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# The multiplicity of purposes in formulations of geometric axiom systems (1959)

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In considering axiomatizations of geometry we have the impression of a great multiplicity of principles according to which such axiomatizations can take place, and already have taken place. The original, simple idea, that one could just speak of *the* axioms of geometry was not only superseded by the discovery of non-Euclidean geometries but, moreover, by the insight into the possibility of different axiomatizations of one and the same geometry. But substantially different methodological principles have also arisen generally, according to which one has undertaken the axiomatization of geometry and whose purposes are in certain respects even antagonistic.

The seed for this multiplicity can already be found in Euclidean axiomatics. For its formulation was determined by the fact that one was led by geometry to the general problem of axiomatics for the first time. Here geometry is simply all of mathematics, so to speak. The methodological relation to number theory is not completely clear. In certain places a bit of number theory is developed using the intuitive idea of number. Moreover the concept of number is used contentfully in the theory of proportions, even with an implicit inclusion of the *tertium non datur*, although it seems that one attempted to avoid its unrestricted use.

While the special methodological position of the concept of number is not especially pronounced here, the concept of magnitude is explicitly put forward as a contentful tool. This is done, incidentally, in a manner that we can no longer accept today, namely by assuming as a matter of course that different objects can have the character of magnitudes. The concept of magnitude is, of course, also subjected to axiomatization; however, in this regard the axioms are explicitly separated from the remaining axioms as antecedent (*κοιναὶ ἐννιαὶ*). These axioms are of a similar kind as those which are used today for Abelian groups. But what remained undone, because of the methodological standpoint at the time, was to determine axiomatically which objects were to be regarded as magnitudes.

Thus it is all the more admirable that one was then already sensitive to the peculiarity of that assumption by which the Archimedean magnitudes, as we call them today, are characterized. The Archimedean (Eudoxean) axiom is then, in the medieval tradition that followed the Greeks, used in particular in the Arabic investigations of the parallel axiom. It also occurs essentially in Saccheri's proof of the elimination of the "hypothesis of the obtuse angle". This elimination is in fact impossible without the Archimedean axiom, since a non-Archimedean, weakly-spherical (resp. weakly-elliptical) geometry is in accordance with the axioms of Euclidean geometry, except for the parallel axiom.

The second axiom of continuity, which was formulated in the late $19^{\text{th}}$ century, does not yet occur in any of these investigations. It could be dispensed with in the proofs for which it came into question—like in the determination of areas and lengths—because of the already mentioned use of the concept of magnitude, according to which it was for example taken for granted that both the area of the circle and the circumference of the circle possess a definite magnitude. In place of the old theory of magnitudes at the beginning of modern times came, as a predominant and super-ordinated discipline, the theory of magnitudes of *analysis*, which developed quite prolifically both formally and contentually still before it reached methodological clarity.

Of course, analysis at first played no significant role in the discovery of non-Euclidean geometry, but it became dominant in the following investigations of Riemann and Helmholz, and later Lie, for the identification of the three special geometries by certain very general, analyticaL conditions. In particular it is characteristic for this treatment of geometry that one not only takes the particular spatial entities as objects, but also the spatial manifold itself. The enormous conceptual and formal means which mathematics had obtained in the meantime showed up in the possibility of carrying out such an investigation. And the conceptual and speculative direction which mathematics took in the course of the $19^{\text{th}}$ century is expressed in the formulation of the general problem.

The differential geometrical treatment of the foundations of geometry was developed further, until very recent times, by Hermann Weyl, as well as Elie Cartan and Levi-Civitá, in connection with Einstein's general relativity theory. Despite the impressiveness and elegance of what has been achieved in this respect, mathematicians were not content with it from a foundational standpoint. At first one tried to free oneself from the fundamental assumption of the methods of differential geometry of the differentiability of the mappings. For this the development of the methods of a general topology was needed, which began at the turn of the century and has taken such an impressive course of development since then. Moreover one strove for independence from the assumption of the Archimedean character of the geometrical magnitudes in general.

This tendency is part of that development by which analysis in some sense lost its previously predominant position. This new stage in mathematical research followed the consequences of the already mentioned conceptual and speculative direction of mathematics of the $19^{\text{th}}$ century, which appeared in particular in the creation of general set theory, in the sharper foundation of analysis, in the constitution of mathematical logic, and in the new version of axiomatics.

