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The Philosophy of Mathematics and Hilbert's Proof Theory (1930)

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Comments:

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1 The Nature of Mathematical Knowledge

Anyone not familiar with mathematical activity may think, when reading and hearing today about the foundational crisis in mathematics or of the debate between "formalism" and "intuitionism," that this science is shaken to its very foundations. In reality mathematics has been moving for a long time on a smooth wake, so that one senses more a lack of bigger sensations, although there is no lack of significant systematic progress and brilliant achievements.

In fact, the current discussion about the foundations of mathematics does not have its origins in a predicament of mathematics itself Mathematics is in a completely satisfactory state of methodological certainty. In particular, the concern caused by the paradoxes of set theory has long been overcome, ever since it was discovered that for the avoidance of the contradictions encountered, one only needs restrictions that do not encroach in the least on the claims of mathematical theories on set theory.

The problematic, the difficulties, and the differences of opinion begin rather at the point where one inquires not simply about the mathematical facts, but rather about the grounds of knowledge and the delimitation [*Abgrenzung*] of mathematics. These questions of a philosophical nature have received a certain urgency since the transformation the methodological approach to mathematics experienced at the end of the nineteenth century.

The characteristic moments of this transformation are: the advance of the concept of set, which aided the rigorous grounding of the infinitesimal calculus, and further the rise of existential axiomatics, that is, the method of development of a mathematical discipline as the theory of a system of things with determinate operations whose properties constitute the content of the axioms. In addition to this we have, as the result of the two aforementioned moments, the establishment of a closer connection between mathematics and logic. |[*Mancosu*: 235] This development confronted the philosophy of mathematics with a completely new situation and entirely new insights and problems. The discussion about the foundations of mathematics has never since come to rest. The debate concerning the difficulties caused by the role of the infinite in mathematics stands in the foreground in the present stage of this discussion. The problem of the infinite, however, is not the only nor the most general question with which one must come to terms in the philosophy of mathematics. The first task is to gain clarity about what constitutes the peculiarity of mathematical knowledge. We would like to concern ourselves first with this question, and, in order to do so, recall the development of [*the different*] views, even though only in broad strokes and without an exact chronological order.

1.1 The Development of the Conceptions of Mathematics

The older conception of mathematical knowledge proceeded from the division of mathematics into arithmetic and geometry; according to this conception, mathematics was characterized as a theory of two kinds of specific domains, that of numbers and that of geometric figures. However, this division was already unsustainable in the face of the advance of the arithmetical method in geometry. Moreover, geometry did not content itself with the study of properties of figures, but rather it expanded to a general theory of manifolds. Klein's Erlangen Program, which systematically combined the various branches of geometry from the

points of view of a group-theoretical formulation, gave concise expression to the completely different situation of geometry. Out of this situation arose the possibility of incorporating geometry into arithmetic, and since the rigorous grounding of the infinitesimal calculus by Dedekind, Weierstrass, and Cantor reduced the more general number concept, as required by the mathematical theory of quantities [*Größenlehre*] (rational number, real number), to the usual ("natural") numbers 1, 2, . . . , the conception arose that the natural numbers constitute the true object of mathematics and that mathematics consists precisely of the *theory of numbers*.

This conception has many supporters. This view is supported by the fact that all mathematical objects can be represented through numbers or combinations of numbers or through formations of higher sets [*höhere Mengenbildungen*], which are derived from the number sequence. In a fundamental aspect, however, the characterization of mathematics as a theory of numbers is unsatisfactory, if only because it remains undecided what one regards as the essence of number. The inquiry into the nature of mathematical knowledge is thereby shifted to the inquiry into the nature of numbers.

However, this question seems completely pointless to the declared proponents of the view of mathematics as the science of numbers. They proceed from the view, which is the common one in mathematical thought, that numbers form a category of things, which by their nature are completely familiar to us, and to such a degree that an answer to the question concerning the nature of numbers could only consist in reducing something familiar to something less familiar. One perceives the reason for the special position of numbers from this point of view in the fact that numbers make up an essential component of the world order; and this order is |[Mancosu: 236] comprehensible to us in a rigorous scientific way exactly insofar as governed by the element [*Moment*] of number.

Opposing this view, according to which number is something completely absolute and fundamental [*Letztes*], soon emerged, in the aforementioned epoch of the development of set theory and axiomatics, a completely different conception, which completely disputes the existence of a particular, peculiar kind of mathematical knowledge and holds that mathematics is to be obtained *from pure logic*. One was naturally led to this conception on the one hand through axiomatics and on the other hand through set theory.

The new methodological turn in axiomatics consisted in giving prominence to the fact that for the development of an axiomatic theory the epistemic status [*Erkenntnischarakter*] of its axioms is irrelevant. Rigorous axiomatics demands that in the proofs no other facts [*Erkenntnisse*] be used from the field that is to be considered than those that are expressly formulated in the axioms. This is already the meaning of axiomatics found in Euclid, even though at certain points the program is not completely carried out.

In accordance with this demand, the logical dependence of the theorems on the axioms is shown through the development of an axiomatic theory. For this logical dependence it does not matter whether or not the axioms placed at the beginning are true statements. The logical dependence represents a purely hypothetical connection: If things are as the axioms claim, then the theorems hold. Such a detachment [*Loslösung*] of deduction from the assertion of the truth of the initial statements is in no way idle hair splitting. An axiomatic development of theories, which. occurs without regard for the truth of the principles assumed at the starting point, can be of high value for our scientific knowledge, since in this way assumptions, on the one hand, whose correctness is doubtful, can be made amenable to a test by means of the facts through the systematic pursuit of their logical consequences and, on the other hand, the *possibilities of theory-formation* [*Theorienbildung*] can be explored a priori through mathematics according to the points of view of systematic simplicity, in advance and all at once [*auf Vorrat durch die Mathematik*]. With the development of such theories mathematics takes over the role of that discipline, which was earlier called *mathematical natural philosophy*.

By completely ignoring the truth of the axioms within the axiom system, the contentual conception of the basic concepts becomes irrelevant, and in this way one arrives at abstracting, in general, *from all intuitive content of the theory*. This abstraction is also supported by a second element, which appears in the recent axiomatics, a prime example being Hilbert's "Foundations of Geometry," and which is essential for the development [*Gestaltung*] of more recent mathematics, namely, the existential formulation of a theory [*die existentiale Fassung der Theorie*].

Whereas Euclid always thinks of the figures to be studied as constructed, contemporary axiomatics proceeds from the idea of a *system of objects* fixed from the outset. In geometry, for example, one considers the points, straight lines, and planes in their totality as such a system of things. Within this system one thinks of the relationships of incidence (a point lies on a straight line, or on a plane), of betweenness (a point lies between two others), and of congruence as being determined from the outset. These relationships can, without regard for their intuitive meaning, be |*Mancosu*: 237 characterized purely abstractly as certain *basic predicates* (we want to use the term "predicate" also in the case of a relationship between several objects, so that we also speak of predicates with several subjects).[1]

[1]This mode of denotation follows a suggestion of Hilbert. It offers certain advantages over the usual distinction between "predicates" and "relations" for the conception of what

In this way, in Hilbert's system the place of the Euclidian construction postulate that demands the possibility of the connection of two points by means of a straight line is taken by the following existence axiom: For any two points there is always a straight line that belongs to each of the two points. "Belonging" [*Zusammengehören*] is here the abstract expression for [*the relationship of*] incidence.

In the sense of this formulation of axiomatics, the axioms as well as the theorems of an axiomatic theory present themselves as statements about one or several predicates, which refer to the things of an underlying system, and the knowledge, provided to us by the proof of a theorem L, which is carried out by means of the axioms $A_1 \ldots A_k$—for the sake of simplicity we assume that in this case only one predicate is at issue—consists in the realization dial if the statements $A_1 \ldots A_k$ hold of a predicate, then the statement L also holds of it.

What we have before us is, however, a very general proposition about predicates, that is, a proposition of pure logic. In this manner, the results of an axiomatic theory, in the sense of the purely hypothetical and existential formulation of axiomatics, present themselves as *theorems of logic*.

These theorems, though, are only meaningful if the conditions formulated in the axioms can at all be satisfied by means of a system of objects with certain predicates that are related to them. If such a satisfaction is unthinkable, that is, logically impossible, then the axiom system does not lead to a theory at all, and the only logically important statement about the system then is

is logical in principle [*des prinzipiell Logischen*] and is also in agreement with the customary meaning of the word "predicate."

the statement [*Feststellung*] of the contradiction that results from the axioms. For this reason there exists for every axiomatic theory the requirement of a proof of *satisfiability*, that is, of the *consistency* of its axioms.

This proof is accomplished in general, unless one can make do with direct finite model constructions, by means of the method of *reduction to arithmetic*, that is, by exhibiting objects and relationships within the realm of arithmetic that satisfy the axioms that are to be investigated. In this way one again faces the question about the epistemic status of arithmetic.

Even before this question became acute in connection with axiomatics, in the connection described, set theory and logistics had already taken a position on this issue in a new manner. Cantor showed that the number concept in the sense of cardinal number (Number [*Anzahl*]) as well as in the sense of ordinal number (order number [*Ordnungszahl*]) can be extended to infinite sets. The theory of natural numbers and the theory of real numbers [*Maßzahlen*] (analysis) were included in general set theory as subdomains. If, in this way, the natural number forfeits something of the essence of its distinct role, nonetheless the number sequence constitutes, also for Cantor's standpoint, something immediately given, from the examination of which set theory originated.

This was not the end of the matter; rather, the logicians soon moved on to the far-reaching claim that sets are nothing other than extensions of concepts [*Begriffsumfänge*] and that set theory is equivalent to extensional logic [*Umfangslogik*]; in particular the theory of numbers is also to be derived from pure logic. |[*Mancosu*: 238] With this thesis that mathematics is to be obtained from pure logic, an old cherished thought of rational philosophy, which had been suppressed by the Kantian theory of pure intuition, was taken up again.

Now the development of mathematics and theoretical physics already demonstrated that the Kantian theory of experience was, in any case, in need of a fundamental revision, and the moment seemed to have arrived for the radical opponents of the philosophy of Kant to refute this philosophy in its initial thesis, namely, in the claim of the synthetic character of mathematics.

This [*attempt*] was, however, not completely successful. A first symptom that showed that the subject is more difficult and entangled than the leaders of the of logistic movement thought became evident in the discovery of the famous set-theoretic paradoxes. This discovery historically constituted the signal for the onset of the critique. If we want to explain the subject philosophically today, then it is more satisfactory to carry out the consideration directly, without introducing the dialectical argument of the paradoxes.

