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The Basic Notions of Pure Geometry in Their Relation to Intuition (1925)

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(Untersuchungen zur psychologischen Vorgeschichte der Definitionen, Axiome und Postulate, by Richard Strohal. Leipzig and Berlin: B. G. Teubner 1925. 137 pp. and 13 images. 13 × 19 cm. Price RM 6.40.)

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Comments:

none

It is possible to discuss the relation between axiomatic geometry and intuition under very different aspects and on the base of different epistemological preconditions.

The present book, composed by R. Strohal with the essential collaboration of Franz Hillebrand, intends to exert a certain methodological and epistemological view of geometry. It is said in the introduction that the object of the investigation is formed by the "psychological prehistory" of geometrical concepts and principles. In fact, however, already the more specific elaboration of the program shows that it does by no means concern questions of genetic psychology, but questions like: in what way do we have to revert to intuition when introducing geometrical concepts, what role does intuition play for the formation of basic concepts and complex concepts as well as for setting up the principles of geometry, and now do we have to evaluate the epistemological character of these principles. |[197r]

In this connection it is also by now means intended by the author to let intuition appear in the most possible extension as determinative for geometry.

On the one hand Strohal intends to desist completely from the question of an application to "our space" (in fact he doesn't behave that extremely), as he mentions in the beginning, for him the foundations of *pure* geometry are concerned. A foundation of geometry by spatial experience doesn't come into consideration for him. But also a rational foundation by appealing to an aprioristic evidence of geometrical intuition is excluded for him, because he doesn't accept any other aprioristic evidence than the analytical one and doesn't award intuition any rational character. He doesn't undertake a closer discussion of the concept of "intuition", but starts, so to say as self-evident, from the view—admittedly also ruling among exact researchers—, according to which intuition is not capable to give us perfectly clear objects, and also not to present us a relation as necessary, so that all idealizations and all insights of strict generality come about only on the way of a conceptual abstraction.

One should now think that in consideration of his epistemological position Strohal should welcome the standpoint of Hilbertean formal axiomatics as |[198l] being in accordance with his opinion and his intention. In fact, however, he by no means agrees to this present-day axiomatics, but turns himself explicitly against it, especially against the Hilbertean foundation of geometry.

The kind of dealing with geometry Strohal aspires to, can hardly be expressed with a few word in a comprehensible way, because in his conception different endeavors cooperate. In any case, this present attempt to diverge in principle from the present-day standpoint of axiomatics and to go back to older tendencies, which can arguably convince some people in a first vague receiving, but which is, in a closer inspection, only suited to put our today's standpoint into a brighter light, and to highlight the motives from which it arose in their justification in an especially precise way. But just under this point of view it doesn't seem to be needless, to present the main points of Strohal's views and to discuss his presentation critically.

Strohal deals especially exhaustively with the *formation of concepts*. First of all, as to the role of intuition, it consists according to Strohal in the following:

1. Elementary concepts are won from intuition by an abstraction process.

2. Intuition serves as a cause (causa occasionalis) for the formation of complex concepts (for "synthetical definitions") by suggesting the formation of certain conceptual syntheses. This occurs in the manner of sharp definitions being posed by combining elementary concepts replacing intuitive concepts, i. e., concepts directly taken from intuition (like the intuitive concept of a straight line or of the circle), whereby, by the way, the extension of a concept formed this way doesn't need to completely coincide with the corresponding intuitive concept.

For one thing, we have thereby to take into account that the intuition spoken about has by no means always to be spatial intuition, e. g., according to STROHAL, the elementary concept of *congruence*, identified, in the style of BOLYAI, with the "indistinguishability irrespective the position," is won in the way that first of all "the intuitive givenness of indistinguishable qualities, colors, sounds, odors etc." leads to a blurred concept of indistinguishability (sameness) from which the rigorous concept of indistinguishability is got as a limit concept by an process of abstraction (pp. 71–72).

