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Problems of Theoretical Logic (1927)

Paul Bernays

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Translation by: *Wilfried Sieg*

Comments:

none

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The topic of this lecture and its title have been chosen in the spirit of Hilbert. What is called theoretical logic here is usually referred to as symbolic logic, mathematical logic, algebra of logic, or logical calculus. The purpose of the following remarks is to present this area of research in a way that justifies calling it theoretical logic.

Mathematical logic is in general not very popular. Most often it is regarded as idle play not suitable for practical reasoning (praktisches Schliessen) and not contributing significantly to our logical insights.

First of all, the charge of the playfulness of mathematical logic is probably justified in view of its initial treatment. The formal analogy to algebra was considered at first to be of major importance and was pursued often as an end in itself. But this was the state of affairs decades ago, and today the problems of mathematical logic are inseparably intertwined with questions concerning the foundations of the exact sciences, so that one can no longer speak of its merely playful character.

Secondly, concerning the application to practical reasoning (praktisches Schliessen), it should first be mentioned that a symbolic calculus could be advantageous only to someone who has sufficient practice in using it. Moreover, one has to take into consideration that—in contrast to most symbolisms, which serve, after all, the purpose of abbreviating and contracting operations—it is the primary task of the logical calculus to analyze inferences into their ultimate constituents and the represent symbolically each individual step and thus bring it into our focus. Consequently, the interest connected with the application of the logical calculus is in the main not technical, but theoretical and fundamental.

This leads me to the third charge; namely that mathematical logic does not further significantly our logical insights. This view is connected with a belief about logic that was expressed by Kant in the second preface to the *Critique of Pure Reason*, where he says: "Logic has furthermore this remarkable feature that up to now it has been unable to advance a single step, and this it seems to be closed and completed."

I intend to show that this view is mistaken. To be sure, Aristotle's formulation of the basic principles of reasoning (Prinzipien des Schliessens) and their immediate consequences is one of the most significant intellectual achievements and one of ht every few items of permanently secured philosophical knowledge. This fact will retain its full appreciation. But it foes not prevent us from recognizing that, even in posing its problems, traditional logic is essentially incomplete and that its arrangement of facts is insufficiently adapted to the demands either of a systematic presentation or of methodological and epistemological insights. Only the newer logic, developed under the name of algebra of logic or mathematical logic, introduced the concepts and the kind of starting point for formal logic that make it possible to meet those demands of a systematic presentation and of philosophy.

The rhelm of logical laws, the world of abstract relations, has been uncovered in its formal structure by this development and the relationship of mathematics and logic has been illuminated in a new way. I will try briefly to give an idea of this transformation and of the results it has brought to light.

In doing this I shall not be concerned with [resenting the historical development and the various forms in which mathematical logic has been pursued. Instead, I want to choose a presentation of the new logic that facilitates connecting and comparing it with traditional logic. As for logical symbols, I shall employ the symbolism Hilbert uses in his lectures and publications.

Traditional logic subdivides its problems into the investigation of concept formation, of judgement, and of inference (Schliessen). It is not advantageous to begin with concept formation, because its essential forms are not elementary but depend already on judgement. Let us begin, therefore, with judgment. Here, right at the beginning, modern logic takes an essential new vantage point and replaces classifications by elementary logical operations. One speaks no longer of the categorical, the hypothetical, the negative judgement, but of the categorical, hypothetical connective, and the negation is called a logical operation. In the same way, one does not classify judgements into universal and particular ones but introduces logical operators for universality and particularity.

This approach is more appropriate than that of classification for the following reason. Different logical processes are in general combined in judgements, so that a unique characterization according to them is not possible at all.

First let us consider the *categorical* relationship, i.e. that of subject and predicate. We have an object here and a statement about it. The symbolical representation for this is

$$P(x),$$

to be read as: "x has the property P".
The connection of the predicate with an object is here explicitly brought out by the variable. This is just a clearer notation. However, the remark that *several objects* can be subjects of a statement is crucial. In that case one speaks of a *relation* between several objects. The notation for this is

$$R(x, y) \text{ or } R(x, y, z) \text{ etc.}$$

Cases and prepositions are used in ordinary language to indicate the different members of relations.

