Bernays Project: Text No. 1

Hilbert's significance for the philosophy of mathematics (1922)

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(Die Bedeutung Hilberts für die Philosophie der Mathematik, Die Naturwissenschaften 10, 1922, pp. 93-99.)

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Comments:

‖[93a] If we look upon the intellectual relationships <*geistige Beziehungen*> of the mathematical sciences to philosophy as they have developed since the times of the Enlightenment, we notice with satisfaction that mathematical thought is now at the point of regaining the powerful influence on philosophical speculation that it possessed up to Kant's time, but then suddenly lost completely. That sudden averting from mathematical thought was influenced by the general estrangement from the spirit of the period of the Enlightenment that took place at the beginning of the nineteenth century.

This detachment of philosophy from the exact sciences was, however, only a unilateral one: While the dominant philosophy became completely estranged[1] from mathematics, a philosophical orientation $\|^{93b}$ evolved more and more among mathematicians.

The most important reason for this was that mathematics had grown far beyond the framework within which it had moved up to the time of Kant. Not only had the domain of investigated facts grown considerably, but the whole form of the investigations became grander and the entire method more encompassing. The concept-formations <*Begriffsbildungen*> rose to a higher level of generality; the meaning of the formula became less important than conceptual abstractions and leading systematic ideas. Furthermore, the attitude toward the foundations and toward the object of the mathematical sciences also changed.

The task of geometry was understood in broader terms. Formation of geometrical concepts became more general and freed themselves more and more from the connection to spatial representation <*Vorstellung*>. In the recently developed geometrical theories, moreover, intuition of space no longer had the significance of an epistemological foundation, but was rather $\|^{94a}$ employed merely in the sense of an intuitive analogy.

In arithmetic, research experienced an essential extension of problem formulation as well. On the one hand, the concepts of number <*Anzahl*> and of order <*Ordnung*> were generalized, through the invention of set theory, in a completely new way and applied to infinite totalities. On the other hand, the development of algebra led to numbers and quantities no longer exclusively

[1]Among those philosophers who represented in this respect a laudable exception, Bolzano must be mentioned in particular; he gave the first rigorous foundation of the theory of real numbers.

viewed as objects of investigation; rather, calculatory formalism itself was made an object of study, and one made, very generally, the consideration of formalisms one's task. Numbers as well as quantities now appeared only as something special, and the more one examined their lawfulness from more general points of view, the more one became unaccustomed to taking this lawfulness for granted.

In this way, the whole development of mathematics moved on: to rob of its appearance of exclusiveness and finality all of that which was previously considered to be the only object of research, and whose basic properties were considered as something to be accepted by mathematics and neither capable nor in need of mathematical investigation. The framework was burst which earlier philosophical views, even Kantian philosophy, had marked out for mathematics. Mathematics no longer allowed philosophy to prescribe the method and the bounds of its research; rather it took the discussion of its methodological problems into its own hands. In this way the axioms of the mathematical theories were investigated in regard to their logical relationships, and the forms of inference were subjected to more precise critique as well. And the more these problems have been pursued, the more mathematical thought has shown its fruitfulness with respect to them, and it has proved itself as an indispensable tool for theoretical philosophy.

David Hilbert has contributed in a significant way to this development, which extends to the present. What he has accomplished in this field will be described in what follows.

When Hilbert applied himself to the problems that were to be solved concerning the foundations of mathematical thought, he had at his disposal not only his comprehensive command of the mathematical methods, but he was also above all, as it were, predestined for the task by his human disposition. For mathematics had for him the significance <*Bedeutung*> of a world view, and he went about those fundamental problems with the attitude of a conqueror who endeavors to secure for mathematical thought a sphere of influence which is as comprehensive as $\|^{94b}$ possible.

The main point, while pursuing this goal, was to avoid the mistake of those extreme rationalist thinkers who thought that complete knowledge of everything real <*alles Wirkliche*> could be attained by pure thought. This could not be a question of, say, incorporating into mathematics all knowledge of the factual [*Tatsächliches*]; rather it was necessary, for extending the realm of mathematics in the widest possible way, to delineate sharply the boundaries between the mathematical and the nonmathematical; that would actually allow one to claim for mathematics all mathematical components in cognition <*alle mathematischen Bestandteile im Erkennen*>.

