Bernays Project: Text No. 2

On Hilbert's thoughts concerning the grounding of arithmetic (1922)

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Comments:

Hilbert's new methodological approach for the grounding of arithmetic, which I would Re to address, presents a modified and more definite version of the plan that Hilbert already had in mind for a long time and to which he first gave expression in his Heidelberg lecture. A sharply outlined and comprehensible programme, the beginnings of which have already been carried out, has now succeeded the previous quite obscure suggestions.

The problem whose solution we are seeking here is that of the proof of the consistency of arithmetic. First we have to bring to mind how one arrives at the formulation of this problem.

The construction of arithmetic (in the wider sense, i.e., encompassing analysis and set theory), as it has been proceeding since the introduction of the rigorous methods, is *axiomatical.* This means that, analogously to the axiomatic grounding of geometry, one begins by assuming a *system of objects* with determinate relational properties [*Verknüpfungseigenschaften*]. In Dedekind's grounding of analysis what is taken as a basis is the system of the elements of the continuum, and in Zermelo's construction of set theory it is the domain of operations **B**. And also in that grounding of analysis that proceeds from the consideration of numerical sequences, the number series is conceived of as a closed, surveyable system, perhaps akin to an infinite keyboard.

In the assumption of such a system with determinate relational properties there lies something *transcendent*, as it were, for mathematics, and there the question arises as to which fundamental position one should take in this regard.

An appeal to an intuitive grasp of the number series as well as to the manifold of magnitudes is certainly to be considered. But this could not be a question of an intuition in the primitive sense, for certainly no infinite manifolds are given to us in the primitive intuitive mode of representation. And even though it might be quite rash to contest any farther-reaching kind of intuitive evidence from the outset, we will nevertheless make allowance for that tendency of exact science that aims to |[*Mancosu*: 216] eliminate the finer organs of cognition [*Organe der Erkenntnis*] as far as possible, and to rely only on the most primitive means of cognition.

According to this viewpoint we will examine whether it is possible to ground those transcendent assumptions in such a way that only *primitive intuitive cognitions come into play.* On account of this restriction of the means of cognition we cannot, on the other hand, demand of this grounding that it allow us to recognize as truths (in the philosophical sense) the assumptions that are to be grounded. Rather, we will be content if we succeed in proving the arithmetic built on those assumptions to be a possible (i.e., consistent) system of thought.

We have hereby already arrived at Hilbert's formulation of the problem. But before we took at the way in which the problem must be tackled, we must first ask ourselves whether there is not a different and perhaps more natural sort of attitude towards the transcendent assumptions.

In fact two different kinds of attempts suggest themselves and have also been undertaken. The first attempt also aims at a demonstration of consistency, not by the means of primitive intuition, but rather with the help of logic. One will recall that the consistency of Euclidian geometry was already proved by Hilbert by the method of reduction to arithmetic. Thus it now also seems appropriate to prove the consistency of arithmetic by reduction to *logic*.

Especially Frege and Russell vigorously attacked the problem of the logical grounding of arithmetic.

As regards the actual aim, the result was negative.

First of all the famous *paradoxes of set theory* showed that no greater certainty of operating whatsoever was achieved by going back to logic. The contradictions of naive set theory could be seen [*ließen sich wenden*] logically as well as set theoretically. And even the control of inferences through the logical calculus, which had been constructed precisely for securing the mathematical inferences, did not help to avoid the contradictions.

When Russell then introduced the very cautious procedure of the calculus of types, it turned out that analysis and set theory in their usual form could not be obtained in this way. And thus Russell and Whitehead, in *Principia Mathematica*, saw themselves forced to introduce an assumption about the system of predicates ”of the first type,” the so-called ”*axiom of reducibility*.” But hereby one again returned to the axiomatic standpoint and gave up the goal of the logical grounding.

Incidentally, the difficulty already appears within the theory of whole numbers. Indeed, by defining the Numbers [*Anzahlen*] logically according to Frege's fundamental idea, one here succeeds in proving the computational laws of addition and multiplication as well as the determinate numerical equations as theorems of logic. However, through this procedure one does not obtain the usual theory of numbers, for one cannot prove that for every number there exists a larger one, unless one expressly introduces some sort of axiom of infinity.

