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A Semantical Analysis of Implicational System I and of the First Degree of Entailment

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A semantics for the implicational system I (of [9]) is described, and the completeness and decidability of I and related systems established. It is shown that I has the same first-degree logic as Anderson-Belnap system E and Ackermann system π and the same positive logic as Lewis system $S3$. A detailed semantical investigation is made of the first degree of entailment systems, and important matrices and algebras are derived and thereby interpreted.

§ 1. System I and Its Semantics

An axiomatisation of I which separates the role of connectives is as follows (dots replace parentheses in accordance with the conventions of Church [4], pp. 74–80):

Implicative axioms:

1. $A \to B \to .\ B \to C \to .\ A \to C.$
2. $(A \to .\ A \to B) \to .\ A \to B.$
3. $A \to A \to B \to B.$
4. $B \to C \to .\ A \to A.$

Positive axioms:

5. $A \,\&\, B \to A.$
6. $A \,\&\, B \to B.$
7. $(A \to B) \,\&\, (A \to C) \to .\ A \to B \,\&\, C.$
8. $A \to A \vee B.$
9. $B \to A \vee B.$
10. $(A \to C) \,\&\, (B \to C) \to .\ A \vee B \to C.$
11. $A \,\&\, (B \vee C) \to (A \,\&\, B) \vee C.$

Negative axioms:

12. $\sim A \to B \to .\ \sim B \to A.$
13. $A \to \sim\sim A.$
14. $A \to \sim A \to .\ \sim A.$

Rules. From A and $A \to B$ to infer B (Modus ponens). From A and B to infer $A \,\&\, B$ (Adjunction).

In the context of the remaining axioms 4 may be replaced by the more appealing (but equally disastrous)

$4'.\ A \to B \to .\ A \to A$

or, equivalently, by the tempting principle of Factor

$4''.\ A \to B \to .\ A \,\&\, C \to B \,\&\, C.$

The strict implication system $S3$ results upon replacing 14 by

$14_I.\ A \to B \to .\ A \to \sim B \to .\ A \to C;$

and the system $S4$ results from I upon replacing (14 by 14_I and) 4 by

$4_I.\ B \to .\ A \to A.$

Alternatively (see [9]) $S3$ is obtained upon replacing 14 by Disjunction Syllogism or by Antilogism.

An *F-model* is a structure $\mathscr{F} = \langle G, K, N, R, *, v\rangle$, where K is a set, N is a subset of K, $G \in N$, R is a binary relation on K, $*$ is a one-one function from K onto K such that $G^* = G$, and v is a valuation function from atomic wff and elements of K to truth-values T and F. An *I-model* is an F-model such that

(i) R is transitive from N and reflexive on N.
(ii) If $H \in N$ then $H = H^*$, for $H, H^* \in K$.
(iii) $H^{**} = H$, for every $H \in K$.
(iv) If $H_1 R H_2$ then $H_1 R H_2^*$, for $H_1, H_2 \in K$.

The valuation function defined for atomic wff and elements of K is extended so that its first domain is the set of all wff, by the following inductive specification:

$$v(A \,\&\, B, H) = T \quad \text{iff} \quad v(A, H) = T \quad \text{and} \quad v(B, H) = T,$$

$$v(A \vee B, H) = T \quad \text{iff} \quad v(A, H) = T \quad \text{or} \quad v(B, H) = T,$$

$$v(\sim A, H) = T \quad \text{iff} \quad v(A, H^*) = F,$$

$$v(A \to B, H) = T \quad \text{iff} \quad H \in N \quad \text{and, for every } H^1, \text{ if } \quad HRH^1 \quad \text{and} \quad v(A, H^1) = T \quad \text{then, materially,} \quad v(B, H^1) = T.$$

Here, as henceforth, H, $H^1, \ldots, H_1, H_2, \ldots$ are understood as variables restricted to K. Where $\vee$ is defined: $A \vee B =_{Df} \sim(\sim A \,\&\, \sim B)$, the evaluation rule for $\vee$ follows.

A wff B is *true in* an I-model $\mathscr{M} = \langle G, K, N, R, *, v\rangle$ iff $v(B, G) = T$, *false in* $\mathscr{M}$ iff $v(B, G) = F$. B is *I-valid* iff B is true in every I-model. I-model $\mathscr{M}$ *satisfies* A (set Γ) iff A (every A in Γ) is true in $\mathscr{M}$. $\mathscr{M}$ is an *I-countermodel* to B iff B is false in $\mathscr{M}$.

In fact I-models can be replaced by simpler structures $\langle K, R, *\rangle$, and the base element G and the set N defined respectively, $G = \xi H(H = H^*)$ and $N = \{H : H = H^*\}$. Here K is a set of atomic situations, where an atomic situation is simply a set of atomic wff or propositions. For each atomic situation a characteristic function v_H indicates whether or not a given atomic wff (proposition, fact) p belongs to H; then $v_H(p) = T$ or $v_H(p) = F$ according as p belongs or does not belong to H. Defining $v(p, H) = v_H(p)$ the set v_H, for $H \in K$, of characteristic functions can be supplanted by a single two-place "valuation" function v.

A situation H (of propositions, facts) is the class of wff (propositions, facts) that are in H. Being in or holding in a situation is defined by recursion:

Where A is atomic A is in situation H (A holds in H) iff A belongs to atomic situation H. Otherwise

$$\begin{array}{rll} A \& B \text{ is in } H & \text{iff} & A \text{ is in } H \text{ and } B \text{ is in } H\,, \\ A \vee B \text{ is in } H & \text{iff} & A \text{ is in } H \text{ or } B \text{ is in } H\,, \\ \sim A \text{ is in } H & \text{iff} & A \text{ is not in } H^*\,, \\ A \to B \text{ is in } H & \text{iff} & H = H^* \text{ and whenever } A \text{ is in a set-up} \\ & & H^1 \text{ such that } HRH^1 \text{ then } B \text{ is in } H^1\,. \end{array}$$

The recursion may be alternatively stated by simply extending the function v. Where v^1 is the extension of v then, $v^1(A, H) = T$ iff A holds in H.

Given set-up H, H^* is the class of wff (propositions, etc.) B such that $\sim B$ is not in H: the specification characterises * completely, given that negation is already understood. The merit of * is that if facilitates the construction of inconsistent and incomplete set-ups. Thereby the distinctness of H and H^* provides for set-ups in which tautologies do not hold and in which the negations of tautologies do hold, and thus it enables the suppression of tautologies in inference and entailment to be revealed (see [10]: V. Routley suggested using inconsistent and incomplete situations in the semantics of entailment, in order to avoid suppression). Because too H^* may differ from H, Carnap's requirements of logical independence and of completeness ([3], p. 70ff.) can be abandoned without the disadvantages of meaning postulates. If H is both consistent, i.e. contains at most one of A and $\sim A$ for every sentence A, and complete, i.e. contains at least one of A and $\sim A$ for every sentence A, then H^* coincides with H. If H is inconsistent with respect to wff B then H^* is incomplete with respect to B, and conversely. Since incompleteness theorems provide situations which are either inconsistent or incomplete, cases where H and H^* differ have to be admitted. For let H be the set of wff provable in Peano arithmetic, then by the Gödel-Rosser theorem H

is either inconsistent or incomplete. Situation G, canonically the set of all and only truths, is neither inconsistent nor incomplete; hence $G = G^*$.

Theorem 1. *If* $\vdash_I B$ *then* B *is* I*-valid.*

Proof is by induction over the length of the proof of B in I.

§ 2. Completeness

Lemma 1. B *is* I*-valid iff* $\sim B$ *is not satisfied in any* I*-model.*

More generally, B is F-valid iff $\sim B$ is not satisfied in any F-model, where F-validity is defined analogously to I-validity.

Where L is a logic, Σ is a set of wff of L and A is a wff of L, $\Sigma \vdash_L A$ iff A is a theorem of the system $L \cup \Sigma$, i.e. the system obtained by adding Σ as axiom to L and closed under the rules of L (in the main case in point where L is system I, under the rules of modus ponens and adjunction). Thus $\Sigma \vdash_I A$, i.e. A is *I-provable* from Σ iff there is a sequence $B_1, \ldots, B_n$ terminating in A each of whose members is a theorem of I, a member of Σ, or the consequence of predecessors by modus ponens or adjunction. Σ is *I-consistent* iff for no wff B of I is both B and $\sim B$ I-provable from Σ. L-consistency is defined analogously. Σ is *$I - A$(bsolutely)-consistent* iff some wff of I is not I-provable from Σ.

A set Σ of wff of I is *I-maximal (I-inflated)* iff

(i) Every theorem of I is in Σ,

(ii) Σ is I-consistent (Σ is $I - A$-consistent),

(iii) Σ is closed under adjunction, i.e. if $A \in \Sigma$ and $B \in \Sigma$ then $A \,\&\, B \in \Sigma$,

(iv) Σ is closed under modus ponens, i.e. if $A \in \Sigma$ and $A \to B \in \Sigma$ then $B \in \Sigma$,

(v) Σ is *prime*, i.e. if $A \vee B \in \Sigma$ then $A \in \Sigma$ or $B \in \Sigma$.

L-maximal and *L-inflated* are defined analogously.

Lemma 2. *If* Σ *is* I*-maximal then*

(a) $A \,\&\, B \in \Sigma$ *iff* $A \in \Sigma$ *and* $B \in \Sigma$

(b) $A \vee B \in \Sigma$ *iff* $A \in \Sigma$ *or* $B \in \Sigma$

(c) *Exactly one of* B *and* $\sim B$, *for each wff* B, *is in* Σ.

(d) *For no* B *is* $B \leftrightarrow \sim B$ *in* Σ, *where* $A \leftrightarrow B =_{Df} (A \to B) \,\&\, (B \to A)$.

If Σ is I-inflated then both (a) and (b) hold.

Similar results hold of course for L-maximal sets of suitable logical systems weaker than I.

Lemma 3. (i) *If* $B \in \Sigma$ *then* $\Sigma \vdash_I A$ *iff* $\Sigma, B \vdash_I A$.

(ii) *If* $\vdash_I B$ *and* $B \vdash_I C$ *then* $\vdash_I C$.

(iii) *If* $\Sigma \subseteqq \Delta$ *and* $\Sigma \vdash_I A$ *then* $\Delta \vdash_I A$.

(iv) *If* $\Sigma, B \vdash_I C$ *and* $\Sigma, C \vdash_I D$ *then* $\Sigma, B \vdash_I D$.

Lemma 4. (i) *If* $\Gamma, A \vdash_I C$ *then* $\Gamma, A \vee B \vdash_I C \vee B$.

(ii) *If* $\Gamma, A \vdash_I C$ *then* $\Gamma, B \vee A \vdash_I B \vee C$.

Proof of (i) is as in Meyer-Dunn [8]. On the hypothesis Γ, $A \vdash_I C$ there is a sequence $A_1, \ldots, A_{m-1}$, $A_m = C$ each of whose members is either a theorem of I, a member of Γ, A itself or a consequence of predecessors by modus ponens or adjunction. Suppose on induction hypothesis

$$\Gamma,\ A \vee B \vdash_I A_i \vee B, \quad \text{for} \quad i<j. \tag{1}$$

It is shown

$$\Gamma,\ A \vee B \vdash_I A_j \vee B, \tag{2}$$

from which (i) follows. There are five cases:

Case 1. $\vdash_I A_j$. Then $\vdash_I A_j \vee B$ by addition, whence (2) follows by a previous lemma.

