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§ 1. The Problem of consistency in axiomatics as a logical decision problem.

The state of research in the field of foundations of mathematics, to which our presentation is related, is characterized by three kinds of investigations:

1. the development of the axiomatic method, especially with the help of the foundations of geometry,
2. the founding of analysis by today's rigorous methods through the reduction of the theory of magnitudes to the theory of numbers and sets of numbers,
3. investigations in the foundations of number theory and set theory.

A deeper set of tasks, linked to the standpoint reached through these investigations, arises on the basis of methods subjected to stricter demands;

these problems involve a new way of dealing with the problem of the infinite. We will introduce these problems by considering axiomatics.

The term ‘axiomatic’ is used partly in a wider, and partly a narrower sense. We call the development of a theory axiomatic in the widest sense of the word, if the fundamental concepts and presuppositions as such are set out on top and marked as such, and the further content of the theory is logically derived from these with the help of definitions and proofs. In this sense the geometry of EUCLID, the mechanics of NEWTON, and the thermodynamics of CLAUSIUS were axiomatically founded.

The axiomatic point of view was made more rigorous in HILBERT’s “Foundations of Geometry”. The greater rigor consists in the fact that in the axiomatic development of a theory one keeps only that portion of the presentational subject matter, from which the fundamental concepts of the theory are formed, that is formulated as an extract in the axioms; one abstracts, however, from all remaining content. Another factor coming along in axiomatics in the narrowest sense is the *existential form*. It serves to distinguish *the axiomatic method* from the *constructive* or *genetic* method of founding a theory.¹ Whereas in the constructive method |²the objects of a theory are introduced merely as a *family* of things,² in an axiomatic theory one is concerned with a fixed system of things (or several such systems) which constitutes a previously *delimited domain of subjects* for all predicates from which the statement of the theory are constituted.

Except in the trivial cases in which a theory has to do just with a finite,

¹See for this comparison appendix VI to HILBERT’s *Grundlagen der Geometrie: Über den Zahlbegriff*, 1900.

²Brouwer and his school use the word “species” in this sense.

fixed totality of things, the presupposition of such a totality, of a “domain of individuals”, involves an idealizing assumption joining the assumptions formulated in the axioms.

It is a characteristic of this sharpened kind of axiomatics that results from abstraction from material content and also the existential form—we will call it “formal axiomatics” for short—that it requires a *proof of consistency*, whereas contentual axiomatics introduces its fundamental concepts by reference to known acts of experience and its basic principles either as obvious facts, which one can make clear to oneself, or as extracts from complexes of experiences, thereby expressing the belief that one is on the track of laws of nature and at the same time intending to support this belief through the success of the theory.

Formal axiomatics as well needs in any case certain evidence in the performance of deductions as well as in the proof of consistency; however, there is the essential difference that this kind of evidence does not depend on any special epistemological relation to special field, but rather it is one and the same for every axiomatization, namely it is that primitive kind of knowledge that is the precondition of every exact theoretical investigation whatsoever. We will consider this kind of evidence more closely.

The following aspects are especially important for a correct evaluation of the significance for epistemology of the relationship between contentual and formal axiomatics:

Formal axiomatics requires contentual axiomatics as a supplement, because only in terms of this supplement can one give instruction in the choice of formalisms and, moreover, in the case of a given formal theory, give an

instruction of its applicability to some domain of reality.

On the other hand we cannot just stay at the level of contentual axiomatics, since in science we are if not always, so nevertheless predominantly, concerned with such theories ³ that get their significance from a *simplifying idealization* of an actual state of affairs rather than from a complete reproduction of it. A theory of this kind cannot get a foundation through a reference to either the evident truth of its axioms or to experience; rather such a foundation can only be given when the idealization performed, i.e., the extrapolation through which the concept formations and the principles of the theory come to overstep the reach either of intuitive evidence or of the data of experience, is understood to be consistent. Furthermore, reference to the approximate validity of the principles is of no use for the recognition of consistency; for an inconsistency could arise just because a relationship which holds only in a restricted sense is taken to hold exactly.

We are therefore forced to investigate the consistency of theoretical systems without considering matters of fact and, with this, we are already at the standpoint of formal axiomatics.

As to the treatment of this problem up until now, both in the case of geometry and in branches of physics, this is done with the help of the *method of arithmetization*: one represents the objects of a theory through numbers and systems of numbers and basic relations through equations and inequalities in such a way that on the basis of this translation the axioms of the theory become either arithmetic identities or provable assertions (as in the case of geometry) or (as in physics) a system of conditions, the simultaneous satisfiability of which can be proved on the basis of certain arithmetic existence

assertions. In this procedure the validity of arithmetic, i.e., the theory of real numbers (analysis) is presupposed; so we come to the question of what kind this validity is.

However, before we concern ourselves with this question we want to see whether there isn't a direct way of attacking the problem of consistency. We want to get the structure of this problem clearly before our minds, anyway. At the same time we already want to take the advantage to familiarize ourselves a bit with *logical symbolism*, which proves to be very useful for the given purpose and which we will have to consider more deeply in the sequel.

