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DIALECTICAL LOGIC, SEMANTICS AND
METAMATHEMATICS*

As with the thesis that God exists, so with the Consistency Hypothesis that the world is consistent, there are three main positions that can be taken; namely, a theistic or classical position which accepts the hypothesis, an agnostic or relevant position which suspends judgement, and an atheistic or dialectical position which rejects the hypothesis.¹ Here the world is Wittgenstein's world, the real or actual world of intensional semantics, that is the class T of all truths, and the Consistency Hypothesis, CH, is the thesis that T is simply consistent, that it contains no contradictory pairs of the form A and $\sim A$, where $\sim A$ is the negation of A . T , whether consistent or not, by no means exhausts the class of statements; for it is an empirical truth that the world is not trivial, that not all statements are true. Neither relevantly nor dialectically, however, does inconsistency lead to triviality; to assume, as classically, that it does is to assume the matter at issue, for the assumption is tantamount to CH (as [1] explains).

The comparison of the Consistency Hypothesis with the theistic thesis that God exists and with such metaphysical theses as the principle of the uniformity of nature is worth pursuing; for although there are, of course, differences between these cases the similarities are striking (as [1] and [3] try to bring out). In each case the assumption underpins the application of extensive theories, classical logic and theories, religions, and theories of induction, respectively; in each case the assumption is not, in its standard sense at least, an analytic one, and would fail in its intended role if it were; in each case, however, it is extremely doubtful that the assumption is an empirical one – the assumptions being impervious to falsifying empirical evidence – yet in each case there are considerations, logical and metaphysical arguments for instance, which ought, I think, to incline philosophers to conclude that they are false; and yet in each case the assumptions can be maintained, despite all counter-considerations, and they can be bolstered or supported by a variety of expedients, by faith, by metaphysical arguments (invariably of a question begging cast), by the purchase of Kantian glasses (the wearing of which makes the assumptions part of one's

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perceptual scheme), or by other familiar political means, e.g. exhortation, persuasion, propaganda, repression, force, and so on. But as Meyer and I reported in [3], §1.7,

we don't have the classical faith, we are not persuaded by the question begging classical arguments for CH, we do not wear Kantian glasses and, in any case consider the whole Kantian business fraudulent, especially as false assumptions can be enforced in this fashion, and we hope we can avoid classical thuggery, for lesser political means are not going to move us.

Our detailed case against the classical arguments for CH is marshalled in [1], [3], [11], and [13]; the case for the dialectical position, set out in [11], rests not on the indefensibility of CH, but appeals, among other things, to the arguments of the logical and semantical "paradoxes", which are construed as proofs that, as a matter of logic, *T* is simply inconsistent.²

The liberating effect of giving up the classical faith – with its commitment to an underlying classical logic, to classical theories and in consequence to CH, to classical limitative theorems and to classical foundations for mathematics and linguistics – is immense: not only do many "paradoxes" generating spurious philosophical problems in a wide range of intensional areas – such as meaning, analysis, information, evidence, confirmation, obligation, to list a few – vanish (see [11], §4); the logical and semantical antinomies cease to be paradoxical and become, what they always seemed to be, proofs; and the limitative theorems of Gödel, Tarski, and others lose their apparent universality and cease to apply. Therewith one is free to return to something like the grand simplicity of naive set theory, to semantically closed natural languages (having abandoned the towering but ill-constructed and mostly unfinished hierarchies of formal languages), and to intuitive accounts of truth, of proof,³ and of many other intensional notions. But making the return leaves much exploring to be done,

And the end of all our exploring
Will be to arrive where we started
And know the place for the first time.⁴

For we should arrive back guided by a dialectical theory which reassembles the intuitive data and sets it in a new light.

The remainder of this paper endeavours to contribute something further

to the discernment of such a dialectical theory, to circumscribing its underlying static logic and furnishing its semantics, to establishing its adequacy, and to outlining certain of its applications in mathematical and foundational studies.

Dialectical logic is commonly regarded as opposed to formal logic. But while there is indeed a sharp opposition between dialectical logic and classical logic, there is no such gulf separating dialectical logic and formal logic. Dialectical logic can be formalised, within the framework of formal non-classical logics, in much the way and to at least the extent that intuitionistic logic which was originally presented, and conceived, as an informal theory, has been formalised. It is hardly to be expected, however, that a formalisation of dialectical logic be uniquely achieved or that the whole of dialectical logic (in some given preanalytic formulation) be formalised at once. To facilitate a comparison with other logics proposed as foundations for mathematics and science and to simplify issues involved in the formalisation of dialectics, while yet retaining the central separating points between dialectical logic and its rivals such as classical and intuitionist logics, let us consider to begin with that part of static dialectical logic that can be represented as a sentential logic with a conventional linguistic structure, that is which consists of wff (well-formed formulae) built up from sentential variables and constants using sentential connectives $\rightarrow$ (implies), $\&$ (and), $\vee$ (or) and $\sim$ (not). The limitation of the discussion that follows to static dialectical logics in no way implies that dynamic dialectical logic which takes account of time and tensing, and accordingly considers such dialectically important matters as change, process, and motion, is beyond the scope of logical techniques; it is not as Apostel [6] reveals, and there is, moreover, a good prospect that the semantical methods devised for relevant logics (outlined below) will apply fruitfully in needed new approaches to dynamic analysis. But (as is clear from Petrov [21] and as [1] explains) several of the most notorious difficulties – from a classical viewpoint – in dialectical logic, for example the truth of contradictions and the acceptance of paradoxes as proofs, can be isolated at the static stage, indeed, in the case of contradictions, in sentential logic. And once the semantics for appropriate sentential dialectical logics have been elaborated (as is shortly done) many of the characteristic features of dialectical logic as now commonly conceived by

the Soviets can be formally established (the detailed arguments appear in [1]). Although the elaboration of dialectical logic attempted is not narrowly historical in orientation – indeed it could hardly be given that the thinking of the historically most significant dialecticians was carried out at a time when the restrictive framework of traditional formal logic dominated the logical scene, and that the crisis in the foundation of mathematics engendered by the logical antinomies had not occurred – it is nonetheless historically controlled, one of the aims being to explicate logically a non-negligible part of informal Soviet logic.

A necessary condition that a sentential logic be *dialectical* is that it is closed under modus ponens, i.e. when A and $A \rightarrow B$ are theses so is B , and that it is simply inconsistent, i.e. contains contradictory theses of the form A and $\sim A$, but non-trivial, i.e. not every wff is a thesis. It follows that none of the mainstream Western logics, such as classical, modal and intuitionist sentential logics, can serve as basis for a dialectical logic; for any inconsistent extension of these logics by the addition of contradictory theses trivialises the logics, by virtue of spread laws, such as $\sim A \rightarrow A \rightarrow B$ and $A \& \sim A \rightarrow B$, which spread contradictions everywhere. The spread laws are, however, nothing but paradoxes of implication, so it is more than a little tempting to base dialectical systems on paradox free logics such as relevant logics (on which see [2] and [3]).

Nor is this the only reason for being tempted to choose a relevant logic for the formalisation of dialectical logic. No dialectical logic which is additive, i.e. $A \rightarrow A \vee B$ is a thesis, and closed under adjunction, i.e. when A and B are theses, so is $A \& B$ – and any dialectical logic with normal conjunction and disjunction connectives will meet these weak requirements – can satisfy that *bête noire* of relevant logic, disjunctive syllogism, $A \& (\sim A \vee B) \rightarrow B$ (or its equivalent, given weak and acceptable conditions, rule antilogism, i.e. when $A \& B \rightarrow C$ is a thesis so is $A \& \sim C \rightarrow \sim B$). For let A and $\sim A$ be the contradictory theses of a dialectical logic; then by the conditions, $\sim A \vee B$ and $A \& (\sim A \vee B)$ are theses, whence, by disjunctive syllogism, B is a thesis, for arbitrary B , i.e. the logic is trivial, contradicting conditions on a dialectical logic.⁵ But the dropping of disjunctive syllogism from the core first degree part of classical and modal logics (i.e. from the common part shared by those logics, the part which contains no nested occurrences of $\rightarrow$) leads exactly to the common first degree part of all relevant logics (the details are given in [3]). The abandon-

ment of disjunctive syllogism is indeed *the* characteristic feature of the *relevant* logic solution to the implicational paradoxes.