Case 2. $A_j = A$. Then (2) is immediate.

Case 3. $A_j \in \Gamma$. Then Γ, $A \vee B \vdash_I A_j$, whence (2) follows.

Case 4. A_j follows by modus ponens from A_k and $A_k \rightarrow A_j$, with $k<j$. By induction hypothesis (1), Γ, $A \vee B \vdash_I A_k \vee B$ and Γ, $A \vee B \vdash_I (A_k \rightarrow A_j) \vee B$. By adjunction, $\Gamma, A \vee B \vdash_I (A_k \vee B) \mathbin{\&}. (A_k \rightarrow A_j) \vee B$. Then Γ, $A \vee B \vdash_I A_k \mathbin{\&} (A_k \rightarrow A_j) \vee B$, whence (2) follows using theorems of I.

Case 5. $A_j (= A_h \mathbin{\&} A_k)$ follows by adjunction from A_h and A_k, with $h, k<j$. By (1) and adjunction Γ, $A \vee B \vdash_I (A_h \vee B) \mathbin{\&} (A_k \vee B)$, whence Γ, $A \vee B \vdash_I A_h \mathbin{\&} A_k \vee B$. Proof of (ii) is similar.

Lemma 5. (i) *Every I-consistent class* Γ *has an I-maximal extension* Γ^+, *such that* $\Gamma^+ \supseteqq \Gamma$.

(ii) *If* A *is not I-provable from* Σ *there is an I-inflated extension* Σ^+ *of* Σ *such that* $A \notin \Sigma$.

Proof of (i). Using some standard method fix an enumeration $B_0, B_1, \ldots, B_j, \ldots$ of wff of I, such that each wff appears exactly once and both C and D precede $(C \mathbin{\&} D)$, $(C \rightarrow D)$ and $(C \vee D)$. Define:

$$\Gamma_0 = \Gamma.$$

$$\Gamma_{j+1} = \begin{cases} \Gamma_j \cup \{B_j\} & \text{if} \quad \Gamma_j \cup \{B_j\} \text{ is } I\text{-consistent} \\ \Gamma_j \ \text{otherwise}. \end{cases}$$

$$\Gamma^+ = \bigcup_{j<\omega} \Gamma_j.$$

Then (a) $\Gamma \subseteqq \Gamma^+$, and Γ^+ is I-consistent. For Γ_0 is I-consistent and, for each $j \geqq 0$, Γ_j is I-consistent implies Γ_{j+1} is I-consistent. For if $\Gamma_{j+1} = \Gamma_j \cup \{B_j\}$ is not I-consistent then $\Gamma_{j+1} = \Gamma_j$. Thus every Γ_j, for $j<\omega$, is I-consistent, so their union Γ^+ is I-consistent. For any proof of I-consistency would be a proof from Γ_k for some k.

(b) Every theorem of I is in Γ^+. Let C be a typical theorem of I; then $C = B_k$ for some unique k. Suppose $\Gamma_k \cup \{C\}$ is not I-consistent. Then for some wff D, $\Gamma_k \cup \{C\} \vdash_I D \& \sim D$. But then by the definition of I-provability $\Gamma_k \vdash_I D \& \sim D$, i.e. Γ_k is I-consistent, contradicting (a). Accordingly $\Gamma_k \cup \{C\}$ is I-consistent, so $C \in \Gamma^+$.

(c) Γ^+ is closed under adjunction. Suppose, on the contrary, $C \in \Gamma^+$, $D \in \Gamma^+$ but $C \& D \notin \Gamma^+$. Then in view of the enumeration of wff, for some k, $C \in \Gamma_k$, $D \in \Gamma_k$ but $\Gamma_k \cup \{C \& D\}$ is not I-consistent. Thus for some wff E,

$$\Gamma_k,\ C \& D \vdash_I E \& \sim E\,.$$

Since however $C \& D$ is I-provable when C, D both are $\Gamma_k, C, D \vdash_I E \& \sim E$, whence since $C, D \in \Gamma_k$ $\Gamma_k \vdash_I E \& \sim E$, contradicting (a).

(d) Γ^+ is closed under modus ponens. Suppose, again on the contrary, $B \in \Gamma^+$, $B \rightarrow C \in \Gamma^+$ but $C \notin \Gamma^+$. Then for some k and some E, Γ_k, $C \vdash_I E \& \sim E$. In view of the arrangement of the enumeration, for some $p > k$ $B \in \Gamma_p$, $B \rightarrow C \in \Gamma_p$. Since then Γ_p, $B, B \rightarrow C \vdash_I C$, by a lemma $\Gamma_p \vdash_I C$. Also since $\Gamma_k \subseteqq \Gamma_p$, Γ_p, $C \vdash_I E \& \sim E$, whence $\Gamma_p \vdash_I E \& \sim E$ contradicting (a).

(e) Γ^+ is prime. Suppose, on the contrary, $B \vee C \in \Gamma^+$, $B \notin \Gamma^+$ and $C \notin \Gamma^+$. Since every wff occurs exactly once in the enumeration there are unique indices j, k, l such that $B \vee C = B_j$, $B = B_k$ and $C = B_l$. Since B and C precede $(B \vee C)$ in the enumeration $j > m = \max(k, l)$, and $\Gamma_j \supseteqq \Gamma_k$ and $\Gamma_j \supseteqq \Gamma_l$. On the suppositions made, then, $\Gamma_j \cup \{B \vee C\}$ is I-consistent, but $\Gamma_j \cup \{B\}$ and $\Gamma_j \cup \{C\}$ are not. Thus for some D and E, Γ_j, $B \vdash_I D \& \sim D$ and Γ_j, $C \vdash_I E \& \sim E$. Applying lemmas it follows in turn, Γ_j, $B \vee C \vdash_I D \& \sim D \vee C$, Γ_j, $D \& \sim D \vee C \vdash_I D \& \sim D \vee E \& \sim E$, Γ_j, $B \vee C \vdash_I D \& \sim D \vee E \& \sim E$. But since $\vdash_I \sim (D \& \sim D \vee E \& \sim E)$, Γ_j, $B \vee C \vdash_I \sim (D \& \sim D \vee E \& \sim E)$, contradicting the I-consistency of $\Gamma_j \cup \{B \vee C\}$.

Proof of (ii) uses the following definition:

$$\Sigma_0 = \Sigma$$

$$\Sigma_{j+1} = \begin{cases} \Sigma_j & \text{if} \quad \Sigma_j \cup \{B_j\} \vdash A \\ \Sigma_j \cup \{B_j\} & \text{otherwise}. \end{cases}$$

$$\Sigma^+ = \bigcup_{j<\omega} \Sigma_j\,.$$

Then (a) Σ^+ is $I - A$-consistent since A is not I-provable from Σ_j, for any j. Proofs of (b) – (e) are the same as, or simplifications of, those given in proof of (1).

Theorem 2. *If Γ is an I-consistent class of wff then there is an I-model $\mathcal{M} = \langle G, K, H, R, *, v\rangle$ such that $\mathcal{M}$ satisfies Γ.*

Proof. Let N be the class of I-maximal sets. For $H \in N$ define: $B \leqq_H C =_{Df} B \to C \in H$; $B =_H C =_{Df} B \to C \in H$.

Lemma 6. *For each $H \in N$,*

$$\langle H, \leqq_H, \sim\rangle, \text{ where } \sim, \&, \vee \text{ provide, respectively,}$$

complementation, intersection and union operations, is a de Morgan lattice.

Proof. (i) $=_H$ is an identity relation; for it is an equivalence relation and full substitutivity of identity holds.

(ii) $\leqq_H$ is a partial order on H. For $A \leqq_H A$ since all theorems of I belong to H. Likewise since H is closed under modus ponens and $\to$ is transitive, $B \leqq_H C$ and $C \leqq_H A$ implies $B \leqq_H A$. Finally $A \leqq_H B$ and $B \leqq_H A$ imply $A \to B \in H$ and $B \to A \in H$, whence by adjunction $A \leftrightarrow B \in H$ and $A =_H B$.

(iii) Remaining distributive lattice postulates are immediate from substitution instances of first degree theorems of I. For example, since $A \,\&\, (B \vee C) \to . (A \,\&\, B) \vee (A \,\&\, C) \in H$, $A \,\&\, (B \vee C) \leqq_H A \,\&\, B \vee A \,\&\, C$, where $\&$ and $\vee$ are now lattice operations.

(iv) The lattice is de Morgan, since $B \leftrightarrow \sim\sim B \in H$ implies $B =_H \sim\sim B$, and $B \leqq_H C$ implies $B \to C \in H$ which implies $\sim C \to \sim B \in H$, i.e. $\sim C \leqq_H \sim B$.

Now define a canonical I-model $\mathcal{M} = \langle G, K, N, R, *, v\rangle$ as follows: $G = \Gamma^+$, where Γ^+ is the I-maximal extension of Γ guaranteed by Lemma 5. N is the class of I-maximal sets. K is the class of prime filters of the lattices defined, as in Lemma 6, by elements of N. $H_1 R H_2$ iff, for every wff B and C, if $B \to C \in H_1$ and $B \in H_2$ then $C \in H_2$. $H^* = \{B : \sim B \notin H\}$. $v(p, H) = T$ iff $p \in H$, for each atomic wff p. Then

(α) $H^* \in K$, for every $H \in K$. For

Lemma 7. (i) *If $H \in N$ then $H = H^*$.*

(ii) *If H is a prime filter then H^* is a prime filter.*

Proof of (i) follows from Lemma 2. When $H \in N$, for no B, $B \in H$ and $\sim B \in H$. Since every theorem of I is in H $A \vee \sim A \in H$. Thus since H is prime $A \in H \vee \sim A \in H$, for every A. Combining these results, for every A, $\sim A \notin H$ iff $A \in H$, whence $H^* = H$.

As for (ii), H^* is a filter since $B \in H^*$ and $C \in H^*$ iff $(B \,\&\, C) \in H^*$ follows using the definition of H^* and the fact that H is prime; and H^* is prime since $B \in H^*$ or $C \in H^*$ iff $(B \vee C) \in H^*$ using the fact that H is a filter.

(β) The canonical I-model $\mathcal{M}$ defined is an I-model. For:

(i) $G \in N$, since Γ^+ is an I-maximal set.

(ii) R is reflexive, since H is a filter.

(iii) R is transitive from N. Suppose $H_1 R H_2$ & $H_2 R H_3$ & $H_1 \in N$ & $H_2 \in N$. Then if $B \to C \in H_1$ and $C \to C \in H_2$, $B \to C \in H_2$, using Exported Syllogism. Since $H_2 \in N$, $C \to C \in H_2$. Thus, using $H_2 R H_3$, if $B \to C \in H_1$ and $B \in H_2$ then $C \in H_2$ for every B and C, i.e. $H_1 R H_3$.

(iv) If $H \in N$ then $H = H^*$, by (α).

(v) $H^{**} = H$.

Since every $H^* \in K$ is a filter of an I-maximal set and $\sim\sim B \leftrightarrow B$ belongs to every I-maximal set, $\sim\sim B \in H$ iff $B \in H$.

