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Remarks on the philosophy of mathematics (1969)

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(Bemerkungen zur Philosophie der Mathematik)

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Comments:

none

When we compare mathematics with logic in regard to the role these two domains of knowledge [*Erkenntnisgebieten*] are assigned to within philosophical thinking, we find a disagreement among the philosophers.

For some logic is primary [*das Ausgezeichnete*]; for them, logic in the wide sense is the $\lambda \acute{o} \gamma o \varsigma$, that what is rational, and logic the narrower sense is the inventory [*Bestand*] of elementary insights which should lie beneath all considerations [*Überlegungen*], i.e., the inventory of those truths that hold independently of any particular matters of fact [*Sachhaltigkeit*]. Thus, logic in the narrower sense ("pure logic") has a primary epistemological status.

A different starting point [*Ausgangspunkt*] takes the method of mathematics as exemplary [*als Vorbild*] for all scientific thinking. While that which is logical [*das Logische*] is regarded as self-evident and unproblematic [*unproblematisch*] from the first point of view, from the second point of view that which is mathematical is regarded as epistemologically unproblematic. Accordingly, understanding is ultimately mathematical understanding. The idea that all rational insight [*Einsicht*] must be of a mathematical kind plays a fundamental role particularly also in the arguments [*Argumentationen*] of David Hume.

From this point of view Euclid's Elements were regarded for a long time as representative [*als Repräsentant*] for the mathematical method. But often times it was not sufficiently clear that the Euclidean axiom system is particular from the axiomatic standpoint (the fact that early commentators had already come up with suggestions for replacing axioms by equivalent ones has been an indication for this). Obviously many were of the opinion — although probably not the authors of the Greek work — that the possibility of a strict and successful proof in geometry is based on the evidence [*? Evidenz*] of the axioms.

Those who philosophized after the axiomatic method, in particular in the school of Christian Wolff, at times understood evidence as conceptual evidence, so that they did not distinguish between the logical and the mathematical. The principle of contradiction [*Satz vom Widerspruch*] (which mostly included the principle of excluded middle [*Satz vom ausgeschlossenen Dritten*]) was regarded as a panacea [*Zaubermittel*] so to speak, from which all scientific [*naturwissenschaftlichen*] and metaphysical knowledge [*Erkenntnisse*] could be obtained with the help of suitable formations of concepts [*Begriffsbildungen*].

As you know, Kant has emphasized the moment [*das Moment*] of the intuitive in mathematics in his theory [*Lehre*] of pure intuition in opposition to this philosophy. But also for Kant the possibility of geometry as a successful deductive science is based on the evidence of the axioms, i.e., on the intuitive [*anschaulichen*] insightfulness [*Einsichtigkeit*] of their postulates of existence [*Existenzpostulate*], according to him. That the discovery of non-Euclidean, Boyai-Lobatschefskian geometry had such a revolutionary [*umwälzend*] effect on the philosophical doctrine [*Lehrmeinung*] is explained by the vagueness [*Undeutlichkeit*] in the epistemological judgment of Euclid's geometry.

But a fundamental change of aspect [*Aspekt*] resulted also for the first of the two mentioned points of view from the development of mathematical logic. It became clear that logic as a discipline (which it was already with Aristotle) does not consist directly in establishing [*Feststellung*] singular logical facts, but rather in investigating the possibilities of proofs [*Beweismoeglichkeiten*] in formally delimited domains of deduction, and should better be called metalogic. Furthermore, the method of such a metalogic is typically mathematical.

Thus it might seem appropriate [*Hiernach moechte es als angezeigt erscheinen*] to classify logic under mathematics. The fact that this has mostly not been done is explained by the lack of a satisfactory epistemological point of view [*Ansicht*] of mathematics. The term "mathematical" was not, so to speak, a sufficiently familiar philosophical term [*Vokabel*]. One tried to understand mathematics itself by classifying it under logic. This is particularly true of Gottlob Frege. You surely know Frege's definition of cardinal number [*Anzahl*] in the framework of his theory of predicates [*Prädikatentheorie*]. The method employed here is still important today for the classification of number theory under set theory. Several objections (that might be discussed with who is interested) can be raised against this view that hereby an epistemological reduction to pure logic has been achieved.

