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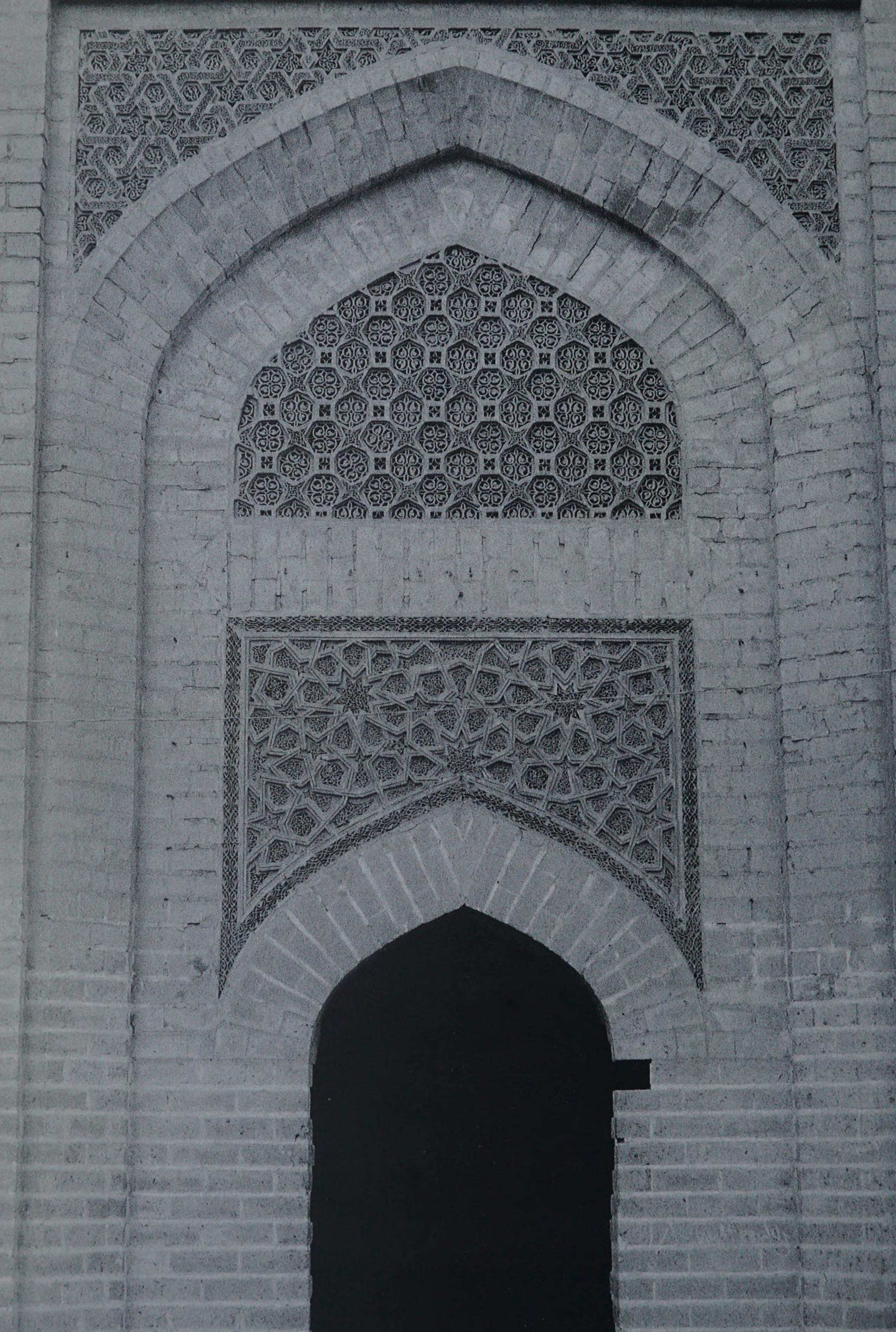
Islamic Art and Architecture
THE SYSTEM OF GEOMETRIC DESIGN



Islamic Art and Architecture

THE SYSTEM OF GEOMETRIC DESIGN

ISSAM EL-SAID

Edited by

Tarek El-Bouri & Keith Critchlow

General Editor: Salma Samar Damluji

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Title Page: Detail of a portal of the courtyard at Al Mustanṣiriyyah School, Baghdad (1980).

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Contents

Preface 6

Foreword 8

Editors' Introduction 10

I. GEOMETRIC PATTERNS IN ISLAMIC ART

Introduction 14

Geometric Patterns based on the Square Repeat Unit and the Root Two System of Proportion

The construction of the square repeat unit and the division of the circle into four, or multiples of four, equal parts 16
Determining the size of the square repeat unit 17
The grid-patterns that underlie the root two system of proportion 19
Expressing the grid-patterns in terms of a selected square module 20
Generating the grid-patterns using numbers 22
Mathematical or numerical rationalization of the root two system of proportion 24
Graphical analyses of geometric patterns based on the root two system of proportion 30

Geometric Patterns based on the Hexagonal Repeat Unit and the Root Three System of Proportion

The construction of the hexagonal repeat unit and the division of the circle into three, or multiples of three, equal parts 56
Determining the size of the hexagonal repeat unit 58
The grid-patterns that underlie the root three system of proportion and their measure 60
Mathematical or numerical rationalization of the root three system of proportion 66
Graphical analyses of geometric patterns based on the root three system of proportion 69

Geometric Patterns based on the Double Hexagon System of Proportion

The grid-patterns that underlie the double hexagon system of proportion 88
Graphical analyses of geometric patterns based on the double hexagon system of proportion 89

Notes 113

II. MAN AND MEASURE

Historical Background 116

Numerals and Arithmetic 117

Measuring and Dividing Dimensions

a. Using the Circle 121
b. Using the Square 124

Geometry and Irrational Numbers 128

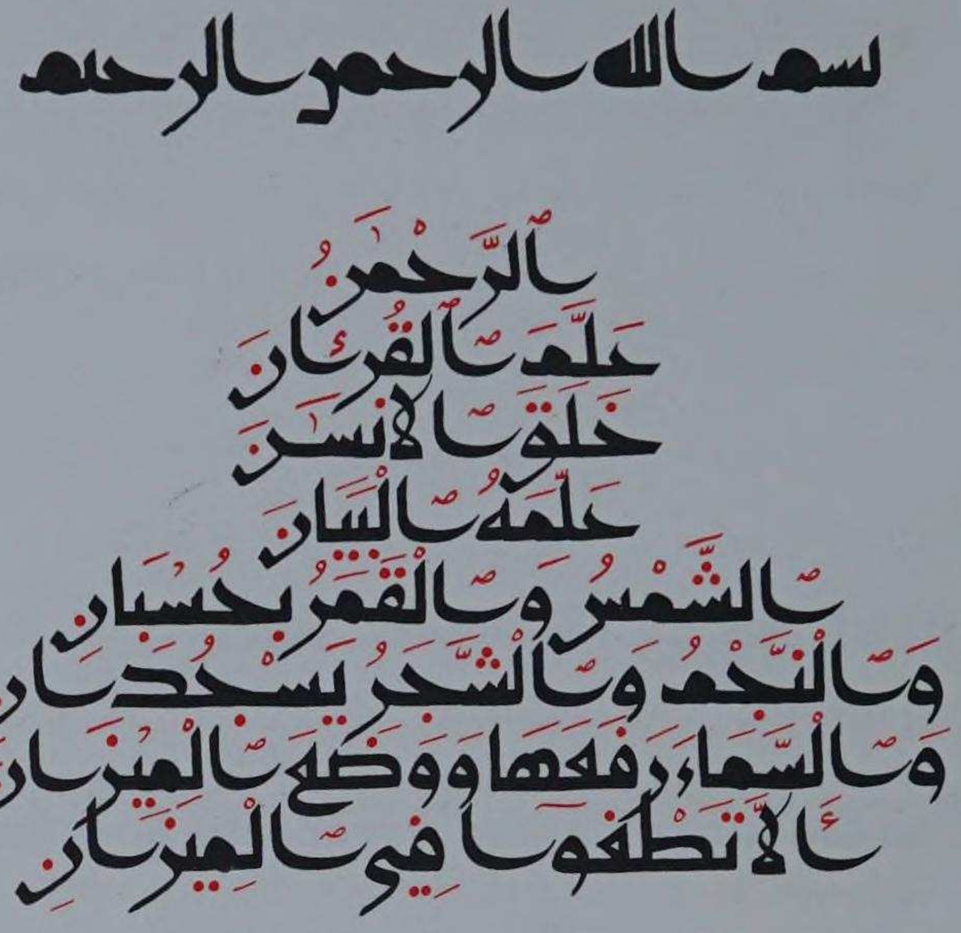
The Remen and the Royal Cubit 130

The Grid-Pattern 132

Notes 133

Bibliography 134

Index 135



Preface

Issam El-Said, artist and architect, died suddenly in December 1988. The last five years of his life were dedicated to his researches on the methodology of geometric proportioning in Islamic architecture, and this was the subject of his PhD thesis, under the supervision of Professor Miles Danby at the University of Newcastle-upon-Tyne. This book presents key extracts from his thesis, which at the time of his death was not complete.

El-Said's obsession with geometry started over two decades earlier when he began formulating the design concepts for constructing geometric patterns, determining a formula used by craftsmen and master builders in Islamic art and architecture, though never recorded as a design process. His research led him to investigate the foundations and principles that he later identified as 'Uṣūl', which he conceived as "the system of creativity in the Islamic artistic heritage". The inherent geometric proportions in Islamic patterns, which he pursued in his designs, paintings and research, were seen by El-Said as the conceptual essence of al Mīzān, the 'balance' mentioned in the Qur'ān:

"And the sky He exalted and established the balance". (Sūrah LV)

Geometric patterns – the vocabulary and language of Islamic art – the structure of architectural design concepts and town planning were the domains where his forte and enigma lay. This involvement developed alongside his understanding of the essence, beauty, perfection and harmony found in the Islamic arts and architecture, irrespective of region or era, and he delved into theory and practice to reconstruct their elements. Whether in music, calligraphy or architecture, he sought to analyse geometrically the underlying abstract layers of patterns woven by a perfected system of design which inspired and enriched the Islamic artistic heritage. In this realm of investigation there was no separation between art, culture and science; nor between the compasses and the computer. Thus he was able to retrace the same systems used from pre-Islamic times to establish the methods that characterized and inspired creative expression.

This also endorsed his theory that "all the Islamic arts were governed by the science of preparations in the realm of metre, rhythm and in the form of interval in music", as developed by Muslim scholars and craftsmen. Seeking to re-establish these sciences and acknowledging their methodology, he considered that "the main reason for the gradual disappearance of the Islamic theory of design in contemporary works and practice was the inaccessibility of the scientific geometric know-how acquired by traditional master craftsmen", the quality of which was manifested in the methods, materials and designs skilfully and cleverly integrated while based on the same geometric principles that generated the creative dimension of design. It was this "geometric method of proportioning, in which the execution and aesthetics of composition were systematically determined" which engaged El-Said in the 'Islamic theory of design' and the infinite possibility of its application in contemporary art and architecture. He has made an exceptional contribution to fathoming the depth, beauty and vision of a distinguished culture.

Salma Samar Damluji

Acknowledgements

Tarek El-Bouri, bore the larger responsibility of researching, writing and assembling this manuscript together with the supervision of Dr. Keith Critchlow. The untiring efforts and contribution of Samir Chorbachi are to be acknowledged for co-ordinating and organizing the work on the book project at the initial stages and through its inception. The drawings, based on originals by Issam El Said, were finely executed by Tarek El-Bouri specifically for this publication.

It was the remarkable dedication and energy of Dr. Ismat El-Said that initiated the research project attached to Issam's original PhD draft, having set up the Issam El-Said Foundation which in turn provided the grant for completing this manuscript. The Foundation would like to acknowledge the assistance of Dr. Archie G.Walls for his advice.

S.S. Damluji

Foreword

The creative work of Issam El-Said in the form of painting, sculpture, printmaking, calligraphy, design and architectural projects shows clearly his dedication to the expression of the Islamic cultural identity. This has been beautifully illustrated in the volume *Issam El-Said: Artist and Scholar* published by the Issam El-Said Foundation in 1989 after his untimely death in 1988. In this important book, his mother, Dr. Esmat El-Said, describes how he succeeded in making an original artistic contribution, using materials and techniques of his time, without any conflict with the historic traditions of Islamic art that he so deeply loved and admired.

Issam El-Said felt that the achievements of Islamic art and design had been greatly underestimated and he expressed this point of view in the introduction to *Geometric Concepts in Islamic Art*, written with Ayşe Parman and published in 1976. The first chapter of the book traced the use of geometric patterns to the earliest methods of the measurement of space in architecture by the Egyptians and Mesopotamians. In later chapters the concept of the 'repeat unit' was developed. This was determined by the use of a circle to demonstrate pattern-making based on the square and the 'root two system' of proportion, followed by the hexagon and the 'root three system'. The use of the pentagon and the decagon produced with the use of golden ratio or section was shown to create yet another series of patterns, and finally the circle was divided into 12 equal parts with two hexagons inscribed within the circle to form a fifth series. The study went on to analyse geometric patterns in the planning of historic monuments, the mathematical basis of Arabic calligraphy, examples of Arabic poetry and music in the Muslim world and patterns used in works of applied art, including carpets, created within the Muslim tradition. The ideas put forward proved a source of reference for many other publications, notably the British Museum *Pattern Book of Islamic Designs* prepared by Eva Wilson.

Issam continued to be fascinated by the geometric principles of design and, being a practising artist and designer, was particularly interested in the practical methods used by generations of craftsmen to create sophisticated and complicated patterns and space compositions. Although they worked within a well-established tradition and often under the direction of masters, he was convinced that the use of compasses, straight lines and whole numbers could account for these beautiful and intricate designs. He wanted to develop and extend these ideas and decided to study for the degree of Doctor of Philosophy at the School of Architecture of the University of Newcastle-upon-Tyne.

He continued to practise as an artist, designer and architect while preparing his thesis and although this added to the pressure of work it also had the effect of enriching his theoretical manuscripts and diagrams, which were regularly submitted for comment and discussion. Theory and practice were ideally balanced. A commissioned design for a mosque in Baghdad was intended as a working demonstration of his thesis, *The Methodology of Geometric Proportioning in Islamic Architecture*. Tragically, the construction of the mosque did not proceed and at the time of

Issam's death, most of his thesis remained in fragmentary draft form. Dr. Esmat El-Said has managed to assemble the surviving documents and enlist the help of Dr. Keith Critchlow and Tarek El-Bouri to edit two of the projected five chapters and prepare them for publication. Fortunately these two chapters comprise the basic ideas and principles of the thesis.

The first, 'Geometric Patterns in Islamic Art', is an extension and development of the concept of his methodology of composition and system of measure which avoids complicated mathematical calculations. Grid patterns and graphical analyses of three of the five series described in his previous work are presented. Perhaps the most important further development is the demonstration of numerical rationalization for every pattern, however elaborate. Innumerable intricate patterns, for use in applied art and architecture, can be created with the use of compasses, simple tools, templates and whole numbers. The second chapter, 'Man and Measure', is an elaboration of the opening chapter of El-Said's 1976 work, describing in greater detail the history of notation and mathematics from Ancient Egyptian times and explaining its relevance to the evolution of the grid pattern in Islamic art and architecture.

This publication provides a practical and theoretical discipline that can be used by designers, architects, builders and craftsmen to restore the religious and cultural context that is lacking in so much of contemporary building in the Muslim world. In this way, unity can again be established among the diversity of building types and materials to maintain historic continuity. The spiritual values of Islam can be integrated with the practice of art and architecture, as Issam El-Said so wonderfully demonstrated in his personal life and creative work.

Miles Danby

Editors' Introduction

This publication brings together two chapters from Issam El-Said's doctoral thesis, *The Methodology of Geometric Proportioning in Islamic Architecture.* The complete contents of the doctoral work originally constituted five chapters:

Chapter I. Artistic Heritage of Islam

Chapter II. Geometric Patterns in Islamic Art

Chapter III. Geometry and Arabic Calligraphy

Chapter IV. Islamic Architecture

Chapter V. Man and Measure

The Chapters presented here, 'Geometric Patterns in Islamic Art' and 'Man and Measure', are considered key extracts from the complete work. Together they present an original and practical study of the basic methods and procedures of the geometrical system of design in Islamic art, as distilled by Issam El-Said.

El-Said's doctoral thesis is an extension of his earlier work, *Geometric Concepts in Islamic Art* (1976), in which he first developed his unique analytic approach to understanding the geometry that inspires Islamic art. At the heart of his original thesis lies the consideration of the relationship between geometry and number, and its application to provide a practical method for pattern-making and developing systems of measure. His aim was "to trace man's approach to measure and to outline the geometric concepts of composition and the process of their application in different fields of Islamic Art."[1] El-Said sees this as being essential not only to the process of Islamic art, but also to the understanding of it. This is the means by which the art has sustained its distinct characteristics throughout history despite the geographic and ethnographic diversity of the Muslim world.

In the Preface to his doctoral thesis, Issam El-Said wrote: "It is the concept and design process common to most artistic fields of Islamic culture which is the subject of the investigations dealt with in this study. The geometric analyses presented do not pretend to reveal all the secrets of the design process but try to establish a deeper insight into the artistic rules of composition as applied in the art and architecture of Islamic culture." He thought that a better understanding of this would, perhaps, enable us to learn that "true artistic freedom is not synonymous with disregard for rules and principles, but entails a mastery of them and an ability to use them correctly."

The text has remained true to the original work with no major omissions or additions. Some slight modifications have been made, including reorganizing the contents of the chapter 'Geometric Patterns in Islamic Art' where the examples of analyses of geometric patterns have been moved to the end of each section, with the respective mathematical analyses preceding them.

[1] El-Said, *Geometric Concepts in Islamic Art*, p.xi

Consequently, the format of the figures illustrating these analyses has been revised to integrate into one figure the two stages of the geometrical construction of the pattern, and its reconstruction on a square grid using numbers. In addition to this, all the figures have been specially re-drawn for the publication. Finally, some technical terms have been modified, or substitutes suggested, for the purpose of clarifying their usage in the text. These are explained in footnotes.

We hope the resulting work does justice to the dedication, scholarship and research of Issam El-Said, who has done so much to raise the objective understanding, and thereby the appreciation, of this major art form and its contribution not only to Islam but to world culture.

Tarek El-Bouri

Keith Critchlow

I

GEOMETRIC PATTERNS IN ISLAMIC ART

Detail of a wall in the courtyard of Madrasat Al ʿAṭṭārīn, Fes, adorned in zillīj cut tiles, with a band of calligraphy in the same technique (1992).

Introduction

One of the chief characteristics that gives the artistic heritage of the Muslim world its distinct identity is the use of geometric patterns. Their universal application in metalwork, woodwork, ceramics, textiles, carpets, miniatures and architectural decoration leads to the belief that they were designed according to a clearly-formulated method incorporating a coherent and adaptable system of composition allowing both variation and innovation. In fact it is inconceivable that artists and craftsmen have continued the tradition of making geometric patterns and designs over the centuries without: **a**) a general concept of composition and principles of design; **b**) a methodology[1] for composition, understood and practised by the majority, which incorporates the use of traditional tools, i.e. rule and compass and the 'mathematical' knowledge of the period; **c**) a system of measure, for this is without doubt a fundamental aspect of composition and design and therefore essential to the process of making geometric patterns.

Having defined the method of composition and principles of measure, examples of the construction of geometric patterns and their systematic reproduction will be used to substantiate the proposed thesis.

To my knowledge, there are no surviving records in the Muslim world of instructions in the theory of making traditional geometric patterns. In the following study an attempt is made to illustrate how the broadly illiterate artist craftsmen[2] at different periods and in different parts of the Muslim world, having little or no knowledge of theoretical geometry, applied geometric principles to the practical problems of making patterns using only a rule and compasses. Although there have been numerous attempts to describe the construction of geometric patterns, no historical work has considered either the concept of what I call the 'repeat unit'[3] of a geometric pattern or the principles of measure involved.

The term 'pattern' connotes the use of a basic unit of composition which is the 'repeat unit', and it is by the systematic application and arrangement of the repeat unit that the pattern is produced. A single composition, however complicated, is not a pattern; but a single shape, an image, or an elaborate composition becomes a pattern if repeated regularly and systematically over a surface.

The examples of geometric patterns analysed at the end of each section of this chapter are grouped into distinct categories according to the system of proportion that dictates the repeat unit of the given patterns. The method of composition employed in constructing each repeat unit will be examined in the light of traditional examples without regard to chronological or geographical occurence. Detailed analyses will show how, in each group of patterns, the repeat units and the 'grid patterns'[4] determining their composition are produced by the manipulation of regular polygons, i.e. squares, equilateral triangles, hexagons, and so on. The grid-patterns are generated by joining with straight lines selected points of intersection of the sides of the polygons. Consequently, the final composition of the repeat unit, and hence the overall pattern, is achieved.

A modular or unit system of measure is indicated in each case, defining the dimensions of the composition of the repeat unit.

From a circle it is possible to generate any regular polygon once the circumference is divided equally to the required number of parts and the points of division are joined with straight lines. The first step of dividing the circumference into three, four and five equal parts, or their multiples, will determine the subsequent grid-patterns(s) and repeat unit, while the radius or diameter of the circle can be taken as a unit measure. The geometries are grouped under the following sections:

Geometric Patterns based on the Square Repeat Unit and the Root Two (√2) System of Proportion

This includes patterns generated by the division of the circle into four, or multiples of four, equal parts. The square is used as a device and mensuration and hence it is called the root two (√2) system of proportion (the ratio of the side of a square to its diagonal = 1:√2).

Geometric Patterns based on the Hexagonal Repeat Unit and the Root Three (√3) System of Proportion

This includes patterns generated by the division of the circle into six, or multiples of six, equal parts. The equilateral triangle or the hexagon is used as a device for composition and mensuration and hence it is called the root three (√3) system of proportion (the ratio of half the base to the height of an equilateral triangle = 1 :√3).

Geometric Patterns based on the Double Hexagon System of Proportion

This includes groups of patterns generated by the double hexagon.

Geometric Patterns based on the Square Repeat Unit and the Root Two System of Proportion

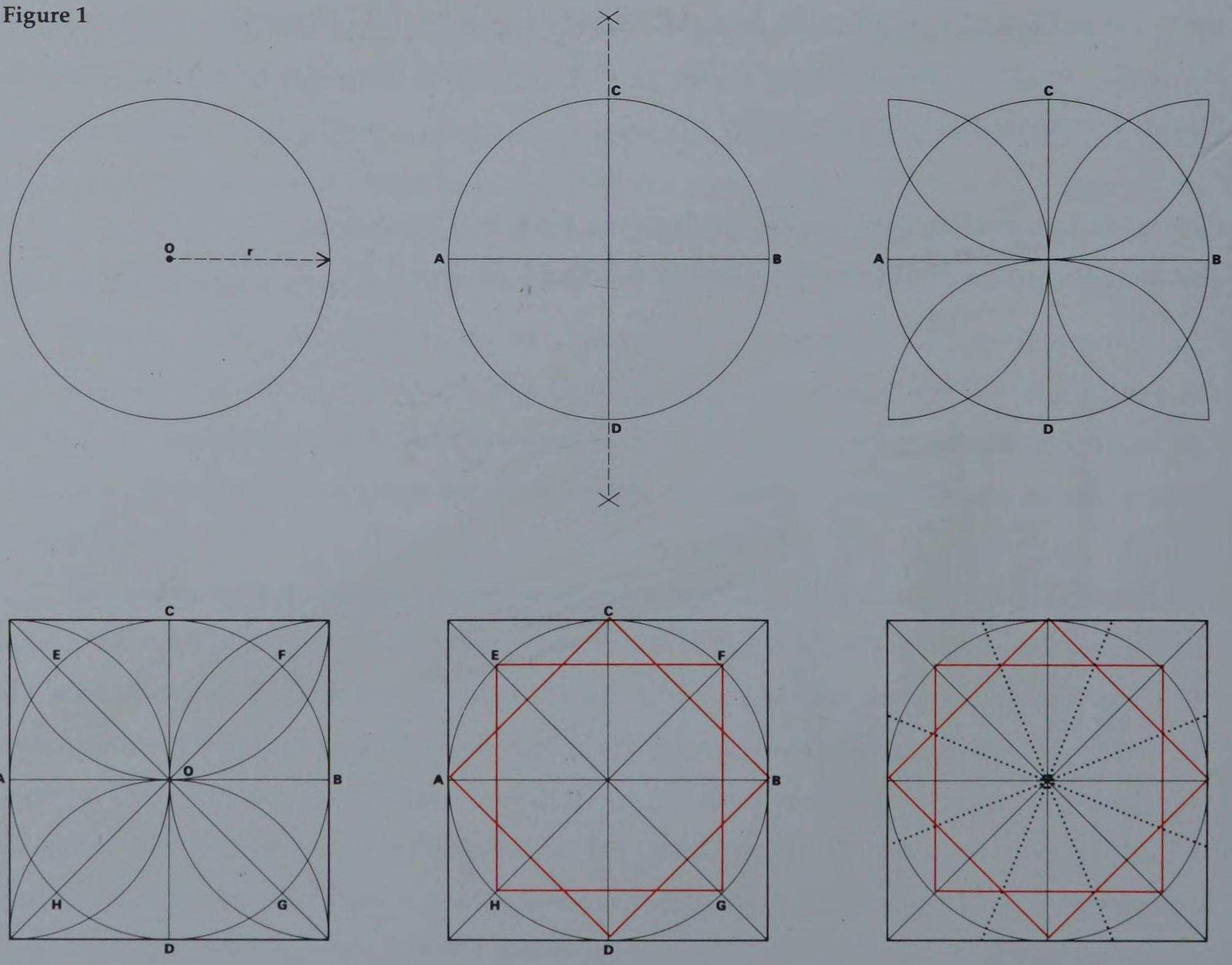
The construction of the square repeat unit and the division of the circle into four, or multiples of four, equal parts

Describe a circle with centre *O* and radius *r* (see **Figure 1**). Construct two perpendicular diameters *AB* and *CD* dividing the circumference into four equal parts. With radius *r* draw four semicircles with centres at the end of the diameters.

Join with straight lines opposite points of the intersecting semicircles through centre *O*, thus cutting the circle at *E*, *F*, *G*, and *H*. This results in the division of the circumference into eight equal parts and also defines the diagonals of the circumscribing square, which is termed the 'square repeat unit', and has sides equal to the diameter.

Join *AC*, *CB*, *BD*, *DA*, and *EF*, *FG*, *GH*, *HE* with straight lines, thus constructing two squares which form an octagonal star inscribed in the circle. Join with straight lines the opposite points of the intersecting sides of the octagonal star through centre *O* and extend them to the circumference, thus dividing the circle into 16 equal parts.

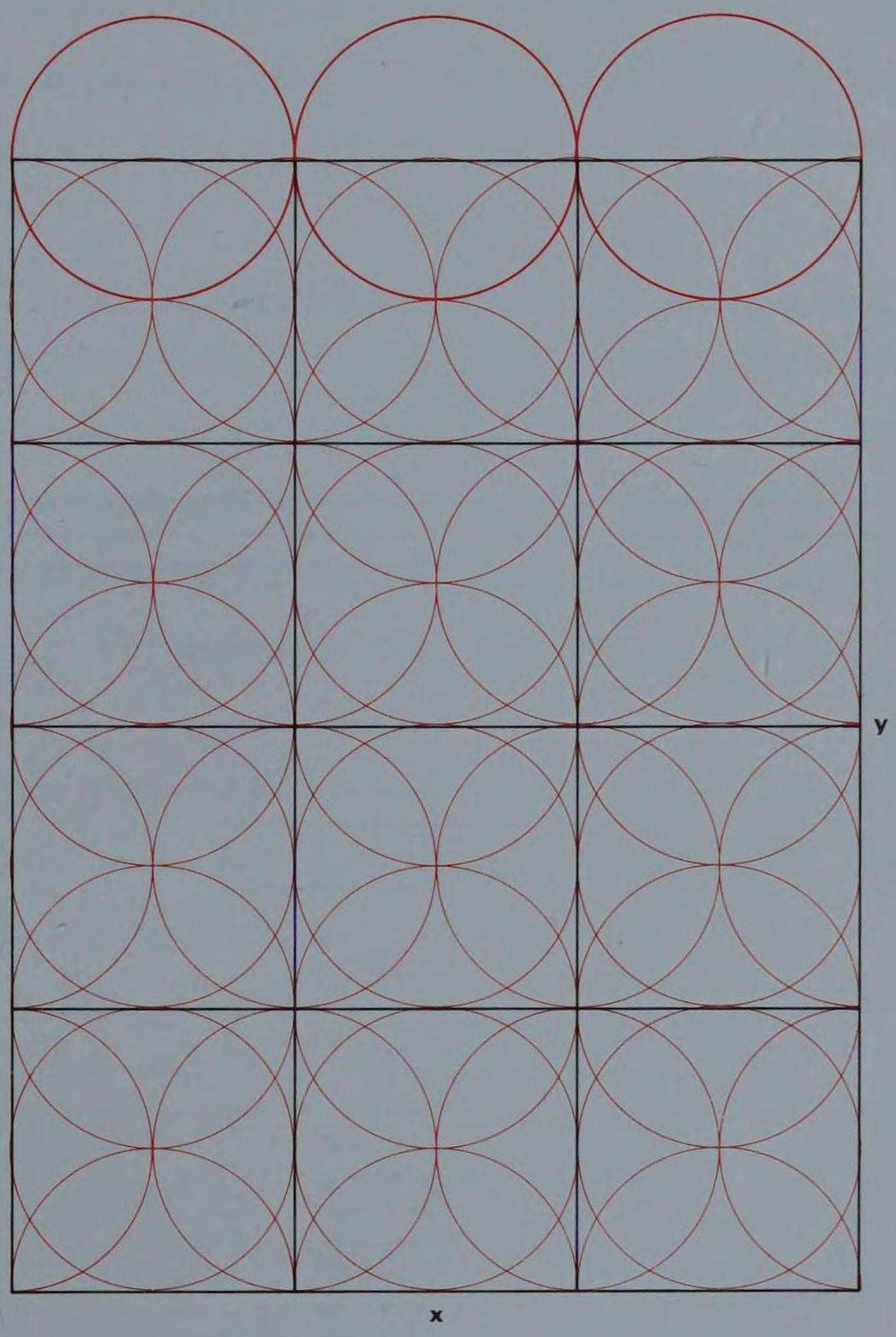
Figure 1



Determining the size of the square repeat unit

The following procedure describes a traditional practical method the determine the size of the repeat unit to be used in creating a pattern for a given area xy (see **Figure 2**). A string can be used to measure side x of the area considered. The required number of repeat units along side x is determined by folding the string into 2, 3, 5 ..., equal folds. Any number of further proportional divisions is achieved by a repeated folding in two. Thus the size and required number of repeat units is obtained by folding, and thereby doubling, the number of the previous length.

Figure 2



In dividing a given length using the folding technique, the result is always in whole units and does not involve fractions (see **Figure 3**). Having determined the size of the square repeat unit which equals the fold, the number of repeat units along side y of the given area can also be determined and the whole area divided into equal squares (providing the original rectangle xy is regularly divisible into whole squares).

Number of parts in first folding:

2, 3, 5, ...

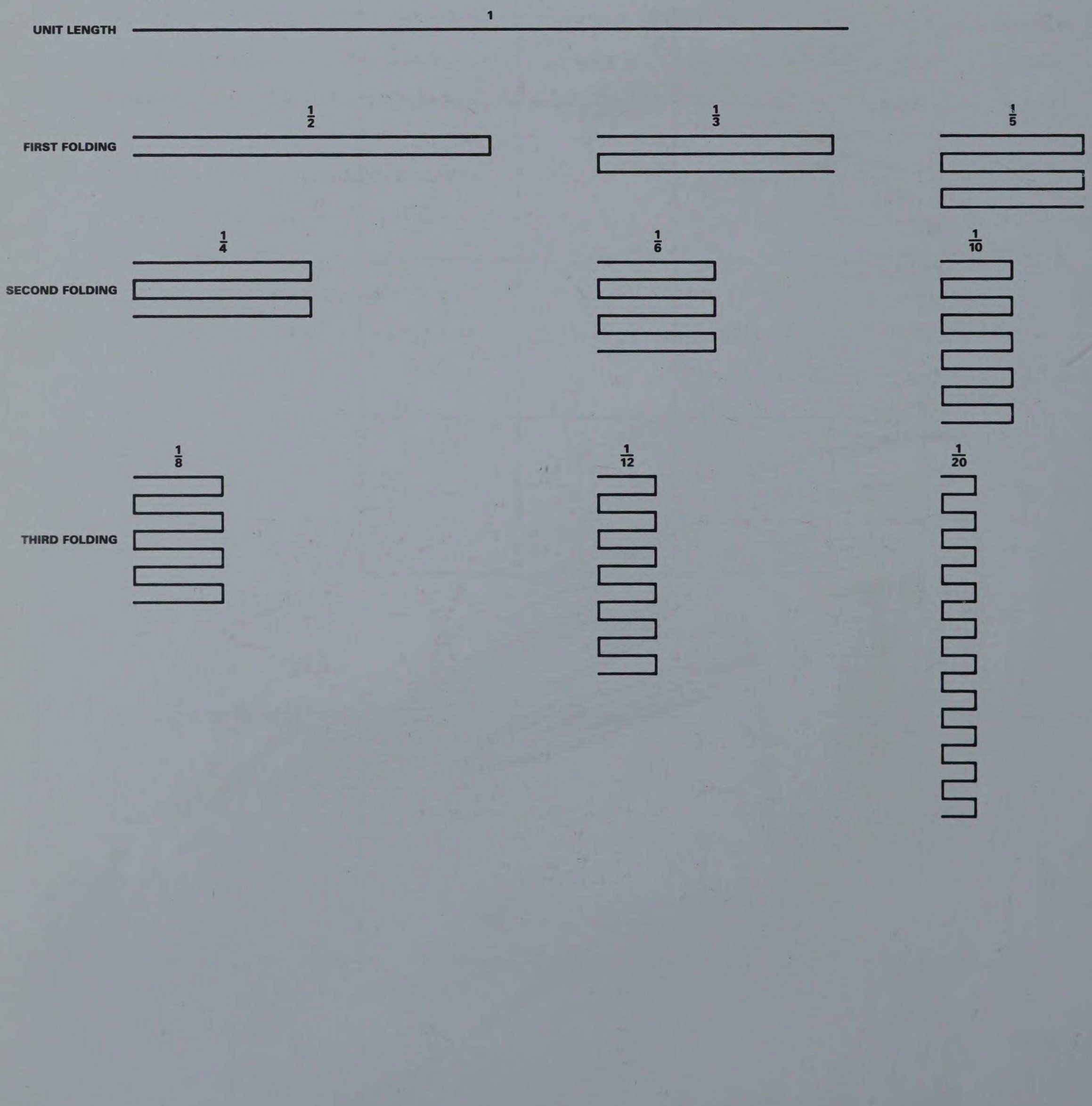
Number of parts in second folding:

4, 6, 10, ...

Number of parts in third folding:

8, 12, 20, ...

Figure 3



The grid-patterns that underlie the root two system of proportion

The repeat units in the examples of patterns given at the end of this section are in most cases based on two types of grid-patterns, to be called 'primary grid A' and 'primary grid B'.[5] From these grid-patterns, secondary and at times tertiary grid-patterns are generated. The grid-patterns establish a system of proportion determined by the properties of the square and root two ($\sqrt{2}$), and thus ensure a harmony of composition for the repeat units.

Primary grid A, an octagonal star, is constructed in a circle by dividing its circumference into eight parts and joining every second point of the eight divisions with straight lines (see **Figure 4**) as for the construction shown in **Figure 1**.

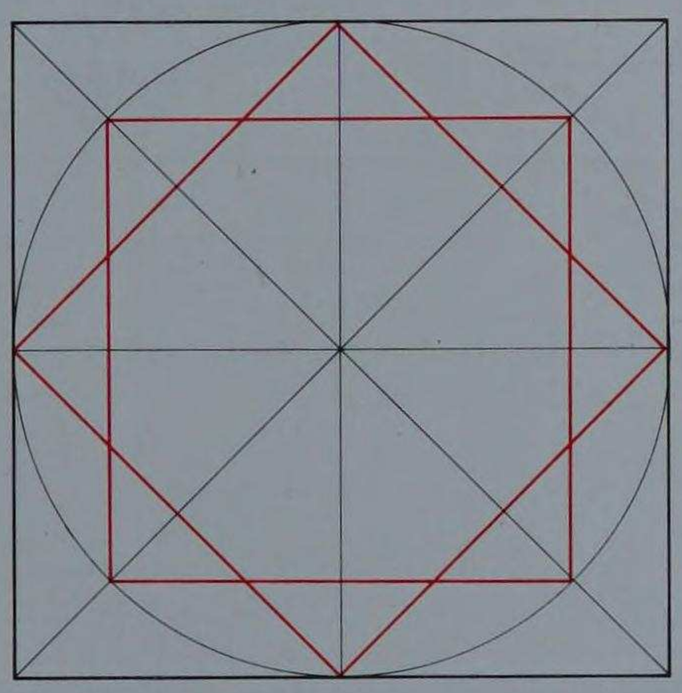
Primary grid B is another form of octagonal star. It is constructed in a circle whose circumference is divided into eight equal parts, by joining with straight lines every third point of the eight divisions.

The 'point-joining method' described here,[6] demonstrates the best practical way to generate geometric patterns without resorting to complicated mathematical calculations. The repeat unit can be constructed to any required scale, depending on the size of the square of the circumscribing circle whose diameter equals the side of the square repeat unit.

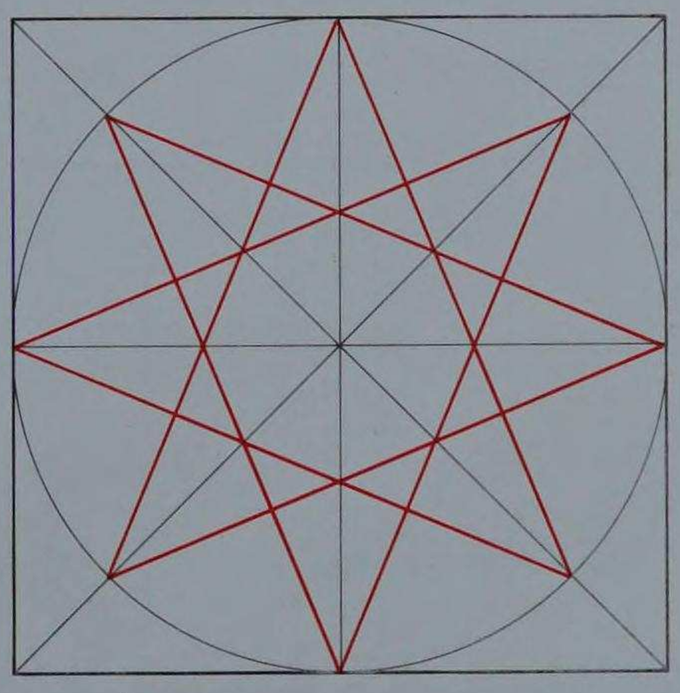
The repeat unit can then be applied with the aid of a full-scale template pierced with holes at the principal points of intersection of the grid-patterns relevant to a particular geometric composition.

In making ceramic tiles, marble and wood inlay, plaster or stone patterns, the dimensions, shapes and required number of component geometric forms can be accurately determined from the construction of a single repeat unit.

Figure 4



PRIMARY GRID A



PRIMARY GRID B

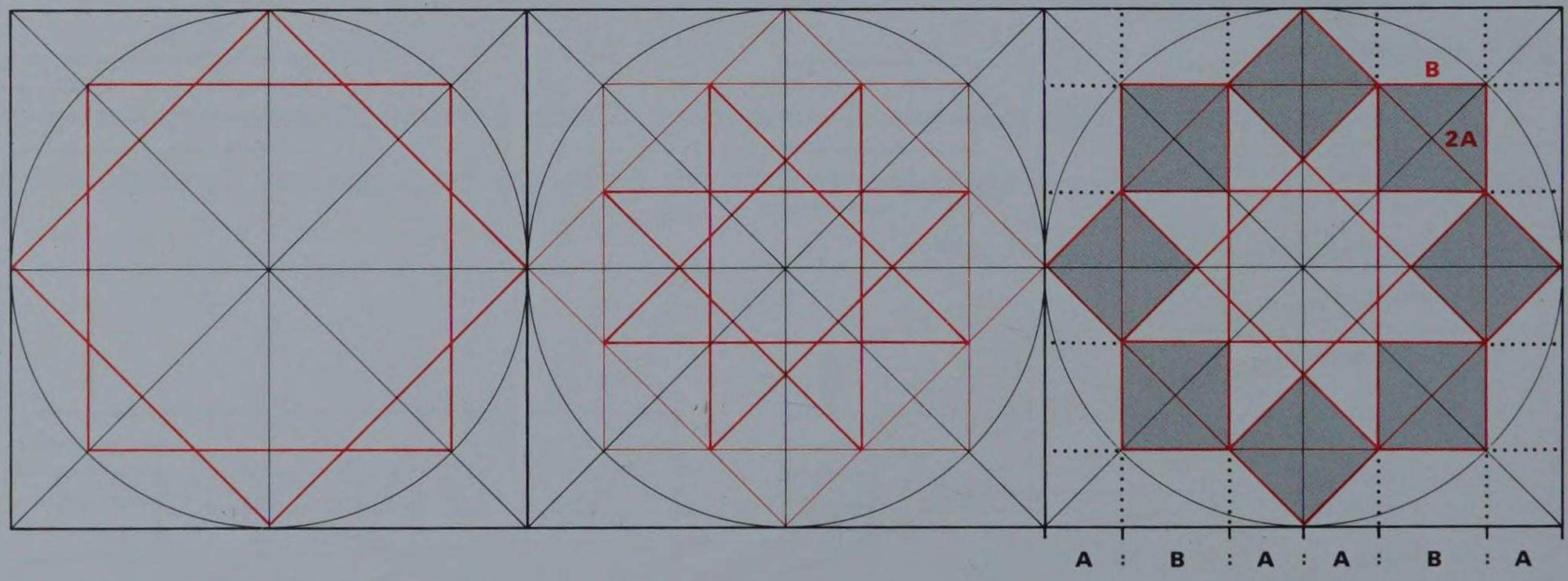
Expressing the grid-patterns in terms of a selected square module

The process of constructing a repeat unit involves ordering the shapes and interrelationships of various geometric forms, thus subjecting the composition to what is called 'proportioning'. This is achieved by establishing the grid-patterns, which create a basic underlying rhythm adhering to the particular system of proportion, regardless of whether the resultant repeat unit is simple or highly complex. No dimension in the repeat unit can be altered independently without destroying the harmony and balance of its composition.

Close examination of primary grid A (see **Figure 5**) shows that the sides of the two inscribed squares cut each other in a certain proportion. When a secondary grid-pattern, 'secondary grid A', is introduced by joining with straight lines opposite points of the intersecting sides of primary grid A, eight identical squares are formed within the repeat unit. If the value *B* is given to the side of one of these squares and *2A* to its diagonal, then it is possible to express the sides of the square repeat unit in terms of a selected square taken as a module. This square module has side *A* and diagonal *B*, and consequently the square repeat unit has a rhythmic division in the proportions of:

$A : B : A : A : B : A$.

Figure 5



Therefore, by using the selected square module (shown shaded), (see **Figure 6**) with side *A* and diagonal *B*, it is possible to express dimensions for all the squares formed by the proportional divisions of primary and secondary grids *A* in terms of *A* and *B* (Table 1).

Figure 6

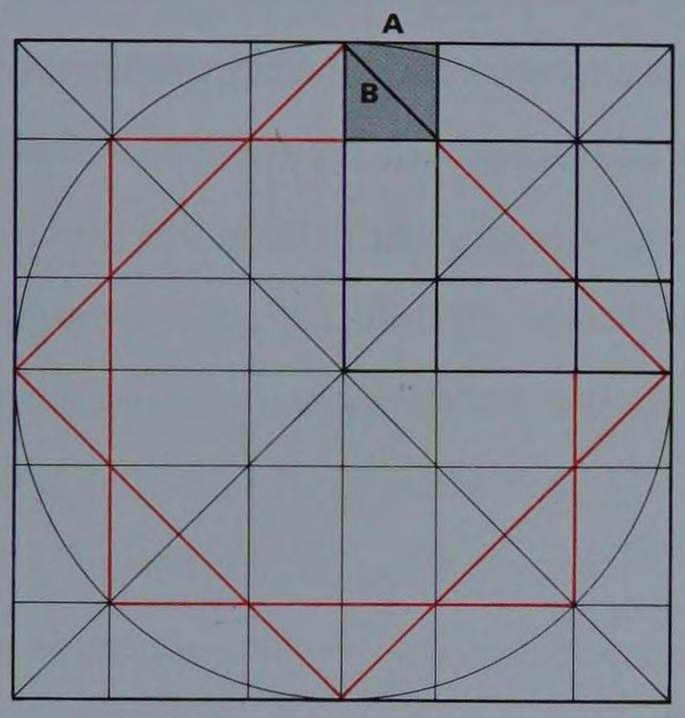


TABLE I

Side of Square	**Diagonal**
1	$\sqrt{2}$
A	B
B	$2A$
$A + B/2$ ($\frac{1}{4}$ repeat unit)	$B + A$
$A + B$	$B + 2A$
$2A + B$ ($\frac{1}{2}$ repeat unit)	$2B + 2A$
$3A + B$	$3B + 2A$
$3A + 2B$	$3B + 4A$
$4A + 2B$	$4B + 4A$

It should be noted that when converting the side of the square to its diagonal, *B* is substituted for *A* and $2A$ for *B*.

The value A in any sized square (see **Figure 7**) can be determined graphically with the use of a compass by subtracting the length of $\frac{1}{4}$ of the diagonal from $\frac{1}{2}$ the side of the square. Similarly *B* can be determined by subtracting $\frac{1}{2}$ of the side of the square from $\frac{1}{2}$ of its diagonal.

Figure 7

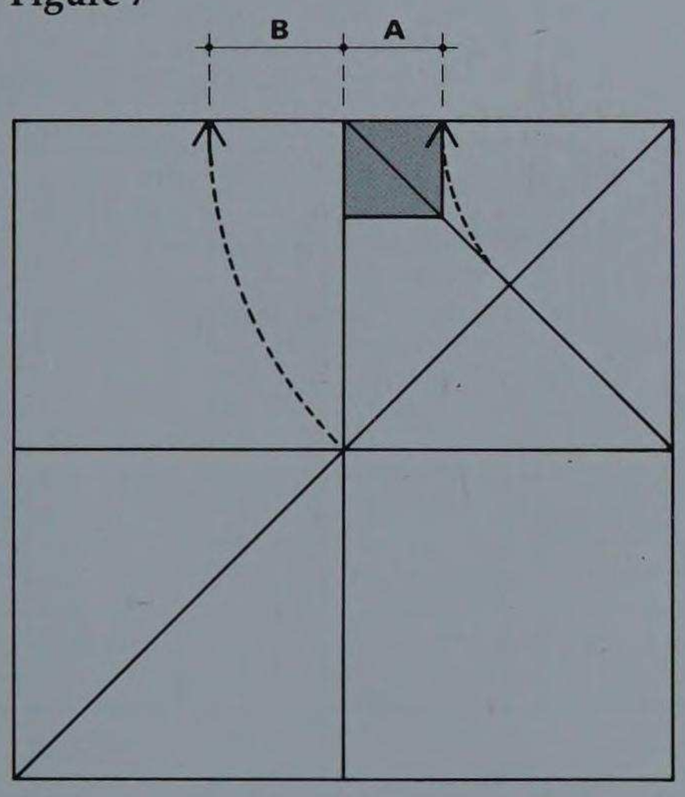
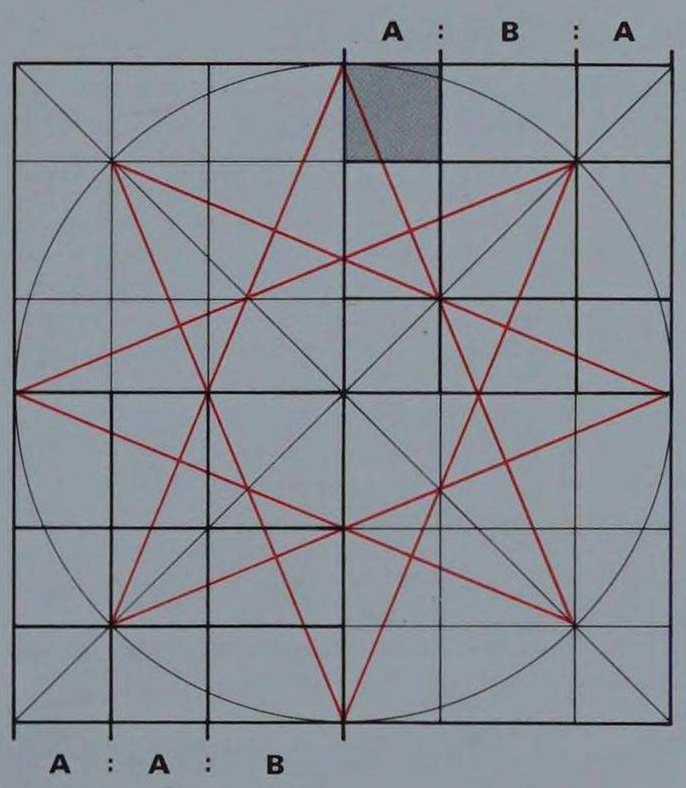


Figure 8

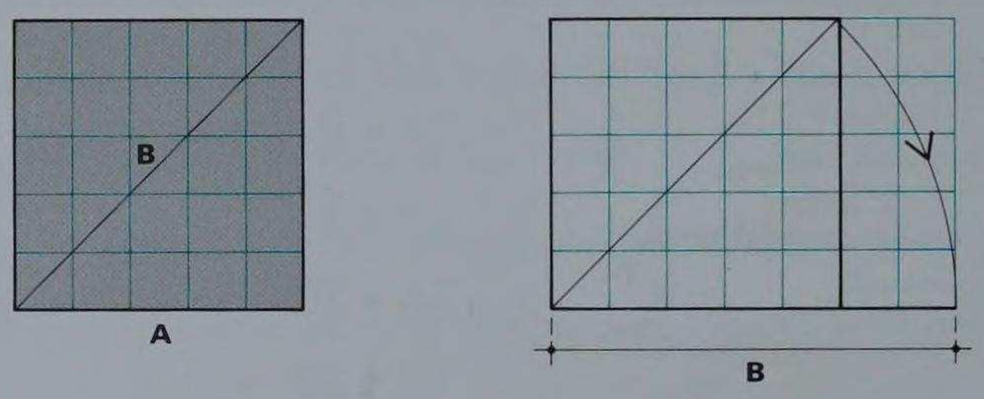


As shown in **Figure 8,** primary grid B also divides the sides of the square repeat unit in proportions expressible in terms of *A* and *B* either as: $A : B : A : A : B : A$ *or* $A : A : B : B : A : A$. Thus the grid-patterns create the proportional rhythm of horizontal and vertical divisions within the repeat unit. These proportions are expressed in terms of *A* and *B*, the side and the diagonal of the selected square module. The beauty of a pattern is not due so much to the nature of its geometric elements as to their correct use as units in a rhythmic scheme. However beautiful a pattern may be, if it strikes a false rhythmic note anywhere, it fails.

Generating the grid-patterns using numbers

Having established the selected square module with a ratio of side to diagonal of $A : B$, this can then be expressed mathematically as $A : B = 1 : \sqrt{2}$ (Theorem of Pythagoras). If A is taken to be 5 units in length (see **Figure 9**), then B equals approximately 7 units, since: $B = A\sqrt{2}$, $= 5\sqrt{2}$, $=7.07$. A difference of one per cent is imperceptible to the eye and acceptable in practical terms for small-scale work. The numerical values of 5 and 7 units are believed to have been employed in Ancient Egypt specifically as architectural units of measurement; they were called the 'remen' and the 'royal cubit' respectively (see page 130).

Figure 9



By translating the geometric values $A : B = 1 : \sqrt{2} = 5 : 7$ units, a simple numerical system of measurement based on counting is realized and this can be used to reconstruct primary grid A on a square grid. This numerical procedure (see Table II) facilitates translating into number the sides and diagonals of all the squares set up in the square repeat unit by the proportional divisions of primary and secondary grids A (Table I).

TABLE II

Side of Square	Diagonal	Mathematical difference
1	√2	
A 5	B 7	-1.0%
B 7	2A 10	+1.0%
A+B/2 8.5	B+A 12	-0.17%
A+B 12	B+2A 17	+0.17%
2A+B 17	2B+2A 24	-0.17%
3A+B 22	3B+2A 31	-0.36%
3A+2B 29	3B+4A 41	-0.03%
4A+2B 34	4B+4A 48	-0.17%

This series is an approximation which becomes more accurate the higher it goes until it nears the conventional decimal value of $\sqrt{2} = 1.4142136$. The mathematical difference gives the percentage by which the value for the ratio of the diagonal to the side differs by more or less to the decimal value of $\sqrt{2}$. For example, $41/29 = 1.4137931$ which is less than 1.4142136 by only 0.03 %.

Secondary grid A introduces a centrally-placed octagonal star composed of two squares (see **Figure 10**), the sides of which also divide each other in the proportion of $a : b : a : a : b : a$, where a equals the side of the square module set up and b its diagonal (indicated by the smaller shaded square). Graphically, the numerical value of $a = 2$ units and $b = 3$ units are deduced, thereby establishing the following values with respect to the selected square module of side $A = 5$ units and diagonal $B = 7$ units:

$a = 2$ units $= 1$,

$b = 3$ units $= \sqrt{2}$,

$A = a + b = 5$ units $= 1 + \sqrt{2}$,

$B = a + b + a = 7$ units $= 2 + \sqrt{2}$.

Figure 10

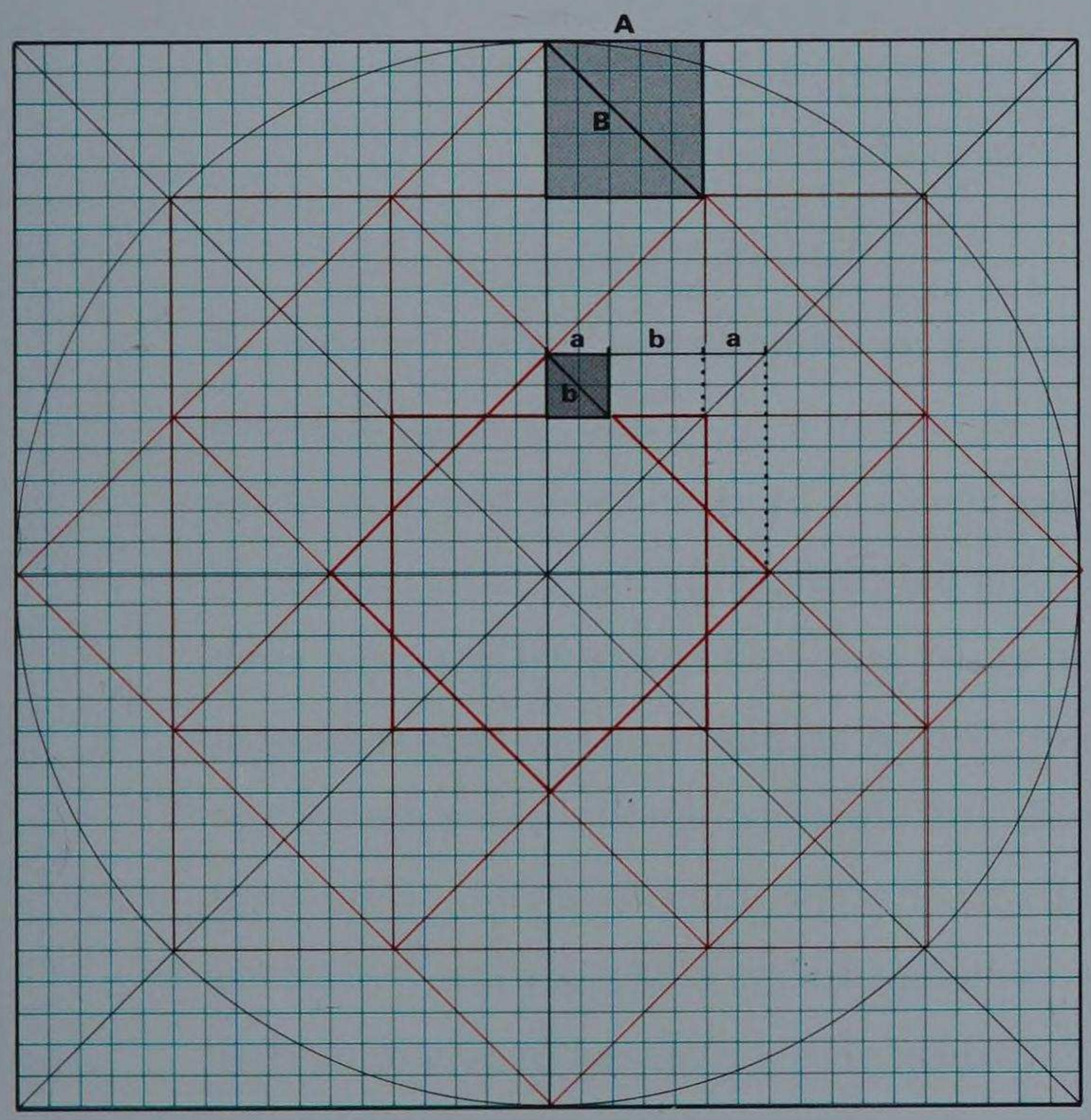
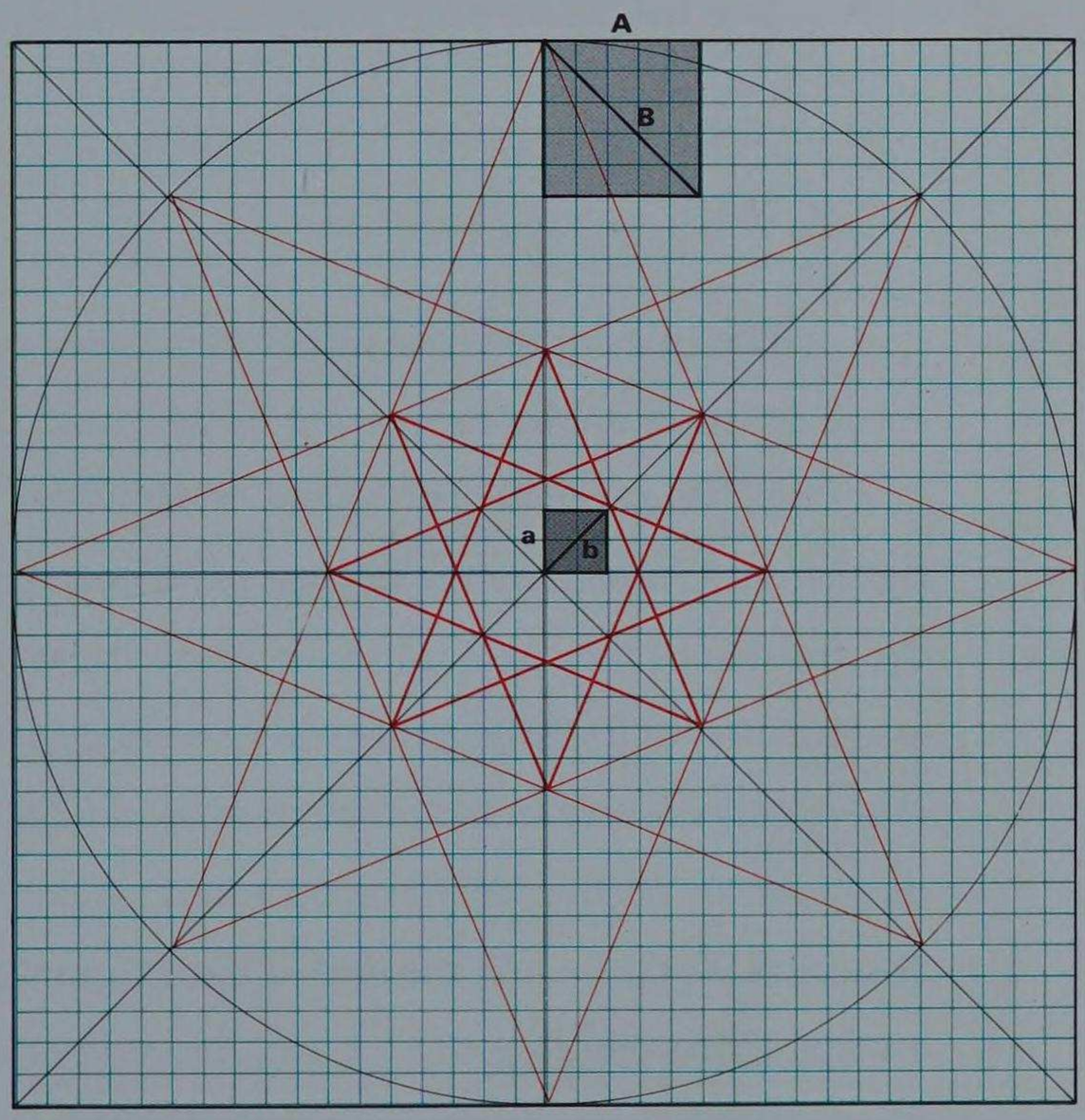


Figure 11



Similarly, when primary grid B with nesting secondary grid B (introduced by joining with straight lines every third point of the central octagon set up by primary grid B) are constructed graphically (see **Figure 11**), using the same numerical values for $A = 5$ units and $B = 7$ units, the values of a and b are again found to be 2 and 3 units respectively.

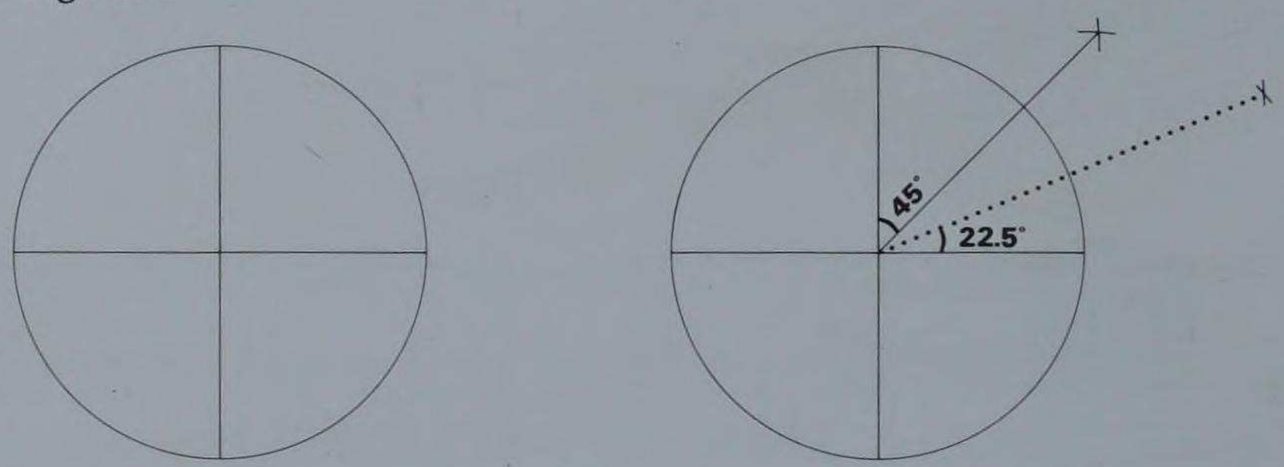
From these results it then becomes feasible to reconstruct geometric patterns whose repeat units are based on primary grids A and B and their secondary grids, simply by substituting the numerical values of a, b, A, and b, and counting to determine the proportions set up by the grid-patterns.

Mathematical or numerical rationalization of the root two system of proportion

The graphic method of point-joining used in constructing geometric patterns does not necessarily require any familiarity with Euclidean geometry. It is a practical geometry, utilized by the broadly illiterate craftsmen, as a means simply to create and reproduce 'geometric art'.[7] Unlike Euclidean geometry, which developed with time into an exact science and a device for reasoning and logical training of a theoretical nature, the point-joining method understood and traditionally practised by artisans was more an applied art from than a theoretical science.

This lack of an apparent abstract mathematical reasoning to the practical methods of geometry employed does not imply that the system is without rules. It must entail techniques and skills for ordering, measuring and calculation of a simple nature, acquired empirically through apprenticeship for the purpose of generating and executing geometric patterns.

Figure 12



The rules in the case of patterns based on the square repeat unit must govern the accurate construction of angles measuring 90°, 45° and 22.5° corresponding to $\frac{1}{4}$, $\frac{1}{8}$ and $\frac{1}{16}$ equal divisions of a circle (**Figure 12**). This could be achieved graphically either by continuous bisection of the right angles formed by the two perpendicular diameters of the circle, or by the method describing the division of the circle into four, or multiples of four, equal parts (see **Figure 1**). However, close examination of a master grid-pattern combining primary grids A and B shows that they cross each other progressively in the proportion of $1 : \sqrt{2}$ (**Figure 13**). It is also apparent that along the vertical diameter, in all the right-angled triangles containing a 22.5° angle, the proportion of the short adjacent side (shown in the dotted line, **Figure 14**), to the long adjacent side along the vertical has a value corresponding to $1 : 1 + \sqrt{2}$, i.e. side of square module: side + diagonal of square module (Table III); the numerical value is given below each dimension. (For the value of the diagonal of a given side in terms of *A* and *B*, see Table I.)

The ratios presented in Table III are approximate trigonometric values of tan 22.5°, thereby enabling the definition $\frac{1}{16}$ of a circle to be expressed in terms of a square module approximately as:
(side of square) / (side+diagonal) = $1/(1+\sqrt{2})$ = tan 22.5° = 0.4142136.
The higher ratios give a very near approximation of tan 22.5°:
$\frac{5}{12}$ = tan 22.620° (a difference of 0.120° or +0.533%),
$\frac{12}{29}$ = tan 22.479° (a difference of 0.021° or -0.093%).

