Bernays Project: Text No. 23

Comments on Ludwig Wittgenstein's *Remarks on the foundations of mathematics* (1959)

Paul Bernays

(*Remarks on the Foundations of Mathematics*, by Ludwig Wittgenstein. Edited by G. H. von Wright, R. Rhees, G. E. M. Anscombe. Basil Blackwell, Oxford, 1956. 37s 6d. Reprinted with the kind permission of the author and the editor from *Ratio*, II, no. I (1959), 1–22. Translated from the German.)

Translation by: *?*

Comments:

none

I

The book with which the following observations are concerned is the second part of the posthumous publications of selected fragments from Wittgenstein in which he sets forth his later philosophy.[1] The necessity of making a selection and the fragmentary character which is noticeable in places are

[1]The book was originally published in German, with the English translation attached. All pages and numbers quoted refer to the German text.

not unduly disconcerting since Wittgenstein in his publications in any case refrains from a systematic treatment and expresses his thoughts paragraph-wise—springing frequently from one theme to another. On the other hand we must admit in fairness to the author that he would doubtless have made extensive changes in the arrangement and selection of the material had he been able to complete the work himself. Besides, the editors of the book have greatly facilitated a survey of the contents by providing a very detailed table of contents and an index. The preface gives information on the origin of the different parts I–V.

Compared with the standpoint of the *Tractatus*, which indeed considerably influenced the originally very extreme doctrine of the Vienna Circle, Wittgenstein's later philosophy represents a rectification and clarification in essential respects. In particular it is the very schematic conception of the |[*Mancosu*: 511] structure of the language of science—especially the view on the composition of statements out of atomic propositions—which is here dropped. What remains is the negative attitude towards speculative thinking and the permanent tendency to disillusionize.

Thus Wittgenstein says himself, evidently with his own philosophy in mind (p. 63, No. 18): 'Finitism and behaviorism are quite similar trends. Both say: all we have here is merely... Both deny the existence of something, both with a view to escaping from a confusion. What I am doing is, not to show that calculations are wrong, but to subject the *interest* of calculations to a test.' And further on he explains (p. 174, No. 16): 'My task is not to attack Russell's logic from *within*, but from without. That is to say, not to attack it mathematically—otherwise I should be doing mathematics—but to attack its position, its office. My task is not to talk about Gödel's proof, for example, but to by-pass it.'

As we see, jocularity of expression is not lacking with Wittgenstein; and in the numerous parts written in dialogue form he often enjoys acting the rogue.

On the other hand he does not lack *esprit de finesse*, and his formulations contain, in addition to what is explicitly stated, many implicit suggestions.

Two problematic tendencies, however, appear throughout. The one is to dispute away the proper rôle of thinking—reflective intending—in a behavioristic manner. David Pole, it is true, in his interesting account and exposition of Wittgenstein's later philosophy,[2] denies that Wittgenstein is a supporter of behaviorism. This contention is justified inasmuch as Wittgenstein certainly does not deny the existence of the mental experiences of feeling, perceiving and imagining; but with regard to thinking his attitude is distinctly behavioristic. Here he tends everywhere towards a short circuit. Images and perceptions are supposed in every case to be followed immediately by behavior. 'We do it like this', that is usually the last word of understanding—or else he relies upon a need as an anthropological fact. Thought, as such, is left out. It is characteristic in this connection that a 'proof' is conceived as a 'picture' or 'paradigm'; and although Wittgenstein is critical of the method of formalizing proofs, he continually takes the formal method of proof in the Russellian system as an example. Instances of proper mathematical proofs, which are not mere calculations, which neither result merely from showing

[2]David Pole, *The Later Philosophy of Wittgenstein*, University of London, The Athlone Press, 1958.

a figure nor proceed formalistically, do not occur at all in this book on the foundations of mathematics, a major part of which treats of the question as to what proofs really are—although the author has evidently concerned himself with many mathematical proofs.

One passage may be mentioned as characterizing Wittgenstein's behavioristic attitude and as an illustration of what is meant here by a short circuit. Having rejected as unsatisfactory various attempts to characterize I |[Mancosu: 512] inference, he continues (p. 8, No. 17): 'Thus it is necessary to see how we perform inferences in the practice of language; what kind of operation inferring is in the language-game. For example, a regulation says: "All who are taller than six foot are to join the ... section." A clerk reads out the names of the men, and their heights. Another allots them to such and such sections. "*X* six foot four." "So *X* to the ... section." That is inference.' Here it can be seen that Wittgenstein is satisfied only with a characterization of inferring in which one passes directly from a linguistic establishment of the premisses to an action, in which, therefore, the specifically reflective element is eliminated. Language, too, appears under the aspect of behavior ('language-game').

The other problematic tendency springs from the program—already present in Wittgenstein's earlier philosophy—of strict division between the linguistic and the factual, a division also present in Carnap's *Syntax of Language*. That this division should have been retained in the new form of Wittgenstein's doctrine does not go without saying, because here the approach, compared with the earlier one, is in many respects less rigid. Signs of a certain change can in fact be observed, as, for instance, on p. 119, No. 18: 'It is clear that mathematics as a technique of transforming signs for the purpose of prediction has nothing to do with grammar.' Elsewhere (p. 125, No. 42) he even speaks of the 'synthetic character of mathematical propositions'. It is said there: 'It might perhaps be said that the synthetic character of mathematical propositions appears most obviously in the unpredictable occurrence of the prime numbers. But their being synthetic (in this sense) does not make them any the less *a priori* ... The distribution of prime numbers would be an ideal example of what could be called synthetic *a priori*, for one can say that it is at any rate not discoverable by the analysis of the concept of a prime number.' As can be seen, Wittgenstein returns here from the concept 'analytic' of the Vienna Circle to a concept-formation which is more in the Kantian sense.

