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CLASSICAL RELEVANT LOGICS. I

In a number of papers, the authors have offered semantic and algebraic analyses of the relevant logics of Ackermann, Anderson, and Belnap, and of logics akin thereto. The most interesting of these logics, in our opinion, are the Anderson-Belnap system R, which we analyzed in [1], and our own system B, studied in [5] and [6]. These are respectively the strongest and the weakest relevant logics, others being intermediate. Now one of the chief features of the relevant logics — indeed, the chief feature on the motivational lines of [7], [8], and [1] — lies in their blocking the so-called paradoxes of material and strict implication. The paradox which has drawn the most ink, particularly in Meyer's remarks, is the old saw that from a contradiction anything follows: i.e.,

(1) $A \,\&\, \neg A \rightarrow B$.

One would have thought, accordingly, that addition of (1), with other classical negation principles, to positive relevant logics would result in their breakdown, just as intuitionist logic fails to accomodate, on pain of breakdown, (1) together with excluded middle.

Like so many things one would have thought, this is false. In fact, on a semantical analysis akin to that of [1] and [5] but in certain respect more natural as regards negation, the positive insights emerge unscathed. The purpose of this paper is to present that semantics and the *classical relevant logics* to which it gives rise. Arising out of this analysis are new algebraic insights as well; classical relevant logics are Boolean and so the algebras of these logics — sufficing in particular for the positive parts of the relevant logics studied to date and in distinction to the algebras developed by Dunn in [9] and by us in [6] — are just Boolean algebras with an additional binary operation. Indeed, as we show, we may limit our considerations to *set algebras* — i.e., algebras whose domain is the power set of some set. This leads to considerable simplification of some outstanding problems for positive relevant logics — in particular, for the long-open *decision question* for R^+ — and we suggest lines along which we believe a successful solution of this question can be obtained.

I

1. *A simplified semantics for* R^+. According to [1], the vehicle for the semantical analysis of the positive part R^+ of the Anderson-Belnap system R of relevant impli-

cation is the *positive relevant model structure* (r^+. m.s.), requisite semantical notions being given as follows:

An r^+. m. s. is a triple $\langle O, K, R\rangle$, where K is a set, $O \in K$, and R is a ternary relation on K subject to the following definitions and postulates:

d1. $a < b =\text{df}\, ROab$

d2. $R^2abcd =\text{df}\, \exists x\,(Rabx \text{ and } Rxcd)$

p1. $ROaa$ (i. e., $a < a$)

p2. $Raaa$

p3. $R^2abcd \Rightarrow R^2acbd$

p4. $R^2Oabc \Rightarrow Rabc$

Let the sentential language SL^+ be a triple $\langle S, O, F\rangle$, where S is a denumerably infinite set of sentential parameters, O is the set whose members are the binary connectives &, $\vee$, $\rightarrow$, $\circ$, and F the set of formulas built up as usual from the parameters in S and the connectives in O. (We use 'p', etc., to refer to parameters in S and 'A' etc., to refer to arbitrary formulas of F, ranking the connectives &, $\circ$, $\vee$, $\rightarrow$ in order of increasing scope for ease in reading formulas and otherwise associating to the left.) Then

(a) v is a valuation of SL^+ in $\langle O, K, R\rangle$ provided that v is a function from $S \times K$ to $\{T, F\}$ that respects $<$, in the sense that for all p in S and a, b in K,

(2) $a < b$ and $v(p, a) = T \Rightarrow v(p, b) = T$.

(b) I is the interpretation associated with v provided that I is a function from $F \times K$ to $\{T, F\}$ which satisfies the following conditions, for all p in S, a in K, and A, B in F.

i. $I(p, a) = v(p, a)$

ii. $I(A \,\&\, B, a) = T$ iff $I(A, a) = I(B, a) = T$

iii. $I(A \vee B, a) = T$ iff $I(A, a) = T$ or $I(B, a) = T$

iv. $I(A \rightarrow B, a) = T$ iff, for all b, c in K, $Rabc$ and $I(A, b) = T \Rightarrow I(B, c) = T$.

v. $I(A \circ B, a) = T$ iff there exist b, c in K such that $Rbca$ and $I(A, b) = T$ and $I(B, c) = T$.

A formula A is *true* on a valuation v, or on the associated interpretation I, at a point a of K, just in case $I(A, a) = T$; otherwise A is *false* at a. Truth at O is what counts in verifying logical truths; accordingly, we say simply that A is *verified* on v, or on the associated I, just in case $I(A, O) = T$. Finally, A is *valid* in an *r^+. m. s.* $\langle O, K, R\rangle$ just in case A is verified on all valuations therein, and A is *R^+-valid* just in case A is valid in all r^+. m. s.

The system called R^+ has varied from formulation to formulation, with respect to the presence or absence of a sentential constant t and the intensional conjunction $\circ$; in the present formulation (called R^{++} in [1]), $\circ$ is in and t is out. Theorems in the crucial connectives &, $\vee$, $\rightarrow$ are of course not affected by these differences, and, letting R^+ be given syntactically by axiom schemes A1–A11, A14–A15 and rules R1–R2 of [1], we borrow the following completeness result from [1].

LEMMA 1. *For all formulas A of SL^+, A is a theorem of R^+ iff A is R^+-valid.*

Proof essentially in [1].

We turn now to the simplification promised in the title of this section. Though in one sense the semantics is simple enough as it stands, the role of the relation $<$ defined by d1 leaves something to be desired. For our condition (2) on what counts as a valuation means that an arbitrary assignment of truth-values to parameters at worlds would not always count as a valuation. This fact, although motivationally justifiable on the understanding of the "worlds" as theories some of which are extensions of others, complicates the techniques of [1] in various places, particularly in the theory of propositions developed in the latter part of that paper. Evidently we might free ourselves of these complications if we could get by with r$^+$. m. s. totally unordered by the binary relation $<$; i.e., such that for all a, b in K, $a < b$ iff $a = b$. (This question occurred to us and was put independently by Peter Woodruff.) Its answer, so far as R^+ is concerned, is affirmative, as we now show.

A *classical relevant model structure* (c. r. m. s.) is a triple $\langle O, K, R\rangle$, where K is a set, $O \in K$, and R is a ternary relation on K subject to definitions d1–d2 and postulates

q1. $ROab$ iff $a = b$,

q2. $Raaa$,

q3. $R^2abcd \Rightarrow R^2acbd$.

