Bernays Project: Text No. 14

Hilbert's investigations of the foundations of arithmetic (1935)

Paul Bernays

(Hilberts Untersuchungen über die Grundlagen der Arithmetik, 1935.)

Translation by: *Dirk Schlimm. Revised by Bill Ewald (4/2/03), and by SA, WS (4/10/03). Bill Ewald (5/8/03).*

Comments:

"Variablengattung" is translated as "type of variables" or "variable type;" "Art von Variablen" as "sort of variables." "Inhaltlich" is rendered as "contentually," "methodisch" as "methodical."

The long quotations from Hilbert are in a `quote` *environment, without quotation marks. The German text (and page numbering) is from Hilbert's* Gesammelte Abhandlungen, *vol. 3, p. 196–216. Many* references *in the footnotes!* Citing *of titles as well as* proper names *have to be unified.*

||[196] Hilbert's first investigations of the foundations of arithmetic follow temporally as well as conceptually his investigations of the foundations of geometry. Hilbert begins the paper "On the concept of number"[1] by applying

[1] Jber. dtsch. Math.-Ver. Bd. 8 (1900); reprinted in Hilbert's "Foundations of Geometry," 3rd-7th ed., as appendix VI.

to arithmetic, just as to geometry, the axiomatic method, which he contrasts to the otherwise usually applied “genetic” method.

> Let us first recall the manner of introducing the concept of number. Starting from the concept of the number 1, usually one thinks at first the further rational positive numbers 2, 3, 4,... as arising through the process of counting, and their laws of calculation as being developed in the same way; then one arrives at the negative number by the requirement of the general execution of subtraction; one further defines the rational number say as a pair of numbers—then every linear function has a zero—, and finally the real number as a cut or a fundamental sequence—thereby obtaining that every whole rational indefinite, and generally every continuous indefinite function has a zero. We can call this method of introducing the concept of number the *genetic method*, because the most general concept of real number is *generated* by successive expansion of the simple concept of number.
>
> One proceeds fundamentally differently with the development of geometry. Here one tends to begin with the assumption of the existence of all elements, i.e., one presupposes at the outset three systems of things, namely the points, the lines, and the planes, and then brings these elements—essentially after the example of Euclid—into relation with each other by certain axioms, namely the axioms of incidence, of ordering, of congruency, and of continuity. Then the necessary task arises of showing the *consistency* and *completeness* of these axioms, i.e., it must be proven that

the application of the axioms that have been laid down can never lead ||[197] to contradictions, and moreover that the system of axioms suffices to prove all geometric theorems. We shall call the procedure of investigation sketched here the *axiomatic method.*

We raise the question, whether the genetic method is really the only one appropriate for the study of the concept of number and the axiomatic method for the foundations of geometry. It also appears to be of interest to contrast both methods and to investigate which method is the most advantageous if one is concerned with the logical investigation of the foundations of mechanics or other physical disciplines.

My opinion is this: *Despite the great pedagogical and heuristic value of the genetic method, the axiomatic method nevertheless deserves priority for the final representation and complete logical securing of the content of our knowledge.*

Already Peano developed number theory axiomatically.[2] Hilbert now

[2]Peano, G. "Arithmetices principia nova methodo exposita." (Torino 1889.) The introduction of recursive definitions is here not unobjectionable; the proof of the solvability of the recursion equations is missing. Such a proof was provided already by Dedekind in his essay "Was sind und was sollen die Zahlen" (Braunschweig 1887). If one bases the introduction of recursive functions on Peano's axioms, it is best to proceed by first proving the solvability of the recursion equations for the sum following L. Kalmár by induction on the parameter argument, then defining the concept "less than" with the help of the sum, and finally using Dedekind's consideration for the general recursive definition. This procedure is presented in Landau's textbook "Grundlagen der Analysis" (Leipzig 1930). Admittedly here the concept of function is used. If one wants to avoid it, the recursion

sets up an axiom system for analysis, by which the system of real number is characterized as a real Archimedean field which cannot be extended to a more extensive field of the same kind.

A few illustrative remarks about dependencies follow the enumeration of the axioms. In particular it is mentioned that the law of commutativity of multiplication can be deduced from the remaining properties of a field and the order properties with the help of the Archimedean axiom, but not without it.

The requirement of non-extendibility is formulated by the "axiom of completeness." This axiom has the advantage of conciseness; however, its logical structure is complicated. In addition it is not immediately apparent from it that it expresses a demand of continuity. If ||[198] one wants, instead of this axiom, one that clearly has the character of a demand of continuity and on the other hand does not already include the requirement of the Archimedean axiom, it is recommended to take Cantor's axiom of continuity, which says that if there is a series of intervals such that every interval includes the following one, then there is a point which belongs to every interval. (The formulation of this axiom requires the previous introduction of the concept of number series).[3]

equations of the sum and product have to be introduced as axioms. The proof of the general solvability of recursion equations follows then by a method by K. Gödel (cf. "Über formal unentscheidbare Sätze ..." [Mh. Math. Physik, Bd. 38 Heft 1 (1931)], and also Hilbert-Bernays Grundlagen der Mathematik Bd. 1 (Berlin 1934) p. 412 ff.)

[3]Concerning the independence of the Archimedean axiom from the mentioned axiom of Cantor, cf. P. Hertz: "Sur les axiomes d'Archimède et de Cantor." C. r. soc. de phys. et d'hist. natur. de Genève Bd. 51 Nr. 2 (1934).