As an example of axiomatics we take the *geometry of the plane*; and for the sake of simplicity we will consider only the axioms of the geometry of position (the axioms that are presented as "axioms of connection" and "axioms of order" in Hilbert's "Grundlagen der Geometrie" together with the parallel axiom. For our purpose it suggests itself to diverge from Hilbert's axiom system by not taking points and lines⁴ as two basic systems of things but rather to *take only points as individuals*. Instead of the relation "points x and y determine the line g ," we use the relation between *three* points " x, y, z lie on one line" for which we use the designation $Gr(x, y, z)$. Betweenness comes as a second fundamental relation to this relation: " x lies between y and z ", which we designate with $Zw(x, y, z)$.³ Moreover, identity of x and y appears in the axioms as a notion belonging to logic, for which we use the usual equality sign $x = y$.

³The method of taking only points as individuals is in particular developed in the axiomatics of OSWALD VEULEN "A system of axioms for geometry". Here furthermore all geometrical relations are defined in terms of the relation "between".

In addition we only need the logical signs for the symbolic presentation of the axioms, namely first the signs for generality and existence: if $P(x)$ is a predicate referring to the object x , then $(x)P(x)$ means, “all x have the property $P(x)$,” and $(Ex)P(x)$ means “there is an x with the property $P(x)$.” (x) is named the “for-all-sign,” and (Ex) the “there-is-sign.” The for-all-sign and there-is-sign can refer to any other variable y, z, u in the same way they can refer to x . The variable belonging to such a sign is “bound” by this sign, in the same way an integration variable is bound by the integration sign, so that the whole statement does not depend on the value of the variables.

Signs for negation and the joining of sentences are added as further logical signs. We designate the negation of a statement by overstriking. In the case of a preceding for-all-sign or there-is-sign the negation stroke is to be set only above this sign, and instead of $\overline{x = y}$ the shorter $x \neq y$ should be written. The sign $\&$ (“and”) between two statements means that both statements hold (conjunction). The sign \vee (“or” in the sense of “vel”) between two statements means that at least one of the two statements holds (“disjunction”).

The sign \rightarrow between two statements means that the holding of the first entails the holding of the second, or with other words, that the first statement does not hold, without the second holding as well (“implication”). An implication $\mathfrak{A} \rightarrow \mathfrak{B}$ between two statements \mathfrak{A} and \mathfrak{B} is accordingly only then wrong, if \mathfrak{A} is true and \mathfrak{B} is false. In all other cases it is true.

The combination of the sign of implication with the for-all-sign results in the presentation of general hypothetical statements. For example, the formula

$$(x)(y) (\mathfrak{A}(x, y) \rightarrow \mathfrak{B}(x, y)) ,$$

with $\mathfrak{A}(x, y)$, $\mathfrak{B}(x, y)$ standing for the presentation of certain relations between x and y , represents the statement “If $\mathfrak{A}(x, y)$ holds, then $\mathfrak{B}(x, y)$,” or also: “for every pair of individuals x, y for which $\mathfrak{A}(x, y)$ holds, $\mathfrak{B}(x, y)$ holds as well.”⁴

We use brackets in the usual way for linking together parts of formulas. For saving brackets we stipulate that for the separation of symbolic expressions \rightarrow takes precedence over $\&$ and \vee , $\&$ over \vee , and that \rightarrow , $\&$, \vee all have precedence over the for-all-sign and the there-is-sign. Brackets are omitted if no ambiguities are possible. We write, for example, instead of the expression

$$(x) ((Ey)R(x, y)),$$

in which $R(x, y)$ designates an arbitrary relation between x and y , simply $(x)(Ey)R(x, y)$ because in this case only one reading is possible: “for every x there is a y for which the relation $R(x, y)$ holds.”—

We are now in position to write down the axiom system considered. To make it easier the first axioms are accompanied by a linguistic version.

The demarcation of the axioms does not correspond completely to that in HILBERT’s “Grundlagen der Geometrie.” We therefore give for each group of axioms the relationship of the axioms here presented as formulas to those of HILBERT.⁵

I. Axioms of connection.

⁴The relation between disjunction and implication defined here and disjunctive and hypothetical junctions of statements in the usual sense will be discussed in § 3.

⁵This information is especially meant for those familiar with HILBERT’s “Grundlagen der Geometrie.” All references are to the seventh edition.

1. $(x)(y)Gr(x, x, y).$

“ x, x, y always lie on one line.”

2. $(x)(y)(z)(Gr(x, y, z) \rightarrow Gr(y, x, z) \ \& \ Gr(x, z, y)).$

“If x, y, z lie on a line, then so do y, x, z as well as x, z, y lie on a line.”

3. $(x)(y)(z)(u)(Gr(x, y, z) \ \& \ Gr(x, y, u) \ \& \ x \neq y \rightarrow Gr(x, z, u)).$

“If x, y are different points and if x, y, z as well as x, y, u lie on a line then also x, z, u lie on a line.”

