Hilbert, David
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HILBERT, DAVID (1862-1943), German mathematician, was born in Königsberg and, except for a semester at Heidelberg, did his university studies there. His dissertation, presented in 1884, was on a problem in the theory of algebraic invariants, and it was to this theory that Hilbert devoted his mathematical researches until l892. Through these penetrating investigations Hilbert obtains many pregnant results, some of them (*Hilbertscher Nullstellensatz, Hilbertscher Irreduzibilitätssatz*) still know by his name. The methods he used in these investigations inaugurated a trend toward treating algebra more conceptually and have since become dominant in the field.

In 1886 Hilbert became a *Privatdozent* and in 1892 an extraordinary professor at the University of Königsberg. In 1893 he was named by the minister of culture Friedrich Althoff to succeed his teacher, Felix Lindemann, as an

ordinary professor at Königsberg. In 1895 Hilbert accepted an invitation initiated by Felix Klein, to the University of Göttingen to succeed Heinrich Weber. Hilbert remained in Göttingen, despite many offers of other chairs, notably from Leipzig in 1898, Berlin in 1902, and Heidelberg in 1904. The invitation from Berlin led to Hilbert's obtaining, through the help of Althoff, a chair at Göttingen for Hermann Minkowski, whom Hilbert had known since they were students at Königsberg. The personal intercourse between the two investigators was highly stimulating to both men but was prematurely ended, to Hilbert's grief, by Minkowski's death in 1909.

Hilbert's most important mathematical investigations were carried out between 1892 and 1909. He simplified the existing transcendence proofs for the numbers e and π. His investigations in the theory of algebraic number field in particular his monumental report "Die Theorie der algebraischen Zahlkörper" (1897), greatly amplified existing theory and directed further research in the field. His famous *Grundlagen der Geometrie* is discussed below. He showed the possibility of directly supporting the Dirichlet principle, that the existence of a conformal mapping may be inferred from the presumed existence of a minimum of a certain integral (which Bernhard Riemann had taken as the basis for his general theorems concerning conformal mappings), by means of an existence proof. This method for giving an existence proof, when worked out by Richard Courant and Hermann Weyl, proved very successful. Hilbert's contributions to the calculus of variations, in particular his statement of the *Unabhängigkeitsatz* ("independence axiom"), constituted an illuminating commentary on Adolf Kneser's textbook in the field. He continued the theory of Ivar Fredholm concerning integral equations. In particular,

he introduced the analysis of infinitely many variables and generalized the transformation to principal axes. The theory thus established has proved highly fruitful in topology and in physics, particularly in quantum mechanics. Utilizing a result of Adolf Hurwitz, Hilbert solved the Waring problem concerning the representation of natural numbers by sums of nth powers.

Hilbert's familiarity with the various domains of mathematics was impressively demonstrated by the address "Mathematische Probleme," which he presented at the Second International Congress of Mathematicians in Paris in 1900. In this address Hilbert surveyed the situation then $\|^{497}$ existing in mathematics, at the same time formulating 23 problems which have much occupied mathematicians since then. A great many of these problems have been solved in the meantime.

After Minkowski's death Hilbert turned to problems of theoretical physics. He first applied the theory of integral equations to the kinetic theory of gases and to the theory of radiation. Immediately after the appearance of Einstein's general theory of relativity, Hilbert published "Die Grundlagen der Physik" (1915-1916), which offered the first proposal of a way to unify gravitational theory and electrodynamics.

After 1916 Hilbert returned to the problems of the foundations of mathematics. These investigations led to the development of proof theory, which will be discussed below.

In his later years Hilbert gave lectures providing careful general surveys of mathematics, such as "Anschauliche Geometrie" (On intuitive geometry), as well as popular philosophical lectures. The spirit of these philosophical lectures can be seen in the speech "Naturerkennen und Logik," which he gave

at the congress of natural scientists in Königsberg in 1930. At this congress his native city named him an honorary citizen.

Hilbert's character was not that of a specialized scientist. He took pleasure in the joy of life, especially in sociability, and also took a vivid interest in political events. He enjoyed the exchange of ideas both in science and in general thought; in discussions he had a predilection for pregnant, sometimes paradoxical, formulations.

Hilbert had a great many pupils, and he was the adviser on many famous dissertations whose themes were suggested by his investigations. He had the satisfaction of seeing his work highly appreciated in his own lifetime.

