MATHEMATICAL LOGIC IN LATIN AMERICA
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PROBLEMS AND SOLUTIONS IN THE SEMANTICS OF QUANTIFIED RELEVANT LOGICS. I.

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ABSTRACT. The main problem investigated is the adequacy of constant domain relational world semantics for quantified relevant logics. The problem is solved, though in a disagreeably circuitous way, for many weaker relevant logics, and an outline of how the solution may be extended to stronger logics such as RQ is given.
Alternative necessity and intensional-conjunction style rules for the evaluation of quantifiers are studied and shown to simply force the main problems above with the usual (extensional conjunction style) quantifier-rule to reappear, unmitigated, at alternative outlets. Finally some philosophical problems allegedly engendered by constant domain world semantics are examined briefly: it is argued that the "problems" are no problems.

In the 'Semantics of entailment I' (Routley and Meyer 1973, p. 238) Routley and Meyer say, after presenting putative constant domain worlds semantics for quantified relevant logic RQ and sketching the semantic soundness argument, that they presume semantic completeness is no problem either. Well, it hasn't exactly worked out that way, and more than 6 years later they are still grappling, off and on, with the problems that have emerged. Naturally the problems exceed solutions. Still there has been some progress. The region of the problematic in quantified relevant semantics is being gradually reduced, though much remains to be accomplished. What one hopes - this is one of the aims of this exposition of where things stand - is that some logicians will come forth and solve some, even better all, of the remaining main problems. The problems are presented for the most part as we come to them, mostly towards the end of the paper. Before that there is much exposition, setting out the framework of the problems, and including a progress report which presents results, positive and negative (i.e. failures), so far achieved-partly for typical display purposes, and partly with the aim of helping problem-solvers on their way. Most of the paper is technical, and only at the end of the final section are philosophical problems the semantics supposedly cause attacked.

§ 1. SYNTAX OF THE SYSTEMS; AND QUANTIFIED RELEVANT AFFIXING LOGICS AND THEIR SEMANTICS.

The common (and standard) syntax of each quantificational logic LQ investigated, is as follows:

Subject variables : $x, y, z, x_1, \ldots$
Subject constants : $\alpha, \beta, \gamma, \alpha_1, \ldots$
Predicate parameters of n-places ($n \geqslant 1$): $f^n, g^n, h^n, f_1{}^n, \ldots$
Sentential parameters (0-place predicate parameters): $p, q, r, p_1, \ldots$
Terms and well-formed formulae (wff) are defined recursively as follows (the basic primitive improper symbols being exhibited in the course of the definition):

(i) A subject variable or constant is a term.

(ii) If $t_1, \ldots, t_n$ are terms and f^n is an n-place predicate symbol, $f^n(t_1, \ldots, t_n)$ is an initial (or atomic) formula.

(iii) If x is an apparent variable and A and B are formulas, $(Ux)A$, $(A \& B)$, $\sim A$, and $(A \rightarrow B)$ are formulas.

(iv) Nothing is a term or a formula except in accordance with (i) - (iii).

An occurrence of a term t in a formula A is a *free* occurrence of t in A if it is not part of a subformula of A of the form $(Ut)B$, and a term t *occurs free in* a formula A if there is a free occurrence of t in A. $A(t/u)$ (also written $\check{S}^{u}_{t} A|$) represents the result of substituting the term t at each free occurrence of the term u in the formula A, provided that there are no free occurrences of u in A in a formula context $(Ut)B$; otherwise $A(t/u)$ is A. For the most part, the notation and substitution and other conventions adopted are familiar ones (those of Church 1956 and Anderson and Belnap 1975).

The central LQ systems have as primitive syncategorematic symbols parentheses and the connectives and quantifier: $\&$, $\sim$, $\rightarrow$, U. But classical LQ systems, the CLQ systems, replace relevant negation $\sim$ by classical negation $\neg$; and some extensions of these systems will include yet other connectives. Further connectives and quantifiers are introduced by definition, in particular in central LQ systems:

$$A \vee B =_{Df} \sim(\sim A \& \sim B); \quad A \supset B =_{Df} \sim(A \& \sim B); \quad A \leftrightarrow B =_{Df} (A \rightarrow B) \& (B \rightarrow A); \quad (x)A =_{Df} (Ux)A; \; (Px)A =_{Df} \sim(x)\sim A .$$

Where $x_1, \ldots, x_n$ are all the free variables of the formula A, $(x_1) \ldots (x_n) A$ is a *(universal) closure* of A. A *sentence* of LQ is a formula in which no variables occur free.

The systems that are central in what follows - they are by no means the only important quantified relevant logics - are the affixing systems, the constant domain quantificational extensions of the affixing systems of Routley and Meyer 1978. The basic affixing system BQ has as postulates these schemes:

A1. $A \rightarrow A$. A2. $A \& B \rightarrow A$.
A3. $A \& B \rightarrow B$. A4. $(A \rightarrow B) \& (A \rightarrow C) \rightarrow . A \rightarrow (B \& C)$.
A7. $(A \rightarrow C) \& (B \rightarrow C) \rightarrow . (A \vee B) \rightarrow C$.
A8. $A \& (B \vee C) \rightarrow (A \& B) \vee C$.
A9. $\sim\sim A \rightarrow A$.
QA1. $(x)A \rightarrow A(t/x)$, where t is a term.

QA2. $(x)(A \to B) \to . A \to (x)B$, where x is not free in A .

QA3. $(x)(A \vee B) \to . A \vee (x)B$, where x is not free in A .

QA5. $(x)(A \to B) \to . (Px)A \to B$, where x is not free in B .

R1. $A, A \to B \Rightarrow B$ (Modus Ponens).

R2. $A, B \Rightarrow A \& B$ (Adjunction).

R3. $A \to B, C \to D \Rightarrow B \to C \to . A \to D$ (Affixing).

R4. $A \to \sim B \Rightarrow B \to \sim A$ (Rule Contraposition).

QR1. $A \Rightarrow (x) A$ (Generalization).

Here $A_1, \ldots, A_n \Rightarrow B$ abbreviates: where $A_1, \ldots, A_n$ are theorems so is B . BQ may be reaxiomatised, along the lines of the axiomatisation of RQ in Meyer, Dunn and Leblanc 1974, to eliminate QR1 as a primitive rule: the procedure is to take the closure of all axioms as axioms and to compensate for theorems lost thereby adding new axioms, such as $(x)A \& (x)B \to (x)(A \& B)$. But as a primitive rule of BQ, QR1 is no worse than R3, for example, which likewise is "abnormal" in having no direct theorem scheme analogue. Nor need it have: it is essentially a device for generating theorems from theorems, not itself a theorem analogue.

Extensions of BQ which have full contraposition

D4. $A \to \sim B \to . B \to \sim A$

as a theorem scheme can eschew axiom-schemes A7 and QA5 which then follow respectively from A4 and QA2.

Additional axiom schemes and rules drawn from the list in Routley and Meyer 1978 may be added to BQ singly or in combination to yield a wealth of stronger systems with the same (constant domain) quantificational structure. Here a fairly short list of optional extras will keep us quite busy enough. The list will however be at least large enough to include such systems as RQ, EQ, TQ and S4Q.

B1. $A \& (A \to B) \to B$.

B3. $A \to B \to . B \to C \to . A \to C$.

B4. $A \to B \to . C \to A \to . C \to B$.

B5. $A \to (A \to B) \to . A \to B$.

B6. $A \to . (A \to B) \to B$.

B10. $A \to . B \to B$.

D1. $A \vee \sim A$.

D3. $(A \to \sim A) \to \sim A$.

D4. $A \to \sim B \to . B \to \sim A$.

D5. $A \& (\sim A \vee B) \to B$.

BR1. $A \Rightarrow (A \to B) \to B$.

Some of the more or less familiar systems which result from BQ in this way are these:- GQ: BQ + D1. TQ: BQ + {B3, B4, B5, D3, D4} . The addition of B3, B4 and D4 means of course that rules R3 and R4 can be derived. EQ: TQ + BR1. RQ: TQ + B6. S4Q: EQ + B10. More compact axiomatisations, of RQ in particular, are presented subsequently.

An LQ *model structure* (LQ *m.s.*) is a structure $S = \langle K; \mathcal{D} \rangle$ where K is an L m.s. and $\mathcal{D}$ is a non-null set of objects (cf. Routley and Meyer 1973, p. 238). In particular, a BQ m.s. S, from which other LQ m.s. are derived, is a structure $S = \langle T, O, K, R, {}^*, \mathcal{D} \rangle$ where O is a subset of K, $T \in O$, R is a 3-place relation on K, $*$ is a 1-place operation on K, and $\mathcal{D}$ is a non-null set, constrained for every a, b, c, $d \in K$, by the following conditions, in which $a \leq b =_{Df} (Px)(Ox \& Rxab)$:-

p1. $a \leqslant a$.

p2. If $a \leqslant d$ and $Rdbc$ then $Rabc$.

p3. $a = a^{**}$.

p4. If $a \leqslant b$ then $b^* \leqslant a^*$.

LQ m.s. result upon adding further modelling conditions to BQ m.s. For the extensions of BQ explicity introduced the modelling conditions corresponding to postulates are as follows (with q_i corresponding to B_i and s_j to D_j for appropriate i and j, and drl to BR1), where $R^2abcd =_{Df} (Px)(Rabx \; \& \; Rxcd)$ and $R^2a(bc)d =_{Df} (Px)(Rbcx \; \& \; Raxd)$:-

q1. $Raaa$.

q3. If R^2abcd then $R^2b(ac)d$.

q4. If R^2abcd then $R^2a(bc)d$.

q5. If $Rabc$ then R^2abbc.

q6. If $Rabc$ then $Rbac$.

q10. If $Rabc$ then $b \leqslant c$.

s1. $x^* \leqslant x$ for x in O.

s3. Raa^*a.

s4. If $Rabc$ then Rac^*b^*.

s5. $a^* \leqslant a$.

drl. $Raxa$ for some x in O.

An LQ *m.s.* is *reduced* iff $O = \{T\}$; is *normal* iff $T = T^*$; and is *fully normal* iff $x = x^*$ for x in O. It is *denumerable* [*countable*] iff its domain D is.

A LQ model adds to a LQ m.s. an interpretation, or valuation function, I. Specifically a LQ *model* M is a structure $M = \langle T, K, R, D, I \rangle$, i.e. $\langle S ; I \rangle$ where the substructure $S = \langle T, K, R, D \rangle$ is a LQ m.s., and I is a function which assigns to each subject term $\mathbf{t}$ (i.e. subject variable or constant) an element $I(\mathbf{t})$ of D, to each n-place predicate parameter f^n at each world a of K, an n-place relation on K (extensionally, a subset of K^n), and to each sentential parameter p at each a in K exactly one of the values $\{1, 0\}$; subject always to the constraints for a and b in K:

(1) if $a \leqslant b$ and $I(p, a) = 1$ then $I(p, b) = 1$; and

(1') if $a \leqslant b$ then $I(f^n, a) \subseteq I(f^n, b)$.

Interpretation I is extended generally to all wff as follows:

$I(f(\mathbf{t}_1, \ldots, \mathbf{t}_n), a) = 1$ iff $\langle I(\mathbf{t}_1), \ldots, I(\mathbf{t}_n) \rangle \in I(f^n, a)$, i. e. iff the interpretations of terms $\mathbf{t}_1, \ldots, \mathbf{t}_n$ instantiate the relation assigned to n-place predicate f^n at world a;

$I(\mathbf{A} \;\&\; \mathbf{B}, a) = 1$ iff $I(\mathbf{A}, a) = 1 = I(\mathbf{B}, a)$;

$I(\sim\mathbf{A}, a) = 1$ iff $I(\mathbf{A}, a^*) \neq 1$;

$I(\mathbf{A} \rightarrow \mathbf{B}, a) = 1$ iff, for every b and c in K, if $Rabc$ and $I(\mathbf{A}, b) = 1$ then materially $I(\mathbf{B}, c) = 1$;

$I((\mathrm{U}x)\,\mathbf{A}, a) = 1$ iff $I^x(\mathbf{A}, a) = 1$ for every x-variant I^x of I, where I^x is an x-variant of I iff I^x differs from I at most in assignments to x (the *elementary* rule).

A wff A is *true in* M just in case $I(A, T) = 1$, and *false in* M otherwise. A is LQ-*valid* iff A is true in all LQ models, and *invalid* otherwise. A set Δ of wff is LQ *simultaneously satisfiable* iff for some LQ model M, every wff A in Δ is true in M.

Truth-valued semantics for LQ systems, which in effect always select D as the domain of terms of LQ and so can delete D, are even simpler. A LQ *TV m.s.* is simply a L m.s.; and a TV *valuation* in such m.s. is a function which assigns to each atomic wff at each a of K and element of $II = \{1, 0\}$. The extension of I for wff compounded by connectives is as before, but the extension to quantified wff becomes:

$I((x)A, a) = 1$ iff $I(A(t/x), a) = 1$ for every term t.

TV truth, validity, and so on, are defined, in terms of TV valuations, as above for truth, etc. The emphasis in what follows is, for the most part, on objectual (i.e. domained) semantics, with the truth-valued semantics simply being noted as a by-product.

Semantical investigation of weaker relevant logics and of systems such as EQ which include a theory (of sorts) of necessity is facilitated by the addition of a sentential constant t (interpreted variously as the conjunction of all truths, of all necessary truths, and of all theorems) subject to the two-way rule

$$A \Leftrightarrow t \rightarrow A \qquad (t\text{-rule})$$

LQt is the LQ system enlarged by constant t. For each system LQ investigated it will turn out that LQt is a conservative extension of LQ, i.e. where A is a wff of LQ, A is a theorem of LQt iff A is a theorem of LQ. Thus, in effect, LQ can be investigated by studying LQt.

The semantical analysis of LQt is precisely the same as that of LQ, except that to accommodate t the following interpretation rule is added:

$$I(t, a) = 1 \quad \text{iff} \quad a \in O \quad \text{(cf. RLR, 4.1).}$$

The enlargement of LQ by t will subsequently provide an examplar for enlargements of other LQ theories by constants representing their truths.

§2. SOUNDNESS OF THE LOGICS, AND THE COMPLETENESS STRATEGY.

None of the relational semantics proposed for quantified relevant logics, such as BQ, GQ, EQ and RQ, has hitherto been proved adequate. Proof of soundness of the semantics is indeed no problem; it is a straightforward elaboration of that for corresponding sentential systems (given in Routley and Meyer 1978, Chapter 3), and succeeds not only for every system introduced but for a great range of additional systems. The preliminaries (elaborated in Routley and Meyer 1978) are these :- Where I is a valuation in a given LQ m.s. S, A *implies* B on I, or A I-*implies* B, just in case for every $a \in K$ if $I(A, a) = 1$ then $I(B, a) = 1$.

A *implies* B in S just in case A I-implies B on every valuation in S; and A LQ-*implies* B iff A implies B in every LQ m.s.

LEMMAS. *Where I is an interpretation in LQ m.s. S, for every wff* A *and* B *and every* a *and* b *in* K;

(1) *if* $a \leqslant b$ *and* $I(\mathbf{A}, a) = 1$ *then* $I(\mathbf{A}, b) = 1$;

(2) *if* $\mathbf{A}$ I-*implies* $\mathbf{B}$ *then* $\mathbf{A} \to \mathbf{B}$ *is true on* I;

(3) *if* $\mathbf{A}$ *implies* $\mathbf{B}$ *in* S *then* $\mathbf{A} \to \mathbf{B}$ *is valid in* S (*i.e. true on all valuations therein*); *and*

(4) $\mathbf{A}$ LQ-*implies* $\mathbf{B}$ *iff* $\mathbf{A} \to \mathbf{B}$ *is* LQ-*valid*.

PROOFS are like the corresponding lemmas of Routley and Meyer 1978. ∎

THEOREM 1. (*LQ Soundness*). *If* $\mathbf{A}$ *is a theorem of* LQ *then* $\mathbf{A}$ *is* LQ-*valid*.