At the same time it was characteristic for this new stage that one returned again to the methods of ancient Greek axiomatics, as happened repeatedly in those epochs in which emphasis was put on conceptual precision. In Hilbert's Foundations of Geometry we find on the one hand this return to the old elementary axiomatics, of course with a fundamentally changed methodological conception, and on the other hand the exclusion, as far as possible, of the Archimedean axiom as a principal theme: in the theory of proportions, in the concept of area, and in the foundation of the line segment calculus. For Hilbert, by the way, this kind of axiomatization was not intended to be exclusive; shortly afterwards he put a different kind of foundation along side it, in which the program of a topological foundation mentioned above was formulated and carried out for the first time.

Around the same time as Hilbert's foundation, the axiomatization of geometry was also cultivated in the school of Peano and Pieri. Shortly afterwards the axiomatic investigations of Veblen and R. L. Moore followed; and by then the directions of research were chosen along which occupation with the foundations of geometry proceeds also today. As is characteristic of it, there are numerous methodological directions.

One of them seeks to characterize the multiplicity of congruent transformations by conditions that are as general and succinct as possible. The second one puts the projective structure of space at the beginning and strives to reduce the metrical structure to the projective with the methods developed by Cayley and Klein. And the third aims at elementary axiomatization of the full geometry of congruences.

Different and fundamentally new points of view were added during the development of these directions. Firstly, the projective axiomatization gained an increased systematization through lattice theory. In addition, one became aware that the set-theoretic and function-theoretic concept formations can be deemphasized in the identification of the group of congruent transformations by identifying the transformations with structures determining them. Therewith the procedure approaches that of elementary axiomatics, since the group relations are now represented as relations between geometric structures.

But I do not want to speak further of these two directions of research in geometrical axiomatics, for which more authentic representatives are present here, and also not of the successes that have been achieved using topological methods, about which the newest essays of Freudenthal give a survey. Instead, I turn to the questions of the direction of axiomatization that was mentioned in the third place.

Even within this direction we find a multiplicity of possible goals. On the one hand one can aim to manage with as few as possible basic elements, perhaps only one basic predicate and one sort of individuals. On the other hand one can especially aim to isolate natural separations of parts of the axiomatics. These viewpoints lead to different alternatives.

So on the one hand the consideration of non-Euclidean geometry suggests the preliminary investigation of an "absolute" geometry. On the other hand something is to be said for a procedure that starts off with affine vector geometry, as is done at the beginning of Weyl's "Space, Time, Matter". The demands of both these viewpoints can hardly be satisfied with a single axiomatic system. Starting with the axioms of incidence and ordering it is a possible and elegant conceptual reduction to reduce the concept of collinearity to the concept of betweenness, in the way of Veblen. On the other hand it is important for some considerations to separate the consequences of the incidence axioms which are independent of the concept of ordering. So it is desirable to realize the independence of the foundation of the line segment calculus on the incidence axioms from the ordering axioms. In the theory of ordering itself one has again realized the possibility of replacing the axioms of linear ordering by applications of the axiom of Pasch; on the other hand in some respect a formulation of the axioms is preferable in which those axioms are separated which characterize the linear ordering.

The multiplicity of the goals that are possible, and are also pursued in fact, is not exhausted in the least by these examples of alternatives. Indeed it is a possible and plausible, but not obligatory, regulative viewpoint that the axioms should be formulated in such a way that they refer only to a limited part of space respectively. This thought is implicitly at work already in Euclidean axiomatics; and it may also be that the offense that has been taken so early at the parallel axiom relies precisely on the fact that the concept of a sufficiently long extension occurs in the Euclidean formulation. The first explicit realization of the mentioned program happened with Moritz Pasch, and it was followed by the introduction of ideal elements by intersection theorems, which is a method for the foundation of projective geometry that has been successively developed since.

A different kind of possible additional task is to imitate conceptually the blurriness of our pictorial imagination as it was done by Hjelmslev. This results not only in a different kind of axiomatization, but in a variant relational system, which has not found much approval because of its complication. But also without moving so far from the customary manner in this direction it is possible to aim at something similar, in some respects, by avoiding the concept of point as a basic term as it is done in various interesting newer axiomatizations, in particular in Huntington's.

