

The American Plan Completed: Alternative Classical-Style Semantics, without Stars, for Relevant and Paraconsistent Logics

Author(s): Richard Routley

Source: *Studia Logica: An International Journal for Symbolic Logic*, Vol. 43, No. 1/2, Paraconsistent Logics (1984), pp. 131-158

Published by: Springer

Stable URL: <http://www.jstor.org/stable/20015155>

Accessed: 15-07-2016 15:22 UTC

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RICHARD
ROUTLEY

The American Plan Completed: Alternative Classical-Style Semantics, Without Stars, for Relevant and Paraconsistent Logics

Abstract. American-plan semantics with 4 values $1, 0, \{1, 0\} \{ \}$, interpretable as True, False, Both and Neither, are furnished for a range of logics, including relevant affixing systems. The evaluation rules for extensional connectives take a classical form: in particular, those for negation assume the form $1 \in \tau(\sim A, a)$ iff $0 \in \tau(A, a)$ and $0 \in \tau(\sim A, a)$ iff $1 \in \tau(A, a)$, so eliminating the star function $*$, on which much criticism of relevant logic semantics has focussed. The cost of these classical features is a further relation (or operation), required in evaluating falsity assignments of implication formulae.

Two styles of 4 valued relational semantics are developed; firstly a semantics using notions of double truth and double validity for basic relevant system **B** and some extensions of it; and secondly, since the first semantics makes heavy weather of validating negation principles such as Contraposition, a reduced semantics using more complex implicational rules for relevant system **C** and various of its extensions. To deal satisfactorily with elite systems **R**, **E** and **T**, however, further complication is inevitable; and a relation of mateship (suggested by the Australian plan) is introduced to permit cross-over from 1 to 0 values and vice versa.

There are two different, if related, semantical stories that can be told about the contradictory truths of dialethic logics (i.e. strongly paraconsistent logics) — stories that can be world, or theory, relativised to the inconsistent statements that hold at some worlds or theories of paraconsistent logics. On one story, some statements, namely true contradictions, are both true, because true, and false, because contradictory. According to the other story, which may (or may not) insist that no statements are both true and false, some statements are such that both they and their negations are true. Most semantics for paraconsistent logics have been of this latter sort, for example, the semantics for relevant logics (such as Anderson-Belnap systems, **P** systems, and dialectical **D** systems), as well as the semantics for irrelevant logics such as the **C_n** systems. However, for some logics, such as the system **FD** of first degree relevant logics, a semantics of the latter type can be translated into one of the former sort. The question accordingly arises: can other paraconsistent logics also be furnished with semantics of the former sort? What is shown in what follows is that a large class of paraconsistent logics, including all relevant logics which can be given a semantics of the latter type, can also be supplied with semantics of the former sort, though nothing quite as simple as the first degree mapping will suffice. Although the methods extend to a wide range of irrelevant paraconsistent logics with non-normal negations, including **C_n** systems (cf. the methods of [7] 6, which can be rendered 4-valued), the investigation

here undertaken is confined to relevant logics (and various of their irrelevant extensions) with normal negations that can be dressed up in classical fashion.

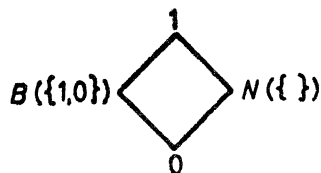
§ 1. The American plan for relevant logic semantics versus the Australian plan ¹

As paraconsistent studies have shown (see [10] and [7]), there are frequently good reasons for taking seriously *theories* or *worlds* that are either inconsistent or incomplete or both (with respect to negation). Let us simply adopt theories, certainly including such inconsistent and incomplete theories, as our guide as to how truth (or holding) values are assigned — whatever we go on to make epistemically or ontically or neutrally of such values as 1 (on, closed, holding, true, ...) and 0 (off, open, failing, false, ...). There are then two conspicuous possibilities for the way the truth value assignment is to be made. One way, the *Australian plan*, simply assigns 1 to a wff that is in the theory and 0 to a wff which fails to be in the theory. Evidently, on the Australian plan, we assign *exactly one* of 1 or 0 to each wff of a theory at that theory (or whatever is taken to *realize* a theory, ontically or neutrally or epistemically). But, where the theory is inconsistent or incomplete, there will be formulae A such that both A , $\sim A$ are assigned value 0, or else both assigned 1. Plainly the classical evaluation of negation, through a rule of the form, $I(\sim A, a) = 1$ iff $I(A, a) = 0$, i.e. $\sim A$ holds in theory a iff A does not hold in a , fails where theory a is inconsistent or incomplete. What generalises and supersedes the classical rule is the star rule, $I(\sim A, a) = 1$ iff $I(A, a^*) = 0$, where $*$ is an operation on theories (which for each theory a gives its opposite a^*). It is such stars, especially in the negation rule, that the rather different *American plan* disposes of entirely. The Americans do not want any (such) stars.

The *American plan* has us adhere resolutely to the classical view that $\sim A$ holds iff A does not hold, everywhere. But it allows that, in addition to being assigned a simple 1 or 0, a formula may be assigned *both* 1 and 0 or *neither*. Specifically, still being guided just by the theory, we now assign 1 to A iff it is in the theory and its negation is not; 0, if A is not in but its negation is. These are the classical circumstances. But if the theory is nonclassical as regards A , the American plan has us assign *both* of 1 and 0 to A if both of A and $\sim A$ are in the theory; *neither*, if neither A nor $\sim A$ is in the theory. Thus on the American plan every formula is assigned at a theory (or whatever is taken to *realize* that theory) one of the four values:

¹The (inappropriate but pictorial) nationalistic labels are taken from Meyer [6] from which initial description of the two rival plans is also drawn.

1, 0, both or neither. The four values are often conveniently represented on (what Belnap calls) the *logical lattice* L_4 :



But, though the usual negation rules apply (in the sense that $\sim A$ is assigned *at least* 1 iff A is assigned at least 0, and dually) the allegedly “strange” Australian situation in which both A , $\sim A$ are assigned 1 gets an American counterpart in which A is assigned both of 1, 0; similarly, the Australian assignment under which neither A nor $\sim A$ hold has an American counterpart in which A is assigned no value at all. Thus *each assignment in each plan has (at an elementary level) a counterpart under the other plan.*

On its own neither the Australian plan nor the American plan affords a semantics for anything. Rather, the plans serve as strategies which may be employed in conjunction with other intuitive and formal insights in arriving at semantics. Each plan has its advantages: the Australian, for its firm 2-valuedness and its success with full relevant logics; the American, it is said, for its more classical generalization of the classical treatment of negation, and removal of the $*$ operator and starred worlds and theories. Nor, for that matter, should the differences between the Australian and the American plans be exaggerated. For relevant logics the hard technical part of the semantics analysis concerns relevant implication, $\rightarrow$, reducing to a less central technical issue the question of whether we be Australians or Americans in our analysis of negation. Or so it seemed from a distance (e.g. to Meyer in [6]); for what will emerge is that under the American plan treatments of implication and negation are not disconnected but interact to some extent. However, as the motivating explanation indicates, we can always, *at the first degree*,² where implication recedes from view to a fair extent, obtain an American evaluation from an Australian one and vice versa.

But for higher degree relevant logics, where nested occurrences of $\rightarrow$ matter, simple correlation between the plans breaks down. And though several North Americans have expended considerable effort in trying to bring the American plan to fruition for full relevant logics, attempts

²The details involved in actually showing this are considerable: they have been supplied by Priest and are presented in [7] 3.2.

hitherto have failed.³ However, with a less simple correlation, and a small Australian input, the American plan can be made good. Showing as much is the task in what follows. What also eventuates is that the American plan is an Ugly American plan, leading to a complex semantics much inferior to the Australian alternative. It is to the same inferior semantics that Dunn's advertised "intuitive semantics" for first degree entailment⁴ also lead, once nesting is duly taken into account.

The American-type semantics given for basic relevant logic **B**, and a large class of extensions thereof, including systems **R**, **E** and **T**, are of classical-style in that they use, as Dunn has explained (in [4]), evaluation rules for $\&$ (conjunction) and $\sim$ (negation) and, derivatively, for $\vee$ (disjunction) that can be made to *look just like* world-relativised classical evaluations; namely, in the form they will be given in what follows:

$\sim$ rules: $1 \in \tau(\sim A, a)$ iff $0 \in \tau(A, a)$; $0 \in \tau(\sim A, a)$ iff $1 \in \tau(A, a)$;

$\&$ rules: $1 \in \tau(A \& B, a)$ iff $1 \in \tau(A, a)$ and $1 \in \tau(B, a)$;

$0 \in \tau(A \& B, a)$ iff $0 \in \tau(A, a)$ or $0 \in \tau(B, a)$;

$\vee$ rules: $1 \in \tau(A \vee B, a)$ iff $1 \in \tau(A, a)$ or $1 \in \tau(B, a)$;

$0 \in \tau(A \vee B, a)$ iff $0 \in \tau(A, a)$ and $0 \in \tau(B, a)$.

