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Comments:

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§ 1. The Problem of consistency in axiomatics as a logical decision problem.

The **state of research** in the field of foundations of mathematics, to which our presentation is related, is characterized by three kinds of investigations:

1. the development of the axiomatic method, especially **with the help of the** foundations of geometry,
2. the founding of analysis by today's rigorous methods through the reduction of **the theory of magnitudes** to the theory of numbers and sets of numbers,
3. investigations in the foundations of number theory and set theory.

A deeper set of **tasks is linked** to the **standpoint** reached through these investigations, arisen **on the base of** methods [~~delete: are~~] subjected to stricter demands; these problems involve a new kind of **dealing with the** problem of the infinite. We will introduce these problems by considering axiomatics.

The term ‘axiomatic’ is used **partly** in a wider, and **partly** a narrower sense. We call the development of a theory axiomatic in the widest sense of the word **if** the fundamental concepts and presuppositions **as such** are put **on top** and marked as such and the further content of the theory is logically derived from these with the help of definitions and proofs. In this sense the geometry of [~~sc here and throughout~~] EUCLID, the mechanics of NEWTON, and the thermodynamics of CLAUSIUS were axiomatically founded.

The axiomatic point of view was made more rigorous in HILBERT’s “Foundations of Geometry”. The greater rigor consists in the fact that in the axiomatic development of a theory one **keeps** of only that portion of the **objective material of intuition** from which the fundamental concepts of the theory are formed which is formulated **as extraction** in the axioms; **abstracting, however,** from all remaining content. Another factor in axiomatics in the narrowest sense **coming along** is the *existential form*. It serves to distinguish *the axiomatic method* from the *constructive* or *genetic* method of founding a theory.¹ **Whereas in** the constructive method |²the objects of a theory are introduced merely as a *family* [**one has to**

¹See for this comparison *Hilbert’s Grundlagen der Geometrie: Über den Zahlbegriff*, 1900.

obey the **Gattung–Art (genus–species) distinction in German**] of things (Brouwer and his school use the word “species” in this sense.); **[omit: but]** in an axiomatic theory one is concerned with a fixed system of things (or several such systems) which constitutes a previously *delimited domain of subjects* for all predicates from which the **statements** of the theory are **constituted**.

Except in the trivial cases in which a theory has to do just with a **finite**, fixed totality of things, the presupposition of such a totality, **of a** “domain of individuals”, an idealizing assumption **joining the assumptions** formulated in the axioms.

It is a characteristic of **this tightened** kind of axiomatics **resulting from** abstraction from material content and also the existential form — **we will call it “formal axiomatics” for short** — that it requires a *proof of consistency*; **whereas** contentual axiomatics introduces its fundamental concepts by reference to known experiences and its basic **principles** either as obvious facts which **one** can make clear to **oneself** or as extracts from **complexes of** experiences; **[footnote with alternative translation omitted]** **thereby** expressing the belief that **one** is on the track of laws of nature and at the same time intending to support this belief through the success of the theory.

[delete: However,] Formal axiomatics, **as well,** needs **in any case** certain **[delete: a ... amount of]** **evidences** in the performance of deductions as well as in the proof of consistency; **[delete: there is]** however, **with the** essential difference **that this kind of** evidence **[delete: required]** does not depend on any special epistemological relation to **special field**, but rather it

is one and the same for every axiomatization, **namely** that primitive kind of knowledge which is the precondition of every exact theoretical investigation whatsoever. We will consider this kind of evidence more closely.

The following **aspects** are especially important for a correct evaluation of the significance for epistemology of the relationship between contentual and formal axiomatics:

Formal axiomatics requires contentual axiomatics as a supplement, because only in terms of this supplement can one give instruction in the choice of formalisms and, **moreover**, in the case of a **given** formal theory, give an **instruction** of its applicability to some **domain** of reality.

On the other hand we cannot just stay at the level of contentual axiomatics, since in science we are, **if not always, so nevertheless predominantly** concerned with **such** theories |³ which get their significance from a *simplifying idealization* of an actual state of affairs rather than from a complete reproduction of it. A theory of this kind **cannot** get a foundation through a reference to either the evident truth of its axioms or to experience; rather such a foundation **can only be** given when the idealization **performed**, i.e. the extrapolation through which the **formations of** concepts and **the principles** of the theory come to overstep the **reach either** of intuitive evidence or of **data of** experience, is **understood** to be **consistent**. **Furthermore**, reference to the approximate **validity** of the **principles** is of no use for the recognition of consistency; for an inconsistency could arise just because a relationship which holds only in a restricted sense is taken to hold exactly.

We are therefore forced to investigate the consistency of theoretical systems without considering matters of fact and, **with this, we are already**

on the standpoint of formal axiomatics.

As to the treatment of this problem up until now, both in the case of geometry and of branches of physics, **this is done with the help of the *method of arithmetizing***: one represents the objects of a theory through numbers and systems of numbers and basic relations through **equations** and inequalities **in such a way that on the base of this** translation the axioms of the theory become either arithmetic identities or provable assertions (as in the case of geometry) or (as in physics) a system of conditions the simultaneous satisfiability of which can be proved on the basis of certain **arithmetic** existence assertions. **In this** procedure the **validity** of arithmetic, i.e. the theory of real numbers (analysis) is presupposed; so we **come to the question of what kind this validity is**.