But while dialectical logics are bound to share important features with relevant logics, the classes only properly overlap, and there are in particular two types of systems – weakened negation systems, characteristically based on Hilbert's positive logic, and many-valued systems, typically functionally incomplete 3-valued logics⁶ – which meet the necessary condition imposed on dialectical logics but which are not even weakly relevant, i.e. they contain theses of the form $A \rightarrow B$ where A and B are wff without constants which share no variables.

A satisfactory sentential logic should include however, especially if philosophical applications are intended, a relevant implication (such at least is one of the main theses argued in [2] and [3]); and thus a satisfactory dialectical logic will be a relevant one. But the irrelevance of the proposed dialectical alternatives is not the only reason for serious dissatisfaction with the alternatives. The other reasons are, firstly, internal systemic reasons, for instance the weakened negation systems lack all forms of contraposition though surely some are correct, and indeed there is little basis for regarding the so-called negations of these systems as genuine negations at all rather than, say, positive modal connectives, e.g. weird truth or necessity connectives; and the many-valued systems all contain bizarre theses forced by the finite matrices. Secondly there are external reasons concerning extensions of the systems. For example, practically none of the alternatives can fulfill the most important business of furnishing a logical basis for naive dialectical set theory and for semantically closed natural languages, because the sentential logics admit principles which allow the derivation of Curry paradoxes and their like which trivialise these extensions (see [3] and, in the case of many-valued logics, [10]). Such at least is an outline of part of the case for choosing relevant dialectical logics over the alternatives; the rest of the case derives from features of the semantics for relevant dialectical logics, in particular the way in which the semantics encompasses inconsistent and incomplete worlds and takes the actual world T as inconsistent.

The reasons sketched for preferring a relevant approach to the formalisation of static dialectical logic by no means however exclude a combination of approaches. And it may be valuable, for instance in the logical explication of historically important dialectical positions, and in designing more philosophically satisfactory logics, to amalgamate the three approaches

discussed. Thus in order to do some logical justice to the dialectical positions according to which the negation of A is richer than A , a non-classical negation $\neg$ which satisfies the principle $\neg\neg A \rightarrow A$ but rejects $A \rightarrow \neg\neg A$ should be added to the core relevant logic adopted; and to take due account of the logics of significance (as discerned, e.g. in [9]) the same logic should be inflated to a 3-valued relevant logic. But these amalgamations are projects for other and more advanced logical times.

Even with the field for choice of satisfactory dialectical logics initially narrowed to relevant logics, there remains considerable scope for choice within the importive, or so-called “weaker”, relevant logics. None of the official relevant logics of Anderson and Belnap [2] is suitable as a foundation for dialectical logic. There are, in the first place, substantial reasons for dissatisfaction with the exportive relevant logics as furnishing analyses of the implication $\rightarrow$ ([3], chapter 2). But additional reasons come into play when these logics are considered as foundations for dialectical logics. Firstly, the stronger systems supply the suspect principles which enable Curry paradoxes to be proved when the abstraction scheme of naive set theory is adjoined ([3], [11]), and likewise in the parallel semantical case when a comparable satisfaction axiom is added to the logic. Secondly, exportive systems such as R are liable to exclude dialectical extensions. In particular, every pure variable inconsistent extension of R is trivial: more precisely, where system L is an extension of R with the same vocabulary, which is closed under the rules of modus ponens, adjunction, and substitution on sentential variables, then, if L is simply inconsistent, L is trivial (this theorem is proved in [2], p. 462). Though the result does not rule out the dialectical extension of R by constant inconsistent sentences, such as $p_0 \ \& \ \sim p_0$, it does exclude such positions as dialectical connexivism, obtained by adding Aristotle’s thesis $\sim(A \rightarrow \sim A)$ to a relevant logic. For them one obtains such pure variable contradictions as the pair $p \ \& \ \sim p \rightarrow \cdot p \vee \sim p$ and $\sim(p \ \& \ \sim p \rightarrow \cdot p \vee \sim p)$, which result in R , by the theorem, in triviality. This is due entirely to the unsatisfactory way in which R permits contraction and permutation of iterated implications, and can accordingly be avoided in importive relevant logics.

Avoidance of the undesirable features and principles of the official relevant logics places an upper bound on the sort of relevant logic to be sought as sentential base. Inclusion of the logical principles that figure in the informal arguments of standard Soviet philosophical textbooks

imposes some minimal requirements on the choice of a base logic, given the intended objective of formalising part of Soviet logic. Thus it can be assumed that conjunction and disjunction behave in classical fashion, e.g. the conjunction satisfies the principles of simplification, i.e. $A \& B \rightarrow A$ and $A \& B \rightarrow B$, without restriction, and of composition, i.e. where $A \rightarrow B$ and $A \rightarrow C$ then $A \rightarrow B \& C$. Indeed any dialectical logic aimed at capturing salient features of Soviet logic should, it seems, be some extension of the logic of distributive lattices (i.e. the logic DLL of [3], chapter 2). Because the logic should include the traditional principles of identity, $A \rightarrow A$, and excluded middle and also dialectical double negation, this narrows the class of base relevant logics still further, in fact to some dialectical extension of the Arruda-da Costa system $\underline{P}$ (of [22] and studied semantically in [23]). That is, $\underline{P}$, together with a representative contradictory thesis $p_0 \& \sim p_0$, is virtually a minimal dialectical logic. The contradiction $p_0 \& \sim p_0$, with p_0 some sentential constant, is added as an axiom simply to guarantee in a minimal trouble-free way that the logic is dialectical. The contradiction is intended to represent some real-life⁷ contradiction, e.g. the conclusion of a logical antinomy or of a Kantian antinomy with both p_0 and $\sim p_0$ proved by pure reason from premises conceded to be true, or the conclusion of a semantical antinomy or of one of Zeno's paradoxes with $\sim p_0$ (e.g. 'Achilles does not overtake the Tortoise') proved by pure reason and p_0 established by observation.

There are several grounds for not remaining satisfied for long with $\underline{P}$ as the base logic. The system does not include the remaining traditional law of thought, Aristotle's law of non-contradiction $\sim(A \& \sim A)$, which the orthodox Soviet position accepts, nor therefore the appropriate negation principles which enable the proof of non-contradiction from excluded middle. And really the negation principles of $\underline{P}$ are inadequate to characterise a *negation*, especially as no contraposition principles are admitted, and they do not appear to have other than an *ad hoc* semantics. There is reason then for going beyond $\underline{P}$ to the following minimal dialectical system DMO, which includes $\underline{P}$ and has the same positive axioms and rules as $\underline{P}$ but has a more satisfactory negation part:

AXIOMS: $A \rightarrow A$, $A \& B \rightarrow A$, $A \& B \rightarrow B$, $A \rightarrow A \vee B$, $B \rightarrow A \vee B$, $A \& (B \vee C) \rightarrow (A \& B) \vee (A \& C)$, $\sim A \& \sim B \rightarrow \sim(A \vee B)$, $\sim(A \& B) \rightarrow \sim A \vee \sim B$, $A \vee \sim A$, $\sim \sim A \rightarrow A$, $p_0 \& \sim p_0$;

RULES: $A, A \rightarrow B \Rightarrow B$; $A \rightarrow B, B \rightarrow C \Rightarrow A \rightarrow C$; $A, B \Rightarrow A \& B$; $A \rightarrow B, A \rightarrow C \Rightarrow A \rightarrow B \& C$; $A \rightarrow C, B \rightarrow C \Rightarrow A \vee B \rightarrow C$; $A \rightarrow B \Rightarrow \sim B \rightarrow \sim A$.

(Worlds semantics for DMO are included in [23]).

DMO is not the long awaited stopping point. There are several reasons for some dissatisfaction with DMO; in particular, it does not appear to include a full classical sublogic, i.e. all tautologies in connectives $\& \vee \sim$, though these principles are conceded on the orthodox Soviet position; it does not admit the rule of substitution of coimplicants though the rule appears desirable for a satisfactory implicational relation; because of the failure of intuitionistic double negation, it does not permit several apparently correct derivations in naive set theory and elsewhere (the effect of a non-classical negation like that of minimal logic on mathematics providing an interesting comparison); and finally it retains in rule form only principles such as composition which appear to be correct as theses.

