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The manifold of directives for the formation [*Gestaltung*] of geometric axiom systems (1959)

Paul Bernays

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Translation by: *Dirk Schlimm*

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none

Looking at [*Bei der Betrachtung*] axiomatizations of geometry we stand under the impression [*stehen wir unter dem Eindruck*] of the great manifold of principles [*Gesichtspunkte*] under which the axiomatization can take place and also already took place. The original [*ursprüngliche*] simple old idea [*Vorstellung*], according to which one could plainly [*schlechtweg*] speak of *the* axioms of geometry, is not only superseded [*Verdrängt*] by the discovery of non-Euclidean geometries and moreover also by the insight into the possibility of different axiomatizations of one and the same geometry, but substantially different methodical [*methodische*] principles have generally arisen under which one has undertaken the axiomatization of geometry and whose purposes [*Zielsetzungen*] are in certain relations [*in gewissen Beziehungen*] even antagonistic.

The seed for this manifold can be found already in the Euclidean axiomatic. For its formation the circumstance was determinant that one was led by means [*an Hand*] of geometry for the first time to the problem [*Problemstellung*] of axiomatic. Here geometry is simply mathematics so to speak. The relation to number theory is methodically [*methodisch*] not a completely clear one. In certain parts a piece of number theory is developed using the intuitive [*anschaulich*] idea of number [*Zahlvorstellung*]. Moreover the concept of number [*Zahlbegriff*] is used contentfully [*inhaltlich*] in the theory of proportions [*Proportionenlehre*], even with an implicit inclusion [*Einschluss*] of the tertium non datur; but it seems that one strived to avoid its full use.

While the methodical exceptional position [*methodische Sonderstellung*] of the concept of number does not step forward explicitly here, the concept of magnitude [*Grössenbegriff*] is explicitly put in front [*an die Spitze gestellt*] as a contentful tool [*inhaltliches Hilfsmittel*], incidentally in a manner which we cannot concede any longer today, namely by assuming as a matter of course [*selbstverständlich*] that different objects [*Gegenständlichkeiten*] have the character of magnitudes [*Grössencharakter*]. The concept of magnitude is sure enough also subjected to axiomatization; however, in this regard the axioms are explicitly separated [*abgesondert*] from the remaining axioms as antecedent (*κοιναὶ ἐννιαὶ*). These axioms are of a similar kind as those which are used [*aufstellt*] today for Abelian groups. But what remained undone because of the methodical [*methodischen*] standpoint then [*damaligen*], was that it was not fixed axiomatically which objects were to be regarded as magnitudes.

The more it is to admire that one already then became attentive [*aufmerksam*] at the peculiarity [*Besondere*] of that assumption by which the Archimedean magnitudes, as we call them today, are characterized [*ausgezeichnet werden*]. The Archimedean (Eudoxean) axiom is then, in the medieval tradition that followed the Greeks, used in particular in the Arabic investigations of the parallel axiom. It also occurs fundamentally [*als wesentlich*] in Saccheri's proof of the preclusion [*Ausschliessung*] of the "hypothesis of the obtuse angle". This preclusion is in fact impossible without the Archimedean axiom, since a non-Archimedean, weakly-spherical (resp. weakly-elliptical) geometry is in accordance with the axioms of Euclidean geometry, except for the parallel axiom.

The second axiom of continuity, which was formulated in the late 19^{th} century, does not yet occur in all these investigations.

It could be dispensed with in the proofs [*Beweisführungen*] for which it came into question — like in the determination of areas and lengths [*Flächeninhalts- und Längenbestimmungen*] — because of the mentioned use of the concept of magnitude, according to which it was for example taken for granted [*selbstverständlich*] that both the area of the circle and the circumference of the circle possess a definite magnitude. In lieu of [*An die Stelle*] the old theory of magnitudes [*Grössenlehre*] came at the beginning of modern times [*Neuzeit*] as predominant and superordinated [*beherrschende übergeordnete*] discipline the theory of magnitudes of *analysis*, which developed formally and contentually [*dem Inhalt nach*] very prolifically, still before it reached methodical clarity [*methodischer Deutlichkeit*].