PROOF. Enlarges the corresponding induction of Routley and Meyer 1978 showing that the axioms are valid and that the rules preserve validity. There are only these new cases:

ad QR1. Suppose $(x)\,\mathbf{A}$ is not LQ-valid. Then for some I in some S, $I((x)\,\mathbf{A}, T) = 0$. Thus, for some x-variant I' of I, $I'(\mathbf{A}, T) = 0$; so $\mathbf{A}$ is not LQ-valid.

ad QA1. Suppose for arbitrary I in S that for arbitrary $a \in K$, $I((x)\,\mathbf{A}, a) = 1$. Then for every x-variant I' of I, $I'(\mathbf{A}, a) = 1$. Hence for I itself, $I(\mathbf{A}(\mathbf{t}/x), a) = 1$ in virtue of the restrictions on substitution (since the relevant metatheoretic restrictions on instantiation exactly parallel systemic ones). Thus $(x)\,\mathbf{A}$ I-implies $\mathbf{A}(\mathbf{t}/x)$, whence by (3) $(x)\,\mathbf{A} \to \mathbf{A}(\mathbf{t}/x)$ is true on I in S.

ad QA2. Suppose for arbitrary I in S that for arbitrary $a \in K$, $I((x)(\mathbf{A} \to \mathbf{B}), a) = 1$. As indicated in the previous case, it suffices to show that $I(\mathbf{A} \to (x)\,\mathbf{B}, a) = 1$, that is, given that $Rabc$ and $I(\mathbf{A}, b) = 1$.

$I((x)\,\mathbf{B}, c) = 1$, i.e. $I'(\mathbf{B}, c) = 1$ for every x-variant I' of I, But as $I((x)(\mathbf{A} \to \mathbf{B}), a) = 1$,

$I'(\mathbf{A} \to \mathbf{B}, a) = 1$ for every x-variant I' of I; whence for every I' since $Rabc$ and $I(\mathbf{A}, b) = 1$

(because, as x is not free in $\mathbf{A}$, $I'(\mathbf{A}, b) = I(\mathbf{A}, b)$, for every x-variant I' of I), $I'(\mathbf{B}, c) = 1$, as required.

ad QA3. Again following the same argument pattern as in the previous two cases, suppose $I((x)(\mathbf{A} \vee \mathbf{B}), a) = 1$. Then for every x-variant I' of I, $I'(\mathbf{A} \vee \mathbf{B}, a) = 1$, so $I'(\mathbf{A}, a) = 1$ or $I'(\mathbf{B}, a) = 1$. Since x is not free in $\mathbf{A}$, $I'(\mathbf{A}, a) = I(\mathbf{A}, a)$ for every x-variant I' of I. Hence either $I(\mathbf{A}, a) = 1$ or else $I'(\mathbf{B}, a) = 1$ for every x-variant I' of I; that is $I(\mathbf{A}, a) = 1$ or $I((x)\,\mathbf{B}, a) = 1$, whence $I(\mathbf{A} \vee (x)\,\mathbf{B}, a) = 1$.

The validation of additional axiom and rule schemes, given that corresponding modelling conditions hold in every model, is exactly as in Routley and Meyer 1978 as no genuine quantificational enrichments have been considered. ∎

COROLLARY 1. *If* $\mathbf{A}$ *is a theorem of* LQ *then* $\mathbf{A}$ *is* TV-LQ-*valid*.

PROOF differs from proof of the theorem only on quantificational postulates, and there the T case simplifies that already given. Consider to illustrate the case of QA1. Suppose once again $I((x)\,\mathbf{A}, a) = 1$. Then $I(\mathbf{A}(\mathbf{t}/x), a) = 1$ for every subject term **t** . ■

COROLLARY 2. *If* A *is a the theorem of* LQt *then* A *is* LQt*-valid and* TV-LQt*-valid.*

PROOF adds to the theorem details concerning t drawn from Routley and Meyer 1978, 4.1. ■

The problems begin with completeness. A direct relevant approach, elaborating the completeness argument for sentential systems (as set out in Routley and Meyer 1978 for instance), has run into serious impasses (1). The problems are all, of course, caused by the presence of quantifiers, since no problems of the sort remain for relevant sentential logics such as R, and they arise from the need to be able to show in the canonical model that for each appropriate RQ - theory a,

$$I((x)\mathbf{A}, a) = 1 \quad \text{iff} \quad (x)\mathbf{A} \in a,$$

a requirement which forces us to make all RQ - theories *rich*, i.e. such that whenever $\mathbf{A}(t/x) \in a$ for every term t then $(x)\mathbf{A} \in a$. From this requirement of richness comes the main problem, that on each occasion on which a new theory is introduced it has to be shown that it is already rich and can be kept so on expanding the theory to incorporate other desired properties, or that it can be made rich without loss of requisite properties. The main problem can be broken down to three sorts of cases for systems such as RQ (which have reduced models):-

P1) In the induction step for $\rightarrow$ it has to be shown that where $\mathbf{B} \rightarrow \mathbf{C} \notin a$ for some b and c appropriately related to *a both rich and with the same domain* as a, $\mathbf{B} \in b$, $\mathbf{C} \notin c$.

The relation on worlds in question, the generalised counterexample relation R, is the source of the second, and most severe, problem. For R is not without conditions which in the case of stronger systems take the form: if R — & ... then, for some x, R — &, e.g. in the case of the modelling requirement for $\mathbf{A} \rightarrow \mathbf{B} \rightarrow . \mathbf{B} \rightarrow \mathbf{C} \rightarrow . \mathbf{A} \rightarrow \mathbf{C}$, if $Rabz$ & $Rzcd$ then, for some x, $Rbxd$ & $Racx$.

P2) It has to be shown that the new theories needed to guarantee the modelling condition are, or can be made, rich.

But, thirdly, in the presence of relevant negation modelled through a star - operation on theories, richness is not enough. To show that a^* is a rich theory when a is, a has to be *saturated*, that is whenever $(\mathrm{P}x)\mathbf{A} \in a$ then $\mathbf{A}(\mathbf{t}/x) \in a$ for some term (or witness) **t** .

P3) The new theories introduced in steps required for P1) and P2) have to be extended, preserving requisite properties, to saturated theories.

Fortunately the problems can be met one at a time; that is the approach here followed. Firstly, problem P3) is avoided by not including [just] a relevant negation, but by considering systems which contain a classical-style negation, the so-called 'classical relevant" logics; and secondly problem P2) is just set aside to begin with by considering weaker relevant logics which do not call for the troublesome semantical postulates that cause the problem.

Thus it is that a rather circuitous approach aimed at, circumventing the impasses (but, regrettably, through classical enemy territory) and isolating the

(1) Meyer and Routley have convinced themselves that they have a direct completeness argument for BQ; but so far they have not recorded the argument.

problems is tried in what follows. The completeness argument will make use in particular of the following systems, related as follows, where $<$ indicates conservative extension and $\leqq$ equivalence:

$$\begin{array}{lllll} & LQ & & CLQ \text{ (direct adequacy proof)} & (2) \\ (1) & \vee & & \vee & \\ & LQ^{\neg} & \underset{(4)}{\leqq} & CLQ^* \text{ (direct adequacy proof)} & (3) \end{array}$$

Hence also CLQ* is a conservative extension of LQ. Now suppose **A** is not a theorem of LQ. Then **A** is not a theorem of CLQ*, whence, by adequacy of CLQ*, **A** is not CLQ* - valid. But the CLQ* countermodel to **A** supplies an LQ countermodel to **A**, so **A** is not LQ - valid.

To establish step (1), that $LQ^{\neg}$ conservatively extends LQ in the absence of a completeness proof for LQ, resort is had to an alternative, more trivial, semantics for LQ, a semantics extending that for first degree quantified system FDQ. On route some other quantified relevant systems of interest are encountered.

§3. EXTENSIONS OF THE SEMANTICS FOR FIRST DEGREE QUANTIFIED RELEVANT LOGIC, FDQ, TO THE HIGHER DEGREE.

The first degree system FDQ - where wff are restricted to the first degree, i.e. nested occurrences of $\rightarrow$ are excluded - may be axiomatised as follows:-

Axiom schemes:

A1. $A \rightarrow A$.

A2. $A \rightarrow B \supset . B \rightarrow C \supset . A \rightarrow C$.

A3. $A \,\&\, B \rightarrow A$.

A4. $A \,\&\, B \rightarrow B$.

A5. $A \rightarrow B \supset . A \rightarrow C \supset . A \rightarrow (B \,\&\, C)$.

A6. $A \,\&\, (B \vee C) \rightarrow (A \,\&\, B) \vee C$.

A7. $\sim\sim A \rightarrow A$.

A8. $A \rightarrow \sim B \supset \cdot B \rightarrow \sim A$.

A9. $A \rightarrow B \supset . A \supset B$.

A10. $A \supset B \supset . \sim(B \,\&\, C) \supset . \sim(C \,\&\, A)$.

A11. $(x) A \rightarrow A\,(t/x)$, for any term t .

A12. $(x)(A \rightarrow B) \supset .\ A \rightarrow (x) B$, where x is not free in A .

A13. $(x)(A \vee B) \rightarrow . A \vee (x) B$, where x is not free in A .

Rules:

R1. $A\ , A \supset B \Rightarrow B$ (Material Detachment).

R2. $A \Rightarrow (x) A$ (Generalisation).

The axiomatisation of FDQ given is a simple extension, to include basic entailment principles, of classical quantificational logic Q. The semantical analysis of FDQ, the adequacy of which is established in Routley 1978c, is likewise a straightforward worlds elaboration of classical quantificational semantics. An FDQ - *model* M is a structure $M = \langle T, K, *, \mathcal{D}, I \rangle$ where K is a set of worlds, $T \in K$, $*$ is an operation on K such that $T^* = T$ and $a^{**} = a$ for every $a \in K$, $\mathcal{D}$ is a non-null set of items or objects, and I is a valuation function (in the

FDQ *model structure* $\langle T, K, *, \mathcal{D} \rangle$) which assigns to each term of FDQ an element of $\mathcal{D}$, to each n-place predicate at each world a of K an n-place relation on $\mathcal{D}^n$, and to each sentential parameter at each $a \in K$ exactly one of the holding values $\{1, 0\}$. That is, an interpretation is an *un*constrained LQ interpretation.

I is extended to all wff of FDQ by the following evaluation rules, which apply for every $a \in K$:

$$I(f^n(t_1, \ldots, t_n), a) = 1 \text{ iff } I(f^n, a)\ (I(t_1), \ldots, I(t_n)).$$

(On an alternative, truth-valued, semantics $I(f^n(t_1, \ldots, t_n), a)$ is again assigned 1 or 0 by the model without analysis of $f^n(t_1, \ldots, t_n)$ into subject and predicate components.)

$I(\mathbf{A} \,\&\, \mathbf{B}, a) = 1$ iff $I(\mathbf{A}, a) = 1 = I(\mathbf{B}, a)$.

$I(\sim\mathbf{A}, a) = 1$ iff $I(\mathbf{A}, a^*) \neq 1$.

$I((x)\mathbf{A}, a) = 1$ iff $I'(\mathbf{A}, a) = 1$ for every x-variant I' of I.

$I(\mathbf{A} \to \mathbf{B}, T) = 1$ iff for every $b \in K$, if $I(\mathbf{A}, b) = 1$ then $I(\mathbf{B}, b) = 1$.

Derived evaluation rules include the following:

$I(\mathbf{A} \vee \mathbf{B}, a) = 1$ iff $I(\mathbf{A}, a) = 1$ or $I(\mathbf{B}, a) = 1$;

$I(\mathbf{A} \supset \mathbf{B}, a) = 1$ iff $I(\mathbf{A}, a^*) \neq 1$ or $I(\mathbf{B}, a) = 1$;

$I((Px)\mathbf{A}, a) = 1$ iff $I'(\mathbf{A}, a) = 1$, for some x-variant I' of I.

Semantical notions such as truth and validity are defined just as for LQ. As shown in Routley 1978c, the theorems of FDQ are precisely the FDQ valid wff, and corollaries such as the Skolem-Löwenheim theorem for FDQ follow from the completeness argument.

The first degree restriction of FDQ is incorporated in the semantics through the interpretation rule for $\to$, which assigns values to implication formulae only at T. To move to the higher-degree this limitation to T has to be removed, something that may be done in a variety of ways, e.g. replacing T by a in the rule would yield a kind of rigid S5, higher degree irrelevant, logic. More interesting is the minimal adjustment - yielding at the sentential level the nonreplacement systems of Routley and Loparić 1978 and Routley and Meyer 1978 - which assigns values from $\{1, 0\}$ arbitrarily to implicational wff at worlds other than T. This simple step furnishes a semantical analysis for the system $\text{FDQ}^\dagger$, which differs from FDQ only in the removal of the formation restriction of FDQ to first degree wff (2).

THEOREM 2. *Where* $\mathbf{A}$ *is a wff of system* $\text{FDQ}^\dagger$ *(i.e. of the higher degree system axiomatised exactly like* FDQ *except that wff are not confined to the first degree),* $\mathbf{A}$ *is a theorem of* $\text{FDQ}^\dagger$ *iff* $\mathbf{A}$ *is* $\text{FDQ}^\dagger$*-valid.*

(2) Observe however that $\text{FDQ}^\dagger$ is not simply a substitutional extension of FDQ. The effect of Material Detachment has also to be considered. Thus, for example, $((p \to p) \to q) \supset q$ is a theorem of $\text{FDQ}^\dagger$ (and valid) but does not result by substitution from a theorem of FDQ (this example is due to J. Slaney).

PROOF is almost the same as that given in Routley 1978c for FDQ. ∎

In the canonical model used for completeness $I(\mathbf{A} \to \mathbf{B}, a)$ is assigned value 1, for each world a distinct from T, iff $\mathbf{A} \to \mathbf{B} \in a$.

There is good reason for dissatisfaction with FDQ† - apart from the weakness of its higher degree logic, and the failure in particular of the rule of Replacement of Equivalence (i.e. coimplications) - namely the excessively classical character of FDQ†, most conspicuously its inclusion of the rule γ of Material Detachment as a primitive rule, and its consequent inadequacy for dialectical purposes. The first stage in removing classical assumptions involves reaxiomatising FDQ† without use of γ (i.e. R1) as a primitive rule, and correspondingly adjusting the semantics so that the classical assumption $T = T^*$ is removed. The semantical adjustment is the simpler. A partial ordering (or inclusion) relation $\leqslant$ is added to FDQ† models, and the requirement $T = T^*$ replaced by the requirement $T^* \leqslant T$, which while ensuring completeness of T of one sort (that symbolised in the law of excluded middle $\mathbf{A} \vee {\sim}\mathbf{A}$) does not preclude inconsistency of T. The semantical adjustment also turns out to have the advantage that it enables various sublogics of FDQ† of interest to be semantically encompassed. Specifically, then, to proceed downwards in the direction of weakness and generality, one moves first to a *model structure* $M = \langle T, K, \leqslant, *, D \rangle$ with $T \in K$, $*$ an (so far unrestricted) operation on K, $\leqslant$ an order relation on K such that if $a \leqslant b$ then $b^* \leqslant a^*$, and D a non null set. The way down leads all the way down to a system P$^+$Q, a quantified version of the minimal positive system P$^+$ (of Arruda-da-Costa: see Routley and Loparić 1978), P$^+$Q, which is a minimal relevant logic with respect to the modelling and the style of completeness proof, has the following postulates:-

P1. $\mathbf{A} \to \mathbf{A}$.

P2. $\mathbf{A}, \mathbf{A} \to \mathbf{B} \Rightarrow \mathbf{B}$.

P3. $\mathbf{A} \to \mathbf{B}, \mathbf{B} \to \mathbf{C} \Rightarrow \mathbf{A} \to \mathbf{C}$.

P4. $\mathbf{A} \,\&\, \mathbf{B} \to \mathbf{A}$.

P5. $\mathbf{A} \,\&\, \mathbf{B} \to \mathbf{B}$.

P6. $\mathbf{A}, \mathbf{B} \Rightarrow \mathbf{A} \,\&\, \mathbf{B}$.

