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1. The Problem of consistency in axiomatics as a decision problem.

The situation in the field of foundations of mathematics, to which our presentation is related, is characterized by three kinds of investigations:

1. the development of the axiomatic method, especially in foundations of geometry,

2. the founding of analysis by today's rigorous methods through the reduction of mathematics [*Grössenlehre*] to the theory of numbers and sets of numbers,

3. investigations in the foundations of number theory and set theory.

A deeper set of problems, related to the situation reached through these investigations, arises when methods are subjected to stricter demands; these problems involve a new kind of analysis of the problem of the infinite. We will introduce these problems by considering axiomatics.

The term 'axiomatic' is used in a wider and a narrower sense. We call the development of a theory axiomatic in the widest sense of the word when the fundamental concepts and presuppositions are put at the beginning and marked as such and the further content of the theory is logically derived from these with the help of definitions and proofs. In this sense the geometry of Euclid, the mechanics of Newton, and the thermodynamics of Clausius were axiomatically founded.

The axiomatic point of view was made more rigorous in Hilbert's *Foundations of Geometry*; The greater rigor consists in the fact that in the axiomatic development of a theory one makes use of only that portion of the representational subject matter [*sachlichen Vorstellungsmaterial*] from which the fundamental concepts of the theory are formed which is formulated in the axioms; one abstracts from all remaining content. Another factor in axiomatics in the narrowest sense is the *existential form*. This factor serves to distinguish the axiomatic method from the constructive or genetic method of founding a theory. (compare Hilbert's "Über den Zahlenbegriff") In the constructive method the objects of a theory are introduced merely as a *species* of things (Brouwer and his school use the word "species" in this sense.); but in an axiomatic theory one is concerned with a fixed system of things (or several such systems) which constitutes a previously *delimited domain of subjects* for all predicates from which the assertions of the theory are constructed.

Except in the trivial cases in which a theory has to do just with a fixed finite totality of things, the presupposition of such a totality, the so-called "domain of individuals", involves an idealizing assumption over and above those formulated in the axioms.

It is a characteristic of the more rigorous kind of axiomatics involving abstraction from material content and also the existential form ("formal axiomatics" for short) that it requires a *proof of consistency*; but contentual axiomatics introduces its fundamental concepts by reference to known experiences and its basic assertions either as obvious facts which a person can make clear to himself or as extracts from complicated experiences;[1] expressing the belief that man is on the track of laws of nature and at the same time intending to support this belief through the success of the theory.

However, formal axiomatics also needs a certain amount of evidence in the performance of deductions as well as in the proof of consistency; there is, however, an essential difference: the evidence required does not depend on any special epistemological relation to the material being axiomatized, but rather it is one and the same for every axiomatization, and it is that primitive kind of knowledge which is the precondition of every exact theoretical investigation whatsoever. We will consider this kind of evidence more closely.

The following points of view are especially important for a correct evaluation of the significance for epistemology of the relationship between contentual and formal axiomatics:

Formal axiomatics requires contentual axiomatics as a supplement, because only in terms of this supplement can one give instruction in the choice

[1] **Alternative translation:** to known experiences and either presents its basic assertions as obvious facts which a person can make clear to himself or formulates them as extracts from complicated experiences,

of formalisms and, in the case of a particular formal theory, give an indication of its applicability to some part of reality.

On the other hand we cannot just stay at the level of contentual axiomatics, since in science we are almost always concerned with theories which get their significance from a *simplifying idealization* of an actual state of affairs rather than from a complete reproduction of it. A theory of this kind does not get a foundation through a reference to either the evident truth of its axioms or to experience; rather such a foundation is given only when the idealization involved, i.e. the extrapolation through which the concepts and fundamental assertions of the theory come to overstep the bounds of intuitive evidence and of experience, is seen to be free of inconsistency. Reference to the approximate correctness of the fundamental assertions is of no use for the recognition of consistency; for an inconsistency could arise just because a relationship which holds only in a restricted sense is taken to hold exactly.

We must then investigate the consistency of theoretical systems without considering matters of fact and, therefore, from the point of view of formal axiomatics.

The treatment of this problem up until now, both in the case of geometry and of branches of physics, involved *arithmetizing*: one represents the objects of a theory through numbers and systems of numbers and basic relations through equalities and inequalities thereby producing a translation of the axioms of the theory; under this translation the axioms become either arithmetic identities or provable arithmetic assertions (as in the case of geometry) or (as in physics) a system of conditions the simultaneous satisfiability of which can be proved on the basis of certain existence assertions of arithmetic. This procedure presupposes the correctness of arithmetic, i.e. the theory of real numbers (analysis); so we must ask what this correctness amounts to.

However, before we concern ourselves with this question we want to see whether there isn't a direct way of attacking the problem of consistency. We want to get the structure of this problem clearly before our minds once and for all. At the same time we want to. be able to familiarize ourselves with *logical symbolism*, which proves to be very useful for our purposes and which we will have to consider more deeply in the sequel.

As an example of axiomatics we take the *geometry of the plane*; for the sake of simplicity we will consider only the parallel axiom plus the axioms for the geometry of position (the axioms which are presented as "axioms of connection" [*Verknüpfung*] and "axioms of order" [*Anordnung*] in Hilbert's *Grundlagen der Geometrie*. For our purpose it is better to diverge from Hilbert's axiom system by not taking points and lines as two basic systems of things but rather to *take only points as individuals*. Instead of the relation "points x and y determine the line g" we use the relation between *three* points "x, y and z lie on one line" ('$Gxyz$' in symbols). A second fundamental relation is betweenness: "x lies between y and z", in symbols '$Zxyz$'. (See Oswald Veblen "A system of axioms for geometry" for an especially thorough axiomatization in which only points are taken as individuals and in which all geometrical relations are defined in term of betweenness.) In addition the logical concept of the identity of x and y, for which we use the usual identity sign '$x = y$', is represented in the axioms.

...... **(There follows a description of the symbolism of first order**

logic.)

The presentation of the axioms does not correspond completely to that in Hilbert's *Grundlagen der Geometrie*. We therefore give in connection with each group of axioms the relationship of the axioms here presented as formulas to those of Hilbert. (We do this especially for those familiar with Hilbert's *Grundlagen der Geometrie*. All references are to the seventh edition.)

I. Axioms of connection

1. $(x)(y)\ Gxxy$.

   "x, x, y always lie on one line."

2. $(x)(y)(z)\ (Gxyz \rightarrow Gyxz\ \&\ Gxzy)$.

   "If x, y, z lie on a line, then so do y, x, z as well as x, z, y lie on a line."

3. $(x)(y)(z)(u)\ (Gxyz\ \&\ Gxyu\ \&\ x \neq y \rightarrow Gxzu)$.

   "If x, y are different points and if x, y, z as well as x, y, u lie on a line then also x, z, u lie on a line."

4. $(\exists x)(\exists y)(\exists z)\ \neg Gxyz$.

   "There are points x, y, z which do not lie on a line."

Of these axioms, 1) and 2) replace the axioms I.1,—because of the changed concept of line; 3) corresponds to the axiom I.2; and 4) corresponds to the second part of I.3.

II. Axioms of order

1. $(x)(y)(z)\ (Zxyz \rightarrow Gxyz)$

2. $(x)(y)\ \neg Zxyy$

3. $(x)(y)(z)\ (Zxyz \rightarrow Zxzy\ \&\ \neg Zyxz)$

4. $(x)(y)\ (x \neq y \rightarrow (\exists z)Zxyz)$
   "If x and y are different points, there is always a point z such that x lies between y and z."

5. $(x)(y)(z)(u)(v)$
   $(\neg Gxyz\ \&\ Zuxy\ \&\ \neg Gvxy\ \&\ \neg Gzuv \rightarrow (\exists w)(Guvw\ \&\ (Zwxz \vee Zwyz)))$

1) and 2) together constitute the first part of Hilbert's axioms II.1; 3) unites the last part of Hilbert's axioms II.1 with II.3; 4) is the axiom II.2; and 5) is the axiom of plane order II.4.

III. Parallel axiom

Since we are not including congruence axioms, we must take the parallel axiom in the following broader sense: "For every straight line there is exactly one line through a point outside it which does not intersect it." (Compare page 83 of Hilbert's *Grundlagen der Geometrie*.)

To make symbolic formulation easier the symbol

$$Pxyuv$$

will be used as an abbreviation for the expression

$$\neg(\exists w)(Gxyw\ \&\ Guvw)$$

"There is no point which lies on a line both with x and y and with u and v."

The axioms is then

$$(x)(y)(z)\ (\neg Gxyz \rightarrow (\exists u)(Pxyzu\ \&\ (v)(Pxyzv \rightarrow Gzuv))).$$

If we go through the axioms here enumerated and unite them, we get a single logical formula which represents an assertion about the predicates 'G', 'Z' and which we designate as

$$\mathfrak{A}(G, Z).$$

In the same way we could represent a theorem of plane geometry involving only position and order relations as a formula

$$\mathfrak{S}(G, Z).$$

This representation still accords with contentual axiomatics in which the fundamental relations are viewed as definite in content because referable to something in experience or in intuition, something about which the statements of the theory make assertions.

On the other hand, in formal axiomatics the fundamental relations are not conceived as previously determined in content; rather they receive their determination implicitly through the axioms; and in any consideration of an axiomatic theory only what is expressly formulated in the axioms is about the fundamental relations is used.

As a result, if the names used for fundamental relations in axiomatic geometry, coincide with those of intuitive geometry, e.g. "lie on" or "between", this is only a concession to custom and a means of simplifying the connection of the theory with intuitive facts. In fact, however, in formal axiomatics the fundamental relations play the role of *variable* predicates.

