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Mathematical Existence and Consistency (1950b)

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It is a familiar thesis in the philosophy of mathematics that existence, in the mathematical sense, means nothing but consistency; this thesis is used to describe the specific character of mathematics. The claim is that there is no philosophical question of existence for mathematics. However, this thesis is neither as simple nor as self-evident as it may seem, and reflecting on it may shed light on several issues current in philosophical discussions.

Let us begin by describing what the thesis is directed against. It opposes quite obviously the view that attributes to mathematical entities an ideal being (i.e., a manner of existence that is independent on the one hand of being thought or imagined and also on the other hand of appearing as the determination of something real); this view claims furthermore that the existential statements of mathematics are to be understood with reference to this ideal being. One fact speaks from the very outset against this view; namely, that without apparent necessity an assumption is introduced here which does not do any methodological work. To make things clear, it may be advantageous to compare this with existential claims in the natural sciences. It is well known that an extreme phenomenalistic philosophy sought to eliminate the assumption of objects that exist independently of perception even from the representation of relations in nature [*Naturzusammenhänge*]. However, even a rough orientation about our experience suffices to show that such an undertaking—apart from the manifold obstacles that confront its implementation—is also inappropriate from a scientific perspective. In terms of perception alone, we do not gain any perspicuous laws. The world of our experience would have to be completely different in order for a theory—founded on notions concerning the purely perceptual—to be successful. Hence, positing the objective existence of entities-in-nature [*Naturgegenständlichkeiten*] is by no means solely an effect of our instinctive attitude, but is appropriate from the standpoint of scientific methodology. (This is also true for contemporary quantum physics, even though according to it states cannot be specified with complete precision.)

When comparing this case with that of mathematical entities, we find the following obvious difference. In the theoretical and concrete use of mathematical objects an independent existence of these objects plays no role (i.e., an existence independent of their respective appearance as determinations of something otherwise objective). The assumption of objective physical entities, by contrast, has explanatory value only because the entities and states in question are posited as existing at particular times and in particular locations.

What we find here concerning mathematical objects holds in general for all those entities that can be called "theoretical objects [*ideelle Gegenstände*]."

Meant are those entities of reflection to which we cannot ascribe, at least not directly, the character of the real, or more precisely, of the independently real; e.g., species, totalities, qualities, forms, norms, relations, concepts. All mathematical entities belong to this realm.

One can hold the view—and this view has indeed been defended by some philosophers—that all statements about theoretical entities are reducible, if made precise, to statements about the real. This kind of reduction would yield, in particular, an interpretation of existence statements in mathematics. However, at this point fundamental difficulties arise. On a somewhat closer inspection, it turns out that the task of reduction is by no means uniquely determined, since several conceptions of the real can be distinguished: The "real" may refer, e.g., to what objectively real, or to what is given in experience, or to concrete things. Depending on the conception of the real, the task of reduction takes on a completely different form. Furthermore, it does not seem that in any one of these alternative ways the desired reduction can be achieved in a satisfactory way.

One has to mention in this connection especially the efforts of the school of logical empiricism towards a "unified language" for science. It is noteworthy that recently the attempt to reduce all statements to those about the concrete has been abandoned. This was prompted in particular by the requirements in the field of semantics (an analysis of meaning of the syntactic forms of language).

We will not base our discussion on any presuppositions regarding the possibility in principle of avoiding the introduction of theoretical entitites in the language of science. In any event, the existing situation is that in areas of research (and even in the approaches of everyday life) we are constantly dealing with theoretical entities, and we adopt this familiar attitude here.

As yet this attitude in no way includes an assumption about an independent existence of ideal entities. It is understandable though that such an assumption has, in fact, often been connected with theoretical entities—particularly if we agree with Ferdinand Gonseth, according to whom the more general concept of an entity arises from a primary cruder notion of an entity that is expressed in a "physique de l'objet quelconque." As regards the cruder objecthood, the character of the objective is most intimately tied to existence independent of our perception and representation. Thus it is easy to understand that for entities of a general kind we are inclined to attribute their objective character to an independent existence. It is not at all necessary to do so, however. Here it is especially significant that refraining from an assumption of ideal existence does not prevent us from using existence statements about theoretical entities: such statements can be interpreted without this particular assumption. Let us bring to mind the main cases of such interpretations:

a) Existence of a theoretical entity may mean the distinct and complete representability of the object.