By taking into account relations logic is extended in an essential way when compared with its traditional form. I shall speak about the significance of this extension when discussing the theory of inference (Lehre von den Schlssen).

The forms of universality and particularity are based on the categorical relationship. Universality is represented symbolically by

$$(x)P(x)$$

"all x have the property P".

The variable x appears here as a "bound variable"; the statement does not depend on x—in the same way as the value of an integral does not depend on the variable of integration.

The particular judgment is first of all sharpened by replacing the somewhat indefinite statement, "some x have the property P", with the existential judgment:

"there is an x with the property P",

written symbolically:

$$(Ex)P(x).$$

Adding *negation*, we obtain the four types of judgement which are denoted by the letters "a, e, i, o" in Aristotelian logic. We represent negation by barring the expression to be negated and thus obtain the following representations of the four types of judgment:

$$\begin{array}{rll} \text{a:} & (x) & P(x) \\ \text{e:} & (x) & \overline{P(x)} \\ \text{i:} & (Ex) & P(x) \\ \text{o:} & (Ex) & \overline{P(x)} \end{array}$$

Already here, in the doctrine of "opposition", it proves useful for comprehending matters to separate out the operations; we recognize, for example, that the distinction between contradictories and contraries lies in the fact that in the first case the whole statement, e.g., $(x)P(x)$, is negated, whereas in that second case only the predicate $P(x)$ is negated.

Let us now turn to *hypothetical relationship.*

$$A \to B \quad \text{“when } A\text{, so } B\text{”}.$$

This includes a combination (Verknpfung) of *two* statements (predications). The members of this combination already have the form of statements, and the hypothetical relationship applies to these statements as *undivided units.* The latter is already true for the negation $\bar{A}$.

There are still other such combinations of statements, in particular:
The conjunctive combination of A and B: $A \,\&\, B$,
and further, the *disjunctive combination*; there we have to distinguish between the exclusive "or", in the sense of the Latin "aut-aut", and the "or" in the same sense of "vel". This latter combination is represented by $A \vee B$ in accordance with Russell's notation.

In ordinary language, conjunctions are used to express such combinations of statements.

An approach analogous to that in the doctrine of opposition suggests itself here, namely to combine the binary connectives with negation in one of two ways, either by negating the individual members of the combination or by negating the latter as a whole. Now lets see what relations of dependency result.

To indicate that the two combinations have materially the same meaning (or are "equivalent"), I will write "eq" between them. (Clearly, "eq" is not a sign of our logical symbolism.)

In particular the following combinations and equivalences result:

$$
\begin{aligned}
\bar{A} \mathbin{\&} \bar{B}&: \quad \text{“neither } A \text{ nor } B\text{”} \\
\overline{A \mathbin{\&} B}&: \quad \text{“}A \text{ and } B \text{ are incompatible”} \\
\overline{A \mathbin{\&} B} &\quad \text{eq } \bar{A} \vee \bar{B} \\
&\quad \text{eq } A \to \bar{B} \\
&\quad \text{eq } B \to \bar{A} \\
\bar{A} \to B &\quad \text{eq } A \vee B \\
\bar{\bar{B}} &\quad \text{eq } B
\end{aligned}
$$

(double negation is equivalent to affirmation).