Here and in the sequel we understand "predicate" in the wider sense so that it applies to predicates with two or more subjects. We speak of "one-place", "two-place",... predicates, using the expressions to indicate the number of subjects.

In the part of axiomatic geometry considered by us there are two variable three-place predicates:

$$Rxyz, \quad Sxyz.$$

The axiom system consists of a stipulation on two such predicates expressed in the logical formula $\mathfrak{A}(R,S)$, which we get from $\mathfrak{A}(G,Z)$ when we replace $Gxyz$ with $Rxyz$, $Zxyz$ with $Sxyz$. The identity relation $x = y$ which appears in this formula along with the variable predicates.is to be interpreted contentually. The acceptance of this predicate as contentually determinate is no violation of our methodological standpoint. For the contentual determination of identity (which is no relation at all in the true sense) does not depend on the particular conceptual content of the field being investigated axiomatically; rather it is only a question of distinguishing individuals which must be taken as already given when the domain of individuals is laid down.

From this point of view an assertion of the form $\mathfrak{S}(G,Z)$ corresponds to the logical statement that for *any* predicates $Rxyzv, Sxyz$ satisfying the stipulation $\mathfrak{A}(R,S)$ the relation $\mathfrak{S}(R,S)$ also holds; In other words, for any two predicates $Rxyz, Sxyz$ the formula

$$\mathfrak{A}(R,S) \to \mathfrak{S}(R,S)$$

represents a true assertion. In this way a geometrical sentence is transformed into a sentence of pure predicate logic.

From this point of view the problem of consistency presents itself in a corresponding way as a problem of sure predicate logic. In fact it is a question of whether two three-place predicates $Rxyz, Sxyz$ can satisfy the conditions expressed in the formula $\mathfrak{A}(R, S)$ or whether, on the contrary the assumption that the formula $\mathfrak{A}(R, S)$ is satisfied for a certain pair of predicates leads to a contradiction so that in general for every pair of predicates R, S the formula $\neg\mathfrak{A}(R, S)$ represents a correct assertion. (This imprecise way of putting the question will be sharpened in the sequel.)

Such questions as these are part of the "*decision problem*". In modern logic this problem is understood to be that of discovering general methods for deciding the "validity" or "satisfiability" of logical formulas. (This explanation is correct only for the decision problem in its narrower sense. We have no need here to consider the broader conception of this decision problem.)

In this connection the formulas investigated are composed of predicate variables and equalities together with variables in subject positions which we call "individual variables"; It is assumed that every variable is bound by a universal or existential quantifier.

A formula of this kind is called valid when it represents a true assertion for *every* determination of the variable predicates; it is called satisfiable when it represents a true assertion for some appropriate determination of the predicate variables.

...... **Examples are given.**

It is to be noticed that along with the determination of the predicates a *domain of individuals* over which the variables $x, y, \ldots$ range has to be fixed. This enters into a logical formula as a kind of *hidden variable.* However, the satisfiability of a logical formula is invariant with respect to a one-one mapping of a domain of individuals onto another, since the individuals enter into the formula only as variable subjects; as a result the only essential determination for a domain of individuals is the *number of individuals*.

Accordingly, we have to distinguish the following questions in relation to validity and satisfiability:

1. The question of validity for *every* domain of individuals (and also of satisfiability for *any* domain of individuals).

2. The question of validity or satisfiability for a given number of individuals.

3. The question for which numbers of individuals is a formula valid or satisfiable.

It should be noted that it is best to leave out of consideration the domain of 0 individuals. Formally this domain has a rather special position, and consideration of it is trivial and worthless for applications. (The stipulation that every domain of individuals should contain at least one thing so that a true generalization must hold of at least one thing ought not to be confused with the convention prominent in Aristotelean logic that a judgment of the form "all S are P" counts as true only if there are in fact things with the property S. This convention has been dropped in modern logic. A judgment of this kind is represented symbolically in the form $(x)(Sx \rightarrow Px)$; It counts as true if a thing x, insofar as it has the property Sx, always has the property Px as well—independently of whether there is anything with the property Sx at all…)

Furthermore one should notice that only the "course of values" of a predicate is relevant to its determination; that is to say, all that is relevant is for which values of the variables in subject positions the predicate holds or does not hold (is "true" or "false").

This circumstance has as a consequence that for a *given finite* number of individuals the validity or satisfiability of a specific logical formula represents a pure *combinatorial fact* which one can determine through elementary testing of all cases.

To be specific, if n is the number of individuals and k the number of subjects ("places") of a predicate, then n^k is the number of different systems of values for the variables; and since for every one of these systems of values the predicate is either true or false, there are

$$2^{(n^k)}$$

different possible courses of value for a k-place predicate.

If then

$$R_1, \ldots, R_t$$

are the distinct predicate variables occurring in a given formula, and

$$k_1, \ldots, k_t$$

the number of their places, then

$$2^{(n^{k_1}+n^{k_2}+\ldots+n^{k_t})}$$

is the number of systems of values to be considered, the number of different possible predicate systems for short.

Accordingly validity of the formula means that for all of these

$$2^{(n^{k_1}+n^{k_2}+\ldots+n^{k_t})}$$

explicitly enumerable predicate systems the formula represents a true assertion; its satisfiability means that the formula represents a true assertion for one of these predicate systems. Moreover, for a fixed predicate system the truth or falsity of the assertion represented by the formula is decidable by a finite testing of cases; the reason is that only n values come into consideration for a variable bound by a universal or existential quantifier so that 'all' signifies the same thing as a conjunction with n members and 'there is' a disjunction with n members.

For example, consider the formulas

$$(x)\ Pxx \rightarrow (x)(\exists y)Pxy$$

$$(x)(y)\ (Pxy\ \&\ Pyx \rightarrow x = y)$$

of which the first is valid the second satisfiable. We refer these formulas to a domain of two individuals.

We can indicate these individuals with the numerals 1, 2. In this case we have $t = 1, n = 2, k_1 = 2$; therefore the number of different predicate systems is

$$2^{(2^2)} = 2^4 = 16.$$

In place of $(x)Pxx$ we can put Pll & $P22$, and in place of $(x)(\exists y)\ Pxy$

$$(P11 \vee P12)\ \&\ (P21 \vee P22),$$

so that the first of the two formulas becomes

$$P11\ \&\ P22 \rightarrow (P11 \vee P12)\ \&\ (P21 \vee P22).$$

This conditional is true for those predicates P for which $P11$ & $P22$ is false, as well as for those for which

$$(P11 \vee P12) \;\&\; (P21 \vee P22)$$

is true. One can now verify that for each of the 16 courses of value that one gets when one assigns one of the truth values "true" or "false" to each of the pairs of values

$$(1,1),(1,2),(2,1),(2,2)$$

one of the two conditions is fulfilled; thus the whole expression always receives the value "true" (Verification is simplified in this example because determination of the values of $P11$ and $P22$ suffices to fix the correctness of the expression.) In this way the validity of our first formula for domains of two individuals can be determined through a direct test [*Ausprobieren*].

For domains of two individuals the second formula signifies the same thing as the conjunction

$$(P11 \;\&\; P11 \rightarrow 1 = 1) \;\&\; (P22 \;\&\; P22 \rightarrow 2 = 2) \;\&$$

$$(P12 \;\&\; P21 \rightarrow 1 = 2) \;\&\; (P21 \;\&\; P12 \rightarrow 2 = 1).$$

Since $1 = 1$ and $2 = 2$ are true the first two members of the conjunction are always true assertions. The last two members are true when and only when $P12$ & $P21$ is false.

Therefore, to satisfy the formula under consideration one has only to eliminate those determinations of value for P in which the pairs $(1,2)$ and $(2,1)$ are both assigned the value "true". Every other determination of value produces a true assertion. The formula is therefore satisfiable in a domain of two elements.

These examples should make clear the purely combinatorial character of the decision problem in the case of a given finite number of individuals. one result of this combinatorial character is that for a prescribed finite number of individuals the validity of a formula F has the same significance as the unsatisfiability of the formula $\neg F$; likewise the satisfiability of of a formula P signifies the same thing as that $\neg P$ is not valid. Indeed P represents a true assertion for those predicate systems for which $\neg F$ represents a false assertion and vice-versa.

We now return to the question of the consistency of an axiom system. We consider an axiom system presented symbolically and combined into one formula like our example.

The question of the satisfiability of this formula for a prescribed finite number of individuals can be decided, in principle at least, through testing. Suppose then the satisfiability of the formula is determined for a definite finite number of individuals. The result is a proof of the consistency of the axiom system; indeed it is a proof by the *method of exhibition* since the finite domain of individuals together with the courses of value chosen for the predicates to satisfy the formula constitutes a model in which we can show concretely that the axioms are satisfied.

We give an example of such an exhibition from axiomatic geometry. We begin from the previously presented axiom system but replace the axiom I 4), which postulates three points not lying on a line, with the weaker axiom

I 4') $(\exists x)(\exists y)\ x \neq y.$

"There are two distinct points."

Furthermore we drop the axiom of plane order II 5); In its place we add to the axioms two sentences which can be proved using II 5); First we expand II 4) to

II 4') $(x)(y)\ (x \neq y \rightarrow (\exists z)Zzxy\ \&\ (\exists z)Zxyz)$,

and then add

II 5) $(x)(y)(z)(x \neq y\ \&\ x \neq z\ \&\ y \neq z \rightarrow Zxyz \vee Zyzx \vee Zzxy)$

We keep the parallel axiom.

