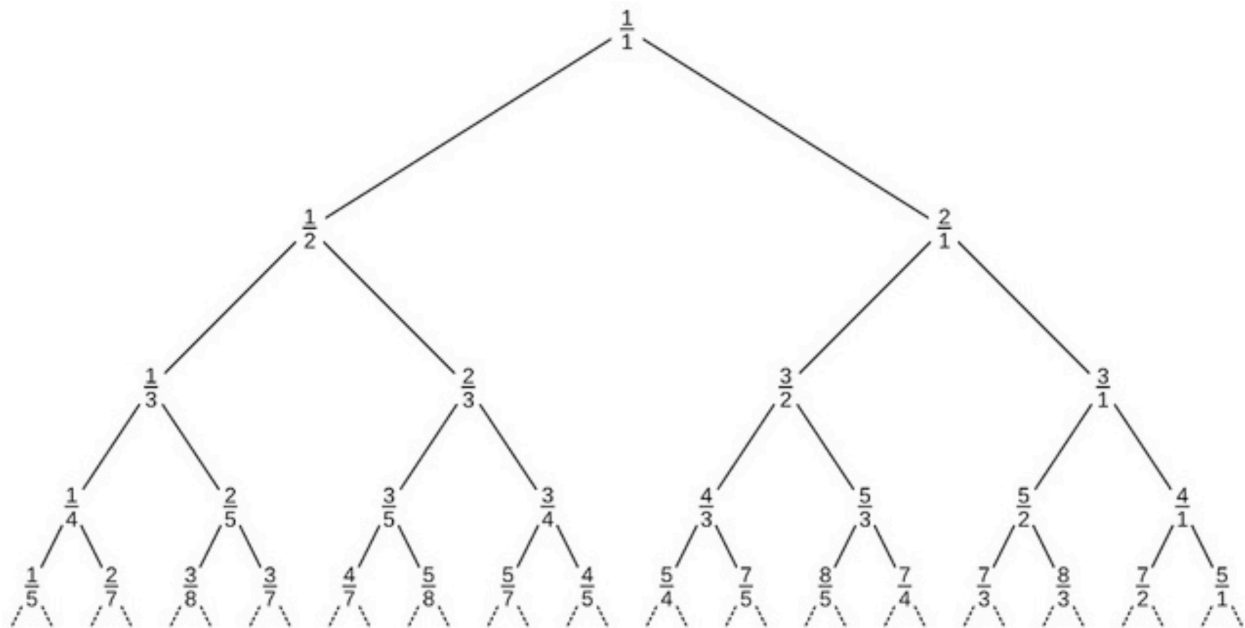
In the years 2000–2004, Marc Sabat and Wolfgang von Schweinitz worked together in Berlin to develop a different accidental based method, the Extended Helmholtz-Ellis JI Pitch Notation.[18] Following the method of notation suggested by Helmholtz in his classic "On the Sensations of Tone as a Physiological Basis for the Theory of Music", incorporating Ellis' invention of cents, and following Johnston's step into "Extended JI", Sabat and Schweinitz consider each prime dimension of harmonic space to be represented by a unique symbol. In particular they take the conventional flats, naturals and sharps as a Pythagorean series of perfect fifths. Thus, a series of perfect fifths beginning with F proceeds C G D A E B F♯ and so on. The advantage for musicians is that conventional reading of the basic fourths and fifths remains familiar. Such an approach has also been advocated by Daniel James Wolf. In the Sabat-Schweinitz design, syntonic commas are marked by arrows attached to the flat, natural or sharp sign, Septimal Commas using Giuseppe Tartini's symbol, and Undecimal Quartertones using the common practice quartertone signs (a single cross and backwards flat). For higher primes, additional signs have been designed. To facilitate quick estimation of pitches, cents indications may be added (downward deviations below and upward deviations above the respective accidental). The convention used is that the cents written refer to the tempered pitch implied by the flat, natural, or sharp sign and the note name. A complete legend and fonts for the notation (see samples) are open source and available from Plainsound Music Edition. For example, the Pythagorean major third on C is C-E♭ while the just major third is C-E♭-arrow-down.



Staff notation of partials 1, 3, 5, 7, 11, 13, 17, and 19 on C using Johnston's notation. One of the great advantages of such notation systems is that they allow the natural harmonic series to be precisely notated.

Just Intonation and the Stern-Brocot Tree

Since I work in Just Intonation I'm always looking for new ways to organize and think about ratios. Recently I came across the mathematical idea called the Stern-Brocot Tree from a very nice video posted by a friend. This is a way to represent all whole number ratios in their lowest terms as a binary tree growing from 1:1. Any ratio can be reached through a number of successive left or right moves down the tree. For specifics please consult the above links of other sources as I will not cover them here.



I decided to implement ways of traversing the tree in a SuperCollider class (this depends on the MathLib and dewdrop_lib Quarks). This works based on three representations of any position: the actual ratio, a matrix, and a continued fraction. By jumping back and forth between these three forms, you can easily move left, right, up, compare properties of different ratios, etc. It is functional but pretty rough and subject to major refactoring whenever I get around to it.

My current focus with the tree involves ratios between 1:1 and 2:1 (otoneality) and the difference tones between them. Since playing with this I've noticed a few interesting things (that I'm sure others have seen before too).

Paths

Within the tree structure we have this notion of a path, which is a sequence of right (R) and left (L) steps downward. In Just Intonation theory, one of the most important types of ratios is the superparticular $((n+1)/n)$. If we look at the tree we can see that each of these are reached by starting from 1:1, moving R to 2:1, followed by consecutive L moves. This means every superparticular ratio falls in the path of RLLLLL.... etc. This can be represented as an array based on the continued fraction form of the ratio. After studying this representation we can see that each element is the number of steps in a direction, starting with a R, and alternating LRLR after that. For example, the path to 4:3 is RLL and the continued fraction is $[1,2]$ (one R followed by 2 Ls). 5:4 would be $[1,3]$, 6:5 is $[1,4]$ and so on.

Difference Tones against 1:1

Say you want to know what difference tone you will get between a given fundamental and some ratio above it where both pitches are in the same octave (between 1:1 and 2:1) as each other. The math for that is simple: just subtract the two. However I've found that there is a correlation between the difference tone of any ratio between 1:1 and 2:1 and its path within the tree. Take $(3/2) - (1/1) = 1/2$. The path to 3:2 is [1,1] and the path to 1:2 is [0,1] (the 0 means that we do not have any R moves, so our first move is an L). How about $(5/3) - (1/1) = 2/3$. 5:3 is [1,1,1] and 2:3 is [0,1,1]. Seeing a pattern? It turns out that the path of the difference tone against 1:1 for any ratio between 1:1 and 2:1 is the path to the ratio with the first R move set to 0. For 7:4 [1,1,2], the difference against 1:1 is [0,1,2].

Axis and Mirroring

The next concept is that of a mirror. All this means is that you take the opposite path (if you moved R before, move L this time) around a central ratio that we will call an axis. If our axis is 1:1, the mirror is simply the reciprocal (the mirror of 2:3 is 3:2). Why do I have a new term if this already has a name? Because if we mirror around ratios other than 1:1 the result is not the reciprocal. Using 1:1 as an axis is the special case where the mirror and the reciprocal are the same thing.

Lets look at this in terms of the path to a ratio first. The mirror of 2:1 [1] around the axis 1:1 is 1:2 [0,1]. The mirror of 3:2 [1,1] around 1:1 is 2:3 [0,1,1]. Like we said earlier, if you want to move L first instead of R, you make the first slot in the array 0. With the difference between a ratio and 1:1 we replaced the first element with 0. Here we insert a 0 before the remaining operations if we are going left of 1:1, and we remove the 0 if we are going right of 1:1.

But why? Well, the path to 1:1 is [0] since we don't need to go anywhere to get to it, but that still doesn't tell us the whole story.

For this we have to understand how a change of direction in the tree is achieved. First lets look at the paths to the whole numbers. 2:1 [1], 3:1 [2], 4:1 [3]; each of these are consecutive R moves from 1:1 [0]. Now look at 3:2 [1,1] and 3:1 [2] in the tree. They are both children of 2:1 and are also mirrors around 2:1 since one is to the L and the other to the R. We can represent this in the path by separating 2:1's path from the paths of both ratios. If we think for a second about the path in terms of Rs and Ls rather than just numbers:

2:1 = [1] = R
3:1 = [2] = RR
3:2 = [1,1] = RL

it becomes rather obvious that we are simply moving from the first R. So 3:2 and 3:1 are mirrors around the axis 2:1 since their paths are inversions of each other starting from the axis. Here's one more example with 3:2 as the axis:

3:2 = [1,1] = RL
7:5 = [1,2,1] = RLLR
8:5 = [1,1,1,1] = RLRL

Now that we can think of it as Rs and Ls, we can manipulate the continues fraction to represent that:

3:2 = {1,1}

$$7:5 = [\{1, (1\}+1), 1]$$

$$8:5 = [\{1, 1\}, 1, 1]$$

Difference Tones against 3:2

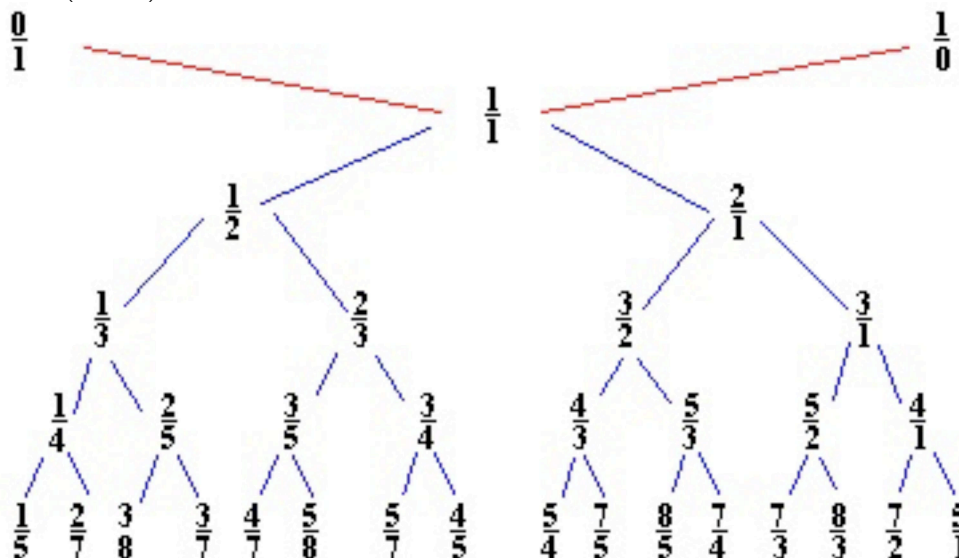
In exploring more of the difference tones within the tree I found a very special property of 3:2. The difference tone of 3:2 and any of its children is the same as the difference between 3:2 and the child's mirror around 3:2. Let's take the ratios we were just looking at. The difference between 3/2 and 7/5 is 1/10. The mirror of 7:5 around the axis 3:2 is 8:5. The difference between 8/5 and 3/2 is 1/10. The same is true for any child of 3:2 and its mirror. Also interesting is that the denominator of all mirrors around 3:2 will be the same (e.g. 7/5, 8/5).

- source: <http://coleingraham.com/tag/just-intonation/>

Stern-Brocot Tree

0/1 is a fraction while 1/0 is not. However, using it as such helps describe a way to get all possible positive fractions arranged in a very nice manner. So disguising 1/0 as a fraction has a very good excuse.

We already know how to obtain the median of two given fractions. The median of two fractions m_1/n_1 and m_2/n_2 is, by definition, the fraction $(m_1 + m_2)/(n_1 + n_2)$. So, for example, from 0/1 and 1/0 we get 1/1. The median of 0/1 and 1/1 is 1/2 while the median of 1/1 and 1/0 is 2/1. On the next stage of the construction, we form four new fractions: 1/3 from 0/1 and 1/2, 2/3 from 1/2 and 1/1, 3/2 from 1/1 and 2/1, and, finally, the median of 2/1 and 1/0 which is 3/1. Continuing this way we get an infinite tree known as the Stern-Brocot tree because it was discovered independently by the German mathematician Moriz Stern (1858) and by the French clock maker Achille Brocot (1860).



A remarkable thing about Stern-Brocot tree is that it contains all possible non-negative fractions expressed in lowest terms and each exactly once. Just to remind, a fraction m/n is in lowest terms iff m and n are coprime, i.e. iff $\gcd(n, m) = 1$. To prove this we'll need a series of facts of which the following is the most fundamental: if m_1/n_1 and m_2/n_2 are two consecutive (consecutive in the sense that one of them is a direct descendant of the other) fractions at any stage of the construction then

(1)

$$m_2 n_1 - m_1 n_2 = 1$$

This relation is true for the initial fractions: $1 \cdot 1 - 0 \cdot 0 = 1$. Also, assuming (1) holds for two fractions m_1/n_1 and m_2/n_2 , their mediant should satisfy the following two:

$$n_1(m_1 + m_2) - (n_1 + n_2)m_1 = 1, \text{ and}$$

$$m_2(n_1 + n_2) - (m_1 + m_2)n_2 = 1$$

both of which are equivalent to (1).

Also, if $m_1/n_1 < m_2/n_2$ then

$$m_1/n_1 < (m_1 + m_2)/(n_1 + n_2) < m_2/n_2,$$

from which it follows that the construction of the tree preserves the natural order between the rationals. Therefore, it's impossible to obtain the same fraction twice.

And there is the last point: let a/b be a fraction in lowest terms with a and b positive. I wish to show that this fraction will appear somewhere on the tree. So long as it did not, we shall have

$$m_1/n_1 < a/b < m_2/n_2$$

for a couple of fractions m_1/n_1 and m_2/n_2 satisfying (1). These are rewritten as

$$n_1a - m_1b > 0 \text{ and } m_2b - n_2a > 0,$$

which imply

$$n_1a - m_1b \geq 1 \text{ and } m_2b - n_2a \geq 1.$$

from where

$$(m_2 + n_2)(n_1a - m_1b) + (m_1 + n_1)(m_2b - n_2a) \geq m_1 + n_1 + m_2 + n_2.$$

An application of (1) yields

$$a + b \geq m_1 + n_1 + m_2 + n_2.$$

with inevitable conclusion that after at most $a+b$ steps of computing mediants one of them will be equal to a/b .

P.S.

Prof. W.McWorter has offered the following clarification to the proof:

At the root of the tree the minimum numerator-denominator sum (nd sum) is 2. At the first level of the tree the minimum nd sum is 3, and at the n -th level of the tree the minimum nd sum is $n + 2$. To see this, every fraction at the n -th level is the mediant of a fraction at the $(n - 1)$ -st level and one at a higher level. The nd sum of a fraction at the $(n - 1)$ -st level is at least $n + 1$ and the nd sum of a fraction at a higher level is at least 1. Hence the nd sum of their mediant is at least $n + 2$.

Thus, if a/b is a fraction in lowest terms not equal to any fraction in the tree, then it lies strictly between two consecutive fractions, one of which is at level $a+b-1$ of the tree (consecutive here is restricted to all fractions constructed up to and including level $a + b - 1$; none constructed beyond this level are considered). Hence by the same calculations $a + b \geq a + b + 1$, a contradiction.

Remark

Pierre Lamothe from Canada informed me recently of a property of the Stern-Brocot tree I was unaware of. Pierre discovered that property in his research on music and harmony.

Let's associate with any irreducible fraction p/q the number $w(p/q) = 1/pq$ - its simplicity. The property discovered by Pierre states that the sum of simplicities of all fractions in any row of the Stern-Brocot tree equals 1! We can easily see that this is true for a few first rows:

Row number	Tree members	Sum of simplicities
0	1/1	1/1
1	1/2, 2/1	1/2 + 1/2
2	1/3, 2/3, 3/2, 3/1	1/3 + 1/6 + 1/6 + 1/3
3	1/4, 2/5, 3/5, 3/4, 4/3, 5/3, 5/2, 4/1	1/4 + 1/10 + 1/15 + 1/12 + 1/12 + 1/15 + 1/10 + 1/4

The inductive proof is based on a fact (proved on later pages) that, for any fraction p/q , two fractions $p/(p+q)$ and $(p+q)/q$ are located just one row below that of p/q . We have an obvious identity:

$$w(p/(p+q)) + w((p+q)/q) = 1/p(p+q) + 1/q(p+q) = 1/pq = w(p/q),$$

from which it follows that if the assertion holds for one row, it holds for the next one as well.