However, before we concern ourselves with this question we want to see whether there isn't a direct way of attacking the problem of consistency. We want to get the structure of this problem clearly before our minds, **anyway**. At the same time we **already** want to **take the advantage** to familiarize ourselves **a bit** with *logical symbolism*, which proves to be very useful for **the given purpose** and which we will have to consider more deeply in the sequel.

As an example of axiomatics we take the *geometry of the plane*; **and** for the sake of simplicity we will consider only [**delete: the parallel axiom plus**] the axioms **of** the geometry of position (the axioms which are presented as "axioms of connection" and "axioms of order" in Hilbert's "Grundlagen der Geometrie") **together with the parallel axiom**. For our purpose it **suggests itself** to diverge from Hilbert's axiom system by not taking points

and lines $|^4$ as two basic systems of things but rather to *take only points as individuals*. Instead of the relation “points x and y determine the line g ” we use the relation between *three* points “ x, y, z [**“and” deleted**] lie on one line” **for which we use the designation** $Gr(x, y, z)$ [**Expression in brackets deleted and formula changed**]. **Betweenness comes as a second fundamental relation to this relation:** “ x lies between y and z ”, which we designate with $Zw(x, y, z)$ [**formula changed**].² Moreover, identity of x and y appears in the axioms as a notion belonging to logic, for which we use the usual equality sign $x = y$ [**quotation marks omitted**].

[From here translation of the introduction of symbolism added.
End will be indicated]

In addition we only need the logical signs for the symbolic presentation of the axioms, namely first the signs for generality and existence: if $P(x)$ is a predicate referring to the object x then $(x)P(x)$ means: “All x have the property $P(x)$.” (x) is named the “for-all-sign”, (Ex) the “there-is-sign”. For-all-sign and there-is-sign can refer to any other variable y, z, u in the same way it refers to x . The variable belonging to such a sign is “bound” by this sign, in the same way an integration variable is bound by the integration sign, so that the whole statement does not depend on the value of the variables.

Signs for negation and the junction of sentences are added as further

²The method of taking only points as individuals is particularly executed in the axiomatics of OSWALD VEULEN “A system of axioms for geometry”. Here are furthermore all geometrical relations defined in terms of the relation “between”.

logical signs. We designate the negation of a statement by overstriking. In the case of a preceding for-all-sign or there-is-sign the negation stroke is to be set only above this sign, and instead of $\overline{x=y}$ it should be written in a shorter way $x \neq y$. The sign $\&$ (“and”) between two statements means that both statements hold. The sign \vee (“or” in the sense of “vel”) between two statements means that at least one of the two statements holds (“disjunction”).

The sign \rightarrow between two statements means that the holding of the first entails the holding of the second, or with other words, that the first statement does not hold, without the second holding as well (“implication”). An implication $\mathfrak{A} \rightarrow \mathfrak{B}$ between two statements \mathfrak{A} and \mathfrak{B} is according to that only then wrong, if \mathfrak{A} is true and \mathfrak{B} is false. In all other cases it is true.

The junction of the sign of implication with the for-all-sign results in the presentation of general hypothetical statements. For example, the formula

$$(x)(y) (\mathfrak{A}(x, y) \rightarrow \mathfrak{B}(x, y)) ,$$

with $\mathfrak{A}(x, y)$, $\mathfrak{B}(x, y)$ standing for the presentation of certain relations between x and y represents the statement “If $\mathfrak{A}(x, y)$ holds, then $\mathfrak{B}(x, y)$ ”, or also: “for every pair of individuals x, y for which $\mathfrak{A}(x, y)$ holds, $\mathfrak{B}(x, y)$ holds as well.”³

We use brackets in the usual way for linking together parts of formulas. For saving brackets we stipulate that for the separation of symbolic expressions \rightarrow takes precedence over $\&$ and \vee , $\&$ over \vee , and that \rightarrow , $\&$, \vee all have precedence over the for-all-sign and the there-is-sign. Brackets are

³The relation between disjunction and implication defined here and disjunctive and hypothetical junctions of statements in the usual sense will be discussed in § 3.

omitted if no ambiguities are possible. We write, for example, instead of the expression

$$(x)((Ey)R(x, y)),$$

in which $R(x, y)$ designates an arbitrary relation between x and y , simply $(x)(Ey)R(x, y)$ because in this case only one way of reading is possible: “for every x there is a y for which the relation $R(x, y)$ exists.”—

We are non in the position to write down the axiom system considered. To make it easier the first axioms are accompanied by a linguistic version.

[End of the translation added.]

The **demarcation** of the axioms does not correspond completely to that in HILBERT’s “Grundlagen der Geometrie”. We therefore give **for** each group of axioms the relationship of the axioms here presented as formulas to those of HILBERT.⁴

I. Axioms of connection **[notation changed]**

1. $(x)(y)Gr(x, x, y).$

“ x, x, y always lie on one line.”

2. $(x)(y)(z)(Gr(x, y, z) \rightarrow Gr(y, x, z) \& Gr(x, z, y)).$

“If x, y, z lie on a line, then so do y, x, z as well as x, z, y lie on a line.”

3. $(x)(y)(z)(u)(Gr(x, y, z) \& Gr(x, y, u) \& x \neq y \rightarrow Gr(x, z, u)).$

“If x, y are different points and if x, y, z as well as x, y, u lie on a line then also x, z, u lie on a line.”