4. $(Ex)(Ey)(Ez)\overline{Gr(x, y, z)}.$

“There are points x, y, z which do not lie on a line.”

Of these axioms, 1) and 2) replace the axioms I 1,—because of the changed concept of line; 3) corresponds to the axiom I 2; and 4) corresponds to the second part of I 3.

II. Axioms of order

1. $(x)(y)(z)(Zw(x, y, z) \rightarrow Gr(x, y, z))$

2. $(x)(y)\overline{Zw(x, y, y)} .$

3. $(x)(y)(z)(Zw(x, y, z) \rightarrow Zw(x, z, y) \ \& \ \overline{Zw(y, x, z)}) .$

4. $(x)(y)(x \neq y \rightarrow (Ez)Zw(x, y, z)) .$

“If x and y are different points, there is always a point z such that x lies between y and z .”

5. $(x)(y)(z)(u)(v) \left(\overline{Gr(x, y, z)} \ \& \ Zw(u, x, y) \ \& \ \overline{Gr(v, x, y)} \ \& \ \overline{Gr(z, u, v)} \right. \\ \left. \rightarrow (Ew)\{Gr(u, v, w) \ \& \ Zw(w, x, z) \vee Zw(w, y, z)\} . \right.$

1) and 2) together constitute the first part of HILBERT's axioms II 1; 3) unites the last part of HILBERT's axioms II 1 with II 3; 4) is the axiom II 2; and 5) is the axiom of plane order II 4.

III. Parallel axiom

Since we are not including congruence axioms, we must take the parallel axiom in the following broader sense: "For every straight line there is exactly one line through a point outside it which does not intersect it."⁶

To make symbolic formulation easier the symbol

$$Par(x, y; u, v)$$

will be used as an abbreviation for the expression

$$\overline{(Ew)}(Gr(x, y, w) \& Gr(u, v, w))$$

"There is no point w which lies on a line both with x and y and with u and v ."

The axiom is then

$$(x)(y)(z) \left(\overline{Gr(x, y, z)} \rightarrow (Eu) \{ Par(x, y; z, u) \& (v) (Par(x, y; z, v) \rightarrow Gr(z, u, v)) \} \right) .$$

If we imagine the axioms here enumerated and unite them, we get a single logical formula which represents an assertion about the predicates ' Gr ', ' Zw ' and which we designate as

$$\mathfrak{A}(Gr, Zw) .$$

In the same way we could represent a theorem of plane geometry involving only position and order relations as a formula

⁶Cf. p. 83 of HILBERT's "Grundlagen der Geometrie."

$$\mathfrak{S}(Gr, Zw).$$

This representation still accords with contentual axiomatics in which the fundamental relations are viewed as something that can be shown in experience or in the intuitive imagination and thus definite in content about which the statements of the theory make assertions.

On the other hand, in formal axiomatics the fundamental relations are not conceived from the beginning as determined in content; rather they receive their determination *implicitly* through the axioms; and in any consideration of an axiomatic theory only what is expressly formulated in the axioms about the fundamental relations is used.

As a result, if in axiomatic geometry the respective names for relations in intuitive geometry like “lie on” or “between” are used, this is only a concession to custom and a means of simplifying the connection of the theory with intuitive facts. In fact, however, in formal axiomatics the fundamental relations play the role of *variable* predicates.

Here and in the sequel we understand “predicate” in the wider sense so that it also applies to predicates with two or more subjects. We speak of “one-place”, “two-place”, . . . predicates according to the number of subjects.

In the part of axiomatic geometry considered by us there are two variable three-place predicates:

$$R(x, y, z), \quad S(x, y, z) .$$

The axiom system consists of a demand on two such predicates expressed in the logical formula $\mathfrak{A}(R, S)$, that we get from $\mathfrak{A}(Gr, Zw)$ when we replace

$Gr(x, y, z)$ with $R(x, y, z)$, $Zw(x, y, z)$ with $S(x, y, z)$. The identity relation $x = y$ which is to be interpreted contentually appears in this formula along with the variable predicates. The acceptance of this predicate as contentually determinate is no violation of our methodological standpoint. For the contentual determination of identity—which is no relation at all in the true sense—does not depend on the particular range of imagination of the field being investigated axiomatically; rather it is only related to a question of distinguishing individuals which must be taken as already given when the domain of individuals is laid down.

From this point of view a sentence of the form $\mathfrak{S}(Gr, Zw)$ corresponds to the logical statement that for *any* predicates $R(x, y, z)$, $S(x, y, z)$ satisfying the demand $\mathfrak{A}(R, S)$, the relation $\mathfrak{S}(R, S)$ also holds; in other words, for any two predicates $R(x, y, z)$, $S(x, y, z)$ the formula

$$\mathfrak{A}(R, S) \rightarrow \mathfrak{S}(R, S)$$

represents a true statement. In this way a geometrical sentence is transformed into a sentence of pure predicate logic.⁸

From this point of view the problem of consistency presents itself in a corresponding way as a problem of pure predicate logic. In fact it is a question of whether two three-place predicates $R(x, y, z)$, $S(x, y, z)$ can satisfy the conditions expressed in the formula $\mathfrak{A}(R, S)$ ⁷ or whether, on the contrary, the assumption that the formula $\mathfrak{A}(R, S)$ is satisfied for a certain pair of predicates leads to a contradiction so that in general for every pair of predicates R, S the formula $\overline{\mathfrak{A}(R, S)}$ represents a correct assertion.