P7. $\mathbf{A} \to \mathbf{B}, \mathbf{A} \to \mathbf{C} \Rightarrow \mathbf{A} \to (\mathbf{B} \,\&\, \mathbf{C})$.

P8. $\mathbf{A} \to \mathbf{A} \vee \mathbf{B}$.

P9. $\mathbf{B} \to \mathbf{A} \vee \mathbf{B}$.

P10. $\mathbf{A} \to \mathbf{C}, \mathbf{B} \to \mathbf{C} \Rightarrow \mathbf{A} \vee \mathbf{B} \to \mathbf{C}$

P11. $\mathbf{A} \,\&\, (\mathbf{B} \vee \mathbf{C}) \to . (\mathbf{A} \,\&\, \mathbf{B}) \vee \mathbf{C}$.

QP1. $(x)\mathbf{A} \to \mathbf{A}(t/x)$.

QP2. $\mathbf{A} \to \mathbf{B} \Rightarrow \mathbf{A} \to (x)\mathbf{B}$, x not free in $\mathbf{A}$.

QP3. $(x)(\mathbf{A} \vee \mathbf{B}) \to . \mathbf{A} \vee (x)\mathbf{B}$, x not free in $\mathbf{A}$.

QP4. $\mathbf{A} \Rightarrow (x)\mathbf{A}$.

QP5. $\mathbf{A}(t/x) \to (\mathrm{P}x)\mathbf{A}$.

QP6. $\mathbf{B} \to \mathbf{A} \Rightarrow (\mathrm{P}x)\mathbf{B} \to \mathbf{A}$, x not free in $\mathbf{A}$.

QP7. $(\mathrm{P}x)(\mathbf{A} \,\&\, \mathbf{B}) \to . \mathbf{A} \,\&\, (\mathrm{P}x)\mathbf{B}$, x not free in $\mathbf{A}$.

Furnishing good semantics for P$^+$Q, unlike the stronger affixing systems, is not a difficult feat (and is carried out in Routley and Loparić 197+).

However the present investigation concerns not the way down from FDQ†, but the way up, the way to relevant affixing systems.

§4. STRENGTHENING THE HIGHER DEGREE; AND TRIVIAL AND LESS TRIVIAL SEMANTICS FOR SUCH SYSTEMS.

The trivial semantics [3] for extensions of such systems as $FDQ^{\dagger}$ simply stipulate semantical postulates for each scheme of the extension beyond schemes of FDQ, e.g. if extension includes the axiom $\mathbf{A} \to \mathbf{B} \to . \mathbf{B} \to \mathbf{C} \to . \mathbf{A} \to \mathbf{C}$, the semantics has the postulate that, for every a, if $I(\mathbf{A} \to \mathbf{B}, a) = 1$ then $I(\mathbf{B} \to \mathbf{C} \to . \mathbf{A} \to \mathbf{C}, a) = 1$. The method, which includes I in the modelling, foregoes recursive specification of I from atomic beginnings, and instead specifies I as a function on wff and worlds which takes values in $\{1,0\}$, and which is subject to a set of conditions, namely all those characterising I in the case of $FDQ^{\dagger}$ and also further conditions for schemes of the extension.

The method may be usefully illustrated by the trivial semantics for RQ. RQ gets selected throughout for illustrative purposes not because there is any special interpretational virtue about it - it fails badly for all the important notions relevant logics aim to explicate such as conditionality, implication, lawlike connection, entailment, propositional inclusion - but because in lands of deviant logics it's moderately well known as logics go, and also because it's technically exasperating. It's so close to classical to be good for practically nought but enthymematic purposes, yet though only one (albeit large and irrelevant) step removed as it were from classical, nothing much classical appears to work either at all or at all well. A *trivial model* M for RQ is an $FDQ^{\dagger}$ model, i.e. $M = \langle T, K, *, D, I \rangle$, with $T \in K$, $*$ an operation on K such that $a^{**} = a$ and $T^* = T$, D a non-null domain, and I a valuation function subject to all FDQ conditions together with the following:-

For B3 : $I(\mathbf{A} \to \mathbf{B}, a) = 1 \supset . \ I(\mathbf{B} \to \mathbf{C} \to . \ \mathbf{A} \to \mathbf{C}, a) = 1$.

For B5 : $I(\mathbf{A} \to . \ \mathbf{A} \to \mathbf{B}, a) = 1 \supset . \ I(\mathbf{A} \to \mathbf{B}, a) = 1$.

For B6 : $I(\mathbf{A}, a) = 1 \supset . \ I(\mathbf{A} \to \mathbf{B} \to \mathbf{B}, a) = 1$.

For A4 : $I(\mathbf{A} \to \mathbf{B}, a) = 1 = I(\mathbf{A} \to \mathbf{C}, a) \supset . \ I(\mathbf{A} \to \mathbf{B} \,\&\, \mathbf{C}, a) = 1$.

For D4 : $I(\mathbf{A} \to \sim \mathbf{B}, a) = 1 \supset . \ I(\mathbf{B} \to \sim \mathbf{A}, a) = 1$.

For QA2.[4] Where x is not free in $\mathbf{A}$, if $I((x)(\mathbf{A} \to \mathbf{B}), a) = 1$ then $I(\mathbf{A} \to (x)\mathbf{B}, a) = 1$.

THEOREM 3. *For each of the higher degree extensions of* FDQ *considered, including* RQ *(and for many other higher degree extensions* LQ *of* FDQ*),* $\mathbf{A}$ *is a theorem of* LQ *iff* $\mathbf{A}$ *is trivially* LQ *valid.*

PROOF enlarges on the proof of Theorem 2. Consider an axiom scheme of the form $\mathbf{C} \to \mathbf{D}$ where $\mathbf{C}$ or $\mathbf{D}$ is an implicational wff. The axiom is reflected semantically by an interpretational requirement if $I(\mathbf{C}, a) = 1$ then $I(\mathbf{D}, a) = 1$. So it is immediate by the evaluation rule for $\to$ at T that the axiom is valid. For completeness set $I(\mathbf{A}, b) = 1$ iff $\mathbf{A} \in b$, and use closure and the fact that T is regular. Observe that if $\mathbf{C} \in a$, then $\mathbf{D} \in a$ in virtue of theory closure under provable implication; hence $I(\mathbf{C}, a) = 1$ implies $I(\mathbf{D}, a) = 1$. ■

In the case of relevant logics such as RQ there is one considerable further

(3) There are more trivial semantics.

(4) This scheme, U distribution, is the only quantificational postulate of higher degree character that occurs in the main axiomatisations considered.

difficulty, that of showing that the logic does extend FDQ†. The main difficulty here is that of showing that the rule (γ) of Material Detachment, i.e. R1, is an admissible rule. But this can be proved, for all the common relevant systems which include $A \vee \sim A$ as a theorem, by the method of metavaluations (for main details see Meyer 197+).

The "trivial" semantics are more trivial than the relational semantics introduced because, at least in the case of higher degree implicational wff, the trivial semantics place no great distance between the axiom schemes and the corresponding semantical conditions on the valuation function. Still the "trivial" semantics (which have been presented as giving a satisfactory semantical analysis of certain relevant logics by Hunter 197+) have merit in that they enable various results to be proved, including one important result needed subsequently.

Let the classical negation enlargement $LQ^{\urcorner}$ of LQ be formed by adding to LQ a negation symbol, $\bar{\ }$, subject to the following schemes:-

C1. $\bar{\bar{A}} \to A$. CR1. $A \mathbin{\&} B \to \bar{C} \Rightarrow . A \mathbin{\&} C \to \bar{B}$.

A (trivial) $LQ^{\urcorner}$ *model* is an LQ model which conforms to the classical negation rule:

$I(\bar{A}, a) = 1$ iff $I(A, a) \neq 1$, for every a in K.

THEOREM 4. *Where* $LQ^{\urcorner}$ *is the classical negation enlargement of any of the extensions of* FDQ *considered,* A *is a theorem of* $LQ^{\urcorner}$ *iff* A *is trivially* $LQ^{\urcorner}$ *valid.*

PROOF. Given the previous theorem, soundness is a matter of verifying the postulates C1 and CR1. C1 and its converse are immediate, since $I(\bar{\bar{A}}, a) = I(A, a)$. If the conclusion of CR1 is not valid then, in some model, $I(A, a) = 1 = I(B, a) \neq I(\bar{C}, a)$. So $I(A, a) = 1 = I(C, a) \neq I(\bar{B}, a)$, i.e. the premiss of CR1 is not valid. Completeness calls for a refinement in previous theorems, namely these theorems can be reproved using nondegenerate theories, i.e. theories that are neither null nor universal, in the canonical modellings. Given nondegeneracy and theorems proved using CL and CR1 (exactly as below) the classical negation rule emerges as follows. It suffices to show $\bar{A} \in a$ iff $A \notin a$. Suppose $\bar{A} \in a$ and $A \in a$ also; so $A \mathbin{\&} \bar{A} \in a$, whence since $\vdash A \mathbin{\&} \bar{A} \to B$, a is universal, contradicting a's nondegeneracy. Suppose, conversely, $A \notin a$ and $\bar{A} \notin a$. Then $A \vee \bar{A} \notin a$. But as a is non-null, for some B, $B \in a$. Hence as $\vdash B \to . A \vee \bar{A}$, $A \vee \bar{A} \in a$, which is impossible. ∎

COROLLARY. *For each* LQ *considered,* $LQ^{\urcorner}$ *is a conservative extension of* LQ; *i.e. if* A *is a wff of* LQ, A *is a theorem of* $LQ^{\urcorner}$ *iff* A *is a theorem of* LQ.

PROOF. One half is immediate from the inclusion of LQ in $LQ^{\urcorner}$. For the converse, suppose A, which is a wff of LQ, is not a theorem of LQ. Then as A is not trivially LQ valid, there is an $LQ^{\urcorner}$ counter-model M to A. Form a new model M' by adding the classical negation rule to M. Then M' is an $LQ^{\urcorner}$ model, *since no other conditions are required*, and A is not true in M' since, as it contains no occurrences of classical negation, its assignments are the same as in M. ∎

But though the trivial semantics do thus yield information they leave much to be desired - in particular, a recursive evaluation rule for implication at nonregular worlds, and semantical conditions which less blatantly mirror corresponding axioms. Less trivial semantics do result upon adding relational or operational evaluation rules. It is here that the main unsolved problems emerge again. It is time to turn to stage (2) of the completeness argument, to the investigation of weaker classical relevant logics.

§5. THE SYNTACTICAL STRUCTURE OF CLQ SYSTEMS.

Primitive are the connectives $\rightarrow$ (implication), & (conjunction) and $^{-}$ (classical negation) and the universal quantifier U. Further syncategorematic symbols are defined thus:

$A \vee B =_{Df} \overline{(\overline{A} \,\&\, \overline{B})}$; $A \supset B =_{Df} \overline{A} \vee B$; $A \leftrightarrow B =_{Df} (A \rightarrow B) \,\&\, (B \rightarrow A)$; $(x)A =_{Df} (Ux)A$; $(Px)A =_{Df} \overline{(x)\overline{A}}$. Note how the CLQ definitions of V and P differ from the relevant definitions. The already familiar postulates of basic system CBQ are divided, for convenience, into three groups.

Postulates of B^{+}, the positive part of systems B and G (cf. Routley and Meyer 1978, Anderson and Belnap 1975):

A1. $A \rightarrow A$.

A2. $A \,\&\, B \rightarrow A$.

A3. $A \,\&\, B \rightarrow B$.

A4. $(A \rightarrow B) \,\&\, (A \rightarrow C) \rightarrow. A \rightarrow B \,\&\, C$.

[A5. $A \rightarrow . A \vee B$.]

[A6. $B \rightarrow . A \vee B$.]

A7. $(A \rightarrow C) \,\&\, (B \rightarrow C) \rightarrow. (A \vee B) \rightarrow C$.

[A8. $A \,\&\, (B \vee C) \rightarrow. (A \,\&\, B) \vee C$.]

R1. $A,\ A \rightarrow B \Rightarrow B$.

R2. $A,\ B \Rightarrow A \,\&\, B$.

R3. $A \rightarrow B,\ C \rightarrow D \Rightarrow B \rightarrow C \rightarrow . A \rightarrow D$.

Negation Postulates for CB (cf. Meyer and Routley 1973, p. 57):

C1. $\overline{\overline{A}} \rightarrow A$.

CR1. $A \,\&\, B \rightarrow \overline{C} \Rightarrow A \,\&\, C \rightarrow \overline{B}$.

Quantificational postulates (cf. Routley and Meyer 1973, Routley 1978):

QA1. $(x)A \rightarrow \check{S}^{x}_{t} A|$, i.e. $(x)A \rightarrow A(t/x)$, with the usual proviso built in.

QA2. $(x)(A \rightarrow B) \rightarrow. A \rightarrow (x)B$, with x not free in A .

QA3. $(x)(A \vee B) \rightarrow. A \vee (x)B$, with x not free in A .

QR1. $A \rightarrow (x)A$.

[QA4. $\check{S}^{x}_{t} A| \rightarrow (Px)A$.]

QA5. $(x)(B \rightarrow A) \rightarrow. (Px)B \rightarrow A$, with x not free in A .

[QA6. $(Px)(A \,\&\, B) \rightarrow. A \,\&\, (Px)B$, with x not free in A .]

With V and P defined the square bracketed schemes are not independent (as will be shown). The status of A7 and QA5 remains a little puzzling: with a normal negation they would be simply derivable, but the failure of full contraposition principles in C systems blocks such derivations. R2 is almost eliminable as well ($A \supset . B \supset A \& B$ follows quickly from T6).

A few theorems and metatheorems of CB and CBQ, derived primarily with a view to establishing completeness theorems, are these:-

T1. $A \& \overline{A} \rightarrow B$.

PROOF. By A2, $\overline{A} \& \overline{B} \rightarrow \overline{A}$, whence by CR1, $\overline{A} \& A \rightarrow \overline{\overline{B}}$.
The result then follows by B^+ theorems and C1. ■

T2. $A \rightarrow \overline{\overline{A}}$; $A \leftrightarrow \overline{\overline{A}}$.

PROOF. By A2, $A \& \overline{A} \rightarrow A$, whence by CR1, $A \& A \rightarrow \overline{\overline{A}}$. ■

T3. $A \& \overline{A \& B} \rightarrow B$; $A \& (\overline{A} \vee B) \rightarrow B$.

PROOF. $A \& B \rightarrow C \Rightarrow A \& B \rightarrow \overline{\overline{C}} \Rightarrow A \& \overline{C} \rightarrow \overline{B}$, by T2, CR1. ■

$A \& B \rightarrow A \& B$, by A1, whence by the rule $A \& \overline{A \& B} \rightarrow \overline{B}$.
$A \& \overline{B} \rightarrow C \Rightarrow A \& \overline{C} \rightarrow B$, as just above.

Since then $A \& \overline{B} \rightarrow \overline{\overline{A}} \& \overline{B}$, by T2 and B^+ principles, $A \& \overline{\overline{\overline{A}} \& B} \rightarrow B$, whence the result by definition of $\vee$. ■

RT1. $A \rightarrow \overline{B} \Rightarrow B \rightarrow \overline{A}$.

PROOF. $A \rightarrow \overline{B} \Rightarrow B \& A \rightarrow \overline{B}$, using A3,
$\Rightarrow B \& B \rightarrow \overline{A}$, by CR1 ,
$\Rightarrow B \rightarrow \overline{A}$, from A7. ■

RT2. $A \rightarrow B \Rightarrow \overline{B} \rightarrow \overline{A}$; by RT1 and T2.

T4. $(x)(A \rightarrow B) \rightarrow . (x)A \rightarrow (x)B$.

PROOF. $(x)(A \rightarrow B) \rightarrow . (x)A \rightarrow B$, by QA1, QA2.
Hence by R3 and QA2, $(x)(A \rightarrow B) \rightarrow (x)((x)A \rightarrow B)$; but by QA2 again $(x)((x)A \rightarrow B) \rightarrow . (x)A \rightarrow (x)B$; whence the result by transitivity. ■

MT1 (Replacement of coentailments). $A \leftrightarrow B \Rightarrow C(A) \rightarrow C(B)$, *where* C(A) *is any context of* CLQ *containing* A *(perhaps vacuously)*.