(Both of these sentences were introduced as axioms in earlier editions of Hilbert's *Grundlagen der Geometrie*. It turns out that they are provable using the axioms of plane order. See pp. 5, 6 of the seventh edition.) The resulting axiom system corresponds to a formula $\mathfrak{A}'(R, S)$ instead of the earlier $\mathfrak{A}(R, S)$; it is satisfiable in a domain of 5 individuals, as Veblen remarked (in "A system of axioms for geometry", p. 350). First of all the courses of value for the predicates R, S is so chosen that the predicate G is determined to be true for every value triple x, y, z. (We can here use the symbols 'G', 'Z' with no danger of misunderstanding.) One sees immediately that all axioms I as well as II 1) and III are satisfied. In order that the axioms II 2), 3), $5'$), and $4'$) be satisfied it is necessary and sufficient that the following three conditions be placed on the predicate Z:

1. Z is always false for a triple $x.y, z$ in which two elements coincide.

2. For any combination of three of the 5 individuals there are 6 possible orderings of the elements; Z is true for 2 of the 6 orderings with a common first element, false for the remaining 4 orderings.

3. . Each pair of distinct elements occurs as an initial as well as a final pair in one of the triples for which Z is true.

The first condition can be directly fulfilled by stipulation. The simultaneous satisfaction of the other two conditions is accomplished as follows: We designate the 5 elements with the numerals 1, 2, 3, 4, 5. The number of value-triples of three distinct elements for which Z still has to be defined is $5 * 4 * 3 - 60$. Six of these belong to a given combination; for two of these 3 should be true and false for the rest. We must therefore indicate those 20 of the 60 triples for which Z is to be defined as true. They are those which one obtains from the four triples

$$(1, 2, 5), (1, 5, 2), (1, 3, 4), (1, 4, 3)$$

by applying the cyclical permutation (1 2 3 4 5).

It is easy to verify that this procedure satisfies all the conditions. Thus the axiom system is recognized as consistent by the method of exhibition. (It follows immediately from the fact that the modified axiom system $\mathfrak{A}'$ is satisfiable in a domain of 5 individuals that the axioms of this system do not completely determine linear ordering.)

The method of exhibition presented in this example has very many different applications in modern axiomatic investigations. It is especially useful for *proofs of independence.* The claim that a sentence $\mathfrak{S}$ is independent of an axiom system $\mathfrak{A}$ signifies the same thing as the claim that the axiom system

$$\mathfrak{A} \; \& \; \neg\mathfrak{S}$$

which we get when we add the negation of the sentence to the axioms of $\mathfrak{A}$, is consistent. A consistency proof by the method of exhibition can be given if this axiom system is satisfiable in a finite domain. (A number of examples of this procedure can be found in the works on linear and cyclical order by E.V. Huntington and his associates. See especially "A new set of postulates for betweenness with proof of complete independence". Here one also finds references to previous works.) Thus this method provides a sufficient extension of the method of deduction for many fundamental investigations in the sense that the unprovability of a sentence from certain axioms can be proved through exhibition, its provability through deduction.

But is the application of the method of exhibition restricted to finite domains of individuals? This does not follow from what we have said up until now. However, we do see that in the case of infinite domains of individuals the possible systems of predicates no longer constitute a surveyable multitude and there can be no talk of testing all courses of value. Nevertheless in the case of given axioms we might be in a position to show their satisfiability by given predicates. And this is actually the case. Consider for example the system of three axioms

$$(x)\ \neg Rxx,$$

$$(x)(y)(z)\ (Rxy\ \&\ Ryz \rightarrow Rxz),$$

$$(x)(\exists y)\ Rxy.$$

Let us clarify what these say: We start with an object a in the domain of individuals. According to the third axiom there must be a thing b for which Rab is true; and because of the first axiom b must be different from a. For b there must further be a thing c for which Rbc is true, and because of the second axiom Rac is also true; according to the third axiom c is distinct from a and b. For c there must again be a thing d for which Rcd is true. For this thing Rad and Rbd are also true, and d is distinct from a, b, c. This process involved here has no end; and it shows us we cannot satisfy the axioms in a finite domain of individuals. On the other hand we can easily show satisfaction in an infinite domain of individuals: We take the integers as individuals and substitute the relation "x is less than y" for Rxy; one sees immediately that all three axioms are satisfied.

It is the same with the axioms

$$(\exists x)(y)\ \neg Syx,$$

$$(x)(y)(u)(v)\ (Sxu\ \&\ Syu\ \&\ Svx \rightarrow Svy),$$

$$(x)(\exists y)\ Sxy.$$

One can easily ascertain that these cannot be satisfied in a finite domain of individuals. On the other hand they are satisfied in the domain of positive integers if we replace Sxy with the relation "y immediately follows x".

However, we notice in these examples that exhibiting in these cases does not conclusively settle the question of consistency; rather the question is *reduced* to that of the *consistency of number theory*. In the earlier examples of finite exhibition we took integers as individuals. There, however, this was only for the purpose of having a simple way to designate individuals. Instead of numbers we could have taken other things, letters for example. And also the properties of numbers which were used could have been established by a concrete proof.

In the cases now before us, however, we cannot produce a concrete representation of the numbers; for we make essential use of the assumption that the *integers* constitute *a domain of individuals* and therefore a complete totality.

We are, of course, quite familiar with this assumption since in modern mathematics we are constantly working with it; one is inclined to consider it self-evident. It was Frege who with a sharp and vigorous critique first established that the recreation of the sequence of integers as a complete totality must be justified by a proof of consistency. (*Grundlagen der Arithmetik* and *Grundgesetze der Arithmetik*) According to Frege such a proof, in the sense of an exhibition, could be carried out only as a proof of existence; and he believed the objects for such an exhibition were to be found in the domain of logic. His exhibition-procedure amounts to defining the totality of integers with the help of the totality (presupposed to exist) of all conceivable one-place predicates. However, this basic assumption, which under impartial consideration seems very suspect anyway, was shown to be untenable by the well-known logical and set-theoretic paradoxes discovered by Russell and Zermelo. And the failure of Frege's undertaking has made us even more conscious of the problematic character of assuming the totality of the sequence of integers than did his dialectical critique.

In the light of this difficulty we might try to use some other infinite domain of individuals instead of the sequence of integers for the purpose of proving consistency; a domain taken from the realm of sense perception or physical reality rather than being a pure product of thought like the sequence of integers. However, if we look more closely we will realize that wherever we think we encounter infinite manifolds in the realm of sensible qualities or in physical reality there can be no question of the actual presence of such a manifold; rather the conviction that such a manifold is present rests on a mental extrapolation, the justification of which is as much in need of investigation as the conception of the sequence of integers.

A typical example in this connection are those cases of the infinite which gave rise to the well-known paradoxes of Zeno. Suppose some distance is traversed in a finite time; the traversal includes infinitely many successive subprocesses the traversal of the first half, then of the next quarter, then the next eighth, and so on. If we are considering an actual motion, then these subtraversals must be genuine processes succeeding one another.

People have tried to refute these paradoxes with the argument that the sum of infinitely many temporal intervals may converge producing a finite duration. However, this reply does not come to grips with an essential point of the paradox, namely the paradox that lies in the fact that an infinite succession, the completion of which we could not accomplish in the imagination either actually or in principle, should be accomplished in reality.

Actually there is much more radical solution of the paradox. It consists in pointing out that it is by no means necessary to believe that the mathematical space-time representation of movement remains physically meaningful for arbitrarily small segments of space and time; rather there is every reason to assume that a mathematical model extrapolates the facts of a certain domain of experience, e.g. movement, below the orders of magnitude accessible to our observation for the purpose of a simplified conceptual structure; for example, continuum mechanics involves an extrapolation in that it is based on the conception of space as filled with matter; it is no more the case that unbounded division of a movement always produces something characterizable as movement than that unbounded spatial division of water always produces water. When this is accepted the paradoxes vanish./conceptual

Its *idealizing structure* notwithstanding, the mathematical model of movement has a permanent value for the purpose of simplified representation. For this purpose it must not only coincide approximately with reality but also the extrapolation it involves must be consistent. From this point of view the mathematical conception of movement is not in the least shaken by Zeno's paradoxes; the mathematical counterargument just referred to has in this case complete cogency. It is another question however, whether we possess a real proof of the consistency of the mathematical theory of motion. This theory depends essentially on the mathematical theory of the continuum; this in turn depend essentially on the conceptualization of the set of all integers as a complete totality. We therefore come back by a roundabout.way to the problem we tried to avoid by referring to the facts about motion.

It is much the same in every case in which a person thinks he can show directly that some infinity is given in experience or intuition for example the infinity of the tone row extending from octave to octave to infinity, or the continuous infinite manifold involved in the passage from one color to another. Closer consideration shows that nothing infinite is given to us in these cases at all; rather it is interpolated or extrapolated through some mental process.

These considerations make us realize that reference to non-mathematical objects can not settle the question whether an infinite manifold exists; the question must be solved within mathematics itself. But how should one begin an such a solution? At first glance it seems that something impossible is being demanded here: to produce infinitely many individuals is impossible in principle; therefore an infinite domain of individuals as such can only be indicated through its structure through relations holding among its elements. In other words: a proof must be given that for this domain certain formal relations can be satisfied. The existence of an infinite domain of individuals can *not be represented in any other way than through the satisfiability of certain logical formulas*; but these are exactly the kind of formulas we were led to investigate in asking about the existence of an infinite domain of individuals; and the satisfiability of these formulas vas to have been demonstrated by the exhibition of an infinite domain of individuals. The attempt to apply the method of exhibition to the formulas under consideration leads then to a vicious circle.

But exhibition should serve only as a means in proofs of the consistency of axiom systems. This procedure enabled us to consider domains with a given finite number of individuals; and we recognized that in such domains the consistency of a formula has the same significance as its satisfiability.