Above all it is essential, however, that we, according to STROHAL's opinion, are not free, to introduce any concept got from intuition by abstraction. He rather claims: a concept may be regarded as an elementary concept only "if an entity falling under the extension of the respective concept, cannot also be given by conceptual marks," or in a shorter formulation: "Where it is *possible* at all to define a concept explicitly, there one *has to* define it."

Sure enough, this "criterion" is completely undetermined; because the possibility to define a concept explicitly depends essentially on the choice of geometrical principles, and listing up the principles acts again in accordance with the choice of elementary concepts.|[198r]

Also the motivation of the criterion is absolutely unsatisfactory. STROHAL asserts that the explanation of a concept has to make it possible,"to decide from a somehow given object whether it falls under the extension of the respective concept or not" (p. 18). E. g., we have to be able to decide whether the geometrical location of all points, being equidistant from two fixed points A, B falls under the extension of the concept of a straight line; one would face such a task helplessly, he thinks, if one would regard the concept of a straight line as a basic concept (p. 19). Again STROHAL doesn't consider, that the extensional relations between geometrical concepts are determined only by the principles of geometry and that on the other hand they can also make it possible to prove a complex concept to be extensional equivalent to an elementary concept. Lacking a closer justification he says "obviously".

On the other hand, despite the indeterminacy of the criterion the aim pursued with it can be recognized: Geometry should—like a philosophical science—advance from the highest generality to the particular on the way of a conceptual synthesis. It is therefore not allowed to pose as elementary concepts the concepts of special, geometrical entities, but only those of very general character.

Because of this methodological demand, STROHAL is forced to deviate completely from the well-known elementary construction of geometry as it can be found in EUKLID and similarly in Hilbert's foundations. He finds a formation of geometrical concepts analogous to his principle in LOBATSCHEFSKIJ and BOLYAI. He follows these two, especially LOBATSCHEFSKIJ, in introducing the elementary concepts. On the base of an exhaustive discussion he arrives at the following system of elementary concepts:

1. the spatial (spatial formations);
2. the contact (the adjoining);
3. the "having it inside" (the relation of the whole to the part);
4. the congruence (indistinguishability not regarding the position).

Obviously we have it here to do with such a construction of geometry according to which the topological properties of space are posed in the beginning and afterwards the *metric* is introduced. This method to construct geometry and its systematic advantages are well-known to the mathematician—especially since the investigations of Riemann and Helmholtz[1] on the foundations of geometry. He will not be satisfied, however, with having only this way of founding at hands. In particular, the common elementary way of founding has the big methodological advantage, that here geometry, like elementary number theory, starts from considering certain simple, easily understandable objects, and that one doesn't need to introduce the concept of continuity and the limitation process from the outset. Anyway, however, |[199l] one will insist on the freedom of choosing the basic concepts relative to the viewpoint according to which geometry is done.

Strohal concedes in principle, however, the possibility that systems differing from the one he gives "in another way follow intuition immediately, i. e., other elementary concepts are taken as a base" (p. 63). In fact, however, almost all other ways of founding are rejected by him.

According to his opinion, e. g., the concept of a straight line should not be taken as a basic concept.[2] He also deliberately avoids introducing the point as basic element. In his systematic the point is defined as the common border of two lines touching each other, the line results accordingly from two

[1]Helmholtz's group-theoretic conception elaborated by Lie and Hilbert doesn't be in the line of Strohal's intention (as being proved in the following). Closer to this is the "derivation of the elementary spatial concepts from the one of equality" sketched by Weyl (in the first paragraph of his book "Raum–Zeit–Materie").

[2]By the way, Strohal considers a straight line only as a spatial object, or the straightness as feature of a straight line. He doesn't consider at all the possibility of introducing collinearity as a relation between *three* points.

touching planes and the plane from two touching bodies.