1.2 The Mathematical Element in Logic—Frege's Number Definitions

In order to deal with the essential points of view, we need only consider the new discipline of theoretical logic, the creation [*Gedankenwerk*] of the important logicians Frege, Schröder, Peano, and Russell and observe what it teaches us about the relationship of mathematics to logic [*des Mathematischen zum Logischen*].

A peculiar double sidedness of this relationship, which is revealed in the different version of the task of theoretical logic, becomes immediately apparent: While Frege strives to subordinate the mathematical concepts to the concept-formations [*Begriffsbildungen*] of logic, Schröder attempts conversely to emphasize the mathematical character of logical relationships and develops his theory as an "algebra of logic."

However, this is only a difference in emphasis. In the various systems of logistics the specifically logical point of view nowhere rules alone; rather it is permeated from the beginning with a mathematical mode of consideration. The mathematical formalism and the mathematical concept formation prove to be, in a way completely analogous to the area of theoretical physics, the proper aid in the representation of the connections and in the achievement of a systematic overview.

It is not the usual formalism of algebra and analysis that is applied here, though, but rather a newly created calculus, which theoretical logic develops with the aid of the language of formulas, by means of which it represents the logical connections. No one familiar with this calculus will doubt that both this calculus and its theory have a pronouncedly mathematical character.

This state of affairs shows that the concept of the mathematical needs to be delimited, independently of the factual stock of mathematical disciplines, by means of a fundamental characterization of the mathematical way of knowledge [*Erkenntnisart*]. If we pursue what we mean by the mathematical character of a consideration, it becomes apparent that the typical characteristic is located in a certain mode of abstraction that comes into play. This abstraction, which may be called *formal* or *mathematical abstraction*, consists in emphasizing and exclusively taking into account the structural elements of an object—"object" here is meant in its widest |[Mancosu: 239] sense—that is, the manner of its composition from its constituent parts. One may, accordingly, define mathematical knowledge as that which rests on the *structural* consideration of objects.

A study of theoretical logic further teaches us that in the relationship between mathematics and logic the mathematical way of consideration, in contrast to the contentual logical way, under certain circumstances constitutes the standpoint of higher abstraction. The aforementioned analogy between theoretical logic and theoretical physics extends in such a way, that just as the mathematical lawlikeness of theoretical physics is contentually specialized by means of its physical interpretation, so the mathematical relationships of theoretical logic also experience a specialization through their contentual logical interpretation. *The lawlikeness of the logical relationships appears here as a special model for a mathematical formalism.*

This peculiar relationship between logic and mathematics, that is, that not only can one subject mathematical judgments and inferences to logical abstraction, but also the logical relationships to a mathematical abstraction, has its reason in the special position the area of "the formal" [*des Formalen*] occupies vis-à-vis logic. Namely, whereas in logic one can usually abstract from the specific determinations of any domain of logic, this is not possible in the area of the formal, because *formal elements enter essentially into logic.*

This is especially true for *logical inference.* Theoretical logic teaches that one can "formalize" a logical proof. The method of formalization consists first of all in the representation of the initial propositions of the proof with the aid of the logical language of formulas by means of specific formulas, and further in the replacement of the principles of logical inference by rules that specify certain procedures, according to which one proceeds from given formulas to other formulas. The result of the proof is represented by an end formula, which expresses the proposition to be proven on the basis of the interpretation of the logical language of formulas.

In this way it becomes evident that all logical inference, observed according to its course [*Verlauf*], can be reduced to a limited number of logical elementary processes that can be exactly and completely enumerated. In this way it becomes possible to pursue systematically the questions of *provability.* There results here a field of theoretical research, within which the theory, developed in traditional logic, of the various possible forms of categorical syllogisms constitutes only a very specific, special problem.

The typically mathematical character of the theory of provability manifests itself especially clearly through the role of logical *symbolism* [*Symbolik*]. Symbolism in this case is the *tool for the accomplishment of the formal abstraction.* The transition from the contentual logical to the formal approach takes place in such a way that we disregard the original meaning of the logical symbols and make the symbols themselves representatives of formal objects and connections.

If, for example, the hypothetical relationship

"if A, then B"

is symbolically represented by

$$A \to B$$

|*Mancosu:* 240 then the transition to the formal point of view consists in abstracting from the mean ing of the symbol $\to$ and in taking the connection by means of the "sign" $\to$ itself as that which is to be contemplated. In this way a figural specialization takes the place of the original contentual specialization

of the connection; this is, however, harmless to the extent that it is readily grasped as an inessential element. Mathematical thought accomplishes the formal abstraction just by means of the symbolic figure.

The method of formal consideration is not artificially introduced, but rather it appears almost by necessity if one wants to pursue more closely the process of logical inference with respect to its outcome.

If we consider, then, why it is that the examination of logical inference is in such need of the mathematical method, we find the following [fact]. In the process of demonstration, there are two significant moments that work together: the clarification of the concepts, that is, the moment of *reflection*, and the mathematical moment of *combination*.

As long as inference rests only on the clarification of the meanings, it is in the strictest sense analytical; the progress to something new takes place only through mathematical combination.

This combinatorial element seems so obvious that it is not at all regarded as a special factor. Especially in philosophy it was always customary to consider as epistemologically problematic and in need of explication only that aspect of a theorem [*an einer deduktiv gewonnenen Erkenntnis*] that is the given for the proof. the initial propositions and the rules of inference. This point of view is, however, inadequate for the the philosophical understanding of mathematics, because the typical achievement of a mathematical proof only begins after the starting propositions and the rules of inference are already fixed, and the astounding thing of mathematical results does not disappear if we modify the provable propositions contentually by introducing the highest assumptions of the theory as premises and besides by also specifying explicitly the rules of inference, in the sense of the formal point of view.

For the clarification of the facts of the matter, Weyl's comparison of a proof carried out purely formally to a chess game can be helpful to us; the initial position in the game corresponds to the initial sentences in the proof, and the rules of the game correspond to the rules of inference. Let us now assume that an astute chess champion has discovered for a certain initial position A the possibility to checkmate his opponent in ten moves. From the point of view of the customary mode of consideration, we have to say that this possibility is *logically* determined by the initial position and the rules of the game. On the other hand, however, one cannot main tain that the claim that in ten moves the opponent can be checkmated is logically contained [*sinnesmäßig enthalten*] in giving [*Angabe von*] the initial position A and the rules of the game. The appearance of a contradiction between these claims disappears if we make it clear to ourselves that the "logical" outcome of the rules of the game rests on *combination*, and it comes to light therefore not by means of a mere analysis of meaning but only through a real demonstration [*Vorführung*].

Every mathematical proof [*Beweis*] is, in this sense, a demonstration [*Vorführung*]. Let us show, by means of a simple special case, how the combinatory element appears in a proof.

We have the inference rule: "If A holds and if from A follows B, then B holds." |[Mancosu: 241] In a formalized proof [*ins Formale übersetzen*] to this inference principle corresponds the rule that from two formulas A, $A \to B$ the formula B can be derived. Now let this rule be applied in a formal derivation and indeed let us assume that A and $A \to B$ do not belong to the initial formulas. Then we have an inference sequence S that leads to A, and a sequence T that leads to $A \to B$; and so the formulas A, $A \to B$ yield, in accordance with the aforementioned rule, the formula B.

If we want to analyze what takes place here, we must take care not to anticipate the decisive point through the manner of denotation. Namely, the end formula of the inference sequence T is first of all only determined as such, and it is a new step for knowledge to establish the coincidence of this formula with the one originating from the other given formula A and from the formula B that is to be derived, by means of connection [*Zusammenfügung*] through "→."

The establishment of an identity is in no way always an identical (tautological) establishment. The coincidence, which is to be found in the present case, cannot be directly read off from the content of the formal inference rules and from the structure of the initial formulas, but rather it can only be read off from that structure obtained through the application of the inference rules, that is, through the carrying out of the inferences. There exists here in fact a combinatorial element.[2]

If we clearly understand in this way the role of mathematics [*des Mathematischen*] in logic, the possibility of the inclusion of arithmetic into the system of theoretical logic will not seem astonishing. However, this inclusion also loses its epistemological significance for the standpoint we have reached. For we know in advance that the formal element is not eliminated by means of

[2]P. Hertz made the claim, in his work "On Thought" [*Über das Denken*] (1923) that the logical inference contains "synthetic elements." His grounding for this claim, which will be expounded in a forthcoming treatise on the nature of logic, includes the point of view articulated here, but relies also on other considerations.

the inclusion of arithmetic into logical systematics [*logische Systematik*]. With reference to the formal [*sphere*], mathematical consideration represents, as we found, the point of view of higher abstraction when compared to conceptually logical consideration. Therefore we cannot gain a higher generality for mathematical knowledge [*mathematische Erkenntnisse*] through its inclusion into logic, but rather on the contrary only a *specialization through logical interpretation*, which is a kind of *logical clothing*.

A typical example of such a logical clothing is represented by the method according to which the natural numbers are defined by Frege, and following him, with a certain modification, by Russell.

Let us recall briefly the train of thought of Frege's theory. Frege introduces the numbers [*Zahlen*] as Numbers [*Anzahlen*] (cardinal numbers). His starting theses are as follows:

The Number applies to a predicate as determination [*Die Anzahl kommt als Bestimmung einem Prädikat zu*]. The Number concept originates from the concept of equinumerosity. Two predicates are said to be equinumerous if the things to which the one predicate applies can be reversibly [*umkehrbar*] and uniquely associated to those to which the other predicate applies.

If the predicates are partitioned into classes with respect to equinumerosity in such a way that all predicates of a class are equinumerous to one another and predicates of different classes are not equinumerous, then each such class represents the Number, which applies to the predicates that belong to it.

In the sense of this general Number definition, the individual finite numbers such as 0, 1, 2, 3 are defined as follows:

0 is the class of predicates which do not apply to anything. 1 is the class of |$^{Mancosu:\ 242}$ "one-valued" [*einzahlig*] predicates, and a predicate P is called "one-valued" if there is a thing x to which P applies, and no other thing (different from x), to which P applies. Correspondingly, a predicate is two-valued [*zweizahlig*], if there is a thing x and a thing y that differs from x, so that P applies to x and to y, and if there is nothing that differs from x and y to which P applies. 2 is the class of two-valued predicates. The numbers 3, 4, 5 and so on are to be defined analogously as classes.

