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Hilbert's significance for the philosophy of mathematics (1922)

Paul Bernays

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Translation by: *Paolo Mancosu*

Comments:

If we look upon the intellectual relationships of the mathematical sciences to philosophy as they have developed since the times of the Enlightenment, we notice with satisfaction that at present mathematical thought is at the point of regaining that powerful influence on philosophical speculation that it possessed up to Kant's time, which it then, suddenly, lost completely. That sudden averting from mathematical thought was influenced by the general estrangement from the spirit of the period of the Enlightenment that took place at the beginning of the nineteenth century.

However, this detachment of philosophy from the exact sciences was only a unilateral one. Namely, while the dominant philosophy became quite estranged[1] from mathematics, a philosophical orientation evolved more and more among mathematicians.

The essential reason for this was that mathematics had grown far beyond the limits within which it had moved up to the time of Kant. Not only had the domain of investigated facts grown considerably larger, but the whole structure of the investigations became grander and the entire method more encompassing. The concept-formations [*Begriffsbildungen*] rose to a higher level of generality; the meaning of the formula became less important compared with conceptual abstractions and systematic fundamental ideas. Furthermore, the attitude toward the foundations and toward the object of the mathematical sciences also changed.

The task of geometry was understood in broader terms. Geometrical concepts became more general and freed themselves more and more from the subordination to spatial intuition [*Vorstellung*]. Moreover, in the recent geometrical theories the intuition of space no longer had the significance of an epistemological foundation, but rather it was employed here merely in the sense of an intuitive analogy.

In arithmetic, research experienced an essential extension of problem formulation as well. On the one hand, the concepts of Number [*Anzahl*] and of order [*Ordnung*] were, through the invention of set theory, generalized in a completely |*Mancosu*: 190 new way and applied to infinite totalities. On the other hand, the development of algebra led to numbers and quantities no

[1] Among those philosophers who in this respect represented a laudable exception, Bolzano, in particular, must be mentioned, who first gave the rigorous foundation of the theory of real numbers.

longer exclusively viewed as objects of investigation; rather, mathematical formalism itself was made an object of study, and one focused, very generally, on the examination of formalisms. Numbers as well as quantities now appeared only as something special, and the more one examined their lawfulness under more general points of view, the more one became unaccustomed to taking this lawfulness for granted. In this way, the whole development of mathematics moved on: to rob of its appearance of exclusiveness and finality all of that which was previously considered to be the only object of research, and whose basic properties were considered as something to be accepted by mathematics and neither capable nor in need of mathematical investigation. The bounds the previous philosophical view, and even the Kantian philosophy, had marked out for mathematics were burst. Mathematics no longer allowed philosophy to prescribe the method and the bounds of its research; rather it took the discussion of its methodological problems into its own hands. In this way the axioms of the mathematical theories were more precisely investigated on the basis of their logical relations, and the forms of inference were subjected to more precise critique as well. And the more these problems have been pursued, the more mathematical thought has shown its fruitfulness with respect to them, and it has proved itself an indispensable tool for theoretical philosophy.

David Hilbert has contributed in a significant way to this development, which extends to the present. What he has accomplished in this field will be described in what follows.

When Hilbert applied himself to the problems that were to be solved concerning the foundations of mathematical thought, he had at his disposal not only his comprehensive command of the mathematical sciences, but he was also above all, as it were, predestined for the task by his human disposition. Mathematics had for him the meaning of a world view, and he went about those fundamental problems with the attitude of a conqueror who endeavors to secure a sphere of influence for mathematical thought that is as comprehensive as possible.

In the pursuit of this goal the main point was to avoid the mistake of those extreme rationalist thinkers who thought that a complete knowledge of all that is real [*alles Wirkliche*] could be attained by pure thought. This could not be a question of, say, including in mathematics all knowledge of the factual [*Tatsächliches*]; rather it was necessary, for the purpose of the widest possible extension of the domain of mathematics, to undertake a sharp delineation of boundaries between the mathematical and the nonmathematical, which would actually allow one to claim for mathematics all mathematical components in cognition [*alle mathematischen Bestandteile im Erkennen*].