Where, as classically and modally, there are just two values, 1 and 0, these are exactly the classical rules. Under the American plan there are, however, four values composed from 1 and 0; as well as 1 and 0 (which can be equated unproblematically with $\{1\}$ and $\{0\}$), there are values $\{1, 0\}$, i.e. both, and $\{\}$, i.e. neither. Part of the point and role of these values has been explained by Belnap [2] and Dunn (e.g. in [4]), and is differently elaborated in [7] (especially 3.2), where it is also argued that neither the epistemic interpretation of values that Belnap and Dunn want to insist

³Semantics of the American type have been provided for irrelevant logics, e.g. by Thomason for Fitch logics (in [9]) and by Dunn for system **RM** (in [5]), but none have been furnished for full relevant logics. As Dunn admits, (in [4]) 'In all frankness I have not had much luck in extending my framework to the whole of systems **E** or **R**'. Belnap's related studies (e.g. [2]) likewise stop short of full higher degree logics. While Meyer's Boolean-valued semantics are adequate for full sentential system **R**, They do *not* eliminate the star operation in the way the American plan demands. (However, on the basis of the results established below this generalised semantics could be reworked without stars. The new relations *S* and *M* would, like relation *R*, operate on algebras.)

It should be observed that the American plan never had the priority over the Australian plan that some Americans have accorded it.

⁴These semantics, which have of course Australian counterparts, are presented in [4] and elsewhere.

upon, nor the ontic interpretation that they see as the alternative, is required or is adequate, and instead a neutral interpretation is canvassed.

There are two well-developed Australian-style semantical analyses for relevant logics, relational semantics and operational (or semi-lattice) semantics (surveyed in [7] 4 and 11), and both can presumably be reworked according to the American plan. In what follows a relational semantics conforming to the American plan will be given. The reasons for not working with the often very attractive and useful operational semantics are primarily that, for the more “orthodox” relevant logics (those studied in Anderson and Belnap [1]), the classical-style evaluation rules given have to be considerably complicated, and secondarily, as a result, the semantics become less “classically intuitive” and much more difficult to keep under control.⁵ The primary problem arises in this way: The operational semantics involve an operation $\oplus$ on theories (pieces of information, or whatever, according to the intuitive interpretation), which figures in the evaluation rule for implication, for example as follows:

$1 \in \tau(A \rightarrow B, a)$ iff for every theory b if $1 \in \tau(A, b)$ then $1 \in \tau(B, a \oplus b)$.

Now $\oplus$, though an operation on theories, is not an operation on prime theories: a and b may be prime theories and $a \oplus b$ not prime (where a theory is prime iff whenever $A \vee B$ holds in it either A holds in it or B holds in it). Thus an operational semantics must take account of both theories *and* prime theories and the important step-down relations between them. But the classical evaluation rules for $\vee$ and $\&$ (the 0 case for $\&$) hold good only for prime theories and assume more complicated forms — forms that have to be provided in the operational framework — for theories. The verification of the rule if $0 \in \tau(A \& B, a)$ then $0 \in \tau(A, a)$ or $0 \in \tau(B, a)$ in the familiar canonical model illustrates the need for primeness. For what has to be shown generally is that if $\sim(A \& B) \in a$, i.e. $\sim A \vee \sim B \in a$, then either $\sim A \in a$ or $\sim B \in a$: but this just amounts to primeness.

Even so, by using step-down from theories to prime theories as under the Australian method (see [7] 12) operational reworkings of the classical-style semantics can no doubt be furnished. *Such reworkings* have substantial philosophical point, for they *demolish at once two common myths*: that not only can relevant logics not be outfitted with semantics with a classical-style negation, but that even if they could be the semantics would be ineluctably committed to use of three-place relations on worlds. On operational

⁵The considerations adduced by no means exclude the possibility that there are less orthodox systems — such as those called Urquhart systems in [7], because they obtained some promotion from Urquhart and arise naturally from the semilattice semantics of which he was one of the originators — for which the classical-style rules can be retained. See proposed repairs to [3] in [11].

semantics such three-place relations are displaced by two-place operations in combination with a lattice ordering relation.

Given the Australian method and its two-valued relational semantics with starred negation, there are, in realising the American plan, these *new moves* (most of which having been seen can be seen to have been glimpsed from the Australian plan):

i) Important features of the American plan at the first degree and first degree entailment stages are taken over, in particular not only the evaluation rules given above, but the determination of truth in a model and validity, not just from value 1's belonging but also from value 0's not belonging (see the definitions below).

ii) A second three-place relation S in addition to relation R is added to model structures to enable the evaluation of falsity in an implicational wff. In part, but only in part S , compensates for the sacrifice of the star operation.

iii) Some worlds, especially the base worlds of model structures, are considered as being related as mates. This is the remaining American compensation for the loss of star.

Move iii) is, perhaps, less important and can, in the case of weaker relevant logics, be avoided entirely.

Although the American plan can be made good, it is *substantially* less satisfactory than the Australian plan, both in the way it forces the doubling up of conditions for less basic axioms, and in its accommodation of negation postulates, especially Contraposition (which is the Achilles' heel of the American approach). Moreover, working through the American plan repeatedly suggests the star operation and negation rule of the Australian plan, and indeed a reasonable treatment of Contraposition and Reductio appears to require some surrogate of star. So perhaps a major virtue of the American approach is that it provides another approach to and explanation of — what it was set up partly in opposition to, and was supposed to be getting away from — the star negation rule and its role in the Australian plan.

§ 2. Basic system B , and some extensions of B such as R .

System B has as primitive connectives $\&$ ('and'), $\sim$ ('not') and ' $\rightarrow$ ' ('implies' or if ... then'). Other connectives are defined in ways familiar from Lewis modal logics, e.g. $A \vee B =_{df} \sim(\sim A \& \sim B)$, $A \leftrightarrow B =_{df} (A \rightarrow B) \& (B \rightarrow A)$. Notational conventions, abbreviations, etc., follow those of [7]. System B is characterized by the following axiom schemes and rules:

- A1. $A \rightarrow A$
- A2. $A \& B \rightarrow A$
- A3. $A \& B \rightarrow B$

- A4. $(A \rightarrow B) \ \& \ (A \rightarrow C) \rightarrow A \rightarrow (B \ \& \ C)$
 A5. $A \rightarrow A \vee B$
 A6. $B \rightarrow A \vee B$
 A7. $(A \rightarrow C) \ \& \ (B \rightarrow C) \rightarrow A \vee B \rightarrow C$
 A8. $A \ \& \ (B \vee C) \rightarrow (A \ \& \ B) \vee C$
 A9. $\sim \sim A \rightarrow A$
 R1. $A, A \rightarrow B / B$ (Modus Ponens)
 R2. $A, B / A \ \& \ B$ (Adjunction)
 R3'. $A \rightarrow B, C \rightarrow D / B \rightarrow C \rightarrow A \rightarrow D$ (Affixing)
 R5. $A \rightarrow \sim B / B \rightarrow \sim A$ (Rule Contraposition)

The Affixing rule combines Suffixing (R3. $A \rightarrow B / B \rightarrow C \rightarrow A \rightarrow C$) and Prefixing (R4. $A \rightarrow B / C \rightarrow A \rightarrow C \rightarrow D$), rules which in stronger relevant logics are replaced by corresponding implications.

Until the mid-seventies, most interest in relevant logics focussed on systems much stronger than **B**, in particular on the three systems selected for most attention by Anderson and Belnap, **E**, **R** and **T**. These systems strengthen **B** in the implication and implication-negation parts. For subsequent use some of the schemes commonly considered for addition to **B** are first set out. In terms of these axiomatisations of **E**, **R** and **T** are readily set down.