Pierre also introduced another function W defined for the members of the tree. For an irreducible fraction p/q , Let $N(p/q)$ denote the row of the tree that contains p/q . We start with $N(1/1) = 0$, $N(1/2) = 1$, $N(2/1) = 1$, $N(1/3) = 2$, and so on. Then the weighted simplicity $W(p/q)$ is defined as

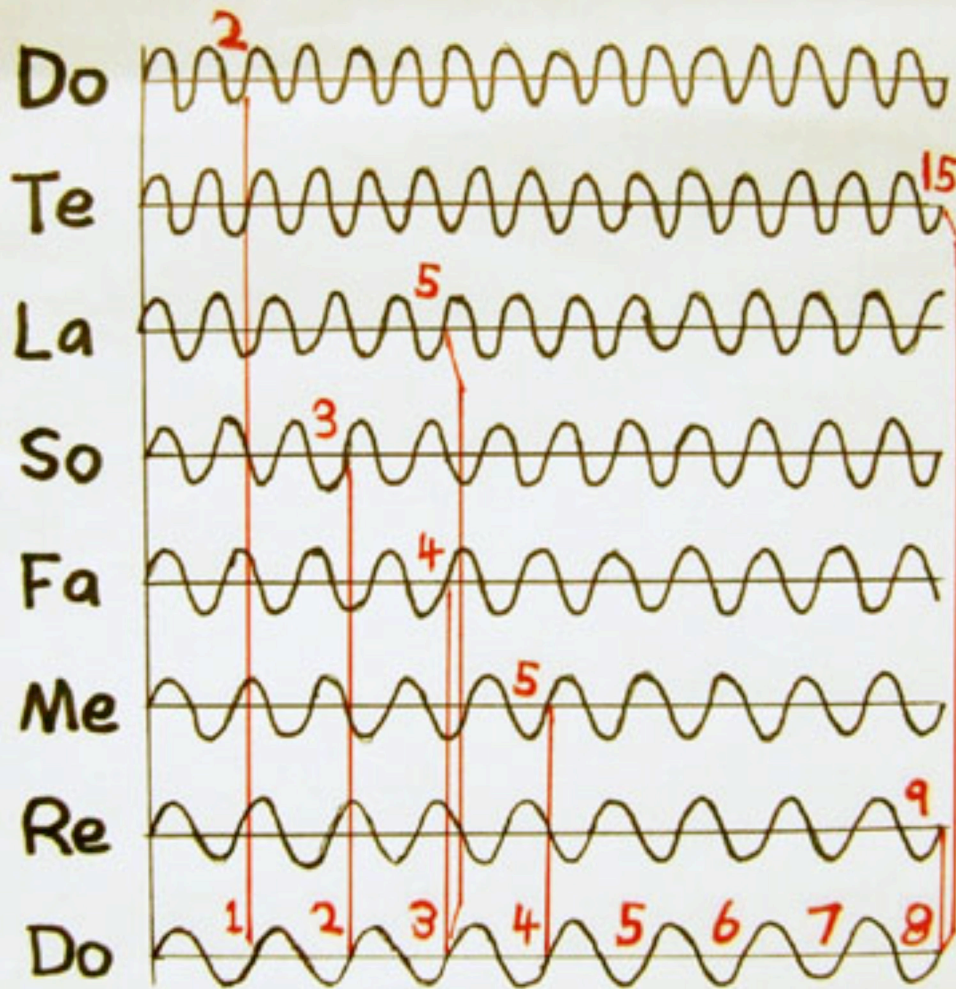
$$W(p/q) = w(p/q) \cdot 2^{N(p/q)-1}$$

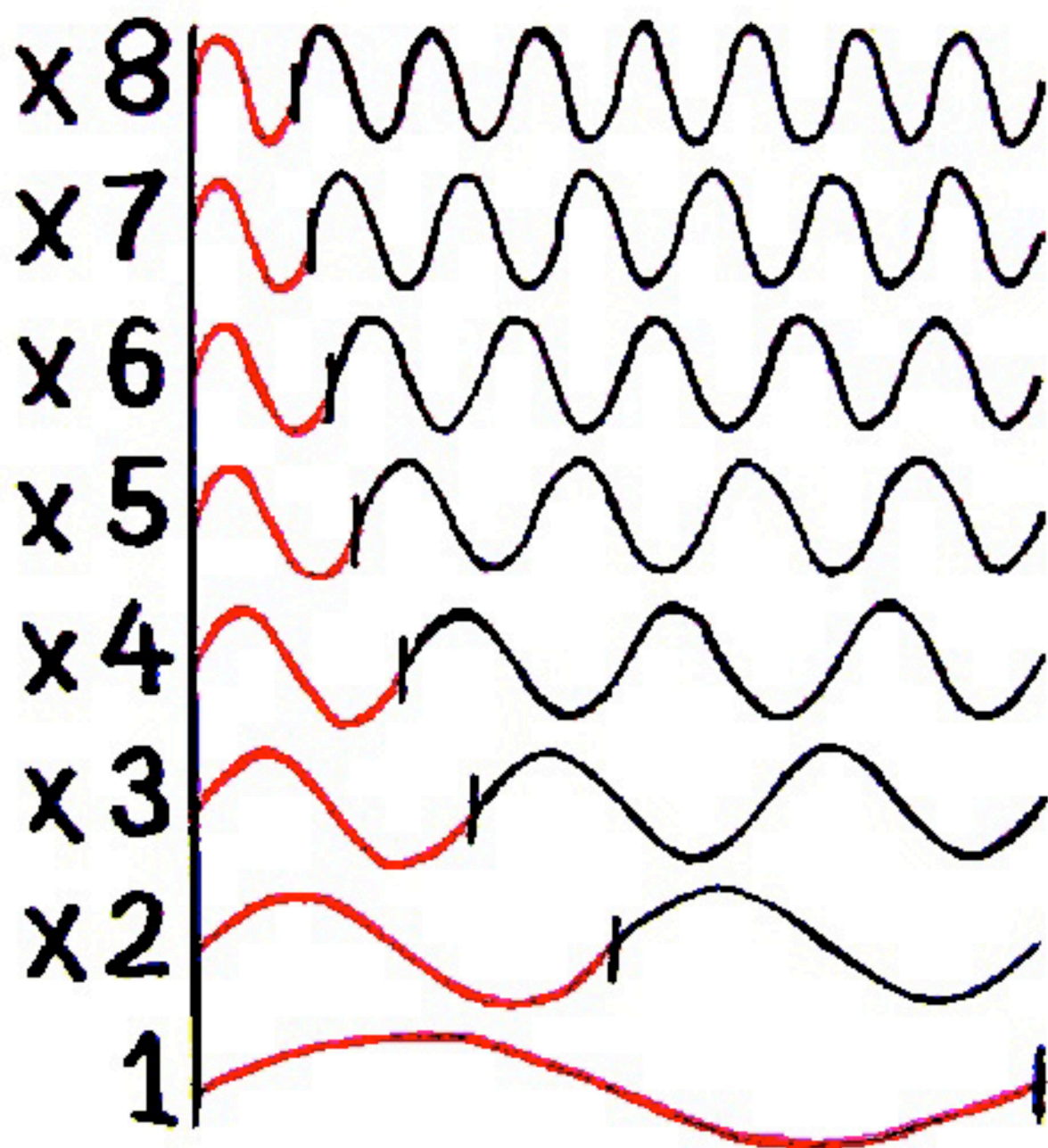
From the previous statement it then follows that the sum of all weighted simplicities of the fractions on the tree is equal to $1!$

References

1. R.Graham, D.Knuth, O.Patashnik, Concrete Mathematics, 2nd edition, Addison-Wesley, 1994.
2. Brian Hayes, Computing Science On the Teeth of Wheels, American Scientist, July-August, Volume 88, No. 4,

-source: <http://www.cut-the-knot.org/blue/Stern.shtml>





Frequencies for equal-tempered scale, A4 = 440 Hz

Note	Frequency (Hz)	Wavelength (cm)
C0	16.35	2109.89
C#0/Db0	17.32	1991.47
D0	18.35	1879.69
D#0/Eb0	19.45	1774.20
E0	20.60	1674.62
F0	21.83	1580.63
F#0/Gb0	23.12	1491.91
G0	24.50	1408.18
G#0/Ab0	25.96	1329.14
A0	27.50	1254.55
A#0/Bb0	29.14	1184.13
B0	30.87	1117.67
C1	32.70	1054.94
C#1/Db1	34.65	995.73
D1	36.71	939.85
D#1/Eb1	38.89	887.10
E1	41.20	837.31
F1	43.65	790.31
F#1/Gb1	46.25	745.96
G1	49.00	704.09
G#1/Ab1	51.91	664.57
A1	55.00	627.27
A#1/Bb1	58.27	592.07
B1	61.74	558.84
C2	65.41	527.47
C#2/Db2	69.30	497.87
D2	73.42	469.92
D#2/Eb2	77.78	443.55
E2	82.41	418.65
F2	87.31	395.16
F#2/Gb2	92.50	372.98
G2	98.00	352.04
G#2/Ab2	103.83	332.29
A2	110.00	313.64
A#2/Bb2	116.54	296.03
B2	123.47	279.42
C3	130.81	263.74
C#3/Db3	138.59	248.93
D3	146.83	234.96
D#3/Eb3	155.56	221.77
E3	164.81	209.33
F3	174.61	197.58
F#3/Gb3	185.00	186.49
G3	196.00	176.02
G#3/Ab3	207.65	166.14
A3	220.00	156.82
A#3/Bb3	233.08	148.02
B3	246.94	139.71
C4	261.63	131.87
C#4/Db4	277.18	124.47
D4	293.66	117.48
D#4/Eb4	311.13	110.89

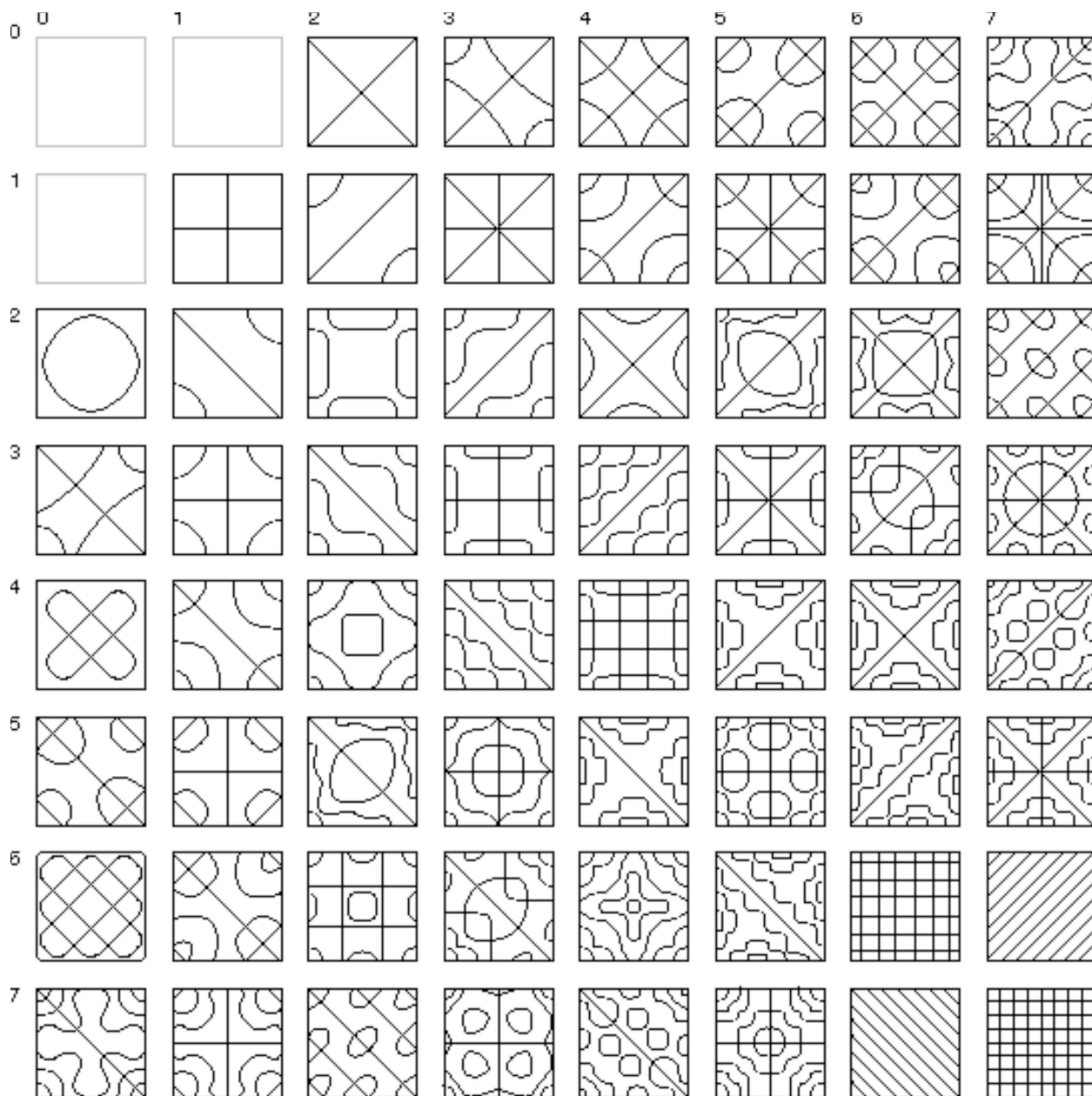
E4	329.63	104.66
F4	349.23	98.79
F#4/Gb4	369.99	93.24
G4	392.00	88.01
G#4/Ab4	415.30	83.07
A4	440.00	78.41
A#4/Bb4	466.16	74.01
B4	493.88	69.85
C5	523.25	65.93
C#5/Db5	554.37	62.23
D5	587.33	58.74
D#5/Eb5	622.25	55.44
E5	659.25	52.33
F5	698.46	49.39
F#5/Gb5	739.99	46.62
G5	783.99	44.01
G#5/Ab5	830.61	41.54
A5	880.00	39.20
A#5/Bb5	932.33	37.00
B5	987.77	34.93
C6	1046.50	32.97
C#6/Db6	1108.73	31.12
D6	1174.66	29.37
D#6/Eb6	1244.51	27.72
E6	1318.51	26.17
F6	1396.91	24.70
F#6/Gb6	1479.98	23.31
G6	1567.98	22.00
G#6/Ab6	1661.22	20.77
A6	1760.00	19.60
A#6/Bb6	1864.66	18.50
B6	1975.53	17.46
C7	2093.00	16.48
C#7/Db7	2217.46	15.56
D7	2349.32	14.69
D#7/Eb7	2489.02	13.86
E7	2637.02	13.08
F7	2793.83	12.35
F#7/Gb7	2959.96	11.66
G7	3135.96	11.00
G#7/Ab7	3322.44	10.38
A7	3520.00	9.80
A#7/Bb7	3729.31	9.25
B7	3951.07	8.73
C8	4186.01	8.24
C#8/Db8	4434.92	7.78
D8	4698.63	7.34
D#8/Eb8	4978.03	6.93
E8	5274.04	6.54
F8	5587.65	6.17
F#8/Gb8	5919.91	5.83
G8	6271.93	5.50
G#8/Ab8	6644.88	5.19
A8	7040.00	4.90

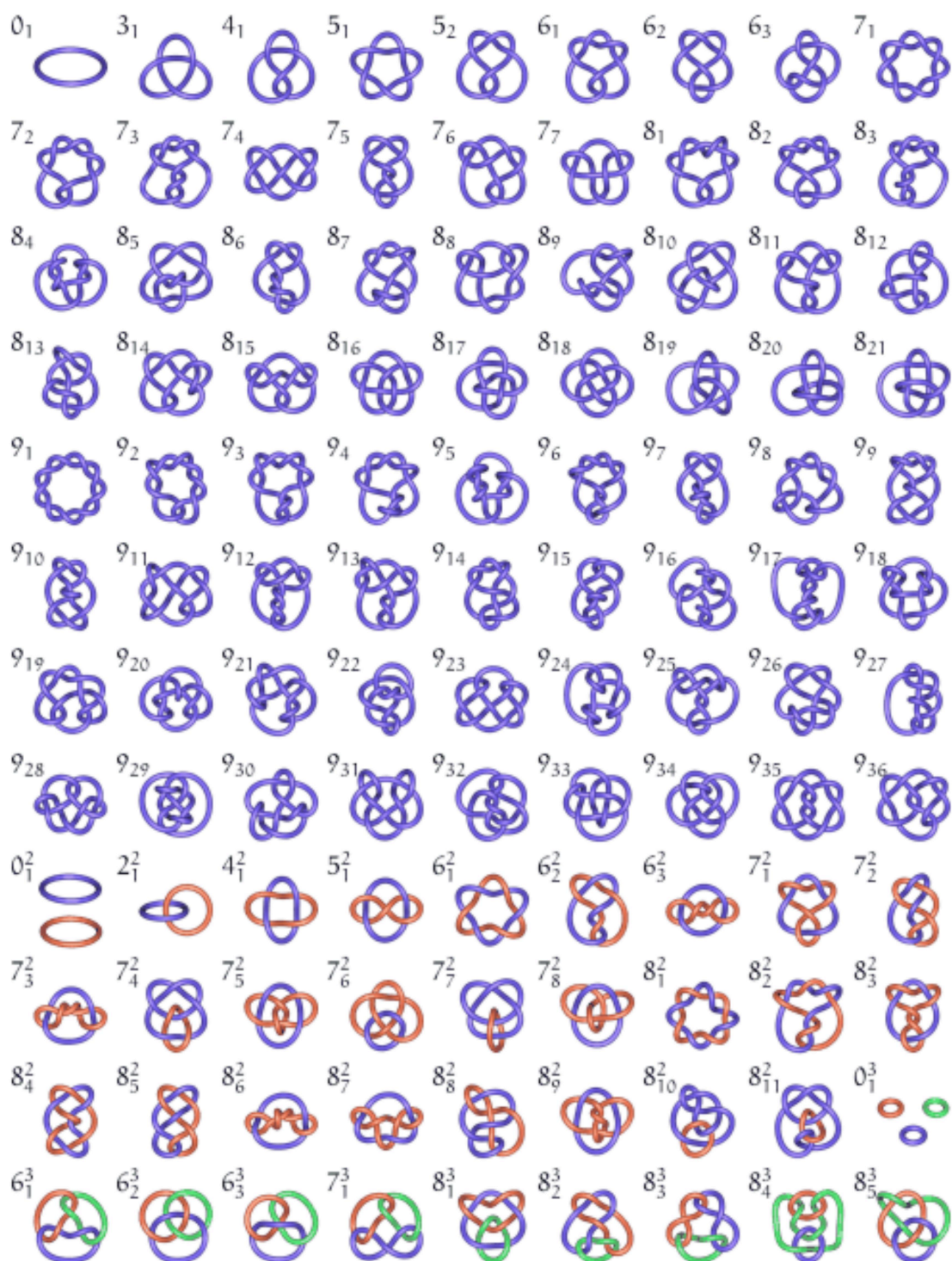
A#8/Bb8
B8

7458.62
7902.13

4.63
4.37

("Middle C" is C4)
(To convert lengths in cm to inches, divide by 2.54)
(Speed of Sound = 345 m/s = 1130 ft/s = 770 miles/hr)





Quarter-comma meantone

Quarter-comma meantone, or **1/4-comma meantone**, was the most common meantone temperament in the sixteenth and seventeenth centuries, and was sometimes used later. This method is a variant of Pythagorean tuning. The difference is that in this system the perfect fifth is flattened by one quarter of a syntonic comma, with respect to its just intonation used in Pythagorean tuning (frequency ratio 3:2). The purpose is to obtain justly intoned major thirds (with a frequency ratio equal to 5:4). It was described by Pietro Aron (also spelled Aaron), in his *Toscanello de la Musica* of 1523, by saying the major thirds should be tuned to be "sonorous and just, as united as possible." Later theorists Gioseffo Zarlino and Francisco de Salinas described the tuning with mathematical exactitude.