(vi) If $H_1 R H_2$ then $H_1 R H_2^*$. For if, for any B, C $B \to C \in H_1$ then $\sim C \to \sim B \in H_1$, since H_1 is a filter of an element of N. Then if $H_1 R H_2$, $\sim C \in H_2$ implies $\sim B \in H_2$, whence $\sim B \notin H_2$ implies $\sim C \notin H_2$. Thus, for every B, C, $B \to C \in H_1$ and $B \in H_2^*$ imply $C \in H_2^*$, given $H_1 R H_2$; i.e. $H_1 R H_2^*$.

(γ) $v(A, H) = T \equiv A \in H$, for every wff A and every $H \in K$.

Proof is by induction from the given basis. Cases are as follows:

$$\begin{aligned}
\underset{=}{\sim} \quad v(A,H) = T &\equiv v(A, H^*) = F \\
&\equiv A \notin H^*, \text{ by induction hypothesis and } (\alpha). \\
&\equiv \sim A \in H
\end{aligned}$$

$$\begin{aligned}
\underset{=}{\&} \quad v(A \,\&\, B, H) = T &\equiv v(A,H) = T = v(B,H) \\
&\equiv A \in H \,\&\, B \in H, \text{ by hypothesis} \\
&\equiv A \,\&\, B \in H, \text{ since } H \text{ is a filter}
\end{aligned}$$

$$\begin{aligned}
\underset{=}{\vee} \quad v(A \vee B, H) = T &\equiv v(A,H) = T \vee v(B,H) = T \\
&\equiv A \in H \vee B \in H \\
&\equiv A \vee B \in H, \text{ since } H \text{ is prime.}
\end{aligned}$$

$\underset{=}{\to}$ (i) if $B \to C \in H$ then $(B \to C, H) = T$.

For, first, $B \to C \in H$ implies that for every H^1 if $H R H^1$ and $B \in H^1$ then $C \in H^1$, by quantification logic using the definition of R. Second, if H contains one implication $(B \to C)$, then, since H is a filter under $\leqq_{H^1}$ for some H^1,

(a) H contains all theorems of I. For if A is a theorem of I of the form $A_1 \to A_2$ (a form which includes all axioms) then in virtue of $\vdash_I B \to C \to . A_1 \to A_2$, $A \in H$. Further the rules of inference of I are mirrored by conditions on filters. Since also

(b) H is a prime filter of a normal lattice, H satisfies the remaining conditions for being an I-maximal set.

Hence, by (a) and (b), $H \in N$.

(ii) if $v(B \to C, H) = T$ then $B \to C \in H$.

Suppose that $B \to C \notin H$ and $H \in N$ – whence $B \nleqq_H C$ – but that $H R H^1$ and $B \in H^1$ formally imply $C \in H^1$. Applying Stone's 1937 theorem, that in a distributive lattice, if $a \nleqq b$ then there is a prime filter P such that $a \in P$ and $b \notin P$, it follows from $B \nleqq_H C$ that there is a prime filter H' of H such that $B \in H'$ and $C \notin H'$. Since H' is a filter of H, $B' \leqq_H C'$ and $B' \in H'$ imply $C' \in H'$, i.e. HRH'. Hence $B \in H'$ implies $C \in H'$. Since $B \in H'$, $C \in H'$ contradicting $C \notin H'$.

(δ) $\mathcal{M}$ satisfies Γ.

For any wff B, if $B \in \Gamma$, then $B \in G$. So by (γ), $v(B, G) = T$.

Theorem 3. *If A is I-valid then* $\vdash_I A$.

Proof. If A is I-valid then, by Lemma 1, $\sim A$ is not satisfied in any I-model. Hence by Theorem 2, $\sim A$ is not I-consistent. Thus for some D, $\sim A \vdash_I D \& \sim D$. By Lemma 4, $A \vee \sim A \vdash_I A \vee D \& \sim D$; hence $\vdash_I A \vee D \& \sim D$. Since $\vdash_I \sim(D \& \sim D)$, by the Meyer-Dunn theorem (of [8]; Factor satisfies the requisite conditions), $\vdash_I A$.

§ 3. Decidability

Where Ψ is a set of wff closed under subformulae define the following relation on K:

$H_1 \equiv_\Psi H_2$ iff, for every $B \in \Psi$, $v(B, H_1) = v(B, H_2)$. Then $\equiv_\Psi$ is an equivalence relation, and equivalence classes can be defined: $(\hat{H})_\Psi =_{Df} \{H^1 : H^1 \equiv_\Psi H\}$. Using these equivalence classes, relative to a given class Ψ of wff and a given I-model $\mathcal{M}$, a new I-model $\hat{\mathcal{M}}$ is defined. Following Segerberg [11], define a *filtration* $\hat{\mathcal{M}} = \langle \hat{G}, \hat{K}, \hat{N}, \hat{R}, \hat{*}, \hat{v} \rangle$ of $\mathcal{M}$ through Ψ, written $\hat{\mathcal{M}} = \mathcal{M}/\Psi$, as follows:

$$\hat{K} = \{\hat{H} : H \in K\}\,; \qquad \hat{N} = \{\hat{H} : H \in N\}\,;$$

$$(\hat{H}^{\hat{*}})_\Psi = \{H^1 : H^1 \equiv_\Psi H^*\}\,; \qquad \hat{v}(p, \hat{H}) = T \equiv .\ v(p, H) = T \,\&\, p \in \Psi\,;$$

$\hat{H}_1 \hat{R} \hat{H}_2$ iff, for every $B, C \in \Psi$, if $v(B \to C, H_1) = T$ and $v(B, H_2) = T$ then $v(C, H_2) = T$.

Lemma 8. (i) *$\hat{G} \in \hat{N}$ iff $G \in N$; $\hat{H} \in \hat{N}$ iff $H \in N$.*
(ii) $\hat{H}^{\hat{*}} = \widehat{H^*}$.
(iii) *$H_1 R H_2$ implies $\hat{H}_1 \hat{R} \hat{H}_2$.*
(iv) *$\hat{R}$ is reflexive & transitive from $\hat{N}$.*
(v) *$\hat{H} \in \hat{N}$ implies $\hat{H} = \hat{H}^{\hat{*}}$.*
(vi) *$(\hat{H}^{\hat{*}})^{\hat{*}} = \hat{H}$.*
(vii) *$\hat{H}_1 \hat{R} \hat{H}_2 \supset \hat{H}_1 \hat{R} \hat{H}_2^{\hat{*}}$.*
Hence $\hat{\mathcal{M}}$ is an I-model.

Proofs. (i) and (ii) are immediate from definitions. (iii) follows by quantification logic and definition of $\hat{R}$ and $\hat{v}$. That $\hat{R}$ is reflexive follows from (iii) since R is reflexive. To show transitivity from $\hat{N}$ it has to be shown: $\hat{H}_1 \hat{R} \hat{H}_2$ and $\hat{H}_2 \hat{R} \hat{H}_3$ and $\hat{H}_1 \in \hat{N}$ and $\hat{H}_2 \in \hat{N}$ jointly imply $\hat{H}_1 \hat{R} \hat{H}_3$. Assuming the antecedent, since $H_1 \in N$ and $H_2 \in N$ by (i), $v(B \to C, H_1) = T$ implies $v(C \to C \to .\ B \to C, H_1) = T$, which implies $v(B \to C, H_2) = T$ (since $v(C \to C, H_2) = T$), which implies that if $v(B, H_3) = T$ then $v(C, H_3) = T$. The result follows by quantification logic and the definition of $\hat{R}$. As to (v), $\hat{H} \in \hat{N}$ implies $H \in N$, which implies $H = H^*$; thus $\hat{H} = \hat{H}^*$. (vi) follows from (ii) and $H^{**} = H$. If $\hat{H}_1 \notin \hat{N}$ then $H_1 \notin N$ and (vii) is vacuously true because $\hat{H}_1 \hat{R} \hat{H}$ for any $\hat{H}$. If instead, $\hat{H}_1 \in \hat{N}$ then $H_1 \in N$; so $v(B \to C, H_1) = T$ implies $v(\sim C \to \sim B, H_1) = T$, which implies that if $v(\sim C, H_2) = T$ then $v(\sim B, H_2) = T$, using $\hat{H}_1 \hat{R} \hat{H}_2$. Thus if $v(B, H_2^*) = T$ then $v(C, H_2^*) = T$. Hence by quantification logic, $\hat{H}_1 \hat{R} \widehat{H_2^*}$, whence $\hat{H}_1 \hat{R} \hat{H}_2^*$.

Lemma 9. *For every $A \in \Psi$, $\hat{v}(A, \hat{H}) = T$ iff $v(A, H) = T$.*

Proof is by induction over the number of connectives in A, using as basis the specification of $\hat{v}(p, \hat{H})$. The induction step for & is immediate, that for $\sim$ follows using (ii) of the previous lemma. That $\hat{v}(B \to C, \hat{H}) = T$ implies $v(B \to C, H) = T$ follows from the induction hypothesis using (i) and (iii) above. For the converse suppose $\hat{v}(B \to C, H) \neq T$. If $\hat{H} \notin \hat{N}$ then by (i) $v(B \to C, H) \neq T$ as required. Otherwise for some $\hat{H}_1$, $\hat{H} \hat{R} \hat{H}_1$ and $\hat{v}(B, \hat{H}_1) = T$ and $\hat{v}(C, \hat{H}_1) = F$. By induction hypothesis $v(B, H_1) = T$ and $v(C, H_1) = F$, whence using $\hat{H} \hat{R} \hat{H}_1$ $v(B \to C, H) \neq T$.

Theorem 4. (i) *If a wff A is false in I-model $\mathcal{M}$ then, where Θ is the subformulae closure of A, A is false in I-model $\mathcal{M}/\Theta$.*
(ii) *I has the finite model property, and accordingly is decidable.*

Proof. By previous lemmata $\mathcal{M}/\Theta$ is an I-model and $\hat{v}(A, \hat{G}) = F$. Further K is finite since Θ is finite.

The completeness and decidability of I may be alternatively proved (constructively) by methods which adapt those used by Kripke (in [7c]) to prove the completeness of intuitionistic sentential logic: and in this case the Meyer-Dunn theorem for I follows as a corollary.

§ 4. Relations to other Systems

I is first of all related to $S3$. An *$S3$-model* is an I-model such that $H^* = H$; clearly conditions on * for I-models are rendered otiose. *$S3$-validity* etc. are defined analogously to I-validity etc. By specialisations of the proofs given for I, Theorems 1–4 for $S3$ follow, i.e.

Theorem 5. (i) $\vdash_{S3} A$ *iff A is $S3$-valid.*
(ii) *$S3$ has the finite model property and is decidable.*

Corollary. *I is a subsystem of $S3$.*

Consider now the positive subsystem, I^+, of I, which has Axioms 1–11 and whose wff do not contain the negation symbol, i.e. whose wff are I^+-*wff*. An I^+-*model* is an I-model; I^+-*validity* etc. are defined just as for I-validity etc., except that the inductive clause specifying $v(\sim A, H)$ is of course eliminated.