⁴**This information is especially meant** for those familiar with HILBERT’s “Grundlagen der Geometrie”. All references are to the seventh edition. **[Footnote created, type-setting changed (sc, style of quotation)].**

$$4. (Ex)(Ey)(Ez)\overline{Gr(x, y, z)}.$$

“There are points x, y, z which do not lie on a line.”

Of these axioms, 1) and 2) replace the axioms I 1,—because of the changed concept of line; 3) corresponds to the axiom I 2; and 4) corresponds to the second part of I 3 [**notation of references changed**].

II. Axioms of order [**Notation changed**]

$$1. (x)(y)(z)(Zw(x, y, z) \rightarrow Gr(x, y, z))$$

$$2. (x)(y)\overline{Zw(x, y, y)} .$$

$$3. (x)(y)(z)(Zw(x, y, z) \rightarrow Zw(x, z, y) \& \overline{Zw(y, x, z)}) .$$

$$4. (x)(y)(x \neq y \rightarrow (Ez)Zw(x, y, z)) .$$

“If x and y are different points, there is always a point z such that x lies between y and z .”

$$5. (x)(y)(z)(u)(v) \left(\overline{Gr(x, y, z)} \& Zw(u, x, y) \& \overline{Gr(v, x, y)} \& \overline{Gr(z, u, v)} \right. \\ \left. \rightarrow (Ew)\{Gr(u, v, w) \& (Zw(w, x, z) \vee Zw(w, y, z))\} \right) .$$

1) and 2) together constitute the first part of [**here and throughout: names in sc**] HILBERT’s axioms II 1; 3) unites the last part of HILBERT’s axioms II 1 with II 3; 4) is the axiom II 2; and 5) is the axiom of plane order II 4. [**Notation of references changed**]

III. Parallel axiom

Since we are not including congruence axioms, we must take the parallel axiom in the following broader sense: “For every straight line there is exactly one line through a point outside it which does not intersect it.”⁵

To make symbolic formulation easier the symbol [**here and throughout the section symbolism is changed**]

$$Par(x, y; u, v)$$

will be used as an abbreviation for the expression

$$\overline{(Ew)}(Gr(x, y, w) \ \& \ Gr(u, v, w))$$

“There is no point w which lies on a line both with x and y and with u and v .”

The axioms is then

$$(x)(y)(z) \left(\overline{Gr(x, y, z)} \rightarrow (Eu)\{Par(x, y; z, u) \ \& \ (v)(Par(x, y; z, v) \rightarrow Gr(z, u, v))\} \right) .$$

If we **imagine** the axioms here enumerated and unite them, we get a single logical formula which represents an assertion about the predicates ‘ Gr ’, ‘ Zw ’ and which we designate as

$$\mathfrak{A}(Gr, Zw) .$$

In the same way we could represent a theorem of plane geometry involving only position and order relations as a formula

$$\mathfrak{S}(Gr, Zw).$$

⁵Cf. p. 83 of HILBERT’s “Grundlagen der Geometrie”. [**Footnote created, notation changed.**]

This representation still accords with contentual axiomatics in which the fundamental relations are viewed as definite in content **which can be shown** in experience or in **intuitive imagination**, about which the statements of the theory make assertions.

On the other hand, in formal axiomatics the fundamental relations are not conceived as **from the beginning** determined in content; rather they receive their determination *implicitly* [**italics added**] through the axioms; and in any consideration of an axiomatic theory only what is expressly formulated in the axioms [**“is” omitted**] about the fundamental relations is used.

As a result, if **in axiomatic geometry the respective names for relations in intuitive geometry like “lie on” or “between” are used** this is only a concession to custom and a means of simplifying the connection of the theory with intuitive facts. In fact, however, in formal axiomatics the fundamental relations play the role of *variable* predicates.

Here and in the sequel we understand “predicate” in the wider sense so that it **also** applies to predicates with two or more subjects. We speak of “one-place”, “two-place”, . . . predicates **according to** the number of subjects.

In the part of axiomatic geometry considered by us there are two variable three-place predicates [**notation!**]:

$$R(x, y, z), \quad S(x, y, z) .$$

The axiom system consists of a **demand** on two such predicates expressed [**notations!**] in the logical formula $\mathfrak{A}(R, S)$, which we get from $\mathfrak{A}(Gr, Zw)$ when we replace $Gr(x, y, z)$ with $R(x, y, z)$, $Zw(x, y, z)$ with $S(x, y, z)$. The identity relation $x = y$ which is to be interpreted contentually appears in this

formula along with the variable predicates [**order of sentence changed**]. The acceptance of this predicate as contentually determinate is no violation of our methodological standpoint. For the contentual determination of identity—which is no relation at all in the true sense— [**dashes for brackets**] does not depend on the particular **range of imaginations** of the field being investigated axiomatically; rather it is only **related to** a question of distinguishing individuals which must be taken as already given when the domain of individuals is laid down.