Construction

In a meantone tuning, we have diatonic and chromatic semitones, with the diatonic semitone larger. In Pythagorean tuning, these correspond to the Pythagorean limma and the Pythagorean apotome, only now the apotome is larger. In any meantone or Pythagorean tuning, a whole tone is composed of one semitone of each kind, a major third is two whole tones and therefore consists of two semitones of each kind, a perfect fifth of meantone contains four diatonic and three chromatic semitones, and an octave seven diatonic and five chromatic semitones, it follows that:

- Five fifths down and three octaves up make up a diatonic semitone, so that the Pythagorean limma is tempered to a diatonic semitone.
- Two fifths up and an octave down make up a whole tone consisting of one diatonic and one chromatic semitone.
- Four fifths up and two octaves down make up a major third, consisting of two diatonic and two chromatic semitones, or in other words two whole tones.

Thus, in Pythagorean tuning, where sequences of just fifths (frequency ratio 3:2) and octaves are used to produce the other intervals, a whole tone is

$$\frac{(3/2)^2}{2} = \frac{9/4}{2} = \frac{9}{8},$$

and a major third is

$$\frac{(3/2)^4}{2^2} = \frac{81/16}{4} = \frac{81}{64} \approx \frac{5}{4} = \frac{5 \cdot 16}{4 \cdot 16} = \frac{80}{64}.$$

An interval of a seventeenth, consisting of sixteen diatonic and twelve chromatic semitones, such as the interval from D4 to F6?, can be equivalently obtained using either

- a stack of four fifths (e.g. D4—A4—E5—B5—F6#), or
- a stack of two octaves and one major third (e.g. D4—D5—D6—F6#).

This large interval of a seventeenth contains $(5 + (5 - 1) + (5 - 1) + (5 - 1) = 20 - 3 = 17$ staff positions). In Pythagorean tuning, the size of a seventeenth is defined using a stack of four justly tuned fifths (frequency ratio 3:2):

$$\left(\frac{3}{2}\right)^4 = \frac{81}{16} = \frac{80}{16} \cdot \frac{81}{80} = 5 \cdot \frac{81}{80}.$$

In quarter-comma meantone temperament, where a just major third (5:4) is required, a slightly narrower seventeenth is obtained by stacking two octaves (4:1) and a major third:

$$\frac{4}{1} \cdot \frac{5}{4} = 5.$$

By definition, however, a seventeenth of the same size (5:1) must be obtained, even in quarter-comma meantone, by stacking four fifths. Since justly tuned fifths, such as those used in Pythagorean tuning, produce a slightly wider seventeenth, in quarter-comma meantone the fifths must be slightly flattened to meet this requirement. Letting x be the frequency ratio of the flattened fifth, it is desired that four fifths have a ratio of 5:1,

$$x^4 = 5,$$

which implies that a fifth is

$$x = \sqrt[4]{5} = 5^{1/4},$$

a whole tone, built by moving two fifths up and one octave down, is

$$\frac{x^2}{2} = \frac{\sqrt{5}}{2},$$

and a diatonic semitone, built by moving three octaves up and five fifths down, is

$$\frac{2^3}{x^5} = \frac{8}{5^{5/4}}.$$

Notice that, in quarter-comma meantone, the seventeenth is 81/80 times narrower than in Pythagorean tuning. This difference in size, equal to about 21.506 cents, is called the syntonic comma. This implies that the fifth is a quarter of a syntonic comma narrower than the justly tuned Pythagorean fifth. Namely, this system tunes the fifths in the ratio of

$$5^{1/4} \approx 1.495349$$

which is expressed in the logarithmic *cents* scale as

$$1200 \log_2 5^{1/4} \text{ cents} \approx 696.578 \text{ cents},$$

which is slightly smaller (or flatter) than the ratio of a justly tuned fifth:

$$\frac{3}{2} = 1.5$$

which is expressed in the logarithmic *cents* scale as

$$1200 \log_2 \frac{3}{2} \text{ cents} \approx 701.955 \text{ cents}.$$

The difference between these two sizes is a quarter of a syntonic comma:

$$\approx 701.955 - 696.578 \approx 5.377 \approx \frac{21.506}{4} \text{ cents}.$$

In sum, this system tunes the major thirds to the just ratio of 5:4 (so, for instance, if A is tuned to 440 Hz, C# is tuned to 550 Hz), most of the whole tones (namely the major seconds) in the ratio $\sqrt{5} : 2$, and most of the semitones (namely the diatonic

semitones or minor seconds) in the ratio $8 : 5^{\frac{5}{4}}$. This is achieved by tuning the seventeenth a syntonic comma flatter than the Pythagorean seventeenth, which implies tuning the fifth a quarter of a syntonic comma flatter than the just ratio of 3:2. It is this that gives the system its name of *quarter-comma meantone*.

12-tone scale^[edit]

The whole chromatic scale (a subset of which is the diatonic scale), can be constructed by starting from a given *base note*, and increasing or decreasing its frequency by one or more fifths. This method is identical to Pythagorean tuning, except for the size of the fifth, which is tempered as explained above. The construction table below illustrates how the pitches of the notes are obtained with respect to D (the *base note*), in a D-based scale (see Pythagorean tuning for a more detailed explanation).

For each note in the basic octave, the table provides the conventional name of the interval from D (the base note), the formula to compute its frequency ratio, and the approximate values for its frequency ratio and size in cents.

Note	Interval from D	Formula	Freq. ratio	Size (cents)
A $\flat$	diminished fifth	$x^{-6} \cdot 2^4 = \frac{16\sqrt{5}}{25}$	1.4311	620.5
E $\flat$	minor second	$x^{-5} \cdot 2^3 = \frac{8\sqrt{5}x}{25}$	1.0700	117.1
B $\flat$	minor sixth	$x^{-4} \cdot 2^3 = \frac{8}{5}$	1.6000	813.7
F	minor third	$x^{-3} \cdot 2^2 = \frac{4x}{5}$	1.1963	310.3
C	minor seventh	$x^{-2} \cdot 2^2 = \frac{4\sqrt{5}}{5}$	1.7889	1006.8
G	perfect fourth	$x^{-1} \cdot 2^1 = \frac{2\sqrt{5}x}{5}$	1.3375	503.4
D	unison	$x^0 \cdot 2^0 = 1$	1.0000	0.0
A	perfect fifth	$x^1 \cdot 2^0 = x$	1.4953	696.6
E	major second	$x^2 \cdot 2^{-1} = \frac{\sqrt{5}}{2}$	1.1180	193.2
B	major sixth	$x^3 \cdot 2^{-1} = \frac{\sqrt{5}x}{2}$	1.6719	889.7
F $\sharp$	major third	$x^4 \cdot 2^{-2} = \frac{5}{4}$	1.2500	386.3
C $\sharp$	major seventh	$x^5 \cdot 2^{-2} = \frac{5x}{4}$	1.8692	1082.9
G $\sharp$	augmented fourth	$x^6 \cdot 2^{-3} = \frac{5\sqrt{5}}{8}$	1.3975	579.5

In the formulas, $x = \sqrt[4]{5} = 5^{1/4}$ is the size of the tempered perfect fifth, and the ratios $x:1$ or $1:x$ represent an ascending or descending tempered perfect fifth (i.e. an increase or decrease in frequency by x), while $2:1$ or $1:2$ represent an ascending or descending octave.

As in Pythagorean tuning, this method generates 13 pitches, but $A?$ and $G?$ have almost the same frequency, and to build a 12-tone scale $A\flat$ is typically discarded (although the choice between these two notes is completely arbitrary).

C- based construction tables

The table above shows a D-based stack of fifths (i.e. a stack in which all ratios are expressed relative to D, and D has a ratio of $1/1$). Since it is centered at D, the base note, this stack can be called *D-based symmetric*:

$A\flat - E\flat - B\flat - F - C - G - D - A - E - B - F\sharp - C\sharp - G\sharp$

Except for the size of the fifth, this is identical to the stack traditionally used in Pythagorean tuning. Some authors prefer showing a C-based stack of fifths, ranging from $A\flat$ to $G\sharp$. Since C is not at its center, this stack is called *C-based asymmetric*:

$A\flat - E\flat - B\flat - F - C - G - D - A - E - B - F\sharp - C\sharp - G\sharp$

Since the boundaries of this stack ($A\flat$ and $G\sharp$) are identical to those of the D-based symmetric stack, the 12 tone scale produced by this stack is also identical. The only difference is that the construction table shows intervals from C, rather than from D. Notice that 144 intervals can be formed from a 12 tone scale (see table below), which include intervals from C, D, and any other note. However, the construction table shows only 12 of them, in this case those starting from C. This is at the same time the main advantage and main disadvantage of the C-based asymmetric stack, as the intervals from C are commonly used, but since C is not at the center of this stack, they unfortunately include an augmented fifth (or **A5**, i.e. the interval from C to $G?$), instead of a minor sixth (**m6**). This A5 is an extremely dissonant wolf interval, as it deviates by 41.1 cents (a diesis of ratio 128:125, almost twice a syntonic comma!) from the corresponding pure interval of $8/5$ or 813.7 cents.

On the contrary, the intervals from D shown in the table above, since D is at the center of the stack, do not include wolf intervals and include a pure m6 (from D to $B?$), instead of an impure A5. Notice that in the above mentioned set of 144 intervals pure m6's are more frequently observed than impure A5's (see table below), and this is one of the reasons why it is not desirable to show an impure A5 in the construction table. A *C-based symmetric* stack might be also used, to avoid the above mentioned drawback:

$G\flat - D\flat - A\flat - E\flat - B\flat - F - C - G - D - A - E - B - F\sharp$

In this stack, $G\flat$ and $F\sharp$ have a similar frequency, and $G\flat$ is typically discarded. Also, the note between C and D is called $D\flat$ rather than $C\sharp$, and the note between G and A is called $A\flat$ rather than $G\sharp$. The C-based symmetric stack is rarely used, possibly because it produces the wolf fifth in the unusual position of $F\sharp - D\flat$ instead of $G\sharp - E\flat$, where musicians using Pythagorean tuning expected it).

Justly intonated quarter-comma meantone

A just intonation version of the quarter-comma meantone temperament may be constructed in the same way as Johann Kirnberger's rational version of 12-TET. The value of $5^{1/8} 35^{1/3}$ is very close to 4, that's why a 7-limit interval 6144:6125 (which is the difference between the 5-limit diesis 128:125 and the septimal diesis 49:48), equal to 5.362 cents, appears very close to the quarter-comma $(81/80)^{1/4}$ of 5.377 cents. So the perfect fifth has the ratio of 6125:4096, which is the difference between three just major thirds and two septimal major seconds; four such fifths exceed the

ratio of 5:1 by the tiny interval of 0.058 cents. The wolf fifth there appears to be 49:32, the difference between the septimal minor seventh and the septimal major second.

Greater and lesser semitones

As discussed above, in the quarter-comma meantone temperament,

- the ratio of a semitone is $S = 8 : 5^{5/4}$,
- the ratio of a tone is $T = \sqrt{5} : 2$.

The tones in the diatonic scale can be divided into pairs of semitones. However, since S^2 is not equal to T , each tone must be composed of a pair of unequal semitones, S , and X :

$$S \cdot X = T.$$

Hence,

$$X = \frac{T}{S} = \frac{\sqrt{5}/2}{8/5^{5/4}} = \frac{5^{1/2} \cdot 5^{5/4}}{8 \cdot 2} = \frac{5^{7/4}}{16}.$$

Notice that S is 117.1 cents, and X is 76.0 cents. Thus, S is the greater semitone, and X is the lesser one. S is commonly called the **diatonic semitone** (or minor second), while X is called the **chromatic semitone** (or augmented unison).

The sizes of S and X can be compared to the just intonated ratio 18/17 which is 99.0 cents. S deviates from it by +18.2 cents, and X by -22.9 cents. These two deviations are comparable to the syntonic comma (21.5 cents), which this system is designed to tune out from the Pythagorean major third. However, since even the just intonated ratio 18/17 sounds markedly dissonant, these deviations are considered acceptable in a semitone.

Size of intervals

The table above shows only intervals from D. However, intervals can be formed by starting from each of the above listed 12 notes. Thus, twelve intervals can be defined for each **interval type** (twelve unisons, twelve semitones, twelve intervals composed of 2 semitones, twelve intervals composed of 3 semitones, etc.).

As explained above, one of the twelve fifths (the wolf fifth) has a different size with respect to the other eleven. For a similar reason, each of the other interval types, except for the unisons and the octaves, has two different sizes in quarter-comma meantone. This is the price paid for seeking just intonation. The table below shows their approximate size in cents. Interval names are given in their standard shortened form. For instance, the size of the interval from D to A, which is a perfect fifth (**P5**), can be found in the seventh column of the row labeled **D**. Strictly just (or *pure*) intervals are shown in **bold** font. Wolf intervals are highlighted in red.

From	Interval											
	A1	A5	A2	A6	A3	P1	P5	M2	M6	M3	M7	A4
E $\flat$	76	773	269	966	462	0	697	193	890	386	1083	579
B $\flat$	76	773	269	966	503	0	697	193	890	386	1083	579
F	76	773	269	1007	503	0	697	193	890	386	1083	579
C	76	773	310	1007	503	0	697	193	890	386	1083	579
G	76	814	310	1007	503	0	697	193	890	386	1083	579
D	117	814	310	1007	503	0	697	193	890	386	1083	579
A	117	814	310	1007	503	0	697	193	890	386	1083	621
E	117	814	310	1007	503	0	697	193	890	386	1124	621
B	117	814	310	1007	503	0	697	193	890	427	1124	621
F $\sharp$	117	814	310	1007	503	0	697	193	931	427	1124	621
C $\sharp$	117	814	310	1007	503	0	697	234	931	427	1124	621
G $\sharp$	117	814	310	1007	503	0	738	234	931	427	1124	621
	m2	m6	m3	m7	P4	P1	d6	d3	d7	d4	d8	d5

Approximate size in cents of the 144 intervals in D-based quarter-comma meantone tuning. Interval names are given in their standard shortened form. Pure intervals are shown in **bold** font. Wolf intervals are highlighted in red.

Surprisingly, although this tuning system was designed to produce pure major thirds, only 8 of them are pure (5:4 or about 386.3 cents).

The reason why the interval sizes vary throughout the scale is that the pitches forming the scale are unevenly spaced. Namely, as mentioned above, the frequencies defined by construction for the twelve notes determine two different kinds of semitones (i.e. intervals between adjacent notes):

1. The minor second (m2), also called diatonic semitone, with size

$$S \text{ (or } S_1) = \frac{8}{5^{5/4}} \approx 117.1 \text{ cents}$$

(for instance, between D and E?)

2. The augmented unison (A1), also called chromatic semitone, with size

$$X \text{ (or } S_2) = \frac{5^{7/4}}{16} \approx 76.0 \text{ cents}$$

(for instance, between C and C?)

Conversely, in an equally tempered chromatic scale, by definition the twelve pitches are equally spaced, all semitones having a size of exactly

$$S_E = \sqrt[12]{2} = 100 \text{ cents.}$$

As a consequence all intervals of any given type have the same size (e.g., all major thirds have the same size, all fifths have the same size, etc.). The price paid, in this case, is that none of them is justly tuned and perfectly consonant, except, of course, for the unison and the octave.

By definition, in quarter-comma meantone 11 perfect fifths (**P5** in the table) have a size of approximately 696.6 cents ($700 - \Sigma$ cents, where $\Sigma = 3.422$ cents); since the average size of the 12 fifths must equal exactly 700 cents (as in equal temperament), the other one must have a size of $700 + 11\Sigma$ cents, which is about 737.6 cents (the wolf fifth). Notice that, as shown in the table, the latter interval, although enharmonically equivalent to a fifth, is more properly called a diminished sixth (**d6**). Similarly,

- 10 major seconds (**M2**) are ≈ 193.2 cents ($200 - 2\Sigma$), 2 diminished thirds (**d3**) are ≈ 234.2 cents ($200 + 10\Sigma$), and their average is 200 cents;
- 9 minor thirds (**m3**) are ≈ 310.3 cents ($300 + 3\Sigma$), 3 augmented seconds (**A2**) are ≈ 269.2 cents ($300 - 9\Sigma$), and their average is 300 cents;
- 8 major thirds (**M3**) are ≈ 386.3 cents ($400 - 4\Sigma$), 4 diminished fourths (**d4**) are ≈ 427.4 cents ($400 + 8\Sigma$), and their average is 400 cents;
- 7 diatonic semitones (**m2**) are ≈ 117.1 cents ($100 + 5\Sigma$), 5 chromatic semitones (**A1**) are ≈ 76.0 cents ($100 - 7\Sigma$), and their average is 100 cents.

In short, similar differences in width are observed for all interval types, except for unisons and octaves, and they are all multiples of Σ , the difference between the 1/4-comma meantone fifth and the average fifth.

Notice that, as an obvious consequence, each augmented or diminished interval is exactly 12Σ cents (≈ 41.1 cents) wider or narrower than its enharmonic equivalent. For instance, the d6 (or wolf fifth) is 12Σ cents wider than each P5, and each A2 is 12Σ cents narrower than each m3. This interval of size 12Σ is known as a diesis, or diminished second. This implies that Σ can be also defined as one twelfth of a diesis.

Triads in the chromatic scale

The major triad can be defined by a pair of intervals from the root note: a major third (interval spanning 4 semitones) and a perfect fifth (7 semitones). The minor triad can likewise be defined by a minor third (3 semitones) and a perfect fifth (7 semitones).