The memory of Hilbert's personality is vivid in all those who knew him, and the impulses he gave to science are still effective in the research of today.

The foundations of geometry

In Hilbert's scientific work, his studies in the foundations of mathematics constitute an important part. These investigations fall into two stages separated by an interval of nearly 13 years. The first period, which extends from about 1893 to 1904, embraces Hilbert's inquiries into geometric axiomatics and is highlighted by the publication of the *Grundlagen der Geometrie* (1899), the work that made Hilbert's name familiar to a wide public of scientists and philosophers. The second period, which began with the publication in 1917 of "Axiomatisches Denken," centers on the foundations of arithmetic and the development of Hilbert's program for proof theory.

Abstract axiomatics. A main feature of Hilbert's axiomatization of geometry is that the axiomatic method is presented and practiced in the spirit of the abstract conception of axiomatics that arose at the end of the nineteenth century and which has been generally adopted in modern mathematics. It consists in abstracting from the intuitive meaning of the terms for the kinds of primitive objects (individuals) and for the fundamental relations and in understanding the assertions (theorems) of the axiomatized theory in a hypothetical sense, that is, as holding true for any interpretation or determination of the kinds of individuals and of the fundamental relations for which the axioms are satisfied. Thus, an axiom system is regarded not as a system of statements about a subject matter but as a system of conditions for what might be called a relational structure. Such a relational structure is taken as the immediate object of the axiomatic theory; its application to a kind of intuitive object or to a domain of natural science is to be made by means of an interpretation of the individuals and relations in accordance with which the axioms are found to be satisfied.

This conception of axiomatics, of which Hilbert was one of the first advocates (and certainly the most influential), has its roots in Euclid's *Elements*, in which logical reasoning on the basis of axioms is used not merely as a means of assisting intuition in the study of spatial figures; rather, logical dependencies are considered for their own sake, and it is insisted that in reasoning we should rely only on those properties of a figure that either are explicitly assumed or follow logically from the assumptions and axioms. This program was not strictly adhered to in all parts of the Elements, nor could it have been, for its system of axioms was not sufficient for the purpose. The

first axiom system meeting the requirements of the program was given by Moritz Pasch in his *Vorlesungen über neuere Geometrie* (Leipzig, 1882).

This abstract kind of axiomatics, which consists in separating out the purely mathematical aspects of a theory, is not the only possible one. Hilbert himself knew that it can be applied advantageously only in domains of science whose theoretical development is sufficiently advanced. But abstract axiomatics is useful wherever the logical dependence or independence of theoretical assumptions is under investigation.

The distinguishing property of Hilbert's axiomatics is frequently described by saying that in it the terms for the kinds of elements (points, straight lines) and for the relations (incidence, betweenness, congruence) are *implicitly defined* by the axioms. This expression, first introduced in 1818 by J. D. Gergonne (Hilbert did not employ it), is often used in a misleading way. The axioms generally impose conditions on the relations and on the kinds of elements of the system; some of these conditions are partial characterizations of the relations or the kinds of elements, others characterize the space with respect to the elements and relations. The entire axiom system—as Hilbert observed in a letter to Gottlob Frege—can be regarded as a single definition. But this is an *explicit* definition of a term denoting the relational structure in question. The defined concept is a predicate of the second type (*zweiter Stufe*, as Frege called it), applying to domains of things and to certain relations between them.

Non-Archimedean systems. Another main feature of Hilbert's *Grundlagen der Geometrie* is the development of geometry, and, in particular, of

plane geometry, independently of the Archimedean axiom. This axiom states that given any two line segments, either may be exceeded by an entire multiple of the other. Thus, it partly compensates for the absence of general commensurability of line segments. It was with the help of this axiom that the theory of $\parallel^{498}$ proportions was established in Book V of Euclid's *Elements* (attributed to Eudoxus). It is also a consequence of this axiom that, once a unit segment is chosen, there corresponds to any line segment a real number (in Richard Dedekind's sense of the term) that is its measure (ratio number); therefore, Hilbert also called the Archimedean axiom the axiom of measurement.

Recourse to the Archimedean axiom introduced an arithmetical element into reasoning, and hence avoidance of it in geometrical proofs amounts to an emancipation from a nongeometrical type of reasoning. The avoidance of nongeometrical reasoning does not preclude an analytic geometry. In fact, Hilbert was able to construct a calculus of line segments, independent of the Archimedean axiom, in two different ways.