THE SQUARE REPEAT UNIT AND ROOT TWO SYSTEM

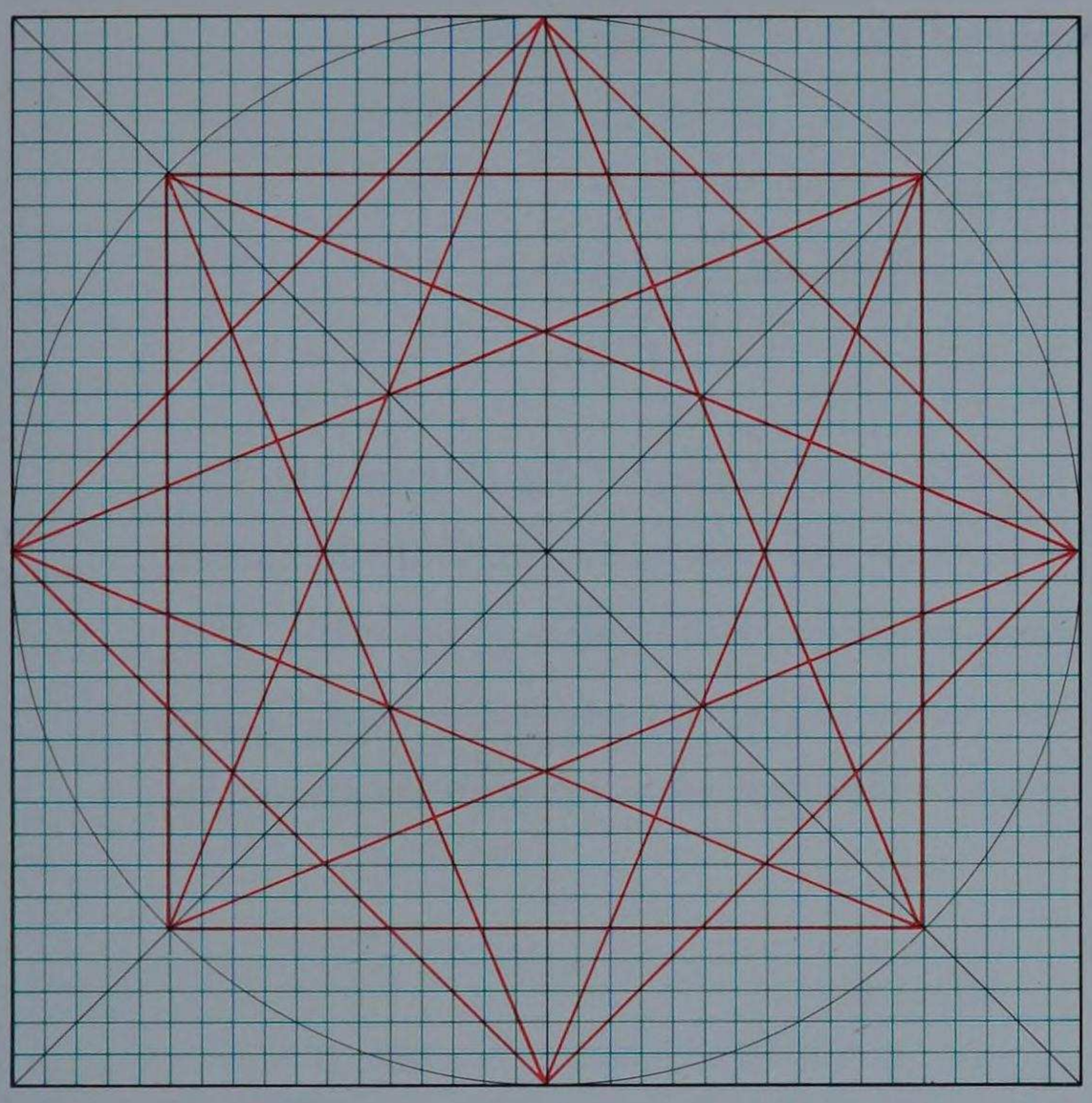


Figure 13

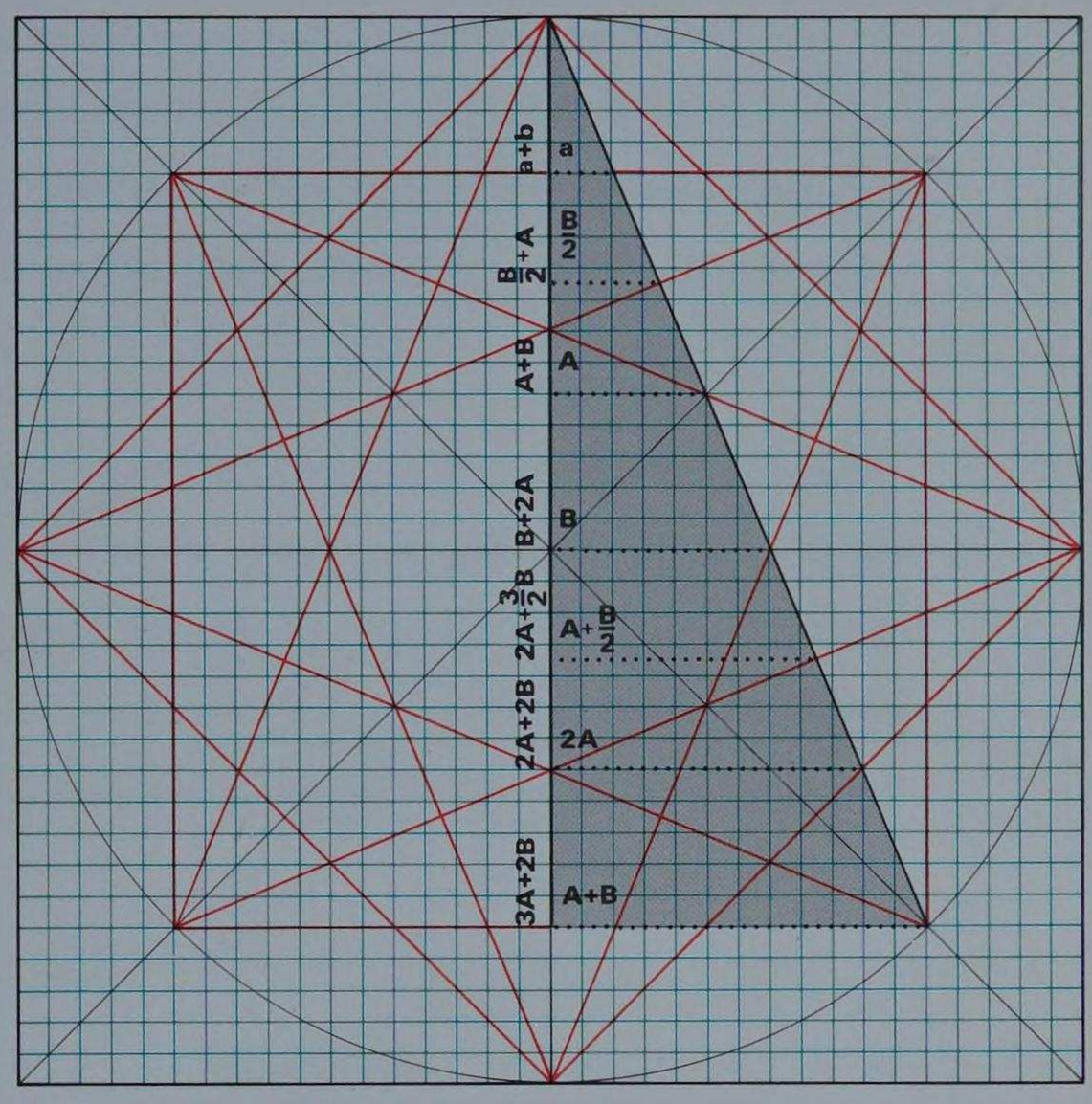


Figure 14

TABLE III

Short adjacent side (side of square module)	Long adjacent side (side + diagonal of square module)	Ratio
1	$1+\sqrt{2}$	$\frac{1}{1+\sqrt{2}}$
a 2	a+b 2+3	$\frac{a}{a+b}=\frac{a}{A}=\frac{2}{5}$
b 3	b+2a 3+4	$\frac{b}{b+2a}=\frac{b}{B}=\frac{3}{7}$
B/2 3.5	B/2+A 3.5+5	$\frac{B/2}{B/2+A}=\frac{3.5}{8.5}$
A 5	A+B 5+7	$\frac{A}{A+B}=\frac{5}{12}$
B 7	B+2a 7+10	$\frac{B}{B+2A}=\frac{7}{17}$
A+B/2 5+3.5	(A+B/2)+(B+A) (5+3.5)+(7+5)	$\frac{A+B/2}{2A+3B/2}=\frac{8.5}{20.5}$
2A 10	2A+2B 10+14	$\frac{2A}{2A+2B}=\frac{10}{24}$
A+B 5+7	(A+B)+(B+2A) (5+7)+(7+10)	$\frac{A+B}{3A+2B}=\frac{12}{29}$

Referring to Table III, tan 22.5° can also be expressed as:

$\frac{a}{a+b}$, $\frac{a}{A}$ and $\frac{A}{A+B}$, which in turn can be interpreted as:

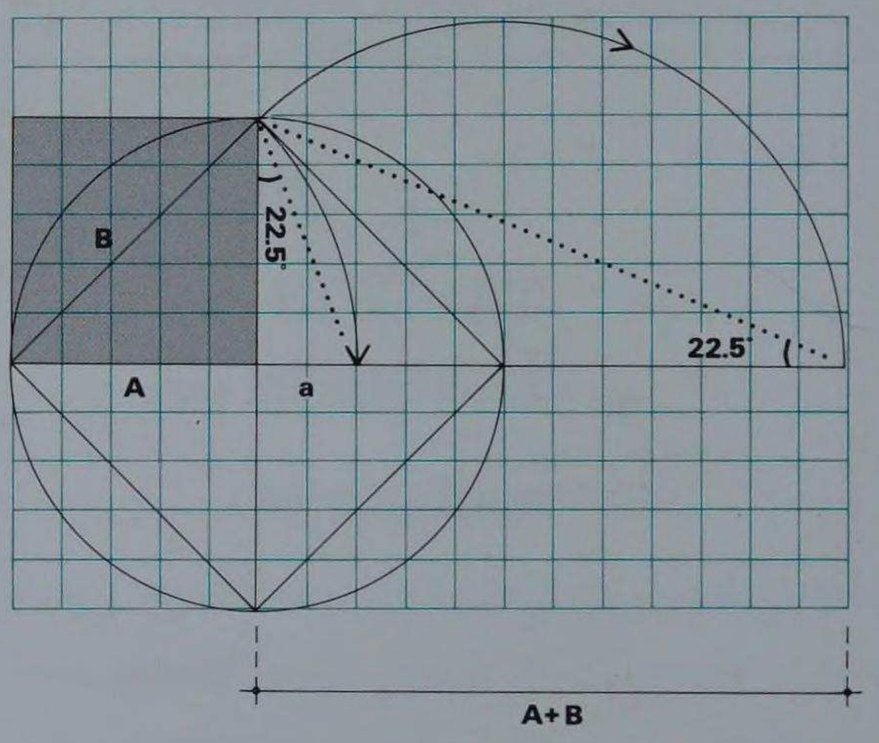
(diagonal–side) / (side) of a square = (√2–1)/1 = 0.4142136,

whereby for a side = A, the diagonal =B and thus tan 22.5° $= \frac{B-A}{A} = \frac{(a+b+a)-(a+b)}{A} = \frac{a}{A}$.

These two expressions of tan 22.5°: $\frac{1}{1+\sqrt{2}}$ and $\frac{\sqrt{2}-1}{1}$ are illustrated graphically (**Figure 15**). The ratio of tan22.5° in the smaller triangle equals $\frac{B-A}{A} = \frac{a}{a}$ while the ratio of tan 22.5° in the larger triangle equals $\frac{A}{A+B}$. Furthermore, the reverse of these ratios gives the value of tan 67.5°. Any value can be given to a, e.g. 4, 7, 10, 12, 29, ... and so on. For more accurate approximations, higher values should be used, and the values for b, A and B can be determined accordingly, as shown in Table II. For example:

a	b	$A=a+b$	$B=a+b+a$
1	√2	1+√2	1+√2+1
5	7	12	17
12	17	29	41
29	41	70	99

Figure 15



Further examination of the right-angled triangle with adjacent sides measuring 5 and 12 units (**Figure 16)** shows the hypotenuse H= 13 units (Theorem of Pythagoras). The hypotenuse can also be expressed in terms of A and B simply by subtracting B from the hypotenuse H, (13-7+6), leaving 6 units which equals $\frac{A+B}{2} = \frac{5+7}{2}$, and thus $H = B + \frac{A+B}{2} = \frac{A+3B}{2}$.

Therefore the proportions of all the sides of a 22.5°, 67.5°, 90° triangle can be expressed in terms of a square module. When the short adjacent side equals the side of the square module, the long adjacent side equals the side plus the diagonal of the square module, and the hypotenuse equals the diagonal plus half the long adjacent side, the application of this formula to the triangles in Table III gives us the respective dimensions of the hypotenuse (Table IV).

Figure 16

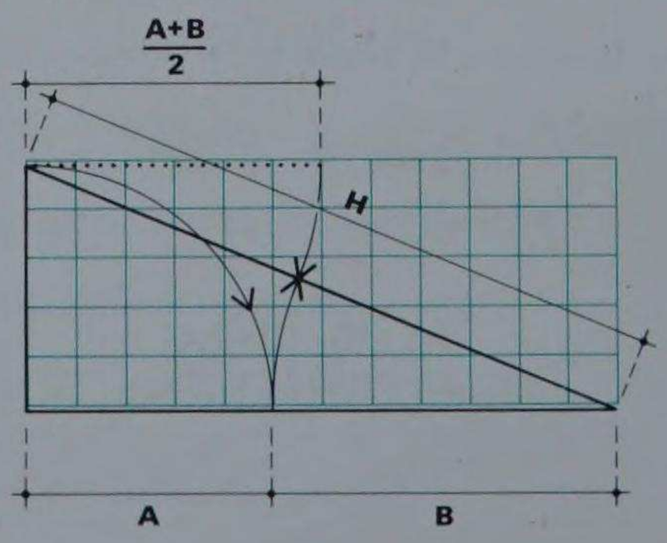


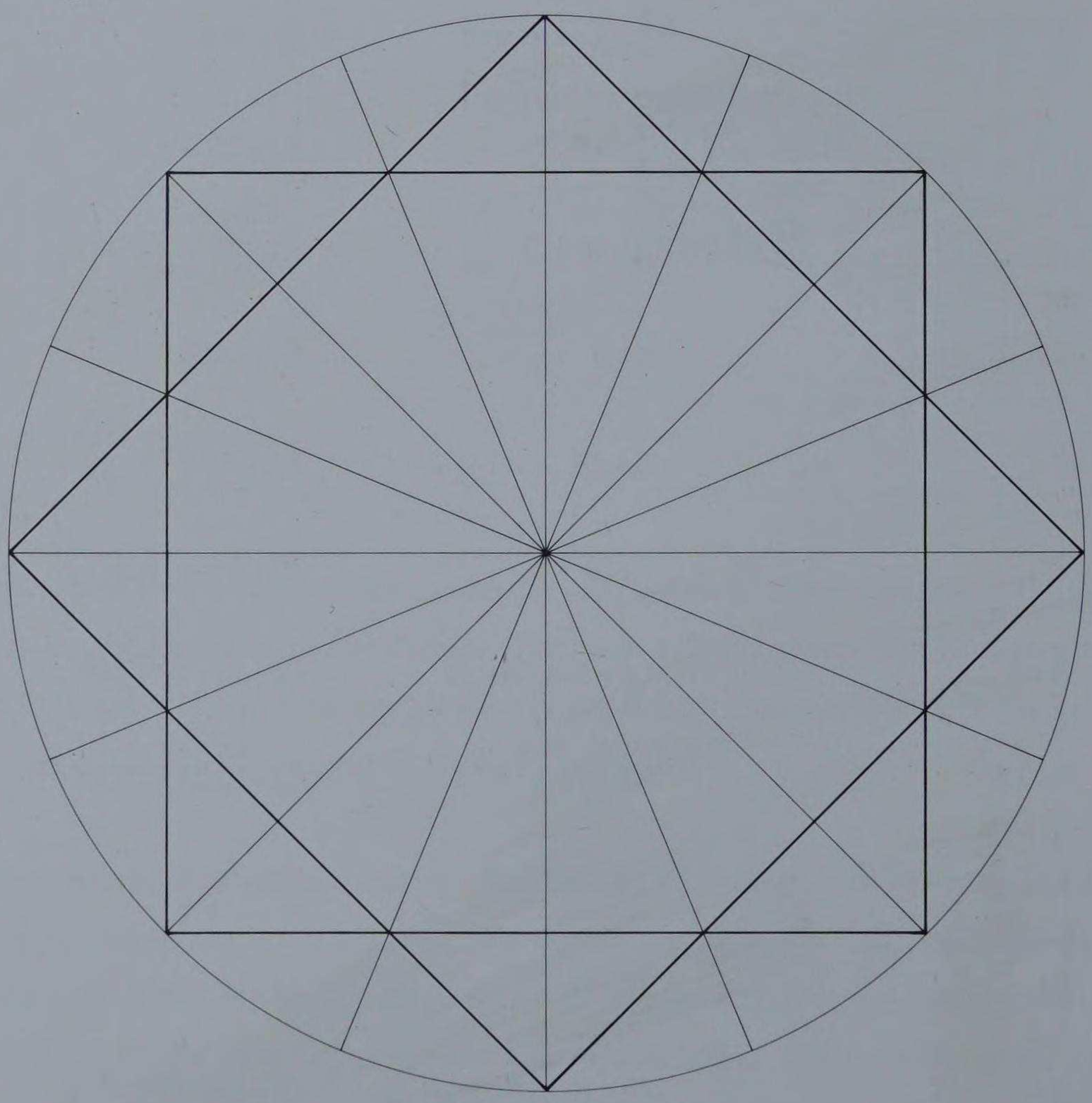
TABLE IV

Short adjacent side (side of square module)	Long adjacent side (side + diagonal of square module)	Hypotenuse (diagonal + half long adjacent side)
1	$1+\sqrt{2}$	$\sqrt{2}+\frac{(1+\sqrt{2})}{2}$
a	$a+b$	$b+\frac{a+b}{2}$
2	$2+3$	$3+2.5=5.5$
b	$b+2a$	$2a+\frac{b+2a}{2}$
3	$3+4$	$4+3.5=7.5$
$\frac{B}{2}$	$\frac{B}{2}+A$	$A+\frac{(B/2)+A}{2}$
3.5	$3.5+5$	$5+4.25=9.25$
A	$A+B$	$B+\frac{A+B}{2}$
5	$5+7$	$7+6=13$
B	$B+2A$	$2A+\frac{B+2A}{2}$
7	$7+10$	$10+8.5=18.5$
$A+\frac{B}{2}$	$(A+\frac{B}{2})+(B+A)$	$(B+A)+\frac{2A+(3B/2)}{2}$
$5+3.5$	$8.5+12$	$12+10.25=22.25$
$2A$	$2A+2B$	$2B+\frac{2A+2B}{2}$
10	$10+14$	$14+12=26$
$A+B$	$(A+B)+(B+2A)$	$(B+2A)+\frac{3A+2B}{2}$
$5+7$	$12+17$	$17+14.5=31.5$

Hence all the relevant dimensions of the master grid-pattern (**Figure 13**) can be extrapolated by simple addition and subtraction. This graphic procedure demonstrates how, by the manipulation of the side and diagonal of a square module, the proportional divisions of a repeat unit can be established and the calculation of dimensions can be performed easily (and within reasonable approximation) without resorting to complicated trigonometric and Pythagorean calculations.

The rules and geometric principles involved ensure the accuracy, symmetry and balance of composition, thus providing a working framework for the artisan or craftsman. However, within this framework, their artistic abilities and skills are put to effect in creating the elements of a composition, that is, its constituent geometric shapes and their arrangement. This is clearly demonstrated in the example of constructing compositions based on a 16-pointed star. A procedure for such compositions is as follows:

Figure 17



Divide the circumference of a circle into 16 equal sections using primary grid A (see **Figure 17**). Join the 16 points of division to each other with straight lines (**Figure 18**), thus generating a principal grid-pattern. This provides unlimited combinations of geometric compositions (see **Figures 19** and **20**).

Figure 18

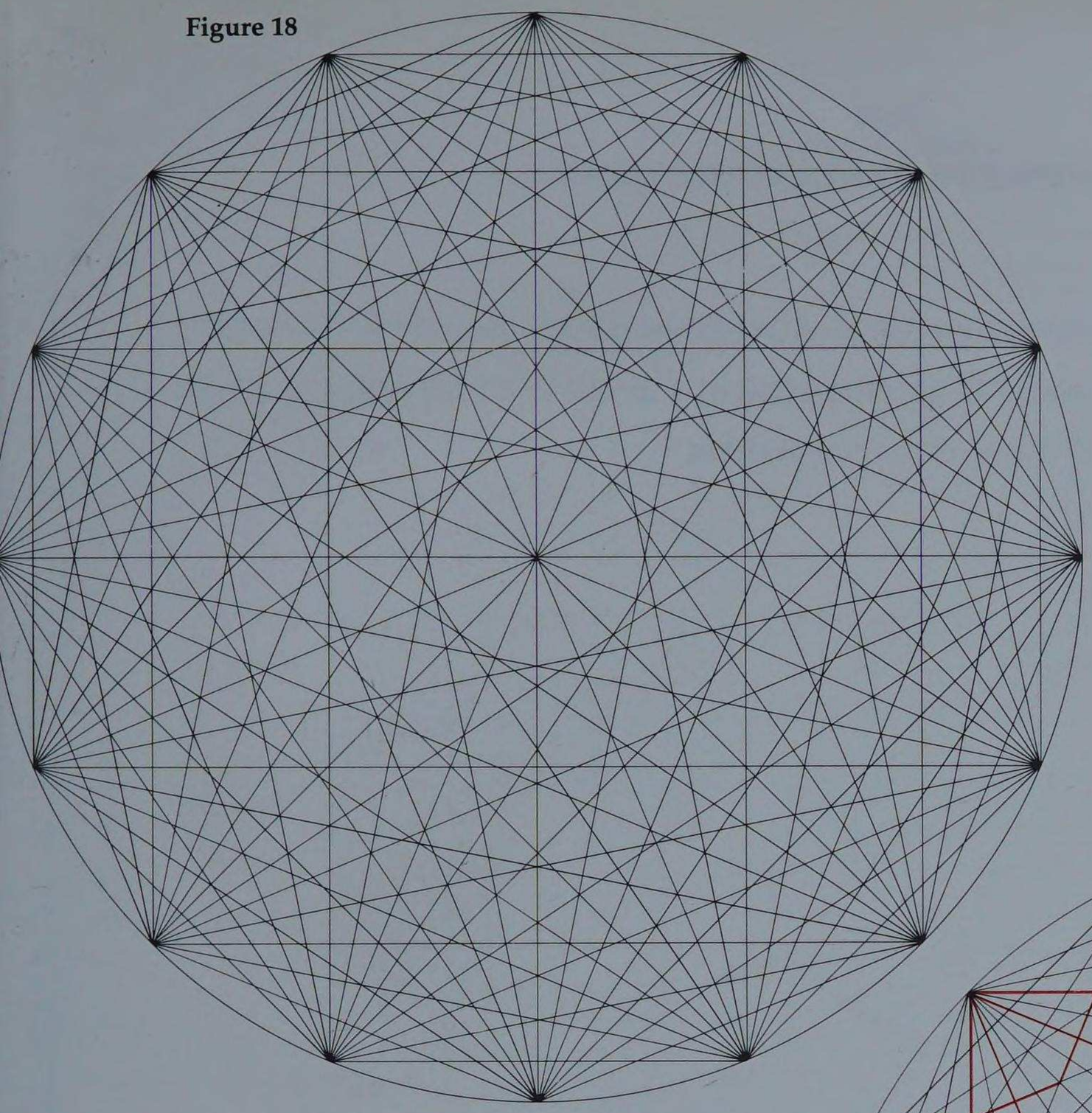


Figure 19

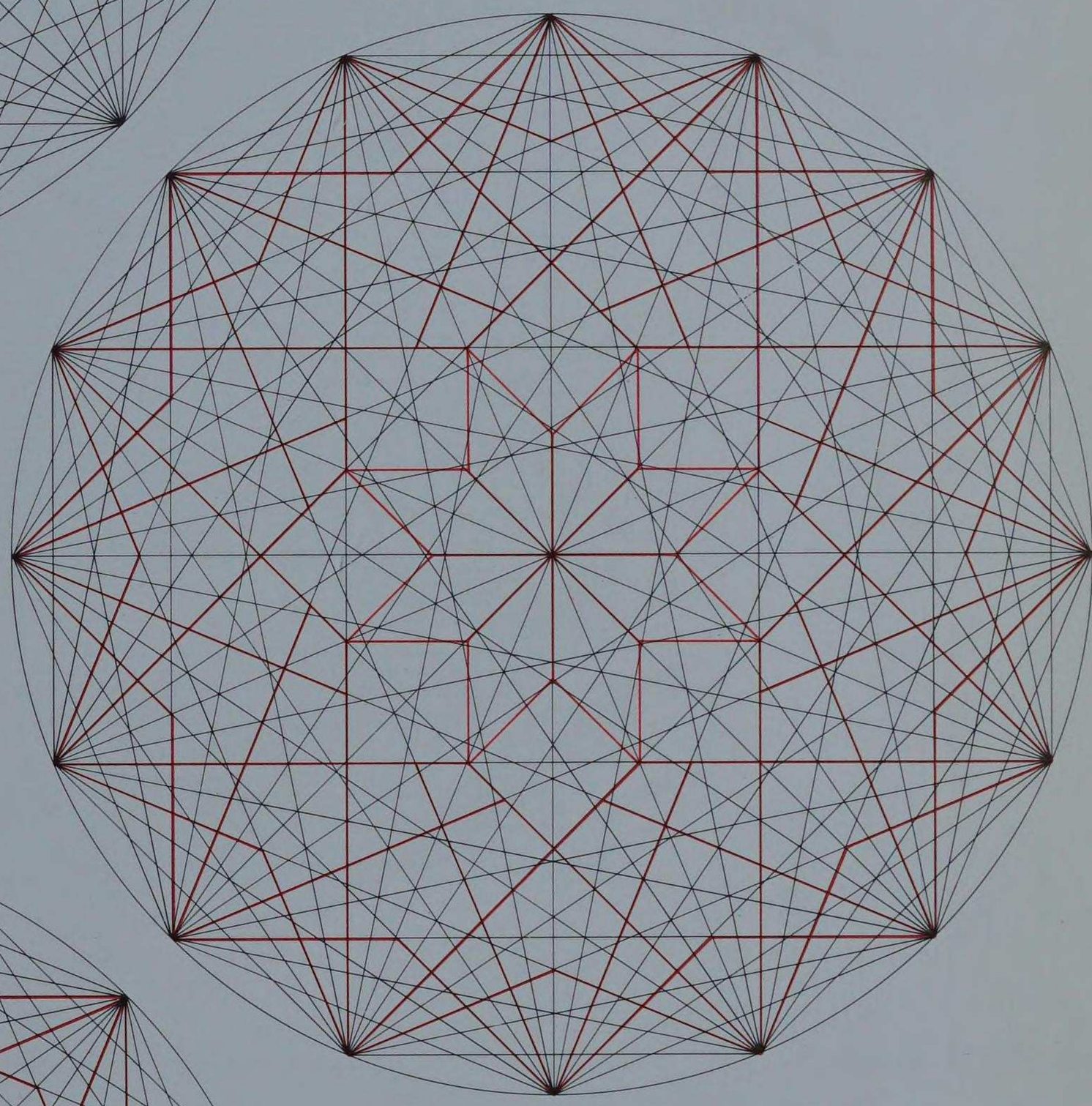
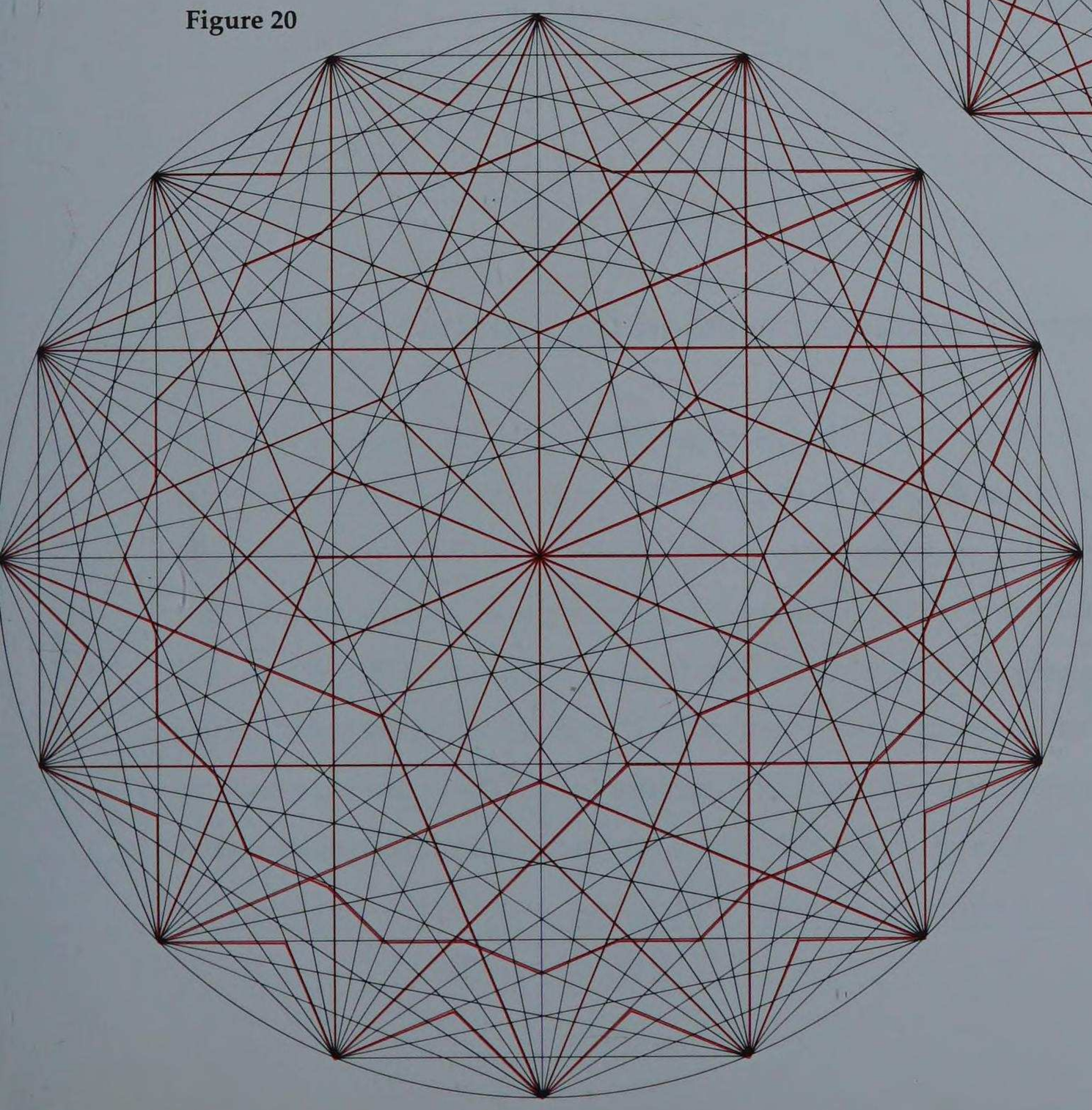
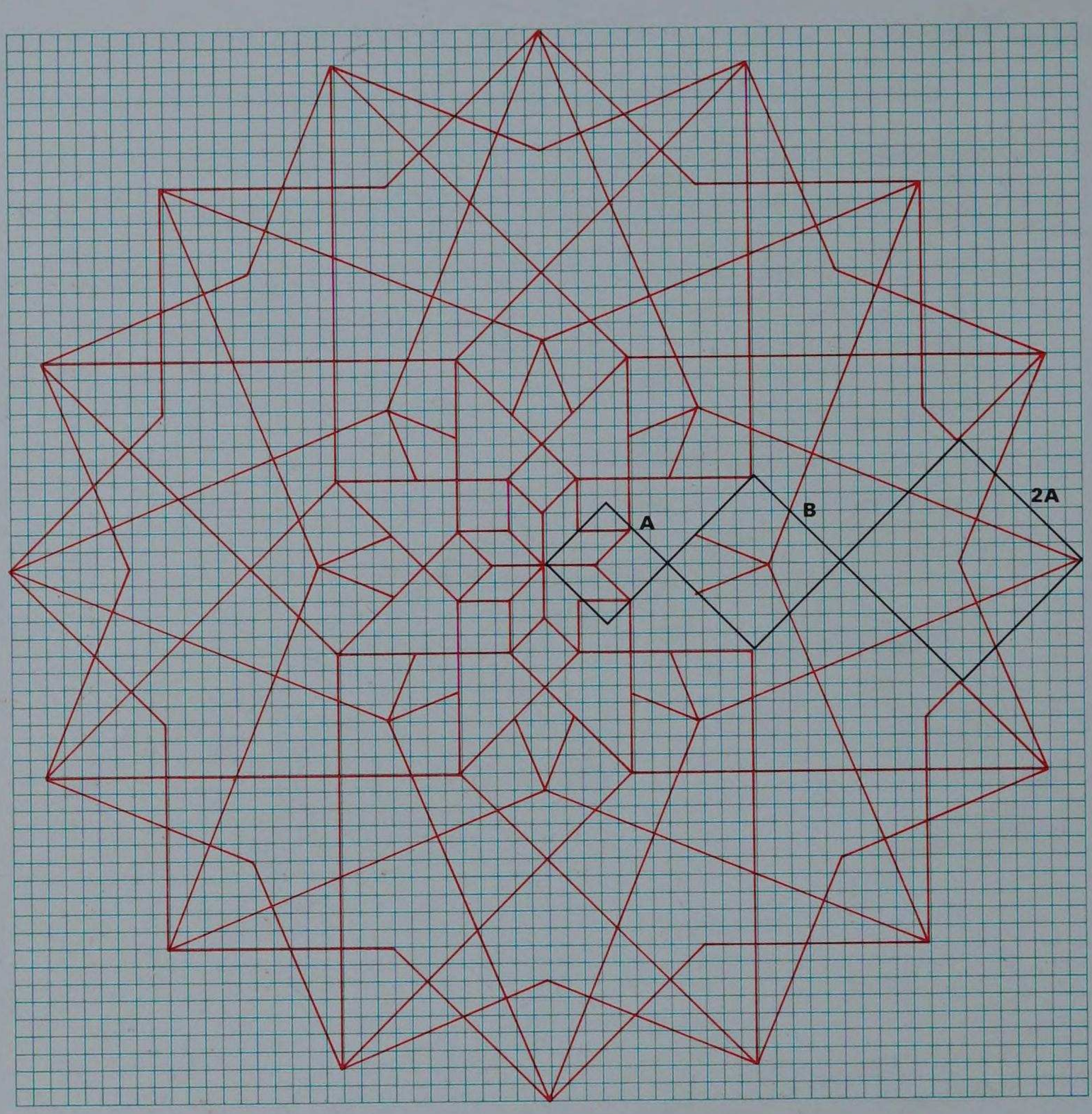


Figure 20



A variation is shown here (**Figure 21**) using a progression of three square modules (shown black). The sides of the three squares are related in the proportion of $A : B : 2A$ or $1 : \sqrt{2} : 2$ thus creating a scale for calculating the harmonic divisions of the composition.

Figure 21



Graphical analyses of geometric patterns based on the root two system of proportion

The format of the presentation of the following examples of geometric patterns (**Figures 22** to **46**) illustrates the progression from the initial division of the circle into eight or 16 equal parts to the final juxtaposition of the repeat units to create the pattern. The top three of each figure show the construction of the repeat unit, with the middle frame giving the type of grid-pattern(s) that generate its composition. The translation of the geometry into number is shown in the bottom left-hand frame by superimposing it on a square grid. Practically, this is achieved by substituting numerical values for the major proportional divisions of the repeat unit. These are expressed in terms of a, b, A and B (see page 23) and are given for the side of one quadrant of the square repeat unit: $a = 2$ units, $b = 3$ units, $A = a + b = 5$ units, $B = a + b + a = 7$ units.

Figure 22. The repeat unit is the octagonal star derived from primary grid A.
$A : B : A = 5 : 7 : 5$.

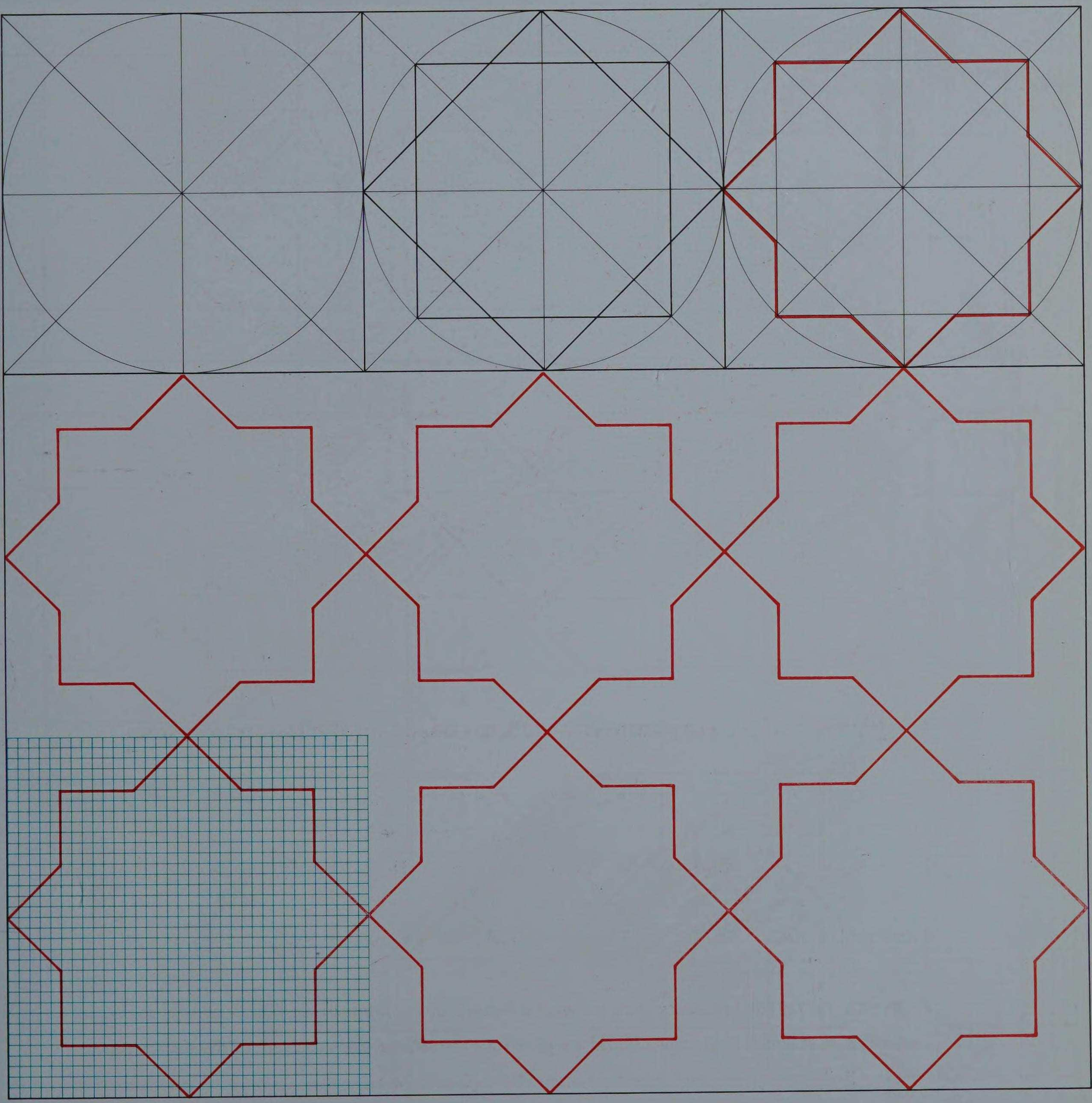


Figure 23. The eight points of intersection of the two squares forming primary grid A are joined diagonally to give a secondary grid-pattern. The composition of the repeat unit is then completed by extending the sides of the regular square.
$A : a : A : A = 5 : 2 : 5 : 5.$

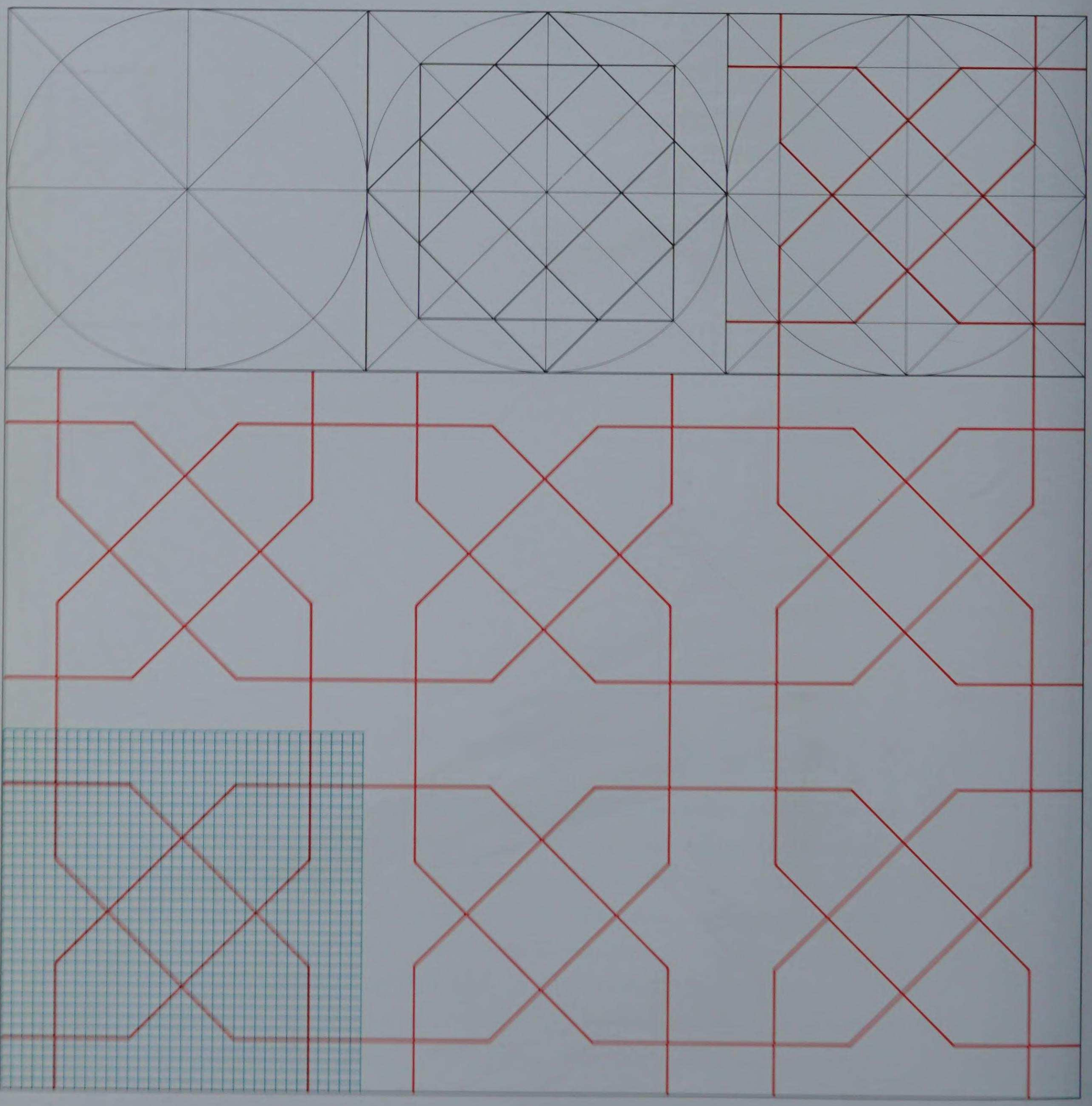


Figure 24. A second grid-pattern, secondary grid A, which is another form of octagonal star or stellated octagon, is introduced nesting within primary grid A, by joining the symmetrically opposite intersection points of the sides of primary grid A. By further extending lines from the centre of the circle through the extreme points of secondary grid A to the perimeter of the square repeat unit, the regular octagon circumscribing the circle is defined.
$a : b : a : A : A : = 2 : 3 : 2 : 5 : 5.$

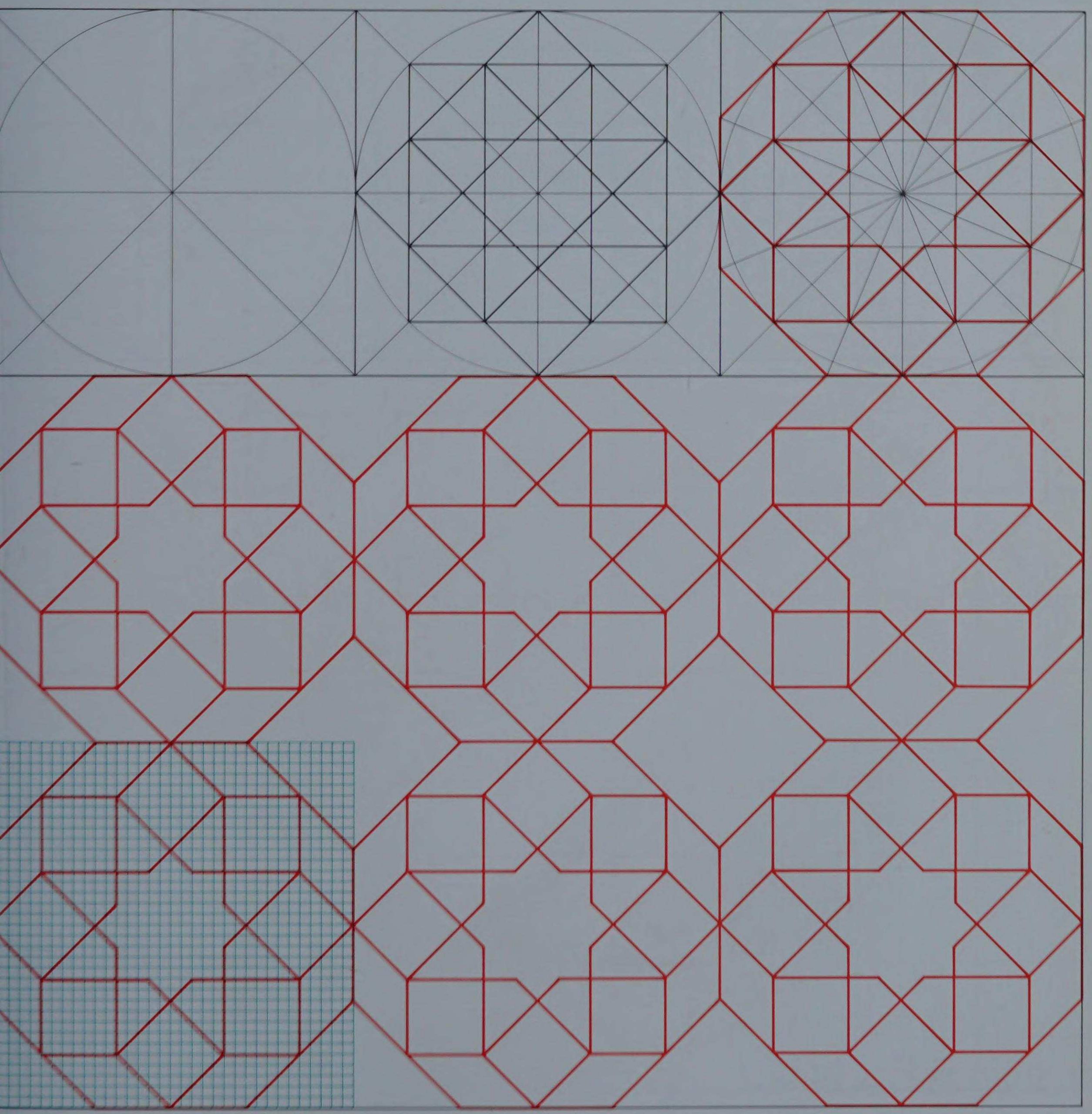


Figure 25. The sides of secondary grid A are extended to the perimeter of the square repeat unit. Further point-joining generates a secondary grid-pattern which provides the framework for the composition.

$a : b : a : b : a : b : a = 2 : 3 : 2 : 3 : 2 : 3 : 2.$

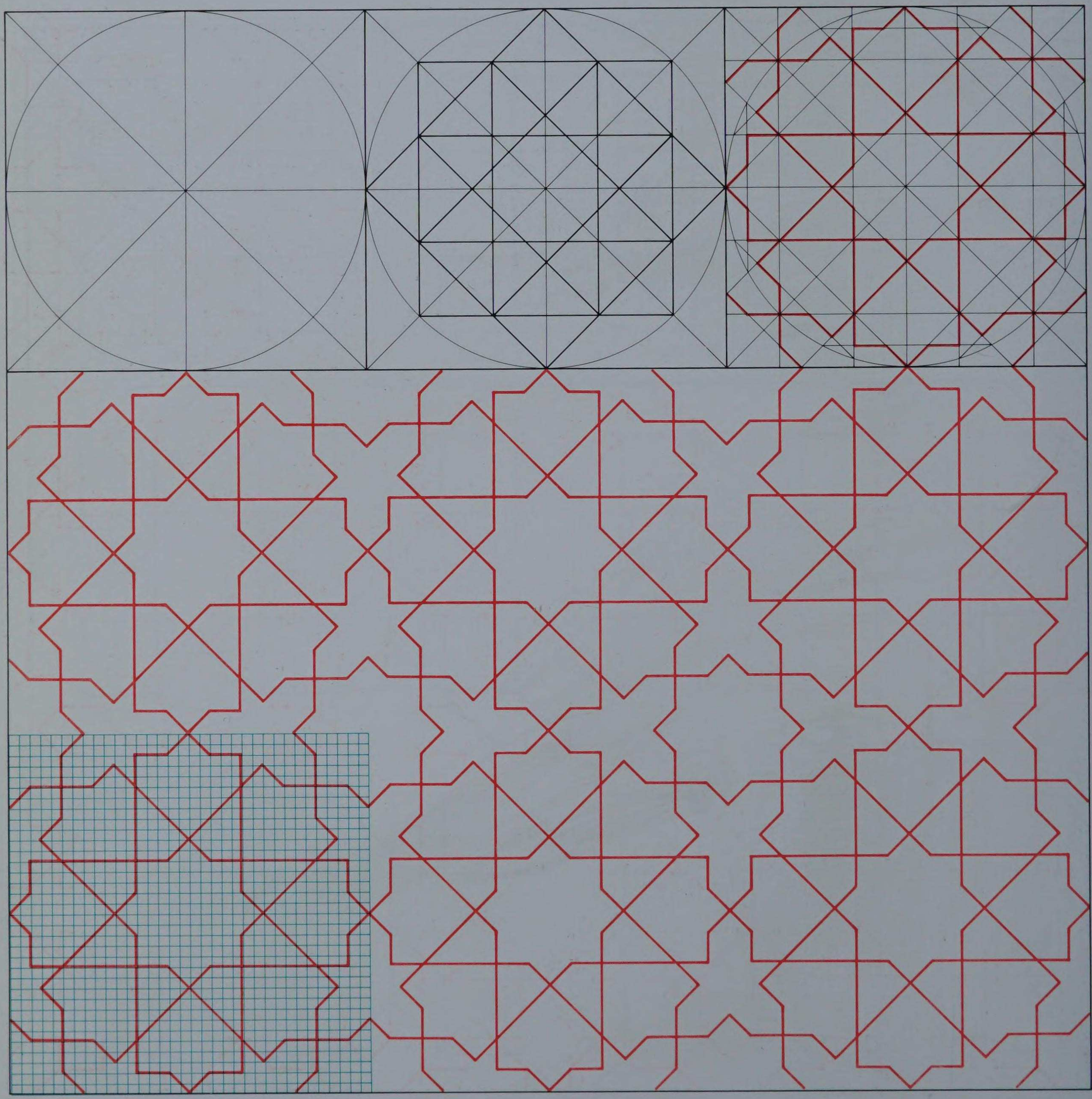


Figure 26. Tertiary grid A, a stellated octagon, is introduced nesting in secondary grid A. The complete grid-pattern generating the composition is developed in a similar way to the previous example, whereby the sides of primary, secondary and tertiary grids A are extended to the perimeter of the square repeat unit.

$a : b : b : 2a : a : a : b = 2 : 3 : 3 : 4 : 2 : 3.$

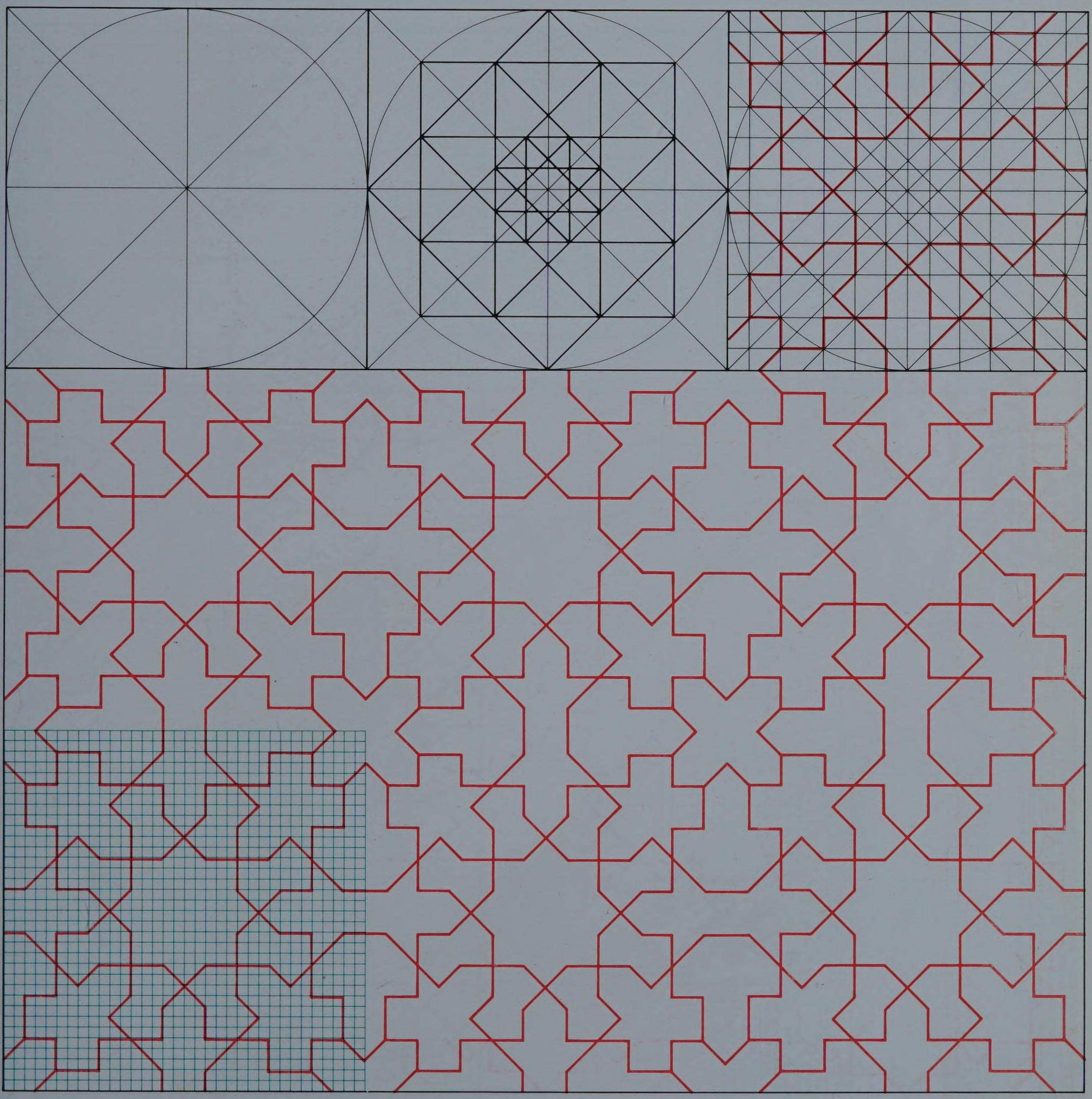


Figure 27. A tertiary grid-pattern nesting within secondary grid A is set up by diagonally joining opposite points of the octagon at the centre of secondary grid A. The lines of this grid pattern are then extended to the perimeter of the square repeat unit to give the complete grid-pattern for the composition.

$a : b : b : a : a : b : a = 2 : 3 : 3 : 2 : 2 : 3 : 2.$

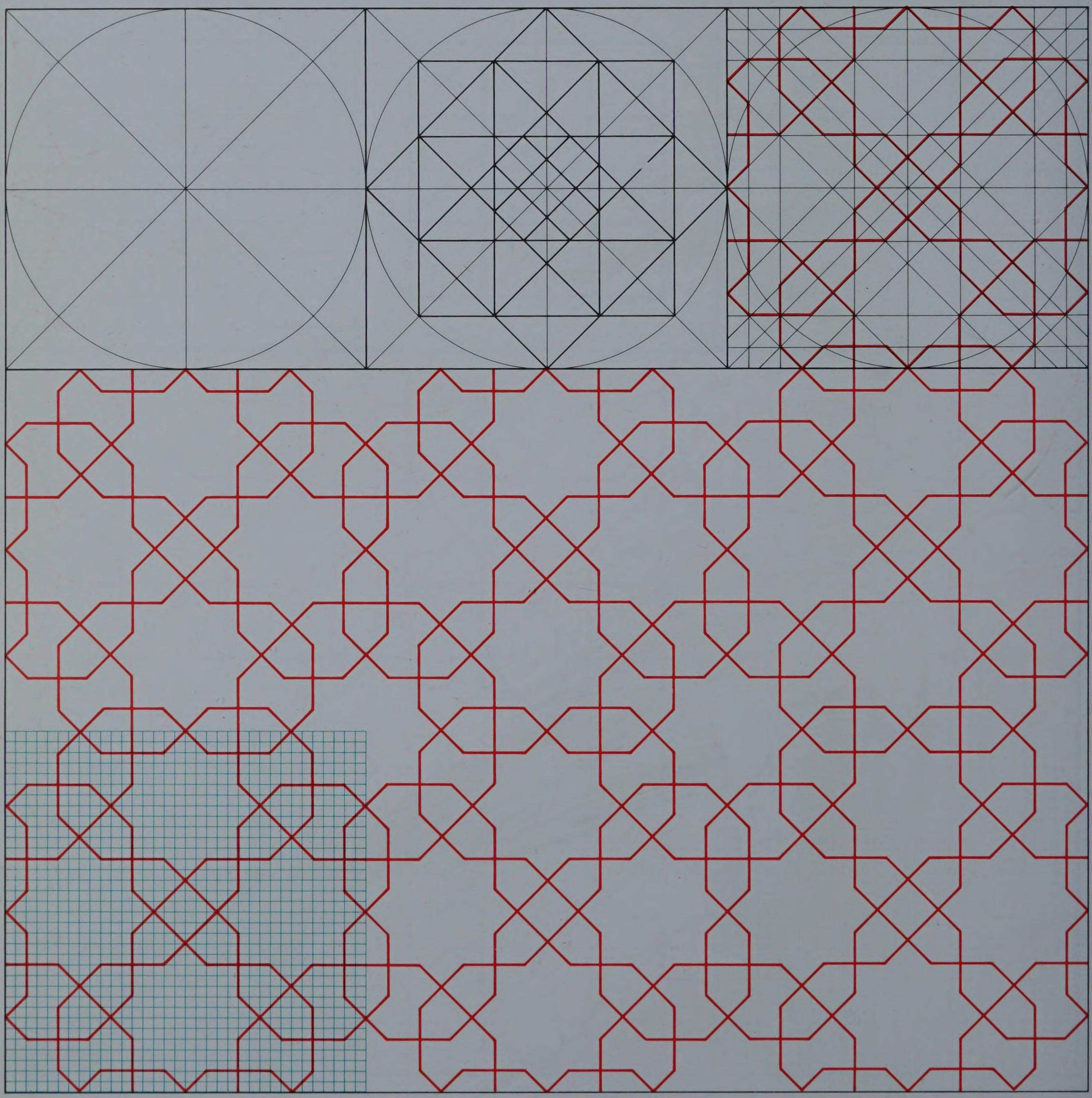


Figure 28. Using primary grid A, and the diagonals of the square repeat unit, a secondary grid-pattern is constructed dividing the sides of the square into four equal parts. Two other superimposed grid-patterns are constructed; the first by joining selected intersection points on primary grid A together and extending the lines to the corners of the square repeat unit; the second by joining the ends of the diameters of the circle to the corners of the regular square in primary grid A, as shown.

$A : B : A = 5 : 7 : 5.$

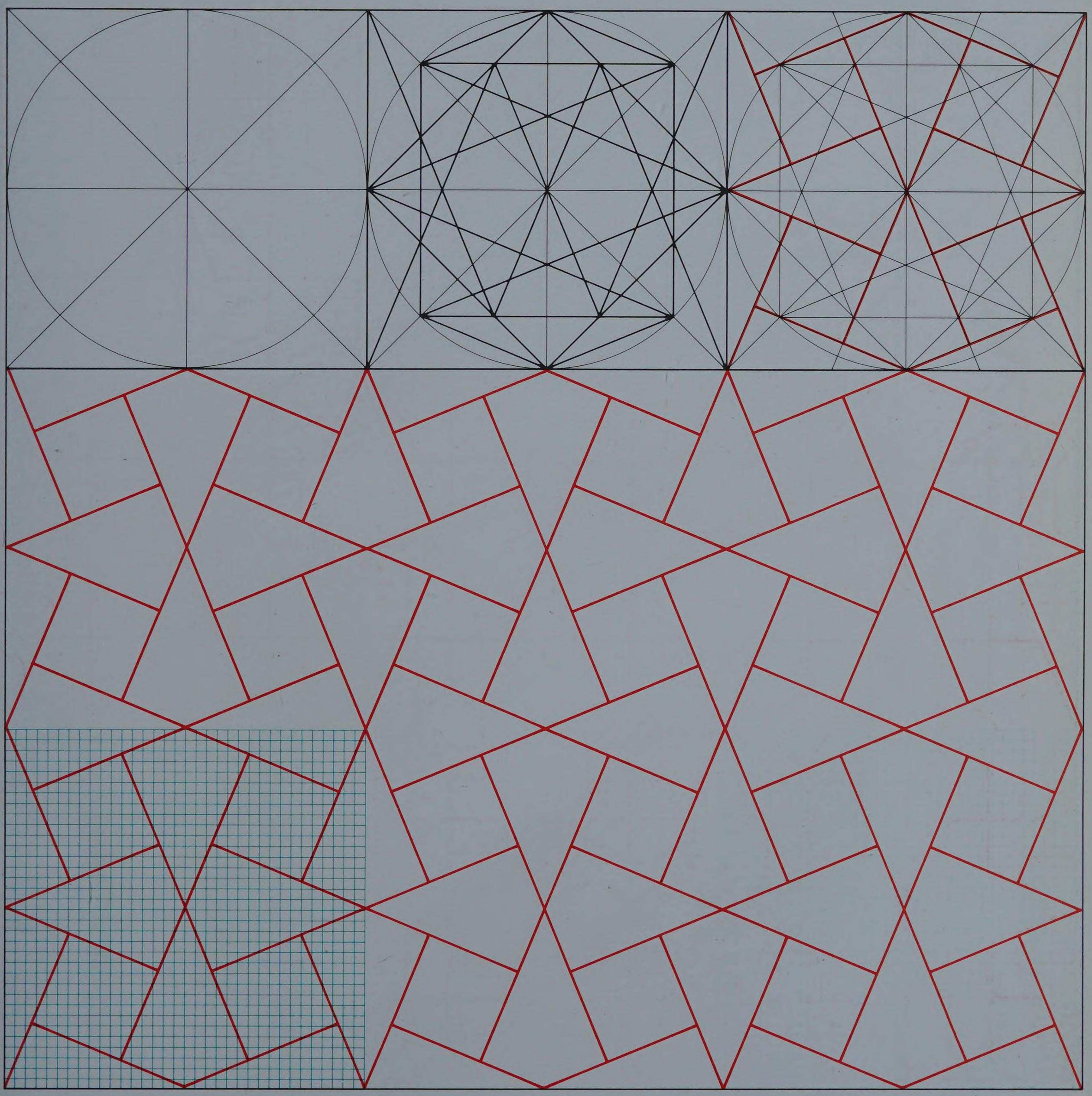


Figure 29. Two of the grid-patterns used for the previous composition are reused in this example. The composition is completed by extending selected lines from the grid-patterns.
$A{:}\frac{B}{2}{:}\frac{B}{2}{:}\,A = 5 : 3.5 : 3.5 : 5.$

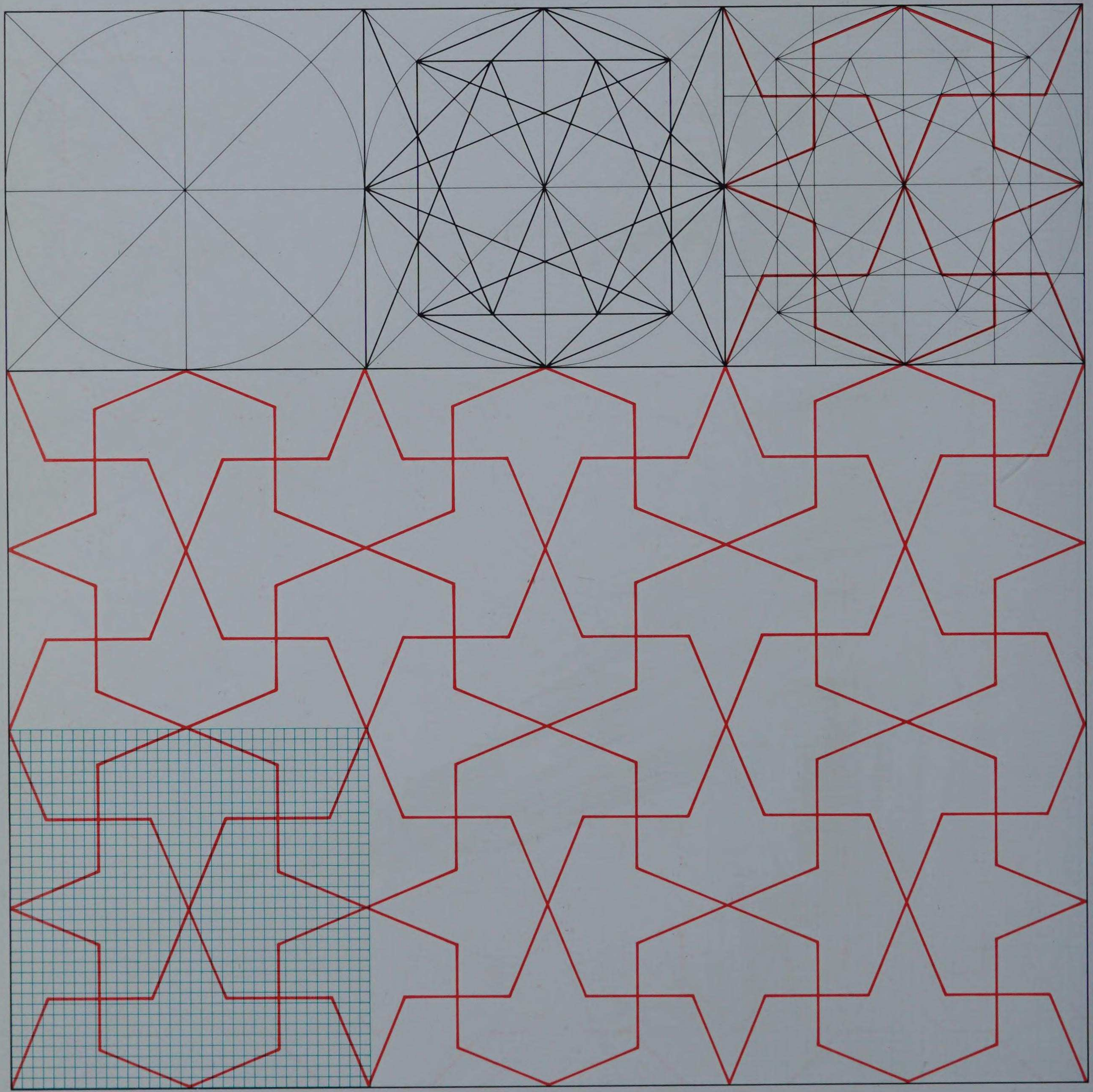


Figure 30. A stellated octagon inscribed in the circle is superimposed on primary grid A, and lines extended from selected intersection points. The sides of primary grid A are also extended to the perimeter of the square repeat unit:
$\frac{A}{2} : 2a : \frac{a}{4} : A : A : = 2.5 : 4 : 0.5 : 5 : 5.$

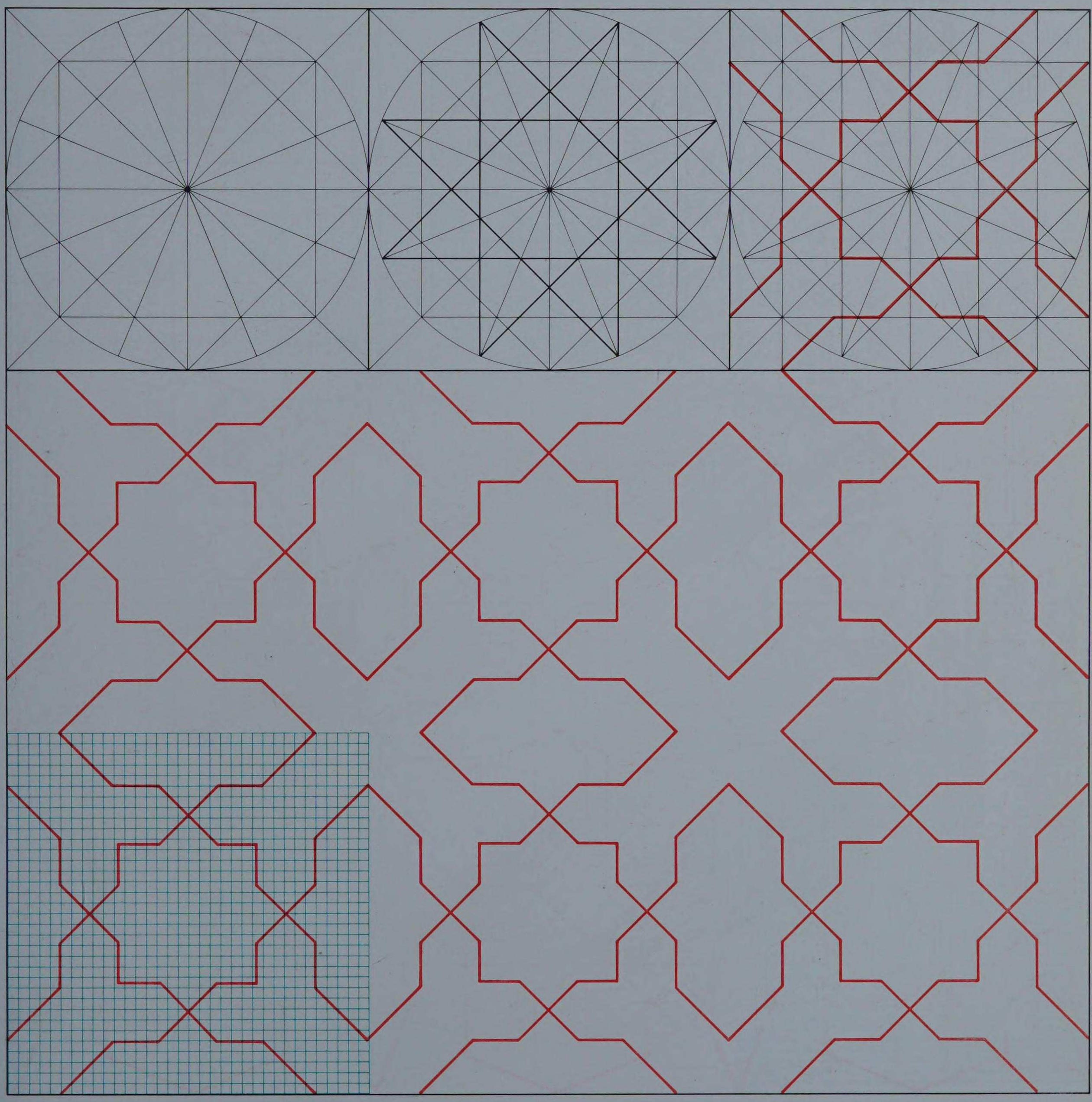


Figure 31. The sides of primary grid A are extended to the perimeter of the square repeat unit from which opposite points are joined. A rotated (dynamic) square is drawn nesting within primary grid A and its sides extended to the perimeter of the repeat unit. Selected intersection points are joined to give the complete grid-pattern.