- | | |
|---|-----------------------------------|
| B1. $A \ \& \ (A \rightarrow B) \rightarrow B$ | q1. $Raaa$ |
| B2. $(A \rightarrow B) \ \& \ (B \rightarrow C) \rightarrow A \rightarrow C$ | q2. If $Rabc$ then $R^2a(ab)c$ |
| B3. $A \rightarrow B \rightarrow B \rightarrow C \rightarrow A \rightarrow C$ | q3. If R^2abcd then $R^2b(ac)d$ |
| B4. $A \rightarrow B \rightarrow C \rightarrow A \rightarrow C \rightarrow B$ | q4. If R^2abcd then $R^2a(bc)d$ |
| B5. $A \rightarrow (A \rightarrow B) \rightarrow A \rightarrow B$ | q5. If $Rabc$ then R^2abbc |
| B6. $A \rightarrow A \rightarrow B \rightarrow B$ | q6. If $Rabc$ then $Rbac$ |
| B10. $A \rightarrow B \rightarrow B$ | q10. If $Rabc$ then $b \leq c$ |
| B11. $B \rightarrow A \rightarrow B$ | q11. If $Rabc$ then $a \leq c$ |
| D2. $A \vee \sim A$ | s2. $x^* \leq x$ for $x \in 0$ |
| D3. $(A \rightarrow \sim A) \rightarrow \sim A$ | s3. Raa^*a |
| D4. $A \rightarrow \sim B \rightarrow B \rightarrow \sim A$ | s4. If $Rabc$ then Rac^*b^* |
| D5'. $A \ \& \ (\sim A \vee B) \rightarrow B$ | s5. $a^* \leq a$ |
| BR1. $A / (A \rightarrow B) \rightarrow B$ | dr1. $Raxa$ for some $x \in 0$ |

The conditions on the right are the modelling conditions for the matching postulates on the left, under the Australian plan. No such simple 1-1 correspondence emerges under the American plan. The following definitions have been applied: d2. $R^2abcd =_{df} (Px)(Rabx \ \& \ Rxcd)$; d3. $R^2a(bc)d =_{df} (Px)(Rbcx \ \& \ Raxd)$.

Each of **E**, **R** and **T** has rules R1 and R2 (R3' and R5 are derivable) and axiom schemes A2-A9 of **B**. To these **R** adds B3, B5, B6 and D4; and **T** adds B4 (or B3), B5, D3 and D4; and **E** adds to **T** rule BR1, the rule analogue of B6. The systems form a chain $\mathbf{T} \subset \mathbf{E} \subset \mathbf{R}$.

§ 3. Classical-style semantics for B

An *American B model structure* (B a.m.s.) is a structure $M = \langle T, U, O, P, K, R, S \rangle$ where K is a set of worlds, O (comprising the regular worlds) and P are subsets of K , T and U are elements (the positive and negative base worlds) of K with T in O and U in P , and S are three-place relations on K , such that the following definitions apply and postulates hold, for all a, b, c, d in K :

- d1. $a \leq b =_{df} (Px)(x \in O \ \& \ Rxab)$ $\bar{p}0$. if for some x in P , $Sxab$, then $a \leq b$.

It follows since $T \in O$ and $U \in P$ that

- dp0. if $RTab$ then $a \leq b$, and $d\bar{p}0$. if $SUab$ then $a \leq b$.
 p1. $a \leq a$. $\bar{p}1$. for some x in P , $Sxaa$.
 p2. if $a \leq b$ and $Rbcd$ then $Racd$. $\bar{p}2$. if $a \leq b$ and $Sacd$ then $Sbcd$.

The model structures can be varied in several ways (discussed in [7] 4) without affecting their adequacy; e.g. T and U can be identified⁶, and in fact both can be removed; O and P can be equated and in stronger systems removed; the ordering relation $\leq$ can be made a partial order; and stronger monotonicity conditions (like p2 and $\bar{p}2$) can be required, for each place in R and S , etc.

A *valuation* τ in M is a function which assigns a value from $III = \{\{1\}, \{0\}, \{1, 0\}, \{\}\}$ to each sentential parameter p at each world a such that for a, b in K

- (1) if $a \leq b$ and $1 \in \tau(p, a)$ then $1 \in \tau(p, b)$, and
 (2) if $a \leq b$ and $0 \in \tau(p, a)$ then $0 \in \tau(p, b)$.

Conditions (1) and (2) are the basic hereditariness conditions. Like most else under the American plan, in contrast with the Australian plan, they come in double form.

Function τ is extended inductively from initial wff (sentential parameters) to all wff by the following rules, which hold generally, for all wff A and B and every a, b in K :

- $\sim$ rules: } as formulated in the introductory section.
 $\&$ rules: }
 $\rightarrow$ rules: $1 \in \tau(A \rightarrow B, a)$ iff, for every b and c in K ,
 if $Rabc$ and $1 \in \tau(A, b)$ then $1 \in \tau(B, c)$;
 $0 \in \tau(A \rightarrow B, a)$ iff, for some b and c in K ,
 $Sabc$ and $0 \in \tau(B, b)$ and $0 \notin \tau(A, c)$.

⁶But the assumption that $T = U$ is effectively a normality assumption that it is preferable to avoid. Though normality does hold for the systems considered, it can then be proved using the semantics, rather than established in showing their adequacy.

HEREDITARINESS LEMMA. Where τ is a valuation in an a.m.s. M , then for every wff A and every a, b , in K ,

- (1†) if $a \leq b$ and $1 \in \tau(A, a)$ then $1 \in \tau(A, b)$, and
 (2†) if $a \leq b$ and $0 \in \tau(A, a)$ then $0 \in \tau(A, b)$.

PROOF is by joint induction from basis (1) and (2). In the inductive argument for (1†) there are these cases:

ad &. Suppose $a \leq b$ and $1 \in \tau(B \& C, a)$. Then $1 \in \tau(B, a)$ and $1 \in \tau(C, a)$ whence by induction hypothesis $1 \in \tau(B, b)$ and $1 \in \tau(C, b)$. So $1 \in \tau(B \& C, b)$.

ad $\sim$. Suppose $a \leq b$ and $1 \in \tau(\sim B, a)$. Then $0 \in \tau(B, a)$, whence by induction hypothesis from (2†), $0 \in \tau(B, b)$. Therefore, by the $\sim$ rule, $1 \in \tau(\sim B, b)$.

ad $\rightarrow$. Suppose $a \leq b$ and $1 \in \tau(B \rightarrow C, a)$. In order to show $1 \in \tau(B \rightarrow C, b)$, suppose for c and d in K that $Rbcd$ and $1 \in \tau(B, c)$. It suffices to show $1 \in \tau(C, d)$. Since $a \leq b$ and $Rbcd$, by p2, $Racd$. Hence as $1 \in \tau(B \rightarrow C, a)$ when $1 \in \tau(B, c)$, $1 \in \tau(C, d)$, as required.

In the argument for (2†) there are these similar cases:

ad &. Suppose $a \leq b$ and $0 \in \tau(B \& C, a)$. Then either $0 \in \tau(B, a)$ or $0 \in \tau(C, a)$. The cases are similar: consider the first alternative. Then, by induction hypothesis, $0 \in \tau(B, b)$, whence $0 \in \tau(B \& C, b)$.

ad $\sim$. Suppose $a \leq b$ and $0 \in \tau(\sim B, a)$. Then $1 \in \tau(B, a)$, so by (1†) $1 \in \tau(B, b)$, and $0 \in \tau(\sim B, b)$.

ad $\rightarrow$. Suppose $a \leq b$ and $0 \in \tau(B \rightarrow C, a)$. Then for some c and d in K , $Sacd$ and etc. It suffices to show for $0 \in \tau(B \rightarrow C, b)$ that $Sbcd$ and (the same) etc. But $a \leq b$ and $Sacd$, so by $\bar{p}2$ $Sbcd$.

The semantical notions defined are based on double, or increased, truth. A wff A is *doubly true* (2-true) on τ in a.m.s. M just in case $1 \in \tau(A, T)$ and $0 \notin \tau(A, U)$. Wff A is (doubly) *valid* in M iff A is 2-true on all valuations therein, and otherwise *invalid* in M and M provides a *countermodel structure* to A (and the a.m.s. and such a valuation concerned, a countermodel to A). A is (doubly) *valid* iff A is (doubly) valid in all a.m.s.

Interconnected with 2-truth and double validity are double implication relations. With respect to a given a.m.s., wff A *doubly* (or 2-) *implies* B on τ iff, for every a in K , if $1 \in \tau(A, a)$ then $1 \in \tau(B, a)$ and if $0 \in \tau(B, a)$ then $0 \in \tau(A, a)$. A *2-implies* B iff, for every valuation τ in every a.m.s., A 2-implies B on τ .

DOUBLE IMPLICATION LEMMA. (1) Where τ is a valuation on M , if A 2-implies B on τ then $A \rightarrow B$ is 2-true on τ .

(2) A 2-implies B iff $A \rightarrow B$ is doubly valid.

PROOF. As to (1), given A 2-implies B in τ it suffices to show (i) $1 \in \tau(A \rightarrow B, T)$ and (ii) $0 \notin \tau(A \rightarrow B, U)$. For (i) suppose $RTab$ and $1 \in \tau(A, a)$, for arbitrary a and b in K . It suffices to show $1 \in \tau(B, b)$. Since $RTab$, by $dp0$, $a \leq b$; hence by hereditariness $1 \in \tau(A, b)$. But A 2-implies B in τ , whence $1 \in \tau(A, b)$. For (ii) suppose $SUab$ and $0 \in \tau(B, a)$, for arbitrary a and b in K ; to show $0 \in \tau(A, b)$. By $dp0$, $a \leq b$, hence by hereditariness $0 \in \tau(B, b)$. But then by 2-implication, $0 \in \tau(A, b)$.