b) Existence of a theoretical entity of a particular kind may mean that it is realized in something that is objectively given in nature. Thus, for instance, the observation that a certain word has different meanings in a language tells us that in the use of this language, the word is employed with different meanings.

c) An existence claim concerning theoretical entities can be made with reference to a structured object [Gebilde] of which that entity is a constituent part. Examples of this are statements about constituent parts of a figure, as when we say, "the configuration of a cube contains 12 edges," or statements about something that occurs in a particular play, or about provisions that are part of Roman law. We are going to call existence in this sense, i.e., existence within a comprehensive structure, "relative existence."

d) Existence of theoretical entities may mean that one is led to such entities in the course of certain reflections. For example, the statement that there are judgments in which relations appear as subjects expresses the fact that we are also led to such "second order" judgments (as they are called) when forming judgments.

In case a) the existence of the theoretical entity is nothing but the representational objectivity (in the sense of representation proper); in case b) existence amounts to a reality in nature [*Naturwirklichkeit*]; case c) is concerned with an immanent fact of a total structure that is under consideration.

In these three cases the interpretation of the existence statement provides a kind of immediate contentual reduction. Case d) is different in that "being led" to entities is not to be understood as a mere psychological fact but as something objectively appropriate. Here reference is made to the development of intellectual situations with the factors of freedom and commitment operative therein—freedom in the sense in which Gonseth speaks of a "charte de nos libertés" (for example, the freedom to add in thought a further element to a totality of elements represented as surveyable) and, on the other hand, commitment which consists, e.g., in the fact that the means we use for the description and intellectual control of entities yield, on their part, new and possibly even more complex entities.

Yet even this interpretation of existence statements does not introduce an assumption of independent existence of theoretical entities.[1] The existence statement is kept within the particular conceptual context, and no philosophical (ontological) question of modality, which goes beyond this context is entered into. Whether such a question is meaningful at all is left open.

These considerations apply to theoretical entities in general. But what of the specific case of mathematical entities, which, as has been noted, are theoretical entities? If we apply our preceding reflections to the case of mathematical entities, we notice that we already have a kind of answer to the question of what existence may mean in mathematics. However, the thesis under discussion—that existence for mathematical entities is synonymous with consistency—is intended to offer a simpler answer. For the discussion of this claim we have already gained several clarifying points. Let us now turn to this discussion.

For this purpose let us first replace the obviously somewhat abbreviated formulation of the claim by a more detailed one. What is meant is surely this: Existence of an entity (of a complex [*Gebilde*], a structure) with certain required properties means in the mathematical sense nothing but the consistency of those required properties. The following simple example may illustrate this. There is an even prime number, but there is no prime number divisible by 6. Indeed, the properties "prime number" and "even" are compatible, but the properties "prime number" and "divisible by 6" contradict each other. Examples like this give the impression that the explanation of

[1]The German text has "ideale Gegenstände" here, but "ideelle" seems to be intended.

mathematical existence in terms of consistency is entirely satisfactory. It must be noted, however, that these examples do not show what this explanation is capable of; they only demonstrate how one infers consistency from the existence of an example and, on the other hand, non-existence from inconsistency, but not how one infers existence from an already established consistency; and that, after all, would be the decisive case.

This one remark suffices to make us hesitate. It draws attention to the fact that in mathematics existential claims are usually not inferred from proofs of consistency but, conversely, that proofs of consistency are given by exhibiting models; the satisfaction of the required properties is always verified in the sense of a positive assertion. In other words, the usual proofs of consistency are proofs [*Nachweise*] of the *satisfiability* of conditions, or more precisely: [they are proofs] of the satisfaction of conditions by a theoretical entity.

An unaccustomed innovation was brought about by Hilbert's proof theory in that it demanded consistency proofs in the sense of showing the impossibility of arriving deductively at an inconsistency. A preconditon of such a proof is that the pertinent methods of deduction to be considered can be clearly delimited. The methods of symbolic logic provide the technique for making the process of logical inference more precise. We are thus in a position to delimit the methods of inference used in mathematical theories, especially in number theory and the theory of functions, by an exactly specified system of rules. This is, however, only a delimitation of the inferences used *de facto* in the theories. In general this does not lead to making an unrestricted concept of consistency more precise, but only consistency in a certain relatively elementary domain of logico-mathematical concept-formation. In this domain the concept of mathematical proof can be delimited in such a way that one can show: each requirement that does not lead deductively to an inconsistency can (in a more precisely specified sense) be satisfied. This completeness theorem of Gödel's makes particularly clear that the claimed coincidence of consistency with satisfiability is far from obvious, but is substantially dependent on the structure of the domain of statements and inferences considered. If one goes beyond this domain, making the methods of proof precise no longer yields the coincidence of consistency and satisfiability. This coincidence—as shown again by Gödel—cannot be achieved in general (if certain natural requirements are imposed on the concept of provability).