One system previously arrived at in the search for a satisfactory dialectical logic – a system which meets the foregoing objections at least, though not the only or minimal system to do so or one chosen by immediate criteria in an entirely non-arbitrary way – is the following system DK (of [11]):

AXIOM SCHEMES: $A \rightarrow A$, $(A \rightarrow B) \& (B \rightarrow C) \rightarrow A \rightarrow C$, $A \& B \rightarrow A$, $A \& B \rightarrow B$, $(A \rightarrow B) \& (A \rightarrow C) \rightarrow A \rightarrow B \& C$, $A \& (B \vee C) \rightarrow (A \& B) \vee C$ with $A \vee B =_{\text{df}} \sim(\sim A \& \sim B)$, $\sim \sim A \rightarrow A$, $A \rightarrow \sim B \rightarrow B \rightarrow \sim A$, $A \vee \sim A$, $p_0 \& \sim p_0$;

RULES: $A, A \rightarrow B \Rightarrow B$; $A, B \rightarrow A \& B$; $A \rightarrow B, C \rightarrow D \Rightarrow B \rightarrow C \rightarrow A \rightarrow D$.

In order to use this dialectical base for the formulation and investigation of non-classical first-order theories such as dialectical set theory, an extension of the language to that of quantification logic (with an added symbol $\rightarrow$) is required along with postulates for quantifiers. There are comparatively few problems as to the language (and that of [9] is adapted), but there are problems concerning the choice of quantificational postulates, and the proper investigation of options in this area is only in its infancy.

The quantificational postulates for DKQ which follows are those arrived at by Anderson and Belnap in their investigations of entailment with quantification, and which result (independently) from a quantificational semantics for rigid, i.e. world invariant, individuals within a relevant semantical framework, together with one axiom designed to ensure a contradiction in things:

AXIOM SCHEMES: $(x)A \rightarrow \check{S}_x^t A$; $(x)(A \rightarrow B) \rightarrow A \rightarrow (x)B$, with x not free in A ; $(x)(A \vee B) \rightarrow A \vee (x)B$, with x not free in A ; $f_0(z_0) \& \sim f_0(z_0)$.

RULE: $A \Rightarrow (x)A$.

The particular quantifier P is defined in the usual way:

$(Px)A =_{df} \sim(x)\sim A$. Also $A \leftrightarrow B =_{df} (A \rightarrow B) \& (B \rightarrow A)$.

Much light is shed on DKQ by its semantical analysis in terms of worlds; for, as will appear, the semantics reveal far more clearly than the corresponding syntactical formulation what is going on philosophically. A DKQ *model structure* (*DQK m.s.*) $\mathbf{M}$ is a structure $\mathbf{M} = \langle T, O, K, <, *, D \rangle$, where K is a class of situations (or worlds), O is a subclass of K comprising the (theorem) regular situations, i.e. those where all theorems hold, T is an element of O , $<$ is a 3-place ordering relation on situations, i.e. on K , $*$ is a 1-place operation on K , and D is a non-null class of items (or individuals). Among the situations in K will be incomplete situations, i.e. situations where for some wff A neither A nor $\sim A$ holds, and situations which are inconsistent, i.e. where for some A both A and $\sim A$ hold. The operation $*$ is a reversal operation which takes a situation into its reverse, and hence incompleteness in a situation into inconsistency, and inconsistency into incompleteness, i.e. the reverse of a situation b where both A and $\sim A$ hold is a situation b^* where neither A nor $\sim A$ hold. It is thus evident that with respect to each wff A at each situation b there are 4 cases, which can be represented respectively by 4 values: A holds at b but $\sim A$ does not, value t ; both A and $\sim A$ hold at b , value i (for inconsistent or overcomplete); neither A nor $\sim A$ hold at b , value n (for neither, neuter, or incomplete); and A does not hold at b but $\sim A$ does, value f . This suggests, what turns out to be the case, that the logic can alternatively be developed as a 4-valued intensional one with value set $\{t, i, n, f\}$, but without operation $*$,

and that $*$ can be explained alternatively as a function which at each world maps $\{t, i, n, f\}$ to $\{t, n, i, f\}$.

The situations of m.s. $\mathbf{M}$ are structured by two relations, $<$, with $b <_a c$ reading ' b precedes c from the perspective of a ', and $\subseteq$, with $b \subseteq c$ reading ' b is included in c ', and defined: $(Px)(Ox \ \& \ b <_x c)$, i.e. b is a subsituation of c if b precedes c as seen from some regular situation.

A DKQ m.s. is constrained by the following conditions:

1. Basic conditions (cf. the basic affixing system B of [3]):

- p1. $a \subseteq a$
- p2. If $a \subseteq d$ and $b <_a c$ then $b <_d c$
- p3. $a = a^{**}$
- p4. if $b <_a c$ then $c^* <_a b^*$

2. Extra conditions:

- p5. (for conjunctive syllogism) If $b <_a c$ then, for some world x , $b <_a x$ and $x <_a c$; i.e. $A \rightarrow B \ \& \ B \rightarrow C \rightarrow \cdot A \rightarrow C$ (which is an optional extra) semantically imposes a density requirement on $<_a$.
- p6. (for rule counterexample) $x^* \subseteq x$, for $x \in O$.
- p7. (for contradiction) $x \neq x^*$, for $x \in O$.

The distinctively dialectical condition is p7 which requires, in virtue of $T \neq T^*$, that the actual world T , where truth is determined, is an inconsistent one. This is the semantical core of the dialectical position. All the remaining conditions are carried over from semantics for relevant logics.

A valuation v in a DKQ m.s. $\mathbf{M}$ is a function which assigns to each subject term t an element of D , to each n -place predicate parameter g^n at each world a of K an n -place relation on D^n , and to each sentence parameter p at each a or K one of $\{1, 0\}$, subject to these general constraints (for $a, b \in K$, etc.):

- (1) If $a \subseteq b$ and $v(p, a) = 1$ then $v(p, b) = 1$, and
- (1') if $a \subseteq b$ then $v(g^n, a) \subseteq v(g^n, b)$.
- (2) $v(p_0, x) = 1 \neq v(p_0, x^*)$, for $x \in O$, and
- (2') $z_0 \in v(f_0, x) \ \& \ z_0 \notin v(f_0, x^*)$, for $x \in O$.

According to (1) and (1') if a is a substitution of b and initial wff A holds in a then it holds in b , while according to (2) and (2') p_0 and $f_0(z_0)$ hold in actual situation T but not in its image T^* , and likewise for other regular situations. Apart from these intuitively warranted constraints, a valuation in a model is defined exactly as in the now familiar constant domains semantical analysis of normal quantified modal logics.

An *interpretation* I in $\mathbf{M}$ is associated with valuation v in the usual relevant logical way, as follows: I is a function from wff and elements of K to $\{1\}$ which satisfies the following conditions everywhere:

- i. $I(p, a) = v(p, a)$ for every sentential symbol p .
- ii. $I(g^n t_1, \dots, t_n, a) = 1$ iff $v(g_n, a)(v(t_1), \dots, v(t_n))$.
- iii. $I(A \& B, c) = 1$ iff $I(A, a) = 1 = I(B, a)$.
- iv. $I(A \rightarrow B, c) = 1$ iff for every $b, c \in K$, if $I(A, b) = 1$ and $b <_a c$ then $I(B, c) = 1$.
- v. $I(\sim A, a) = 1$ iff $I(A, a^*) \neq 1$.
- vi. $I((x)A, a) = 1$ iff $I'(A, a) = 1$ for every x -variant I' of I , where valuations v and v' are x -variants iff they agree on all assignments except possibly at x , and I and I' are x -variants iff v and v' are. As derived conditions one has:

$$I(A \vee B, c) = 1 \text{ iff } I(A, a) = 1 \text{ or } I(B, a) = 1;$$

$$I((Px) A, a) = 1 \text{ iff } I'(A, a) = 1 \text{ for some } x\text{-variant } I' \text{ of } I.$$

A *DKQ model* $\mathbf{M}$ is a structure $\mathbf{M} = \langle T, O, K, <, *, D, v \rangle$ where v is a valuation in $\langle T, O, K, <, *, D, \rangle$. Thus notions such as holding, truth, etc., which are now defined will extend automatically to models. A wff A *holds on* a valuation v , or on associated interpretation I , in DKQ m.s. *at* an element $a \in K$ just in case $I(A, a) = 1$; A is *true on* v , or on I , in $\mathbf{M}$ just in case $I(A, T) = 1$; and otherwise A is *falsified on* v , or on I , in $\mathbf{M}$. A is *valid in* $\mathbf{M}$ just in case A is true on all valuations therein; and otherwise *invalid in* $\mathbf{M}$. Finally A is *DKQ-valid* just in case A is valid in all DKQ m.s., i.e. A is true in every DKQ model; otherwise A is *DKQ-invalid*.