One method operates within the framework of metric plane geometry. It is based on the axioms of incidence (for the plane), those of order, those of congruence, and the parallel axiom. Hilbert defines the sum of segments in the usual way and the product of segments, after establishing a unit segment, by a parallel construction; he then shows that by these definitions the usual computation laws for sum and product are satisfied.

By this segment calculus an elementary foundation of the theory of proportions and thereby also of analytic geometry is obtained. Hilbert further showed how with the aid of the segment calculus the theory of the areas of

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polygons can be set up without supposing, as is assumed in Euclid, that to any polygon there corresponds its area as a quantity, i.e., in agreement with Euclid's axioms of quantities. Thus he showed that no accessory reliance on intuition is required for the theory of areas of polygons.

Hilbert conjectured that the theory of the volumes of polyhedrons is not fully analogous to the theory of the areas of polygons. He posed the problem of showing that tetrahedrons of equal volume cannot always be obtained from one another by a series of processes of pairwise additions and subtractions of congruent polyhedrons, a problem solved by Max Dehn ("Über raumgleiche Polyeder" and "Über den Rauminhalt"). Various investigations have derived from this problem.

Hilbert's second calculus of line segments independent of the Archimedean axiom is for *affine* geometry of the plane. A difficulty here is that the axioms of plane affine geometry do not suffice for the foundation of this geometry. The same holds for plane projective geometry.

Hans Wiener stated at the Naturforscherversammlung in Halle (1891) that it is impossible to give autonomous foundations to both plane projective geometry and plane affine geometry by adjoining to the axioms of incidence the Desargues theorem and a specialized form of the Pascal theorem on conic sections (with the conic section degenerated to a pair of straight lines). Hilbert was impressed by these statements and gave a proof of them for affine geometry by means of a calculus of segments. Here sum and product of segments are defined by elementary parallel constructions, and, with the aid of the Desargues theorem, the computation laws, with the exception of the commutative law for the product, are proved to be satisfied. These proofs

were simplified by Arnold Schmidt in the seventh edition of the *Grundlagen*.

This calculus of segments leads to an analytic geometry over a skewfield—as it is called today—for the plane. This geometry can be extended, as Hilbert showed, to an analytic geometry of three-dimensional space satisfying the incidence axioms and the parallel axiom for the space. This is the extent of the role of the Desargues theorem. The specialized Pascal theorem is needed to prove that the segment calculus satisfies the commutative law for multiplication. This law, as Hilbert showed, can be inferred from the other computation laws and the laws of order with the aid of the Archimedean axiom, but not without it. (Gerhard Hessenberg proved, somewhat later, that the Desargues theorem is a consequence of the specialized Pascal theorem.)

Hilbert's positive treatment of the Archimedean axiom and, in particular, the question of its independence complemented his elimination of it from the foundations of geometry, The possibility of a non-Archimedean geometry was first considered in detail by Giuseppe Veronese in his *Fondamenti di Geometria* (Padua, 1891). This possibility can be inferred, by the methods of analytic geometry, from the existence of a (generalized) number system for which the operations of sum and product and their inverses, as well as the operation $\sqrt{1 + a^2}$ and the relation "smaller than," can be defined in such a way that the familiar computation laws, but not the Archimedean axiom, are satisfied.

Hilbert gave as an instance of such a non-Archimedean system a system whose elements are algebraic functions of an argument t. But the instance he presented in "Über den Satz von der Gleichheit der Basiswinkel im gleich-schenkligen Dreieck" is easier to operate with. (This essay is one of a series of

studies closely connected with the *Grundlagen* and added to it as appendixes in the second and later editions; this essay forms Appendix II.) It deals with the possibility of restricting, in plane geometry, the last congruence axiom concerning triangles to the case of triangles assigned to one another in equal orientation. The effect of this restriction is to admit as congruences only those transformations obtained by translations and plane rotations, thus excluding symmetry from the notion of congruence. Two kinds of questions arise, those concerning the anomalies which can occur in a model of the restricted axiom system and those relating to the ways of compensating for the weakening of the triangle congruence axiom. Many anomalies are stated by Hilbert to occur in two models which he ingeniously constructed. Concerning different methods of compensating for the restriction of the triangle congruence axiom, see Supplement V2 of the ninth edition of the *Grundlagen* (pp. 264-268) and the literature mentioned there in the footnote on p. 265.