As regards (2), one half (LH to RH) follows from (1) simply by universal quantification. For the converse, suppose $A \rightarrow B$ is doubly valid. Then for arbitrary $M = \langle T, U, O, P, K, R, S \rangle$ and τ in M , $1 \in \tau(A \rightarrow B, T)$ and $0 \notin \tau(A \rightarrow B, U)$. Suppose further (i) for arbitrary $a \in K$, $1 \in \tau(A, a)$. It suffices for one half of 2-implication to show $1 \in \tau(B, a)$. Since by $p1$, $a \leq a$, for some $d \in O$, $Rdaa$. Form a new a.m.s. $M' = \langle d, U, O, P, K, R, S \rangle$, i.e. an m.s. differing from M only at first base; and define an identical valuation τ' in M' , i.e. for a in K , $\tau'(p, a) = \tau(p, a)$ for every sentential parameter p . Then $\tau = \tau'$ by induction; and M' is indeed an a.m.s.⁷

As $A \rightarrow B$ is valid, it is 2-true in M' , so $1 \in \tau'(A \rightarrow B, d)$, i.e. $1 \in \tau(A \rightarrow B, d)$. But $Rdaa$, hence as $1 \in \tau(A, a)$, $1 \in \tau(B, a)$ as required. Suppose next (ii) for arbitrary $a \in K$, $0 \in \tau(B, a)$. It remains to show $0 \in \tau(A, a)$. The argument is like that of (i) but uses $\bar{p}1$ in place of $p1$. By $\bar{p}1$, for some e in P $Seaa$. Form a new a.m.s. M'' with e as second base in place of U (alternatively, the mated bases T, U can be replaced together by the pair d, e), and determine an identical valuation to τ on M'' . Since $A \rightarrow B$ is valid, it is 2-true in M'' , so $0 \notin \tau(A \rightarrow B, e)$. But $Seaa$, so as $0 \in \tau(B, a)$, $0 \in \tau(A, a)$ as required.

§ 4. Adequacy of the semantics for B

The main preliminaries in showing soundness have already been proved.

SOUNDNESS THEOREM. *Every theorem of B is doubly valid.*

PROOF. is by the usual inductive argument, for which it is enough to show that the rules preserve validity and the axioms are valid.

(I) The rules of B preserve validity.

ad R1. Suppose first for arbitrary τ in M , $1 \in \tau(A, T)$ and show $1 \in \tau(B, T)$. But, as $A \rightarrow B$ is valid, A 2-implies B , whence the result. Suppose further $0 \notin \tau(A, U)$ and show $0 \notin \tau(B, U)$. But again A 2-implies B , so if $0 \in \tau(B, U)$ then $0 \in \tau(A, U)$, whence the result.

ad R2. The result is immediate from the valuation rules for $\&$.

⁷For none of the modelling conditions involve T and U essentially; other elements of O and P will do as well. But ensuring this has complicated formulation of the conditions, e.g. $\bar{p}0$ in place of $dp0$ and hence involvement of class P .

ad R3. Suppose $B \rightarrow C \rightarrow A \rightarrow C$ is not valid. Then there is a countermodel τ in $\mathcal{M}$. There are two alternatives to be dealt with: either (i) $1 \notin \tau(B \rightarrow C \rightarrow A \rightarrow C, T)$ or (ii) $0 \in \tau(B \rightarrow C \rightarrow A \rightarrow C, U)$. Suppose first (i). Then for some $a, b \in K$, $RTab$ and $1 \in \tau(B \rightarrow C, a)$ but $1 \notin \tau(A \rightarrow C, b)$. By $dp0$, $a \leq b$, so $1 \notin \tau(A \rightarrow C, a)$ by hereditariness. Hence for some $c, d \in K$, $Racd$ and $1 \in \tau(A, c)$ but $1 \notin \tau(C, d)$. But $1 \in \tau(B \rightarrow C, a)$, so $1 \notin \tau(B, c)$. Therefore A does not 2-imply B , and $A \rightarrow B$ is not valid. Suppose next (ii). Then for some $b, c \in K$, $SUbc$ and $0 \in \tau(A \rightarrow C, b)$ but $0 \notin \tau(B \rightarrow C, c)$. By $d\bar{p}0$, $b \leq c$, so $0 \in \tau(A \rightarrow C, c)$ by hereditariness. Hence for some $d, e \in K$, $Sede$ and $0 \in \tau(C, d)$ but $0 \notin \tau(A, e)$. But $0 \notin \tau(B \rightarrow C, c)$ so as $0 \in \tau(C, d)$, $0 \in \tau(B, e)$. Therefore A does not 2-imply B and again $A \rightarrow B$ is not valid. Hence since it is not valid under either alternative, it is not valid.

ad R4: similar to R3.

ad R5. Given that $A \rightarrow B$ is valid, and hence that A 2-implies B , in showing that $\sim B \rightarrow \sim A$ is valid, by showing that $\sim B$ 2-implies $\sim A$, there are two cases, namely i) if $1 \in \tau(\sim B, a)$ then $1 \in \tau(\sim A, a)$, i.e. if $0 \in \tau(B, a)$ then $0 \in \tau(A, a)$, and ii) if $0 \in \tau(\sim A, a)$ then $0 \in \tau(\sim B, a)$, i.e. if $1 \in \tau(A, a)$ then $1 \in \tau(B, a)$. But these cases are guaranteed by the fact that A 2-implies B .

Verification of R5 reveals the point of doubling up: the whole complication of doubling is required to yield Contraposition principles in a natural way under the American plan.

(II) The axioms of $\mathbf{B}$ are valid. Three examples illustrate the method completely.

ad A2. To show that $A \& B$ 2-implies B it is enough to show that (i) if $1 \in \tau(A \& B, a)$ then $1 \in \tau(A, a)$ and (ii) if $0 \in \tau(A, a)$ then $0 \in \tau(A \& B, a)$. But these connections are immediate from the evaluation rule for $\&$.

ad A4. It is enough to show (i) if $1 \in \tau(A \rightarrow B \& A \rightarrow C, a)$ then $1 \in \tau(A \rightarrow B \& C, a)$ and (ii) if $0 \in \tau(A \rightarrow B \& C, a)$ then $0 \in \tau(A \rightarrow B \& A \rightarrow C, a)$. As to (i), on the hypothesis, by $\&$ and $\rightarrow$ rules, for every b and c in K , if $Rabc$ and $1 \in \tau(A, b)$ then $1 \in \tau(B, c)$ and also $1 \in \tau(C, c)$, whence $1 \in \tau(B \& C, c)$; so $1 \in \tau(A \rightarrow B \& C, a)$. As to (ii), on the hypothesis, for some b and c in K , $Sabc$ and $0 \in \tau(B \& C, b)$ and $0 \notin \tau(A, c)$. So either $0 \in \tau(B, b)$ or $0 \in \tau(C, b)$. If $0 \in \tau(B, b)$ then $0 \in \tau(A \rightarrow B, a)$; while if $0 \in \tau(C, b)$ then $0 \in \tau(A \rightarrow C, a)$. So in either case, $0 \in \tau(A \rightarrow B \& A \rightarrow C, a)$.

ad A9. If $1 \in \tau(\sim\sim A, a)$ then $0 \in \tau(\sim A, a)$ and $1 \in \tau(A, a)$. Similarly $0 \in \tau(\sim\sim A, a)$ iff $0 \in \tau(A, a)$. Hence $\sim\sim A$ 2-implies A , and also conversely.

The completeness argument uses canonical model structures whose elements are characterised in terms of theories and prime theories. As most of the many preliminaries are like those in [7] 4.6, only a condensed

presentation is given. An $\mathbf{L}$ -theory a , for a logic $\mathbf{L}$ extending $\mathbf{B}$, is a set of wff closed under adjunction and provable $\mathbf{L}$ -implication, i.e. whenever B and $C \in a$ then $B \& C \in a$, and whenever $B \in a$ and $\vdash_{\mathbf{L}} B \rightarrow C$ then $C \in a$. A $\mathbf{L}$ -theory a is *regular* iff all theorems of $\mathbf{L}$ are in a , and *prime* iff whenever $A \vee B \in a$ either $A \in a$ or $B \in a$.

Let $\bar{K}_{\mathbf{L}}$ be the class of $\mathbf{L}$ -theories and $K_{\mathbf{L}}$ the class of prime $\mathbf{L}$ -theories. More generally unbarred notions are the prime restrictions (restrictions to prime theories) of barred notions. $\bar{O}_{\mathbf{L}}$ is the class of regular $\mathbf{L}$ -theories, $O_{\mathbf{L}}$ of prime regular $\mathbf{L}$ -theories. For a, b, c in $\bar{K}_{\mathbf{L}}$, $\bar{R}_{\mathbf{L}}abc$ iff whenever $A \rightarrow B \in a$ and $A \in b$ then $B \in c$, for all wff A and B ; and $\bar{S}_{\mathbf{L}}abc$ iff whenever $\sim B \in b$ and $\sim A \notin c$ then $\sim(A \rightarrow B) \in a$ for all A and B . Where x is a prime theory in $O_{\mathbf{L}}$, let y be the class of wff B such that $\sim B \notin x$; then $P_{\mathbf{L}}$ is the class of all such y . $P_{\mathbf{L}}$ is demonstrably a prime $\mathbf{L}$ -theory.