$\frac{A}{2} : \frac{B}{2} : \frac{A}{2} : \frac{A}{2} : \frac{B}{2} : \frac{A}{2} = 2.5 : 3.5 : 2.5 : 2.5 : 3.5 : 2.5.$

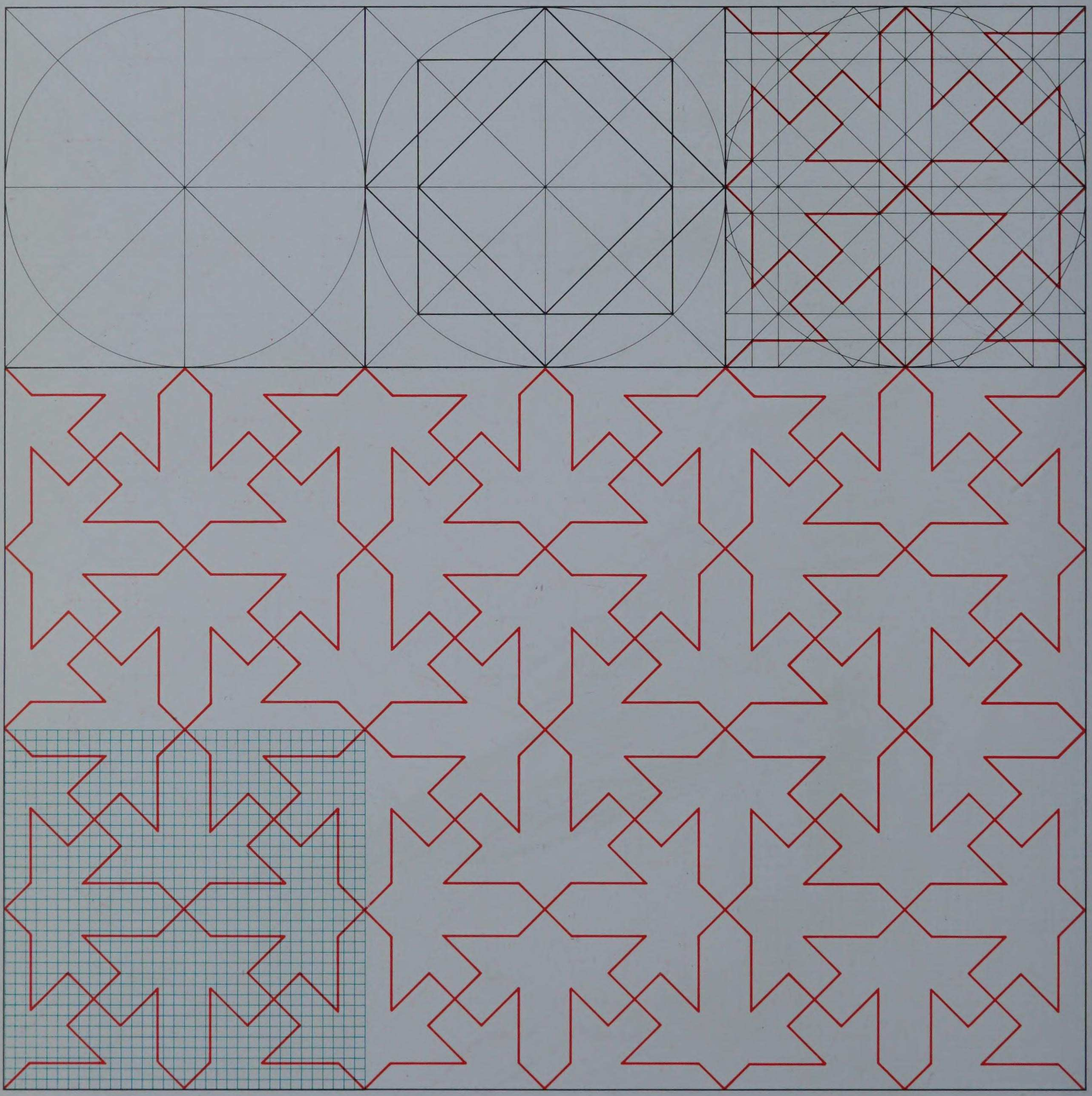


Figure 32. Using primary grid A, the circle is divided into 16 equal parts and a 'rotated' octagonal star is inscribed. In turn a secondary grid-pattern is drawn nesting within it. A regular octagon is inscribed in the circle and the sides of the secondary grid-pattern are extended to it.
$A : a : b : a : b : a = 5 : 2 : 3 : 2 : 3 : 2$.

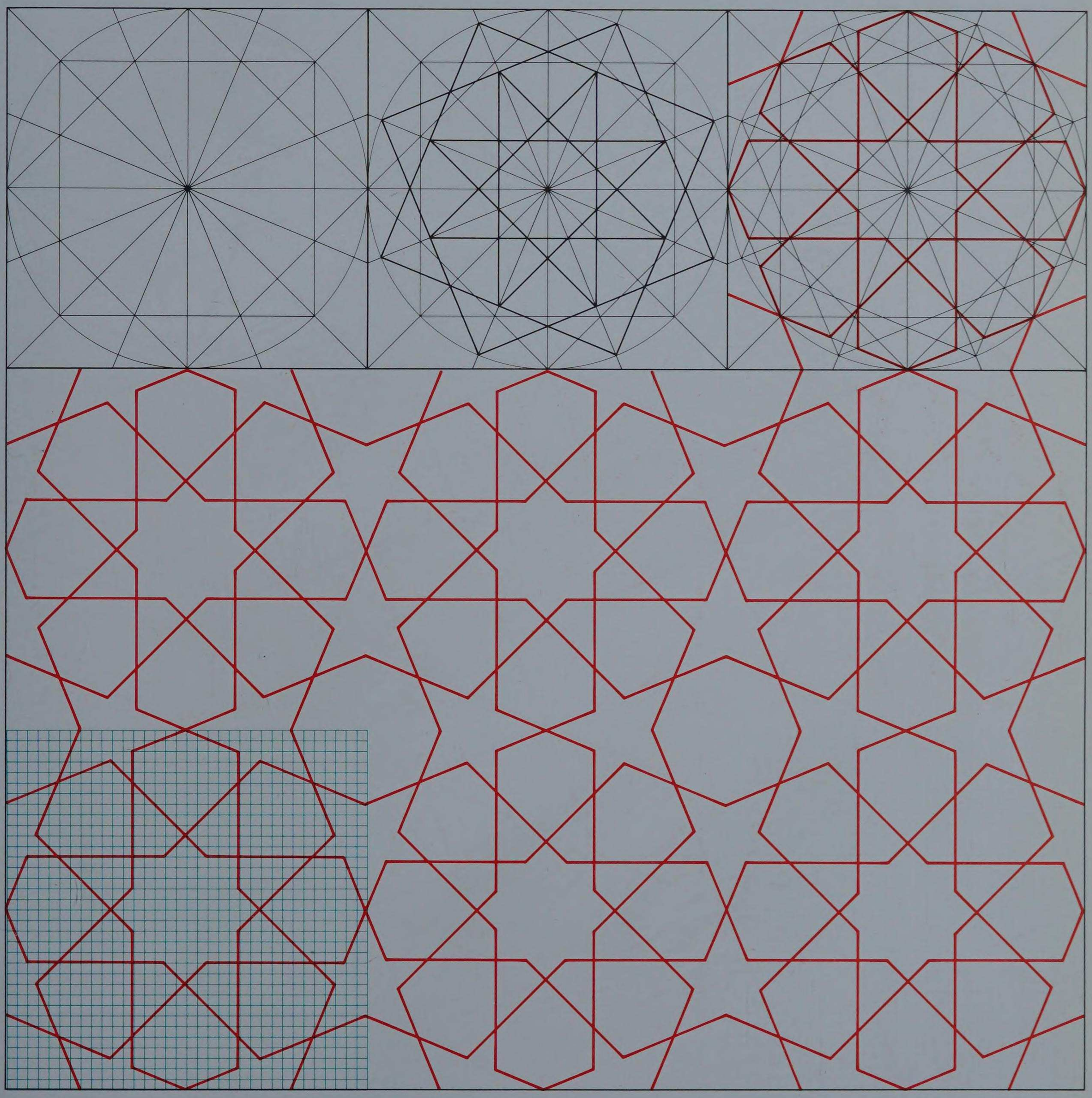


Figure 33. This and the following two examples are based on primary grid B. In this case the repeat unit consists of a four-pointed star derived from primary grid B by highlighting the sides on the vertical and horizontal axes. An octagon is inscribed in the circle by joining its eight points of division.

$A : B : A = 5 : 7 : 5$.

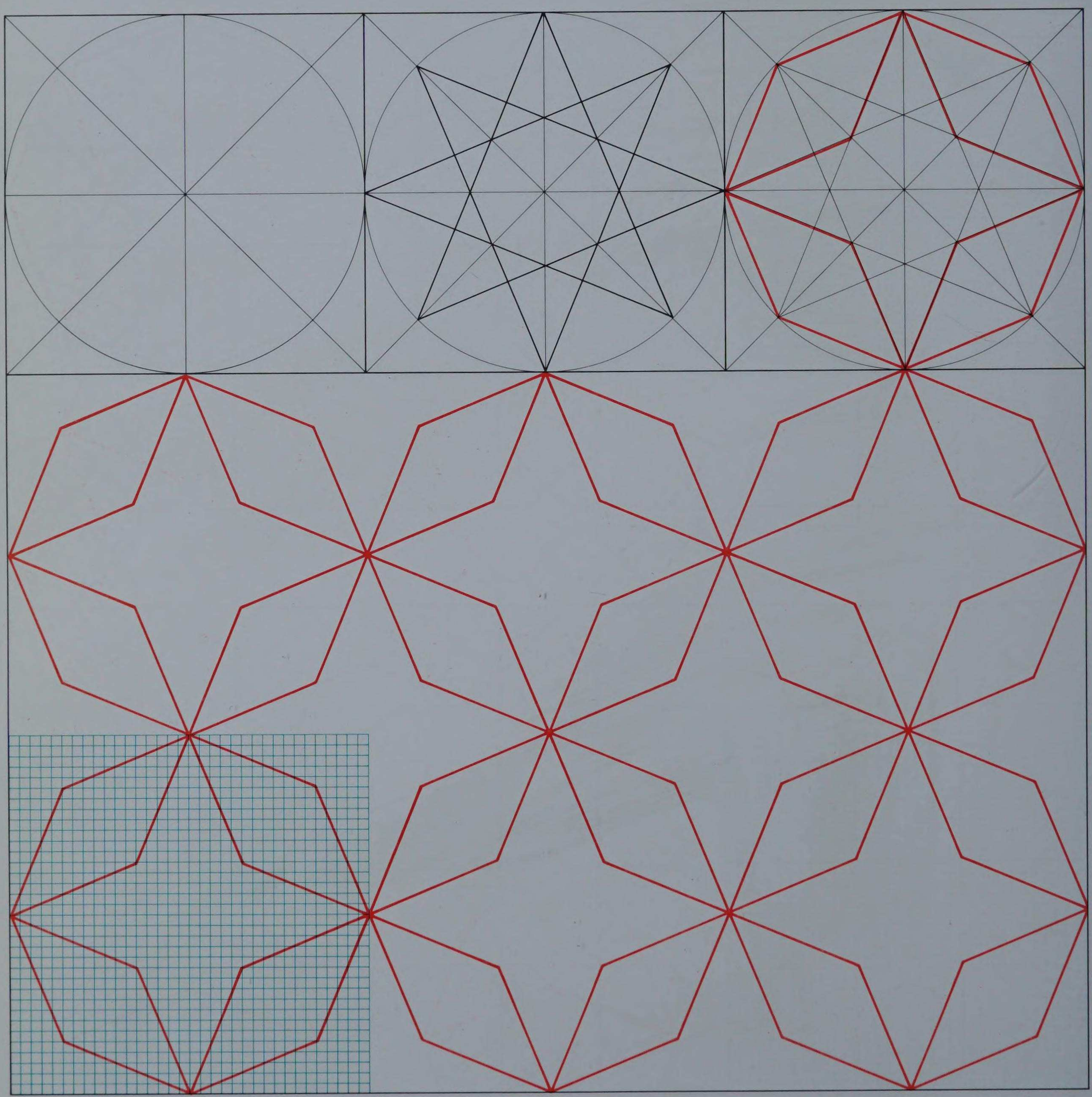


Figure 34. The elements of the repeat unit here are made different from the last example by highlighting the sides of primary grid B on the diagonal axes of the square. This results in a variation on the form.

$B : A : A = 7 : 5 : 5.$

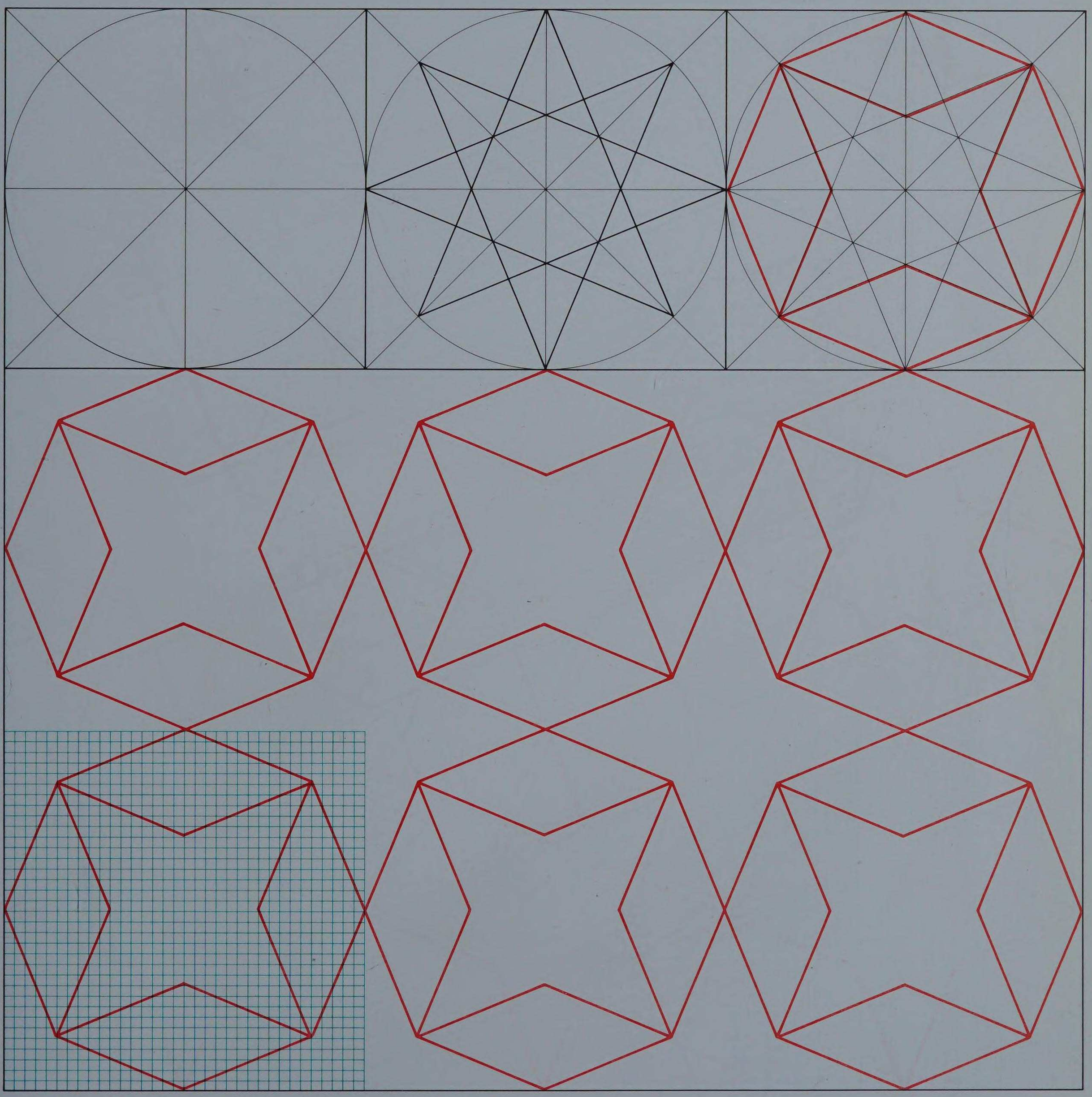


Figure 35. This composition uses the star of primary grid B as its main element. Where the star intersects with the diagonals of the square, its sides are extended to the perimeter of the square repeat unit.

$A : a : A : a : b = 5 : 2 : 5 : 2 : 3$.

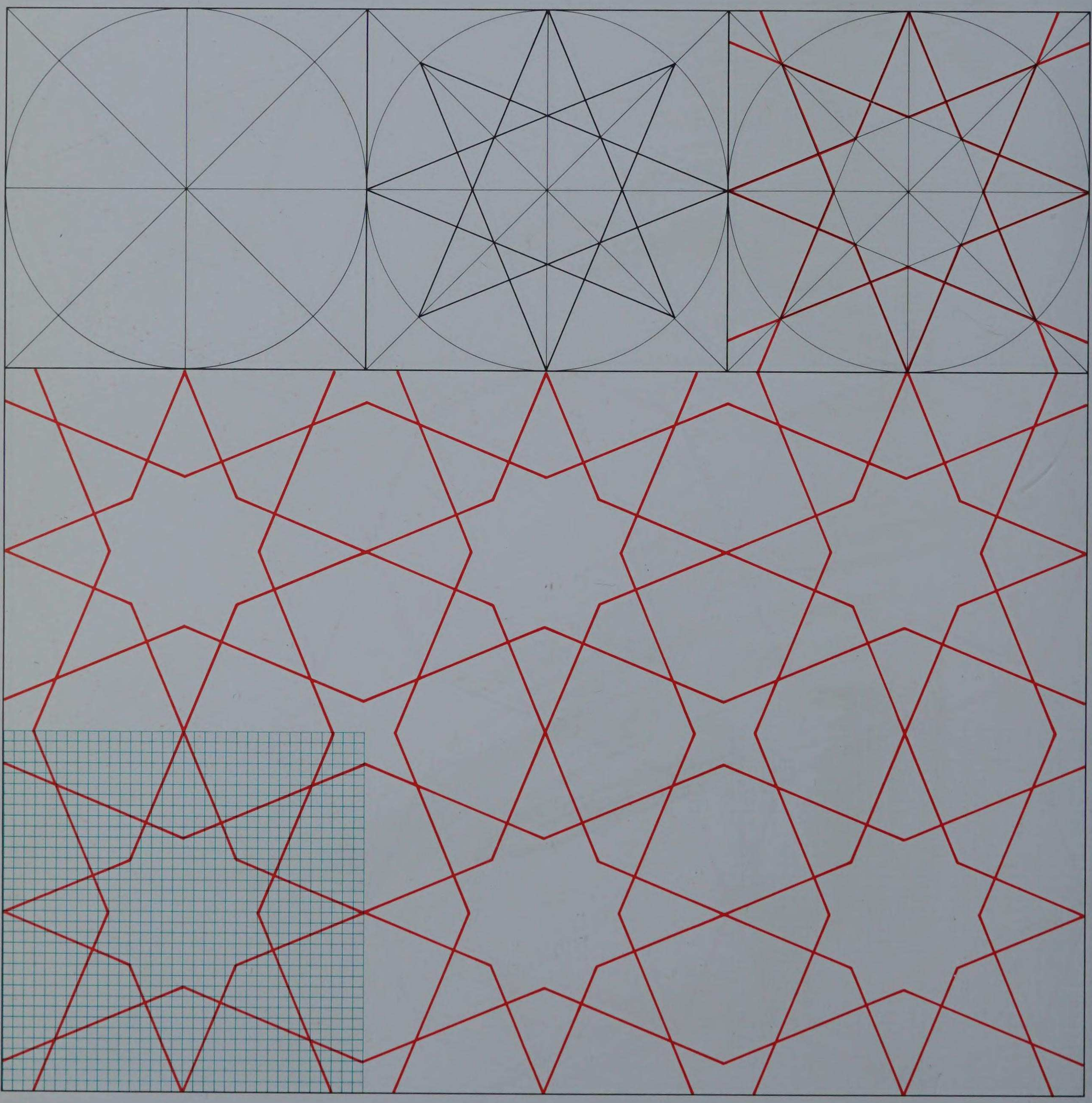


Figure 36. This and the following two examples are based on primary grid B, secondary grid B nesting within it, and an octagon inscribed in the circle. In this case the sides of secondary grid B along the diagonals of the square are extended to the perimeter of the square repeat unit.
$A : a : b : a : A = 5 : 2 : 3 : 2 : 5$.

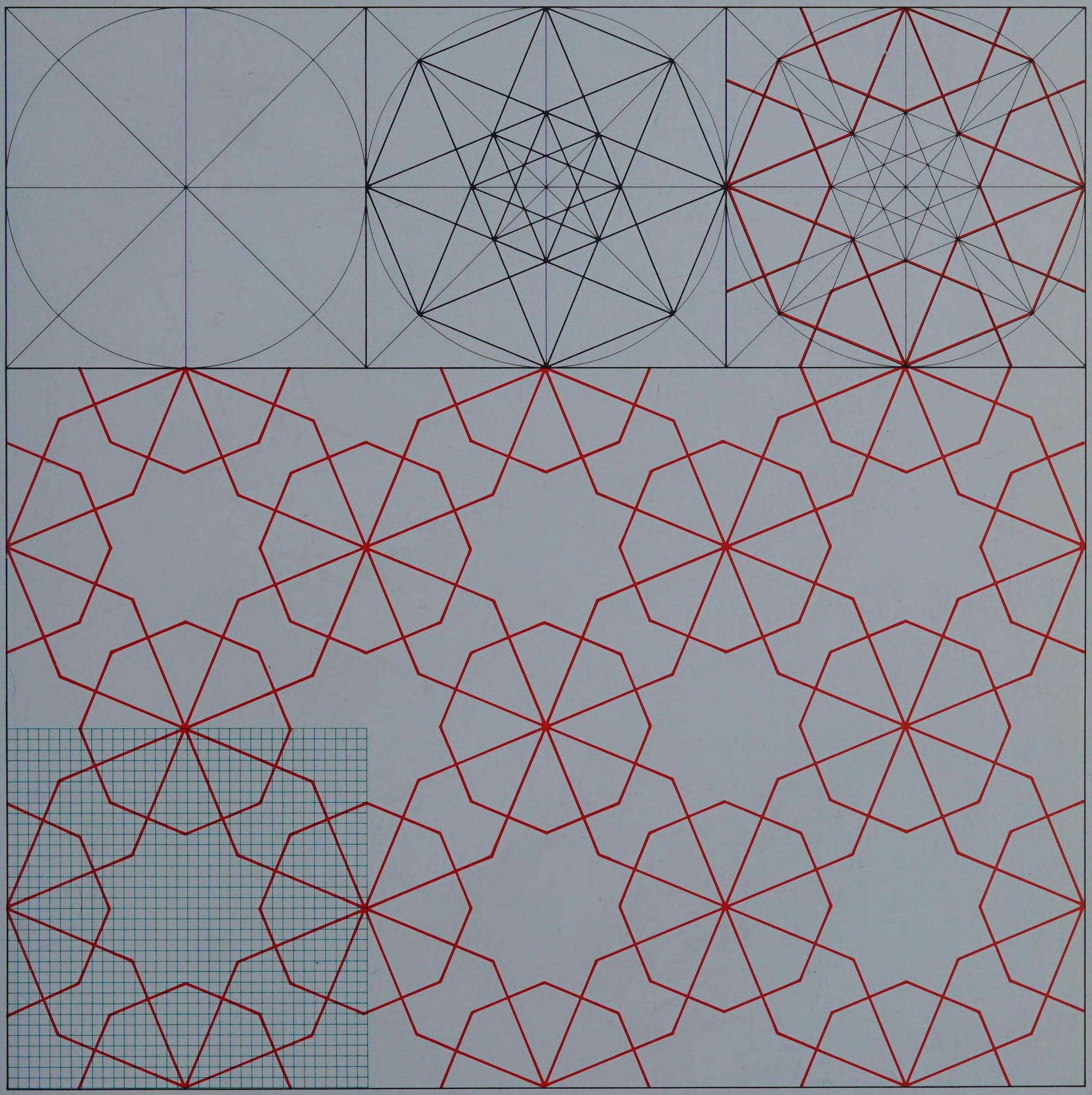


Figure 37. In this example all eight sides of secondary grid B are extended to intersect with the octagon inscribed in the circle.

$a : b : a : b : a : A = 2 : 3 : 2 : 3 : 2 : 5$.

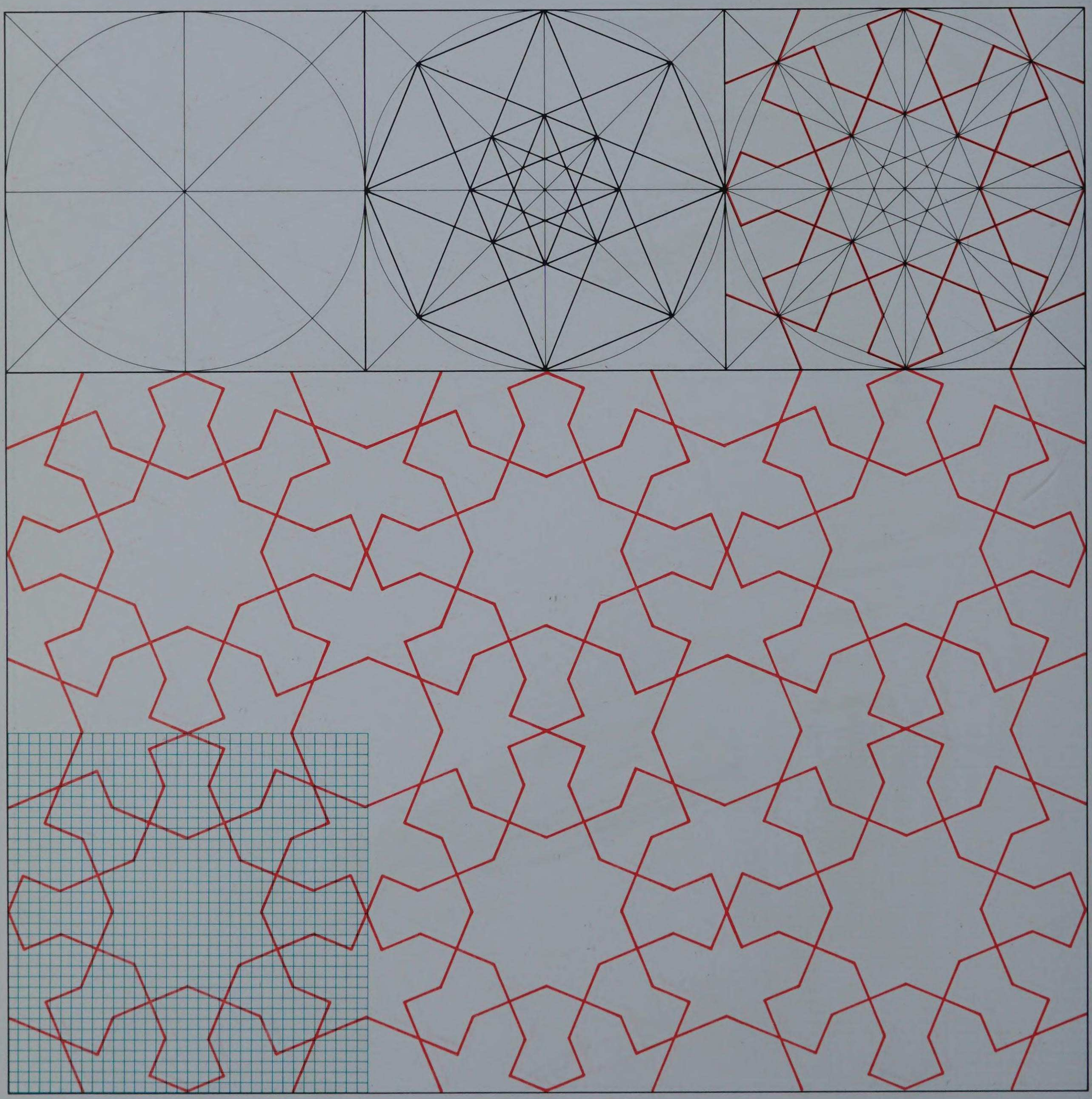


Figure 38. The grid-pattern for this repeat unit is the same as for the previous example, but its composition has as its central element the outline star form of secondary grid B.
$A : a : b : a : A = 5 : 2 : 3 : 2 : 5$.

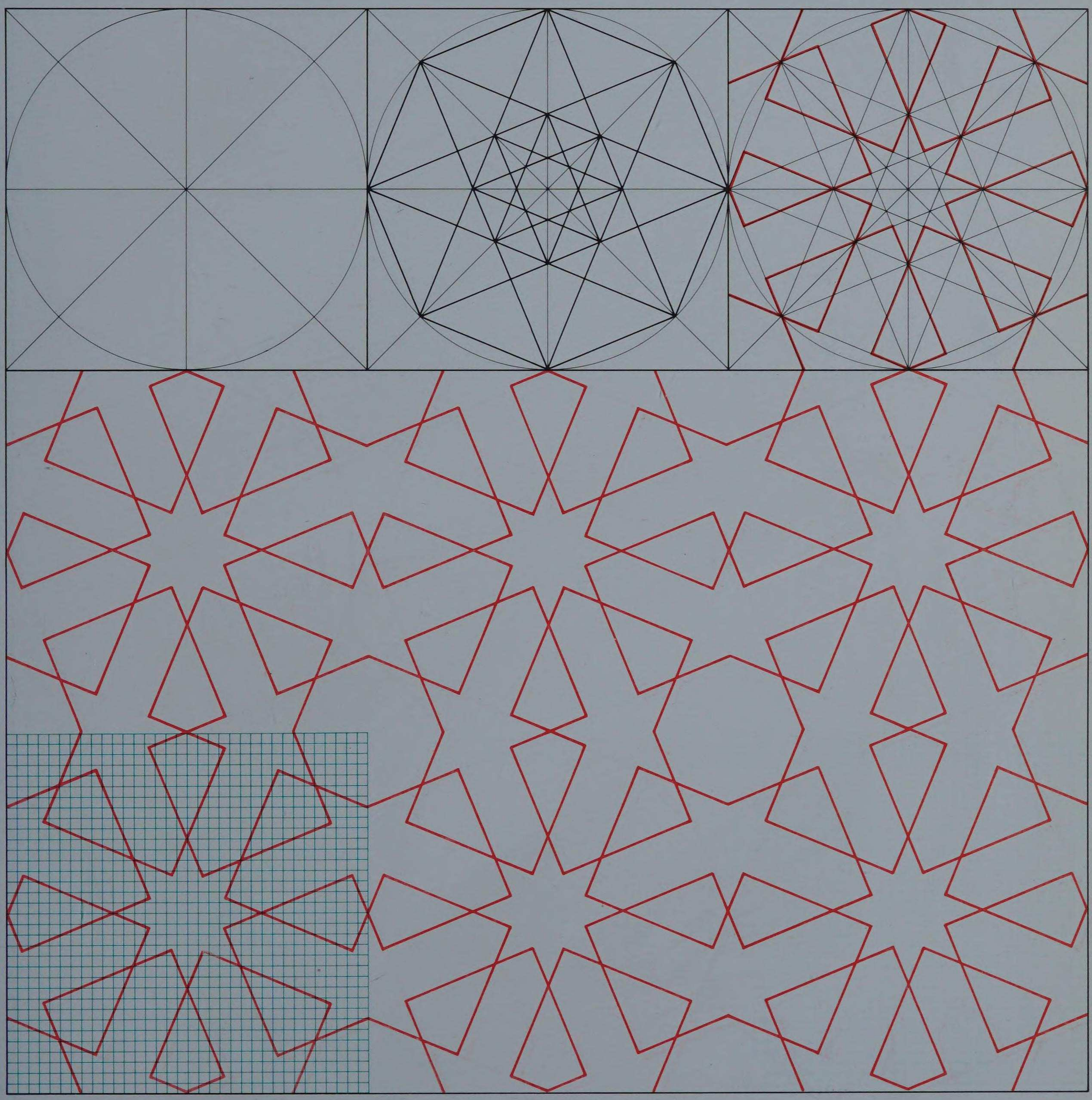


Figure 39. The grid-pattern and repeat unit is in this case an octagon inscribed in the circle, achieved by consecutively joining the eight points of division of the circumference.
$A + B : A = 12 : 5$.

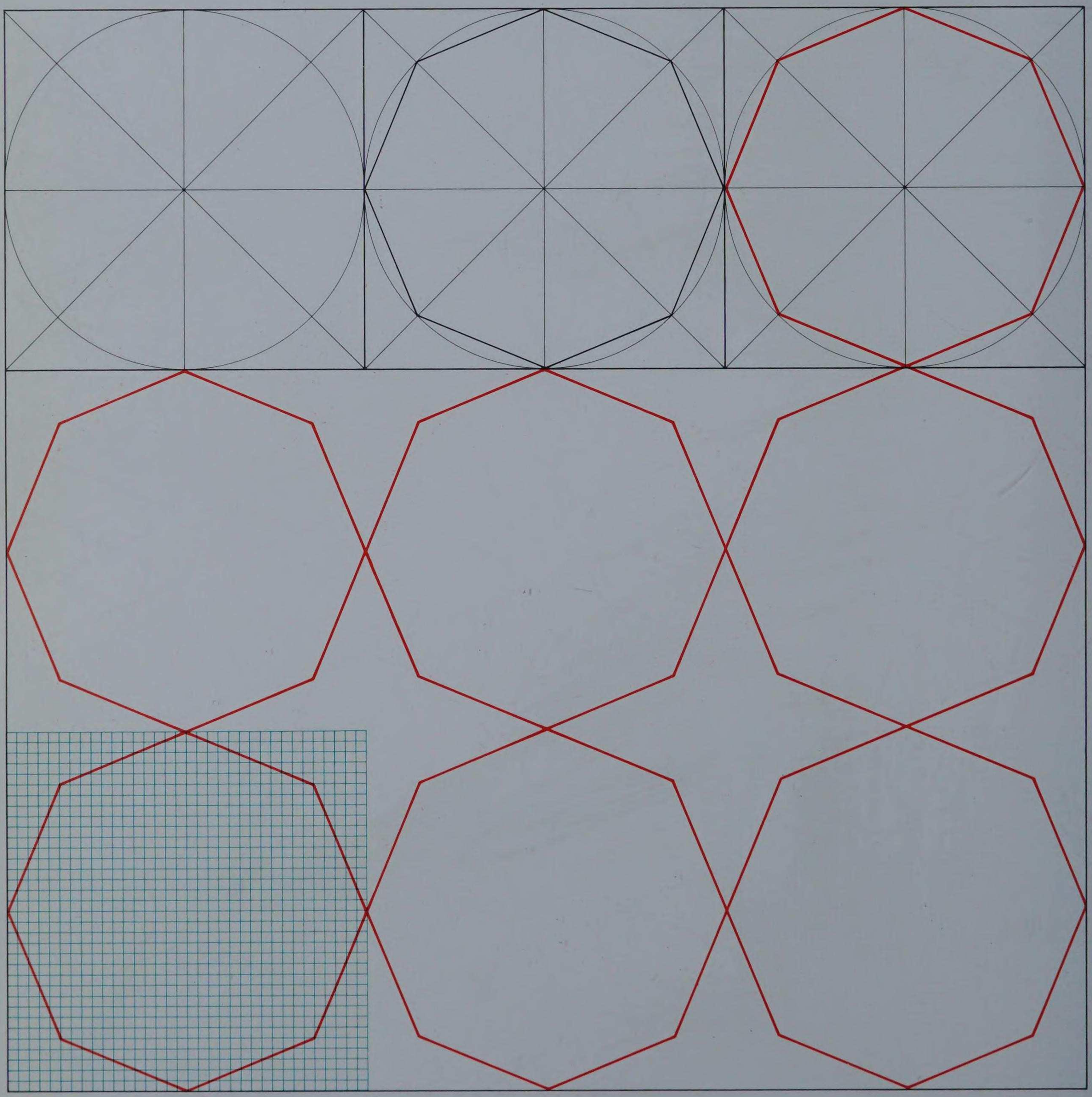


Figure 40. The circle is divided into 16 equal parts by joining opposite points of the intersecting sides of primary grid A through its centre. These lines are extended to the perimeter of the square repeat unit. At these points a regular octagon is inscribed in the square.
$B : 2A = 7 : 10$.

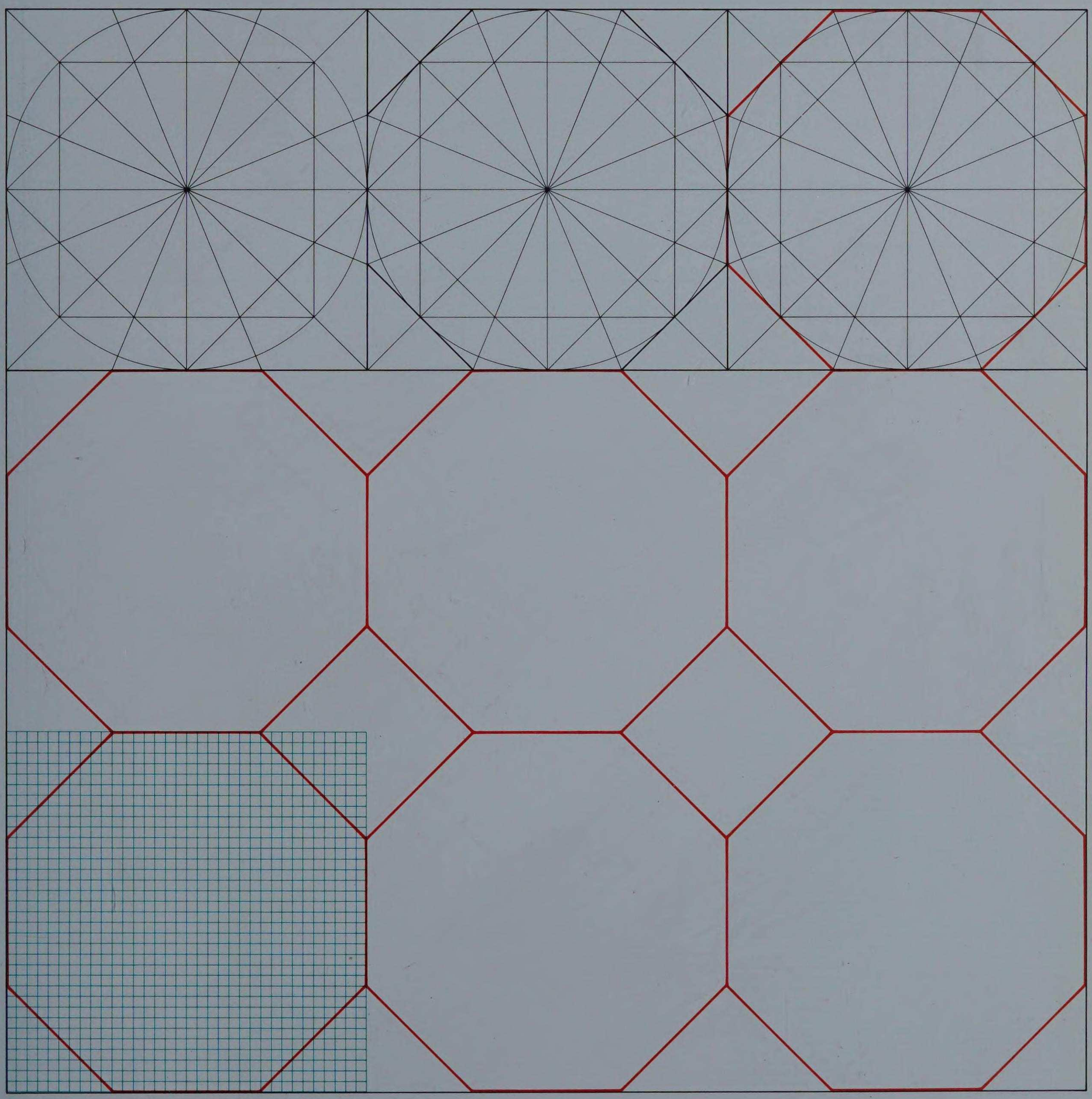


Figure 41. The side of the square repeat unit is divided into four equal parts by drawing lines through the intersection points of the diagonals of the square and a rotated square inscribed in the circle. Further point-joining achieves the complete grid-pattern.
$A + \frac{B}{2} : A + \frac{B}{2} = 8.5 : 8.5.$

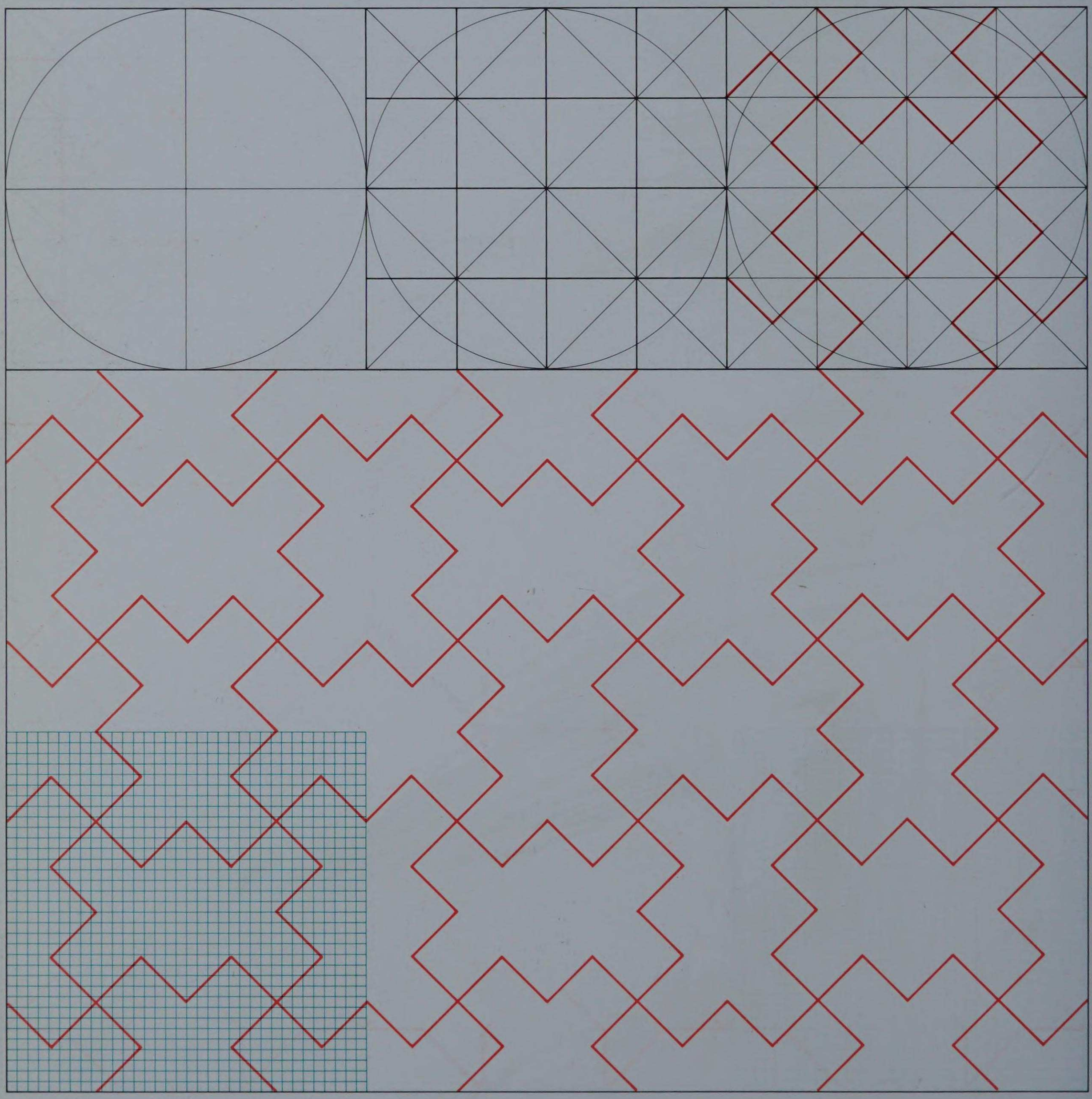


Figure 42. A rotated square is inscribed in the circle and four isosceles triangles are constructed in the square repeat unit with the side of the square repeat unit as base and the mid-point of its opposite side as apex. The intersection points of the triangles, the rotated square and the diagonals are projected onto the side of the square repeat unit, dividing it equally into six. Further point-joining gives the complete grid-pattern.

$2a : 2a : 2a = 6 : 6 : 6$.

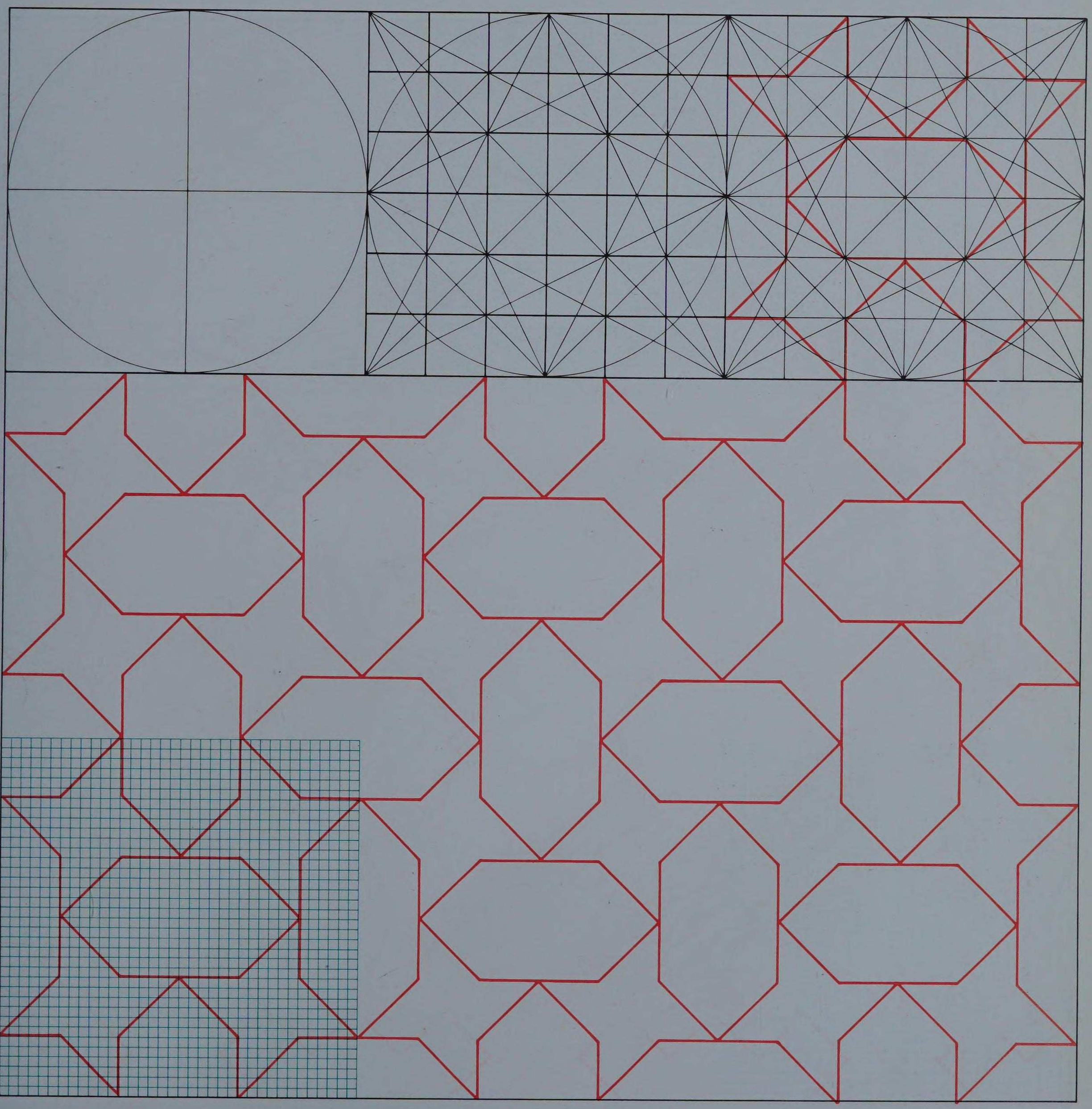


Figure 43. This and the following two repeat units are based on the division of the circle into 16 equal parts. In this example, the lines dividing the circle are extended to the perimeter of the square repeat unit from where every second point is joined to give a 'rotated' octagonal star inscribed in the square.

$B : b : 2a : b = 7 : 3 : 4 : 3$.

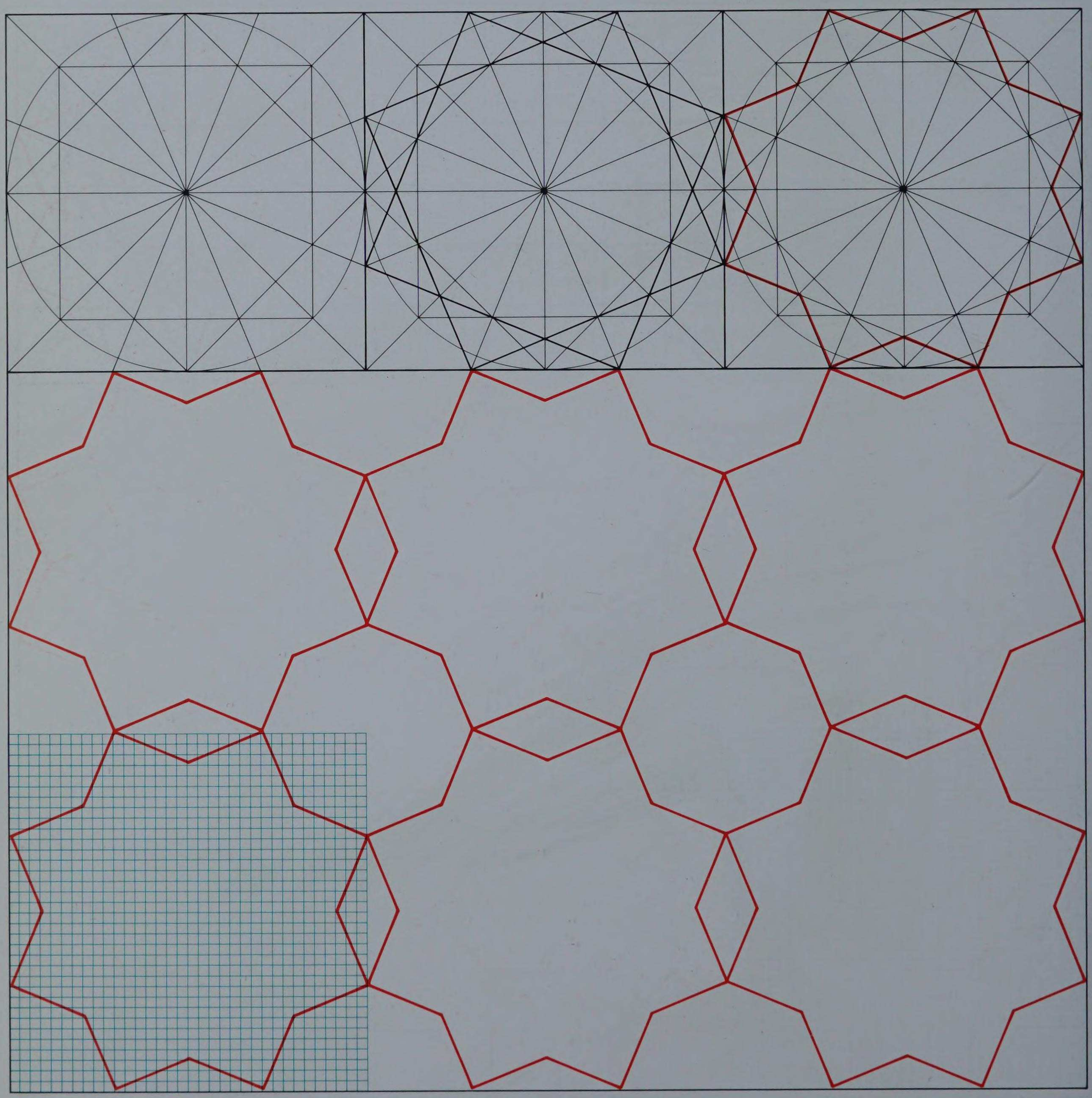


Figure 44. As a development from the last example, a secondary grid-pattern is drawn nesting within the 'rotated' octagonal star to give the composition of the repeat unit.
$2a : b : b : 2a : b = 4 : 3 : 3 : 4 : 3$.

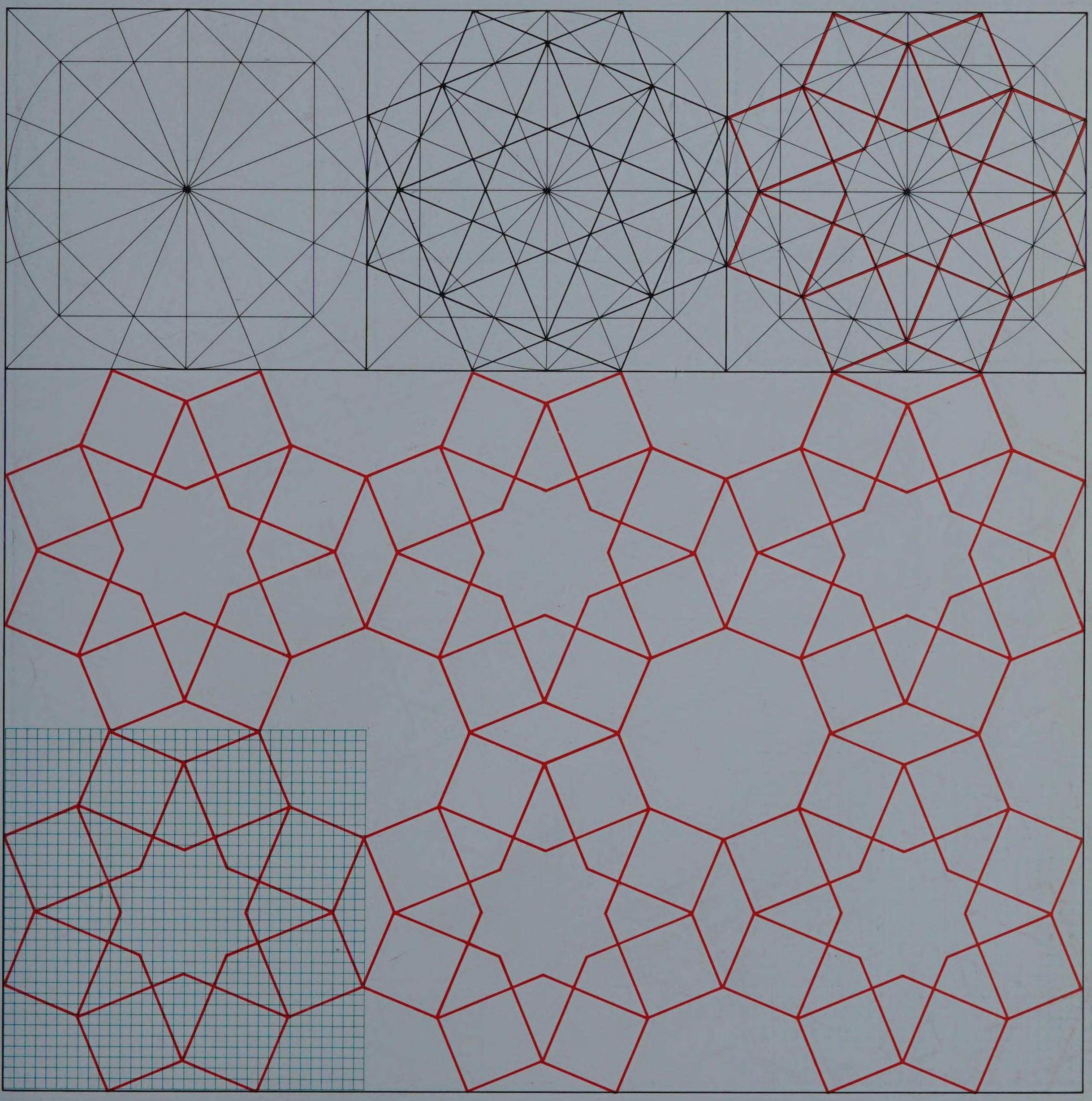


Figure 45. As in the last example, the grid-pattern is generated from a 'rotated' octagonal star, which in this case has a secondary grid-pattern superimposed on it. Selected points of intersection are joined to give the composition.

$b : 2a : b : b : 2a = 3 : 4 : 3 : 3 : 4$.

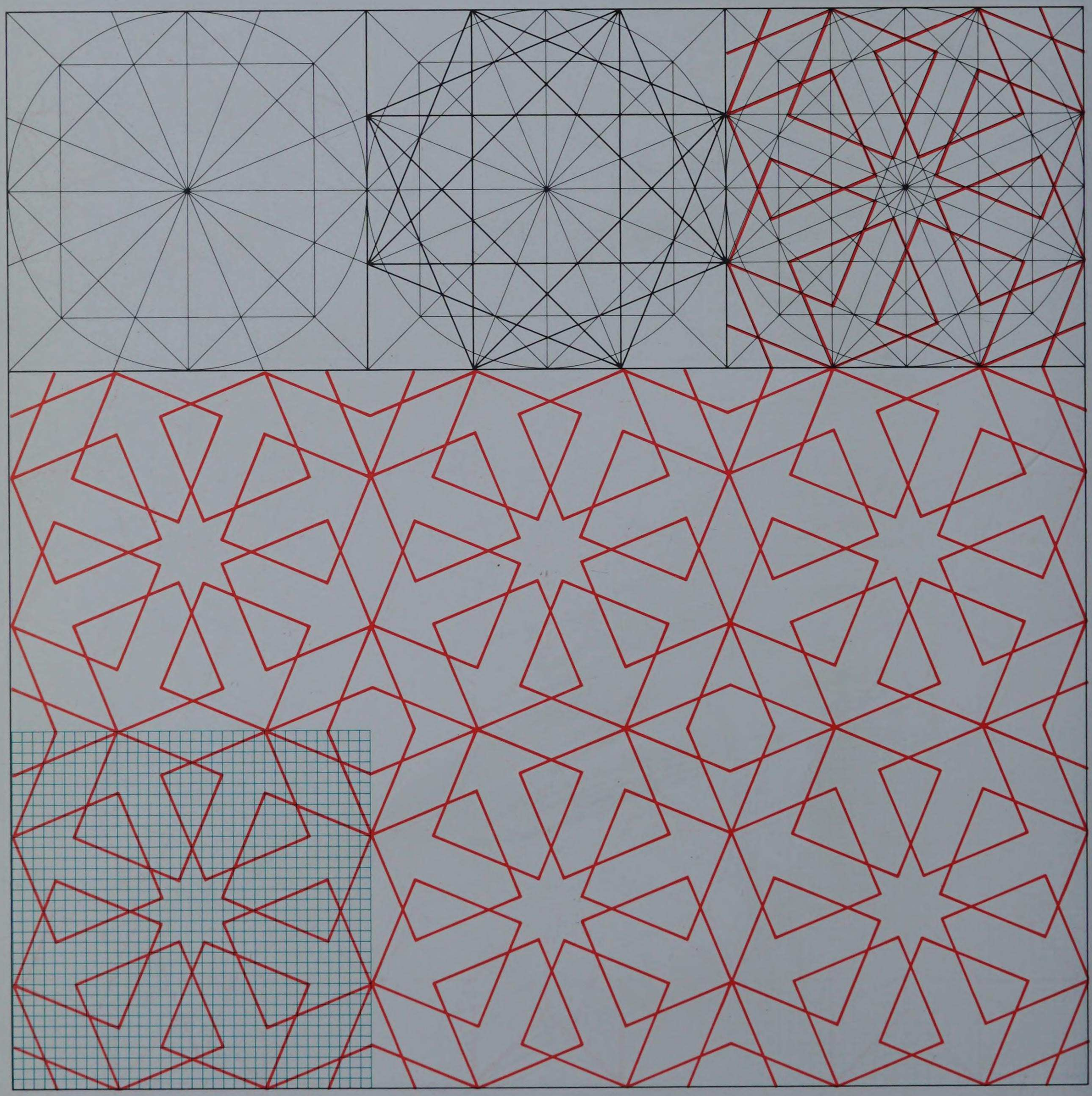
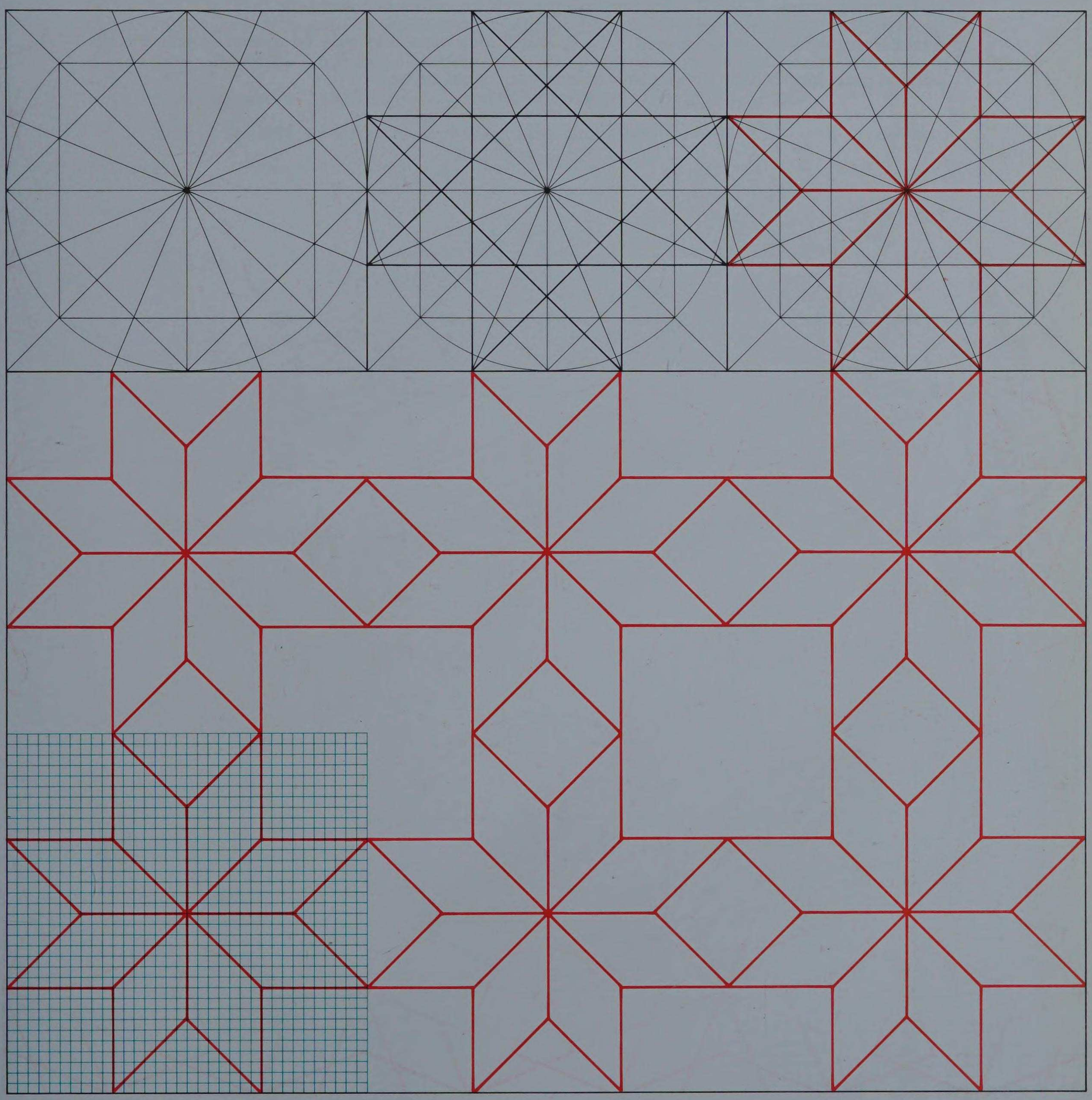


Figure 46. The lines dividing the circle into 16 equal parts are extended to the perimeter of the square repeat unit. A stellated octagon as grid-pattern, inscribed in the square, forms the composition and is drawn by joining every third point along the perimeter of the repeat unit.
$B : b : B = 7 : 3 : 7$.