PROOF is by the usual induction over the number of occurrences of the primitive connectives and quantifiers in C(A). The induction step for classical negation is provided by RT2 and for the universal quantifier by QR1 and T4. ■

T5. $B \rightarrow . A \vee \overline{A}$; by T1 and RT1.

T6. $A \vee \overline{A}$; from T5.

T7. $(\overline{A \& B}) \leftrightarrow \overline{A} \vee \overline{B}$; by MT1, T2, and definition of $\vee$.

T8. $(\overline{A \vee B}) \leftrightarrow \overline{A} \& \overline{B}$; T2 and definition of $\vee$.

RT3. $A \& B \rightarrow C \Rightarrow A \rightarrow . \overline{B} \vee C$.

PROOF. $A \& B \rightarrow C \Rightarrow \overline{C} \rightarrow . \overline{A} \vee \overline{B}$; by RT2, T7 ,

$\Rightarrow B \,\&\, \overline{C} \rightarrow . \; B \,\&\, (\overline{A} \vee \overline{B})$, by B^+ theorems,

$\Rightarrow B \,\&\, \overline{C} \rightarrow \overline{A}$, using T3,

$\Rightarrow A \rightarrow . \; \overline{B} \vee C$, by RT1, T7, MT1, T2. ■

RT4. $A \,\&\, B \rightarrow . \; C \vee D \Rightarrow A \,\&\, \overline{D} \rightarrow . \; \overline{B} \vee C$.

PROOF. $A \,\&\, B \rightarrow . \; C \vee D \Rightarrow A \,\&\, \overline{C \vee D} \rightarrow \overline{B}$, using CR1, T2, MT1,

$\Rightarrow A \,\&\, \overline{C} \,\&\, \overline{D} \rightarrow \overline{B}$, by T8, MT1,

$\Rightarrow A \,\&\, \overline{D} \rightarrow . \overline{B} \vee C$, by RT3, B^+ theorems, T2, MT1. ■

RT5. $A \,\&\, D \rightarrow B$, $A \rightarrow . \; B \vee D \Rightarrow A \rightarrow B$.

PROOF. $A \,\&\, D \rightarrow B$, $A \rightarrow . \; B \vee D \Rightarrow A \,\&\, D \rightarrow B$, $A \rightarrow (B \vee D) \,\&\, A$,

$\Rightarrow B \vee (A \,\&\, D) \rightarrow B$, $A \rightarrow . \; B \vee (A \,\&\, D)$, using A8,

$\Rightarrow A \rightarrow B$. ■

T9. (i.e. A5, A6). $A \rightarrow . \; A \vee B$, $B \rightarrow A \vee B$.

T10. (i.e. A8). $A \,\&(B \vee C) \rightarrow (A \,\&\, B) \vee C$.

PROOF. $A \,\&\, \overline{\overline{A} \,\&\, \overline{B}} \,\&\, \overline{C} \rightarrow \overline{B} \,\&\, \overline{C}$, by T3, B^+:

$A \,\&\, \overline{\overline{B} \,\&\, \overline{C}} \rightarrow \overline{\overline{A} \,\&\, \overline{B} \,\&\, \overline{C}}$, from CR1. ■

T11. $(A \rightarrow C) \,\&\, (B \rightarrow D) \rightarrow . \; A \vee B \rightarrow . \; C \vee D$.

PROOF. $A \rightarrow C \rightarrow . \; A \rightarrow C \vee D$, $B \rightarrow D \rightarrow . \; B \rightarrow C \vee D$, from T9,
$(A \rightarrow C) \,\&(B \rightarrow D) \rightarrow . \; A \rightarrow . \; C \vee D$, $(A \rightarrow C) \,\&(B \rightarrow D) \rightarrow . \; B \rightarrow . \; C \vee D$, by A5 and A6,
$(A \rightarrow C) \,\&\, (B \rightarrow D) \rightarrow . \; (A \rightarrow (C \vee D)) \,\&\, (B \rightarrow (C \vee D))$, by A7.
The result then follows by A7 and Rule Syllogism. ■

T12. (i.e. QA4). $\check{S}^{x}_{t} \, A| \rightarrow (Px) A$.

PROOF from QA1 by Rule Contraposition. ■

T13. (i.e. QA6). $(Px)(A \,\&\, B) \rightarrow . \; A \,\&\, (Px) A$, *with* x *not free in* A.

PROOF from QA3. ■

MT2. (Rule of change of bound variable). Statement and proof as in Church 1956, p. 192.

A little easier to accommodate semantically than CBQ is its extension CAQ obtained by flattening R3 to material form and adding a material form of Ass, i.e. of $A \,\&\, (A \rightarrow B) \rightarrow B$. CAQ has as postulates that is, all postulates of CBQ except R3 together with the schemes:

MR3. $A \rightarrow B \,\&\, C \rightarrow D \supset . \; B \rightarrow C \rightarrow . \; A \rightarrow D$. C2. $A \,\&\, (A \rightarrow B) \supset B$.

In the context of classical negation MR3 is difficult to resist given R3. Also C2[1], unlike its arrow strengthening, Ass, is hard to dispute. And it is tantamount to the very appealing principle of material counterexample: $A \rightarrow B \supset . \overline{A} \vee B$.

Proof of coimplication is as follows:-

$$\begin{aligned} A \,\&\, (A \rightarrow B) \supset B &\leftrightarrow \overline{A \,\&\, (\overline{A \rightarrow B})} \vee B ,\\ &\leftrightarrow \overline{A} \vee (\overline{\overline{A \rightarrow B}}) \vee B ,\\ &\leftrightarrow (\overline{\overline{A \rightarrow B}}) \vee \overline{A} \vee B ,\\ &\leftrightarrow A \rightarrow B \supset . \overline{A} \vee B . \end{aligned}$$

In the only classical relevant system that has so far received much news coverage, CR, both schemes MR3 and C2 are derivable. System CR results in fact from adding to the positive system R^+ the postulates C1 and CR1. Specifically - since the system is important in what follows - CR has the following postulates:

Postulates of R^+, the positive part of R (cf. §1 and Anderson and Belnap 1975, p. 453):

A1 - A8 and R1 - R2 (i.e. B^+), together with:

B3. $A \rightarrow B \rightarrow . B \rightarrow C \rightarrow . A \rightarrow C$. B5. $A \rightarrow (A \rightarrow B) \rightarrow . A \rightarrow B$.

B6. $A \rightarrow . A \rightarrow B \rightarrow B$.

Postulates for CR (cf. Meyer and Routley 1973, p. 57):

C1. $\overline{\overline{A}} \rightarrow A$. CR1. $A \,\&\, B \rightarrow \overline{C} \Rightarrow A \,\&\, C \rightarrow \overline{B}$.

CRQ results by adding the quantificational postulates already given for CBQ to CR. Between CBQ and CRQ there is a wealth of systems, some of which will be considered subsequently. Proof that CR` includes CA uses the following theorems of CR:-

T14. $A \,\&\, (A \rightarrow B) \rightarrow B$.

PROOF which uses B5 is as in Routley and Meyer 1978. ∎

T15. $A \rightarrow \overline{A} \rightarrow \overline{A}$.

PROOF. $A \,\&(A \rightarrow \overline{A}) \rightarrow \overline{A}$, by T14,
$A \,\&\, A \rightarrow \overline{A \rightarrow \overline{A}}$, using CR1 ,
$A \rightarrow \overline{A} \rightarrow \overline{A}$, by R^+ principles and Rule Contraposition. ∎

T16. $A \rightarrow B \rightarrow . \overline{A} \vee B$.

PROOF that T15 is interderivable with T16 is as in Routley and Meyer 1978, 3.2. ∎

T17. (i.e. C2). $A \,\&\, (A \rightarrow B) \supset . B$.

PROOF. Apply T16 to T14. ∎

T18. (i.e. MR3). $(A \rightarrow B) \,\&\, (C \rightarrow D) \supset . B \rightarrow C \rightarrow . A \rightarrow D$.

PROOF. Apply T16 to $(A \rightarrow B) \,\&\, (C \rightarrow D) \rightarrow . B \rightarrow C \rightarrow . A \rightarrow D$, a theorem of R^+. ∎

To cope semantically with CBQ (= CGQ), it simplifies matter to extend the system (conservatively as it turns out) by the sentential constant t, subject as always to the two-way rule

Rt. $\mathbf{A} \Leftrightarrow t \rightarrow \mathbf{A}$ (t rule).

The resulting system is called CBQt.

§6. CONSTANT DOMAIN MODELLINGS FOR THE CLQ SYSTEMS.

Finding a modelling for CRQ is simply a matter of building on the semantics for CR of Meyer and Routley 1973 (showing the adequacy of the modelling is another matter). A CRQ *model structure* (CRQ m.s.) adds to a CR r.m.s. (as previously explained in Meyer and Routley 1973) a non-null domain $\mathcal{D}$ of objects. Precisely, a CRQ m.s. S is a structure $S = \langle T, K, R, \mathcal{D} \rangle$, where K and $\mathcal{D}$ are non-null sets, $T \in K$, R is a three-place relation on K, subject generally to these requirements:

q1. $RTab$ iff $a = b$;

q2. $Raaa$;

q3. if R^2abcd then R^2acbd, where $Rabcd =_{Df} (Px)(Rabx \,\&\, Rxcd)$.

A CRQ model adds to a CRQ m.s. an interpretation, or valuation function, I, which is defined as for FDQ systems. Specifically a CRQ *model* M is a structure $M = \langle T, K, R, \mathcal{D}, I \rangle$, where the substructure $\langle T, K, R, \mathcal{D} \rangle$ is a CRQ m.s., and I is a function which assigns to each subject term t (i.e. subject variable or constant) an element, $I(\mathrm{t})$, of $\mathcal{D}$, to each n-place predicate parameter at each world a of K, and n-place relation on K (extensionally, a subset of K^n), and to each sentential parameter at each a in K exactly one of the values $\{1, 0\}$. Interpretation I is extended generally to all wff as follows:

$I(f(\mathbf{t}_1, \ldots, \mathbf{t}_n), a) = 1$ iff $\langle I(\mathbf{t}_1), \ldots, I(\mathbf{t}_n) \rangle \in I(f^n, a)$, i.e. iff the interpretations of terms $\mathbf{t}_1, \ldots, \mathbf{t}_n$ instantiate the relation assigned to n-place predicate f^n *at* world a;

$I(\mathbf{A} \,\&\, \mathbf{B}, a) = 1$ iff $I(\mathbf{A}, a) = 1 = I(\mathbf{B}, a)$;

$I(\overline{\mathbf{A}}, a) = 1$ iff $I(\mathbf{A}, a) \neq 1$;

$I(\mathbf{A} \rightarrow \mathbf{B}, a) = 1$ iff, for every b and c in K, if $Rabc$ and $I(\mathbf{A}, b) = 1$ then materially $I(\mathbf{B}, c) = 1$;

$I((\mathrm{U}x)\mathbf{A}, a) = 1$ iff $I^x(\mathbf{A}, a) = 1$ for every x-variant I^x of I, where as before I^x is an x-variant of I iff I^x differs from I at most in assignments to x.

To model other CLQ systems, i.e. classical "relevant" systems *with the same quantificational structure*, it is largely a matter of varying the modelling conditions. For systems which admit reduced modellings (in the sense of Routley and Meyer 1978)a basic system is CAQ for which only the one semantical postulate q1 is required. That is, a CAQ model M is a structure subject just to the requirement q1, or equivalently to:

q1a. if $RTab$ then $a = b$; and q1b. $RTaa$.

In each case the interpretation rules are extended precisely as for CRQ. Where, however, the CLQ system (strictly CLQt system) includes t both the model structure and the interpretation rules have to be enlarged.

A CLQt *model* M is a structure $M = \langle T, O, K, R, D \rangle$ which differs from a CLQ model primarily in containing a set O such that $T \in O \subseteq K$. The rule for evaluating t is again simply

$$I(t, a) = 1 \quad \text{iff} \quad Oa.$$

For many systems the semantical postulates have then to be adjusted. Those for CBQt, on which other systems are built, are as follows:

q1a'. For $x \in O$ if $Rxab$ then $a = b$.

q1b'. For some $x \in O$, $Rxaa$.

A wff A is *true* in M just in case $I(A, T) = 1$, and *false in* M otherwise. A is CLQ-*valid* iff A is true in all CLQ-models, and *invalid* otherwise. A set S of wff is CLQ *simultaneously satisfiable* iff for some CLQ-model M, every wff A in S is true in M.

Truth-valued semantics for CLQ systems are again simpler. A CLQ TV *m.s.* is simply a CL m.s.; and a TV *valuation* in such m.s. is a function which assigns to each atomic wff at each a of K an element of II. The extension of I for wff compounded by connectives is as before, but the extension to quantified wff becomes:

$$I((x)\,A, a) = 1 \quad \text{iff} \quad I(A(t/x), a) = 1 \quad \text{for every term } t.$$

TV truth, validity, and so on, are defined, in terms of TV valuations, as above for truth, etc.

§7. ADEQUACY OF THE SEMANTICS FOR WEAKER CLQ SYSTEMS.

Soundness is straightforward and succeeds not only for every system considered but for a great range of additional systems.

THEOREM 5 (CLQ *and* CLQt *Soundness*). *Every theorem of* CLQ *is* CLQ-*valid, and also* CLQ-TV-*valid. Similarly for* CLQt.

PROOF is, for the most part, straightforward case by case verfication, showing that the axioms are valid and that the rules preserve validity. Some strategic examples serve to illustrate the method.

ad A2. Suppose $I((A \,\&\, B) \to A, T) \neq 1$. Then for some a, b in K, $RTab$ and $I(A \,\&\, B, a) = 1 \neq I(A, b)$. By q1, $a = b$, so $I(A, a) = 1$, but $I(A, b) = 1$ which is impossible.

More generally whenever $I(A \to B, T) \neq 1$, for some a, $RTaa$ and $I(A, a) = 1 \neq I(B, a)$. Thereafter the procedure can follow the details of Routley and Meyer 1973. Thus, for example, the R^+ axioms schemes can be verified as in Routley and Meyer 1973 or 1978.

ad R1. Suppose $I(A, T) = 1 = I(A \to B, T)$. Since $RTTT$, by q1b, $I(B, T) = 1$.

ad R3. Suppose $I(\text{R3}, T) \neq 1$. Then $I(\mathbf{A} \to \mathbf{B}, T) = 1 = I(\mathbf{C} \to \mathbf{D}, T) \neq I(\mathbf{B} \to \mathbf{C} \to . \mathbf{A} \to \mathbf{D}, T)$. Hence for some a, $RTaa$, $I(\mathbf{B} \to \mathbf{C}, a) = 1 \neq I(\mathbf{A} \to \mathbf{D}, a)$. Thus for some b, c, $Rabc$ and $I(\mathbf{A}, b) = 1 \neq I(\mathbf{D}, c)$. As $Rabc$ and $I(\mathbf{B} \to \mathbf{C}, a) = 1$, either $I(\mathbf{B}, b) \neq 1$ or $I(\mathbf{C}, c) = 1$. Since $I(\mathbf{A} \to \mathbf{B}, T) = 1$, $I(\mathbf{A}, b) = 1$ and, by q1b, $RTbb$, $I(\mathbf{B}, b) = 1$. Hence $I(\mathbf{C}, c) = 1$. Similarly then, as $I(\mathbf{C} \to \mathbf{D}, T) = 1$, $I(\mathbf{D}, c) = 1$, which is impossible.

ad CR1. Suppose that in some model, $I(\mathbf{A} \,\&\, \mathbf{C} \to \overline{\mathbf{B}}, T) \neq 1$. Then for some a, $RTaa$ and $I(\mathbf{A}, a) = 1 = I(\mathbf{C}, a) \neq I(\overline{\mathbf{B}}, a)$. Thus $I(\overline{\mathbf{C}}, a) \neq 1 = I(\mathbf{B}, a)$. Hence as $RTaa$, $I(\mathbf{A} \,\&\, \mathbf{B} \to \overline{\mathbf{C}}, T) \neq 1$. In sum, if $\mathbf{A} \,\&\, \mathbf{C} \to \overline{\mathbf{B}}$ is not CRQ valid neither is $\mathbf{A} \,\&\, \mathbf{B} \to \overline{\mathbf{C}}$.