Hilbert actually understood the problem also in this way. His first and largest work in the field of methodological questions is the *Foundations of Geometry*, which appeared in 1899. In this work, Hilbert laid out a new system of axioms for geometry that he chose according to the criteria of simplicity and logical completeness, following Euclid's concept-formations as closely as possible. He divided the whole system of axioms into five groups of axioms and then investigated more precisely the share the different groups of axioms (as well as single axioms) have in the logical development of geometry.

Through its wealth of new, fruitful methods and viewpoints, this investigation has exercised a powerful influence on the development of mathematical research. However, the significance of Hilbert's foundations of geometry by no means lies only in its purely mathematical content. Rather, what made this book popular and Hilbert's name renowned, far beyond the circle of his colleagues, was the new methodological turn that was given to the idea of axiomatics.

The essence of the axiomatic method, i.e., the method of logically developing a science from axioms and definitions, consists according to the familiar conception in the following: One starts with a few basic propositions <*Grundsätze*>, of whose truth one is convinced, puts them as axioms at the top, and derives from them by means of logical inference theorems; their truth is then as certain as that of the axioms, precisely because they follow logically from the axioms. In this view, attention is focused above all on the epistemic character [*Erkenntnischarakter*] of the axioms. Indeed, originally one considered as axioms only propositions whose truth was evident a priori. And Kant still held the view that the success and the fruitfulness of the axiomatic method in geometry and mechanics essentially rested on the fact ||[95a] that in these sciences one could proceed from a priori knowledge (the axioms of pure intuition and the principles of pure understanding [*Verstand*]).

However, the demand that each axiom expresses an a priori knowable truth was soon abandoned, for, in the manifold occasions that presented themselves for the axiomatic method, especially in the further development of physics, it followed, so to speak, automatically that one chose both empirical statements and also mere hypotheses as axioms of physical theories. The axiomatic procedure turned out to be especially fruitful in cases where one succeeded in encompassing the results of multifarious experiences in a statement of general character through positing an axiom. A famous example of this is that of the two propositions about the impossibility of a *perpetuum mobile* of the first and second type ; Clausius put them as axioms at the top of the theory of heat.

In addition, the belief in the a priori knowledge of the geometrical axioms was increasingly lost among the researchers in the exact sciences, mainly as a consequence of non-Euclidian geometry and under the impression of Helmholtz' arguments. Thus the empirical viewpoint, according to which geometry is nothing but an empirical science, found more and more supporters. However, this departure from a prioricity <*vom Apriorismus*> did not alter essentially the perspective on the axiomatic method.

A more powerful change, however, was brought about by the systematic development of geometry. Mathematical abstraction had, starting with elementary geometry, raised itself far above the domain of spatial intuition; it had led to the construction of comprehensive systems <*Lehrgebäude*>, in which ordinary Euclidian geometry could be incorporated and within which its lawlikeness appeared only as one particular among others of equal mathematical rights. This opened up a new sort of mathematical speculation by means of which one could consider the geometrical axioms from a higher standpoint. It immediately became apparent, however, that this type of consideration had nothing to do with the question of the epistemic character of the axioms, which had formerly been considered, after all, as the only significant feature of the axiomatic method. Accordingly, the necessity of a clear separation between the mathematical and the epistemological problems of axiomatics ensued. The demand for such a separation of the problems had already been stated with full ||[95b] rigor by Klein in his Erlangen Program.[2]

What was essential, then, about Hilbert's foundation of geometry was that here, from the beginning and for the first time, in the laying down of the axiom system, the separation of the mathematical and logical realm from the spatial-intuitive realm, and thereby from the epistemological foundation of geometry, was completely carried out and expressed with complete rigor <*mit voller Schärfe*>.

To be sure, in the introduction to his book Hilbert does express the thought that laying down the axioms for geometry and the investigation of their relationships is a task that amounts "to the logical analysis of our spatial intuition," and likewise he remarks in the first section that each single of these groups of axioms expresses "certain basic facts of our intuition which belong together." But these statements are located completely outside the axiomatic development <*Aufbau*>, which is carried out without any reference to spatial intuition.

A rigorous axiomatic grounding <*Begründung*> of geometry has of course always to satisfy the demands that the proofs should exclusively appeal to what is formulated in the axioms, but that they must not draw, in any way, on spatial intuition. More recently, it was especially Pasch who, in his foundation of geometry,[3] has emphazised satisfying this demand <*auf die Durchfürhung dieser Idee Gewicht gelegt hat*> and has done so in a consistent way <*und ihr auch vollständig entsprochen hat*>.

[2] Klein [1872].
[3] Pasch [1882].