Other semantical notions are as given above, noting that the restriction (2) is no longer any restriction at all. And we have

THEOREM 1. *For all formulas A of SL^+, A is a theorem of R^+ iff A is valid in all c. r. m. s.*

Proof. q2 and q3 being just p2 and p3, and p1 and p4 following trivially from q1, evidently every c. r. m. s. is an r$^+$. m. s. So by lemma 1, all theorems of R^+ are valid in all c. r. m. s. To establish the converse, we recall from [1] that there is a particular r$^+$. m. s. $\langle O^+, \mathcal{H}^{+\prime}, R'\rangle$,got from the collection $\mathcal{H}^{+\prime}$ of all prime R^+-theories, and a *canonical valuation* v^+ therein, such that *every* non-theorem A of R^+ is false on v^+. We note also

(3) $B \vee C$ is false on v^+ iff both of B, C are false on v^+, at O

(4) $B \circ C$ is true on v^+ iff both of B, C are true on v^+, at O^+.

(3) just follows from the definition of a valuation, but (4) requires a little argument. That argument can be supplied using either the Dunn Gentzen-technique presented in [15] or by extending the metacompleteness arguments developed in [16]. Using the latter, we define the *canonical quasi-valuation* for R^+ as the function v_c defined on the set F of formulas of R^+ with values in T, F determined recursively thus:

(i) $v_c(B) = F$ if B is a sentential variable,

(ii) $v_c(B \,\&\, C) = T$ iff $v_c(B) = v_c(C) = T$,

(iii) $v_c(B \vee C) = T$ iff $v_c(B) = T$ or $v_c(C) = T$,

(iv) $v_c(B \to C) = T$ iff both $B \to C$ is a theorem of R^+ and either $v_c(B) = F$ or $v_c(C) = T$,

(v) $v_c(B \circ C) = T$ iff both $B \circ C$ is a theorem of R^+ and $v_c(B) = v_c(C) = T$.

As in [16], which differs from the present treatment only in not considering $\circ$, an inductive argument on length of proof establishes that all theorems of R^+ are true on the canonical quasi-valuation v_c, after a preliminary induction on length of formula to the effect that everything true on v_c is a theorem of R^+. It then follows immediately from (v) that if $B \circ C$ is a theorem of R^+, so are both B and C; the converse is trivial, since $B \,\&\, C$ relevantly implies $B \circ C$. So, for all formulas B of R^+, $v_c(B) = v^+(B, O)$, all of which together suffices for (4).

(4) in particular having been established, we can now play with our r+. m. s. $\langle O^+, \mathcal{K}^{+\prime}, R' \rangle$ and our valuation v^+ to produce a c. r. m. s. $\langle O, \mathcal{K}, R \rangle$ and a valuation v therein which continues to refute all non-theorems of R^+. In fact, let $\mathcal{K}$ result from $\mathcal{K}^{+\prime}$ by adding the single new element O. Let R be the extension of R' got by making $ROab$ and $RaOb$ true iff $a = b$, for all a, b in $\mathcal{K}$. Verify that $\langle O, \mathcal{K}, R \rangle$ is indeed a c. r. m. s.; the only hard part is verifying q3, for which one uses the fact that p3 holds in $\langle O^+, \mathcal{K}^{+\prime}, R' \rangle$ and argues by cases, assuming for a, b, x, c, d in $\mathcal{K}$ that $Rabx$ and $Rxcd$ and finding a y in $\mathcal{K}$ such that $Racy$ and $Rybd$. Since this holds trivially unless one of a, b, x, c, d is O, one looks successively at $a = O$, $b = O$, etc., finding in each case the desired y, a task left to the interested reader.

We conclude by defining the valuation v in $\langle O, \mathcal{K}, R \rangle$, letting it agree with v^+ where v^+ is defined and setting $v\,(p, O) = F$ for all sentence parameters p. Let I be the interpretation associated with v and I^+ the interpretation associated with v^+. We wish to show, for all a in $\mathcal{K}^+$ and all formulas B of R^+, that $I\,(B, a) = I^+(B, a)$ and moreover that $I\,(B, O) = I^+(B, O^+)$. Proof is by induction on length of formula. The base case, where B is a sentential variable, is trivial, by definition of v. For the inductive part of the argument, the &, $\vee$ cases are immediate. Suppose then that B is of the form $C \circ D$. Suppose first, for a in $\mathcal{K}^+$, that $I^+(C \circ D, a) = T$; then there are b, c in $\mathcal{K}^+$ such that $R'bca$, $I\,(C, b) = T$, and $I\,(D, c) = T$, on inductive hypothesis, whence $I\,(C \circ D, a) = T$; if in particular $a = O^+$, by (4) we have $I^+(C, O^+) = I^+(D, O^+) = T$, whence on inductive hypothesis and given $ROOO$, $I\,(C \circ D, O) = T$. Suppose on the other hand that $I^+(C \circ D, a) = F$, for a in $\mathcal{K}^+$. Then we cannot find b, c in $\mathcal{K}^+$ such that $I^+(C, b) = T$, $I^+(D, c) = T$ and $R'bca$. If we can nevertheless find b, c in $\mathcal{K}$ such that $I\,(C, b) = I\,(D, c) = T$ and $Rbca$, using the inductive hypothesis evidently either $b = O$ or $c = O$. Suppose without loss of generality that $b = O$; then, by q1, $c = a$. But, since $\langle O^+, \mathcal{K}^{+\prime}, R' \rangle$ is an r$^+$. m. s., by p1 $R'O^+ca$ and, on inductive hypothesis, $I^+(C, O^+) = T = I\,(D, a)$, whence *contra hypothesis* $I+(C \circ D, a) = T$. So for all a in $\mathcal{K}^+$, $I\,(C \circ D, a) = I^+(C \circ D, a)$. Our remaining task in the $\circ$ case is to show that if $I\,(C \circ D, O^+) = F$, $I\,(C \circ D, O) = F$. Suppose the contrary. Then, since $\langle O, \mathcal{K}, R \rangle$ obeys q1, inasmuch as $I\,(C \circ D, O) = T$, $I\,(C, O) = I\,(D, O) = T$. (For $RbcO$ iff $b = c = O$). Then on inductive hypothesis $I^+(C, O^+)$

$= I^+(D, O^+) = T$, whence again $I^+(C \circ D, O^+) = T$, which was not supposed to happen. So we conclude also that $I(C \circ D, O) = I(C \circ D, O^+) = I^+(C \circ D, O^+)$ on inductive hypothesis. The last case occurs when B is of the form $C \to D$. Reversing the order of the $\circ$ case, we show first that $I(C \to D, O) = I^+(C \to D, O^+)$. Indeed, by theorem 1 of [1], this amounts to showing that the following are equivalent.