Even though the development of mathematical logic did not in principle lead beyond the axiomatic standpoint, an impressive systematic construction of arithmetic as a whole, equal in rank to the system of Zermelo, has nonetheless emerged in this way. |[Mancosu: 217] Moreover, symbolic logic has taken us further in methodological knowledge: While previously one only justified the *assumptions* of the mathematical theories, now the *inferences* are specified as well. And it turns out that one can replace mathematical inference—insofar as it is only a matter of the results proceeding from it—by a purely formal manipulation according to determinate rules in which actual thinking is completely eliminated.

However, as was already said, mathematical logic does not achieve the goal of a logical grounding of arithmetic. And it is not to be assumed that the reason for this failure lies in the particular form of Frege's approach. It seems rather to be the case that the problem of reducing mathematics to logic is in general wrongly posed, namely, because mathematics and logic do not really stand to each other in the relationship of particular to general.

Mathematics and logic are based on two different directions of abstraction. V;While logic deals with the *contentually* most general [*das inhaltlich Allgemeinste*], (pure) mathematics is the general theory of the *formal* relations and properties, and so on the one hand each mathematical reflection is subject to the laws of logic, and on the other hand every logical construct of thought falls into the domain of mathematical reflection on account of the outer structure that is necessarily inherent in it.

In view of this situation, one is impelled to attempt an investigation that is, in a certain way, opposed to the logical grounding of arithmetic. Because we are unsuccessful in proving the mathematically transcendent basic assumptions as logically necessary, we then ask ourselves whether these assumptions cannot in fact be dispensed with.

Indeed, a possibility for the elimination of the axiomatic basic assumptions seems to consist in removing, without exception, the existential form of the axioms and putting *construction postulates* in place of the existential assumptions.

Such a replacement procedure is not new to the mathematician; especially in elementary geometry the constructive formulation of the axioms is often applied. For example, instead of laying down the axiom that any two points determine a straight line, one postulates as a possible construction the connection of two points by a straight line. One can also proceed in the same way with the arithmetical axioms. For example, instead of saying "each numbers has a successor," one introduces progression [*Fortschreiten*] by one, or the affixing of +1, as a basic operation.

One thus arrives at the attempt of a *purely constructive development* [*rein konstruktiver Aufbau*] of arithmetic. And indeed the goal for mathematical thought is a very tempting one: Pure mathematics ought to construct its own edifice and not be dependent on the assumption of a certain system of things.

This constructive tendency, which was first brought to bear very forcefully by Kronecker, and later by Poincaré in a less radical form, is currently pursued by Brouwer and Weyl in their new grounding of arithmetic.

Weyl first checks the higher modes of inference in regard to the possibility of a constructive reinterpretation; that is, he investigates the procedures of analysis, as well as those of Zermelo's set theory, as to whether or not they can be interpreted as constructive. He finds that this is not possible, for in the attempt to carry out a |[*Mancosu*: 218] thoroughgoing replacement of the existential axioms by constructive methods, one falls into logical circles at every turn.

Thus Weyl draws the conclusion that the modes of inference of analysis and set theory have to be restricted to the extent that no logical circles come about in their constructive interpretation. In particular, he feels compelled to give up the theorem of the existence of the upper bound.

Brouwer goes even further in this direction by also applying the constructive principle to large numbers. If one wants, as Brouwer does, to avoid the assumption of a closed given totality of all numbers and takes as a foundation only the unlimited performable act of progressing by one, then statements of the form "There are numbers of such and such a type..." do not necessarily have a meaning. And thus one is also not in general justified in putting forward, for every number theoretical statement, the alternative that either it holds for all numbers or that there is a number (respectively, a pair of numbers, a triple of numbers,...) by which it is refuted. This sort of application of the "tertium non datur" is then at least questionable.

Thus we find ourselves in a great predicament:: The most successful, most elegant, and most established modes of inference ought to be abandoned just because, from a specific standpoint, one has no grounds for them.

The considerations through which Weyl tries to show that the concept of the mathematical continuum, which lies at the basis of usual analysis, does not correspond to the pictorial [*bildhaft*] representation of continuity, also does not help us get over the unsatisfactoriness of such a procedure. For an exact analogy to the content of perception is not at all necessary for the applicability and the fruitfulness of analysis; rather, it is perfectly sufficient that the method of idealization and conceptual interpolation used therein be consistently practicable. Concerning the question of pure mathematics, what matters is only whether the usual, axiomatically characterized mathematical continuum is in itself a possible, that is, a consistent, structure [*Gebilde*].