However, Hilbertian axiomatics goes even one step further in the elimination of spatial intuition. Drawing on <*Heranziehung*> spatial representation is completely avoided here, not only in the proofs but also in the axioms and the concept-formations. The words “point,” “line,” “plane” serve only as names for three different sorts of objects, about which nothing else is directly assumed except that the objects of each sort constitute a fixed determinate system. Any further characterization is, then, carried out through the axioms. In the same way, expressions like “the point A lies on the line a” or “the point A lies between B and C” are not associated with the usual intuitive meanings; rather these expressions will designate only certain, at first indeterminate, relations, which *are implicitly characterized*[4] only through the axioms in which these expressions occur.

According to this conception, the axioms are in no way judgments that can be said to be true or false; after all <*überhaupt*>, they have a sense only in the ||[96a] context of the whole axiom system. And even the axiom system as a whole does not state a truth; rather, the logical structure of axiomatic geometry in Hilbert’s sense—completely analogous to that of abstract group theory—is a purely hypothetical one. If there are anywhere in reality three systems of objects, as well as determinate relations between these objects, such that the axioms of geometry hold of them (this means that by an appropriate assignment of names to the objects and relations, the axioms turn into true assertions), then all theorems of geometry hold of these objects and relations as well. Thus the axiom system itself does not express something factual; rather, it presents only a possible form of a system of connections

[4] One speaks in this sense of “implicit definition.”

that must be investigated mathematically according to its *internal* properties.

Accordingly, the axiomatic treatment of geometry amounts to separating the purely mathematical part of knowledge from geometry—originally considered as a science of spatial figures—and investigating it in isolation on its own. The spatial relationships are, as it were, mapped into the sphere of the mathematical-abstract in which the structure of their interconnections appears as an object of pure mathematical thought. This structure is subjected to a mode of investigation that concentrates only on the logical relations and is indifferent to the question of the *factual* truth, that is, the question whether the geometrical connections determined by the axioms are found in reality (or even in our spatial intuition).

This sort of interpretation of the axiomatic method was presented in Hilbert's *Foundations of Geometry*; it offered the particular advantage of not being restricted to geometry but indeed of being transferable to other disciplines without further ado. From the beginning, Hilbert envisaged the point of view of the uniformity of the axiomatic method in its application to the most diverse domains, and guided by this viewpoint, he tried to bring this method to bear as widely as possible. In particular, he succeeded in grounding axiomatically the kinetic theory of gases as well as the elementary theory of radiation in a rigorous way.

In addition, many mathematicians subscribed to Hilbert's axiomatic mode of investigation and worked in the spirit of his endeavors. In particular, it was a success of axiomatics when Zermelo, in the field of set theory, overcame the existing uncertainty of inference by a suitable axiomatic delimitation of the $\|^{96b}$ inferential modes and, at the same time, created with his system a common foundation for number theory, analysis, and set theory.[5]

In his Zurich lecture on "Axiomatic Thought" (1917),[6] Hilbert gave a summary of the leading methodical thoughts <*methodischen Leitgedanken*> and an overview of the results of axiomatic research. Here he characterizes the axiomatic method as a general procedure of scientific thinking. In all areas of knowledge where one has already come to the point of setting up a theory, or, as Hilbert says, to an arrangement of the facts by means of a framework of concepts [*Fachwerk von Begriffen*], this procedure sets in. Then, it always becomes obvious that a few propositions suffice for the logical construction of the theory, and through this the axiomatic foundation of the theory is made possible. This will at first take place in the sense of the old axiomatics. However, one can always—as within geometry—move on to Hilbert's axiomatic standpoint by disregarding the epistemic character of the axioms and by considering the whole framework of concepts [*Fachwerk von Begriffen*] only (as a *possible* form of a system of interconnected relations <*Verknüpfungszusammenhang*>) in regard to its internal structure.

Thus, the theory turns into the object of a purely mathematical investigation, just what is called *axiomatic* investigation. Namely, the same main questions must always be considered for any theory: First of all, in order to represent a possible system of interconnected relations <*Verknüpfungszusammenhang*>, the axiom system must satisfy the condition of *consistency*; i.e., the relations expressed in the axioms must be logically compatible with

[5]Zermelo [1907].