Hilbert has actually also understood the problem in this way. His first and largest work in the field of methodological questions is the *Foundations of Geometry*, which appeared in 1899. In this work, Hilbert laid out a new system of axioms for geometry that he chose according to the criteria of simplicity and logical completeness, following Euclid's concepts as closely as possible. He subdivided the whole system of axioms into five groups of axioms and then investigated more precisely the part that the different groups of axioms (as well as single axioms) have in the logical construction of geometry. |*Mancosu*: 191

Through the wealth of new, fruitful methods and viewpoints it presented, this investigation has exercised a powerful influence on the development of mathematical research. However, the significance of Hilbert's "Foundations of Geometry" by no means lies only in purely mathematical contents. What conferred popularity on this book and made Hilbert's name renowned far beyond the circle of his colleagues was the new methodological turn that was given to the idea of axiomatics.

The essence of the axiomatic method, that is, the method of logically developing a science from axioms and definitions, consists, according to the familiar conception, in the following: One starts with a few basic propositions [*Grundsätze*], of whose truth one is convinced, puts them as axioms at the top, and from them, by means of logical inference, one derives theorems whose truth is then as certain as that of the axioms, precisely because they follow logically from the axioms. In this view, attention is focused above all on the epistemic character [*Erkenntnischarakter*] of the axioms. Indeed, originally one recognized as axioms only propositions whose truth would be clear a priori. And still Kant held the view that the success and the fruitfulness of the axiomatic method in geometry and mechanics essentially rested on the fact that in these sciences one could proceed from a priori knowledge (the axioms of pure intuition and the principles of pure understanding [*Verstand*]).

This demand that each axiom should express an a priori knowable truth was soon abandoned, for, in the manifold occasions that presented themselves for the axiomatic method, especially in the further development of physics, it followed, so to speak, automatically that one chose both empirical statements and also mere hypotheses as axioms of physical theories. The axiomatic procedure turned out to be especially fruitful in the cases where one succeeded in encompassing the results of multifarious experiences in a statement of general character through the positing of an axiom. A famous example of this is the two propositions about the impossibility of a *perpetuum mobile* of the first and second type, which Clausius put at the top as axioms of the theory of heat.

Moreover, the belief in the a priori knowledge of the geometrical axioms was increasingly lost among the researchers in the exact sciences, mainly as a consequence of non-Euclidian geometry and the persuasiveness of the arguments by Helmholtz. Thus the empirical viewpoint, according to which geometry is nothing but an empirical science, found more and more supporters. However, this departure from a prioricity [*vom Apriorismus*] did not essentially alter the view under which one considered the axiomatic method.

A more powerful change, however, was brought about by the systematic development of geometry. Mathematical abstraction had, starting with elementary geometry, raised itself far above the domain of spatial intuition and had led to the construction of comprehensive systems, in which ordinary Euclidian geometry could be incorporated and within which its lawlikeness appeared only as one particular among others of equal mathematical rights. With this a new sort of mathematical speculation opened up by means of which one could consider the geometrical axioms from a higher standpoint. It immediately became apparent, however, that this mode of consideration had nothing to do with the question of the epistemic character of the axioms, which had, after all, formerly been considered as the only significant feature of the axiomatic method. Accordingly, the necessity of a clear separation between the mathematical and the epistemological problems of axiomatics

|*Mancosu: 192* ensued. The demand for such a separation of the problems had already been stated with full rigor by Klein in his Erlangen Programme.[2] The important thing, then, about Hilbert's "Foundations of Geometry" was that here, from the beginning and for the first time, in the laying down of the axiom system, the separation of the mathematical and logical [*spheres*] from the spatial-intuitive [*sphere*], and with it from the epistemological foundation of geometry, was completely carried out and expressed with complete rigor.

To be sure, in the introduction to his book Hilbert does express the thought that laying down the axioms of geometry and the investigation of their relationships is a task that amounts "to the logical analysis of our spatial intuition," and likewise he remarks in the first paragraph that each single group of axioms expresses "certain related basic facts of our intuition." However, these statements are located completely outside the axiomatic construction, which takes place without any reference whatsoever to spatial intuition.

Of course demands have always been placed upon a rigorous axiomatic grounding of geometry that the proofs should exclusively appeal to what is formulated in the axioms, and they must not rely upon spatial intuition in any way. More recently, Pasch, in his foundation of geometry,[3] has placed importance on the carrying out of this demand and has been completely consistent in doing so.