As shown above, a chromatic scale has twelve intervals spanning seven semitones. Eleven of these are perfect fifths (P5), while the twelfth is a diminished sixth (d6). Since they span the same number of semitones, P5 and d6 are considered to be enharmonically equivalent. In an equally tuned chromatic scale, P5 and d6 have exactly the same size. The same is true for all the enharmonically equivalent intervals spanning 4 semitones (M3 and d4), or 3 semitones (m3 and A2). However, in the meantone temperament this is not true. In this tuning system, enharmonically equivalent intervals may have different sizes, and some intervals may markedly deviate from their justly tuned ideal ratios. As explained in the previous section, if the deviation is too large, then the given interval is not usable, either by itself or in a chord.

The following table focuses only on the above-mentioned three interval types, used to form major and minor triads. Each row shows three intervals of different types but which have the same root note. Each interval is specified by a pair of notes. To the right of each interval is listed the formula for the interval ratio. The intervals d4, d6 and A2 may be regarded as wolf intervals, and have been marked in red. S and X denote the ratio of the two above-mentioned kinds of semitones (m2 and A1).

3 semitones (m3 or A2)		4 semitones (M3 or d4)		7 semitones (P5 or d6)	
Interval	Ratio	Interval	Ratio	Interval	Ratio
C—E $\flat$	$S^2 \cdot X$	C—E	$S^2 \cdot X^2$	C—G	$S^4 \cdot X^3$
C $\sharp$ —E	$S^2 \cdot X$	C $\sharp$ —F	$S^3 \cdot X$	C $\sharp$ —G $\sharp$	$S^4 \cdot X^3$
D—F	$S^2 \cdot X$	D—F $\sharp$	$S^2 \cdot X^2$	D—A	$S^4 \cdot X^3$
E $\flat$ —F $\sharp$	$S \cdot X^2$	E $\flat$ —G	$S^2 \cdot X^2$	E $\flat$ —B $\flat$	$S^4 \cdot X^3$
E—G	$S^2 \cdot X$	E—G $\sharp$	$S^2 \cdot X^2$	E—B	$S^4 \cdot X^3$
F—G $\sharp$	$S \cdot X^2$	F—A	$S^2 \cdot X^2$	F—C	$S^4 \cdot X^3$
F $\sharp$ —A	$S^2 \cdot X$	F $\sharp$ —B $\flat$	$S^3 \cdot X$	F $\sharp$ —C $\sharp$	$S^4 \cdot X^3$
G—B $\flat$	$S^2 \cdot X$	G—B	$S^2 \cdot X^2$	G—D	$S^4 \cdot X^3$
G $\sharp$ —B	$S^2 \cdot X$	G $\sharp$ —C	$S^3 \cdot X$	G $\sharp$ —E $\flat$	$S^5 \cdot X^2$
A—C	$S^2 \cdot X$	A—C $\sharp$	$S^2 \cdot X^2$	A—E	$S^4 \cdot X^3$
B $\flat$ —C $\sharp$	$S \cdot X^2$	B $\flat$ —D	$S^2 \cdot X^2$	B $\flat$ —F	$S^4 \cdot X^3$
B—D	$S^2 \cdot X$	B—E $\flat$	$S^3 \cdot X$	B—F $\sharp$	$S^4 \cdot X^3$

First, look at the last two columns on the right. All the 7-semitone intervals except one have a ratio of

$$S^4 \cdot X^3 \approx 1.4953 \approx 696.6 \text{ cents}$$

which deviates by -5.4 cents from the just 3:2 of 702.0 cents. Five cents is small and acceptable. On the other hand, the d6 from G $\sharp$ to E $\flat$ has a ratio of

$$S^5 \cdot X^2 \approx 1.5312 \approx 737.6 \text{ cents}$$

which deviates by +35.7 cents from the just fifth. Thirty-five cents is beyond the acceptable range.

Now look at the two columns in the middle. Eight of the twelve 4-semitone intervals have a ratio of

$$S^2 \cdot X^2 = 1.25 \approx 386.3 \text{ cents}$$

which is exactly a just 5:4. On the other hand, the four d4 with roots at C $\sharp$, F $\sharp$, G $\sharp$ and B have a ratio of

$$S^3 \cdot X = 1.28 \approx 427.4 \text{ cents}$$

which deviates by +41.1 cents from the just M3. Again, this sounds badly out of tune.

Major triads are formed out of both major thirds and perfect fifths. If either of the two intervals is substituted by a wolf interval (d6 instead of P5, or d4 instead of M3),

then the triad is not acceptable. Therefore major triads with root notes of C#, F#, G# and B are not used in meantone scales whose fundamental note is C. Now look at the first two columns on the left. Nine of the twelve 3-semitone intervals have a ratio of

$$S^2 \cdot X \approx 1.1963 \approx 310.3 \text{ cents}$$

which deviates by -5.4 cents from the just 6:5 of 315.6 cents. Five cents is acceptable. On the other hand, the three augmented seconds whose roots are E?, F and B? have a ratio of

$$S \cdot X^2 \approx 1.1682 \approx 269.2 \text{ cents}$$

which deviates by -46.4 cents from the just minor third. It is a close match, however, for the 7:6 septimal minor third of 266.9 cents, deviating by +2.3 cents. These augmented seconds, though sufficiently consonant by themselves, will sound "exotic" or atypical when played together with a perfect fifth.

Minor triads are formed out of both minor thirds and fifths. If either of the two intervals are substituted by an enharmonically equivalent interval (d6 instead of P5, or A2 instead of m3), then the triad will not sound good. Therefore minor triads with root notes of E♭, F, G# and B♭ are not used in the meantone scale defined above.

The following major triads are usable: C, D, E♭, E, F, G, A, B♭.

The following minor triads are usable: C, C#, D, E, F♭, G, A, B.

The following root notes are useful for both major and minor triads: C, D, E, G and A. Notice that these five pitches form a major pentatonic scale.

The following root notes are useful only for major triads: E♭, F, B♭.

The following root notes are useful only for minor triads: C#, F#, B.

The following root note is useful for neither major nor minor triad: G#.

Alternative construction

As discussed above, in the quarter-comma meantone temperament,

- the ratio of a greater (diatonic) semitone is $S = 8 : 5^{5/4}$,
- the ratio of a lesser (chromatic) semitone is $X = 5^{7/4} : 16$,
- the ratio of most whole tones is $T = \sqrt{5} : 2$,
- the ratio of most fifths is $P = 5^{1/4}$.

It can be verified through calculation that most whole tones (namely, the major seconds) are composed of one greater and one lesser semitone:

$$T = S \cdot X = \frac{8}{5^{5/4}} \cdot \frac{5^{7/4}}{16} = \frac{5^{2/4}}{2} = \frac{\sqrt{5}}{2}.$$

Similarly, a fifth is typically composed of three tones and one greater semitone:

$$P = T^3 \cdot S = \frac{5^{3/2}}{2^3} \cdot \frac{8}{5^{5/4}} = 5^{6/4-5/4} = 5^{1/4},$$

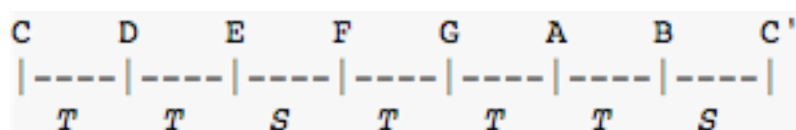
which is equivalent to four greater and three lesser semitones:

$$P = T^3 \cdot S = S^4 \cdot X^3.$$

Diatonic scale

A diatonic scale can be constructed by starting from the fundamental note and

multiplying it either by T to move up by a tone or by S to move up by a semitone.

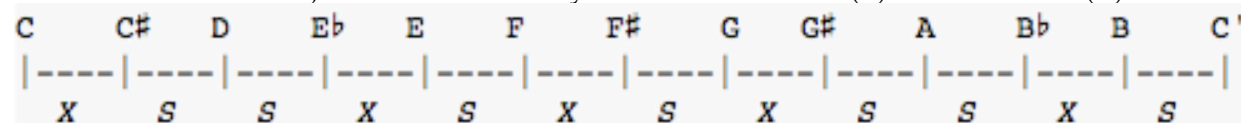


The resulting interval sizes with respect to the base note C are shown in the following table:

Note	Formula	Ratio	Cents	Pythagorean cents	EQT Cents
C	1	1.0000	0.0	0.0	0
D	T	1.1180	193.2	203.9	200
E	T^2	1.2500	386.3	407.8	400
F	$T^2 S$	1.3375	503.4	498.0	500
G	P	1.4953	696.6	702.0	700
A	$P T$	1.6719	889.7	905.9	900
B	$P T^2$	1.8692	1082.9	1109.8	1100
C'	$P T^2 S$	2.0000	1200.0	1200.0	1200

Chromatic scale

Construction of a 1/4-comma meantone chromatic scale can proceed by stacking a series of 12 semitones, each of which may be either diatonic (S) or chromatic (X).



Notice that this scale is an extension of the diatonic scale shown in the previous table. Only five notes have been added: C#, Eb, F#, G# and Bb (a pentatonic scale).

As explained above, an identical scale was originally defined and produced by using a sequence of tempered fifths, ranging from Eb (five fifths below D) to G# (six fifths above D), rather than a sequence of semitones. This more conventional approach, similar to the *D-based* Pythagorean tuning system, explains the reason why the X and S semitones are arranged in the particular and apparently arbitrary sequence shown above.

The interval sizes with respect to the base note C are presented in the following table. The frequency ratios are computed as shown by the formulas. Delta is the difference in cents between meantone and 12-TET. 1/4-c is the difference in quarter-commas between meantone and Pythagorean tuning.

Note	Formula	Ratio	Cents	12TET	Delta	1/4-c
C	1	1.0000	0.0	0	0.0	0
C#	X	1.0449	76.0	100	-24.0	-7
D	T	1.1180	193.2	200	-6.8	-2
E $\flat$	$T S$	1.1963	310.3	300	+10.3	3
E	T^2	1.2500	386.3	400	-13.7	-4
F	$T^2 S$	1.3375	503.4	500	+3.4	1
F#	T^3	1.3975	579.5	600	-20.5	-6
G	P	1.4953	696.6	700	-3.4	-1
G#	$P X$	1.5625	772.6	800	-27.4	-8
A	$P T$	1.6719	889.7	900	-10.3	-3
B $\flat$	$P T S$	1.7889	1006.8	1000	+6.8	2
B	$P T^2$	1.8692	1082.9	1100	-17.1	-5
C'	$P T^2 S$	2.0000	1200.0	1200	0.0	0

Comparison with 31 equal temperament

The perfect fifth of quarter-comma meantone, expressed as a fraction of an octave, is $1/4 \log_2 5$. This number is irrational and in fact transcendental; hence a chain of meantone fifths, like a chain of pure $3/2$ fifths, never closes (i.e. never equals a chain of octaves). However, the continued fraction approximations to this irrational fraction number allow us to find equal divisions of the octave which do close; the denominators of these are 1, 2, 5, 7, 12, 19, 31, 174, 205, 789 ... From this we find that 31 quarter-comma meantone fifths come close to closing, and conversely 31 equal temperament represents a good approximation to quarter-comma meantone. See: schisma.

Circle of Fifth and Fourth Intervals

The circle of fifths and fourths can also be very helpful to learn intervals. Although intervals have already been explored, they have not been discussed from the perspective of the circle of fifths. The figures in this chapter are going to be very helpful visual aids in learning and remembering intervals by utilizing geometrical patterns.

The figures below show major second and minor seventh intervals. The major second intervals rotate clock-wise, and the minor seventh intervals rotate counter clock-wise. The first figure 6.1 shows the first iteration starting from C. C to D is a major second interval, and C to Bb is a minor seventh interval. The second figure 6.2 completes six major second intervals and six minor seventh intervals creating the hexagon shape.

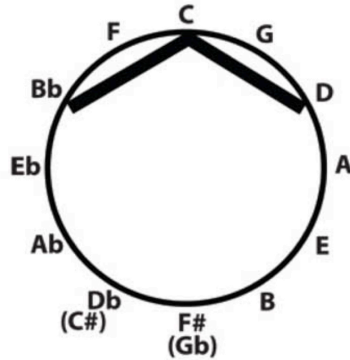


Figure 6.1 – C major second and minor seventh intervals

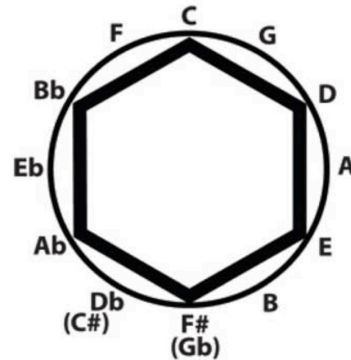


Figure 6.2 – Six major second and minor seventh intervals

By rotating the hexagon to the right (1/12 clock-wise turn), the remaining six interval sets can be constructed. The first figure 6.3 below shows one iteration from G to A (major second) clock-wise and G to F (minor seventh) counter-clockwise. The second figure 6.4 completes the cycle of major seconds and minor sevenths.