Theorem 6. (i) $\vdash_{I^+} A$ *iff* A *is* I^+*-valid.*

(ii) I^+ *has the finite model property and is decidable.*

Proof of completeness. If A is not a theorem of I^+, A is not I^+-provable from the null set Δ. But then, by Lemma 5, there is an I^+-inflated extension G of Δ such that $A \notin G$. Now define a canonical I^+-model $\mathscr{M} = \langle G, K, N, R, v\rangle$ with base G just as for I in Theorem 2 except that N is now the class of I^+-inflated sets and that * and its associated conditions are omitted. Then just as in the proof of Theorem 2, for every wff A of I^+ and every $H \in K$, $v(A, H) = T$ iff $A \in H$. Thus since $\mathscr{M}$ is an I^+-model, $\mathscr{M}$ falsifies A.

A similar proof may be produced for Theorem 5 since $S3$-maximal sets coincide with $S3$-inflated sets.

Theorem 6 facilitates proof of the following separation theorem which shows that the positive parts of I and $S3$ coincide.

Theorem 7. *If A is provable in $S3$ or in I, then $\vdash_{I^+} A$ iff A is an I^+-wff.*

Proof. Since if A is provable in $S3$, A is provable in I, suppose A is provable in I. Then $v(A, G) = T$ in all I-models. Hence $v(A, G) = T$ in all I^+-models iff A is an I^+ wff. So by the previous theorem $\vdash_{I^+} A$ iff A is an I^+-wff.

By varying the modelling conditions semantics for I-like systems with the same positive logic as other strict implicational systems can be obtained. For example, by relinquishing the requirement that R be transitive a semantics can be obtained for a system $IS2$ of $S2$ strength. $IS2$ may be axiomatised by amending the axiomatisation of I as follows:

1^1. $(A \to B) \& (B \to C) \to .\ A \to C$,

and the rule from $A \to B$ to infer $B \to C \to .\ A \to C$, replace 1. The strict implicational system $S2$ results upon replacing 14 by 14_{II}.

$$A \& (\sim A \vee B) \to B \quad \text{(Disjunctive Syllogism)}$$

or by Antilogism. The positive part $S2^+$ of $S2$ results when negative Axioms 12 and 13 and 14_{II} are dropped.

Theorem 8. (i) $\vdash_{IS2(S2, S2^+)} A$ *iff* A *is* $IS2$-($S2$-, $S2^+$-)*valid.*

(ii) $IS2$, $S2$ *and* $S2^+$ *are decidable.*

(iii) *Where* $\vdash_{IS2} A$ *or* $\vdash_{S2} A$,
$\vdash_{S2^+} A$ *iff* A *is an* $S2^+$*-wff.*

Terminology and proofs are like that in the case of I and $S3$.

System $IS4$ does not differ from $S4$. $IS4$ is obtained from I by replacing axiom scheme 4 of I by the scheme

4_I. $B \rightarrow . \ A \rightarrow A$.

An $IS4$-model is an I-model which satisfies also the closure condition:
(c) $H_1 R H_2$ and $H_1 \in N$ imply $H_2 \in N$.
$IS4$- and $S4$-validity are defined in the expected way.

Theorem 9. (i) $\vdash_{IS4(S4)} A$ *iff* A *is* $IS4$-($S4$) *valid.*
(ii) $\vdash_{IS4} A$ *iff* $\vdash_{S4} A$.

Proofs. As to (i) it is straightforwardly verified that 4_I is $IS4$-valid (and $S4$-valid, where $H^* = H$). Conversely, assuming $H_1 R H_2$, then $(B \rightarrow . \ A \rightarrow A) \in H_1$ and $B \in H_1$ imply $A \rightarrow A \in H_2$ in the canonical model. Hence if $H_1 \in N$, $A \rightarrow A \in H_2$; thus $H_1 \in N$ implies $H_2 \in N$.

Ad (ii). If $\vdash_{IS4} A$ then $\vdash_{S4} A$, since every postulate of $IS4$ is a theorem of $S4$. Conversely, if $\sim \vdash_{IS4} A$ then A is not $IS4$-valid by (i). Hence A has a connected $IS4$-countermodel (extending Kripke [6], p. 71); i.e. $v(A, G) = F$ in $IS4$-model $\mathcal{M} = \langle G, K, N, R, *, v \rangle$ where for every $H \in K$, $G R H$ (for as R is transitive, then $G R^* H$, so $G R H$). Since, however, $G \in N$, $H \in N$ for every $H \in K$; thus $H = H^*$ always, that is the model is an $S4$-model. Thus A is not $S4$ valid, so $\sim \vdash_{S4} A$.

The following properties of interest follow from the semantics:

Theorem 10. (1) *Where* A *and* B *are implicational wff of* I *or of* $IS2$, $\vdash A \vee B$ *iff* $\vdash A$ *or* $\vdash B$.

(2) *Where* A *and* B *are negation-free wff of* I *or* $S3$, *or of* $IS2$ *or* $S2$, $\vdash A \vee B$ *iff* $\vdash A$ *or* $\vdash B$.

Proof of (1) is a simple extension of Kripke [7], p. 94–95, and proof of non-immediate half (2) starts off in the same way. Suppose A and B are negation-free wff of I or $S3$ (or of $S2^+$) such that neither A nor B is provable. Let $\mathcal{M}_1 = \langle G_1, K_1, R_1, N_1, v_1 \rangle$ and $\mathcal{M}_2 = \langle G_2, K_2, R_2, N_2, v_2 \rangle$ be I^+-countermodels to A and B respectively, where K_1 and K_2 are disjoint. Define a model $\mathcal{M} = \langle G, K, R, N, v \rangle$ where G is a new situation not in K_1 or K_2, thus: $K = K_1 \cup K_2 \cup \{G\}$; $H_1 R H_2$ iff either $H_1, H_2 \in K_1$ and $H_1 R H_2$, or $H_1, H_2 \in K_2$ and $H_1 R_2 H_2$, or $H_1 = G$. Then R is reflexive and is transitive iff R_1 and R_2 are. Further $H \in N$ iff $H \in N_1$ or $H \in N_2$: $v(p, H) = v_1(p, H)$ for $H \in K_1$, $v(p, H) = v_2(p, H)$ for $H \in K_2$, and $v(p, G) = F$ iff, for some $H \in K_1 \cup K_2$, $v(p, H) = F$. Then by induction on connectives, (*) $v(C, G) = T$ implies $v(C, H) = T$, for $H \in N$. Hence, since $G_1 \in N$ and $G_2 \in N$ and $v(A, G_1) = F$ and $v(B, G_2) = F$, $v(A, G) = F = v(B, G)$. Thus $v(A \vee B, G) = F$, and $\mathcal{M}$ provides an I^+ countermodel to $A \vee B$.

Corollary. *I^+ and $S2^+$ are reasonable in the sense of Halldén.* In view of this corollary the serious semantical incompleteness of $S3$ and $S2$, which Halldén pointed out in [6], can be ascribed to the behaviour of negation in these systems. I does not suffer from this incompleteness (see Theorem 20).

§ 5. The First Degree of Entailment and Deducibility

Entailment is a propositional relation distinct from strict implication. (This thesis has been defended by Anderson and Belnap in a valuable series of papers, listed in Anderson [1]; it is also argued for in [10].) Whatever propositional systems turn out to provide the best formulations of entailment in combination with truth functional relations they will, I believe, have the same first-degree logic as that of E. As a step towards vindicating this viewpoint it is shown that an extensive class of systems, some with as good a claim as E to formulating entailment, have the same first degree logic as E and I.

The system E (of [1]) differs from I in deleting Axiom 4 and adding the positive axiom

15. $$NA \,\&\, NB \rightarrow N(A \,\&\, B)$$

where

$$NC =_{Df} (C \rightarrow C) \rightarrow C\,.$$

Since $I = E +$ Factor, every first-degree theorem of E is a first-degree theorem of I, where a wff is *first degree* (*fd*) iff it contains no arrows (i.e. occurrences of symbol $\rightarrow$) nested within arrows. That, conversely, the first degree of I does not outrun that of E follows either from Theorem 15 (below), or thus using the results of Belnap (in [2]). An *I-lattice* is a structure $\langle A, \leqq, N, T, \rightarrow \rangle$ where $\langle A, \leqq, N, T \rangle$ is a Belnap lattice, i.e. an intensionally complemented distributive lattice with truth filter (*icdlwtf*, for short), and $\rightarrow$ is a two-place operation on A such that $A \rightarrow b = \cap T$ or $A \rightarrow b = \cap A$, according as $a \leqq b$ or $a \nleqq b$ for every $a, b \in A$. Since $a \rightarrow b \in T$ iff $a \leqq b$ for every $a, b \in A$, an I-lattice is an implicative intensional lattice, and validity for I lattices is defined as for these implicative lattices (see [2]).

Lemma 10. *Every theorem of I is valid in every I-lattice.*

Corollary. *Every first-degree theorem of I is valid in M^c, for every c.*

Theorem 11. *Given a sequence S of first-degree wff of I, the following statements are equivalent:*

(1) *S is valid in every propositional frame.*

(2) *S is valid in M^c, with c the number of occurrences of the symbol $\rightarrow$ in S.*

(3) *S has no non-tautological full normal branch.*
(4) *S is valid in every complete intensional lattice.*
(5) *S is provable in E.*
(6) *S is provable in system R, i.e.* $E + A \rightarrow .\ A \rightarrow B \rightarrow B$.
(7) *S is provable in Ackermann's system* π.
(8) *S is provable in I.*
(9) *S is I-valid.*

Proof. The equivalence of (1)–(6) is established in Belnap [2], and that of (7) follows using [8]. Since E is a sublogic of I, (5) implies (8), and by the corollary above (8) implies (2). By the main results on I, (8) is equivalent to (9).

Succeeding results do not presuppose Belnap [2].

Theorem 12. *Where A is a first-degree (fd) wff with k occurrences of the symbol* $\rightarrow$, $\vdash_I A$ *iff*

(i) *A is true in every I-model on set*

$$K = \{G, H_1, H_1^*, \ldots, H_c, H_c^*\} \quad \textit{for every } c, \quad 0 \leqq c \leqq k\ ;$$

(ii) *A is true in every I-model on set*

$$K = \{G, H_1, \ldots, H_k, H_k^*\}\ .$$

Proof. One half of each follows from the Theorem 1. For the converse, suppose $\sim \vdash_I A$. Then $v(A, G) = F$ in some I-model $\mathscr{M} = \langle G, K, N, R, {}^*, v\rangle$. Let $K_0 = \{H_1, H_1^*, \ldots, H_c, H_c^*, G\}$, with $c \leqq k$, be the class of situations in K used in falsifying A. That there is such a subclass of K can be shown by considering the first-degree form of A. For A is a truth-function of k first degree entailments; and that truth-function is evaluated on G. Each first degree entailment which is assigned value F in this evaluation leads to the introduction of a pair $\{H_i, H_i^*\}$ on which the truth-functional components of the first degree entailment are evaluated. Now the restriction $\mathscr{M}_0 = \langle G, K_0, N_0, R_0, {}^*{}_0, v_0\rangle$ of $\mathscr{M}$ [where N_0 is the intersection of K_0 with N, R_0 and ${}^*{}_0$ are the restrictions of R and $*$ to K_0, and $v_0(p, H)$ is $v(p, H)$ with H restricted to K_0] is an I-model, and $v_0(A, G) = v(A, G) = F$, by induction over the connectives in A (as below). Hence (i) is established. For (ii) a new I-model $\mathscr{M}_1 = \langle G, K_1, N_1, R_1, {}^*{}_1, v_1\rangle$ is defined as follows: $K_1 = K_0 \cup J$, where $J = \{H_{c+1}, \ldots, H_k, H_k^*\}$ are new situations not included in K. Further R_1 is the restriction of R to K_0 together with reflexivity of elements of J, ${}^*{}_1$ is the restriction of $*$ to elements of K_0 together with a suitable correlation on elements of J (most simply $H_j = H_j^*$, for each j), N_1 is N_0 together with elements of J such that $H_j = H_j^*$, $v_1(p, H) = v(p, H)$ for $H \in K_0$ and is arbitrarily assigned on J, say $v_1(p, H) = T$ for $H \in J$. Then, by inspection, M_1 is an I-model. Also

(α) for every subformulae B of A, and for every $H \in K_0$, $v_1(B, H) = v(B, H)$.