The *canonical a.m.s.* $M_{\mathbf{L}}$ for $\mathbf{L}$ on bases T, U is the structure $\langle T, U, O_{\mathbf{L}}, P_{\mathbf{L}}, K_{\mathbf{L}}, R_{\mathbf{L}}, S_{\mathbf{L}} \rangle$ defined. It remains to determine T and U appropriate in $O_{\mathbf{L}}$ and $P_{\mathbf{L}}$ respectively. An auxiliary notion, that usefully enables previous work (in [7] and elsewhere) to be taken over, is that of the *opposite*, or *mate*, of a prime theory. Where $a \in K$, a^* , the opposite of a , is defined as $\{B: \sim B \notin a\}$. Then $a^* \in K$. In these terms, $P_{\mathbf{L}} = \{x^*: x \in O_{\mathbf{L}}\}$ and, as it will turn out, $U = T^*$. So it remains to determine T , which is done in terms of the classes of wff to be verified and refuted.

Set T of wff is $\mathbf{L}$ -derivable from set S , written $S \vdash_{\mathbf{L}} T$, iff, for some $A_1, \dots, A_m$ in S and some $B_1, \dots, B_n$ in T , $\vdash_{\mathbf{L}} A_1 \& \dots \& A_m \rightarrow B_1 \vee \dots \vee B_n$.

STRONG COMPLETENESS THEOREM. *Where set W is not $\mathbf{L}$ -derivable from regular set S of wff, there is a (canonical) valuation τ in a (canonical) $\mathbf{B}$ a.m.s. $M_{\mathbf{B}}$ which 2-verifies (i.e. makes 2-true) every member of S and 2-falsifies every member of W (i.e. for B in W , $1 \notin \tau(B, T)$ and $0 \in \tau(B, U)$ where T, U are the base elements of $M_{\mathbf{B}}$).*

PROOF. Let S_1 be the theory-closure of S , i.e. $S_1 = \{D: (PB_1, \dots, B_n \in S) \vdash_{\mathbf{L}} B_1 \& \dots \& B_n \rightarrow D\}$ and let W_1 be the ideal-closure of W , i.e. $W_1 = \{C: (PA_1, \dots, A_m \in U) \vdash_{\mathbf{L}} C \rightarrow A_1 \vee \dots \vee A_m\}$. Then (as in [7], theorem 4.10) S_1 is an $\mathbf{L}$ -theory containing S , W_1 which contains W is closed under disjunction, and S_1 is disjoint from W_1 . Under these conditions, there is a prime regular $\mathbf{B}$ -theory T which includes S_1 but is disjoint from W_1 . Let $U = T^*$, and take the canonical a.m.s. $M_{\mathbf{B}}$ for $\mathbf{B}$ on T, U so defined.

(I) $M_{\mathbf{B}}$ is a $\mathbf{B}$ a.m.s. Since it is immediate that $T \in O_{\mathbf{L}}$ and $U \in P_{\mathbf{L}}$, it remains to establish the modelling conditions. p1 and p2 are proved as in previous work, e.g. [7] or [8], and previous work can be used to establish propositions concerning $S_{\mathbf{L}}$ given the following connection

RS. $S_{\mathbf{L}}abc$ iff $R_{\mathbf{L}}a^*c^*b^*$, which can serve as a guiding connection in linking Australian and American plans.

(III) Since for $A \in S$, $A \in T$, by (II), $1 \in \tau(A, T)$. But $0 \notin \tau(A, U)$ iff $\sim A \notin U$, i.e. iff $\sim A \notin T^*$, i.e. iff $A \in T$, i.e. if $1 \in \tau(A, T)$. Hence A is 2-true on τ . Since further for $B \in W$, $B \notin T$, $1 \notin \tau(B, T)$. Hence B is not 2-true on τ , and in fact τ in M_B 2-falsifies every member of W .

COROLLARIES 1. *Completeness: Every doubly valid wff of B is a theorem.*

PROOF. Let A be a non-theorem of B . Let $W = \{A\}$ and $S = B$, and apply the theorem, since A is not B -derivable from B . Then A is 2-falsified, and hence not doubly valid. Other corollaries such as *compactness* follow as in [7] 4.

§ 5. Some positive extensions of B

Without much doubt, virtually any extension of B by positive axiom schemes that can be accommodated under the Australian plan can be handled by the American plan upon mirroring the required semantical postulate involving R by one involving S . Such positive schemes are those involving connectives $\rightarrow$, $\&$ or $\vee$ (and perhaps others such as fusion, $\circ$, and fission, $+$), but not negation or its derivatives such as material implication. So long as there is no "crossing" or "mixing" of 1s and 0s, mirroring of R conditions by S conditions is straightforward; but when negation enters "crossing" occurs and the straightforward method is lost, as we shall see. Here only a few sample positive extensions of B will be given by axiom schemes like those required in reaching systems like the big three R , T and E ; but any of the positive postulates listed earlier (in § 2) would be similarly accommodated.

B3.	$A \rightarrow B \rightarrow . B \rightarrow C \rightarrow . A \rightarrow C$	q3. If R^2abcd then $R^2b(ac)d$; $\overline{q}3$. If $Sazb$ & $Szcd$ then (Px) ($Saxd$ and $Sbcx$).
B5.	$A \rightarrow (A \rightarrow B) \rightarrow . A \rightarrow B$	q5. If $Rabc$ then R^2abbc ; $\overline{q}5$. If $Sabc$ then (Px) ($Saxc$ and $(Sxbe)$
B6.	$A \rightarrow . A \rightarrow B \rightarrow B$	q6. If $Rabc$ then $Rbac$; $\overline{q}6$. If $Sabc$ then $Scba$.
BR1	$A / (A \rightarrow B) \rightarrow B$	dr1 $Raxa$ for some $x \in O$; $\overline{dr}1$ $Saax$ for some $x \in O$.

For every postulate there are two cases, a 1 case and a 0 case. The 1 case is the same, *both* for soundness *and* completeness, as under the Australian method (of [7] 4), while the 0 case is an S dual of the 1 case. Some examples of details for adequacy will make this clear.

ad B3. What has to be shown is 1) If $1 \in \tau(A \rightarrow B, a)$ then $1 \in \tau(B \rightarrow C \rightarrow . A \rightarrow C, a)$, and 2) If $0 \in \tau(B \rightarrow C \rightarrow . A \rightarrow C, a)$ then $0 \in \tau(A \rightarrow B, a)$. The

first 1) is but a transcription of what happens under the Australian method: for the transcription write $1 \in \tau(D \rightarrow E, a)$ for $I(D \rightarrow E, a) = 1$. For 2) suppose $0 \in \tau(B \rightarrow C \rightarrow A \rightarrow C, a)$ but $0 \notin \tau(A \rightarrow B, a)$. Then for some z and b in K , $Sazb$ and $0 \in \tau(A \rightarrow C, z)$ and $0 \notin \tau(B \rightarrow C, b)$. Similarly for some c and d $Szcd$ and $0 \in \tau(C, c)$ and $0 \notin \tau(A, d)$. Now by $\bar{q}3$, for some x $Saxd$ and $Sbcx$. Since $Sbcx$, $0 \in \tau(B, x)$ but then as $Saxd$, $0 \in \tau(A, d)$ which is impossible.

ad $q3$ and $\bar{q}3$. Proof of $q3$ is as under the Australian method (see [7] 4), and $\bar{q}3$ is a variant on $q3$ after translation. For $\bar{q}3$ translates to: $Ra*b*z*$ and $Rz*d*c*$ implies (Px) ($Ra*d*x*$ and $Rb*x*c*$), which is a rewrite of $q3$ (interchange c and d and star all elements).

ad B5. In the second case to show 2) If $0 \in \tau(A \rightarrow B, a)$ then $0 \in \tau(A \rightarrow A \rightarrow B, a)$, suppose otherwise. Then for some b and c , $Sabc$ and $0 \in \tau(B, b)$ and $0 \notin \tau(A, c)$. By $\bar{q}5$. $Saxc$ and $Sabc$. Since $Saxc$ and $0 \notin \tau(A \rightarrow A \rightarrow B, a)$, if $0 \in \tau(A \rightarrow B, x)$ then $0 \in \tau(A, c)$; so $0 \notin \tau(A \rightarrow B, x)$. Hence as $Sabc$ and $0 \in \tau(B, b)$, $0 \in \tau(A, c)$ which is impossible.

ad $\bar{q}5$. The condition translates to: if $Ra*c*b*$ then (Px) ($Ra*c*x*$ and $Rx*c*b*$), which is a starred rewrite of $q5$.