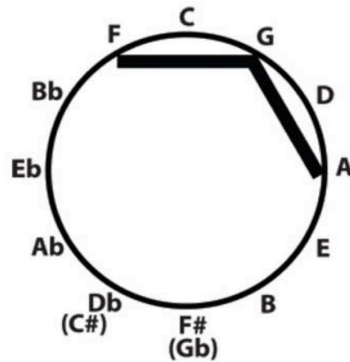


Figure 6.3 – G major second and minor seventh intervals

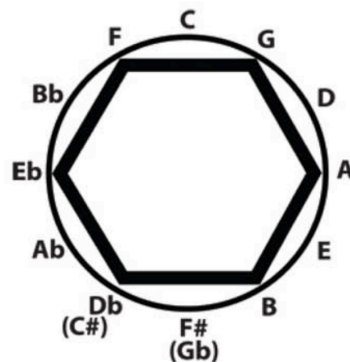
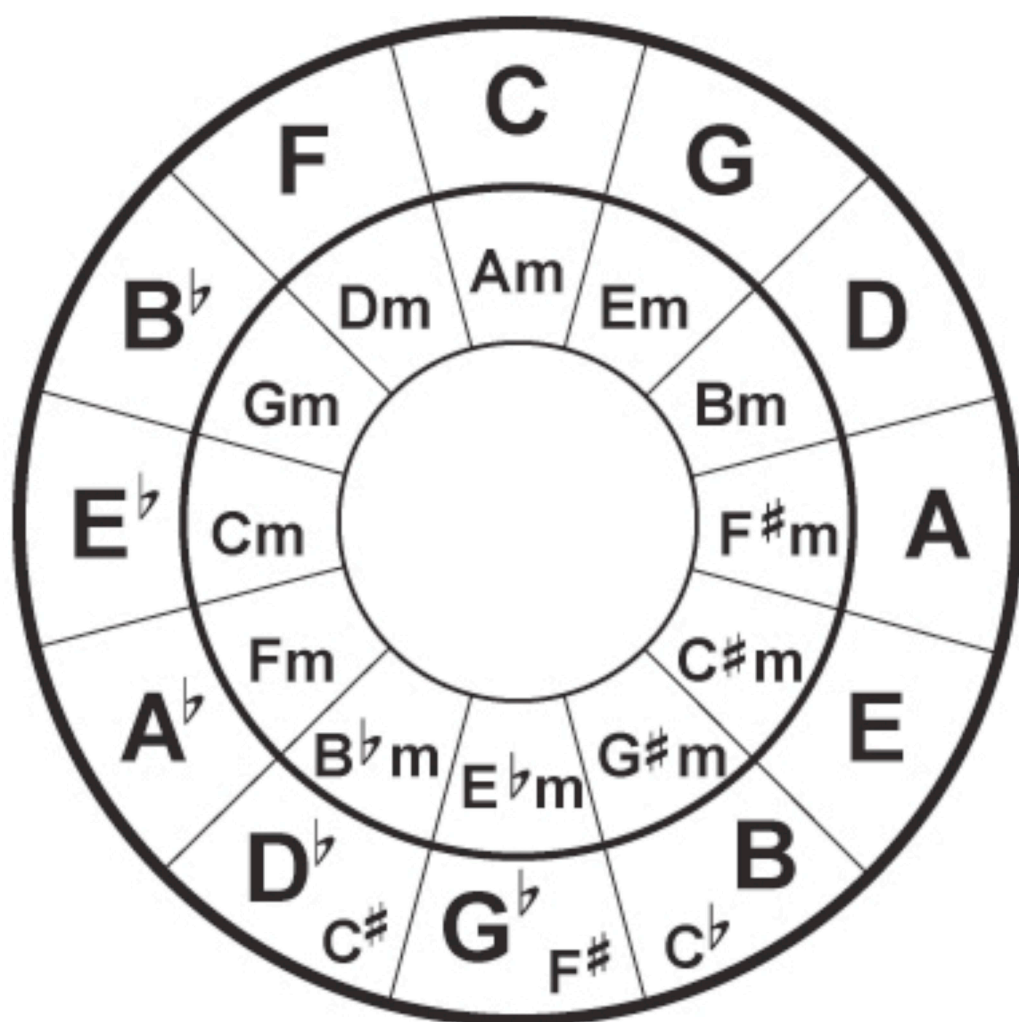
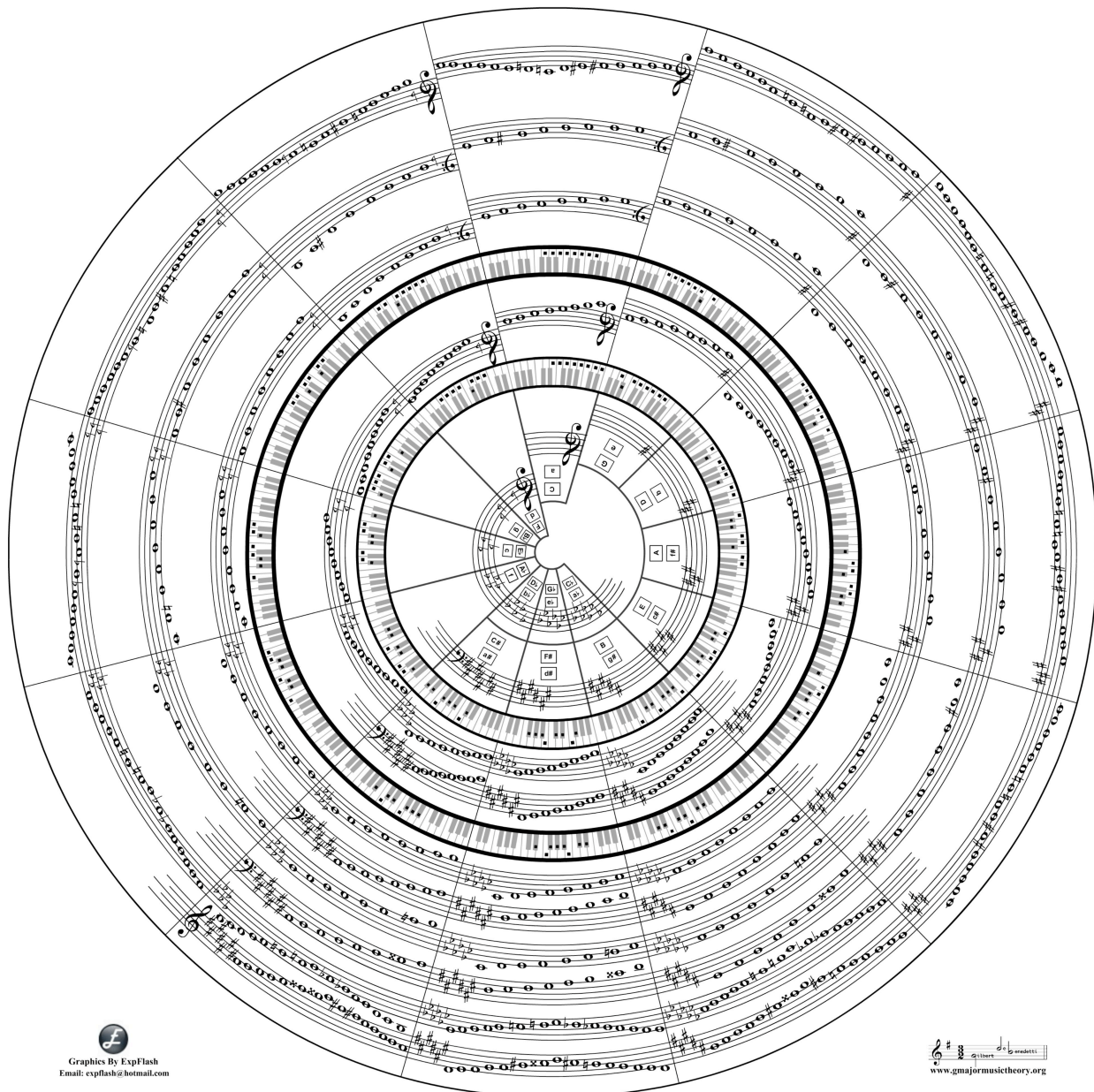
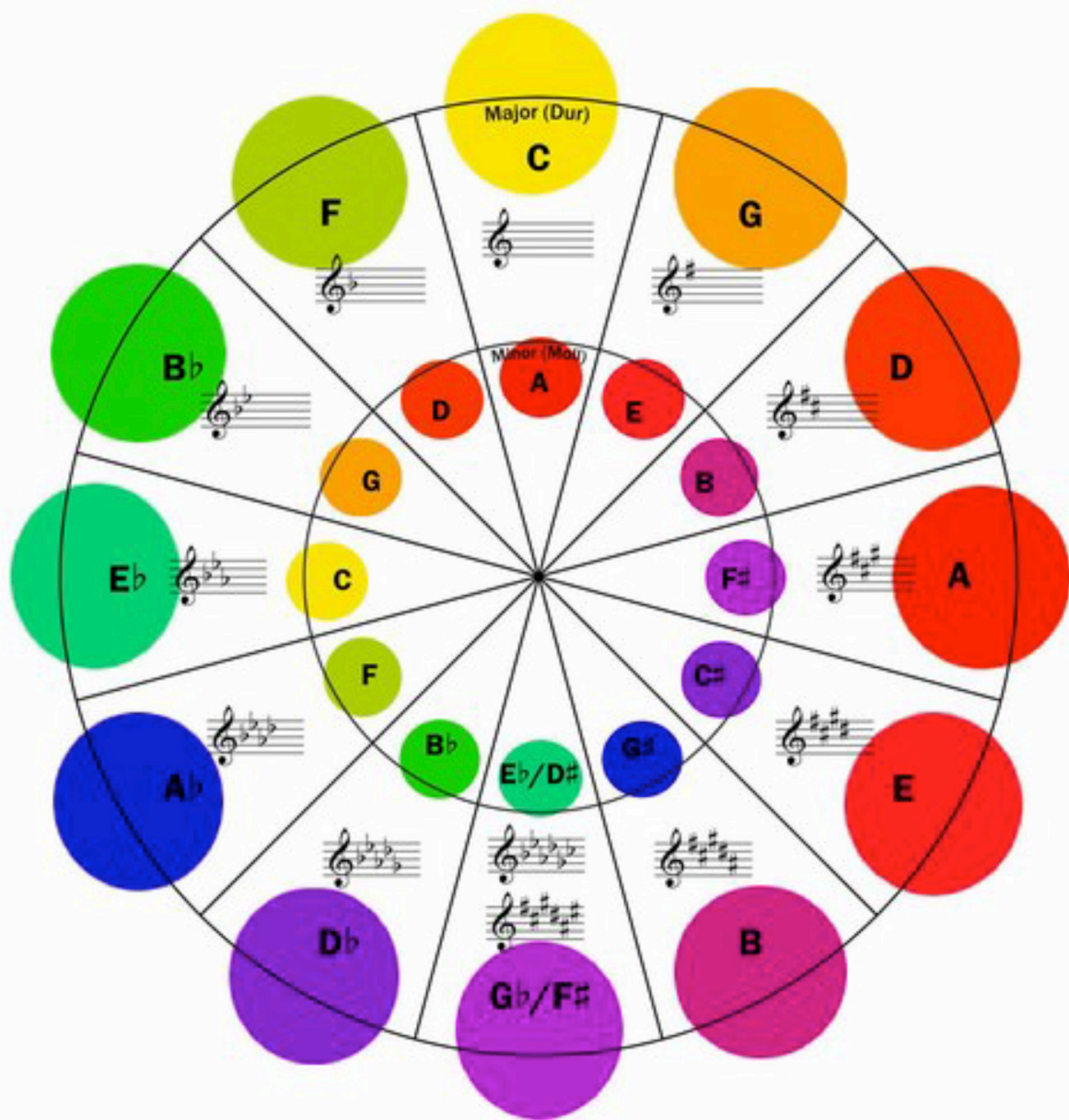


Figure 6.4 – Six major second and minor seventh intervals

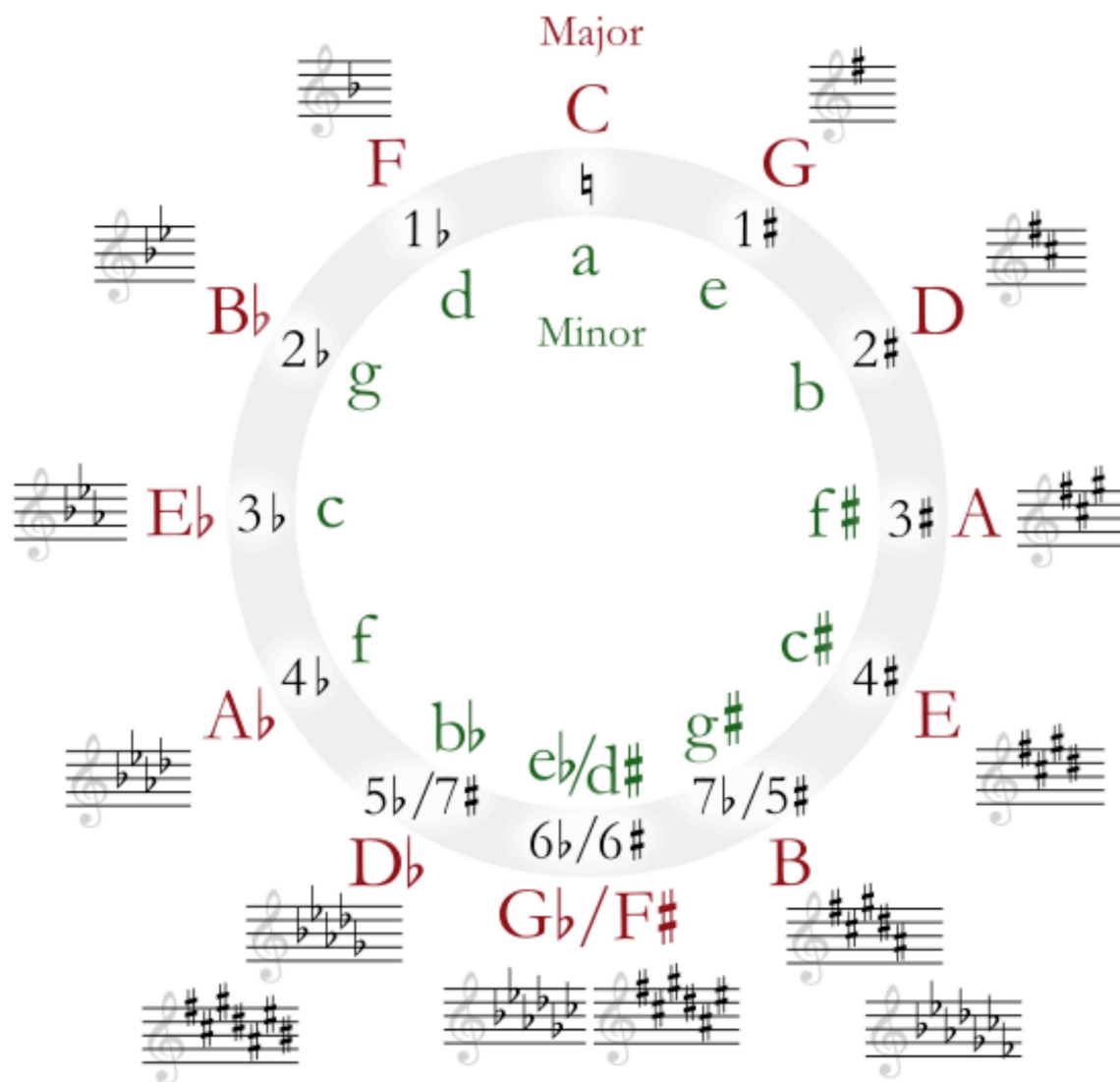
The four figures below show major third and minor sixth intervals. The major third intervals rotate clock-wise, and the minor sixth intervals rotate counter clock-wise. Major thirds cycle are very helpful when building augmented chords and have only four iterations. Remember, the formula to build augmented chords is 1, 3, #5, or two major third intervals. The four figures below show the four iterations. The first one shows the notes to build the Bbaug, Daug, and F#aug chords. The Bbaug chord has the notes Bb, D, F#. Bb to D is a major third interval, as well as D to F#. The Daug chord has the notes D, F#, and Bb, and the F#aug chord has the notes F#, Bb, and D. Notice all three chords use the same







Circle of fifths



Circle of fifths showing major and minor keys

Первое колесо выста вообразъ веселой мѣсн.
 ии наставляе же егда вообразъ печальной
 мѣсн ии се же вѣжда и же во вса челоукомъ
 писменѣ мощно ти нѣкоже нѣсть оуво-
 рнѣти и **д**, и **р**, и **м**, и **ф**, и **с**, и **л**.

Вообразъ:



Образъ колесо прѣидѣе низемни.

Nikolay Diletsky's circle of fifths in *Idea grammatiki musikiyskoy* (Moscow, 1679)

In music theory, the **circle of fifths** (or **circle of fourths**) is a visual representation of the relationships among the 12 tones of the chromatic scale, their corresponding key signatures, and the associated major and minor keys. More specifically, it is a geometrical representation of relationships among the 12 pitch classes of the chromatic scale in pitch class space.

Definition

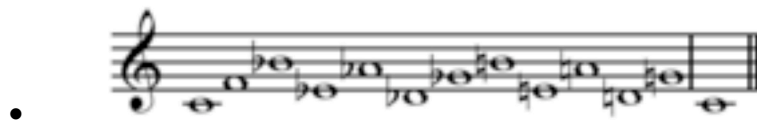
The term 'fifth' defines an interval or mathematical ratio which is the closest and most consonant non-octave interval. The circle of fifths is a sequence of pitches or key tonalities, represented as a circle, in which the next pitch is found seven semitones higher than the last. Musicians and composers use the circle of fifths to understand and describe the musical relationships among some selection of those pitches. The circle's design is helpful in composing and harmonizing melodies, building chords, and modulating to different keys within a composition.

At the top of the circle, the key of C Major has no sharps or flats. Starting from the apex and proceeding clockwise by ascending fifths, the key of G has one sharp, the key of D has 2 sharps, and so on. Similarly, proceeding counterclockwise from the apex by descending fifths, the key of F has one flat, the key of B $\flat$ has 2 flats, and so on. At the bottom of the circle, the sharp and flat keys overlap, showing pairs of enharmonic key signatures.

Starting at any pitch, ascending by the interval of an equal tempered fifth, one passes all twelve tones clockwise, to return to the beginning pitch class. To pass the twelve tones counterclockwise, it is necessary to ascend by perfect fourths, rather than fifths. (To the ear, the sequence of fourths gives an impression of settling, or resolution. (see cadence))



circle of fifths clockwise within one octave



circle of fifths counterclockwise within one octave

Structure and use

Pitches within the chromatic scale are related not only by the number of semitones between them within the chromatic scale, but also related harmonically within the circle of fifths. Reversing the direction of the circle of fifths gives the **circle of fourths**. Typically the "circle of fifths" is used in the analysis of classical music, whereas the "circle of fourths" is used in the analysis of Jazz music, but this distinction is not exclusive. Since fifths and fourths are intervals composed respectively of 7 and 5 semitones, the circumference of a circle of fifths is an interval as large as 7 octaves (84 semitones), while the circumference of a circle of fourths equals only 5 octaves (60 semitones).



Octaves ($7 \times 1200 = 8400$) versus fifths ($12 \times 700 = 8400$), depicted as with Cuisenaire rods (red (2) is used for 1200, black (7) is used for 700).

Diatonic key signatures

The circle is commonly used to represent the relationship between diatonic scales. Here, the letters on the circle are taken to represent the major scale with that note as tonic. The numbers on the inside of the circle show how many sharps or flats the key signature for this scale has. Thus a major scale built on A has 3 sharps in its key signature. The major scale built on F has 1 flat.

For minor scales, rotate the letters counter-clockwise by 3, so that, e.g., A minor has 0 sharps or flats and E minor has 1 sharp. (See relative key for details.) A way to describe this phenomenon is that, for any major key [e.g. G major, with one sharp (F#) in its diatonic scale], a scale can be built beginning on the sixth (VI) degree (relative minor key, in this case, E) containing the same notes, but from E - E as opposed to G - G. Or, G-major scale (G - A - B - C - D - E - F# - G) is enharmonic (harmonically equivalent) to the e-minor scale (E - F# - G - A - B - C - D - E).

When notating the key signatures, the order of sharps that are found at the beginning of the staff line follows the circle of fifths from F through B. The order is F, C, G, D, A, E, B. If there is only one sharp, such as in the key of G major, then the one sharp is F sharp. If there are two sharps, the two are F and C, and they appear in that order in the key signature. The order of sharps goes clockwise around the circle of fifths. (The major key you are in is one half-step above the last sharp you see in the key signature. This does not work if you are in a minor key.)

For notating flats, the order is reversed: B, E, A, D, G, C, F. This order runs counter-clockwise along the circle of fifths; in other words they progress by fourths. If you follow the major keys from the key of F to the key of C flat (B) counter-clockwise around the circle of fifths, you will see that as each key signature adds a flat, they are these seven flats always in this order. D flat in the key signature is always found in the fourth position among flats, and G is always in the fifth position. If there are three flats in the key signature, they are only B, E, and A, while the other positions are empty. (The major key you are in is the penultimate flat you find in the key signature; of course, there must be at least two flats in the key signature for this to work. It does not work if you are in a minor key.)

Modulation and chord progression

Tonal music often modulates by moving between adjacent scales on the circle of fifths. This is because diatonic scales contain seven pitch classes that are contiguous on the circle of fifths. It follows that diatonic scales a perfect fifth apart share six of their seven notes. Furthermore, the notes not held in common differ by only a semitone. Thus modulation by perfect fifth can be accomplished in an exceptionally smooth fashion. For example, to move from the C major scale F - C - G - D - A - E - B to the G major scale C - G - D - A - E - B - F?, one need only move the C major scale's "F" to "F#".

In Western tonal music, one also finds chord progressions between chords whose roots are related by perfect fifth. For instance, root progressions such as D-G-C are common. For this reason, the circle of fifths can often be used to represent "harmonic distance" between chords.



IV-V-I, in C

According to theorists including Goldman, harmonic function (the use, role, and relation of chords in harmony), including "functional succession", may be "explained by the circle of fifths (in which, therefore, scale degree II is closer to the dominant than scale degree IV)". In this view the tonic is considered the end of the

line towards which a chord progression derived from the circle of fifths progresses.



ii-V-I turnaround, in C Play subdominant, supertonic seventh, and supertonic chords illustrating the similarity between them.