⁷This imprecise way of putting the question will be sharpened in the sequel.

A question like the one given here is part of the “*decision problem.*” In newer logic this problem is understood to be that of discovering general methods for deciding the “validity” or “satisfiability” of logical formulas.⁸

In this connection the formulas investigated are composed with the help of logical signs out of predicate variables and equalities—together with variables in subject positions which we call “individual variables”—, and it is assumed that every variable is bound by a for-all sign or there-is sign.

A formula of this kind is called logically valid when it represents a true assertion for *every* determination of the variable predicates; it is called satisfiable when it represents a true assertion for some *appropriate* determination of the predicate variables.

Simple examples for logically valid formulas are the following:

$$\begin{aligned} (x)F(x) \ \& \ (x)G(x) \ \rightarrow \ (x)(F(x) \ \& \ G(x)) \\ (x)P(x, x) \ \rightarrow \ (x)(Ey)P(x, y) \\ (x)(y)(z)(P(x, y) \ \& \ y = z \ \rightarrow \ P(x, z)). \end{aligned}$$

Examples for satisfiable formulas are:

$$\begin{aligned} (Ex)F(x) \ \& \ (Ex)\overline{F(x)} \\ (x)(y)(P(x, y) \ \& \ P(y, x) \ \rightarrow \ x = y) \\ (x)(Ey)P(x, y) \ \& \ (Ey)(x)\overline{P(x, y)} . \end{aligned}$$

These formulas result, e. g., in true assertions for the domain of individuals of the numbers 1, 2, if in the first formula for $F(x)$ “ x is even” is set, in the second formula for $P(x, y)$ the predicate $x \leq y$, and in the third formula for $P(x, y)$ the predicate $x \leq y \ \& \ y \neq 1$.

⁸This explanation is correct only for the decision problem in its narrower sense. We have no need here to consider the broader conception of this decision problem.

It is to be observed that along with the determination of the predicates the *domain of individuals* over which the variables x, y, \dots range has to be fixed. This enters into a logical formula as a kind of *hidden variable*. However, the logical formula in respect to satisfiability is invariant with respect to a one-one mapping of a domain of individuals onto another, since the individuals enter into the formulas only as variable subjects; as a result the only essential determination for a domain of individuals is the *number of individuals*.

Accordingly, we have to distinguish the following questions in relation to logical validity and satisfiability:

1. The question of logical validity for *every* domain of individuals, and also of satisfiability for *any* domain of individuals respectively.
2. The question of logical validity or satisfiability for a given number of individuals.
3. The question for which numbers of individuals is a formula logically valid or satisfiable.

It should be noted that it is best to leave out of consideration the domain of 0 individuals on principle, since formally zero-numbered domains of individuals have a special status, and on the other hand consideration of them is trivial and worthless for applications.⁹

⁹The stipulation that every domain of individuals should contain at least one thing, so that a true general judgement must hold of at least one thing, ought not to be confused with the convention prominent in ARISTOTELEAN logic that a judgment of the form “all S are P ” counts as true only if there are in fact things with the property S . This convention has been dropped in newer logic. A judgment of this kind is represented symbolically in

Furthermore one should take into account that only the “value-range” of a predicate is relevant to its determination; that is to say, all that is relevant is for which values of the variables in subject positions the predicate holds or does not hold (is “true” or “false”).

This circumstance has as a consequence that for a *given finite* number of individuals the logical validity or satisfiability of a specific given logical formula represents a pure *combinatorial fact* which one can determine through elementary testing of all cases.

To be specific, if n is the number of individuals and k the number of subjects (“places”) of a predicate, then n^k is the number of different systems of values for the variables; and since for every one of these systems of values the predicate is either true or false, there are

$$2^{(n^k)}$$

different possible value-ranges for a k -place predicate.

If then

$$R_1, \dots, R_t$$

are the distinct predicate variables occurring in a given formula, with arities

$$k_1, \dots, k_t$$

then

$$2^{(n^{k_1} + n^{k_2} + \dots + n^{k_t})}$$

the form $(x)(S(x) \rightarrow P(x))$; it counts as true if a thing x , insofar as it has the property $S(x)$, always has the property $P(x)$ as well—independently of whether there is anything with the property Sx at all. We will take up this topic again in connection with the deductive construction of predicate logic. (See § 4 pp. 106–107.)

is the number of systems of value-ranges to be considered, or the number of different possible predicate systems for short.