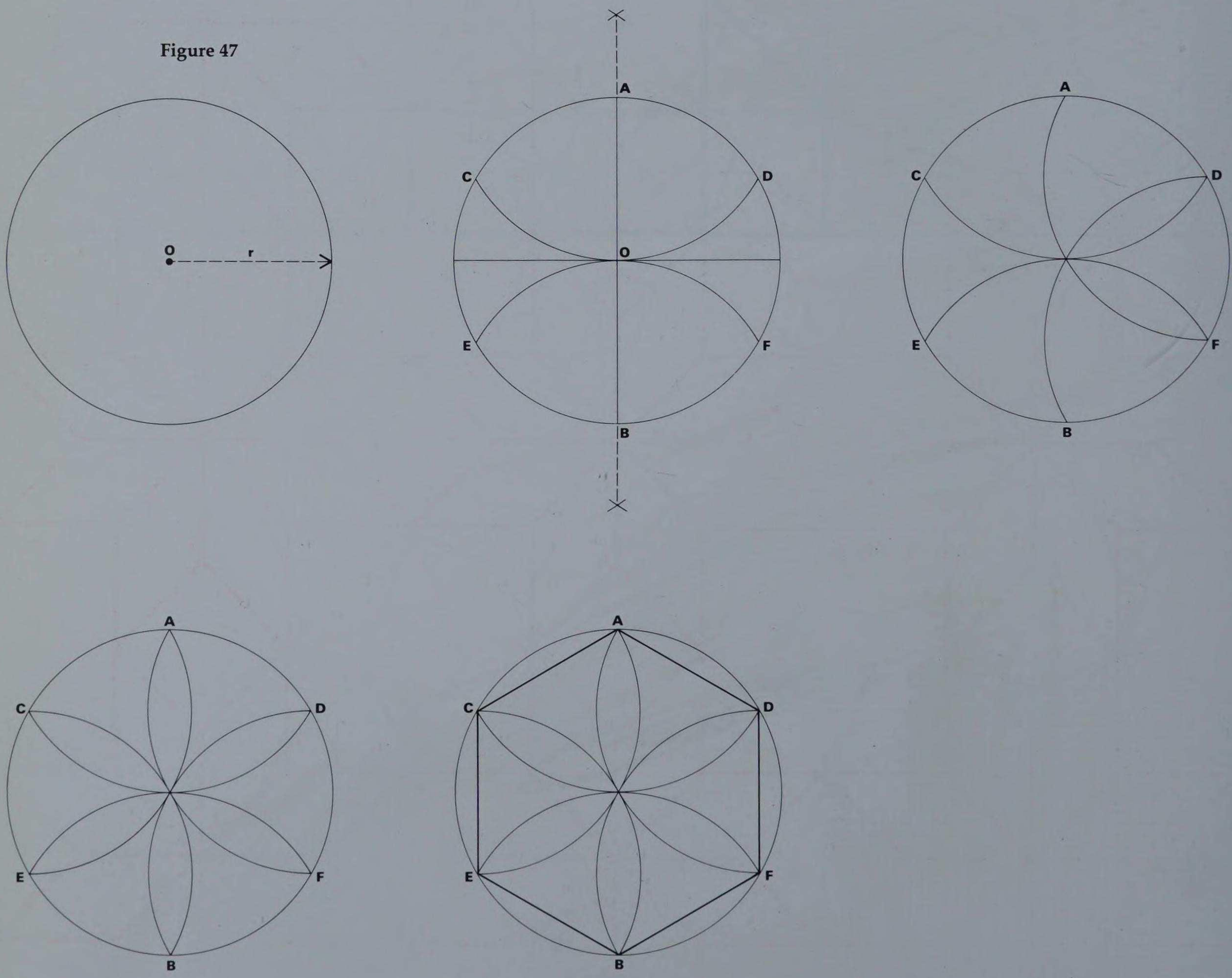


Geometric Patterns based on the Hexagonal Repeat Unit and the Root Three System of Proportion

The construction of the hexagonal repeat unit and the division of the circle into three, or multiples of three, equal parts

This is illustrated in **Figure 47**. Describe a circle with centre *O* and radius *r*. Draw the horizontal diameter and from it construct the vertical diameter cutting the circle at points *A* and *B*. With radius *r* and centres *A* and *B* draw two arcs through the centre *O* cutting the circle at C and *D* and *E* and *F* respectively. With centres *C*, *D*, *E*, and *F* four more arcs may be drawn to give the six-pointed petal form which divides the circle into six equal parts.

Figure 47



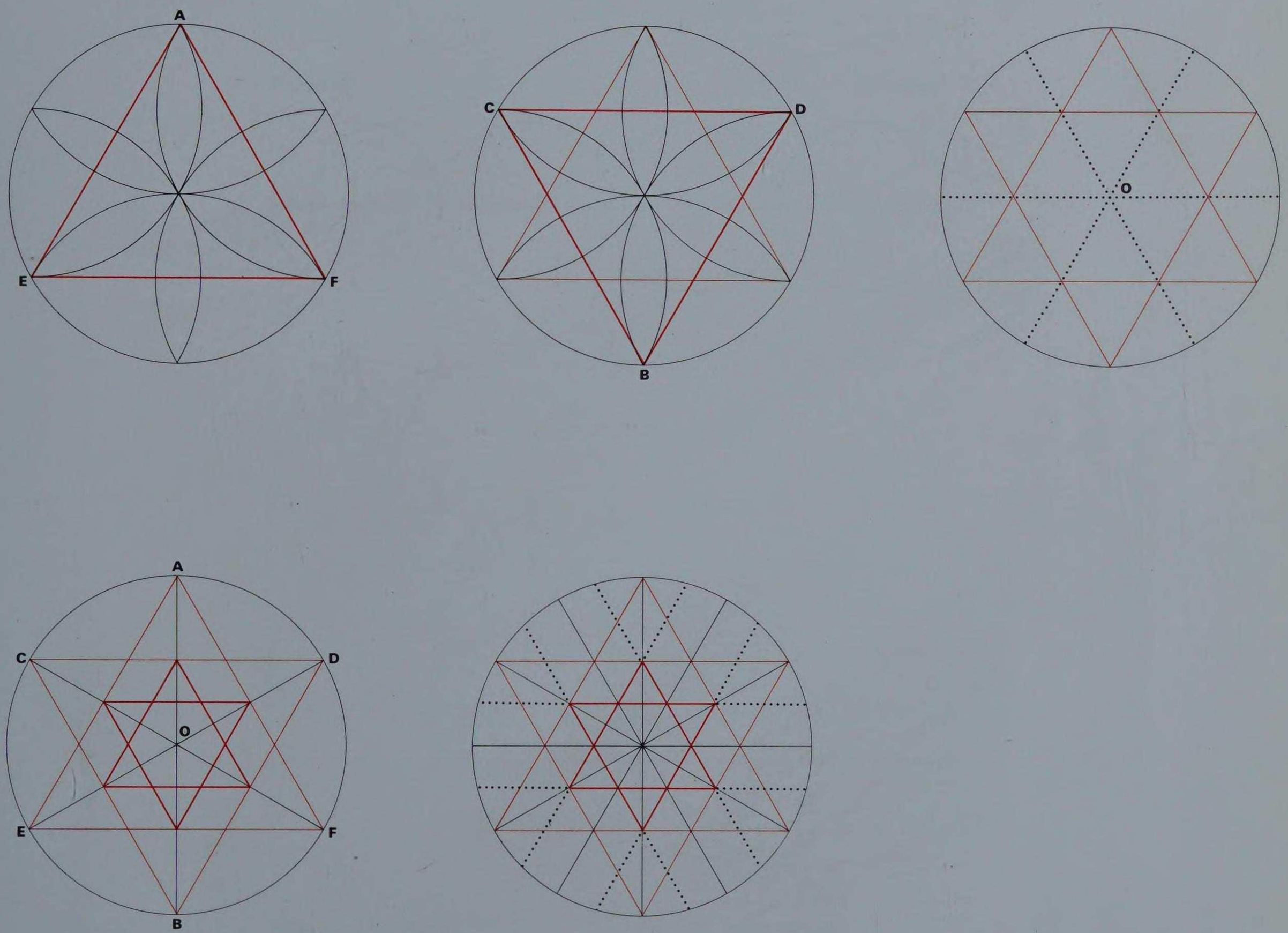
Join with straight lines points *ADFBEC* to define the hexagon (the 'hexagonal repeat unit') with side equal to radius *r*. Join every second point of division of the circumference with straight lines (**Figure 48**) starting from *A* to give an equilateral triangle *AFE* which divides the circumference into three equal parts.

Join the other three points of division *CDB* with straight lines to give a six-pointed star composed of two equilateral triangles. Join opposite intersection points of the sides of the six-pointed star through the centre *O* with straight lines and extend the lines to cut the circumference giving 12 equal divisions of the circle.

Introduce a secondary six-pointed star nesting within the first by joining every second mid-point of the sides of the large six-pointed star. The mid-points are established by drawing straight lines through the centre *O* joining *AB, DE,* and *FC* which equally divide all the sides.

Extend the sides of the secondary star to the circumference to give 24 equal divisions of the circle.

Figure 48



Determining the size of the hexagonal repeat unit

This is illustrated in **Figures 49i, 49ii,** and **49iii.** The basic procedure for determining the size of the hexagonal repeat unit is similar to that described for the square repeat unit. Likewise, with the use of a string or compasses and the folding technique, side x of a given area is subdivided into the number of parts equivalent to the number of repeat units to be used on the given area. By representing the required number of repeat units as circles along side x, the whole area is then further divided systematically using the same circle, and the adjacent inscribed hexagons are drawn by the point-joining method. The hexagonal repeat units, the sides of which equal the radius of the circumscribing circle, will completely cover the given area.

Figure 49i

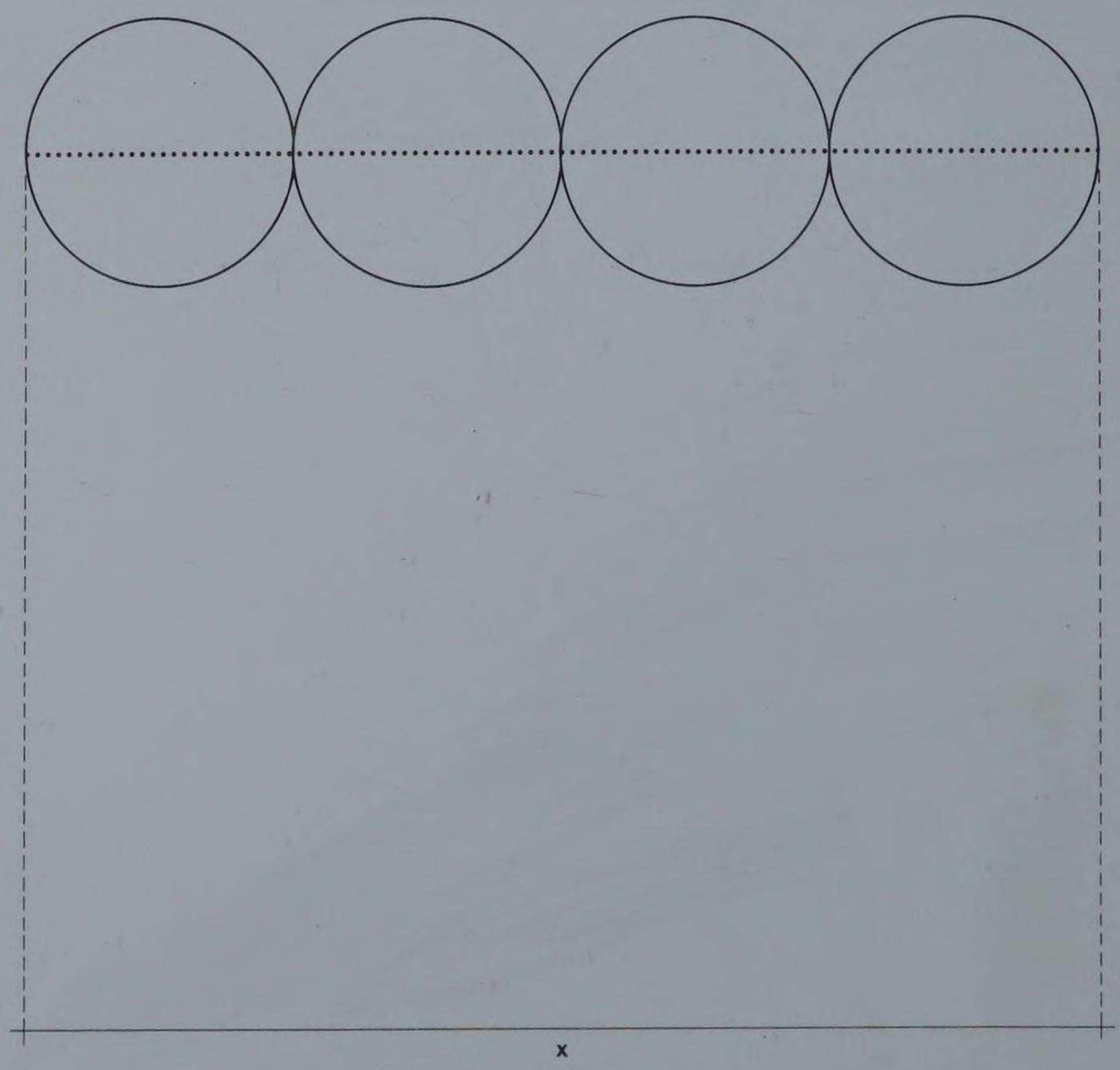


Figure 49ii

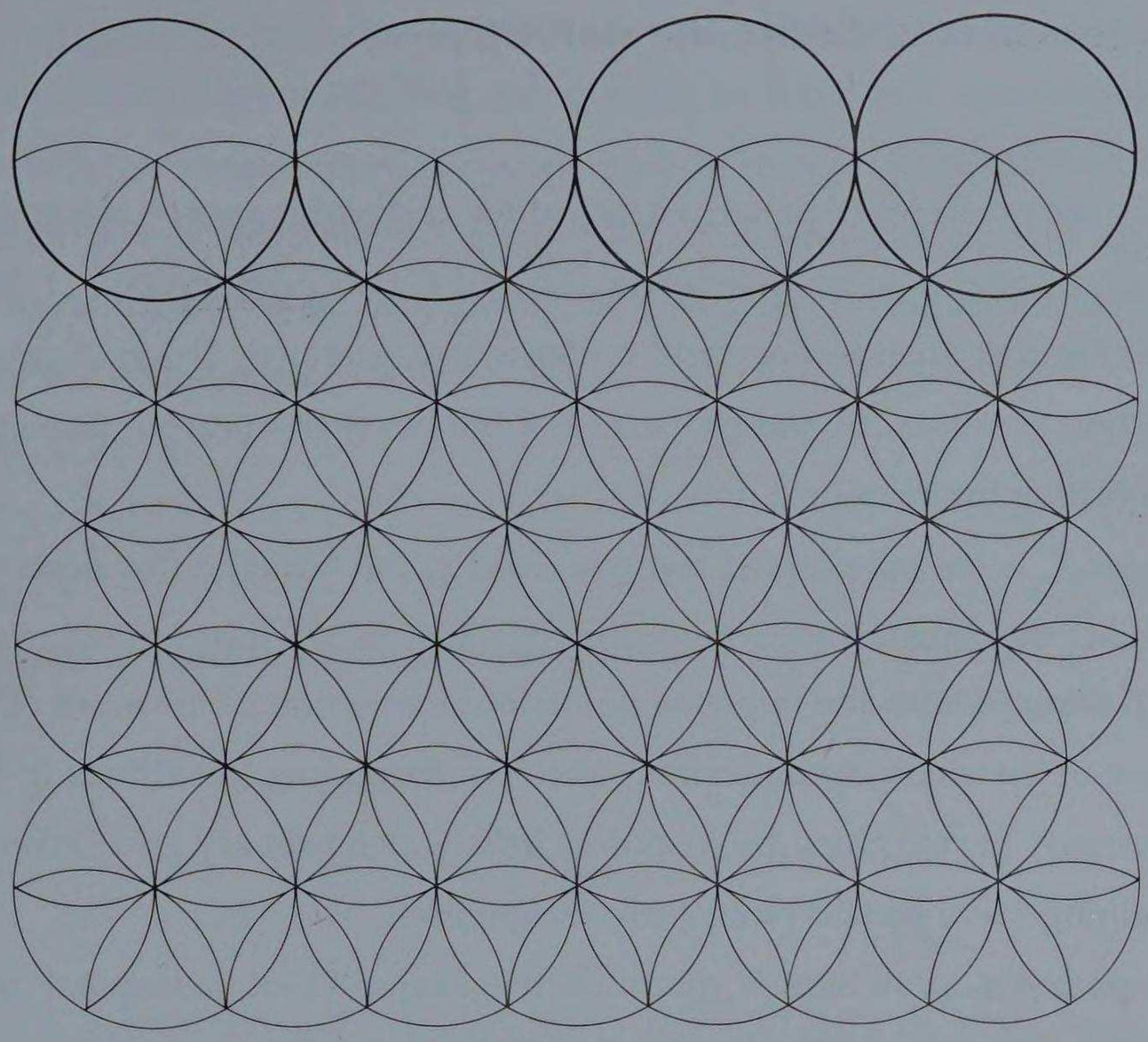
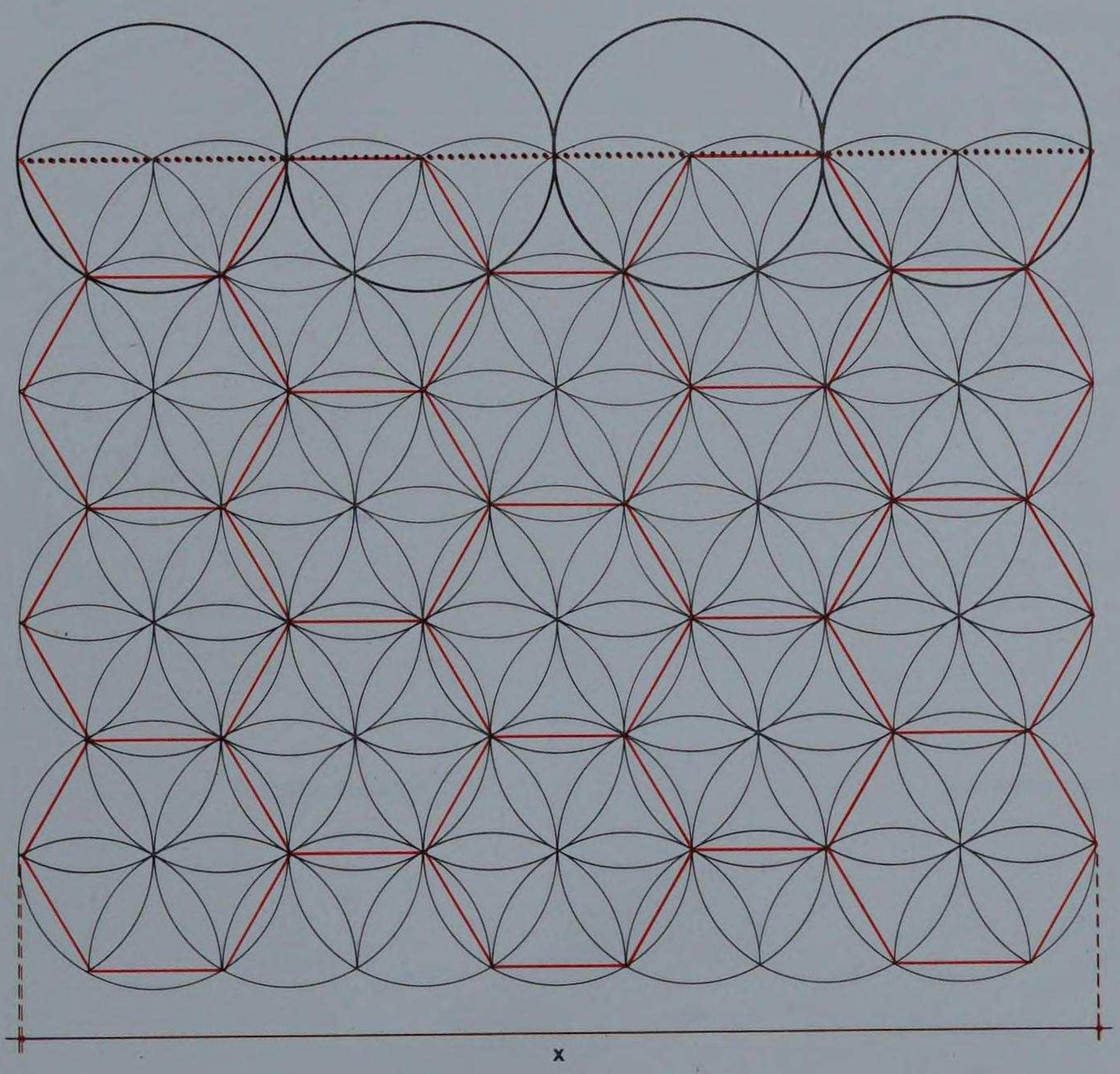


Figure 49iii



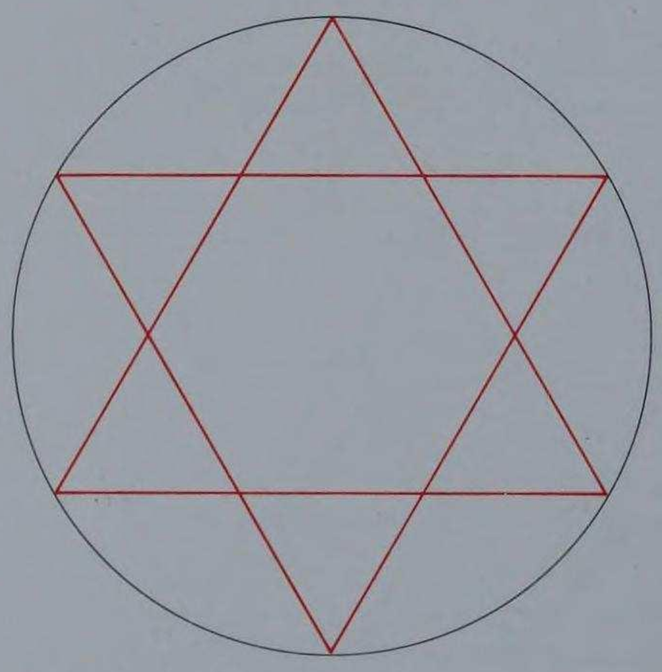
The grid-patterns that underlie the root three system of proportion and their measure

The repeat units in the examples of patterns given at the end of this section are in most cases based on two types of grid-patterns to be called primary-grid C and primary-grid D. From these grid-patterns, secondary and at times tertiary grid-patterns are generated. The grid-patterns establish a system of proportion governed by the properties of the hexagon and root three thus ensuring a harmony of composition for the repeat units.

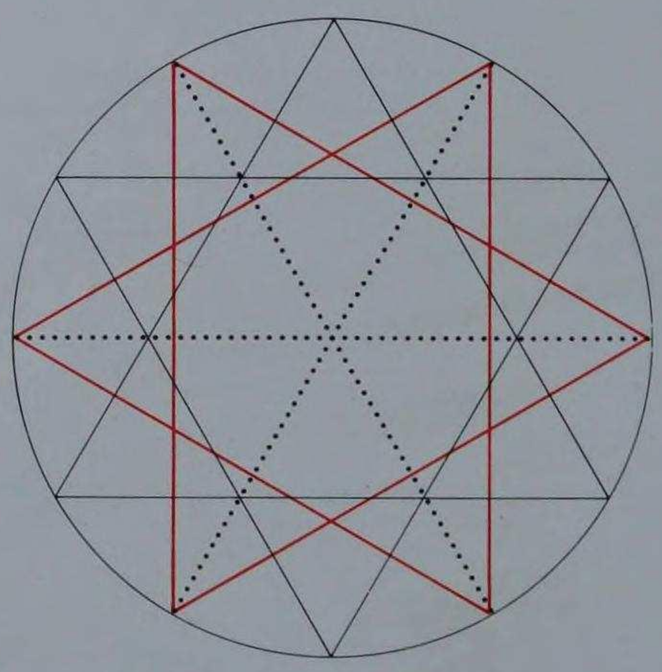
Primary grid C is the six-pointed star inscribed in the hexagon (for description see p.57). Primary grid D is another form of six-pointed star (see **Figure 50**). It is derived from primary grid C by joining with straight lines the opposite points of intersection of the sides of primary grid C through the centre of the circle circumscribing the star. Extending these lines to cut the hexagon circumscribing primary grid C gives the six points from which primary grid D can be constructed. This is done by joining every second point with straight lines.

The grid-patterns upon which many of the examples of hexagonal repeat units are constructed are themselves generated by manipulating the equilateral triangle on which the six-pointed star (primary grid C) is based. The procedure results in dividing the whole area of the hexagon into equilateral triangles.

Figure 50



PRIMARY GRID C



PRIMARY GRID D

When the diameters of the hexagon are drawn in, the 30°, 60°, 90° triangle becomes the basic unit of composition (**Figure 51**). This triangle will be taken as a unit of measure or module, and to ensure its accurate construction, the proportions of its sides have to be secured accordingly. It is evident that the 30°, 60°, 90° triangle equals $\frac{1}{2}$ an equilateral triangle, and since the equilateral triangle has equal sides then the sides of the 30°, 60°, 90° triangle (**Figure 52**) are in the proportion of 1:√3:2 (Theorem of Pythagoras), whereby:

1 = short adjacent side = a

√3 = long adjacent side = c

2 = hypotenuse = $2a$.

Hence the angles can be defined as the following: $\tan 30° = \frac{1}{\sqrt{3}} = \frac{a}{c}$; $\tan 60° = \frac{\sqrt{3}}{1} = \frac{c}{a}$.

Figure 51

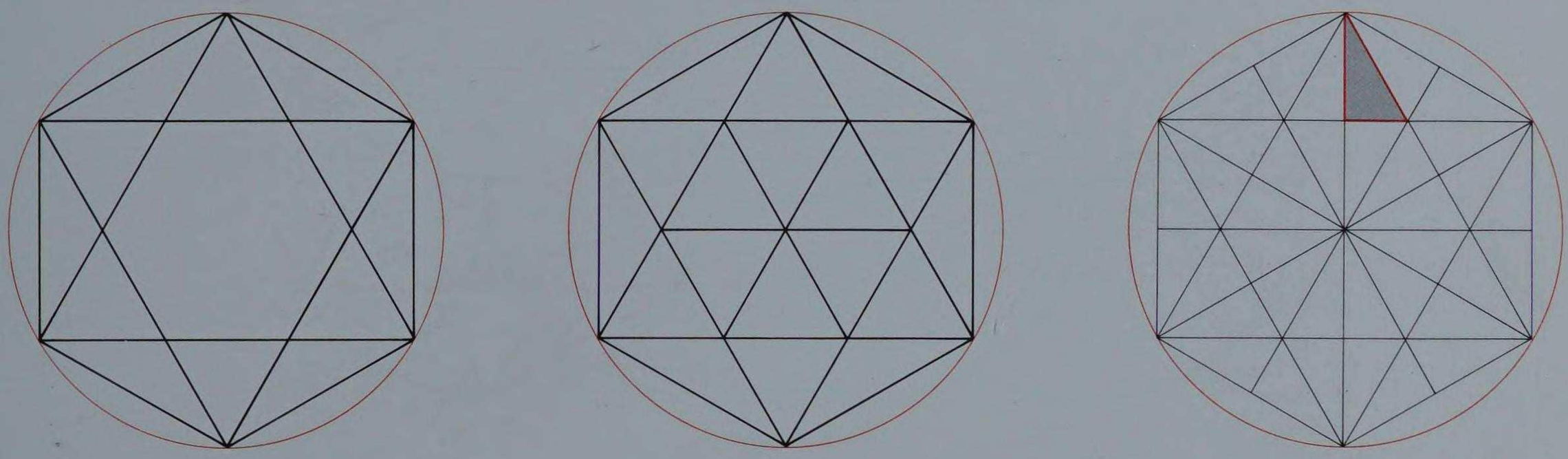
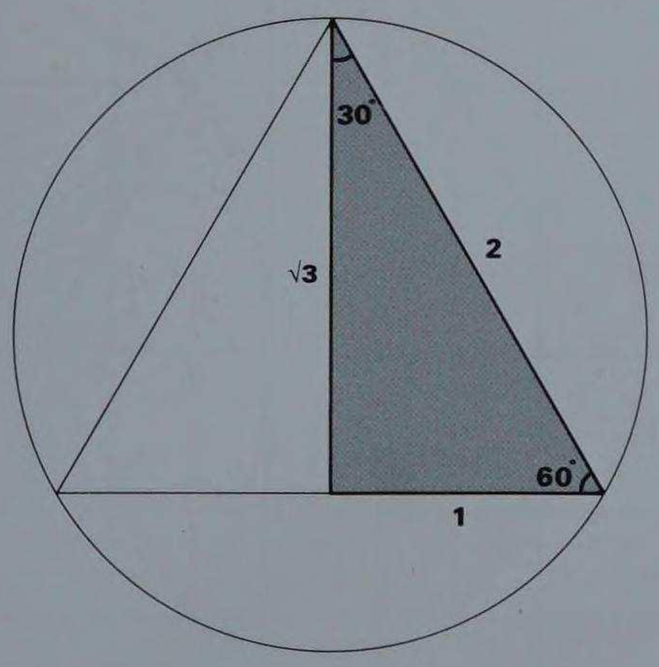


Figure 52



Examining the sequence for constructing the square repeat unit (see **Figure 1**) reveals that at the third stage of the sequence the circumference of the circle is divided into 12 equal parts (**Figure 53**). This makes it possible to reconstruct within it the hexagonal repeat unit with primary grid C and thus provide a means to express its relative dimensions in terms of $a = 2$, $b = 3$, $A = 5$ and $B = 7$ units (**Figure 54**). For example, the sides of all the 30°, 60°, 90° triangles contained between the vertical diameter of the repeat unit and one side of the six-pointed star (shown in red) (**Figure 55**) can be measured in terms of these units. In Table V, the dimensions of the short adjacent side, long adjacent side and the hypotenuse, which are in the proportion of $1 : \sqrt{3} : 2$ respectively, are presented in terms of $a = 2$, $b = 3$, $A = 5$ and $B = 7$ units.

Figure 53

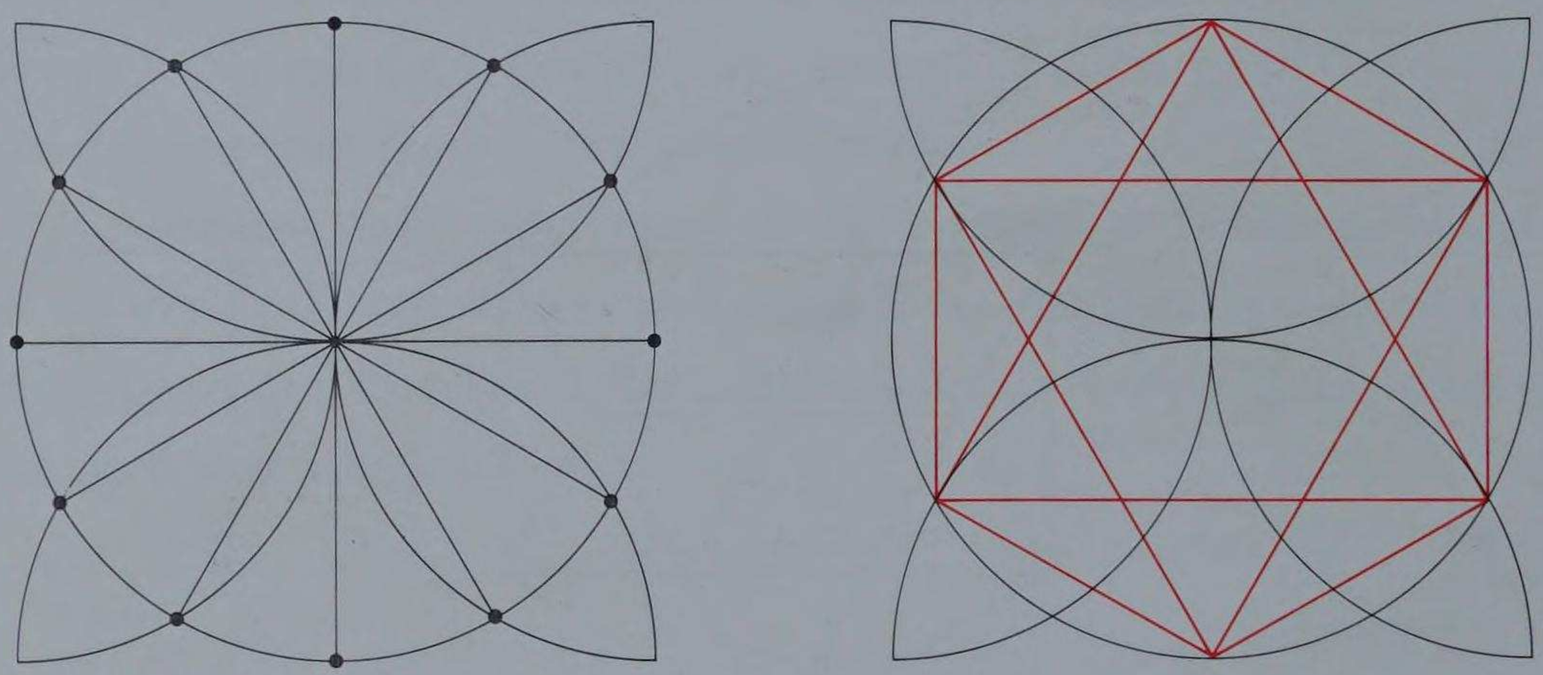


Figure 54

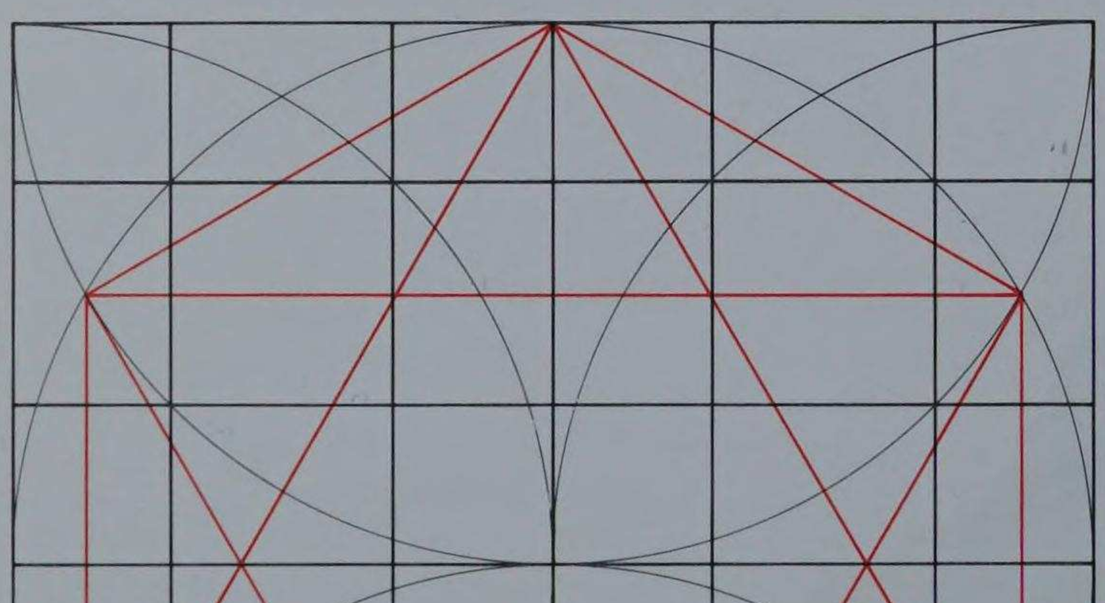


Figure 55

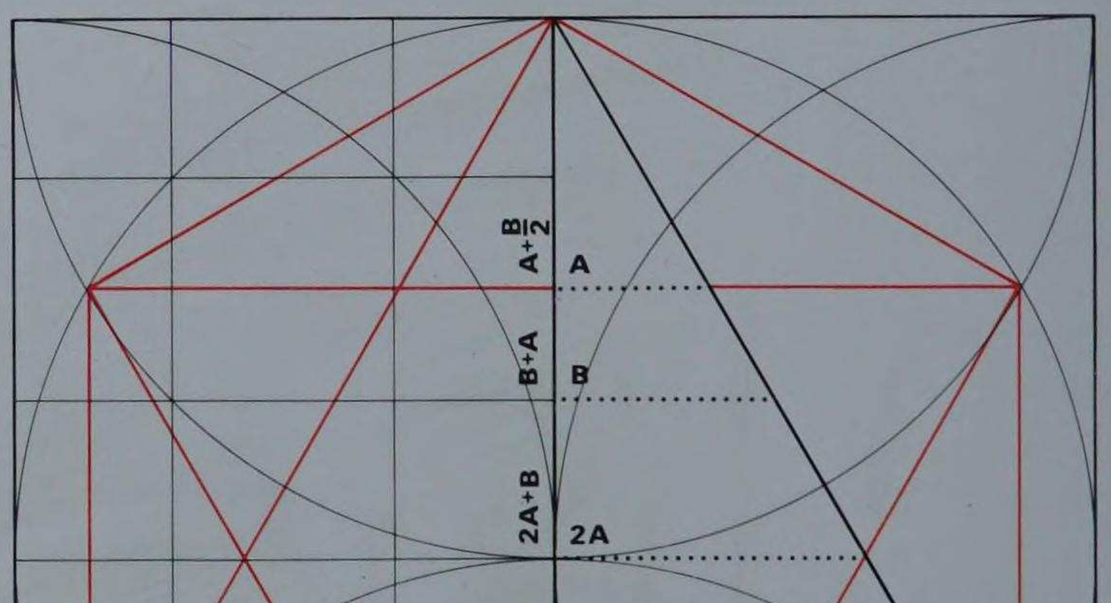


TABLE V

Short adjacent side	Long adjacent side	Hypotenuse
1	$\sqrt{3}$	2
a	c	2a
a	a+b/2	2a
2	2+1.5 = 3.5	4
b	b+2a/2	2b
3	3+2 = 5	6
A	A+B/2	2A
5	5+3.5 = 8.5	10
B	B+2A/2	2B
7	7+5 = 12	14
A+B/2	$(A+B/2)+\frac{B+A}{2}$	2A+B
8.5	8.5+6 = 14.5	17
A+B=6a	6a+6b/2	2(A+B)
12	12+9 = 21	24
2B	2B+4A/2	4B
14	14+10 = 24	28
3A	3A+3B/2	6A
15	15+21/2 = 25.5	30
2A+B	$(2A+B)+\frac{2A+2B}{2}$	2(2A+B)
17	17+12 = 29	34

From these results it can be resolved that $\sqrt{3} = c$ can be expressed approximately as $1+\frac{\sqrt{2}}{2}$. This can be verified mathematically by comparing the angles formed by the tangents of these values $\frac{\sqrt{3}}{1}=\frac{c}{a}$=1.732 = tan 60°;

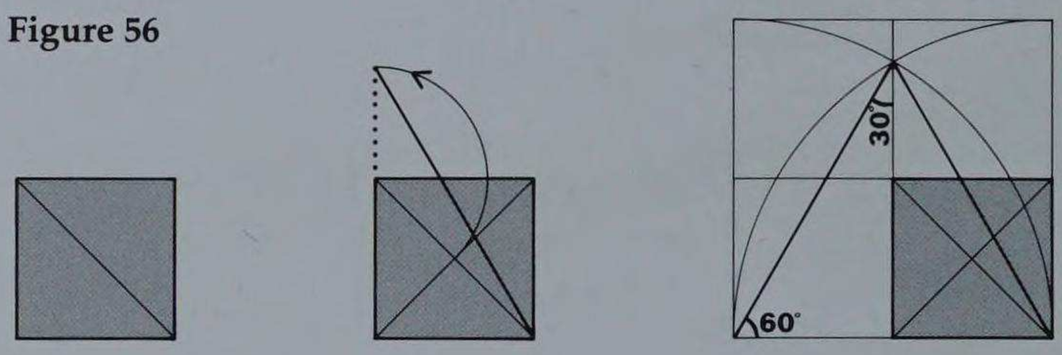
$\frac{1+\sqrt{2}/2}{1}$=1.707 = tan 59.63° (difference of -0.37°);

$\frac{1}{\sqrt{3}}=\frac{a}{c}$= 0.5773 = tan 30°;

$\frac{1}{1+\sqrt{2}/2}$= 0.5858 = tan 30.36° (difference of +0.36°).

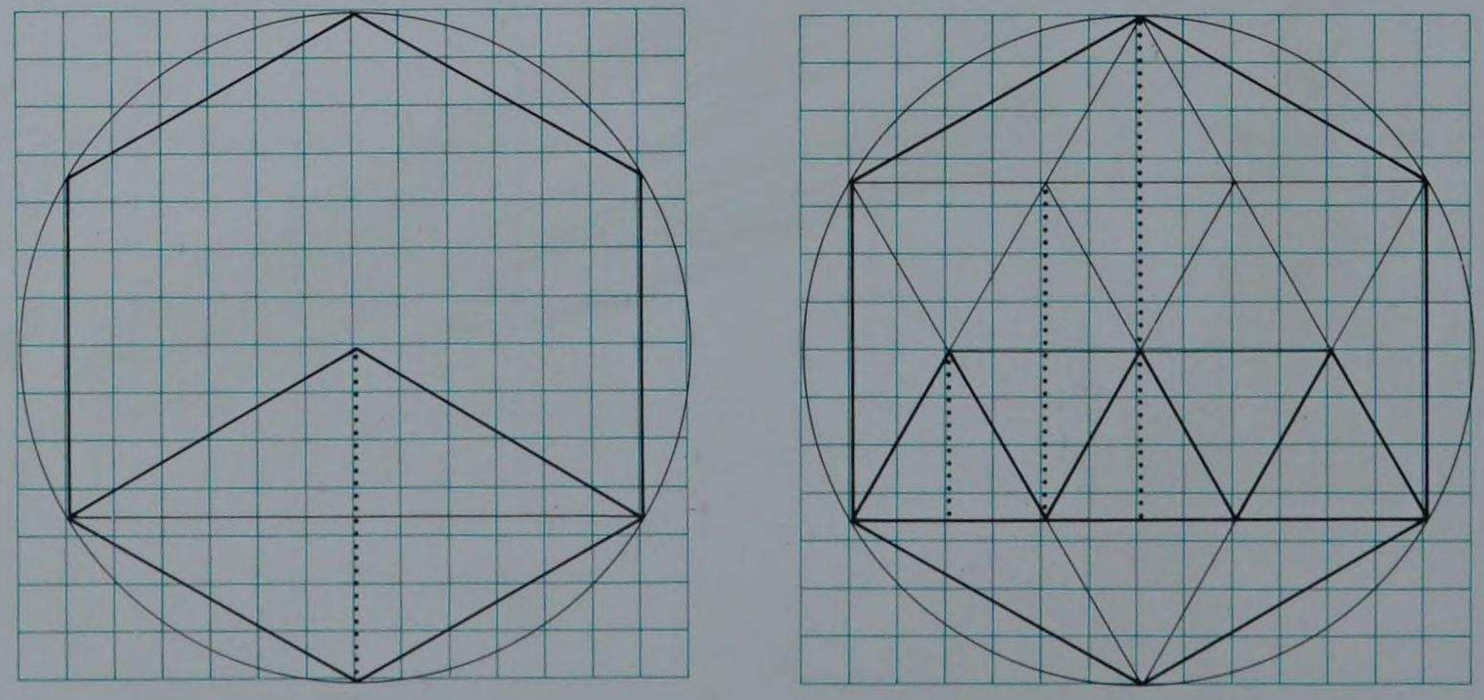
In practical terms this means that if (as shown in **Figure 56**), in any 30°, 60°, 90° triangle, the short adjacent side which equals half the side of an equilateral triangle is taken as the side of a square module, then the long adjacent side which is the height of the equilateral triangle equals the side of the square module plus half its diagonal, and the hypotenuse equals twice the side of the square module.

Figure 56



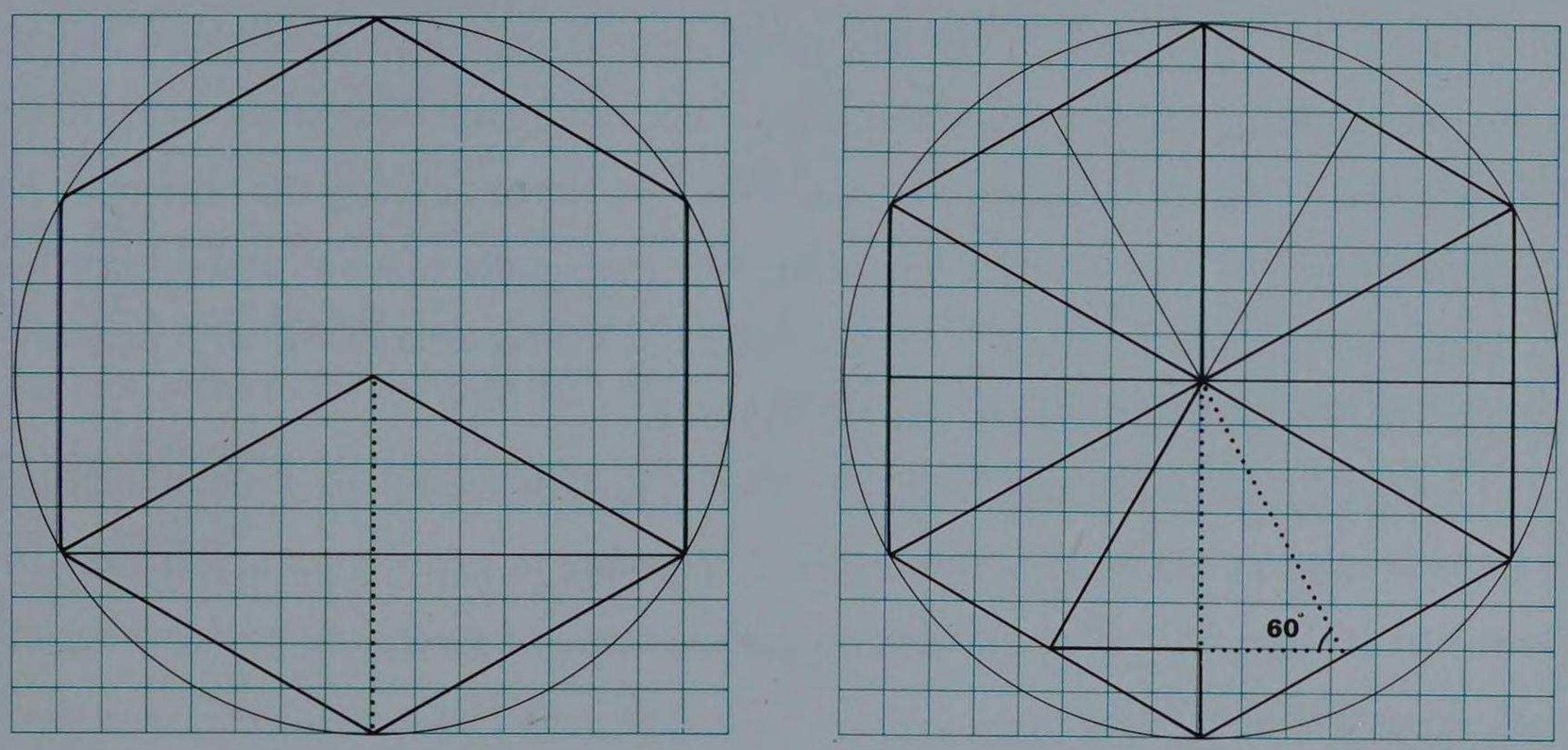
In **Figure 57**, the side of the hexagonal repeat unit which equals the radius of the circle is given the value of 7 units. This defines the side of primary grid C as 12 units or twice the apothem (the perpendicular distance from the centre to the side of the hexagon). The sides of primary grid C cut each other into three equal parts, each part defining a side of the resultant equilateral triangles (shown in dark line) which equals $\frac{12}{3}$ = 4 units and the height of the triangles (shown in dotted line) equals 3.5 units. This procedure provides a simple way of reckoning the various linear dimensions within the repeat unit. For example: height $/(\frac{1}{2}\text{base})=\frac{c}{a}=\frac{3.5}{2}=\frac{7}{4}=\frac{10.5}{6}$ = tan 60.255°. A difference of 0.255%, or 15 minutes, is tolerable for this purpose.

Figure 57



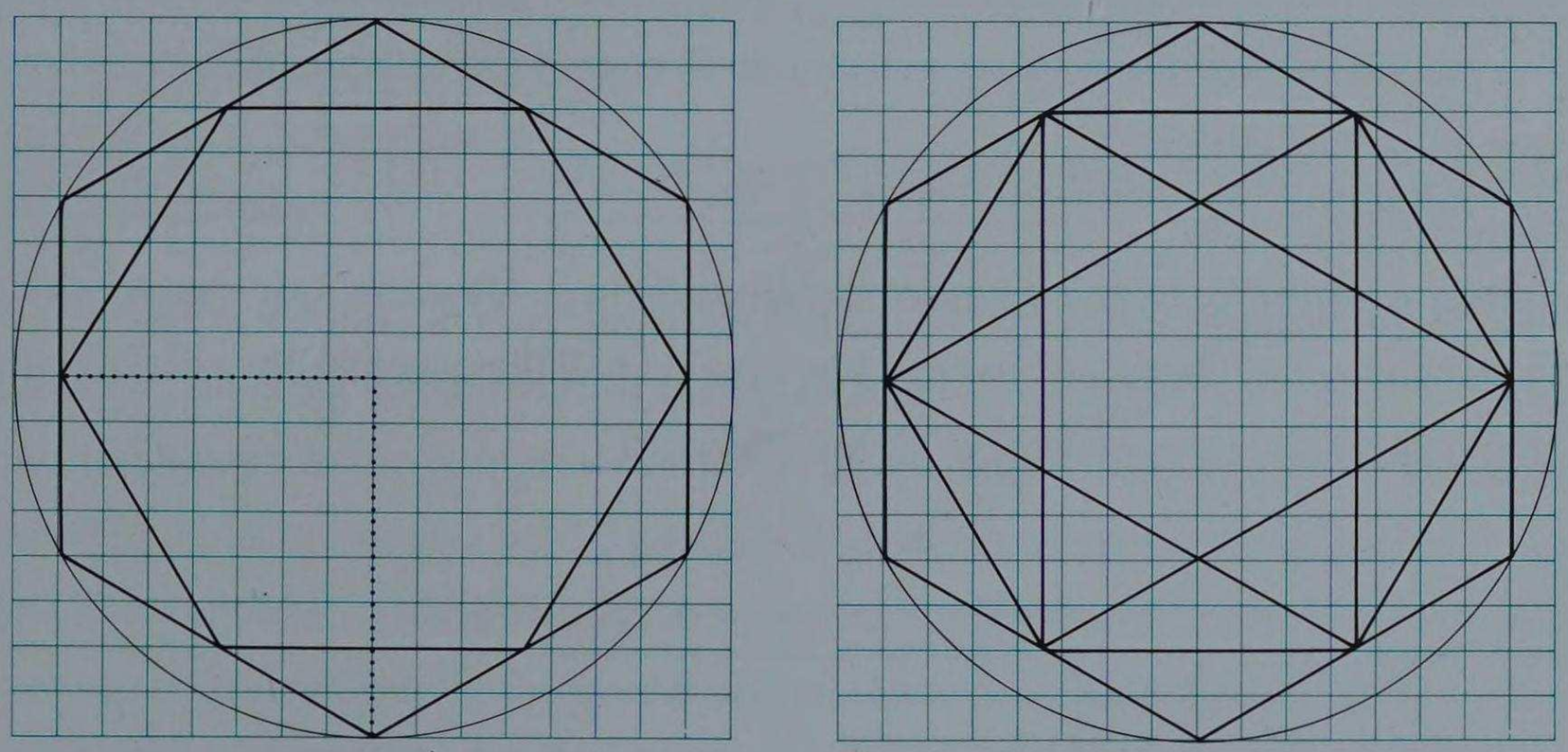
Similarly (in **Figure 58**), where the radius is given the value of 8 units, the apothem is defined as 7 units, and again tan 60° is expressed as: $\frac{3.5}{2} = \frac{6}{3.5} = \frac{7}{4} = \frac{12}{7} = \frac{14}{8}$ units, $\frac{c}{a} = \frac{3a}{c} = \frac{2c}{2a} = \frac{6a}{2c} = \frac{4c}{4a}$.

Figure 58



When these two figures are combined (see **Figure 59**), they generate primary grid D, thus providing a means by which to measure the linear dimensions of the repeat unit.

Figure 59



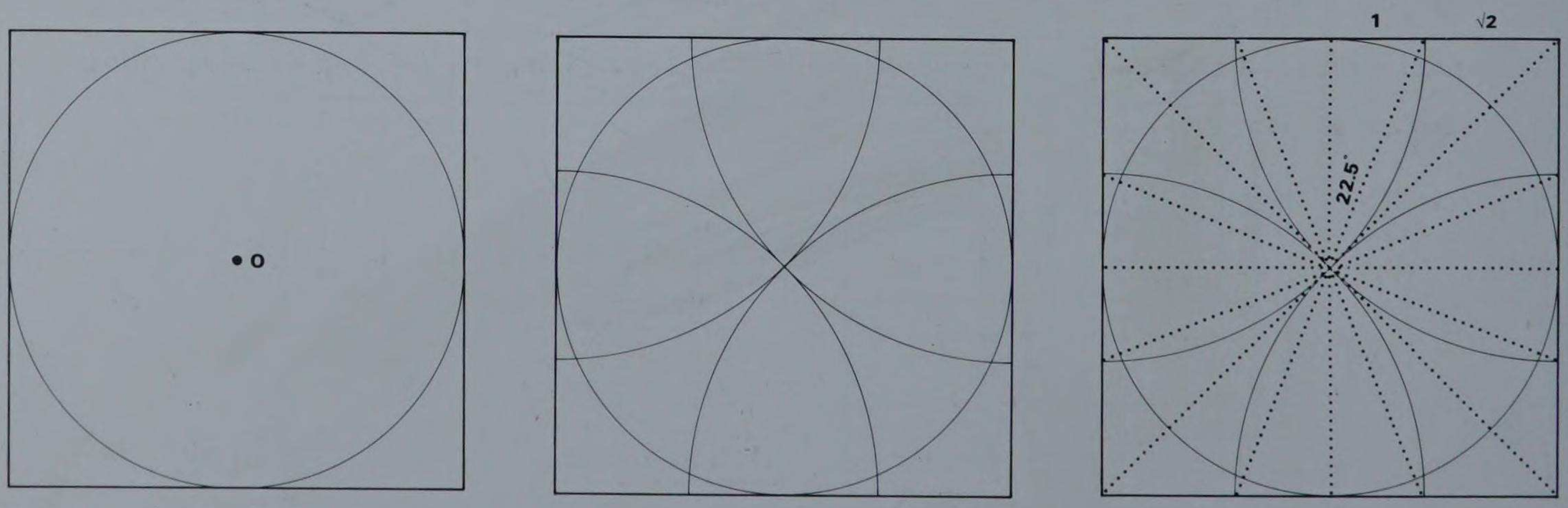
Mathematical or numerical rationalization of the root three system of proportion

The rules employed in this system entail the accurate construction of 30°, 60° and 90° angles. This is made possible by subdividing the hexagonal repeat unit into equilateral triangles by triangulation using the point-joining method, and in turn giving the six-pointed star grid-patterns. The measure of the equilateral triangle was established by approximating the ratio of its height to half its base $\sqrt{3} : 1$ as $1+ \frac{\sqrt{2}}{2} :1$, where 1 equals the side of a square module, and $\frac{\sqrt{2}}{2}$ one half of its diagonal. Using the numerical values, $a = 2$ units and $b = 3$ units established with primary grid A for defining the ratios of $1 : \sqrt{2}$ (see page 23), the value of:
$1+\sqrt{2} \ / \ 2 : 1 = (a+b/2) \ / \ a = (2+1.5) \ / \ 2 = 3.5/2 : \cong \sqrt{3}/1$, thereby establishing the ratio as tan 60.25° and the inverse ratio as tan 29.75°. This makes it possible to express approximately in whole numbers the linear dimensions of the constituent elements of geometric patterns based on the hexagonal repeat unit (see page 69).

It has been demonstrated that the ratio of tan 22.5° can be expressed approximately as $\frac{1}{1+\sqrt{2}}$... thus giving a series using a, b, A and B as approximate units of measure, i.e. $\frac{a}{a+b}, \frac{b}{b+2a}, \frac{A}{A+B}, \frac{B}{B+2A}$,... can be expressed as $\frac{2}{5}, \frac{3}{7}, \frac{5}{12}, \frac{7}{17}$,...(see Table III). Although the numerical value of $\frac{5}{12}$ was suitably taken as an accepted approximate substitute, any dimension can be used for the values of 1 and $\sqrt{2}$ in the expression $\frac{1}{1+\sqrt{2}}$. The practical significance of this formula is that it provides a graphic method for dividing a circle into 16 equal parts utilizing the proportions of the square as a measuring device.

Draw a circle with centre O and a square circumscribing it. Describe four quarter circles with centres at each corner of the square and a radius equal to half the diagonal, thus dividing each half side of the square in the proportion of $1 : \sqrt{2}$ (see **Figure 60**).

Figure 60

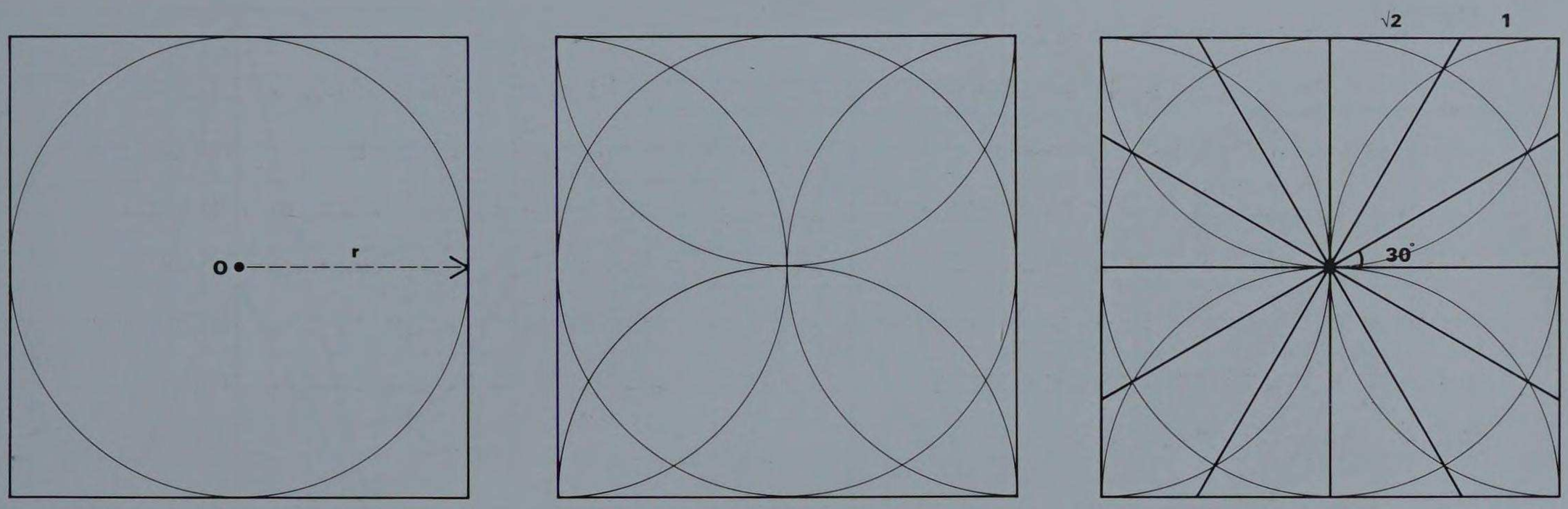


Join with straight lines the opposite points of division through the centre O and draw the diagonal, vertical and horizontal diameters to divide the circumference into 16 equal parts and therefore define tan 22.5° as $\frac{1}{1+\sqrt{2}}$ and the inverse value of tan 67.5° as $\frac{1+\sqrt{2}}{1}$.

Similarly, the division of a circle into 12 equal parts can be achieved by dividing half the sides of the circumscribing square in the proportion of $\sqrt{2} : 1$.

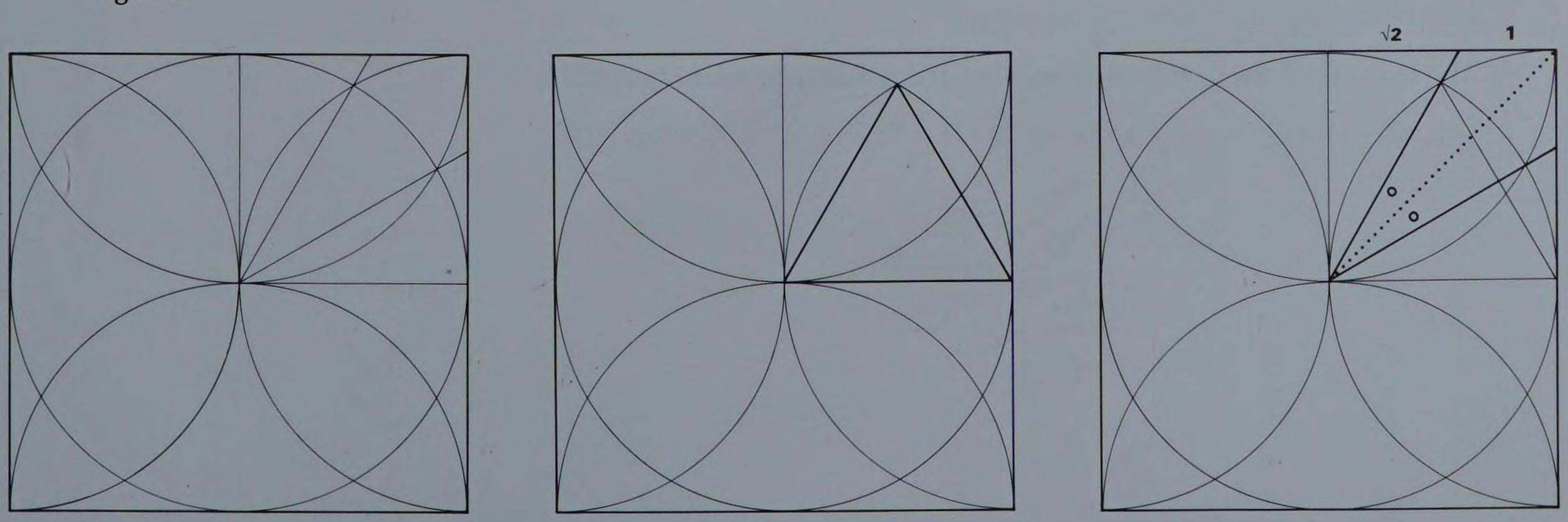
Draw a circle with centre *O* and a square circumscribing it. Describe four semicircles with radius *r* and centres at the points where the horizontal and vertical diameters meet the circumference (see **Figure 61**). Join with straight lines the opposite points of intersection of the circle and the semicircles through the centre *O*, thus dividing the circumference into 12 equal parts. This defines tan 30° approximately as $\frac{\sqrt{2}}{1+\sqrt{2}} = \frac{b}{a+b}$, and the inverse ratio gives an approximate value of tan 60°, i.e. $\frac{1+\sqrt{2}}{\sqrt{2}}$.

Figure 61



Combining these graphic systems (see **Figure 62**) provides a method of extrapolating the 15° angle using the angular properties of the square and the hexagon. By dividing two sides of one quadrant of a square in the proportion of $\sqrt{2} : 1$, an equilateral triangle may be constructed with sides equal to the radius of the circle. By further drawing the diagonal of the quadrant square, the 15° angle is extrapolated. This is possible by either subtracting the 45° made by the diagonal of the quadrant square from the 60° of the equilateral triangle or by subtracting 30°, half the angle of the equilateral triangle, from 45°.

Figure 62



The measure of tan 15° can be resolved by applying the geometric fact that 'an angle that an arc subtends at the centre of a circle is twice that it subtends at the circumference' (**Figure 63**). The side of the described hexagon is given the value of 7 units and an equilateral triangle is drawn with its apex at the centre of the circle. Since the angle of an equilateral triangle is 60°, then the angle made at the circumference and subtended by the same arc is 30°. By drawing the vertical diameter of the circle the 30° angle is halved thus defining tan15° as:

($\frac{1}{2}$side of equilateral triangle) / (height of triangle + side of triangle).

This gives the value of 1 : ($\sqrt{3}$ +2). Mathematical verification confirms the value exactly: $\tan \frac{1}{\sqrt{3}+2} = 0.2679 = 15°$.

When the numerical values at hand are applied, the tangent of the angle defined by this ratio = $\frac{3.5}{(6+7)} = 0.2692 = 15.07°$ (a difference of 0.07° or just over 4 minutes).