According to Goldman's *Harmony in Western Music*, "the IV chord is actually, in the simplest mechanisms of diatonic relationships, at the greatest distance from I. In terms of the [descending] circle of fifths, it leads away from I, rather than toward it." Thus the progression I-ii-V-I (an authentic cadence) would feel more final or resolved than I-IV-I (a plagal cadence). Goldman[4] concurs with Nattiez, who argues that "the chord on the fourth degree appears long before the chord on II, and the subsequent final I, in the progression I-IV-viio-iii-vi-ii-V-I", and is farther from the tonic there as well. (In this and related articles, upper-case Roman numerals indicate major triads while lower-case Roman numerals indicate minor triads.)



IV vs. ii7 with root in parenthesis, in C.

Goldman argues that "historically the use of the IV chord in harmonic design, and especially in cadences, exhibits some curious features. By and large, one can say that the use of IV in final cadences becomes more common in the nineteenth century than it was in the eighteenth, but that it may also be understood as a substitute for the ii chord when it precedes V. It may also be quite logically construed as an incomplete ii7 chord (lacking root)." The delayed acceptance of the IV-I in final cadences is explained aesthetically by its lack of closure, caused by its position in the circle of fifths. The earlier use of IV-V-I is explained by proposing a relation between IV and ii, allowing IV to substitute for or serve as ii. However, Nattiez calls this latter argument "a narrow escape: only the theory of a ii chord without a root allows Goldman to maintain that the circle of fifths is completely valid from Bach to Wagner", or the entire common practice period.

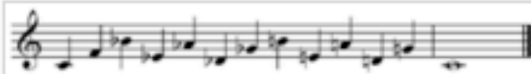
Circle closure in non-equal tuning systems

When an instrument is tuned with the equal temperament system, the width of the fifths is such that the circle "closes". This means that ascending by twelve fifths from any pitch, one returns to a tune exactly in the same pitch class as the initial tune, and exactly seven octaves above it. To obtain such a perfect circle closure, the fifth is slightly flattened with respect to its just intonation (3:2 interval ratio).

Thus, ascending by justly intonated fifths fails to close the circle by an excess of approximately 23.46 cents, roughly a quarter of a semitone, an interval known as the Pythagorean comma. In Pythagorean tuning, this problem is solved by markedly shortening the width of one of the twelve fifths, which makes it severely dissonant. This anomalous fifth is called wolf fifth as a humorous metaphor of the unpleasant

sound of the note (like a wolf trying to howl an off-pitch note). The quarter-comma meantone tuning system uses eleven fifths slightly narrower than the equally tempered fifth, and requires a much wider and even more dissonant wolf fifth to close the circle. More complex tuning systems based on just intonation, such as 5-limit tuning, use at most eight justly tuned fifths and at least three non-just fifths (some slightly narrower, and some slightly wider than the just fifth) to close the circle.

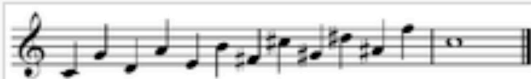
In lay terms



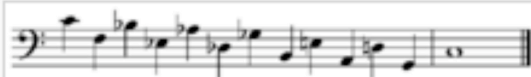
1 octave, fourths, descending



1 octave, fifths, ascending



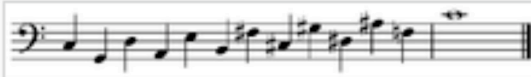
2 octave, fifths, ascending



2 octave, fourths, descending



2 octave, fourths, ascending



2 octave, fifths, descending



multi octave, fifths, ascending



multi octave, fourths, descending



multi octave, fourths, ascending

Playing the circle of fifths

A simple way to see the musical interval known as a fifth is by looking at a piano keyboard, and, starting at any key, counting seven keys to the right (both black and white) to get to the next note on the circle shown above. Seven half steps, the distance from the 1st to the 8th key on a piano is a "perfect fifth", called 'perfect' because it is neither major nor minor, but applies to both major and minor scales and chords, and a 'fifth' because though it is a distance of seven semitones on a keyboard, it is a distance of five steps within a major or minor scale.

A simple way to hear the relationship between these notes is by playing them on a piano keyboard. If you traverse the circle of fifths backwards, the notes will feel as though they fall into each other. This aural relationship is what the mathematics describe.

Perfect fifths may be justly tuned or tempered. Two notes whose frequencies differ by a ratio of 3:2 make the interval known as a justly tuned perfect fifth. Cascading twelve such fifths does not return to the original pitch class after going round the circle, so the 3:2 ratio may be slightly detuned, or tempered. Temperament allows perfect fifths to cycle, and allows pieces to be transposed, or played in any key on a piano or other fixed-pitch instrument without distorting their harmony. The primary tuning system used for Western (especially keyboard and fretted) instruments today, twelve-tone equal temperament, uses an irrational multiplier, $2^{1/12}$, to calculate the frequency difference of a semitone. An equal-tempered fifth, at a frequency ratio of $2^{7/12}$:1 (or about 1.498307077:1) is approximately two cents narrower than a justly tuned fifth at a ratio of 3:2.

History

Musicalischer Circul.

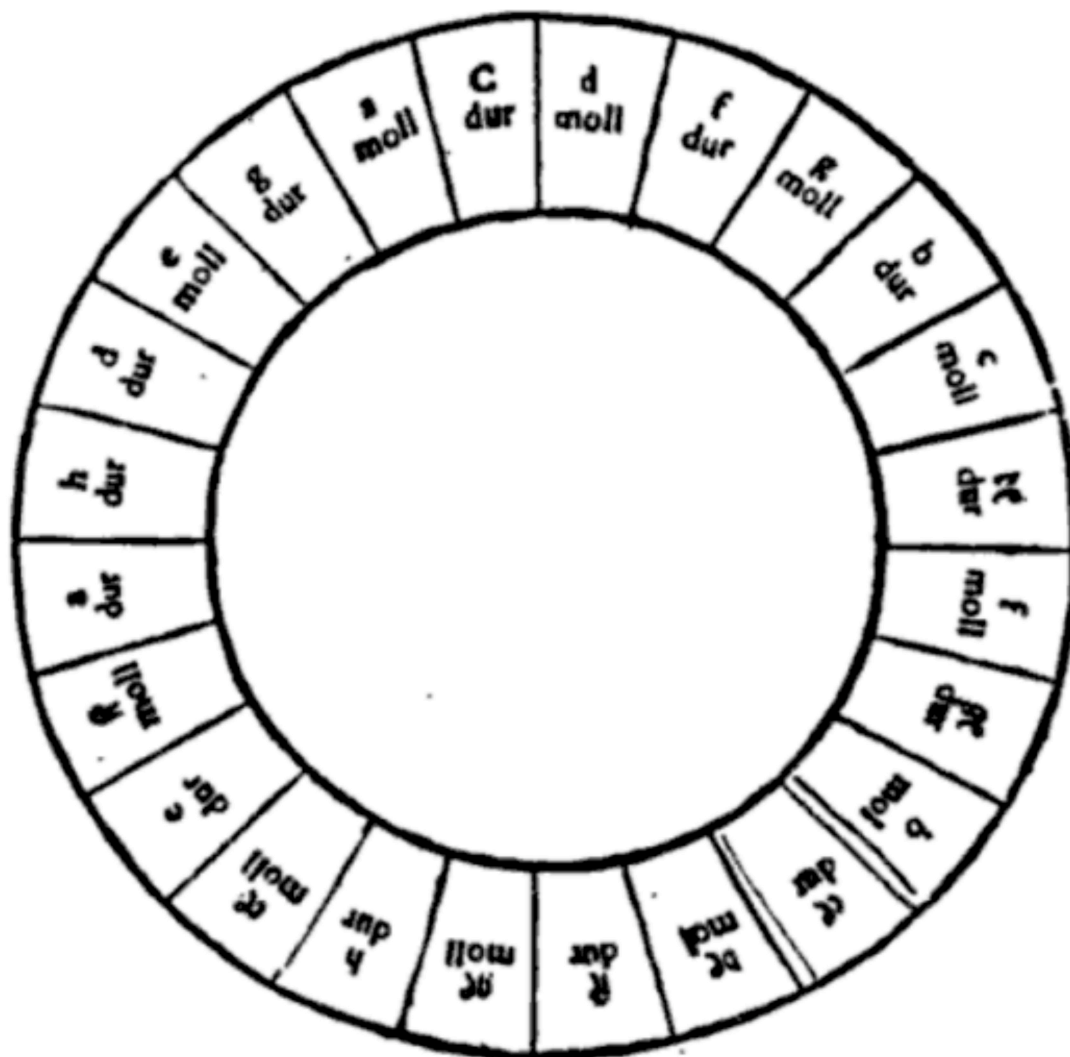


Fig. 261.

Heinichen's musical circle (German: *musicalischer circul*)(1711)

In the late 1670s a treatise called *Grammatika* was written by the composer and theorist Nikolai Diletskii. Diletskii's *Grammatika* is a treatise on composition, the first of its kind, which targeted Western-style polyphonic compositions. It taught how to write kontserty, polyphonic a cappella, which were normally based on liturgical texts and were created by putting together musical sections that have contrasting rhythm, meters, melodic material and vocal groupings. Diletskii intended his treatise to be a guide to composition but pertaining to the rules of music theory. Within the *Grammatika* treatise is where the first circle of fifths appeared and was used for students as a composer's tool.

Related concepts

Diatonic circle of fifths

The diatonic circle of fifths is the circle of fifths encompassing only members of the

diatonic scale. Therefore, it contains a diminished fifth, in C major between B and F. See structure implies multiplicity.

The circle progression is commonly a circle through the diatonic chords by fifths, including one diminished chord and one progression by diminished fifth:



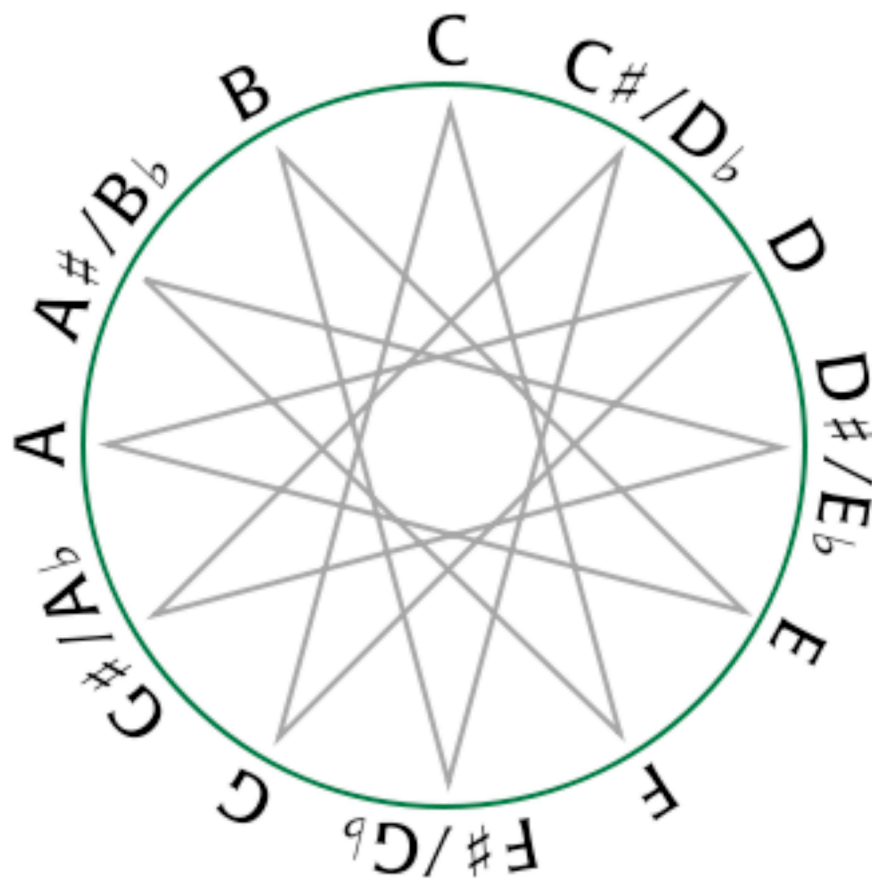
I-IV-viio-iii-vi-ii-V-I (in major)

Chromatic circle

The circle of fifths is closely related to the chromatic circle, which also arranges the twelve equal-tempered pitch classes in a circular ordering. A key difference between the two circles is that the chromatic circle can be understood as a continuous space where every point on the circle corresponds to a conceivable pitch class, and every conceivable pitch class corresponds to a point on the circle. By contrast, the circle of fifths is fundamentally a *discrete* structure, and there is no obvious way to assign pitch classes to each of its points. In this sense, the two circles are mathematically quite different.

However, the twelve equal-tempered pitch classes can be represented by the cyclic group of order twelve, or equivalently, the residue classes modulo twelve, $\mathbb{Z}_{12}$. The group has four generators, which can be identified with the ascending and descending semitones and the ascending and descending perfect fifths. The semitonal generator gives rise to the chromatic circle while the perfect fifth gives rise to the circle of fifths.

Relation with chromatic scale



The circle of fifths drawn within the chromatic circle as a star dodecagram.

The circle of fifths, or fourths, may be mapped from the chromatic scale by multiplication, and vice versa. To map between the circle of fifths and the chromatic scale (in integer notation) multiply by 7 (M7), and for the circle of fourths multiply by 5 (P5).

Here is a demonstration of this procedure. Start off with an ordered 12-tuple (tone row) of integers

(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)

representing the notes of the chromatic scale: 0 = C, 2 = D, 4 = E, 5 = F, 7 = G, 9 = A, 11 = B, 1 = C#, 3 = D#, 6 = F#, 8 = G#, 10 = A#. Now multiply the entire 12-tuple by 7:

(0, 7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77)

and then apply a modulo 12 reduction to each of the numbers (subtract 12 from each number as many times as necessary until the number becomes smaller than 12):

(0, 7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5)

which is equivalent to

(C, G, D, A, E, B, F#, C#, G#, D#, A#, F)

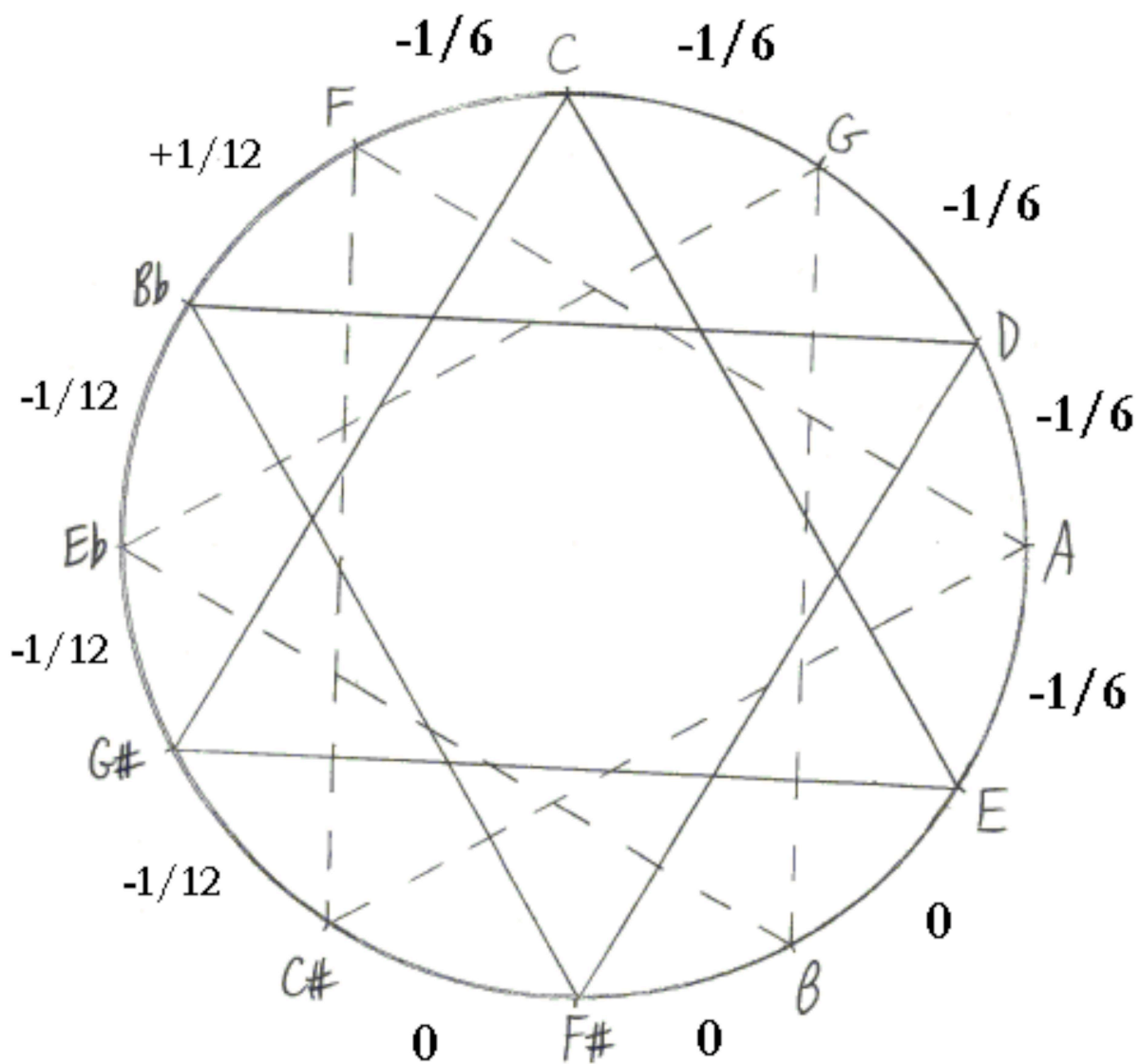
which is the circle of fifths. Note that this is enharmonically equivalent to:

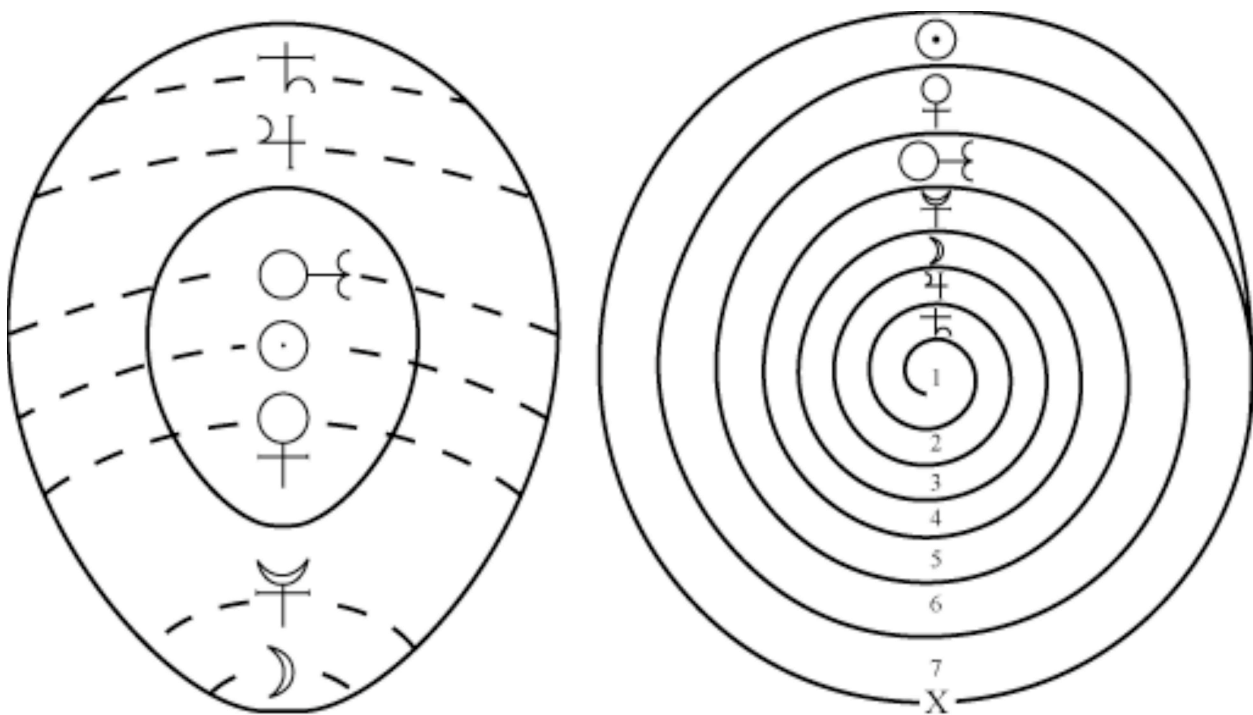
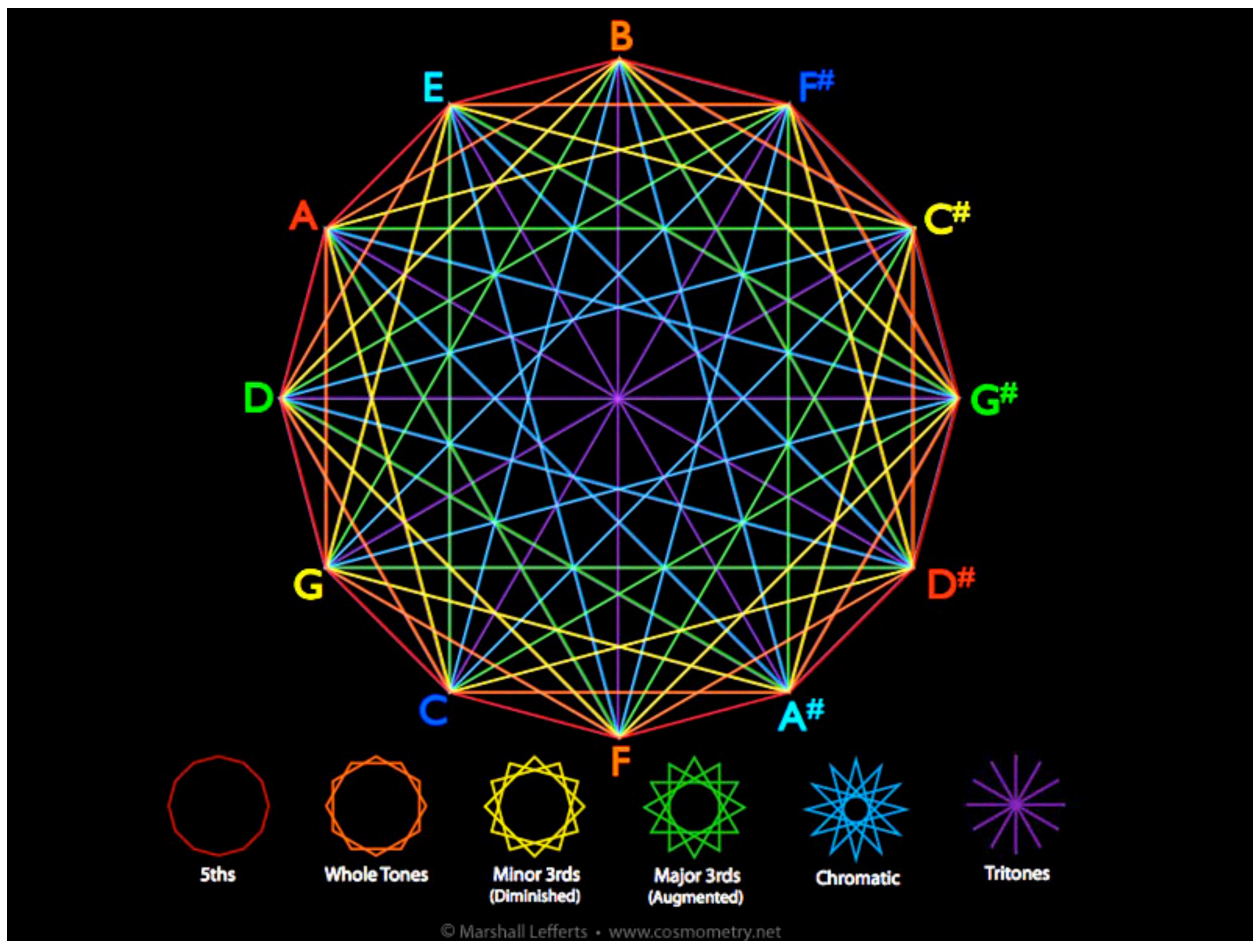
(C, G, D, A, E, B, Gb, Db, Ab, Eb, Bb, F).

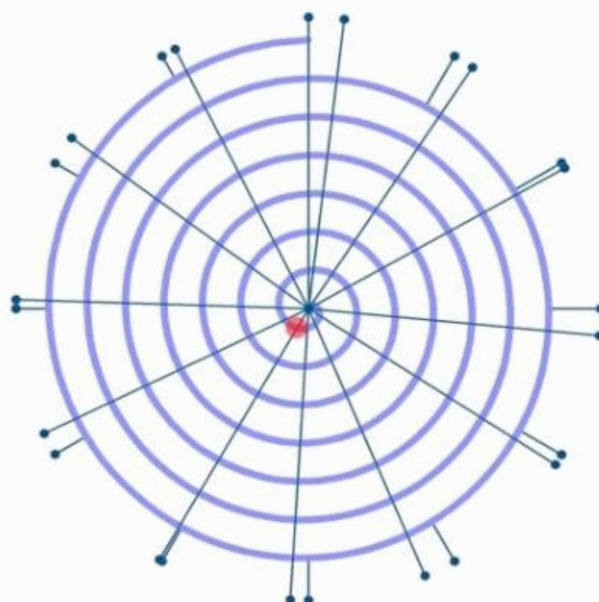
Enharmonics

The “bottom keys” of the circle of fifths are often written in flats and sharps, as they are easily interchanged using enharmonics. For example, the key of B, with five sharps, is enharmonically equivalent to the key of Cb, with 7 flats. But the circle of fifths doesn’t stop at 7 sharps (C#) or 7 flats (Cb). Following the same pattern, one can construct a circle of fifths with all sharp keys, or all flat keys.

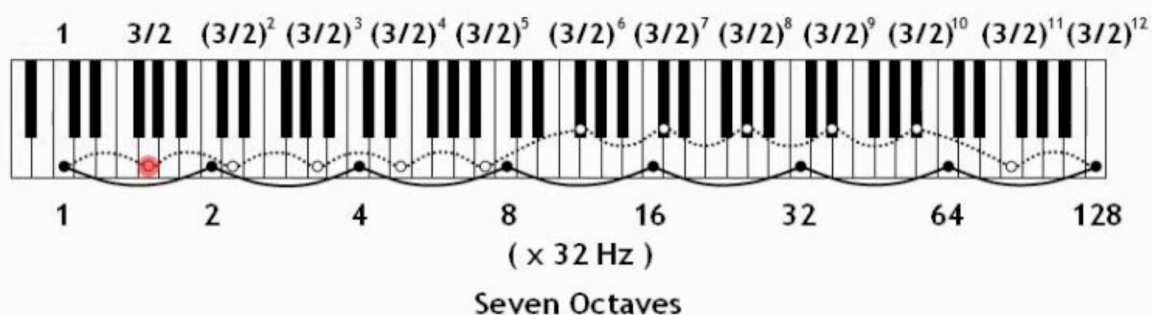
After C# comes the key of G# (following the pattern of being a fifth higher, and, coincidentally, enharmonically equivalent to the key of A♭). The “8th sharp” is placed on the F#, to make it F. The key of D#, with 9 sharps, has another sharp placed on the C#, making it C. The same for key signatures with flats is true; The key of E (four sharps) is equivalent to the key of F♭ (again, one fifth below the key of C♭, following the pattern of flat key signatures). The last flat is placed on the B♭, making it B.



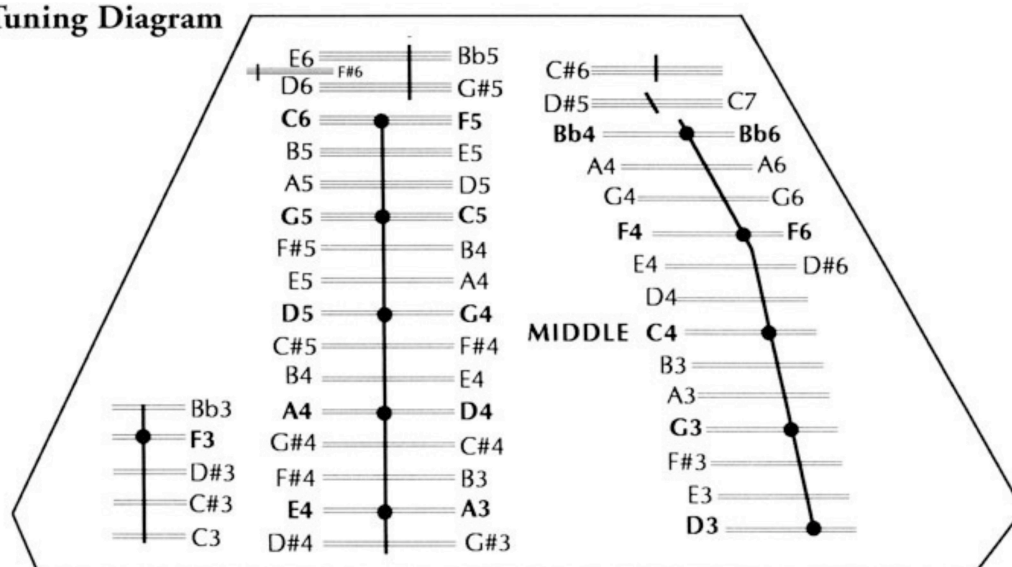




Twelve Perfect Fifths

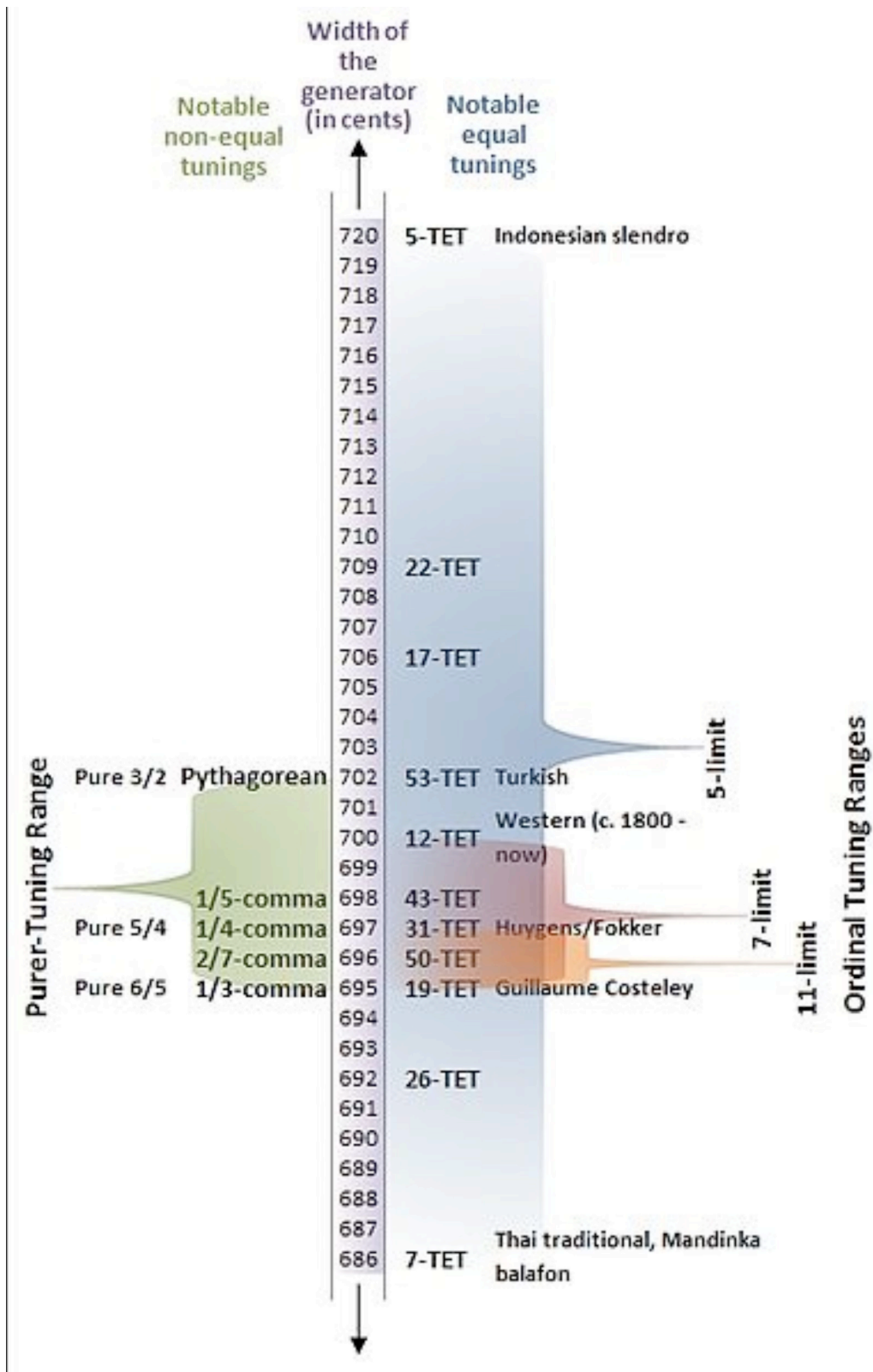


4-0 Tuning Diagram



Note	G $\flat$	D $\flat$	A $\flat$	E $\flat$	B $\flat$	F	C	G	D	A	E	B	F $\sharp$
Ratio	1024:729	256:243	128:81	32:27	16:9	4:3	1:1	3:2	9:8	27:16	81:64	243:128	729:512
Cents	588	90	792	294	996	498	0	702	204	906	408	1110	612

Pythagorean tuning is a tuning of the syntonic temperament in which the generator is the ratio 3:2 (i.e., the untempered perfect fifth), which is 702 cents wide (see the figure labelled "The syntonic tuning continuum" below).



The syntonic tuning continuum, showing Pythagorean tuning at 702 cents. Hence, it is a system of musical tuning in which the frequency ratios of all intervals are based on the ratio 3:2, "found in the harmonic series." [2] This ratio, also known as

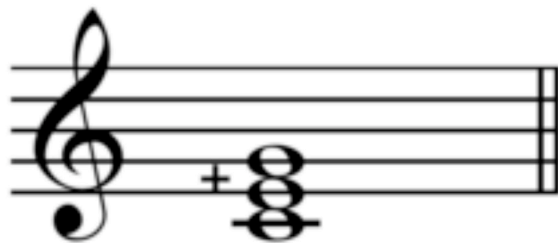
the "pure" perfect fifth, is chosen because it is one of the most consonant and easy to tune by ear.



Pythagorean diatonic scale on C.



Diatonic scale on C 12-tone equal tempered and just intonation.



Pythagorean (tonic) major chord on C (compare equal tempered and just).

The system had been mainly attributed to Pythagoras (sixth century BC) by modern authors of music theory, while Ptolemy, and later Boethius, ascribed the division of the tetrachord by only two intervals, called "semitonium", "tonus", "tonus" in Latin ($256:243 \times 9:8 \times 9:8$), to Eratosthenes. The so-called "Pythagorean tuning" was used by musicians up to the beginning of the 16th century. Leo Gunther wrote, that "the Pythagorean system would appear to be ideal because of the purity of the fifths, but other intervals, particularly the major third, are so badly out of tune that major chords [may be considered] a dissonance."

The **Pythagorean scale** is any scale which may be constructed from only pure perfect fifths (3:2) and octaves (2:1) or the gamut of twelve pitches constructed from only pure perfect fifths and octaves, and from which specific scales may be drawn (see Generated collection). For example, the series of fifths generated above gives seven notes, a diatonic major scale on C in Pythagorean tuning, shown in notation on the top right. In Greek music it was used to tune tetrachords and the twelve tone Pythagorean system was developed by medieval music theorists using the same method of tuning in perfect fifths, however there is no evidence that Pythagoras himself went beyond the tetrachord.