Sure enough [*freilich*], analysis played at first no significant role in the discovery of non-Euclidean geometry, but it became dominant in the following investigations of Riemann and Helmholz, and later Lie, for the identification [*Kennzeichnung*] of the three special [*ausgezeichneten*] geometries by certain very general, analytically subsumable [*fassbar*] conditions. In particular it is characteristic for this treatment of geometry that one does not only take the single space entities [*Raumgebilde*] as objects [*zum Gegenstand nehmen*], but also the space manifold [*Raummannigfaltigkeit*] itself. The enormous conceptual and formal means which mathematics had obtained [*gewonnen*] in the meantime showed up in the possibility of carrying out such an inspection [*Durchführung einer solchen Betrachtung*]; and the conceptual-speculative direction which mathematics took in the course of the 19^{th} century is expressed in the formulation of the problem [*Anlage der Problemstellung*].

The differential geometrical [*differentialgeometrische*] treatment of the foundations of geometry has been developed further until recent times [*bis in die neueste Zeit*] by Hermann Weyl as well as Elie Cartan and Levi-Civitá, following up [*in Anknüpfung an*] Einstein's general relativity theory. Despite the impressiveness and elegance of what has been achieved in this respect, mathematicians were not content with it [*haben sich nicht damit zufrieden gegeben*] from a foundational [*grundlagentheoretischen*] standpoint. At first one tried to free oneself from the fundamental assumption of the methods of differential geometry of the differentiability of the mapping functions [*Abbildungsfunktionen*]. For this the developmen [*Ausbildung*] of the methods of a general topology was needed, which began at the turn of the century and has taken such an impressive development since then. Moreover one strived for the independence from the assumption of the Archimedean character of the geometrical magnitudes in general.

This tendency stands in the light [*im Zeichen*] of that development by which analysis in some sense lost its previously predominant position. This new stage in mathematical research followed the consequences of the already mentioned conceptual-speculative direction of mathematics of the 19^{th} century, which appeared in particular in the creation of general set theory, in the sharper foundation of analysis, in the constitution of mathematical logic, and in the new version of axiomatic.

At the same time it was characteristic for this new stage that one returned again to the methods of the ancient Greek axiomatic, like it happened repeatedly in those epochs in which emphasis was put on conceptual precision. In Hilbert's Foundations of Geometry we find on one hand this return to the old elementary axiomatic, sure enough in a fundamentally changed methodical perception [*Auffassung*], and on the other hand the as wide as possible exclusion of the Archimedean axiom as a principal theme: in the theory of proportions, in the concept of area [*Flächeninhaltsbegriff*], and in the foundation of the line segment calculus [*Streckenrechnung*]. For Hilbert by the way this kind of axiomatization was not intended as being exclusive [*hatte nicht den Sinn der Ausschliesslichkeit*]; shortly afterwards he put a different kind of foundation on its side, in which the program of a topological foundation mentioned above was formulated [*aufgestellt*] and carried out for the first time.

Around the same time as Hilbert's foundation the axiomatiatization of geometry was also cultivated [*gepflegt*] in the school of Peano and Pieri. Shortly afterwards the axiomatic investigations of Veblen and R. L. Moore followed; and by then the directions of research [*Forschungsrichtungen*] were chosen [*eingeschlagen*] in which the occupation with the foundations of geometry moves along also today. As characteristic for it we have a multiplicity of methodical directions.

One of them seeks to characterize the manifold of congruent transformations by conditions that are as general and succinct as possible, the second one puts the projective structure of space at the beginning and strives for reducing the metric to the projective with the methods developed by Cayley and Klein, and the third aims at [*auf ... ausgeht*] elementary axiomatization of the full geometry of congruences [*Kongruenzgeometrie*].

Different fundamentally new points of view were added during the development of these directions. Firstly, the projective axiomatization gained an increased systematization [*verstärkte Systematisierung*] through lattice theory [*Verbandstheorie*]. In addition one became aware that one can leave behind [*zurücktreten lassen*] the set-theoretic and function-theoretic concept formations [*Begriffsbildungen*] in the identification [*Kennzeichnung*] of the groups of congruent transformations by fixing the transformations through determining entities [*Gebilde*]. Therewith the procedure approaches that of elementary axiomatic, since the group relations are now represented as relations between geometric entities.

But I do not want to speak further of these two directions of research [*Forschungsrichtungen*] of geometrical axiomatic [*geometrischen Axiomatik*], for which more authentic representatives are present here, and also not of the successes that have been achieved using topological methods, about which the newest essays [*Abhandlungen*] of Freudenthal give a survey, but to turn to the questions of the direction of axiomatization that was mentioned in the third place.