(5) For all x in $\mathcal{K}$, if $I(C, x) = T$ then $I(D, x) = T$.

(6) For all x in $\mathcal{K}^{+\prime}$, if $I^+(C, x) = T$ then $I^+(D, x) = T$.

Since $\mathcal{K}$ results by construction from $\mathcal{K}^{+\prime}$ just by throwing in the new element O, and since we may on inductive hypothesis suppose that I and I^+ agree where both are defined, evidently (5) implies (6). Given moreover on inductive hypothesis that $I^+(C, O^+) = I(C, O)$ and $I^+(D, O^+) = I(D, O)$, it's also clear that (6) implies (5), ending the proof that $I(C \to D, O) = I^+(C \to D, O^+)$. We may end the inductive argument by showing, for all a in $\mathcal{K}^{+\prime}$, $I^+(C \to D, a) = I(C \to D, a)$. Suppose first that $I^+(C \to D, a) = F$. Then there are b, c such that $R'abc$, $I^+(C, b) = T$ and $I^+(D, c) = F$, whence applying the inductive hypothesis and the definition of R, $I(C \to D, a) = F$. Suppose on the other hand that $I(C \to D, a) = F$. Then there are b, c in $\mathcal{K}$ such that $Rabc$, $I(C, b) = T$, and $I(D, c) = F$. Since $a \neq 0$, $c \neq O$, by q1. If $b \neq O$, then straightforward inductive argument shows $I^+(C \to D, a) = F$; if $b = O$, then $c = a$, and we have $I(C, O) = T$, $I(D, c) = F$ and $RaOc$; since $\langle O^+, \mathcal{K}^{+\prime}, R' \rangle$ is an r$^+$. m. s., we have $R' aO^+c$, and, on inductive hypothesis, $I^+(C, O^+) = F$, whence $I^+(C \to D, a) = F$. This ends the argument that v and v^+ agree wherever the latter is defined, and that moreover the formulas verified on v are exactly the formulas verified on v^+. Since all non-theorems of R^+ are in particular unverified on v^+, they are refuted on v as well; so all non-theorems of R^+ are invalid in some c. r. m. s., ending the proof of theorem 1.

On review of the proof of theorem 1, we see that its import is that we can always trivialize the defined relation $<$ by adding a new O, and that we can get away with such trivialization provided that $B \circ C$ is verified on a critical interpretation v^+ in the old structure iff B & C is verified. But, it should be noted, this condition by no means reduces the intensional conjunction $\circ$ to its truth-functional relative &, for it does not follow from the fact that B & C and $B \circ C$ are *verified* — i.e. true at O — under the same circumstances that the same holds for all "worlds" a; it does not, on pain of collapse.

2. *Classical negation added to* R^+. In [1], an r$^+$. m. s. $\langle O, K, R \rangle$ became an r. m. s. (relevant model structure) on addition of a unary operation * and postulates

p5. $Rabc \Rightarrow Rac^*b^*$

p6. $a^{**} = a$

and the addition of the following recursive condition to (i)–(v) on what counts as a valuation

vi. $I(\bar{A}, a) = T$ iff $I(A, a^*) = F$.

The condition (vi) permits that both of A, $\bar{A}$ may be true together, or false together, at a given world a (though without change in the set of R-valid formulas we may lay it down that this never happens at the preferred world O). So long as our purpose is the noble one of blocking paradoxes of the ilk of (1) above, some condition like (vi), and the * operation which gives rise to it, is certainly in order. On the other hand, it is not altogether nonsensical to dump * and to lay down, for a classical negation $\neg$ which we distinguish typographically from — above,

vii. $I(\neg A, a) = T$ iff $I(A, a) = F$.

Henceforth, except for comparative purposes, we shall not examine the relevant negation — of R; rather we add $\neg$, and whatever axioms are appropriate for it, directly to R^+, given the semantical principle vii. The result is a system CR, *Classical relevant implication*, which has in common with R the same positive fragment R^+.

Before we simplified the semantics of R^+ in section 1, there would have been little point in adding $\neg$ together with its governing condition vii. The reason is the following: The condition (2) on what counts as a valuation applies only to parameters p, q, etc. It was in [1] and succeeding papers an important lemma, proved by induction on length of formula, that for all A, and all interpretations I in a given model structure,

(7) $I(A, a) = T$ and $a < b$ imply $I(A, b) = T$.

The importance of (7) lies in the fact that, without it, not even $A \rightarrow A$ can be shown valid in general, destroying semantical insights before they even get off the ground. And, in particular, when $\neg$ is characterized by vii, the inductive argument used to prove (7) fails when A is of the form $\neg B$. For this reason, we did not propose vii to characterize any kind of negation in previously published papers, in the series [1]–[5].

Given theorem 1, and the semantical justification of R^+ via classical relevant model structures, the considerations which ban vii no longer operate. For since in c. r. m. s. $<$ is always trivial, of course (7) will hold for formulas of the form $\neg B$ as for any others; for all that (7) says for such formulas in c. r. m. s. is that if $\neg B$ is true at a, it is true at a, and to this no objection can be entered.

All of this suggests the natural extension of our machinery. The sentential language $CSL = \langle CS, CO, CF \rangle$ shall result from SL^+ by setting $CS = S$, forming CO from O by adding the connective $\neg$, and forming CF from F closing the set of formulas under the connectives in CO. Semantical notions shall be defined as before for R^+, adding the recursive condition vii; in particular, a formula of CSL shall be CR-valid iff it is valid in all classical relevant model structures $\langle O, K, R \rangle$. We already know by theorem 1 that, among positive formulas, the CR-valid ones are just the theorem of R^+. We turn now to the problem of finding axioms and rules to be added to those for R^+ to provide a consistent and complete axiomatization of CR.

The following lemma, which required work for R^+ and R, is immediate for CR:

Lemma 2. *Let $\langle O, K, R \rangle$ be a c. r. m. s., and I an interpretation therein. Then for all formulas A and B, $I(A \rightarrow B, O) = T$ iff A I-entails B*; i.e., *iff, for all x in K, if $I(A, x) = T$ then $I(B, x) = T$.*

Proof immediate by q1.