A certain approach to the Kantian conception is embodied also in Wittgenstein's view that it is mathematics which first forms the character, 'creates the forms of what we call facts' (see p. 173, No. 15). In this sense Wittgenstein also strongly opposes the opinion that the propositions of mathematics have the function of empirical propositions. On the other hand he emphasizes on various occasions that the applicability of mathematics, in particular of arithmetic, rests on empirical conditions; on p. 14, No. 37, for example, he says: 'This is how our children learn sums, for we make them put down three beans and then another three beans and then count what is there. If the result were at one time five, at another time seven ..., then the first thing we should do would be to declare beans to be unsuitable for teaching sums. But if the same thing happened with sticks, fingers, strokes and most other things, then that would be the end of doing sums.—"But wouldn't it then still be that $2 + 2 = 4$?”—This sentence would then have become unusable.’ |[Mancosu: 513] Statements like the following, however, remain important for Wittgenstein’s conception (p. 160, No. 2): ‘He who knows a mathematical proposition is supposed still to know *nothing*.’ (The words in the German text are: ‘soll noch nichts wissen’.) He repeats this twice at short intervals and adds: ‘That is, the mathematical proposition is only to supply the scaffolding for a description.’ In the manner of Wittgenstein we could here ask: ‘Why is the person in question *supposed* to still know nothing?’ What need is expressed by this ‘supposed to’? It appears that only a preconceived philosophical view determines this requirement, the view, namely, that there can exist but one kind of factuality: that of concrete reality. This view conforms to a kind of nominalism as it figures also elsewhere in the discussions on the philosophy of mathematics. In order to justify such a nominalism Wittgenstein would have to go back further than he does in this book. At all events he cannot here appeal to our actual mental attitude. For indeed he attacks our tendency to regard arithmetic, for example, ‘as the natural history of the domain of numbers’ (see p. 117, No. 13 and p. 116, No. 11). However, he is not fully at one with himself on this point. He asks himself (p. 142, No. 16) whether ‘mathematical alchemy’ is characterized by the mere fact that mathematical propositions are regarded as statements about mathematical objects. ‘In a certain sense it is not possible, therefore, to appeal to the meaning of signs in mathematics, because it is mathematics itself which first gives them their meaning. What is typical for the phenomenon about which I am speaking is that the *mysteriousness* about any mathematical concept is not *straight away* interpreted as a misconception, as a fallacy, but as something which is at all events not to be despised, which should perhaps even be respected. All that I can do is to show an easy escape from this obscurity and this glitter of concepts. It can be said, strangely enough, that there is so to speak a solid core in all these glistening concept-formations. And I should like to say that it is this which makes them into mathematical products.'

One may doubt whether Wittgenstein has succeeded here in showing 'an easy escape from this obscurity'; one may be more inclined to suspect that here the obscurity and the 'mysteriousness' actually have their origin in the philosophical concept-formation, i.e. in the philosophical language used by Wittgenstein.

The fundamental division between the sphere of mathematics and the sphere of facts appears in several passages in the book. In this connection Wittgenstein often speaks with a certitude which strangely contrasts with his readiness to doubt so much of what is generally accepted. The passage on p. 26, No. 80 is characteristic of this; he says here: 'But of course you cannot get to know any property of the material by imagining.' Again we read on p. 29, No. 98: 'I can calculate in the imagination, but not experiment.' From the point of view of common experience all this certainly does not go without saying. An engineer or technician has doubtless just as lively an image of materials as a mathematician has of geometrical curves, and the image which |[*Mancosu*: 514] any one of us may have of a thick iron rod is no doubt such as to make it clear that the rod could not be bent by a light pressure of the hands. And in the case of technical inventing, a major r6le is certainly played by experimenting in the imagination. Wittgenstein apparently uses here without being aware of it a philosophical schema which distinguishes the *a* *priori* from the empirical. To what extent and in what sense this distinction, which is so important particularly in the Kantian philosophy, is justified will not be discussed here; but in any case its introduction, particularly at the present time, should not be taken too lightly. With regard to the *a priori*, Wittgenstein's viewpoint differs from the Kantian viewpoint particularly by the fact that it includes the principles of general mechanics in the sphere of the empirical. Thus he argues, for example (p. II 4, No. 4): 'Why are the Newtonian laws not axioms of mathematics? Because we could quite well imagine things being otherwise ... To say of a proposition: "This could be imagined otherwise" . . . ascribes the rôle of an empirical proposition to it.' The concept of 'being able to imagine otherwise', also used by Kant, has the inconvenience of ambiguity. The impossibility of imagining something may be understood in various senses. This difficulty occurs particularly in geometry. This will be discussed later.

The previously mentioned tendency of Wittgenstein to recognize only one kind of fact becomes evident not only with regard to mathematics, but also with respect to any phenomenology. Thus he discusses the proposition that white is lighter than black (p. 30, No. 105) and explains it by saying that black serves us as a paradigm of what is dark, and white as a paradigm of what is light, which makes the statement one without content. In his opinion statements about differences in brightness have content only when they refer to specific visible objects and, for the sake of clarity, differences in the brightness of colours should not be spoken about at all. This attitude obviously precludes a descriptive theory of colours.

Actually one would expect Wittgenstein to hold phenomenological views.

This is suggested by the fact that he often likes to draw examples from the field of art for the purpose of comparison. It is only the philosophical program which prevents the development of an explicitly phenomenological viewpoint.

This case is an example of how Wittgenstein's method aims at eliminating a very great deal. He sees himself in the part of the free-thinker combating superstition. The latter's goal, however, is freedom of the mind; whereas it is this very mind which Wittgenstein in many ways restricts, through a mental asceticism for the benefit of an irrationality whose goal is quite undetermined.