Verification of the quantificational postulates is like that given in § 2 (and in Routley and Meyer 1973 and Routley 1978 a).

For the weaker systems containing t there are some complications.

ad $\mathbf{A} \Rightarrow t \to \mathbf{A}$. Suppose $t \to \mathbf{A}$ is not valid: then in some model, $I(t \to \mathbf{A}, T) \neq 1$. Then for some a, $RTaa$ and $I(t, a) = 1 \neq I(\mathbf{A}, a)$. Then for some a, $O\,a$ and $I(\mathbf{A}, a) \neq 1$. Form a new model with base a in place of T. This is permissible since Oa, given that all semantical postulates are stated, not in terms of T, but generally for x in O. Then $\mathbf{A}$ is not valid.

ad $t \to \mathbf{A} \Rightarrow \mathbf{A}$. The rule is tantamount to Axiom t.

For it yields t since $t \to t$; and t yields the rule by Modus Ponens. And t is valid in virtue of OT.

Also verification of the rules is a little more complex.

ad R1. Suppose $I(\mathbf{A}, T) = 1$ but $I(\mathbf{B}, T) = 1$ for some model.

Then, since by q1b' for some x in O, $RxTT$, $I(\mathbf{A} \to \mathbf{B}, x) \neq 1$. Form a new model M' on base x. Since it is indeed a model $\mathbf{A} \to \mathbf{B}$ is not valid.

ad R3. Suppose $\mathbf{B} \to \mathbf{C} \to . \mathbf{A} \to \mathbf{D}$ is not valid, i.e. for some model $I(\mathbf{B} \to \mathbf{C} \to . \mathbf{A} \to \mathbf{D}, T) \neq 1$. Then as in the previous case for R3, $I(\mathbf{A}, b) = 1 \neq I(\mathbf{D}, c)$ and either $I(\mathbf{B}, b) \neq 1$ or $I(\mathbf{C}, c) = 1$. Since for some x in O, $Rxbb$ and $\mathbf{A} \to \mathbf{B}$ is valid, $I(\mathbf{A}, b) = 1$ materially implies $I(\mathbf{B}, b) = 1$. Similarly, as $\mathbf{C} \to \mathbf{D}$ is valid, $I(\mathbf{C}, c) = 1$ materially implies $I(\mathbf{D}, c) = 1$; and contradiction results. ■

Proofs of completeness are somewhat more arduous, and require many preliminaries. (For indications of the origins of these preliminaries see Routley 1978 a.) Since the same notions will recur in completeness proofs for a range of quantified implication systems LQ, the preliminaries are, as in Routley 1978 a , stated more generally than required simply for CLQ systems. The definitions are intended to apply both to LQ - a representative relevant system without perhaps a classical negation - and to linguistic extensions of LQ - also designated on occasion by LQ, though maintaining distinctions here is of critical importance in avoiding fallacious argument - obtained by adding further (at most denumerably many more) variables or constants to LQ (and accordingly inflating the supply of wff and logical axioms).

An LQ-*theory* T is any set of wff of LQ which is closed under adjunction and provable LQ-implication, i.e. for any wff $\mathbf{A}, \mathbf{B}$ if $\mathbf{A} \in T$ and $\mathbf{B} \in T$ then $\mathbf{A} \,\&\, \mathbf{B} \in T$, and if $\mathbf{A} \in T$ and $\vdash_{LQ} \mathbf{A} \to \mathbf{B}$ then $\mathbf{B} \in T$. More generally, a *theory* (linguistically construed) is a set of wff closed under certain operations. An LQ-theory T is *regular* iff all theorems of LQ are in T; a theory is *prime* ($\vee$ - complete) iff whenever $\mathbf{A} \vee \mathbf{B} \in T$ either $\mathbf{A} \in T$ or $\mathbf{B} \in T$; *rich* (U-complete) iff, whenever

$A(t/x) \in T$ for every subject term t of LQ, $(x)A \in T$; *saturated* (P-complete) iff whenever $(Px)A \in T$, $A(t/x) \in T$ for some term t of LQ. A theory T (for LQ) is *quantifier-complete* iff both rich and saturated; *straight* iff prime and quantifier-complete; and *adequate* iff straight and regular. T is *non-degenerate* (n.d.) iff T is neither null nor universal (i.e. contains every wff). For systems with a classical negation $^-$, such as CRQ, the canonical model is, in one way or another, built out of n.d. straight theories. The way depends on whether the modelling is reduced or not. Consider the unreduced case first.

Let $\tilde{K}_{LQ}$ be the class of LQ-theories, and K_{LQ} the class of n.d. straight theories; $\tilde{O}_{LQ}$ the class of regular LQ-theories, and O_{LQ} the class of n.d. adequate LQ-theories. For $a, b, c \in \tilde{K}_{LQ}$, $\tilde{R}_{LQ}\, abc$ iff whenever $A \to B \in a$ and $A \in b$, $B \in c$. R_{LQ} is the restriction of $\tilde{R}_{LQ}$ to n.d. straight LQ-theories. D_{LQ} is the class of terms of LQ; thus D_{LQ} is denumerable.

Where T_{CLQ} is any adequate CLQ-theory, the (*unreduced*) *canonical* CLQ *m.s. on* T_{CLQ} is the structure $S_c = \langle T_{CLQ}, O_{CLQ}, K_{CLQ}, R_{CLQ}, D_{CLQ} \rangle$. The *canonical interpretation* I in S_c is defined as follows:

$I(p, a) = 1$ iff $p \in a$, for every sentential parameter p;

$I(t, a) = t$, for every subject term t of CLQ;

$I(f, a) = \hat{t}_1, \ldots, \hat{t}_n\ (f(t_1, \ldots, t_n) \in a)$ for every n-place predicate parameter f, for each n.

The (*unreduced*) *canonical* CLQ *model* on T_{CLQ} is the structure $M_c = \langle T_{CLQ}, O_{CLQ}, K_{CLQ}, R_{CLQ}, D_{CLQ}, I \rangle$.

In reduced modellings LQ-theories are replaced uniformly by T-theories, for some chosen adequate LQ-theory T. A set of wff a is a *T-theory* iff a is closed under adjunction and under T-implication, i.e. whenever $A \in a$ and $A \to B \in T$ (also written $\vdash_T A \to B$) then $B \in a$. Where T_{CLQ} is any adequate CLQ-theory the *reduced canonical m.s.* [*model*] *on* T is the structure $\langle T_{CLQ}, K_T, R_T, D_{CLQ} \rangle$ [$\langle T_{CLQ}, K_T, R_T, D_{CLQ}, I \rangle$] where K_T is the class of n.d. straight T-theories and R_T is defined on T-theories.

Where S, T and U are sets of wff of LQ, T is LQ-*derivable from* S, written $S \vdash_{LQ} T$, iff for some $A_1, \ldots, A_m$ in S and $B_1, \ldots, B_n$ in T, $\vdash_{LQ} A_1 \,\&\, \ldots \,\&\, A_m \to . B_1 \vee \ldots \vee B_n$; and the pair $\langle S, T \rangle$ is (LQ-) *sound* iff T is not LQ-derivable from S. A pair $\langle S, T \rangle$ is (LQ-) *constant* iff whenever $\langle S, T \cup \{(x)B\} \rangle$ is sound, then, for some subject term u of LQ, $\langle S, T \cup \{(x)B, B(u/x)\} \rangle$ is sound.

A pair $\langle S, T \rangle$ is (LQ-) *maximal* iff

(1) $\langle S, T \rangle$ is sound;

(2) $S \cup T$ = LQ wff, i.e. every wff of LQ belongs to either S or T (but not both); and

(3) (a) if $(Px)A \in S$ then $A(t/x) \in S$ for some term t of LQ, and
(b) if $(x)A \in T$ then $A(t/x) \in T$ for some term t.

LEMMA. *Suppose* $\langle S, T \rangle$ *is* LQ-*maximal. Then*

(i) *S is an* LQ-*theory ;*
(ii) *S is prime;*
(iii) *S is saturated and rich;*
(iv) *$\langle S, T \rangle$ is constant.*

PROOF is exactly as in Routley 1978 a. Certain minimal properties of LQ are invoked, e.g. $\vdash_{LQ} \mathbf{A} \,\&\, \mathbf{B} \to \mathbf{A}$. CRQ meets all the requisite conditions. ■

EXPANSION LEMMA. *Where $\langle S, T \rangle$ is* LQ-*sound, it can be extended to an* LQ'-*maximal pair $\langle S', T' \rangle$ in a logic* LQ' *with at most denumerably more subject terms, i.e. there is an* LQ'-*maximal pair with $S \subseteq S'$ and $T \subseteq T'$.*

PROOF again depends on properties of LQ, e.g. the inclusion of the theorem, $\mathbf{A} \,\&\, (\mathbf{B} \vee \mathbf{C}) \to. (\mathbf{A} \,\&\, \mathbf{B}) \vee \mathbf{C}$, which CRQ certainly has. ■

EXPANSION COROLLARY. *Where U is a non-null set of wff of relevant system* LQ *and S is a regular set of wff if U is not* LQ-*derivable from S, then there is an n.d. adequate* LQ'-*theory T_{LQ}, which includes S but excludes U.*

PROOF. By the expansion lemma there is an LQ'-maximal pair $\langle S', U' \rangle$ extending $\langle S, U \rangle$. It suffices to show that $T_{LQ'} = S'$ is non-degenerate and regular. But the first point is immediate since both S' and U' are non-null. ■

The corollary may alternatively be established using Zorn's lemma (cf. the procedure in Routley and Meyer 1973). The corollary records the only occasion on which inflation of the domain of terms is required. Henceforth available terms are recycled in ensuring quantifier completeness. For once maximality is attained, by a single inflation of terms, it need never be sacrificed, as subsequent lemmas will reveal. The lemmas also reveal that conservation of terms is a rather more demanding feat than term profligacy - a feat we still do not know how to accomplish in a direct way for systems such as RQ which (correctly) lack a classical negation.

So far little use has been made of the properties of classical negation; in the lemmas that follows that situation changes, properties of classical negation being heavily exploited.

TRANSFER LEMMA. *Henceforth let $\overline{U} = \{\overline{\mathbf{A}} : \mathbf{A} \in U\}$. Now let $S = S_1 \cup \{\mathbf{A}_1, \ldots, \mathbf{A}_m\}$ and $T = T_1 \cup \{\mathbf{B}_1, \ldots, \mathbf{B}_n\}$ with $m,n \geq 1$. Then if $S \vdash_{LQ} T$, $S_1 \cup \overline{T}_1 \vdash_{LQ} \overline{\mathbf{A}_1 \,\&\, \ldots \,\&\, \mathbf{A}_m} \vee . \mathbf{B}_1 \ldots \vee \mathbf{B}_n$.*

PROOF. Suppose $S \vdash_{LQ} T$. Then for some $\mathbf{C}_1, \ldots, \mathbf{C}_j \in S_1$ and $\mathbf{D}_1, \ldots, \mathbf{D}_k \in T_1$, $\vdash \mathbf{C}^{\&} \,\&\, \mathbf{A}^{\&} \to . \mathbf{D}^{\vee} \vee \mathbf{B}^{\vee}$, where $\mathbf{C}^{\&}$ and $\mathbf{A}^{\&}$ are conjunctions of C's <u>and</u> A's <u>respectively</u> and $\mathbf{D}^{\vee}$ and $\mathbf{B}^{\vee}$ are similar disjunctions. By RT4, $\vdash \mathbf{C}^{\&} \,\&\, \overline{\mathbf{D}^{\vee}} \to . \overline{\mathbf{A}^{\&}} \vee \mathbf{B}^{\vee}$. The result follows by repeated application of T8. ■

CONSTANCY LEMMA. *If* $\langle S, T\rangle$ *is (sound and) constant then if* $\langle S \cup \{\mathbf{A}\}, T \cup \{\mathbf{B}\}\rangle$ *is sound, it is constant.*

PROOF. Suppose $\langle S \cup \{\mathbf{A}\}, T \cup \{\mathbf{B}, (x)\mathbf{F}(x)\}\rangle$ is sound. Then $\langle S \cup \overline{T}, \{\overline{\mathbf{A}} \vee \mathbf{B} \vee (x)\mathbf{F}(x)\}\rangle$ is sound. For suppose not. Then for some conjunctions $S^{\&}$ and $\overline{T}^{\&}$ of elements of S and $\overline{T}$, $\vdash S^{\&} \& \overline{T}^{\&} \to . \overline{\mathbf{A}} \vee \mathbf{B} \vee (x)\,\mathbf{F}(x)$, whence by RT4 $\vdash S^{\&} \& \mathbf{A} \to . \mathbf{B} \vee (x)\,\mathbf{F}(x) \vee T^{\vee}$, where $T^{\vee}$ is a disjunction of elements of T, so contradicting the soundness of $\langle S \cup \{\mathbf{A}\}, T \cup \{\mathbf{B}, (x)\,\mathbf{F}(x)\}\rangle$.

Now for appropriately chosen y, i.e. y is not free in $\mathbf{A}$, $\mathbf{B}$ or $(x)\,\mathbf{F}(x)$, $\overline{\mathbf{A}} \vee \mathbf{B} \vee (x)\,\mathbf{F}(x) \leftrightarrow (y)(\overline{\mathbf{A}} \vee . \mathbf{B} \vee (x)\,\mathbf{F}(x) \vee \mathbf{F}(y/x))$ (= $\mathbf{F}$ say). The, since $\langle S \cup \overline{T}, \{\mathbf{F}\}\rangle$ is sound, $\langle S, T \cup \{\mathbf{F}\}\rangle$ is sound. Alternatively, a little more simply, by RT4 $\langle S, T \cup \{\overline{\mathbf{A}} \vee \mathbf{B} \vee (x)\,\mathbf{F}(x)\}\rangle$ is sound, and so $\langle S, T \cup \{\mathbf{F}\}\rangle$ is sound. Hence as $\langle S, T\rangle$ is constant, $\langle S, T \cup \{\overline{\mathbf{A}} \vee \mathbf{B} \vee (x)\,\mathbf{F}(x) \vee \mathbf{F}(\mathbf{u}/x)\}\rangle$ is sound for some $\mathbf{u}$. Therefore $\langle S \cup \{\mathbf{A}\}, T \cup \{\mathbf{B}, (x)\,\mathbf{F}(x), \mathbf{F}(\mathbf{u}/x)\}\rangle$ is sound for some $\mathbf{u}$, as required. ∎

CONSTANT EXTENSION LEMMA. *Let* S *and* T *be sets of wff of system* LQ *such that* $\langle S, T\rangle$ *is sound and constant. Then there is an* LQ*-maximal pair* $\langle S', T'\rangle$ *- in the same vocabulary - with* $S \subseteq S'$ *and* $T \subseteq T'$.

PROOF. Let the set of wff of LQ be enumerated: $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n, \ldots$. Define sets S_i, T_i recursively for each non-negative integer i, as follows:- $S_0 = S$, $T_0 = T$. Given that S_i and T_i have been defined *and* that $\langle S_i, T_i\rangle$ is sound and constant, then S_{i+1} and T_{i+1} are defined thus:-

(i) Suppose $S_i \cup \{\mathbf{A}_{i+1}\} \vdash T_i$. Then $S_{i+1} = S_i$.

(ia) If $\mathbf{A}_{i+1}$ is of the form $(x)\mathbf{A}$ then $T_{i+1} = T_i \cup \{\mathbf{A}_{i+1}, \mathbf{A}(\mathbf{u}/x)\}$, where $\mathbf{u}$ is some paremeter such that $S_i \nvdash T_i \cup \{(x)\mathbf{A}, \mathbf{A}(\mathbf{u}/x)\}$.