Given lemma 2, what sorts of formulas are *CR*-valid. First, evidently, all classical tautologies in $\&$, $\vee$, $\neg$, $\circ$, since the evaluation of such formulas at O on an interpretation I does not lead out of O; conversely, non-tautologies are of course *CR*-invalid. Second, the most conspicuous theorems involving classical negation, including some paradoxical ones, hold as well: double negation, DeMorgan laws, and such formulas as $A \,\&\, \neg A \to B$, $A \to B \vee \neg B$, $A \,\&\, (\neg A \vee B) \to B$ are all immediate by lemma 2; the last of these establishes the rule $\vee$ of detachment for material implication, which again required considerable work when the negation involved was the relevant negation —. Going further afield, we see that the principle $(A \to \neg A) \to \neg A$ of *reductio ad absurdum* is also *CR*-valid; for if $A \to \neg A$ is true on I at a point x of a c. r. m. s., then $\neg A$ is also true at x (for otherwise A is true at x, whence since $Rxxx$, so is $\neg A$ after all, which is impossible).

We are in the neighborhood of apparent breakdown, since all principles mentioned thus far valid for the relevant negation — are also valid for $\neg$, which also admits paradoxes; but if paradoxes are added to R, it breaks down all the way to classical logic. Conclusion: there must be some classical negation principle which is valid for the relevant negation — in R which is not valid for $\neg$ in *CR*. That's true, and further inspection reveals the principle in question to be contraposition. This holds for R in the form

$$(8) \qquad (A \to \bar{B}) \to (B \to \bar{A}) .$$

(8) is verified in the semantics of R using the contraposition postulate p5, which allows us to trade in $Rabc$ for Rac^*b^*; the important point is permuting the b and c, with stars. Stars are gone in *CR*, and an analogous argument for the analogue of (8) would require some permuting not licensed by the postulates q1–q3. Nevertheless, in *rule* form we do have for *CR*

$$(9) \qquad (A \to \neg B) \Rightarrow (B \to \neg A) .$$

For suppose, in support of (9), that $A \to \neg B$ is *CR*-valid. By lemma 2, for an arbitrary interpretation I, wherever A is true on I, B isn't. So wherever B is true on I, A isn't, ending the argument for the *CR*-validity of the rule (9).

Our examination thus far has accordingly shown that in one direction, *CR* has almost the strength of R, being weaker only in that contraposition must be taken in rule form but otherwise validating all conspicuous classical principles, just like R itself; in the other direction, admitting the paradoxes, *CR* is of course much stronger than R, though without collapsing positive insights. Frankly, this is amazing, for one would have thought that one could not go so far in the direction of paradox, sacrificing so little of the classical theory of negation, without disturbing the paradox-free positive insights of R^+. So, however, it turns out. What, now, are our axioms for *CR* to be.

In fact, added to an axiom set sufficient for R^+ (e.g., that suggested above), one axiom scheme and one rule suffice; i.e.,

B1. $\neg\neg A \to A$ (Double negation)

R3. $A \,\&\, B \to \neg C \Rightarrow A \,\&\, C \to \neg B$ (Antilogism)

The principle R3 is of course invalid for R, with the relevant negation —, on pain of just those paradoxes which CR buys; on the other hand, that R3 preserves CR-validity is established, *mutatis mutandis*, like (9) above. So, immediately,

LEMMA C. *All theorems of the system CR, formulated by adding* B1 *and* R3 *to the basis for* R^+ *in lemma* 1, *are CR-valid.*

Proof by verifying axioms and showing that rules preserve CR-validity.

The hard part, as usual, is completeness, though this is eased *vis a vis* previous papers by the highly classical character of CR. Let us sketch the proof of some theorems.

1.	$A \mathbin{\&} \neg B \to A$	A5
2.	$A \mathbin{\&} \neg A \to \neg\neg B$	1, R3
3.	$A \mathbin{\&} \neg A \to B$	2, A1, transitivity of $\to$
4.	$A \mathbin{\&} B \to B$	A6
5.	$A \mathbin{\&} (\neg A \vee B) \to B$	3, 4, A10, distribution, transitivity
6.	$A \mathbin{\&} \neg A \to \neg A$	A6
7.	$A \to \neg\neg A$	6, R3, idempotence of &

Suppose $A \to \neg B$ is a theorem. So, by transitivity and A5, is $B \mathbin{\&} A \to \neg B$. By R3, so is $B \mathbin{\&} B \to \neg A$, whence so by the elementary positive properties is $B \to \to \neg A$, whence the rule

8. $A \to \neg B \Rightarrow B \to \neg A$

9. $\neg (A \mathbin{\&} B) \to \neg A \vee \neg B$ B1, 7, 8, and elementary properties of & and $\vee$.

10.	All DeMorgan laws	Proof as of 9.
11.	$B \to A \vee \neg A$	3, 8, 9, double negation
12.	$A \vee \neg A$	11, *modus ponens* for B a theorem

13. All classical tautologies in &, $\vee$, $\neg$. (Use 12 to prove a conjunctive normal form, and transform using elementary properties of &, $\vee$, $\neg$ already established.)

We now turn to the development of a calculus of intensional theories as in [1]. Specifically, a *CR-theory* shall be any non-empty set of formulas of CR closed under adjunction and provable CR-entailment; a CR-theory shall be *regular* if it contains all theorems of CR; *prime* if whenever it contains $A \vee B$ it contains one of A, B; *consistent* if for no formula A does it contain both of A, $\neg A$; *complete* if for each A it contains at least one of A, $\neg A$; *maximal* if both consistent and complete; and *normal* if regular and maximal (alternatively, regular, prime, and consistent).

The treatment of theories is so similar to that of [1] that we pause only over differences. A trivial difference is that, for technical convenience, we do not count the null set a theory here. More important differences lie in the more classical character of CR. For example, in virtue of 11, every prime theory is complete here; in virtue of 3, the only inconsistent theory is the set of all formulas of CR; so maximal theories are important here, as they were not in [1], since except for the trivial, inconsistent theory the prime theories, the complete theories, and the maximal theories all coincide. (Algebraically, the point might be made by saying that we have passed from the general distributive lattice properties of $R+$ to the special properties of Boolean algebras in introducing CR-style negation.)

Let then the *calculus of CR-theories* be the quadruple $\mathcal{H} = \langle H, \subseteq, \circ, O \rangle$, where H is the collection of all CR-theories ordered by set inclusion, O is the set of theorems of CR, and $\circ$ is defined on H by setting, for all a, b, $a \circ b = \{C: \exists A \exists B\, (A \in a \;\&\; B \in \in b \;\&\; A \circ B \to C$ is a theorem of $CR)\}$.