Characterization of the plane. As Blumenthal's biography indicates, Hilbert was led to the problems of Appendix II by investigations (reprinted in Appendix IV) in which he gave a very different foundation for plane geometry from that presented in the main part of the book. The problem here is to characterize the plane by means of the properties of the manifold of congruent motions. It was first treated by Hermann von Helmholtz and soon after by Sophus Lie, who emphasized its group-theoretic aspects. Both Helmholtz and Lie proceeded by the methods of the differential calculus. Hilbert sought to avoid any assumption $\|^{499}$ concerning differentiability. His arguments in Appendix IV are within the framework of the theory of point sets. They rely

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especially on Camille Jordan's theorem concerning simply closed continuous curves (Jordan curves) in the "number plane," which generalizes the theorem on the decomposition of the plane by a simple polygon. Hilbert starts from a characterization of the geometric plane as a two-dimensional manifold by means of the concept of neighborhoods introduced in an axiomatic way—this is now a familiar method in topology, but at that time it was scarcely known at all.

Two characterizations of the "plane" are offered. According to the narrower definition the plane is topologically equivalent to a connected domain in the number plane; according to the wider definition it is locally equivalent (homeomorphic) to the interior of a Jordan curve and is globally connected. Hilbert chose the narrower characterization for simplicity.

The concept of continuous transformation can be defined by means of the representation of the geometric plane in the number plane. The motions are then taken as special continuous one-to-one transformations of the geometric plane onto itself such that in the representation each Jordan curve preserves its orientation. This provisional characterization of the geometric plane is then completed by three axioms on motions: (1) The motions constitute a group with respect to their composition; (2) given two different points, A and B, there are infinitely many points into which B can be transformed by a motion keeping A fixed; (3) if A, B, C and A', B', C' are triples of points in the geometric plane (the members of a triple not necessarily being different) and if in an arbitrary proximity of A, B, C there exist triples P, Q, R and in an arbitrary proximity of A', B', C' there exist triples P', Q', R' such that P, Q, R is transformed by a motion into F', Q', R', then A, B, C is transformed by a

motion into A', B', C'.

In a valuable discussion that made use of set-theoretic, topological, and group-theoretic arguments, Hilbert proved that from these axioms, with the obvious definition of congruence by means of the concept of motion and a suitable set-theoretic definition of straight line, it follows that the geometric plane under consideration satisfies the axioms of plane geometry as stated in the main part of the *Grundlagen*, with two exceptions: (1) the triangle congruence axiom is obtained only in the restricted form relating to motions, and (2) the parallel axiom does not result. Two possibilities then remain: the plane satisfies either Euclidean geometry or Bolyai-Lobachevski geometry.

Hilbert's handling of these problems disclosed a new direction of investigation, which is still being pursued. His results have been extended in three ways: (1) by weakening the topological assumptions through the adoption of the wider characterization, mentioned above, of a two-dimensional manifold, (2) by generalizing the discussion to higher dimensions, and (3) by modifying the axioms on the motions. (See the surveys of these researches in the introduction to Freudenthal's "Neuere Fassung des Riemann-Helmholz-Lieschen Raumproblems" and his "Im Urkreis der sogenannten Raumprobleme.")

Continuity. A final aspect of Hilbert's axiomatization of geometry in the *Grundlagen* is his treatment of continuity. The Archimedean axiom is stated as an axiom of continuity, yet it excludes only a particular kind of discontinuity. In fact, if this axiom alone is added to the Hilbert axioms of incidence, order, and congruence (including the parallel axiom), then the axiom system is satisfied by an analytic geometry constructed over a *restricted* number sys-

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tem consisting only of algebraic numbers and not including the square root of each positive number.

In this respect Hilbert's axioms differ from those of Euclid's *Elements*. Euclid explicitly postulated the construction of a circle around a given point with a given radius (and implicitly made assumptions about the intersection of circles and of circles with straight lines). However, in order to realize by constructions the existence statements of Hilbert's axioms, it is sufficient to have, in addition to a ruler, not a compass but an "*Eichmass*"—that is, an instrument for determining a given distance on a given straight line from a given point in a prescribed direction. Hilbert showed, in Chapter 7 of the *Grundlagen*, that the *Eichmass* and the ruler allow us to perform all the constructions corresponding to the existence axioms.