From this point of view **a sentence** of the form $\mathfrak{S}(Gr, Zw)$ [**notation changed throughout!**] corresponds to the logical statement that for *any* predicates $R(x, y, z)$, $S(x, y, z)$ satisfying the **demand** $\mathfrak{A}(R, S)$ the relation $\mathfrak{S}(R, S)$ also holds; **in** other words, for any two predicates $R(x, y, z)$, $S(x, y, z)$ the formula

$$\mathfrak{A}(R, S) \rightarrow \mathfrak{S}(R, S)$$

represents a true **statement**. In this way a geometrical sentence is transformed into a sentence of pure predicate logic.⁸

From this point of view the problem of consistency presents itself in a corresponding way as a problem of **pure** predicate logic. In fact it is a question of whether two three-place predicates [**notation throughout!**] $R(x, y, z)$, $S(x, y, z)$ can satisfy the conditions expressed in the formula $\mathfrak{A}(R, S)$ ⁶ or whether, on the contrary the assumption that the formula $\mathfrak{A}(R, S)$ is satisfied for a certain pair of predicates leads to a contradiction so that in general for every pair of predicates R, S the formula $\overline{\mathfrak{A}(R, S)}$ represents a correct

⁶This imprecise way of putting the question will be sharpened in the sequel. [**Footnote placed and order changed**]

assertion.

A question like the one given here is part of the “*decision problem*”. In **newer** logic this problem is understood to be that of discovering general methods for deciding the “validity” or “satisfiability” of logical formulas.⁷

In this connection the formulas investigated are composed of predicate variables and equalities—together with variables in subject positions which we call “individual variables”—, **and is** assumed that every variable is bound by a **for-all sign** or **there-is sign**.

A formula of this kind is called **logically** valid when it represents a true assertion for *every* determination of the variable predicates; it is called satisfiable when it represents a true assertion for some *appropriate* [**changed to italics**] determination of the predicate variables.

[**The following translated has been added.**]

Simple examples for logically valid formulas are the following:

$$\begin{aligned} (x)F(x) \ \& \ (x)G(x) \ \rightarrow \ (x)(F(x) \ \& \ G(x)) \\ (x)P(x, x) \ \rightarrow \ (x)(Ey)P(x, y) \\ (x)(y)(z)(P(x, y) \ \& \ y = z \ \rightarrow \ P(x, z)). \end{aligned}$$

Examples for satisfiable formulas are:

$$\begin{aligned} (Ex)F(x) \ \& \ (Ex)\overline{F(x)} \\ (x)(y)(P(x, y) \ \& \ P(y, x) \ \rightarrow \ x = y) \\ (x)(Ey)P(x, y) \ \& \ (Ey)(x)\overline{P(x, y)} . \end{aligned}$$

These formulas result, e. g., in true assertions for the domain of individuals of the numbers 1, 2, if in the first formula is set for $F(x)$ “ x is even”, in the

⁷This explanation is correct only for the decision problem in its narrower sense. We have no need here to consider the broader conception of this decision problem.

second formula for $P(x, y)$ the predicate $x \leq y$, in third formula form $P(x, y)$ the predicate $x \leq \& y \neq 1$. [**End of new translation**]

It is to be **observed** that along with the determination of the predicates **the domain of individuals** over which the variables x, y, \dots range has to be fixed. This enters into a logical formula as a kind of *hidden variable*. However, the **logical formula in respect to** satisfiability is invariant with respect to a one-one mapping of a domain of individuals onto another, since the individuals enter into the formulas only as variable subjects; as a result the only essential determination for a domain of individuals is the *number of individuals*.

Accordingly, we have to distinguish the following questions in relation to **logical**] validity and satisfiability:

1. The question of **logical** validity for *every* domain of individuals, and also of satisfiability for *any* domain of individuals **respectively**.
2. The question of **logical** validity or satisfiability for a given number of individuals.
3. The question for which numbers of individuals is a formula **logically** valid or satisfiable.

It should be noted that it is best to leave out of consideration the domain of 0 individuals **on principle, since formally zero-numbered domains of individuals have a special status, and on the other hand consideration of them** is trivial and worthless for applications.⁸

⁸The stipulation that every domain of individuals should contain at least one thing so that a true **general judgement** must hold of at least one thing ought not to be confused

Furthermore one should **take into account** that only the “**value range**” of a predicate is relevant to its determination; that is to say, all that is relevant is for which values of the variables in subject positions the predicate holds or does not hold (is “true” or “false”).

This circumstance has as a consequence that for a *given finite* number of individuals the **logical** validity or satisfiability of a specific **given** logical formula represents a pure *combinatorial fact* which one can determine through elementary testing of all cases.

To be specific, if n is the number of individuals and k the number of subjects (“places”) of a predicate, then n^k is the number of different systems of values for the variables; and since for every one of these systems of values the predicate is either true or false, there are

$$2^{(n^k)}$$

different possible value-ranges for a k -place predicate.

If then

$$R_1, \dots, R_t$$

with the convention prominent in ARISTOTELEAN [**small caps**] logic that a judgment of the form “all S are P ” counts as true only if there are in fact things with the property S . This convention has been dropped in **newer** logic. A judgment of this kind is represented symbolically in the form $(x)(S(x) \rightarrow P(x))$ [**notation changed throughout!**]; it counts as true if a thing x , insofar as it has the property $S(x)$, always has the property $P(x)$ as well—independently of whether there is anything with the property Sx at all. **We will take up this topic again in connection with the deductive construction of predicate logic. (See § 4 pp. 106–107.)**

are the distinct predicate variables occurring in a given formula, and

$$k_1, \dots, k_t$$

the number of their places, then

$$2^{(n^{k_1} + n^{k_2} + \dots + n^{k_t})}$$

is the number of systems of value-**ranges** to be considered, **or** the number of different possible predicate systems for short.