Figure 63

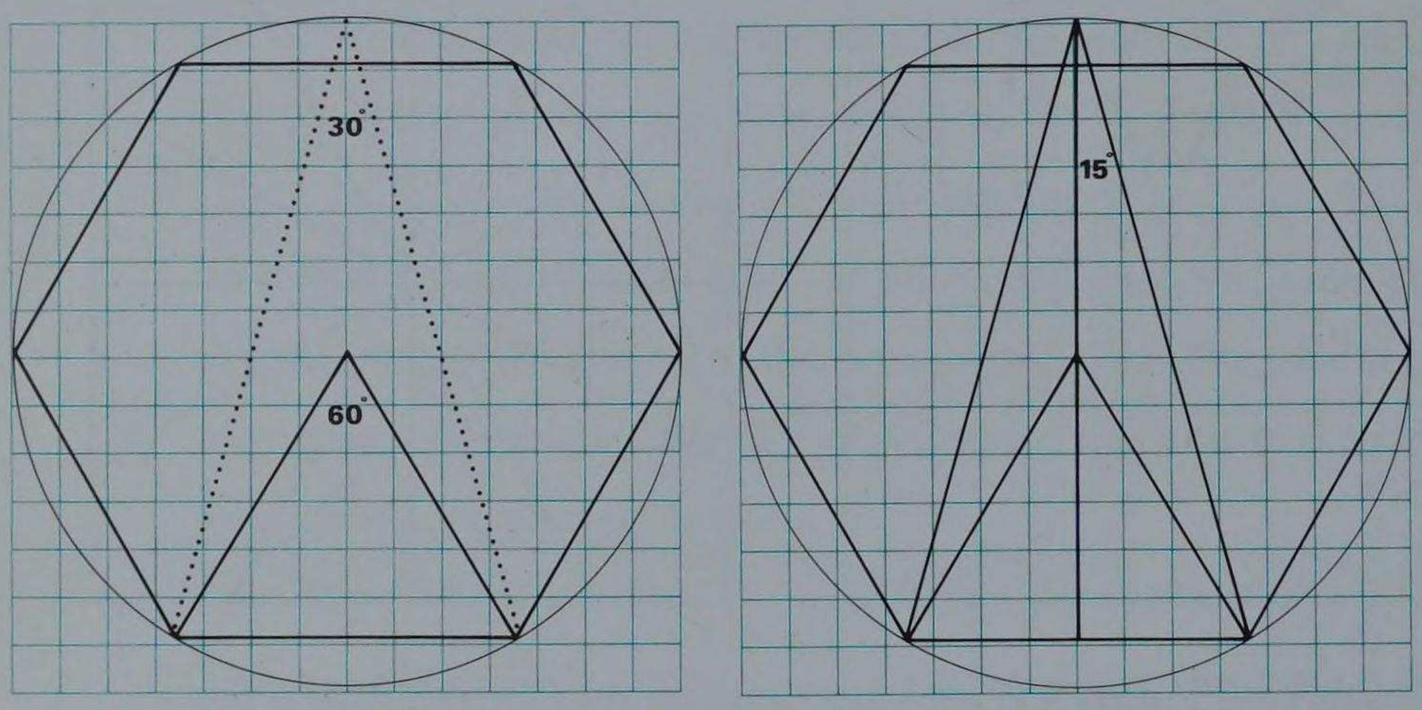
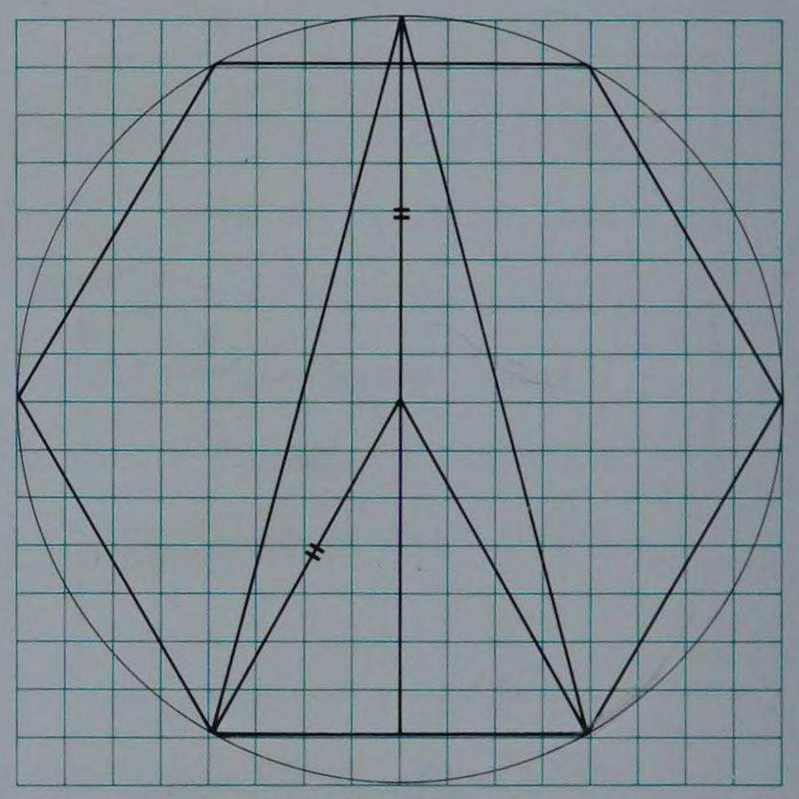


Figure 64



Similarly, when the radius is 8 units (see **Figure 64**), the ratio gives an angular value of $\frac{1}{\sqrt{3}+2} \cong \frac{4}{7+8} = \frac{4}{15}$ = tan 14.93° (less 0.07°). Thus the average gives the ratio $\frac{7.5}{28}$ as tan 14.995°, which is a very close approximation.

It has already been shown that the value of $\sqrt{3}$ can be expressed approximately as $1+\frac{\sqrt{2}}{2}$.

It follows that the ratio $\frac{1}{\sqrt{3}+2}$ for tan 15° can be interpreted as: $\frac{1}{1+\sqrt{2}/2+2} = \frac{1}{3+\sqrt{2}/2}$ (measuring an angle of 15.09°). This allows the definition of tan 15° in terms of a square module as:

side of square / 3 × side + $\frac{1}{2}$diagonal.

The inverse of these ratios defines an accepted approximation of tan 75°,

i.e. $\tan 75° = (2+\sqrt{3}) / 1 \cong (3+\sqrt{2}/2) / 1 \cong 13/3.5 \cong 15/4 = \tan 75.07°$.

Graphical analyses of geometric patterns based on the root three system of proportion.

The format of the presentation of the following examples of geometric patterns (**Figures 65** to **82**) illustrates the progression from the initial division of the circle into six equal parts to the final juxtaposition of the repeat units to create the pattern. The top three frames of each figure show the construction of the hexagonal repeat unit with the middle frame giving the type of grid-pattern(s) that generate its composition. The translation of the geometry into number is shown in the bottom left-hand frame by superimposing it on a square grid. Practically, this is achieved by giving the value of $2a = 4$ units to the side of the smallest equilateral triangle that occurs in the repeat unit, while the height of the triangle = 3.5 units (see **Figure 57**). For each example, the measures of the side and the apothem of the hexagonal repeat unit are given.

Figure 65. The repeat unit is the six-pointed star derived from primary grid C.

Side of hexagonal repeat unit = 7 units.

Apothem = 6 units.

Figure 66. As in the previous example, the repeat unit is derived from primary grid C. However, its main compositional element is a central hexagon.

Side of hexagonal repeat unit = 7 units.

Apothem = 6 units.

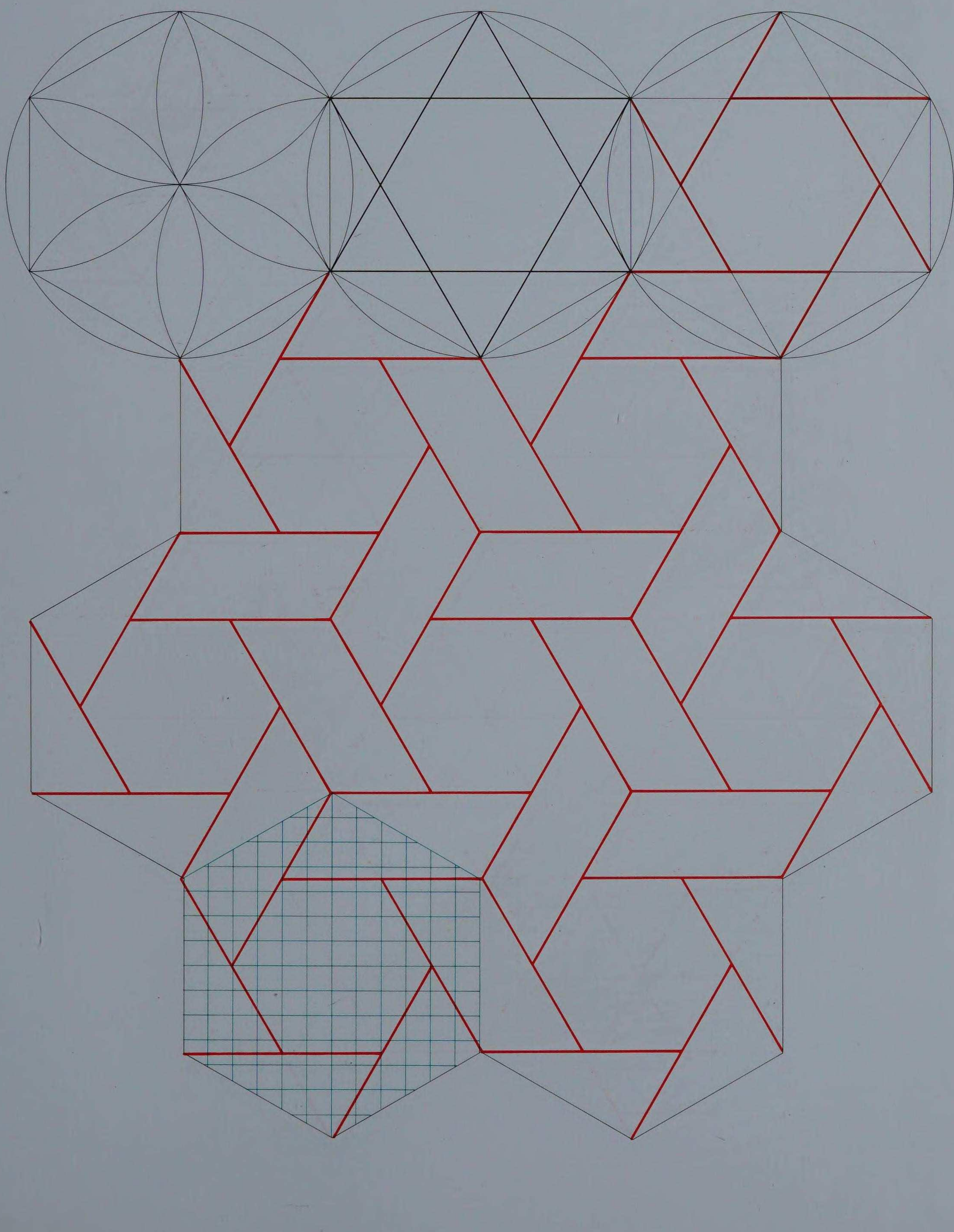


Figure 67. The intersection points of the sides of primary grid C are joined through the centre of the circle to give the complete grid-pattern.

Side of hexagonal repeat unit = 7 units.

Apothem = 6 units.

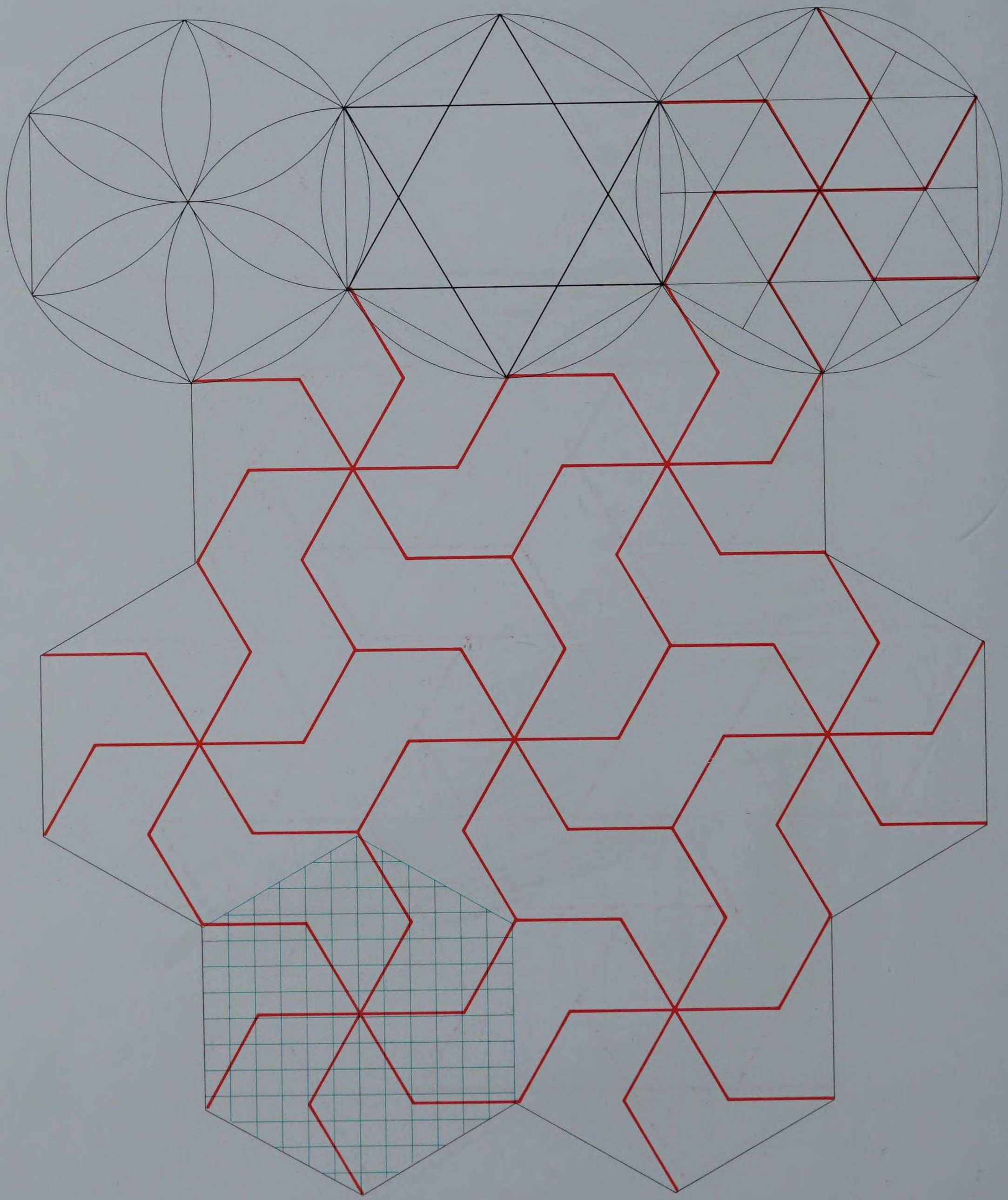


Figure 68. This example develops the grid-pattern of the previous example by extending the lines running through the centre of the circle and the intersection points of the sides of primary grid C to bisect the sides of the hexagonal repeat unit. Joining the mid-points of the sides of the repeat unit gives an inscribed hexagon and the required grid-pattern.

Side of hexagonal repeat unit = 14 units.

Apothem = 12 units.

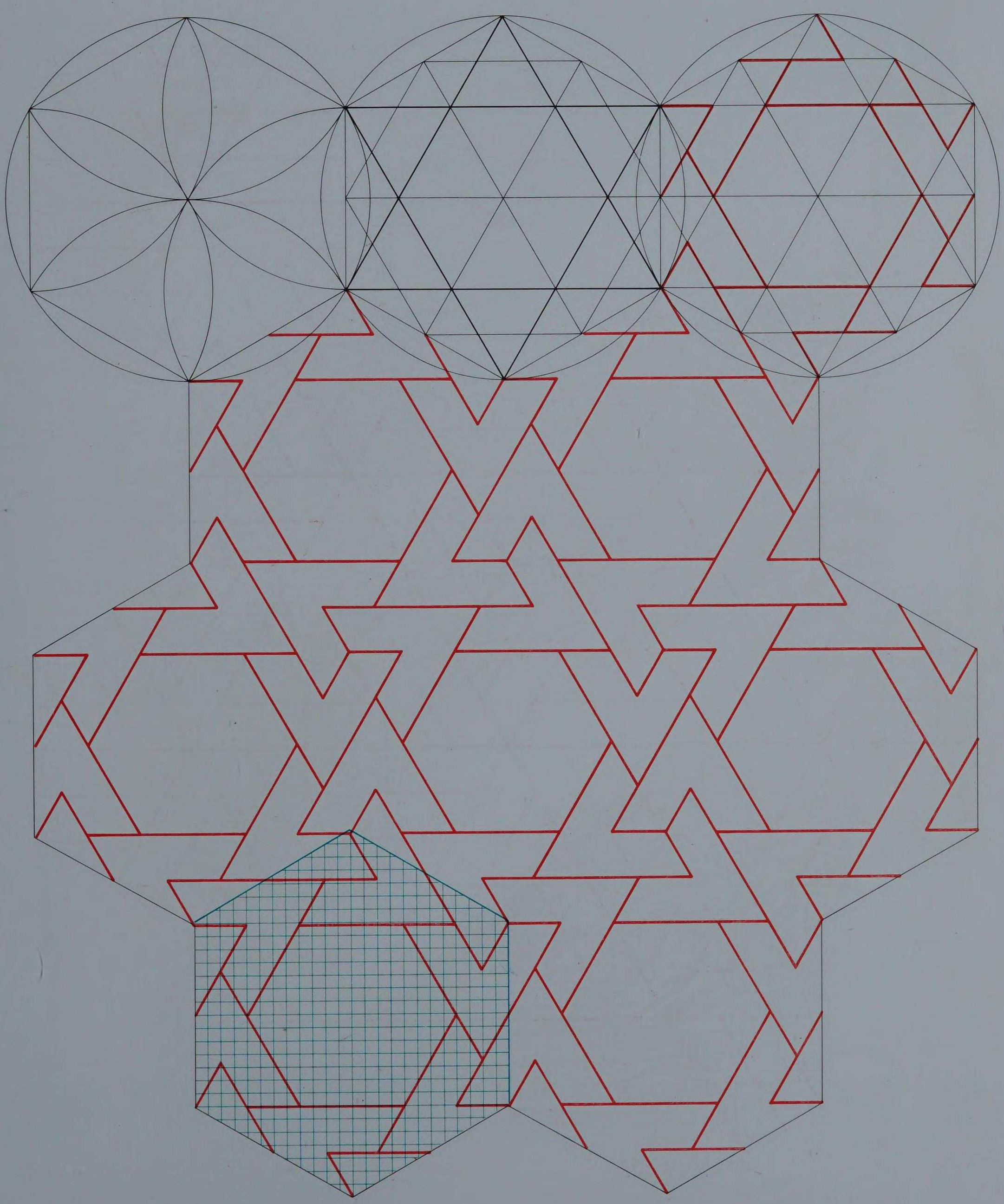


Figure 69. In this example the grid-pattern is a further development of the last one whereby lines are drawn parallel to the inscribed hexagon joining the intersection points of the hexagon and primary grid C. These lines are further extended to the perimeter of the hexagonal repeat unit.

Side of the hexagonal repeat unit = 14 units.

Apothem = 12 units.

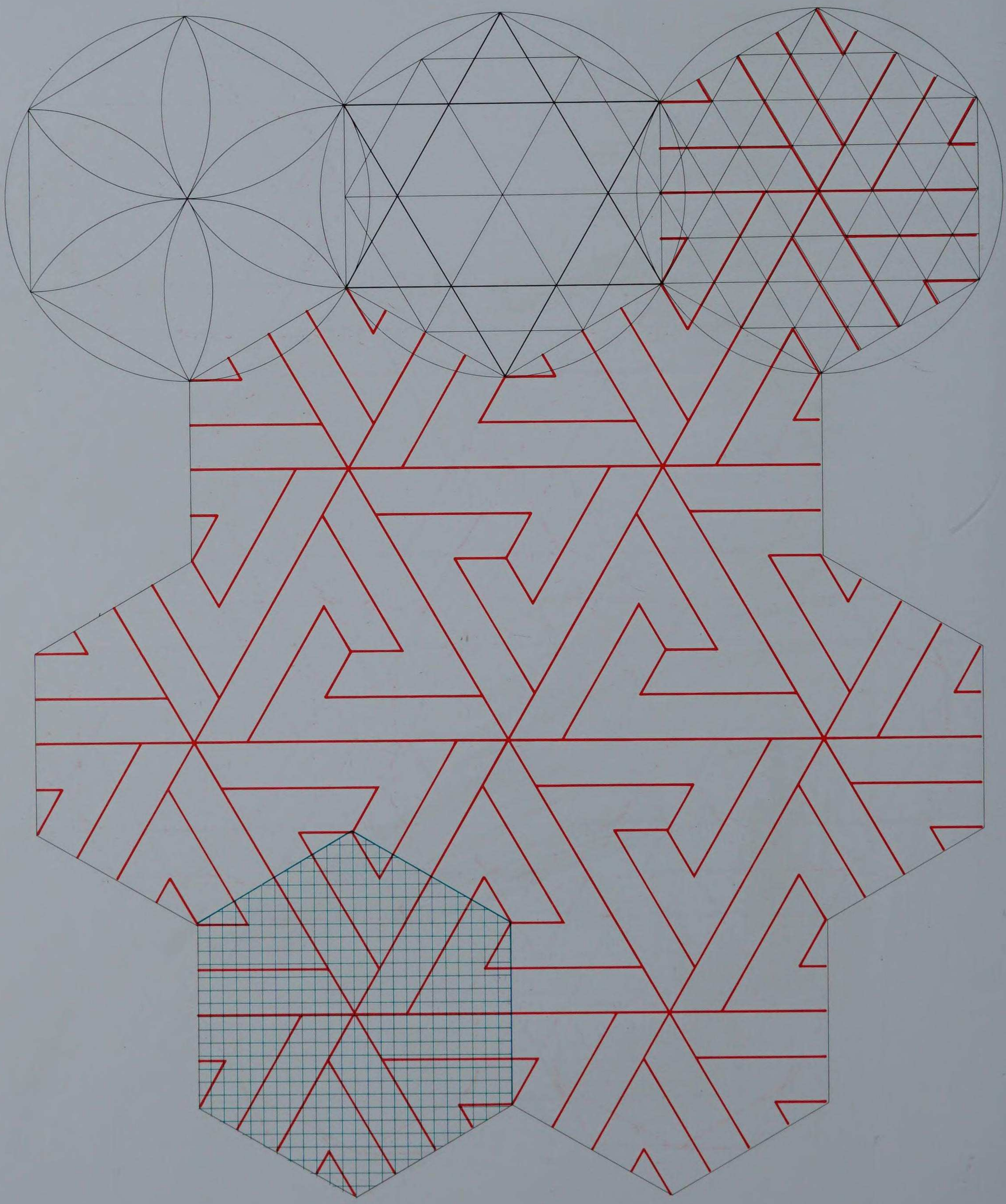


Figure 70. The grid-pattern for this repeat unit consists of two nesting hexagons inscribed in the hexagonal repeat unit. The first is generated by joining the mid-points of the sides of the hexagonal repeat unit and the second by further joining the mid-points of the sides of the first nesting hexagon.

Side of the hexagonal repeat unit = 14 units.

Apothem = 12 units.

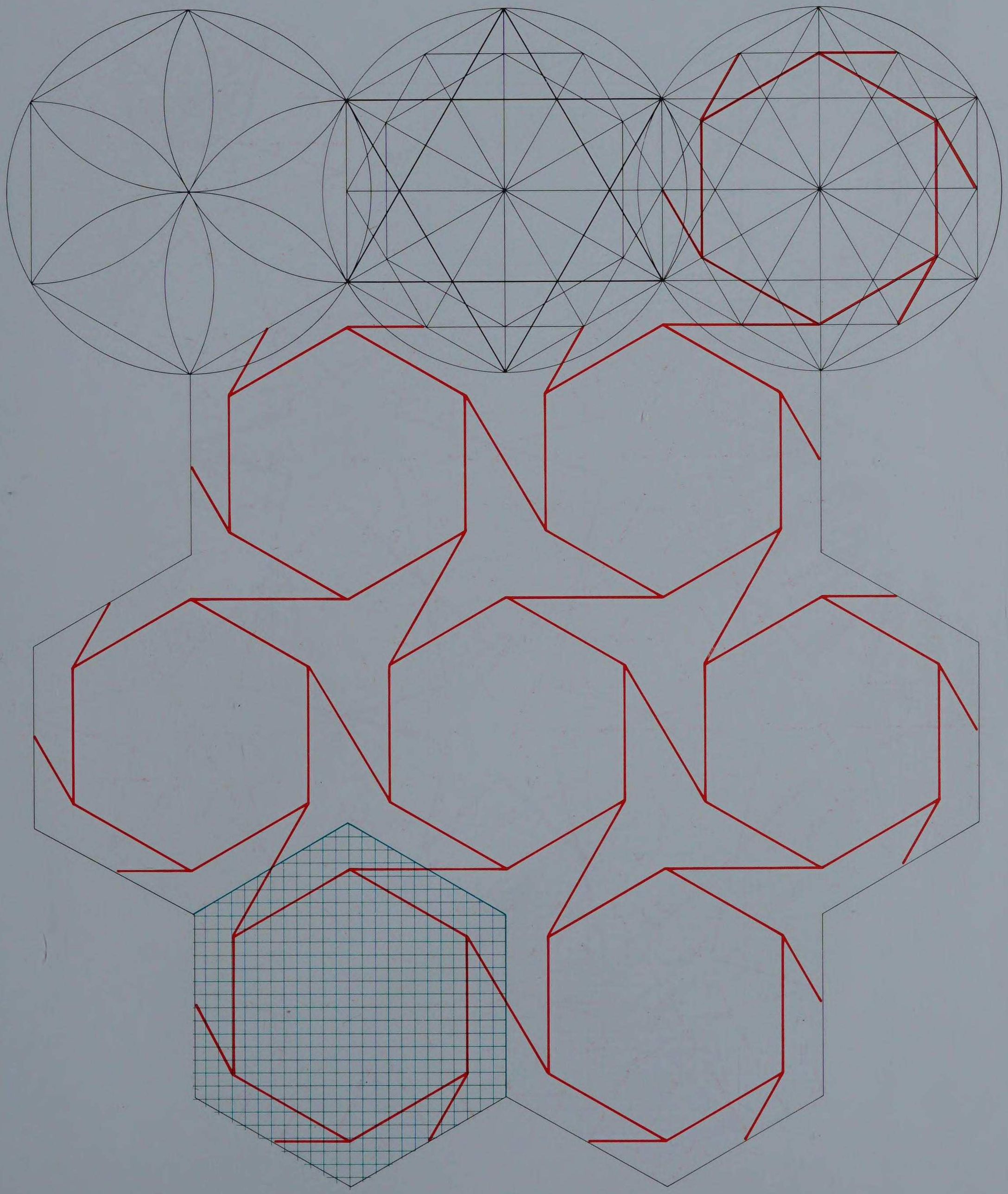


Figure 71. The grid-pattern is based on primary grid C. The sides of the hexagonal repeat unit are bisected by drawing lines through the intersection points of the sides of primary grid C and the centre of the circle. From each of these mid-points two lines are drawn to the corners of the side directly opposite.

Side of hexagonal repeat unit = 16 units.

Apothem = 14 units.

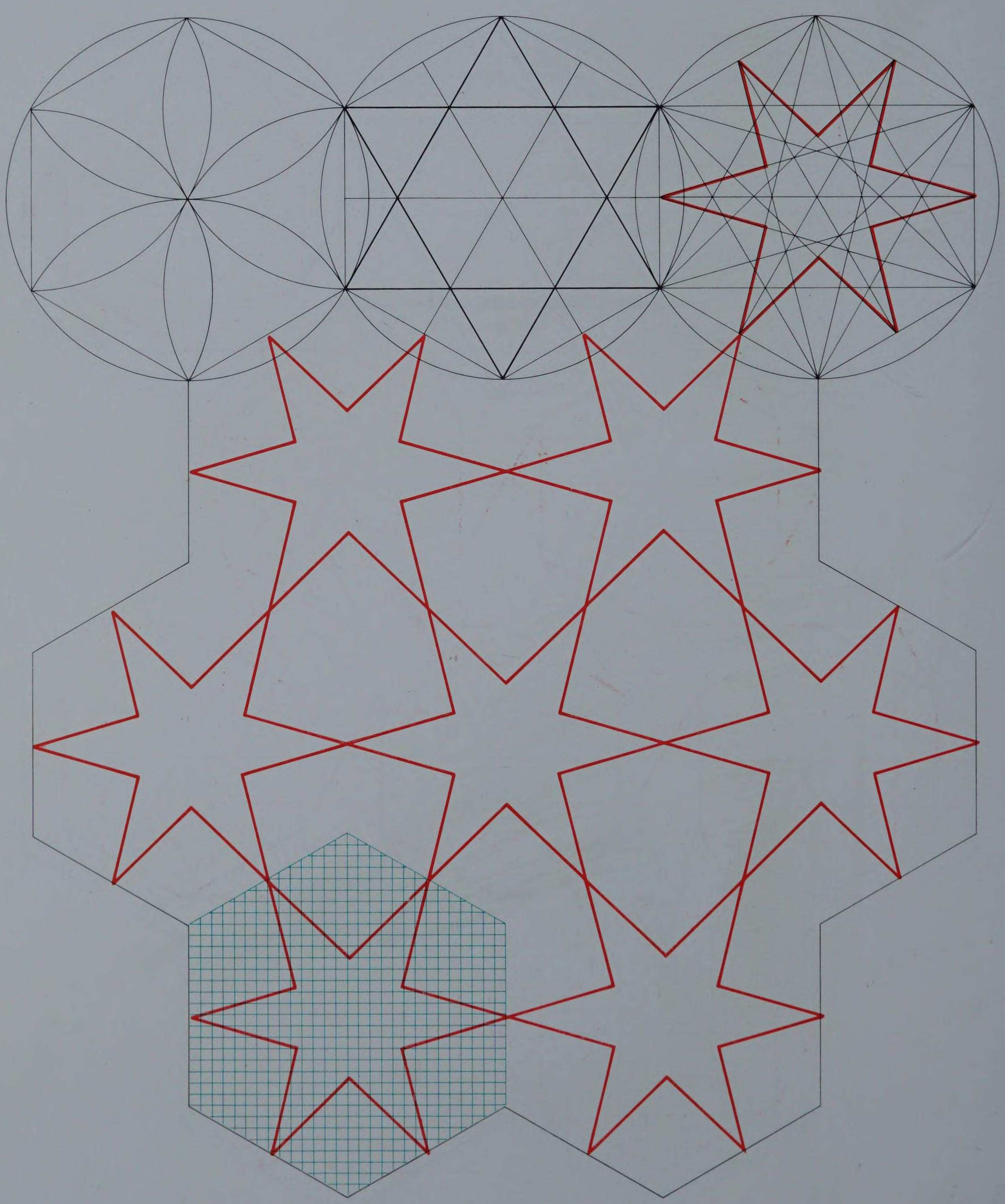


Figure 72. The repeat unit is the six-pointed star of primary grid D which is introduced by joining every second point of the mid-points of the sides of the hexagonal repeat unit.
Side of the hexagon repeat unit = 14 units.
Apothem = 12 units.

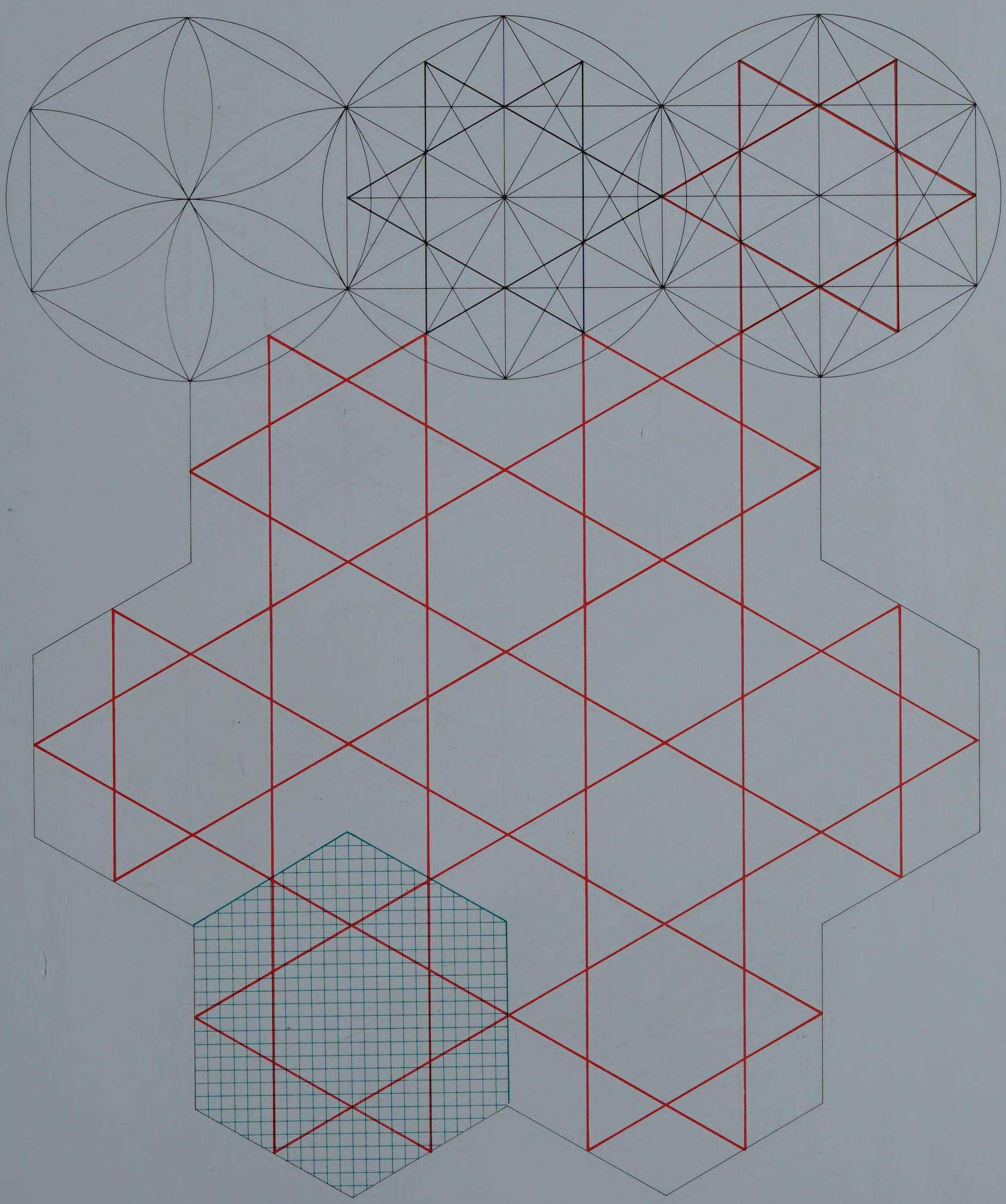


Figure 73. The grid-pattern in this example is primary grid D from which a spiral form is generated as the composition of the repeat unit.

Side of hexagonal repeat unit = 14 units.

Apothem = 12 units.

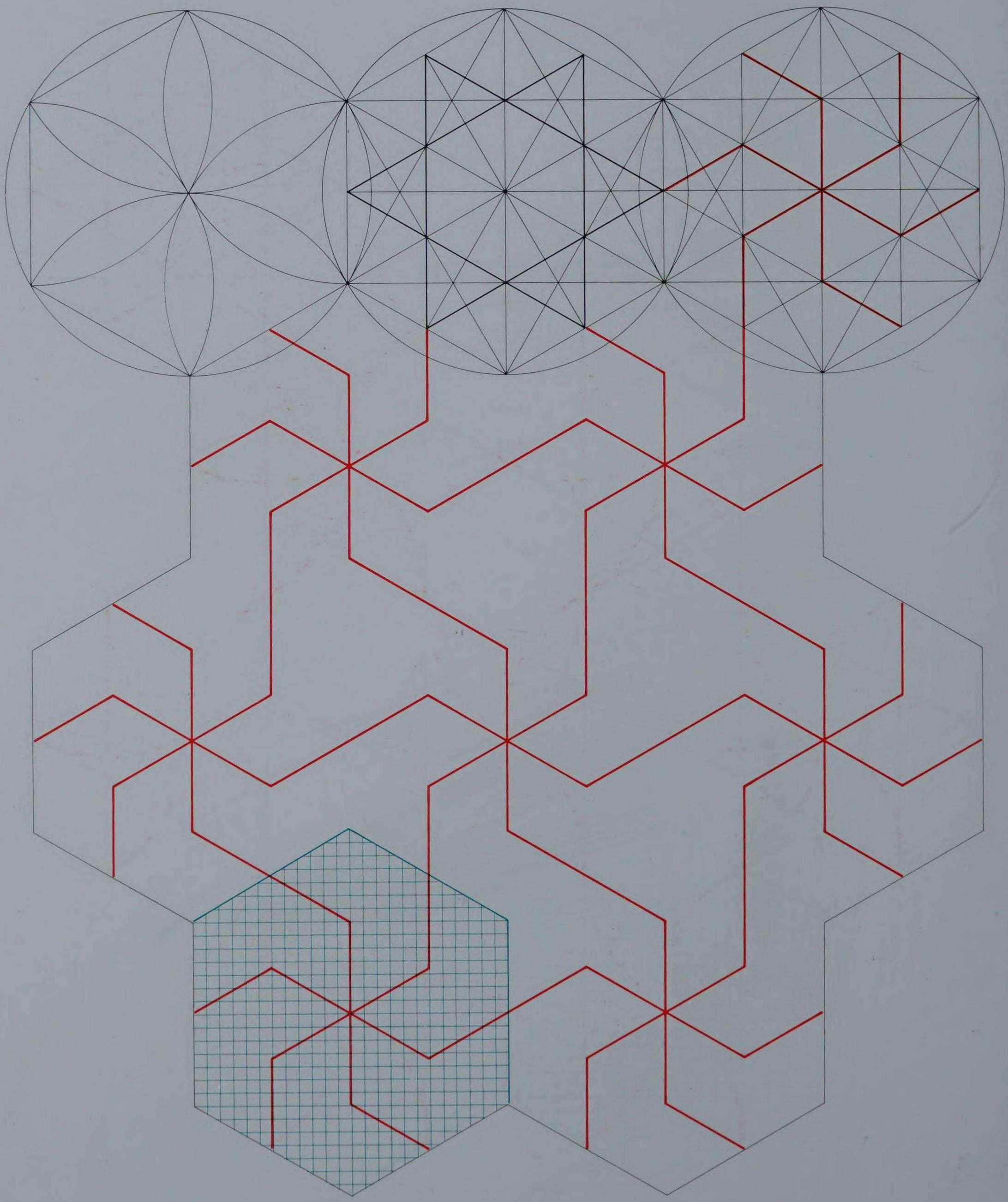


Figure 74. Taking the apothem of the hexagonal repeat unit as radius and the six points of the star of primary grid D as centres, six consecutive arcs are drawn clockwise, each joining one point of the star to the centre. The result is a curved spiral form.
Side of hexagonal repeat unit = 14 units.
Apothem = 12 units.

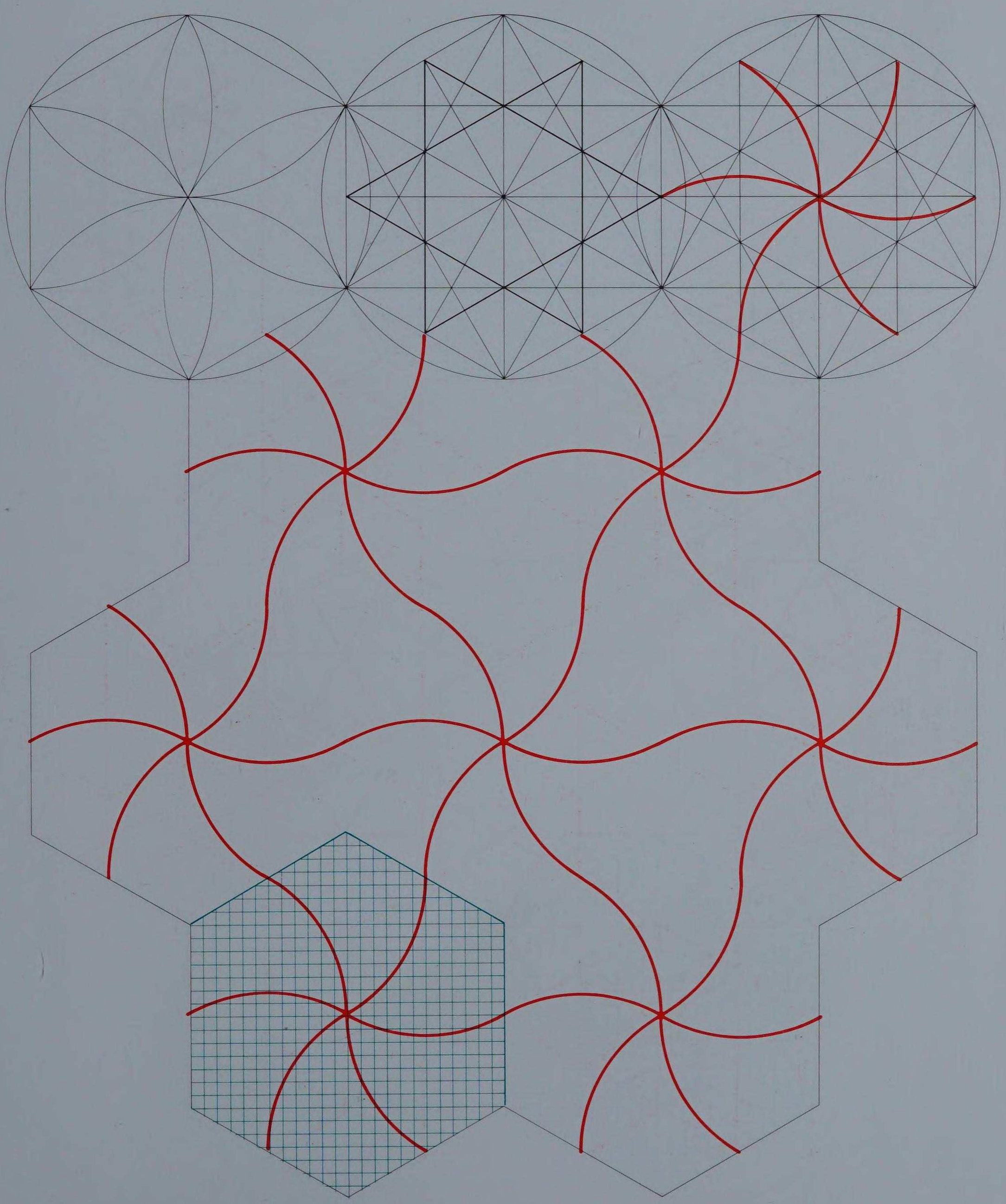


Figure 75. The resultant grid-pattern is generated by superimposing primary grid C onto primary grid D.

Side of hexagonal repeat unit = 14 units.

Apothem = 12 units.

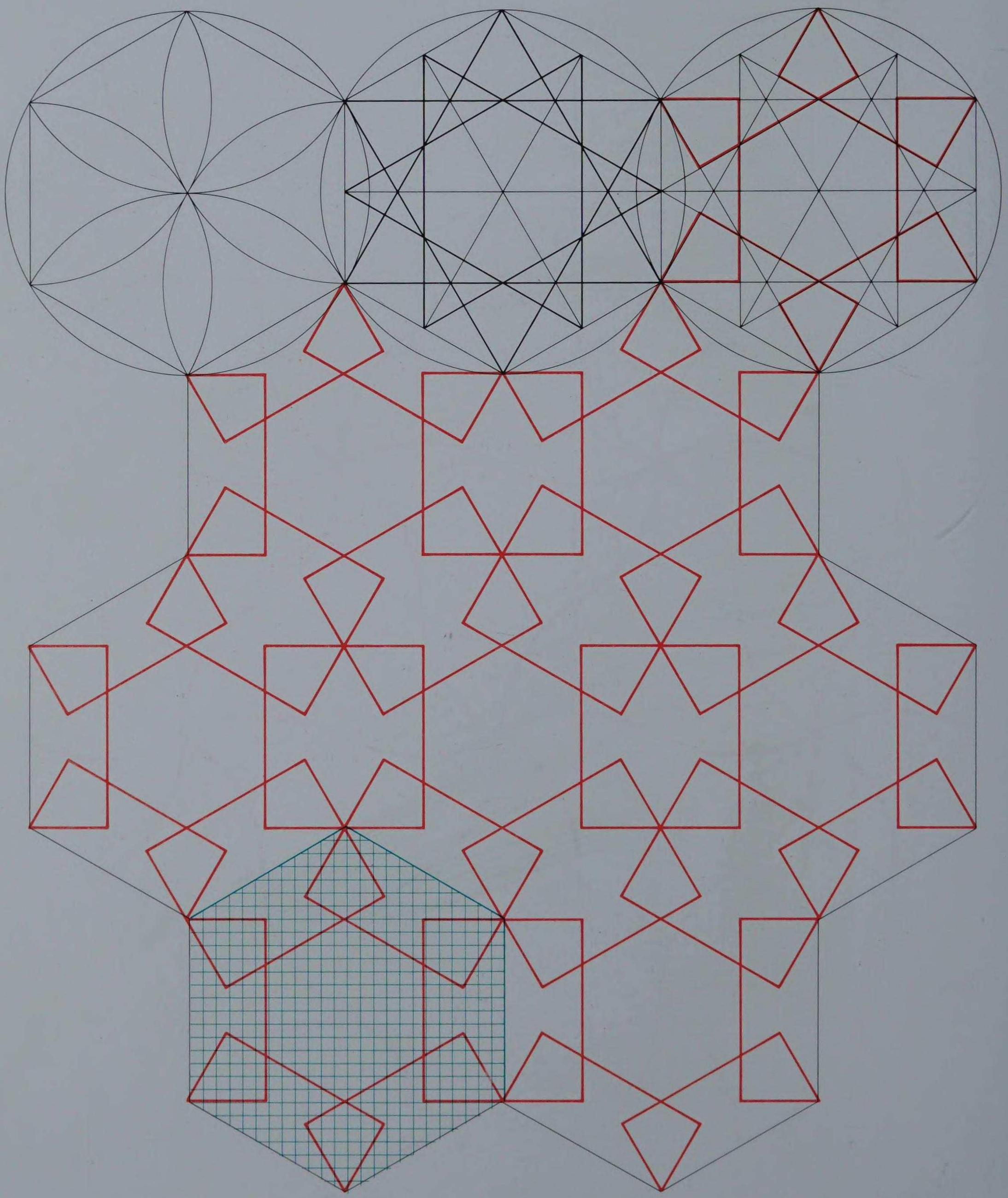


Figure 76. This and the following two examples are based on primary grid C and secondary grid D nesting within it. In this case the resultant grid-pattern is generated by extending the sides of secondary grid D to the perimeter of the hexagonal repeat unit.

Side of hexagonal repeat unit = 12 units.

Apothem = 10.5 units.

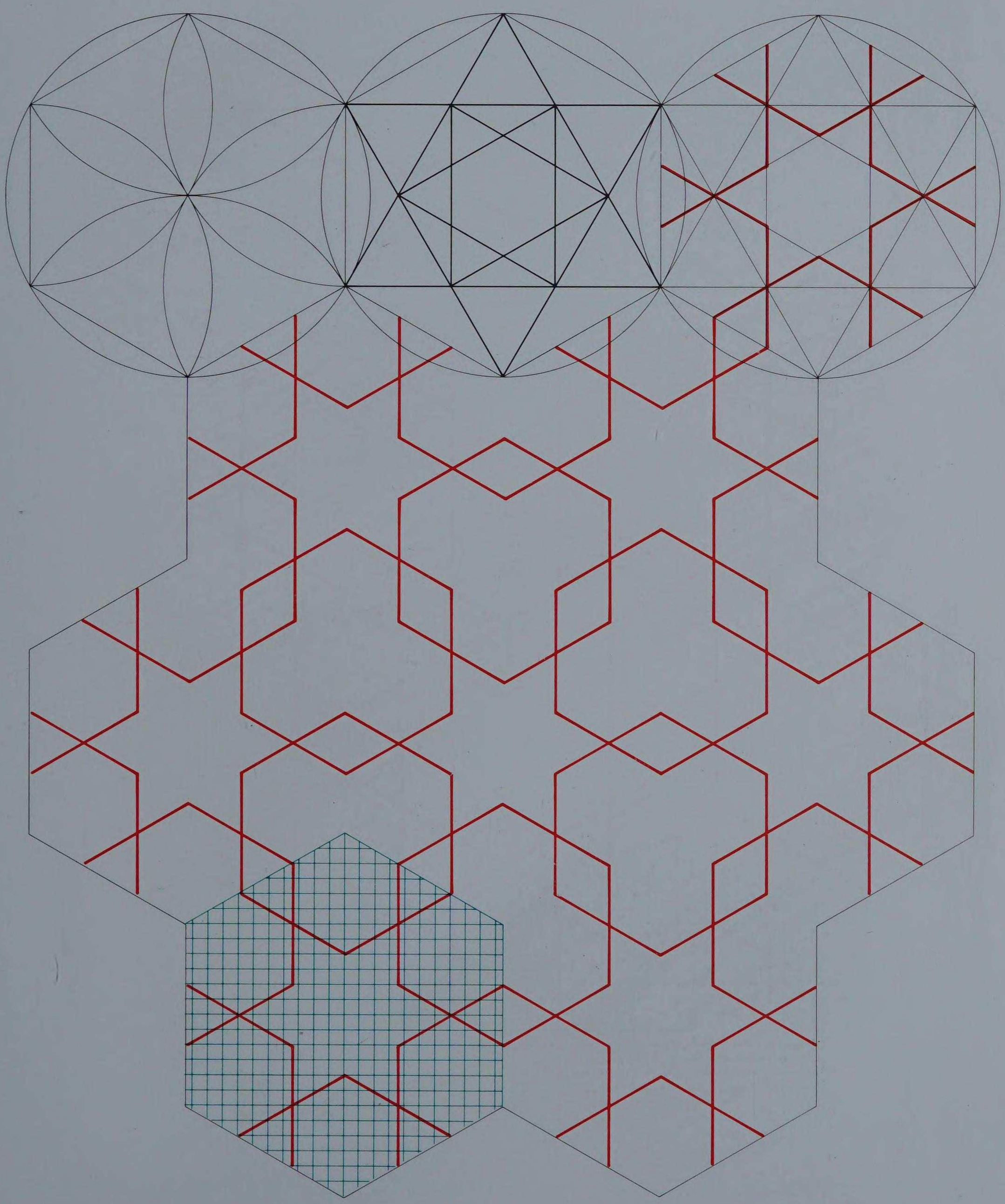


Figure 77. In this example the previous grid-pattern is developed further by joining the intersection points of the extended sides of secondary grid D with the hexagonal repeat unit. These are drawn parallel to the hexagonal repeat unit and give a hexagon as the central element of the composition.

Side of the hexagonal repeat unit = 12 units.

Apothem = 10.5 units.

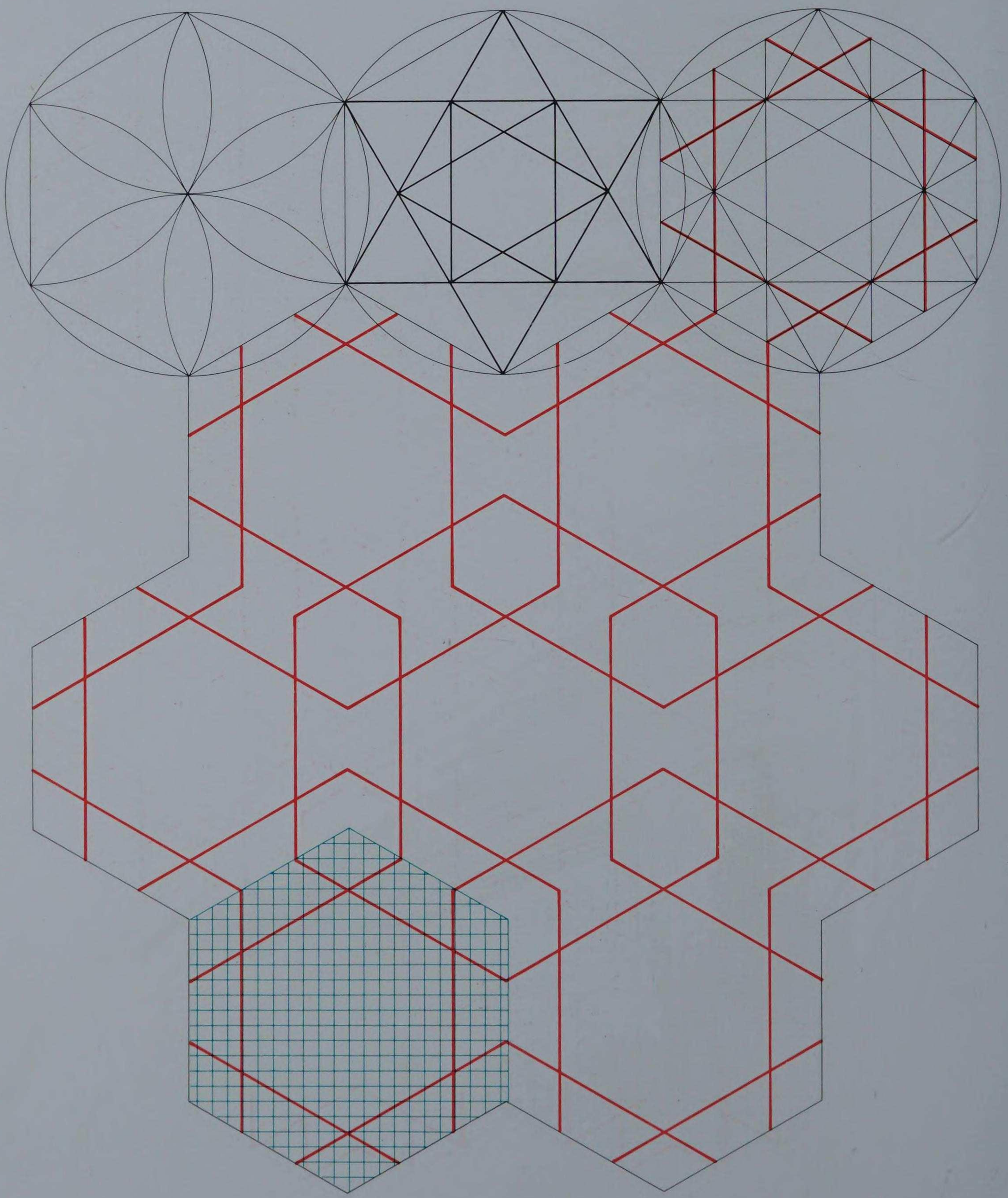


Figure 78. The grid-pattern is identical to the previous one, but the repeat unit has a six-pointed star as its central element.

Side of hexagonal repeat unit = 12 units.

Apothem = 10.5 units.

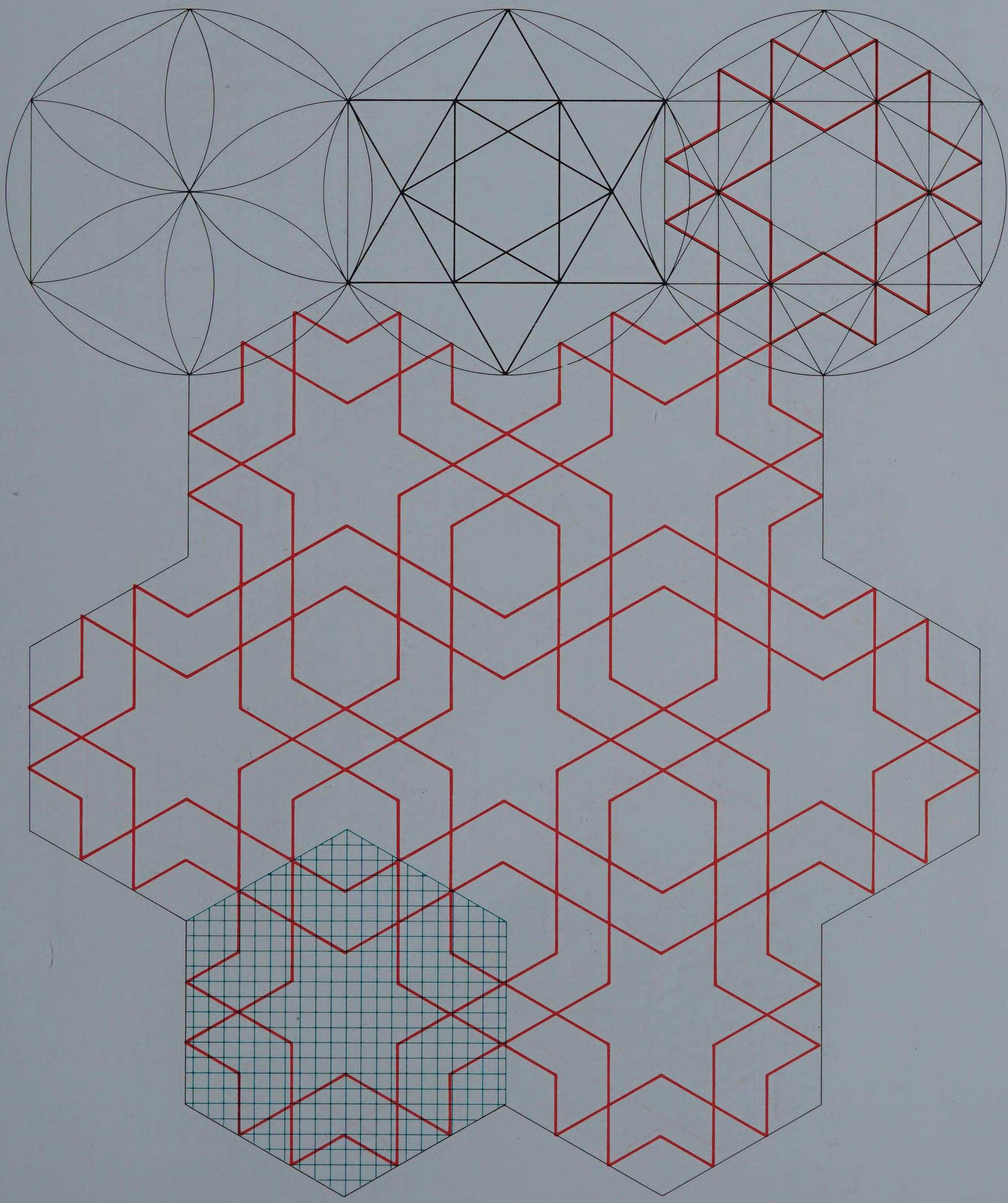


Figure 79. The resultant grid-pattern is generated from primary and secondary grids D. The sides of secondary grid D are extended to the perimeter of the hexagonal repeat unit and at the points where they intersect with primary grid D, lines are drawn through, parallel to the sides of the hexagonal repeat unit.

Side of the hexagonal repeat unit = 16 units.

Apothem = 14 units.

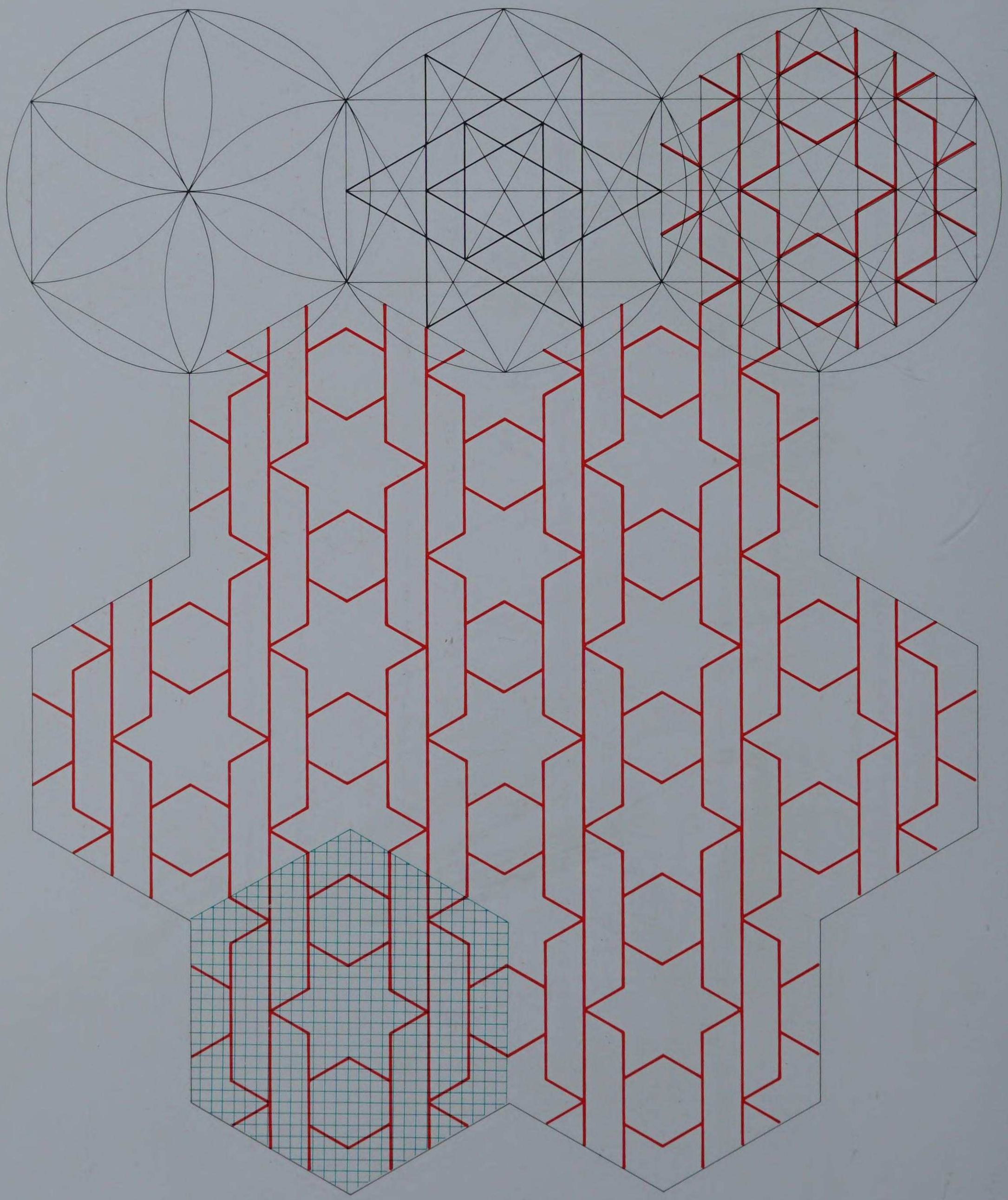


Figure 80. The same grid-pattern as for the previous example is used here. However, by highlighting certain areas, the resultant composition of the repeat unit varies.
Side of the hexagonal repeat unit = 16 units.
Apothem = 14 units.

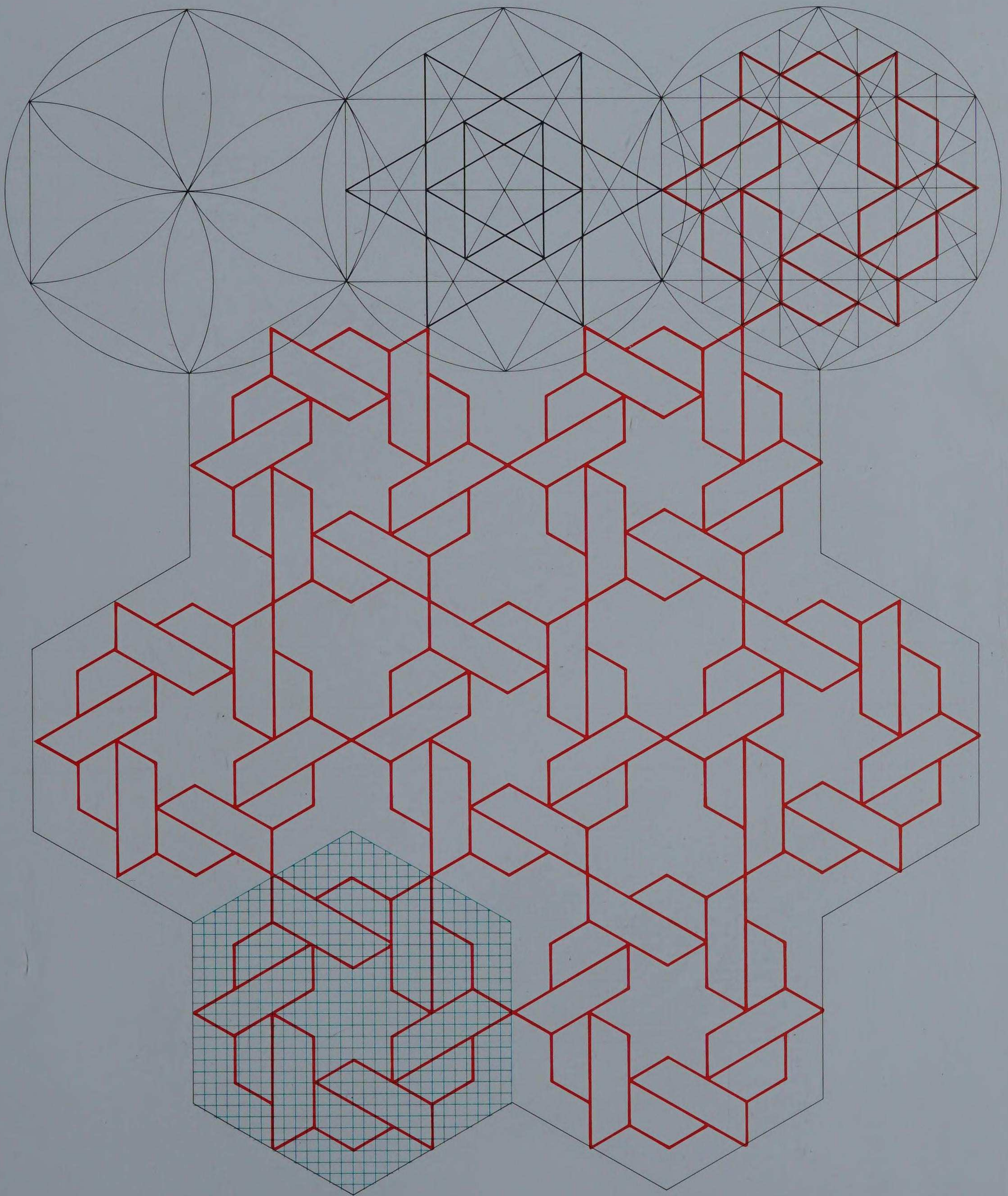


Figure 81. This composition is based on three grid-patterns: primary grid C, secondary grid D and a tertiary grid C nesting within secondary grid D. The sides of both secondary grid D and tertiary grid C are extended to the perimeter of the hexagonal repeat unit to give the resultant grid-pattern.
Side of the hexagonal repeat unit = 21 units.
Apothem = 18 units.