It is, of course, possible to extend the concept of proof by means of a more general concept of "consequence," following a method developed by Carnap and Tarski, so that for the resulting concepts of logical validity and contradictoriness (leading to a contradiction) we have the alternative that every purely mathematical proposition is either logically valid or contradictory. Consequently, every requirement on a mathematical entity is either inconsistent or satisfied by an entity.

Thus the identification of existence with consistency appears to receive exact confirmation. On closer inspection, however, one notices that the decisive factor is anticipated, so to speak, by the definitions. For, on the basis of the definitions, a mathematical requirement on a mathematical entity is already contradictory if it is not satisfied by any entity. Accordingly, in the field of mathematics the coincidence of consistency of a requirement and satisfaction by an entity says no more than that an entity of species G satisfying a condition B exists if and only if not every entity of species G violates condition B.

Of course—from the standpoint of classical mathematics and logic—this is a valid equivalence. But using this equivalence to interpret existence statements is surely unsatisfactory: If the claim that there is an exception to a universal proposition is considered to be in need of a contentual explanation, since it is an existential statement, then the negation of that universal proposition certainly is not clearer as to its content. The equivalence between the negation of a universal proposition and an existential proposition serves (in classical mathematics), among other things, to explicate more clearly the sense of the negation of a universal proposition. This is also indicated by Brouwer's intuitionism, which does not recognize this equivalence. At the same time, it denies that simple negation of a universal mathematical proposition has any sense at all, and introduces a sharpened negation—absurdity—which contains an existential aspect (since "absurdity" is to be understood as an effective possibility of a refutation).

The difficulties to which we have been led here ultimately arise from the fact that the concept of consistency itself is not at all unproblematic. The common acceptance of the explanation of mathematical existence in terms of consistency is no doubt due in considerable part to the circumstance that on the basis of the simple cases one has in mind, one forms an unduly simplistic idea of what consistency (compatibility) of conditions is. One thinks of the compatibility of conditions as something the complex of conditions wears on its sleeve [*etwas gleichsam direkt Anhaftendes*], as it were, such that one need only sort out the content of the conditions clearly in order to see whether they are in agreement or not. In fact, however, the role of the conditions is that they affect each other in functional use and by combination. The result obtained in this way is not contained as a constitutent part of what is given through the conditions. It is probably the erroneous idea of such inherence that gave rise to the view of the tautological character of mathematical propositions.

Leaving aside the difficulties connected with the concept of consistency and with the relation between consistency and satisfiability, there is another aspect which points to the fact that it is not always appropriate to interpret existence as consistency in mathematics. Let us consider the case of existence axioms of an axiomatic mathematical theory. Interpreting the existence statement as an assertion of consistency in this case, yields confusion insofar as in an axiomatic theory consistency relates to the system of axioms as a whole. The condition of consistency may well function as a prior postulate for the design of any axiom system. The axioms themselves, however, are intended to generate commitments, at least in the usual form of axiomatics. An existence axiom does not say that we *may* postulate [*ansetzen*] an entity under certain circumstances, but that we are committed to postulate [*ansetzen*] it under these circumstances.

On the basis of our initial reflections, we also have an appropriate understanding of axiomatic existence statements available. That is to say, if we consider that an axiom system as a whole may be regarded as a description of a certain structured complex—for example, an axiom system of Euclidean geometry [may be regarded] as describing the structure of a Euclidean manifold—then we recognize that the existence claims within an axiomatic theory can be understood as statements about *relative existence*:

Just as each corner in the configuration of a cube is incident to three edges, so through any two distinct points in the manifold of Euclidean space passes a straight line; and the theorem of Euclidean geometry which states that for any two points there exists a straight line through both expresses this fact of relative existence.[2]

It must be admitted, to be sure, that the perspective of relative existence, as appropriate as it is for the practical application of the existence concept in mathematics, only shifts, as it were, the philosophical question of mathematical existence. For relative existence is scientifically significant only insofar as the particular total structure, on which the relativeness is based, is to be regarded as mathematically existent. The question thus arises: what is the status of the existence of those total structures; for example, the existence of the number series, the existence of the continuum, the existence of the Euclidean space-structure and also of other space-structures?