Let T be a regular CR-theory. A CR-theory a is a T-theory provided that a is closed under T-entailment; i.e., when $A \in a$ and $A \to B \in T$, $B \in a$. As in [1], let the *calculus of T-theories* be the quadruple $\mathcal{H}_T = \langle H_T, \subseteq, \circ, O_T \rangle$, where H_T is the collection of all T-theories, $\subseteq$ and $\circ$ are as above, and O_T is T. (Note that $\mathcal{H}$ itself is an $\mathcal{H}_T$, since CR itself is the smallest regular T-theory.) We then have

LEMMA 3. *Let T be any regular CR-theory. Then the calculus $\mathcal{H}_T$ of T-theories is a partially ordered square-decreasing commutative monoid*; *i.e.*, $\circ$ *is commutative and associative*, O_T *is an identity with respect to* $\circ$, H_T *is closed under* $\circ$, *and if* $a \subseteq b$ *then* $a \circ c \subseteq b \circ c$, *for all* a, b, c *in* H_T; *moreover*, $a \circ a \subseteq a$. *And, in particular*, $\mathcal{H}_T$ *is a sub-semigroup of* $\mathcal{H}$, *the calculus of all CR-theories.*

Proof. Like lemmas 9 and 10 of [1], *mutatis mutandis*. (Since $\varnothing$ counted as a theory in [1], note here that for a, b in H, by definition $a \circ b \neq \varnothing$.)

LEMMA 4. *Suppose A is not a theorem of CR. Then there is a normal CR-theory T such that $A \notin T$.*

Proof. As these things go. Lindenbauming CR (i.e., identifying provably equivalent formulas), CR becomes a Boolean algebra (in virtue of theorems 1–13 above), and the set of theorems becomes a filter in this Boolean algebra, not containing the equivalence class of A. By Stone's maximal filter theorem for Boolean algebras, the theorem filter may be extended to a maximal filter, which still lacks the equivalence class of A. Let T be the set of all formulas of CR whose equivalence classes belong to the maximal filter just formed. Evidently T is regular, consistent, and lacks A; moreover it is closed under adjunction, and is complete, by familiar arguments. We end the proof of the lemma by showing T closed under CR-entailment. In fact, since T is an extension of CR, if $A \to B$ is a theorem of CR and $A \in T$, then $(A \to B) \;\&\; A$ belongs to T. Since T corresponds to a filter in a Boolean algebra, $B \vee [(A \to B) \;\&\; A]$ is easily seen to belong to T as well; but this last is provably equivalent to B in CR, ending the verification that T is closed under CR-entailment and hence the proof of lemma 4.

Let us return now to the task of providing a completeness proof for CR relative to our semantics, with the aid of the above lemmas. Let A be a non-theorem of CR. Lemma 4 grinds out a normal CR-theory T to which A does not belong. By working on the calculus of T-theories defined in lemma 3, for that T, we shall produce a c. r. m. s. in which A is invalid. Adaptation and simplification of the arguments of [1] is again our principal instrument.

Let $\boldsymbol{M} = \langle M, \leqslant, \circ, O \rangle$ be any commutative, partially ordered, square-decreasing monoid in the sense of lemma 3. By the r^+. m. s. *associated with* $\boldsymbol{M}$, we mean the structure $\langle O, M, R \rangle$, where R is a ternary relation on M such that $Rabc$ holds iff $a \circ$

$\circ\, b \leqslant c$ in $\boldsymbol{M}$. Lemma 11 of [1] states that the r^+. m. s. associated with $\boldsymbol{M}$ is indeed an r^+. m. s., satisfying p1–p4 above. Thus every calculus $\mathcal{H}_T$ of T-theories gives rise in this manner to an r^+. m. s. by lemma 3. By sifting all but the maximal T-theories out of H_T, our next lemma sets up the promised completeness proof.

Lemma 5. *Let T be a normal* CR-*theory and $\mathcal{H}_T = \langle H_T, \subseteq \circ, O_T\rangle$ the calculus of T-theories. Let $\langle O_T, H_T, R_T\rangle$ be the r^+. m. s. just associated with $\mathcal{H}_T$, Let H_T' be the subset of H_T consisting of all the maximal theories in H_T, and let R_T' be the restriction of R_T to H_T'. Then $\langle O_T, H_T', R_T'\rangle$ satisfies* q1–q3; *i.e., it's a c. r. m. s.*

Proof. We note first that since O_T is the identity of $\mathcal{H}_T$, by definition $R_T\, O_T\, ab$ iff $a \subseteq b$, for all a, b in H_T. But for a, b in H_T' — i.e., for *maximal* theories in H_T — evidently $a \subseteq b$ iff $a = b$. So q1 holds for $\langle O_T, H_T', R_T'\rangle$. That q2 holds is immediate from the fact that p2 holds for $\langle O_T, H_T, R_T\rangle$. q3 for $\langle O_T, H_T', R_T'\rangle$ likewise follows from p3 for $\langle O_T, H_T, R_T\rangle$, though the argument is much trickier; for suppose $R_T'^2\, abcd$. There is by p3 and x in H_T, but not known to be in H_T', such that $R_T\, acx$ and $R_T\, xbd$. x cannot be the CR-theory consisting of all formulas of CR; for if it were, then where B is any member of b, and F is any contradiction $C\ \&\ \neg\, C$, $F \circ B \in d$; but F implies anything, $B \rightarrow F$ in particular; $\circ$ imports, so $F \circ B \rightarrow F$ is a theorem of CR; but d, being a CR-theory, is closed under provable CR-entailment, whence $F \in d$; but d, since it belongs to H_T', is consistent, ending the proof that the x, or any x, such that $R_T acx$ and $R_T\, xbd$ is consistent. With this guarantee, the argument of lemma 13 of [1] then produces a prime CR-theory x' such that $R_T\, acx'$ and $R_T\, x'bd$; as observed x' cannot be inconsistent, whence since all other prime CR-theories are maximal there is an x in H_T', namely x', connecting ac and bd; i.e., $R_T'^2\, acbd$ holds when $R_T'^2\, abcd$ does, ending the verification of q3 and the proof of lemma 5.

The completeness of CR is at hand.

Theorem 2. *For all formulas A in the vocabulary of CR, A is a theorem of CR iff A is CR-valid.*

Proof. That theoremhood implies validity has already been noted. It remains to be proved that non-theoremhood implies invalidity. Let A accordingly be a non-theorem of CR, and let T be a normal CR-theory without A, whose existence is guaranteed by lemma 4 and in accordance with the strategy laid down above. Let $\langle O_T, H_T', R_T'\rangle$ be the c. r. m. s. associated with T by the conclusion of lemma 5. Define a valuation v therein by setting, for each maximal theory a in H_T' and each sentential variable p, $v(p, a) = T$ iff $p \in a$. We may end the proof of the theorem by showing, for the associated interpretation I and each A and a,

(i) $\quad I(A, a) = T$ iff $A \in a$.