[6]Hilbert [1918].

one another. Consequently, the task of proving the consistency of the axiom system arises. The old conception of axiomatics does not know this problem, since here indeed every axiom counts as stating a truth. Then it is a question of gaining an overview of the logical *dependencies* among the different theorems of the theory. A particular focus of investigation must be whether the axioms are logically independent of each other or whether, say, one or more axioms can be proved from the remaining ones and are thus superfluous in their role as axioms. In addition, there remains the task of investigating the possibilities of a "deepening of the foundations" of the theory, i.e., examining whether the given axioms of the theory might not be reduced to propositions of a more fundamental character that would then constitute "a deeper layer of axioms" for the framework of concepts [*Fachwerk von Begriffen*] under consideration.

||[97a] This sort of investigation, which is of a mathematical character throughout, can now be applied to any domain of knowledge that is at all suitable for theoretical treatment, and its execution is of the highest value for the clarity of knowledge and for a systematic overview. Thus, through the idea of axiomatics, mathematical thought gains a universal significance for scientific cognition [*Erkennen*]. Hilbert can indeed claim: "Everything whatsoever, that can be the object of scientific thought is subject, as soon as it is ripe for the formation of a theory, to the axiomatic method and thereby of mathematics."

Now, by means of this comprehensive development [*Ausgestaltung*] of the axiomatic idea, a sufficiently wide framework for the mathematical formulation of problems was indeed obtained, and the epistemological fruitfulness of mathematics was made clear. But with regard to the *certainty* of the mathematical procedure, a fundamental question still remained open.

Namely, the task of proving the consistency of the axiom system was indeed recognized as first and foremost in the axiomatic investigation of a theory. In fact, the consistency of the axioms is the vital question for any axiomatic theory; for whether the framework of concepts represents a system of interconnected relations at all or only the appearance of such a system depends on this question.

If we now examine how things stand with the proof of consistency for the several geometrical and physical theories that have been axiomatically grounded, then we find that this proof is produced everything only in a relative sense: The consistency of the axiom system to be investigated is proved by exhibiting a system of objects and of relations *within mathematical analysis* that satisfies the axioms. This "method of reduction" to analysis (i.e., to arithmetic in the wider sense) presupposes that analysis itself—independently of whether it is considered as a body of knowledge [*Inbegriff von Erkenntnissen*] or only as an axiomatic structure <*Gebäude*> (i.e., as a merely possible system of relations)—constitutes a consistent system.

However, the consistency of analysis is not as immediately evident as one would like to think at first. The modes of inference applied in the theory of real numbers and real functions do not have that character of direct evidence

[*Charakter des unmittelbar Handgreiflichen*] which is characteristic, for instance, of the inferences of elementary number theory. And if one wants to free the methods of proof from everything that is somehow problematic, then one is compelled to axiomatically set up analysis. Thus it turns out to be necessary to provide also a consistency proof for ‖[97b] analysis.

From the beginning, Hilbert recognized and emphasized the need for such a proof to guarantee the certainty of the axiomatic method and of mathematics in general. And although his efforts concerning this problem have not yet reached the ultimate goal, he has nonetheless succeeded in finding the methodological approach by which the task can be mathematically undertaken.

Hilbert presented the main ideas of this approach already in 1904 in his Heidelberg lecture "On the Foundations of Logic and Arithmetic."[7] However, this exposition was difficult to understand and was subject to some objections. Since then, Hilbert has pursued his plan further and has given a comprehensible form to his ideas, which he recently presented in a series of lectures in Hamburg.

The line of thought on which Hilbert's approach to the foundations of arithmetic and analysis is based is the following: The methodical difficulties of analysis, on the basis of which in this science one is compelled to go beyond the framework of what is concretely representable, result from the fact that

[7] Appendix VII to the *Grundlagen der Geometrie*.

here continuity and infinity play an essential role. This circumstance would also constitute an insuperable obstacle to the consistency proof for analysis, if this proof had to be carried out by showing that a system of things as assumed by analysis, say the system of all finite or infinite sets of whole numbers, is logically possible.

However, the claim of consistency needs not at all be proved in this way. Rather, one can give the following entirely different twist to the claim: The modes of inference of analysis can never lead to a contradiction or, what amounts to the same thing: It is impossible to derive the relation $1 \neq 1$ ("1 is not equal to 1") from the axioms of analysis and by means of its methods of inference. Here it is not a question concerning the possibility of a continuous, infinite manifold of certain properties, but concerning the impossibility of a mathematical proof with determinate properties. A mathematical proof is, however, unlike a continuous infinite manifold, a concrete object surveyable in all its parts. A mathematical proof must, at least in principle, be completely communicable from beginning to end. Moreover, the required property of the proof (i.e., that it proceeds according to the principles of analysis and leads to the final result $1 \neq 1$) is also a concretely ||[98a] determinable property. This is why there is also, in principle, the possibility of furnishing a proof of consistency for analysis by means of elementary, and evidently certain, considerations. We only have to take the standpoint that the object of investigation is not constituted by the objects to which the proofs of analysis refer, but rather these proofs themselves.