Since every theorem of DKQ is DKQ-valid but wff such as p and $g^n(x_1, \dots, x_n)$ are not, it follows that DKQ is non-trivial – in case anyone entertained doubts about the consistency of a logic containing real contradictions.

Using the semantics some of the leading and most controversial theses of dialectical logic may be derived.⁸ Firstly *dialectical logic*, *DKQ*, has a *classical truth definition*. This is important both because the lack of *any* truth definition in most dialectical writings has been a major point of criticism of dialectic, and also because on the Soviet view the main object of dialectical logic is the problem of truth. Among the models of DKQ defined, a *factual* model $\underline{M}$ is distinguished; as in [24], $\underline{M}$ is an arbitrarily selected model $\mathbf{M}$ such that, for every wff C , $\underline{I}(C, \underline{T}) = 1$ iff C . Then a statement A is (*classically*) *true* (in DKQ) iff $\underline{I}(A, \underline{T}) = 1$, and A is (*classically*) *false* (in DKQ) iff $\underline{I}(A, \underline{T}) = 0$, where $\underline{T}$ and $\underline{I}$ are components of factual model $\underline{M}$; and *truth* (in DKQ) is the property of all and only those statements which are true (in DKQ), i.e. $\lambda D (D \text{ is true})$. Most standard results on truth are forthcoming.⁹ Thus every theorem statement of DKQ is true but false statements are unprovable, no statement is both true and false but every statement is either true or false, and so on. Furthermore, however, *there are contradictory statements which are simultaneously true, indeed valid*. For both p_0 and $\sim p_0$ are valid: yet p_0 and $\sim p_0$ are contradictory, and demonstrably inconsistent. For as $A \rightarrow A$ is a theorem, so is $\sim(A \Diamond \sim A)$ where $\Diamond$ is Lewis's consistency relation, i.e. A and $\sim A$ are provably inconsistent. $p_0 \& \sim p_0$ is accordingly a real contradiction, and real contradictions hold in the actual case. This sets dialectical logic at odds with mainstream Western philosophy from the pre-Socratics onwards, since, following Parmenides, Plato, and Aristotle, it has been assumed that freedom from contradiction is a crucial determinant of reality, that CH holds. It is a virtual corollary of the dialectical fact, incorporated into DKQ, that reality is inconsistent, that freedom from contradiction does not provide an acceptable necessary condition for rationality or on rational belief or rational inquiry. The Kantian idea that consistency imposes a bound on rationality has, like Hilbert's idea that consistency ensures mathematical correctness and other Kantian evaluations that would put an end to dialectical investigations, to go.

Nevertheless, despite the correctness of contradictions, *Aristotle's principle of non-contradiction is correct, both in syntactical and semantical formulations*. For Aristotle's syntactical principle $\sim(A \& \sim A)$ is a theorem, hence valid, hence true (although various principles of traditional logic, which were commonly regarded as adequately supported by Aristotle's principle, are not valid). In its semantical formulation the principle asserts

that no statement is both true and false, that is, that it is never the case that $\underline{I}(A, \underline{T}) = 1$ and $\underline{I}(A, \underline{T}) = 0$, a point guaranteed by the bivalent features of the semantics. Thus DKQ vindicates the Soviet position according to which Aristotle 'correctly formulated the principle of contradiction, but erroneously inferred that the real could not be contradictory' and that 'there are no contradictions in things'. *There are genuine contradictions in things.* The item z_0 both has property $\lambda x f_0 x$ and also does not have it.

Classical quantificational logic is a special case of dialectical logic DKQ; more precisely DKQ has a theorem all theorems of classical quantificational logic Q, i.e. of predicate logic. The basic difference between the classical sublogic of DKQ, and Q as ordinarily formulated, lies in the rule γ of material detachment, i.e. $A, \sim A \vee B \Rightarrow B$ which is not an admissible rule of DKQ. Thus classical logic can be supplemented by, and, in an important sense we will come to, *completed* by dialectical logic – just as the orthodox Soviet position maintains.

The features of DKQ adduced are already enough to refute the conventional Western wisdom (which has many Soviet sympathisers) according to which so-called dialectical logics are not logically viable. It should be evident that the differences between the orthodox Western and Soviet positions cannot be satisfactorily represented as logic, or formal or classical logic, on the one side, and anti-logic or the rejection of formal logic on the other. Each position can furnish, so it has been argued, viable, equally formal, but competing logical theories; and the differences between these positions come down to metaphysical differences over such highly contentious matters as the consistency of the world. In recent times however classical logic – which is Western mainstream logic – has been strongly on the offensive in the ideological logical warfare between East and West and dialectical logic has been very much on the defensive. It would be pleasant to think that furnishing dialectical logic with an axiomatic base and viable semantics may do something to upset the ideological power structure and, at the same time, shatter the imperviousness of mainstream Western logic, thereby also assisting the cause of that new and minor branch of logic, relevant logic.

Dialectical logic, as formally and semantically based, opens the way both to novel theories and to the reassessment of older discarded, though often intuitive, theories. The effect of serious adoption of dialectical logic

is in fact pervasive, and influences not just most facets of philosophy and the foundations of science and mathematics, but also theoretical investigations in mathematics and physics, in linguistics, psychology and the other social sciences. This may appear to be an extravagant claim, so some illustrations at least are desirable. The general theoretical point behind all the applications of dialectical logic in theory reassessment is however simply made, namely that in any case where an apparently true theory is threatened by paradox or contradiction revision may take the form of dialectical formulation of the theory instead of the classical method of attempting to render the theory consistent by amputation, e.g. of apparent truths as not really true at all, of apparently truth-valued statements as not really truth-valued because indeterminate or not statements, and so on. And the point is similar in the case of novel applications, namely that a genuine attempt is made to take account of inconsistencies that are there, in the facts, in language, in apparently coherent belief sets or theories or philosophies or wherever, not by dismissing them but by incorporating them in the dialectical theory construction. Such dialectical methods have a profound effect on theory construction and adjustment, and on accounts of theory testing and theory change, and accordingly they have significant effects in the sciences and also, and consequently, in the philosophy of science.

Dialectical logic likewise has a major impact on argumentation and reasoning, and thereby on standards of rationality, an impact that reaches into every discipline but essentially into social sciences. It is not, or not just, that paradoxes may cease to be paradoxical and become simply contradictory parts of the theoretical or factual scenery; it is rather that arrival at a contradiction in reasoning or a theory will not put an automatic end to that line of reasoning or to that theory. For instance an intuitive theory such as naive set theory or a natural language like English will not be given up or, what comes to the same, subject to drastic mutilation, because contradictions, the logical and semantical paradoxes respectively, are encountered or tolerated.¹⁰ Thus too a theory will no longer be condemned as irrational because inconsistent, or a person as irrational or insane because he adheres to patently inconsistent beliefs; and avoidance of recognisably inconsistent beliefs will not any longer be a mark of sanity.

It would be well to anticipate the floodgate objection, that by removing

the inconsistency sanction – and thereby undercutting a wide range of philosophical positions, from main forms of modern pragmatism to modern Kantianism and formalism – the floodgates have been opened so that anything, any sort of nonsense or triviality, goes, since all standards of correctness have been abolished. This is false: sanctions, and standards of correctness, such as non-triviality, of course remain; and the floodgates objection is no better here than in most moral cases. It is worth considering why inconsistency has been thought to be so devastating and to put an end to a line of argument or a theory. The reason is *not* that, because of the paradoxes of implication, an inconsistent theory is classically trivial, but (as Tarski [16], p. 77 observes) because an inconsistent theory contains, on classical and constructive perceptions, false statements, since at least one of a contradictory pair A and $\sim A$ must be false. While dialectical logic undermines the view that one of a contradictory pair must be false, it need not – in the case of the logics argued for, does not – repudiate the basic test of correctness relied upon, that *a correct theory can contain only true statements*. In short, the essential test of correctness is, as always, truth, and the exclusion of non-paradoxical falsehood.¹¹ The way the test operates where inconsistency emerges is worth indicating, as it reveals how it is that the floodgates are not open. The absolutely naive theory of perception which takes all perception as veridical quickly runs into difficulties because of illusions: according to it a straight stick in water is both straight, because it feels straight and looks straight out of water, and also not straight, because it looks bent. But dialectically the theory can be rejected as incorrect not (just) because it is inconsistent, but because the statement that the stick is bent is not true – something that can be established in familiar ways by a combination of observation and reason. Thus, dialectically, there are two sorts of contradictions, admissible (welcome, or tolerable) contradictions which do not involve incorrectness, and inadmissible contradictions whose assertion entails non-paradoxical falsehood. It is evident that dialectical positions will differ as to whether contradictions are merely tolerable or actually welcome and as to which contradictions are admissible.