He considers it as excluded to antepose the *concept of direction* as an elementary concept. He declares that if one intends to use the concept of direction for defining the straight line, this would only be possible by considering the concept "equidirected" as an elementary concept not further reducible, and with this to go on from the intuition of "straightness" itself. "That is to say, because no other intuition can help to achieve this elementary concept as the one of an intuitive straight line, it is the same as to regard the straight line itself as an elementary concept." (p. 56). In contrast to that, one has to remark that the distinction of directions starting a point can be intuitively got independently from the imagination of straightness by considering the *different* parts of the visual field and by the imaginations of directions bound to our impulses of motion. And moreover, as to comparing of directions starting from different points, Strohal, according to his methodological principles, would have to accept their synthetic introduction by linking the concept of direction to the concept of "indistinguishability", because he arrives at the comparison of the lengths of distances different in their position on a very similar way. Especially in recent times it has been clarified by Weyl's pure closeness geometry ["reine Nahegeometrie"] that, indeed, the a priori comparability of distinct distances is by no means more easily understandable than the comparability of the distances starting from different points. Here Strohal only repeats an old prejudice.

Strohal also rejects as a circular method the characterization of the relation of congruence with the help of the concept of *rigid motion* "The concept of a rigid body which occurs in this connection again cannot be explained in another way as by presupposing the congruence of the different positions of this body. If one intends to understand the rigid body as an elementary concept, however, one will find that for obtaining it no other intuitions will help than those which give us the concept of congruence itself, so that the deviation via the concept of a rigid body looses any sense" (pp. 17–18). This argumentation would be justified only if the concept of a rigid body would have to be formed as a common generic concept|^{199r}, e. g., in such a way that starting from an empirical intuition of the rigid body one arrives by abstraction at the concept of the perfectly rigid body. In fact, however, it is possible to carry out instead of this abstraction process a completely different one consisting in sharpening by abstraction the intuitive matters of fact, found at rigid bodies concerning the freedom of motion and coincidences to a strict lawfulness, and forms in respect to this *lawfulness* the geometrical concept of a rigid body. In the mathematical formulation this way of concept formation becomes effective in considering from the outset not isolated rigid motions, but the group of rigid motions.

Strohal doesn't consider this thought, originating in Helmholtz, with any word, which was path-breaking for a whole direction of geometrical research and which got an increased actuality faced with relativity theory.

Now, if that many ways adopted by mathematics in order to erect geometry are excluded, one could expect that the way of justification that definitely preferred by Strohal is presented to us as a paragon of methodology. In fact, however, the considerations with the help of which Strohal explains the method, following Lobatschefskij, how to come from the elementary concepts of the spatial, the contact and the having-it-inside to the distinction of dimensions and to the concepts of plane, line and point, are far away from the precision we are used to in dealing with such topological questions; by means of these comments one can by no means determine whether it is possible at all to get by with those three elementary concepts for the topological characterization of space.—

Up to now we have only regarded that part of STROHAL's considerations dealing with geometrical *concept formation*. STROHAL's standpoint, however, becomes really clear in his opinion of the *principles* of geometry.

It is essential for this opinion that STROHAL sticks to the distinction of the (communes animi conceptiones) and the (postulata), as it is to be found in EUCLID's "Elements." STROHAL regards this distinction as principally significant, and sees an essential shortcoming of newer foundations of geometry in the deviation from this distinction.

To this it has to be remarked in the beginning that deviating from EUKLID in this point is not a result of pure sloppiness, but of full intention. EUKLID anteposes to the *specific geometrical* postulates the statements of the *theory of magnitude* gathered under the title as statements of higher than geometrical generality which are to be *applied* to geometry.

The way of application, however, gives the reason for principal objections, because the subordination of geometrical relations under the concepts occurring in the is repeatedly tacitly presupposed in cases where the possibility of such subordination is by no means a self-evident geometrical law.

