

Bernays Project: Text No. 5

**Methods for demonstrating consistency and  
their limitations  
(1932)**

Paul Bernays

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Translation by: *Dirk Schlimm*

Comments:

*Proofread by Buldt and Schlimm.*

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||<sup>201</sup> The methods that were used to prove the consistency of formalized theories from the finitist standpoint can be surveyed according to the following classification.

||<sup>202</sup>1. *Method of valuation.* It has obtained its essential development by Hilbert's procedure of trial valuation. Using this procedure Ackermann and v. Neumann demonstrated the consistency of number theory—admittedly, under the restrictive condition that the application of the inference from  $n$  to  $(n + 1)$  is only allowed to formulas with only free variables.

2. *Method of integration.* This can only be applied to such domains that are completely mastered mathematically. For these, it allows to answer not only the question of consistency, but also those of completeness and decidability, in a completely positive sense. Such domains are in particular:

- a) the monadic function calculus, which was treated conclusively by Löwenheim, Skolem, and Behmann.
- b) Fragments of number theory. To such [formalisms] Herbrand and Presburger have applied the method. Thereby it becomes obvious that the Peano axioms, using the function calculus of “first order” (with the axioms for equality) as a foundation, do not yet suffice for the development of number theory. Only by adding the recursive equations for addition and multiplication do we arrive at full number theory<sup>1</sup>.

3. *Method of elimination.* It can be found already foreshadowed in Russell and Whitehead, in particular in the application to the concept “such that.” However, the actual implementation of the idea is cumbersome. An essential simplification is effected by an approach of Hilbert, which follows the introduction of the “ $\epsilon$ -symbol.”

This approach yields, first, in a simpler way again the result of the method of valuation—as has been shown by Ackermann.

Moreover, one arrives from here at a new proof of a theorem, that was discovered and proved for the first time by Herbrand. It is a reversal of Löwenheim’s famous theorem about the satisfiability in the countable domain and it also yields a general procedure for the treatment of questions about consistency.

Despite the insights obtained in multiple ways the present limitation of the results presents itself as a fundamental one; this is because of Gödel’s

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<sup>1</sup>The situation is different if one, like Dedekind, takes the standpoint of the logic of classes as basic from the outset; this standpoint, however, contains stronger assumptions than are needed for number theory.

new theorem on the limits of decidability in formal systems, in conjunction with v. Neumann's conjecture connected to it.