On the basis of this consideration, the task arises then for Hilbert to examine more precisely the forms of mathematical proofs. We must, so he says in his lecture on axiomatic thought, “make the concept of specific mathematical proof itself the object of an investigation, just as the astronomer takes into account the motion of his location, the physicist concerns himself with the theory of his apparatus, and the philosopher criticizes reason itself.” The general forms of logical inference are decisive for the structure of mathematical proofs. That is why the required investigation of mathematical proofs must, in any case, also concern the logical forms of inference. Accordingly, already in the Heidelberg lecture, Hilbert explained that “a partially simultaneous development of the laws of logic and arithmetic [is] necessary.”

With this thought Hilbert turned to/built upon <*aufnehmen*> *mathematical logic*. This science, whose idea goes back to Leibniz, emerged from primitive beginnings and developed into a fruitful field of mathematical thought in the second half of the nineteenth century. It has build/formed the methods how to arrive at a mathematical mastery of the forms of logical inference through a symbolic notation for the simplest logical relations (as “and,” “or”, “not,” and “all”). It turned out that by this “logical calculus” only one gains the complete overview of the system of logical forms of inference. The inferential figures, which are dealt with in traditional logic, constitute only a relatively small subfield of this system. Peano, Frege, and Russell in particular, succeeded in developing the logical calculus in such a way that the mental inferences [*gedankliche Schlüsse*] of mathematical proofs can be perfectly imitated by means of symbolic operations.

This procedure of the logical calculus forms a natural supplement to the method of the axiomatic grounding of a science to the following extent: It makes possible, along with the exact determination of the *presuppositions*—as it is brought about by the axiomatic method—also an exact pursuit of the *modes of inference* by which one proceeds from the principles of a science to its conclusions.

In adopting the procedure of mathematical logic, Hilbert reinterpreted ||[98b] it as he had done with the axiomatic method. Just as he had formerly stripped the basic relations and axioms of geometry of their intuitive content, he now eliminates the intellectual content of the inferences from the proofs of arithmetic and analysis which he takes as the object of his investigation. He achieves this by taking the systems of formulas, by which those proofs are represented in the logical calculus—detached from their contentual-logical interpretations—, as the immediate object of consideration, and thereby replacing the proofs of analysis with a purely formal manipulation of definite signs according to fixed rules.

Through this mode of consideration, in which the separation of the specifically mathematical from everything contentual reaches its peek, Hilbert's view on the nature of mathematics and on the axiomatic method finally finds its actual completion. For we recognize at this point that that sphere of the mathematical-abstract, into which the method of thought of mathematics translate all that is theoretically comprehensible, is not the sphere of the contentual-logical [*inhaltlich Logisches*] but rather the domain of pure formalism. Mathematics turns out to be the general doctrine of formalisms, and by understanding it as such its universal significance also becomes clear without further ado.

This meaning of mathematics as a general doctrine of forms has come to light in recent physics in a most splendid way, especially in Einstein's theory of gravitation, in which the mathematical formalism gave Einstein the leading idea for setting up his law of gravitation, whose more exact form could never have been found without enlisting mathematical tools <*Hilfsmittel*>. And here it was once again Hilbert who first brought this law of gravitation to its simplest mathematical form. And by showing the possibility of a harmonious combination of the theory of gravitation with electrodynamics, he initiated the further mathematical speculations connected to Einstein's theory which were brought to systematic completion by Weyl with the help of his geometrical ideas. If these speculations should stand the test in physics, then the triumph of mathematics in modern science would thereby be a perfect one.

If we now look at the ideas yielded by Hilbert's philosophical investigations as a whole, as well as the effect caused by these investigations, and if we, on the other hand, bear in mind the unfolding of mathematics in the recent times as outlined above, then, as we will see, what is essential of Hilbert's philosophical accomplishment shows in the following. By developing a broad-minded $\|^{99b}$ philosophical conception of mathematics, which made it possible to do justice to the significance and scope of its method, Hilbert has established, with force and success, the claim to a ||[99a] universal intellectual influence in science that mathematics has gained by its inner deepening and its reshapening on a large scale. For this the friends of mathematics will be always indebted to him.