R. Baldus has recently called attention to Cantor's axiom. See his essay "Zur Ax-

The aim which Hilbert pursues with the axiomatic version of analysis appears particularly clearly at the end of the essay in the following words:

> The objections that have been raised against the existence of the totality of all real numbers and infinite sets in general lose all their legitimacy with the view identified above: we do not have to conceive of the set of real numbers as, say, the totality of all possible laws according to which the elements of a fundamental sequence can proceed, but rather—as has just been explained—as a system of things whose relations between each other are given by the *finite and completed* system of axioms I–IV, and about which new propositions are valid only if they can be deduced from those axioms in a finite number of logical inferences.

But the methodical benefit which this view brings also involves a further requirement: for the axiomatic formulation necessarily entails the task of proving the consistency of the axiom system in question.

Therefore, the problem of the proof of consistency for the arithmetical axioms was mentioned in the list of problems that Hilbert posed in his lecture in Paris "Mathematische Probleme."[4]

iomatik der Geometrie": "I. Über Hilberts Vollständigkeitsaxiom." Math. Ann. Bd. 100 (1928), "II. Vereinfachungen des Archimedischen und des Cantorschen Axioms," Atti Congr. Int. Math. Bologna Bd. 4 (1928), "III. Über das Archimedische und das Cantorsche Axiom." S.-B. Heidelberg. Akad. Wiss. Math.-nat. Kl. 1930 Heft 5, as well as the following essay by A. Schmidt: "Die Stetigkeit in der absoluten Geometrie." S.-B. Heidelberg. Akad. Wiss. Math.-nat. Kl. 1931 Heft 5.

[4]Held at the International Congress of Mathematicians 1900 in Paris, published in Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl. 1900, cf. also this volume essay no. 17.

To accomplish the proof Hilbert thought to get by with a suitable ||199 modification of the methods used in the theory of real numbers.

But in the more detailed engagement with the problem he was immediately confronted with the considerable difficulties that exist for this task. In addition, the set theoretic paradox that was discovered in the meantime by Russell and Zermelo prompted increased caution in the inference rules. Frege and Dedekind were forced to withdraw their investigations in which they thought they had provided unobjectionable foundations of number theory—Dedekind using the general concepts of set theory, Frege the framework of pure logic[5]—since it resulted from that paradox that their considerations contained inadmissible inferences.

The talk[6] "Über die Grundlagen der Logik und der Arithmetik" held in 1904 shows us a completely novel point of view. Here first the fundamental difference is pointed out between the problem of the consistency proof for arithmetic and for geometry. The proof of consistency for the axioms of geometry uses an arithmetical interpretation of the geometric axiom system. However, for the proof of consistency of arithmetic "it seems that the appeal to another foundational discipline is not allowed."

To be sure, one could think of a reduction to logic.

> But by attentive inspection we become aware that certain arithmetical basic concepts are already used in the traditional formulation of the laws of logic, e,g., the concept of set, in part also

[5]R. Dedekind: "Was sind und was sollen die Zahlen?" Braunschweig 1887. G. Frege: "Grundgesetze der Arithmetik" (Jena 1893).

[6]At the International Congress of Mathematicians in Heidelberg 1904, printed in "Grundlagen der Geometrie," 3-7 ed., as appendix VII.

> the concept of number, in particular cardinal number. So we get into a quandary, and to avoid paradoxes a partly simultaneous development of the laws of logic and arithmetic is required.

Hilbert now presents the plan of such a joint development of logic and arithmetic. This plan contains already in great part the leading viewpoints for proof theory, in particular the idea of transforming the proof of consistency into a problem of elementary-arithmetic character by translating the mathematical proofs into the formula language of symbolic logic. Also rudiments of the consistency proofs can be already found here.

But the execution remains still in its beginnings. ||[200] In particular, the proof for the "existence of the infinite" is carried out only in the framework of a very restricted formalism.

The methodical standpoint of Hilbert's proof theory is also not yet developed to its full clarity in the Heidelberg talk. Some passages suggest that Hilbert wants to avoid the intuitive idea of number and replace it with the axiomatic introduction of the concept of number. Such a procedure would lead to a circle in the proof theoretic considerations. Also the viewpoint of the restriction in the contentual application of the forms of the existential and general judgment is not yet brought to bear explicitly and completely.

In this preliminary state Hilbert interrupted his investigations of the foundations of arithmetic for a long period of time.[7] Their resumption is found

[7] A continuation of the direction of research that was inspired by Hilbert's Heidelberg talk was carried out by J. König, who, in his book "Neue Grundlagen der Logik, Arithmetik und Mengenlehre" (Leipzig 1914), surpasses the Heidelberg talk both by a more exact formulation and a more thorough presentation of the methodical standpoint, as well as

announced in the 1917 talk[8] "Axiomatisches Denken."

This talk comes in the wake of the manifold successful axiomatic investigations that had been pursued by Hilbert himself and other researchers in the various fields of mathematics and physics. In particular in the field of the foundations of mathematics the axiomatic method had led in two ways to an extensive systematization of arithmetic and set theory. Zermelo formulated in 1907 his axiom system for set theory[9] by which the processes of set forma-

by the execution. Julius König died before finishing the book; it was edited by his son as a fragment. This work, which is a precursor of Hilbert's later proof theory, exerted no influence on Hilbert. But later J. v. Neumann followed the approach of König in his investigation "Zur Hilbertschen Beweistheorie" [Math. Z. Bd. 26 Heft 1 (1927)]

[8]At Naturvorscherversammlung Zürich, published in Math. Ann. Bd. 78 Heft 3/4; see also this volume essay no. 9.