Even within this direction we find a manifold of possible goals [*Zielsetzungen*]. On the one hand one can aim to manage with as few as possible basic elements [*Grundelementen*], perhaps only one basic predicate [*Grundpredikat*] and one sort [*Gattung*] of individuals. On the other hand one can especially aim to isolate [*hervortreten lassen*] natural separations [*Absonderung*] of parts of the axiomatic. These viewpoints [*Gesichtspunkte*] lead to different alternatives.

So on the one hand the consideration [*Betrachtung*] of non-Euclidean geometry suggests the assumption [*? Voranstellung*] of an "absolute" geometry. On the other hand also such a setup has something for it [*manches für sich*], which starts off with [*vorangestellt wird*] affine vector geometry, like it is done at the beginning of Weyl's "Space, Time, Matter". The demands of both these viewpoints can hardly be satisfied [*Genüge leisten*] with a single axiomatic. Starting with the axioms of incidence and ordering [*Anordnung*] it is a possible and elegant conceptual reduction to reduce the concept of collinearity to the concept of betweenness, after the proceeding of Veblen. On the other hand it is important for some considerations to separate the consequences of the incidence axioms which are independent of the concept of ordering; so it is desirable to realize the independence of the foundation of the line segment calculus [*? Streckenrechnung*] on the incidence axioms from the ordering axioms. In the theory of ordering itself one has again realized the possibility of replacing [*Ersparung*] the axioms of linear ordering by applications of the axiom of Pasch; on the other hand in some respect an arrangement of the axioms is preferable in which those axioms are separated [*abgesondert*] which characterize the linear ordering [*für die lineare Ordnung kennzeichnenden*].

The manifold of the goals [*Zielsetzungen*] that are possible and also pursued in fact is not exhausted in the least [*nicht annähernd erschöpft*] by these examples of alternatives. So it is a possible and plausible [*sinngemässer*], but not obligatory regulative viewpoint that the axioms should be formulated in such a way that they refer only to a limited part of space [*Raumstück*] respectively. This thought [*Gedanke*] is implicitly effective [*? mitbestimmend*] already in the Euclidean axiomatic; and it may also be that the offense that has been taken so early at the parallel axiom relies just on the fact that the concept of a sufficiently long extension occurs in the Euclidean formulation. The first explicit realization of the mentioned program [*Programmpunktes*] happened through Moritz Pasch, and it was followed by the introduction of ideal elements by intersection theorems [*? Schnittpunktsätzen*] which is a method for the foundation of projective geometry that has been successively developed since.

A different kind of the possible additional task [*Aufgabestellung*] is to imitate conceptually the blurriness of our pictorial imagination [*bildhaften Vorstellens*] as it was done by Hjelmslev. This results not only in a different kind of axiomatization, but in a deviating relational system [*abweichendes Beziehungssystem*], which has not found much approval because of its complication [*Komplikation*]. But also without moving so far in this direction from the customary manner [*soweit von dem Üblichen zu entfernen*] it is possible to aim at something similar in some respect [*kann man in gewisser Weise etwas Ähnliches anstreben*], by avoiding the concept of point as genetic term [*? Gattungsbegriff*] as it is done in various interesting newer axiomatizations, in particular in Huntington's.

By this means [*In solcher Weise*] you see [*zeigt sich*] in the greatest number of ways [*auf mannigfachste Art*] that there is no definite optimum for the formation of a geometric axiom system. As regards the reductions in respect to the basic concepts [*Grundbegriffe*] and the kinds of things [*Dingarten*], it has to be always reminded, regardless of the general interest that any such possibility of reduction has, that a real application of such a reduction is only recommended when it leads to a clear formation [*übersichtliche Gestaltung*] of the axiom system.

After all certain directives for reductions can be stated which we can generally accept. Let us take for example the Hilbertian version of axiomatic. In it, on the one hand, lines are taken as a kind of things [*Dinggattung*], on the other hand the ... [*Halbstrahlen*] are introduced as point sets [*Punktmengen*] and afterwards the angles are explained as ordered pairs of two ... [*Halbstrahlen*] that originate in the same point, thus as a pair of sets. Here real [*tatsächlich*] possibilities of simplifying reductions are given. One may be of different opinion, whether one wants to start with only one kind of points instead of the different kinds "point, line, plane", whereby the the relations of collinearity and coplanarity [*Komplanarität*] of points replace the relations of incidence. In the lattice theoretical [*verbandstheoretischen*] treatment the lines and planes are taken to be on par [*gleichstehend*] with points as things. Here again there is an alternative. Whereas to introduce the ... [*Halbstrahlen*] as point sets transcends in any case the scope [*überschreitet den Rahmen*] of elementary geometry and is not necessary for it. Generally we can take as directive that higher kinds should not be introduced without need [*Erfordernis*]. This can be avoided in the case of the definition of angle by reducing the statements about angles by statements about point triples [*Punkttripel*], as it was carried out by R. L. Moore. Even a further reduction is achieved here by explaining the congruence of angles using congruence of line segments [*Streckenkongruenz*], but here again a certain loss takes place. Namely, the proofs [*Beweisführungen*] rest substantially on the congruence of ... assigned [*ungleichsinnig zugeordneten*] triangles. Thus this kind of axiomatization is not suitable for the kind of problems [*Problemkreis*] of the Hilbertian investigations which refer to the relationship [*Verhältnis*] between the ... [*gleichsinnig*] congruence and symmetry. This remarks concerns also most of the axiomatizations that start [*and der Spitze steht*] with the concept of reflection [*Spiegelung*].