This tendency, however, is by no means so extreme here in the later philosophy of Wittgenstein's as in the earlier form. One may already gather from the few passages quoted that Wittgenstein was probably on the way to giving mental contents more of their due.

A fact that may be connected with this is that, in contrast to the assertive |*Mancosu*: 515 form of philosophical statement throughout the *Tractatus*, a mainly aporetical attitude prevails in the present book. There lies here, it is true, a danger for philosophical pedagogics, especially as Wittgenstein's philosophy exerts a strong attraction on the younger minds. The old Greek observation that philosophical contemplation frequently begins in philosophical astonishment[3] today misleads many philosophers into holding the view that the cultivation of astonishment is in itself a philosophical achievement. One may certainly have one's doubts about the soundness of a method which requires young philosophers to be trained as it were in wondering. Wondering is heuristically fruitful only where it is the expression of an instinct of research. Naturally it cannot be demanded of any philosophy that it should

[3] *θαυμάζειν*.

make comprehensible all that is astonishing. Perhaps the various philosophical viewpoints may be characterized by what they accept as ultimate in that which is astonishing. In Wittgenstein's philosophy it is, as far as epistemological questions are concerned, sociological facts. A few quotations may serve to illustrate this (p. 13, No. 35): '... how does it come about that all men ... accept these figures as proofs of these propositions? Indeed, there is here a great—and interesting —agreement.' (p. 20, No. 63): '... it is a peculiar procedure: I *go through* the proof and then accept its result.—I mean: this is simply how we *do* it. This is the custom among us, or a fact of our natural history.' (p. 23, No. 74): 'When one talks about essence one is merely noting a convention. But here one would like to retort: "There is no greater difference than between a proposition about the depth of the essence and one about a mere convention." What, however, if I reply: "To the *depth* of the essence there corresponds the deep need for the convention." ' (p. 122, No. 30): 'Do not look on the proof as a procedure that *compels* you, but as one that *guides* you ... But how does it come about that it so guides *each one* of us in such a way that we are influenced by it conformably? Now how does it come about that we agree in *counting*? "That is just how we are trained", one may say, "and the agreement produced in this way is carried further by the proof." '

II

So much then for the general characterization of the present observations by Wittgenstein. Their contents, however, is by no means exhausted in the general philosophical aspects that are here raised: various specific questions concerning the foundations of philosophy are discussed in detail. We shall deal in what follows with the principal viewpoints occurring here.

Let us begin with a question which concerns the problem previously touched on, that of the distinction between the *a priori* and the empirical: the question of geometrical axioms. Wittgenstein does not deal specifically with geometrical axioms as such. Instead, he raises generally the question as |*Mancosu*: 516 to how far the axioms of a mathematical system of axioms should be self-evident. He takes as his example the parallel axiom. Let us quote a few sentences from his discussion of the subject (p. 113, No. 2): 'What do we say when such an axiom is presented to us, for example, the parallel axiom? Has experience shown us that this is how it is? ... Experience plays a part, but not the one we should *immediately expect*. For we have not, of course, made experiments and found that actually only one straight line through the given point fails to cut the other straight line. And yet the proposition is evident.—Suppose I now say: "it is quite indifferent why it should be evident. It is sufficient that we accept it. All that is important is how we use it" ... When the wording of the parallel axiom, for example, is given ..., the way of using this proposition, and hence its sense, is still quite undetermined. And when we say that it is evident to us, then we have by doing so already chosen, without realizing it, a certain way of using the proposition. The proposition is not a mathematical axiom if we do not employ it precisely *for this purpose*. The fact, that is, that here we do not make experiments, but accept the self-evidence, is enough to decide its use. For we are not so naïve as to let the self-evidence count instead of the experiment. It is not the fact that it appears to us self-evidently true, but the fact that we let the self-evidence count, which makes it into a mathematical proposition.'

In discussing these statements it must first be borne in mind that we have to distinguish two things: whether we recognize an axiom as geometrically valid, or whether we choose it as an axiom. The latter, of course, is not determined by the wording of the proposition. But here we are concerned with a rather technical question of deductive arrangement. However, what interests Wittgenstein here is surely the recognition of the proposition as geometrically valid. It is on this light that Wittgenstein's assertion ('that the recognition is not determined by the wording') must be considered, and it is in any case not immediately evident. He puts it so simply: 'We have not, of course, made experiments.' Admittedly, there has been no experimenting in connection with the formulation of the parallel axiom here considered, and this formulation does not lend itself to this purpose anyway. However, within the scope of the other geometrical axioms the parallel axiom is equivalent to one of the following statements of metrical geometry: 'In a triangle the sum of the angles is equal to two right angles. In a quadrilateral in which three angles are right angles the fourth angle is also a right angle. Six congruent equilateral triangles with a common vertex P (lying consecutively side by side) exactly fill up the neighborhood of point P.' Such propositions—in which, it will be noted, there is no mention of the infinite extension of a straight line—can certainly be tested by experiment. As is known, Gauss did in fact check experimentally the proposition about the sum of the angles of a triangle, making use, to be sure, of the assumption of the linear diffusion of light. This is, however, not the only possibility of such an experiment. Thus Hugo Dingler in particular has emphasized that for the concepts of the straight |[Mancosu: 517] line, the plane and the right angle there exists a natural and, so to speak, compulsory kind of experimental realization. By means of such an experimental realization of geometrical concepts statements like in particular the second one above can be experimentally tested with great accuracy. Moreover in a less accurate way they are continually being implicitly checked by us in the normal practice of drawing figures. Our instinctive estimation of lengths and of the sizes of angles can also be considered as the result of manifold experiences, and propositions which are to serve as axioms of elementary geometry must at all events agree with that instinctive estimation.