A different way of approaching the question of the relation between mathematics and logic is — as is done in particular by R. Carnap — to regard both as being analytic. Thereby the Kantian concept of analyticity is extended in principle [*grundsätzlich*], which has been pointed out especially by E. W. Beth. The same character of self-evidence [*Selbstverständlichkeit*] is mostly attributed to analyticity in this extended sense, that is ascribed [*zukommt*] to analytic sentences in the Kantian sense.

As you know, W. V. Quine was fundamentally opposed to the distinction between the analytic and synthetic. Although his arguments contain many appropriate points [*Zutreffende*], they do not make justice to the circumstance that by the distinction between the analytic in the wide sense and the synthetic, a fundamental distinction is hit upon, namely the distinction between mathematical facts [*Sachverhalt*] and facts about the actuality of nature [*Naturwirklichkeit*]. Just to mention something in this regard: Mathematical statements are justified [*begründet*] in a different sense than statements in physics. The mathematical magnitudes of analysis are relevant for physics only approximately. For example, the question whether the speed of light is measured in the centimeter-second-system by a rational or an irrational number has hardly any physical sense.

Sure enough [*freilich*], the fundamental difference between that which is mathematical and that which belongs to the actuality of nature [*Naturwirklichkeit*] is not a sufficient [*hinlänglich*] reason to equate mathematics with logic. It appears natural to count to logic only what results from the general conditions and forms of discourse [*Diskursivität*] (concept and judgment [*der Begrifflichkeit und des Urteilens*]). But mathematics is about [*handelt von*] possible structures, in particular about idealized structures.

Herewith, on the one hand the methodical importance of logic becomes apparent, but on the other hand also that its role is in some sense anthropomorphic. This does not hold in the same way for mathematics, where we are prompted to transcend the domain of what is surveyable in intuition [*des vorstellungsmäßig Überblickbaren*] in various directions. The importance of mathematics for science results already from the fact that we are concerned with structures in all areas of research (structures in society, structures in the economy, structure of the earth [*Erdkörper*], structures of plants, of processes of life [*Lebensvorgänge*], etc.). The methodical importance of mathematics is also due to the fact that a kind of idealization of the objecthood [*Gegenständlichkeit*] is applied in in most sciences, in particular the theoretical ones. In this sense F. Gonseth speaks of the schematic character of the scientific description. What differentiates the theoretically exact from the concrete is emphasized especially also by Stephan Körner. As you know, science has succeeded to understand the connections in nature [*Naturzusammenhänge*] largely [*in einem großen Maße*], and the applicability of mathematics to the identification [*Kennzeichnung*] and explanation of the processes in nature [*Naturvorgänge*] reaches much further than humanity [*Menschheit*] had once anticipated.

But the success and scope of mathematics is something entirely different than its pretended [*vermeintliche*] self-evidence. The concept of self-evidence is philosophically questionable in general. We can speak of something being relatively self-evident in the sense in which, for example, the mathematical facts are self-evident for the physicist, the physical laws for the geologist, and the general psychological properties of man [*Menschen*] for the historian. It may be clearer to speak here of the procedural [*? Vorgängigem*] (according to Gonseth's expression "préalable") instead of that which is self-evident.

At all events mathematics is not self-evident in the sense that it has no problems, or at least no fundamental problems. But consider for instance, that there was no clear method [*Methodik*] for analysis for a long time despite its great formal [*? im Formalen*] success, but the researchers had to rely more or less upon their instinct. Only in the 19th century precise and clear methods have been achieved here. Considered from a philosophical point of view the theory of the continuum by Dedekind and Cantor, which brought the justifications of these methods to an end, is not at all easy. It is not about the bringing to consciousness [*Bewußtmachung*] of an apriori cognition [*Erkenntnis*]. One might rather say that here a very good compromise between the intuitive [*dem Anschaulichen*] and the demands of precise concepts [*präziser Begrifflichkeit*] has been achieved. You also know that not all mathematicians agree with this theory of the continuum and that the Brouwerian Intuitionism advocates a different description of the continuum — of which one can surely find that it overemphasizes the viewpoint of the strict arithmetization at the disfavor of the geometrically satisfactory one.