Method

Pythagorean tuning is based on a stack of intervals called perfect fifths, each tuned in the ratio 3:2, the next simplest ratio after 2:1. Starting from D for example (*D-based* tuning), six other notes are produced by moving six times a ratio 3:2 up, and the remaining ones by moving the same ratio down:

E $\flat$ —B $\flat$ —F—C—G—**D**—A—E—B—F $\sharp$ —C $\sharp$ —G $\sharp$

This succession of eleven 3:2 intervals spans across a wide range of frequency (on a piano keyboard, it encompasses 77 keys). Since notes differing in frequency by a factor of 2 are given the same name, it is customary to divide or multiply the

frequencies of some of these notes by 2 or by a power of 2. The purpose of this adjustment is to move the 12 notes within a smaller range of frequency, namely within the interval between the **base note** D and the D above it (a note with twice its frequency). This interval is typically called the **basic octave** (on a piano keyboard, an octave encompasses only 13 keys).

For instance, the A is tuned such that its frequency equals 3:2 times the frequency of D — if D is tuned to a frequency of 288 Hz, then A is tuned to 432 Hz. Similarly, the E above A is tuned such that its frequency equals 3:2 times the frequency of A, or 9:4 times the frequency of D — with A at 432 Hz, this puts E at 648 Hz. Since this E is outside the above-mentioned basic octave (i.e. its frequency is more than twice the frequency of the base note D), it is usual to halve its frequency to move it within the basic octave. Therefore, E is tuned to 324 Hz, a 9:8 (= one epogdoon) above D. The B at 3:2 above that E is tuned to the ratio 27:16 and so on. Starting from the same point working the other way, G is tuned as 3:2 below D, which means that it is assigned a frequency equal to 2:3 times the frequency of D — with D at 288 Hz, this puts G at 192 Hz. This frequency is then doubled (to 384 Hz) to bring it into the basic octave.

When extending this tuning however, a problem arises: no stack of 3:2 intervals (perfect fifths) will fit exactly into any stack of 2:1 intervals (octaves). For instance a stack such as this, obtained by adding one more note to the stack shown above

A $\flat$ —E $\flat$ —B $\flat$ —F—C—G—**D**—A—E—B—F $\sharp$ —C $\sharp$ —G $\sharp$

will be similar but not identical in size to a stack of 7 octaves. More exactly, it will be about a quarter of a semitone larger (see Pythagorean comma). Thus, A $\flat$ and G $\sharp$, when brought into the basic octave, will not coincide as expected. The table below illustrates this, showing for each note in the basic octave the conventional name of the interval from D (the base note), the formula to compute its frequency ratio, its size in cents, and the difference in cents (labeled ET-dif in the table) between its size and the size of the corresponding one in the equally tempered scale.

Note	Interval from D	Formula	Frequency ratio	Size (cents)	ET-dif (cents)
A $\flat$	diminished fifth	$\left(\frac{2}{3}\right)^6 \times 2^4$	$\frac{1024}{729}$	588.27	-11.73
E $\flat$	minor second	$\left(\frac{2}{3}\right)^5 \times 2^3$	$\frac{256}{243}$	90.22	-9.78
B $\flat$	minor sixth	$\left(\frac{2}{3}\right)^4 \times 2^3$	$\frac{128}{81}$	792.18	-7.82
F	minor third	$\left(\frac{2}{3}\right)^3 \times 2^2$	$\frac{32}{27}$	294.13	-5.87
C	minor seventh	$\left(\frac{2}{3}\right)^2 \times 2^2$	$\frac{16}{9}$	996.09	-3.91
G	perfect fourth	$\frac{2}{3} \times 2$	$\frac{4}{3}$	498.04	-1.96
D	unison	$\frac{1}{1}$	$\frac{1}{1}$	0.00	0.00
A	perfect fifth	$\frac{3}{2}$	$\frac{3}{2}$	701.96	1.96
E	major second	$\left(\frac{3}{2}\right)^2 \times \frac{1}{2}$	$\frac{9}{8}$	203.91	3.91
B	major sixth	$\left(\frac{3}{2}\right)^3 \times \frac{1}{2}$	$\frac{27}{16}$	905.87	5.87
F $\sharp$	major third	$\left(\frac{3}{2}\right)^4 \times \left(\frac{1}{2}\right)^2$	$\frac{81}{64}$	407.82	7.82
C $\sharp$	major seventh	$\left(\frac{3}{2}\right)^5 \times \left(\frac{1}{2}\right)^2$	$\frac{243}{128}$	1109.78	9.78
G $\sharp$	augmented fourth	$\left(\frac{3}{2}\right)^6 \times \left(\frac{1}{2}\right)^3$	$\frac{729}{512}$	611.73	11.73

In the formulas, the ratios 3:2 or 2:3 represent an ascending or descending perfect fifth (i.e. an increase or decrease in frequency by a perfect fifth), while 2:1 or 1:2 represent an ascending or descending octave.

The major scale based on C, obtained from this tuning is:

Note	C	D	E	F	G	A	B	C	
Ratio	1/1	9/8	81/64	4/3	3/2	27/16	243/128	2/1	
Step	—	9/8	9/8	256/243	9/8	9/8	9/8	256/243	—

In equal temperament, pairs of enharmonic notes such as A $\flat$ and G $\sharp$ are thought of as being exactly the same note — however, as the above table indicates, in Pythagorean tuning they have different ratios with respect to D, which means they are at a different frequency. This discrepancy, of about 23.46 cents, or nearly one quarter of a semitone, is known as a *Pythagorean comma*.

To get around this problem, Pythagorean tuning ignores A $\flat$, and uses only the 12 notes from E $\flat$ to G $\sharp$. This, as shown above, implies that only eleven just fifths are used to build the entire chromatic scale. The remaining fifth (from G $\sharp$ to E $\flat$) is left badly out-of-tune, meaning that any music which combines those two notes is unplayable in this tuning. A very out-of-tune interval such as this one is known as a *wolf interval*. In the case of Pythagorean tuning, all the fifths are 701.96 cents wide, in the exact ratio 3:2, except the wolf fifth, which is only 678.49 cents wide, nearly a quarter of a semitone flatter.

If the notes G $\sharp$ and E $\flat$ need to be sounded together, the position of the wolf fifth can be changed. For example, a C-based Pythagorean tuning would produce a stack of fifths running from D $\flat$ to F $\sharp$, making F $\sharp$ -D $\flat$ the wolf interval. However, there will always be one wolf fifth in Pythagorean tuning, making it impossible to play in all keys in tune.

Size of intervals

The table above shows only intervals from D. However, intervals can be formed by starting from each of the above listed 12 notes. Thus, twelve intervals can be defined for each **interval type** (twelve unisons, twelve semitones, twelve intervals composed of 2 semitones, twelve intervals composed of 3 semitones, etc.).

From	Interval											
	A1	A5	A2	A6	A3	P1	P5	M2	M6	M3	M7	A4
E$\flat$	$3^7/2^{11}$	$3^8/2^{12}$	$3^9/2^{14}$	$3^{10}/2^{15}$	$3^{11}/2^{17}$	1/1	3/2	9/8	27/16	81/64	243/128	729/512
B$\flat$	$3^7/2^{11}$	$3^8/2^{12}$	$3^9/2^{14}$	$3^{10}/2^{15}$	4/3	1/1	3/2	9/8	27/16	81/64	243/128	729/512
F	$3^7/2^{11}$	$3^8/2^{12}$	$3^9/2^{14}$	16/9	4/3	1/1	3/2	9/8	27/16	81/64	243/128	729/512
C	$3^7/2^{11}$	$3^8/2^{12}$	32/27	16/9	4/3	1/1	3/2	9/8	27/16	81/64	243/128	729/512
G	$3^7/2^{11}$	128/81	32/27	16/9	4/3	1/1	3/2	9/8	27/16	81/64	243/128	729/512
D	256/243	128/81	32/27	16/9	4/3	1/1	3/2	9/8	27/16	81/64	243/128	729/512
A	256/243	128/81	32/27	16/9	4/3	1/1	3/2	9/8	27/16	81/64	243/128	$2^{10}/3^6$
E	256/243	128/81	32/27	16/9	4/3	1/1	3/2	9/8	27/16	81/64	$2^{12}/3^7$	$2^{10}/3^6$
B	256/243	128/81	32/27	16/9	4/3	1/1	3/2	9/8	27/16	$2^{13}/3^8$	$2^{12}/3^7$	$2^{10}/3^6$
F$\sharp$	256/243	128/81	32/27	16/9	4/3	1/1	3/2	9/8	$2^{15}/3^9$	$2^{13}/3^8$	$2^{12}/3^7$	$2^{10}/3^6$
C$\sharp$	256/243	128/81	32/27	16/9	4/3	1/1	3/2	$2^{16}/3^{10}$	$2^{15}/3^9$	$2^{13}/3^8$	$2^{12}/3^7$	$2^{10}/3^6$
G$\sharp$	256/243	128/81	32/27	16/9	4/3	1/1	$2^{18}/3^{11}$	$2^{16}/3^{10}$	$2^{15}/3^9$	$2^{13}/3^8$	$2^{12}/3^7$	$2^{10}/3^6$
	m2	m6	m3	m7	P4	P1	d6	d3	d7	d4	d8	d5

Frequency ratio of the 144 intervals in D-based Pythagorean tuning. Interval names are given in their shortened form. Pure intervals are shown in **bold** font. Wolf intervals are highlighted in red. Numbers larger than 999 are shown as powers of 2 or 3.

From	Interval											
	A1	A5	A2	A6	A3	P1	P5	M2	M6	M3	M7	A4
E$\flat$	114	816	318	1020	522	0	702	204	906	408	1110	612
B$\flat$	114	816	318	1020	498	0	702	204	906	408	1110	612
F	114	816	318	996	498	0	702	204	906	408	1110	612
C	114	816	294	996	498	0	702	204	906	408	1110	612
G	114	792	294	996	498	0	702	204	906	408	1110	612
D	90	792	294	996	498	0	702	204	906	408	1110	612
A	90	792	294	996	498	0	702	204	906	408	1110	588
E	90	792	294	996	498	0	702	204	906	408	1086	588
B	90	792	294	996	498	0	702	204	906	384	1086	588
F$\sharp$	90	792	294	996	498	0	702	204	882	384	1086	588
C$\sharp$	90	792	294	996	498	0	702	180	882	384	1086	588
G$\sharp$	90	792	294	996	498	0	678	180	882	384	1086	588
	m2	m6	m3	m7	P4	P1	d6	d3	d7	d4	d8	d5

Approximate size in cents of the 144 intervals in D-based Pythagorean tuning. Interval names are given in their shortened form. Pure intervals are shown in **bold** font. Wolf intervals are highlighted in red.

As explained above, one of the twelve fifths (the wolf fifth) has a different size with respect to the other eleven. For a similar reason, each of the other interval types, except for the unisons and the octaves, has two different sizes in Pythagorean tuning. This is the price paid for seeking just intonation. The tables on the right and below show their frequency ratios and their approximate sizes in cents. Interval names are given in their standard shortened form. For instance, the size of the interval from D to A, which is a perfect fifth (**P5**), can be found in the seventh column of the row labeled **D**. Strictly just (or pure) intervals are shown in **bold** font. Wolf intervals are highlighted in red.

The reason why the interval sizes vary throughout the scale is that the pitches forming the scale are unevenly spaced. Namely, the frequencies defined by construction for the twelve notes determine two different semitones (i.e. intervals between adjacent notes):

1. The minor second (**m2**), also called diatonic semitone, with size

$$S_1 = \frac{256}{243} \approx 90.225 \text{ cents}$$

(e.g. between D and E?)

2. The augmented unison (**A1**), also called chromatic semitone, with size

$$S_2 = \frac{3^7}{2^{11}} = \frac{2187}{2048} \approx 113.685 \text{ cents}$$

(e.g. between E $\flat$ and E)

Conversely, in an equally tempered chromatic scale, by definition the twelve pitches are equally spaced, all semitones having a size of exactly

$$S_E = \sqrt[12]{2} = 100.000 \text{ cents.}$$

As a consequence all intervals of any given type have the same size (e.g., all major thirds have the same size, all fifths have the same size, etc.). The price paid, in this case, is that none of them is justly tuned and perfectly consonant, except, of course, for the unison and the octave.

For a comparison with other tuning systems, see also this table.

By definition, in Pythagorean tuning 11 perfect fifths (**P5** in the table) have a size of approximately 701.955 cents (700+ Σ cents, where $\Sigma \approx 1.955$ cents). Since the average size of the 12 fifths must equal exactly 700 cents (as in equal temperament), the other one must have a size of 700-11 Σ cents, which is about 678.495 cents (the wolf fifth). Notice that, as shown in the table, the latter interval, although enharmonically equivalent to a fifth, is more properly called a diminished sixth (**d6**). Similarly,

- 9 minor thirds (**m3**) are ≈ 294.135 cents (300-3 Σ), 3 augmented seconds (**A2**) are ≈ 317.595 cents (300+9 Σ), and their average is 300 cents;
- 8 major thirds (**M3**) are ≈ 407.820 cents (400+4 Σ), 4 diminished fourths (**d4**) are ≈ 384.360 cents (400-8 Σ), and their average is 400 cents;
- 7 diatonic semitones (**m2**) are ≈ 90.225 cents (100-5 Σ), 5 chromatic semitones (**A1**) are ≈ 113.685 cents (100+7 Σ), and their average is 100 cents.

In short, similar differences in width are observed for all interval types, except for unisons and octaves, and they are all multiples of Σ , the difference between the Pythagorean fifth and the average fifth.

Notice that, as an obvious consequence, each augmented or diminished interval is exactly 12 Σ (≈ 23.460) cents narrower or wider than its enharmonic equivalent. For

instance, the d6 (or wolf fifth) is $12\S$ cents narrower than each P5, and each A2 is $12\S$ cents wider than each m3. This interval of size $12\S$ is known as a Pythagorean comma, exactly equal to the opposite of a diminished second (≈ -23.460 cents). This implies that Σ can be also defined as one twelfth of a Pythagorean comma.

Pythagorean intervals

Main articles: Pythagorean interval and Interval (music)

Four of the above mentioned intervals take a specific name in Pythagorean tuning. In the following table, these specific names are provided, together with alternative names used generically for some other intervals. Notice that the Pythagorean comma does not coincide with the diminished second, as its size (524288:531441) is the reciprocal of the Pythagorean diminished second (531441:524288). Also *ditone* and *semiditone* are specific for Pythagorean tuning, while *tone* and *tritone* are used generically for all tuning systems. Interestingly, despite its name, a semiditone (3 semitones, or about 300 cents) can hardly be viewed as half of a ditone (4 semitones, or about 400 cents). All the intervals with prefix *sesqui-* are justly tuned, and their frequency ratio, shown in the table, is a superparticular number (or epimoric ratio). The same is true for the octave.

Number of semitones	Generic names			Specific names		
	Quality and number		Other naming conventions	Pythagorean tuning	5-limit tuning	1/4-comma meantone
	Full	Short				
0			comma	Pythagorean comma (524288:531441)		diesis (128:125)
0	diminished second	d2		(531441:524288)		
1	minor second	m2	semitone, half tone,	limma (256:243)		
1	augmented unison	A1	half step chromatic semitone, major semitone	apotome (2187:2048)		
2	diminished third	d3	tone, whole tone, whole step			
2	major second	M2		sesquioctavum (9:8)		
3	minor third	m3		semiditone (32:27)	sesquiquintum (6:5)	
4	major third	M3		ditone (81:64)	sesquiquartum (5:4)	
5	perfect fourth	P4	diatessaron	sesquitercium (4:3)		
6	diminished fifth	d5	tritone			
6	augmented fourth	A4				
7	perfect fifth	P5	diapente	sesquialterum (3:2)		
12	(perfect) octave	P8	diapason	duplex (2:1)		

History

Because of the wolf interval, this tuning is rarely used nowadays, although it is thought to have been widespread. In music which does not change key very often, or which is not very harmonically adventurous, the wolf interval is unlikely to be a problem, as not all the possible fifths will be heard in such pieces.

Because most fifths in Pythagorean tuning are in the simple ratio of 3:2, they sound very "smooth" and consonant. The thirds, by contrast, most of which are in the relatively complex ratios of 81:64 (for major thirds) and 32:27 (for minor thirds), sound less smooth.[8] For this reason, Pythagorean tuning is particularly well suited to music which treats fifths as consonances, and thirds as dissonances. In western classical music, this usually means music written prior to the 15th century.

From about 1510 onward, as thirds came to be treated as consonances, meantone

temperament, and particularly quarter-comma meantone, which tunes thirds to the relatively simple ratio of 5:4, became the most popular system for tuning keyboards. At the same time, syntonic-diatonic just intonation was posited by Zarlino as the normal tuning for singers.

However, meantone presented its own harmonic challenges. Its wolf intervals proved to be even worse than those of the Pythagorean tuning (so much so that it often required 19 keys to the octave as opposed to the 12 in Pythagorean tuning). As a consequence, meantone was not suitable for all music.

From around the 18th century, as the desire grew for instruments to change key, and therefore to avoid a wolf interval, this led to the widespread use of well temperaments and eventually equal temperament.

In 2007, the discovery of the syntonic temperament [1] exposed the Pythagorean tuning as being a point on the syntonic temperament's tuning continuum.

Discography[edit]

- Bragod is a duo giving historically informed performances of mediaeval Welsh music using the crwth and six-stringed lyre using Pythagorean tuning
- Gothic Voices - *Music for the Lion-Hearted King* (Hyperion, CDA66336, 1989), directed by Christopher Page (Leech-Wilkinson)
- Lou Harrison performed by John Schneider and the Cal Arts Percussion Ensemble conducted by John Bergamo - *Guitar & Percussion* (Etceter Records, KTC1071, 1990): *Suite No. 1* for guitar and percussion and *Plaint & Variations* on "Song of Palestine"

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