Accordingly logical validity of the formula means that for all of these

$$2^{(n^{k_1} + n^{k_2} + \dots + n^{k_t})}$$

explicitly enumerable predicate systems the formula represents a true assertion; and its satisfiability means that the formula represents a true assertion for one of these predicate systems. Moreover, for a fixed predicate system the truth or falsity of the assertion represented by the formula is again decidable by a finite testing of cases; the reason is that only n values come into consideration for a variable bound by a for-all sign or there-is sign so that ‘all’ has the same meaning as a conjunction with n members and ‘there is’ a disjunction with n members.

For example, consider the formulas mentioned above

$$(x)P(x, x) \rightarrow (x)(Ey)P(x, y)$$

$$(x)(y)(P(x, y) \& P(y, x) \rightarrow x = y)$$

of which the first has been referred to as a logically valid, the second as a satisfiable, formula. We refer these formulas to a domain of two individuals.

We can indicate both individuals with the numerals 1, 2. In this case we have $t = 1, n = 2, k_1 = 2$; therefore the number of different predicate systems is

$$2^{(2^2)} = 2^4 = 16.$$

In place of $(x)P(x, x)$ we can put

$$P(1, 1) \& P(2, 2)$$

in place of $(x)(Ey)P(x, y)$

$$P(1, 1) \vee P(1, 2) \ \& \ P(2, 1) \vee P(2, 2) ,$$

so that the first of the two formulas becomes

$$P(1, 1) \ \& \ P(2, 2) \rightarrow P(1, 1) \vee P(1, 2) \ \& \ P(2, 1) \vee P(2, 2) .$$

This implication is true for those predicates P for which $P(1, 1) \ \& \ P(2, 2)$ is false, as well as for those for which

$$P(1, 1) \vee P(1, 2) \ \& \ P(2, 1) \vee P(2, 2)$$

is true. One can now verify that for each of the 16 value-ranges that one gets when one assigns one of the truth values “true” or “false” to each of the pairs of values

$$(1, 1), (1, 2), (2, 1), (2, 2)$$

one of the two conditions is satisfied; thus the whole expression always receives the value “true.” [Verification is simplified in this example because already the determination of the values of $P(1, 1)$ and $P(2, 2)$ suffices to fix the correctness of the expression.] In this way the validity of our first formula for domains of two individuals can be determined through directly trying it out.

For domains of two individuals the second formula has the same meaning as the conjunction

$$\begin{aligned} & (P(1, 1) \ \& \ P(1, 1) \rightarrow 1 = 1) \ \& \ (P(2, 2) \ \& \ P(2, 2) \rightarrow 2 = 2) \\ & \ \& \ (P(1, 2) \ \& \ P(2, 1) \rightarrow 1 = 2) \ \& \ (P(2, 1) \ \& \ P(1, 2) \rightarrow 2 = 1) . \end{aligned}$$

Since $1 = 1$ and $2 = 2$ are true the first two members of the conjunction are always true assertions. The last two members are true if, and only if,

$$P(1, 2) \ \& \ P(2, 1)$$

is false.

Therefore, to satisfy the formula under consideration one has only to eliminate those determinations of value for P in which the pairs $(1, 2)$ and $(2, 1)$ are both assigned the value “true.” Every other determination of value produces a true assertion. The formula is therefore satisfiable in a domain of two elements.

These examples should make clear the purely combinatorial character of the decision problem in the case of a given finite number of individuals. One result of this combinatorial character is that for a prescribed finite number of individuals the logical validity of a formula \mathfrak{F} has the same meaning as the unsatisfiability of the formula $\overline{\mathfrak{F}}$; likewise the satisfiability of a formula \mathfrak{F} has the same meaning as that $\overline{\mathfrak{F}}$ is not valid. Indeed \mathfrak{F} represents a true assertion for those predicate systems for which $\overline{\mathfrak{F}}$ represents a false assertion and vice-versa.

Let us return to the question of the consistency of an axiom system. Let us consider an axiom system written down symbolically and combined into one formula like our example.

The question of the satisfiability of this formula for a prescribed finite number of individuals can be decided, in principle at least, through trying it out. Suppose then the satisfiability of the formula is determined for a definite finite number of individuals. The result is a proof of the consistency of the axiom system, namely a proof by the *method of exhibition*, since the

finite domain of individuals together with the value-ranges chosen for the predicates (to satisfy the formula) constitutes a model in which we can show concretely that the axioms are satisfied.

We give an example of such an exhibition from axiomatics in geometry. We start from the axiom system presented in the beginning, but replace the axiom I 4), which postulates the existence of three points not lying on a line, with the weaker axiom

$$\text{I } 4') \quad (Ex)(Ey)(x \neq y) .$$

“There are two distinct points.”