Frege defines the general concept of a finite number, after he previously introduced the concept of a number that immediately follows another number [*the successor*], in the following way: A number n is said to be finite if any predicate that applies to 0 and that, when it applies to the number a, also applies to the number that immediately follows it, also applies to n.

The concept of a number that belongs to the number sequence from 0 to n is explained in a similar way. The derivation of the principles of number theory from the concept of finite number is based on these concept-formations.

Now we want to examine in particular Frege's definitions of the individual finite numbers. Take the definition of the number 2, which is defined as the class of two-valued predicates. Against this definition goes the objection that the membership of a predicate in the class of two-valued predicates depends on extralogical conditions, and therefore the class does not constitute a logical object.

This objection is taken care of, however, if we adopt the point of view of Russell's theory regarding the conception of the classes (respectively, sets, extensions of concepts). According to this [*theory*] the classes (extensions of concepts) do not at all constitute real objects; rather, they function only as dependent expressions within a paraphrased proposition. If, for example, K is the class of things with the property E, that is, the extension [*Umfang*] of the concept E, then we are to consider, according to Russell, the statement that a thing a belongs to class K only as a paraphrase of the statement that a thing a has the property E.

If we combine this view with Frege's Number definitions, we arrive at the point of defining the number 2, not by the class of two-valued predicates, but rather by that concept, whose extension constitutes this class. The number 2 is then identified with the *property of two-valuedness* of a predicate, that is, with the property of a predicate to apply to a thing x and to a thing y, which is different from x, but not to a thing that is different from x and y.[3]

For the evaluation of this definition it is essential in which sense the definition [*das Definieren*] is meant and what it is supposed to achieve [*mit welchem Anspruch es geschieht*]. What should be shown here is that this definition cannot be regarded as an account of the true meaning of the Number concept "two," by which this concept, freed from all inessential elements, would be unveiled in its logical purity, but rather that exactly the specifically logical element of the definition is an inessential addition.

Namely, the two-valuedness of a predicate P means nothing other than that there are two things to which the predicate P applies. Here three conceptual elements are separate from each other: the concept "two things," the existential element, and the fact that the predicate P applies [*das Zutreffen*

[3]For the sake of simplicity the discussions about the concept of difference, or more precisely the concept of identity, that contradicts it will be passed over.

des Prädikates P]. The conceptual content of "two things" is not logically dependent on one of the two other conceptual contents. "Two things" already means something, even without the claim of existence of two things, and also without reference to a predicate, |[*Mancosu*: 243] which applies to the two things; it means, simply, "a thing and one more thing."

With respect to this simple definition, the Number concept turns out to be an elementary *structural concept*. The appearance that this concept is obtained from the elements of logic emerges in the logical definitions of Number we have considered by joining the concept with logical elements, namely, the existential form and the subject-predicate relationship, which in themselves are inessential for the number concept. Here we actually intend a *logical clothing* of a formal concept.

The result of these considerations is that the claim of the logicists [*Logisten*] that mathematics is purely logical knowledge turns out to be blurred and misleading upon closer observation of theoretical logic. That claim is only correct when we take over the concept of the mathematical in the sense of the historical definition and we systematically extend, in contrast, the concept of the logical. However, through this definition what is epistemologically essential is concealed, and what is peculiar to mathematics is overlooked.

1.3 Formal Abstraction

We have established formal abstraction as the defining characteristic of the mathematical mode of knowledge, that is, the focusing on the structural side of objects, and thereby we have delimited the field of mathematics in a fundamental way. If we likewise want to grasp epistemologically the concept of the logical, then we are prompted to select from the entire field of the theory of concepts, judgments, and inferences, which is generally denoted as logic, a more narrow subfield, *reflective or philosophical logic*, which is the area of properly analytical knowledge, that is, knowledge that originates from the pure *consciousness of meaning.* Systematic logic is connected to this philosophical logic in that it gathers its initial elements and principles from the results of philosophical logic and develops from these a theory according to mathematical method.

In this way the part of genuine analytical knowledge is clearly separated from that of mathematical knowledge, and in this way it is emphasized what is justified, on the one hand, in Kant's *Theory of Pure Intuition*, and, on the other hand, in the claim of the logicists. We can separate Kant's basic idea that mathematical knowledge, and in general the *successful application of logical inference*, rests on intuitive evidence, from the particular formulation that Kant gave to this idea in his theory of space and time. In this way we simultaneously gain the possibility of doing justice to the very elementary character of mathematical evidence and to the level of abstraction [*Abstraktionshöhe*] of the mathematical attitude. The claim concerning the logical character of mathematics aims at emphasizing these two aspects.

The view we have reached also gives a simple piece of information about the role of number in mathematics: We have defined mathematics as the knowledge that rests on the formal (structural) consideration of objects. The numbers, however, as Numbers constitute the *simplest formal determinations* and as ordinal numbers the *simplest formal objects.*

The Number concepts offer a particular difficulty to philosophical discussion on account of their categorial special position, which also shows itself in language in the necessity for a separate category of terms, that is, the numerals [*Zahlworte*]. |$^{Mancosu: 244}$ We do not need to get more deeply involved with the discussion, but rather we only need to pay attention to the fact that the determinations of Number [*Anzahlbestimmungen*] concern the composition from components of a total complex of that which is given or represented, that is, exactly what constitutes the structural side of an object. And indeed it is the most elementary structural characteristics that are given by the Numbers. In this way the Numbers appear in all areas that are accessible to formal consideration; in particular, we come across the Number in the most diverse ways within theoretical logic, for example, as Number of subjects of a predicate (or as is said: as Number of arguments of a logical function), as the Number of the variable predicates that go into a logical proposition, as the Number of the applications of a logical operation within a concept-formation or within a proposition, as a Number of the propositions within an inferential figure, as the number of the type [*Stufenzahl*] of a logical expression, that is, as the maximum of nestings of the subject-predicate relationships that occur in it (in the sense of the rise from the objects of a theory to the predicates, from the predicates to the predicates of predicates, from these again to their predicates, etc.).

The Numbers give us, however, only formal determinations but not yet formal objects. For example, in the representation [*Vorstellung*] of the number three the combination of the three things into one object is still not present. The connection of several things into one object requires a form of ordering. The simplest ordinal form [*Ordnungsform*] is that of the simple succession, which leads to the concept of *ordinal number.* The ordinal number is in and of itself also not determined as object; it is only a place marker [*Stellenzeiger*]. We can objectively [*gegenständlich*] standardize it by choosing as a *place marker the simplest structure from those that originate in the form of the succession.* Corresponding to the twofold possibility of beginning the number sequence with I or with 0, two kinds of standardization may be considered. The first one is based on a sort of things and a form of addition of a thing; the objects are figures that begin and end with a thing of the appropriate sort, and in which every thing, which does not yet constitute the end of the figure, is followed by an added thing of that kind. In the other kind of standardization we have an initial thing and a process; the objects are then the initial thing itself and further the figures one obtains, beginning with the initial thing, through a single or repeated application of that process.

If we want to have, in the sense of the one or the other standardization, the ordinal numbers as definite [*eindeutig*] objects free from all inessential elements, then in each case we have to take the mere scheme of the relevant figure of repetition [*Wiederholungsfigur*] as an object; this requires a very high abstraction. We are free to represent these purely formal objects through concrete objects ("numerals" [*Zahlzeichen*]); these contain then inessential, arbitrarily added properties that, however, are also easily grasped as such.[4]

[4]The philosopher is inclined to identify this relationship of representation as a meaning connection. One should, however, be aware that in contrast to the customary relationship of word to denotation, the important difference consists here in the fact that the representing object contains in its composition the essential qualities of the represented object, so that the relationships of the represented objects that are to be investigated are to be found in the representatives and can themselves be established by means of an examination of

This procedure takes place each time on the basis of a certain convention, which must be adhered to in the context of one and the same consideration. Such a convention, in the sense of the first standardization, is that according to which the first ordinal numbers are represented by the figures 1, 11, 111, 1111. In accordance with one of the conventions corresponding to the second standardization, the first ordinal numbers are represented by the figures $0, 0', 0'', 0''', 0''''$.

By thus finding easy access to the numbers from the structural side, our |*Mancosu*: 245 conception about the character of mathematical knowledge receives a new validation. For the dominating role of number in mathematics becomes comprehensible from this conception, and our characterization of mathematics as the theory of structures seems to be the appropriate extension of the claim mentioned at the beginning, that the numbers constitute the proper object of mathematics.

The satisfactoriness of the point of view we have attained must not tempt us to the opinion that we have already achieved all required basic insights required for the problem of the foundations of mathematics. We have indeed so far only treated the preliminary question, about which we first wanted to gain clarity, namely, where the specific character of mathematical knowledge is to be seen. Now, however, we must turn to that problem that causes the main difficulties in the foundations of mathematics, the problem of the infinite.

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the representatives.

2 The Problem of the Infinite and Mathematical Idea-formations [*Ideenbildungen*]

2.1 The Postulates of the Theory of the Infinite; the Impossibility of Their Grounding by Means of Intuition—the Finitistic Point of View

The mathematical theory of the infinite is analysis (infinitesimal calculus) and its extension by means of general set theory. We can limit ourselves here to the consideration of the infinitesimal calculus, since the step from it to general set theory requires the addition of assumptions but no fundamental modification of the philosophical conception.

The grounding of the infinitesimal calculus by Cantor, Dedekind, and Weierstrass shows that the rigorous construction of this theory succeeds if, in addition to the elementary inference modes of mathematics, the following two are included:

1. The application of the method of existential inference [*das existentiale Schließen*] to the whole numbers, that is, taking the *system* of whole numbers as basic in the manner of a domain of objects of an axiomatic theory—as it is explicitly brought to expression in *Peano's axiom system of number theory*;

2. The idea of the totality of all sets of whole numbers as a combinatorially surveyable [*übersehbar*] manifold. A set of whole numbers is determined by a distribution [*Verteilung*] of the values 0 and 1 to the positions in

> the number sequence. The number n either does or does not belong to the set depending on whether 1 or 0 is in the nth position of the distribution. Now, just as the totality of the possible distribution of the values 0, 1 is completely surveyable for a finite number of positions, say five, this is analogously assumed for the whole number sequence.