In particular HILBERT has criticized in this sense the application that EUCLID made of the principle that the whole is bigger than the part in the theory of the areas of plane figures—an application which would only be justified, if |^{200l} one could presuppose without hesitation that one could assign to every straight-linedly limited plane figure [*geradlinig begrenzten ebenen Figur*] its area (in such a way that congruent figures have the same area and that by joining the planes the areas are summed up).[3]

While considering such a case one recognizes that the essential point for applying the lies always in the conditions of the applicability. If these conditions are recognized as appropriate, the application of the respective principle becomes in most cases entirely superfluous, and sometimes the statement to be proved by applying the general principle belongs itself to these conditions of applicability.

Preposing the appears to be therefore as a permanent seduction to commit logical mistakes and to be more capable to veil the true geometrical state of affairs than to clear it, and therefore one has completely abstained from this method.

Strohal seems to know nothing about these thoughts; in any case he doesn't mention Hilbert's criticism with any syllable. He intends to exert the distinction between the two kinds of principles anew. Above all, this appears to him already as necessary because, according to his opinion, the have a completely different nature of knowledge than postulates, namely the same as evident analytical statements, whereas postulates do not form the expression of a knowledge at all, but are only *suggested* to us by certain experiences.

Therefore Strohal calls the the "real axioms." He sees a particular

[3]Hilbert has shown by constructing a special "non-Archimedian" and "non-Pythagorean" geometry that this presupposition may in fact not always be valid.

success of his theory of geometrical concept formation in making the analytical nature of the understandable. He finds this understandability therein that these axioms as statements on every one single elementary relation have the sense of a *rule*, saying from which statements of relations one has to abstract the elementary concept, "in order to make the axiom concerned just to an identical statement" (p. 70). This characterization means that the axioms regarded form logical identities on the base of a contentual view of the elementary concepts.

It appears to be remarkable that such geometrically inexpressive statements should be regarded as "real axioms" of geometry, and one furthermore asks oneself to what purpose one has to specially post these statements as principles at all, after having introduced them in a contentual way.

For instance, as one of these axioms the statement is named that if a cannot be distinguished from b and b not from c, then a cannot be distinguished from c. This statement is, because of the meaning of "indistinguishability", a consequence of the purely logical statement: if two things a, b behave the same in respect to the applicability or the non-applicability of a predicate P and also b, c behave in this respect the same, then a and c as well behave in this respect the same.

We have now the following alternative: either the concept "indistinguishable" is used in its contentual meaning, then we face a statement which can be understood purely logically, and there is no reason to list such a statement as an axiom, since in geometry we regard the laws of logic $|^{200r}$ as self-evident base, anyway. Or, however, the concept "indistinguishable" and also other elementary concepts will by no means applied contentually, but in a first step only concept *names* are introduced, concerning which the axioms give certain *instructions*. Then we are on the standpoint of formal axiomatics, and the are nothing else than *implicit definitions* as they are called according to HILBERT.

Those places where he stresses that do not provide "real definition" or "explicit definitions" of an elementary relation (pp. 68 and 72) indicate that this is indeed STROHAL's opinion—who is, sure enough, carefully aware of using the term "implicit definition" anytime.

From this standpoint is is not suitable, however, to ascribe the nature of *evidence* to the respective axioms. Then they simply form *formal requisitions* for certain, initially undetermined relations, and then there is also no principal constraint of separating these axioms from the "postulates."

So either the posing of axioms which have according to STROHAL the role of is superfluous at all, or the separation of these axioms as analytically evident statements from the other principles is not justified.

Furthermore, however, in the application of these axioms in STROHAL we find the same mischiefs again that discredited the EUCLIDean : the formulation of these statements which can easily be confused with geometrically contentful statements, seduces to logical mistakes, and these are really committed.