This question could only be rejected if there was at our disposal a simpler and clearer conceptual structure that would supersede the current mathematical continuum. But if one examines more closely the new approaches by Weyl and Brouwer, one notices that a gain in simplicity cannot be hoped for here; rather, the complications required in the concepts and forms of inference are only increased instead of decreased.

There is thus no justification in rejecting the question of consistency of the usual axiom system of arithmetic. And what we are to draw from Weyl's and Brouwer's investigations is the result that a consistency proof is not possible by way of replacing existential axioms by construction postulates.

Hereby we come back to Hilbert's idea of a theory of consistency based on a primitive-intuitive foundation. And I would now Re to describe the plan, according to which Hilbert conceives of the construction of such a theory, and the guiding principles he follows to this end.

Hilbert adopts what is positively fruitful in the two foundational attempts discussed above. From the logical theory he takes the method of the rigorous formalization of inference. The necessity of this formalization follows directly from the formulation of the problem. For the mathematical proofs are to be made the object |*Mancosu*: 219 of a concrete-intuitive form of consideration. To this end it is, ho,;@ever, necessary that they are projected, as it were, into the domain of the formal. Accordingly, in Hilbert's theory we have to distinguish sharply between the formal image [*Abbild*] of the arithmetical statements and proofs as *object* of the theory, on the one hand, and the contentual thought about this formalism, as *content* of the theory, on the other hand. The formalization is done in such a way that formulas take the place of contentual mathematical statements, and a sequence of formulas, following each other according to certain rules, takes the place of an inference. And indeed no meaning is attached to the formulas; the formula does not count as the expression of a thought, but it corresponds to a contentual judgment only insofar as it plays, within the formalism, a role analogous to that which the judgment plays within the contentual consideration.

More basic than this connection to symbolic logic is the contiguity of Hilbert's approach to the constructive theories of Weyl and Brouwer. For Hilbert in no way wants to abandon the constructive tendency that aims at the self-reliance of mathematics; rather, he is especially eager to bring it to bear in the strongest way. In light of what we stated with respect to the constructive method, this appears at first to be incompatible with the goal of a consistency proof for arithmetic. In fact, however, the obstacle to the unification of both goals lies only in a preconceived opinion from which the advocates of the constructive tendency have until now always proceeded, namely, that within the domain of arithmetic every construction must indeed be a *number construction* (set construction, respectively). Hilbert considers this view to be a prejudice. A constructive reinterpretation of die existential axioms is possible not only in such a way that one transforms them into generating principles for the construction of numbers, but the inference rule made possible by such an axiom can be replaced as a whole by a formal procedure in a such a way that determinate signs stand for general concepts such as number, function, etc.

Whenever concepts are missing, a sign is introduced at the right moment. This is the methodological principle of Hilbert's theory. An example should explain what is meant. The existence axiom "each number has a successor" holds in number theory. In keeping with the restriction to the concretely intuitive, the question is now to avoid the general concept of number as well as the existential form of the statement.

The usual constructive reinterpretation in this case consists (as already remarked) in replacing the existential axiom by the procedure of progression by one. This is a procedure of *number construction.* Hilbert, on the other hand, replaces the concept of number by the symbol Z and puts forward the formula:

$$Z(a) \to Z(a+1)$$

Here a is a variable for which any mathematical expression can be substituted, and the sign $\to$ represents the hypothetical propositional connective "if-then," that is, the rule "if two formulas **A** and **A** $\to$ **B** are written down, then **B** can also be written down," holds.

On the basis of this stipulation, the mentioned formula accomplishes, within the framework of the formalism, exactly what the existence axiom accomplishes in the contentual inference.

|*Mancosu*: 220 Here we see how Hilbert utilizes the method of formalization of inferences keeping with the constructive tendency; in no way does it constitute for him merely a tool for the consistency proof. This method at the same time also provides the approach to a *rigorous constructive development* [*streng konstruktiver Aufbau*] of arithmetic. And indeed the methodological idea of construction is here so broadly conceived, that all higher mathematical modes of inference can also be included in the constructive development.

After having characterized the aim of Hilbert's theory, I would now like to describe the main features of the structure of the theory. The following three questions are to be answered:

1. The constructive development must represent the formal image [*Ab-*

bild] of the system of arithmetic and at the same time must constitute the object for the intuitive theory of consistency. How does such a development take shape?