For if (i) holds, since A is not in our chosen theory, A goes unverified on I and is hence CR-invalid. So we content ourselves with proof of (i), by induction on the number of connectives in A. The case where A is a sentential variable has been settled by fiat, while the inductive step is trivial if the principal conncetive is &. Since H_T' is limited to maximal theories, all of which are prime, and which by definition are consistent and complete, the inductive step remains trivial the main connective is $\vee$ or $\neg$. The two harder cases we examine in detail.

Hard case 1. A is of the form $B \circ C$. Suppose $I(A, a) = T$. There are b, c in H'_T, by semantic definition, such that $I(B, b) = I(C, c) = T$ and $R_T\,bca$; i.e., $b \circ c \subseteq a$, in the calculus $\mathcal{H}_T$. On inductive hypothesis $B \in b$ and $C \in c$, whence by definition of $\circ$ on $\mathcal{H}_T$, $B \circ C \in a$. Suppose conversely $B \circ C \in a$. Letting [B], [C] be respectively the smallest CR-theories containing B, C respectively, evidently $[B] \circ [C] \subseteq a$ in $\mathcal{H}_T$. Again the argument of lemma 12 of [1] produces prime extensions of $[B]$, $[C]$ respectively such that, calling them b, c, $b \circ c \subseteq a$; as in the last lemma, neither of b, c can be inconsistent, whence both belong to H'_T and $R'_T\,bca$; since $B \in b$ and $C \in c$, applying the inductive hypothesis $I(B \circ C, a)$ meets the conditions to have the value T, ending the inductive argument for the first hard case.

Hard case 2. A is of the form $B \to C$. This reduce on inductive hypothesis to a proof that (ii) $B \to C \in a$ iff (iii) for all b, c in H'_T, if $a \circ b \subseteq c$ holds in $\mathcal{H}_T$ and $B \in$ $\in b$ then $C \in c$. Passage from (ii) to (iii) is trivial, since $(B \to C) \circ B \to C$ is a theorem of CR. Suppose that (ii) fails, and let $[B]$ be as above. Then, anyway, $C \notin a \circ [B]$, for otherwise there would be an X in a such that $X \circ B \to C$ (exporting, $X \to (B \to C)$) is a theorem of CR, whence by closure of a under CR-entailment, $B \to C$ would be in a. So Stone-style maximal filtering, a la lemma 4, finds us a maximal c in H_T — i.e., a c in H_T' — such that $a \circ [B] \subseteq c$ and $C \notin c$. As in the previous hard case we may then extend $[B]$ to a b in H_T' such that $a \circ b \subseteq c$, $B \in b$, and $C \notin c$. This shows that the denial of (ii) implies the denial of (iii), ends our examination of the last hard case, and completes the proof of (i) and hence of theorem 2.

Theorem 2 ends our principal business, semantically, with the system CR. We turn now to the promised algebraic payoff for CR and its familiar negation-free fragment R^+.

II

3. *The algebra of propositions of CR is a set algebra.* As we noted at the outset, the nice thing about dealing with classical relevant model structures is that the burdensome condition (a) of the first section on what counts as a valuation is automatically satisfied by any assignment of truth-values to parameters at worlds. Let $\langle O, K, R\rangle$ be a c. r. m. s. According to [1], near enough, a *proposition* in $\langle O, K, R\rangle$ is any subset of K; an additional condition, that if $a \in J$ and $a < b$ then $b \in J$, if $J \subseteq K$ is to be a proposition, is now vacuously satisfied for any subset J of K, by the trivialization of $<$.

That settled, we accordingly characterize the *algebra of propositions* $\Pi(\boldsymbol{K})$ determined by the c. r. m. s. $\langle O, K, R\rangle = \boldsymbol{K}$ as the structure $\Pi(\boldsymbol{K}) = \langle P(K), \circ, \bigcup, ', 1\rangle$, where

(i) $P(K)$ is the power set of K.

(ii) $\bigcup$ is the generalized operation of set-theoretical union on subsets $Q(K)$ of $P(K)$.

(iii) For F, G in $P(K)$, $F \circ G = \{c : \exists a\, \exists b\, (a \in F\ \&\ b \in G\ \&\ Rabc\}$.

(iv) $'$ is the operation of set-theoretic complement.

(v) $1 = \{0\}$.

We set out immediately some of the nice properties of $\Pi(\boldsymbol{K})$.

THEOREM 3. *Let* $\boldsymbol{K}$ *be a c. r. m. s. and let the associated algebra of propositions* $\Pi(\boldsymbol{K})$ *be as just defined. Then, under* $\bigcup$ *and* $'$, $\Pi(\boldsymbol{K})$ *is a set algebra — i.e., a complete, atomic Boolean algebra. Moreover, under the operations* $\bigcup$ *and* $\cap$ *defined as usual, with* $\bigcirc$ *and* 1, $\Pi(\boldsymbol{K})$ *is a complete Dunn monoid — i.e.,*

(I) $P(K)$ *is a complete lattice under* $\cap$ *and* $\bigcup$,

(II) $P(K)$ *is a commutative monoid under* $\bigcirc$ *and* 1,

(III) $P(K)$ *is completely lattice-ordered, in the sense that for all* $a \in P(K)$ *and* $S \subseteq P(K)$, $a \bigcirc \bigcup S = \bigcup \{a \bigcirc s : s \in S\}$

(IV) $P(K)$ *satisfies the infinite distributive law* $a \cap \bigcup S = \bigcup \{a \cap s : s \in S\}$

(V) $P(K)$ *satisfies the square-increasing law* $a \subseteq a \bigcirc a$.