The logical antinomies, which furnish (so it is argued in [11]) a class – a vitally important class – of admissible contradictions, lead to the very obvious applications of dialectical logic with which the slightly more detailed illustrations begin:

Dialectical set, attribute and ensemble theories. These theories add comprehension (i.e. abstraction) axiom schemes to dialectical logic, but the schemes differ (though inessentially), those for sets and ensembles supplying elements of sets while that for attributes supplies instances or exemplifications. A much emphasized difference between sets and ensembles on the one side, and properties (i.e. 1-place attributes) on the other, lies in their identity conditions, which are governed in each case by corresponding axioms, so-called extensionality axioms in the case of ensembles.

To keep things simple – much simpler than they really are, since important categorical distinctions between individuals, sets, and attributes are wiped out – and to facilitate comparisons with more orthodox theories, the ensemble and attribute theories are formulated as orthodox first-order theories, ensemble theory DET adding the membership predicate $\in$ to DKQ (reformulated with no constants other than $\in$ and no predicate variables, and so with no initial contradictions), while attribute theory DAT adds the instantiation constant ι to reformulated DKQ. The leading feature of the dialectical theories proposed (they are argued for in [11]) is the adoption of the following intuitive unrestricted comprehension schemes:

$$(Pw)(x)(x \in w \leftrightarrow A) \quad (\text{set and ensemble abstraction}),$$

and

$$(Pu)(x_1, \dots, x_n)[(x_1, \dots, x_n) \iota u \leftrightarrow A] \quad (\text{attribute abstraction}),$$

i.e. using vector notation $(Pu)(\mathbf{x})(\mathbf{x} \iota u \leftrightarrow A)$. Usual antinomy arguments show that the theories are inconsistent. For example, by the impredicativity antinomy, $\mathbf{u} \iota u \leftrightarrow \sim(\mathbf{u} \iota u)$, and so $\mathbf{u} \iota u \ \& \ \sim(\mathbf{u} \iota u)$, upon taking A as $\sim(\mathbf{x} \iota x)$.¹²

The theories, which can of course be combined, diverge with identity. On by no means unquestionable orthodox perceptions, ensembles (and sets) are identical if they have, as a matter of fact, precisely the same members, whereas attributes are identical if all their instances are logically equivalent (cf. Carnap [25]). This leads to the definitions:

$$x \simeq y =_{\text{df}} (z)(z \in x \equiv z \in y),$$

and

$$x = y =_{\text{df}} (z)(z \iota x \leftrightarrow z \iota y).$$

More problematic are the substitutivity conditions these identities satisfy. It is evident that an extensional ensemble identity such as $x \simeq y$ should not support replacement in intensional frames, nor should it warrant an interconnection of entailment strength (consider the conditions under which the class of featherless bipeds can substitute for the class of humans). Accordingly the replacement postulate for ensemble identity should take the form:

$$\text{if } x \simeq y \text{ then } A(x) \equiv A(y),$$

where frame $A()$ contains only extensional operators, i.e. contains no occurrences of $\rightarrow$, whereas the replacement postulate for attribute identity will take the form:

$$\text{if } x = y \text{ then } A(x) \leftrightarrow A(y),$$

with $A()$ satisfying only usual variable freedom requirements. But the informal conditionals cannot be taken up by relevant implication, $\rightarrow$, without violating conditions of correctness in the shape of relevance requirements, for they lead to irrelevance (see [11], §7). They can however be taken up by rule connections, or better in one of the following two ways, which the property case, with $z \iota x =_{\text{df}} z \iota x$, will serve to illustrate:

$$\text{a) } \quad x = y \ \& \ t \rightarrow \cdot x \iota z \rightarrow y \iota z \quad (t\text{-ed property replacement}),$$

where t is the constant representing the conjunction of all theorems, and thus conforming to the semantical rule $I(t, a) = 1$ iff Oa , which satisfies the two way rule: $t \rightarrow A \Leftrightarrow A$; and

$$\text{b) } \quad x = y \ \& \ z = z \rightarrow \cdot x \iota z \rightarrow y \iota z \quad (\text{property replacement}).$$

Axiom b) is stronger than a): the axiom (suggested by Brady) corresponds to the minimal correct form of the principle of factor in relevant affixing logics, namely $p \leftrightarrow q \ \& \ r \leftrightarrow r \rightarrow \cdot p \ \& \ r \rightarrow q \ \& \ r$.

Dialectical attribute theory results then by adding attribute abstraction and attribute replacement principles to reformulated DKQ, and dialectical ensemble theory by similar addition of ensemble principles. So far we have carried along two theories where there is only a need for one – in a sense only one – theory; and it is dialectical attribute theory with its more

satisfactory replacement principles that is decidedly preferable for dialectical foundational investigations. Moreover property theory can readily supply a corresponding (intensional) set theory, in the following sort of way: first define property abstraction thus: $B(\lambda y A(y)) =_{\text{df}} (Pz) \cdot (x)(x \iota z \leftrightarrow A(x)) \& B(z)$, subject to appropriate scope conventions and variable rewriting. Then, for example,

$$\lambda y A(y) = z \leftrightarrow (x)(x \iota z \leftrightarrow A(x)) \& z = z,$$

and

$$x \iota \lambda y A(y) \leftrightarrow A(x).$$

Now introduce contextual definitions which enable the replacement of property talk in *some* contexts by set talk, in particular $x \xi \{y : A(y)\} =_{\text{df}} x \iota \lambda y A(y)$, $\{y : A(y)\} = \{z : B(z)\} =_{\text{df}} \lambda y A(y) = \lambda z B(z)$, etc. Extensional (contingent) identity between sets may be defined as for ensembles, namely $\{x : A(x)\} \simeq \{x : B(x)\} =_{\text{df}} (x)(A(x) \equiv B(x))$, contingent identity of sets differing from logical identity. Then $x \xi \{y : A(y)\} \leftrightarrow A(x)$, $x = y \& \{z : B(z)\} = \{z : B(z)\} \rightarrow x \xi \{z : B(z)\} \leftrightarrow y \xi \{z : B(z)\}$, and so forth. The objective is of course to allow the replacement of ι by ξ and $\lambda y A(y)$ by $\{y : A(y)\}$ so long as no such non-significance as $B(x \iota \{y : A(y)\})$ and $C(x \xi \lambda y D(y))$ or falsity as $\lambda y A(y) = \{y : A(y)\}$ is produced as truth.¹³ But the contextual definitions do not permit appropriate quantification over sets, and for this the simplest – *not* the only – remedy is to add the postulate: $B \Rightarrow S_{\xi} B$, i.e. the result of replacing ι uniformly by ξ in a thesis B in primitive notation is a thesis – so effectively duplicating the property postulates as set postulates.

Among the questions of adequacy that arise for the dialectical theories outlined, a most important initial one is: Are the theories non-trivial? The answer appears affirmative: both attribute theory (with or without set theory adjoined) and ensemble theory are non-trivial. Non-triviality does not however ensure correctness, so further questions of adequacy arise. One inexact question to which a satisfactory answer is not yet available is: Is DAT relevant, i.e. are there any theses of the form $A \rightarrow B$ where B is irrelevant to A ? Another more pressing question concerns the recovery of traditional mathematics: Is DAT adequate for informal arithmetic and analysis, for example? Again the question is not sharp and an answer is not known, though it is apparent (e.g. from [11]) that DAT is in some

respects a powerful theory, yielding forms of the axioms of choice and infinity *without* supplementary postulates. If arithmetic and analysis can be developed, as there is reason to expect, the reduction will be much more satisfactory than standard set-theoretical reductions, since numbers will appear on the theory as what they are, properties, and not as what they are not, sets (cf. [26]). It is likely to be argued however that, given a non-triviality proof for DAT (“within” DAT), Gödel’s consistency theorem affords an immediate negative answer to even the issue of adequacy for arithmetic. But, as [11] explains, Gödel’s consistency theorem has not been established for non-classical theories: nor is there any reason presented by limitative theorems why dialectical theories, since inconsistent, cannot be completed.