(ib) Otherwise (where $\mathbf{A}_{i+1}$ is not of the form $(x)\mathbf{A}$), $T_{i+1} = T_i \cup \{\mathbf{A}_{i+1}\}$.

(ii) Suppose T_i is not LQ-derivable from $S_i \cup \{\mathbf{A}_{i+1}\}$. Then $T_{i+1} = T_i$.

(iia) If $\mathbf{A}_{i+1}$ is of the form $(\mathrm{P}x)\mathbf{A}$ then $S_{i+1} = S_i \cup \{\mathbf{A}_{i+1}, \mathbf{A}(\mathbf{u}/x)\}$, where $\mathbf{u}$ is some parameter such that $S_i \cup \{(\mathrm{P}x)\mathbf{A}, \mathbf{A}(\mathbf{u}/x)\} \nvdash T_i$.

(iib) Otherwise (where $\mathbf{A}_{i+1}$ is not of the form $(\mathrm{P}x)\mathbf{A}$), $S_{i+1} = S_i \cup \{\mathbf{A}_{i+1}\}$.

Finally $S' = \bigcup_i S_i$, $T' = \bigcup_i T_i$.

To show S_{i+1} and T_{i+1} (and thereby S' and T') are well-defined it has to be shown

(A) Some constant $\mathbf{u}$ satisfies the requirements for clause (ia);
(B) some constant $\mathbf{u}$ satisfies the requirements for clause (iia);
(C) $\langle S_{i+1}, T_{i+1} \rangle$ is sound;
(D) $\langle S_{i+1}, T_{i+1} \rangle$ is constant.

But (C) is established exactly as in the Expansion Lemma, and (D) follows from the Constancy Lemma. It remains to prove (A) and (B).

ad (A). Since $S_i \cup \{(x)A\} \vdash T_i$, $S_i \nvdash T_i \cup \{(x)A\}$; for otherwise $S_i \vdash T_i$ (using, as in the Expansion Lemma, the derived rule RT5), contradicting $\langle S_i, T_i\rangle$'s soundness. Since $\langle S_i, T_i \rangle$ is by assumption constant, and $\langle S_i, T_i \cup \{(x)A\} \rangle$ is sound, by definition of constancy, for some parameter $\mathbf{u}$, $\langle S_i, T_i \cup \{(x)A, A(\mathbf{u}/x)\} \rangle$ is sound.

ad (B). Given that $\langle S_i \cup \{(Px)A\}, T_i \rangle$ is sound, to show that for some parameter $\mathbf{u}$, $\langle S_i \cup \{(Px)A, A(\mathbf{u}/x)\}, T_i \rangle$ is sound. Suppose not, i.e. for every term $\mathbf{u}$, $S_i \cup \{(Px)A, A(\mathbf{u}/x)\} \vdash T_i$. Then, by the transfer lemma, $S_i \cup \overline{T_i} \vdash \overline{(Px)A \ \& \ A(\mathbf{u}/x)}$, for every $\mathbf{u}$. Set $\mathbf{F} = (y)\ \overline{(Px)A \ \& \ A(y/x)}$ where y is a variable not free in $(Px)A$. Suppose $\langle S_i, T_i \cup \{\mathbf{F}\} \rangle$ is sound. Then by constancy of $\langle S_i, T_i \rangle$, for some parameter $\mathbf{t}$, $\langle S_i, T_i \cup \{\mathbf{F}, \overline{(Px)A \ \& \ A(\mathbf{t}/x)}\} \rangle$ is sound, contradicting $S_i \cup \overline{T_i} \vdash \overline{(Px)A \ \& \ A(\mathbf{u}/x)}$ for every $\mathbf{u}$. So $\langle S_i, T_i \cup \{\mathbf{F}\} \rangle$ is not sound, i.e. $S_i \vdash T_i \cup \{\mathbf{F}\}$. But then, since $\mathbf{F} \leftrightarrow . \overline{(Px)A} \vee (x)\overline{A(x)}$ by confinement and change of variable, i.e. $\mathbf{F} \leftrightarrow \overline{(Px)A}$, using definition of P, $S_i \cup \{(Px)A\} \vdash T_i$, contradicting soundness of $\langle S_i \cup \{(Px)A\}, T_i \rangle$. ∎

THEOREM 6. (Strong completeness for weaker CLQ systems). *Where U is a non-null set of wff of CLQ which is not CLQ-derivable from regular set S, there is a [canonical] denumerable CLQ-model under which every member of S is true and every member of U false.*

PROOF. By the corollary to the Expansion Lemma there is an adequate CLQ'-theory T which includes S but excludes U. Form the canonical CLQ' model M_T on T. It is denumerable. The separation between unreduced and reduced models is made wherever required, in particular where important in establishing modelling conditions.

It suffices to show

(α) $I(\mathbf{D}, a) = 1$ iff $\mathbf{D} \in a$, for every $a \in K_T$ and every wff $\mathbf{D}$ of CLQ', and

(β) M_T is a CLQ' model with base T.

For then the theorem quickly follows.

ad (α). Proof is by induction. Since

$I(f(\mathbf{t}_1, \ldots, \mathbf{t}_n), a) = 1$ iff $\langle I(\mathbf{t}_1), \ldots, I(\mathbf{t}_n)\rangle \in I(f, a)$

iff $\langle \mathbf{t}_1, \ldots, \mathbf{t}_n \rangle \in \hat{\mathbf{t}}_1, \ldots, \hat{\mathbf{t}}_n$ $(f(\mathbf{t}_1, \ldots, \mathbf{t}_n) \in a)$

iff $f(t_1,\ldots, t_n) \in a$,

the induction basis is established. The induction step for connective & is straightforward (and as in Anderson and Belnap 1975).

ad $\bar{}$. The result is straightforward once it is shown that $\bar{A} \in a$ iff $A \in a$. Suppose firstly $\bar{A} \in a$ and $A \in a$. Then $\bar{A} \,\&\, A \in a$. But, by T1, $\vdash \bar{A} \,\&\, A \rightarrow B$, so $B \in a$ for arbitrary B contradicting the non-degeneracy of a. Suppose, for the converse, $A \notin a$ and $\bar{A} \notin a$. Since a is non-null some wff $D \in a$. Hence as $\vdash D \rightarrow . A \vee \bar{A}$, by T5, $A \vee \bar{A} \in a$, whence as a is prime $A \in a$ or $\bar{A} \in a$, which is impossible.

ad $\rightarrow$. D is of the form $(B \rightarrow C)$. If $B \rightarrow C \in a$ then $I(B \rightarrow C, a) = 1$ in virtue of the definition of R_T and the induction hypothesis.

For the converse suppose $B \rightarrow C \notin a$. It has to be shown that for some $b, c \in K_c$, $B \in b$, $C \notin c$ and $R_c abc$, for then that $I(B \rightarrow C, a) \neq 1$ follows applying the induction hypothesis and the interpretation rule for $\rightarrow$. For the unreduced case define $b_1 = \{D : \vdash_{CLQ} B \rightarrow D\}$; $c_2 = \{C : (PA)(A \rightarrow C \in a \,\&\, A \in b_1\}$; $U_2 = \{C\}$. Then

(i) $\langle c_2, U_2 \rangle$ is sound and constant.

ad soundness. Suppose not: then for some $D_1, \ldots, D_m$ in c_2, $\vdash D_1 \,\&\ldots\&\, D_m \rightarrow C$, since a disjunction of C reduces to C in CLQ. As $D_i \in c_2$ for some $A_i \in b_1$, $A_i \rightarrow D_i \in a$; and as $A_i \in b_1$, $\vdash B \rightarrow A_i$. Hence, by Rule Suffixing, $\vdash A_i \rightarrow D_i \rightarrow . B \rightarrow D_i$, and $B \rightarrow D_i \in a$. But this is for each i with $1 \leq i \leq m$. Hence by &-composition, $B \rightarrow D_1 \,\&\ldots\&\, D_m \in a$; and so as $\vdash D_1 \,\&\ldots\&\, D_m \rightarrow C$, $\vdash B \rightarrow D_1 \,\&\ldots\&\, D_m \rightarrow . B \rightarrow C$ by Rule Prefixing; whence $B \rightarrow C \in a$ contradicting the hypothesis that $B \rightarrow C \notin a$. It is a corollary that $C \notin c_2$.

ad constancy. Suppose not again: that is

$\langle c_2, U_2 \cup \{(x) D(x)\} \rangle$ is sound (a)

but for every term u

$c_2 \vdash_{CLQ} C \vee (x)\, D(x) \vee D(u/x)$. (b)

Consider, for an appropriate variable y chosen as not free in C, B or $(x)\, D(x)$, $D = C \vee (x)\, D(x) \vee D(y/x)$. Note that $\vdash C \rightarrow D$, by addition. Now suppose $D \in c_2$. Then for some $A, A \rightarrow D \in a$ with $A \in b_2$, i.e. $\vdash B \rightarrow A$. Hence, by Rule Suffixing (from R3), $\vdash A \rightarrow D \rightarrow . B \rightarrow D$, so $B \rightarrow D \in a$. Thus, as a is rich, $(y)\,(B \rightarrow D) \in a$, whence by A10 and A11, $B \rightarrow . C \vee (x)\, D(x) \vee (y)\, D(y/x) \in a$; and so $B \rightarrow . C \vee (x)\, D(x) \in a$. (Observe that appeal to the richness of a can be avoided by considering instead of D, $D' = (y)(C \vee (x)\, D(x) \vee D(y/x))$.) Hence as $B \in b_1$, $C \vee (x)\, D(x) \in c_2$, contradicting (a). Accordingly $D \notin c_2$. Thus for every $A \in b_1$, $A \rightarrow D \notin a$, so in particular $B \rightarrow D \in a$. But this is impossible, since because $B \rightarrow C \in a$ and $\vdash B \rightarrow C \rightarrow . B \rightarrow D$ (from $\vdash C \rightarrow D$ by Rule Prefixing, from R3), $B \rightarrow D \in a$.

In virtue of (i) the Constant Extension Lemma applies to yield a CLQ - maximal pair $\langle c, U_2' \rangle$ with $c_2 \subseteq c$. Since $C \in U_2'$, $C \notin c$.

(ii) $\langle b_1, U_1 \rangle$ is sound and constant, where $U_1 = \{A: (PB)(A \to B \in a \& B \in c)\}$.

ad soundness. Suppose not: then for some $E_1, \ldots, E_n \in b_1$ and some $C_1, \ldots, C_m \in U_1$, $\vdash E_1 \& \ldots \& E_n \to . C_1 \vee \ldots \vee C_m$. Since $\vdash B \to E_i$ for each $E_i \in b_1$, $\vdash B \to . C_1 \vee \ldots \vee C_m$, by B^+ principles. As $C_j \in U_1$, for some $D_j \notin c$, $C_j \to D_j \in a$, for each j, $1 \leqslant j \leqslant m$. Thus by repeated application of T11, $C_1 \vee \ldots \vee C_m \to D_1 \vee \ldots \vee D_m \in a$; and so, using Rule Suffixing, $B \to D_1 \vee \ldots \vee D_m \in a$. Since $B \in b_1$, $D_1 \vee \ldots \vee D_m \in c_2 \subseteq c$. So by primeness of c, $D_1 \in c$ or $D_2 \in c_2$ or ... or $D_m \in c$, contradicting $D_j \notin c$ for each j, $1 \leqslant j \leqslant m$.

ad constancy. Suppose not again: that is

$\langle b_1, U_1 \cup \{(x) D(x)\} \rangle$ is sound (a)

but for every term u

$b_1 \vdash U_1 \cup \{(x) D(x), D(u/x)\}$ (b) .

i.e., since B generates b_1, $B \vdash U_1 \cup \{(x) D(x), D(u/x)\}$, i.e. by the Transfer Lemma $\overline{U}_1 \vdash \overline{B} \vee (x) D(x) \vee D(u/x)$. Now consider for appropriate z not free in $\overline{B}$, C and $(x) D(x)$, the formula $E = (z)(\overline{B} \vee (x) D(x) \vee D(z/x))$.(Alternatively the unquantified formula $\overline{B} \vee (x) D(x) \vee D(z/x)$ can be looked at.) Note that $\vdash \overline{B} \to E$ by T9, R3, A10 and A11. E cannot belong to $\overline{U}_1$; for if it did $\overline{U}_1 \vdash E$, that is confining and contracting on bound variables, $\overline{U}_1 \vdash \overline{B} \vee (x) D(x)$; and so transferring $B \vdash U_1 \vee (x) D(x)$, contradicting (a). Hence $E \notin \overline{U}_1$, i.e. $\overline{E} \notin U_1$. Thus for every B, when $\overline{E} \to B \in a$ then $B \in c$. Accordingly as $C \notin c$, $\overline{E} \to C \notin a$. But by Rule Contraposition, $\vdash \overline{E} \to B$, so $\vdash B \to C \to . \overline{E} \to C$; hence as $B \to C \in a$, $\overline{E} \to C \in a$, which is impossible.

In virtue of (ii) the Constant Extension Lemma applies to yield a maximal pair $\langle b, U_1' \rangle$, with $b_1 \subseteq b$. Since $B \in b_1$, $B \in b$. It remains to prove (iii) $R_c abc$. Suppose for arbitrary A and B, $A \to B \in a$ and $A \in b$; to show $B \in c$. Since $A \in b$, $A \notin U_1$ since U_1 is disjoint from b. Thus by definition of U_1, $B \in c$.

It is worth observing that arguments used in establishing constance have rather more general application, and perhaps merit drawing into a separate lemma.

The reduced case differs only in a few details. Firstly, and this leads to the differences, $b_1 =_{Df} \{D: \vdash_T B \to D\}$. Secondly, versions of Rule Suffixing and Rule Prefixing are required for T, namely if $A \to B \in T$ then $B \to C \to . A \to C \in T$ and if $C \to D \in T$, $B \to C \to . B \to D \in T$. But these forms follow using R 3 given that T is closed under $\supset$. It is:-

SUBLEMMA. *If* $\vdash_{CLQ} A \supset B$ *and* $A \in T$ *then* $B \in T$.

PROOF. Suppose $\mathbf{A} \in T$ and $\vdash \mathbf{A} \supset \mathbf{B}$. Then as T is regular $\mathbf{A} \supset \mathbf{B} \in T$, so $\mathbf{A}$ & $(\overline{\mathbf{A}} \vee \mathbf{B}) \in T$. Hence, as T is closed under $\rightarrow$, by T3, $\mathbf{B} \in T$. Otherwise, since T is CLQ regular, all results for CLQ and CLQ theories extend to T and T theories.

ad U. For every a in K_c,

$I((\mathrm{U}x)\mathbf{A}, a) = 1$ iff $I^x(\mathbf{A}, a) = 1$ for every x-variant I^x of I,
iff $I(\mathbf{A}(\mathbf{t}/x), a) = 1$ for every $I(\mathbf{t}) \in \mathcal{D}_c$,
iff $\mathbf{A}(\mathbf{t}/x) \in a$ for every term $\mathbf{t}$ of CLQ', by applying the induction hypothesis and the equation $I(\mathbf{t}) = \mathbf{t}$,
iff $(\mathrm{U}x)\mathbf{A} \in a$, since a is rich and closed under CLQ' implication.

For the truth-valued semantics the matter is still simpler. For $a \in K_c$,
$I((x)\mathbf{A}, a) = 1$ iff $I(\mathbf{A}(\mathbf{t}/x), a) = 1$ for every subject term $\mathbf{t}$;
iff $\mathbf{A}(\mathbf{t}/x) \in a$ for every $\mathbf{t}$,
iff $(x)\mathbf{A} \in a$. For if $\mathbf{A}(\mathbf{t}/x) \in a$ for every $\mathbf{t}$ then, by richness, $(x)\mathbf{A} \in a$; and the converse follows by instantiation and CLQ'-closure.

ad t (where applicable). $I(t, a) = 1$ iff $O_{\mathrm{CLQ}}\, a$
iff $t \in a$.