Furthermore we drop the axiom of plane order II 5); in its place we add to the axioms¹⁰ two sentences which can be proved using II 5) by, firstly, expanding II 4) to

$$\text{II } 4') \quad (x)(y)\{x \neq y \rightarrow (Ez)Zw(z, x, y) \& (Ez)Zw(x, y, z)\} ,$$

and, secondly, adding

$$\text{II } 5) \quad (x)(y)(z)\{x \neq y \& x \neq z \& y \neq z \rightarrow Zw(x, y, z) \vee Zw(y, z, x) \vee Zw(z, x, y)\} .$$

We keep the parallel axiom. The resulting axiom system corresponds to a formula $\mathfrak{A}'(R, S)$ instead of the earlier $\mathfrak{A}(R, S)$; it is satisfiable in a domain of individuals of 5 things, as O. Veblen remarked.¹¹ The value-ranges for the

¹⁰Both of these sentences were introduced as axioms in earlier editions of HILBERT's “Grundlagen der Geometrie.” It turned out that they are provable using the axioms of plane order. See pp. 5–6 of the seventh edition.

¹¹In the investigation already mentioned “A system of axioms for geometry,” Trans. Amer. Math. Soc. vol. 5, p. 350.

predicates R, S are so chosen that first of all the predicate Gr is determined to be true for every value triple x, y, z —we can here use the symbols ‘ Gr ’, ‘ Zw ’ with no danger of misunderstanding. One sees immediately that then all axioms I as well as II 1) and III are satisfied. In order that the axioms II 2), 3), 5’), and 4’) be satisfied it is necessary and also sufficient that the following three conditions be placed on the predicate Zw :

1. Zw is always false for a triple x, y, z in which two elements coincide.
2. For any combination of three different of the 5 individuals, Zw is true for 2 orderings with a common first element (of 6 possible orderings of the elements), false for the remaining 4 orderings.
3. Each pair of different elements occurs as an initial as well as a final pair in one of the triples for which Zw is true.

The first demand can be directly fulfilled by stipulation. The joint satisfaction of the other two conditions is accomplished as follows: We designate the 5 elements with the numerals 1, 2, 3, 4, 5. The number of value-triples of three distinct elements for which Zw still has to be defined is $5 \cdot 4 \cdot 3 = 60$. Every six of these belong to a combination; for two of these Zw should be true and false for the rest. We must therefore indicate those 20 of the 60 triples for which Zw will be defined as true. They are those which one obtains from the four triples

$$(1\ 2\ 5), (1\ 5\ 2), (1\ 3\ 4), (1\ 4\ 3)$$

by applying the cyclical permutation $(1\ 2\ 3\ 4\ 5)$.

It is easy to verify that this procedure satisfies all the conditions. Thus the axiom system is recognized as consistent by the method of exhibition.¹²

The method of exhibition presented in this example has very many different applications in newer axiomatic investigations. It is especially used for *proofs of independence*. The assertion that a sentence \mathfrak{S} is independent of an axiom system \mathfrak{A} has the same meaning as the assertion of the consistency of the axiom system as the claim that the axiom system

$$\mathfrak{A} \ \& \ \overline{\mathfrak{S}}$$

which we get when we add the negation of the sentence \mathfrak{S} as an axiom to \mathfrak{A} . The consistency can be determined by the method of exhibition if this axiom system is satisfiable in a finite domain domain of individuals.¹³ Thus this method provides a sufficient extension of the method of progressive inferences for many fundamental investigations in the sense that the unprovability of a sentence from certain axioms can be proved through exhibition, its provability through inference.

But is the application of the method of exhibition restricted in its application to finite domains of individuals? We cannot derive this from what we have said up until now. However, we do see immediately that in the case of an infinite domain of individuals the possible systems of predicates no

¹²It follows immediately from the fact that the modified axiom system \mathfrak{A}' is satisfiable in a domain of 5 individuals that the axioms of this system do not completely determine linear ordering.

¹³A great number of examples of this procedure can be found in the works on linear and cyclical order by E. V. Huntington and his collaborators. See especially “A new set of postulates for betweenness with proof of complete independence”, Trans. Amer. Math. Soc. vol. 26 (1924) pp. 257–282. Here one also finds references to previous works.

longer constitute a surveyable multitude and there can be no talk of testing all value-ranges. Nevertheless in the case of given axioms we might be in a position to show their satisfiability by given predicates. And this is actually the case. Consider for example the system of three axioms

$$\begin{aligned} & (x)\overline{R(x,x)} , \\ & (x)(y)(z)(R(x,y) \& R(y,z) \rightarrow R(x,z)) \\ & (x)(Ey)R(x,y). \end{aligned}$$

Let us clarify what these say: We start with an object a in the domain of individuals. According to the third axiom there must be a thing b for which $R(a,b)$ is true; and because of the first axiom, b must be different from a . For b there must further be a thing c for which $R(b,c)$ is true, and because of the second axiom $R(a,c)$ is also true; according to the third axiom c is distinct from a and b . For c there must again be a thing d for which $R(c,d)$ is true. For this thing $R(a,d)$ and $R(b,d)$ are also true, and d is distinct from a, b, c . The method of this consideration here has no end; and it shows us we cannot satisfy the axioms with a finite domain of individuals. On the other hand we can easily show satisfaction by an infinite domain of individuals: We take the integers as individuals and substitute the relation “ x is less than y ” for $R(x,y)$; one sees immediately that all three axioms are satisfied.