Accordingly **logical** validity of the formula means that for all of these

$$2^{(n^{k_1} + n^{k_2} + \dots + n^{k_t})}$$

explicitly enumerable predicate systems the formula represents a true assertion; **and** its satisfiability means that the formula represents a true assertion for one of these predicate systems. Moreover, for a fixed predicate system the truth or falsity of the assertion represented by the formula is **again** decidable by a finite testing of cases; the reason is that only n values come into consideration for a variable bound by a **for-all sign** or **there-is sign** so that ‘all’ **has the same meaning** as a conjunction with n members and ‘there is’ a disjunction with n members.

For example, consider the formulas **mentioned above** [**notation changed throughout**]

$$(x)P(x, x) \rightarrow (x)(Ey)P(x, y)$$

$$(x)(y)(P(x, y) \& P(y, x) \rightarrow x = y)$$

of which the first is **has been referred to as logically valid, the second as satisfiable formula**. We refer these formulas to a domain of two individuals.

We can indicate **both** individuals with the numerals 1, 2. In this case we have $t = 1, n = 2, k_1 = 2$; therefore the number of different predicate systems is

$$2^{(2^2)} = 2^4 = 16.$$

In place of $(x)P(x, x)$ [**notation changed throughout!**] we can put $P(1, 1) \& P(2, 2)$, and in place of $(x)(Ey)P(x, y)$

$$P(1, 1) \vee P(1, 2) \& P(2, 1) \vee P(2, 2) ,$$

so that the first of the two formulas becomes

$$P(1, 1) \& P(2, 2) \rightarrow P(1, 1) \vee P(1, 2) \& P(2, 1) \vee P(2, 2) .$$

This **implication** is true for those predicates P for which $P(1, 1) \& P(2, 2)$ is false, as well as for those for which

$$P(1, 1) \vee P(1, 2) \& P(2, 1) \vee P(2, 2)$$

is true. One can now verify that for each of the 16 **value-ranges** that one gets when one assigns one of the truth values “true” or “false” to each of the pairs of values

$$(1, 1), (1, 2), (2, 1), (2, 2)$$

one of the two conditions is **satisfied**; thus the whole expression always receives the value “true” [Verification is simplified in this example because **already the** determination of the values of $P(1, 1)$ and $P(2, 2)$ [**notation!**] suffices to fix the correctness of the expression.] In this way the validity of our first formula for domains of two individuals can be determined through **directly trying out**.

For domains of two individuals the second formula signifies **has the same meaning** as the conjunction **[notation!]**

$$(P(1, 1) \ \& \ P(1, 1) \rightarrow 1 = 1) \quad \& \quad (P(2, 2) \ \& \ P(2, 2) \rightarrow 2 = 2) \\ \& \ (P(1, 2) \ \& \ P(2, 1) \rightarrow 1 = 2) \quad \& \quad (P(2, 1) \ \& \ P(1, 2) \rightarrow 2 = 1) .$$

Since $1 = 1$ and $2 = 2$ are true the first two members of the conjunction are always true assertions. The last two members are true **if and only if** **[notation!]**

$$P(1, 2) \ \& \ P(2, 1)$$

is false.

Therefore, to satisfy the formula under consideration one has only to eliminate those determinations of value for P in which the pairs $(1, 2)$ and $(2, 1)$ are both assigned the value “true”. Every other determination of value produces a true assertion. The formula is therefore satisfiable in a domain of two elements.

These examples should make clear the purely combinatorial character of the decision problem in the case of a given finite number of individuals. **One** result of this combinatorial character is that for a prescribed finite number of individuals the **logical** validity of a formula **[notation!]** \mathfrak{F} has the same **meaning** as the unsatisfiability of the formula $\overline{\mathfrak{F}}$; likewise the satisfiability of **[one “of” omitted]** a formula \mathfrak{F} **[B. uses in this section \mathfrak{F} for P throughout] has the same meaning** as that $\overline{\mathfrak{F}}$ is not valid. Indeed \mathfrak{F} represents a true assertion for those predicate systems for which $\overline{\mathfrak{F}}$ represents a false assertion and vice-versa.

Let us return to the question of the consistency of an axiom system. **Let us** consider an axiom system **written down** symbolically and combined into one formula like our example.

The question of the satisfiability of this formula for a prescribed finite number of individuals can be decided, in principle at least, through **trying out**. Suppose then the satisfiability of the formula is determined for a definite finite number of individuals. The result is a proof of the consistency of the axiom system; **namely** a proof by the *method of exhibition* since the finite domain of individuals together with the value-ranges chosen for the predicates (to satisfy the formula) constitutes a model in which we can show concretely that the axioms are satisfied.

We give an example of such an exhibition from **geometrical axiomatics**. We **start from** the axiom system **presented in the beginning** but replace the axiom I 4), which postulates **the existence of** three points not lying on a line, with the weaker axiom [**notation! real primes added in items**]

I 4') $(Ex)(Ey)(x \neq y)$.

“There are two distinct points.”