In particular, the validity of *Zermelo's principle of choice* [*Auswahlprinzip*] for collections of sets of numbers results from this analogy. We wish, however, to put aside the discussion of this principle for the time being; it will easily fit in later.

If we study these demands from the point of view of our general characterization of mathematical knowledge, then it seems, at first, that no kind of fundamental difficulty exists for their grounding through mathematical knowledge. For with |*Mancosu*: 246 the number sequence as with the set formations derived from it, it is a question of *structures* that differ from the structures dealt with in elementary mathematics only in that they are structures of infinite manifolds. Existential inference applied to the numbers also appears to be justified through the object character of the numbers as formal objects, whose existence, however, cannot depend on the contingency of the factual representations of the numbers.

However, in opposition to this argument it is to be noted that it is rash to infer from the character of the formal objects, that is, from the detachment [*Loslösung*] present in them from empirical contingencies, that the formal objects must be related to a realm of the formal existent [*des existierenden Formalen*]. We could mention the set-theoretical paradoxes as an argument against this view; however, it is simpler to refer directly to the fact that in primitive mathematical evidence there is no positing of such an area of existing formal objects, but rather that the tie to that which was actually represented is essential as a point of departure for the formal abstraction. In this sense the Kantian proposition, that pure intuition is the form of empirical intuition, is valid.

It also corresponds to this that in the disciplines that proceed from elementary mathematical evidence, existential statements only have an improper meaning. Particularly in elementary number theory we are concerned only with existential statements that relate to a quite determined, presentable collection [*vorweisbare Gesamtheit*] of numbers or to a determined, intuitively performable [*vorführbar*] process, or with the two together, that is, to a collection of numbers that is to be arrived at by means of a performable process.

Examples of this kind of existential claims are: "There is a prime number between 5 and 10"; namely, 7 is a prime number.

"For every number there is a larger one"; namely, when n is a number, then one forms $n + 1$; this number is larger than n.

"For every prime number there is a larger one"; namely, if a prime number p is given, then one forms the product of this prime number with all prime numbers smaller than it and adds 1; if k is the resulting number, then among the numbers from $p + 1$ to k there exists in any case a prime number.

In each of these cases the existential statement is made precise by means of further information; the existential claim keeps to the formation processes that can be carried out [*vollziehbar*] in intuitive representation and does not refer to a manifold of all numbers. We wish to follow Hilbert and designate this elementary way of looking at things, linked to the conditions of basic representability, as the *finitistic* point of view [*der finite Standpunkt*] and in the same sense speak of finitary methods, finitary considerations, and finitary inferences [*von finiten Methoden, finiten Überlegungen undfiniten Schlüssen*].

It is now easy to determine that the existential inference goes beyond the finitistic point of view. This already takes place in every existential proposition that is established without further specification of the existential claim, as, for example, in the statement that every unbounded arithmetic sequence

$$a * n + b, \quad n = 0, 1, 2, 3, \ldots$$

in which a, b are relatively prime, contains at least one prime number. |*Mancosu*: 247

A particularly common and important case of going beyond the finitistic point of view is the inference from the failure of the universal validity of a proposition (for all numbers) to the existence of a counterexample, or, in other words, the principle, according to which for every number predicate $P(n)$ the alternative holds: Either the universal proposition is valid (i.e., $P(n)$ holds of all numbers n), or there is a number n, such that $P(n)$ does not hold. This principle results, from the point of view of existential inference, as a direct application of the principle of the excluded middle, that is, from the meaning of negation. That this logical consequence does not hold for the finitistic point of view lies in the fact that here the claim of the validity of $P(n)$ for all numbers has the purely hypothetical meaning of validity for each given number, so that the negation of this claim does not yield a positive existential statement.

However, with this the discussion of the possibilities of an evident [*einsichtig*] mathematical grounding of the assumptions of analysis is still not closed; granted, that positing a total domain of formal objects as a basis does not correspond to the point of view of primitive mathematical evidence, however, the requirements of the infinitesimal calculus could be motivated by the fact that the totalities of numbers and of sets of numbers are *structures of infinite sets*. In particular, the application of existential inference to the numbers should, accordingly, not be derived from the idea of the totality of numbers in the realm of formal objects, but rather from the consideration of the structure of the number sequence in which the single numbers occur as members of the sequence. We have in fact not yet dealt with the aforementioned argument [*to the effect*] that mathematical knowledge could also concern structures of infinite manifolds.

In this way we come to the question of the *actual infinite*. For the infinite that is at issue in infinite manifolds is the proper actual infinite, in contrast to the "potential infinite," which does not signify an infinite object but rather merely the unboundedness in the progress from the finite to an always new finite [*zu immer neuem Endlichen*], as is the case, for example, for the numbers even from the finitistic point of view, insofar as for every number one can always form a larger number.

The question that we have to pose first of all as regards the actual infinite refers to whether the actual infinite is given to us as an object of intuitive mathematical knowledge.

One could be of the opinion, in agreement with our previous statements, that we are actually capable of an intuitive knowledge of the actual infinite. For even though it is certain that we have a concrete representation only of finite objects, an accomplishment of formal abstraction could consist exactly in the fact that it frees itself from the limitation of the finite and that it completes, as it were, the passage to the limit in certain processes that can be arbitrarily continued. In particular, one will be tempted to refer to geometric intuition and adduce examples of intuitively given infinite manifolds from the domain of geometric objects.

However, first of all, geometric examples do not prove anything. One is easily mistaken here in interpreting the intuitive-spatial [*das Anschaulich-Räumliche*] in the sense of an existential conception. A segment, for example, is given intuitively not as an ordered manifold of points but rather as a homogeneous whole, though as an extended whole within which *positions* can be differentiated. The representation of a position on the segment is an intuitive representation, but the totality *of all positions* |[Mancosu: 248] on the segment is only a conceptual totality [*gedanklicher Inbegriff*]. By means of intuition we come here only to the potential infinite, in that to each position on the segment corresponds a division into two parts [*Teilstrecken*], in which every part can again be divided into parts.

What further concerns infinitely extended entities, such as the infinite line, the infinite plane, the infinite space, is that these cannot be exhibited as objects of an intuitive representation. In particular, space as a whole is not given to us in intuition. We imagine every spatial object as if it were contained in space. But this relationship of the single three-dimensional object [*des einzelnen Räumlichen*] to the whole of space is only objective in intuition as far as with every spatial object a spatial environment is intuitively represented at the same time. Moreover, the inclusion in the whole of space [*Gesamtraum*] is—we have to claim this in opposition to Kant—*only* *intellectually* graspable [*gedanklich fassbar*].

The main argument, which Kant adduces in favor of the intuitive character of our representation of space as a whole, proves' in fact only that one cannot arrive at the concept of a single inclusive space by means of a mere generalizing abstraction. But with the claim of the merely intellectual accessibility of our representation of the whole of space [*Raumganzes*], it should not be suggested that we are concerned here with some kind of mere general concept.

What is meant is rather a more complicated state of affairs, namely, that in the representation of the whole of space two kinds of different thought-formations [*Gedankenbildungen*] are given, both of which go beyond the standpoint of intuition as well as beyond that of reflective [*reflektierende*] logic. The one relies ou the concept of the connection of things to the world totality [*Weltganzen*]; it originates then from our belief in reality. The other is a *mathematical idea-formation* [*Ideenbildung*] that certainly starts out with experience, but, however, does not remain in the realm of that which can be intuitively represented; it is the representation of space as a point manifold subject to the laws of geometry.[5]

In these two ways of representing space as a whole, this totality is not recognized as given but rather only experimentally posited [*versuchsweise ange-*

[5]In the view of nature of Newtonian physics these two representations of space are united with each other and do not clearly contrast with one another. Euclidian geometry constitutes here the law for the spatial association of things in the universe [*Weltganzen*]. Only through the present development in geometry and physics did the necessity arise to differentiate between space as something physical and space, as an ideal manifold, determined by geometric laws.

setzt]. The representation of the physical whole of space is fundamentally problematic; nevertheless, exactly from the standpoint of present-day physics the possibility exists of giving this very vague idea a more narrow and precise formulation, through which it can become accessible and systematically important for research. The geometric idea-formations of spatial manifolds are, to be sure, from the outset precise; however, they do require proof of their consistency.

We have then no reason to assume that we possess an intuitive representation of space as a whole. We cannot directly exhibit such a representation, and there is also no necessity to introduce this assumption as an explanatory ground. But if we deny the intuitiveness of the whole of space, then we also will not claim that infinitely extended spatial entities are intuitively represented.

It should also be noted that the original intuitive conception of elementary Euclidian geometry does not even require a representation of infinite entities. There we always have to deal only with finitely extended figures. Also, infinite manifolds never occur, since no general existence assumptions are used as a basis; rather every existence claim consists in the claim of the possibility of a geometrical constructions. [Translator's notes: *Konstruktion*; the 1976 reprint reads *Konjunktion* but this is clearly a misprint.] For example, from this standpoint, that every segment has |*Mancosu*: 249 a midpoint says nothing other than that for every segment a midpoint can be constructed.[6]

[6]In Euclid's axioms this point of view is, however, not always consistently carried out as the concept of the sufficiently large extension [*Verlängerung*] of a segment appears in them. This concept is in fact avoidable; one only has to give a different formulation of the parallel axiom.

With this the appearance of the exhibitability of the actual infinite in the realm of objects of geometric intuition turns out to be deceptive. However, we can make clear to ourselves in a more general way that a removal of the condition of finiteness through formal abstraction, as would be required for the intuition of the actual infinite, is here out of the question. The condition of finiteness is not an accidental empirical restriction, but rather an essential characteristic of a formal object.

The empirical restriction still lies within the realm of the finite, where the formal abstraction must aid us beyond the borders of the factual faculty of imagination [*faktische Vorstellungskraft*]. A clear example of this is the unlimited divisibility of a segment. Our actual imaginative faculty fails here, as soon as the division exceeds a certain degree of fineness. This threshold is physically accidental, and we can get beyond it with the aid of optical apparatus. However, all optical apparatus fail at a certain degree of minuteness, and in the end our spatial-metric representations become physically meaningless. With the representation of the unlimited divisibility we already abstract from the conditions of factual representation as well as from those of physical reality.