Suppose, first, $O_{\mathrm{CLQ}}\, a$. Then a is regular, but $\vdash_{\mathrm{CLQ}} t$, so $t \in a$. Conversely, suppose $t \in a$. Then since for every theorem $\mathbf{A}$, $\vdash_{\mathrm{CLQ}} t \rightarrow \mathbf{A}$, $\mathbf{A} \in a$, so $O_{\mathrm{CLQ}}\, a$.

ad (β). It is at this stage that results thus far achieved become conspicuously more system dependent.

ad q1. There are two cases:-

ad q1b. Reduced case only. Then $R_c Taa$ holds in virtue of the restriction of theories in the model to T-theories. Such a condition fails in the unreduced case.

ad q1a. Suppose $R_c Tab$. If further $\mathbf{A} \in a$, then, since $\mathbf{A} \rightarrow \mathbf{A} \in T$, $\mathbf{A} \in b$, whence $a \subseteq b$ follows. But then $a = b$ by the following:

SUBLEMMA. *Where a and b are n.d. straight* (CLQ-) *theories (in fact $\bar{}$ consistency and $\bar{}$ - completeness are enough), if $a \subseteq b$ then $a = b$, i.e. the theories are maximal in another important sense.*

For suppose $a \subseteq b$ but $\mathbf{A} \in b$ and $\mathbf{A} \notin a$. Since $\overline{\mathbf{A}} \in a$ (by completeness, i.e. primeness and $\mathbf{A} \vee \overline{\mathbf{A}} \in a$), $\overline{\mathbf{A}} \in b$ contradicting consistency and so non-degeneracy.

ad $T \in K_c$. For many relevant systems this is a problematic condition: not however for classical systems which include $\mathbf{A}$ & $(\mathbf{A} \rightarrow \mathbf{B}) \supset \mathbf{B}$. The problem has been to show that T is closed under provable LQ-implication: it is immediate from the expansion corollary that *otherwise* T has the right properties.

Suppose $\mathbf{A} \in T$ and $\vdash \mathbf{A} \rightarrow \mathbf{B}$; to show $\mathbf{B} \in T$. Since T is regular $\mathbf{A} \rightarrow \mathbf{B} \in T$, so $\mathbf{A}$ & $(\mathbf{A} \rightarrow \mathbf{B}) \in T$. But then, by $\mathbf{A}$ & $(\mathbf{A} \rightarrow \mathbf{B}) \supset \mathbf{B}$ and closure of T under $\supset$, $\mathbf{B} \in T$.

This completes the completeness argument for system CAQ. Also many extensions of CAQ can be rather automatically handled by transferring results established at the sentential case, for example all those where the modelling conditions can be stated without use of quantifiers, as the extension of CAQ by $\mathbf{A} \,\&\, (\mathbf{A} \to \mathbf{B}) \to \mathbf{B}$, for which the modelling condition is q2, illustrates:-

ad q2. Suppose $\mathbf{A} \to \mathbf{B} \in a$, $\mathbf{A} \in a$, for $a \in \widetilde{K}_{\mathrm{CLQ}}$. Then $\mathbf{A} \,\&\, (\mathbf{A} \to \mathbf{B}) \in a$ whence by T12, $\mathbf{A} \,\&\, (\mathbf{A} \to \mathbf{B}) \to \mathbf{B}$, $\mathbf{B} \in a$. Hence $\widetilde{R}aaa$, whence $Raaa$ follows the restriction.

When the modelling condition contains quantifiers uneliminably, establishing the condition may prove a much more arduous business, as will appear. But first let us consider the unreduced system CBQ.

ad q1a'. Suppose $x \in O_{\mathrm{CLQ}}$ and $R_{\mathrm{CLQ}}xab$. Then as before $a \subseteq b$, since $\mathbf{A} \to \mathbf{A} \in x$; whence as before $a = b$.

ad q1b'. Define $x_1 = \{\mathbf{A}: \vdash_{\mathrm{CLQ}t} t \to \mathbf{A}\}$. Then, by the t rule, $x_1 = \{\mathbf{A}: \vdash_{\mathrm{CLQ}t} \mathbf{A}\}$ i.e. CLQt. Also for $a \in K_{\mathrm{CLQ}\,t}$, $R_{\mathrm{CLQ}\,t}x_1aa$, by definition of $R_{\mathrm{CLQ}\,t}$. Now define $U_1 = \{\mathbf{A}: (\mathrm{P}\mathbf{B}, \mathbf{C}) \vdash \mathbf{A} \to . \mathbf{B} \to \mathbf{C} \,\&\, \mathbf{B} \in a \,\&\, \mathbf{C} \notin a\}$. Then

(i) $\langle x_1, U_1 \rangle$ is sound and constant.

ad Soundness. Suppose otherwise. Then for some $\mathbf{D}_1, \ldots, \mathbf{D}_m$ in x_1 and $\mathbf{E}_1, \ldots, \mathbf{E}_n \in U_1$, $\vdash \mathbf{D}_1 \,\&\, \ldots \,\&\, \mathbf{D}_m \to \mathbf{E}_1 \vee \ldots \vee \mathbf{E}_n$. But, as shown in Routley and Meyer 1978, U_1 is closed under disjunction, and by definition of x_1, $\vdash t \to \mathbf{D}_i$ for each i, $1 \leqslant i \leqslant m$. Hence $\vdash t \to \mathbf{E}$, for $\mathbf{E} \in U_1$; so $\vdash \mathbf{E}$, whence for some $\mathbf{B}$, $\mathbf{C} \vdash \mathbf{B} \to \mathbf{C}$, $\mathbf{B} \in a$ and $\mathbf{C} \in a$, which is impossible.

ad Constancy. Suppose again otherwise, that is

$\langle x_1, U_1 \cup \{(x)\mathbf{D}(x)\} \rangle$ is sound (a) ;

but for every term $\mathbf{u}$

$\langle x_1, U_1 \cup \{(x)\mathbf{D}(x), \mathbf{D}(\mathbf{u}/x)\} \rangle$ is not sound (b),

i.e. $\overline{U}_1 \vdash \overline{t} \vee (x)\mathbf{D}(x) \vee \mathbf{D}(\mathbf{u}/x)$.

Consider, for appropriate y, $\mathbf{D} = \overline{t} \vee (x)\mathbf{D}(x) \vee \mathbf{D}(y/x)$.

Since $\overline{U}_1 \vdash \mathbf{D}$, for some

$\overline{\mathbf{A}}_1, \ldots, \overline{\mathbf{A}}_n \in \overline{U}_1$, $\vdash \overline{\mathbf{A}}_1 \,\&\, \ldots \,\&\, \overline{\mathbf{A}}_n \to \mathbf{D}$, i.e. $\vdash \overline{\mathbf{D}} \to . \mathbf{A}_1 \vee \ldots \vee \mathbf{A}_n$ for $\mathbf{A}_1 \ldots \mathbf{A}_n \in U_1$. Thus, as U_1 is closed under disjunction, $\vdash \overline{\mathbf{D}} \to \mathbf{A}$ for some $\mathbf{A}$ in U_1. By T11 and T6, $\vdash \mathbf{D} \vee \mathbf{A}$, whence, by QR1 and QA3, $\vdash \overline{t} \vee (x)\mathbf{D}(x) \vee \mathbf{A}$, and by T3 and $\vdash t$, $\vdash (x)\mathbf{D}(x) \vee \mathbf{A}$, contradicting (a).

As (i) holds, the Constant Extension Lemma yields a CLQ-maximal pair $\langle x, U_1' \rangle$ with $x_1 \subseteq x$ and $U_1 \subseteq U_1'$.

Since Ox_1, Ox. To show $Rxaa$, suppose $\mathbf{A} \to \mathbf{B} \in x$ and $\mathbf{A} \in a$. Then $\mathbf{A} \to \mathbf{B} \notin U_1$, so as $\vdash \mathbf{A} \to \mathbf{B} \to . \mathbf{A} \to \mathbf{B}$ and $\mathbf{A} \in a$, $\mathbf{B} \in a$ as required.

This completes the argument for CBQ$\mathcal{t}$.

It is a straightforward matter to enlarge the completeness result to cover extensions of basic systems such as CAQ and CBQ$\mathcal{t}$ whose additional semantical postulates can be formulated (using the generality interpretation) without bound variables. In particular, the system obtained by adding $A \& (A \to B) \to B$ to CAQ is readily encompassed since the modelling condition q2, $Raaa$, is unproblematic. For $\widetilde{R}aaa$ follows using the definition of $\widetilde{R}$, and then $Raaa$ comes by restriction to n.d. straight theories.

To make the theorem quite specific characterise a weaker quantified relevant logic as any extension of basic systems (BQ, BQ$\mathcal{t}$, CBQ, CBQ$\mathcal{t}$, CAQ) all of whose additional relational semantical postulates can be expressed in quantifier - free from (eventual adequacy of the postulates thus being presumed). ■

§8. ACCOMODATING RELEVANT NEGATION AND REMOVING $\mathcal{t}$: ADEQUACY FOR WEAKER QUANTIFIED RELEVANT LOGICS.

So far relevant negation, ~ , has been left out of the nontrivial semantics. It may however be introduced, definitionally, into the so-called classical- relevant scheme of things by appeal to the star connective, * , of Meyer and Routley 1973. Then $\sim A =_{Df} \overline{A^*}$. The star connective, which is a syntactical surrogate of the star operation of relevant semantics, is construed by Meyer (in Meyer 197+) as a weak assertion functor, in part on account of its definition in $LQ^{\urcorner}$ systems as $A^* =_{Df} \overline{\sim A}$.

The basic postulates on * , to be added to CBQ and extensions are these:-

D1. $A^* \& B^* \to (A \& B)^*$.

D2. $(A \vee B)^* \to . A^* \vee B^*$.

D3. $A^{**} \leftrightarrow A$.

RD1. $A \to B \Rightarrow A^* \to B^*$.

QD1. $(x) A^* \to ((x) A)^*$.

QD2. $((Px) A)^* \to (Px) A^*$.

Semantics for starred classical relevant systems add an operation * on K to classical model structures. That is, a CBQ* m.s. M is a structure $M = \langle T, O, K, R, *, \mathcal{D} \rangle$, where $\langle T, O, K, R, \mathcal{D} \rangle$ is a CLQ m.s., and * is an operation on K such that for a in K,

s1. $a^{**} = a$. (5)

In the case of reduced modellings, O is again elided.

As usual extensions of CBQ* are modelled by imposing corresponding modelling conditions. A few star extensions of immediate interest are tabulated:-

Axiom scheme	Modelling condition
OD2. $A \vee \overline{A^*}$ (LEM).	s2. $x = x^*$ for $x \in O$

(5) For De Morgan systems of Routley and Meyer 1978, Chapter 4, this condition and its corresponding axiom, $A^{**} \leftrightarrow A$, are deleted.

OD3. $A \to B \to . \overline{B^*} \to \overline{A^*}$ (Contraposition). s3. $Rabc \supset Rac^*b^*$.

OD4. $(A \to \overline{A^*}) \to \overline{A^*}$ (Reductio). s4. Raa^*a.

These optional extras suffice to cover the negation logics of GQ, TQ, EQ and RQ.

THEOREM 7. (Adequacy Theorems for starred systems CBQt* and CAQ* and various extensions, including all the star axioms). *The statements for soundness and strong completeness are as for* CLQ *systems considered.*

PROOF (1) Soundness is a matter of verifying new postulates. Some examples illustrate the cases

ad D1. Suppose $I(A^* \mathbin{\&} B^*, a) = 1$, i.e. $I(A, a^*) = 1 = I(B, a^*)$; to show $I((A \mathbin{\&} B)^*, a) = 1$, i.e. $I(A, a^*) = 1 = I(B, a^*)$. Immediate.

ad RD1. Suppose in some model, $I(A^*, a) = 1 \neq I(B^*, a)$, i.e. $I(A, a^*) = 1 \neq I(B, a^*)$. Then in the same model, but with counterworld a^*, $A \to B$ is falsified.

ad QD1. Suppose $I((x)A^*, a) = 1 \neq I(((x)A)^*, a)$. Then for every x-variant I', $I'(A^*, a) = I'(A, a^*) = 1$; but $1 \neq I((x)A, a^*)$, so for some x-variant I', $I'(A, a^*) \neq 1$.

(2) Completeness. In canonical CLQ* models define $a^* = \{A : A^* \in a\}$. New details required are these:

ad (α). There is one new case, that for connective *, but the result is immediate:

$I(A^*, a) = 1$ iff $I(A, a^*) = 1$, i.e. iff $A \in a^*$, i.e. iff $A^* \in a$.

ad (β). It has to be shown that * is well defined, primarily that $a^* \in K$ when $a \in K$, and that modelling conditions on * are satisfied.

ad a^* is closed under provable implication. For suppose $\vdash A \to B$ and $A \in a^*$, i.e. $A^* \in a$. Then as $\vdash A^* \to B^*$, by RD1, $B^* \in a$, i.e. $B \in a^*$, as required.

ad a^* is adjunction closed. Suppose $A, B \in a^*$. Then $A^*, B^* \in a$, so $A^* \mathbin{\&} B^* \in a$. But $\vdash A^* \mathbin{\&} B^* \to (A \mathbin{\&} B)^*$, i.e. D1; so $(A \mathbin{\&} B)^* \in a$, i.e. $A \mathbin{\&} B \in a^*$ as required.

ad a^* is prime, given a is. Suppose $A \vee B \in a^*$; then $(A \vee B)^* \in a$, and so, by D2, $A^* \vee B^* \in a$. As a is prime, $A^* \in a$ or $B^* \in a$, i.e. $A \in a^*$ or $B \in a^*$.

ad a^* is rich. Suppose $A(t/x) \in a^*$ for every t, i.e. $A(t/x)^* \in a$ for every t. As a is rich, $(x)A^* \in a$. But by QD1, $\vdash (x)A^* \to ((x)A)^*$; thus $((x)A)^* \in a$, and so $(x)A \in a^*$.

ad a^* is saturated. Suppose $(Px)A \in a^*$. Then $((Px)A)^* \in a$, so by QD2, $(Px)A^* \in a$. As a is saturated, $A(t/x)^* \in a$ for some term t, i.e. $A(t/x) \in a^*$ for some t.

It has further to be shown that $a^{**} = a$, but this is guaranteed by D3. For optional extras there are these details.

ad s2. Suppose $x \in O$. Then $A \vee \overline{A^*} \in x$, so if $A \in x^*$ then materially $A \in x$, i.e. $x^* \subseteq x$. Hence by a sublemma $x^* = x$.

ad s3. Suppose $Rabc$, $A \to B \in a$ and $A \in c^*$; to show $B \in b^*$, i.e. $B^* \in b$. By OD3, $\overline{B^*} \to \overline{A^*} \in a$, so as $Rabc$ if $\overline{B^*} \in b$, $\overline{A^*} \in a$, i.e. if $A^* \in c$ then $B^* \in b$, as required. Then restrict to straight theories.

ad s4. Suppose $A \to B \in a$, $A \in a^*$ but $B \notin a$. Then $A^* \in a$ and $\overline{B} \in a$, so $A^* \& \overline{B} \in a$.

To make the final linkage in these circuitous adequacy arguments for relevant LQt systems, observe that LQ$^\urcorner$ is essentially the same system as CLQ* (correspondences between extensions have of course to be kept straight). ∎

THEOREM 8. (Equivalence theorem). *A is a theorem of* LQ$^\urcorner$ *iff (under translation) A is a theorem of* CLQ*.

PROOF. The translation is through the definitions given. Connective * is defined in LQ$^\urcorner$ by: $A^* =_{Df} \overline{\sim A}$, and $\sim$ is defined in CLQ* by $\sim A =_{Df} \overline{A^*}$: otherwise the notation of the system is the same. Proof of the equivalence can be purely syntactical: but appeal to Theorem 7 simplifies things.