Besides the general viewpoints I want to mention as something particular a special possibility of the arrangement of an elementary axiom system, namely an axiomatic in which the concept "the triple of points a, b, c forms a right angle at b" is taken as the only basic relation and points as the only basic kind, a program which has been pointed out recently in a paper of Dana Scott. The mentioned relation satisfies the necessary condition for a single sufficient basic predicate for ... [*Planimetrie*], ascertained [*festgestellten*] by Tarski. In comparison with Pieri's technique [*Verfahren*], which has become exemplary for an axiomatic of this kind, and which took an axiomatization of the relation "b and c have the same distance from a" as basic predicate, it seems to exist an easing here, inasmuch the concept of the collinearity of points is closer affiliated [*enger ... anschliesst*] to the one of a right angle than to Pieri's basic concept. As respects the concept of congruency [*Kongruenzbegriff*] there seems to be no simplification for the axioms of congruency from the relation considered. By the way, this axiomatization is, just as the one by Pieri, one of those which do not yield a separation of ... [*gleichsinnigen*] congruency[1].

For an elementary axiomatization of geometry the particular question is raised [*stellt sich als besondere Frage*] of obtaining a completeness [*Vollständigkeit*] in the sense of categoricity. In most axiom systems this is obtained [*erwirkt*] by the continuity axioms [*Stetigkeitsaxiome*]. But, the introduction of these axioms means, as one knows, a transgression of the framework [*Überschreitung des Rahmens*] of usual [*gewöhnlich*] concepts of predicates and sets. Since then [*? seither*] we have learned from Tarski's investigations [*Untersuchungen*] that a completeness, at least in the deductive sense, can be obtained in an elementary framework, where it is striking [*das Bemerkenswerte*] that the cut axiom [*Schnittaxiom*] is preserved [*erhalten bleibt*] in a particular formalization whereas the Archimedean axiom is omitted [*vom ... abgesehen wird*]. The Archimedean axiom is insofar formally unusual [*fällt ja insofern formal aus dem sonstigen Rahmen heraus*], in that in logical formalization it has the form [*Gestalt*] of an infinite alternative, whereas the cut axiom is representable by an axiom schema due [*auf Grund*] to its form of generality [*Form der Allgemeinheit*] and thus it can be adapted in its use to the respective formal framework, — whereby then for the elementary framework of predicate logic the provability of the Archimedean

[1]Some details on the definitions of the concepts of incidence, ordering, and congruency, from the concept of a right angle, as well as on a part of the axiom system follow in the appendix.

axiom from the cut axiom is lost. Sure enough [*Freilich*], such a restriction to a framework of predicate logic has as consequence that some considerations [*verschiedene Überlegungen*] are possible only meta-theoretically, for example, the proof of the theorem that a simple closed polygon decomposes [*zerlegt*] the plane, and also the consideration [*Betrachtung*] about equality of supplementation [*Ergänzungsgleichheit*] and decomposition [*Zerlegungsgleichheit*] of polygons. Here one stands again in front of an alternative, namely whether one wants to begin [*voranstellen*] with the viewpoint of the elementarity [*Elementarität*] of the logical framework, or whether does not want to restrict oneself with respect to the logical framework whereby incidentally [*übrigens*] different gradations [*Abstufungen*] come into consideration [*in Betracht kommen*].