It cannot be maintained, therefore, that our experience plays no part in the acceptance of propositions as geometrically valid. But Wittgenstein does not mean that either. This becomes clear from what follows immediately after the passage quoted (p. 114, Nos. 4 and 5): 'Does experience teach us that a straight line is possible between any two points? ... One could say: *Imagination* teaches us it. And this is where the truth lies; one has only to understand it aright. *Before* the proposition the concept is still pliable. But might not experience cause us to reject the axiom? Yes. And nevertheless it does not play the part of an empirical proposition ... Why are the Newtonian laws not axioms of mathematics? Because one could quite well imagine things being otherwise ... Something is an axiom, *not* because we recognize it as extremely probable, indeed as certain, but because we assign it a certain function, and one which conflicts with that of an empirical proposition ... The axiom, I would say, is another part of speech.' Further on (p. 124, No. 35) he says: 'What about, for example, the fundamental laws of mechanics? Whoever understands them must know on what experiences they are based. It is otherwise with the propositions of pure mathematics.'

In favour of these statements it must certainly be conceded that experience alone does not determine the theoretical recognition of a proposition. A more exact theoretical statement is always something which must be conceived beyond the facts of experience.

The view, however, that there exists in this respect such a fundamental difference between mathematical propositions and the principles of mechanics is scarcely justified. In particular the last quoted assertion that, in order to understand the basic laws of mechanics, the experience on which they are based must be known, can hardly be maintained. Of course, when mechanics is taught at the university, it is desirable that the empirical foundations should be made clear; this is, however, not for the purpose of the theoretical and practical manipulation of the laws, but for the epistemological consciousness and with an eye to the possibilities of eventually necessary modifications of the theory. Yet an engineer or productive technician in order to become skilled in mechanics and capable of handling its laws does not need to bother about how we came upon these laws. To these laws also applies what Wittgenstein so frequently emphasizes in reference to mathematical laws: that the facts of |[*Mancosu*: 518] experience which are important for the empirical motivation of these propositions by no means make up the contents of that which is asserted in the laws. It is important for the manipulation of mechanical laws to become familiar with the concept-formations and to obtain some sort of evidence for these laws. This way of acquiring it is not only practically, but also theoretically significant: the theory is fully assimilated only by the process of rational shaping to which it is subsequently subjected. With regard to mechanics most philosophers and many of us mathematicians have little to say here, not having acquired mechanics in the said manner.-What distinguishes the case of geometry from that of mechanics is the (philosophically somewhat accidental) circumstance that the acquisition of the world of concepts and of the evidence is for the most part already completed in an (at least for us) unconscious stage of mental development.

Ernst Mach's opposition to a rational foundation of mechanics has its justification insofar as such a foundation endeavours to pass over the rôle of experience in arriving at the principles of mechanics. We must keep in mind that concept-formations and the principles of mechanics comprise as it were an extract of experience. On the other hand it would be unjustified to reject outright on the basis of this criticism the efforts towards a construction of mechanics that is as rational as possible.

What is specific about geometry is the phenomenological character of its laws, and hence the important rôle of intuition. Wittgenstein points only in passing to this aspect: '*Imagination* teaches us it. And this is where the truth lies; one has only to understand it aright' (p. 8). The term 'imagination' is very general, and what is said at the end of the second sentence is a qualification which shows that the author feels the theme of intuition to be a very ticklish one. In fact it is very difficult to characterize satisfactorily the epistemological rôle of intuition. The sharp separation of intuition and concept, as it occurs in the Kantian philosophy, does not appear on closer examination to be justified. In considering geometrical thinking in particular it is difficult to distinguish clearly the share of intuition from that of conceptuality, since we find here a formation of concepts guided so to speak by intuition, which in the sharpness of its intentions goes beyond what is in a proper sense intuitively evident, but which separated from intuition has not its proper content. It is strange that Wittgenstein assigns intuition no definite epistemological role although his thinking is dominated by the visual. A proof is for him always a picture. At one time he gives a mere figure as an example of a geometrical proof. It is also striking that he never talks about the intuitive evidence of topological facts, such as for instance the fact that the surface of a sphere divides the (remaining) space into an inner and an outer part in such a way that the curve joining up an inside point with an outside one always passes over one point on the surface of the sphere.

Questions relating to the foundations of geometry and its axioms still belong primarily to the field of inquiry of general epistemology. What is today |[Mancosu: 519] called in the narrower sense mathematical foundational research is mainly directed towards the foundations of arithmetic. Here one tends to eliminate as far as possible what is specific about geometry by splitting it up into an arithmetical and a physical side. We shall leave the question open whether this procedure is justified; this question is not discussed by Wittgenstein. On the other hand he deals in great detail with the basic questions of arithmetic. Let us now take a closer look at his observations concerning this field of questions.

The viewpoint from which Wittgenstein regards arithmetic is not the usual one of the mathematician. Wittgenstein has concerned himself more with the theories on the foundations of arithmetic (in particular with the

Russellian one) than with arithmetic itself. Particularly with regard to the theory of numbers, his examples seldom go beyond the numerical. An uninformed reader might well conclude that the theory of numbers consists almost entirely of numerical equations, which indeed are normally regarded not as propositions to be proved, but as simple statements. The treatment is more mathematical in the sections where he discusses questions of set theory, such as denumerability and non-denumerability, as well as the Dedekind cut theory.

Wittgenstein maintains everywhere a standpoint of strict finitism. In this respect he considers the various types of problem concerning infinity, such as exist for a finitist viewpoint, in particular the problem of the *tertium non datum* and that of impredicative definitions. The very forceful and vivid account he gives is well suited to conveying a clearer idea of the finitist's conception to those still unfamiliar with it. However, it contributes hardly anything essentially new to the argumentation; and those who consciously hold the view of classical mathematics will scarcely be convinced by it.