[9]Zermelo E. "Untersuchungen über die Grundlagen der Mengenlehre I.: Math. Ann. Bd. 65. More recently there have been various investigations building on this axiom system. A. Fraenkel added the axiom of replacement, an extension of the admissible formation of sets in the spirit of Cantor's set theory; J. v. Neumann added an axiom, which rules out that the process of going from a set to one of its elements can, for any given set, be iterated arbitrarily many times. Moreover, Th. Skolem, Fraenkel, and J. v. Neumann have made more precise, all in a different way, in the sense of a sharper implicit characterization of the concept of set, the concept of "definite proposition" which was used by Zermelo in vague generality. The result of these refinements is presented in the most concise way in v. Neumann's axiomatic; namely it is achieved here, that all axioms are of the "first order" (in the sense of the terminology of symbolic logic). Zermelo rejects such a refinement of the concept of set, in particular in the light of the consequence that was first discovered by Skolem that such a sharper axiom system of set theory can be realized in the domain of individuals of the whole numbers.—A presentation of these investigations up to the year 1928, with detailed references, is contained in the textbook by A. Fraenkel: "Einleitung in die Mengenlehre," third edition (Berlin 1928). See also:

tion ||[201] are delimited in such a way that on the one hand the set theoretic paradoxes are avoided and on the other hand the set theoretic inferences that are customary in mathematics are retained. And Frege's project of a logical foundation of arithmetic—for which to be sure the method that Frege employed himself turned out to be faulty—was reconstructed by Russell and Whitehead in their work "Principia Mathematica."[10]

Hilbert says about this axiomatization of logic that one could "see the crowning of the work of axiomatization in general" in the completion of this enterprise. But this praise and acknowledgment is immediately followed by the remark that the completion of the project "still needs new work on many fronts."

In fact, the viewpoint of Principia Mathematica contains an unsolved problematic. What is supplied by this work is the elaboration of a clear system of assumptions for a simultaneous deductive development of logic and mathematics, as well as the proof that this set-up in fact succeeds. For the reliability of the assumptions, besides their contentual plausibility (which also from the point of view of Russell and Whitehead does not yield a guarantee of consistency), only their testing in the deductive use is put forward. But this testing too provides us in regard to consistency only an empirical confidence, not complete certainty. The complete ||[202] certainty of consistency, however,

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J. v. Neumann "Über eine Widerspruchsfreiheitsfrage in der axiomatischen Mengenlehre" J. reine angew. Math. Bd. 160 (1929), Th. Skolem: "Über einige Grundlagenfragen der Mathematik" Skr. norske Vid.-Akad., Oslo. I. Mat. Nat. Kl. 1929 Nr. 4, E. Zermelo: "Über Grenzzahlen und Mengenbereiche." Fund. math. Bd. XVI, 1930.

[10] Russell, B., and Whitehead, A. N.: Principia Mathematica. Cambridge, vol. I 1910, vol. II 1912, vol. III 1913.

is regarded by Hilbert as a requirement of mathematical rigor.

Thus the task of providing a consistency proof remains also for those assumptions, according to Hilbert. To handle this task as well as various further fundamental questions, e.g., "the problem of the solvability in principle of every mathematical question" or "the question of the relation between content and formalism in mathematics and logic," Hilbert thinks it necessary to make "the concept of specifically mathematical proof itself the object of investigation."

In the following years, in particular since 1920, Hilbert devoted himself especially to the plan, hereby taken up anew, of a proof theory.[11] His drive in this direction was strengthened by the opposition which Weyl and Brouwer directed at the usual procedure in analysis and set theory.[12]

Thus Hilbert begins his first communication about his "Neubegründung der Mathematik"[13] by discussing the objections of Weyl and Brouwer. It is noteworthy in this dispute that Hilbert, despite his energetic rejection of the

[11]To collaborate on this enterprise Hilbert then invited P. Bernays with whom he has regularly discussed his investigations since then.

[12]H. Weyl: "Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis" (Leipzig 1918).—"Der circulus vitiosus in der heutigen Begründung der Analysis" Jber. dtsch. Math.-Ver. Bd. 28 (1919).—"Über die neue Grundlagenkrise der Mathematik" Math. Z. Bd. 10 (1921). L. E. J. Brouwer: "Intuitionisme en formalisme." Inaugural address. Groningen 1912.—"Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten." I and II. Verh. d. Kgl. Akad. d. Wiss. Amsterdam, 1. Sekt., part XII no. 5 and 7 (1918/19).—"Intuitionistische Mengenlehre." Jber. dtsch. Math.-Ver. Bd. 28 (1919).—"Besitzt jede reelle Zahl eine Dezimalbruchentwicklung?" Math. Ann. Bd. 83 (1921).

[13]Talk, given in Hamburg 1922, published in Abh. math. Semin. Hamburg. Univ. Bd. 1 Heft 2, see also this volume essay no. 10.

objections that have been raised against analysis, and despite his advocacy for the legitimacy of the usual inferences, agrees with the opposing standpoint that the usual treatment of analysis is not immediately evident and does not conform to the requirements of mathematical rigor. The "legitimacy" that Hilbert, from this point of view, grants to the usual procedure is not based on evidence, but on the reliability of the axiomatic method, of which Hilbert explains that if it is appropriate anywhere at all, then it is here. This is a conception from which the problem of a proof of consistency for the assumptions of analysis arises.

||[203] Moreover, as for the methodical attitude on which Hilbert bases his proof theory and which he explains using the intuitive treatment of number theory, there is a great drawing near to the standpoint of Kronecker[14]—despite the position Hilbert took against Kronecker. This consists in particular in the application of the intuitive concept of number, and also in the fact that the intuitive form of complete induction (i.e., the inference which is based on the intuitive idea of the "setup" of the numerals) is regarded as acceptable and as not requiring any further reduction. By deciding to adopt this methodical assumption Hilbert also got rid of the basis of the objection that Poincaré had raised at that time against Hilbert's enterprise of the foundation of arithmetic based on the exposition in the talk in Heidelberg.[15]

[14]In a later talk "Die Grundlegung der elementaren Zahlenlehre" (held in Hamburg 1931. Math. Ann. Bd. 104 Heft 4, an excerpt of it in this volume no. 12) Hilbert has spoken more clearly about this. After mentioning Dedekind's investigation "Was sind und was sollen die Zahlen?" he explains: "Around the same time, thus already more than a generation ago, Kronecker clearly articulated a view which today in essence coincides with our finite attitude, and illustrated it with many examples.