A strong adequacy theorem now follows for these positive extensions of B .

§ 6. The problem of negative extensions of B under the American plan, and some resolutions thereof

The easy duplicating of modelling conditions that worked for positive extensions of B breaks down when negation enters. Since focal relevant system R requires only one additional negative scheme, namely Contraposition, $A \rightarrow B \rightarrow \sim B \rightarrow \sim A$, this scheme is critical, as well as representative. Consider then what is required to 2-verify it, that i) wherever $1 \in \tau(A \rightarrow B, a)$ then $1 \in \tau(\sim B \rightarrow \sim A, a)$ and ii) wherever $0 \in \tau(\sim B \rightarrow \sim A, a)$ then $0 \in \tau(A \rightarrow B, a)$. As to i), suppose then $1 \in \tau(A \rightarrow B, a)$ and for arbitrary b and c in K , $Rabc$ and $1 \in \tau(\sim B, b)$, and try to show $1 \in \tau(\sim A, c)$, i.e. $0 \in \tau(A, c)$. Then $0 \in \tau(B, b)$, and using remaining data, if $1 \in \tau(A, b)$ then $1 \in \tau(B, c)$. There is no way in the semantical framework so far developed that a conditional concerning 1's memberships and a premiss regarding 0's membership will yield a conclusion concerning 0's membership: 0s and 1s do not cross or interact. The same impasse confronts us as regards ii). There is no direct way through modelling requirements, e.g. on R and S , that Contraposition can be nailed down.

There are however several ways of adjusting modellings so that Contraposition is verified.

	$0 \in \tau(\sim B, b^M)$	iff	$1 \in \tau(B, b^M)$
		iff	$0 \notin \tau(B, b)$
		iff	$1 \notin \tau(\sim B, b)$
ad &	$1 \in \tau(B \& C, b^M)$	iff	$1 \in \tau(B, b^M)$ and $1 \in \tau(C, b^M)$
		iff	$0 \notin \tau(B, b)$ and $0 \notin \tau(C, b)$
		iff	$0 \notin \tau(B \& C, b)$
	$0 \in \tau(B \& C, b^M)$	iff	$0 \in \tau(B, b^M)$ or $0 \in \tau(C, b^M)$
		iff	$1 \notin \tau(B, b)$ or $1 \notin \tau(C, b)$
		iff	$1 \notin \tau(B \& C, b)$
ad $\rightarrow$	$1 \in \tau(B \rightarrow C, b)$	iff	(c, d) (if $Rbcd$ either $1 \notin \tau(B, c)$ or $1 \in \tau(C, d)$)
		iff	(c, d) (if $Sb^M d^M c^M$ either $0 \notin \tau(C, d^M)$ or $0 \in \tau(B, c^M)$)
		iff	$0 \notin \tau(B \rightarrow C, b^M)$
	$0 \in \tau(B \rightarrow C, b)$	iff	(Pc, d) ($Sbcd$ and $0 \in \tau(C, c)$ and $0 \notin \tau(B, d)$)
		iff	(Pc, d) ($Rb^M d^M c^M$ and $1 \in \tau(B, d^M)$ and $1 \notin \tau(C, c^M)$)
		iff	$1 \notin \tau(B \rightarrow C, b^M)$.

Proofs of soundness are not affected by the further requirements associated with M , since what is valid in all models is valid in all M restricted ones (nothing depending on the generality of the models). For completeness, set $b^M = b^*$ for every $b \in K_L$. Then all modelling conditions associated with M are satisfied. Thus M -restricted models are adequate, and in terms of these it is a straightforward matter to model various negation postulates. Again only two sample postulates are tabulated, for reasons that will become evident (if not already so):

Axiom scheme	Corresponding modelling conditions
$A \rightarrow B \rightarrow. \sim B \rightarrow \sim A$	$\begin{cases} \text{if } Rabc \text{ then } Rac^M b^M \\ \text{if } Sabc \text{ then } Sac^M b^M \end{cases}$
$A \rightarrow \sim A \rightarrow. \sim A$	$\begin{cases} Raa^M a \\ Sac^M a \end{cases}$

Proofs of completeness are much as in [7] 4. As to soundness there are the following sorts of details:

ad Contraposition. Continuing the argument in i) above, it follows from $0 \in \tau(B, b)$ that $1 \notin \tau(B, b^M)$. Since $Rabc$, $Rac^M b^M$, hence as $1 \in \tau(A \rightarrow B)$, if $1 \in \tau(A, c^M)$ then $1 \in \tau(B, b^M)$. So $1 \notin \tau(A, c^M)$, whence $0 \in \tau(A, c)$ as required. As to ii), assuming $0 \in \tau(\sim B \rightarrow \sim A, a)$ for some b, c , $Sabc$ and $0 \in \tau(\sim A, b)$ and $0 \notin \tau(\sim B, c)$, whence $1 \in \tau(A, b)$ and $1 \notin \tau(B, c)$, and so $0 \in \tau(B, c^M)$ and $0 \notin \tau(A, b^M)$. Hence as $Sac^M b^M$, $0 \in \tau(A \rightarrow B, a)$.

ad Reductio. For i) assuming $1 \in \tau(A \rightarrow \sim A, a)$ it has to be shown $1 \in \tau(\sim A, a)$, i.e. $0 \in \tau(A, a)$. But as $Raa^M a$, either $1 \notin \tau(A, a^M)$ or $1 \in \tau(\sim A, a)$. If $1 \notin \tau(A, a^M)$ then $0 \in \tau(A, a)$. So either alternative yields the required result. As to ii), suppose $0 \in \tau(\sim A, a)$ but $0 \notin \tau(A \rightarrow \sim A, a)$. Then as $Saa^M a$, either $0 \notin \tau(A, a^M)$ or $0 \notin \tau(\sim A, a)$. But if $0 \notin \tau(A, a^M)$ then $1 \in \tau(A, a)$ so $0 \notin \tau(\sim A, a)$. Hence both options contradict $0 \in \tau(\sim A, a)$.

In fact were it simply required that M be an involution, which it might as well be, then both arguments for ii) reduce to those for i) using the lemma. Indeed S conditions can disappear and therewith the need for S . And of course the whole semantical theory can be considerably simplified using the star rules for negation given in the following

LEMMA. $1 \in \tau(\sim A, b)$ iff $1 \notin \tau(A, b^M)$
 $0 \in \tau(\sim A, b)$ iff $0 \notin \tau(A, b^M)$.

Among other things then, the semantics provides an indirect vindication of the star rules from a more classical viewpoint.

An *improved* version of method 2 through which **R**, **T** and **E** can be given American-style semantics without so blatantly imitating star, will be given in § 8.

Method 3. → rule doubling. Much as we complicated the notion of semantic implication to 2-implication to ensure Rule Contraposition, so we can strengthen systemic implication, through the evaluation rules for $\rightarrow$, in a parallel way, to validate Contraposition. Such a strengthening of the $\rightarrow$ -rules is readily suggested by the argument given in i) above; for what is required to close the argument for Contraposition is that $1 \in \tau(A \rightarrow B, a)$ also guarantees that wherever $0 \in \tau(B, b)$ then $0 \in \tau(A, c)$ provided $Rabc$. The amended $\rightarrow$ -rules which emerge are these:

amended $\rightarrow$ rules $\left\{ \begin{array}{l} 1 \in \tau(A \rightarrow B, a) \text{ iff, for every } b \text{ and } c \text{ in } K, \text{ if } Rabc \text{ then if} \\ 1 \in \tau(A, b) \text{ then } 1 \in \tau(B, c) \text{ and if } 0 \in \tau(B, b) \text{ then } 0 \in \tau(A, c) \\ 0 \in \tau(A \rightarrow B, a) \text{ iff, for some } b \text{ and } c \text{ in } K, Sabc \text{ and either} \\ 0 \in \tau(B, b) \text{ and } 0 \notin \tau(A, c) \text{ or else } 1 \in \tau(A, b) \text{ and } 1 \notin \tau(B, a). \end{array} \right.$

LEMMA. i) $1 \in \tau(\sim B \rightarrow \sim A, a)$ iff $1 \in \tau(A \rightarrow B, a)$
 ii) $0 \in \tau(\sim B \rightarrow \sim A, a)$ iff $0 \in \tau(A \rightarrow B, a)$.

PROOF. is by the amended $\rightarrow$ -rules.

Since the amended rules thus force Contraposition, adoption of the amended rules will not render — assuming the rules work — earlier labour superfluous, since that catered primarily for systems without Contraposition.

Finding rules which might succeed is one thing, showing that the rules do indeed work is quite another, and more laborious, task.