However, Hilbertian axiomatics goes one step further in the elimination of spatial intuition. Reliance on spatial representation is completely avoided here, not only in the proofs but also in the axioms and the concepts. The

[2] "Vergleichende Betrachtungen über neue geometrische Forschungen," Mathematische Annalen, Bd. 43, 1872.

[3] "Vorlesungen über neuere Geometrie," 1882.

words "point," "line," "plane" serve only as names for three different sorts of objects, about which nothing else is assumed directly except that the objects of each sort constitute a fixed determinate system. Any further characterization is carried out only through the axioms. In the same way, expressions Re "the point A lies on the line a" or "the point A lies between B and C" will not be associated with the usual intuitive meanings; rather these expressions will designate only certain, at first indeterminate, relations, which *are implicitly characterized*[4] only through the axioms in which these expressions occur.

According to this conception, the axioms are in no way judgments that can be said to be true or false; they have a sense only in the context of the whole axiom system. And even the axiom system as a whole does not constitute the statement of a truth; rather, the logical structure of axiomatic geometry in Hilbert's sense—analogously to that of abstract group theory—is a purely hypothetical one. If there are anywhere in reality three systems of objects, as well as determinate relations between these objects, such that the axioms of geometry hold of them (this means that by an appropriate assignment of names to the objects and relations, the axioms turn into true statements), then all theorems of geometry hold of these objects and relationships as well. Thus the axiom system itself does not express something factual; rather, it presents only a possible form of a system of connections that must be investigated mathematically according to its *internal* [innere] properties.

Accordingly, the axiomatic treatment of geometry consists in separating the purely mathematical part of knowledge from geometry, considered

[4]One speaks in this sense of "implicit definition."

as a science of spatial figures, and investigating it on its own in isolation. The spatial relationships are, as it were, projected into the sphere of the mathematical-abstract in which the structure of their connections appears as an object of pure mathematical, thought. |Mancosu: 193 This structure is subjected to a mode of investigation that concentrates only on the logical relations and is indifferent to the question of the *factual* truth, that is, the question whether the geometrical connections determined by the axioms are found in reality (or even in our spatial intuition).

This sort of interpretation of the axiomatic method presented in Hilbert's "Foundations of Geometry" offered the particular advantage of not being restricted to geometry but of being directly applicable to other disciplines. From the beginning, Hilbert envisaged the point of view of the uniformity of the axiomatic method in its application to the most diverse domains, and guided by this viewpoint, he tried to bring this method to bear as widely as possible. In particular, he succeeded in grounding axiomatically the kinetic theory of gases as well as the elementary theory of radiation in a rigorous way.

Many mathematicians subscribed to Hilbert's axiomatic mode of investigation and worked in the spirit of his endeavors. In particular, it was a great success of axiomatics when Zermelo, in the field of set theory, overcame the existing uncertainty of inference by a suitable axiomatic delimitation of the inferential modes and, at the same time, created with his system a common foundation for number theory, analysis, and set theory.[5]

[5] "Untersuchungen über die Grundlagen der Mengenlehre," Mathematische Annalen, Bd. 65, 1907.

In his Zurich lecture on "Axiomatic Thought" (1917),[6] Hilbert has given a summary of the methodological guiding principles and an overview of the results of axiomatic research. In his lecture he characterizes the axiomatic method as a general procedure for scientific thinking. This procedure can be applied in all areas of knowledge where one has already come to the point of setting up a theory, or, as Hilbert says, to an arrangement of the facts by means of a framework of concepts. Thus each time we see that a few propositions suffice for the logical construction of the theory, and through this the axiomatic foundation of the theory is made possible. This will at first take place in the sense of the old axiomatics; however, one can always move on to Hilbert's axiomatic standpoint by disregarding the epistemic character of the axioms and by considering the whole framework of concepts only (as a possible form of a connection of relations) in its internal structure.