The problematic [*Problematik*] that is connected to the antinomies of set theory is especially well known and often discussed. As you know, different suggestions have been brought forward to repair [*Behebung*] the antinomies. In particular, axiomatic set theory should be mentioned, which shows that such a small restriction of the set theoretic procedure [*Verfahrens*] suffices to avoid the antinomies that all of Cantor's proofs can be maintained [*aufrecht* *erhalten*]. Zermelo's original axiom system for set theory has been, as you surely know, on the one hand extended, on the other hand formally made sharper [*verschärft*]. The method [*Verfahren*] of solving the antinomies using axiomatic set theory can be interpreted philosophically in the sense that the antinomies are taken as an indication that mathematics as a whole is not a mathematical object and therefore mathematics can only be understood as an open manifold [*Mannigfaltigkeit*].

The application of the methods of making formally precise [*der formalen Präzisierung*] to set theory resulted in a split of the set theoretic considerations into the formulation and deductive development of formal systems, and a model theory. As a result of this split the semantic paradoxes, that could be disregarded for the resolution of the purely set theoretic paradoxes at first, received new formulation [*Ausgestaltung*] and importance. So today we face a new fundamental problematic [*Problematik*] — which surely, as did once the set theoretic antinomies, does not bother [*behelligt*] mathematics in its actual [*eigentlich*] research, that rather unfolds [*entfaltet*] itself in the different disciplines with great success.

The above remarks suggest the following viewpoints [*Gesichtspunkte*] for philosophy of mathematics, which are also relevant for epistemology in general:

1. It appears appropriate to ascribe to mathematics factual content [*Sachhaltigkeit*], which is different than that of the actuality of nature [*Naturwirklichkeit*]. That other kinds of objectivity are possible than the objectivity of the actuality of nature [*Naturwirklichkeit*] is already obvious from the objectivity in the phenomenological areas [*Gebieten des Phänomenalen*]. Mathematics is

not phenomenological insofar, as has been said before, it is about idealized structures on the one hand, and on the other hand it is governed [*beherrscht*] by the method of deduction. By idealizing the intuition [*Anschaulichkeit*] comes into contact with the concepts [*Begrifflichkeit*]. (Therefore, it is not appropriate to oppose intuition [*Anschaulichkeit*] and concepts [*Begrifflichkeit*] so heavily as it is done in Kantian philosophy).

The importance of mathematics for theoretical physics consists in the fact that therein the processes of nature [*Naturvorgänge*] are represented approximatively by mathematical objects [*Gegenständlichkeiten*].

2. It does not follow from the difference between mathematics and empirical research that we have knowledge in mathematics that is secured at the outset (apriori). It seems necessary to concede that we also have to learn in the fields of mathematics and that we here, too, have an experience sui generis (we might call it "mental [*geistige*] experience"). This does not diminish [*Abbruch geschehen*] the rationality of mathematics. Rather, the assumption that rationality is necessarily connected with certainty appears to be a preconception. We almost nowhere have certain knowledge in the simple, full sense. This is the old Socratic insight which is emphasized [*zur Geltung bringen*] today especially also in the philosophies of F. Gonseth and K. Popper.

We have certainly to admit that in mathematical considerations, in particular in those of elementary mathematics, we possess a particular kind of security, because on the one hand the objects are intuitively clear [*anschaulich deutlich*] and, on the other hand, almost everything is stripped off [*abgestreift*] by the idealization of the objecthood [*Gegenständlichkeit*] that could lead to [*An-*

laß geben] subjectivity. — But when we talk about the certainty of $2 * 2 = 4$ in the popular sense, we think at the concrete applications of this statement. But the application of arithmetical statements to the concrete is based on empirical conditions, and for their compliance we only have an empirical, even if [*wenn auch*] practically sufficient certainty.

By dropping the coupling [*Koppelung*] of rationality and certainty we gain, among other things, the possibility to appreciate the *heuristic rationality*, which plays an essential role for scientific inquiry [*Erkenntnis*].

The acknowledgment of heuristic rationality provides in particular the solution to the epistemological difficulty that has been made a problem by David Hume: we can acknowledge the rational character of the assumption of necessary connections in nature, without having to claim that the basic approach [*? Ansatz*] of such connections guarantees the success; with regard to this success we depend in fact on experience.