The induction basis is immediate, and likewise the induction steps for connectives &, $\vee$, and $\sim$ since $H^* \in K_0$ iff $H \in K_0$. For $\rightarrow$ the following points are needed: for $H \in K_0$, $H \in N_1$ iff $H \in N$; for $H \in K_0$, $H R_1 H^1$ iff $H R H^1$. Then for $H \in K_0$, $v_1(B \rightarrow C, H) = T$ iff $v(B \rightarrow C, H) = T$. By ($\alpha$) since $G \in K_0$, $v_1(A, G) = F$, and (ii) follows.

This theorem provides a simple decision procedure for the first degree of R, I and eventually of a number of other systems. Simple as this semantics for the first-degree is however, it can be progressively simplified. A *first-degree strong (fds) model* is a structure $\mathscr{F} = \langle G, K, R_0, *, v \rangle$ where G belongs to set K, R_0 is a relation where first-place is restricted to G such that $G R_0 G$, $*$ is a $1-1$ function such that $G^* = G$, $H^{**} = H$ and $G R_0 H$ implies $G R_0 H^*$. Valuation v is extended as before in the case of truth-functional connectives but for $\rightarrow$ the following fds specification is needed:

$v(B \rightarrow C, G) = T$ iff, for every H, $G R_0 H$ and $v(B, H) = T$ imply $v(C, H) = T$.

For B and C are of course truth-functional wff. A first-degree wff B is *true in* in a *fds* model $\mathscr{F}$ iff $v(B, G) = T$, etc. $\mathscr{F} = \langle G, K, R_0, *, v \rangle$ is a first-degree model *on* set K. Note that for the first-degree the *only* requirement on R_0 is reflexivity at G.

The relation R_0 is also dispensible in the case of purely first-degree wff. A *first-degree* (fd) *model* $\mathscr{F} = \langle G, K, *, v \rangle$ is obtained from an fds model by deleting relation R_0 and the conditions it satisfies. Accordingly $*$ is simply a $1-1$ function such that $G^* = G$ and $H^{**} = H$. Valuation function v is fd extended so that

$v(B \rightarrow C, G) = T$ iff, for every H, $v(B, H) = T$ implies $v(C, H) = T$.

A *strict fd model* $\mathscr{S} = \langle G, K, v \rangle$ (for Lewis strict implicational systems formulated with $\{\&, \vee, \sim, \rightarrow\}$ as primitive set) differs from an fd model just in deleting $*$ and the conditions it satisfies; thus situations are restricted to certain possible situations. The strict fd extension of v is like the fd extension except in the case of negation, where $v(\sim A, H) = T$ iff $v(A, H) \neq T$.

Theorem 13. *Where A is a fd wff with k occurrences of the symbol $\rightarrow$,*

(1) *A is a theorem of I, or of E or R, iff*

(i) *A is true in every fds model on set $K = \{G, H_1, H_1^*, \ldots, H_c, H_c^*\}$ for every c, $0 \leqq c \leqq k$, or, simply for $c = k$,*

(ii) *A is true in every fd model on K;*

(2) *A is a theorem of S3, or of any of the Lewis systems of strict implication, iff A is true in every strict* fd *model on set* $K = \{G, H_1, \ldots, H_c\}$ *for every c*, $0 \leqq c \leqq k$, *or simply for* $c = k$. *Hence*

(3) *where A is a positive* fd *wff, A is a theorem of E iff A is a theorem of S4.*

Proof of (1) (i) applies Theorem 12. Given an I-model $\mathscr{M} = \langle G, K, R, N, {*}, v\rangle$ with $v(A, G) = T$, then $\mathscr{F} = \langle G, K, R_0, {*}, v\rangle$, where $H_1 R_0 H_2$ holds iff both $H_1 = G$ and $G R H_2$, provides an fds model such that $v^1(A, G) = T$ where v^1 is the fds extension of v. Conversely given an fds model $\mathscr{F}$ a corresponding I-model results upon defining $N = \{H \in K : H^* = H\}$ and R by cases thus:

GRH iff GR_0H, and for $H \neq G$, $H_1 R H_2$ iff $H_1 = H_2$. The result then follows by induction over the extension of v. As to (ii), an fds model contracts to an fd model simply by replacing K by $\{H \in K : GR_0H\}$ and omitting R_0; and an fd model furnishes an fds model on defining R_0 so that $H_1 R_0 H_2$ iff $H_1 = G$.

Semantical characterisations of provable fde (first-degree entailments) and of fd strict implications follow as corollaries.

Corollaries. (1) *Where A is an fde, A is provable (in I or R) iff A is true in every fd model on set* $K = \{G, H, H^*\}$.

(2) *Where A is a fd strict implication, A is provable iff A is true in every strict fd model on* $K = \{G, H\}$.

Since, however, G now does no work a further reduction can be made in the size of K. Call a structure $\mathscr{F} = \langle K_R, {*}, v\rangle$ with $K_R = \{H, H^*\}$ an *fde model*; and a structure $\mathscr{S} = \langle\{G\}, v\rangle$ a *pc model*. Extend v as before for truth-functional connectives and as follows for "$\rightarrow$":

$v(A \rightarrow B, H_1) = T$ iff $v(A, H_2) = T$ implies $v(B, H_2) = T$ for every H_2, i.e. in the case of the fde models iff ($v(A, H_1) \neq T$ or $v(B, H_1) = T$) and ($v(A, H_1^*) \neq T$ or $v(B, H_1^*) = T$), and in the case of pc models iff $v(A, H_1) \neq T$ or $v(B, H_1) = T$. *A* wff *B* is *true at* H_1 in an fde (pc) model iff $v(A, H_1) = T$; *B* is *fde (pc) valid* iff *B* is true at every *H* in every fde (pc) model.

Theorem 14. *Where A is a fde (fd strict implication), A is a provable entailment iff A is fde valid, and A is a provable strict implication iff A is pc valid.*

The same first degree semantics serves not only for *E*, *I* and *R* but also for weaker and more satisfactory systems. The system *D*, of deducibility, is among these systems. *D* formulated with connective set $\{\sim, \&, \rightarrow\}$, and with "$\vee$", "$\supset$" and "$\leftrightarrow$" defined as usual, has these postulates:

D1. $A \rightarrow B \,\&\, B \rightarrow C \rightarrow .\ A \rightarrow C.$
D2. $A \,\&\, (A \rightarrow B) \rightarrow B.$
D3. $A \,\&\, B \rightarrow A.$
D4. $A \rightarrow .\ A \,\&\, A.$
D5. $A \rightarrow B \,\&\, C \rightarrow D \rightarrow .\ A \,\&\, C \rightarrow D \,\&\, B.$
D6. $A \,\&\, (B \vee C) \rightarrow .\ (A \,\&\, B) \vee C.$
D7. $A \rightarrow B \rightarrow .\ \sim B \rightarrow A.$
D8. $\sim\sim A \rightarrow A.$
D9. $A \rightarrow B \supset .\ \sim (A \,\&\, B).$

As well as Modus Ponens and Adjunction, D has the material rules of intersubstitutivity of coentailments (SE) and of material detachment

(γ) from A and $A \supset B$ to infer B.

D may be weakened without changing the first degree, *eg* to the system $D0$ where D1 and D5 appear only in rule form. D and D0 may also be amended, to systems C and $C0$ which delete (γ) as a primitive rule, by replacing D9 by

D9¹. $A \rightarrow B \rightarrow .\ \sim (A \,\&\, \sim B).$

In fact for any system between $C0$ and I closed under the rules of $C0$, the Meyer-Dunn theorem holds, i.e. (γ) is admissible, by essentially the proof in [8]. Now K be any such system between $C0$ and I or between $D0$ and I.

Theorem 15. (1) *If Γ is a K-consistent class of fd wff then there is an fds model $\mathscr{F}_s = \langle G, K, R_0, *, v \rangle$ such that $\mathscr{F}_s$ fds satisfies Γ, and an fd model $\mathscr{F} = \langle G, K, *, v \rangle$ such that $\mathscr{F}$ fd satisfies Γ.*

(2) *Where A is an fd wff, $\vdash_K A$ iff A is fds valid, and iff A is fd valid.*

Proof. As to (2), if $\vdash_K A$ then $\vdash_I A$; hence since A is I-valid and fd, A is fds valid and fd valid. Conversely, if A is fds valid then $\sim A$ is not fds satisfied in any fds model. Hence, by (1), $\sim A$ is not K-consistent. Thus for some wff $D, \sim A \vdash_K D \,\&\, \sim D$, whence by ($\gamma$) $\vdash_K A$. Similarly for an fd valid A.

The proof of (1) is like that of Theorem 2. System K has sufficient theorems to guarantee that Lemmas 1–7, reformulated with "K" supplanting "I", hold (N may be defined: $N = \{H : H = H^*\}$). Given Γ define $\mathscr{F}$ as follows: G is the K-maximal extension of Γ; I is the class of prime filters of G formed as follows:

(i) If $B \rightarrow C$ is an fd wff not in G, put the prime filter H such that $B \in H$ and $C \notin H$ in K,

(ii) if H is in K, put $H^* = \{B : \sim B \notin H\}$ in K. GR_0H iff, for every B and C, $B \to C \in G$ and $B \in H$ imply $C \in H$; $v(p, H) = T$ iff $p \in H$, for each atomic p and $H \in K$. Since it follows, as for I, that GR_0G, $G = G^*$, $H = H^{**}$ and GR_0H implies GR_0H^*, $\mathscr{F} = \langle G, K, R_0, {}^*, v\rangle$ is an fds model. An fd model is obtained simply by omitting R_0.

(†) $v(A, H) = T$ iff $A \in H$, for every fd wff A and for every $H \in K$ for which v is defined. The proof, by induction, is like that for I; in fact the steps for truth-functional connectives are identical, and the remaining case, showing that $B \to C \in G \equiv.\ v(B \to C, G) = T$, is similar to that for I in the case of fds model $\mathscr{F}$. For fd model $\mathscr{F}$, $B \to C \in G$ implies that for every H $B \in H$ implies $C \in H$ follows since H is a filter of G; hence $B \to C \in G$ implies $v(B \to C, G) = T$. Conversely, if $B \to C \in G$ then $B \nleq_G C$; so applying Stone's theorem, there is an H, assigned to K, such that $B \in H$ and $C \notin H$; whence $v(B \to C, G) \neq T$.

Since when $B \in \Gamma$, B is an fd wff, by (†) $v(B, G) = T$; hence $\mathscr{F}$ fds-(fd-) satisfies Γ.