Accordingly, the theory turns into the object of a purely mathematical investigation, exactly what is called *axiomatic* investigation. The same main questions must always be considered for all theories: First of all, the axiom system, with which a system of connection of relations is represented, must satisfy the condition of *consistency*; that is, the relations expressed in the axioms must be logically compatible with one another. Consequently, the task of proving the consistency of the axiom system arises. The old axiomatics was not familiar with this problem because for this conception every axiom indeed counts as a statement of a truth. After that it is a question of gaining an overview of the logical *dependencies* among the different theorems of the theory. A particular focus of investigation must be whether the axioms are

[6]Mathematische Annalen, Bd. 78

logically independent of each other, or whether, say, one or more axioms can be proven from the remaining ones and are thus superfluous in their role as axioms. In addition, there remains the task of investigating the possibilities of a "deepening of the foundations," that is, examining whether the given axioms of the theory might not be reduced to propositions of a more fundamental character that would then constitute "a deeper layer of axioms" for the framework of concepts under consideration.

|[Mancosu: 194] This sort of investigation, which is definitely of a mathematical character, can now be applied to any domain of knowledge that is at all suitable for theoretical treatment, and its realization is of the highest value for the clarity of knowledge and for a systematic overview. Thus mathematical thought gains a universal significance for scientific cognition [*Erkennen*] through the idea of axiomatics. Hilbert can indeed claim: "everything that can be the object of scientific thought is subject, as soon as it is ripe for the construction of a theory, to the domain of axiomatics and thereby of mathematics."

By means of this comprehensive development [*Ausgestaltung*] of axiomatic thought, a sufficiently wide context was indeed obtained for the mathematical formulation of problems, and the epistemological fruitfulness of mathematics was made clear. But with regard to the *certainty* of the mathematical procedure, a fundamental question still remained open.

Of course, the task of proving the consistency of the axiom system was indeed recognized as first and foremost in the axiomatic investigation of a theory. In fact, the consistency of the axioms represents the vital question for any axiomatic theory, for whether.the framework of concepts represents a connection of relations or only the appearance of such a connection at all depends on this question.

If we now examine how things stand with the proof of consistency for the several geometrical and physical theories that have been axiomatically grounded, then we find that this proof is produced in general only in a relative sense: The consistency of the axiom system to be investigated is proven by exhibiting a system of objects and of relations *within mathematical analysis* that satisfies the axioms. This "method of reduction" to analysis (i.e., to arithmetic in the wider sense) presupposes that analysis itself—independently of whether it is considered as a body of knowledge [*Inbegriff von Erkenntnissen*] or only as an axiomatic structure (i.e., as a merely possible system of relations)—constitutes a consistent system.

However, the consistency of analysis is not as immediately evident as one would first like to think. The modes of inference that are used in the theory of real numbers and real functions do not have that character of direct evidence [*Charakter des unmittelbar Handgreiflichen*], which is characteristic, for instance, of the inferences of elementary number theory. And if one wants to free the methods of proof from everything problematic, then one is compelled to set up a theory of analysis axiomatically. Thus it turns out to be necessary also to provide a consistency proof for analysis.

From the beginning, Hilbert has recognized and emphasized the need for such a proof for the certainty of the axiomatic method and of mathematics in general. And although his efforts concerning this problem have not yet reached the ultimate goal, he has nonetheless succeeded in finding the methodological approach by which the task can be mathematically undertaken. Hilbert stated the main idea of this approach by 1904 in his Heidelberg lecture "On the Foundations of Logic and Arithmetic."[7] However, these comments were difficult to understand and were subject to some objections. Since then, Hilbert has pursued his plan further and has given a comprehensible form to his ideas, which he recently presented in a series of lectures in Hamburg. ne reasoning on which Hilbert's approach to the foundation of arithmetic and analysis is based is as follows: the methodological difficulties of analysis, on the |*Mancosu*: 195 basis of which in this science one is compelled to go beyond the scope of what is concretely representable, follow from the fact that continuity and infinity play here an essential role. This circumstance would also constitute an insuperable obstacle to the proof of the consistency of analysis, if this proof had to be carried out in the sense that one shows that a system of things as assumed in analysis, say the system of all finite or infinite sets of natural numbers, is logically possible.

However, the claim of consistency need not at all be proven in this sense. Rather, one can give the following new twist to the claim: The modes of inference of analysis can never lead to a contradiction or, what amounts to the same thing: It is impossible to derive the relation $1 \neq 1$ ("1 is different from 1") from the axioms of analysis and by means of its methods of inference. Here it is not a question of the possibility of a continuous, infinite manifold of certain properties, but of the impossibility of a mathematical proof with determinate properties. A mathematical proof is, however, unlike a continuous infinite manifold, a concrete object surveyable in all its parts. A mathematical proof must, at least in principle, be completely communicable from

[7] Appendix VII to the "Grundlagen der Geometrie."