From this further results are obtained:

$$
\begin{aligned}
A \to B &\quad \text{eq } \overline{A \mathbin{\&} \bar{B}} \\
&\quad \text{eq } \bar{A} \vee B \\
\overline{A \vee B} &\quad \text{eq } \overline{\bar{A} \to B} \\
&\quad \text{eq } \bar{A} \mathbin{\&} \bar{B}.
\end{aligned}
$$

These equivalences make it possible to express some of the logical connectives

$$ {}^{-}, \to, \&, \vee $$

by means of others. In fact, according to the above equivalences one can express

$$
\begin{array}{ccl}
\to & \text{by} & \vee \text{ and } {}^{-} \\
\vee & \text{by} & \& \text{ and } {}^{-} \\
\& & \text{by} & \to \text{ and } {}^{-}
\end{array}
$$

so that each of

$$
\begin{array}{cl}
 & \& \text{ and } {}^{-} \\
\text{or} & \vee \text{ and } {}^{-} \\
\text{or} & \to \text{ and } {}^{-}
\end{array}
$$

suffice alone as basic connectives. One can get along even with a single basic connective, but not with one of those for which we have a sign. If we introduce for the combination of incompatibility $\overline{A \mathbin{\&} B}$ the sign

$$A|B$$

then the following equivalences obtain:

$$\begin{aligned} A|A \quad & \text{eq } \bar{A} \\ A|\bar{B} \quad & \text{eq } \overline{A \mathbin{\&} \bar{B}} \\ & \text{eq } A \rightarrow B. \end{aligned}$$

This shows that with the aid of this connective one can represent negation as well as $\rightarrow$ and, consequently, the remaining connectives. In place of the connective of incompatibility, the connective

$$\text{“neither — nor”} \ \bar{A} \mathbin{\&} \bar{B}$$

can be taken as the only basic connective. Of we introduce for that connective the sign

$$A \parallel B,$$

then we have

$$\begin{aligned} A \parallel A \quad & \text{eq } \bar{A} \\ \bar{A} \parallel \bar{B} \quad & \text{eq } A \mathbin{\&} B; \end{aligned}$$

thus negation as well as & are expressible by means of $\parallel$.

These reflections border on the playful. Nevertheless, it is remarkable that the discovery of so simple a fact as that of the reduction of all connectives to a single one was reserved for the 20th century. The equivalences between combinations of statements were not at all systematically investigated in the old logic.[1] There are only a few observations like, for example, that of the equivalence of

$$A \to \bar{B} \text{ and } B \to \bar{A}$$

on which the inference by "contraposition" is based. The systematic search for equivalences is, however, all the more rewarding as one reaches here a self-contained and entirely perspicuous part of logic, the so-called *propositional calculus.* I want to explain in some detail the value of this calculus for reasoning (Schliessen).

Let us consider what the sense of equivalence is. When I say

$$\overline{A \;\&\; B} \text{ eq } \bar{A} \vee \bar{B}$$

I claim not that the two combinations of statements are identical in sense (sinnesgleich) but only that they are *identical in truth value* (wahrheitsgleich). That is, no matter how the individual statements A, B are chosen, $A \vee B$ and $A \;\&\; B$ are simultaneously true or false, and consequently these two expressions can represent each other with respect to truth.

Indeed, any combination of statements A and B can be viewed as a mathematical function assigning to each pair A, B one of the values "true"

[1]Today these historical remarks stand in need of correction. In the first place, the reducibility of all connectives to a single one was already discovered in the 19th century by Charles S. Peirce, a fact which became more generally known only with the publication of his collected works in 1933. Further, the equivalences between combinations of statements were considered systematically in the old logic—clearly not in Aristotelian logic, but rather in other Greek schools of philosophy. (See Bochenski's book Formal Logic.) *Remark:* This footnote, as well as the next three, are subsequent additions occasioned by the republication of this lecture.

or "false." The exact content of the statements A, B does not matter at all. What matters is whether A is true or false and whether B is true or false. So we are dealing with *truth functions*: to a pair of truth values another truth value is assigned. Every function of this kind can be given by a schema; the four possible combinations of two truth values (assigned to the statements A, B) are represented by four squares, and in each of these the corresponding truth value of the function ("true" or "false") is written down.

The schemata for $A \mathbin{\&} B, A \vee B, A \rightarrow B$ are given here.