Let us discuss a few points in more detail. Wittgenstein deals with the question whether in the infinite expansion of π a certain sequence of numbers ϕ such as, say, '777', ever occurs. Adopting Brouwer's viewpoint he draws attention to the possibility that to this question there may not as yet be a definite answer. In this connection he says (p. 138, No. 9): 'However strange it sounds, the further expansion of an irrational number is a further development of mathematics.' This formulation is obviously ambiguous. If it merely means that the determination of a not yet calculated decimal place of an irrational number is a contribution to the development of mathematics, then every mathematician will agree with this. But since the assertion is held to be a 'strange sounding' one, certainly something else is meant, perhaps that the course of the development of mathematics at a given time is undecided and that this undecidedness can influence also the progress of the expansion of an irrational number given by definition, so that the decision as to what figure is to be put at the ten-thousandth decimal place of π would depend on the course of the history of thought. Such a view, however, is not appropriate even according to the conception of Wittgenstein himself, for he says (p. 138, No. 9): 'The question … changes its status when it becomes decidable.' |$^{Mancosu:\ 520}$ Now the digits in the decimal fraction expansion of π can be determined up to any chosen decimal place. Hence the view about the further development of mathematics does not contribute anything to the understanding of the situation in the case of the expansion of π. In this regard we can even say the following. Suppose we could maintain with certainty that the question of the occurrence of the sequence of numbers ϕ is undecidable, then this would imply that the figure ϕ never occurs in the expansion of π; for if it did occur, and if k were the decimal place which the last digit of ϕ has on the first occurrence in the decimal fraction expansion of π, then the question whether the figure ϕ occurs before the $(k+1)$th place would be a decidable question and could be answered positively, and thus the initial question would be answerable. (This argument by the way does not require the principle of the *tertium non datur*.)

Further on Wittgenstein repeatedly reverts to the example of the decimal fraction expansion of π; in one place in particular (p. 185, No. 34) we find an assertion which is characteristic of his view: 'Suppose that people go on and on calculating the expansion of π. An omniscient God knows, therefore, whether by the end of the world they will have reached a figure "777". But can his *omniscience* decide whether they *would* have reached this figure after the end of the world? It cannot do so ... For him, too, the mere rule of expansion cannot decide anything that it does not decide for us.'

That is certainly not convincing. If we conceive the idea of a divine omniscience at all, then we would certainly ascribe to it the attribute of being able to survey at *one* glance a totality of which every single element is in .principle accessible to us. We must pay here particular heed to the double rôle of the recursive definition of the decimal fraction expansion: on the one hand as the definitory fixing of the decimal fraction, and on the other as a means for the 'effective' calculation of decimal places. If we here take 'effective' in the usual sense, then even a divine intelligence can *effectively* calculate nothing other than what we are able to effectively calculate (no more than it world be capable of carrying out the trisection of an angle with a ruler and compass, or of deriving Gödel's underivable proposition in the related formal system); however, it is not inconceivable that this divine intelligence should be able to survey in another (not humanly effective) manner all the possible calculation results of the application of a recursive definition.

In his criticism of the theory of Dedekind's cut, Wittgenstein's main argument is that the extensional approach is mixed up in this theory with the intensional approach. This criticism is applicable in the case of certain versions of the theory where the tendency is to create the impression of a stronger character of the procedure than is actually achieved. If one wants to introduce the cuts not as mere sets of numbers, but as defining arithmetical laws of such sets, then either one must utilize a quite vague concept of the ‘law’, thus gaining little; or, if one sets about clarifying the concept, one meets with the difficulty which Hermann Weyl termed the vicious circle in the foundation |[Mancosu: 521] of analysis and which for some time back was sensed instinctively by various mathematicians, who thereupon advocated a restriction of the procedure of analysis. This criticism of impredicative concept-formation even today plays a considerable rôle in the discussions on the foundations of mathematics. However, difficulties are not encountered if the extensional standpoint is consistently retained, and Dedekind’s conception can certainly be understood in this sense and was probably so understood by Dedekind himself. All that is required here is that we should recognize, besides the concept of number itself, also the concept of a set of natural numbers (and in consequence of this the concept of a set of fractions, too) as an intuitively significant concept not requiring reduction. This implies a certain renunciation in respect of the goal of arithmetizing analysis, and thus geometry, too. ‘But’—one could here ask in the Wittgensteinian manner—‘must geometry be entirely arithmetized ?’ Scientists are often very dogmatic in their attempts at reductions. They are frequently inclined to treat such an attempt as completely successful even when it succeeds not in the manner intended but only in some measure and with a certain degree of approximation. Where such standpoints are encountered, considerations of the kind suggested by Wittgenstein’s book can be very valuable.

Wittgenstein’s detailed discussion of Dedekind’s proof is not satisfactory. Some of his objections can be disposed of simply through a clearer account of Dedekind’s line of thought. In the discussion of denumerability and non-denumerability, the reader must bear in mind that Wittgenstein always understands by cardinal number a finite cardinal number, and by a series one of the order type of the natural numbers. The polemics against the theorem of the non-denumerability of the totality of real numbers is unsatisfactory insofar as the analogy between the concepts 'non-denumerable' and 'infinite' is not brought out clearly. Corresponding to the way in which 'infinity of a totality G' can be defined as the property whereby to a finite number of things out of G there can always be assigned a further one, the non-denumerability of a totality G is defined by the property that to every denumerable sub-totality there can be assigned an element of G not yet contained in the sub-totality. In this sense the non-denumerability of the totality of real numbers is demonstrated by the diagonal procedure, and there is nothing foisted in here, as would appear to be the case according to Wittgenstein's argument. The theorem of the non-denumerability of the totality of real numbers is attainable without comparison of the transfinite cardinal numbers. Besides—this is often disregarded—there exist for that theorem other proofs more geometrical than the one provided by the diagonal procedure. From the point of view of geometry we have here a rather gross fact.