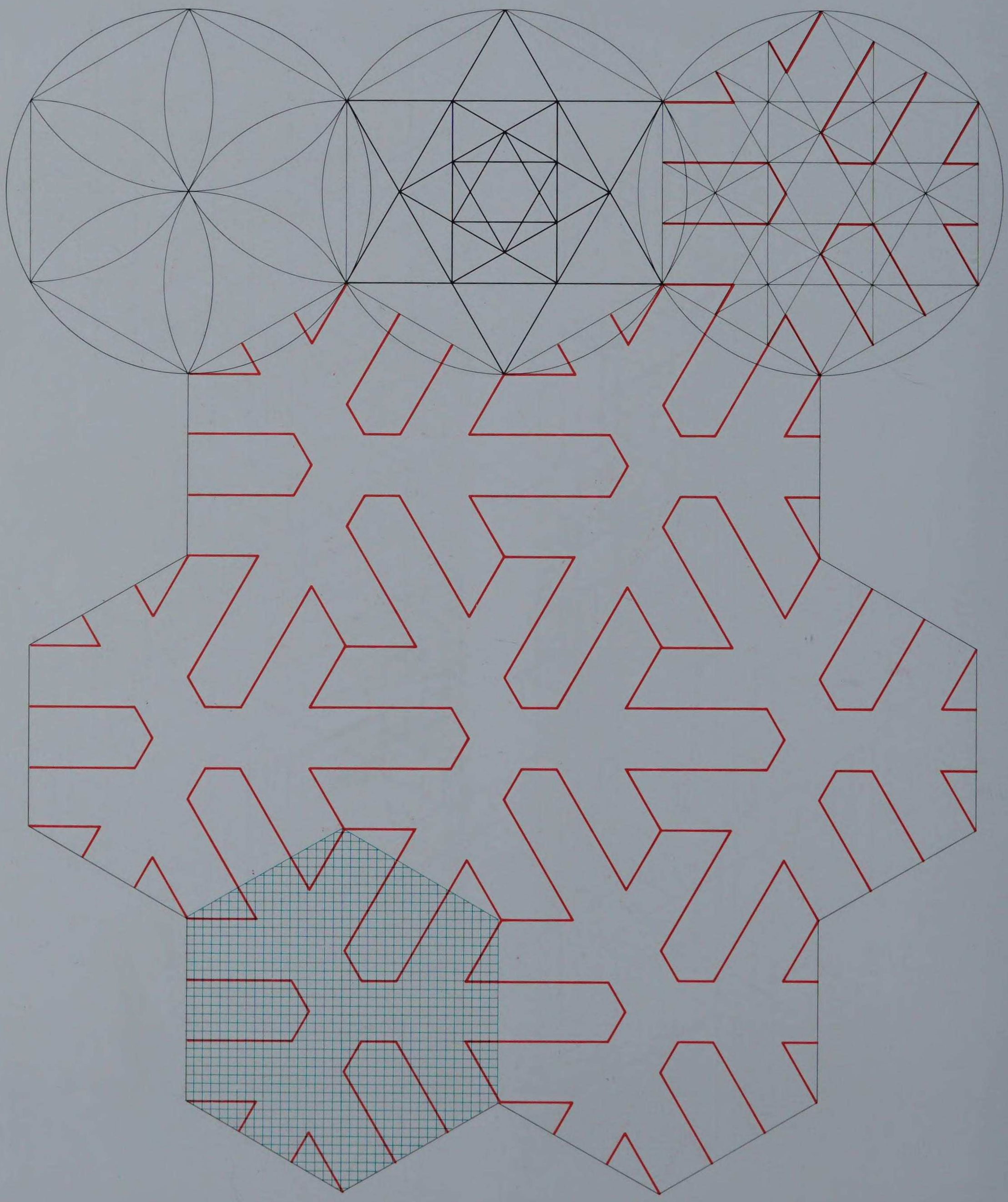
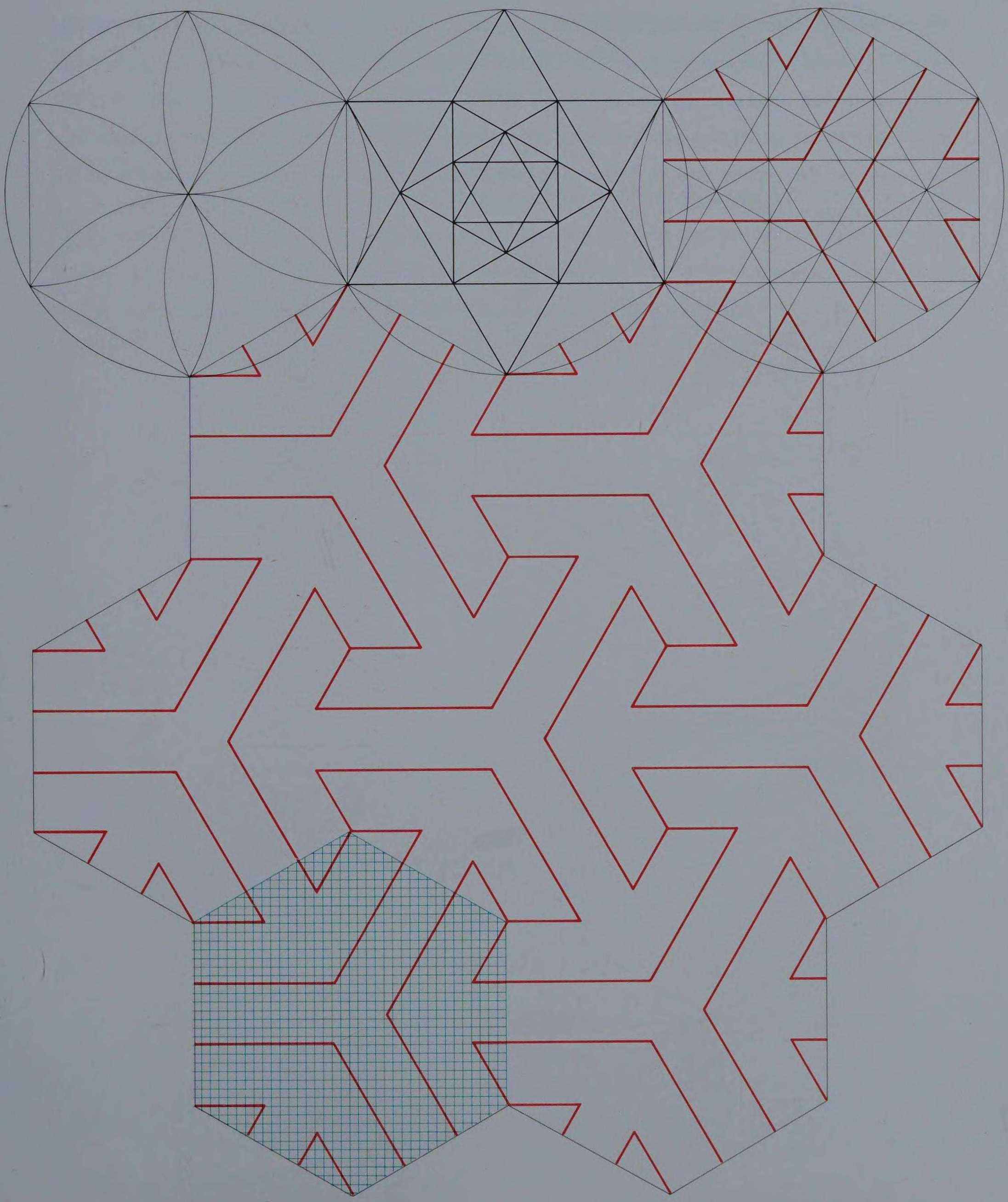


Figure 82. This example has the same combination of grid-patterns as the previous one and the elements of the composition are also similar, giving a subtle variation in the final pattern.
Side of the hexagonal repeat unit = 21 units.
Apothem = 18 units.

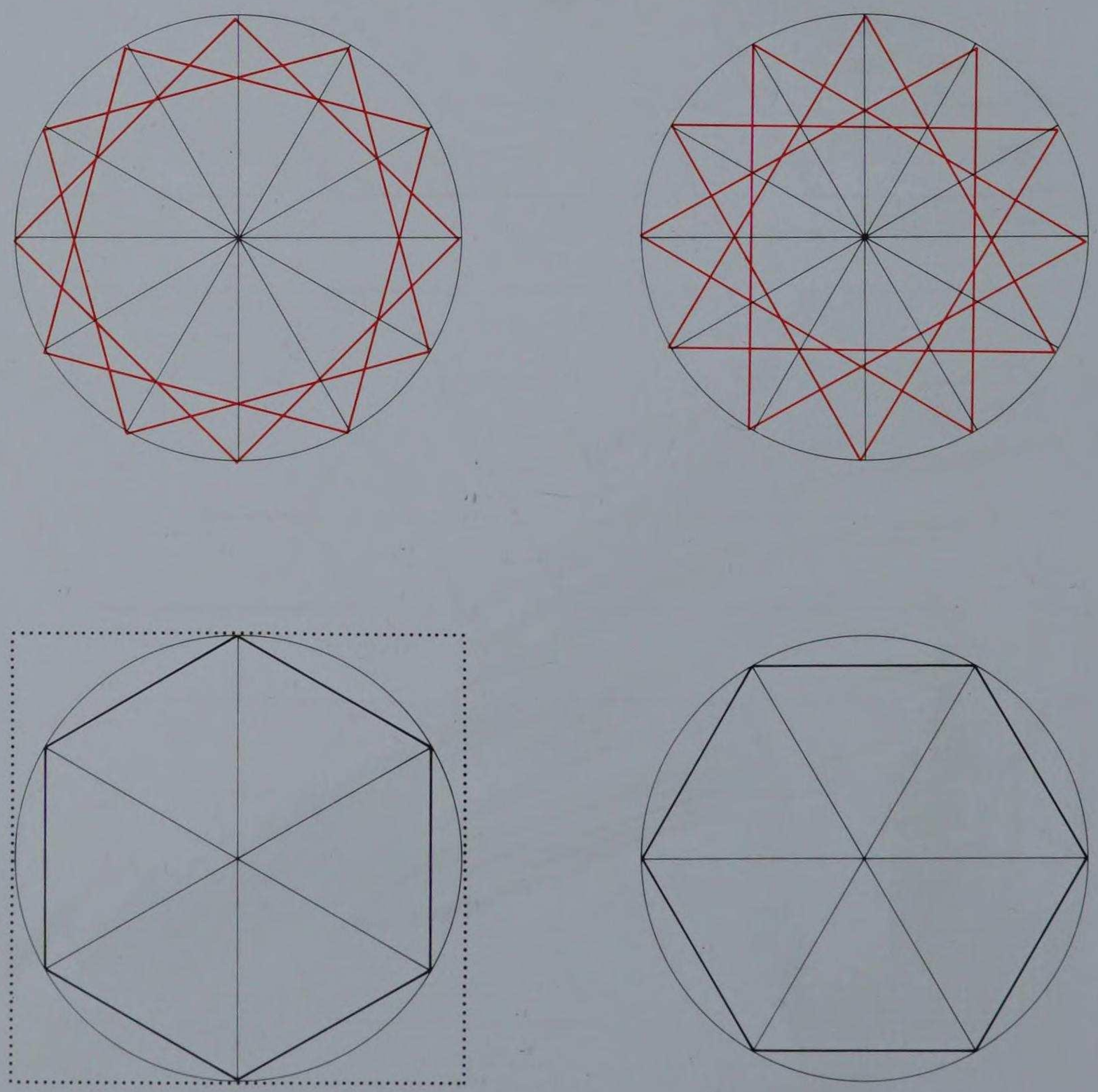


Geometric Patterns based on the Double Hexagon System of Proportion

The grid-patterns that underlie the double hexagon system of proportion

The patterns based on the double hexagon (two equal superimposed hexagons with their axes shifted through 30°) are generated by dividing the circumference of the circle into 12 equal parts, thus combining the root two and root three systems of proportion and utilizing the 15° angle. The resultant repeat units are composed of equilateral triangles, squares, hexagons, octagons and dodecagons and their stars (**Figure 83**), thus creating characteristically rich visual harmonies. The grid-patterns are generated by describing either three squares or four equilateral triangles in a circle, while the repeat unit is defined by either one of the two inscribed hexagons or the circumscribing square.

Figure 83



Graphical analyses of geometric patterns based on the double hexagon system of proportion

The format of the presentation of the following examples of geometric patterns is similar to that for patterns based on the hexagon. However, for the purpose of clarity, the reconstruction of the repeat unit using numerical ratios is illustrated separately for each example.

In the reconstructions, the component geometric figures within the repeat unit, i.e. equilateral triangles, squares, hexagons, and their combinations, are reduced to right-angled triangles containing 15°, 30°, and 45°, thus making it possible to access the relative dimensions of the right angled triangles in terms of the following tangents:

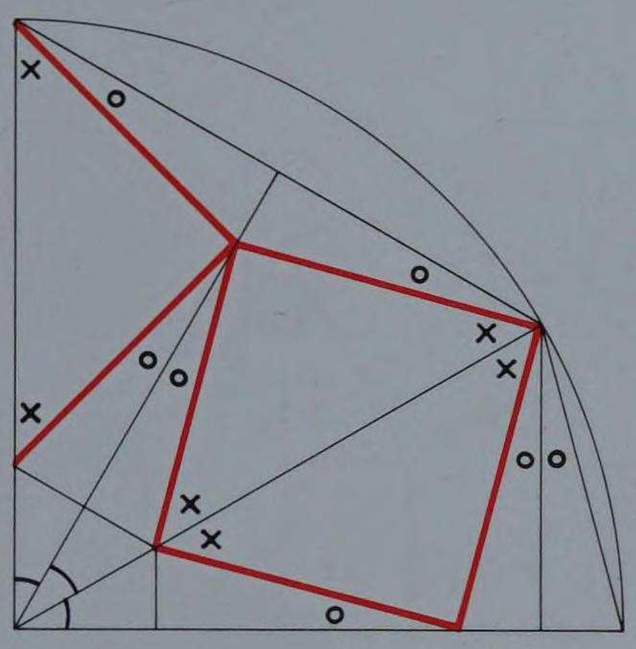
tan 30° = $\frac{1}{\sqrt{3}} = \frac{a}{c}, \frac{c}{3a}$, and their combination;

tan 15° = $\frac{1}{2+\sqrt{3}} = \frac{a}{2a+c}, \frac{c}{2c+3a}$, and their combination.

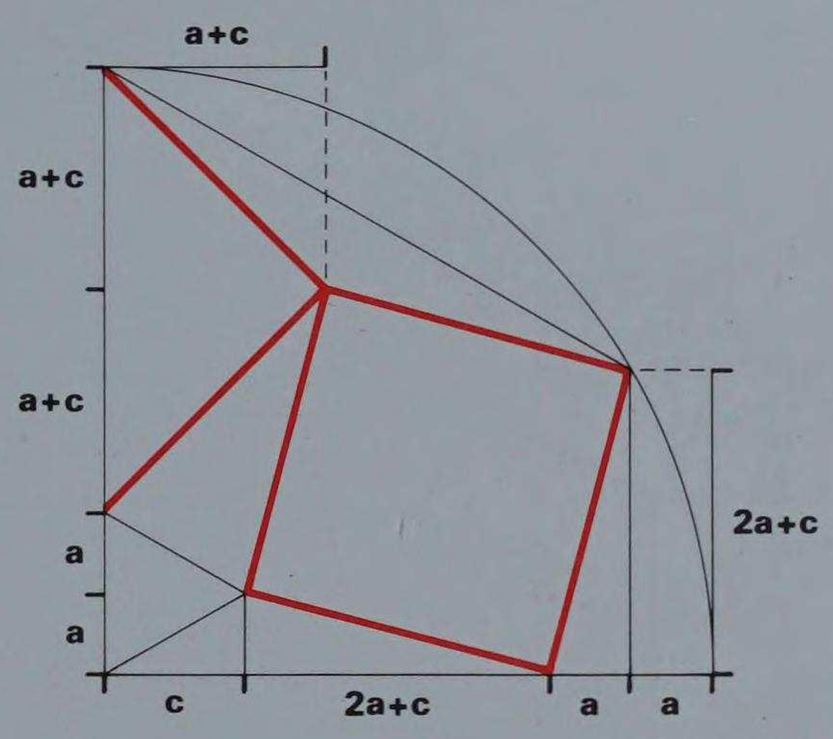
The inverse values of the above give tan 60° and 75° respectively. When the numerical values of $a = 2$ units and $c = 3.5$ units are applied on a square grid the reconstruction of the repeat units can be achieved. This makes it possible to measure all the relevant dimensions of the composition for each repeat unit. It must be noted that here any value for $\frac{a}{c}$ ($\frac{1}{\sqrt{3}}$) can be used (see Table V).

Figure 84i. The grid-pattern is generated by three squares with their apices on the 12 equal divisions of the circumference. This gives a 12-sided polygon. A 12-pointed star nesting within it is constructed by joining every fifth point from each corner of the polygon.

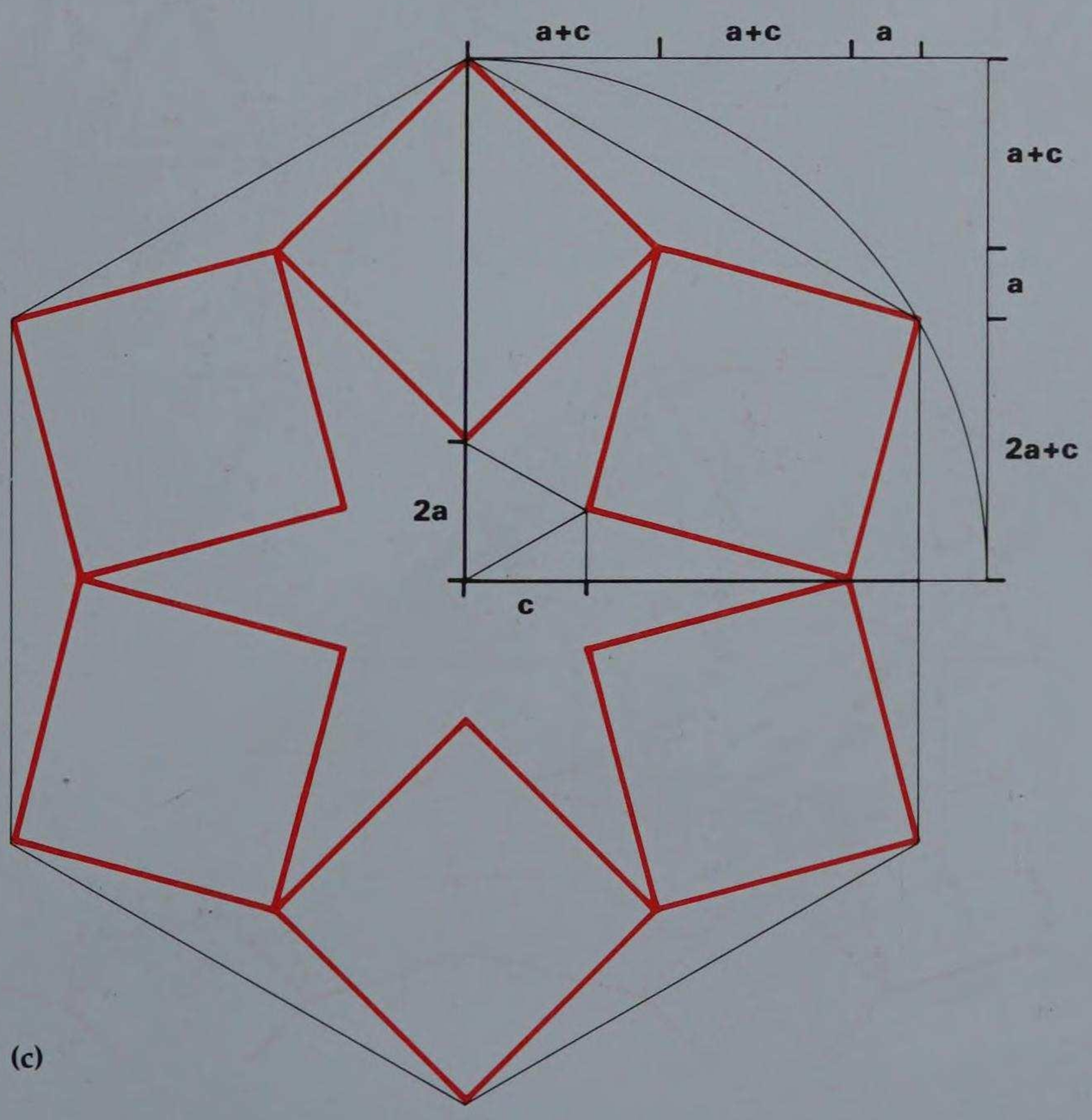
Close examination of the quadrant composition within the hexagonal repeat unit (**Figure 84ii**) reveals that it consists of right-angled triangles with 15°, 30°, and 45° angles as indicated by a circle, an arc and a cross respectively (a). By using the following established values: $\tan 15° = \frac{a}{2a+c}$ and $\tan 30° = \frac{a}{c}$, all the linear dimensions of the composition can be determined (b). When the numerical values of $a = 2$ units and $c = 3.5$ units are applied the dimensions can be translated on a square grid (c), thus establishing the following dimensions: radius of the repeat unit = $4a + 2c = 15$ units and the apothem = $3a + 2c = 13$ units. Half the diagonal of the squares within the composition equals $a + c = 5.5$ units, thus making the diagonals 11 units.



(a)



(b)



(c)

Figure 85i. The apices of the double hexagons define the 12 equal divisions of the circumference of the circle. From the mid-points of their sides, three squares generate the grid-pattern which gives a 12-sided polygon, within which a hexagon is constructed.

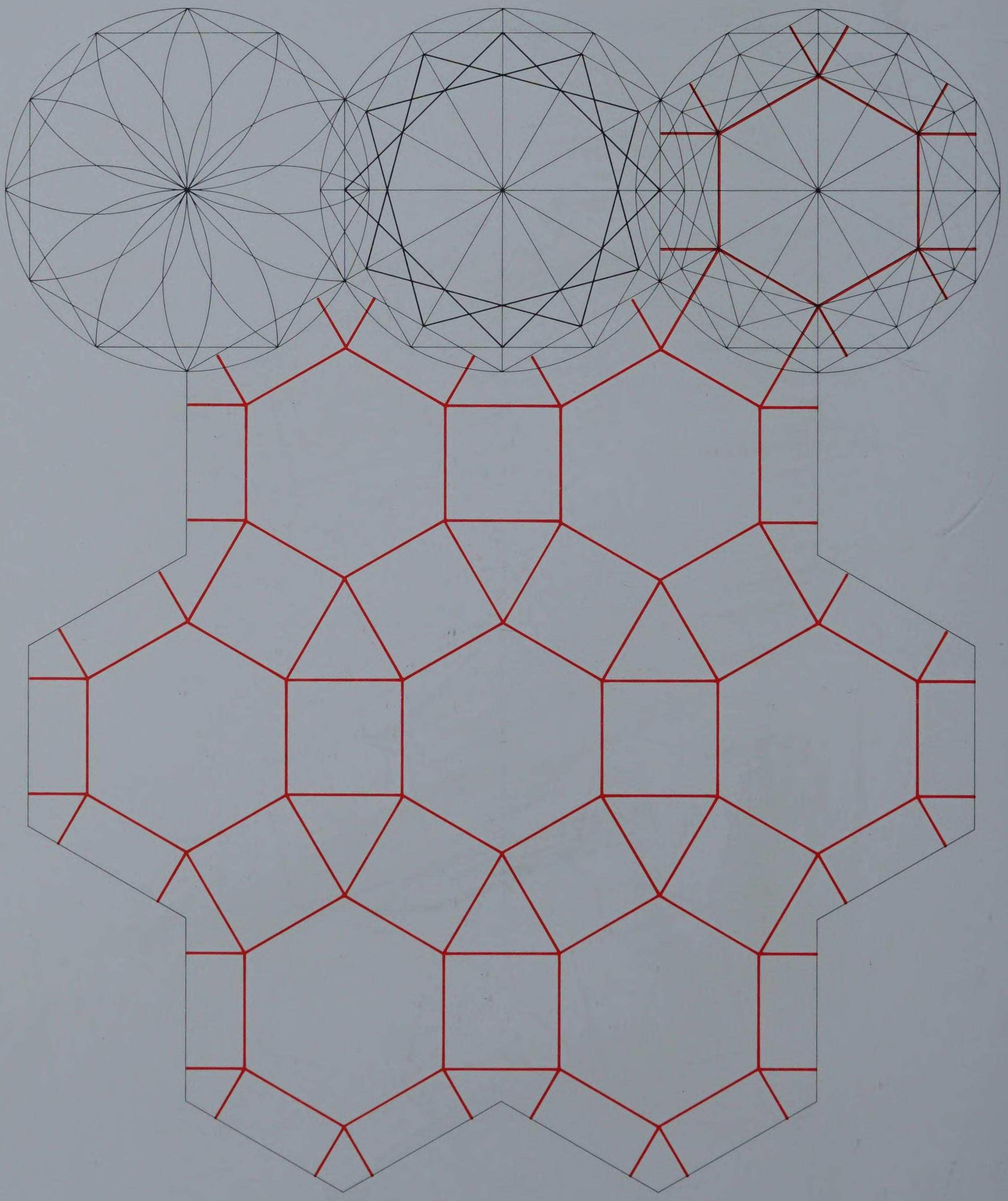
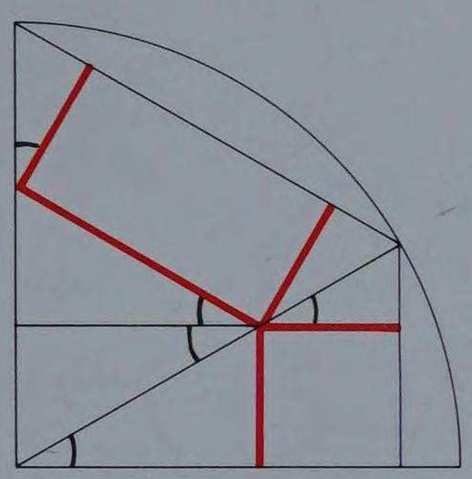
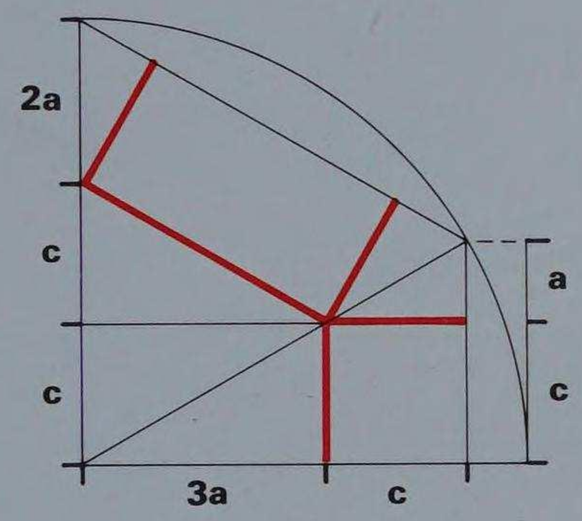


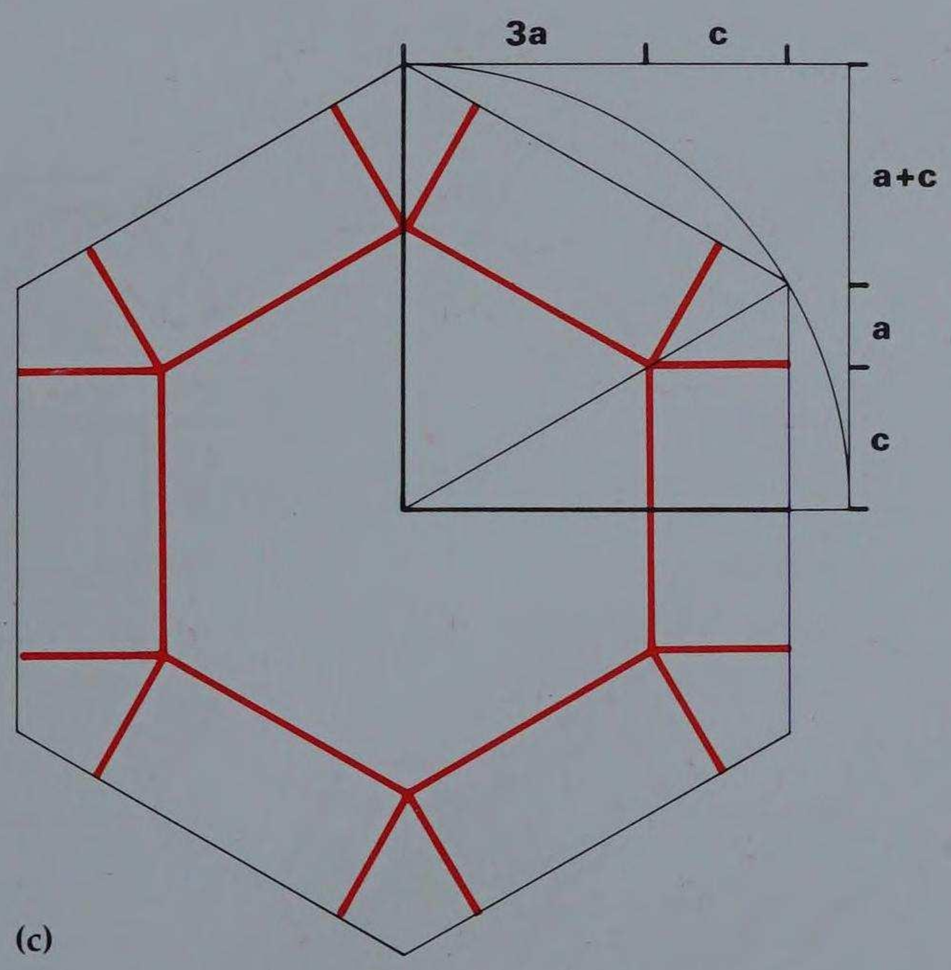
Figure 85ii. The quadrant composition (a) shows that the apothem of the hexagonal repeat unit consists of a right-angled triangle and a square. The triangle has a 30° angle which is indicated by an arc. The radius consists of three right-angled triangles with 30° angles, each indicated by an arc. When the values of tan 30°= $\frac{a}{c}$ and $\frac{c}{3a}$ are applied, the linear dimensions can be determined (b). The numerical values of a and c are applied, thereby reconstructing the repeat unit (c).



(a)

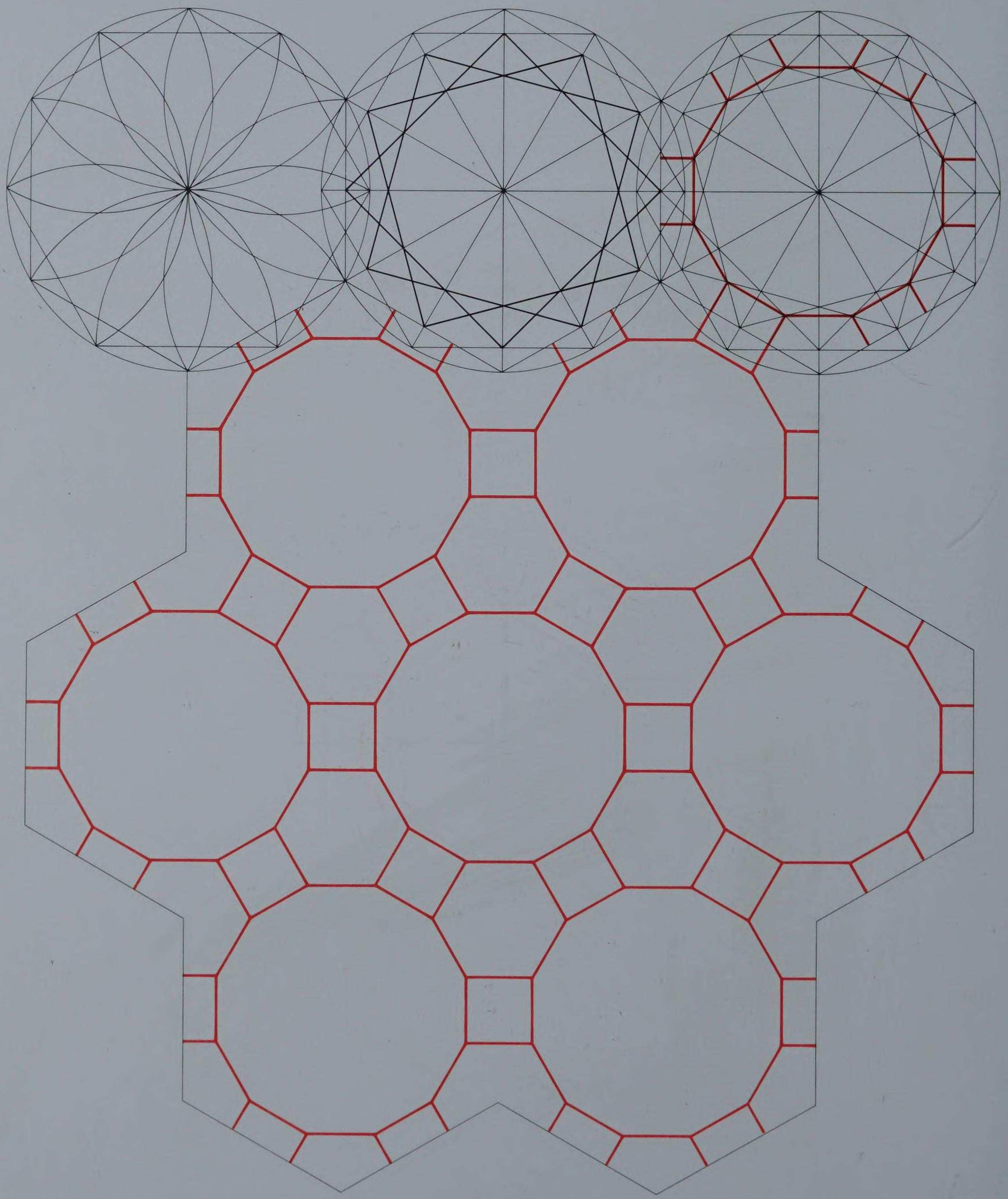


(b)

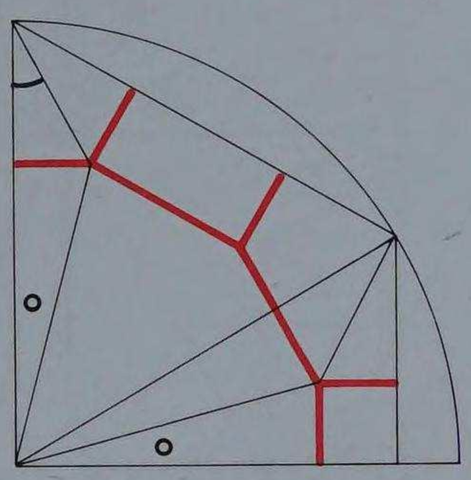


(c)

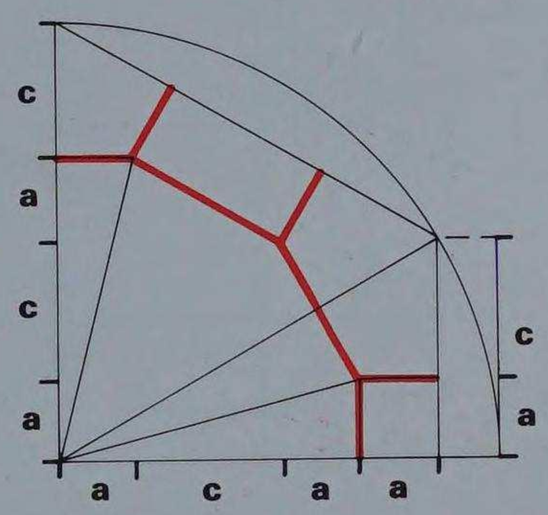
Figure 86i. The grid-pattern for this repeat unit is similar to the previous example. However, the composition has as its central element a 12-sided polygon.



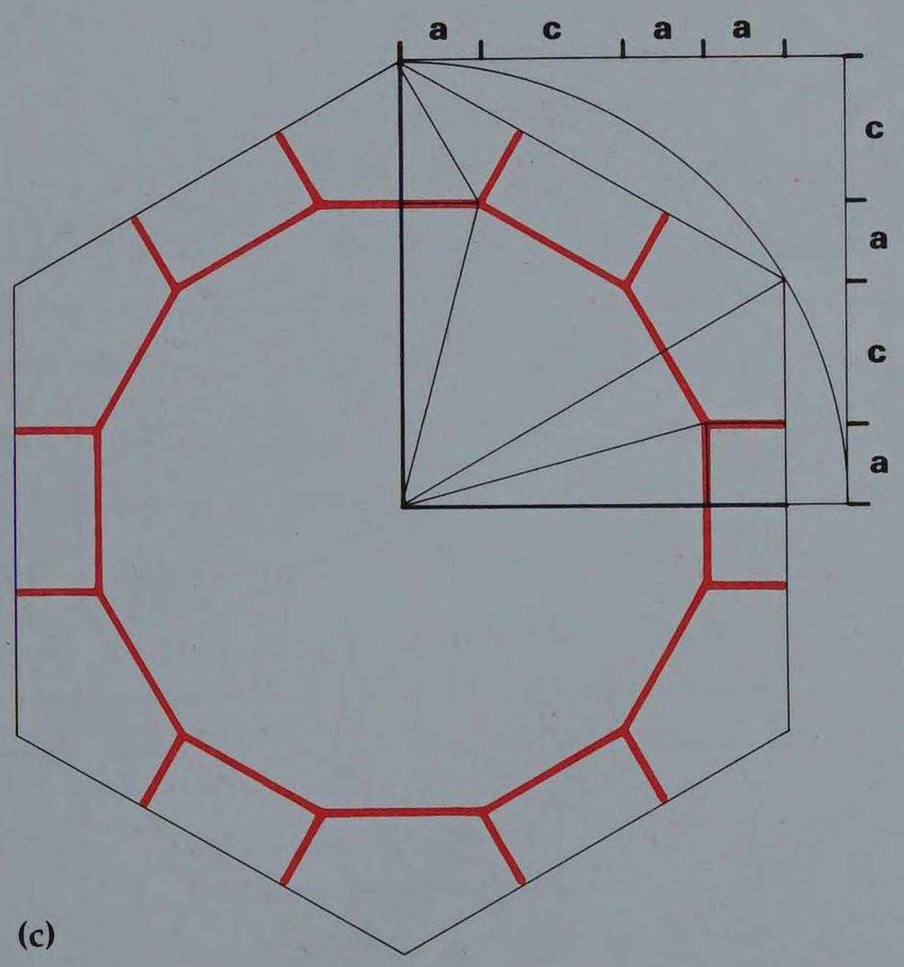
The quadrant composition (**Figure 86ii**) shows that the radius of the hexagonal repeat unit consists of two right-angled triangles with 15° and 30° angles indicated by a circle and an arc respectively, while the apothem consists of a right-angled triangle and a square, the triangle having a 15° angle (a). The value of tan 15° = $\frac{a}{2a+c}$ and tan 30° = $\frac{a}{c}$ are indicated, thus establishing the relevant dimensions in terms of *a* and *c* (b). The numerical values for *a* and *c* are applied on a square grid to reconstruct the composition (c).



(a)

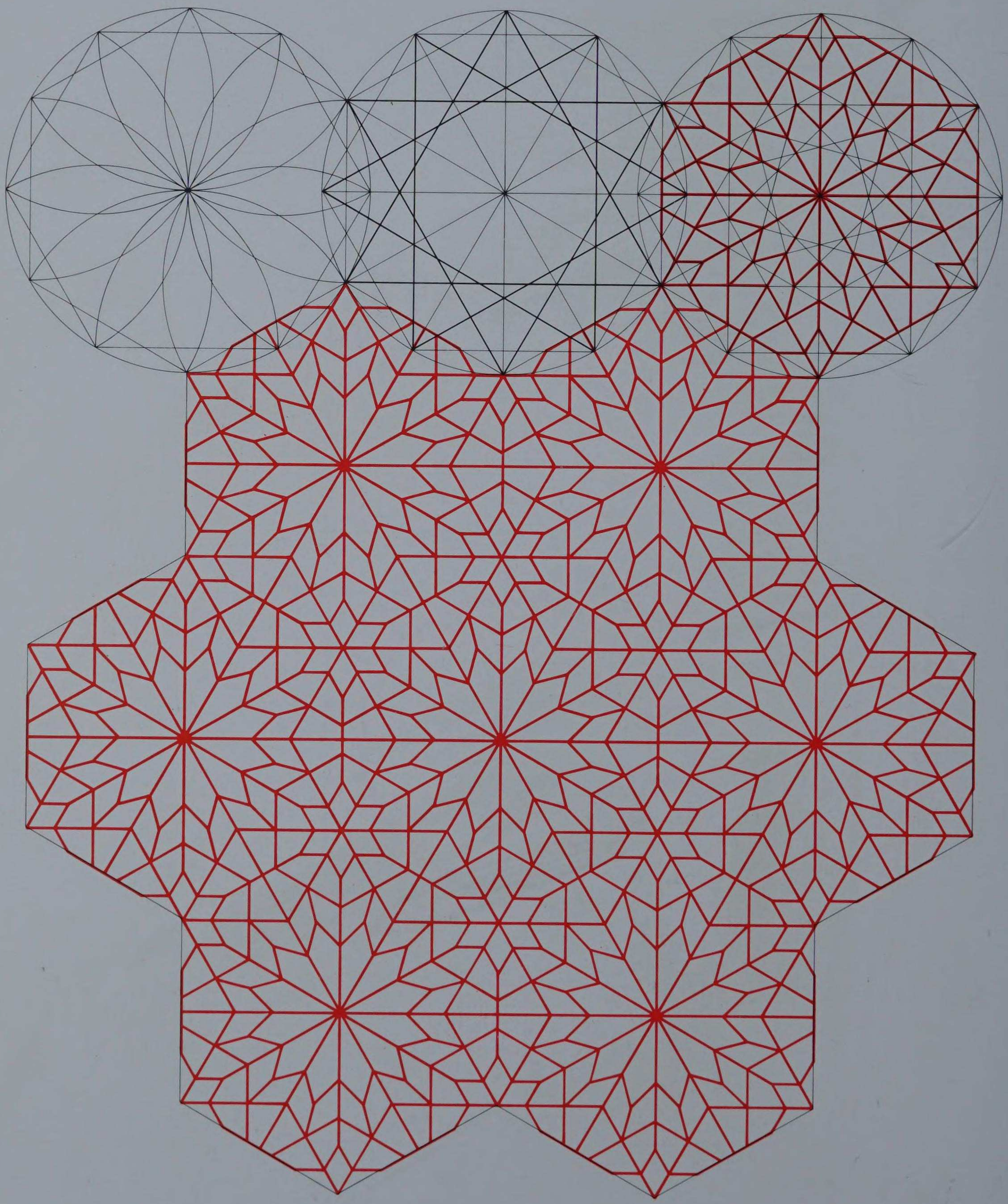


(b)



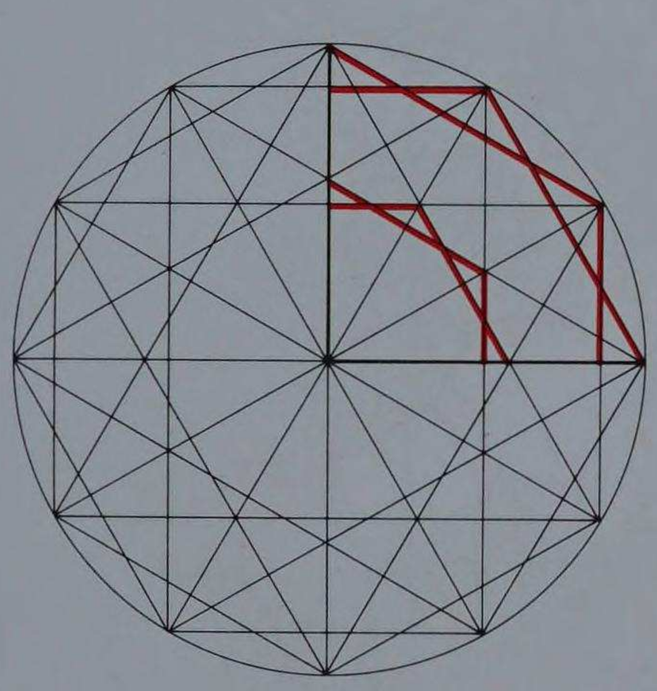
(c)

Figure 87i. In this example the grid-pattern is generated by four equilateral triangles with their apices lying on the 12 points of division of the circumference of the circle. This generates a 12-sided polygon in which a 12-pointed star is constructed by joining every fourth point of the polygon.

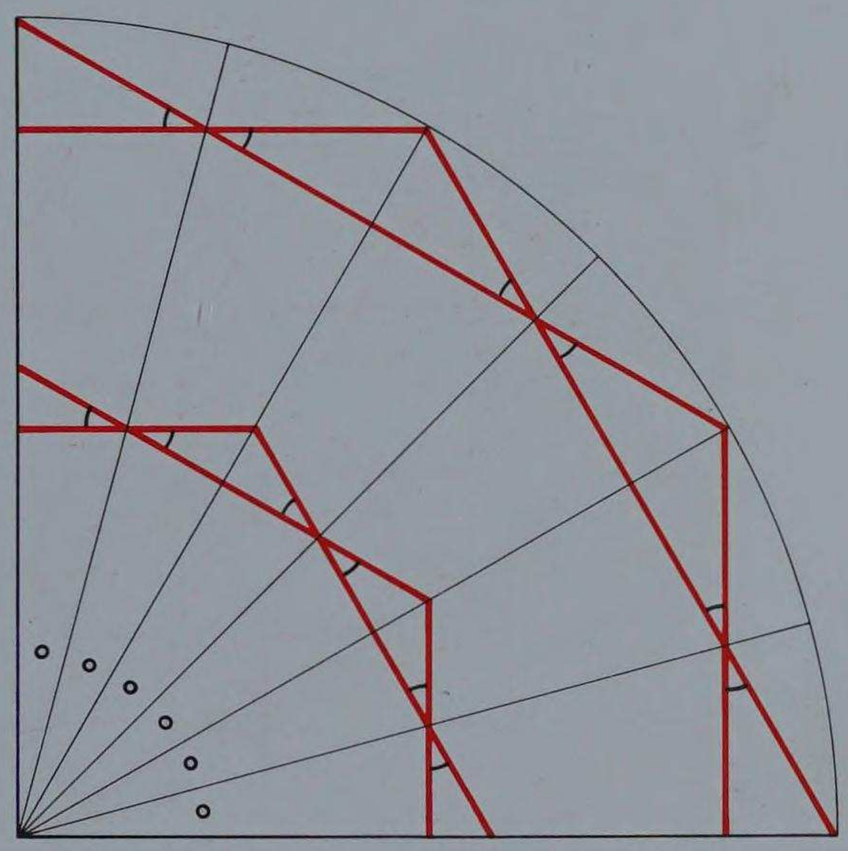


THE DOUBLE HEXAGON

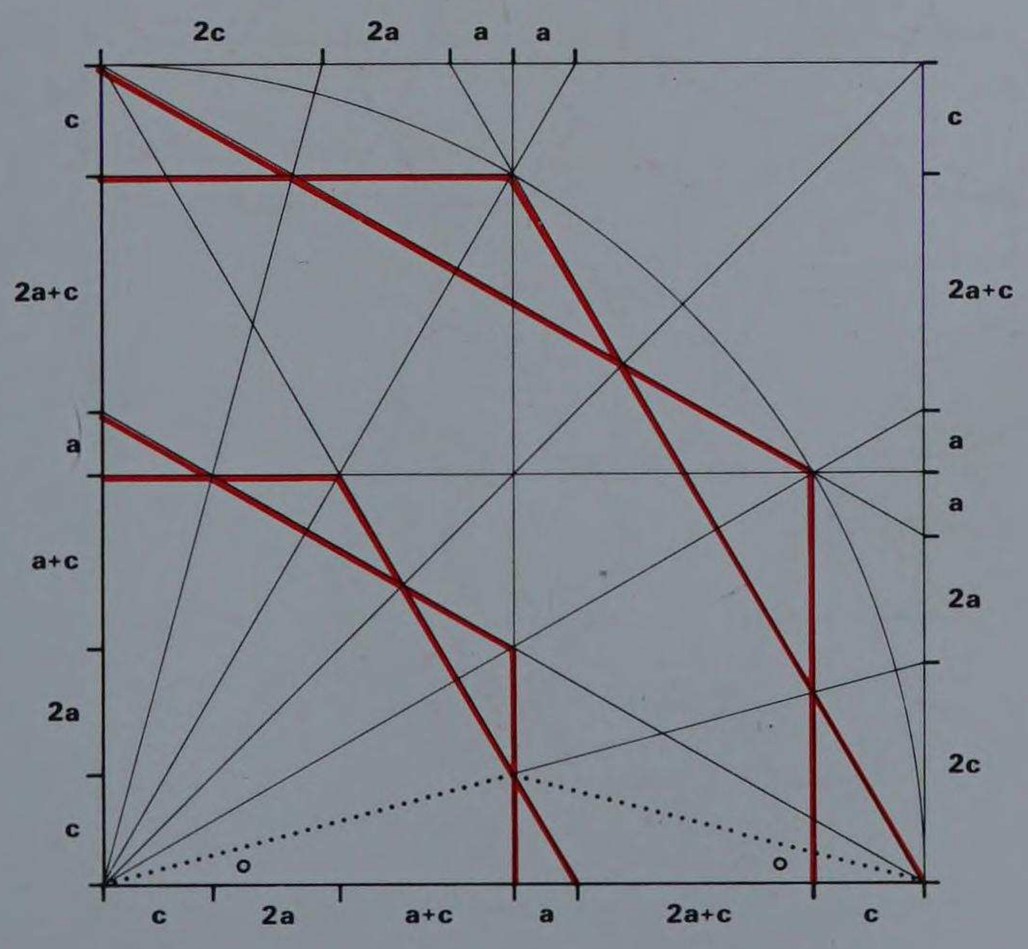
This example (**Figure 87ii**) makes apparent the significance of the double hexagon. When the grid-pattern is formed by triangulating the 12 equal divisions of the circumference, two double hexagons are formed (a). The quadrant shows that joining the points of intersection of the sides of the double hexagons to the centre of the circle divides the quadrant circle into six equal angles of 15°, indicated by circles, while the areas bounded by the intersecting sides of the hexagons are divided into right-angled triangles with 30° angles indicated by arcs (b). By applying the value of $\tan 15° = \frac{c}{2c+3a} = \frac{3.5}{13}$ units, it becomes possible to translate all linear dimensions of the inner and outer double hexagons onto a square grid (c). In this case, the two long adjacent sides of the 15° right-angled triangles, indicated by dotted lines at the base of the quadrant, define the value of the radius as 26 units. By joining further points of intersection and selecting certain sections of the grid-pattern as indicated, the composition can be finally constructed. The apothem of the hexagonal repeat unit equals the radius less *c* (26 - 3.5 = 22.5 units). All other major points of intersection can be simply calculated by counting (d).



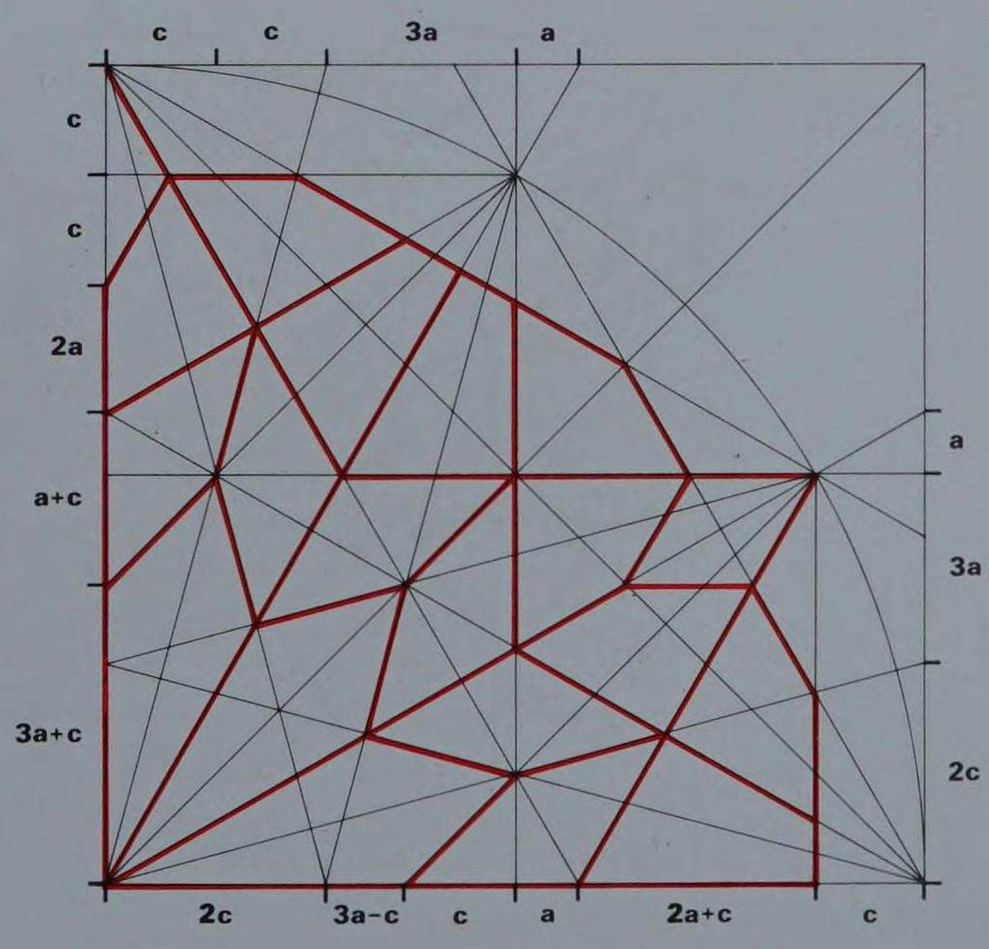
(a)



(b)

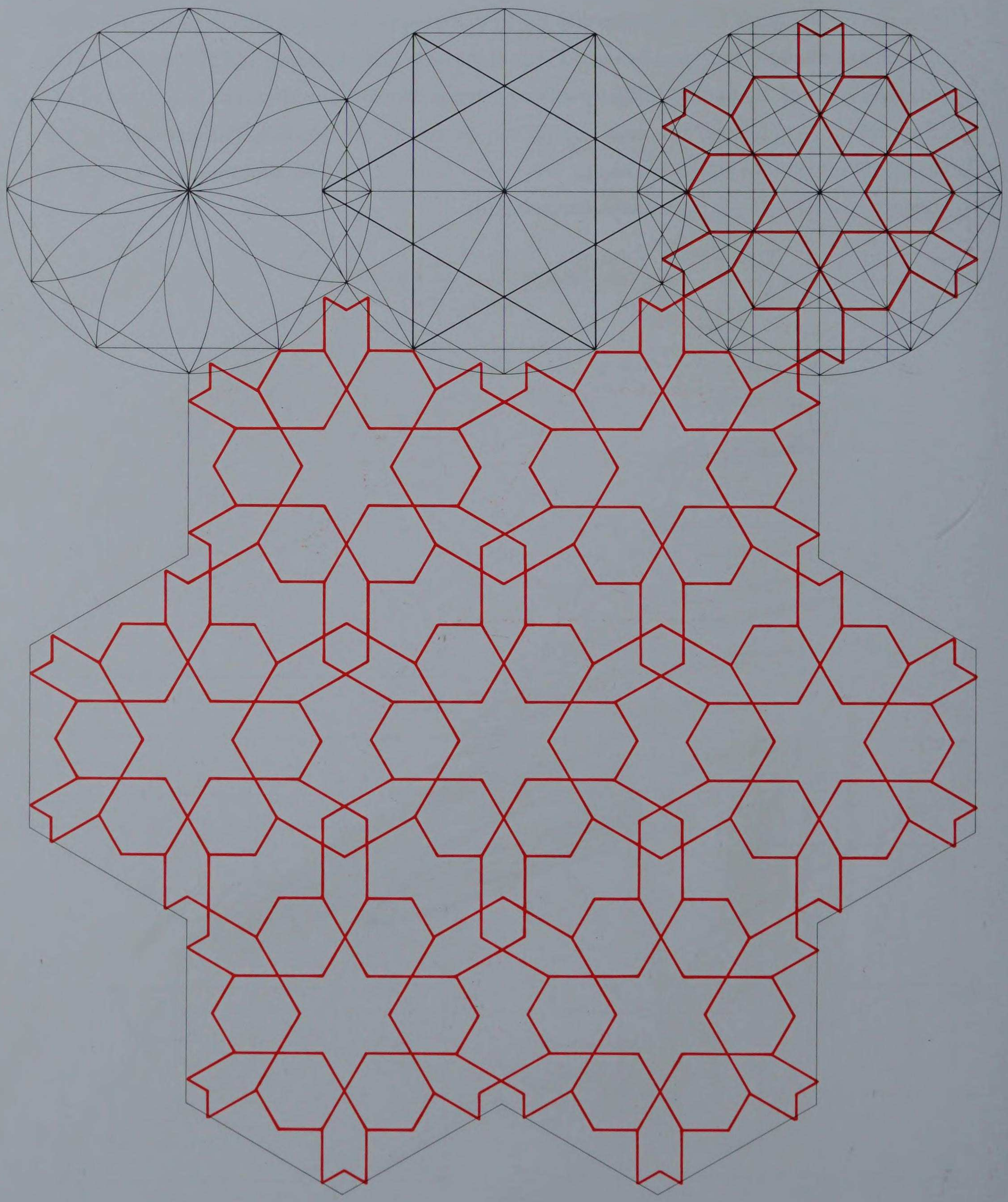


(c)



(d)

Figure 88i. The resultant grid-pattern for this repeat unit is generated by inscribing primary grid C in one hexagon, while the other hexagon defines the repeat unit. The intersection points of primary grid C and the second hexagon are joined to give a central hexagon in which a six-pointed star is inscribed.



Here, (**Figure 88ii**), the constituent sections making up the lengths of the radius and the apothem can be reduced to right-angled triangles with 30° angles, indicated by arcs. This process facilitates the definition of $\tan 30° = \frac{1}{\sqrt{3}}$ as $\frac{a}{c}$ or $\frac{c}{3a}$.

By applying the values $a = 2$ units and $c = 3.5$ units, the major points of the intersecting lines of the grid-pattern can be located, thus determining the composition and its reconstruction on a square grid.

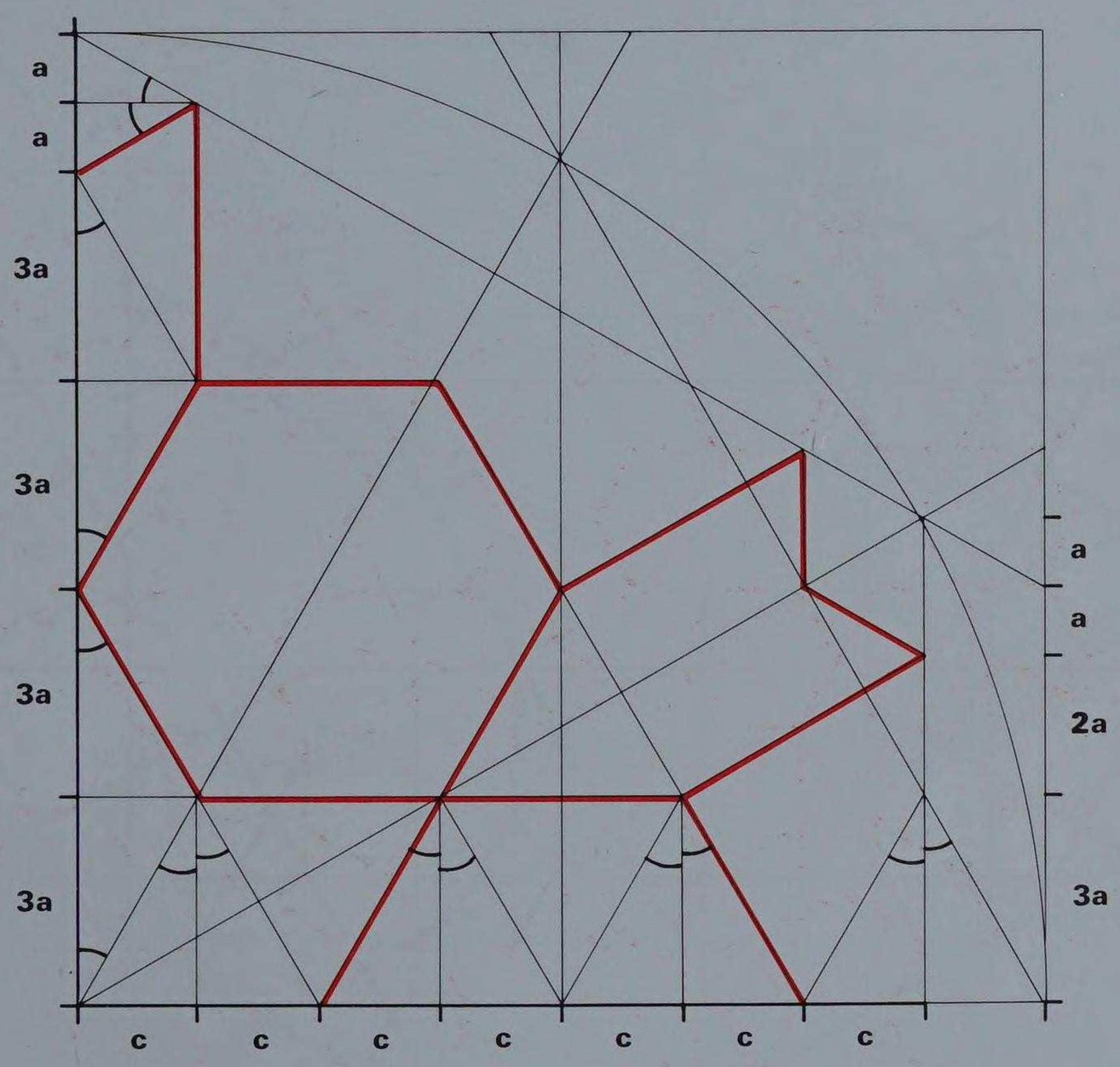
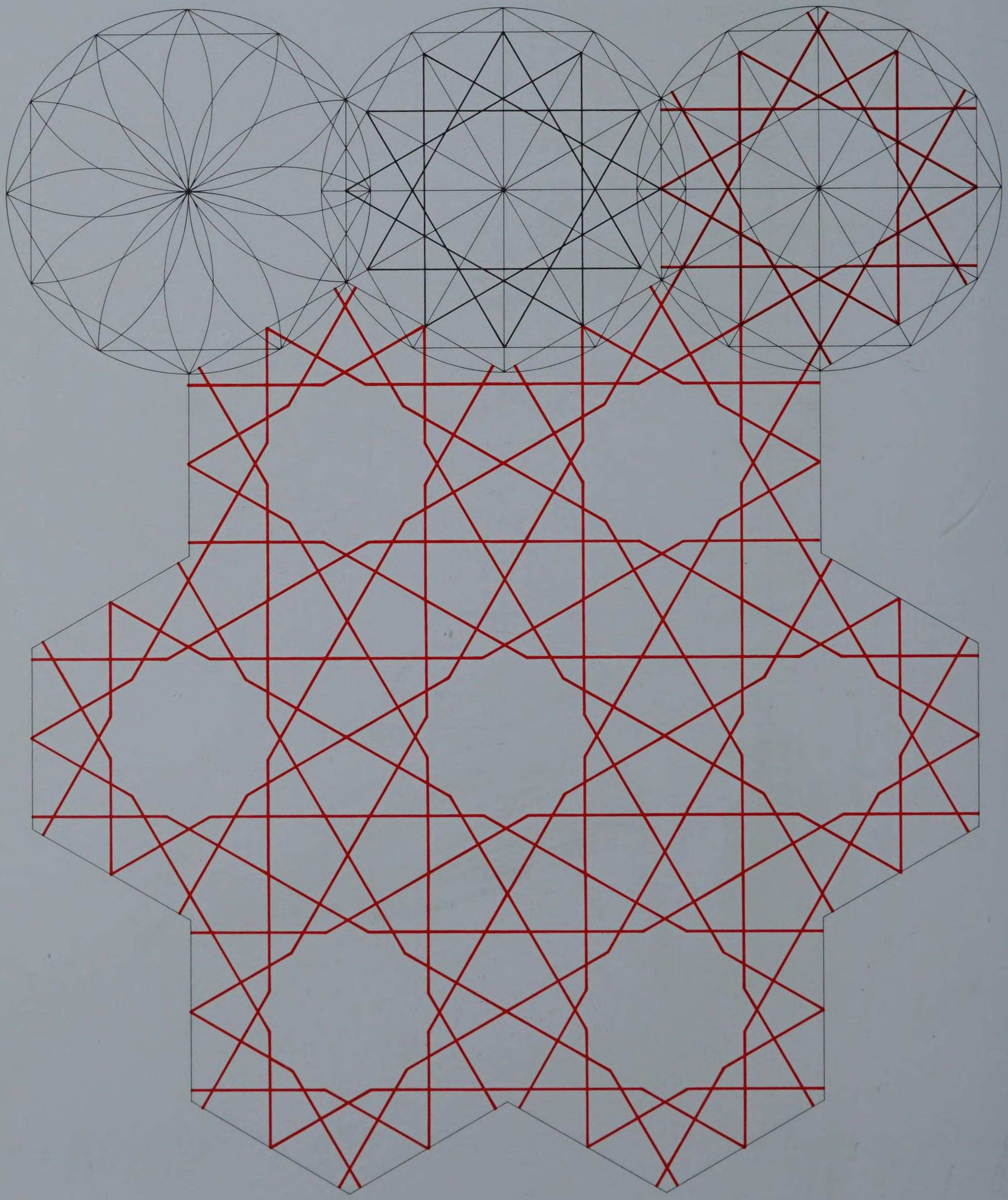
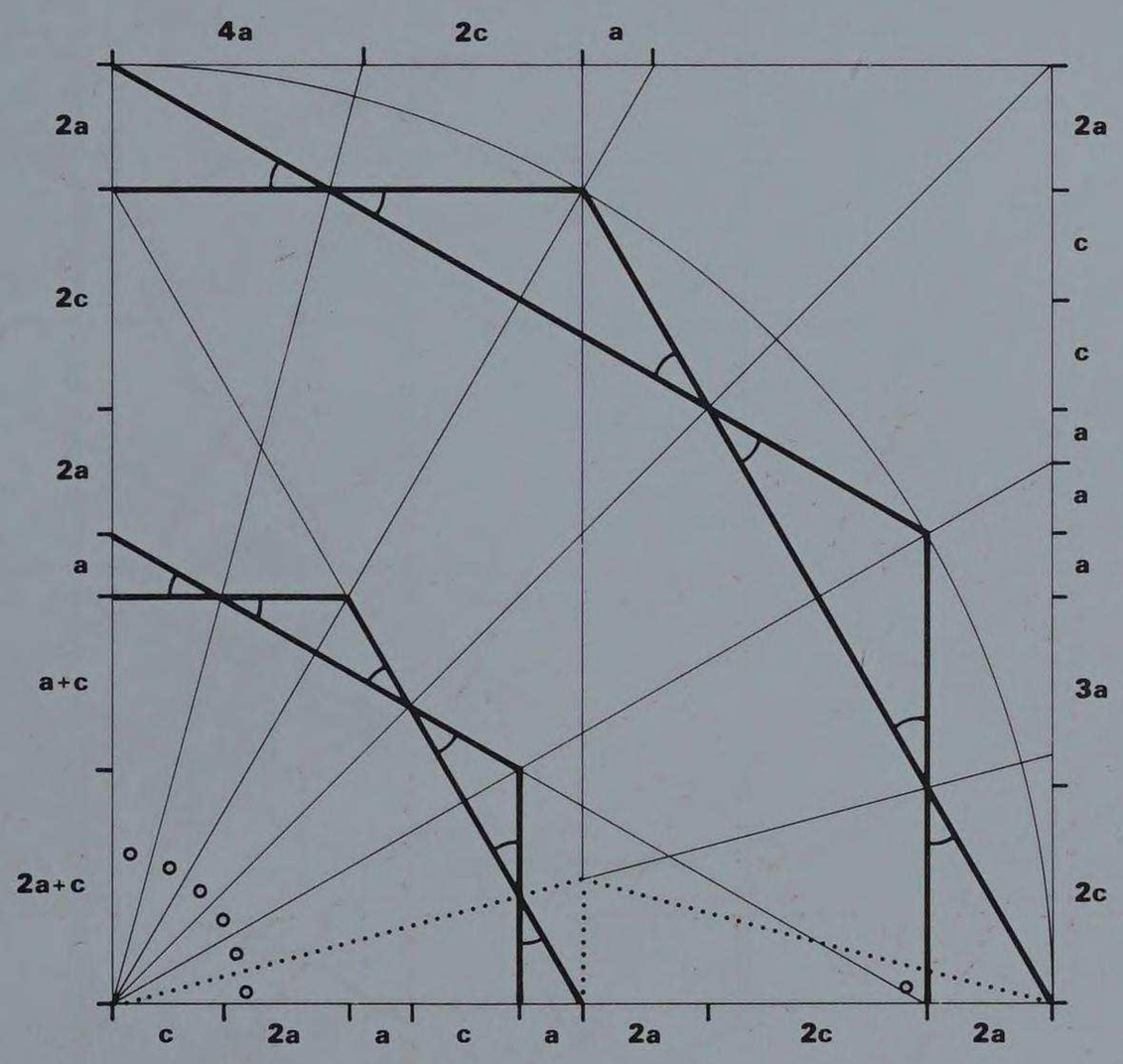


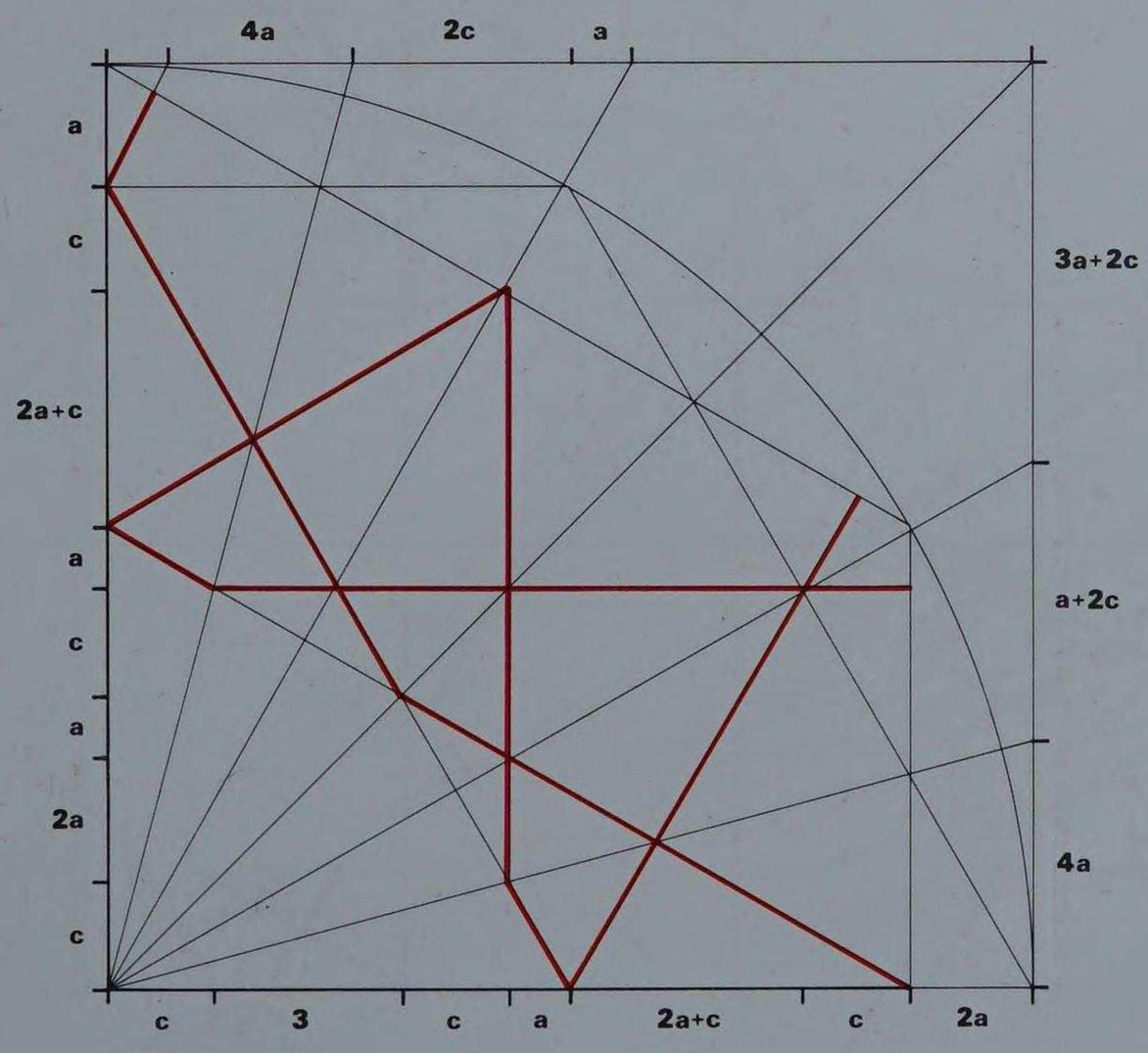
Figure 89i. From the mid-points of the sides of the double hexagons, four equilateral triangles are constructed generating the grid-pattern.