$A \mathbin{\&} B$:

A \ B	true	false
true	true	false
false	false	false

$A \vee B$:

A \ B	true	false
true	true	true
false	true	false

$A \rightarrow B$:

A \ B	true	false
true	true	false
false	true	true

One can easily determine that there are exactly 16 different such functions. The number of different functions of n truth values

$$A_1, A_2, \ldots\ldots, A_n$$

is, correspondingly, $2^{(2^n)}$.

To each function of two or more truth values corresponds a class of intersubstitutable[2] combinations of statements. One class among them is distinguished, namely the class of those combinations that are always true.

These combinations represent all valid logical sentences in which the individual propositions occur only as undivided units.[3] We are going to call the expressions representing valid sentences *valid formulas*.

We master prepositional logic if we know the valid formulas (among combinations of statements), or if we can decide for a given combination whether or not it is valid. After all, the task for reasoning (Schliessen) in prepositional logic is formulated as follows:

Certain combinations

$$V_1, V_2 \ldots\ldots, V_k,$$

are given; they are built up from elementary statements $A, B, \ldots\ldots$, and represent true sentences for a certain interpretation of the elementary statements. The question is whether (the truth of) another given combination W of these elementary statements follows logically from the truth of $V_1, V_2 \ldots\ldots, V_k$, and, in fact, without considering the exact content of the statements $A, B, \ldots\ldots$.

This question has an affirmative answer if and only if

$$(V_1 \,\&\, V_2 \,\&\, \ldots\ldots \,\&\, V_k) \rightarrow W,$$

expressed by $A, B, \ldots\ldots$, represents a valid formula.

[2]Editorial footnote: Ref. To H.B, I, 47;line 19.

[3]Editorial footnote: conc: allgemein gültig versus allgemeingültig

The decision concerning the validity of a combination of statements can in principle always be reached by trying out all possible truth values. The method of considering equivilances, however, provides a more convenient procedure. That is to say, by means of equivalent transformations each formula can be put into a certain *normal form* in which only the logical symbols $\&, \vee, ^{-}$ occur, and from this normal form one can read off directly whether or not the formula is valid.

The rules of transformation are also quite simple. In particular, one can manipulate $\& and \vee$ in analogy to $+$ and $\cdot$ in algebra. Indeed, matters are here even simpler, for & and $\vee$ can be treated symmetrically.

By considering the equivalences we enter the domain of inferences, as already mentioned. But we made inferences here in a naive way, so to speak, on the basis of the meaning of the logical connectives, and we transformed the task of reasoning (Schliessen) into a decision problem.

But for logic there remains the task of *systematically* presenting the rules of reasoning.

Aristotelian logic lays down the following principles of reasoning (Schliessen):

1. Rule of categorical reasoning: the "dictum de omni et nullo" : what holds universally, holds in each particular instance.

2. Rule of hypothetical reasoning: if the antecedent is given, then the consequent is given, i.e. if A and if $A \to B$, then B.

3. Laws of negation: Law of contradiction and of excluded middle: A and $\bar{A}$ cannot both hold, and at least one of the two statements holds.

4. Rule of disjunctive reasoning: if at least one of A or B holds and if $A \to C$ as well as $B \to C$, then C holds.

One can say that each of these laws represents the implicit definition for a logical process: 1. For universality, 2. For the hypothetical relationship, 3. For negation, 4. For disjunction ($\vee$).

These laws contain indeed what is essential for reasoning. But for a complete analysis of inferences this does not suffice. For this we demand that nothing need be considered anymore, once the principles of reasoning have been laid down. The rules of reasoning must be constituted in such a way that they eliminate logical thinking. Otherwise we would have to have logical rules specifying in turn how to apply those rules.

This demand to drive out the intellect (to dispense with thinking, to eliminate logical thinking—Austreibung des Geistes) can indeed be satisfied. The structure of the system of inference obtained in this way is analogous to the axiomatic structure of a theory. Certain logical laws written down as formulas correspond to the axioms, and acting externally (on formulas) according to fixed rules, whose application leads from the initial formulas to further ones, corresponds to contextual reasoning that usually leads from axioms to theorems.