Two cases are especially characteristic. 1. As an example for a real axiom the statement is given,[4] that in a "cut," i. e., in the contact of two adjoining parts of a body (spatial entity), always *two sides of the cut* have to be distinguished (p. 64). This statement is tautological, however, since because

[4]In this example STROHAL follows some considerations of LOBATSCHEFSKIJ.

the two adjoining parts are called “sides” of the cut (p. 23), it says nothing else than if two parts of a body touch each other (adjoin), two adjoining parts have to be distinguished. Also, this statement is completely irrelevant for geometry. However, it seems to mean something geometrically important, because given this wording one thinks of another statement expressing the topological properties of space.

The following fault shows that Strohal himself was not immune from confusions of a similar kind. He raises the following question (on the occasion of a discussion of the concept of congruence): “Is it possible to find two bodies connected by a continuous series of such bodies which have one the same side in common, i. e., which touch themselves all in *one* plane?” “We have to deny this question,” he continues, “because it follows from the explanation of the plane that only *two* bodies are able to touch themselves in one and the same plane” (pp. 42–43).

2. The famous axiom: “The whole is bigger than the part,” which became, as mentioned, the source for a mistake for Euclid, is interpreted by Strohal in the following way: The axiom hints at an elementary concept “bigger” “which can be obtained by abstraction from a $|^{201l}$ divided body.” The procedure of abstraction is characterized “by examining that relation which exists between the totality of all partial bodies (the *whole*) and one of them (the *part*). For the concept ‘bigger’ won on this way, the statement ‘Totum parte maius est’ an identical one” (p. 77). At this place we would like to disregard that in this interpretation the “whole” is wrongly identified with the totality of all partial bodies. In any case, from this interpretation follows that the proposition “a is bigger than b” is only another expression for b

being a part of a. So we have again a perfect tautology, from which one can take nothing for geometry; in particular it is impossible to derive from this the statement that a body cannot be congruent to one of his parts—which also follows from the fact that this statement is generally valid only under certain restrictions, anyway. (For instance, a half line can turn into a part by a congruent move, and equally a spatial octant into a partial octant by a congruent move.)

In fact, however, Strohal would have to have this statement at his disposal, in some formulation, for the theory of congruence which he still doesn't develop in this respect. Because if not it would not be certain whether this "indistiguishability irrespective the location" only means *topological equality*. Indeed, in the conceptual system taken as basis by Strohal the first three elementary concepts: spatial object, adjoining, having-it-inside, all belong to the domain of topological determinations, and only by the concept of congruence the *metric* is introduced to geometry. Therefore, the concept of congruence has to contain a *new distinguishing property* besides the element of correspondence. In the concept of indistinguishability irrespective the location[5] such a distinguishing property as such, however, is not given; for this, a principle is needed according to which certain objects which are in the beginning only determined as different in respect to the position but not as topologically different, can also be recognized as *distinguishable irrespective the location*. With other words: it matters to introduce the *difference in size*. The principle that the whole is bigger than the part should actually

[5]The "location" of a body is, according to the definition Strohal took from Lobatschefskij with a certain revision (pp. 24 and 93), synonymous to the limitation of the body.

help us to achieve this. This will be, however, impossible, if we interpret the statement in the way Strohal does; because from this interpretation it cannot be derived that an object a which is bigger than b, is *distinguishable* from it, even in respect to the location.

Maybe that Strohal didn't pay attention to this circumstance; because otherwise he might have got aware of the fact that his concept of indistinguishability irrespective the location by no means provides yet the geometrical congruence. Thus, we find here a gap very similar to that in Euklid's doctrine of the area.

The result of this consideration is that the method of anteposing the becomes the more disputable through the modified interpretation given to it by Strohal, and, in any case, it doesn't appear to be an exemplary model.

At the same time Strohal's charcterization of these axioms has led us to the assumption, that he doesn't keep the contentual view of elementary concepts within geo|[201r]metry itself, repectively he doesn't make use of it for geometrical proofs. We are confirmed in this assumption by what Strohal points out concerning the *postulates* of geometry.