One thus sees in a great number of ways that there is no definite optimum for the formulation of a geometric axiom system. As regards the reductions with respect to the basic concepts and the sorts of things, it must always be recalled that, regardless of the general interest any such possibility of reduction may have, a real application of such a reduction is only recommended when it leads to a clear formulation of the axiom system.

Certain directives for reductions which are generally acceptable can, however, be stated. Let us take for example the Hilbertian version of axiomatics. In it, on the one hand, lines are taken as a kind of things, on the other hand the rays are introduced as point sets and afterwards the angles are explained as ordered pairs of two rays that originate in the same point, thus as a pair of sets. Here real possibilities of simplifying reductions are given. One may be of different opinion whether one wants to start with only one sort of points instead of the different sorts "point, line, plane", whereby the the relations of collinearity and coplanarity of points replace the relations of incidence. In the lattice theoretical treatment the lines and planes are taken to be on par with points as things. Here again there is an alternative. Whereas to introduce the rays as point sets transcends in any case the scope of elementary geometry and is not necessary for it. Generally we can take as a directive that higher types should not be introduced without need. This can be avoided in the case of the definition of angle by reducing the statements about angles by statements about point triples, as was carried out by R. L. Moore.[1] An even further reduction is achieved here by explaining the congruence of

[1]R.L. Moore: "Sets of metrical hypothese for geometry," *Trans. Amer. Math. Soc.*, vol. 9 (1908), pp. 487–512. [Footnotes 1–4, 7, 8 were added later]

angles using congruence of line segments, but here again a certain loss takes place. Namely, the proofs rest substantially on the congruence of differently oriented triangles. Thus this kind of axiomatization is not suitable for the kind of problems of the Hilbertian investigations, which refer to the relationship between oriented congruence and symmetry. This remark concerns also most other axiomatizations, which turn on the concept of reflection .

Besides the general viewpoints, I want to mention as something particular a special possibility of the formulation of an elementary axiom system, namely one in which the concept "the triple of points $a$, $b$, $c$ forms a right angle at $b$" is taken as the only basic relation and the points are the only basic sort, a program which has recently been called attention to in a paper by Dana Scott.[2] The mentioned relation satisfies the necessary condition ascertained by Tarski for a single sufficient basic predicate for plane geometry. In comparison with Pieri's technique, which has become exemplary for an axiomatic of this kind, and which took an axiomatization of the relation "$b$ and $c$ have the same distance from $a$" as basic predicate, it seems to permit a simplification, inasmuch the concept of the collinearity of points is closer to that of a right angle than Pieri's basic concept. As respects the concept of congruence there seems to be no simplification for the axioms of congruence from the relation considered. By the way, this axiomatization is one of those, like the one by Pieri, which do not distinguish oriented congruence[3].

[2]Dana Scott: "A symmetric primitive notion for Euclidean geometry," *Indagationes Mathematicae*, vol. 18 (1956), pp. 456–461.

[3]Some details on the definitions of the concepts of incidence, ordering, and congruence from the concept of a right angle, as well as of part of the axiom system, follow in the appendix.

For an elementary axiomatization of geometry the special question presents itself of obtaining completeness, in the sense of categoricity. In most axiom systems this is obtained by the continuity axioms. But the introduction of these axioms involves, as is known, a transgression of the usual framework of concepts of predicates and sets. We have, however, learned from Tarski's investigations that completeness, at least in the deductive sense, can be obtained in an elementary framework, where it is noteworthy that the [Dedekind] cut axiom is preserved in a particular formalization, whereas the Archimedean axiom is omitted. The Archimedean axiom is insofar formally unusual, in that in logical formalization it has the form of an infinite disjunction, whereas the cut axiom is representable by an axiom schema, due to its general form. Thus it can be adapted in its use to the formal framework, whereby for the elementary framework of predicate logic the provability of the Archimedean axiom from the cut axiom is then lost. Of course, such a restriction to the framework of predicate logic has as a consequence that some considerations are possible only meta-theoretically, for example, the proof of the theorem that a simple closed polygon decomposes the plane, and also the considerations about equality of supplementation and decomposition of polygons. Here one is again faced with an alternative, namely whether to begin with the viewpoint of an elementary logical framework, or then again not to restrict oneself with respect to the logical framework, whereby incidentally different gradations can be considered.