Things are similar with the representation of unlimited addition in number theory. Here also there exist thresholds for the execution of the iterations both in the sense of actual representability as well as in the sense of physical realization. Let us consider, for example, the number $10^{10^{1000}}$. We can arrive at this number in a finitistic way as follows: We start from the number 10 which we represent, according to one of our earlier standardizations, by the figure

$$1111111111$$

Let now z be any number that is represented by a corresponding figure. If in the previous figure we replace every 1 by the figure z, then we obtain again, as we can intuitively make clear to ourselves, a number figure that for the purposes of communication is denoted by "$10 \times z$."[7] In this manner we obtain the process of a decuplication of a number. From this we arrive at the process of transition from a number a to 10^a, as follows. We let the number 10 correspond to the first I in a and to every affixed I we apply the process of decuplication, and we keep going until we exhaust the figure a. The number obtained by means of the last process of decuplication is denoted by 10^a.

This procedure offers basically no difficulty at all for the intuitive view. If we want to visualize the process in detail, then our representation already fails with quite small numbers. We can aid ourselves somewhat with apparatus or with the use [*Heranziehen*] of objects of external nature in which very large Number determinations occur. But even with all this we soon come to a threshold: We can easily represent the number 20; 10^{20} exceeds by far our actual power of representation but lies entirely in the realm of physical realization; finally, it is highly questionable whether the number $10^{10^{20}}$ exists in any way as a ratio of quantities or as a Number determination in physical reality.

Intuitive abstraction is not concerned with such thresholds for the possibility of the realization. For these thresholds are accidental from the point of view of the formal consideration. Formal abstraction, as it were, does not

[7] We are here dealing with a sign "with meaning."

find an earlier |$^{Mancosu: 250}$ position for a delimitation in principle than the difference between finite and infinite.

This difference is in fact fundamental. If we consider more carefully how an infinite manifold can be characterized as such, we find that this is not at all possible in the form of an intuitive exhibition but rather only by way of a claim (or an assumption or determination) of a lawlike relationship. Infinite manifolds are therefore accessible to us only through *thought*. This thought is, to be sure, also a manner of representation, but in this way what is represented is not the manifold as an object but rather conditions that are satisfied (or more precisely, are to be satisfied) by a manifold.

The essential dependence of formal abstraction on the element of finiteness particularly asserts itself in that in considerations of totalities and of figures the property of finiteness does not constitute a particularly restrictive characteristic from the point of view of intuitive evidence. The restriction to the finite is accomplished automatically from this point of view, as it were, *tacitly*. We do not need here any special definition of finiteness, for the finiteness of objects is a matter of course for formal abstraction. So, for example, the intuitive-structural introduction of the numbers is appropriate only for the finite numbers. From the point of view of intuitive-formal consideration, "iteration" is *eo ipso* finite iteration.

This representation of the finite, implicitly given in the formal view, contains the epistemic grounds for the principle of complete induction and for the admissibility of definition by recursion, both procedures being understood in their elementary form as "finite induction" and "finite recursion."

Of course, this introduction to the representation of the finite does not belong to that which necessarily enters from intuitive evidence into logical inference. It corresponds rather to a point of view, according to which one already *reflects* on the general characteristic features of the intuitive objects. The application of the intuitive representation of the finite can be avoided for number theory if one forgoes treating this theory in an elementary way. However, the intuitive representation of finiteness shows up of necessity as soon as one makes formalism itself the object of consideration, in particular in the systematic theory of logical inference. In this way it is expressed that finiteness is an essential element of the entities [*Gebilde*] of every formalism. The boundaries of the formalism are, however, none other than those of representability, particularly of intuitive compositions [*Zusammensetzungen*].

Thus our answer to the question about the intuitive knowability of the actual infinite is negative. Another result is that the method of finitistic consideration is the appropriate method from the point of view of intuitive mathematical knowledge.

However, in this way we do not achieve a verification of the mentioned assumptions for the infinitesimal calculus.

2.2 Intuitionism—Arithmetic as Theoretical Frame

How should we react in the face of this state of affairs? The views are divided as to how to answer this question. A similar conflict of views takes place to the one we have encountered in the question of the characterization of mathematical knowledge. The proponents of the point of view of primitive intuition [*primitive* |[Mancosu: 251] *Anschaulichkeit*] easily conclude from the circumstance that analysis and set theory go beyond the finitistic point of view on account of their postulates that these mathematical theories are to be abandoned in their present form and must be revised from their very foundations. The supporters of the point of view of theoretical logic, on the other hand, seek either to ground those postulates of the theory of the infinite by means of logic, or they dispute in general the problematical nature of the postulates in that they do not attribute any fundamental significance to the difference between the finite and the infinite.

The former view was, already at the time of the first appearance of the method of existential inference, represented by Kronecker, who was probably the first to examine closely and strongly emphasize the importance of the methodical point of view, which we term the finitistic. His attempts towards the fulfillment of this methodological demand in the area of analysis remained, however, fragmentary; moreover, there was a lack of an exact philosophical explanation of the point of view. In particular, the often-cited remark of Kronecker, that God created the whole numbers and that everything else is the work of man, is not at A suited to the motivation of the demands represented by Kronecker[8]: If the whole numbers are created by God, then one should think that the existential inference is admissible for application to the numbers, while Kronecker, however, excludes exactly the existential way of consideration in the case of the whole numbers.

Brouwer extended Kronecker's point of view in two directions: on the one hand with regard to philosophical motivation by putting forward his theory of "intuitionism,"[9] and on the other hand by showing how one can apply the

[8]The methodical point of view that is appropriate to this remark is the point of view that Weyl has taken in his text "The Continuum" (1918).

[9]It seems to me appropriate in the interest of the clarification of the discussion to

finitistic point of view in the area of analysis and set theory and how one can ground these theories at least to a considerable extent in a finitistic way through a radical reformulation of the concept formations and the modes of inference.

The result of this investigation naturally has a negative side in that it becomes apparent that in this process of finitistic treatment of analysis and set theory one has to accept not only considerable complications but also heavy losses in systematics [*Systematik*].

The complications appear already in the first concepts of the infinitesimal calculus, as in those of boundedness, the convergence of a number sequence, the difference between rational and irrational. Let us take, for example, the concept of boundedness of a sequence of whole numbers. According to the usual view the following alternative exists: Either the sequence exceeds every bound, it is then unbounded, or all numbers of the sequence lie below a bound, then the sequence is bounded. In order to retain a finitistic conceptual determination, we must sharpen the definition of boundedness and unboundedness in the following way: A sequence is said to be bounded if we can exhibit a bound for the numbers of the sequence either directly or by specifying a procedure; a sequence is said to be unbounded if there is a rule, according to which every bound of the sequence is necessarily exceeded, that is, if the assumption of a bound for the sequence leads to an absurdity.

By means of this formulation of the concepts the finitistic character of the definitions is obtained, but we now do not have a complete disjunction

use the expression "intuitionism" to denote a philosophical view, in contrast to the term "finitistic," which signifies a certain mode of inference and concept formation.

between the case of boundedness and the case of unboundedness. Therefore, from a proof that shows the untenability of the assumption of the unboundedness of a sequence, we |[Mancosu: 252] still cannot derive the boundedness of the sequence. By the same token, a proposition that is proved on the one hand under the assumption of the boundedness of a certain number sequence and, on the other hand, under the assumption of its unboundedness, cannot be considered as proven.

In addition to these kinds of complications, which pervade the whole theory, there is also the more important disadvantage that the general theorems through which mathematics gains its systematic clarity become for the most part invalid. Thus, for example, in Brouwer's analysis, not even the following theorem is valid: Every continuous function has a maximum (in a finite, closed interval).

It appears as an unjustified demand upon mathematics on the part of philosophy that mathematics should give up its simpler and more powerful methods in favor of an inconvenient method that is lacking in systematics [*Systematik*], without being led to do so by an inner necessity. The point of view of intuitionism becomes suspicious to us because of this unreasonable request.

Let us consider what the main points of the philosophical view developed by Brouwer amount to. It contains first of all a characterization of mathematical evidence. Our preceding remarks about formal abstraction are in agreement with this characterization on the essential points, particularly in taking its starting point in Kant's theory of pure intuition.

A difference, of course, consists in the fact that according to Brouwer's view, the temporal element belongs essentially to mathematical objectivity. However, we do not need to go into a discussion of this point here, since which way we decide on this issue has no influence on the formulation of the methodological question concerning mathematics: What for Brouwer follows as a consequence of the time dependence [*Zeit-Gebundenheit*] of mathematics is nothing other than what we have derived from the dependency of formal abstraction on the concrete-intuitive point of departure, namely, the methodical limitation of the finitistic procedure.

The crucial consequences of intuitionism follow from the further claim that any mathematical thinking that should be able to claim scientific validity must be carried out on the basis of mathematical evidence; that is, that the boundaries of math ematical evidence are simultaneously boundaries for mathematical thinking.

This demand of the restriction of mathematical thinking to that which is intuitively evident seems at first to be completely justified. Indeed, it corresponds to the view of mathematical certainty familiar to us. We must, however, consider that this common view of mathematics originally belonged together with a philosophical view, for which the intuitive evidence of the foundations of the infinitesimal calculus was not in question. We have departed from such a view because we found that the postulates of analysis cannot be verified by intuition, that is, that the idea of an infinite totality, taken as a basis in analysis, is not graspable in intuition but rather only in the sense of an idea-formation.

We cannot expect that this new view of the boundaries of intuitive evidence is readily consistent with the traditional view of the epistemic character of mathematics. Indeed, our remarks are grounds for suspicion that the widely held view of mathematics represents the facts of the matter too simplistically, and that we cannot account for that which takes place in mathematics from the point of view of evidence alone, but rather we must still grant *thought its own role.* |*Mancosu*: 253

We thus come to a differentiation between the elementary mathematical point of view and a systematic point of view that goes beyond it. This differentiation is not drawn artificially or merely ad hoc, but rather it corresponds to the duality of the points of departure that lead to arithmetic, namely, on the one hand the combinatorial activity with ratios in discrete quantities [*mit Verhältnissen im Diskreten*] and on the other hand the theoretical demand that is placed on mathematics from geometry and physics.[10] The system of arithmetic does not emerge only from a constructive and intuitively contemplating activity [*konstruierende und anschaulich betrachtende Tätigkeit*], but rather mostly from the task to grasp exactly and master theoretically the geometric and physical ideas of set, area, tangent, velocity, etc. The method of arithmetization is a means to this end. In order to serve this purpose, however, arithmetic must extend its methodical point of view from the original ele-

[10]It is remarkable that Jakob Friedrich Fries, who attributed to mathematical evidence a realm that went far beyond the finite—in particular, according to his view the "continuous sequence of the larger and smaller" is given in pure intuition—nevertheless drew a methodical differentiation between "arithmetic as a theory," which conceptualizes the intuitive representation of magnitude and develops it scientifically, and the "theory of combination or syntax [*Syntaktik*]," which rests solely on the postulate of arbitrary arrangement of given elements and their arbitrary repetition; it requires no axioms since its operations are "in themselves immediately comprehensible" (cf. J. F. Fries, *Mathematical Philosophy of Nature*, 1822).

mentary point of view of number theory to a *systematic view* in the sense of the aforementioned postulates.