It is the same with the axioms

$$\begin{aligned} & (Ex)(y)\overline{S(y,x)} , \\ & (x)(y)(u)(v)(S(x,u) \& S(y,u) \& S(v,x) \rightarrow S(v,y)) , \\ & (x)(Ey)S(x,y) . \end{aligned}$$

One can easily ascertain that these cannot be satisfied with a finite domain of individuals. On the other hand they are satisfied in the domain of positive

integers if we replace $S(x, y)$ with the relation “ y immediately follows x .”

However, we notice in these examples that exhibiting in these cases does by no means conclusively settle the question of consistency; rather the question is *reduced* to that of the *consistency of number theory*. In the earlier example of finite exhibition we took integers as individuals. There, however, this was only for the purpose of having a simple way to designate individuals. Instead of numbers we could have taken other things, letters for example. And also the properties of numbers which were used could have been established by a concrete exhibition.

In the case now before us, however, a concrete idea of number is not enough; for we essentially need the assumption that the *integers* constitute *a domain of individuals* and therefore a ready totality.

We are, of course, quite familiar with this assumption since in newer mathematics we are constantly working with it; one is inclined to consider it perfectly natural. It was FREGE who vigorously and with a sharp and witty critique first established that the idea of the sequence of integers as a ready totality must be justified by a proof of consistency.¹⁴ According to FREGE, such a proof had to be carried out in the sense of an exhibition, as an existence proof; and he believed he could find the objects for such an exhibition in the domain of logic. His method of exhibition amounts to defining the totality of integers with the help of the totality (presupposed to exist) of all conceivable one-place predicates. However, the underlying assumption, which under impartial consideration seems very suspect anyway, was shown

¹⁴GOTTLOB FREGE, “Grundlagen der Arithmetik”, Breslau 1884, and “Grundgesetze der Arithmetik”, Jena 1893.

to be untenable by the famous logical and set-theoretic paradoxes discovered by RUSSELL and ZERMELO. And the failure of FREGE's undertaking has made us even more conscious of the problematic character of assuming the totality of the number sequence than did his dialectic.

In the light of this difficulty we might try to use some other infinite domain of individuals instead of the sequence of integers for the purpose of proving consistency, a domain taken from the realm of sense perception or physical reality rather than being a pure product of thought like the sequence of integers. However, if we look more closely we will realize that wherever we think we encounter infinite manifolds in the realm of sensible qualities or in physical reality there can be no talk of the actual presence of such a manifold; rather the conviction that such a manifold is present rests on a mental extrapolation, the justification of which is as much in need of investigation as the conception of the totality of the sequence of integers.

A typical example in this connection is those cases of the infinite which gave rise to the well-known paradox of ZENO. Suppose some distance is traversed in a finite time; the traversal includes infinitely many successive subprocesses: the traversal of the first half, then of the next quarter, then the next eighth, and so on. If we are considering an actual motion, then these subtraversals must be real processes succeeding one another.

People have tried to refute this paradox with the argument that the sum of infinitely many time intervals may converge producing a finite duration. However, this reply does not come to grips with an essential point of the paradox, namely the paradoxical aspect that lies in the fact that an infinite succession, the completion of which we could not accomplish in the imagina-

tion either actually or in principle, should be accomplished in reality.

Actually there is a much more radical solution of the paradox. It consists in considering that we are by no means forced to believe that the mathematical space-time representation of movement remains physically meaningful for arbitrarily small segments of space and time; rather there is every reason to assume that a mathematical model extrapolates the facts of a certain domain of experience, e.g., just the movements, within the range of magnitudes accessible to our observation up to now for the purpose of a simple conceptual structure; this is similar to continuum mechanics which carries out an extrapolation in taking as a basis the idea of space as filled with matter; it is no more the case that unbounded division of a movement always produces something characterizable as movement than that unbounded spatial division of water always produces quantities of water. When this is accepted the paradox vanishes.

Notwithstanding, the mathematical model of movement has, as an *idealizing concept* formation, its value for the purpose of simplified representation. For this purpose it must not only coincide approximately with reality, but it has to meet the condition that the extrapolation it involves must be consistent in itself. From this point of view the mathematical conception of movement is not in the least shaken by ZENO's paradox; the mathematical counterargument just referred to has in this case complete validity. It is another question however, whether we possess a real proof of the consistency of the mathematical theory of motion. This theory depends essentially on the mathematical theory of the continuum; this in turn depend essentially on the idea of the set of all integers as a ready totality. We therefore come

back by a roundabout way to the problem we tried to avoid by referring to the facts about motion.

It is much the same in every case in which a person thinks he can show directly that some infinity is given in experience or intuition, for example the infinity of the tone row extending from octave to octave to infinity, or the continuous infinite manifold involved in the passage from one color quality to another. Closer consideration shows in every case that in fact no infinity is given at all; rather it is interpolated or extrapolated through some mental process.