With respect to the application of a second-order logic I only want to recall here that it can be made precise in the framework of axiomatic set theory, and that no noticable restriction of the methods of proof result. Also the Skolem paradox does not present a real inconvenience in the case of geometry, since it can be eliminated in the model theoretic considerations by equating the concept of set which occurs in one of the higher axioms with the concept of set of model theory.

Finally I want to emphasize that the fact, which I have stressed in my remarks, that there is no definite optimum for the systems of axiomatics, does not at all mean that the results of geometric axiomatics necessarily have an imperfect or fragmentary character. As you know, in this field a number of systems of great perfection and elegance have been achieved. The multiplicity of possible goals is responsible for the older systems not generally being simply outdated by newer ones, and at the same time every perfection attained still leaves room for further efforts.

**Appendix.** *Remarks on the task of an axiomatizing Euclidean plan geometry with a single basic relation $R(a,b,c)$: "the triple of points a, b, c forms a right angle at b."* The axiomatization succeeds as it does, in a simple way, because only the relations of collinearity and parallelism are considered. The following axioms suffice for the theory of collinearity:

A1 $\neg R(a,b,a)$

A2 $R(a,b,c) \to R(c,b,a)\ \&\ \neg R(a,c,b)$[4]

A3 $R(a,b,c)\ \&\ R(a,b,d)\ \&\ R(e,b,c) \to R(e,b,d)$

A4 $R(a,b,c)\ \&\ R(a,b,d)\ \&\ c \neq d\ \&\ R(e,c,b) \to R(e,c,d)$

[4]Already this axiom excludes elliptic geometry.

A5 $a \neq b \to (Ex)R(a,b,x)$

The definition of the relation $Coll(a,b,c)$ is added: "the points $a,s,c$ are collinear:"

**Definition 1.** $Coll(a,b,c) \leftrightarrow (x)(R(x,a,b) \to R(x,a,c)) \vee a = c.$

Then the following theorems are provable:

(1) $Coll(a,b,c) \leftrightarrow a = b \vee a = c \vee b = c \vee (Ex)(R(x,a,b)\ \&\ R(x,a,c))$

(2) $Coll(a,b,c) \to Coll(a,c,b)\ \&\ Coll(b,a,c)$

(3) $Coll(a,b,c)\ \&\ Coll(a,b,d)\ \&\ a \neq b \to Coll(b,c,d)$

(4) $R(a,b,c)\ \&\ Coll(b,c,d)\ \&\ b \neq d \to R(a,b,d)$

(5) $R(a,b,c) \to \neg Coll(a,b,c)$

(6) $R(a,b,c)\ \&\ R(a,b,d) \to Coll(b,c,d)$

(7) $R(a,b,c)\ \&\ R(a,b,d) \to \neg R(a,c,d).$
Proof: $Coll(c,d,b)\ \&\ c \neq b \to (R(a,c,d) \to R(a,c,b))$

(8) $R(a,b,c)\ \&\ R(a,b,d)\ \&\ R(a,e,c)\ \&\ R(a,e,d) \to c = d \vee b = e.$
Proof:
$Coll(b,c,d)\ \&\ Coll(e,c,d)\ \&\ c \neq d \to Coll(b,c,e)$
$Coll(b,c,e)\ \&\ b \neq e\ \&\ R(a,b,c) \to R(a,b,e)$
$Coll(e,c,b)\ \&\ b \neq e\ \&\ R(a,e,c) \to R(a,e,b)$
$R(a,b,e) \to \neg R(a,e,b).$

For the theory of parallelism, we add two further axioms:

A6 $a \neq b \;\&\; a \neq c \rightarrow$

$(Ex)(R(x,a,b) \;\&\; R(x,a,c)) \vee$

$(Ex)(R(a,x,b) \;\&\; R(a,x,c)) \vee R(a,b,c) \vee R(a,c,b)$

In plain language, the axiom says that it is possible to draw a perpendicular to a line $bc$ from a point $a$ lying off from it. The unique determination of a perpendicular depending on a point $a$ and a line $bc$ results with the help of (4) and (8).