With respect to the application of a second order logic [*Logik der zweiten Stufe*] I only want to remind here that a such can be made precise in the framework of axiomatic set theory, and that no palpable [*fühlbare*] restriction of the methods of proof [*Beweismethoden*] results [*erfolgt*]. Also the Skolem paradox does not present a real embarrassment [*eigentliche Verlegenheit*] in the case of geometry, since it can be eliminated [*ausschalten*] in the model theoretic considerations by equating [*gleichsetzen*] the concept of set which occurs in one of the higher axioms with the concept of set of model theory.

At the end I want to emphasize [*hervorheben*] that the fact which I stressed in my remarks [*Ausführungen*], that there is no definite [*eindeutig*] optimum for the formation of the axiomatic [*Gestaltung der Axiomatik*], not at all means that the products [*Erzeugnisse*] of geometrical axiomatic [*der geometrischen Axiomatik*] necessarily bear an imperfect and fragmentary character. You know that in this field a number of formations [*Gestaltungen*] of great perfection [*Vollkommenheit*] and rounding off [*? Abrundung*] have been attained [*erreicht worden sind*]. It is [*Gerade*] the multiplicity [*Vielheit*] of the possible goals [*Zielrichtungen*] that effects that the former [*das Frühere*] is generally not just [*schlechtweg*] outdated [*überholt*] by the newer [*das Neuere*], but at the same time [*während andererseits*] also every attained perfection [*Vollkommenheit*] leaves room for further tasks [*weitere Aufgaben*].

Appendix

Remarks to the task of an axiomatization of Euclidean ... [Planimetrie] *with a single basic relation R(a,b,c): "the triple of points* [Punktetripel] *a, b, c forms a right angle at b".* The axiomatization succeeds [*gelingt*] insofar in a simple way [*auf einfache Art*], because only the relations of collinearity and parallelism are considered [*betrachtet werden*]. The following axioms suffice for the theory of collinearity:

A1 $\neg R(a,b,c)$

A2 $R(a,b,c) \rightarrow R(c,b,a) \;\&\; \neg R(a,c,b)$[2]

A3 $R(a,b,c) \;\&\; R(a,b,d) \;\&\; R(e,b,c) \rightarrow R(e,b,d)$

A4 $R(a,b,c) \;\&\; R(a,b,d) \;\&\; c \neq d \;\&\; R(e,c,b) \rightarrow R(e,c,d)$

A5 $a \neq b \rightarrow (Ex)R(a,b,x)$

The definition of the relation $Koll(a,b,c)$ is added [*Dazu tritt*]: "the points a,s,c are collinear":

[2]Already this axiom excludes elliptic geometry.

Definition 1. $Koll(a,b,c) \leftrightarrow (x)(R(x,a,b) \rightarrow R(x,a,c)) \vee a = c.$

Then the following theorems are provable:

(1) $Koll(a,b,c) \leftrightarrow a = b \vee a = c \vee b = c \vee (Ex)(R(x,a,b) \ \& \ R(x,a,c))$

(2) $Koll(a,b,c) \rightarrow Koll(a,c,b) \ \& \ Koll(b,a,c)$

(3) $Koll(a,b,c) \ \& \ Koll(a,b,d) \ \& \ a \neq b \rightarrow Koll(b,c,d)$

(4) $R(a,b,c) \ \& \ Koll(b,c,d) \ \& \ b \neq d \rightarrow R(a,b,d)$

(5) $R(a,b,c) \rightarrow \neg Koll(a,b,c)$

(6) $R(a,b,c) \ \& \ R(a,b,d) \rightarrow Koll(b,c,d)$

(7) $R(a,b,c) \ \& \ R(a,b,d) \rightarrow \neg R(a,c,d).$
To proof [*Zum Beweis*]: $Koll(c,d,b) \ \& \ c \neq b \rightarrow (R(a,c,d) \rightarrow R(a,c,b))$

(8) $R(a,b,c) \ \& \ R(a,b,d) \ \& \ R(a,e,c) \ \& \ R(a,e,d) \rightarrow c = d \vee b = e.$
To proof [*Zum Beweis*]:
$Koll(b,c,d) \ \& \ Koll(e,c,d) \ \& \ c \neq d \rightarrow Koll(b,c,e)$
$Koll(b,c,e) \ \& \ b \neq e \ \& \ R(a,b,c) \rightarrow R(a,b,e)$
$Koll(e,c,b) \ \& \ b \neq e \ \& \ R(a,e,c) \rightarrow R(a,e,b)$
$R(a,b,e) \rightarrow \neg R(a,e,b).$

For the theory of parallelism we add two further axioms:

A6 $a \neq b \ \& \ a \neq c \rightarrow$
$(Ex)(R(x,a,b) \ \& \ R(x,a,c)) \vee$
$(Ex)(R(a,x,b) \ \& \ R(a,x,c)) \vee R(a,b,c) \vee R(a,c,b)$

In its usual diction [*Ausdrucksweise*] the axiom says [*besagt*] that it is possible to draw a perpendicular on a line bc from a point a lying outside it. The definite determination [*eindeutige Bestimmtheit*] of a perpendicular depending on a point a and a line bc results with the help of (4) and (8).