beginning to end. Moreover, the required property of the proof (i.e., that it proceeds according to the principles of analysis and leads to the final result $1 \neq 1$) is also a concretely determinable property. seat is why there is definitely, also, in principle, the possibility of furnishing a proof of consistency for analysis by means of elementary, and obviously certain, considerations. We only have to take the standpoint that the object of investigation is not constituted by the objects to which the proofs in analysis refer, but rather these proofs themselves. On the basis of this consideration, the task arises then for Hilbert to examine more precisely the forms of mathematical proofs. We must, he says in his lecture on axiomatic thought, "make the concept of specific mathematical proof itself the object of our investigation, exactly as the astronomer takes into account the motion of his location, the physicist concerns himself with the theory of his apparatus, and the philosopher criticizes reason itself." The general forms of logical inference are decisive for the structure of mathematical proofs. That is why the required investigation of mathematical proofs also concerns the logical forms of inference. And thus Hilbert pointed out already in the Heidelberg lecture that "a partially simultaneous development of the laws of logic and arithmetic [is] necessary."

With this thought Hilbert took up *mathematical logic*. This science, the main idea of which goes back to Leibniz, emerged from primitive beginnings and developed into a fruitful field of mathematical thought in the second half of the nineteenth century. Mathematical logic has developed the methods for acquiring a mathematical mastery of the forms of logical inference through a symbolic denotation of the simplest logical relations (as "and," "or", "not," and "all"). It turned out that only by means of this "logical calculus" does one gain the complete overview of the system of logical forms of inference. The inferential figures, which are dealt with in traditional logic, constitute only a relatively small subfield of this system. In particular, Peano, Frege, and Russell succeeded in setting up the logical calculus in such a way that the intellectual inferences [*gedankliche Schlüsse*] of mathematical proofs can be perfectly reproduced by means of symbolic operations.

This procedure of the logical calculus supplements the method of the axiomatic grounding of a science, to the extent that such a procedure makes possible, along with the exact laying down of the *presuppositions* as it is brought about by the|*Mancosu*: 196 axiomatic method, an exact pursuit of the *inference modes* with the aid of which one proceeds from the principles of a science to its conclusions.

In adopting the procedure of mathematical logic, Hilbert reinterpreted it as he had done with the axiomatic method. Just as he had formerly stripped the basic relations and axioms of geometry of their intuitive content, he now eliminates the intellectual content of the inference from the proofs of arithmetic and analysis that he makes the object of his investigation. He obtains this by taking the systems of formulas that represent those proofs in the logical calculus, detached from their contentual-logical interpretations, as the immediate object of study, and by replacing the proofs of analysis with a purely formal manipulation that takes place with certain signs according to definite rules.

Through this mode of consideration, in which the separation of what is specifically mathematical from everything contentual reaches its high point, Hilbert's view on the nature of mathematics and on the axiomatic method then finds its actual conclusion. For we recognize at this point that the sphere of the mathematical-abstract, into which the methods of thought of mathematics translate all that is theoretically comprehensible, is not that of the contentual-logical [*inhaltlich Logisches*] but rather that of the domain of pure formalism. Mathematics turns out to be the general theory of formalisms, and by understanding it as such, its universal meaning also becomes clear.

This meaning of mathematics as general theory of forms has come to light in recent physics in a most splendid way, especially in Einstein's gravitational theory, in which the mathematical formalism gave Einstein the guidelines for setting up his gravitational laws, the exact form of which could never have been found without enlisting mathematical tools. And it was once again Hilbert who first brought gravitational law to its simplest mathematical form by showing the possibility of a harmonious combination of gravitational theory with electrodynamics, which has further opened mathematical speculations that are connected with Einstein's theory and were then systematically completed by Weyl by means of his geometrical ideas. If these speculations should prove to be worthwhile in physics, then the triumph of mathematics in modem science would thereby be complete.

If we now observe the ideas yielded by Hilbert's philosophical investigations as a whole, as well as the effect they have had, and if we, on the other hand, bear in mind the development of mathematics in the recent past as initially outlined, then the important thing about Hilbert's philosophical accomplishment appears to us to consist of the following. By developing a broad-minded philosophical conception of mathematics, which made it possible to do justice to the meaning and scope of its method, Hilbert has shown, with force and success, the claim to a universal influence in science that mathematics has gained by its inner consolidation and its ambitious formulation. The friends of mathematics will always be indebted to him for this.