NOTE. Dunn monoids stem in principle from Dunn's dissertation, though their first explicit introduction, to algebraize R^+, occurs in [10]. Complete Dunn monoids are newly introduced here, in analogy to the complete DeMorgan monoids of [1]. We observe moreover that there is some slack, first pointed out by Belnap in [11], as to what one might want to count as reasonable qualifications for the honorific 'complete' in the characterization of algebraic structures suitable for relevant logics, particularly with respect to the distribution of $\bigcirc$ and $\cap$ respectively over $\bigcup$; we choose (III) and (IV), though so far as $\Pi(\boldsymbol{K})$ is concerned the option is open to follow Belnap by stating, e.g., (IV) for arbitrary intersections of arbitrary unions. Finally, we note that in a *complete* Dunn monoid, the important residual $\rightarrow$ is definable by $a \rightarrow b = \mathrm{df} \bigcup \{c : c \bigcirc a \leqslant b\}$ and that the residual so defined has the property, for all c, that $c \bigcirc a \leqslant b$ iff $c \leqslant a \rightarrow b$. Proof of theorem 3. First, since $P(K)$ is the power set of K, we note that it is trivial that $\Pi(\boldsymbol{K})$ is a set algebra. Second, proof that (I)–(V) holds is simply a matter of verifying postulates, using q1–q3, and is accordingly left to the reader, ending the proof of theorem 3.

To get a full day's work for a full day's pay from theorem 3, it is necessary to furnish an algebraic counterpart of *CR* itself. We shall call the resulting structures *Boolean monoids*, calling attention thus to their dual character as Dunn monoids and Boolean algebras. Specifically, a Boolean monoid is a structure $\boldsymbol{B} = \langle B, \bigcirc, \rightarrow, \vee, ', 1\rangle$, where

(i) B is a Boolean algebra under the binary operation $\vee$ and the unary operation $'$, and

(ii) B is a Dunn monoid under the binary operations $\bigcirc$, $\rightarrow$, $\vee$, and $\wedge$ (the last defined by $a \wedge b = (a' \vee b')'$ with monoid identity 1; i.e., $\bigcirc$ is commutative and associative and $1 \bigcirc a = a$ for all a in B; B is lattice-ordered in the sense that $a \bigcirc (b \vee c) = (a \bigcirc b) \vee (a \bigcirc c)$; B is residuated in the sense that $c \leqslant a \rightarrow b$ iff $c \bigcirc a \leqslant b$ (where $a \leqslant b$ is defined by $a \vee b = b$, as usual); and B is square-increasing in the sense that $a \leqslant a \bigcirc a$.

Note that as a Boolean algebra, a Boolean monoid has of course least and greatest elements $\varnothing$ and I respectively, but that these elements are by no means to be confused with the monoid identity 1. In particular, the definitions assign to 1 no particular place in B under the ordering $\leqslant$. It is, however, convenient often to assume

(iii) 1 is an atom; i.e., for all a in B, if $a \leqslant 1$, $a = \varnothing$ or $a = 1$.

Boolean monoids satisfying the additional condition (iii) shall be called *regular Boolean monoids*.

A Boolean monoid — i.e., a structure satisfying (i) and (ii) above — which is complete as a Boolean algebra and which satisfies the complete lattice-ordering postulate (III) and the infinite distributive law (IV) of theorem 3 shall be called a complete Boolean monoid. We have as an immediate consequence of theorem 3

COROLLARY 3.1. *Every algebra of propositions associated with a classical relevant model structure is a regular, complete Boolean monoid.*

Proof. Immediate from theorem 3, noting that, to establish regularity, it suffices to observe that 1 was defined for a $\Pi(\boldsymbol{K})$ as a unit set in $P(K)$ and hence as an atom in the sense of (iii).

We now go on to show that Boolean monoids do offer an algebraic counterpart of CR. Let an *interpretation* of CR in a Boolean monoid $\boldsymbol{B}$ be a homomorphism from the algebra of formulas of CR into B. A formula A of CR shall be *true* on an interpretation I in a Boolean monoid $\boldsymbol{B}$ just in case $1 \leqslant I(A)$; otherwise A shall be *false* on I. (Note that truth on I means being bigger than (or equal to) the identity 1, and has nothing to do with the greatest element of $\boldsymbol{B}$; put otherwise, as a matrix *every element* in the principal filter determined by 1 is designated in $\boldsymbol{B}$, and all other elements are undesignated.) Finally, a formula A of CR is CR-valid iff A is true on all interpretations in all Boolean, monoids.

THEOREM 4. *The theorems of CR are exactly the CR-valid formulas, in the algebraic sense just given.*

Proof. To show that all theorems are valid, it suffices to show that axioms have this property and that it is preserved under the rules of CR. To show that all valid formulas are theorems, it suffices to Lindenbaum, noting that the Lindenbaum algebra of CR is a Boolean monoid,[1] identifying formulas such that $A \to B$ and $B \to A$ are both theorems of CR; the canonical interpretation which assigns each sentential variable the class of formulas to which it is provably equivalent makes as usual all non--theorems false; so all valid formulas are theorems, ending the proof of theorem 4.

We shall go no further here into the algebraic study of Boolean monoids; the kinds of results obtained by Dunn and by us for related algebras — e.g., in [1] — may be got by ringing the changes on known proofs. But we do wish to call attention to the simplicity of Boolean monoids, particularly regular ones, among the kinds of structures

[1] This is not literally accurate, since no element of the Lindenbaum algebra just formed will do for the identity 1. The cure is to extend CR to a system CR^t, with the sentential constant t added to the language and new axioms t and $t \to (A \to A)$, as in previous papers [e.g., [3]]. The equivalence class of t works as an identity, and its addition is conservative by the argument of [14]. We make this remark parenthetical because we do not really want t in CR; its presence interferes with our preference for *regular* Boolean monoids, and so we sneak it in and sneak it out just to conclude this proof in the fastest way.

which have been proposed for the algebraic investigation of relevant logics. As Dunn has pointed out in conversation, it often turns out that, beginning with more general kinds of structures, one regularly discovers that one can do all one's work with structures that, among other things, are Boolean algebras. Thus, for example, the most important concrete structures to arise thus far in these investigations — for example, the structure $\boldsymbol{M}_0$ of [11], central in many investigations of Belnap, Anderson, and Dunn, is in fact a Boolean algebra, though with extra structure. But previous investigations, though they treated & and $\vee$ in familiar ways, treated —, $\rightarrow$, and $\circ$ when present in novel ways; the glory and the shame of the present analysis is that it adds negation to the connectives to be analyzed familiarly; for confirmation, note that *truth-functional* tautologies and contradictions go respectively to greatest and least elements of Boolean monoids on *all* interpretations; put otherwise, algebraic complementation here produces the true complement of an element, not the special *intensional* complement introduced by Belnap in [11]. That is the glory of the analysis — it's simple; the shame is that the greater complication of R as opposed to CR makes room for a certain logical and philosophical profundity, desirable for its own sake. Still, it is amazing that, with so little distortion, relevant implication can be made part of a truly classical system; the loss, however, is that contradiction again becomes trivial, trivializing γ as well (*modus ponens* for *material* implication) and robbing relevant implication of many of the interesting uses suggested for it in [1] and in the related papers cited there.