The double hexagon is again used here (**Figure 89ii**) as a means to structure the composition of the repeat unit. It divides one quadrant of the repeat unit into right-angled triangles with 15° and 30° angles indicated by a circle and an arc, respectively (a). Applying the values of tan 30° = $\frac{1}{\sqrt{3}}$ as $\frac{a}{c}$, $\frac{2a}{2c}$, ... and tan 15° = $\frac{1}{2+\sqrt{3}}$, as $\frac{c}{2c+3a}$, $\frac{2a}{2(2a+c)}$..., all the relevant dimensions of the grid pattern can be expressed in terms of a and c (a). When the numerical ratios of tan 30° and tan 15° (a = 2 units and c = 3.5 units) are applied on a square grid, and further points of intersection are joined up, the composition can be reconstructed (b), thus making it possible to express the linear dimensions of the composition.

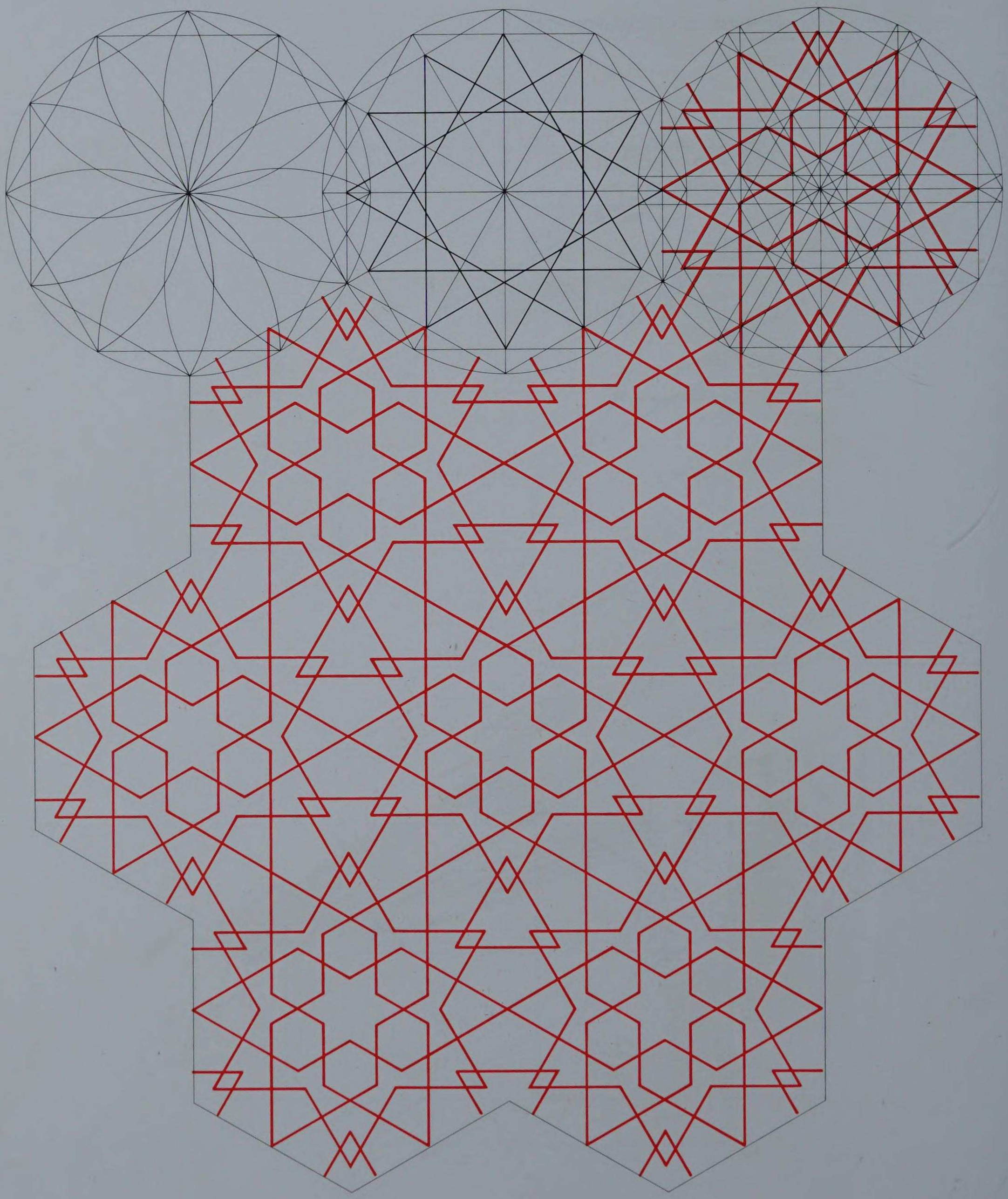


(a)



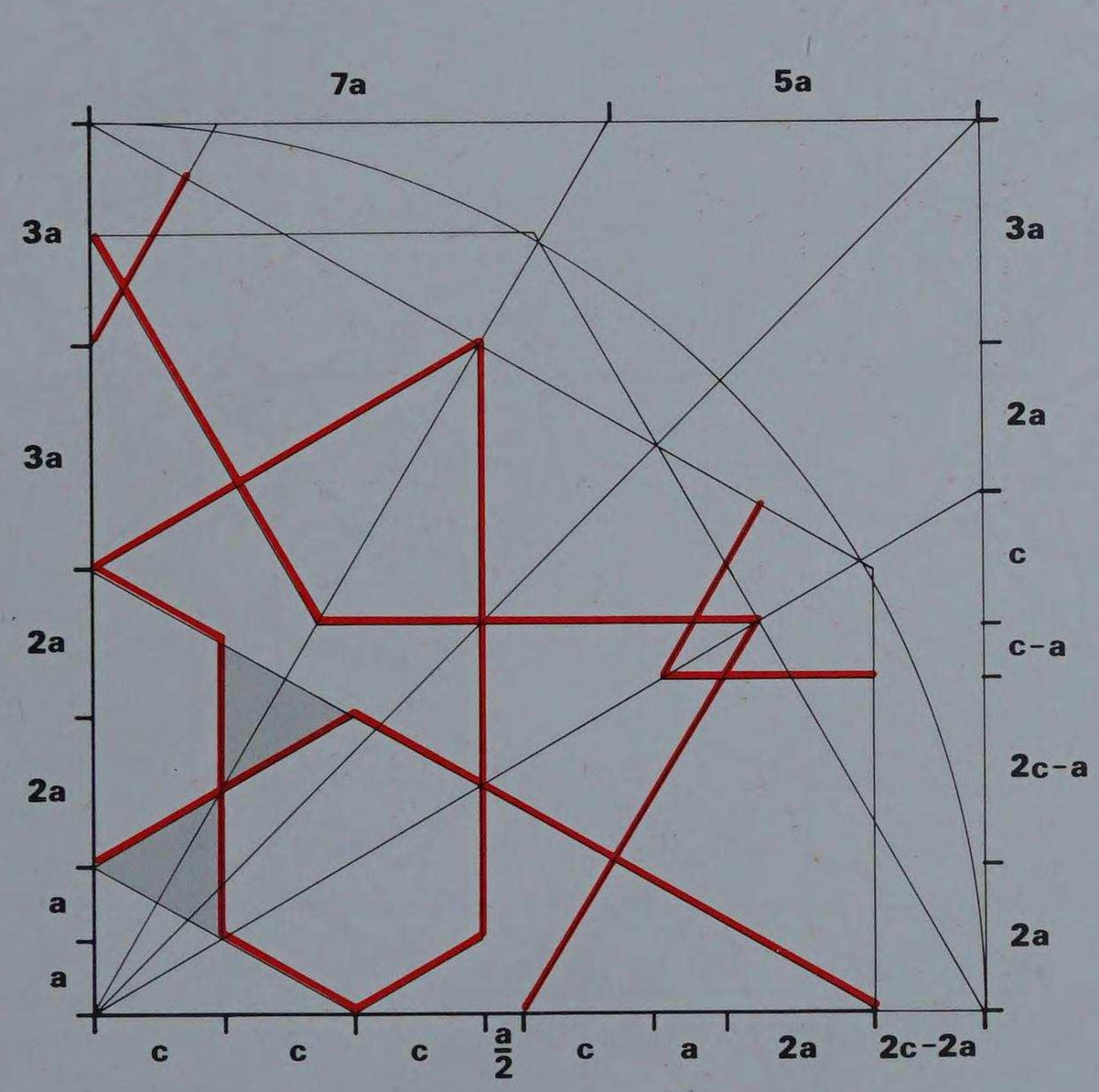
(b)

Figure 90i. This example uses the same grid-pattern as for the previous one, developing it by further point-joining.



THE DOUBLE HEXAGON

Having established the double hexagon as a means for composition, it is possible to construct right-angled triangles with 15° and 30° angles using the numerical ratios that have been derived (**Figure 90ii**). This enables the reconstruction of the composition by, firstly, determining the basic right-angled triangles that form the constituent geometric elements within the composition (i.e. equilateral triangles, squares, hexagons,... and so on, and their combinations) and, secondly, by determining the ratios of each element. In this example the module taken is the equilateral triangle with side $2a$ and height c (shown shaded). The radius, the apothem and the rest of the composition can be measured accordingly in terms of a and c.



Figures 91i, 91ii. For the two examples given here, the grid-pattern is generated by inscribing three squares in the hexagon. This gives a 12-sided polygon in which a 12-pointed star is drawn. In the first case the repeat unit is a hexagon, while in the second it is a square; hence the variation in the resultant patterns.

Figure 91i

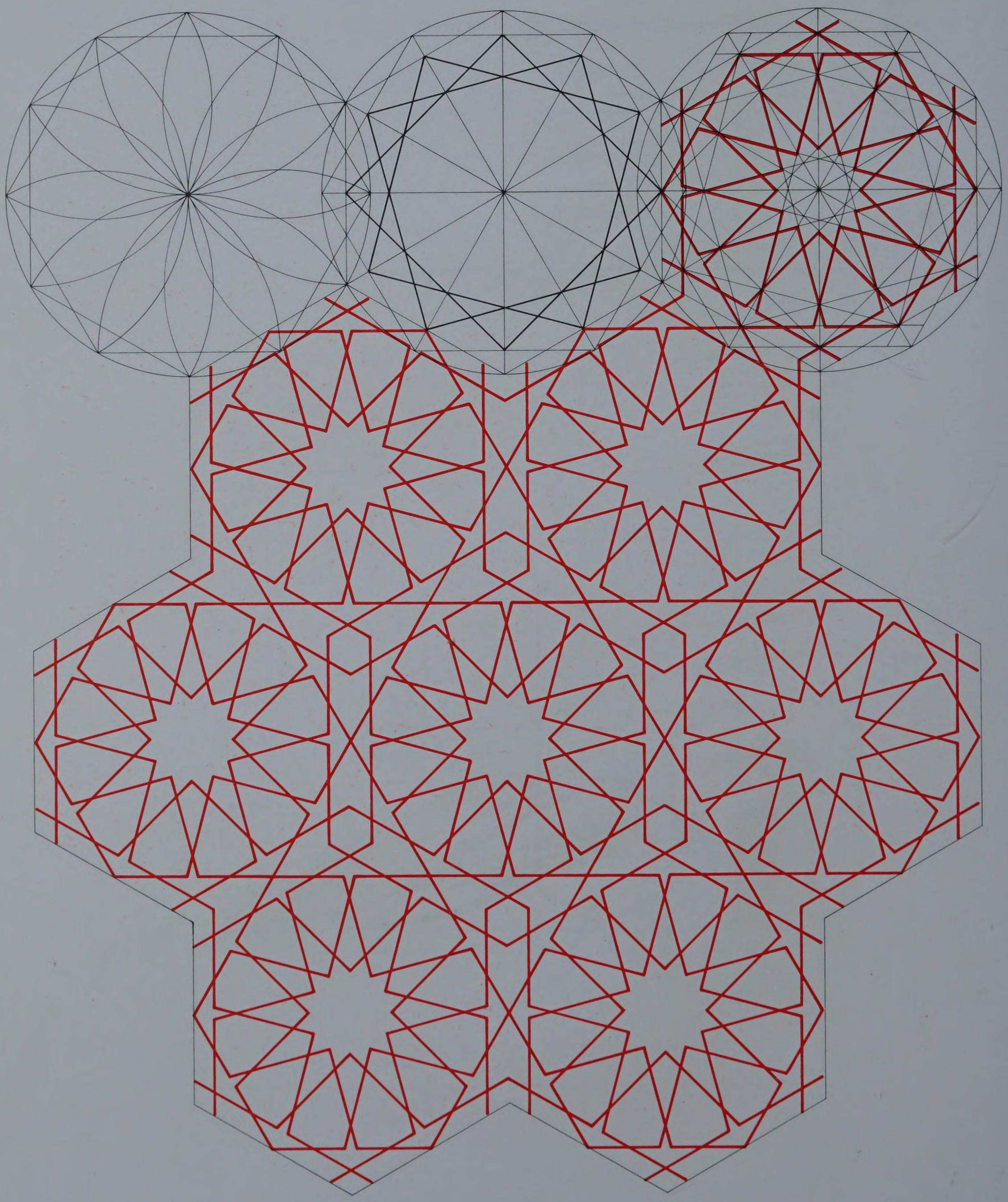
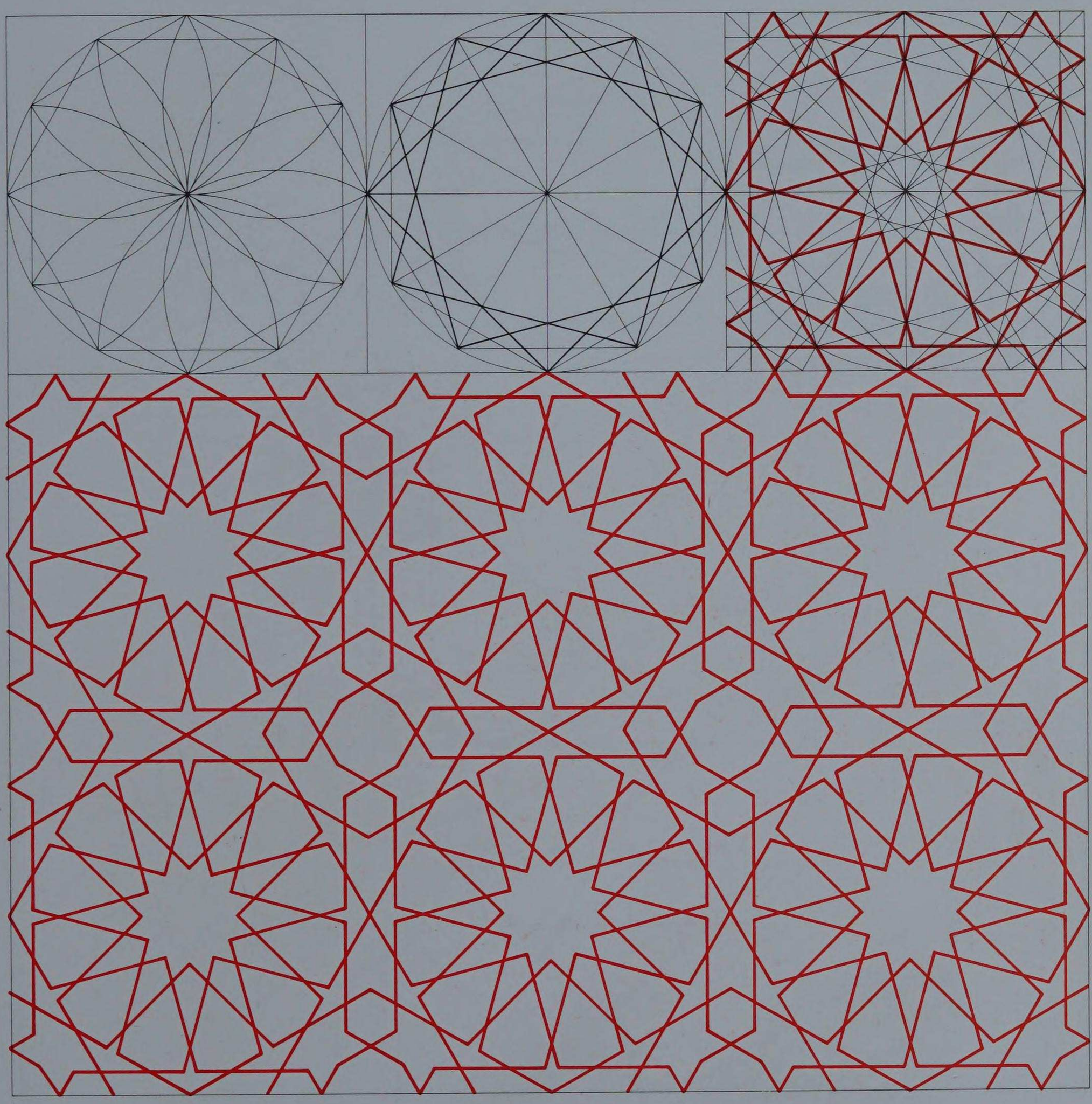
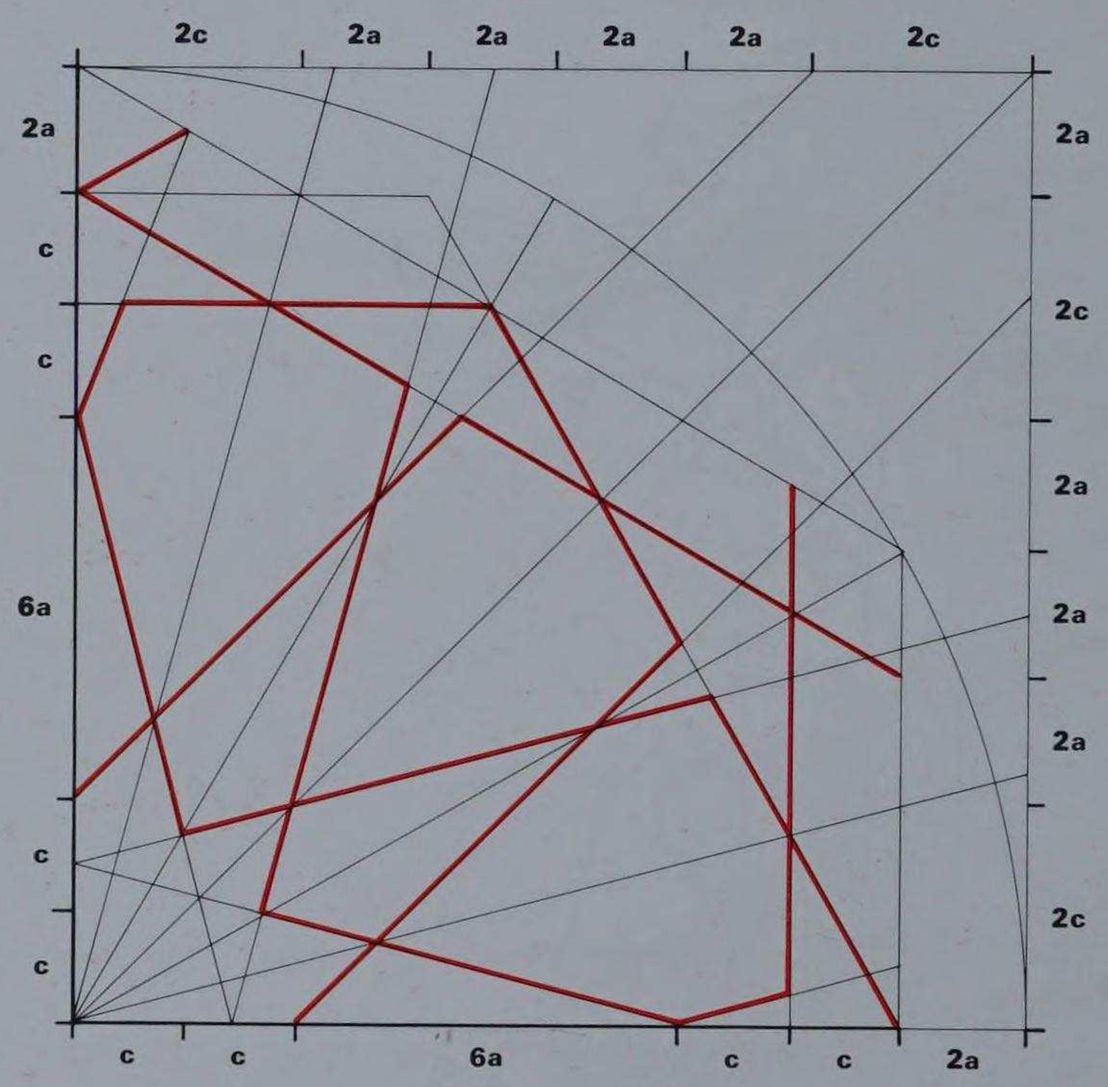


Figure 91ii

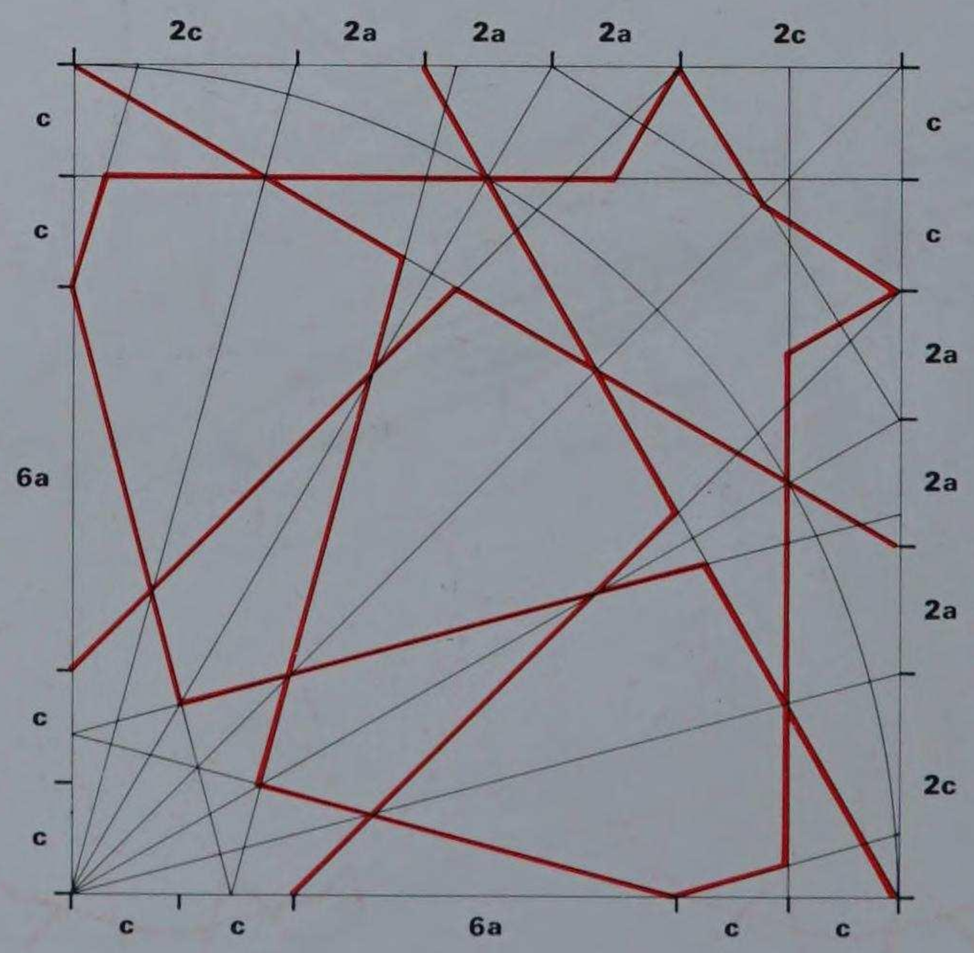


Once the ratios of tan 15° and tan 30° are established (**Figure 91iii**), in this case as $2c / (4c+6a)$ and $(4a+2c) / (4c+6a)$ respectively, the linear dimensions of the radius = $4c + 8a = 30$ units and the apothem as radius less $2a = 30 - 4 = 26$ units can be ascertained. All the relevant points of intersection in the grid-pattern can be located using *a* and *c* as units of measure. In (b), the apothem of the original hexagonal repeat unit in (a) which equals 26 units becomes half the side of the square repeat unit, and all the relevant linear dimensions of the composition can be determined in terms of *a* and *c*.

Figure 91iii



(a)



(b)

Figures 92i,92ii. The grid-pattern for these two examples is generated by four equilateral triangles. In the first case, the apices of the triangles lie on the mid-points of the sides of the double hexagons, one of which becomes the square repeat unit. In the second case, the apices lie on the 12 points of equal division of the circumference of the circle and the repeat unit is the circumscribing square.

Figure 92i

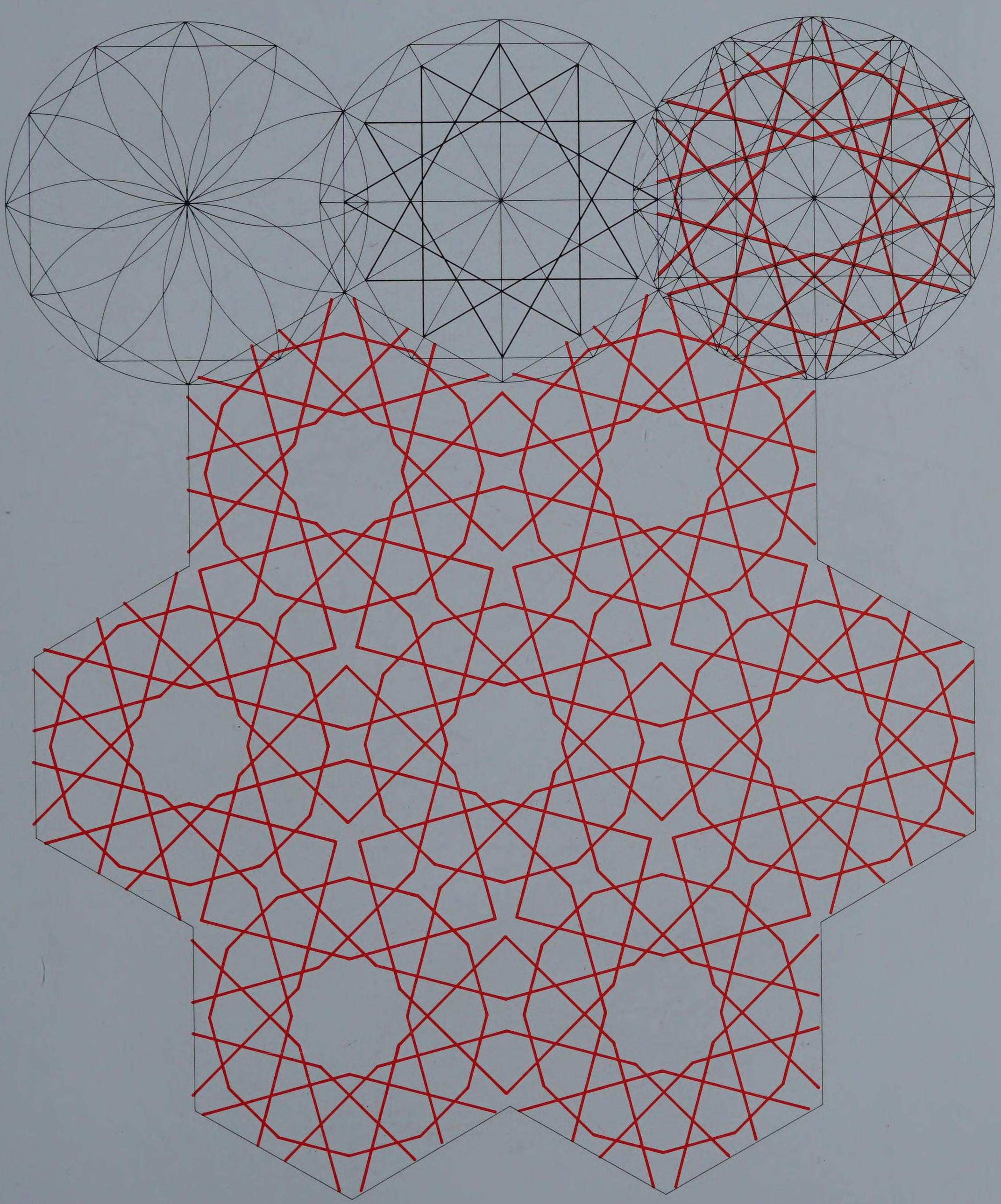
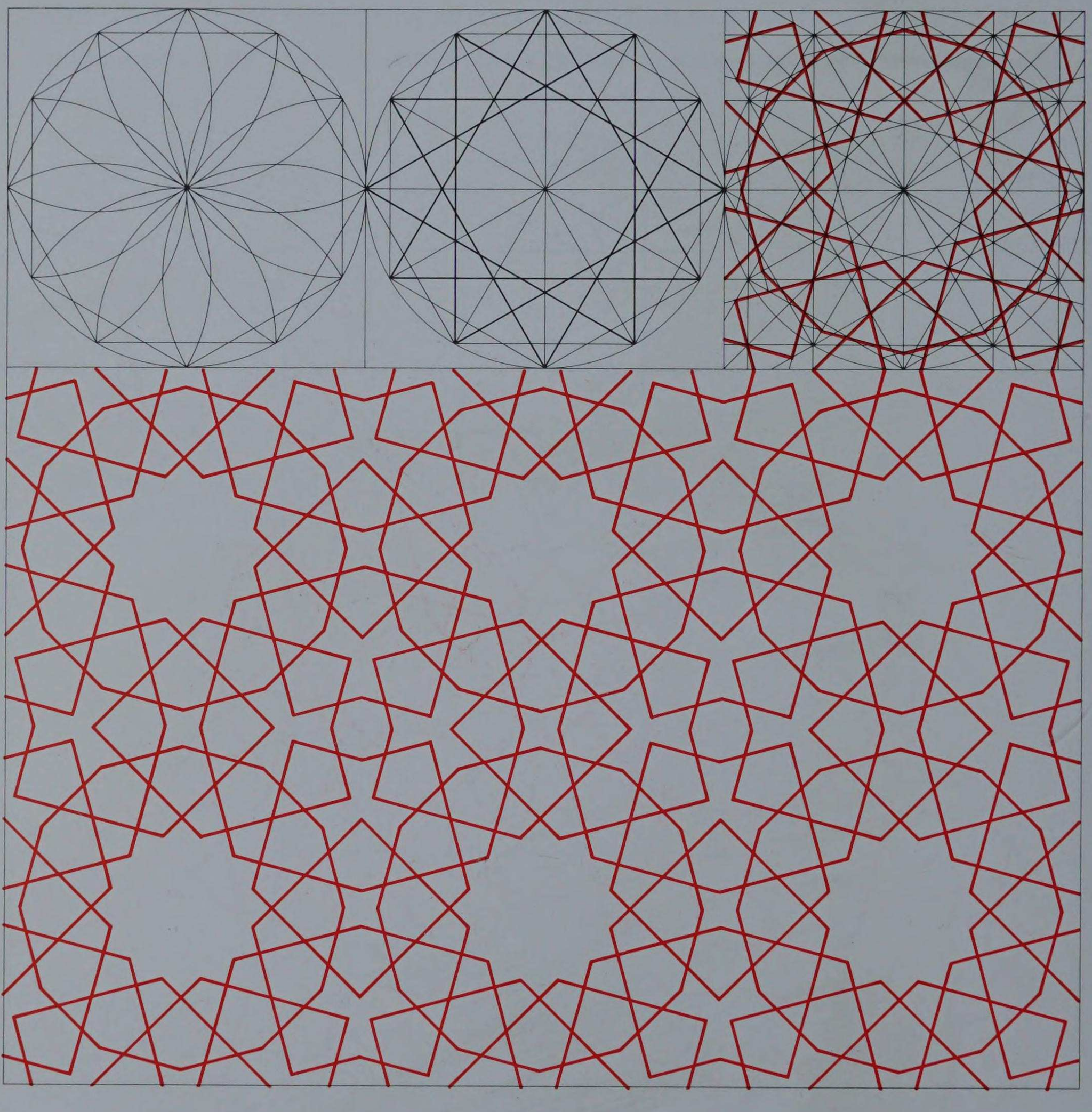
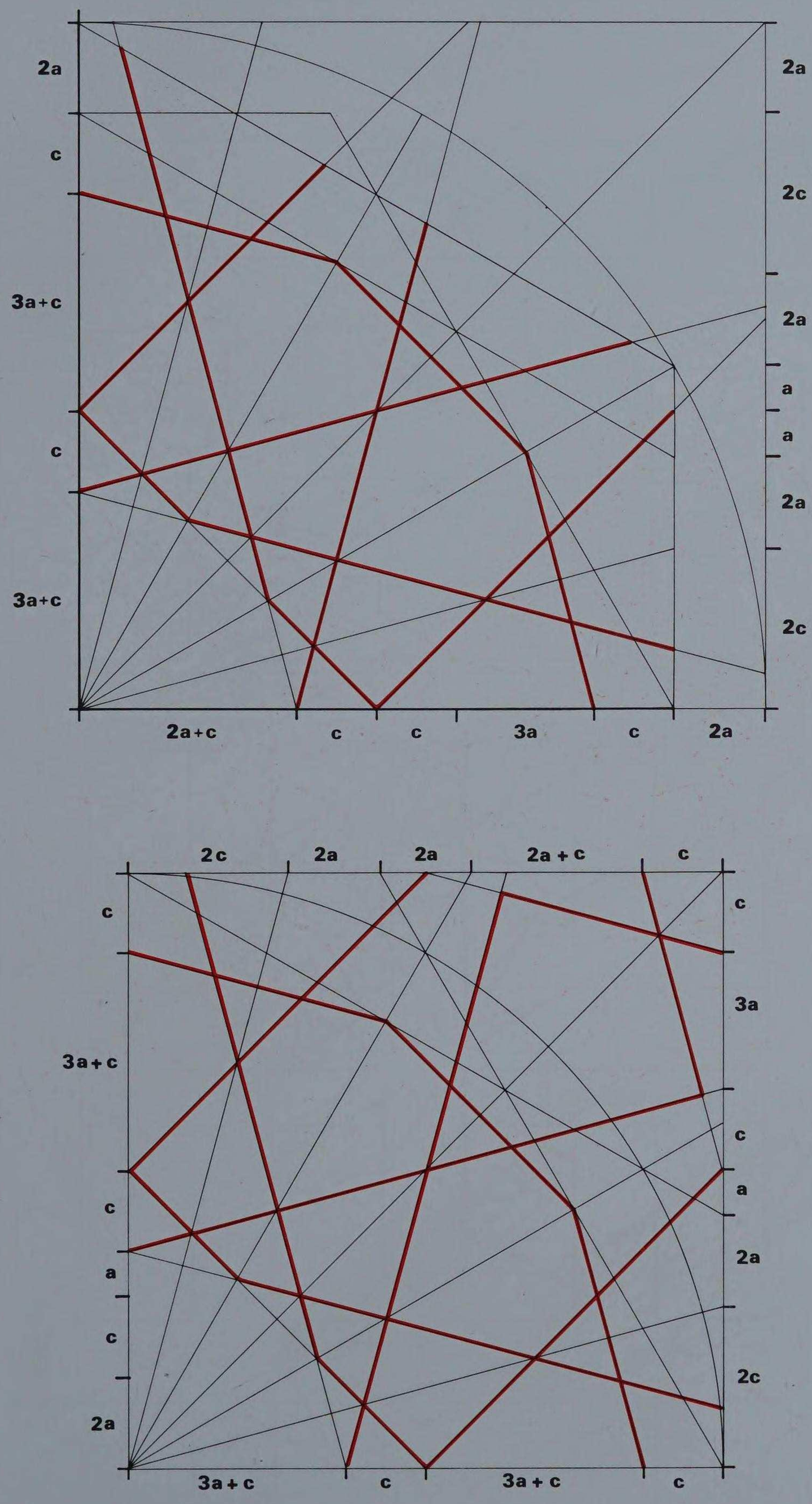


Figure 92ii



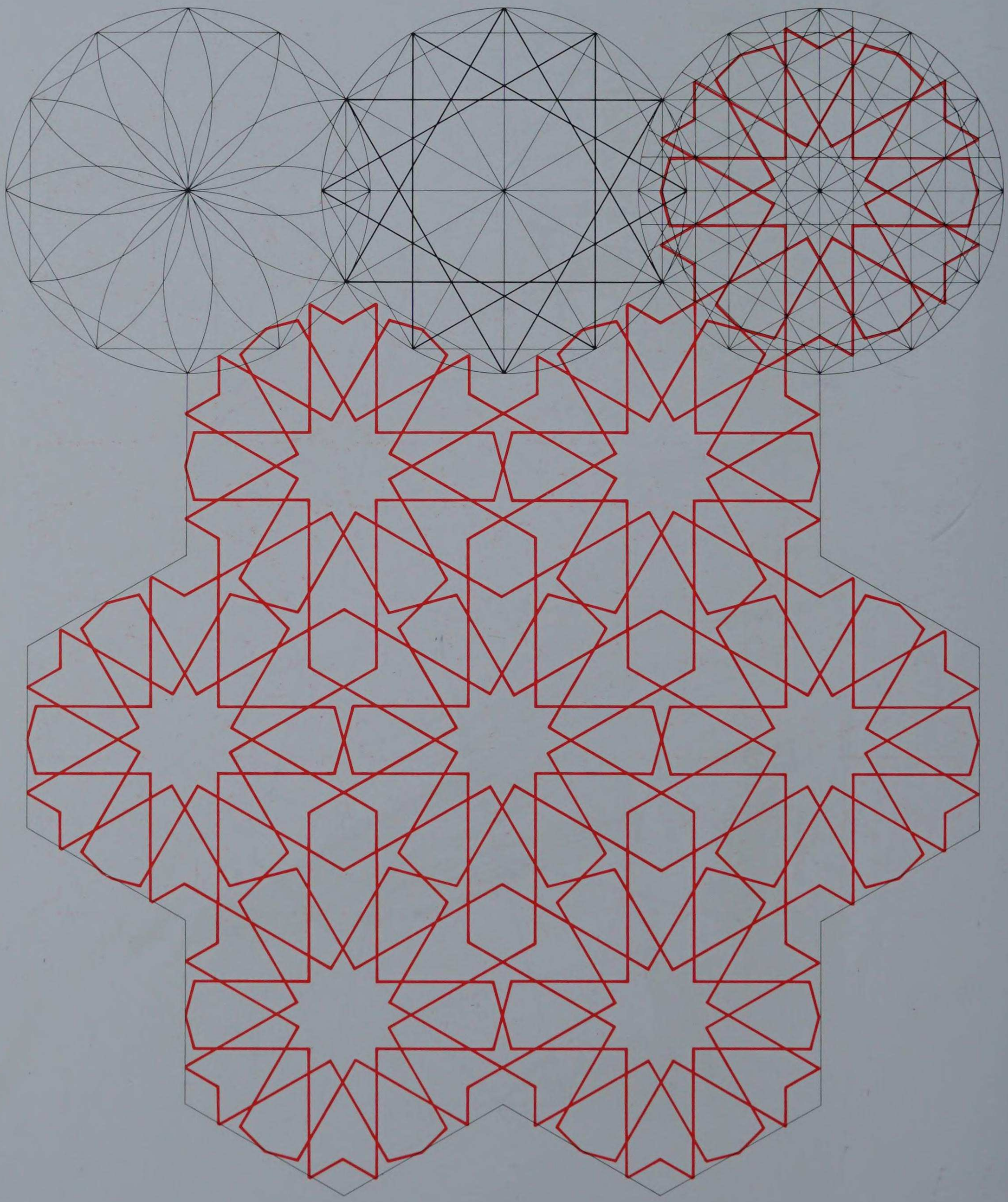
Here (**Figure 92iii**), as elsewhere, the same principle is used to measure the relevant proportional dimensions of the composition using *a* and *c* as modules.

Figure 92iii



Figures 93i, 93ii. In these two examples, the grid-pattern is generated by four equilateral triangles whose extremities coincide with the double hexagon. By further selected point-joining, a 12-pointed star is generated as the central element of the composition. The first example has a hexagonal repeat unit, while the second has a square repeat unit.

Figure 93i



THE DOUBLE HEXAGON

Figure 93ii

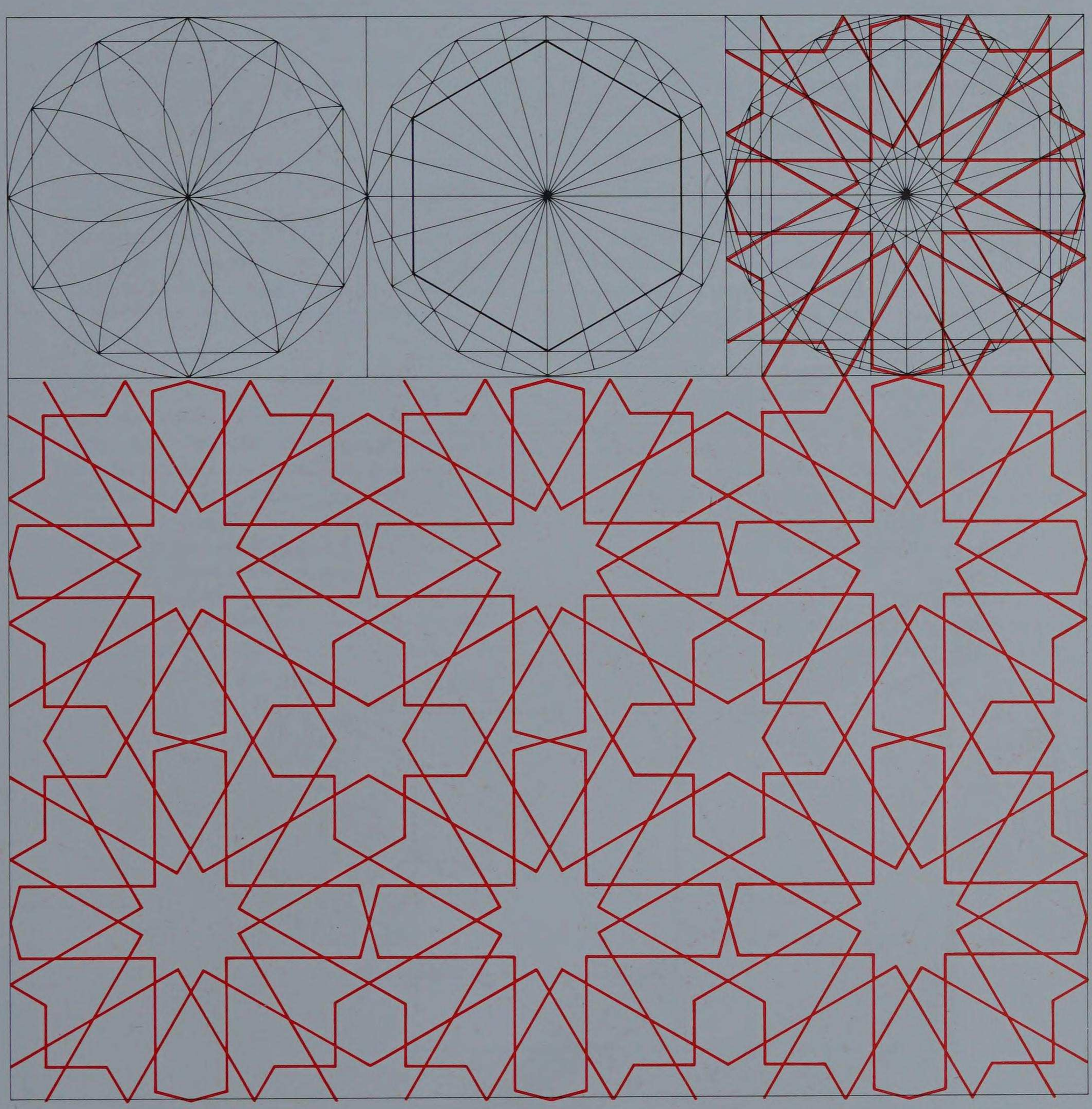
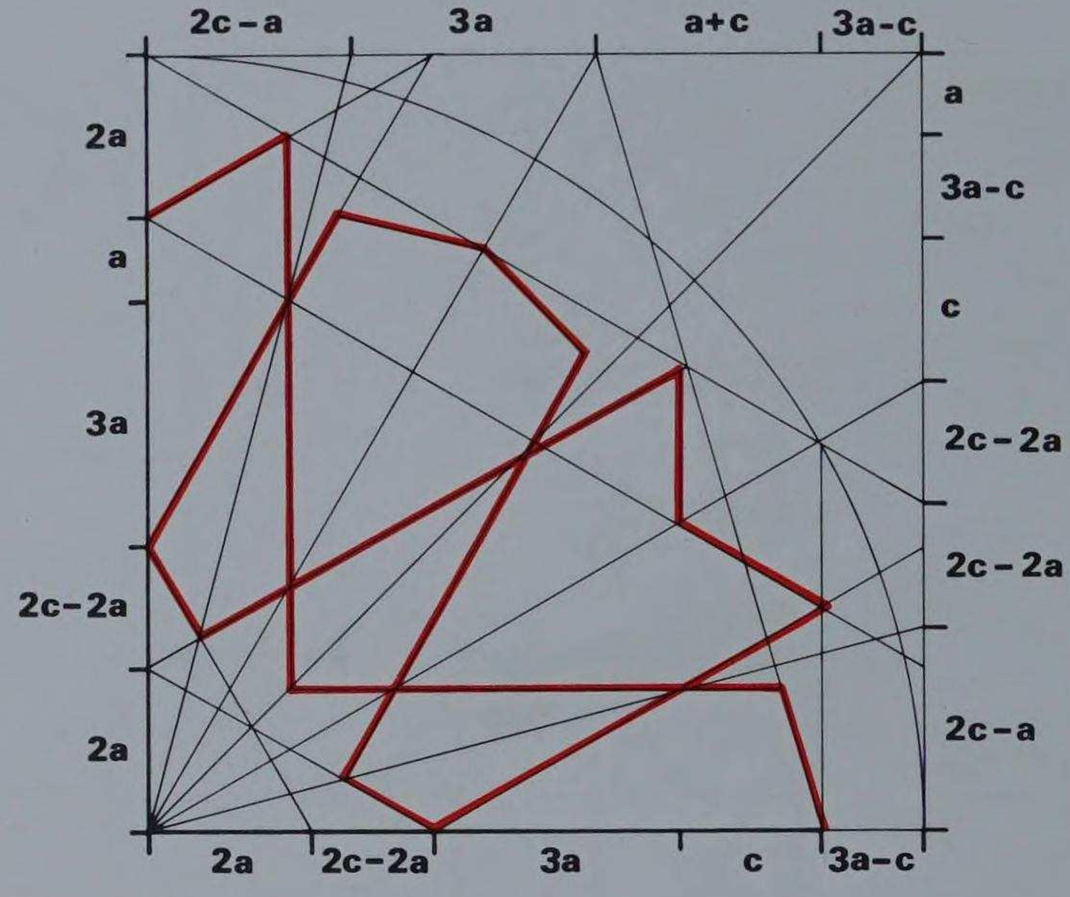


Figure 93iii. The initial division of the quadrant repeat unit by the double hexagon results in the formation of the same central composition within a hexagonal or a square repeat unit, as shown in (a) and (b), respectively. With the relevant dimensions of tan 15° and tan 30° established in terms of *a* and *c*, all linear dimensions of the repeat units can be calculated.

Figure 93iii



(a)

(b)

Notes

1 Methodology can be defined as the outcome of experience gained by experiments logically pursued along definite lines to achieve definite results, which are therefore replicable.

2 Although the art of memory is the basis of the necessary skill we should not exclude the possibility of crafts masters being equally good theoretical mathematicians. This stresses the importance of the oral tradition and the role of recitation in training the memory. (K.C. & T.E.B.)

3 The repeat unit is defined as a single geometric figure or composition which, when applied repeatedly and systematically, produces a repeating geometric pattern. The unit may be a square and based on the root two (√2) system of proportion (square repeat unit) or it may be a hexagon and based on the root three (√3) system of proportion (hexagonal repeat unit).

4 The term 'grid-pattern' has been introduced by the editors as a substitute for the term 'grid' used in the original text. The 'grids' referred to throughout the text are in most cases stellated polygons, as in the example of an octagonal star (e.g. primary grid A, see **Figure 4**). Unlike the more commonly adopted meaning of grid or tessellation (i.e. a framework which can extend indefinitely), a grid-pattern may be defined as a geometric figure or composition which is a finite form existing within the bounds of the repeat unit. The grid- pattern is thus used to generate the composition of a repeat unit, i.e. it becomes its supporting framework or 'grid'. Repeat units having complex compositions may be generated by more than one grid-pattern, which could be superimposed or nested. These are referred to as secondary and tertiary grid-patterns (see examples of geometric patterns in **Figure 27** (√2 proportion), **Figure 81** (√3 proportion) and **Figure 91i** (double hexagon proportion)). The secondary or tertiary grid-patterns generated are not necessarily stellated polygons but could, for example, be square or triangular grids (see examples of geometric patterns **Figure 41** (√2 proportion) and **Figure 67** (√3 proportion), respectively).
(K.C. & T.E.B.)

5 Primary grids A and B are two examples of grid patterns (see p.19). Both are stellated octagons. For the purpose of setting out a mathematical analysis in order to translate the geometry or proportion to whole numbers, these particular compositions have been identified as primary grids A and B to act as reference points. Other grid patterns throughout the text are referenced similarly.

6 The 'point-joining' method is the term given to the practical means employed in geometric constructions where intersection points made by geometrical figures are joined by straight lines to give the required construction.

7 Art here means the act of accomplishing; therefore the proof of 'geometric art' is in the doing. (K.C. & T..E.B.)

II

MAN AND MEASURE

Detail of the stalactite forms lining the arches of the
riwāq of the ʻAbbāsid Palace, Baghdad

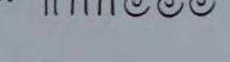
Historical Background

There is evidence for the existence in the Near East, sometime during the fourth millennium BC, of two societies, each with an entirely different structure: the Mesopotamian state-cities and Egypt, united under the rule of a divine king. These societies developed systems of government, a social order, private ownership and commerce, which depended on accepted systems to organize, codify and record them. By 3000 BC, in both regions, there were calendrical astronomers, temple surveyors, architects and irrigation engineers who based their work on a mathematical foundation. Sumer and Egypt both produced monumental modes of artistic expression (e.g. temples, palaces, sculpture) which were without precedent. These innovations are important markers of the transformation of prehistoric cultures into the first great civilizations. The architectural feats of Mesopotamia and Egypt encourage the supposition that their creators had elaborate rules of mensuration.[1]

Numerals and Arithmetic

Most of what is known of Egyptian mathematics is derived from two papyri. The earliest extant document is the *Ahmes Papyrus*, which was named in honour of the scribe by whose hand it had been copied in about 1650 BC, but it is thought to have been based on much earlier work.[2] The other is the *Moscow Papyrus* and is perhaps two centuries older.[3] Both contain rules for the construction of plane figures and the determination of their areas, as well as problem-solving involving simple equations. The earliest records of decipherable numerals found were largely concerned with the accounts of temple estates of the Tigris-Euphrates Valley. These display well-developed number systems and geometry.[4]

In Egypt, numerals from one to nine were represented by simple strokes, 𓏺 𓏺𓏺 𓏺𓏺𓏺,... and special signs were given for the numbers ten, 100, 1,000, and other large numbers.[5]

Thus the number 645 was written as: 

Based on this system, the Egyptians developed an arithmetic mainly of an additive character where all multiplications were reduced to repeat additions. For example, the multiplication of 11 by 11 was obtained by first multiplying 11 by 1 and then progressively doubling 11 by 2, 4, 8, ...and so on. The products corresponding to the multipliers that together add up to 11 are then summed up to give the result.[6]

1	**11**
2	**22**
4	44
8	**88**
11	121

Adding the numbers shown in **red** gives 121.

Division was performed by a process similar to multiplication. For example, the division of 221 by 17 would look exactly the same as the multiplication of 17 by 23. The difference is that the scribe would first discover which products add up to 221 and then sum the corresponding multipliers to find the answer, 13.

1	17
2	34
4	68
8	136
13	221

Adding the multipliers shown in **red** gives 13

This technique was extended to deal with divisions that do not result in whole numbers, for example $\frac{17}{8}$:

1	8		
2	16		
$\overline{2}$	4	(half 8)	
$\overline{4}$	2	(quarter 8)	Adding the multipliers
8	1	(eighth 8)	shown in red gives 28
28	17		$(2\overline{8} = 2\frac{1}{8})^7$

The scribe doubles the denominator (8) until he reaches 16, which is less than the required number, the numerator (17) by 1. Then he successively halves until he again reaches 1. In fact, division was looked on rather as multiplication by a fraction.

The reference to fractions, and their application in examples given in the Ahmes Papyrus (*Papyrus Rhind,*) would suggest that the Ancient Egyptians had made extensive use of them. The numerator of the fraction was always equal to one:

$\frac{1}{5}, \frac{1}{7}, \frac{1}{8}$, ..., and the difference of the fraction from unity;

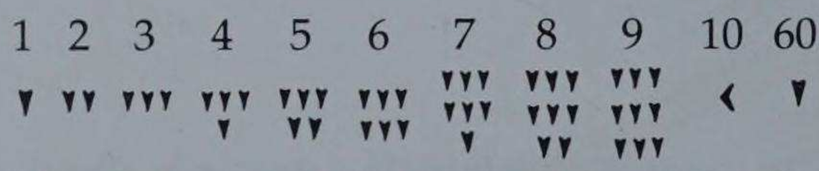
$\frac{4}{5}, \frac{6}{7}, \frac{7}{8}$, ..., was represented as a subtraction from unity;

$1-\frac{1}{5}, 1-\frac{1}{7}, 1-\frac{1}{8}$,..., respectively.

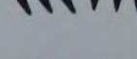
The only exception to this rule was the fraction $\frac{2}{3}$, which was referred to by a special term and represented by the symbol . Otherwise, fractions were denoted by writing over the corresponding whole number.[8] For example: $\frac{1}{12}$ $\frac{1}{32}$. All fractions in Egyptian arithmetic were expressed as the sum of fractions having unity as numerator, i.e. $\frac{2}{5} = \frac{1}{10} + \frac{1}{30}$.

Sources of Mesopotamian mathematics are profuse. Hundreds of clay tablets in cuneiform script have been discovered. One of the richest finds, some 50,000 tablets, comes from the temple site at Nippur, destroyed by the Elamites in c.2000 BC. They deal with geometric relationships, reciprocals, fractions and tables of squares interpolating square roots and cubes.[9] Sumerians and Babylonians used a sexagisimal number system:

1 2 3 4 5 6 7 8 9 10 60

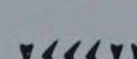


Numbers up to 59 were written in the Egyptian manner, e.g. 34 would be written as:



(10 + 10 + 10 + 4) = 34.

However, 60 had exactly the same symbol as 1; the number 94 would be written as:



(60 + 10 + 10 + 10 + 4) =94.

Thus the same symbol for one and 60 represented different values depending on its position in the number.[10] The principle involved, that of place value, is the one used today: e.g. in the number 947, the 7 represents 7 x 1 units, the 4 represents 4 x 10 units and the 9 represents 9 x 100 units. In other words, the position of the number determines the value it represents, and this

value is a multiple of ten (10^1, 10^2,...10^n) depending on the position of the digit. The number ten is called the base of our decimal system.

The Babylonians also made use of a decimal system of numeration. 100 was represented by a compound sign, 𒐕 , and 1,000 was expressed as 10 x 100 units by 𒌋𒐕 , also regarded as one sign; the prefixed 𒌋 (10) being here multiplicative. However, 𒌋𒌋𒐕 denoted not 2,000 but 10,000 the prefixed 𒌋𒌋 being again a multilpicative (10 x 10 x 100).

The Babylonian sexagesimal system of place value in connection with base 60 survives today in the diversion of angles and hours into 60 minutes and 60 seconds. It was found in use on the *Tables of Senkereh*, discovered by W.K. Loftus in 1854, and dated between 2300–1600 BC.[11] Using this system, fractions could be expressed in units (degrees or hours), by minutes, seconds, and so on. For example $1° = 60'$, $\frac{1°}{2} = 30'$, $\frac{1°}{8} = 7'30''$, $\frac{1°}{16} = 3'45''$, $\frac{1°}{32} = 1'52''30'''$, $\frac{1°}{60} = 1'$.

The method of calculation is demonstrated as follows:

$$\frac{1°}{7} = \frac{1\times 60'}{7} = 8\frac{4'}{7}$$

$$\frac{4'}{7} = \frac{4\times 60''}{7} = 34\frac{2''}{7},$$

$$\frac{2''}{7} = \frac{2\times 60'''}{7} = 17\frac{1'''}{7},$$

$$\frac{1'''}{7} = \frac{1\times 60''''}{7} = 8\frac{4''''}{7}\text{...etc.}$$

Therefore:

$$\frac{1°}{7} = 8'\ 34''\ 17'''\ 8''''...$$

In the same way:

$$\frac{3°}{11}\ 16'\ 21''\ 49'''...,$$

$$\frac{5°}{13}\ 23'\ 4''\ 36'''...,$$

$$\frac{1°}{17}\ 3'\ 31''\ 45'''....$$

The Mesopotamian and Egyptian civilizations used mathematics for many human needs and interests. Simple formulae and numerous elementary rules and techniques were accumulated to answer questions arising in certain situations. They were practical means for working out specific problems as set out in the *Ahmes Papyrus*. No explanations or reasons for the mathematical operations were given, and there is no evidence to suggest that theories or general principles were developed.

Although the Greeks employed various numeral systems, they also conceived the idea of using an alphabetic notation for their decimal system which was later adopted as the official system.[12]

The Greek system of numeration, unlike the Babylonian method, did not use positional notation. Units were denoted by the first nine letters of the Greek alphabet which was taken from the Phoenicians. Tens were expressed by ten further letters and the process was repeated for hundreds, thousands, and so on. They had no symbol denoting zero.

A= 1	I= 10	P= 100
B= 2	K= 20	Σ= 200
Γ= 3	Λ= 30	T= 300
Δ= 4	M= 40	Υ= 400
E= 5	N= 50	Φ= 500
[F] = 6	Ξ= 60	X= 600
Z= 7	O= 70	Ψ= 700
H= 8	Π= 80	Ω= 800
Θ= 9	[Q] = 90	[S] = 900

For example : ΨNΔ = 754 and Σ[Q]Γ = 293

In mathematical calculations, the Greeks handled fractions in three different ways. They used the reciprocals of integers (aliquot parts) like the Egyptians, together with proper fractions, such as $\frac{7}{9}$ and $\frac{3}{11}$, and in astronomical work they used sexagesimal fractions derived from the Babylonians.[13]

Roman numerals were also based on a decimal system of numeration with a special sign for each higher decimal unit (unit of ten).

1	2	3	4	5	6	7	8	9	10	50	100	500	1000
I	II	III	IV	V	VI	VII	VIII	IX	X	L	C	D	M

Number four is expressed as (5–1) IV, six as (5+1) VI, seven as (5+2) VII, nine as (10–1) IX, 40 as (50–10) XL, and so on. Thus 1844 is represented by:

MDCCCXLIV

1000 + 500 + 100 + 100 + 100 + (50–10) + (5–1).

It was not easy to use these systems of numeration in mathematical calculations; hence mathematical tables, and/or the abacus, were used as aids.