Dialectical mathematics and physics and foundations thereof. If a dialectical reduction of traditional mathematics should succeed, a version of logicism would be reinstated. For postulates like those of DAT can be derived from an untyped relevant predicate logic without use of additional non-logical postulates. Moreover the reduction, being dialectical, so far escapes limitative theorems. Accordingly many of the orthodox objections to logicism – the intrusion of non-logical axioms, limitations of logical formalism for intuitive mathematics, etc. – would be avoided. But the mere adoption of a dialectical solution of logical paradoxes does not determine how even a dialectical predicate logic will look, much less how a dialectical theory need to about meeting objections to logicism, if it should. In fact there is room within the confines of what has been sketched out for a variety of different dialectical “foundations”, and there is, as always, room to supplement logicism with other congenial theories, for example, as I should wish to do, with a Meinongian account of mathematical objects and their non-existence combined with a semantical (‘true in virtue of the meanings of constituent terms’) account of mathematical truth.

Almost as speculative in their current state of development as a dialectical foundation for mathematics are the dialectical projects envisaged in mathematics and physics, for example:

(1) A dialectical theory of the infinite. This would start from a formal reconstruction, controlled by dialectical logic, of Cantor’s original informal theory of transfinite sets. (A beginning on such an enterprise is made in [11].)

(2) A dialectical theory of infinitesimals, of Zeno's paradoxes, and a dialectical calculus. Such a theory – an alternative to Robinson's ingenious but artificial non-standard analysis – could begin from the original calculus, based on infinitesimals of Leibniz. The theory, again governed by a dialectical logic, would simply accept the contradictions that led to the abandonment of the intuitive theory of infinitesimals.

(3) Dialectical microphysics. Here too there is more than one line yet to be investigated in proper detail – for instance, the Everett-Wheeler interpretation of quantum mechanics in its initial dialectical form, and the suggestion (looked at in Petrov [21], and in [11]) that micro-objects just do have inconsistent properties, such as behaving in both wave-like and corpuscular fashion. Petrov astutely observes that in the case of the physical sciences, as well as in mathematics and logic, the exclusion of antinomies can manifest itself in incompleteness of theories or in indeterminacy or undecidability – because the consistencizing always amputates truths along with apparent falsehoods – and that such may be the case with the alleged incompleteness of quantum theory as with the celebrated incompleteness of the formalisation of mathematical theories or, more generally, of knowledge.

Dialectical linguistics and the formalisation of natural languages. A major obstacle standing in the way of non-trivial formalisation of natural languages has been Lesniewski-Tarski critique of universal languages: there has seemed no getting around the fact that natural languages appear to be universal (i.e. semantically closed) and accordingly, given admissible logical principles, permit the derivation of semantical antinomies. Essentially the same arguments, based on transcription of semantical paradoxes, that showed the incompleteness of rich formalised theories, showed, so it also seemed, the indefinability of semantical notions such as truth for these theories and the irremediable inconsistency of universal languages. Once again the arguments reckoned (among other things) without non-classical formalisation, which reveals how the main obstacles to formalisation are at root classical logical assumptions. There are two related non-classical approaches which enable the design of non-trivial universal systems, the way of incompleteness (cf. [27], [18]), and the dialectical way of inconsistency.¹⁴ As before, the argument is that the dialectical way is to be preferred, since it fits the data much better ([11], §§5–6). Some of the further

consequences of the relevant dialectical way for linguistics – e.g. on the deep logical structure for natural languages, and for language learning of rich languages – will begin to emerge from the dialectical critique and reshaping of orthodox metamathematics.

The dialectical critique of orthodox metamathematics can profitably begin with a re-examination of the now classic arguments for the metalogical edifice, and there is perhaps nowhere better to begin than with the lucid arguments of Tarski (in [14] and [16]) which support not just the semantical theory of truth but much else in metamathematics (e.g. much that is encompassed in Kleene [17]). As Tarski emphasizes ([16], p. 59)

antinomies have played a pre-eminent role in establishing the foundations of modern deductive sciences. And just as class-theoretical antinomies . . . were the starting point for the successful attempts at a consistent formalisation of logic and mathematics¹⁵, so the antinomy of the liar and other semantic antinomies give rise to the construction of theoretical semantics.

The semantical antinomies have occasioned the inevitable classical rejection of semantically closed languages, the division of object language and metalanguage and the consequent hierarchy of languages, the famous limitative theorems, and soon – in short, a substantial part of the classical gospel. The dialectical acceptance of the semantical antinomies changes all this. Tarski's argument from the semantical antinomies to the rejection of some of the assumptions involved and so to the rejection of semantically closed languages does not even reach first base. Tarski's move from the Liar antinomy is

that we have been compelled to assert a false sentence (since . . . an equivalence between two contradictory sentences is necessarily false). If we take our work seriously we cannot be reconciled with this fact ([16], p. 58).

But this is to read in classical logical assumptions at the bottom. For dialectically the antinomies do not compel the assertion of classical falsehoods. On the dialectical account the antinomy arguments lead to conclusions of the form s (or ' s ' is true) and $\sim s$ (or ' s ' is not true), and so to *true* statements, not just false statements, of the form: s is materially equivalent to $\sim s$, since $s \ \& \ \sim s \rightarrow \cdot s \equiv \sim s$. And dialecticians who are quite reconciled with such contradictory facts, can take their work as seriously as anyone else. Tarski however makes zero allowance for dialectical positions, though they were hardly unknown when we wrote;

through classical neglect of dialectical alternatives he is led to the following statement of faith ([16], p. 77):

I believe everybody agrees that one of the reasons which may compel us to reject an empirical theory is the proof of its inconsistency: a theory becomes untenable if we succeeded in deriving from it two contradictory sentences. . . . It seems to me that the real reason of our attitude is . . . : We know (if only intuitively) that an inconsistent theory must contain false sentences.

Thus classical consistency assumptions are embedded in the very foundations of orthodox semantics, though the assumptions are eminently rejectable. For it is not difficult to envisage inconsistent empirical theories which are not rejected: indeed quantum theory, under certain formulations, may well be such a theory. Nor do we know intuitively, without important classical assumptions, that an inconsistent theory must contain false sentences: indeed a *correct* theory of the semantical antinomies will, on dialectical perceptions, be inconsistent.

Thus of the main directions open in the face of the antinomies, orthodox semantics has taken the wrong direction. It has, in fact, discerned only two of the three directions open, excluding the dialectical option – of accepting *all* the principles involved in the antinomic derivations – as not a logically possible case. The options discerned by the orthodox semantics are determined by the assumptions it takes to be necessary and sufficient ([14], p. 165, [16], p. 54) for a language to admit the semantical antinomies, namely:

(I) The language is *semantically closed* (or, more sweepingly as in [14], *universal*), that is (a) the language contains in addition to its expressions the names of these expressions, (b) the language contains some semantic expressions, e.g. such expressions as ‘true sentence’, ‘name’, or ‘denote’, and (c) all sentences which determine the adequate usage of the semantic expressions, i.e. in the case of ‘true’ all instances of convention *T*, can be asserted in the language; and

(II) The ordinary laws of logic hold in the language. The orthodox approach ‘considers only the possibility of rejecting the assumption (I)’ and accordingly ‘decides not to use any language which is semantically closed’.¹⁶ Mostly the option of rejecting (II) – which is the option most philosophers have preferred, and which under a broad construal can include the dialectical option – is not even considered (cf. [14], [15]), and when it is it tends to be dismissed in a cavalier fashion, e.g. as follows:

It would be superfluous to stress here the consequences of rejecting assumption (II), that is, of changing our logic (supposing this were possible) even in its more elementary and fundamental parts ([16], p. 59).

Logic is not something God-given: it is possible to change elementary logic, in a great many ways (cf. [3], chapter 1), and, in particular, to choose a different logic, such as a dialectical logic, for foundational investigations, e.g. in semantics. The consequences of such a different choice, which rejects assumption (II) but not assumption (I), is precisely our concern here. The first outcome is that we do *not* (contrary to [16], p. 60)

have to use two different languages in discussing the problem of the definition of truth and, more generally, any problems in the field of semantics,

since the orthodox case for these limitations depends on the rejection of semantically closed languages, such as natural languages presumably are. Of course we can retain the distinction between object-language and metalanguage, basic to metamathematics; but we are free to characterize 'true in English', for example, within English, something we can readily do – at least to an extent which meets the orthodox requirement of adequacy encapsulated in convention *T* – by the scheme

«*p*» is true in English if, and only if, *p*

where the Spanish quotation marks represent the quotation function of English (details of this function are set out in [18], where too use of the function is defended against Tarski's objections of [14], pp. 161–3). Truth may then be defined by property abstraction, that is, truth (in English) is the property of *p* such that «*p*» is true in English, i.e. the property of *p* such that *p*.