Chapter 7 also discusses the question of the analytical representation of the constructions with ruler and *Eichmass*. It turns out that the ratio numbers of line segments constructible from a given unit length with ruler and *Eichmass* are the real numbers obtainable by the elementary arithmetical operations together with the operation $\sqrt{1 + c^2}$. This domain of numbers is narrower than that obtained when the operation $\sqrt{1 + c^2}$ is replaced by that of extracting the square root of an arbitrary positive number. The latter domain is the one composed of the ratio numbers of the lengths constructible by ruler and compass, but by no means does it contain all algebraic numbers. Yet, whereas the set of all algebraic numbers is denumerable, the set of all ratio numbers has a higher infinity. Hence, in order to characterize the geometric continuum a further axiom is required. It then becomes apparent that geometric continuity is related to continuity in the theory of real numbers.

When Hilbert wrote the *Grundlagen* the question of conceptually formulating the continuity property of an ordered set had been settled by the Dedekind axiom of *Lückenlosigkeit* and its equivalent, the principle of the least upper bound. For a metrical set, each of these axioms implies the Archimedean property.

Completeness. In direct connection with his work on the foundations of geometry, Hilbert undertook an axiomatization of theory of real numbers. In the paper "Über den Zahlbegriff" (published in 1900 and reprinted as Appendix VI of the *Grundlagen*), he presents an axiom system characterizing the system of real numbers as an ordered Archimedean field that cannot be extended to a wider ordered Archimedean field. He thus replaced the continuity axiom by (1) the Archimedean axiom and (2) a condition of maximality which he called the axiom of completeness.

Hilbert introduced into geometry a corresponding axiom of completeness (which first appears in the second edition of the *Grundlagen*) stating that the space characterized by the axiom system including the axiom of completeness constitutes a maximal (that is, not extensible) model of the other axioms. The connection between the geometrical and $\|^{500}$ the arithmetical completeness axiom is given by the circumstance that any model of the axioms of incidence, order, and congruence and of the parallel and the Archimedean axiom can be represented by an analytic geometry over an ordered Archimedean number field, which again is isomorphic with respect to sum, product, and order to a subfield of the field of all real numbers.

The statement of the completeness axiom is very suggestive, and it was

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with Hilbert's introduction of this axiom that the notion of a maximal model was first conceived. Yet, because of its reference to other axioms, the completeness axiom offers difficulties, particularly with respect to questions of independence. The possibility of decomposing the full continuity axiom into the Archimedean axiom and another axiom which does not entail it is given by Cantor's continuity axiom. (See Federigo Enriques, "Prinzipien der Geometrie," and Richard Baldus, "Zur Axiomatik der Geometrie III: Über das Archimedische und das Cantorsche Axiom.")

Consistency. In "Über den Zahlbegriff" Hilbert recommended substituting an axiomatic presentation of the theory of real numbers for the "genetic" method of treating them. Despite the great pedagogical value of the genetic method, he said, the axiomatic method is to be preferred for the definitive formulation and logical precision of the theory.

This point of view has decisive consequences for the problem of consistency. Hilbert proved the consistency of the geometrical axiom system by using the arithmetical model provided by analytic geometry. But if arithmetic is set up as an axiomatic theory, then Hilbert's proof establishes only a relative consistency. This, of course, is a valuable result, since the structure described by the axioms for the arithmetical continuum is much simpler than that of Euclidean space. The reduction to arithmetic, how ever, cannot then be regarded as a kind of direct verification by intuitive evidence, for the task of proving the consistency of the axiomatic theory of real numbers remains. This problem was one of those Hilbert posed in "Mathematische Probleme" (*Gesammelte Abhandlungen*, Vol. III, pp. 290-329).

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At that time Hilbert thought that a suitable modification of the methods of Dedekind and Weierstrass in the theory of irrational numbers would suffice to obtain the desired proof of consistency. Not long after, however, in the address "Über die Grundlagen der Logik und der Arithmetik" to the Heidelberg Congress of Mathematicians, Hilbert presented an essentially altered view. This alteration was no doubt brought about through the discovery by Russell and Zermelo of very significant forms of the logical paradoxes which gave a more fundamental aspect to the difficulties that Cantor had earlier found with respect to "inconsistent sets." These difficulties showed that in set theory we cannot in general assign to a predicate P "the set of all those things for which P holds" as an object belonging to the universe of discourse.