The numeral system used today, consisting of the positional decimal system with the nine Hindu-Arabic numerals from one to nine, and the zero, was introduced by the Arabs from its origins in India in the ninth century. Its value and use were described early in the ninth century by the Muslim mathematician Al-Khwarazmi in his book *Addition and Subtraction in Indian Arithmetic*, which was translated into Latin as *Algorismi de numero indorum*.[14] Using this system, any number, large or small, can be represented by ten symbols only (1-9 and 0) and calculation can be performed in writing. This numerical system reached Europe sometime in the twelfth century under the name of 'Algorism' or 'Algorithm' (derived from Al-Khwarazmi) and was first described by Leonardo of Pisa in the *Liber Abaci* (1202 AD).[15]

Measuring and Dividing Dimensions

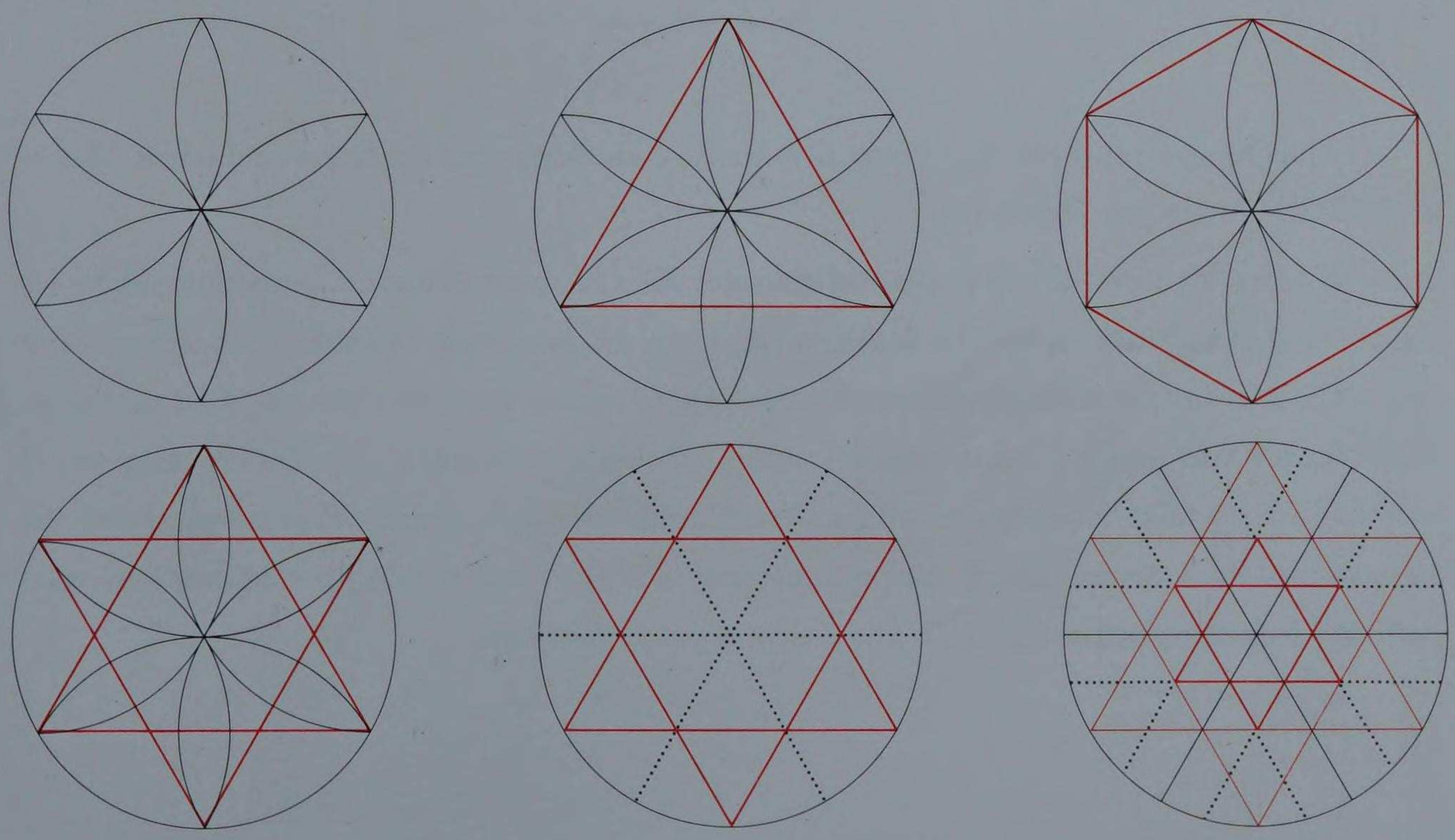
Since ancient times, in societies where mathematical illiteracy was prevalent, man has used familiar units of length as a unit scale, e.g. finger width, palm, forearms (cubit), nose to fingertip (yard), and so on. For a given length of rope, any convenient unit could be used to determine its dimension. Moreover, irrespective of its dimension, the rope could be divided into any number of folds of 3, 4, 5, ... and so on. Any number of further proportional divisions could be made by repeated folding into two, thus continuously doubling the previous number of folds, to 6, 8, 10, ... and then to 12, 16, 20, ... and so on. Any number of further proportional divisions could be made by repeated folding into two, thus continuously doubling the previous number of folds to 6, 8, 10, … and then to 12, 16, 20, … and so on. The result of any division by the folding technique would always be in whole units.

a. Using the Circle

In principle, the Ancient Egyptians' rope-stretching technique for surveying, using a rope and peg, is comparable to that of the compasses, used in architecture. Geometry made possible an ideal method of shaping areas without resorting to complicated mathematics. Even with the development of numerals and the positional decimal system (which are both in current use), the practical application of geometry, which is complete in itself, appears to have survived through the ages and is still practised by builders and craftsmen in parts of the Middle East.

From a circle, it is possible to generate any regular polygon by dividing its circumference equally into the required number of sections and then joining the points of division with straight lines (see **Figure 94**). The geometrical property of the radius of a circle, dividing its circumference

Figure 94



into six equal parts, makes possible the construction of an equilateral triangle, a hexagon or a six-pointed star, simply by joining, with straight lines, the points of division in the relevant combinations. Further point-joining within the six-pointed star figure results dividing the circumference into 12 equal divisions. By inscribing a second six-pointed star, and extending its sides, the circumference is further divided into 24 equal parts (see also page 56).

It is this geometric possibility of the radius being used to subdivide the circumference into equal divisions of 6, 12, 24,... and so on, that encourages the belief that this was probably the basis of the Mesopotamian sexagisimal system of measure. Furthermore, it is possibly also related to the astrological system of the 12 signs of the Zodiac, the calendrical division of the year into 12 months, and measuring systems based on six, such as the dozen and carat.

The division of the circumference of a circle into four equal parts is achieved by constructing perpendicular diameters (see **Figure 95**).

Figure 95

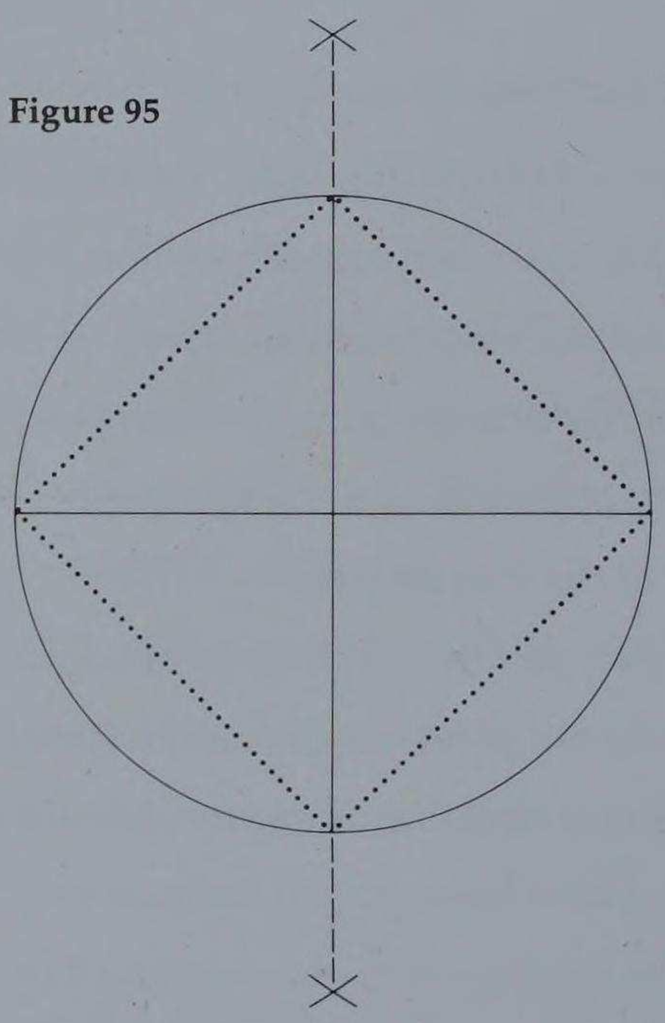
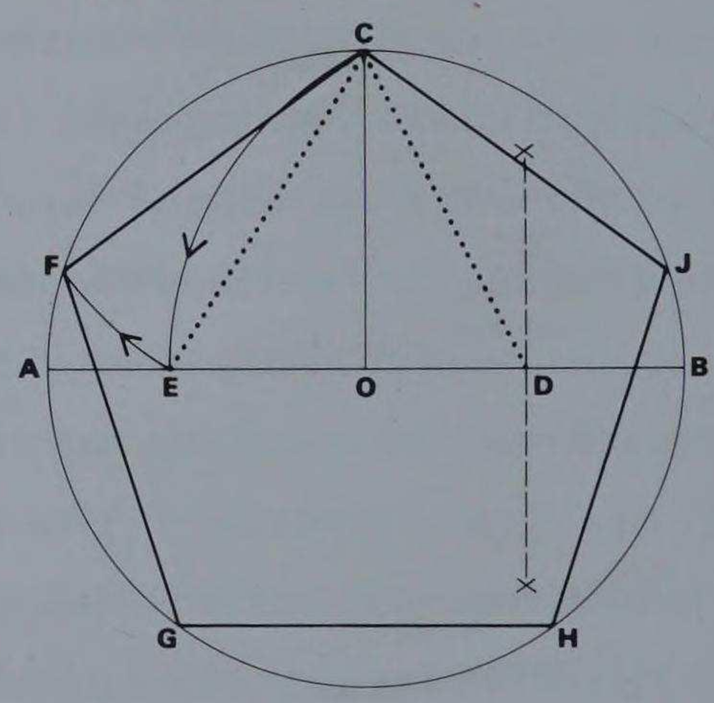


Figure 96



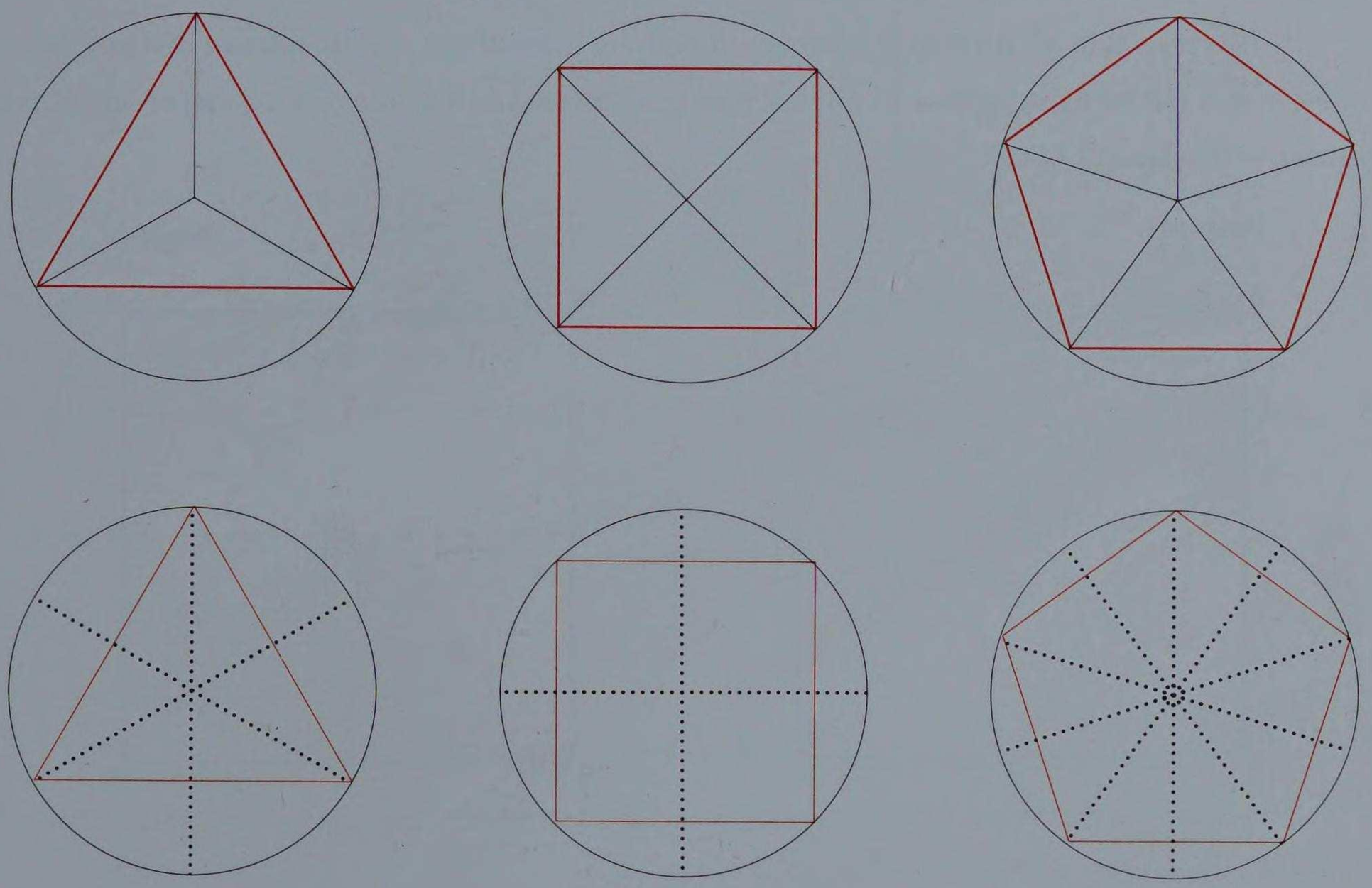
Inscribing a pentagon in a circle divides its circumference into five equal parts. This is achieved as follows (see **Figure 96**):

Draw a circle with centre *O* and horizontal diameter *AB*. Construct radius *OC* perpendicular to *AB*. Bisect *OB* at *D* and with centre *D* and radius *DC*, draw an arc cutting *OA* at *E*. With centre *C* and radius *CE*, draw an arc cutting the circumference at *F*. *CF* equals the side of the inscribed pentagon. With centre *F* and radius *FC*, draw an arc cutting the circumference at *G*. Similarly, with centre *G* and the same radius, draw an arc cutting the circumference at *H*. Finally with centre *H* and the same radius, draw an arc cutting the circumference at *J*. The circle is thus divided into five equal parts and by joining points *C*, *F*, *G*, *H* and *J*, a pentagon is inscribed.

The division of the circumference of a circle into three, four or five equal parts can be achieved by the method just described, using a rule and compasses. When the sides of the polygons are bisected (by joining the centre and a corner of the polygon with a straight line and extending it to bisect the opposite side, see **Figure 97**), the subdivisions of the circumference are doubled to six, eight and ten equal parts, respectively (see middle row).

By 'rotating' the inscribed polygon, the subdivisions of the circumference are further increased by a factor of two (see bottom row), whereby the first divisions of three, four and six are here redoubled to 12, 16 and 24 respectively. Thus the division of the circumference of the circle by the 'point-joining method' is in principle similar to the subdivision of any linear dimension by the folding technique. Regular polygons with equal sides of 3, 4, 5, 6, 8, 10, 12, 16, ... and so on, can be obtained by joining, with straight lines, the points of division of the circumference.

Figure 97



b. Using the Square

A straight line can be divided into any number of aliquot parts by applying the geometrical properties of the square.

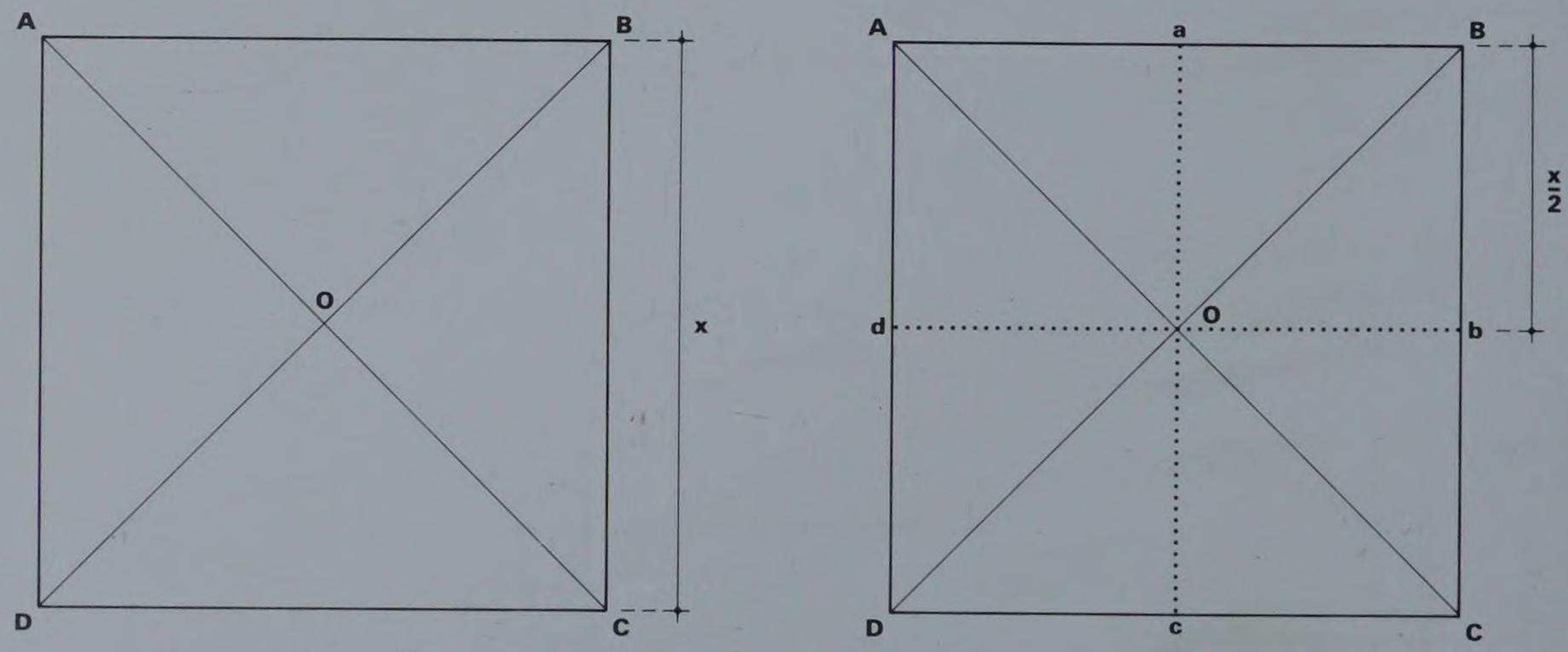
To divide a straight line x into 2, 4, 8, 16, ...equal parts (see Figure 98):

Let side of square *ABCD* equal *x*. Draw the diagonals *AC* and *BD* intersecting at centre *O*. Join the opposite mid-points *ac* and *bd*, thus dividing square *ABCD* into four identical squares, *OdAa*, *OaBb*, *ObCc* and *OcDd*. The side of each square equals half *AB*, and the area of each equals a quarter of square *ABCD*.

Draw diagonals *ab*, *bc*, *cd*, and *da* and join their centres *e*, *f*, *g*, and *h* with straight lines. The extension of these lines divides the sides of the square *ABCD* into quarters. Thus, by repeatedly joining the centres of the newly-formed squares, the sides and diagonals of square *ABCD* are progressively halved into equal parts of 2, 4, 8, 16, 32, ... and so on.

This results in a square grid where each point of intersection and its relative distance from other points can be expressed as an aliquot part proportional to the side or diagonal of any sized square within square *ABCD*.

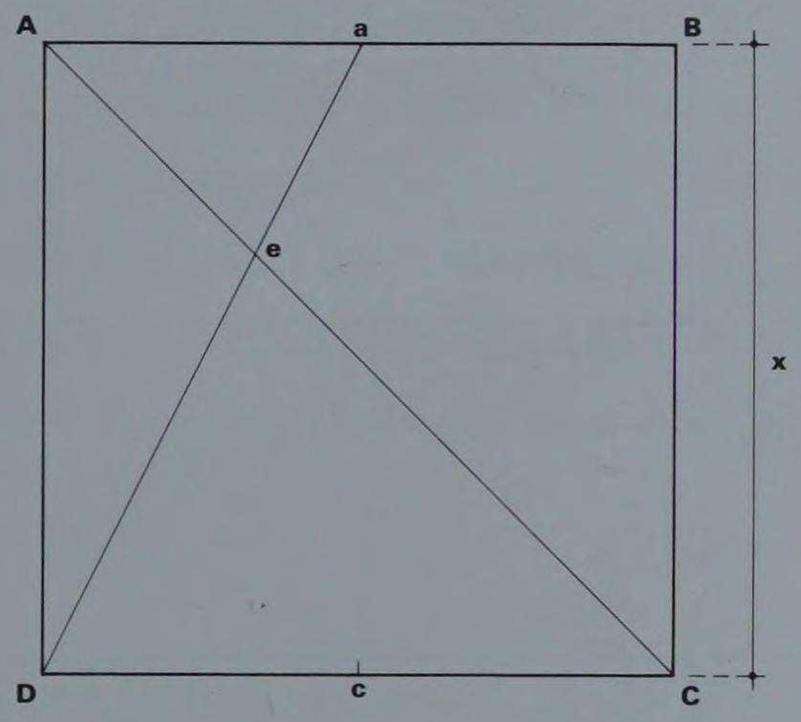
Figure 98

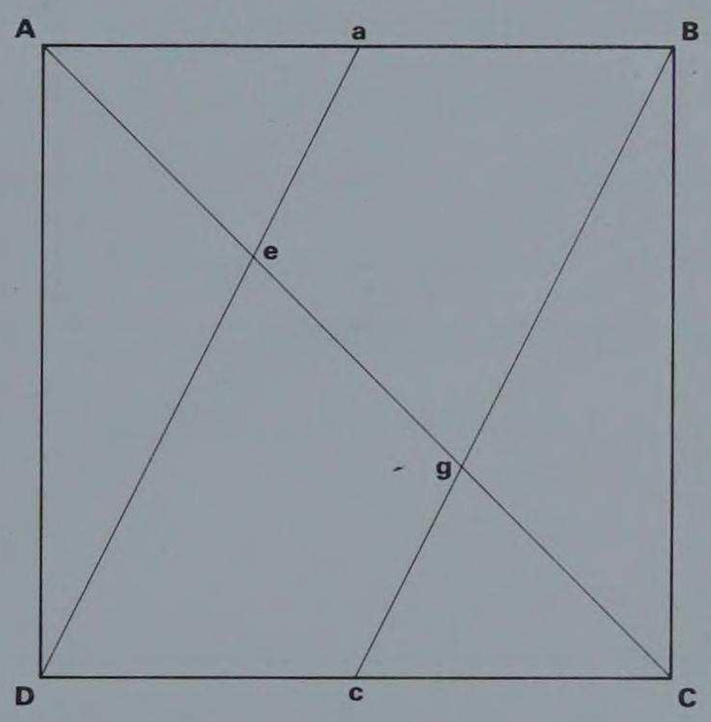


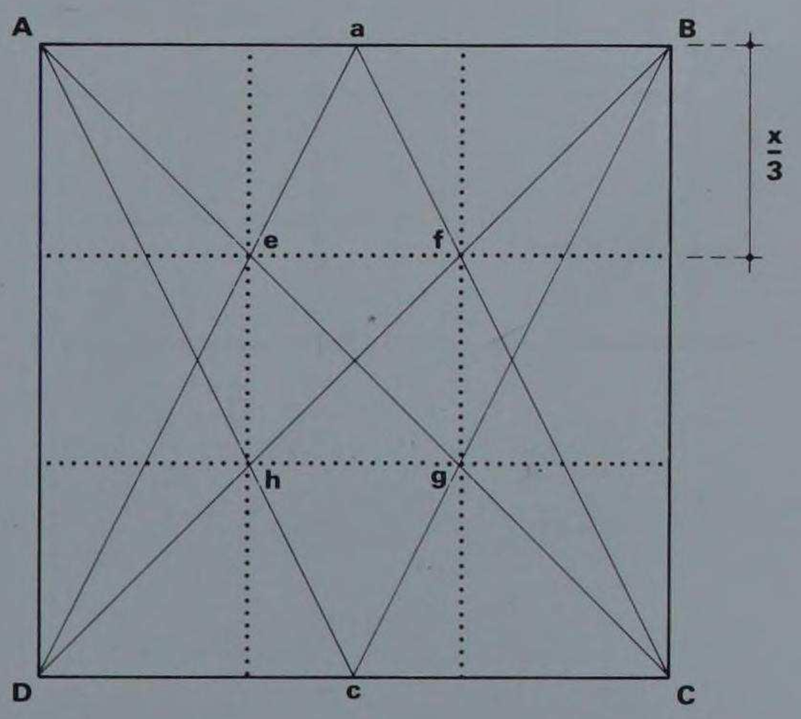
To divide a straight line x into 3, 6, 12, 24, ...(see Figure 99):

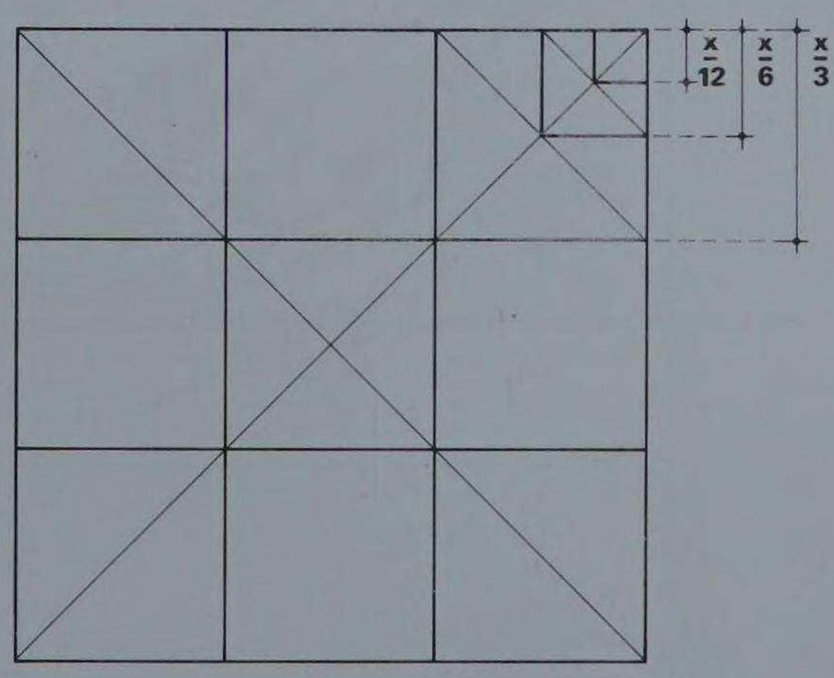
Let side of the square *ABCD* equal *x* and establish mid-point a on side *AB* and mid-point *c* on side *CD*. Draw diagonal *AC* and diagon *aD*.[16] The point of intersection *e* divides the lines in the proportion of 1:2. Draw diagon *Bc*, thus dividing *AC* into three equal parts. Draw diagonal *BD* and diagons *aC* and *Ac* and join the points of intersection, *e*, *f*, *g*, *h* with straight lines which, when extended, divide the sides of square *ABCD* into three equal parts. By repeatedly joining the centres of the newly-formed squares, the initial division of the sides and diagonals of square *ABCD* into three is progressively doubled into equal parts of 6, 12, 24, ... and so on. This results in a square grid in which the side of each unit is $\frac{1}{3}$ and the area $\frac{1}{9}$ of the square *ABCD*.

Figure 99







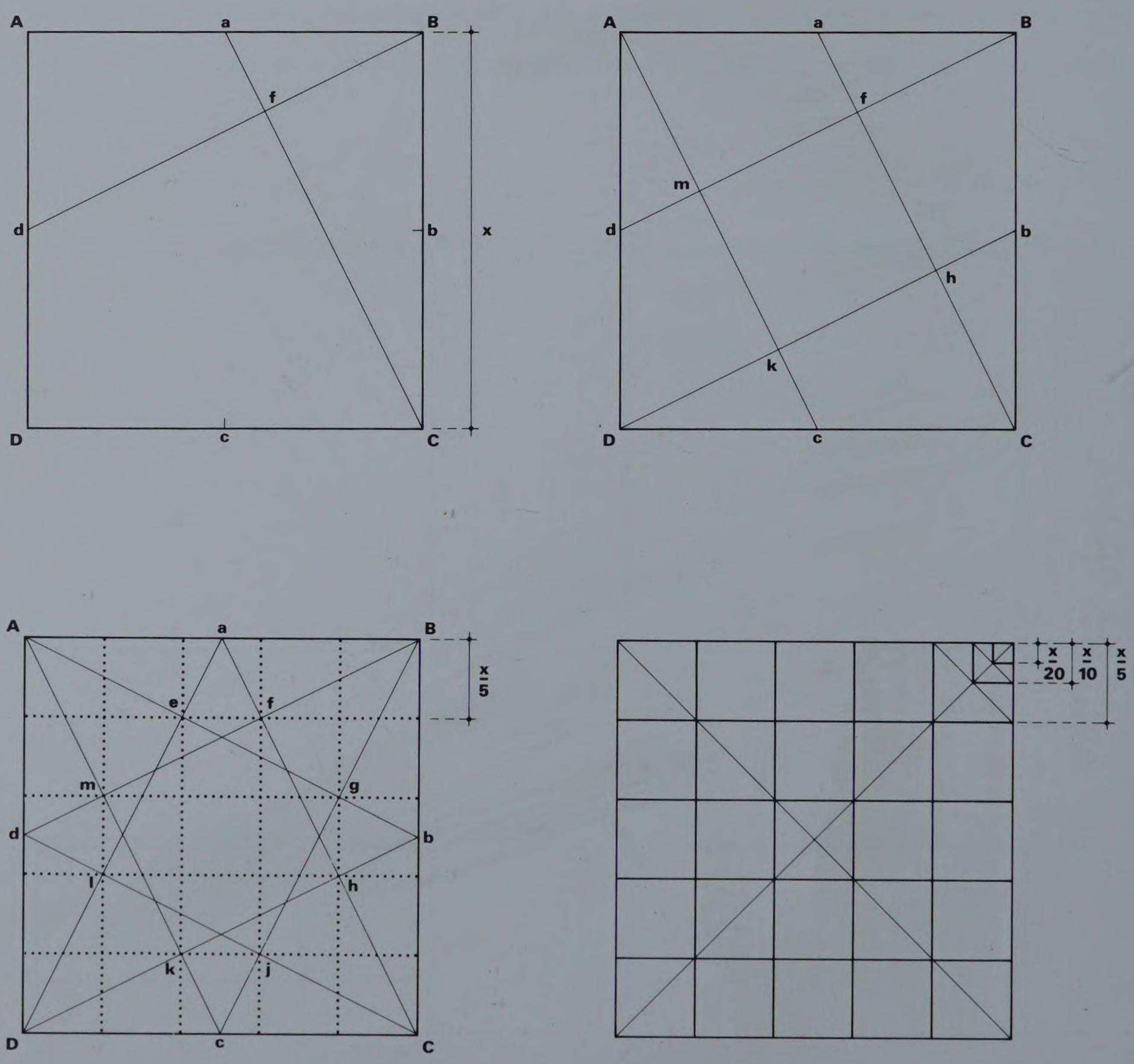


To divide a straight line x into 5, 10, 20, ... equal parts (see Figure 100):

Let side of square *ABCD* equal *x* and establish mid-point a on side *AB* and mid-point *d* on side *DA*. Draw diagons *aC* and *Bd*. The point of intersection *f* divides *aC* and *Bd* in the proportion of 1:4 and 2:3 respectively. Draw diagons *Ac* and *bD*. These intersect each other in the proportion of 1:2:2.

Draw the remaining diagons *Ab*, *Bc*, *Cd* and *Da* and join the points of intersection *e*, *f*, *g*, *h*, *j*, *k*, *l*, *m* with horizontal and vertical lines which, when extended, divide the sides of *ABCD* into five equal parts. By repeatedly joining the centres of the newly-formed squares, the initial division of the sides and diagonals of square *ABCD* into five is progressively doubled into equal parts of 10, 20, ... and so on. This results in a square grid having 25 squares each with sides equal to $\frac{x}{5}$.

Figure 100



To divide a straight line *x* into approximately seven equal parts (see Figure 101):

Let side of square *ABCD* equal *x*. Draw diagonals *AC* and *BD* and inscribe a circle. Join the points of intersection *e, f, g,* and *h* of the diagonals and the circle with straight lines to give an inscribed square *efgh*.

Extend the corners of square *efgh* to cut the sides of square *ABCD* in the proportion of approximately 1:6. This is demonstrated mathematically as follows:

Construct square *abcd*, where *a*, *b*, *c*, and *d* are the mid-points of *AB*, *BC*, *CD*, and *DA* respectively, thus making *abcd* and *efgh* identical squares (inscribed in the same circle) and their diagonals equal to the side of square *ABCD* (*ac*=*BC*). Therefore, the lengths of the side of square *ABCD* and those of squares *abcd* and *efgh* are in the proportion of 1:√2, (triangle *adc*, Theorem of Pythagoras) while their areas are in the proportion of 1:2.

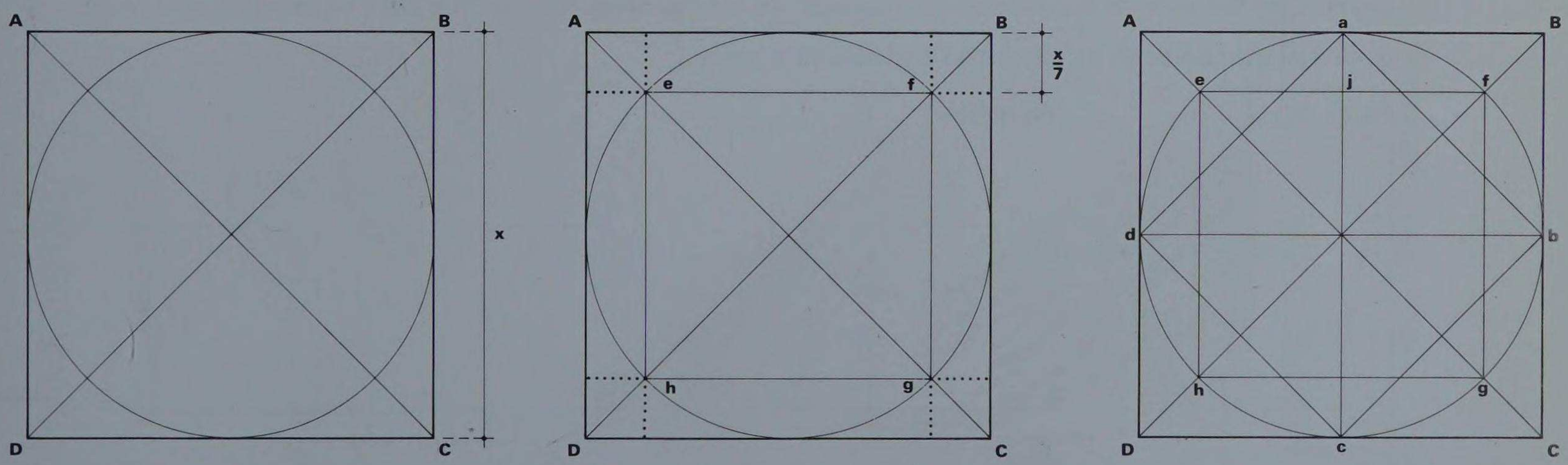
If the side of square *efgh* equals 1 unit:

$BC = fg\sqrt{2} = \sqrt{2}$,

$aj = \frac{BC - fg}{2}$, where *j* is the intersection point of *ac* and *ef* $= \frac{\sqrt{2}-1}{2} = \frac{1.4142-1}{2} = 0.2071$.

Therefore $\frac{BC}{aj} = \frac{1.4142}{0.2071} = 6.83$, which is approximately 7; a deviation of –2.4%. Also: $\frac{fg}{aj} = \frac{1}{0.2071}$ which is approximately 5; a deviation of -3.4%. Since $\frac{BC}{fg} = \frac{\sqrt{2}}{1} = 1.4142$, then the approximation $\frac{7}{5} = 1.4$; a deviation of -1.0%. This leads to the doctrine of the incommensurable, which simply means that in theory, rational and irrational numbers cannot be exactly expressed using the same scale, i.e. a ratio of two integers. The Greeks could not explain the meaning or concept of irrational numbers because they only knew rational numbers. This is possibly a reason why the Pythagoreans called irrational numbers (e.g. √2, √3, √5, ...) 'unspeakable numbers' (*surds*), which could be drawn but not expressed numerically. This made Greek mathematicians turn away from numbers to the use of lines and areas in which such logical difficulties did not arise.[17]

Figure 101



Geometry and Irrational numbers

One fundamental mathematical discovery which came from the Pythagorean school in the fifth century BC states that if every measure of length can be expressed by a number, then the proportions between two different measures should be expressible as the ratio of two numbers. However, a very simple case shows that this cannot be done; if one unit is used to express the length of the side of a square, then that of its diagonal cannot be exactly expressed as another rational number (**Figure 102**), whole or fraction, which when multiplied by itself (squared) equals exactly two (Theorem of Pythagoras, solved practically by use of the remen in Ancient Egypt). Such a number, which is irrational, is called the 'square root of two' and indicated as $\sqrt{2}$. Its approximate value lies between two numbers, one small, the other large, i.e.

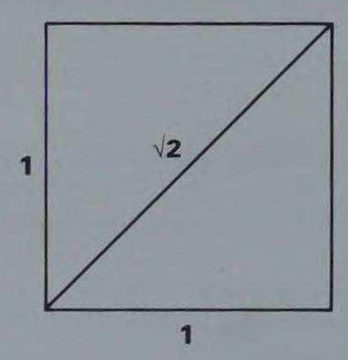
1.4 and 1.5

1.41 and 1.42

1.414 and 1.415

1.4142 and 1.4143 ...etc.

Figure 102

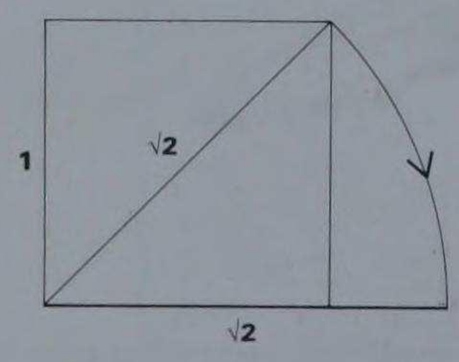


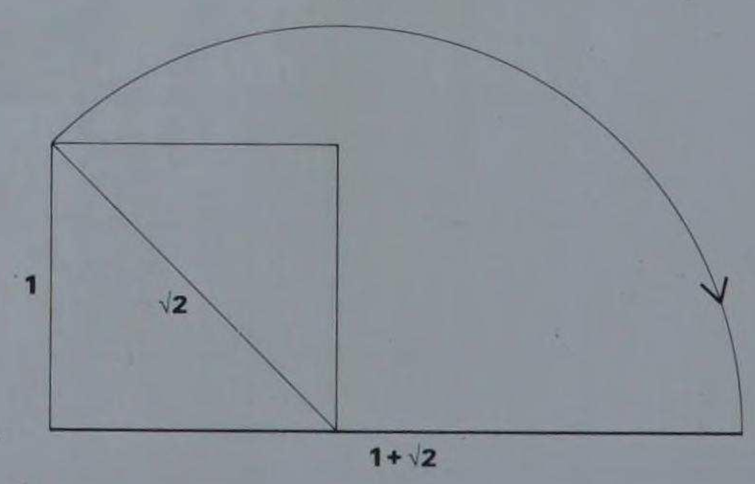
Although the Greeks also approached geometry in a rational way by introducing the notion of proof into mathematics, as in the *Elements of Euclid*,[18] written in the third century BC, they continued to use 'form' in the service of numbers, and solved numerical problems by practical geometry.

In order to conceive numbers with exactness, the Greeks worked with all numbers geometrically. If a length were chosen to represent the number one (**Figure 103**), other numbers were represented in terms of multiples of this length (e.g. 2, 3, 4, ...) and subdivision as aliquot parts of the length (e.g. $\frac{1}{2}, \frac{1}{5}, \ldots$). To represent $\sqrt{2}$ (**Figure 104**), a length equal to the length of the diagonal of a square with side of one unit in length is used; the sum of one and $\sqrt{2}$ is a length formed by the side and diagonal of a square. In this geometric form the sum of rational and irrational numbers can be conceived and resolved easily.

Figure 103

Figure 104





Similarly, as shown in Figures **105, 106 and 107**, the product of multiplication could be expressed geometrically as an area of a rectangle whose sides represent any number. The number of squares within each figure could then be counted to represent the result of the multiplication. This geometric system provides an exact way of working with rational and irrational numbers. The most frequently used irrational numbers are invariably dimensions of certain geometric figures. Hence, when measurements are not commensurable, it can be assumed that they were determined by a geometrical method.

Figure 105

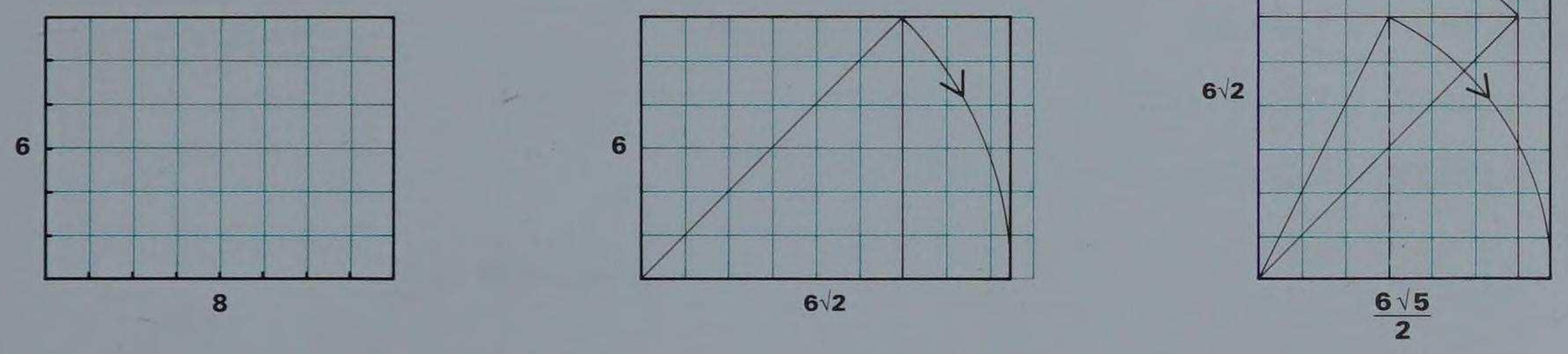


Figure 106

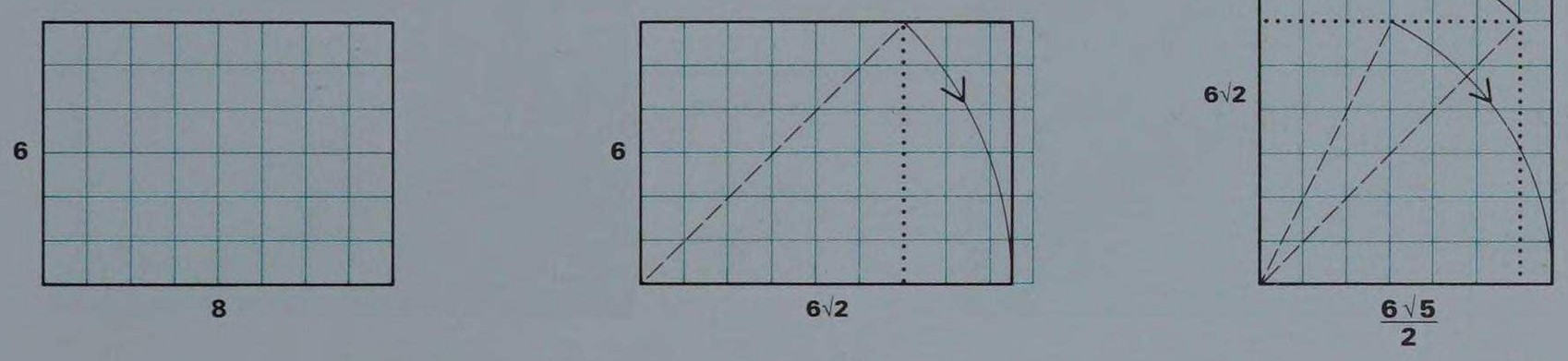
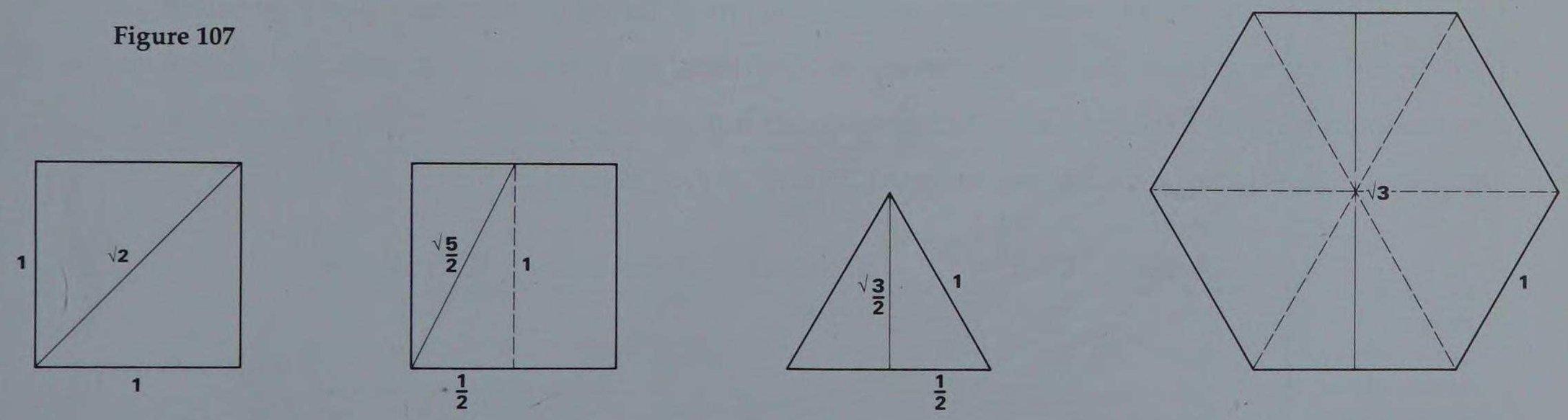


Figure 107



The Remen and the Royal Cubit

The development of writing and mural decoration after 3500 BC provides pictorial as well as material evidence of the development of measuring systems. In ancient Egypt, royal cubit linear measures have been found inscribed on narrow rods of ballast or wood, thought to have been used by masons in building work.[19]

Some fine specimens are preserved in the Louvre. A mural painting of the period c.1400 BC from the tomb of Menna at Abd el Qurna shows Egyptians measuring a corn field with a long cord, knotted at intervals of five cubits.[20] Land measure was based on the remen,[21] double remen and the royal cubit (used primarily in architecture).[22]

These measures had the following proportions (see **Figure 108**):

5 palms = 1 remen =20 fingers

7 palms = 1 royal cubit = 28 fingers

10 palms = 1 double remen = 40 fingers

Figure 108

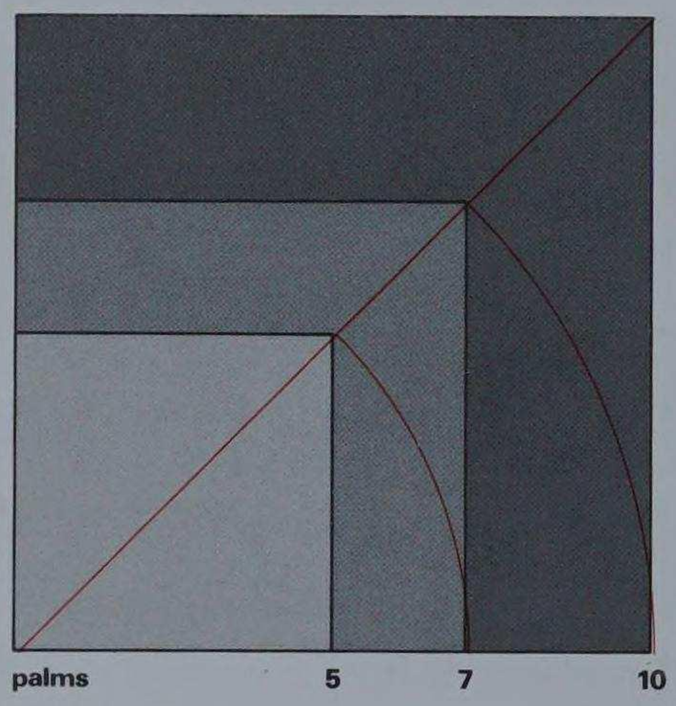
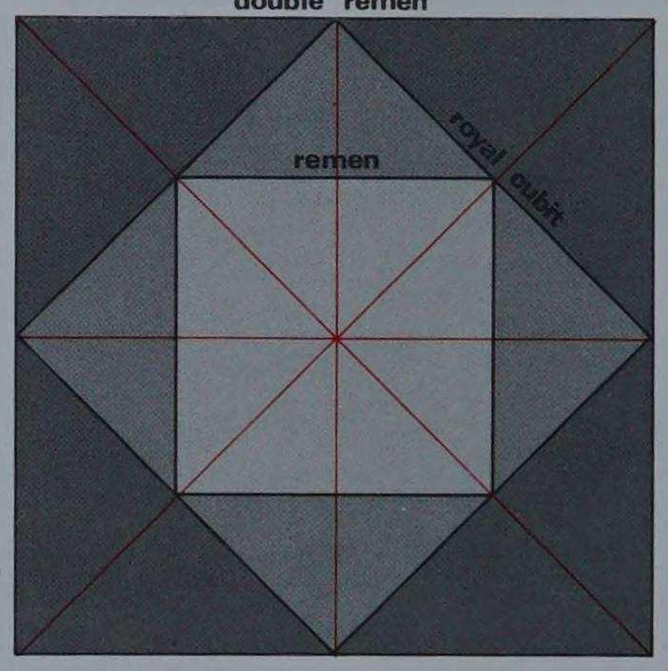


Figure 109



Geometrical figures with remen as sides and royal cubits as diagonals (see **Figure 109**), or with royal cubits as sides and double remens as diagonals, are taken to be squares. Therefore it can be assumed that, by trial and error, an approximate but practical ratio for $\sqrt{2}$:1 was concluded; the diagonal to the side of a square was taken as 7:5 and 10:7, as below.

Ratio	Unit	Decimal value	Difference	%Deviation
$\frac{\sqrt{2}}{1}$	$\frac{\text{diagonal}}{\text{side}}$	1.4142136	—	—
$\frac{7}{5}$	$\frac{\text{royal cubit}}{\text{remen}}$	1.4	0.0142	–1%
$\frac{10}{7}$	$\frac{\text{double remen}}{\text{royal cubit}}$	1.4285	0.0143	+1%

The same squares can also be presented as a series of nesting squares (see **Figure 110**). Drawing concentric octagonal stars, by means of inscribing a circle in the outer square and using the point-joining method, establishes a geometric system of concentric squares in which the consecutive parallel sides are related in the proportion of 1:√2, while the alternate parallel sides are related in the proportion of 1:2. The areas of the consecutive squares are also related in the proportion of 1:2. This results in a harmonic progression (**Figure 111**) where all the sides and diagonals of concentric squares can be expressed as a series: $\sqrt{2} = \frac{7}{5} = \frac{10}{7} = \frac{14}{10} = \frac{20}{14} = \frac{28}{20} = \frac{40}{28} = \frac{56}{40}$ ...

This was the basis of Ancient Egyptian land measure and by this system of metrology it was possible to denote areas in squares equal to one half or twice the area of another (an early form of Pythagoras' Theorem). Therefore it can be assumed that in Egyptian architecture the square was employed as the basic unit of measure.

Figure 110

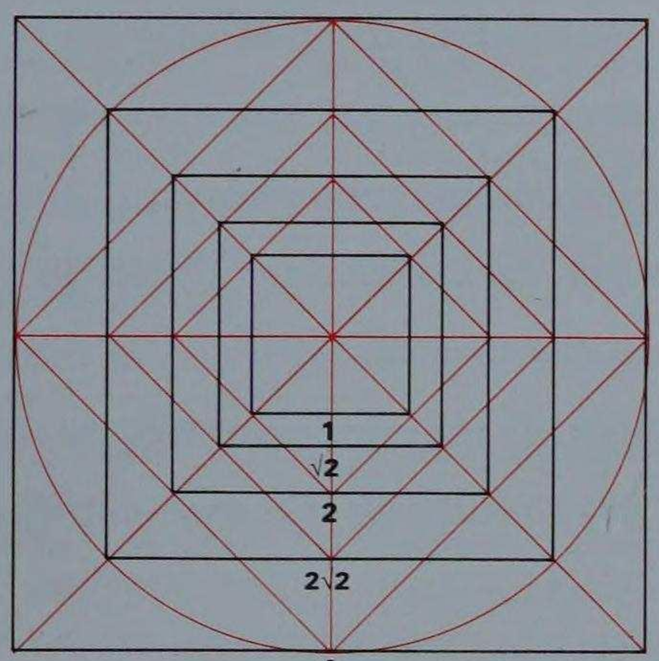
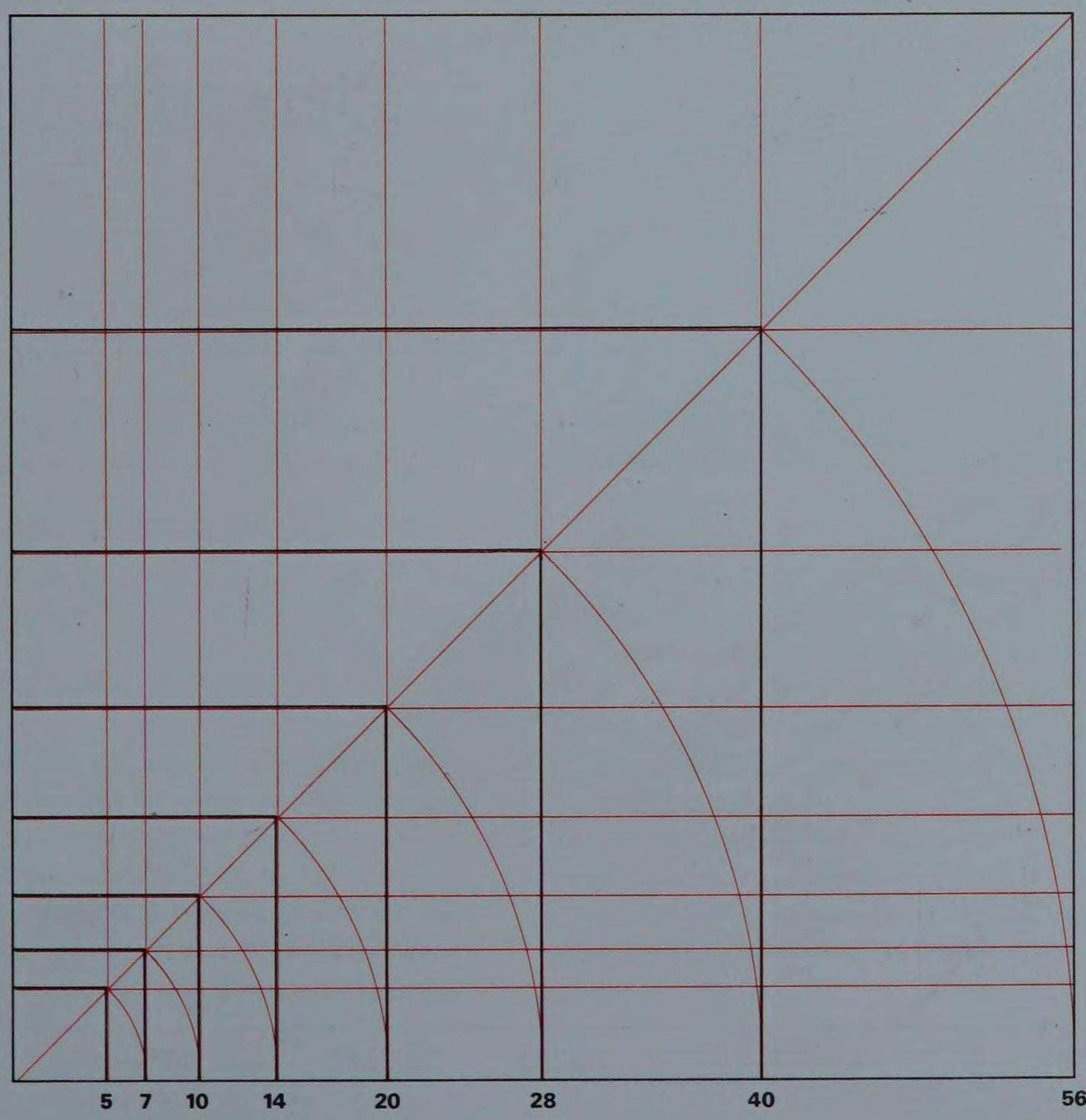


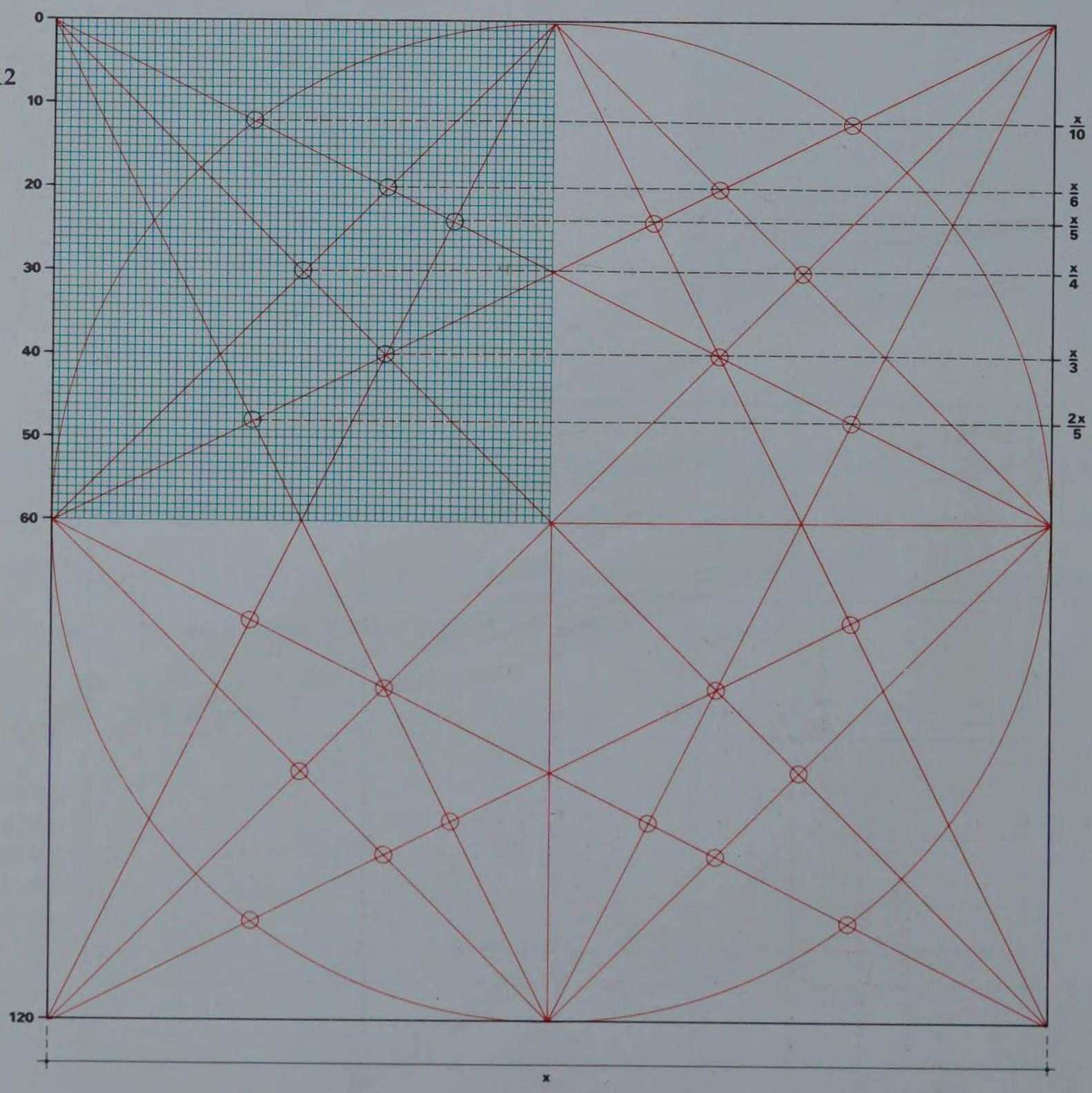
Figure 111



The Grid-Pattern

When the appropriate geometric elements used to divide the sides, diagonals and areas of the square into aliquot parts are drawn (e.g. diagonals, diagons, inscribed squares or circles), they generate a 'grid-pattern'.[23] This grid-pattern has constant proportional relationships among the linear dimensions and points of intersection (shown marked with circles, **Figure 112**). These dimensions can be expressed as aliquot parts of x, the side of the circumscribing square; i.e. the values read on the scale on the right-hand side may be compared to their numerical values given on the scale on the left-hand side where x is taken to be 120 units long. The proportions of these dimensions, therefore, can be considered not as a part of a geometric theory, but as forming convenient rules for arithmetic solutions. If a plan of a building were to be based on such a grid, then the size of specific parts of the building could be derived as an aliquot part of a convenient measure of x. If applied in this geometric way, the result is a logical, repeatable method controlled by a highly-co-ordinated system of geometric proportioning. Thus geometry, which etymologically means the object of land measure, could be considered as an essential practical tool for mensuration, its rules relying on geometrical figures and numerical ratios with virtually no analytical reasoning or calculation. It may be assumed that these rules were based on the experience and observation of many centuries,[24] and that empirical evidence constituted the best proof of correctness.

Figure 112



Notes

1 "Herodotus held that geometry had originated in Egypt, for he believed that the subject had arisen from the practical need for surveying after the annual flooding of the river valley. Aristotle argued that it was the existence of a priestly leisure class in Egypt that had prompted the pursuit of geometry". (C.B. Boyer and U.C. Herzbach, *A History of Mathematics,* 1968, p.6). Their views present opposing theories of the beginnings of mathematics (geometry and arithmetic); one holding to an origin in practical necessity, the other to an origin in priestly leisure and ritual. However, it is clear that the beginnings of mathematics are older that the oldest civilizations.

2 The *Ahmes Papyrus* was bought in 1858 by the Scottish antiquary, Alexander Henry Rhind. It is often known, therefore, as the *Papyrus Rhind* and is now in the British Museum. (D.J. Struik, *A Concise History of Mathematics,* 1987, p.21). Petrie dates the *Papyrus Rhind:* 'The papyrus is well dated in the thirty-third year of the fourth of the great Hyksos Kings of the XVth dynasty, or 2239 BC' (W.Petrie, *The Wisdom of the Egyptians*, p.32).

3 Struik, op. cit., p.22.

4 Chambers Encyclopaedia, Vol. IX, p.148.

5 A.H.Gardiner, *Egyptian Grammar,* 1950, pp.191, 198.

6 Struik, op. cit.., p.22.

7 Ibid.

8 A. Badawy, *Ancient Egyptian Architectural Design,* 1965, pp.8–9.

9 L. Hogben, *Mathematics in the Making,* 1960, p.56.

10 P. Dedron and J. Itard, *Mathematics and Mathematicians*, 1974, Vol.ii, pp.13–15.

11 T.L. Heath, *A History of Greek Mathematics*, 1921, Vol.i, p.28.

12 Dedron and Itard, op. cit., pp18–19.

13 Heath, op. cit., pp41–45.

14 S.H. Nasr, *Islamic Science*, pp.78–79.

15 "Leonardo, also called Fibonacci, travelled in The Orient as a merchant. On his return he wrote his *Liber Abaci,* filled with arithmetical and algebraic information which he had collected on his travels. In the *Practica Geometriae* (1220), Leonardo described in a similar way whatever he had discovered in geometry and trigonometry. The *Liber Abaci is* one of the means by which the Hindu-Arabic system of numeration was introduced into Western Europe ... 'the oldest dated European manuscript containing the numerals is the *Codex Vigilanus,* written in Spain in 976'. Nevertheless, the Roman system of numeration remained in use. Computations were often performed on the abacus and Roman numerals were used to register the results throughout the Middle Ages. The introduction of the Hindu-Arabic numerals was met with opposition at first. In the statues of the *Arte del Cambio* of 1299, the banks of Florence were forbidden to use Arabic numerals and were obliged to use cursive Roman ones. It is from 1494 that only Hindu-Arabic numerals were used in the Medici account books." (Struik, op. cit., pp.80–81).

16 ' Diagon' is the name given to the diagonal of a half-square by investigators attached to the National Museum at Ljubljana, Yugoslavia. (K.J. Conant, "The After-life of Vitruvius in the Middle Ages", *Journal of the Society of Architectural Historians*, 27, 1968, p.34).

17 M. Kline, *Mathematics in Western Culture,* pp.55–56.

18 Euclid was the first to show geometrical theorems as part of a system of connected truths, thus using geometry as a theoretical device for deductive reasoning.

19 Chambers Encyclopaedia, op. cit., p.175.

20 Ibid.

21 Ibid., p.176.

22 Griffith, when Professor of Assyriology at Oxford in 1891, identified the diagonal of the square of the remen with the cubit that Herodotus called 'royal'.

23 The term 'grid-pattern' has been introduced by the editors (see page 113, note 4). In this context, El-Said originally uses the term 'geometric pattern (grid)'.

24 The earliest extant architectural treatise advocating geometry as a device for mensuration is found in *De Architectura (The Ten Books on Architecture)* by Vitruvius, c.30 BC. In the chapter entitled 'The Education of the Architect', Vitruvius writes: "Geometry, also, is of much assistance in architecture, and in particular it teaches us the use of the rule and compasses, by which especially we acquire readiness in making plans for buildings in their grounds... It is true that it is by arithmetic that the total cost of building is calculated and measurements are computed, but difficult questions involving symmetry are solved by means of geometrical theories and methods." (Vitruvius, *The Ten Books on Architecture*, p.6.)

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Index

Ahmes Papyrus (Papyrus Rhind) 117 118 119 133
Ancient Egyptian numeration 22 116 118 –121 130 – 133
Babylonian numeration 118 119
Diagon 125 126 133
Division 18
- Using the circle 121
- Using the square 124
Double Hexagon System 14 88 –111
- Grid patterns underlying... 88
- Graphical analyses of geometric patterns... 89
Double remen 120 130
Euclidean Geometry 24
Folding technique 18 121 123
Greek numeration 119 120 127 128 130
Grid-patterns 14 113 132 133
- Underlying the root two system of proportion 19
- Expressed in terms of the selected square module 20
- Generated using numbers 22
- Underlying the root three system of proportion 60
- Underlying the double hexagon system of proportion 88
Hexagonal repeat unit 14 56 – 87 91 106
- Construction of... 56
- Determining the size of... 58
Hindu-Arabic numerals 120 133
Irrational numbers 127 128 129
Mesopotamian numeration 116 118 119
Moscow Papyrus 117
Octagon 41 42 45 46 48 49
Octagonal Star (stellated octagon) 19 33 35 36 39 55
Phoenician alphabet 119
Point-joining method 19 24 113 121
Polygon 14 121 123
- Twelve sided 90 92 94 96 104
Primary Grid A 19 20 21 22 24 28 31 32 33 37 39 40 41 49 66
Primary Grid B 19 21 23 24 42 – 55
Primary Grid C 60 64 70 – 74 76 80 81 86 87 98
Primary Grid D 60 65 77 – 81 84 85
Proportioning 20
Pythagoras' Theorem 22 26 61 127 128 131
Rational numbers 127 128
Remen 22 128 130 133
Root Two system of proportion 14 16 – 55
- Grid - patterns underlying... 19
- Mathematical and numerical rationalization... 24
- Graphical analyses of geometric patterns... 30 – 55
- Combining root two and root three systems... 88
Root Three system of proportion 14 56 – 87
- Grid patterns underlying... 60
- Mathematical and numerical rationalization... 66
- Graphical analyses of geometric patterns... 69
Rotated (dynamic) square 40 51
Rotated octagonal star 41 52 – 5 4
Royal Cubit 22 130
Secondary Grid A 20 – 23 33 – 36
Secondary Grid B 23 45 – 47
Secondary Grid D 82 84 – 87
Selected square module 20 23
Sexagesimal system 119
Six pointed star 57 62 66 83 98 121 122
Sixteen pointed star 28
Square repeat unit (s) 14 16 – 55 107 110
- Construction of ... 16
- Determining the size of... 17
- Determining the number of... 18
Sumerian mathematics 116
Tertiary Grid A 35 36
Tertiary Grid C 86 87
Theorem of Pythagoras: see Pythagoras' Theorem

P9-DAW-756