Here we encounter examples where the identification of existence with consistency is justified. Thus it is justified when we say that the existence of non-Euclidean (Bolyai-Lobachevsky) geometry lies in its consistency. But even in such a case, the situation surely is that the consistency proof is given

[2]Bruno von Freytag-Löringhoff has emphasized what is unproblematic, so to speak, about relative existence in his article, "Die ontologischen Grundlagen der Mathematik," (Halle 1937) to which the present investigation owes a number of suggestions. In this connection the author speaks of the "small existence problem." His point of view, however, differs from the one presented here in that he regards the identification of existence with consistency as appropriate for the small existence problem, whereas in this presentation the viewpoint of relative existence is offered as a correction of the view equating existence with consistency.

by exhibiting a model and that thereby the consistency claim is strengthened to the assertion that a model satisfying the axioms exists—"exists" understood here relative to the domain of the arithmetic of real and complex numbers. In analogous ways many consistency proofs in the sense of establishing satisfiability can be given; for example, the proof of the consistency of a non-Archimedean geometry (i.e., a geometry with infinitely small segments); further, the consistency of calculating with imaginary magnitudes, taking the theory of real numbers as a basis. Most model constructions of this sort are carried out within the framework of the theory of the mathematical continuum (the theory of real numbers). The satisfiability of the axioms of the continuum itself can again be seen, starting from the number series, by essential use of set-theoretical construction processes.

But where do all these reductions lead? We finally reach the point at which we make reference to a theoretical framework [*ideeller Rahmen*]. It is a thought-system that involves a kind of methodological attitude; in the final analysis, the mathematical existence posits [Existenz-Setzungen] relate to this thought system.

We can state descriptively that the mathematician moves with confidence in this theoretical framework and that here he has at his disposal a kind of acquired evidence (for which constructions, even of a more complicated nature, such as infinite sequences of numbers, present themselves as something objectual). The consistency of this methodology has been tested so well in the most diversely combined forms of application that there is *de facto* no doubt about it; it is, of course, the precondition for the validity of the existence posits made within the theoretical framework. But we notice here again that we cannot simply identify existence with consistency, for consistency applies to the framework as a whole, not to the individual thing being posited as existent.

Let us consider the situation more closely, using the example of the number sequence. The postulation [*Setzung*] of the number sequence is included in the framework of our operating mathematically. But what does consistency of the number sequence mean? If we are content with an answer to this question that appeals to conceiving the unbounded continuation of the process of counting in an idealizing form of representation, then we understand existence in an objectual way. We view it in this way, whether we regard the number sequence merely as a domain of theoretical entities or, in accordance with a stronger idealization, as a structured complex in itself. And only from this objectual understanding do we infer consistency. If, however, consistency is to be recognized from the point of logic, then, on the one hand, the conditions contained in the idea [*Vorstellung*] of the number sequence must be understood conceptually and, on the other hand, we must base our considerations on a more precise notion of logical consequence.

In this connection we also come to realize that the concept of logical consequence gives rise to an unbounded manifold similar to that of the number sequence; that is due to the possibilities of combining inference processes. Furthermore, it becomes apparent that the domain of logic can be understood in a narrower or a wider sense, and that therefore its appropriate delimitation is problematic.

At this point we come to the area of mathematical-logical research in foundations. Its controversial character contrasts sharply with the aforementioned confidence in operating mathematically within the framework of the usual methods.

The difficulties we are facing here are as follows: The usual framework for operating mathematically is adequately determined for use in the classical theories; at the same time, however, certain indeterminacies with regard to the demarcation and the method of giving a foundation remain. If one tries to eliminate these, one faces several alternatives, and in deciding between them different views emerge. The differences of opinion are reflected in particular in the effort to obtain the foundation of mathematics from a standpoint without any substantive assumptions, such that one relies solely on what is absolutely trivial or absolutely evident. It becomes apparent here that there is no unanimity concerning the question of what is to be considered obvious or completely evident.