A7 $R(a,b,c)$ & $R(b,c,d)$ & $R(c,d,a) \to R(d,a,b)$

This is a form of the Euclidean parallel axiom in the narrower, angle-metric [*winkelmetrischen*] sense.

The parallelism [*Parallelität*] is now defined by:

Definition 2. $Par(a,b;c,d) \leftrightarrow a \neq b$ & $c \neq d$ & $(Ex)(Ey)(R(a,x,y)$ & $R(b,x,y)$ & $R(c,y,x)$ & $R(d,y,x))$

As provable theorems the following arise:

(9) $Par(a,b;c,d) \to Par(b,a;c,d)$ & $Par(c,d;a,b)$

(10) $Par(a,b;c,d) \to a \neq c$ & $a \neq d$ & $b \neq c$ & $b \neq d$

(11) $Par(a,b;c,d) \leftrightarrow a \neq b$ & $c \neq d$ & $(Ex)(Eu)((R(a,x,u) \vee x = a)$ & $(R(b,x,u) \vee x = b)$ & $(R(x,u,c) \vee u = c)$ & $(R(x,u,d) \vee u = d))$

For the proof of the implication from right to left one has to show that there are at least five different points lying on the line a, b which succeeds with the help of axioms A1–A6.

(12) $Par(a,b;c,d) \to (x)((R(a,x,c) \vee x = a)$ & $(R(b,x,c) \vee x = b) \to R(x,c,d))$

(13) $Par(a,b;c,d)$ & $Koll(a,b,e)$ & $b \neq e \to Par(b,e;c,d)$

and hence [*daraus*] in particular:

(14) $Par(a, b; c, d) \to \neg Koll(a, b, c)$;

moreover

(15) $Par(a, b; c, d) \mathbin{\&} Koll(a, b, e) \to \neg Koll(c, d, e)$

(16) $\neg Koll(a, b, c) \to (Ex)Par(a, b; c, x)$

(17) $Par(a, b; c, d) \mathbin{\&} Par(a, b; c, e) \to Koll(c, d, e)$

(18) $Par(a, b; c, d) \mathbin{\&} Par(a, b; e, f) \to$
$Par(c, d; e, f) \vee (Koll(e, c, d) \mathbin{\&} Koll(f, c, d))$.

The concept of vector equality [*Vektorgleichheit*] is also tied up to [*knüpft sich an*] the concept of parallelism: "a, b and c, d are the opposite sides of a parallelogram".

Definition 3. $Pag(a, b; c, d) \leftrightarrow Par(a, b; c, d) \mathbin{\&} Par(a, c; b, d)$

Herewith one can prove:

(19) $Pag(a, b; c, d) \to Pag(c, d; a, b) \mathbin{\&} Pag(a, c; b, d)$

(20) $Pag(a, b; c, d) \mathbin{\&} Pag(a, b; c, e) \to d = e$

(21) $Pag(a, b; c, d) \to \neg Koll(a, b, c)$.

For the proof of the existence theorem [*Existenzsatz*]

(22) $\neg Koll(a, b, c) \to (Ex)Pag(a, b; c, x)$

one needs a further axiom:

A8 $R(a,b,c) \to (Ex)(R(a,c,x) \mathbin{\&} R(c,b,x))$.

It is generally provable with the help of this axiom that two different, not parallel lines possess a point of intersection:

(23) $\neg Koll(a,b,c) \mathbin{\&} \neg Par(a,b;c,d) \to$
$(Ex)(Koll(a,b,x) \mathbin{\&} Koll(c,d,x))$. —

It is left open [*bleibe dahingestellt*] whether it is possible to achieve altogether [*im Ganzen*] a clear [*übersichtlich*] axiomatic using the basic concept R. Here we content ourself with stating definitions for the fundamental [*wesentlichen*] further concepts. For these it is after all [*immerhin*] possible to attain a certain clarity [*Übersichtlichkeit*].