Each formula that can be derived in this way represents a valid logical sentence.

Once again it is advisable to separate out *propositional logic*, which rests on the principles 2., 3., and 4. Only the following rules are needs when representing the elementary propositions by variables

$$X, Y, \ldots$$

The first rule states: any combination of statements can be substituted for a variable (substitution rule).

The second rule is the inference schema

$$\frac{\begin{array}{c}\mathfrak{S}\\ \mathfrak{S} \to \mathfrak{T}\end{array}}{\mathfrak{T}}$$

according to which the formula $\mathfrak{T}$ is obtained from two formulas $\mathfrak{S}, \mathfrak{S} \to \mathfrak{T}$.

The initial formulas can be chosen in a number of quite different ways. One has tried very hard to get by with the smallest number of axioms, and in this respect the limits of what is possible have indeed been reached. For the purposes of logical investigation it is better, however, to separate, as in the axiomatic presentation of geometry, various *groups of axioms* from one another such that each of them expresses the roles of one logical operation. The following list then emerges:

I		Axioms of implication
II	a)	Axioms for &
II	b)	Axioms for $\vee$
III		Axioms of negation

This system of axioms[4] generates through application of the rules *all* valid formulas of propositional logic.[5] The *completeness* of the axiom system

[4]Editorial footnote: fixing the axioms as in HB I, p. 65.

[5]We refer only to those formulas that can be built up with the operations $\to, \&, \vee$ and with negation. If further operational symbols are added, then they can be introduced by replacement rules. Of course, one is not bound to distinguish the four mentioned operations in this particular way.

can be characterized even more sharply by the following fact: if we add any underivable formula to the axioms, then we can deduce any arbitrary formula with the help of the rules.

A particular advantage of dividing the axioms into groups is that it allows one to separate out *positive logic*. That is the system of combinations of statements that are valid without assuming that every statement can be negated.[6] Examples of such are:

$$(A \mathbin{\&} B) \rightarrow A$$
$$(A \mathbin{\&} (A \rightarrow B)) \rightarrow B.$$

The system of these formulas is obtained in out axiomatization as the totality of the formulas is obtained in our axiomatization as the totality of the formulas that are derivable without using axiom group III. This system is not nearly so perspicuous as the system of all valid formulas. And no decision method is known that allows one to determine in accordance with a definite procedure whether a formula is in the system.[7] In particular, it is not correct that every formula expressible with $\rightarrow, \&, \vee$, which is valid and therefor derivable on the basis of I–III, is already derivable from I–II. One can rigorously prove that this is not the case.

One example is provided by the formula

$$A \vee (A \rightarrow B).$$

Representing $\rightarrow$ by $\vee$ and $^{-}$ this formula turns into

$$A \vee (\bar{A} \vee B),$$

[6]Editorial footnote: Ref. to HB I, p. 67.

[7]Since then decision methods for positive logic have been given by Gerhard Gentzen and Mordechaj Wajsberg (editorial footnote).

and this formula is immediately recognized as valid. However, it can be shown that the formula is not derivable within positive logic, i.e. on the basis of axioms I–II. Hence, it does not represent a law of positive logic.

We recognize here quite clearly that negation plays the role of an *ideal element* whose introduction aims at rounding off the logical system to a totality with a simpler structure, just as the system of real numbers is extended to a more perspicuous totality by the introduction of imaginary numbers and just as the ordinary plane is completed to a manifold with a simpler projective structure by the addition of points at infinity. Thus the method of ideal elements, fundamental to science, is already encountered in logic, even if we are usually not aware of its significance here.