The situation is more complicated in the case of infinite domains of individuals. It is true in this case also that an axiom system represented by a formula $\mathfrak{A}$ is inconsistent if and only if the formula $\neg\mathfrak{A}$ is valid. But since we are no longer dealing with a surveyable supply of courses of value for the variable predicates, we can no longer conclude that if $\neg\mathfrak{A}$ is not valid there is some model for satisfying the axiom system $\mathfrak{A}$ at our disposal.

Accordingly, when we are dealing with infinite domains of individuals, the satisfiability proves to be a sufficient but not a necessary condition of consistency. We cannot therefore expect that in general a proof of consistency can be accomplished by means of a proof of satisfiability. On the other hand we are not forced to prove consistency by establishing satisfiability; we can just hold to the original negative sense of inconsistency. That is to say—if we again imagine an axiom system. again represented by a formula $\mathfrak{A}$—we do not have to show that satisfiability of the formula $\mathfrak{A}$, but only need to prove that the assumption that $\mathfrak{A}$, is satisfied by certain predicates cannot lead to a logical inconsistency.

To attack the problem in these terms we must first get an overview of the possible logical inferences that can be made from an axiom system. The *formalization of logical inference* as developed by Frege, Schröder, Peano, and Russell presents itself as an appropriate means to this end.

We have now arrived at the following tasks: 1. to formalize rigorously the principles of logical inference and turn them into a completely surveyable system of rules; 2. to show for a given axiom system $\mathfrak{A}$, (which is to be proved consistent) that starting with this *system* $\mathfrak{A}$ *no inconsistency can arise via logical deductions*, that is to say, no two formulas of which one is the negation of the other can be proved.

However, we do not have to carry out this proof for each axiom system individually; for we can make use of the method of *arithmetizing* to which we referred at the beginning. From the point of view we have reached now this procedure can be characterized as follows: we want an axiom system $\mathfrak{A}$ which on the one hand has a sufficiently surveyable structure so that we can give a proof of consistency (in the sense of the second task); on the other hand, however, $\mathfrak{A}$ must be sufficiently comprehensive to enable us to derive the satisfiability of axiom systems for geometry and branches of physics from the presupposition that $\mathfrak{A}$ is satisfied by a system $\mathfrak{S}$ of things and relations; such a derivation would involve the representation of the objects,pf an axiom system $\mathfrak{B}$ by individuals or complexes of individuals from $\mathfrak{S}$ and the substitution for the fundamental relations of predicates constructed from the fundamental relations of $\mathfrak{S}$ using logical operations.

This suffices to show that the axiom system $\mathfrak{B}$ is in fact consistent; for any inconsistency produced in this system -would represent an inconsistency derivable from the axiom system $\mathfrak{A}$ even though the axiom system $\mathfrak{A}$ is known to be consistent.

Axiomatic arithmetic presents itself as such an $\mathfrak{A}$.

The "reduction" of axiomatic theories to arithmetic does not depend upon arithmetic being a set of facts presentable to the intuition; arithmetic need be no more than an ideal structure which we can prove consistent and which provides a systematic framework encompassing the axiom systems of the theoretical sciences; because they are encompassed in this framework the idealizations of what is actually given which they involve will also be proved consistent.

We now summarize the results of our latest considerations: The problem of the satisfiability of an axiom system or a logical formula can be positively solved in the case of a finite domain of individuals by exhibition; but in the case where the satisfaction of the axioms requires an infinite domain of individuals this method is no longer applicable because it is not determined whether an infinite domain of individuals exists; the introduction of such an infinite domain is not justified prior to a proof of the consistency of an axiom system characterizing the infinite.

Because of the refutation of a positive method of deciding, there remains only one possibility: to give a proof of consistency in the negative sense, i.e. a *proof of impossibility*; such a proof requires a formalization of logical inference.

If we are going to approach the task of giving such a proof of impossibility we must be clear that it cannot be carried out using axiomatic-existential methods.of inference. Rather we must use only those kinds of inferences which are free from idealizing assumptions of existence.

As a result of these considerations the following thought comes to mind: If this proof of impossibility can be carried out without axiomatic-existential assumptions, shouldn't it also be possible to found arithmetic directly in the same way and make the proof of impossibility completely superfluous? We will consider this question in the following paragraphs.

2. Elementary number theory.—Finitistic inference and its limits.

The question raised at the end of the previous paragraph was whether we couldn't found arithmetic directly by a method independent of axiomatics and make a special proof of consistency superfluous; this question gives us reason to recall that the technique of rigorous axiomatics and its pre- supposition of a fixed domain of individuals (and especially existential inference) is by no means the original mathematical method.

Geometry was indeed built up axiomatically from the beginning. But Euclid's axiomatization is intended to be contentual and intuitive. In this axiomatization there is no abstraction from the intuitive significance of figures. Moreover the axioms are not in existential form. Euclid does not presuppose that points and lines constitute any fixed domain of individuals whatsoever. And that is why he does not formulate any existence-axioms but only construction-postulates.

An example of such a postulate is: one can join two points with a straight line; or, one can draw a circle around a given point with a prescribed radius.

The methodological standpoint involved here can only be carried out if the postulates are looked on as the expression of known facts or as immediately evident. As is well known the question which arises in this connection, namely ‘In what domain do the axioms of geometry hold, is a very controversial and difficult one; and it is an essential advantage of formal axiomatics that it makes the foundation of geometry independent of deciding this question.

In arithmetic we are free of this whole set of problems which are connected with the character of geometric knowledge; in fact we find in this domain, in elementary number theory and algebra, the purest examples of direct contentual investigation without axiomatic assumptions. The mark of such an investigation is that deliberations are in the form of *thought experiments* involving objects assumed to be concretely given. In number theory the concretely given objects are numbers, in algebra expressions involving letters with given numerical coefficients.

We wish to consider the methods involved here more closely and make the methodological principles somewhat more precise. In number theory we have a beginning object and a method of continuing. We must determine both intuitively. The particular determination is inessential here, but once it is made it must be maintained for the whole theory. We choose the numeral 1 as beginning object and the adjunction of 1 as the method of continuing. The things which we obtain by applying the method of continuing, be- ginning from the numeral 1 are, for example,

$$1, 11, 1111;$$

what are obtained in this way are figures beginning with 1 and ending with 1; on every 1 which is not the end of the figure there follows an adjoined 1. These figures are obtained by applying the method of continuing and,. therefore by a concrete *construction* which terminates; this construction [*Aufbau*] can, therefore be reversed in terms of a step-by-step *decomposition* [*Abbau*].

These figures constitute a kind of numeral; we propose to use the word "numeral" to designate *these* figures only.

As is usual we imagine that a certain amount of latitude is allowed concerning the exact formal configuration of numerals; for example, small differences in construction such as the shape of the 1 or its size or the distance between 1's, will not be taken into consideration. All that we require is (1) that both 1 and the affixing of 1 are intuitive objects which can always be recognized in an unambiguous way, and (2) that we can always survey the discrete parts from which a numeral is constructed.

In addition to numerals we introduce further signs, "communication-signs". These signs are fundamentally different from numerals which constitute the *objects* of number theory.

In itself a communication-sign is also a figure; and we presuppose that it can always be recognized in an unambiguous way and that small differences in its construction are irrelevant. However, within the theory itself it is not considered as an object; it is only a means for formulating facts, assertions, and assumptions briefly and clearly.

In number theory we use the following kinds of communication-signs:

1. Small Latin letters to designate any indeterminate numeral;

2. the usual number-signs abbreviating definite numerals, e.g. 2 for 11, 3 for 111;

3. Signs for certain operations which we perform with numerals and for certain processes of construction by which we get from given numerals other numerals;

4. the sign $=$ to indicate coincidence between figures, the sign $\neq$ to indicate the difference between two figures: the sign $<$ to indicate a relation of magnitude between numbers which has not yet been explained;

5. Parentheses as signs for the order of operations when there is a possibility of ambiguity.

How these signs are manipulated and how contentual deliberations are carried out becomes clearest if we develop number theory in its basic features somewhat further.

The first thing we determine for numbers is the relationship of magnitude. Let a be a numeral different from a numeral b. Let us consider how this is possible. Both begin with 1, and the construction continues in the same way for both unless one of the numerals comes to an end while the construction of the other continues. This must in fact occur at some time, and so the one numeral coincides with a *portion* of the other; in more precise terms: the construction of one numeral coincides with an initial portion of the construction of the other.

If a numeral a coincides with a portion of b, we say that a is smaller than b or that b is bigger than a; in symbols

$$a < b, \quad b > a.$$

Our considerations show that for numerals a and b one of the relations

$$a = b, \quad a < b, \quad b < a$$

must hold; and it is obvious from the intuitive meaning that these relations exclude one another. Similarly it is an immediate consequence that if $a < b$ and $b < c$ then also $a < c$.

Addition is very closely bound up with the relationship of magnitude. If a numeral b coincides with a portion of a, the remainder is also a numeral c; one gets the numeral a by affixing c to b in such a way that the 1 with which c begins is attached to the 1 with which b ends in conformity with the method of continuing. This kind of concatenation of numerals we call *addition*; and we use the sign + for it.

We conclude directly from this definition of addition: if $b < a$ then from the comparison of b with a one gets a representation of a in the form $b + c$, with c a numeral. And if one starts with any numerals b, c then addition produces another numeral a, such that

$$a = b + c;$$

in this case we have

$$b < a.$$

The significance of numerical equalities and inequalities such as $2 < 3$, $2+3 = 5$ is clear from the above definitions. $2 < 3$ says that the numeral 11 coincides with a portion of 111; $2 + 3 = 5$ says that the numeral 11111 results from the affixing of 111 with 11.