The following two different definitions of the relation "a is the center of the line segment b,c" are tied up to the figure of the parallelogram:

Definition 4_1. $Mp_1(a;b,c) \leftrightarrow (Ex)(Ey)(Pag(b,x;y,c) \mathbin{\&}$
$Koll(a,b,c) \mathbin{\&} Koll(a,x,y))$

Definition 4_2. $Mp_2(a;b,c) \leftrightarrow (Ex)(Ey)(Pag(x,y;a,b) \mathbin{\&} Pag(x,y;c,a))$.

In the sense of the second definition one can prove the possibility of doubling a line segment [*Verdoppelung einer Strecke*]:

(24) $a \neq b \to (Eu)Mp_2(a;b,u)$.

The existence of the center of a line segment in the sense of Df. 4_1, i.e.,

(25) $b \neq c \to (Eu)Mp_1(u;b,c)$,

is provable if one adds the axiom:

A9 $Par(a,b;c,d)$ & $Par(a,c;b,d) \to \neg Par(a,d;b,c)$.
(In a parallelogram the diagonals intersect.)

By specializing the figure pertaining to the definition of Mp_1 we obtain the definition of the relation: "a, b, c form a isosceles [*gleichschenkliges*] triangle with the tip [*Spitze*] in a":

Definition 5_1. $Ist_1(a;b,c) \leftrightarrow (Eu)(Ev)(Pag(a,b;c,v)$ & $R(a,u,b)$ & $R(a,u,c)$ & $R(b,u,v))$.

With the help of Mp_1 and Ist_1 we can define Pieri's basic concept: "a has the same distance from b and c":

Definition 6. $Is_1(a;b,c) \leftrightarrow b = c \vee Mp_1(a;b,c) \vee Ist_1(a;b,c)$.

A different kind of definition of the concept Is is based on the use of symmetry. The following helper concept [*? Hilfsbegriff*] serves for this [*Hierzu dient*]: "a, b, c, d, e form a 'normal' quintuple [*Quintupel*]":

Definition 7. $Qn(a,b,c,d,e) \leftrightarrow R(a,c,b)$ & $R(a,d,b)$ & $R(a,e,c)$ & $R(a,e,d)$ & $R(b,e,c)$ & $c \neq d$.

With the help of Qn we obtain a further way for defining Mp and Ist:

Definition 4_3. $Mp_3(a;b,c) \leftrightarrow (Ex)(Ey)Qn(x,y,b,c,a)$

Definition 5_2. $Ist_2(a;b,c) \leftrightarrow (Ex)(Ey)Qn(a,x,b,c,y)$,

from which Is_2 can be defined respectively like Is_1.

Moreover also the definition of the mirror image [*Spiegelbildlichkeit*] of points a, b in relation to a line c, d follows [*schließt sich hieran*]:

Definition 8. $Sym(a, b; c, d) \leftrightarrow c \neq d$ &

$(Ex)(Ey)(Ez)(Koll(x, c, d)$ & $Koll(y, c, d)$ & $Qn(x, y, a, b, z))$. —

For the definition of congruency of line segments [*Streckenkongruenz*] we finally still need the concept of ... [*? gleichsinnigen*] congruency on a line: "the line segments a b and c d are collinear, congruent, and directed in the same direction [*? gleichgerichtet*]":

Definition 9₁. $Lg_1(a, b; c, d) \leftrightarrow Koll(a, b, c)$ &

$(Ex)(Ey)(Pag(a, x; b, y)$ & $Pag(c, x; d, y))$,

or also:

Definition 9₂. $Lg_2(a, b; c, d) \leftrightarrow Koll(a, b, c)$ & $a \neq b$ & $(Ex)(Mp(x; b, c)$ &

$Mp(x; a, d)) \vee (a = d$ & $Mp(a; b, c)) \vee (b = c$ & $Mp(b; a, d))$,

(where any of the three definitions above can be taken for Mp.) Now the congruency of line segments can be defined altogether (with any of the two definitions of Lg):

Definition 10. $Kg(a, b; c, d) \leftrightarrow Lg(a, b; c, d) \vee Lg(a, b; d, c) \vee$

$(a = b$ & $Is_1(a; b, d)) \vee (Ex)(Pag(a, b; c, x)$ & $Is_1(c; x, d))$.

By a definition analogous to that of Lg_2 it is possible to introduce the congruency of angles with same ... [*? Scheitelpunkt*] as a six-place relation, after one has introduced before the concept of ... [*? Winkelhalbierenden*]: "$d \neq a$ lies on the ... [*? Halbierenden*] of the angle b a c":

Definition 11. $Wh(a, d; b, c) \leftrightarrow \neg Koll(a, b, c)$ &

$(Ex)(Ey)(Ez)(Koll(a, c, x)$ & $Koll(a, d, y)$ & $Qn(a, y, b, x, z))$.