A7 $R(a,b,c) \;\&\; R(b,c,d) \;\&\; R(c,d,a) \rightarrow R(d,a,b)$

This is a form of the Euclidean parallel axiom in the narrower, angular metrical sense.

Parallelism is now defined by:

**Definition 2.** $Par(a,b;c,d) \leftrightarrow a \neq b \;\&\; c \neq d \;\&\; (Ex)(Ey)(R(a,x,y) \;\&\; R(b,x,y) \;\&\; R(c,y,x) \;\&\; R(d,y,x))$

As provable theorems the following arise:

(9) $Par(a,b;c,d) \rightarrow Par(b,a;c,d) \;\&\; Par(c,d;a,b)$

(10) $Par(a,b;c,d) \rightarrow a \neq c \;\&\; a \neq d \;\&\; b \neq c \;\&\; b \neq d$

(11) $Par(a,b;c,d) \leftrightarrow a \neq b \;\&\; c \neq d \;\&\; (Ex)(Eu)((R(a,x,u) \vee x = a) \;\&\; (R(b,x,u) \vee x = b) \;\&\; (R(x,u,c) \vee u = c) \;\&\; (R(x,u,d) \vee u = d))$

For the proof of the implication from right to left one has to show that there are at least five different points lying on the line $a,b$, which succeeds with the help of axioms A1–A6.

(12) $Par(a,b;c,d) \rightarrow (x)((R(a,x,c) \vee x = a) \;\&\; (R(b,x,c) \vee x = b) \rightarrow R(x,c,d))$

(13) $Par(a,b;c,d) \;\&\; Coll(a,b,e) \;\&\; b \neq e \rightarrow Par(b,e;c,d)$

and thus in particular:

(14) $Par(a,b;c,d) \rightarrow \neg Coll(a,b,c)$;

moreover

(15) $Par(a,b;c,d) \;\&\; Coll(a,b,e) \rightarrow \neg Coll(c,d,e)$

(16) $\neg Coll(a,b,c) \rightarrow (Ex)Par(a,b;c,x)$

(17) $Par(a,b;c,d) \;\&\; Par(a,b;c,e) \rightarrow Coll(c,d,e)$

(18) $Par(a,b;c,d) \;\&\; Par(a,b;e,f) \rightarrow$
$Par(c,d;e,f) \vee (Coll(e,c,d) \;\&\; Coll(f,c,d))$.

The concept of vector equality is also tied up with the concept of parallelism: "$a,b$ and $c,d$ are the opposite sides of a parallelogram".

**Definition 3.** $Pag(a,b;c,d) \leftrightarrow Par(a,b;c,d) \;\&\; Par(a,c;b,d)$

Herewith one can prove:

(19) $Pag(a,b;c,d) \rightarrow Pag(c,d;a,b) \;\&\; Pag(a,c;b,d)$

(20) $Pag(a,b;c,d) \;\&\; Pag(a,b;c,e) \rightarrow d = e$

(21) $Pag(a,b;c,d) \rightarrow \neg Coll(a,b,c)$.

For the proof of the existence theorem

(22) $\neg Coll(a,b,c) \rightarrow (Ex)Pag(a,b;c,x)$

one needs a further axiom:

A8 $R(a,b,c) \rightarrow (Ex)(R(a,c,x)\ \&\ R(c,b,x))$.

It is generally provable with the help of this axiom that two different, non-parallel lines have a point of intersection:

(23) $\neg Coll(a,b,c)\ \&\ \neg Par(a,b;c,d) \rightarrow$
$(Ex)(Coll(a,b,x)\ \&\ Coll(c,d,x))$. —

It is left open whether it is possible to achieve altogether a clear axiom system using the basic concept $R$. Here we content ourself with stating definitions for the fundamental further concepts. For these it is in any case possible to attain a certain clarity.