Hilbert stated that these paradoxes seemed to show that the views and methods of logic "conceived in the traditional sense" ("*im hergebrachten Sinne aufgefasst*") are not equal to the strong requirements of set theory. And, although he strongly opposed Leopold Kronecker's tendency to restrict mathematical methods, he nevertheless admitted that Kronecker's criticism of the usual way of dealing with the infinite was partly justified.

The resulting point of view was not yet explicitly developed in Hilbert's Heidelberg address. However, Hilbert presented there the following programmatic ideas: (1) One must include in the arithmetical theory whose consistency is to be demonstrated the methods of logical reasoning used in the theory; (2) the methods of symbolic logic for representing mathematical sentences by formulas are to be applied; (3) the sequences of formulas representing mathematical proofs can be made the object of intuitive elementary reasoning regarding their structural properties and relations, and in

this way proofs of consistency can be carried out. Various devices for proving consistency were also exhibited.

Hilbert's investigations of the foundations of arithmetic remained in this provisional state for a long time. During the interval major developments took place in the foundations of mathematics and in mathematical logic. Zermelo proved the well-ordering theorem and published his axiom system for set theory in 1908. Two years later the first volume of Russell and Whitehead's *Principia Mathematica* appeared. Julius König attempted to carry out Hilbert's plan, but his work was interrupted by his premature death and appeared only in fragmentary form, edited by his son, in 1914 (*Neue Grundlagen der Logik, Arithmetik und Mengenlehre*, Leipzig, 1914). In this work some steps of the later Hilbert proof theory are already carried out, but Hilbert did not know of it when he again took up his investigation of the foundations of arithmetic.

Proof theory

Hilbert's return to the problem of the foundations of arithmetic was announced by his delivery at Zurich in 1917 of the lecture "Axiomatisches Denken." In the latter part of this lecture he pointed out several epistemological questions which, as he said, are connected with that of the consistency of number theory and set theory: the problem of the solubility in principle of every mathematical question; that of finding a standard of simplicity for mathematical proofs; that of the relation of contents and formalism in mathematics; and that of the decidability of a mathematical question by a finite

procedure. Questions of this kind, he observed, seem to constitute a domain that should be investigated, and to carry out this investigation it will be necessary to inquire into the concept of mathematical proof. The general idea and the aims of proof theory were thus proclaimed, but the means of investigation were not thereby fixed, for indeed the theory was not to rely on the current mathematical methods.

At the time of his Zurich lecture Hilbert tended to restrict the methods of proof-theoretic reasoning to the most primitive evidence. The apparent needs of proof theory induced him to adopt successively those suppositions which constitute what he then called the "*finite Einstellung.*"

Consistency. In his first publication on proof theory "Neubegründung der Mathematik, Erste Mitteilung," Hilbert explains how number theory can be treated in a $\|^{501}$ finitist way, whereas mathematics in general transcends finitist methods. But, Hilbert argues, we can regain an elementary kind of mathematical objectivity by formalizing the statements and proofs, using the methods of symbolic logic, and by taking the representing formulas and proofs directly as objects. In the same paper Hilbert also gives indications of the nature of formalization and presents an instance of a proof of consistency—as yet for only a very restricted system.

A more advanced stage is reached in Hilbert's lecture at the Leipzig congress of the Deutsche Naturforscher Gesellschaft in 1922, "Die logische Grundlagen der Mathematik."

In this speech the method is presented of dealing in proof theory with the logical forms of generality and existence (quantifiers) by means of a logical

choice function which assigns to any predicate A an object τA for which A holds only if it is generally satisfied. This idea is formally expressed by the "transfinite axiom," $A(\tau A) \rightarrow A(a)$, in which a predicate expression can be substituted for A and any term representing an individual can be substituted for a. A slight modification, soon applied, replaced the function τA by the function ϵA, dual to it, whose axiom is $A(a) \rightarrow A(\epsilon A)$.

By means of the choice function the quantifiers can be eliminated from a formalized proof in such a way that the rules for the use of "all" and "exists" are reduced to applications of the transfinite axiom, so that the explicit logical structure of the proof becomes transformed into an elementary one, consisting only in applications of the propositional calculus and substitutions.