(1) CLQ* includes LQ$^\urcorner$. The relevant negation postulates of LQ have to be established for the basic case BQ.

ad Double Negation, $A \leftrightarrow \sim\sim A$, i.e. $A \leftrightarrow \overline{(\overline{A^*})^*}$. This follows using $\overline{A^*} \leftrightarrow (\overline{A})^*$ as follows:- By relettering and $A^{**} = A, \overline{A} \leftrightarrow (\overline{A^*})^*$, whence by Rule Contraposition $A \leftrightarrow \overline{(\overline{A^*})^*}$.

ad $\overline{A^*} \leftrightarrow (\overline{A})^*$. Most simply use the adequacy Theorem 7.

ad Rule Contraposition, $A \to B \Rightarrow \sim B \to \sim A$.
$A \to B \Rightarrow A^* \to B^* \to \overline{B^*} \to \overline{A^*}$.

(2) LQ$^\urcorner$ includes CLQ*. Star postulates have to be established.

ad D1, i.e. $\overline{\sim A} \,\&\, \overline{\sim B} \to \overline{\sim(A \,\&\, B)}$. For the present let $\dot{\vee}$ be defined $A \dot{\vee} B =_{Df} \sim(\sim A \,\&\, \sim B)$: In fact $A \vee B \leftrightarrow A \dot{\vee} B$. Now $\overline{A} \,\&\, A \to B$ and $\overline{A} \,\&\, B \to B$, so $\overline{A} \,\&\, A \dot{\vee} \overline{A} \,\&\, B \to B$, whence distributing $\overline{A} \,\&\, (A \dot{\vee} B) \to B$, i.e. $\overline{A} \,\&\, \sim(\sim A \,\&\, \sim B) \to B$. Relettering $\overline{\sim A} \,\&\, \sim(A \,\&\, B) \to \sim B$, so by CR1, $\overline{\sim A} \,\&\, \overline{\sim B} \to \overline{\sim(A \,\&\, B)}$.

ad D2, i.e. $\overline{\sim(A \vee B)} \to .\ \overline{\sim A} \vee \overline{\sim B}$. Similar.

ad D3. By Double Negation for each negation.

ad RD1. Suppose $A \to B$; to show $\overline{\sim A} \to \overline{\sim B}$. Apply Rule Contraposition twice. ∎

Proof that t may be eliminated, i.e. that it conservatively extends systems to which it has been added, may take either syntactical or semantical form. Here a semantical version using the trivial semantics is given, subsequently in the case of stronger systems a syntactical version is outlined.

THEOREM 9. (t removal). LQt *is a conservative extension of* LQ *and* CLQt

of CLQ.

PROOF. Since $LQ \subseteq LQt$ one half is immediate. For the converse suppose **A** is a wff of LQ but not a theorem of LQ: to show **A** is not a theorem of LQt, it suffices to furnish a countermodel to **A** in an adequate semantics for LQt. This can be done in a straightforward reworking of the trivial semantics in which world T is replaced by set of worlds O and truth is determined at an arbitrary element of O (or alternatively validity in a model is characterised in terms of holding at all elements of O). Theorem 3 goes through as before, and it has a corollary an exactly analogous result for LQt, where t, subject to the t rule, is evaluated by the semantical rule:

$I(t, a) = 1$ iff Oa. Now, as **A** is not a theorem of LQ, there is by the reworking of Theorem 3 and LQ countermodel, built on O, to **A**. But this countermodel is in fact an LQt countermodel to **A**; hence by the corollary to Theorem 3, **A** is not a theorem of LQt. The argument for classical systems is similar. ■

COROLLARY (Completeness for CBQ and weaker extensions). *Where* CLQ *is a weaker extension of* CBQ, *if* **A** *is* CLQ-*valid then* **A** *is a theorem of* CLQ.

PROOF. Suppose **A**, which is a wff of CLQ, is not a theorem of CLQ. Then, by Theorem 9 it is not a theorem of $CLQt$. Hence, by Theorem 6, it is not $CLQt$-valid. But **A** is a wff of CLQ, so, as its semantical assessment does not involve t, **A** is not CLQ-valid. (Strictly the last move involves a step like that used in Theorem 9, namely:- Reformulate CLQ semantics with O in place of T, and show that the semantics is equivalent in that the valid wff are the same. Then **A** is not CLQ-valid in the O based semantics, so it is not CLQ valid in the T based semantics.)

THEOREM 10. (Adequacy theorem for weak relevant logics). **A** *is a theorem of* LQ *iff* **A** *is* LQ-*valid, where* LQ *is any weaker extension of* BQ *or* BQT.

PROOF. Soundness is by the usual induction. Proof of completeness assembles earlier results. Suppose $\sim\vdash_{LQ} \mathbf{A}$.

Then **A** is not a theorem of $LQ^{\neg}$, by the corollary to Theorem 4. Hence, by Theorem 8, **A** is not a theorem of CLQ*; and so **A** is not CLQ*-valid; that is, there is a valuation I on CLQ* m.s. $M = \langle T, O, K, R, *, D \rangle$ for which $I(\mathbf{A}, T) \neq 1$. But 1) the m.s. M is an LQ m.s. and 2) the valuation I is an LQ interpretation in M; hence **A** is not LQ valid. It remains to prove 1) and 2).

ad (2). Apply the reduction: $a \leqslant b$ iff $a = b$. Trivially, if $a = b$ then $a \leqslant b$. Conversely suppose $a \leqslant b$, i.e. $(Px)(Ox \ \& \ Rxab)$. By q1a', $a = b$.

ad (1). It suffices to establish the semantical postulates. Postulate p2, $a \leqslant d \ \& \ Rdbc \supset Rabc$ is immediate. p1, i.e. $a \leqslant a$, is guaranteed by q16'; and the * postulates in LQ m.s. are taken care of by the corresponding postulates in CLQ* m.s.

Hence **A** is not LQ-valid. In case LQ does not involve t there is a further detail: steps from Theorem 9 are applied. ■

As usual a strong completeness result for a system facilitates the proof of many results about the system and some of its extensions, e.g. Skolem-Löwenheim theorems and compactness, and the completeness of strong identity theory; but in the case of relevant quantificational theories it also facilitates the proof of many other results, e.g. (where $A \vee \sim A$ is a theorem) the admissibility of the rule γ of material detachment.

§9. THE QUEST FOR ADEQUACY PROOFS FOR STRONGER QUANTIFIED RELEVANT LOGICS.

Only validation of q3 is outstanding in the semantical quest for CRQ and so RQ. Outstanding it seems however determined to remain, given the methods so far adopted. Hence problem P2). Suppose, to see where the trouble lies, that $Rabz$ and $Rzcd$. Define $x' = \{C: (PB)(B \to C \in a \ \& \ B \in c)\}$. Then it is not difficult to show that $x' \in \widetilde{K}_{CRQ}$, $\widetilde{R}acx'$ and $\widetilde{R}bx'd$. What has not been shown, so far, is that for some appropriate U, $\langle x', U \rangle$ is constant, so that the Constant Extension Lemma can be applied. The problem could be avoided were x' rich. But there is no guarantee, even though a and c are rich, that x' is rich. Suppose that $A(t/x) \in x'$ for each term t. Thus for each term t there is some wff B^t say, such that $B^t \to A(t/x) \in a$ and $B^t \in c$. But there need be no uniformity in the form of B^t from one term to another; and nothing to enable a move to $(x)(B^t \to A(t/x)) \in a$. The problem does suggest, firstly, including an infinitary conjunction in the framework, and then, alternatively and at least cost, including some sentential constant (like t in the case of CRQ itself) which can represent appropriate infinite conjunctions. It is easy enough to see how an infinite conjunction might solve the problem at hand (even if at the same time storing up new ones elsewhere). Let $\& B$ be the conjunction of all the wff B^t, and suppose theories are also closed under denumerable adjunction. Then $\& B \in c$. Also since $\& B \to B^t$ for each term t, by Generalised Simplification, $\& B \to A(t/x) \in a$ for every term t. Hence by Generalisation and Distribution of U, $(x) \ \& \ B \to (x) A \in a$. But as c too is rich $(x) \ \& \ B \in a$, should $\& B$ contain x free. Hence $(x) A \in x'$. To show that x' is a CRQ-theory when closure under generalised conjunction is adopted, a postulate of Generalised &-Composition is required. In short, exactly the standard postulates for the generalised conjunction of infinitary logics are already motivated.

Naturally all this had occurred sometime ago to Meyer who asked Helman to investigate (though without much reported success: see Helman 197+) infinitary R^+. A slightly different approach which commends itself is to define the theories used in the CRQ canonical model in terms of an infinitary extension CRQI - essentially $CR_{\omega_1 \omega}$ - of CRQ. An infinitary CRQ theory, a CRQI-*theory*, is a class of wff of CRQI closed under CRQI provable implication and under generalised adjunction. Regularity, primeness, richness, and so on, are defined as before. To show that when $\langle S, T \rangle$ is CRQI maximal S is closed under generalised adjunction, derivability has to be generalised to admit infinite conjunctions on the left hand side. The question arises as to whether to admit infinite disjunctions - not needed in the completeness argument - and suitable generalised distribution principles. Symmetry suggests so, but it may not be compulsory. But this approach encounters problems in the extension lemmas, problems strongly suggesting a cut-down to countably many wff, out of the infinitary scene and back to sentential constants representing the infinitary conjunctions that are required. This takes us to the generalised t method, which is in outline as follows.

The basic point exploited is that the completeness construction need employ only countably many theories. If we can add conservatively to each theory c a constant t_c, subject to the axioms $\vdash t_c \to B$ iff $B \in c$, then the work will be done. And this can be done given an appropriate construction.

§10. CONCLUSION : OUTSTANDING ISSUES.

Nothing upsets a bonafide relevant logician so much as having to appeal to classical proof theory to push his results through. In establishing the adequacy of the semantics for relevant sentential logics, classical methods are the main ones so far used; but it has been shown (primarily by Meyer) that, at least in the case of strong relevant systems such as R, the use of classical methods is inessential, that relevant semantics using proof principles R itself supplied can be given. Similarly weaker relevant logics can be equipped with a relevant semantics though in some cases the proof methods, though relevant, may have to exceed those the system acknowledges. But in the case of the semantics for quantified relevant logic there is a double indignity. Not only are the methods classical but the argument is indirect and circuits through rather overtly classical territory. What is wanted, in particular, is a completeness argument that avoids this second circuit. Direct attempts with stronger systems at avoiding this detour have so far mostly led into complex bogs that we have been unable to fight our way through.

But there are perhaps some alternative, if not completely direct routes, through the swamps. One, already noted, involves an appeal to infinitary methods to deal with *both* main difficulties in the completeness proof. Another, not noted, is to vary the semantics to fit with the facts of the noninfinitary-based canonical models. This involves widening the field of investigation and considering alternative quantifier rules and inconstant domains. Some (temporary) relief from the problem of establishing richness of R related theories can be had by amendment of the rules for evaluating quantifiers. While the elementary evaluation rule is retained no relief can be had of course, since for the completeness argument, that $(x)\mathbf{A} \in a$ iff $\mathbf{A}(\mathbf{t}/x) \in a$ for every term $\mathbf{t}$, to succeed a must be rich. But the elementary rule has no monopoly; different rules are adopted in semantics for intuitionistic logic, and are required where domains are not constant from world to world; and there is much scope for the investigation of different context rules in a relevant context. The elementary rule takes it for granted that the universal quantifier behaves semantically like a conjunction, an extensional conjunction. There are, however, other analogies with sentential connectives that can be taken equally seriously, in particular that the universal quantifier behaves like

i) an intensional conjunction, or

ii) a necessity connective.

In stating these rules, and in subsequent initial investigations, let us switch, to make life easier, to a substitutional-style (6) approach to the interpretation of quantifiers; for this aids (as Dunn pointed out to me) in the understanding of quantifiers and also eliminates further steps in the completeness arguments we are trying to force (since the canonical model connects quantifiers with terms of a theory). The *intensional* rule i) may then be stated (in part) as follows:

$I((x)\mathbf{A}, a) = 1$ iff, for some b such that Uba, $I(\mathbf{A}(\mathbf{t}/x, b) = 1$ for every term $\mathbf{t}$, i.e., in symbols, $(\mathrm{P}x)(Uba \;\&\; (\mathbf{t})(I(\mathbf{A}(\mathbf{t}/x), b) = 1)$, where U is a two-place relation on K.

The *necessity* rule ii) takes (in its main part) the very different form:

$I((x)\mathbf{A}, a) = 1$ iff, for every world b such that Vab, for every term $\mathbf{t}$, $I(\mathbf{A}(\mathbf{t}/x), b) = 1$, or equivalently in symbols, iff $(b, \mathbf{t})(Vab \supset I(\mathbf{A}(\mathbf{t}/x), b) = 1)$.

There is one striking difference between universality and necessity, namely that there is no equivalent with necessity to vacuous quantification. The differ-

(6) Remember however that terms include variables.

ence emerges conspicuously in the vacuous quantification principle, $A \to (x)A$ with x not free in A, which extended to necessity would collapse modalities; it also emerges in the confinement principle QA3, which is *not* matched by an acceptable necessity principle $\Box(A \vee B) \to . A \vee \Box B$. In order not to fail confinement the best strategy appears to split the necessity rule into two parts, that already given in the case of nonvacuous quantification, and the following rule for vacuous quantification, namely

$$I((x)A, a) = 1 \text{ iff } I(A, a) = 1 \text{ when } x \text{ is not free in } A.$$

There is then the matter of ensuring that the rules fit together properly.

A similar two-part rule also proves advantageous in the case of the intensional rules. Specifically, the rule is as follows:

When x is not free in A, $I((x)A, a) = 1$ iff $I(A, a) = 1$; and otherwise,

when x is free in A, $I((x)A, a) = 1$ iff $(Pb)(Uba \;\&\; (t)(I(A(t/x), b) = 1))$.

The rule fits well into the following semantical framework. A (basic) reduced LQ *i.m.s.* M is a structure $M = \langle T, K, R, *, U, D \rangle$, where $\langle T, K, R, * \rangle$ is an L.m.s. (as in Routley and Meyer 1978), D is a nonnull set, and U a two-place relation on K, which conforms to the following conditions (where $a \leqslant b =_{Df} RTab$):

Ui)	$Uba \;\&\; a \leqslant c \supset . Ubc$	(for Hereditariness).
Uii)	$Uab \supset . a \leqslant b$	(for Instantiation and Confinement)
Uiii)	UTT	(for Generalisation).
Uiv)	$Uda \;\&\; Rabc \supset . (Px)(Rdbx \;\&\; Uxc)$	(for U-Distribution).

An *interpretation* I *in* an LQ m.s. is a function defined as before for classical systems but subject to the restrictions:

(1) if $a \leqslant b$ and $I(p, a) = 1$ then $I(p, b) = 1$ for every sentential parameter p; and

(1') if $a \leqslant b$ then $I(g^n, a) \subseteq I(g^n, b)$, for every predicate parameter g^n (for each n).

I is extended to all wff by the rules already given for classical systems *except* that the intensional quantifier rule is used, and the rule for classical negation is supplanted by the relevant rule:

$$I(\sim A, a) = 1 \text{ iff } I(A, a^*) \neq 1.$$

Validity, etc., are defined as before.

But working out these semantics and semantics for the necessity rule, and taking advantage of the considerable simplifications conferred by the new rules in the completeness arguments (since richness is no longer obligatory), are problems for sequels.

Some of you - only some of you - may be disappointed, or disgusted, that I have concentrated pretty well exclusively on techno-logical problems in the semantics of quantified relevant logics, leaving aside problems as to the applications of these semantics and philosophical problems the semantics themselves raise. For example, the semantics are worlds semantics, and there is alleged to be a problem about worlds other than the actual one, and especially about the incomplete and inconsistent worlds the quantifiers of relevant semantics catch. Furthermore the semantics aimed for are constant domains semantics, to which a range of objections are often made, e.g. domains of objects can vary from world to world, that essentialism is involved in such semantics, and so on. There is no time to meet these objections to constant domain worlds semantics here: but I believe that it can be shown, on the basis of the two main facets of noneism explained in Routley 1978 b, nonexistential quantification and extensional identity, that the objections are, for the most part, thoroughly misguided.

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