[15]H. Poincaré: "Les mathématiques et la logique." Rev. de métaph. et de morale,

The beginning of proof theory, as it is laid down in the first communication, already contains the detailed formulation of the formalism. In contrast to the Heidelberg talk, the sharp separation of the logical-mathematical formalism and the contentful "metamathematical" consideration is prominent, and is expressed in particular by the distinction of signs "for communication" and symbols and variables of the formalism.

But the formal restriction of negation to inequalities appears as a remnant of the stage when this separation had not yet been performed, while a restriction is really only needed in the metamathematical application of negation.

A characteristic of Hilbert's approach, the formalization of the "tertium non datur" by transfinite functions, appears already in the first communication. In particular, the "tertium non datur" for the whole numbers is formalized with the function function $\chi(f)$, whose argument is a number theoretic function, and which has the value 0 if $f(a)$ has the value 1 for all number values a, but otherwise represents the smallest number value a for which $f(a)$ has a value different than 1.

The leading idea for the proof of the consistency of the transfinite functions (i.e., of their axioms), which Hilbert already ||[204] possessed, is not presented in this communication. A proof of consistency is rather provided here only for a certain part of the formalism; but this proof is only important as an example of a metamathematical proof.[16]

vol. 14 (1906).

[16]The method of proof rests here mainly on the fact that the elementary inference rules for the implication, which are formalized by the "Axioms of logical inference" (numbered 10 through 13), are not included in the part of the formalism under consideration.

In the Leipzig talk “Die logischen Grundlagen der Mathematik,”[17] which followed soon after the first communication, we find the approach and realization of proof theory developed further in various respects. I want to mention briefly the main respects in which the presentation of the Leipzig talk goes beyond those of the first communication:

1. The fundamental way in which ordinary mathematics goes beyond the intuitive approach (which consists in the unrestricted application of the concepts “all,” “there exists” to infinite totalities) is pointed out and the concept of “finite logic” is elaborated. Furthermore, a comparison between the role of “transfinite” formulas and that of ideal elements is carried out here for the first time.

2. The formalism is freed from unnecessary restrictions (in particular the avoidance of negation).

3. The formalization of the “tertium non datur,” and also of the principle of choice using transfinite functions, is simplified.

4. The main features of the formalism of analysis are developed.

5. The proof of consistency is provided for the elementary number theoretic formalism that results from the exclusion of bound variables. The task of proving the consistency of number theory and analysis is then focused on the treatment of the “transfinite axiom”

$$A(\tau(A)) \to A(a),$$

[17]Held at the Deutschen Naturforscher-Kongreß 1922. Math. Ann. Bd. 88 Heft 1/2, this volume essay no. 11.

which is employed in two ways, since the argument of A is related on the one hand to the domain of ordinary numbers and on the other hand to the number series (functions).

6. A method (which is successful at least in the simplest cases) is stated for the treatment of the "transfinite axiom" in the consistency proof.

The basic structure of proof theory was reached with its formulation as presented in the Leipzig talk.

||[205] Hilbert's next two publications on proof theory, the Münster talk "Über das Unendliche"[18] and the (second) talk in Hamburg "Die Grundlagen der Mathematik,"[19] in which the basic idea and the formal approach of proof theory is presented anew and in more detail, still show various changes and extensions in the formalism. However, they serve only in smaller part the original goal of proof theory; they are used mainly with respect to the plan to solve Cantor's continuum problem, i.e., the proof of the theorem that the continuum (the set of real numbers) has the same cardinality as the set of numbers of the second number class.

Hilbert had the idea of ordering the number theoretic functions, i.e., the functions that map every natural number to another—(the elements of the continuum surely can be represented by such functions) —in accordance with the type of the variables which are needed for their definition, and to achieve a mapping of the continuum to the set of numbers of the second number class on the basis of the ascent of the variable types, which is analogous to that

[18]Presented in 1925 on the occasion of a meeting organized in honor of the memory of Weierstrass, published in Math. Ann. vol. 95.

[19]Presented in 1927, published in Abh. math. Semin. Hamburg. Univ. Bd. IV Heft 1/2.

of the transfinite ordinal numbers. But the pursuit of this goal did not get beyond a sketch, and Hilbert therefore left out the parts which refer to the continuum problem in the reprints of both mentioned talks in "Grundlagen der Geometrie."[20]

Hilbert's considerations about the treatment of the continuum problem have nevertheless produced various fruitful suggestions and viewpoints.

Thus W. Ackermann has been inspired to his investigation "Zum Hilbertschen Aufbau der reellen Zahlen"[21] by the considerations regarding the recursive definitions. Hilbert lectures in his talk in Münster on the question and the result of this paper (which had not been published at the time):

> Consider the function
>
> $$a + b;$$
>
> ||[206] by iterating n times and equating it follows from this:
>
> $$a + a + \ldots + a = a \cdot n.$$
>
> Likewise one arrives from
>
> $$a \cdot b \text{ to } a \cdot a \cdot \ldots a = a^n,$$
>
> further from
>
> $$a^n \text{ to } a^{(a^a)}, a^{(a^{(a^a)})}, \ldots.$$

[20] Both talks are included in the seventh edition of "Grundlagen der Geometrie" as appendix VIII and IX. Other than the omissions also small editorial changes have been done, in particular with respect to the notation of the formulas.