The task of proving the consistency of a formalized domain of arithmetic is thus essentially reduced. This task—in virtue of the law "ex falso quodlibet"—amounts to showing that the formula $0 \neq 0$ cannot be derived in the domain; in other words, to showing that in any formal derivation of the formalized domain having a numerical end formula, this end formula differs from the formula $0 \neq 0$. Consideration of formalized proofs can now be restricted to those obtained by the transformation using the function ϵA. The main problem is then to eliminate the formulas resulting from the transfinite axiom by substitution (the "critical formulas").

The method that Hilbert indicates for attacking this problem consists—after first removing the free variables, which is possible—of a sequence of steps. In each step the terms that occur are replaced by numerical values. Then, either all critical formulas turn into true numerical formulas, and the attempted elimination is effected, or the result of the step determines a next

step. It must still be shown that the process has an end, and this, at least in the simple cases, can be seen to hold.

This method is not in principle restricted to cases where the predicates to which the logical choice function applies are number predicates and where the individuals are therefore natural numbers; it can also be used for individuals of higher types. The particular case in which number functions are taken as individuals is essential to the formalization of the theory of real numbers. In the Leipzig lecture, Hilbert gave several indications of how this formalization can be performed; in particular, he showed how some form of the Zermelo choice principle (used in the theory of functions of real numbers) can be derived from the transfinite axiom related to the type of real numbers (as individuals).

Thus, it seemed that carrying out proof theory was only a question of mathematical technique. Such an expectation, however, turned out to be illusory. An indication was that the first substantial consistency proof following Hilbert's scheme of reasoning by Wilhelm Ackermann (in his thesis, "Begründung des 'tertium non datur' mittels der Hilbertschen Theorie der Widerspruchsfreiheit") required an essential restriction of the formal system not envisaged in the original plan. Similarly, in John von Neumann's inquiry "Zur Hilbertschen Beweistheorie," where a formal system for the logic of first and second order (including the first four Peano axioms) was set up and a consistency proof using Hilbert's method was given, the consistency proof did not apply to the full system but excluded the comprehension axiom, which provides the manipulation of substitutions for variables of second type. Thus, two highly able investigators did not succeed in obtaining a con-

sistency proof for a formal system of the theory of real numbers by means of the above-mentioned Hilbert method (connected with the logical choice function) of eliminating the critical formulas.

A second method of eliminating the critical formulas, devised by Hilbert and elaborated by Ackermann, yields the proof of a general theorem which states that any axiomatic system, formalized within the frame of standard logic (that is, propositional logic and the rules governing quantifiers), whose axioms have a finitist interpretation is consistent (see Hilbert and Bernays, *Grundlagen der Mathematik*, Vol. II, Sec. 1, esp. pp. 18-38). The method is one of the easiest for proving an important theorem of mathematical logic (first stated by Jacques Herbrand in his doctoral dissertation) which yields a kind of normal form for derivations in pure logic and which also can be applied to decision problems. But this method is not sufficient to demonstrate the consistency of the proper formal system of number theory and therefore is the less sufficient for the systems of infinitesimal analysis.

Completeness. Ackermann revised and simplified the proof presented in his thesis. It was thought that by this modified proof and by that of von Neumann the consistency of formalized number theory, at least, had been proved. Such was the situation when Hilbert presented, at the International Congress of Mathematicians in Bologna in 1928, his "Probleme der Grundlegung der Mathematik." To the problem of proving consistency he here added two problems of completeness: the problem of showing that every universally valid logical schema is derivable by the rules of the predicate calculus and the problem of showing the completeness of formalized number theory, in the

sense that the formal system of number theory contains no formula which, together with its negation, can be shown to be underivable in the system.

Gödel's results. Kurt Gödel soon took up both these problems of completeness, but he stated completeness only for the case of the predicate calculus (first-order functional calculus), whereas he proved the incompleteness of formalized number theory even in the strong sense that ∥[502] no strictly formal system is possible in which each true number-theoretic proposition is derivable. At the same time Gödel proved a theorem from which it follows that a finitist proof of consistency for a formal system strong enough to formalize all finitist reasonings is impossible ("Über formale unentscheidbare Sätze der Principia Mathematica und verwandte Systeme I"). Von Neumann was convinced that this last condition holds for the formal system of num ber theory, and hence he inferred that Gödel's result implies the impossibility of a finitist consistency proof not only for the broader systems discussed by Gödel but even for the formal system of number theory.