Arithmetic, which forms the large frame in which the geometric and physical disciplines are incorporated [*eingeordnet*], does not simply consist in the elementary intuitive treatment of the numbers, but rather it has itself the character of a *theory* in that it takes as a basis the idea of the totality of numbers as a system of things as well as of the idea of totality of the sets of numbers. This systematic arithmetic fulfills its task in the best way possible, and there is no reason to object to its procedure, as long as we are clear about the fact that we do not take the point of view of elementary intuitiveness but that of thought-formation, that is, that point of view Hilbert calls the *axiomatic* point of view.

The reproach of arbitrariness is not justified against this axiomatic process, for in the foundations of systematic arithmetic we are not dealing with an arbitrary system of axioms, set up according to need, but rather with a *natural systematic extrapolation of elementary number theory.* And the analysis and set theory that develop from this foundation constitute a theory that is already *purely intellectually* distinguished [*rein gedanklich ausgezeichnet*], that is, suited to be taken as the $\kappa\alpha\tau\epsilon\eta o\xi\nu\eta$ in which we incorporate the systems and the theoretical statements of geometry and physics.

We therefore cannot recognize the veto that intuitionism directs against the procedure of analysis. The statement, about which we are in agreement with intuitionism, that the infinite is not given to us intuitively, compels us to modify our philosophical view of mathematics but not to reshape mathematics itself.

However, the problem of the infinite returns. For by taking a thought-formation as the point of departure for arithmetic we have introduced something problematic. An intellectual approach, however plausible and natural from the systematic point of view, still does not contain in itself the guarantee of its consistent realizability [*Durchführbarkeit*]. By grasping the idea of the infinite totality of numbers and the sets of numbers, it is still not out of the question that this idea could lead to a contradiction in its consequences. Thus it remains to investigate the question of freedom of contradiction [*Widerspruchsfreiheit*], of the "consistency" [*Konsistenz*] of the system of arithmetic.[11]

Intuitionism wants to spare us this task by limiting mathematics to the realm of finitistic consideration; this elimination of the problem, however, asks too high |*Mancosu*: 254 a price: The problem disappears, but the systematic simplicity and clarity of analysis are also lost.

2.3 The Problematic of Logistic Theory—Value of the Logistic

Incorporation [*Einordnung*] of Arithmetic The proponents of the logistic point of view believe to be able to come to terms with this problem in a completely different way. With the discussion of this point of view we take up our earlier reflections on logistics. There it was a matter of recognizing that intuitive evidence already enters into deductive logic and that the logical Number definitions do not prove the Number concepts as such to be of a specifically

[11]It may be suggested here to use this expression, which is used by Cantor specifically with regard to set formations, in general, in reference to any theoretical approach.

logical nature (as pure concepts of reflection), but are rather only logical standardizations of elementary structural concepts.

These considerations concern the separation of the [*realm of the*] logical in the more narrow sense from the [*realm of the*] formal. But with the recognition of the formal element in logic the methodological question of logistics is still by no means resolved. Logistics is not content with the theoretical development of the theory of inferences, but rather, as already mentioned, it takes as its task moreover to incorporate all of arithmetic into the logical formalism. This incorporation takes place in the following way: First of all, the Numbers are introduced as properties of predicates in the manner described earlier, and in addition—in a way that cannot be explained in more detail here—one expresses the modes of formation of the sets of numbers by means of the logical formalism, in doing so replacing every set with a predicate that defines it. The totality of number predicates thus takes the place of the totality of all sets of numbers.

In this way one indeed succeeds in assigning to every arithmetic statement a statement from the realm of theoretical logic, in which apart from the variables only "logical constants" occur, that is, logical basic operations, such as conjunction, negation, the form of universality, etc.

It is now clear that the problem of the infinite still cannot be solved only by means of this translation of arithmetic into the logical formalism. If theoretical logic deductively produces the system of arithmetic, then either explicit or hidden assumptions, through which the introduction of the actual infinite is brought about, must be contained in its procedures. The justification of these assumptions and the positions on them constituted right from the beginning the weak point of logistics. In this way Frege and Dedekind, whose arguments and considerations are otherwise marked by extreme precision and rigor, were uncritical in basing the point of view of general logic on a supposedly evident assumption, namely, the idea of a completed totality of all conceivable logical objects.

This idea, if it were tenable, would of course be systematically more satisfactory than the specific postulates of arithmetic. As is well known, this idea had to be dismissed on account of the contradictions to which it gave rise. Since then logistics does not attempt to prove the existence of an infinite totality; rather, it explicitly posits an *axiom of infinity*.

This axiom of infinity still does not suffice as an assumption for obtaining arithmetic as conceived logically. With it we would only obtain that which results from |[Mancosu: 255] the application of our first postulate, that is, from the admissibility of existential inference with regard to the whole numbers. In order to be in keeping with our second postulate, something more is required, namely, the application of existential in ference with *regard to the predicates*. The justification of this procedure may appear at first to be logically obvious, and for the view Frege and Dedekind took as a point of departure, it does not actually come into question. With the abandonment of the idea of the totality of all logical objects, the idea of a totality of all predicates also becomes problematic, and upon closer examination a particular fundamental difficulty becomes apparent.

Namely, it corresponds to the logical point of view that we conceive the totality of predicates as something, which for the most part come into being only in the context of the system of logic in such a way that the logical formation processes are applied to certain prelogical initial predicates that are derived, say, from intuition. By referring to the totality of predicates, more predicates are in turn obtained. An example of this is Frege's definition, mentioned earlier, of finite number: "A number n is said to be finite, if every predicate that applies to the number 0 and that, if it applies to the number a, also applies to its successor, also applies to n." Here the predicate of finiteness is defined with reference to the totality of all predicates.

Such definitions-termed "impredicative"[12]—appear everywhere in the foundations of arithmetic, and indeed in decisive places.

In itself there is nothing to object to in the fact that one determines a thing from a totality by a property that is related to this totality. So, for example, in the totality of numbers, a certain number is defined by the property that it is the largest of all the prime numbers, whose product by 1000 is larger than the product of the previous prime number by 1001.[13]

But here it is presupposed that the totality concerned is determined *independently* of the definitions that refer to it; otherwise we enter into a vicious circle.

[12]The term stems from Poincaré, who in contrast to the other critics of set theory, almost all of whom were only focusing on the axiom of choice, brought the view point of the impredicative definition into the discussion and attached importance to it. However, his critique was open to criticism insofar as he represents the use of impredicative definitions as an innovation introduced by set theory. Zermelo could object to him that the impredicative definitions substantially appear in the usual inferential modes of analysis, fully recognized by Poincaré. Since then Russell and Weyl, in particular, have thoroughly explained and brought to full clarity the role of impredicative definitions in analysis.

[13]The example is chosen so that the reference to the totality of numbers cannot be easily eliminated, as is usually the case in most simpler examples.

This precondition cannot easily be fulfilled, however, and precisely in the case of the totality of the predicates and of the impredicative definitions that refer to them, because the field of predicates is determined—according to the view discussed here—by means of the logical laws of formation [*Bildungsgesetze*], and the im predicative definitions also belong to this group.

In order to avoid the vicious circle, it would, however, suffice if it could be shown that every predicate introduced by an impredicative definition can also be defined in another way "predicatively." Indeed, one would even manage with a weaker statement. Since within the logical foundation of arithmetic a predicate is only considered with regard to its extension, that is, with regard to the set of things to which it applies, we would only need to know that every predicate introduced by an impredicative definition is co-extensional with a predicate that is predicatively defined.

Russell, who recognized with complete clarity the difficulty present in the im predicative definitions, posited this postulate as "*axiom of reducibility*" next to the axiom of infinity.

How, then, are we to understand this axiom of reducibility? It does not emerge from its formulation whether what is supposed to be expressed by it is a logical law or an extralogical assumption. |*Mancosu*: 256

In the first case, that is, if the reducibility axiom were the expression of a logical law, its validity would have to be independent of what kind of domain of prelogical initial predicates is used as a basis—provided at least that this domain satisfies the axiom of infinity. That would, however, mean that an axiomatic theory in which the forms of universal and existential judgment (existential inference) are applied only to the objects and not to the predicates is not capable of an extension of its predicate domain by means of the introduction of impredicative definitions provided only that the system of axioms is such that it requires an infinite system of objects to be satisfied.

However, the validity of such a statement is out of the question. One can easily construct examples that disprove this claim.

Such an example is provided by Dedekind's introduction of the number concept. Dedekind proceeds from a system in which an object 0 is distinguished and admits a reversible, one-to-one mapping onto a subset, to which that thing 0 does not belong. If we represent this mapping by means of a predicate with two subjects and formulate the required properties of this predicate as axioms, then we obtain an elementary axiom system, which in its axioms contains no reference to the totality of predicates and that furthermore can only be satisfied by an infinite system of objects. Let us now consider Dedekind's concept of number; its definition can be formulated quite analogously to Frege's definition of finite number, by translating it from the language of set theory into the language of theory of predicates: "a thing n of our system is a number, if every predicate that applies to 0 and that, if it applies to a thing a of our system, also applies to that thing that is assigned to the thing a in the reversible one-one mapping, also applies to n." This definition is impredicative; and one can see that it is not possible to obtain a co-extensive predicate to that just defined of "being a Number by means of a predicative definition from the basic elements of the theory."[14]

We find consequently that we only need to consider the second interpre-

[14]Waismann has given another example (in a note on "The Nature of the Reducibility Axiom," 1928). This, however, is in need of modification.

tation of the reducibility axiom, according to which the axiom expresses a *demand on the initial domain of prelogical predicates* [*eine Anforderung an den Ausgangsbereich der vorlogischen Prädikate*].

With the introduction of such an assumption, however, one renounces the ide that the domain of the predicates is produced by the logical processes. The goal of an essentially logical theory of predicates is thereby abandoned.