The following two different definitions of the relation "$a$ is the center of the line segment $b,c$" are related to the figure of the parallelogram:

**Definition $4_1$.** $Mp_1(a;b,c) \leftrightarrow (Ex)(Ey)(Pag(b,x;y,c)\ \&$
$Coll(a,b,c)\ \&\ Coll(a,x,y))$

**Definition $4_2$.** $Mp_2(a;b,c) \leftrightarrow (Ex)(Ey)(Pag(x,y;a,b)\ \&\ Pag(x,y;c,a))$.

According to the second definition one can prove the possibility of doubling a line segment:

(24) $a \neq b \rightarrow (Eu)Mp_2(a;b,u)$.

The existence of the center of a line segment according to Df. $4_1$, i.e.,

(25) $b \neq c \rightarrow (Eu)Mp_1(u;b,c)$,

is provable if one adds the axiom:

A9 $Par(a, b; c, d)$ & $Par(a, c; b, d) \rightarrow \neg Par(a, d; b, c)$.
(In a parallelogram the diagonals intersect.)

By specializing the figure pertaining to the definition of $Mp_1$ we obtain the definition of the relation: "$a, b, c$ form a isosceles triangle with the peak at $a$":

**Definition $5_1$.** $Ist_1(a; b, c) \leftrightarrow (Eu)(Ev)(Pag(a, b; c, v)$ & $R(a, u, b)$ & $R(a, u, c)$ & $R(b, u, v))$.

With the help of $Mp_1$ and $Ist_1$ we can define Pieri's basic concept: "$a$ has the same distance from $b$ and $c$":

**Definition 6.** $Is_1(a; b, c) \leftrightarrow b = c \vee Mp_1(a; b, c) \vee Ist_1(a; b, c)$.

A different kind of definition of the concept $Is$ is based on the use of symmetry. The following auxiliary concept is used for this: "$a, b, c, d, e$ form a 'normal' quintuple":

**Definition 7.** $Qn(a, b, c, d, e) \leftrightarrow R(a, c, b)$ & $R(a, d, b)$ & $R(a, e, c)$ & $R(a, e, d)$ & $R(b, e, c)$ & $c \neq d$.

With the help of $Qn$ we obtain a further way of defining $Mp$ and $Ist$:

**Definition $4_3$.** $Mp_3(a; b, c) \leftrightarrow (Ex)(Ey)Qn(x, y, b, c, a)$

**Definition $5_2$.** $Ist_2(a; b, c) \leftrightarrow (Ex)(Ey)Qn(a, x, b, c, y)$,

from which $Is_2$ can be defined respectively like $Is_1$.

Moreover also the definition of the reflection of points $a, b$ with respect to a line $c, d$ follows:

**Definition 8.** $Sym(a,b;c,d) \leftrightarrow c \neq d$ &

$(Ex)(Ey)(Ez)(Coll(x,c,d)$ & $Coll(y,c,d)$ & $Qn(x,y,a,b,z))$. —

Finally, for the definition of congruence of line segments we still need the concept of oriented congruence on a line: "the line segments $a\ b$ and $c\ d$ are collinear, congruent, and oriented in the same direction":

**Definition 9$_1$.** $Lg_1(a,b;c,d) \leftrightarrow Coll(a,b,c)$ &

$(Ex)(Ey)(Pag(a,x;b,y)$ & $Pag(c,x;d,y))$,

or also:

**Definition 9$_2$.** $Lg_2(a,b;c,d) \leftrightarrow Coll(a,b,c)$ & $a \neq b$ & $(Ex)(Mp(x;b,c)$ &

$Mp(x;a,d)) \vee (a = d$ & $Mp(a;b,c)) \vee (b = c$ & $Mp(b;a,d))$,

(where any of the three definitions above can be taken for $Mp$.) Now the congruence of line segments can be defined altogether (with any of the two definitions of $Lg$):

**Definition 10.** $Kg(a,b;c,d) \leftrightarrow Lg(a,b;c,d) \vee Lg(a,b;d,c) \vee$

$(a = b$ & $Is_1(a;b,d)) \vee (Ex)(Pag(a,b;c,x)$ & $Is_1(c;x,d))$.

By a definition analogous to that of $Lg_2$ it is possible to introduce the congruence of angles with the same vertex as a six-place relation, after one has already introduced the concept of angle bisection: "$d \neq a$ lies on the bisection of the angle $b\ a\ c$":

**Definition 11.** $Wh(a,d;b,c) \leftrightarrow \neg Coll(a,b,c)$ &

$(Ex)(Ey)(Ez)(Coll(a,c,x)$ & $Coll(a,d,y)$ & $Qn(a,y,b,x,z))$.