In both of these cases it is a question of the representation of a correct assertion, but, for example $2+3 = 4$ is the representation of a false assertion.

We now have to determine that computational laws hold for intuitively defined addition.

These laws will be here conceived as propositions about arbitrary given numerals and investigated in terms of intuitive deliberation.

The associative law, according to which if a, b, c are any numerals,

$$a + (b + a) = (a + b) + c,$$

is directly inferred from the definition of addition. The commutative law, which says that

$$a * b = b * a$$

always holds is not so directly given. We need here the method of proof by *complete induction*. We first make clear how this kind of inference is to be understood from our elementary point of view: Consider any assertion about a numeral which has an elementary intuitive content. The assertion holds for 1, and one knows that if it holds for a numeral n then in every case it also holds for the numeral $n + 1$. One infers that the expression holds for every given numeral a.

In fact the numeral a is constructed when one applies the process of adjoining 1 beginning from 1. If one establishes that the assertion under consideration holds for 1 and that it holds for the result of adjoining a 1 to a numeral for which it is supposed to hold, then, with the completion of the construction of a one determines that the assertion holds for a.

Complete induction then is not an independent principle but a consequence of the concrete construction of numerals.

Using this method of inference we can now show in the usual way that for every numeral

$$1 + a = a + 1,$$

and from this that one always has

$$a + b = b + a.$$

We will now sketch briefly the introduction of multiplication and division and the method of constructing.concepts connected with it,

Multiplication can be defined in the following way: $a * b$ signifies the numeral which one gets from b when one replaces every 1 with an a in constructing b; thus one first constructs a and affixes a in every place where a 1 is introduced in the construction of b.

The associative law for multiplication as well as the distributive law

$$a * (b + c) = (a * b) + (a * c)$$

are immediate from the definition. The other distributive law

$$(b + c) * a = (b * a) + (c * a)$$

is seen to hold with the help of complete induction using the laws of addition. In this way one also gets the commutative law of multiplication.

In order to get at division we must first introduce some preliminary considerations. The construction of a numeral is such that the adjunction of a numeral always produces a new numeral. The formation of a numeral a then involves the formation of a concrete row of numerals beginning with 1 and ending with a in which every numeral arises from its predecessor through the adjoining of l. One sees equally that this row contains except for a itself only numerals which are $< a$ and that a numeral which is $< a$ must occur in this row. We call this sequence of numerals "the row of numerals from 1 to a" for short.

Let b be a numeral different from 1 and $< a$. b has the form $1 + c$; thus

$$b * a = (1 * a) + (c * a) = a + (c * a),$$

and therefore

$$a < b * a.$$

If we multiply b successively with the numerals in the row from 1 to a we get the row of numerals

$$b * 1, b * 11, \ldots, b * a;$$

the first of these is $< a$, and the last $> a$. We go through this row until we first come upon a numeral $> a$; its predecessor (call it $b * q$) is either $= a$ or $< a$, while

$$b * (q + 1) = (b * q) + b > a.$$

Then either

$$a = b * q$$

or we have a representation

$$a = (b * q) + r;$$

in the latter case we have

$$(b * q) + r < (b * q) + b$$

and so

$$r < b.$$

In the first case a is "divisible by b" ("b divides a"), and in the second case there is division with a remainder.

In general we say a is divisible by b if the numeral a occurs in the row

$$b * 1, b * 11, \ldots, b * a;$$

This occurs if $b = 1$ or if $b = a$ or in the first case just described.

From the definition of divisibility it follows immediately that if a is divisible by b the determination that it is yields a representation

$$a = b * q.$$

But the converse also holds; the divisibility of a by b follows from an equation $a = b * q$ since the numeral q must belong to the row of numerals from 1 to a.

If $a \neq 1$ and no divisor of a other than 1 and a occurs in the row of numerals from 1 to a, then every product $m * n$ in which m and n belong to the row of numerals from 2 to a is distinct from a; in such a case we call a a *prime number*.

If a is a numeral different from 1 then there is a first numeral in the row from 1 to n which is distinct from 1 and a divisor of n. It is easy to show that this "least divisor of n distinct from 1" is a prime number.

Now we can also prove in the same way as Euclid did that for any numeral a a prime number $> a$, can be determined: One multiplies together the numbers from the row from 1 to a, adds 1, and then gets the least divisor t

distinct from 1 of the numeral thus obtained. This is a prime number, and one sees easily that t cannot occur in the row of numbers from 1 to a and so is $> a$.

The further development of elementary number theory is clear; only one point still requires fundamental discussion, the method of *recursive definition*. We recall the nature of this method: A new function-symbol, say, ϕ, is introduced, and the definition of the function involves two equations which in the simplest case have the form

$$\begin{aligned} \phi(1) &= a \\ \phi(n+1) &= \psi(\phi(n), n). \end{aligned}$$

Here a is a numeral and ψ is a function constructed from functions already known so that $\psi(b, c)$ can be computed for given numerals b, c and has as value a numeral.

For example the function

$$\rho(n) = 1 * 2 * \ldots n$$

is defined by the equations

$$\begin{aligned} \rho(1) &= 1 \\ \rho(n+1) &= \phi(n) * (n+1). \end{aligned}$$

What sense this method of definition has is not self-evident. To clarify it we must first make precise the concept of function. We understand a *function* to be intuitive instructions on the basis of which a numeral can be paired with a given numeral. A pair of equations of the above kind, which we call a “recursion”, we interpret as an *abbreviated indication* of the following instructions:

Let m be some numeral. If $m = 1$, then m is to be paired with the numeral a. Otherwise m has the form $b + l$. First one writes schematically

$$\psi(\phi(b), b).$$

If $b = 1$ then one replaces $\phi(b)$ here with a; otherwise b has the form $c + 1$ and one replaces $\phi(b)$ with

$$\psi(\phi(c), c).$$

Again, either $a = 1$ or a is of the form $d + 1$. In the first case one replaces $\phi(c)$ with a and in the second with

$$\psi(\phi(d), d).$$

This procedure terminates in every case. For the numerals

$$b, c, d, \ldots,$$

which we obtain in sequence arise from the *decomposition of the numeral* m; and this, like the construction of m must terminate. When the decomposition reaches 1, $\phi(1)$ is replaced with a; the resulting configuration no longer contains the sign ϕ; rather only ψ occurs as a function sign, perhaps in several strata, and the innermost arguments are numerals. We have then obtained a computable expression; for ψ is supposed to be an already known function. This computation has to be done from the inside out; the numeral obtained is the numeral to be paired with m.

The content of these instructions shows us first that in principle they can be carried out for any case for a given numeral m and that the result is uniquely fixed. At the same time we see that for a given numeral n the equation

$$\phi(n+1) = \psi(\phi(n), n)$$

is satisfied if we replace $\phi(n)$ and $\phi(n+l)$ with the numerals paired with n, and $n+1$ according to our directions and then substitute for the known function ψ its definition.

The somewhat more general case in which one or more undetermined numerals occur as "parameters" in the function being defined is handled in much the same way. In the case where there is one parameter t the recursion equations have the form

$$\begin{aligned} \phi(t,1) &= \alpha(t) \\ \phi(t,n+1) &= \psi(\phi(t,n),t,n); \end{aligned}$$

here both α and ψ are known functions. For example, the function $\phi(tn) = t^n$ is defined by the recursion

$$\begin{aligned} \phi(t,1) &= t \\ \phi(t,n+1) &= \phi(t,n) * t. \end{aligned}$$

Definition by recursion does not involve an independent principle of definition; within the framework of elementary number theory it has only the significance of a convention for abbreviating the description of certain construction-processes through which one gets from one or more given numerals another numeral.

We take as an example to indicate that we can carry out a *proof of impossibility* in the framework of intuitive number theory the assertion expressing the irrationality of $\sqrt{2}$: There cannot be two numerals m, n such that

$$m * m = 2 * n * n.$$

(We follow customary procedure in writing products involving several factors without parentheses; this procedure is permissible because of the associative laws for multiplication.) As is well known, the proof proceeds as follows: One shows first that every numeral is either divisible by 2 or of the form $(2 * k) + 1$; therefore $a * a$ is divisible by 2 only if a is divisible by 2.

If a pair of numbers m, n satisfying the above equation were given, we could examine all number-pairs a, b with

$$a \text{ belonging to the row } 1, \ldots, m,$$
$$b \text{ belonging to the row } 1, \ldots, n,$$

and determine whether or not

$$a * a = 2 * b * b.$$

We choose from among the pairs of values satisfying the equation the one in which b has the smallest value. There can only be one such; call it m', n'. In accordance with our previous remark, it follows from the equation

$$m' * m' = 2 * n' * n'$$

that m' is divisible by 2:

$$m' = 2 * k';$$

therefore we have

$$\begin{aligned} 2 * k' * 2 * k' &= 2 * n'n', \\ 2 * k' * k' &= n' * n'. \end{aligned}$$

But then n', k', would be a pair of numbers satisfying our equation with $k' < n'$. This however is inconsistent with the way in which n' was picked.

Of course, the sentence just proved can be expressed positively: If m and n are any two numerals $m * m$ is different from $2 * n * n$.

Let this much suffice as a characterization of the elementary treatment of number theory. We have developed it as a theory of numerals, i.e. of a certain kind of especially simple figure. The significance of this theory for epistemology depends upon the relation of numerals to the ordinary *concept of ordinal number* [*Anzahl-Begriff*]. We obtain this relation in the following way:

Imagine a concrete (and therefore finite) collection of things. One considers the things in the collection successively and correlates them in a row with the numerals 1, 11, 111, ... as numbers. When no thing is left a certain numeral has been reached. This numeral is then the *ordinal number* of the collection of things taken in the sequence chosen.