If one decides to do this, then it appears more natural and more appropriate to return to that idea of *logical function* that corresponds to Schröder's point of view: One considers a logical function as a distribution [*Verteilung*] of the values "True" and "false" to the things in the domain of individuals. Every predicate defines such a distribution, but the totality of the value distributions is conceived, in analogy to the finite, as a *combinatorial manifold existing independently of the conceptual definitions.*

By means of this conception, the circularity of the impredicative definitions of theoretical logic is removed; we only need to replace every statement about the totality of predicates with the corresponding statement about the totality of logical functions. Consequently, the axiom of reducibility becomes dispensable.

The logicist school took this step at the suggestion [*Anregung*] of Wittgenstein and Ramsey. In particular, they pointed out that in order to avoid the contradictions that are connected to the concept of the set of all mathematical objects, it is not |*Mancosu:* 257 necessary to carry out a differentiation of the predicates according to the form of their definition as Whitehead and Russell did in "Principia Mathematica"; rather, it is sufficient to delimit clearly the domain of definition of the predicates, so that one differentiates between predicates of individuals, predicates of predicates, predicates of predicates of predicates, and so on.

In this way one returns from the type theory [*Stufentheorie*] of "Principia Mathematica" to the simpler conceptions of Cantor and Schröder.

One should, however, not deceive oneself into thinking that one thereby has not left the point of view of logical self-evidence. The assumptions, which in this way are put at the basis of theoretical logic, are in principle of the same type as the basic postulates of analysis and are completely analogous to them in content: The axiom of infinity in the logical theory corresponds to the idea of the number sequence as infinite totality, and instead of the totality of all sets of numbers, the totality of all logical functions (related to the "domain of individuals" or to a definite domain of predicates) is postulated here.

Thus by incorporating arithmetic in the system of theoretical logic, one does not cut down on assumptions. This incorporation does not at all have, as one might have thought at first, the significance of a reduction of the postulates of arithmetic to more basic assumptions; its value lies rather in the fact that the mathematical theory is placed on a wider basis by means of its union with the logical formalism.

In the first place, the theory gains, in this way, a higher degree of methodological distinction in that it is shown that its assumptions are obtained not only from the intuitive number theory by means of a natural extrapolation, but rather also equally result by *extrapolating extensional logic* in the sense of an extension to infinite totalities.

In addition, through the joining of arithmetic to theoretical logic we gain an insight into the connection of the process of set formation with the logical basic operations, and the logical structure of the concept-formation and of the inferences stands out more clearly.

In particular, the meaning of the principle of choice becomes in this way completely comprehensible only through the logical formalism. We can express the principle in the following form: If $B(x, y)$ is a predicate with two subjects (defined in a certain domain) and if for every thing x of the domain of definition there is at least a thing y of this domain for which $B(x, y)$ holds, then there is (at least) one function $y = f(x)$ with the property that for every thing x of the domain of definition of $B(x, y)$ the value $f(x)$ is again a thing of this domain, and indeed such that $B(x, f(x))$ holds.

If one considers what this claim means for the special case of a subject domain with two individuals, whose things we can represent by the numbers 0, 1 and for which only four different courses of value of functions $y = f(x)$ come into consideration, then one finds that the claim turns out to be a simple application of one of the distributive laws that is valid for the relationship between conjunction and disjunction, namely, an application of the following elementary-logical proposition: "If A holds and if B or C holds as well, then either A and B hold, or A and C hold."[15]

In the case of any given finite Number of things of the subject domain, the statement of the principle of choice also follows from this distributive law. The uni versal statement of the principle of choice is then nothing other than the extension of an elementary-logical law for conjunction and

[15] The "or" is meant here in both cases not in the sense of an exclusive "or," but rather in the sense of the Latin "vel." However, the proposition is valid also for the exclusive "or."

disjunction to infinite totalities, and the principle of choice constitutes thus a completion of the logical rules that |[Mancosu: 258] concerns the universal and the existential judgment, that is, of the rules of existential inference, whose application to infinite totalities also has the meaning that certain elementary laws for conjunction and disjunction are transferred to the infinite.

In comparison with these rules of existential inference, the principle of choice is entitled to a special position only to the degree that the *concept of function* is required for its formulation; and this concept in turn receives an adequate implicit characterization only through the principle of choice.

This concept of function corresponds to the concept of logical function but with the difference that "true" and "false" are not taken as function values, but rather the things of the subject domain. The totality of functions, with which we are concerned here, is then the totality of all possible self-coverings [*Selbstbelegungen*] of the subject domain.

In the sense of this concept of function, the existence of a function with property E in no way guarantees the existence of a concept-formation through which a determinate function with property E is uniquely fixed. In considering this circumstance the usual objections against the principle of choice become invalid; these objections are for the most part based on the fact that one is misled by the name "principle of choice" to believe that the principle claims the possibility of a choice.

Simultaneously we recognize that the assumption, which finds expression in the principle of choice, does not fundamentally go beyond the conception upon which we otherwise already had to base the procedure [*Verfahren*] of theoretical logic in order to be able to interpret it in a noncircular way without the introduction of an axiom of reducibility.

Naturally we can give this statement the opposite emphasis: The contentiousness of the principle of choice, which is in keeping with the aim of a consistent presentation of the point of view of theoretical logic, makes us realize the problematic nature of this point of view in an especially forceful fashion.

This is the result to which the consideration of the logicist foundation of arithmetic has led us as well, that is, that this procedure of incorporating arithmetic into theoretical logic does certainly create a wider foundation for arithmetical theory and contributes to the contentual grounding [*Motivierung*] of its assumptions; it does not, however, go beyond the methodical point of view of the ideal approach, which is beyond the point of view of axiomatics.

In this way, the problem of the infinite is formulated, but not solved. For it remains to be seen whether the analogies, postulated as assumptions for the construction of analysis and set theory, between the finite and the infinite form an admissible, that is, a consistent and feasible, theoretical approach [*Gedankenansatz*].

This question, which intuitionism wants to avoid by eliminating the problematic assumptions, and whose justification is for the most part challenged by the logicists, in that they do not at all recognize a basic difference between the finite and the infinite, is tackled positively by *Hilbert's Proof Theory.*

2.4 Hilbert's Proof Theory

In order better to grasp the main ideas of proof theory, we need first of all to recall what kind of problem we have to solve. It is a matter of proving the consistency of the mathematical idea-formation on which the edifice of arithmetic rests. |[Mancosu: 259]

From the side of philosophy the question has been repeatedly raised whether a proof of consistency suffices as a justification for this idea-formation. This formulation of the question is misleading; it does not take into account the fact that the scientific grounding of the theoretical approach to arithmetic has been achieved for the most part through science, and that the proof of consistency is indeed the only desideratum that still needs to be fulfilled.

The edifice of arithmetic is constructed on the basis of thoughts that are of important significance for scientific systematics in general, namely, on the principle of the *conservation ("Permanence") of the laws* [*Prinzip der Erhaltung ("Permanenz") der Gesetzlichkeiten*], which here appears as the postulate of the unlimited applicability of the usual logical forms of judgment and inference, and on the demand of a purely *objective* formulation of the theory through which the latter is detached from every reference to our *knowledge*.

In the fundamental methodological meaning of these demands lies the *inner* grounding and specificity of the approach of the arithmetical theory.

To this inner grounding is added the magnificent way in which the conceptual system of arithmetic in the sense of its deductive fruitfulness, its systematic success, and the unanimity of its consequences has proved its worth. The suitability of this conceptual system for the mastering of ratios, both of Numbers and of magnitudes, is spectacular. The order [*Systematik*] of the magnificent edifice, which emerges through the union of function theory with number theory and algebra, is unequalled. And as a comprehensive conceptual apparatus for theory-formation in the natural sciences, arithmetic turns out to be not only suitable for the formulation and development of the laws, but it is also invoked with great success, to a degree earlier undreamt of, in the search for the laws.

As for what further concerns the unanimity of the consequences, this is tested, in the best possible way, through the intensive theoretical development and the frequent numerical application of analysis.

What is still missing here is the achievement of a true insight into the consistency of arithmetic, that is, into the constant agreement of its results, in place of the simple empirical confidence on the consistency obtained by repeated testing. To bring this about is the task of the consistency proof.

The situation is not that the conceptual system of arithmetic should be first established by means of the proof of consistency, but rather the task of the proof consists exclusively in creating for us the complete, insightful certainty that this conceptual system, which is already motivated by inner reasons of systematics and tested in its implementation as intellectual apparatus in the best possible way, can not collapse [*zu Falle kommen*] on account of an inconsistency of its consequences.

If this is successful, then we know that the idea of the completed infinity can be carried out in a consistent way. And we can then rely on the results of the application of the basic postulates of arithmetic just as if we were in the position to verify these intuitively. For by recognizing the consistency of the application of these postulates, it is established at the same time that an intuitive proposition that is interpretable ire the finitistic sense, which follows from them, can never contradict an intuitively recognizable fact. In the case of a finitistic proposition, however, the determination of its irrefutability is equivalent to the determination of its truth.

What in particular emerges from this consideration about the requirement and |$^{Mancosu:\ 260}$ the purpose of the consistency proof is that this proof is only a matter of seeing [*einsehen*] the consistency of arithmetic theory in the literal sense of the word, that is, *the impossibility of its immanent refutation.*

The novelty of Hilbert's approach is that he limited himself to this formulation of the problem, while formerly one had always carried out the consistency proof for an axiomatic theory in the sense that by means of it one showed positively at the same time the satisfiability of the axioms by means of certain objects. For this method of demonstration [*Aufweisung*] the case of arithmetic gives no handle; in particular, Frege's idea, to take the objects to be exhibited [*aufzuweisen*] from the field of logic, therefore falls short of its goal, because, as we have made clear, the application of ordinary logic to the infinite is just as problematic as arithmetic, which is to be proven consistent. The basic postulates of the theory of arithmetic concern exactly the extended application of the usual forms of judgment and inference.

With the realization of this fact, we are led directly to the *first guiding principle of Hilbert's proof theory:* This states that in the consistency proof of arithmetic we are to include the laws of logic, as they are applied in arithmetic, into the do main of that which is to be proven consistent, so that the proof of consistency applies *jointly to logic and arithmetic.*

The first important step in the implementation of this idea has been already taken by means of the incorporation of arithmetic in the system of theoretical logic. On the basis of this incorporation, the task of the consistency proof of arithmetic amounts to recognizing theoretical logic as consistent, or, in other words, establishing the consistency of the axiom of infinity, of impredicative definitions, and of the principle of choice.