According to Strohal we are *forced* to posit the postulate neither by intuition nor by logical reasons, "but caused by certain experiences" (p. 97). For pure geometry they have the meaning of determinations; they are "tools for definitions for the geometrical space, their totality forms the definition of the geometrical space" (p. 103). Contentually they are characterized as "exclusions of certain combinations of elementary concepts being a priori possible" (p. 103).

The sense of this characterization results from the opinion Strohal has of the deductive development of geometry. According to Strohal this development proceeds in continuously combining properties, i. e., by forming synthetical definitions. In forming the first syntheses one is only bounded to those restrictions resulting from the . “By the way, one can proceed completely arbitrarily in combining elementary concepts,” i. e., the decision , “whether one should intend to unite certain elementary concepts to a synthesis or to exclude them,” is caused by motives, “which are outside of pure geometry.” “However, in arbitrarily excluding the existence of a certain combination, one introduces a statement into pure geometry which has to serve as a norm for further syntheses. Statements of this kind are called demands, , postulates.” “In forming higher syntheses” one has to show that these “do not contradict to postulates already established. One has, concisely said, to prove the *possibility*, the *existence*, of the defined object. Here existence and possibility mean the same, nothing else than *consistency* with the postulates” (pp. 98–99 and p. 102).

It is most striking in this description of the geometrical procedure that here, contrary to all known kinds of geometrical axiomatics, only a *negative* content is ascribed to the postulates, namely the one of the exclusion of possibilities, whereas all geometrical existence statements are only interpreted as statements on consistency.

This opinion of Strohal corresponds to the direction of his philosophical school which includes as an essential element Brentano's doctrine of the judgement. According to this, all general judgements are negative existence judgements of the content that the matter of a judgement (a combination of the contents of intuitions) is rejected (excluded).

Indeed, every general judgement can be brought into this logical form. By producing such a normal form, however, the existential moment is not removed, but transferred into the formation of the matters of judgements.

So one doesn't succeed in geometry as well, to completely exclude existential claims or rather to reduce them to claims on consistency. One can hide an existential claim only by a double application of negation. In this way, e. g., Strohal proceeds in speaking of something which is excluded by the assumption “that, say, in dividing a geometrical body the congruence of parts cannot appear anytime”’ (p. 93). Exactly such an example we find in his discussion of Dedekind's continuity |[202l] axioms. After having spoken of the divisions of a line segment AB that has the cut property, and furthermore of the construction of a cut in the point C, he continues: “In *excluding* the possibility of such a division of some line segment AB on which such a point C cannot be found, I express the of continuity for the line segment” (p. 113). Whether speaking of “occurring”, “being found”, or “existence”, that all comes to the same thing. And anyway here, where the position of postulates is concerned, the interpretation of existence in the sense of consistency with postulates is not suitable.

The identification of existence and consistency is justifiable in a twofold sense: firstly in respect to the geometrical space whose existence consists indeed only in the consistency of the postulates defining it, secondly also in respect to the geometrical objects, but only under the condition of the *completeness of the systems of postulates*.

If the system of postulates is complete, i. e., if, already by the postulates, for every combination (every synthesis) of elementary concepts it is decided whether they are accepted or excluded, then, indeed, the possibility (consistency) of an object coincides with its existence .

However, as long as one is on the way of gaining a system of postulates, i. e., of stepwise determining the geometrical space, one has to distinguish between existence and consistency. From the proof of the consistency of a synthesis it only follows that it is in harmony with the postulates *already posed*; it could nevertheless be possible to exclude this synthesis by a further postulate. Compared to it, an *existential proof* says, that already by the prior postulates one is logically *forced* to accept the respective synthesis.

Let's take as an example "absolute geometry" resulting from common geometry by excluding the parallel axiom. In this geometry one can assume, without contradiction to the postulates, a triangle with an angular sum of a right angle; if we would like to identify consistency with existence in this context, we would get the statement: "In absolute geometry there is a triangle with the angular sum of one right angle." Then the following statement would be equally valid: "In absolute geometry there exists a triangle with an angular sum of two right angles." Hence, in absolute geometry both, a triangle with an angular sum of a right angle and one with an angular sum of two right angles would have to exist. This consequence contradicts, however, a theorem proved by LEGENDRE, according to which in absolute geometry it follows from the existence of a triangle with an angular sum of two right angles follows that *every* triangle has this angular sum.