But now we easily convince ourselves that the resulting numeral n is in no way dependent on what sequence is chosen. For let

$$a_1, a_2, \ldots, a_n$$

be the things of the collection in the sequence chosen,

$$b_1, b_2, \ldots, b_k$$

the things in some other sequence. We can go from the first enumeration to the second by a succession of interchanges of numbers in the following manner: If a_1, is different from b_1, then we interchange the number r which b_1 had in the first enumeration with 1, that is to say we correlate a_r with the number r, a_1 with the number 1. In the resulting enumeration b_1, has the number 1; following it and correlated to the number 2 is the thing b_2 unless this thing has in this enumeration some other number s, (necessarily distinct from 1); if it does we exchange this number s with 2 in the enumeration; the result is an enumeration in which b_1 has the number 1, b_2 the number 2. In the resulting enumeration b_3 has the number 3 or some other, t,distinct from 1 and 2; we exchange the latter with 3.

This procedure must terminate; for with every interchange the enumeration is brought at least one step closer to agreement with the enumeration

$$b_1, b_2, \ldots, b_k;$$

as a result one will eventually get the number 1 for b_1, the number 2 for b_2, …, the number k for b_k; and then there is no other thing left. On the other hand the stock of numbers used remains exactly the same with every interchange; for all that happens is that the number of one thing is exchanged with that of another. Therefore in every case the enumeration goes from 1 to n .and as a result we have

$$k = n.$$

Thus the numeral n is assigned to the collection under consideration independently of any sequence; in this sense we can correlate n with the collection as its *cardinal number* [Anzahl]. We say that the collection consists of n things. (These considerations were put forward by von Helmholtz in his essay “Zahlen und Messen”.)

If two concrete collections have a common cardinal number, we get from an enumeration of each a one-one correlation of the things in one collection with those in the other. On the other hand, if we have such a correlation between two given collections of things the two have the same cardinal number; this is an immediate consequence of our definition of cardinal number.

Contentual considerations enable us to pass from the definition of cardinal number to the fundamental theorems of the *theory of cardinal numbers*, e.g. to the assertion that the uniting of two collections of cardinality a and b and without a common element gives rise to a collection of $a + b$ things.

After the representation of elementary number theory we would like to indicate briefly the character of the elementary contentual point of view in *algebra*. We shall deal with the elementary theory of integral functions of one or more variables with integers as coefficients.

The objects of the theory are again certain figures, "polynomials"; they are constructed from a determinate stock of letters, $x, y, z, \ldots$, called "variables" and the numerals with the help of the signs $+, -, *$ and parentheses. In this case then the signs $+, *$ belong to the objects of the theory; they should not be construed as communication-signs as in elementary number theory.

We again use small Latin letters as communication-signs, not just for numerals but also for arbitrary polynomials.

The construction of polynomials out of the signs indicated above follows these rules:

A variable and also a numeral can be considered in itself a polynomial. From two polynomials a, b the polynomials

$$a + b, \quad a - b, \quad a * b$$

can be constructed; from a polynomial a $(-a)$ can be constructed. The usual rules for parentheses hold here. As communication-signs we also introduce:

- the numbers $2, 3, \ldots$,
- the number-signs $2, 3, \ldots$, as in elementary number theory;
- the sign 0 for $1 - 1$;
- the usual signs for powers: for example, if z is a numeral x^z signifies the polynomial which results from z when x is put in place of every 1 and the sign "$*$" is put between every two successive x's;
- the sign $=$ indicates the mutual *substitutability* of 'two polynomials.

Substitutability is determined by the following contentual rules:

1. The associative and commutative rules for "+" and "$*$".
2. The distributive law $a * (b + c) = (a * b) + (a * c)$.
3. Rules for "$-$": $a - b = a + (-b)$, $(a + b) - b = a$.
4. $1 * a = a$
5. If neither variables nor "$-$" occur in two polynomials m, n and the equality $m = n$ holds *in the sense of equality for elementary number theory*, then n is substitutable form.

These rules of substitutability relate to polynomials occurring as *parts* of other polynomials. It is possible to derive from them further assertions about substitutability, sentences which constitute the "identities" and theorems of elementary algebra. As examples of simple provable identities we mention

$$\begin{array}{rclrcl} a+0 &=& a & \qquad -(a-b) &=& b-a \\ a-a &=& 0 & -(-a) &=& a \\ a*0 &=& 0 & (-a)*(-b) &=& a*b \end{array}$$

Of the theorems which can be established from contentual considerations we mention the following fundamental assertions:

a) If a, b are two mutually substitutable polynomials of which at least one contains the variable x and if a_1, b_1 result from a, b when the variable x is replaced throughout with the polynomial c, then b_1 is substitutable for a_1.

b) Substitution of numerals for variables in a correct equation between polynomials yields a correct numerical equation in the sense of number theory (if we suppose computation with negative numbers is introduced into number theory).— The meaning of b) may be illustrated by a simple example: The equation

$$(x+y)*(x+y) = x^2 + 2*x*y + y^2$$

says nothing except that we have determined that $x^2 + 2*x*y + y^2$ is substitutable for $(x+y)*(x+y)$. On the basis of sentence b), however, we can infer that if m and n are number signs $(m+n)*(m+n)$ and $m*m+2*m*n+n*n$ coincide in the number-theoretic sense.

c) Every polynomial can be substituted for either by 0 or by a sum of powers of variables, each conjoined with a positive or negative numerical multiplicand; in this connection the polynomial 1 counts as a sum of products of powers of variables.

This normal form makes possible a procedure for deciding for two given polynomials whether or not they are mutually substitutable. For the following assertion holds:

d) A polynomial which is the sum of different products of powers with numerical factor 0 is not substitutable for a polynomial which is the sum of different products of powers with numerical multiplicands, and two such polynomials are mutually substitutable if and only if they coincide when the order of the summands and the order of the factors in the products of powers with their numerical multiplicands is overlooked.

The second part of this assertion follows from the first; and the first can be proved with the help of b) by considering suitable substitutions of numerals.

The following is a particular consequence of d):

e) if a numeral n is substitutable for a numeral m, conceived as a polynomial, m and n coincide.

A methodological point about these assertions should be noted: The substitutability of polynomials assumed in a), e) is to be understood as the assumption that the substitutability has been determined according to the rules. The assertion of substitutability in c) is more closely determined because a method of establishing it is given and described in the proof of the theorem.

And so here we are just as much in the domain of elementary contentual inferences as we were in the case of elementary number theory. And the same is true of the other assertions and proofs of elementary algebra.

The consideration of the principles of number theory and algebra has served to show us the application and use of direct contentual inference carried out in thought experiments performed on intuitively imagined objects and free from axiomatic assumptions. We will call this kind of inference " [*finitistic*]" [*finit*] inference in order to have a short expression; likewise we shall call the attitude underlying this kind of inference the "finitistic" attitude or point of view. We will speak of finitistic concepts or assertions in the same sense; in using the word "finitistic" we convey the idea that the consideration, assertion or definition in question remains within the limits of objects which it is in principle possible to observe and of processes which it is in principle possible to complete; that it is carried out in the framework of concrete thought.

To characterize the finitistic point of view further we consider certain general aspects relating to the use of the forms of judgment as classified in logic in finitistic thinking; we consider assertions about *numerals* as examples.

A *universal* judgment about numerals can only be interpreted in a hypothetical sense from a finitistic point of view; that is to say, it can only be interpreted as an assertion about any given numeral. Such a judgment expresses a law which must be confirmed in every particular case.

An *existential sentence* about numerals, i.e. a sentence of the form "there is a numeral n with the property $A(n)$" is to be construed finitistically as a "partial judgment"; it is an incomplete communication of a more precisely determined assertion, either a direct indication of a numeral with the property $A(n)$ or the indication of such a procedure a definite bound must be given to the number of operations involved.

Judgments in which a universal assertion is conjoined with an existential one are to be interpreted in a corresponding way. So. for example, a sentence of the form "for every numeral k with the property $A(k)$ there is a numeral 1 such that $B(k, 1)$" is constructed finitistically as an incomplete communication of a procedure which for any given numeral k makes possible the finding of a numeral 1 with the property $A(k)$ makes possible the finding of a numeral 1 which stands to k in the relation $B(k, 1)$.

The application of *negation* demands special attention.

Denial is unproblematic in the case of "elementary" judgments, which involve a question decidable by a direct intuitive determination (an "inspection"). For example, if k, 1 are particular numerals it can be directly determined whether or not

$$k + k = 1,$$

i.e. whether or not $k + k$ and 1 coincide or are different.

The negation of such an elementary judgment says simply that the result of the intuitive decision diverges from the situation asserted to obtain by the judgment; and one has directly for an elementary judgment a dichotomy: either it or its negation is correct.

On the other hand it is not immediately clear what should count as the negation of a universal or existential judgment in the finitistic sense.

We consider first assertions of existence. The assertion that there is not a numeral with the property $A(n)$ might mean (in an imprecise sense) that we are unable to indicate a numeral with this property; no such numeral is at our disposal. But such an assertion has no objective significance because it involves an accidental epistemological condition. If, however, one wishes to maintain the unavailability of a numeral with the property $A(n)$ independently of epistemological conditions, he can do it in a finitistic sense only with an assertion of impossibility, an assertion that a numeral can not have the property $A(n)$.