In consideration of the very composite character of this congruence relation Kg one will reduce in the axiomatization the laws about Kg to the concepts that occur as parts of the defining expression. Because of the plurality of definitions for Mp, Ist, Is there are alternatives depending on whether one employs more the relations of parallelism or of symmetry. In any case the axiom of vector geometry [*Vektorgeometrie*]

A10. $Pag(a, b; p, q)$ & $Pag(b, c; q, r) \rightarrow$
$Pag(a, c; p, r) \vee (Koll(a, c, p)$ & $Koll(a, c, r))$

or an equivalent one should be advisable [*zweckmässig*]. On the whole one could set oneself as a target to represent [*zur Darstellung bringen*] the interaction [*Zusammenspiel*] of parallelism and reflection [*Spiegelung*] that occurs in Euclidean ... [*? Planimetrie*] in a most symmetric way.

Finally, with respect to the interrelation [*Zwischenbeziehung*], the figure for the definition of the relation "a lies between b and c" is already contained as part in that of Qn. Namely, we can define:

Definition 12. $Zw(a; b, c) \leftrightarrow (Ex)(R(b, a, x)$ & $R(c, a, x)$ & $R(b, x, c))$.

For this concept at first is provable:

(26) $\neg Zw(a; b, b)$

(27) $Zw(a; b, c) \rightarrow Zw(a; c, b)$

(28) $Zw(a; b, c) \rightarrow Koll(a, b, c)$

and also using A5, A6, and A8

(29) $a \neq b \rightarrow (Ex)Zw(x; a, b)$ & $(Ex)Zw(b; a, x)$.

To obtain further properties of the in-between concept [*Zwischenbegriffes*] the following axioms can be used:

A11 $R(a, b, c)$ & $R(a, b, d)$ & $R(c, a, d)$ & $R(e, c, b) \to \neg R(b, e, d)$

A12 $R(a, b, d)$ & $R(d, b, c)$ & $a \neq c \to Zw(a; b, c) \vee Zw(b; a, c) \vee Zw(c; a, b)$

A13 $Zw(a; b, c)$ & $Zw(b; a.d) \to Zw(a; c, d)$

A14 $R(a, b, d)$ & $R(d, b, c)$ & $R(a, c, e)$ & $Zw(d; a, e) \to Zw(b; a, c)$

From this axiom it is possible to obtain the more general theorem in a few steps:

(30) $Zw(b; a, c)$ & $Koll(a, d, e)$ & $Par(b, d; c, e) \to Zw(b; a, c)$

This succeeds using the theorem

(31) $R(a, b, e)$ & $R(e, b, c)$ & $R(b, a, d)$ & $R(b, c, f)$ & $R(b, e, d)$ & $R(b, e, f)$ & $Zw(b; a, c) \to Zw(e; d, f)$.

which can be derived from the aforementioned axiom A10.

With the help of (30) and axiom A13 one can prove:

(32) $\neg Koll(a, b, c)$ & $Zw(b; a, d)$ & $Zw(e; b, c) \to$ $(Ex)(Koll(e, d, x)$ & $Zw(x; a, c))$.

i.e., Pasch's axiom in the narrower formulation of Veblen. —

Subsequently [*Anschließend*], I want to mention the following definition of Kg using the concepts Is and Zw which is based on a construction of Euclid:

Definition 13. $Kg^*(a, b; c, d) \leftrightarrow (Ex)(Ey)(Ez)(Is(x, a; c)$ & $Zw(y; a, x)$ & $Zw(z; c, x)$ & $Is(a; b, y)$ & $Is(c; d, z)$ & $Is(x; y, z))$.

(For Is either Is_1 or Is_2 can be taken at will.)

One surely can not demand from an axiomatic as the one described here, in which the collinearity and the in-between relation are coupled with orthogonality, that it provides a separation of the axioms of the linear [*des Linearen*]. Moreover the arrangement is limited from the outset to the ... [*? Planimetrie*], since the definition of collinearity is not applicable in the multi dimensional. The restriction to Euclidean geometry is also introduced at an early stage. On the other hand this axiomatization may be particularly suited to show the great simplicity and elegance of the lawfulness [*Gesetzlichkeit*] of Euclidean ... [*? Planimetrie*].