To corroborate this inference he was able to show that in the proof of consistency of the formal system of number theory by the elimination of critical formulas, the demonstration that the process of elimination has an end did not apply in full generality (see Hilbert and Bernays, *Grundlagen*, Vol. II, pp. 123-125). It thus became clear that in two respects Hilbert's program demanded more than can be fulfilled: mathematical theories cannot be formalized with full adequacy, and consistency proofs cannot be strictly finitist in the essential cases.

Broadening of proof theory.   It soon became apparent that proof theory could be fruitfully developed without fully keeping to the original program. It was discovered that a proof of consistency for the formal system of number theory, although not a finitist one, is possible by methods of proof admitted by L. E. J. Brouwer's intuitionism.

Arend Heyting, in two papers of 1930, set up a formal system of intuitionistic number theory. And, as Gödel and Gerhard Gentzen independently observed, there is a relatively simple method of showing that any contradiction derivable in the formal system of classical number theory would entail a contradiction in Heyting's system. Hence, from the consistency of Heyting's system the consistency of the classical system follows (Kurt Gödel, "Zur intuitionistischen Arithmetik und Zahlentheorie"—Gentzen withdrew his own paper, already in print, because of the appearance of Gödel's paper).

In this way it appeared that intuitionistic reasoning is not identical with finitist reasoning, contrary to the prevailing views at that time. In particular, intuitionistic reasoning deals with concepts not admitted as methods in finitist proofs, such as the quite general concept of consequence when it is not delimited by any rules of proof. It thus became apparent that the "*finite Standpunkt*" is not the only alternative to classical ways of reasoning and is not necessarily implied by the idea of proof theory. An enlarging of the methods of proof theory was therefore suggested: instead of a restriction to finitist methods of reasoning, it was required only that the arguments be of a constructive character, allowing us to deal with more general forms of inference.

By this modification of the program, various proofs of consistency for the

formal system of number theory were obtained, the first by Gentzen ("Die Widerspruchsfreiheit der reinen Zahlentheorie," "Die gegenwärtige Lage in der mathematischen Grundlagenforschung," and "Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie"). Ackermann was then able to complete the consistency proof proceeding by the method of eliminating the critical formulas ("Zur Widerspruchsfreiheit der Zahlentheorie"). The broadened methods also permitted a loosening of the requirements of formalizing. One step in this direction, made by Hilbert himself, was to replace the schema of complete induction by the stronger rule later called infinite induction ("Die Grundlegung der elementaren Zahlenlehre" and "Beweis des Tertium non datur").

However, going beyond finitist methods is not generally required in proof theory; many important results have been obtained by finitist methods, results concerning the following topics: pure logic, the combinatorial calculus. completeness (the completeness of a system of real algebra), undecidability, and relative consistency.

One main result concerning relative consistency is connected with Hilbert's attempt at a positive solution of Cantor's continuum problem in the paper "Über das Unendliche." The sketch of a proof given in this work contains many detailed arguments, and it stimulated the study of recursive definitions. As a whole, however, the sketch was fragmentary, and there were in principle hindrances to its completion. Twelve years later Gödel connected the ideas of Hilbert's paper with the concepts of axiomatic set theory and proved the consistency of Cantor's continuum hypothesis in its generalized form on the assumption that axiomatic set theory (without the axiom of choice) is con-

sistent. (Nevertheless, this result, which is obtained by a powerful method of proof, does not settle the continuum problem. In fact, from results obtained by Paul Cohen it appears that axiomatic set theory, at least in its formal delimitation, leaves this problem fully undecided.)

On the whole, Hilbert's idea of making mathematical proof an object of mathematical research by means of formalization has proved to be very fruitful. And although Hilbert's work in the foundations of arithmetic has not had the effect he sought, "to remove once and for all the questions of foundations in mathematics" ("die Grundlagenfragen in der Mathematik als solche endgültig aus der Welt zu schaffen——"Die Grundlagen der Mathematik," p. 65, and "Die Grundlagen der elementaren Zahlenlehre," p. 489), he did establish proof theory as a valuable domain of mathematical investigation, and thus Hilbert was a pioneer in the newer mathematical foundation theory, as he was in many other fields of mathematics.

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