These considerations make us realize that reference to non-mathematical objects can not settle the question whether an infinite manifold exists; the question must be solved within mathematics itself. But how should one make a start with such a solution? At first glance it seems that something impossible is being demanded here: to present infinitely many individuals is impossible in principle; therefore an infinite domain of individuals as such can only be indicated through its structure, i. e., through relations holding among its elements. In other words: a proof must be given that for this domain certain formal relations can be satisfied. The existence of an infinite domain of individuals can *not be represented in any other way than through the satisfiability of certain logical formulas*; but these are exactly the kind of formulas we were led to through investigating the question of the existence of an infinite domain of individuals; and the satisfiability of these formulas was to have been demonstrated by the exhibition of an infinite domain of individuals. The attempt to apply the method of exhibition to the formulas under consideration leads then to a vicious circle.

But exhibition should serve only as a means in proofs of the consistency of axiom systems. We were led to this procedure through considering domains with a given finite number of individuals, and just through recognizing that in such domains the consistency of a formula has the same significance as its satisfiability.

The situation is more complicated in the case of infinite domains of individuals. It is true in this case also that an axiom system represented by a formula \mathfrak{A} is inconsistent if, and only if, the formula $\overline{\mathfrak{A}}$ is logically valid. But since we are no longer dealing with a surveyable supply of value-ranges for the variable predicates, we can no longer conclude that if $\overline{\mathfrak{A}}$ is not logically valid, there is some model for satisfying the axiom system \mathfrak{A} at our disposal.

Accordingly, when an infinite domain of individuals is under consideration, the satisfiability of an axiom system is a sufficient condition for its consistency, but it is not proved to be a necessary condition. We cannot therefore expect that in general a proof of consistency can be accomplished by means of a proof of satisfiability. On the other hand we are not forced to prove consistency by establishing satisfiability; we can just hold to the original negative sense of inconsistency. That is to say—if we again imagine an axiom system represented by a formula \mathfrak{A} —we do not have to show that satisfiability of the formula \mathfrak{A} , but only need to prove that the assumption that \mathfrak{A} is satisfied by certain predicates cannot lead to a logical contradiction.

To attack the problem in these terms we must first aim at an overview of the possible logical inferences that can be made from an axiom system. The *formalization of logical inference* as developed by FREGE, SCHRÖDER, PEANO, and RUSSELL presents itself as an appropriate means to this end.

We have thus arrived at the following tasks: 1. to formalize rigorously the principles of logical inference and by this turn them into a completely surveyable system of rules; 2. to show for a given axiom system \mathfrak{A} (which is to be proved consistent), that starting with this *system* \mathfrak{A} *no contradiction can arise via logical deductions*, that is to say, no two formulas of which one is the negation of the other can be proved.

However, we do not have to carry out this proof for each axiom system individually; for we can make use of the method of *arithmetizing* to which we referred at the beginning. From the point of view we have reached now this procedure can be characterized as follows: we chose an axiom system \mathfrak{A} that on the one hand has a structure surveyable to such an extent that we can give a proof of consistency (in the sense of the second task); that, on the other hand, is so rich that we can derive the satisfiability of axiom systems for the branches of geometry and physics from the *presupposition* that \mathfrak{A} is satisfied by a system \mathfrak{S} of things and relations in such a way that we represent the objects of such an axiom system \mathfrak{B} by individuals or complexes of individuals from \mathfrak{S} and put as fundamental relations such predicates which can be formed from the fundamental relations of \mathfrak{S} using logical operations.

This suffices to show that the axiom system \mathfrak{B} is in fact consistent; for any contradiction arising from this axiom system as conclusion would represent a contradiction derivable from the axiom system \mathfrak{A} even though the axiom system \mathfrak{A} is known to be consistent.

Arithmetic (axiomatically constructed) presents itself as such an \mathfrak{A} .

The “method of reduction” of axiomatic theories to arithmetic does not depend upon arithmetic being a set of facts presentable to the intuition;

arithmetic need rather be no more than a formation of ideas which we can prove consistent and which provides a systematic framework encompassing the axiom systems of the theoretical sciences; because they are encompassed in this framework, the idealizations of what is actually given which, executed in them will also be proved consistent.—

We now summarize the results of our latest considerations: The problem of the satisfiability of an axiom system (or a logical formula) can be positively solved in the case of a finite domain of individuals by exhibition; but in the case where the satisfaction of the axioms requires an infinite domain of individuals this method is no longer applicable because it is not determined whether an infinite domain of individuals cannot be considered as settled; rather, the introduction of such infinite domains is only justified by a proof of the consistency of an axiom system characterizing the infinite.

Because of the failure of a positive decision method, there remains only one possibility: there is only the way of proving consistency in the negative sense, i.e., a *proof of impossibility*; such a proof requires a formalization of logical inference.—

If we are going to approach the task of giving such a proof of impossibility we must be clear that it cannot be carried out using axiomatic-existential methods of inference. Rather we may use only those kinds of inferences which are free from idealizing assumptions of existence.

As a result of these considerations the following thought comes at once to mind: If this proof of impossibility can be carried out without axiomatic-existential assumptions, shouldn't it also be possible to found all of arithmetic directly in the same way thereby making the proof of impossibility completely

superfluous? We will consider this question in the following paragraph.