To be sure, these differences of opinion are less irritating if one frees oneself from the idea that an assumptionless foundation, obtained from a starting point determined entirely *a priori*, is necessary. Instead, one can adopt the epistemological viewpoint of Gonseth's philosophy which does not restrict the character of a duality—due to the combination of rational and empirical factors—to knowledge in the natural sciences, but rather finds it in all areas of knowledge. For the abstract fields of mathematics and logic this means specifically that thought-formations are not determined purely *a priori*, but grow out of a kind of intellectual experimentation. This view is confirmed when we consider the foundational research in mathematics. Indeed, it becomes apparent here that one is forced to adapt the methodological framework to the requirements of the task [at hand] by trial and error. Such experimentation, which must be judged as an expression of failure according to the traditional view, seems entirely appropriate from the viewpoint of intellectual experience. In particular, from this standpoint experiments that turned out to be unfeasible cannot *eo ipso* be considered methodological mistakes. Instead, they can be appreciated as stages in intellectual experimentation (if they are set up sensibly and are carried out consistently). Seen in this light, the variety of competing foundational undertakings is not objectionable, but appears in analogy with the multiplicity of competing theories encountered in several stages of development of research in the natural sciences.

If we now examine more closely the—at least partial—methodological analogy between these foundational speculations and theoretical research in the natural sciences, we are led to think that with each more precise delimitation of a methodological framework for mathematics (or for an area of mathematics) a certain domain of mathematical reality is intended, and that this reality is to a certain degree independent of the particular configuration of that framework. This can be made clear by means of the geometric axiomatics. As we know, the theory of Euclidean geometry can be developed axiomatically in various ways. The resulting structural laws of Euclidean geometry, however, are independent of the particular way in which this is done. The relations in the theory of the mathematical continuum and the disciplines associated with it are in a similar sense independent of the particular way in which the real numbers are introduced, and even more so of the particular method of a theoretical foundation. In a foundational investigation those relations, which are, as it were, forced on us as soon as we settle on certain versions of the calculus and of operating mathematically, have the role of the given, and it more precise theoretical fixation is the task at hand. The method of this fixation can contain problematic elements which do not affect what is, so to speak, given.

The viewpoint gained in this way places a mathematical reality face to face with a methodological framework constructed for the fixation of this reality. This is also quite compatible with the results of the descriptive analysis to which Rolin Wavre has subjected the relationship of invention and discovery in mathematical research. He points out that two elements are interwoven, the invention of concept formations, and the discovery of lawlike relations between the conceived entities, and furthermore that the conceptual invention is aimed at discovery.

With respect to the latter, it is frequently the case that the invention is guided by a discovery already more or less clearly available and that it serves the purpose of making it conceptually definite, thereby making it also accessible to communication. The necessity of adapting the concepts to the demands of giving expression to something objective exists in this situation as much as it does in similar situations in the theoretical natural sciences. Thus the concepts of the differential quotient and of a field were introduced with a view to giving expression to something objective in the same way as the concepts of entropy and the electrical field.

In the constitution of a framework for mathematical deduction we assume to have a case of the same methodological type, when we speak of a mathematical reality that is to be explicated by that framework.

If we now apply this perspective to our question of mathematical existence, we obtain an essential addition to our earlier observation that the existence statements in our mathematical theories are, in the final analysis, relative to a system of thought that functions as a methodological framework. This relativity of the existence statements now seems compensated to a large extent by the fact that the essential properties of the reality intended by the methodological framework are invariant, so to speak, with respect to the particulars (the invented aspects) of that framework.

Furthermore, it must be noted here that the mathematical reality also stands out from any delimited methodological framework insofar as it is never fully exhausted by it. On the contrary, the conception of a deductive framework always results in further mathematical relations which go beyond that framework.

Do we not—so one may ask—return with such a view of mathematical reality to the assumption of an ideal existence of mathematical entities which we rejected as unmotivated at the outset of our reflections? To respond to this question we must recall the limits of the analogy between mathematical and physical reality. We are dealing here with something very elementary.

It is inherent in the purpose of scientific concept formation that it seeks to provide us with an orienting interpretation of the environment. In the natural sciences the modality of the factually real plays therefore a distinguished role, and in comparison with this reality all other existence one can talk about appears as mere improper existence—as when we speak of the existence [*Bestehen*] of the relations given by laws of nature. This is, in fact, true even though the statements concerning the existence of laws of nature contentually go beyond what can be ascertained in the domain of the factual.