We have worked only on R, but there are of course many relevant logics. Evidently, as the special semantics of R^+ was modified here, so the general semantic of positive relevant logics presented in [3] may be analogously modified to produce from E^+, T^+, B^+, etc., systems CE, CT, CB. And just as R^+ is the common positive fragment of R and of CR, by modifying the proof of theorem 1 in obvious ways we can show that our other classical relevant logics have the same positive fragments as the familiar ones. This may lead, we hope, to new light on the long-open decision question for R, E, T, and related logics, to which in conclusion we turn.

III

4. *Last thoughts on decidability*. We review the status of the decision problem for R and for E. This problem was solved by Kripke for the implicational fragments R_I and E_I of these logics in 1959 by Kripke as reported in [12]. These results are straightforwardly extendible in some directions — e.g., addition of conjunction; the most considerable extension was by Belnap and Wallace in [13], where they gave a Kripke-style decision method for the system E_I^- of entailment with negation, but without & or $\vee$; the analogous system R_I^- is also decidable. In [11], using semantic arguments, Belnap produced a decision procedure for the common first-degree fragment of E and of R. And there, essentially, progress has stopped cold, despite a lot of work and some minor victories of Anderson, Belnap, Routley, Meyer, et. al. in the interim.

Of the systems whose decidability is still in question, the system both most likely to be decidable and most likely to facilitate further progress is R^+, the $\rightarrow$, &, $\vee$, $\circ$

fragment of R.[2] The present analysis, for its part, has produced the best behaved model of $R+$ to date, which is the content of our last theorem.

THEOREM 5. *There is a complete, regular, atomic Boolean monoid, and an interpretation I therein, such that for every formula A of R^+, A is true on I iff A is a theorem of R^+.*

Proof. The proof of theorem 1 provides a c. r. m. s. $\langle O, \mathcal{K}, R\rangle$ and a valuation v therein such that exactly the theorems of R^+ are true at O on v. Consider now the algebra of propositions $\Pi(\mathcal{K})$ associated with $\langle O, \mathcal{K}, R\rangle$. By theorem 3 and corollary 3.1, $\Pi(\mathcal{K})$ is evidently a complete, regular, atomic Boolean monoid. We can complete the proof of theorem 5 by finding an interpretation I of CR therein such that, if A is a formula of R^+, $1 \subseteq I(A)$ in $\Pi(\mathcal{K})$ iff A is a theorem of R^+. In fact, for each sentential variable p, let $I(p) = \{a : v(p, a) = T\}$, where v is the aforementioned valuation in $\langle O, \mathcal{K}, R\rangle$. Defining I on sentential variables of course determines it for all formulas of R^+; we show by induction that, for all a in $\mathcal{K}$ and for each formula A, $a \in I(A)$ iff A is true on v at a. The hardest case is when A is of the form $B \to C$, others being left to the reader. $B \to C$ is T at a on v iff, for all x, y in $\mathcal{K}$, $Raxy$ & B is T at x on v imply C is T at y on v; iff, for all x, y in $\mathcal{K}$, $Raxy$ and $x \in I(B)$ imply $y \in I(C)$, on inductive hypothesis; iff $\{y : \exists x\, (Raxy \ \&\ x \in I(B))\} \subseteq I(C)$; iff $\{a\} \circ I(B) \subseteq I(C)$; iff $a \in \bigcup \{X : X \in P(\mathcal{K}) \ \&\ X \circ I(B) \subseteq I(C)\}$; iff $a \in B \to C$.[3] Since exactly the theorems are true on v at O, for all formulas A of R^+, the inductive argument has established that $O \in I(A)$, iff $\{O\} = 1 \subseteq I(A)$, iff A is a theorem of R^+, ending the proof of theorem 5.

The import of theorem 5 for decidability is, we think, the following. Every non--theorem of R^+ is refutable in a set algebra, with an additional operation $\circ$. (The elements, of course, can be thought of simply as infinite strings of 1's and 0's.) Moreover, for any elements s and r, the value of $s \circ r$ is determined as the join of all the $s_i \circ r_j$, s_i and r_j atoms contained in s and r respectively. So, it would seem plausible to suppose, given any formula A we ought to be able to select out a finite subset of the atoms that enter into the rejection of A, and make do with them and possibly their products, finitizing in an appropriate sense the rejection of A and rendering it effective. At writing, however, we have not discovered the technique that will enable us to complete this positive solution of the decision question for R^+ (and presumably for CR also).[4,5]

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[2] $\circ$ as noted above is optional, but we think its presence more likely to facilitate than to block the attainment of desired results.

[3] Defining $\to$ on $\Pi(\mathcal{K})$ in accordance with the note to theorem 3.

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KLASYCZNE LOGIKI RELEWANTNE

(Streszczenie)

Można by sądzić, że dodanie do relewantnych logik pozytywnych zasady $A \,\&\, \neg A \rightarrow B$ oraz innych klasycznych zasad dotyczących negacji, spowoduje zburzenie tych logik. Ale tak nie jest. Przy analizie semantycznej, pokrewnej do przedstawionej w [1] i [5], ale naturalniejszej w stosunku do negacji, pozytywne intuicje okazują się nietknięte. Celem obecnego artykułu jest przedstawienie takiej właśnie semantyki oraz klasycznych logik relewantnych, które powstają na jej gruncie. Wynikiem naszych analiz są także pewne nowe intuicje algebraiczne, które pozwalają na znaczne uproszczenie niektórych ważnych problemów związanych z pozytywnymi logikami relewantnymi.

Р. К. Мейер, Р. Раутлей

РЕЛЕВАНТНЫЕ КЛАССИЧЕСКИЕ ЛОГИКИ

(Резюме)

Можно бы полагать, что прибавление к позитивным релевантным логикам приц累па $A \& \neg A \rightarrow B$ и других классических принципов, касающихся отрицания приведет к разрушению этих логик. Но это не так. При семантическом анализе, родственном представленному в [1] и [5] но более естественному относительно отрицания, позитивные интуиции остаются без изменения. Целью этой статьи является представление такой именно семантики и классических релевантных логик, которые возникают на ее почве. Результатом наших анализов являются также некоторые новые алгебраические интуиции, которые дают возможность заметно упростить некоторые важные проблемы, связанные с позитивными релевантными логиками.