It is also strange to find the author raising a question like this: 'How then do we make use of the proposition: "There is no largest [scil. finite] cardinal number."? ... First and foremost it is to be noticed that we put the |*Mancosu*: 522 question at all, which indicates that the answer is not obvious' (p. 57, No. 5). We might think that one need not spend long searching for the answer here. Our entire analysis with its applications in physics and technology rests on the infinity of the series of numbers. The theory of probability and statistics make continually implicit use of this infinity. Wittgenstein argues as though mathematics existed almost solely for the purposes of housekeeping.

The finitist and constructive attitude on the whole taken by Wittgenstein towards the problems of the foundations of mathematics conforms to the general tendency of his philosophizing. However, it can hardly be said that he finds a confirmation for his viewpoint in the situation of the foundational investigations. All that he shows is how this standpoint has to be applied when engaging in the questions in dispute. It is generally characteristic of the situation with regard to the foundational problems that the results obtained hitherto clearly favor neither the one nor the other of the main two opposing philosophical views—the finitist-constructive and the 'Platonic'-existential view. Either of the two views can advance arguments against the other. The existential conception, however, has the advantage of enabling us to appreciate the investigations directed towards the establishment of elementary constructive methods (just as in geometry the investigation of constructions with ruler and compass has significance even for a mathematician who admits other methods of construction), while for the strictly constructivist view a large part of classical mathematics simply does not exist.

To some extent independent of the partisanship in the mentioned opposition of viewpoints are those observations of Wittgenstein's which concern the role of formalization, the reduction of number theory to logic, and the question of consistency. His views here show more independence, and these considerations are therefore of greater interest.

With regard to the question of consistency he asserts in particular what has meanwhile also been stressed by various other investigators in the field of foundational research: that within the bounds of a formal system the contradiction should not be considered solely as a deterrent, and that a formal system as such can still be of interest even when it leads to a contradiction. It should be observed, however, that in the former systems of Frege and Russell the contradiction already arises within a few steps, almost directly from the basic structure of the system. Furthermore, much of what Wittgenstein says in this connection overshoots the mark by a long way. Particularly unsatisfactory is the frequently quoted example of the producibility of contradictions by admitting division by nought. (One need only consider the foundation of the rule of reduction in order to see that this is not applicable in the case of the factor nought.)

Wittgenstein recognizes at all events the importance of demonstrating consistency. Yet it is doubtful whether he is sufficiently well aware of the rôle played by the condition of consistency in the reasoning of proof-theory. Thus the discussion of Gödel's theorem of non-derivability in particular |[*Mancosu*: 523] suffers from the defect that Gödel's quite explicit premiss of the consistency of the considered formal system is ignored. A fitting comparison, which is drawn by Wittgenstein in connection with the Gödelian proposition, is that between a proof of formal unprovability and a proof of the impossibility of a certain construction with ruler and compass. Such a proof, Wittgenstein says, contains an element of prediction. The remark which follows, however, is strange (p. 52, No. 14): 'A contradiction is unusable as such a prediction.' Such proofs of impossibility in fact always proceed by the deduction of a contradiction.

In his considerations on the theory of numbers Wittgenstein shows a noticeable reserve towards Frege's and Russell's foundation of number theory, such as was not to be found in the earlier stages of his philosophy. Thus he says on one occasion (p. 67, No. 4): '... the logical calculus is only — frills tacked on to the arithmetical calculus.' This thought had hardly been formulated previously as pregnantly as here. It might be appropriate to reflect on the sense in which the assertion holds good. There is no denying that the attempt at incorporating the arithmetical, and in particular, the numerical propositions into logistic has been successful. That is to say, it has proved possible to formulate these propositions in purely logical terms and to prove them within the domain of logistic on the basis of this formulation. Whether this result may be regarded as yielding a proper philosophical understanding of the arithmetical proposition is, however, open to question, When we consider the logistical proof of an equation such as $3 + 7 = 10$. we observe that within the proof we have to carry out quite the same comparative verification which occurs in the usual counting. This necessity shows itself particularly clearly in the formalized form of logic; but it is also present when we interpret the content of the formula logically. The logical definition of three-numberedness (Dreizahligkeit), for example, is structurally so constituted that it to some extent contains within itself the element of three-numberedness. The property possessed by the predicate P (or by the class that forms the extension of P) of being three-numbered is indeed defined by the condition that there exist things x, y, z having the property P and differing each from the others, and that further everything having the property P is identical with x or y or z. Now the conclusion that for a three-numbered predicate P and a seven-numbered predicate Q, in the case that the predicates do not apply in common to one thing, the alternative $P \vee Q$ is a ten-numbered predicate, requires for its foundation just the kind of comparison that is used in elementary arithmetic—only that now an additional logical apparatus (the 'frills') comes into operation. When this is clearly realized, it appears that the proposition of the logical theory of predicates is valid because $3 + 7 = 10$, and not vice versa.

Thus in spite of the possibility of incorporating arithmetic into logistic, arithmetic constitutes the more abstract ('purer') schema; and this appears paradoxical only because of a traditional, but on closer examination |[*Mancosu*: 524] unjustified view according to which logical generality is in every respect the highest generality.

Yet it might be good to took at yet another aspect of the matter. According to Frege a number (Anzahl) is to be defined as the property of a predicate. This view already presents difficulties for the normal use of the number concept; for in many contexts where a number occurs, the indication of a predicate of which it is the property proves to be highly forced. In particular it should be noted that numbers occur not only in statements: they also occur in directions and in demands or requests—such as when a housewife says to an errand-boy: 'Fetch me ten apples.'