It is here advisable to replace Russell's axiom of infinity by Dedekind's characterization of the infinite.

Russell's axiom of infinity demands for every finite number n (in the sense of Frege's definition of finite Number) the existence of an n-valued predicate, whereby the infinity of the domain of individuals (of the starting domain of things) is also implicitly demanded. It is an unnecessary complication that ought to be criticized in principle, that three infinities occur here side by side at different levels: that of the infinitely many things of the domain of individuals; furthermore, that of the in finitely many predicates; and then finally, that of the infinitely many Numbers resulting from the above, which are, after all, defined as predicates of predicates.

We can avoid this multiplicity by determining the infinity of the domain of individuals by means of a single predicate with two subjects rather than by an infinite series of predicates with one subject. Such a predicate provides a reversible one-to-one mapping of the domain of individuals onto a proper (that is, at least excluding one thing) subset of the domain. The introduction of this Dedekindian characterization of the infinite takes shape in the most simple and most elementary way, if we postulate the reversible, one-to-one mapping not by means of an existence axiom but rather introduce it immediately in an explicit way, by taking an initial object and a basic process as basic elements of the theory.

In this way it is achieved that the numbers occur not only as predicates of predicates but already as things of the domain of individuals. |[Mancosu: 261]

This consideration already refers to the peculiar kind of implementation of the systematic construction, concerning which numerous ways are open. However, we must still orient ourselves in general as to how a consistency proof might be carried out in the desired sense at all. This possibility is not immediately apparent. For how can one survey all possible inferences that result from the assumptions of arithmetic, or from those of theoretical logic?

Here the investigation of mathematical proofs by means of the logical calculus comes to bear decisively. The logical calculus has shown that the concept-formations and inferences that are applied in the theories of analysis and set theory are reducible to a limited Number of processes and rules, so that one succeeds in completely formalizing these theories in the context of a precisely defined symbolism.

From the possibility of this formalization, which was originally pursued only for the purpose of a more exact logical analysis of the proofs, Hilbert has drawn the conclusion—this is the *second leading thought of his proof theory*—that the task of the consistency proof of arithmetic is a *finitistic problem.*

A contradiction in the contentual theory must show itself on the basis of the formalization in such a way that according to the rules of the formalism two formulas are derivable, such that one of them originates from the other by means of that process that forms the formal representation [*Abbild*] of negation. The consistency statement is therefore equivalent to the statement that two formulas, which are in the aforementioned relationship, cannot both be derived according to the rules of the formalism. However, this statement has fundamentally the same character as any general statement of finitistic number theory, for example, the statement that it is impossible to give three whole numbers a, b, c (different from 0) for which the relationship $a^3+b^3 = C^3$ holds.

So the consistency proof for arithmetic amounts indeed to a finitistic problem of the theory of inference. Hilbert calls the finitistic investigation, which has as object the formalized theories of mathematics, *metamathematics*. The task, which is the role of metamathematics vis-à-vis the system of mathematics, is analogous to that which Kant assigned to the critique of reason vis-à-vis the system of philosophy.

In the sense of this methodological program, proof theory has already made considerable progress,[16] though there are still considerable mathematical difficulties to be overcome. Through the proofs carried out by Ackermann and von Neumann, the consistency for the first postulate of arithmetic, that is, the applicability of existential inference to whole numbers, is established. There is also a more developed attempt by Ackermann for the further problem of consistency of the general concept of sets of numbers (of numerical function, respectively), including the associated principle of choice.

[16]Hilbert delivered a first outline of a proof theory in 1904 in his Heidelberg lecture "On the Foundations of Logic and Arithmetic." The first leading thought of the joint treatment of logic and arithmetic is already explicitly formulated in it; the methodical principle of the finitistic point of view is also intended but not explicitly articulated. Between this lecture and Hilbert's later publications on proof theory falls the investigation of Julius Koenig, "New Foundations of Logic, Arithmetic and Set Theory" (published in 1914), which closely approximates Hilbert's point of view and in which a consistency proof in the sense of proof theory is already carried out. This proof concerns only a very narrow domain of the formal realm [*des formalen Operierens*], so that its significance is only methodological.

With the solution of this problem, almost the entire domain of existing mathematical theories would be proven to be consistent.[17] In particular, this proof would be sufficient to recognize the geometrical and physical theories as consistent.

One can proceed further in the formulation of the problem and investigate the consistency for more comprehensive systems, for example, for axiomatic set theory. Axiomatic set theory, as first put forward by Zermelo and supplemented and extended by Fraenkel and von Neumann, reaches beyond what is factually needed $|^{Mancosu:\ 262}$ in mathematics in its formation processes, and with the establishment of its consistency, the system of theoretical logic would also be proven consistent.

But even this does not achieve an absolute completion of concept-formation. For, formalized set theory gives once more occasion to a metamathematical consideration, which has as its objects the formal formations of set theory and thereby also goes beyond these formations.[18]

This possibility of the extension of the concept-formation notwithstanding, a formalized theory can nevertheless have the character of completeness, namely, if by means of the extension of the concept-formation, no new results in the domain of the laws that can be formulated by the concepts of

[17] Cantor's theory of numbers of the second number class is also included here.

[18] The more precise discussions of this fact refer to Richard's paradox, which has recently received a sharper version by Skolem. Insofar as these considerations take place in the context of a nonfinite mathematics, they do not possess a definitive character. A final clarification of the question discussed here would only be brought about if it were possible to give a set of numbers in a finitistic way, of which it could be demonstrated that it does not occur in axiomatic set theory.

the theory come into being.

This condition is then in any case fulfilled if the theory is in general *deductively complete* [*abgeschlossen*], that is, if it is impossible to add to it a new axiom, expressible in the concepts of the theory that is not already derivable in such a way that no contradiction emerges—or what amounts to the same thing: if every statement that can be formulated in the context of the theory is either provable or refutable.[19]

We believe that number theory, as it is defined by Peano's axioms with the addition of definition by recursion, is deductively complete in this sense; the problem of an actual proof for this is, however, still completely unsolved. The question becomes even more difficult if we proceed beyond the domain of number theory, to analysis and the further concept-formations of set theory.

In the area of this and related questions, a considerable field of problems still remains open. These problems, however, are not of the kind that they represent an objection to the point of view we have adopted. We must only keep in mind the fact that the formalism of statements and proofs, with which we represent our idea-formation, does not coincide with the formalism of that structure we intend in the concept-formation. The formalism is sufficient to formulate our ideas about infinite manifolds and to draw from these the logical consequences, but it is, in general, not capable of producing the manifold, as it were, combinatorially from within.

The view at which we have arrived concerning the theory of the infinite

[19]One should observe that this demand of deductive completeness does not go as far as the demand of decidability of every question of the theory, which means that there should be a procedure [*Verfahren*] for deciding for any given pair of two assertions of the theory contradicting each other, which of the two is provable ("correct").

can be seen as a kind of philosophy of the "as if." However, it differs entirely from the so-called philosophy of Vaihinger in the fact that it emphasizes the consistency and the stability [*Beständigkeit*] of the idea-formations, whereas Vaihinger regards the demand of consistency to be a prejudice and straightforwardly states that the contradictions in the infinitesimal calculus are "not only not to be argued away, but rather ... exactly, the very means by which progress is achieved."[20]

Vaihinger's observation is exclusively focused on scientific *heuristics.* He knows only "fictions," which appear as mere temporary tools of thought, through whose introduction thought violates itself and whose contradictory character (if it is a matter of "true fictions") is rendered harmless only by means of skillful compensation of the contradictions.

The idea-formations in our sense are the enduring property of the mind [*bleibendes Eigentum des Geistes*]. They are outstanding forms of systematic extrapolation and of the idealized approximation to the factual [*das Tatsächliche*]. They are in no way something arbitrary, nor are they forced upon thought; on the contrary, they form a world, in which our thought is at home and from which the human mind that becomes absorbed in them draws satisfaction and joy. |*Mancosu*: 263

Appendix

On the basis of various insights that have emerged since the publication of the above essay, there are a few corrections to be made to what has been

[20]Vaihinger, "Philosophy of As If," 2nd edition, Chap. XII.

stated here.

First of all, as regards intuitionism, it was thought at first that the methodology of intuitionistic proofs coincided with that of Hilbert's "finitistic point of view." However, it has become clear that the methods of intuitionism go beyond the finitistic proof procedures intended by Hilbert. In particular, Brouwer makes use of the universal concept of contentual proof, to which the concept of "absurdity" is also connected, and which, however, is not made use of in finitistic inference.

As for what then concerns Hilbert's proof theory, the opinion that the consistency proof for arithmetic boils down to a finitistic problem, is well founded only in the sense that the statement of consistency can be formulated in a finitistic sense. However, from this it does not follow at all that the problem is solvable with finitistic methods. On the basis of a theorem of Gödel, the possibility of a finitistic solution has been made highly implausible for number theory, if not completely ruled out, and moreover it turned out that the mentioned consistency proofs that were at hand at the time did not suffice for the total formalism of number theory. The methodical point of view of proof theory was consequently extended, and different consistency proofs have been carried out, first of all for formalized number theory and then also for formal systems of analysis, whose proof methods are certainly not limited to finitistic, that is, to the elementary, combinatorial consideration, but which, however, also do not require the usual methods of existential inference or, on the other hand, the general concept of contentual proof.

In connection with the mentioned theorem of Gödel, the assumption that the axiomatically defined and formalized number theory is deductively complete turned out to be wrong. More generally, it has been demonstrated by Gödel that formalized theories, which satisfy certain, very general conditions of expressiveness as well as sharpness of formalization, as long as they are consistent, cannot be deductively complete.

On the whole the situation is such that Hilbert's proof theory, in connection with the uncovering of the possibilities of formalization of mathematical theories, has created a rich area of research, and, however, the epistemological points of view, from which its establishment started out, have become problematic.

This gives cause to revise the epistemological observations of this essay. Of course, the positive observations, in particular the demonstration of the mathematical element in logic and the emphasis on elementary arithmetical evidence, are hardly in need of revision. However, the sharp distinction between the intuitive and the not intuitive, as it is employed in the treatment of the problem of the infinite, can apparently not be implemented so strictly, and the consideration of mathematical idea-formations is in this regard indeed in need of closer elaboration. The following essays contain different considerations for such elaborations. [This refers to the remaining essays in the collection Bernays 1976. Transl.]