[21] Math. Ann. Bd. 29 Heft 1/2 (1928).

So we successively obtain the functions

$$\begin{aligned} a+b &= \varphi_1(a,b), \\ a\cdot b &= \varphi_2(a,b), \\ a^b &= \varphi_3(a,b). \end{aligned}$$

$\varphi_4(a,b)$ is the b^{th} value in the series:

$$a, a^a, a^{(a^a)}, a^{(a^{(a^a)})}, \ldots.$$

In analogous way one obtains $\varphi_5(a,b), \varphi_6(a,b)$ etc.

It would now be possible to define $\varphi_n(a,b)$ for variable n by substitution and recursion; but these recursions would not be ordinary successive ones, but rather one would be led to a crossed recursion of different variables at the same time (simultaneous), and it is only possible to resolve this into ordinary successive recursions by using the concept of a function variable: the function $\varphi_a(a,a)$ is an example for a function of the number variable a, which can not be defined by substitution and ordinary successive recursion alone, if one allows only for number variables.[22] How the function $\varphi_a(a,a)$ can be defined using function variables is shown by the following formulas:

$$\begin{aligned} \iota(f,a,1) &= a, \\ \iota(f,a,n+1) &= f(a,\iota(f,a,n)); \\ \varphi_1(a,b) &= a+b \\ \varphi_{n+1}(a,b) &= \iota(\varphi_n,a,b). \end{aligned}$$

[22]W. Ackermann has provided a proof for this claim. (Footnote in Hilbert's text.)

Here ι stands for an individual function with two arguments, of which the first one is itself a function of two ordinary number variables.

The investigation of recursive definitions has been recently carried forward by Rozsa Péter. She proved that all recursive definitions which proceed only after the values of *one* variable and which do not require any other sort of variables than the free number variables, can be reduced to the simplest recursion schema. Using this result ||[207] she also simplified substantially the proof of the paper of Ackermann just mentioned.[23]

These results concern the use of recursive definitions to obtain number theoretic functions. In Hilbert's proof plan recursive definitions also occur in a different way, namely, as a procedure for constructing numbers of the second number class and also types of variables. Here Hilbert presupposes certain general ideas concerning the sorts of variables, of which he gives the following short summary in the talk "Die Grundlagen der Mathematik":

The *mathematical variables* are of two sorts:

1. the *basic variables*
2. the *types of variables*

1. While one gets by with the ordinary whole number as the only basic variable in all of arithmetic and analysis, now a basic variable for each one of Cantor's transfinite number classes is

[23]See R. Péter: "Über den Zusammenhang der verschiedenen Begriffe der rekursiven Funktionen." Math. Ann. Bd. 111 Heft 4 (1935).

added, which is able to assume the ordinal numbers belonging to this class. To each basic variable there accordingly corresponds a proposition that characterizes it as such; this is defined implicitly by axioms.

To each basic variable belongs a kind of recursion, which is used to define functions whose argument is such a basic variable. The recursion belonging to the number variable is the "usual recursion" by which a function of a number variable n is defined by specifying which value it has for $n = 0$ and how the value for n' is obtained from the value at n.[24] The generalization of the usual recursion is transfinite recursion, whose general principle is to determine the value of the function for a value of the variable using the previous values of the function.

2. We derive further kinds of variables from the basic variables by applying logical connectives to the propositions for the basic variables, e.g., to Z.[25] The so defined variables are called types of variables, and the statements defining them are called type-statements; for each of these a new individual symbol is introduced. Thus the formula

$$\Phi(f) \sim (x)(Z(x) \rightarrow Z(f(x)))$$

yields the simplest example of a type of variables; this formula defines ||[208] the type of function variables (being a function). A

[24]Here n' is the formal expression for "the number following n".

[25]The formula $Z(a)$ corresponds to the proposition "a is an ordinary whole number".

further example is the formula

$$\Psi(g) \sim (f)(\Psi(f) \to Z(g(f)));$$

it defines "being a function-function"; the argument g is the new function-function variable.

For the construction of higher variable types the type-statements have to be equipped with indices which enables a method of recursion.

These concept formations are applied in particular in the theory of numbers of the second number class. Here a new suggestion emerged from Hilbert's conjecture that every number of the second number class can be defined without transfinite recursion, but using ordinary recursion alone—assuming a basic element 0, the operation of progression by one ("stroke-function") and the limit process, as well as the number variable and the basic variable of the second number class —.

The first examples of such definitions that go beyond the most elementary cases, namely the definition of the first ε-number (in Cantor's terminology) and the first critical ε-number,[26] have already been given by P. Bernays and

[26]An ε-number is a transfinite ordinal number α with the property $\alpha = \omega^\alpha$. The first ε-number is the limit of the series

$$\alpha_0, \alpha_1, \alpha_2, \ldots,$$

where $\alpha_0 = 1$, $\alpha_{n+1} = \omega^{\alpha_n}$. the first critical ε-number is the limit of the series

$$\beta_0, \beta_1, \beta_2, \ldots,$$

where $\beta_0 = 1$, β_{n+1} is the β_n-th ε-number.

J. v. Neumann. Hereby already recursively defined types of variables are used.[27]

But these various considerations, which refer to the recursive definitions, already go beyond the narrower domain of proof theoretic questions. Since Hilbert's Leipzig talk it was the task of this narrower field of investigation of proof theory to prove consistency according to Hilbert's approach, including the transfinite axiom. Shortly after the talk in Leipzig the transfinite axiom was brought into the form of the logical "ε-axiom"

$$A(a) \to A(\varepsilon_x A(x))$$

$\|^{209}$ by the introduction of the choice function $\varepsilon(A)$ (in detail: $\varepsilon_x A(x)$) replacing the earlier function $\tau(A)$. The role of this ε-axiom is explained by Hilbert in his talk in Hamburg in with following words:

> The ε-function is applied in the formalism in three ways.
>
> a) It is possible to define "all" and "there exists" with the help of ε, namely as follows:[28]

$$(x)A(x) \sim A(\varepsilon_x \overline{A(x)}),$$
$$(Ex)A(x) \sim A(\varepsilon_x A(x)).$$

[27]Cf. the statement in Hilbert's talk "Die Grundlagen der Mathematik" ("Grundlagen der Geometrie," 7 ed. appendix IX, p. 308.—The examples mentioned have not been published yet.