Since a "formally correct" definition of the predicate 'is true' can be constructed along these lines for very rich languages such as English, doubt immediately gets transferred to Tarski's claim that the problem of the definition of truth sometimes has a negative solution, namely when the metalanguage is not essentially richer than the object-language ([16], pp. 61–2). A second, most important, outcome is that Tarski's case for a negative solution fails. Let us suppose that

an interpretation of the metalanguage in the object-language is possible. . . . As a result of this interpretation, the hypothesis that a satisfactory definition of truth has been formulated in the metalanguage turns out to imply the possibility of reconstructing in that language the antinomy of the liar; and this in turn forces us to reject the hypothesis in question ([16], p. 62; two progressively fuller sketches of the argument are offered in [14], p. 247 ff.)

But, dialectically, the coding into a language of a semantical antinomy forces no such rejection of a truth definition. Only if further crucial assumptions are added, namely that the class of all provable sentences of the metalogic is consistent ([14], p. 247, p. 251) *and* that the metalogic is classical, does Tarski's negative solution result, 'that it is impossible to construct an adequate definition of truth in the sense of convention *T* on the basis of the metatheory'. The crucial assumptions are however dialectically unacceptable, so the negative conclusion never ensues.

There is a singularly unfortunate tendency for expositors of limitative theorems to omit the requisite qualifications under which the theorems hold – not that those qualifications have ever been worked out with due care (see [20]) – a fault from which authors of the theorems such as Tarski and Gödel are not immune. Thus, for example, Tarski claims that his fundamental negative result (Theorem I)

shows that it is impossible to define in the metatheory a class of sentences of the language studied which consists exclusively of materially true sentences and is at the same time complete ([14], p. 254).

But it shows much less. For the crucial assumptions already mentioned have been left off, along with the essential, but dialectically rectifiable, assumption that any set of inconsistent statements must include a false statement.

If Tarski's limitative theorems concerning the definability of semantical notions such as truth fail to establish results of the generality that has commonly been accorded to them, and do not stand up without essential but dialectically unacceptable assumptions, how, it is likely to be asked, do Gödel's limitative theorems, and all the results that bear on arithmetic and its extensions, fare? Since all these results are permitted on consistency assumptions, it is evident that, correct or not, they can be dialectically escaped. This elementary observation is virtually enough to dispose of a great many exaggerated claims made on the strength of limitative theorems – claims encouraged by Gödel's informal statements, e.g. [19] p. 41 and elsewhere. Thus Meltzer in his preface to [19] asserts:

Kurt Gödel's astonishing discovery and proof... that even in elementary parts of arithmetic there exist propositions which cannot be proved within the system, is one of the most important contributions to logic since Aristotle. Any formal logical system which disposes of sufficient means to compass the addition and multiplication of positive integers and zero is subject to this limitation, so that one must consider this kind of incompleteness an inherent characteristic of formal mathematics as a whole.

Similarly Braithwaite in introducing [19] announces:

Every system of arithmetic contains arithmetical propositions, by which is meant propositions concerned solely with relations between whole numbers, which can be neither proved nor disproved within the system.

These claims are not only false, they are decidedly misleading, since they fail to take due account of essential qualifications on Gödel's first, undecidability, theorem. Even Rosser's extension of Gödel's result, applies at best to *consistent* arithmetics and Gödel's results only to ω -consistent ones. None of the results show that a dialectical arithmetic has to contain undecidable statements, *when* such a system is negation inconsistent. But if such an arithmetic is rich enough to admit the transcription of variations of the semantical antinomies its inconsistency is exactly what should be expected. Consider 'the main lines of the proof' Gödel has set out ([19], p. 38–41), and let ' G ' abbreviate Gödel's $[R(q); q]$. What is shown to establish the undecidability of G is simply:

G is provable if, and only if, G is not provable,

from which it does follow, in the dialectical systems presented, that $\vdash G$ (i.e. G is provable) and $\vdash \sim G$, and so $\vdash G \ \& \ \sim G$. But it does *not* follow, unless the metalogic is excessively classical, that $\sim \vdash G$. Moreover suppose, on the contrary, that G were not provable, and allow that dialectical arithmetic is – like everyday arithmetic – syntactically closed, so that it includes at least assertions as to its own provability; that is to say, dialectical arithmetic includes the symbol $\vdash$, subject to appropriate axioms such as those of modal system $S4^0Q$ and a sufficient part of the logic of arithmetisation. By the construction of G

$\sim \vdash G$ if, and only if, G ,

i.e. G "asserts" its own unprovability. Since, by hypothesis, $\sim \vdash G$, so G , i.e. G is true. Since G has thus been proved true, $\vdash G$ (using the $S4$ rule). Hence, in any case, $\vdash G$, and so also $\vdash \sim G$. Thus if we do not exclude syntactically closed arithmetics the dialectical inconsistency conclusion – not the classical undecidability conclusion – is inevitable. It might be objected that the argument from G 's truth to G 's provability is a little swift. Ask: how is G 's truth established? The answer is: by a metamathematical *proof* of G . A little more specifically, G is a (metamathematical description of (a) wff of the form $(n)F(n)$. By Gödel's argument, for every numeral n , $\vdash F(n)$, i.e. $(n) \vdash F(n)$. But then, by the Barcan formula of $S4^0Q$

(effectively an ω -rule), $\vdash (n)F(n)$, and so $\vdash G$. (The orthodox metamathematical argument takes a semantical detour at this stage, appealing to truth in the standard model for arithmetic to get from the truth of $F(n)$ for each n to the truth of $(n)F(n)$.) But if arithmetic is suitably closed such a proof will be a proof within arithmetic, and so a proof of G . The escape from paradox of the classical limitative theorems is achieved essentially by the rejection of closed languages. It thus becomes evident also that *one* important strategy for obtaining a formal arithmetic which is not subject to Gödel incompleteness is to syntactically (and semantically) close the arithmetic: the resulting system will normally be a dialectical one. Such a closed formalisation would not be something remote and arithmetically unreal; for, as remarked, everyday arithmetic as presented within a natural language like English appears, unlike say first-order Peano arithmetic, appropriately closed: 'is provable in arithmetic' and 'is arithmetically true' are English, and in a good sense arithmetical, predicates.

But has not Tarski established that 'true in arithmetic' is not definable in arithmetic and Kleene ([17], p. 298 ff) that there is no correct formal system for certain intuitive arithmetical predicates and hence no complete formalisation of informal arithmetic? The answer is: not with the generality that has commonly been accorded these results, but only under classical (or in Kleene's case, intuitionistic) and other assumptions. For example Tarski's theorem for arithmetic depends on assuming that arithmetic truth is truth in the standard model where the standard model for arithmetic is taken to be simply consistent (cf. [28], proof of corollary 3.38). But if arithmetic really is dialectical – or for that matter should be formalised within a suitable non-classical logic – then Tarski's argument fails to prevent representation, to exclude determination of a one-place predicate T such that every instance of $T(x) \leftrightarrow A$ holds, where A is a sentence of arithmetic and x its numerical code-name. For all the standard diagonal paradox argument shows, in locating a sentence $\sim T(k)$ whose code-name is k , is $T(k) \leftrightarrow \sim T(k)$ – something which suitable non-classical (non-intuitionistic) logics can perfectly well admit: it shows only that $T(k)$ is either indeterminate or inconsistent. Similarly Kleene's generalised Gödel theorem ([17], pp. 302–3) depends on excluding indeterminate and inconsistent assignments as represented by theses of the form $T(k) \leftrightarrow \sim T(k)$; for proof of a crucial step (of [12], Theorem V, pp. 282–3) relies upon the principle $\sim (A \leftrightarrow \sim A)$ – more precisely, on the rule $A \rightarrow \sim B \Rightarrow \sim (A \leftrightarrow B)$,

not a principle of DK – which in the assumed framework of intuitionistic logic with its spread law, $\sim C \rightarrow \cdot C \rightarrow D$, trivialises any system containing theses of the form $A \leftrightarrow \sim A$, that is containing paradoxes.