In this way we arrive at a rigorous negation; however, it is not exactly the contradictory opposite of an assertion of existence, "there is a numeral n with the property $A(n)$", which, as a partial judgment, points to a known numeral with this property or to a procedure for producing such a numeral which we possess.

Unlike an elementary assertion and its negation, an existential assertion and its rigorous negation are not assertions about the only two possible results of *one and the same* decision; they correspond to two distinct epistemological possibilities: on the one hand the discovery of a numeral with a given property, on the other the discernment of a general law about numerals.

It is not logically obvious that one of these two possibilities must be realized. From the finitistic point of view then we cannot use the alternative: either there is a numeral n such that $A(n)$ or the holding of $A(n)$ for a numeral is precluded.

The situation is much the same in the case of a universal judgment of the form "for every numeral n $A(n)$ holds" with respect to finitistic negation. The denial that such a judgment holds does not have a direct finitistic sense; if it is made precise as the assertion that the universal truth of $A(n)$ can be refuted by a counterexample, then this rigorous negation no longer constitutes the contradictory opposite of a universal judgment; for again it is not logically obvious that either a universal judgment or its rigorous negation must hold, i.e. that either $A(n)$ holds for every given numeral n or that a numeral can be given for which $A(n)$ does not hold.

Of course, it must be noted that the discovery of a counterexample is not the only possibility for refuting a universal judgment. Pursuing the consequences of a universal judgment can lead to a contradiction in other ways. This circumstance, however, does not eliminate any difficulties but only increases the complications. For it is no more obvious that logically obvious that a universal judgment about numerals must either hold or lead to a contradiction in its consequences and therefore be refutable than it is that such a judgment if refutable is refutable through a counterexample.

The complicated situation that we find here with respect to the denial of judgments from the finitistic point of view corresponds to the thesis of Brouwer that the law of the excluded middle does not hold for infinite collections. This thesis also holds from the finitistic point of view insofar as we are unable to find a notion of negation with finitistic content which satisfies the law of the excluded middle for existential and universal judgments.

These considerations will perhaps suffice as an indication of the finitistic point of view. If we look at customary treatments of arithmetic we realize that our methodological point of view does not obtain; arithmetic inferences and concepts often go beyond the limits of the finitistic way of thinking in many ways.

The inferences of number theory already go be yond the finitistic point of view; for here assertions about the existence of integers are permitted, and no attention is paid to the possibility of actually determining the number in question; (in ordinary mathematics we speak of integers instead of "numerals" (more exactly we speak of "positive integers" or of "numbers" for short) and also use is made of the alternative that an assertion about integers either holds for all integers or there is a number for which it does not hold.

This alternative, the "tertium non datur" for integers is implicit in the application of the "least number principle"; "If an assertion about integers holds for at least one number, there is a least number for which it holds".

In its *elementary* applications the least number principle has a finitistic character. Indeed, if $A(a)$ is the assertion about a number a and m is a definite number for which $A(m)$ holds, then one need only go through the numbers from 1 to m; for one must come in this way to a first number k for which $A(k)$ holds since m is such a number. k is then the least number with the property A.

But these considerations depend on two presuppositions which are not always fulfilled in non-elementary applications of the least number principle. In the first place it is presupposed that A holds of a number in the sense that a number m with the property $A(m)$ is actually given; but in applications the existence of a number with the property A is often proved using the "tertium non datur" so that no actual determination of such a number is possible. The second presupposition is that it can be decided for any number k in the row from 1 to m whether or not $A(k)$ holds; of course, it is possible to decide this for elementary assertions $A(m)$; on the other hand, for a non-elementary expression $A(a)$ the question whether it holds for a given number k may constitute an unsolved problem.

For example, let $\psi(a)$ be a function defined by a sequence of recursions and substitutions, and so admissible in finitistic number theory; and let $A(b)$ stand for the assertion that there is a number a for which $\psi(a) = b$. Then, for a given number k, the question whether $A(k)$ holds is not in general (i.e. when the function ψ is not especially simple) decidable by direct inspection; rather it has the character of a mathematical problem. For the recursions which enter into the definition of ψ give the value of the function only *for a given* *argument*; but the question whether there is a number a for which $\psi(a)$ has the value k involves the whole course of values of the function *psi*.

In any case then where these presuppositions of the finitistic justification of the least number principle are not fulfilled, the justification of the principles requires reference to the "tertium non datur" for integers.

We give some examples of number-theoretic disjunctions which result from the tertium non datur for integers but which are not provable in a finitistic way, given our present knowledge:

"Either every even number greater than 2 is representable as the sum of two prime numbers or there is an even number greater than 2 and not representable as the sum of two prime numbers."

"Either every integer of the form $2^{(2^k)} + 1$ with $k > 4$ is divisible into two factors greater than 1 or there is a prime number of the form $2^{(2^k)} + 1$ with $k > 4$."

"Either every sufficiently great integer is representable as the sum of less than 8 cubes or for every integer n there is an integer m greater than n and not representable as the sum of less than 8 cubes."

"Either there are arbitrarily great prime numbers p with the property that $p + 2$ is also a prime number or there is a greatest prime number with this property."

"Either for every integer $n > 2$ and arbitrary positive integers a, b, c the inequality $a^n + b^n \neq c^n$ holds or there is a least integer $n > 2$ for which the equation $a^n + b^n = c^n$ has a solution in the positive integers." This kind of example is appropriate for making clear the simplest forms of non-finitistic argumentation. However, we will not really feel the need to go beyond the finitistic point of view in number theory; for there is hardly any number-theoretic proof in which the non-finitistic inferences that happen to be made can not be circumvented with rather easy simple modifications.

It is quite different in the case of analysis (infinitesimal calculus); here non-finitistic methods of constructing concepts and carrying out proofs are a basic tools of the theory.

We first recall the fundamental concept of analysis, the concept of a real number. The real numbers are defined either as a strictly increasing sequence of rational numbers

$$r_1 < r_2 < r_3 < \dots$$

which are all less than a given bound ("fundamental sequence"), or as an infinite decimal, or fraction, or as a partition of the rational numbers into two classes, every member of the first class being smaller than every member of the second ("Dedekind cut").

In all of these definitions the idea of the rational numbers as a determinate totality which can be considered as a *domain of individuals* is fundamental. And in analysis the totality of possible sequences of rational numbers or of possible partitions of all rational numbers is also conceived as a domain of individuals.

Of course, it is sufficient to consider the totality of integers rather than the totality of rational numbers and the totality of all partitions of integers rather than that of all rational numbers. For in fact every positive rational number is given by a pair of numbers m, n and every rational number whatever can be represented as the difference between two positive rational numbers, i.e. as a pair of pairs of numbers $(m, n; p, q)$. Also every fraction of the form

$$0, a_1 a_2 a_3 \dots$$

with $a_1, a_2, a_3, \dots$ all either $= 0$ or $= 1$ can be interpreted as a partition of all integers: namely the partition into those numbers k for which $a_k = O$ and into those for which $a_k = 1$. From this point of view there is a one-one correspondence between the partitions of the positive integers and the binary fractions of the above form; and on the other hand every real number can be represented as the sum of a positive integer and binary fraction of this form.

It is possible to consider *sets* of integers instead of partitions; for every set of integers determines the partition into the numbers which belong to the set and those which do not; and equally every set of integers is completely determined by such a partition. The same remark holds for the Dedekind cut; it likewise can be represented by a *set* of rational numbers, namely the set containing the smaller rational numbers. This set is characterized by the following properties: 1. it contains at least one and not every rational number; 2. if it contains a rational number it contains all smaller and at least one bigger rational number.

These transformations, however, weaken the existential presupposition which must be at the basis of analysis in an unessential way only. It is still necessary to construe the manifold of the integers and that of the sets of integers as a fixed domain of individuals; the "tertium non datur" is taken to hold for these domains, and an assertion of the existence of an integer or set of integers with a property P is taken to be meaningful whether or not it has significance as a partial judgment. So even though the infinitely large and the infinitely small in any genuine sense are excluded by these theories of real numbers and remains only as a way of speaking, still the *infinite as a totality* is retained. One can even say that the rigorous foundation of analysis introduced and validated the representation of infinite totalities systematically for the first time.

We wish to convince ourselves that the presupposition of the totality of the domain of integers or rational numbers and of the domain of sets (partitions) of integers or rational numbers has an essential application in the founding of analysis; for this purpose we need to introduce certain fundamental concepts and ideas.

If the reals are defined as increasing sequences of rational numbers

$$r_1 < r_2 < r_3 < \ldots,$$

the concept of equality for real numbers is already non-finitistic, For whether or not two such sequences of rational numbers define the same real number depends upon whether or not for every number in one of the sequences there is a larger in the other and vice-versa. But we do not have a general procedure for deciding this question.

If, however, we begin with the definition of real numbers via Dedekind cuts, we have to prove that every bounded increasing sequence of rational numbers gives rise to a cut representing the upper bound of the sequence. One gets this cut by partitioning the rational numbers into those which are less than at least one member of the sequence and those which are not. That is to say: a rational number r is said to be in the first or second class according to whether there is in the sequence a number $> r$ or whether all numbers in the sequence are $\leq r$. This, however, is no finitistic distinction.