The theoretical elaboration of this conception is not without complications either. A definite number does not generally belong as such to a predicate, but only with reference to a domain of things, a universe of discourse (apart from the many cases of extra-scientific predicates to which no definite number at all can be ascribed). Thus it would be more accurate to characterize a number as a relation between a, predicate and a domain of individuals. In Frege's theory, it is true, this complication does not arise since he presupposes what might be called an absolute domain of individuals. But, as we know, it is precisely this starting point which leads to the contradiction noted by Russell. Apart from this, the Fregian conception of his predicate theory, in which the value distributions (extensions) of the predicate are treated as things quite on a par with ordinary individuals, already implies a clear deviation from our customary logic in the sense of a theoretical construction of a formal derivative frame. The idea of such a frame has retained its methodological importance, and the question as to the most favorable formation of it is still one of the main problems in foundational theory. However, with respect to such a frame one can speak of a 'logic' only in an extended sense; logic in its usual sense, stating merely the general rules for deductive reasoning, must be distinguished from the latter.

Wittgenstein's criticism of the incorporation of arithmetic into logic is, it is true, not advanced in the sense that he recognizes arithmetical theorems as stating facts *sui generis*. His tendency is rather to deny altogether that such theorems express facts. He even declares it to be the 'curse of the invasion of mathematics by mathematical logic that any proposition can now be represented in mathematical notation and we thus feel obliged to understand it, although this way of writing is really only the translation of vague, ordinary prose' (p. 155, No. 46). Indeed he recognizes calculating only as an acquired skill with practical utility. In particular, he seeks to explain away in a definitory manner what is factual about arithmetic. Thus he asks, for instance (p. 33, No. 112): 'What do I call "the multiplication 13×13"? Only the correct pattern of multiplication at the end of which comes 169? Or a "wrong multiplication" too?' Likewise, the question often arises as to what it is that we 'call calculating' (p. 97, No. 73). And on p. 92, No. 58 he argues: 'Suppose one were to say that by calculating we become acquainted with the properties |[Mancosu: 525] of numbers. But do the properties of numbers exist outside of calculating?' The tendency is apparently to take the correct additions and multiplications as defining calculating and to characterize them as 'correct' in a trivial manner. But one cannot succeed in this way, i.e. one cannot express in this way the many facts of relatedness which appear in the numerical computations. Let us take, say, the associativity of addition. It is certainly possible to fix by definition the addition of the single figures. But then the strange fact remains that the addition $3 + (7 + 8)$ gives the same result as $(3+7)+8$, and that the same holds whatever numbers replace $3, 7, 8$. The number-theoretic expressions are, from the definitory point of view, so to speak, over-determined. It is indeed on this kind of over-determinateness that the many checks are based of which we may make use in calculating.

Occasionally Wittgenstein raises the question as to whether the result of a calculation carried out in the decimal system is also valid for the comparison of numbers carried out by means of the direct representation with sequences of strokes. The answer to this is to be found in the usual mathematical foundation of the method of calculating with decadic figures. Yet here Wittgenstein touches upon something fundamental: the proofs to be given for the justification of the decadic rules of calculation rest, if they are obtained in a finitist way, upon the assumption that every number we can form decadically is producible also in the direct stroke notation, and that the operations of concatenation, etc., as also of comparison, are always performable with such stroke sequences. From this it appears that the finitistic theory of numbers, too, is not in the full sense 'concrete', but utilizes idealizations.

The previously mentioned assertions in which Wittgenstein speaks of the synthetic character of mathematics are in a certain apparent contrast with the tendency to regard numerical calculating as being characterized merely by way of definition and to deny that arithmetical propositions have the character of facts. In this connection the following passage may be noted (p. 160, No. 3): 'How can you maintain that "... 625 ..." and "... 25×25 ..." say the same thing?—It is only through our arithmetic that they *become one*.'

What is meant here is about the same thing that Kant had in mind in the argument against the view that $7+5=12$ is a merely analytical proposition, and where he contends that the concept 12 'is by no means already conceived through my merely conceiving this union of 7 and 5', and then adds: 'That 7 are to be added to 5, I have, it is true, conceived in the concept of a sum $= 7 + 5$, but not that this sum is equal to the number 12' (*Critique of Pure Reason*, B 14ff.). The Kantian argument could be expressed in a modern form somewhat as follows. The concept '$7 + 5$' is an individual concept (in accordance with Carnap's terminology) expressible by the description 1_x $(x = 7+5)$, and this concept is different from the concept '12'; the only reason for this not being so obvious is that we involuntarily carry out the addition of the small numbers 7 and 5 directly. We have here the case, so |[Mancosu: 526] often discussed in the new logic following Frege, of two terms with a different 'sense' but the same 'Bedeutung' (called 'denotation' by A. Church); in order to determine the synthetic or analytic character of a judgment one must, of course, always go by the sense, not the 'Bedeutung'. The Kantian thesis that mathematics is of a synthetic character does not at all conflict with what the Russellian school maintains when it declares the propositions of arithmetic to be analytic. We have here two entirely different concepts of the analytic- a fact which in recent times has been pointed out in particular by E. W. Beth.[4]

A further intrinsic contrast is to be found in Wittgenstein's attitude towards logistic. On the one hand, he often tends to regard proofs as formalized proofs. Thus he says on p. 93, No. 64: 'Suppose I were to set someone the problem: "Find a proof of the proposition ..."—The solution would surely be to show me certain signs.' The distinctive and indispensable rôle of everyday language compared with that of a formalized language is not given prominence in his remarks. He often speaks of the 'language game' and by no means restricts the use of this expression to the artificial formal language, for which alone it is indeed appropriate. Our natural language has in no way the character of a game; it is peculiar to us, almost in the way our limbs are. Apparently Wittgenstein is still governed by the idea of a language of science comprehending the whole of scientific thought. In contrast with this are the highly critical remarks on usual mathematical logic. Apart from the one already mentioned concerning 'the curse of the invasion of mathematics by mathematical logic', the following in particular is worthy of notice (p. 156,

[4] "Over Kants Onderscheiding von synthetische en analytische Oordeelen," *De Gids*, vol. 106, 1942. Also: The "Foundations of Mathematics," *Studies in Logic*, Amsterdam, 1959, pp. 41–47.