In consideration of the composite character of this congruence relation $Kg$, one will reduce the laws about $Kg$ in the axiomatization to the concepts that occur as parts of the defining expression. Because of the variety of definitions for $Mp, Ist, Is$ there are alternatives depending on whether one employs the relations of parallelism or of symmetry more. In any case, the axiom of vector geometry

A10. $Pag(a, b; p, q)$ & $Pag(b, c; q, r) \rightarrow$
$Pag(a, c; p, r) \vee (Coll(a, c, p)$ & $Coll(a, c, r))$

or an equivalent one should be useful. On the whole one could set oneself as a goal to represent the interaction of parallelism and reflection that occurs in Euclidean plane geometry in a most symmetric way.

Finally, with respect to the betweenness relation, the form of the definition of the relation "$a$ lies between $b$ and $c$" is already contained as a part in that of $Qn$. Namely, we can define:

**Definition 12.** $Bt(a; b, c) \leftrightarrow (Ex)(R(b, a, x)$ & $R(c, a, x)$ & $R(b, x, c))$.

For this concept, at first, is provable:

(26) $\neg Bt(a; b, b)$

(27) $Bt(a; b, c) \rightarrow Bt(a; c, b)$

(28) $Bt(a; b, c) \rightarrow Coll(a, b, c)$

and also using A5, A6, and A8

(29) $a \neq b \rightarrow (Ex)Bt(x; a, b)$ & $(Ex)Bt(b; a, x)$.

To obtain further properties of the betweenness concept the following axioms can be used:

A11 $R(a,b,c)$ & $R(a,b,d)$ & $R(c,a,d)$ & $R(e,c,b) \to \neg R(b,e,d)$

A12 $R(a,b,d)$ & $R(d,b,c)$ & $a \neq c \to Bt(a;b,c) \vee Bt(b;a,c) \vee Bt(c;a,b)$

A13 $Bt(a;b,c)$ & $Bt(b;a.d) \to Bt(a;c,d)$

A14 $R(a,b,d)$ & $R(d,b,c)$ & $R(a,c,e)$ & $Bt(d;a,e) \to Bt(b;a,c)$

From this axiom it is possible to obtain the more general theorem in a few steps:

(30) $Bt(b;a,c)$ & $Coll(a,d,e)$ & $Par(b,d;c,e) \to Bt(b;a,c)$

This succeeds using the theorem

(31) $R(a,b,e)$ & $R(e,b,c)$ & $R(b,a,d)$ & $R(b,c,f)$ & $R(b,e,d)$ & $R(b,e,f)$ & $Bt(b;a,c) \to Bt(e;d,f)$.

which can be derived from the aforementioned axiom A10.

With the help of (30) and axiom A13 one can prove:

(32) $\neg Coll(a,b,c)$ & $Bt(b;a,d)$ & $Bt(e;b,c) \to$ $(Ex)(Coll(e,d,x)$ & $Bt(x;a,c))$.

i.e., Pasch's axiom in the narrower formulation of Veblen. —

In conclusion, I want to mention the following definition of $Kg$ using the concepts $Is$ and $Bt$, which is based on a construction of Euclid:

**Definition 13.** $Kg^*(a,b;c,d) \leftrightarrow (Ex)(Ey)(Ez)(Is(x,a;c)$ & $Bt(y;a,x)$ & $Bt(z;c,x)$ & $Is(a;b,y)$ & $Is(c;d,z)$ & $Is(x;y,z))$.

(For $Is$ either $Is_1$ or $Is_2$ can be taken.)

One surely can not demand from an axiomatic system like the one described here, in which the collinearity and the betweenness relation are coupled with orthogonality, that it provides a derivation of the axioms of linearity. Moreover the formulation is limited from the outset to plane geometry, since the definition of collinearity is not applicable in the multi-dimensional case. The restriction to Euclidean geometry is also introduced at an early stage. On the other hand this axiomatization may be particularly suited to showing the great simplicity and elegance of the lawfulness of Euclidean plane geometry.