The case is similar if real numbers are defined via infinite decimal or binary fractions. Again it must be shown that a bounded sequence of rational numbers

$$r_1 < r_2 < r_3 < \ldots,$$

determines a decimal (or binary) fraction. For simplicity let us suppose we are dealing with a sequence of positive proper fractions:

$$0 < b_1 < b_2 < \ldots < 1;$$

we wish to determine the binary fraction which represents the upper bound of the sequence of fractions. This is done as follows:

a_1 is 0 or 1 depending on whether or not all fractions b_n are $< 1/2$;

a_{m+1} is 0 or 1 depending on whether or not all fractions b_n are less than

$$\frac{a_1}{2} + \frac{a_2}{4} + \ldots + \frac{a_m}{2^m} + \frac{1}{2^{m+1}}.$$

In each case one considers the alternative whether all rational numbers in a given sequence

$$r_1, r_2, r_3, \ldots$$

satisfy a certain inequality or whether there is at least one exception to this inequality. This alternative depends upon the "tertium non datur" for integers; for it is presupposed that for every integer n the rational number r_n, satisfies the inequality in question or there is at least one integer n such that r_n fails to satisfy it.

Moreover, the use of the *totality of integers* as a domain of individuals is not sufficient for analysis; we need in addition the *totality of real numbers.* As we saw, this totality is essentially equivalent to that of the sets of integers.

The need for the real numbers as a domain of individuals is made clear in connection with the theorem of the upper bound of a bounded set of real numbers. Consider the proof of the existence of the upper bound of a bounded set of real numbers, e.g. reals in the interval between 0 and 1, for real numbers defined on the basis of Dedekind cuts; in this proof one considers the partition of the rational numbers into those which are and those which are not exceeded by a real number in the set. Thus one counts a rational number r as being in the first class if and only if there is a real number $a > r$ in the set.

Now one must be clear that in analysis a set is usually given only by a defining property; that is to say, the set is introduced as the collection of those real numbers which satisfy a certain condition B. Therefore the question whether there is a real number greater than r in a set under consideration amounts to the question whether there is a real number greater than r and also satisfying a certain condition B. Construing it in the latter way makes clear that the totality of real numbers is presupposed as a. domain of individuals. (This situation was made especially clear by Weyl in his monograph "Das Kontinuum".)

It should also be remarked that the process just described for constructing an upper bound amounts to taking a *set-theoretic union*. In fact every real number is defined by a partition of the rational numbers into larger and smaller (or it is defined as the set of the smaller rational numbers). The given set of real numbers is therefore represented as a set M of sets of rational numbers, And the upper bound of the set M is constructed from the set of those rational numbers which belong to at least one of the sets in M. The collection of these rational numbers is, however, the union of M.

Defining the real numbers by means of fundamental sequences or binary fractions instead of using Dedekind's definition does not make it possible to circumvent the introduction of the real numbers as a. domain of individuals. In deed these definitions make the procedure even more complicated because an additional recursive technique enters. We will indicate what is involved in the case of the definition of real numbers via binary fractions. We are then concerned with a set of binary fractions

$$0.a_1a_2a_3\ldots$$

which is again determined by a certain criterion B; and the upper bound is represented by a binary fraction

$$0.b_1b_2\ldots$$

defined in the following way:

$b_1 = 0$ if 0 stands in the first binary position in all binary fractions satisfying the condition B; otherwise $b_1 = 1$;

$b_{n+1} = 0$ if 0 stands in the $(n+1)$th position in all binary fractions satisfying the condition B and having the first n numerals coincident with $b_1, b_2, \ldots, b_n$ respectively; otherwise $b_{n+1} = 1$.

Here the totality of the real numbers enters as the totality of all binary fractions; and we use the assumption that the "tertium n6n datur" holds for infinite sequences of zeros and ones; otherwise $b_{n+1} = 1$.

But this presupposition of the totality of all real numbers (all binary fractions) is not sufficient. This can be seen in the following simple case: Let a be the upper bound of a set of real numbers. We want to show that there is a sequence of real numbers *from the set* which converges toward a. To do this we argue as follows:

It follows.from the properties of an upper bound that for every integer n there is a number c_n in the set such that

$$a - \frac{1}{n} < c_n < a;$$

and so

$$|a - c_n| < \frac{1}{n}.$$

The numbers c_n constitute therefore a sequence which converges toward a, and they all belong to the set under consideration.

When we argue in this way our manner of expression hides an important point in the proof. For when we write c_n we presuppose that a for each number n a definite one of the real numbers c belonging to the set under consideration and satisfying the inequality

$$a - \frac{1}{n} < c \leq a$$

has been distinguished for each number n.

There is an assumption involved here. All we can immediately infer is this: for every number n there is a subset M_n of the set under consideration which consists of those numbers satisfying the above inequality; and for every n this subset has at least one element. Now what is assumed is that in each of these sets

$$M_1, M_2, M_3, \ldots$$

we can distinguish an element c_1, in M_1, c_2 in M_2, $\ldots c_n$ in M_n and thereby get a determinate infinite sequence of real numbers.

We have here a special case of the *axiom of choice*; its general formulation is the following: "If for every thing x in a species G_1 there is at least one thing y in the species G_2 which stands to x in the relation $B(x, y)$, then there is a function ϕ which correlates each thing x in the species G_1 with one thing $\phi(x)$ in the species G_2 which stands to x in the relation $B(x, \phi(x))$."

In the case at hand the species G_1 is that of the positive integers, G_2 that of the real numbers; the relation $B(x, y)$ is the inequality

$$a - \frac{1}{x} < y \leq a$$

and the function ϕ the existence of which is derived from the axiom of choice, correlates the real number c_x with its number x.

Zermelo was the first to recognize the axiom of choice as a special assumption and to formulate it set-theoretically; its use involves a further overstepping of the finitistic point of view and goes beyond the application of the "tertium non datur". The above consideration of methodological examples teaches us that the founding of analysis foundations of the infinitesimal calculus, as they have been given since the discovery of rigorous techniques, does not involve a reduction to *finitistic* number-theoretic thought. In this sense the *arithmetizing* of mathematics is not *complete* since it involves certain systematic and fundamental conceptions which do not belong to the domain of intuitive arithmetic thinking. The insight which the rigorous foundation of analysis has brought us is that these few fundamental assumptions already suffice to build up mathematics as a theory of sets of numbers.

Large areas of mathematics, such as function theory, differential geometry, and topology (analysis situs) are governed dominated by the methods of analysis. General *set theory*, the methods of which have penetrated modern abstract algebra and topology, goes well beyond the presuppositions of analysis and makes the most extensive use of non-finitistic assumptions.

The usual treatment of arithmetic is by no means an expression of the finitistic point of view but depends essentially upon additional principles of inference. We see then that if we want to keep arithmetic in its current form while acknowledging the demands of the finitistic point of view with respect to evidence, we are confronted with the task of justifying the application of those principles which transcend finitistic thinking by means of a consistency proof. The success of such a proof of the consistency of the techniques of argument customary in arithmetic would give us a guarantee that the results of using these techniques could never be contradicted by a finitistic determination or reflection; for finitistic methods are a part of ordinary arithmetic, and a finitistic contradiction of an assertion proved by the techniques of ordinary arithmetic would indicate a contradiction within ordinary arithmetic.

We return.then to the problem raised in #1. It remains to answer the question from which the considerations of this section began: whether instead of using the formalization of inference to prove the impossibility of a contradictions arising in arithmetic, we couldn't found all of arithmetic directly without additional assumptions and make the proof of impossibility superfluous.

The answer to this question is partly positive, partly negative. The investigations of Kronecker and Brouwer have shown what is involved in the possibility of a direct finitistic founding of enough arithmetic for practical applications.

Kronecker was the first to insist on the requirements of the finitistic point of view; he intended to eliminate completely non-finitistic modes of inference from mathematics. He succeeded with the theory of algebraic numbers and number fields. Maintaining the finitistic point of view was possible in this case without giving up anything essential in the way of theorems or proof-techniques. (Kronecker did not systematically publish the results of these investigations but only presented them in lectures.)

The criticisms of Kronecker were completely rejected for a long time; later Brouwer set himself the task of founding arithmetic independently of the law of the excluded middle and developed important parts of analysis and set theory in terms of this program. Of course, his methods require giving up essential theorems and putting up with considerable complications in the construction of concepts. (A detailed list of Brouwer's publications in this area is found in A. Fraenkels textbook *Einleitung in die Mengenlehre*, third edition.)

The methodological standpoint of "Intuitionism" which Brouwer makes fundamental constitutes something of an *extending of the finitistic position*; Brouwer permits the introduction of the assumption that an inference or a proof is given even though the intuitive nature of the.inference or proof is not determined. For example, from Brouwer's point of view assertions of the following forms are permissible: "if on the assumption that A the sentence B holds, then C holds too" or "the assumption that A is refutable leads to a contradiction" (as Brouwer puts it: "the absurdity of A is absurd").

An extension of the finitistic point of view of this kind turns out to be necessary if one is going to go beyond a certain elementary domain on the basis of finitistic considerations; from an epistemological point of view this extension amounts recounting as intuitive judgments and considerations of a general logical character. We will be led to the necessity of this extension at a late stage of our investigations.

The investigations we have been discussing indicate a way by which one can go far into mathematics without using non-finitistic techniques of inference; Nevertheless they in no way make a proof of the consistency of the methods of ordinary arithmetic superfluous. For they do not achieve an avoidance of non-finitistic methods of inference in the sense of completely replacing these methods with other processes; the avoidance succeeds in analysis and related areas of mathematics only at the cost of an essential loss in systematization and proof-technique.

A mathematician, however, cannot be expected to accept such a loss without compelling reasons. The methods of analysis have been tested to a greater extent than almost any other scientific assumption, and they have been validated most impressively. If we criticize these methods from the point of view of evidence, then we face the task of tracing the ground of their applicability, just as we do everywhere in mathematics where a successful technique is applied on the basis of conceptions which, in terms of evidence, leave much to be desired.

Insofar as we accept the finitistic point of view we cannot escape the problem of obtaining a clear understanding about the applicability of non-

Ian Mueller's translation ends here.