Furthermore we drop the axiom of plane order II 5); **in** its place we add to the axioms⁹ two sentences which can be proved using II 5) **by, firstly, expanding** II 4) to [**notation!**]

⁹Both of these sentences were introduced as axioms in earlier editions of HILBERT'S “Grundlagen der Geometrie”. It **turned** out that they are provable using the axioms of plane order. See pp. 5–6 of the seventh edition. [**Footnote added and replaced. Style of typesetting changed.**]

II 4') $(x)(y)\{x \neq y \rightarrow (Ez)Zw(z, x, y) \& (Ez)Zw(x, y, z)\}$,

and, **secondly**, add

II 5) $(x)(y)(z)\{x \neq y \& x \neq z \& y \neq z \rightarrow Zw(x, y, z) \vee Zw(y, z, x) \vee Zw(z, x, y)\}$

We keep the parallel axiom. The resulting axiom system corresponds to a formula $\mathfrak{A}'(R, S)$ instead of the earlier $\mathfrak{A}(R, S)$; it is satisfiable in a domain of **individuals 5 things**, as Veblen remarked.¹⁰ [**“First of all” replaced**] The value-ranges for the predicates R, S **are** so chosen that **first of all** the predicate [**notation!**] Gr is determined to be true for every value triple x, y, z —**we** can here use the symbols ‘ Gr ’, ‘ Zw ’ with no danger of misunderstanding. One sees immediately that **then** all axioms I as well as II 1) and III are satisfied. In order that the axioms II 2), 3), 5'), and 4') be satisfied it is necessary and **also** sufficient that the following three conditions be placed on the predicate Zw :

1. Zw is always false for a triple x, y, z in which two elements coincide.
2. **For any combination of three different of the 5 individuals, Zw is true for 2 orderings with a common first element (of 6 possible orderings of the elements), false for the remaining 4 orderings.**
3. [**dot deleted**] Each pair of **different** elements occurs as an initial as well as a final pair in one of the triples for which Zw is true.

¹⁰In the investigation already mentioned ‘...’, Trans. Amer. Math. Soc. vol. 5, p. 350.

The first **demand** can be directly fulfilled by stipulation. The **joint** satisfaction of the other two conditions is accomplished as follows: We designate the 5 elements with the numerals 1, 2, 3, 4, 5. The number of value-triples of three distinct elements for which Zw still has to be defined is [**formula changed**] $5 \cdot 4 \cdot 3 = 60$ **Every six** [?; “**Je sechs**”] of these belong to a [“**given**” **deleted**] combination; for two of these Zw should be true and false for the rest. We must therefore indicate those 20 of the 60 triples for which Zw **will be** defined as true. They are those which one obtains from the four triples [**notation!**]

$$(1\ 2\ 5), (1\ 5\ 2), (1\ 3\ 4), (1\ 4\ 3)$$

by applying the cyclical permutation (1 2 3 4 5).

It is easy to verify that this procedure satisfies all the conditions. Thus the axiom system is recognized as consistent by the method of exhibition.¹¹

The method of exhibition presented in this example has very many different applications in **newer** axiomatic investigations. It is especially **used** for *proofs of independence*. The **assertion** that a sentence \mathfrak{S} is independent of an axiom system \mathfrak{A} **has the same meaning as the assertion of the consistency of the axiom system** as the claim that the axiom system [**notation!**]

$$\mathfrak{A} \ \& \ \overline{\mathfrak{S}}$$

which we get **when we add the negation of the sentence \mathfrak{S} as an axiom to \mathfrak{A} . The consistency can be determined by the method of**

¹¹It follows immediately from the fact that the modified axiom system \mathfrak{A}' is satisfiable in a domain of 5 individuals that the axioms of this system do not completely determine linear ordering. **Footnote added.**

exhibition if this axiom system is satisfiable in a finite domain **domain of individuals**.¹² Thus this method provides a sufficient extension of the method of **progressive inferences** for many fundamental investigations in the sense that the unprovability of a sentence from certain axioms can be proved through exhibition, its provability through **inference**.

But is the application of the method of exhibition **is restricted in its application** to finite domains of individuals? This **cannot be derived** from what we have said up until now. However, we do see **immediately** that in the case of **an infinite domain** of individuals the possible systems of predicates no longer constitute a surveyable multitude and there can be no talk of testing all value-ranges . Nevertheless in the case of given axioms we might be in a position to show their satisfiability by given predicates. And this is actually the case. Consider for example the system of three axioms **[notation! typesetting changed]**

$$\begin{aligned} & (x)\overline{R(x, x)} , \\ & (x)(y)(z)(R(x, y) \ \& \ R(y, z) \rightarrow R(x, z)) \\ & (x)(Ey)R(x, y). \end{aligned}$$

Let us clarify what these say: We start with an object a in the domain of individuals. According to the third axiom there must be a thing b for which $R(a, b)$ **[notation throughout]**is true; and because of the first axiom b must be different from a . For b there must further be a thing c for which $R(b, c)$

¹²A **great** number of examples of this procedure can be found in the works on linear and cyclical order by E. V. Huntington and his **collaborators**. See especially “A new set of postulates for betweenness with proof of complete independence”, **Trans. Amer. Math. Soc. vol. 26 (1924) pp. 257–282**. Here one also finds references to previous works.

is true, and because of the second axiom Rac is also true; according to the third axiom c is distinct from a and b . For c there must again be a thing d for which $R(c, d)$ is true. For this thing $R(a, d)$ and Rbd are also true, and d is distinct from a, b, c . **The method of this consideration** here has no end; and it shows us we cannot satisfy the axioms **with** a finite domain of individuals. On the other hand we can easily show satisfaction **by** an infinite domain of individuals: We take the integers as individuals and substitute the relation “ x is less than y ” for $R(x, y)$ [**notation!**]; one sees immediately that all three axioms are satisfied.