Theorem 15 also gives an alternative proof of Theorem 13. By combining Theorem 15 with preceding theorems semantical characterisations of fd theorems of E and of tautological entailments follow.

§ 6. Matrices and Lattices

A model $\mathscr{M}$ can be resolved into a pair $\langle \mathscr{S}, v\rangle$, where $\mathscr{S}$ is (as in Kripke [7]) a *model structure (m.s.)* and v is a *valuation on* m.s. $\mathscr{S}$. In the case of I-models, $\mathscr{S} = \langle G, K, R, N, {}^*\rangle$ provides the m.s.; in the case of fd models $\mathscr{S} = \langle G, K, {}^*\rangle$ is the relevant m.s. We consider now, in effect, the class $\mathscr{N}$ of models obtained by taking all two-valued valuations v on model structure $\mathscr{S}$ on set K. Following Kripke ([7], p. 92), define a *matrix value* as a mapping whose domain is K and whose range is the set $\{T, F\}$. Accordingly a matrix-value is tantamount to a subset of K; for each matrix value determines that unique subset of K which it maps into T and conversely such subset of K specifies a mapping. Furthermore valuation v maps propositional variables into matrix values; for, given variable p, v assigns as matrix-value $\{H : v(p, H) = T\}$. Thus the class $\mathscr{N}$ of models provides a matrix. Where ϱ and σ are matrix values, define matrix operations "$\sim$", "&", "v" and "$\to$" as follows:

$$(\sim\varrho)(H) = T \quad \text{iff} \quad \varrho(H^*) \neq T;$$
$$(\varrho \,\&\, \sigma)(H) = T \quad \text{iff} \quad \varrho(H) = T = \sigma(H);$$
$$(\varrho \vee \sigma)(H) = T \quad \text{iff} \quad \varrho(H) = T \quad \text{or} \quad \sigma(H) = T;$$

and given an I-model, $(\varrho \to \sigma)(H) = T$ iff $H \in N$ and, for every H', if HRH' and $\varrho(H') = T$ then $\sigma(H') = T$. For fd models this reduces, for relevant H, to $(\varrho \to \sigma)(H) = T$ iff, for every H' $\varrho(H') \neq T$ or $\sigma(H') = T$. It follows: $\sim\varrho = \{H : \varrho(H^*) = F\}$ and $\varrho \,\&\, \sigma = \varrho \cap \sigma$, and $\varrho \vee \sigma = \varrho \cup \sigma$, where $\cap$ and $\cup$ are set or lattice operations. Finally, in models where truth is evaluated at G, ϱ is a *designated value* iff $\varrho(G) = T$. For fde models ϱ is *designated* iff $\varrho(H) = T = \varrho(H^*)$.

Where $\mathscr{S}$ is an m.s. the *matrix* (or *algebra*) $\mathscr{S}^+ = \langle M, D, \sim, \&, v, \to \rangle$ *on* $\mathscr{S}$ is defined as follows: M is the class of matrix values, D is the class of designated matrix values, and $\sim$, &, v, $\to$ are the class-closing matrix operations defined above. Then $\mathscr{S}^+$ is a matrix in the standard sense.

Lemma 11. *Where $\mathscr{S}^+$ is the matrix on m.s. $\mathscr{S}$, and where truth is evaluated at G, $\mathscr{S}^+$ satisfies A iff, for every v on $\mathscr{S}$, $v(A, G) = T$, for every wff A. Where $\mathscr{S}^+$ is defined on fde model $\mathscr{S}$, $\mathscr{S}^+$ satisfies A iff, for every v on $\mathscr{S}$, $v(A, H) = T = v(A, H^*)$, for every wff A.*

Proof. Let $A^+(p_1^+, \ldots, p_n^+)$ be the matrix function of $p_1^+, \ldots, p_n^+$ corresponding to wff $A(p_1, \ldots, p_n)$ under the replacement of connectives by corresponding matrix operations, where v maps p_1 to $p_1^+, \ldots, p_n$ to p_n^+. Thus $p_i^+ = \{H : v(p_i, H) = T\}$. Let A^+ be the matrix value of A. Then, it follows by induction over connectives (or operations),

(*) $H \in A^+$ iff $v(A, H) = T$.

Applying (*), $v(A, G) = T$ iff $G \in A^+$ iff A^+ is designated iff for the given assignment $\mathscr{S}^+$ satisfies A. Generalising, $\mathscr{S}^+$ satisfies A iff for every assignment v on $\mathscr{S}$, $v(A, G) = T$. Similarly $v(A, H) = T = v(A, H^*)$ iff $H \in A^+ \,\&\, H^* \in A^+$ iff A^+ is designated; and hence $\mathscr{S}^+$ satisfies A for fde iff for every v, $v(A, H) = T = v(A, H^*)$.

Theorem 16. *Where A is an fde, A is a provable entailment (of K) iff the Smiley matrices,*

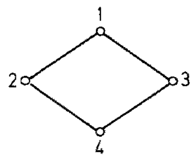
&	1	2	3	4	$\sim$
*1	1	2	3	4	4
2	2	2	4	4	2
3	3	4	3	4	3
4	4	4	4	4	1

$\to$	1	2	3	4
*1	1	4	4	4
2	1	1	4	4
3	1	4	1	4
4	1	1	1	1

satisfy A; i.e. these matrices are characteristic for fde.

Proof. A is provable iff A is fde valid. Define matrix values as follows: $1 = \{H, H^*\} = K_R$; $2 = \{H\}$; $3 = \{H^*\}$; $4 = \Lambda$. Then, with matrix operations defined as before for fde models, the Smiley matrices result. It also follows that 1 is the only designated value. Applying the previous lemma, A is fde valid iff it is true at H and H^* for every valuation function on $K = \{H, H^*\}$ iff the derived Smiley matrices satisfy A.

Thus the semantics provides an interpretation of the Smiley matrices. In fact K_R generates the fundamental de Morgan lattice D_0 with diagram,



and with N defined: $N1=4$, $N4=1$, $N2=2$ and $N3=3$. As well as a matrix on m.s. $\mathcal{S}$ a de Morgan lattice $L(\mathcal{S})=\langle M, \cap, \cup, N, \leqq\rangle$ may be defined on $\mathcal{S}$, and $L(K)=L(\mathcal{S})$ on K, as follows: $M=\mathcal{P}K$, i.e. the set of subsets of K; for $\varrho, \sigma \in M$, $\varrho\cap\sigma=\{H\in K: H\in\varrho \;\&\; H\in\sigma\}$, $\varrho\cup\sigma=\{H\in K; H\in\varrho \vee H\in\sigma\}$ $N\varrho=\{H\in K: H^*\notin\varrho\}$; $\varrho\leqq\sigma$ iff $\varrho\subseteq\sigma$.

Hence $\varrho\leqq\sigma$ iff for every H $\varrho(H)=T$ implies $\sigma(H)=T$, iff $(\varrho\to\sigma)(G)=T$ when $\mathcal{S}$ is a fdms $L(\mathcal{S})=\langle M, \cap, \cup, N\rangle=\langle M, N, \leqq\rangle$.

Lemma 12. (1) *Where $\mathcal{S}$ is an I, fds, fd or fde model structure, $L(\mathcal{S})$ is a de Morgan lattice*

(2) *The direct product $L_1\times\cdots\times L_n$ of de Morgan lattices $L_1,\ldots,L_n$ is a de Morgan lattice.*

(3) *Where $L(K_1),\ldots,L(K_n)$ are de Morgan lattices on mutually disjoint *-complemented sets K_i $(1\leqq i\leqq n)$ $L\left(\bigcup_i K_i\right)$ is a de Morgan lattice.*

(4) *Where $\{K_i\}_{1\leqq i\leqq n}$ is a set of mutually disjoint *-complemented sets and $L(K_i)$ is a de Morgan lattice on K_i, $L\left(\bigcup_i K_i\right)$ is isomorphic to $L(K_1)\times\cdots\times L(K_n)$. [In fact n need not be finite].*

(5) *Where $\mathcal{S}$ is an fde model, i.e. $K=\{H, H^*\}$, $L(\mathcal{S})=L(K)=D_0$.*

Proof. (1) $L(\mathcal{S})$ without operation N is the subset lattice of K (as defined in Szasz [12], p. 34), and accordingly is a distributive lattice. Further, $N(N\varrho)=\varrho$ since $H^{**}=H$, and $N\varrho\leqq N\sigma$ iff $\sigma\leqq\varrho$; hence $L(\mathcal{S})$ is de Morgan.

(2) is a consequence of the componentwise definition of operations in a direct product (see Szasz [12], p. 197). In particular since, by definition $N\langle\varrho_1,\varrho_2,\ldots,\varrho_n\rangle=\langle N\varrho_1, N\varrho_2,\ldots,N\varrho_n\rangle$ for $\varrho_i\in\mathcal{P}(K_i)$, and since $\langle\varrho_1,\ldots,\varrho_n\rangle\leqq\langle\sigma_1,\ldots,\sigma_n\rangle$ iff for every $1\leqq i\leqq n$ $\varrho_i\leqq\sigma_i$, $NN\langle\varrho_1,\ldots,\varrho_n\rangle=\langle\varrho_1,\ldots,\varrho_n\rangle$ and $N\langle\varrho_1,\ldots,\varrho_n\rangle\leqq N\langle\sigma_1,\ldots,\sigma_n\rangle$ iff $\langle\sigma_1,\ldots,\sigma_n\rangle\leqq\langle\varrho_1,\ldots,\varrho_n\rangle$.

(3) is like (1); for the union of disjoint *-complemented sets is a *-complemented set.

(4) Where K_i has n_i elements then the elements of $L\left(\bigcup_i K_i\right)$, given by $\mathcal{P}\left(\bigcup_i K_i\right)$ are $2^{\sum_i n_i}$ in cardinal number, and the elements of

$L(K_1)\times\cdots\times L(K_n)$, given by $\mathscr{P}(K_1)\times\cdots\times\mathscr{P}(K_n)$, are $2^{n_1+\cdots+n_m}$ in number. Hence there is a $1-1$ correspondence between the sets concerned given by a $1-1$ function. Now order $\mathscr{P}(K_1)\times\cdots\times\mathscr{P}(K_n)$ and represent the i^{th} member $\alpha_i=\langle \mathrm{sub}_i K_1,\ldots,\mathrm{sub}_i K_n\rangle$ with $\mathrm{sub}_i K_j$ the i^{th} subset of $\mathscr{P}(K_j)$; and order $\mathscr{P}\left(\bigcup_i K_i\right)$ so that the i^{th} member $\beta_i=\mathrm{sub}_i K_1\cup\cdots\cup\mathrm{sub}_i K_n$.