In order to characterize therefore the existence of geometrical objects with the help of their consistency with the postulates, as STROHAL intends to, one has to have a *complete* system of postulates for which no decision concerning the admission of a synthesis is open anymore. This prerequisite of completeness isn't mentioned by STROHAL at any place, and furthermore, it cannot be derived from his description of the progressive method of forming and excluding syntheses whether one comes on this way to an ending anyway. |[202r]

Desisting from all these objections, however, which concern the special kind of characterizing postulates and the progressive method to obtain them, it has to be remarked, above all, that, according to the description of geometry, as STROHAL gives it here in the section on the postulates, geometry turns out to be pure conceptual combinatorics,—as it cannot be performed in formal axiomatics in a more extreme way: combinations of elementary concepts are tested; the content of these concepts is thereby not taken into account, but only certain axioms representing this content, acting as first rules of the game. Moreover, certain combinations are excluded by arbitrary stipulations, and now one waits and sees what remains as possible.

Here, the dissolution from the contentual formation of concepts is executed to the same degree as in the HILBERTean axiomatics; the initial contentual introduction of elementary concepts doesn't exert in this development; they are, so to speak, eliminated with the help of the .

Here as well, we have—similar to EUKLID's foundation of geometry—the state of affairs that the contentual determination of the elementary concepts completely runs dry, i. e., just that state of affairs for the sake of which one desists in the new axiomatics from a contentual setting of the elementary concepts.

In EUKLID's foundation, however, the state of affairs is that aspect insofar different as here the postulates are still absolutely given in an intuitive way. Especially in the first three postulates the close dependence on geometrical drawing is especially apparent. The constructions demanded here are nothing else than idealizations of graphical procedures. This contentual formation of the postulates allows that interpretation according to which the postulates are positive existential claims concerning intuitively obvious possibilities which get their verification on grounds of the intuitive content of the elementary concepts. For STROHAL, such a standpoint of *contentual axiomatics* doesn't come into consideration, because he considers an intuitively obvious verification of the postulates as impossible and therefore he can adjudicate the postulates only the character of stipulations.

So STROHAL's sketch of the geometrical axiomatics ends in a discord between the intuitive introduction of concepts and the completely non-intuitive way the geometrical system of doctrines is to be developed as a purely conceptual science starting from the definition of the geometrical space given by the postulates,—a discrepancy which is only scantily veiled by the twofold role of the the , on the one hand as analytically discernible statements, on the other hand as first restricting conditions for conceptual syntheses.

In the light of these unsatisfying results one wonders what reasons STROHAL has to reject the simple and consequent standpoint of the HILBERTean axiomatics. This question is the more justified as STROHAL knows very well the reasons leading to the HILBERTean standpoint. So he himself says: "The intuitions representing the causa occasionalis for forming the syntheses, do not pass … into geometry in the sense that one could immediately prove a statement as correct by referring to intuition;" moreover, shortly afterwards:

"As soon as the axioms"—Strohal means here only the —"are formulated, |[203l] the specific nature of elementary concepts has no influence on the further development of the geometrical deduction" (pp. 132–133).

Indeed Strohal as well has nothing objectively sound to advance against the Hilbertean foundation of geometry which can be found in the final section of his book.

Here his main argument is that in the Hilbertean conception of axiomatics the contentual element is only *pushed back* to the formal properties of the basic relations, i. e., relations of higher order. The preconditions for the basic relations expressed in the axioms would themselves have to be regarded contentually, and the contentual intuitions necessary for this could again be obtained only by abstraction from respective relational intuitions. Thus, concerning the higher relations existing in the demanded properties of geometrical basic relations one "has arrived at the reference to intuition which just was to be avoided by the axiomatics" (p. 129).