It is the same with the axioms [**notation! typesetting changed**]

$$\begin{aligned} & (Ex)(y)\overline{S(y, x)} , \\ & (x)(y)(u)(v)(S(x, u) \ \& \ S(y, u) \ \& \ S(v, x) \rightarrow S(v, y)) , \\ & (x)(Ey)S(x, y) . \end{aligned}$$

One can easily ascertain that these cannot be satisfied **with** a finite domain of individuals. On the other hand they are satisfied in the domain of positive integers if we replace $S(x, y)$ [**notation!**] with the relation “ y immediately follows x ”.

However, we notice in these examples that exhibiting in these cases does **by no means** conclusively settle the question of consistency; rather the question is *reduced* to that of the *consistency of number theory*. In the earlier **example** of finite exhibition we took integers as individuals. There, however, this was only for the purpose of having a simple way to designate individuals. Instead of numbers we could have taken other things, letters for example. And also the properties of numbers which were used could have been established by a concrete **exhibition**.

In the **case** now before us, however, **a concrete idea of number is not enough; for we need essentially the assumption** that the *integers* constitute *a domain of individuals* and therefore a **finished** totality.

We are, of course, quite familiar with this assumption since in **newer** mathematics we are constantly working with it; one is inclined to consider it **perfectly natural**. It was FREGE [sc!] who with a sharp and **witty** critique first established **insistently** that the **idea** of the sequence of integers as a complete totality must be justified by a proof of consistency.¹³ According to FREGE [sc] such a proof, **had to be carried out** in the sense of an exhibition, **as existence proof**; and he believed **to find** the objects for such an exhibition in the domain of logic. His **method of exhibition** amounts to defining the totality of integers with the help of the totality (presupposed to exist) of all conceivable one-place predicates. However, **the underlying** assumption, which under impartial consideration seems very suspect anyway, was shown to be untenable by the **famous** logical and set-theoretic paradoxes discovered by RUSSELL and ZERMELO. And the failure of FREGE's [sc] undertaking has made us even more conscious of the problematic character of assuming the totality of the sequence of integers than did his **dialectic**.

[NB: here and at other places. Is “Zahlenreihe” really “sequence of integers”? I would write more generally “number sequence”.]

In the light of this difficulty we might try to use some other infinite domain of individuals instead of the sequence of integers for the purpose of proving consistency; a domain taken from the realm of sense perception or

¹³GOTTLOB FREGE, “Grundlagen der Arithmetik”, Breslau 1884, and “Grundgesetze der Arithmetik”, Jena 1893.

physical reality rather than being a pure product of thought like the sequence of integers. However, if we look more closely we will realize that wherever we think we encounter infinite manifolds in the realm of sensible qualities or in physical reality there can be no talk of the actual presence of such a manifold; rather the conviction that such a manifold is present rests on a mental extrapolation, the justification of which is as much in need of investigation as the conception of **of the totality of** the sequence of integers.

A typical example in this connection are those cases of the infinite which gave rise to the well-known **paradox** of ZENO. Suppose some distance is traversed in a finite time; the traversal includes infinitely many successive subprocesses: the traversal of the first half, then of the next quarter, then the next eighth, and so on. If we are considering an actual motion, then these subtraversals must be **real** processes succeeding one another.

People have tried to refute **this paradox** with the argument that the sum of infinitely many **time** intervals may converge producing a finite duration. However, this reply does not come to grips with an essential point of the paradox, namely the **paradoxical aspect** that lies in the fact that an infinite succession, the completion of which we could not accomplish in the imagination either actually or in principle, should be accomplished in reality.

Actually there is much more radical solution of the paradox. It consists in **considering that that we are by no means forced to believe** that the mathematical space-time representation of movement remains physically meaningful for arbitrarily small segments of space and time; rather there is every reason to assume that a mathematical model extrapolates the facts of a certain domain of experience, e.g. **just the movements, within the**

range of magnitudes accessible to our observation up to now for the purpose of a **simple** conceptual structure; **this is similar to continuum mechanics which carries out an extrapolation in taking as a basis the idea of space** as filled with matter ; it is no more the case that unbounded division of a movement always produces something characterizable as movement than that unbounded spatial division of water always produces **quantities of water**. When this is accepted the **paradox vanishes**. [delete “/conceptual”]

Notwithstanding, the mathematical model of movement has, as a idealizing concept formation, its value for the purpose of simplified representation. For this purpose it must not only coincide approximately with reality but **it has to meet the condtion that** also the extrapolation it involves must be consistent **in itself**. From this point of **view** the mathematical conception of movement is not in the least shaken by ZENO's [sc] **paradox**; the mathematical counterargument just referred to has in this case complete **validity**. It is another question however, whether we possess a real proof of the consistency of the mathematical theory of motion. This theory depends essentially on the mathematical theory of the continuum; this in turn depend essentially on the **idea** of the set of all integers as a **finished** totality. We therefore come back by a roundabout way [full stop deleted] to the problem we tried to avoid by referring to the facts about motion.

It is much the same in every case in which a person thinks he can show directly that some infinity is given in experience or intuition for example the infinity of the tone row extending from octave to octave to infinity, or the continuous infinite manifold involved in the passage from one color **quality**

to another. **Closer consideration shows in every case that in fact no infinity is given at all**; rather it is interpolated or extrapolated through some mental process.