§ 7. Reduced classical-style semantics for system *C*, and extensions and complications.

A *reduced* model structure is one where *O* contracts to *T*, i.e. only a single regular world, the factual world *T*, is required. Relevant system *C* is a minimal system as regards reduced models. *C* is axiomatised by adding to the axiom schemes of *B* — but without A1, and with A9 replaced by A9'. $A \rightarrow \sim \sim A$ — the following

- B1. $A \ \& \ (A \rightarrow B) \rightarrow B$
 B3. $A \rightarrow B \rightarrow . B \rightarrow C \rightarrow . A \rightarrow C$
 D4'. $\sim A \rightarrow B \rightarrow . \sim B \rightarrow A,$

and just the rules of Modus Ponens and Adjunction. As Affixing and Rule Contraposition are derived rules, *C* properly includes *B*.

An *American C model structure* (*C* a.m.s.) is a structure $M = \langle T, K, R, S \rangle$ where *K* is a set of worlds, *T* belongs to *K* and *R* and *S* are three-place relations on *K*, such that the following definitions apply and postulates hold, for all *a, b, c, d, e*, in *K*:

- | | |
|---|---|
| d1. $a \leq b =_{Df} RTab$ | d4. $S^2a[bc]d =_{Df} (Px)(Saxd \ \& \ Sxbe)$ |
| d5. $S^2a(bc)d =_{Df} (Px)(Saxd \ \& \ Sbcx)$ | d6. $R/Sa[bc]d =_{Df} (Px)(Raxd \ \& \ Sxbe), \text{ etc.}$ |
| d7. $S/Ra(bc)d =_{Df} (Px)(Saxd \ \& \ Rbcx)$ | |
| p1. $a \leq a$ | |
| p2. If $a \leq b$ and $Rbcd$ then $Racd$ | $\bar{p}2.$ If $a \leq b$ and $Sacd$ then $Sbcd$ |
| [For B1] q1. $Raaa$ | $\bar{q}1.$ $Saaa$ |
| [For B3] q3. If R^2abcd then $R^2b(ac)d$ | $\bar{q}3.$ If $S^2a[bc]d$ then $S^2d(ab)c$ |
| q4. If R^2abcd then $R^2a(bc)d$ | $\bar{q}4.$ If $S^2a[bc]d$ then $S^2a(bc)d$ |
| qm3. If $R/Sa[bc]d$ then $S/Rd[ab]c$ | qm4. If $R/Sa[bc]d$ then $R/Sa(db)c$ |
| qm5. If $S/Rabcd$ then $R/Sb(ac)d$ | qm6. If $S/Rabcd$ then $S/Ra(bc)d$ |

Theorems of *C* include A1, A9, Rule Syll, $A \rightarrow B/B \rightarrow C \rightarrow . A \rightarrow C$, $A \rightarrow B \rightarrow . \sim B \rightarrow \sim A$, $\sim B \rightarrow \sim A \rightarrow . A \rightarrow B$, $A \rightarrow B/C \rightarrow A \rightarrow . C \rightarrow B$, D4, B4 (proofs are for the most part duals of those given in [1], p. 141).

A *valuation* τ in *M* is defined as before. Function τ is extended to all wff by the $\sim$ and $\&$ rules previously given and the amended $\rightarrow$ rules. The

HEREDITARINESS LEMMA is formulated as before, but the proof involves some new cases:

New $\rightarrow$ cases. Assuming $1 \in \tau(B, c)$ it can be shown as before that $1 \in \tau(C, d)$. Now assume $0 \in \tau(C, c)$; to show $0 \in \tau(B, d)$. As before by p2, *Racd*, so as $1 \in \tau(B \rightarrow C, a)$ when $0 \in \tau(C, c)$, $0 \in \tau(B, d)$. The generalised argument given in the $\rightarrow$ -step for (2 $\dagger$) already suffices.

Truth and validity can now be defined in the usual way, that is (as on the Australian plan) *doubling is not required*. A wff A is *true on τ in M* just in case $1 \in \tau(A, T)$; A is *valid in M* iff it is true in all valuations τ in M ; and A is *valid* iff it is valid in all C a.m.s. While 2-implication is still required, it now emerges naturally from the amended $\rightarrow$ -rules.

IMPLICATION LEMMA FOR REDUCED MODELLINGS (1) A 2-implies B in τ iff $A \rightarrow B$ is true on τ .

(2) A 2-implies B iff $A \rightarrow B$ is valid (double validity is not required).

PROOF. (2) follows from (1) by quantificational logic. As to (1), suppose $A \rightarrow B$ is true in τ , i.e. $1 \in \tau(A \rightarrow B, T)$. Then as *RTaa*, for every $a \in K$, when $1 \in \tau(A, a)$ $1 \in \tau(B, a)$ and when $0 \in \tau(B, a)$ $0 \in \tau(A, a)$ i.e. A 2-implies B on τ .

The converse *requires* 2-implication. Otherwise there is no guarantee of the 0 clause in the $\rightarrow$ -rule.

Suppose A 1-implies B on τ . To show $1 \in \tau(A \rightarrow B, T)$, suppose further *RTab* for arbitrary a and b in K . It has to be shown that (i) when $1 \in \tau(A, a)$ then $1 \in \tau(B, b)$ and (ii) when $0 \in \tau(B, a)$ then $0 \in \tau(A, b)$. Since *RTab*, $a \leq b$. Then both (i) and (ii) follow by hereditariness combined with 2-implication.

SOUNDNESS THEOREM. *Every theorem of C is valid.*

PROOF enlarges on the corresponding result for **B**, and postulates drawn from **B** are treated precisely as before: so it remains to validate B1, B3 and D4'.

ad B1. In virtue of the Implication Lemma it suffices to show

- 1) if $1 \in \tau(A \ \&. \ A \rightarrow B, a)$ then $1 \in \tau(B, a)$, and
- 2) if $0 \in \tau(B, a)$ then $0 \in \tau(A \ \&. \ A \rightarrow B, a)$.

As to 1), $1 \in \tau(A, a)$ and $1 \in \tau(A \rightarrow B, a)$, so as *Raaa*, $1 \in \tau(B, a)$.

As to 2), suppose $0 \notin \tau(A \ \&. \ A \rightarrow B, a)$. Then $0 \notin \tau(A, a)$ and $0 \notin \tau(A \rightarrow B, a)$. Since *Saaa*, if $0 \in \tau(B, a)$ then $0 \in \tau(A, a)$, so $0 \notin \tau(B, a)$. 1) follows by Contraposition.

ad B3. What is to be shown is

- 1) if $1 \in \tau(A \rightarrow B, a)$ then $1 \in \tau(B \rightarrow C \rightarrow. A \rightarrow C, a)$ and
- 2) if $0 \in \tau(B \rightarrow C \rightarrow. A \rightarrow C, a)$ then $0 \in \tau(A \rightarrow B, a)$.

The argument is case-ridden: there are 8 eventual subcases. These cases will simply be illustrated:

ad 2) Suppose $0 \in \tau(B \rightarrow C \rightarrow A \rightarrow C, a)$ but $0 \notin \tau(A \rightarrow B, a)$. Then for some b and c in K for which $Sabc$

either 2.1) $0 \in \tau(A \rightarrow B, b)$ and $0 \notin \tau(B \rightarrow C, c)$

or 2.2) $1 \in \tau(B \rightarrow C, b)$ and $1 \notin \tau(A \rightarrow C, c)$

ad 2.2) Then for some d and e in K such that $Rcde$

either 2.21) $1 \in \tau(A, d)$ and $1 \notin \tau(C, e)$

or 2.22) $0 \in \tau(C, d)$ and $0 \notin \tau(A, e)$.

But $Sabc$ & $Rcde$ imply that for some x , $Sadx$ and $Rbxe$ and also imply, for some other x , $Rbdx$ & $Saxe$. Hence, for 2.21), since $0 \notin \tau(A \rightarrow B, a)$, as $Sadx$, $1 \in \tau(B, x)$. Since further $1 \in \tau(B \rightarrow C, b)$, as $Rbxe$, $1 \in \tau(C, e)$, which is impossible. Similarly, for 2.22), as $Rbdx$, $0 \in \tau(B, x)$. But $Saxe$, so $0 \in \tau(A, e)$, which is again impossible.

ad D4. The argument is as in the previous section from amended $\rightarrow$ rules. No conditions are required.

The completeness argument differs from that given for B in § 4 in that T - L theories replace L -theories. Where T is any regular L -theory, a T - L -theory a is any set of wff of L which is closed under adjunction and under T -implication, i.e. wherever $A \rightarrow B \in T$ (also written $\vdash_T A \rightarrow B$) and $A \in a$ then $B \in a$. Regularity, primeness, etc. are defined for T - L -theories just as for L -theories. Furthermore, $\bar{K}_T$, $\bar{R}_T$, $\bar{S}_T$, etc., are now defined in terms of T - L -theories, where T is some (prime) regular theory. Where T is a prime regular theory the T canonical L a.m.s. is the structure $M_T = \langle T, K_T, R_T, S_T \rangle$ where as before prime restrictions are taken to obtain unbarred predicates.