No. 48): ' "Mathematical logic" has completely distorted the thinking of mathematicians and philosophers by declaring a superficial interpretation of the forms of our everyday language to be an analysis of the structures of facts. In this, of course, it has only continued to build on the Aristotelian logic.'

We shall come closer to the idea which probably underlies this criticism if we bear in mind that logical calculus was intended by various of its founders as a realization of the Leibnizian conception of the *characteristica universalis*. As to Aristotle, Wittgenstein's criticism, if we look at it more closely, is not directed against him. For all that Aristotle wanted to do with his logic was to fix the usual forms of logical argumenting and to test their legitimacy. The task of the *characteristica universalis*, however, was intended to be a much larger one: to establish a concept-world which would make possible an understanding of all connections existing in reality. For an undertaking aimed at this goal, however, it cannot be taken for granted that the grammatical structures of our language have to function as the basic framework of the theory; for the categories of this grammar have a character that is at least partially anthropomorphic. Yet nothing even approaching the same value has hitherto been devised in philosophy to replace our usual logic. What |[*Mancosu*: 527] Hegel in particular put in place of the Aristotelian logic in his rejection of it is a mere comparison of universals by way of 'analogies and associations, without any clearly regulative procedure. This certainly cannot pass as any sort of approach to the fulfilment of the Leibnizian ideas.

From Wittgenstein, however, we can obtain no guidance on how conventional logic may be replaced by something philosophically more efficient.

He probably considered an 'analysis of the structures of facts' to be a task wrongly set. Indeed he did not look for a procedure determined by some directive rules. The 'logical compulsion', the 'inexorability of logic', the 'hardness of the logical must' are always a stumbling block for him and ever again a cause of amazement. Perhaps he does not always realize that all these terms have the character of merely a popular comparison which in many respects is inappropriate. The strictness of the logical and the exact does not limit our freedom. Our very freedom enables us to achieve precision through thought in a perceptive world of indistinctness and inexactness. Wittgenstein speaks of the 'must of kinematics' being 'much harder that the causal must' (p. 37, No. 121). Is it not an aspect of freedom that we can conceive virtual motions subject merely to kinematic laws, as well as real, causally determined motions, and can compare the former with the latter?

Enlightened humanity has sought in rational definiteness its liberating refuge from the dominating influence of the merely authoritative. At the present time, however, this has for a large part been lost to consciousness, -and to many people scientific validity that has to be acknowledged appears .as an oppressing authority.

In Wittgenstein's case it is certainly not this aspect which evokes his critical -attitude towards scientific objectivity. Nevertheless, his tendency is to understand the intersubjective unanimity in the field of mathematics as an heteronomous one. The agreement, he believes, is to be explained by the fact that we are in the first place 'trained' in common in elementary technique and that the agreement thus created is continued through the proofs (cf. quotation on p. 195). That this kind of explanation is inadequate might occur to anybody not attracted by the impression of originality of the aspect. The mere possibility of the technique of calculating with its manifold possibilities of decomposing a computation into simpler parts cannot be regarded merely as a consequence of agreement (cf. remark on pp. 17 and 18). Furthermore, when we think of the enormously rich concept-formations towered up on each other, as for instance in function theory—where one can say of the theorems obtained at any stage what Wittgenstein once said: 'We lean on them or rest on them' (p. 124, No. 35)—we see that the conception mentioned does not in any way explain why these conceptual edifices are not continually collapsing. Considering Wittgenstein's viewpoint, it is, in fact, not surprising that he does not feel the contradiction to be something odd; but what does not appear from his account is that contradictions in mathematics are to be found only in quite peripheral extrapolations and nowhere else. In this sense |[*Mancosu*: 528] one can say that the fact of mathematics does not become at all understandable through Wittgenstein's philosophy. And it is not his anthropological point of view which gives rise to the difficulty.

Where, however, does the initial conviction of Wittgenstein's arise that in the region of mathematics there is no proper knowledge about objects, but that everything here can only be techniques, standards and customary attitudes? He certainly reasons: 'There is nothing here at all to which knowing could refer.' That is bound up, as already mentioned, with the circumstance that he does not recognize any kind of phenomenology. What probably induces his opposition here are such phrases as the one which refers to the 'essence' of a colour; here the word 'essence' evokes the idea of hidden properties of the color, whereas colors as such are nothing other than what is evident in their manifest properties and relations. But this does not prevent such properties and relations from being the content of objective statements; colors are not just a nothing. Even if we do not adopt the pretensions of the philosophy of Husserl with regard to 'intuition of the essence', that does not preclude the possibility of an objective phenomenology. That in the region of colors and sounds the phenomenological investigation is still in its beginnings is certainly bound up with the fact that it has no great importance for theoretical physics, since in physics we are induced, at an early stage, to eliminate colors and sounds as qualities. Mathematics, however, can be regarded as the theoretical phenomenology of structures. In fact, what contrasts phenomenologically with the qualitative is not the quantitative, as is taught by traditional philosophy, but the structural, i.e. the forms of being aside and after, and of being composite, etc., with all the concepts and laws that relate to them.

Such a conception of mathematics leaves the attitude towards the problems of the foundations of mathematics still largely undetermined. Yet, for anyone proceeding from the Wittgensteinian conception, it can open the way to a viewpoint that does greater justice to the peculiarity and the comprehensive significance of mathematics.