A special part of positive logic is the theory of *chain inferences* (Kettenschlüsse) discussed already in Aristotelian logic. In this area there are also natural problems and simple results, not known to traditional logic and requiring again the use of specifically mathematical considerations. I am thinking here of Paul Hertz's investigations of sentence-systems (Satzsysteme).—[8]

So far our axiomatization is concerned only with those inferences which depend solely on the rules for the conditional, for disjunction and negation. There remains the task of incorporating *categorical reasoning* into our axiomatization. How this is done I will describe only briefly.

Of the *dictum de omni et nullo* we need also the converse: "what holds in each particular instance, holds generally." Furthermore, we have to take into account the particular judgement. For it we have similarly:

"If a statement $A(x)$ is true of some object x, then there is an object of

[8] Editorial footnote.

which it is true, and vice versa."

Thus we obtain four principles of reasoning that are represented in the axiomatization by two new initial formulas and two rules. A substitution rule for the individual variables $x, y, \ldots$ is added.

Moreover, the substitution rule concerning variables for statements $X, Y, \ldots$ has to be extended now in such a way that the formulas of propositional logic can also be used for expressions containing individual variables.

Let us now see how the typical Aristotelian inferences are worked out from this standpoint. For that it is necessary first to say something about the interpretation of the universal judgment "all S are P".

According to Aristotelian view, such a judgment presupposes that there are certain objects with property S, and it is then claimed that all these objects have property P. This interpretation, to which Franz Bretano in particular objected from a philosophical point of view, is as a matter of fact quite correct in itself. But it is suited neither to the purposes of theoretical science not to the formalization of logic, since its implicit presupposition creates unnecessary complications. We shall therefor restrict the content of the judgment, "all S are P", to the assertion, "an object having property S has also property P."

Accordingly, such a judgment is simultaneously universal and hypothetical. It is represented in the form

$$(x)(S(x) \to P(x)).$$

The so-called categorical inferences contain consequently a combination of categorical and hypothetical modes of inference. I want to illustrate this by a classical example:

"All men are mortal, Cajus is a man, therefor Cajus is mortal."

If we represent "x is human" and "x is mortal" in our notation by $H(x)$ and $M(x)$ respectively[9], then the premises are

$$(x)(H(x) \to M(x)),$$
$$H(Cajus),$$

And the conclusion is: $M(Cajus)$.

With the inference from the general to the particular one deduces from

$$(x)(H(x) \to M(x))$$

the formula

$$H(Cajus) \to M(Cajus).$$

This statement together with

$$H(Cajus)$$

yields with the schema of the hypothetical inference:

$$M(Cajus).$$

It is characteristic that in this representation of the inference no quantitative interpretation is given to the categorical judgement (in the sense of subsumption). One recognizes here particularly clearly that mathematical logic does not in the least depend upon being a logic of extensions.

Our rules and initial formulas permit us to derive all the familiar Aristotelian inference moods that agree with our interpretation of the universal

[9]Editorial footnote.

judgment—that leaves just 15. Thus it is recognized that one is dealing with a very small number of genuinely different kinds of inferences. Furthermore, one gets the impression that the underlying problem is delimited in a quite arbitrary way.

A more general problem, which is indeed solved in mathematical logic, consists in finding a decision method that allows one to determine whether or not a predicate formula is valid. With that, one masters deductive reasoning in the domain of predicates, just as one masters propositional logic with the decision method mentioned earlier.

But our rules of inference extend much further. The real wealth of logical connections is revealed only when we consider *relations* (predicates with several subjects). Only then is it possible to give a complete logical analysis of *mathematical proofs*.

However, here one is led to use additional *extensions* which are suggested to us also by ordinary language.

The first extension consists in introducing a formal expression for "x is the same object as y", or "an object different from y". For this purpose the "*identity* of x and y" has to be formally represented as a particular relation, the properties of which are to be formulated as axioms.

Secondly, we need a symbolic representation of the logical relation we express linguistically with the aid of the genitive or the relative pronoun in such phrases as "the son of Mr. X" or "the object which". This relation forms the basis of the *function concept* in mathematics. It is crucial here that an object that uniquely has a certain property or satisfies a certain relation to particular objects is characterized by this property or relation.