[28]Instead of the double arrow used by Hilbert the symbol of equivalence $\sim$ is applied in both following formulas; the remarks on the introduction of the symbol $\sim$ in Hilbert's text are thus dispensable.

Based on this definition the ε axioms yields the valid logical notations for the ‘for all’ and ‘there exists’ symbols, like

$$(x)A(x) \to A(a) \quad \text{(Aristotelian axiom)},$$
$$\overline{(x)}A(x) \to (Ex)\overline{A(x)} \quad \text{(Tertium non datur)}.$$

b) If a proposition $\mathfrak{A}$ is true of one and only one thing, then $\varepsilon(\mathfrak{A})$ is *that thing* for which $\mathfrak{A}$ holds.

Thus, the ε-function allows one to resolve such a proposition $\mathfrak{A}$ that holds of only one thing into the form

$$a = \varepsilon(\mathfrak{A}).$$

c) Moreover, the ε plays the role of a choice function, i.e., in the case that $\mathfrak{A}$ holds of more than one thing, $\varepsilon(\mathfrak{A})$ is *any* of the things a of which $\mathfrak{A}$ holds.

The ε-axiom can be applied to different types of variables. For a formalization of number theory the application to number variables suffices, i.e., the type of natural numbers. In this case the number theoretic axioms

$$a' \neq 0,$$
$$a' = b' \to a = b,$$

as well as the recursion equations for addition and multiplication[29] and the principle of inference of complete induction, have to be added to the the

[29]Cf. footnote 1 on p. 197 of this report.

logical formalism and the axioms of equality. This principle of inference can be formalized using the ε symbol by the formula

$$\varepsilon_x A(x) = b' \rightarrow \overline{A(b)}$$

in connection with the elementary formula

$$a \neq 0 \rightarrow a = (\delta(a))'.$$

$\|^{210}$ The additional formula for the ε symbols corresponds to a part of the statement of the least number principle [30] and the added elementary formula represents the statement that for every number different than 0 there is a preceeding one.

For the formalization of analysis one has to apply the ε-axiom also to a higher type of variables. Different alternatives are possible here, depending on whether one prefers the general concept of predicate, set, or function. Hilbert chooses the type of function variables, i.e., more precisely, of the variable number theoretic function of one argument.

The introduction of higher types of variables allows for the replacement of the inference principle of complete induction by a definition of the concept of natural number following the method of Dedekind.

The essential factor in the extension of this formalism is based on the connection between the ε-axiom and the replacement rule for the function variable, whereby the "impredicative definitions" of functions, i.e., the definitions of functions in reference to the totality of functions, are incorporated into the formalism.

[30] I.e., the principle of the existence of a least number in every nonempty set of numbers.

The task of proving consistency for the number theoretic formalism and for analysis is hereby mathematically sharply delimited. For its treatment one had Hilbert's approach at one's disposal, and at the beginning it seemed that only an insightful and extensive effort was needed to develop this approach to a complete proof.

However, this vision has been proved mistaken. In spite of intensive efforts and a multitude of contributed proof ideas the desired goal has not been achieved. The expectations that had been entertained have been disappointed step by step, and in the same process it also became apparent that the danger of mistake is particularly great in the domain of metamathematical considerations.

At first the proof of the consistency of analysis seemed to succeed, but this appearance soon revealed itself as an illusion. Thereafter it was believed that the goal had been reached at least for the number theoretic formalism. Hilbert's talk in Hamburg "Die Grundlagen der Mathematik" falls in this stage, where at the end he cites a report on a consistency proof by Ackermann, as well as the talk "Probleme der Grundlegung der Mathematik,"[31] held in 1928 in Bologna, where Hilbert gave an overview of the situation in proof theory at that time and put forward in part problems of consistency and in part problems of completeness.

||[211] Here Hilbert connects all problems of consistency to the ε-axiom, presenting the mathematical domains that are encompassed in place of the various formalisms.

In this presentation is expressed the view, shared at that time by all

[31] Math. Ann. Bd. 102 Heft 1.

parties, that the proof for the consistency of the formalism of number theory had been given already by the investigations of Ackermann and v. Neumann.

That in fact this goal had not yet been achieved was only realized when it became dubious, based on a general theorem of K. Gödel, whether it was at all possible to provide a proof for the consistency of the number theoretic formalism with elementary combinatorial methods in the sense of the "finite standpoint".

The theorem mentioned is one of the various important results of Gödel's paper "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I,"[32] which has clarified in a fundamental way the relation between content and formalism—the investigation of which was mentioned by Hilbert in "Axiomatisches Denken" as one of the aims of proof theory.

The basic message of the theorem is that a proof of the consistency of a consistent formalism encompassing the usual logical calculus and number theory cannot be represented in this formalism itself; more precisely: it is not possible to deduce the elementary arithmetical theorem which represents the claim of the consistency of the formalism—based on a certain kind of enumeration of the symbols and variables and an enumeration of the formulas and of the finite series of formulas derived from it—in the formalism itself.