This argumentation misses the essential point. What should be avoided by the Hilbertean axiomatics is the reference to *spatial intuition*.

The sense of this method is that only that of intuitive contents is kept which *essentially enters into* geometrical proofs. By fulfilling this demand we free ourselves from the special sphere of intuition in the subject of the spatial, and what we use of contentual intuition is only that primitive kind of intuition which concerns the elementary forms of the combination of discrete, bound objects, and which is the common precondition for all scientific thinking—as it was especially stressed by Hilbert in his recent investigations on the foundations of mathematics.[6]

This methodological dissolution from spatial intuition may not be identified with ignoring the spatial intuitive starting point of geometry. It is also not connected with the intention—as described by STROHAL—"to act as if these and exactly these axioms had found together to the system of geometry due to some inner necessity" (p. 131). On the contrary, the names of spatial objects and of spatial connections of the respective objects and relations are maintained deliberately in order to give a visual expression to the correlation with spatial intuitions and facts, and to keep them continuously present.

The inadequacy of STROHAL's polemics becomes especially apparent when he additionally generates artificially the occasion for an objection. While reporting on the procedure of proving the consistency of the geometrical axioms, he states: "For this purpose one chooses as an interpretation, e. g., the concepts of common geometry; by this, the HILBERTean axioms transform into certain |[203r] statements of common geometry whose compatibility, i. e., consistency is already certain from another source. Or one interprets the symbols by numbers or functions; then the axioms fade into certain relations of numbers whose compatibility can be stated according to the laws of arithmetic" (p. 127).

STROHAL himself added the first kind of interpretation; in HILBERT there is not one single syllable speaking of an interpretation by "common geometry." STROHAL has, however, the nerve to connect with this high-handedly added explanation an objection to HILBERT's method: "If one, say, proves the consistency of HILBERTean axioms by interpreting its "points", "lines",

[6]Cf. especially the treatise: "Neubegründung der Mathematik". Hamburg 1922.

"planes" as points, lines, planes of EUCLIDean geometry whose consistency is certain, then one presupposes ... these objects as defined from another side" (p. 130).

In sum one gets the impression that STROHAL blocks himself instinctively against the acceptance of the HILBERTean standpoint, due to a resistance against the methodological innovation which is given by the formal standpoint of axiomatics compared with the contentual-conceptual opinion.

STROHAL shows this behaviour, however, not only against the HILBERTean axiomatics, but also against the most that newer science contributed of independent and important thoughts to the topic treated. This spirit of hostility is expressed in the present book not only by the distribution of praise and blame, but even more in the fact that essential achievements, thoughts and results are simply withholded. STROHAL passes over, e. g. (as already mentioned earlier), the famous investigation of HELMHOLTZ, which concerns the present topic in the closest sense, likewise the KANTean doctrine of spatial intuition with complete silence. And as to the strict mathematical proof of the independence of the parallel axiom from the other geometrical axioms, STROHAL presents this as if there is still an unsolved problem: "This question will finally be clarified only if one shows that a consequence from the other postulates *can* never collide with a refusal of the parallel postulate" (p. 101). And this statement cannot be explained as being due to ignorance; because, as follows from other passages, STROHAL has knowledge of KLEIN's projective determination of the measure of magnitudes, and also knows (from a review by WELLSTEIN) POINCARÉ's presentation of non-Euclidean geometry by a spherical geometry within the EUCLIDean space. The explanation is rather due to Strohal's oppositional emotional attitude, who refuses to accept the great achievements of newer mathematics in their significance.

So, an unacquainted reader can only receive from Strohal's book a distorted picture from the development of the geometrical science. The one who is oriented about our today's science, might take Strohal's disastrous enterprise, in view of the different methodological tendencies working together in it, as an advantage to think again through the principal questions of axiomatics.

P. Bernays