Let $\hbar$ be the $1-1$ function which maps α_i to β_i; then $\hbar$ is the required isomorphism. For, first,

$$\begin{aligned}\hbar(\alpha_i\cap\alpha_j) &= \hbar(\langle \mathrm{sub}_i K_1\cap\mathrm{sub}_j K_1,\ldots,\mathrm{sub}_i K_n\cap\mathrm{sub}_j K_n\rangle)\\ &= (\mathrm{sub}_i K_1\cap\mathrm{sub}_j K_1)\cup\cdots(\mathrm{sub}_i K_n\cap\mathrm{sub}_j K_n)\\ &= (\mathrm{sub}_i K_1\cup\cdots\cup\mathrm{sub}_i K_n)\cap(\mathrm{sub}_j K_1\cup\cdots\cup\mathrm{sub}_j K_n),\\ &\quad\ \text{since } \mathrm{sub}_i K_\alpha\cap\mathrm{sub}_j K_\beta=\Lambda \quad \text{for} \quad \beta\neq\alpha.\\ &= \hbar(\alpha_i)\cap\hbar(\alpha_j).\end{aligned}$$

$$\hbar(\alpha_i\cup\alpha_j)=\hbar(\alpha_i)\cup\hbar(\alpha_j)\quad \text{directly}.$$

Also

$$\hbar(N\alpha_i)=N\hbar(\alpha_i)$$

For

$$N\hbar(\alpha_i)=N(\mathrm{sub}_i K_1\cup\cdots\,\mathrm{sub}_i K_n)$$

and

$$\hbar N(\alpha_i)=N\,\mathrm{sub}_i K_1\cup\cdots\cup N\,\mathrm{sub}_i K_n.$$

Let $H_j\in K_i$, then $H_j\in\hbar(N\alpha_i)$ iff $H_j\in N\,\mathrm{sub}_j K_j$; and also $H_j\in N\hbar(\alpha_i)$ iff $H_j\in N(\mathrm{sub}_i K_1\cup\cdots\,\mathrm{sub}_i K_n)$ iff $H_j\in N\,\mathrm{sub}_i K_j$. Hence, generalising, for every $H\in\cup K_i$ $H\in\hbar(N\alpha_i)$ iff $H\in N\hbar(\alpha_i)$, i.e. $\hbar(N\alpha_i)=N\hbar(\alpha_i)$ as required. Thus $\hbar$ is a homomorphism.

(5) The lattice diagram of $L(\mathscr{S})$ is immediate from inclusion relations among the elements of $M=\{1,2,3,4\}$; and the properties of N follow immediately from the definition of "N" in $L(\mathscr{S})$.

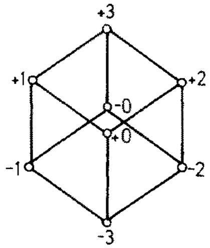
The de Morgan lattice $L(\mathscr{S})$ on $\mathscr{S}$ may, of course, be read off the matrix $\mathscr{S}^+$ on $\mathscr{S}$ by replacing mappings by their extensions, the subclass of K they map to T, operations $\sim$, &, $\vee$ on mappings by operations N, $\cap$, $\cup$ on their extensions, and defining $\varrho\leqq\sigma$ as $(\varrho\to\sigma)(G)=T$ (except for fde models), i.e. as $\varrho\to\sigma\in D$. Conversely matrix operations $\sim$, &, $\vee$ may be read off (or identified with) corresponding de Morgan lattice operations, and designated values of $\varrho\to\sigma$ read off $\varrho\leqq\sigma$ *when* D is determined. Intermediate between $L(\mathscr{S})$ and $\mathscr{S}^+$, and determining D but not $\to$, is the *icdlwtf (Belnap lattice)* $I(\mathscr{S})=\langle M, D, \cap, \cup, N, \leqq\rangle$ *on* m.s. $\mathscr{S}=\langle G, K, *, \ldots\rangle$, and likewise $I(K)$ on K, where $I(\mathscr{S})$ is as for $L(\mathscr{S})$ except for D which is defined: $D=\{\varrho; G\in\varrho\}$.

Lemma 13. (1) *$I(\mathscr{S})$ on m.s. $\mathscr{S}$ is an icdlwtf, i.e. $L(\mathscr{S})$ is a de Morgan lattice and $D\subsetneqq M$ is a truth-filter.*

(2) *If for some* $H \in K$, $H = H^*$, *de Morgan lattice* $L(K)$ *on* K *has a truth filter* $D = \{\varrho : H \in \varrho\}$.

(3) *Where* $G \in K_j$ *define a (G-) truth filter* T_p *for the direct product* $P = L(K_1) \times \cdots I(K_j) \times \cdots L(K_n)$ *of de Morgan lattices as the class of ordered sets of* P *whose* j^{th} *elements belong to the truth-filter* T_j *of* $I(K_j)$. *Then the filtered product* P *is T-isomorphic to* $I\left(\bigcup_i K_i\right)$ *(where T-isomorphism is defined as in Dunn and Belnap* [5], *p.* 29*).*

(4) $I(\{G, H, H^*\})$, *i.e.* $I(\mathscr{S})$ *on m.s.* $\mathscr{S} = \langle G, \{G, H, H^*\}, *, \ldots\rangle$, *is the fundamental icdlwtf* M_0, *which has the following diagram, where* $N(\pm\varrho) = \mp\varrho$ *and* $D = \{+0, +1, +2, +3\}$:



(5) *Where* $\mathbf{2} = I(\{G\})$, *the two element Boolean algebra with elements* $1 = \{G\}$ *and* $0 = \Lambda$ *and truth-filter* 1, M_0 *is T-isomorphic to the filtered product* $\mathbf{2} \times D_0$.

(6) *Where* M^n *is the product icdlwtf (defined in Belnap* [2] *and Dunn and Belnap* [5]*) and* $D^n = D_0 \times \cdots \times D_0$ *n times,* M^n *is T-isomorphic to the filtered product* $\mathbf{2} \times D^n$. *(Note: each* D_0 *can be taken to be generated by a different set* $\{Hj, Hj^*\}$*.)*

Proof. (3) Let $\hbar$ be the isomorphism given in Lemma 12 (3). It suffices to show that if $\alpha_i \in T_p$ then $\hbar(\alpha_i) \in T$, where $T = \left\{\varrho \in I\left(\bigcup_i K_i\right) : G \in \varrho\right\}$. Now supposing $\alpha_i \in T_p$, then $\text{sub}_i K_j \in T_j$. Hence $G \in \text{sub}_i K_j$. Hence $G \in \hbar(\alpha_i)$, and $\hbar(\alpha_i) \in T$.

(4) Set
$$\begin{array}{ll} +3 = \{G, H, H^*\} & -3 = \Lambda \\ +2 = \{G, H\} & -2 = \{H\} \\ +1 = \{G, H^*\} & -1 = \{H^*\} \\ +0 = \{G\} & -0 = \{H, H^*\}. \end{array}$$

The Belnap 8-valued relevance matrices for matrix operations &, $\vee$, $\sim$ can be read off this icdlwtf and the designated places in the $\rightarrow$ matrix are determined. Hence the semantics provides a partial interpretation of these matrices.

(6) The mapping for $a=\{a_i\}_{i<n}$ of M^n (as given in Belnap's miscellaneous notes) is:

$\mathscr{h}(a)=\langle\mathscr{sg}(a),\{\mathscr{f}(a_i)\}_{i<n}\rangle$, where homomorphisms $\mathscr{f}$ and $\mathscr{sg}$ are defined from M_0 onto D_0 as follows:

$$\mathscr{f}(-0)=\mathscr{f}(+3)=1\,;\qquad \mathscr{f}(-3)=\mathscr{f}(+0)=4$$

$$\mathscr{f}(-1)=\mathscr{f}(+1)=2\,;\qquad \mathscr{f}(-2)=\mathscr{f}(+2)=3$$

$$\mathscr{sg}(a)=1 \quad \text{iff} \quad a\in D(M_0)\,;\qquad \mathscr{sg}(a)=0 \quad \text{iff} \quad a\notin D\,.$$

Then $\mathscr{h}$; $M^n\to\mathbf{2}\times D^n$ is a T-isomorphism.

Given an icdlwtf I, an *evaluation* v (for an fde A) is an assignment of lattice elements [matrix values] of M to sentential variables (of A). It is extended as follows for zero degree wff B and C:

$$v(\sim B)=Nv(B)\,;\qquad v(B\vee C)=v(B)\cup v(C)\,;\qquad v(B\,\&\,C)=v(B)\cap v(C)\,.$$

I verifies fde $C_1\to C_2$ *for* evaluation v iff $v(C_1)\leqq v(C_2)$. *I verifies* fde A iff I verifies A for every evaluation; otherwise *I falsifies A*.

Lemma 14. *$I(\mathscr{S})$ on $\mathscr{S}=\langle G,K,{}^*,\ldots\rangle$ verifies fde A iff $\upsilon(A,G)=T$ for every valuation υ on $\mathscr{S}$.*

Proof. Given $\upsilon(p,H)$ define $\upsilon(p)=\{H:\upsilon(p,H)=T\}$. Conversely given $v(p)=\varrho$ define $\upsilon(p,H)=T$ iff $H\in\varrho$. Then since $\varrho=\{H:H\in\varrho\}$ $=\{H:\upsilon(p,H)=T\}$, $v(p)=\{H:\upsilon(p,H)=T\}$. Thus, in any case, $v(p)$ $=\{H:\upsilon(p,H)=T\}$, from which it follows by induction, as in Lemma 11, $v(B)=\{H:\upsilon(B,H)=T\}$ for any zero degree wff B. Then, v verifies $C_1\to C_2$ iff $v(C_1)\leqq v(C_2)$ iff $\{H:\upsilon(C_1,H)=T\}\subseteq\{H:\upsilon(C_2,H)=T$ iff $\upsilon(C_1\to C_2,G)=T$.

Theorem 17. *Where A is an fde, A is a provable entailment (of K) iff M_0 verifies A, i.e. M_0 is characteristic for fde.*

Proof. A is provable iff A is true for every valuation on m.s. $\mathscr{S}$ $=\langle G,\{G,H,H^*\},{}^*\rangle$, i.e. iff $I(\mathscr{S})$ verifies A. But $I(\mathscr{S})$ on $\mathscr{S}$ is M_0.

Actually M_0, unlike D_0, is characteristic for any truth-functional compound formed from a single fde.

Given an icdlwtf I, an *evaluation* for fd wff (for fd wff A) is a pair $\langle v,\omega\rangle$ where v is, as before, an assignment of lattice elements to sentential variables (of A), and ω is a valuation assigning truth-values T and F to wff as follows, given the extension of v: Where A is a zero degree wff $\omega(A)=T$ iff $v(A)\in D$; where A has the form $C_1\to C_2$, $\omega(A)=T$ iff $v(C_1)\leqq v(C_2)$; and otherwise $\omega(\sim B)=T$ iff $\omega(B)\neq T$, $\omega(B\,\&\,C)=T$ iff $\omega(B)=T=\omega(C)$, and $\omega(B\vee C)=T$ iff $\omega(B)=T$ or $\omega(C)=T$. (Compare Belnap [2].) *I verifies A for* evaluation $\langle v,\omega\rangle$ (for A) iff $\omega(A)=T$; and *I verifies A* iff I verifies A for every I-evaluation.

Lemma 15. *$I(\mathscr{S})$ on $\mathscr{S} = \langle G, K, *, \ldots\rangle$ verifies fd wff A iff $\upsilon(A, G) = T$ for every valuation υ on m.s. $\mathscr{S}$.*

Proof extends the preceding lemma for fdes. Given υ, define $\omega(A) = \upsilon(A, G)$ and given $\langle v, \omega\rangle$ define $\upsilon(A, G) = \omega(A)$. The adequacy of these definitions follows from the following points

(i) Where A is a zero degree wff $\upsilon(A, G) = T$ iff $v(A) \in D$. For $v(A) \in D$ iff $\{H : \upsilon(A, H) = T\} \in D$ iff $G \in \{H : \upsilon(A, H) = T\}$ iff $\upsilon(A, G) = T$.