To be sure, nothing is said hereby directly about the possibility of finite consistency proofs; but a criterion follows, which every proof of the consistency for a formalism of number theory or a more comprehensive formalism has to meet: a consideration must occur in the proof which can not be

[32]Mh. Math. Physik Bd. 38 Heft 1 (1931).

represented—based on the arithmetical translation—in the given formalism.

By means of this criterion one became aware that the existing consistency proofs were not yet sufficient for the full formalism of number theory.[33]

Moreover, the conjecture was prompted that ||[212] it was in general impossible to provide a proof for the consistency of the number theoretic formalism within the framework of the elementary intuitive considerations that conformed to the "finite standpoint" upon which Hilbert had based proof theory.

This conjecture has not been disproved yet.[34] However, K. Gödel and G. Gentzen have noticed[35] that it is rather easy to prove the consistency of the usual formalism of number theory[36] assuming the consistency of intuitionistic arithmetic[37] as formalized by A. Heyting.

From the standpoint of Brouwer's Intuitionism the proof of the consistency of the formalism of number theory has hereby been achieved. But this does not disprove the conjecture mentioned above, since intuitionistic arithmetic goes beyond the realm of intuitive, finite considerations by having also

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[33]V. Neumann's proof referred to a narrower formalism from the outset; but it appeared that the extension to the entire formalism of number theory would be without difficulties.

[34]But see postscriptum on p. 216.

[35]K. Gödel: "Zur intuitionistischen Arithmetik und Zahlentheorie". Erg. math. Kolloqu. Wien 1933 Heft 4. G. Gentzen has withdrawn his paper about the subject matter which was already in print because of the publication of Gödel's note.

[36]Namely, it is possible to show that every formula that is deducible in the usual formalism of number theory, which does not contain any formula variable, disjunction, or existential quantifier, can be deduced also in Heyting's formalism.

[37]A. Heyting: "Die formalen Regeln der intuitionistischen Logik" and "Die formalen Regeln der intutionistischen Mathematik," S.-B. preuß. Akad. Wiss. Phys.-math. Klasse 1930 II.

contentful proofs as objects besides the proper mathematical objects, and therefore needs the abstract general concept of an intelligible inference.—

A brief compilation of various finite consistency proofs for formalisms of parts of number theory that have been given will be presented here. Let the formalism which is obtained from the logical calculus (of first order) by adding axioms for equality and number theory, but where the application of complete induction is restricted to formulas without bound variables, be denoted by F_1; with F_2 we denote the formalism that results from F_1 by adding the ε-symbol and the ε-axiom,—whereby the formulas and schemata for the universal and existential quantifiers can be replaced by explicit definitions of the universal and existential quantifiers.[38] A consistency proof for F_2 immediately results in the consistency of F_1.

‖[213] The consistency of F_2 is shown:

[38]See in this paper p. 209.—With regard to the axioms of equality it is to be observed that they appear in the formalism in the more general form

$$a = a, \ a = b \rightarrow (A(a) \rightarrow A(b))$$

so that in particular the formula

$$a = b \rightarrow \varepsilon_x A(x, a) = \varepsilon_x A(x, b)$$

can be deduced. In the formalism F_1 the formula

$$a = b \rightarrow (A(a) \rightarrow A(b))$$

can be replaced by the more special axioms

$$a = b \rightarrow (a = c \rightarrow b = c), \ a = b \rightarrow a' = b'.$$

1. by a proof of W. Ackermann, which proceeds from the approach presented in Hilbert's Leipzig talk "Die logischen Grundlagen der Mathematik"[39];

2. by a proof by J. v. Neumann, who proceeds from the same assumptions[40];

3. using a second so far unpublished approach of Hilbert's executed by Ackermann; the idea behind this approach consists in applying a disjunctive rule of inference to eliminate the ε symbol instead of replacing the ε by number values.[41]

The consistency of F_1 is shown:

1. by a proof of J. Herbrand which rests on a general theorem about the logical calculus[42] that has been stated for the first time and proved by Herbrand in his thesis "Recherches sur la théorie de la démonstration"[43];

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[39]The concluding portion of the proof is not yet carried out in detail in Ackermann's dissertation "Begründung des 'tertium non datur' mittels der Hilbertschen Theorie der Widerspruchsfreiheit" [Math. Ann. Bd. 93 (1924)]. Later Ackermann provided a complete and at the same time more simple proof. This definitive version of Ackermann's proof has not been published yet; so far only Hilbert's already mentioned report in his second talk in Hamburg "Die Grundlagen der Mathematik" and the more detailed "Appendix" by P. Bernays which appeared with the talk in Abh. math. Semin. Hamburg Univ. Bd. 6 (1928) are available. (The remark at the end of the appendix with regard to the inclusion of complete induction has to be abandoned.)

[40]J. v. Neumann: "Zur Hilbertschen Beweistheorie." Math. Z. Bd. 26 (1927).

[41]Cf. the statement in the talk "Methoden des Nachweises von Widerspruchsfreiheit und ihre Grenzen" Verh. d. int. Math.-Kongr. Zürich 1932, second volume, by P. Bernays.

[42]J. Herbrand: "Sur la non-contradiction de l'arithmétique." J. reine angew. Math., vol. 166 (1931).

[43]Thèse de l'Univ. de Paris 1930, published in Travaux de la Soc. Sci Varsovie 1930.

||[214]

2. by a proof of G. Gentzen, which results from a sharpening and extension of Herbrand's theorem mentioned above found by Gentzen.[44]

For the time being one has not gone beyond these results, which are important mainly for theoretical logic and elementary axiomatics, and for the uncovering mentioned above of the relation between the usual number theoretic formalism and that of intuitionistic arithmetic.

But all the problems of completeness which Hilbert posed in his talk "Probleme der Grundlegung der Mathematik" have been treated in various directions.