These considerations make us realize that reference to non-mathematical objects can not settle the question whether an infinite manifold exists; the question must be solved within mathematics itself. But how should one **make a start with** such a solution? At first glance it seems that something impossible is being demanded here: to **present** infinitely many individuals is impossible in principle; therefore an infinite domain of individuals as such can only be indicated through its structure, **i. e.**, through relations holding among its elements. In other words: a proof must be given that for this domain certain formal relations can be satisfied. The existence of an infinite domain of individuals can *not be represented in any other way than through the satisfiability of certain logical formulas*; but these are exactly the kind of formulas we were led **led to through investigating the question** about the existence of an infinite domain of individuals; and the satisfiability of these formulas **was** to have been demonstrated by the exhibition of an infinite domain of individuals. The attempt to apply the method of exhibition to the formulas under consideration leads then to a vicious circle.

But exhibition should serve only as a means in proofs of the consistency of axiom systems. **We were led to this procedure through considering** domains with a given finite number of individuals, and **just through recognizing** that in such domains the consistency of a formula has the same significance as its satisfiability.

The situation is more complicated in the case of infinite domains of in-

dividuals. It is true in this case also that an axiom system represented by a formula \mathfrak{A} is inconsistent if and only if the formula $\bar{\mathfrak{A}}$ [notation!] is valid. But since we are no longer dealing with a surveyable supply of value-ranges for the variable predicates, we can no longer conclude that if $\bar{\mathfrak{A}}$ [notation!] is not **logically** valid there is some model for satisfying the axiom system \mathfrak{A} at our disposal.

Accordingly, when **an infinite domain of individuals is under consideration**, the satisfiability of an axiom system is a **sufficient condition for its consistency, but it is not proved to be a necessary condition**. We cannot therefore expect that in general a proof of consistency can be accomplished by means of a proof of satisfiability. On the other hand we are not forced to prove consistency by establishing satisfiability; we can just hold to the original negative sense of inconsistency. That is to say—if we again imagine an axiom system [**“ . again” deleted**] represented by a formula \mathfrak{A} —we do not have to show that satisfiability of the formula \mathfrak{A} , but only need to prove that the assumption that \mathfrak{A} [**“ ,” deleted**] is satisfied by certain predicates cannot lead to a logical **contradiction**.

To attack the problem in these terms we must first **aim at** an overview of the possible logical inferences that can be made from an axiom system. The *formalization of logical inference* as developed by FREGE, SCHRÖDER, PEANO, and RUSSELL [**names in sc**] presents itself as an appropriate means to this end.

We have **thus** arrived at the following tasks: 1. to formalize rigorously the principles of logical inference and **by this** turn them into a completely surveyable system of rules; 2. to show for a given axiom system \mathfrak{A} , (which is

to be proved consistent) that starting with this *system* \mathfrak{A} *no contradiction can arise via logical deductions*, that is to say, no two formulas of which one is the negation of the other can be proved.

However, we do not have to carry out this proof for each axiom system individually; for we can make use of the method of *arithmetizing* to which we referred at the beginning. From the point of view we have reached now this procedure can be characterized as follows: we **chose** an axiom system \mathfrak{A} which on the one hand **has a structure surveyable to such an extent** that we can give a proof of consistency (in the sense of the second task); **which is, on the other hand, so rich that we can** derive the satisfiability of axiom systems for **the branches of geometry and physics** from the *presupposition* that \mathfrak{A} is satisfied by a system \mathfrak{S} of things and relations; **in such a way that we represent the objects of such an axiom system \mathfrak{B} by individuals or complexes of individuals from \mathfrak{S} and put as fundamental relations such predicates which can be formed from the fundamental relations of \mathfrak{S} using logical operations.**

This suffices to show that the axiom system \mathfrak{B} is in fact consistent; for any **contradiction arising from this axiom system as conclusion** [“-” **deleted**] would represent a **contradiction** derivable from the axiom system \mathfrak{A} even though the axiom system \mathfrak{A} is known to be consistent.

Arithmetic (axiomatically constructed) presents itself as such an \mathfrak{A} .

The “**method of reduction**” of axiomatic theories to arithmetic does not depend upon arithmetic being a set of facts presentable to the intuition; arithmetic need **rather** be no more than a **formation of ideas** which we can prove consistent and which provides a systematic framework encompassing

the axiom systems of the theoretical sciences; because they are encompassed in this framework the idealizations of what is actually given which **executed in them** will also be proved consistent.

We now summarize the results of our latest considerations: The problem of the satisfiability of an axiom system (or a logical formula) can be positively solved in the case of a finite domain of individuals by exhibition; but in the case where the satisfaction of the axioms requires an infinite domain of individuals this method is no longer applicable because it is not determined whether an infinite domain of individuals **cannot be considered as settled; rather, the introduction of such infinite domains is only justified by a** a proof of the consistency of an axiom system characterizing the infinite.

Because of the **failure** of a positive **decision method**, there remains only one possibility: **there is only the way of proving** consistency in the negative sense, i.e. a *proof of impossibility*; such a proof **requiring** a formalization of logical inference.

If we are going to approach the task of giving such a proof of impossibility we must be clear that it cannot be carried out using axiomatic-existential methods [**full stop deleted**] of inference. Rather we **may** use only those kinds of inferences which are free from idealizing assumptions of existence.

As a result of these considerations the following thought comes **at once** to mind: If this proof of impossibility can be carried out without axiomatic-existential assumptions, shouldn't it also be possible to found **all of** arithmetic directly in the same way **thereby making** the proof of impossibility completely superfluous? We will consider this question in the following **para-**

graph.