2. How is the consistency statement to be formulated?

3. What are the means of contentual consideration through which the consistency proof is to be carried out?

First, the constructive development takes place in the following way. with the different kinds of signs are introduced, and thereby the substitution are layed down. Furthermore, certain formulas will be put forward as las. And the question is now that of forming "proofs."

What counts as a proof is a concretely written-down sequence of formulas in which for every formula occurring in the sequence the following holds: Either the formula is identical with a basic formula, or it is identical with a preceding formula, or results from such a formula by a valid substitution; or, alternatively, it constitutes the end formula in an "inference," that is, in a sequence of formulas of the type

$$\frac{\begin{array}{c}\mathbf{A}\\ \mathbf{A}\rightarrow\mathbf{B}\end{array}}{\mathbf{B}}$$

Hence a "proof" is nothing else than a figure with determinate concrete properties and the formal image [*Abbild*] of arithmetic consists of such figures.

This answer to the first question makes the urgency of the second quite evident. For what should the statement of consistency express within the pure formalism? Is it not the case that mere formulas cannot contradict themselves?

The simple reply goes as follows: The contradiction is simply formalized as well. Faithful to his principle Hilbert introduces the letter Ω for the contradiction; and the role of this letter within the formalism is determined by putting forward basic formulas so that from any two formulas to which contrary statements correspond, Ω can be deduced. More precisely, by adding two such formulas to the basic formulas, a proof can be constructed with Ω as the end formula.

Specifically the following basic formula

$$a = b \to (a \neq b \to \Omega)$$

|[Mancosu: 221] where $\neq$ is the usual sign of inequality, serves us here. (The relation of inequality is taken by Hilbert as a genuine arithmetical relation, just as equality is, and by no means as the logical negation of equality. Hilbert does not introduce a sign for negation at all.)

The statement of consistency is now simply formulated as follows: Ω cannot be obtained as the end formula of a proof.

It is then necessary to provide a proof for this claim.

Now the only question still remaining concerns the means by which this proof should be carried out. In principle this question is already decided. For our whole problem originates from the demand of taking only the concretely intuitive as a basis for mathematical considerations. Thus the matter is simply to realize which tools are at our disposal in the context of the concrete-intuitive mode of reflection.

This much is certain: We are justified in using the elementary ideas of sequence and ordering, as well as the us" counting, to the full extent. (For example, we can determine whether there are three occurrences of the sign $\rightarrow$ in a formula or fewer.)

However, we cannot get by in this way alone; rather, it is absolutely necessary to apply certain forms of complete induction. Yet, by doing so we still do not go beyond the domain of the concretely intuitive.

In this regard, two types of complete induction are to be distinguished: the narrower form of induction, which relates only to something completely and concretely given, and the wider form of induction, which uses either the general concept of whole number or the operating with variables in an essential manner.

Whereas the wider form of complete induction is a higher form of inference whose justification constitutes one of the tasks of Hilbert's theory, the narrower form of inference belongs to the primitive intuitive mode of cognition and can therefore be applied as a tool of contentual inference.

As typical examples of the narrower form of complete induction, as they are used in the argumentations of Hilbert's theory, let us adduce the following two inferences:

1. If the sign $+$ occurs at all in a concretely given proof, then in reading the proof one finds a place where it occurs for the first time.

2. If one has a general procedure for eliminating from a proof with a certain concretely describable property **E** the first occurrence of the sign Z, without the proof losing the property **E** in the process, then

> one can, by repeated application of the procedure, completely remove the sign Z from such a proof, without its losing the property **E**.

(Notice that here it is exclusively a question of formal proofs, i.e., proofs in the sense of the definition given above.)

The method the theory of consistency must follow is hereby set forth in its essentials. The development of this theory is currently still in its beginnings; most of it still has to be carried out. Certainly the basic possibility and the feasibility of the modes of reflections demanded can already be recognized from what has been said so far; and one also sees that the considerations to be employed here are *mathematical* in a very genuine sense.

The great advantage of Hilbert's procedure rests precisely on the fact that the problems and difficulties that present themselves in the grounding of mathematics |*Mancosu*: 222 are transferred from the epistemologico-philosophical domain into the domain of what is properly mathematical.

Mathematics here creates a court of arbitration for itself, before which all fundamental questions can be settled in a specifically mathematical way, without having to rack one's brain about subtle logical dilemmas [*Gewissensfragen*] such as whether judgments of a certain form have a meaning or not.

Therefore, it is also to be expected that the enterprise of Hilbert's new theory will soon find resonance as well as participation in the circles of mathematicians.

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