(ii) $\upsilon(C_1 \to C_2, G) = T$ iff $v(C_1) \leqq v(C_2)$.

(iii) $\upsilon(\sim B, G) = T$ iff $\upsilon(B, G) \neq T$ *etc.*

Where $K = \{G, H_1, \ldots, H_k^*\}$ define $N^k = I(K)$. N^k plays the role with respect to fd wff that M_0 plays for fde, and of course $M_0 = N^1$.

Theorem 18. *Where A is an fd wff with k occurrences of symbol $\to$, A is provable in system K iff*

(1) *N^k verifies A.*

(2) *every icdlwtf verifies A.*

(3) *the filtered product $\mathbf{2} \times D^k$ verifies A.*

(4) *M^k verifies A.*

Proof. (1) A is provable iff A is true for every valuation on m.s. $\mathscr{S} = \langle G, \{G, H_1 \ldots H_k^*\}, *\rangle$ iff $I(\mathscr{S})$ verifies A iff N^k verifies A.

(2) If A is provable then $\vdash_I A$, so by Lemma 10 for any I-lattice J, A is valid in J; and hence, since A is fd, J verifies A. (For proof, where v^1 is the implicative valuation, as defined in Belnap [2], extending v, define $\omega(B) = T$ iff $v^1(B) \in T$. It then follows that $\langle v, \omega\rangle$ is an evaluation, and that it verifies a fd wff iff v^1 does.) Conversely if A is not provable N^k falsifies A, so not every icdlwtf verifies A.

(3) and (4) follow from (1) by Lemma 13 (3) and (6). For N^k is T-isomorphic to $\mathbf{2} \times D^k$ and thereby to M^k. Then if M^k falsifies A the evaluation obtained by composing the T-isomorphism function with the falsifying evaluation falsifies N^k.

Corollary. *Every finite icdlwtf is isomorphic to a sub-lattice of N^k for some finite k, and to a sublattice of M^c for some finite c.*

Proof is the same as the stronger corollary in Belnap [2], except that given the inverse diagram $ID(A)$ of A, $D(A)$ the disjunction of all members of $ID(A)$ is defined.

Had the results been carried through (as they could have: see Theorem 23) for sequences of fd wff Belnap's corollary would go through intact. A better embedding theorem can however be obtained by algebraic means.

Theorem 19. *Where $I = \langle M, D, \cap, \cup, N, \leqq\rangle$ is an icdlwtf, there is an m.s. $\mathscr{S} = \langle G, K, *\rangle$ such that I is T-isomorphic to a sublattice of $I(\mathscr{S})$.*

Proof. Let K be the set F of all prime filters of I. For $H \in K$, define $H^* = \{m \in M : Nm \notin H\}$ and $G = D$. Then $G \in K$, since D is a prime filter, $G = G^*$ since D is consistent; and H^* is a prime filter. For, $m, n \in M$,

$$\begin{aligned} m \cap n \in H^* \quad & \text{iff} \quad N(m \cap n) \notin H \\ & \text{iff} \quad Nm \cup Nn \notin H \\ & \text{iff} \quad Nm \notin H \text{ and } Nn \notin H \text{ iff } m \in H^* \text{ and } n \in H^* \\ m \cup n \in H^* \quad & \text{iff} \quad m \in H^* \text{ or } n \in H^*, \end{aligned}$$

by a similar argument.

Now for $m \in M$, define $\hbar(m) = \{H \in F : m \in H\}$, i.e. as in Stone's representation theorem, the mapping $\hbar$ associates with each element m the set of prime filters to which m belongs. For a T-homomorphism it is required,

(i) for $m, n \in M$, $\hbar(m \cap n) = \hbar(m) \cap \hbar(n)$,

(ii) for $m, n \in M$, $\hbar(m \cup n) = \hbar(m) \cup \hbar(n)$,

(iii) for $m \in M$, $\hbar(Nm) = N\hbar(m)$,

(iv) iff $m \in D$ then $\hbar(m) \in D^1 = \{\varrho \in M^1 : G \in \varrho\}$.

Proofs of (i) and (ii) are standard. As to (iii), for any $H \in K$;

$$\begin{aligned} H\hbar(Nm) \quad & \text{iff} \quad Nm \in H \\ & \text{iff} \quad \sim (m \in H^*) \\ & \text{iff} \quad \sim (H^* \in \hbar(m)) \\ & \text{iff} \quad H \in N\hbar(m), \text{ by definition of } N\varrho, \text{ for } \varrho \in M^1. \end{aligned}$$

For (iv), given $m \in D$, $m \in G$. Thus, since $G \in K$, $G \in \{H \in F : m \in H\}$, i.e. $G \in \hbar(m)$. Hence $\hbar(m) \in \{\varrho \in M^1 : G \in \varrho\}$, i.e. $\hbar(m) \in D^1$.

Now define $\overline{I(\mathscr{S})} = \langle \bar{M}, \bar{D}, \bar{\ }, \bar{\ }, \bar{N}, \leqq \rangle$ as follows: $\bar{M} = \{\hbar(m) : m \in M\}$ and $\bar{D} = \{\hbar(m) : m \in D\}$; and the operations of $\overline{I(\mathscr{S})}$ are the operations of I restricted to $\bar{M}$. Since M^1, the power set of the set of all prime filters of I, includes $\bar{M}$, and since $\overline{I(\mathscr{S})}$ is closed under operations, $\overline{I(\mathscr{S})}$ is a sublattice of $I(\mathscr{S})$. Accordingly $\hbar$ is a T-homomorphism of I into $I(\mathscr{S})$. Further, since $\hbar$ is $1-1$, it is a T-isomorphism. For if $m \neq n$ then $m \leqq n$ say. By Stone's prime filter theorem there is a prime filter H_1 say such that $m \in H_1$ and $n \notin H_1$, and hence $\hbar(m) \neq \hbar(n)$.

Corollaries. 1. *Every icdl is isomorphic to an N-complemented ring of sets.*

2. *Every icdlwtf of cardinality* α *is T-isomorphic to a sublattice of* M^c *for some* $c \leqq 2^{2^\alpha}$.

Theorem 9 provides the basis for a more algebraic proof of the completeness results established above.

Theorem 20. *Where K is an entailment system (as defined in § 5),*

(1) *K has a normal characteristic matrix (algebra), where $\mathcal{N} = \langle M, D, \&^1, \vee^1, \sim^1, \rightarrow^1 \rangle$ is normal iff for $\varrho, \sigma \in M$, $\varrho \,\&^1 \sigma \in D$ iff $\varrho \in D \,\&\, \sigma \in D$, $\varrho \vee^1 \sigma \in D$ iff $\varrho \in D \vee \sigma \in D$, and $\sim^1 \varrho \in D$ iff $\varrho \notin D$.*

(2) *K has a completely normal characteristic matrix, where $\mathcal{N}$ is completely normal iff $\mathcal{N}$ is normal and, for $\varrho, \sigma \in M$, $\varrho \leftrightarrow^1 \sigma \in D$ iff $\varrho = \sigma$.*

(3) *K is reasonable in the sense of Halldén, i.e. where wff A and B do not share a variable, if $\vdash_K A \vee B$ then $\vdash_K A$ or $\vdash_K B$.*

Proof. (3) follows from (1) by a result of Kripke [7b], p. 207; and (1) follows by extending Meyer and Dunn [8], Theorem 3. (2) follows from (1) by identifying equivalent wff. Much as in the Lindenbaum construction, given normal characteristic K-matrix

$$\mathcal{N} = \langle M, D, \&_1, \vee_1, \sim_1, \rightarrow_1 \rangle,$$

define: $\varrho \equiv \sigma$ iff $\varrho \leftrightarrow_1 \sigma \in D$; $\|\varrho\| = \{\sigma : \sigma \equiv \varrho\}$; and $L(K) = \langle \bar{M}, \bar{D}, \cap, \cup, N, \Rightarrow \rangle$ where $\bar{M} = \{\|\varrho\| : \varrho \in M\}$; $\bar{D} = \{\|\varrho\| : \varrho \in D\}$; $N\|\varrho\| = \|\sim_1 \varrho\|$; $\|\varrho\| \cap \|\sigma\| = \|\varrho \,\&_1 \sigma\|$; $\|\varrho\| \cup \|\sigma\| = \|\varrho \vee_1 \sigma\|$; $\|\varrho\| \Rightarrow \|\sigma\| = \|\varrho \rightarrow_1 \sigma\|$. Then

(i) $L(K)$ is a characteristic K-matrix;

(ii) $L(K)$ is completely normal.

$$\begin{aligned}
\|\varrho\| \Leftrightarrow \|\sigma\| \in \bar{D} \quad &\text{iff} \quad \|\varrho \leftrightarrow_1 \sigma\| \in \bar{D} \quad \text{iff} \quad \varrho \leftrightarrow_1 \sigma \in D \\
&\text{iff} \quad \varrho \equiv \sigma \quad \text{iff} \quad \|\varrho\| = \|\sigma\| \\
\|\varrho\| \cap \|\sigma\| \in \bar{D} \quad &\text{iff} \quad \|\varrho \,\&_1 \sigma\| \in \bar{D} \quad \text{iff} \quad \varrho \,\&_1 \sigma \in D \\
&\text{iff} \quad \varrho \in D \,\&\, \sigma \in D \quad \text{iff} \quad \|\varrho\| \in \bar{D} \,\&\, \|\sigma\| \in \bar{D}.
\end{aligned}$$

That the systems K under investigation are reasonable in Halldén's sense, and not semantically incomplete in the way Halldén has shown $S2$ and $S3$ are, is an important point in favour of these entailment systems, especially as compared with Lewis's proposed analyses of entailment summed up in $S2$ and $S3$.

Lemma 16. *Where $\mathcal{N} = \langle M, D, \cap, \cup, N, \Rightarrow \rangle$ is the completely normal characteristic matrix of implicative system K*

(1) *$\mathcal{N}$ is an implicative lattice.*

(2) *$\mathcal{N}_1 = \langle M, D, \cap, \cup, N, \leqq \rangle$ (with the redundant "$\leqq$" defined: $\varrho \leqq \sigma$ iff $\varrho \cap \sigma = \varrho$, for $\varrho, \sigma \in M$) is an icdlwtf which verifies every fd wff which $\mathcal{N}$ verifies.*

Theorem 21. *A sequence Γ of fd wff is provable in implication system K iff Γ is fd valid.*

Proof. The consistency half follows as in Theorem 15. Conversely, if sequence Γ is not provable in K then it is falsifiable in a completely normal K-matrix, by Theorem 20; hence it is falsifiable in an icdlwtf, by Lemma 16; so by Theorem 19 it is falsifiable in $I(\mathscr{S})$ on m.s. $\mathscr{S} = \langle G, K, {}^* \rangle$ on composing the evaluations with the T-isomorphism; and therefore, by extending Lemma 15, Γ is not fd valid.

(Added in proof) The decidability result for I, though correct, requires further details: these may be provided, for example, by amending the definition of $\hat{R}$ and either weakening the operation requirement on * or amending the definition of $\equiv_{\Psi}$.

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