One of these problems deals with the proof of the completeness of the system of logical rules which are formalized in the logical calculus (of first order). This proof has been given by K. Gödel in the sense that he showed[45]: if it can be shown that a formula of the first order logical calculus is not deducible, then it is possible to give a counterexample to the universal validity of that formula in the framework of number theory (using "tertium non datur," in particular in the form of the least number principle).

The other problem of completeness regards the axioms of number theory; it is to be shown: If a number theoretic statement can be shown to be consistent (on the basis of the axioms of number theory), then it is also provable. This claim contains also the following: "If it can be shown that

[44] G. Gentzen: "Untersuchung über das logische Schließen" Math. Z. Bd. 39 Heft 2 u. 3 (1934).

[45] K. Gödel: "Die Vollständigkeit der Axiome des logischen Funktionenkalküls." Mh. Math. Pysik Bd. 37 Heft 2 (1930).

a sentence[46] $\mathfrak{S}$ is consistent with the axioms of number theory, then the consistency with those axioms cannot also be shown for the sentence $\overline{\mathfrak{S}}$ (the converse of $\mathfrak{S}$)."

This problem is so far indeterminate, in that it is not specified on which formalism of logical inference it should be based. However, it was shown that the claim of completeness is justified for all logical formalisms, as long as one maintains the requirement of the rigorous formalization of the proofs.

This result stems again from K. Gödel, who ||[215] proved the following general theorem in the paper mentioned above "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I:" If a formalism $\mathfrak{F}$ is consistent in the sense that it is impossible to deduce the negation of a formula $(x)\mathfrak{A}(x)$ provided that the formula $\mathfrak{A}(\mathfrak{z})$ can be deduced in $\mathfrak{F}$ for all numerals $\mathfrak{z}$, and if the formalism is sufficiently comprehensive to contain the formalism of number theory (or an equivalent formalism), then it is possible to state a formula with the property that neither it itself nor its negation is deducible.[47] Thus, under these conditions, the formalism $\mathfrak{F}$ does not have the property of deductive completeness (in the sense of Hilbert's claim for

[46]A sentence is meant which can be represented in the formalism of number theory without free variables.

[47]Moreover this formula has the special form

$$(x)(\varphi(x) \neq 0),$$

where $\varphi(x)$ is a function defined by elementary recursion, and the non-deducibility of this formula as well as the correctness and deduciblity of the formula $\varphi(\mathfrak{z}) \neq 0$ for every given numeral $\mathfrak{z}$ follows already from consistency in the ordinary sense without the more restricted requirement mentioned above.

the case of number theory).[48]

Even before this result of Gödel was known Hilbert already had given up the original form of his problem of completeness. In his talk "Die Grundlegung der elementaren Zahlenlehre"[49] he treated the problem for the special case of formulas of the form $(x)\mathfrak{A}(x)$, which do not contain any bound variables other than x. He modified the task by adding an inference rule which says that a formula $(x)\mathfrak{A}(x)$ of the kind under consideration can be always taken as a basic formula if it is possible to show that the formula $\mathfrak{A}(\mathfrak{z})$ represents a true statement (according to the elementary arithmetic interpretation) for all numerals $\mathfrak{z}$.

||[216] With the addition of this rule the result follows very easily from the fact that if a formula of the special form under consideration is consistent,

[48]A different kind of incompleteness has been shown recently by Th. Skolem for the formalism of number theory ("Über die Unmöglichkeit einer vollständigen Charakterisierung der Zahlenreihe mittels eines endlichen Axiomensystems." Nordk. Mat. Forenings Skifter, Ser. II Nr. 1–12 1933). The formalism is not "categorical" (the term is used in analogy to O. Veblen's expression), as it is possible to state an interpretation of the relations $=, <$ and of the functions a', $a+b$, $a \cdot b$ in relation to a system of things (they are number theoretic functions)—using "tertium non datur" contentually for whole numbers—, such that on the one hand every number theoretic theorem that can be deduced in the formalism of number theory remains true also for that interpretation, but on the other hand that the system is by no means isomorphic to the number sequence (with regard to the relations under consideration), but that it contains in addition to the subset that is isomorphic to the number sequence also elements that are *greater* (in the sense of the interpretation) than all elements of that subset.

[49]Held 1930 in Hamburg, published in Math. Ann. Bd. 104 Heft 4; in this collection, essay no. 12.

then it is also true under the contentful interpretation.[50]

The method by which Hilbert enforces, so to speak, the positive solution of the completeness problem (for the special case that he considers) means a deviation from the previous program of proof theory. In fact, the requirement for a complete formalization of the rules of inference is abandoned by the introduction of the additional inference rule.

One does not have to regard this step as final. But in light of the difficulties that have arisen with the problem of consistency, one will have to consider the possibility of widening the previous methodical framework of the metamathematical considerations.

This previous framework is not explicitly required by the basic ideas of Hilbert's proof theory. It will be crucial for the further development of proof theory if one succeeds in developing the finite standpoint appropriately, such that the main goal, the proof of the consistency of usual analysis, remains achievable—regardless of the restrictions of the goals of proof theory that follow from Gödel's results—.

During the printing of this report the proof for the consistency of the full number theoretic formalism has been presented by G. Gentzen,[51] using a method that conforms to the fundamental demands of the finite standpoint. Thereby the mentioned conjecture about the range of the finite methods

[50] Hilbert had already mentioned earlier this fact in his second Hamburg talk "Die Grundlagen der Mathematik". There he used it to show that the finite consistency proof for a formalism also yields a general method for obtaining a finite proof from a proof of an elementary arithmetical theorem in the formalism, for example of the character of Fermat's theorem.

[51] This proof will be published soon in the Math. Ann.

(p. 212) is disproved.