The dialectical criticism made of the theorems of Tarski, Gödel, and Kleene only initiates the important task of reappraisal of a whole series of similarly established limitative theorems and fundamental results in orthodox metamathematics (e.g. theorems of Kleene concerning non-recursive enumerability). In this way the dialectical critique reopens many fundamental questions, not merely in metamathematics and the foundations of mathematics, but elsewhere, for instance linguistics. It reopens such questions as: Can a complete (dialectical) axiomatisation of arithmetic be obtained, and if so what would it look like (apart from being a closed language), and how would its completeness be established? Are classically undecidable theories such as dialectical quantification theory really decidable? Are the truths of mathematics recursively enumerable? Priest conjectures (in [8]) that the set of true sentences of arithmetic is recursively enumerable, and suggests, furthermore, that the class of truths of mathematics is recursively enumerable. With this answer in hand the straightforward answer can be given, as Priest, goes on to say, to the question as to how one generation of mathematicians learns what counts as true from the previous generation, namely they learn certain basic mathematical truths and how to prove others by making deductions. Similar answers – that certain sets classically considered to be non-recursive or not recursively enumerable really are, in the larger dialectical scheme of things with closed languages, recursive or recursively enumerable – are important for the program of formalisation of natural languages and for straightforward solutions to central problems of language learning.

To proceed with these ambitious tasks will require, in the first place, substantial extensions of dialectical logics. Formalisation of what used to be (called) the semantical metalanguage will, for one thing, be needed. This can take two directions, both meriting investigation. One is a direct untyped protothetic extension of DK by expected analogues of the attribute theory postulates such as

$$(Pf)(q) \cdot f(q) \leftrightarrow A(q) \quad (\text{protothetic comprehension}),$$

and supplementary postulates of a logistic cast for specific constants, e.g. $\langle p \rangle Np$ (where N is the naming or aboutness relation), ANp & $ANq \rightarrow \cdot$

$\sim(p \ \& \ \sim q)$, etc. (cf. Davis [29] and also [9]). The other is a reduction of such direct talk by coding to, e.g. arithmetical talk, and extension of DKQ with arithmetic by way of satisfaction postulates such as

$$x \text{ satisfies } N_A(\mathbf{y}) \leftrightarrow A(x, \mathbf{y})$$

(where $N_A(\mathbf{y})$ is the code-name of the formula $A(x, \mathbf{y})$ with one free variable x and numerical parameters $\mathbf{y}$), and again supplementary logical postulates. There is a good prospect that the non-triviality results for attribute theory DAT will adapt to these semantical extensions of DK by abstraction-style postulates. But regarding the adequacy of these theories little else is known, except that, in being inconsistent, they reflect the semantical facts.

NOTES

* I am much indebted, in ways that may not be obvious from [8], to Graham Priest. I also owe much to my constant but always cordial critic, Newton da Costa. And the paper has been much improved through the suggestions and criticism of E. M. Barth, C. Mortensen, and J. K. Slaney.

¹ Whereas in [1] adoption of the relevant position was argued for as the rational course of action, in [3] and [11] the dialectical position is defended. The case for the shift from the agnosticism of 1973–1974 to atheism is explained in [3]. [1] and [11] are desirable background reading for the present paper.

² It is assumed that the semantical versions considered appeal only to purely logical premisses. The unintended lessons that some leading representatives of the classical faith show to be taught by empirical forms of the semantic antinomies are worth recording. Tarski, for instance, remarks [14], p. 158, concerning the one substitution that yields Łukasiewicz's version of the liar antinomy:

...no rational ground can be given why such substitutions should be forbidden in principle,

which suggests, in its context, that it is not really rational to reject the antinomy argument. And Quine concludes [15], p. 133:

Then Epimenides' statement becomes false if true, and true if false – an impossible situation.

Exactly so.

³ Priest indeed (in [8]) advances, as a major reason for accepting the logical antinomies, the ability to make this return:

...there are theoretical reasons why the paradoxes must be accepted; must, that is, if we are to get to grips with the notion of mathematical provability (in the naive sense).

But I prefer to see this as a very appealing outcome of the dialectical acceptance of the antinomies, which is made for quite different reasons: see [11], §5, 6.

⁴ T. S. Eliot, *Little Gidding*, in [12].

⁵ Note that a rule form of *reductio* at least is a thesis of the relevant dialectical logics considered; the rule form is tantamount to the principle of excluded middle, $A \vee \sim A$.

⁶ The weakened negation systems have been studied, under such headings as *discussive* logics, *contradictory* logics, *inconsistent* logics and *paraconsistent* logics, by Jaśkowski (e.g. [4]), by da Costa (e.g. [5]), Arruda and many coinvestigators in Latin America and some in Poland, and by Apostel ([6]; Apostel's sentential logic KP just is da Costa's system C_ω). These systems are distinguished from minimal logics – which are likewise based on positive logic and lack the negative paradoxes of implication – by the dialectical double negation principle $\sim \sim A \rightarrow A$ (whereas of course minimal logic lacks this principle and has, what discussive logics lack, the intuitionistic form $A \rightarrow \sim \sim A$) and (though inessentially) by the principle of excluded middle.

The three valued systems have been studied, under such titles as logics of *antinomies* and of *paradoxes*, independently by Asenjo (see [7]), Priest [8], Goddard and Routley [9], and others. The motivation for these systems is primarily semantical – in contrast to the weakened negation systems where the motivation is primarily syntactical and the semantics are somewhat *ad hoc*. The idea is that as well as statements that are true whose negations are not true and statements that are not true whose negations are true there are statements that are paradoxical in that both they and their negations are true (or, making the metalogic non-classical also, are both true and false). This semantical motivation is also part of the case for relevant dialectical logics some of which allow however not just for inconsistency or paradoxicalness but for incompleteness, i.e. for statements such that neither they nor their negations hold.

The three valued systems are not really competitors with relevant systems; for in an important sense the relevant systems already contain the three-valued systems (see [3], 4.3).

⁷ To say 'real life' in this sense is *not* to say that famous arguments for the antinomies are valid: the fallaciousness of Kant's arguments for his antinomies, for example, has deservedly obtained modern notoriety.

⁸ Some of these theses are presented in a rather summary form, since a much fuller development, at the sentential level however, may be found in [1].

⁹ But not all. In particular it is not always the case that $\sim A$ is classically true iff A is classically false. Thus the Tarski *T*-conventions according to which A is classically true iff A and A is classically false iff $\sim A$ break down. Classical truth of the classical semantical metalanguage diverges from systemic (dialectical) truth as explicated through the *T*-conventions: it is such a divergence that renders the classical meta-theoretic translation of dialectical logic possible.

¹⁰ The very substantial bearing of dialectical acceptance of the semantical paradox arguments on orthodox metamathematics will be explained in detail in the next section.

¹¹ The qualification to *non-paradoxical* falsehood is however different, and important. When truth and falsity connectives and semantical paradoxes are incorporated *within* DLQ, the logic will contain some statements (e.g. those that generate the paradoxes) which are, systemically, both true and false. However, systemic falsity has to be carefully distinguished from metalogical falsity as introduced in the classical semantics. The exclusion of all classical falsehood, as seen from the metalanguage, can be retained as a test of correctness. All this helps reveal how the classical translation of

dialectical logic is obtained: it succeeds by severing some connections between systemic and metalogical notions.

In the end of course the classical metalogical crutch here adopted should be thrown away. The modern object language/metalanguage dichotomy grew, in part, out of the classical treatment of the semantical paradoxes; and accordingly to this extent it is dialectically otiose. Ultimately dialectical logics and theories, which are set up within ordinary language (taken as U language), should contain their own semantics.

¹² While the idea of a many-valued set theory appears to have originated in Russia and Poland with the work of Bochvar and others, the very different idea of an anti-nomic set theory, though having antecedents in Russia and Poland, seems to have developed in South America from the work of Asenjo and da Costa.

¹³ A more adequate theory, such as a significant attribute theory, would enable the assertion that $x \notin \lambda yA(y)$ is non-significant.

¹⁴ But semantically the ways are linked on the relevant picture. For when $x \neq x^*$ one of the situations x and x^* is inconsistent and the other is incomplete. Correlatively when $x = x^*$ both x and x^* are consistent and complete, i.e. the model structure also allows for classical consistent-and-complete situations (possible worlds).

¹⁵ This is an exaggeration, as well as historically inaccurate. The work of Frege, Dedekind, Peano, and Russell was well advanced *before* the set-theoretical paradoxes upset classical developments. And, in view of results in metamathematics, there can be no certainty, classically, that a sufficient rich logical theory is consistent, and so no adequate basis for Tarski's claim as to successful attempts.

¹⁶ The orthodox approach in fact imposes on the languages it is prepared to consider much stronger requirements than are necessary: it would suffice to reject in part one of (a) or (b) or (c).

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