In mathematics we do not have such a marked difference in modality. From the viewpoint of the mathematician, the individual mathematical entity does not present itself as something that exists in a more eminent sense than the relations given by laws. Indeed, one can say that there is no clear difference at all between something directly objectual and a system of laws [*Gesetzlichkeit*] to which it is subject, since a number of laws [*Gesetzlichkeiten*] present themselves through formal developments which possess the character of the directly objectual. Even axiom systems may be considered as structured objects. In mathematics, we therefore have no reason to assume existence in a sense fundamentally different from that in which we assume the existence [*Bestehen*] of relations given by laws.

This eliminates the various reservations that seem to oppose our view of the relativity of mathematical existence statements to a system of the conceptual (to a deductive framework): Irrespective of the various possibilities of constructing such a system of the conceptual, this view does not amount to relativism. On the contrary, we can form the idea of a mathematical reality that is independent in each case of the particulars of the construction of the deductive framework. The thought of such a mathematical reality, on the other hand, does not mean a return to the view of an independent existence of mathematical entities. It is not a question of being [*Dasein*] but of relational, structural connections and of the emergence (being induced) of theoretical entities from other such entities.

In order not to be one-sided, however, our reflection on mathematical existence still requires a complementary prespective. We have carried out this reflection in accord with the attitude of the mathematician who directs his attention purely toward the objectual. If we bear in mind, however, our methodological comparison between the mathematical (foundational) starting points and those of physics, then we might realize that this analogy also applies to a point we have not noted yet: Just as the theoretical language and the theoretical attitude of physics is substantially complemented by the attitude and language of the experimentalist, so is the theoretical attitude in mathematics also complemented by a manner of reflection that is directed toward the procedural aspect of mathematical activity. Here we are concerned with existence statements that do not refer to abstract entities but to arithmetical expressions, to formal developments, operations, definitions, methods for finding solutions, etc. The significance of such a constructive mode of reflection and expression—as it comes to the fore especially in Brouwer's intuitionism and for the method of Hilbert's proof theory—is also acknowledged by mathematicians who are not willing to be content with an exclusively constructive mathematics and, therefore, just as little with an action-language [*Tätigkeits-Sprache*] of mathematics as the only form of mathematical expression.

In this context it should be emphasized with respect to Hilbert's proof theoretic project which is based on an operative (constructive) standpoint: the interest of this project for the philosophy of science is not at all tied to that philosophical doctrine of "formalism" which arose from the original formulation of the aim of proof theory. In order to appreciate the methodological fruitfulness of proof theory, there is in particular no need to take the position that the theories subjected to symbolic formalization (for proof theoretic purposes) should be simply identified from then on with the schema of their symbolic formalism and thus should be considered merely as a technical apparatus.

We must also bear in mind that the motivation for the conceptual system of contemporary mathematics does not lose its significance through the proof theoretic investigation of consistency; this motivation results from the connection to the problems that gave rise (in several stages) to the conceptual system in the first place. Such a motivation is indeed assumed to have already been obtained before the proof theoretic investigation begins.[3]

Finally it should be remembered—as regards the methods of constructive proof theory and also those of Brouwer's intuitionism—that with these methods one does not remain in the domain of the representationally objectual, properly so called [*im Bereich des eigentlich Vorstellungs-Gegenständlichen*]. The concept of the effective is idealized and extended here in the sense of an adaptation to theoretical demands—of course in a way which is in principle more elementary than it is done in ordinary mathematics. The methodological standpoint also in this case is thus not without pre-conditions, but we are concerned, once again, with a theoretical framework that includes general kinds of positing [*Setzungen*]. Our preceding reflections are, therefore, also applicable to this constructive mathematics.

On the whole our considerations point out that it is not indicated either to exaggerate the methodological difference between mathematics and the sciences of the factual, which undeniably exists, or to underestimate the

[3]As regards the task of a *systematic* motivation of the concept formations of classical mathematics, we are led to the problem already mentioned for obtaining a deductive framework that is as appropriate and as satisfactory as possible. This problem constitutes a major topic of contemporary foundational research in mathematics.

philosophical problems associated with mathematics.