The most significant extension, however, is brought about by the circumstance that we are led to consider predicates and relations themselves as objects, just as we do in ordinary language when we say, for example, "patience is a virtue". We can state properties of predicates and relations, and furthermore, relationships between predicates and also between relations. The forms of universality and particularity can also be used with respect to predicates and relations. In this way we arrive at *second order* logic. For its formalization the laws of categorical reasoning have to be extended in an appropriate way to the domain of predicates and relations.

We have enlarged the range of logical relationships by the inclusion of relations and by the other extensions mentioned above. Here the solution of the decision problem—which, incidentally, is in a natural way subsumed under a more general problem—presents an enormous task. Its solution would mean that we have a method that, at least in principle, permits us to decide for any given mathematical statement whether or not it is provable from a given list of axioms. And we are indeed far from a given solution to this problem. Nevertheless, several considerable results of a very general character have been obtained in this area through the investigations of Löwenheim and Behmann; in particular the decision problem for second order *predicate logic* was solved.[10]

We see here that the traditional theory of inferences comprises only a miniscule part of what really belongs to the domain of logical reasoning.

[10]Notice that one speaks here of "predicate logic" in the sense of the distinction between predicates and relations. By "predicate logic" is meant what today is mostly called the logic of monadic predicates. Even the first order logic of polyadic predicates is not generally decidable, as has been shown by Alonzo Church.

As yet I have not even mentioned *concept formation*. And, for lack of time, I cannot consider it in detail. I will say just this much: a truly penetrating logical analysis of concept formation becomes possible only on the basis of the theory of relations. It is only by means of this theory that one recognizes what kind of complicated combinations of logical expressions (relationships, existence statements, etc.) are concealed by short expressions of ordinary language. Such an analyses of concept formation has been begun, especially by Bertrand Russell, and it has led to the recognition of general logical processes of concept formation. Through their clarification, the understanding of the methods of science is furthered considerably.

I now come to the end of my remarks. I have tried to show that logic, that is to say, the real classical logic as it was always intended, obtains its genuine rounding off, its proper development and systematic completion, only through its mathematical treatment. The mathematical viewpoint is introduced here not artificially, but rather arises entirely on its own in the further pursuit of problems.

The resistance to mathematical logic is widespread, particularly among philosophers; it has—apart from the reasons mentioned earlier—an additional basic reason. Many approve of letting mathematics be absorbed into logic. Here, the opposite is seen, namely, that the system of logic is absorbed into mathematics. Logic appears as a specific interpretation and as an application of a mathematical formalism, exhibiting the same relation to the formalism which obtains, for example, between the theory of electricity and mathematical analysis, when the former is treated according to Maxwell's theory.

That does not contradict the generality of logic; it does, however, contradict the view that logic is of greater generality than mathematics. Logic treats of certain contents (Inhalte) that find application to any subject matter whatsoever, insofar as it is thought about. Mathematics, on the other hand, treats of the most general laws of any combination whatsoever. This is also a kind of highest generality namely, in the direction towards the *formal*. All reflections, including the mathematical ones, are subject to logical laws; but equally well, all structures, all manifolds however primitive (and thus also the manifold given in the combination of sentences or parts of sentences) fall under the laws of mathematics.

If we wanted a logic free of mathematics, no theory at all would be left, but only pure reflection on the most simple connections of meaning. Such purely contentual considerations—which can be comprised in 'philosophical logic"—are, in fact, indispensable and decisive as a starting point for the logical theory; in the same way in which the purely physical considerations serving as the starting point for a physical theory constitute the fundamental conceptual achievement for that theory. But such considerations do not form the theory itself. Its development requires the mathematical formalism. Exact systematic theory of a subject is mathematical treatment, and it is in this sense that Hilbert's dictum holds: "Whatever can be the object of scientific thought at all, as soon as it is ripe for the formation of a theory... it will fall into mathematics." Even logic cannot escape this fate.