In order to exploit previous work the operator $*$ is again introduced as an inessential auxiliary, with $a^* = \{B: \sim B \notin a\}$ for $a \in K_T$. It follows that (RS) $S_T abc$ iff $R_T a^* c^* b^*$, i.e. iff, in virtue of Contraposition, $R_T a^* bc$.

These interconnections greatly reduce the task of verifying modelling conditions, especially those for B3.

STRONG COMPLETENESS THEOREM. *Where W is not C -derivable from regular set S , there is a canonical valuation τ in a canonical C a.m.s. M which verifies every member of S and falsifies every member of W .*

PROOF is like that for the corresponding theorem for system B , but with differences as remarked. A prime regular C theory T is obtained as before, but now a T canonical C a.m.s. $M_T = \langle T, K_T, R_T, S_T \rangle$ on base T is formed.

(I) M_T is a C a.m.s. Much is proved as in theorem 4.10 of [7]. In particular, q1, q2 and q4 may be established precisely as there. Note that B1 is used in showing that $T \in K_T$.

ad $\bar{q}1$. By (RS) this amounts to $R_T a^* a^* a^*$ which follows from q1.

ad $\bar{q}3$. Expanding this is: if S_Tazd & S_Tzbc then for some x , S_Tdxc & S_Tabx , i.e. by (RS) if $R_Ta*d*z*$ & R_Tz*bc then for some x R_Td*xc & R_Ta*bx . But this is $q3$.

ad $qm3$, i.e. if R_Tazc & S_Tzde then, for some x , R_Tadx & S_Tcxe . That is by (RS) if R_Tac*z* & R_Tz*de then for some x , R_Tadx & R_Tc*xe , i.e. $q3$ relettered. The other conditions for B3 are similarly verified.

A canonical valuation τ is defined as in the corresponding theorem for **B**, and (II) is established in much the same way. But note that as in theorem 4.10 of [7], both B3 and B4 are required on the induction step for $\rightarrow$.

ad amended $\rightarrow$ rules: By theorem 4.10,

$B \rightarrow C \in a$ iff $(b, c \in K_T)$ if R_Tabc and $B \in b$ then $C \in c$, i.e. by Contraposition iff $(b, c \in K_T)$ if R_Tabc then if $B \in b$ then $C \in c$ and if $\sim C \in c$ then $\sim B \in b$. By induction hypothesis, this suffices to show

$$B \rightarrow C \in a \quad \text{iff} \quad 1 \in \tau(B \rightarrow C, a).$$

$\sim(B \rightarrow C) \in a$ iff $B \in C \notin a^*$, i.e. iff the negation of the above but with S_Tabc in place of R_Ta*bc using (RS), i.e.

$$\text{iff } 0 \in \tau(B \rightarrow C, a).$$

Corollaries such as completeness, compactness and implicational adequacy follow as before.

Some extensions of **C** are moderately straightforward, though again case-ridden, for instance Absorption:

$$\text{B5. } A \rightarrow (A \rightarrow B) \rightarrow A \rightarrow B \quad \text{q5 set} \quad \left\{ \begin{array}{l} \text{q6. If } Rabc \text{ then } R^2abbc \\ \bar{\text{q6. If } Sabc \text{ then } S^2a[bc]c \\ \text{qm7. If } Rabc \text{ then } R/Sa[bc]c \\ \text{qm8. If } Sabc \text{ then } S/Rabbc \end{array} \right.$$

The really disconcerting new phenomenon is that with remaining axioms required for **R** and **T** problems emerge which beset not merely negative axioms like Reductio, D3, but also positive axioms such as EAssertion, B6. Much as previously with Contraposition, a 1-to-0 cross-over is called for, which however the rules do not sanction. Consider, to illustrate, the putative validation of B6, the only scheme outstanding in an American semantics for system **R**. It has to be shown, for one case, that if $1 \in \tau(A, a)$ then $1 \in \tau(A \rightarrow B \rightarrow B, a)$. Suppose otherwise; so both $1 \in \tau(A, a)$ and for some b and c in K for which $Rabc$ either 1.1) $1 \in \tau(A \rightarrow B, b)$ and $1 \notin \tau(B, c)$ or 1.2) $0 \in \tau(B, b)$ and $0 \notin \tau(A \rightarrow B, c)$. Consider 1.2). Since $0 \in \tau(A \rightarrow B, c)$ if $Sabc$ then if $0 \in \tau(B, a)$ then $0 \in \tau(A, b)$ and if $1 \in \tau(A, a)$ then $1 \in \tau(B, b)$. But without some crossing from belonging to 0 to not belonging to 1 or vice versa, no trouble with the remaining assumptions $1 \in \tau(A, a)$ and $0 \in \tau(B, b)$ can emerge. Thus **C** a.m.s. strongly counter Ackermann fallacies such as B6. With Reductio there is a similar "problem".¹⁰

¹⁰Some might view these outcomes not as problems but as desirable features. B6 is certainly a dubious principle (even when entailment is not in question), and there are substantial grounds for being suspicious of D3 (on both see [7] 3).

- &. $0 \notin \tau(A \& B, x)$ only if $0 \notin \tau(A, x) \& 0 \notin \tau(B, x)$
 only if $1 \in \tau(A, x) \& 1 \in \tau(B, x)$
 only if $1 \in \tau(A \& B, x)$
- $\rightarrow$. If $0 \notin \tau(A \rightarrow B, x)$ then for every $b, c \in K$ if $Sxbc$ then both
 if $0 \in \tau(B, b)$ then $0 \in \tau(A, c)$ and
 if $1 \in \tau(A, b)$ then $1 \in \tau(B, c)$.

But by c1, $Sxbc$ can be replaced by $Rxbc$, whence $1 \in \tau(A \rightarrow B, x)$.

The adequacy theorem for C is unaffected by change to enlarged models. Soundness is as before. For completeness, define $C = \{a: a^* \subseteq a\}$. Then if $x \in C, x^* \subseteq x$, so if $R_T xbc$ by p2 effectively, $R_T x^*bc$, i.e. $S_T xbc$, confirming c1. For c2, $0 \notin \tau(P, x)$ implies $\sim A \notin x$, which implies $A \in x^*$, which implies $A \in x$, as $x \in C$, which implies $1 \in \tau(A, x)$.

This enables simple, but important, extensions of C like the following to be accommodated:

$$D2. \quad A \vee \sim A \qquad s2(T) \quad T \in C$$

For soundness, suppose otherwise $1 \notin \tau(A \vee \sim A, T)$ for some C -enlarged m.s. Then $1 \notin \tau(A, T)$ and $1 \notin \tau(\sim A, T)$ i.e. $0 \notin \tau(A, T)$. But since by s2(T), $T \in C$, by the lemma, $1 \in \tau(A, T)$ also, which is impossible. For completeness it is enough to show $T^* \subseteq T$. But for every A , because $A \vee \sim A \in T$, $A \in T$ or $\sim A \in T$, i.e. $A \notin T^*$ or $A \in T$, as required.

World completeness does not however suffice for the validation of Reductio and EAssertion, for which the much more demanding matter of world consistency is also involved. As regards modelling, consistency is the dual of completeness. An H -enlarged C a.m.s. is a structure $M = \langle T, N, K, R, S \rangle$ where further $H \subseteq K$ and

- h1. For $x \in H$, if $Sxbc$ then $Rxbc$;
 h2. For $x \in H$, if $0 \in \tau(p, x)$ then $1 \notin \tau(p, x)$, for every sentential parameter p .

CONSISTENCY CROSS-OVER LEMMA. For every wff A and every $x \in N$, if $0 \in \tau(A, x)$ then $1 \notin \tau(A, x)$.

Again the adequacy theorem for C is unaffected.

For the complete possible worlds of modal logic, C and H are simply combined (i.e. the *only ifs* of h1, h2 and the lemma are replaced by *iffs*). But life, particularly attempts to prove the hard parts of the adequacy theorem, is made easier by introducing completeness and possibility separately, not jointly. A CH -enlarged C a.m.s. is a structure $M = \langle T, C, H, K, R, S \rangle$ where $C \subseteq K$ and $H \subseteq K$ and all of c1, c2, h1 and h2 hold. Adequacy is again unaffected, and putative modelling conditions for the outstanding schemes of R and T can now be supplied.

$$\begin{aligned}
 \text{B6. } A \rightarrow . A \rightarrow B \rightarrow B \quad \text{q6} \quad \text{set} & \left\{ \begin{array}{l} \text{If } Rabc \text{ then } Rbac; \\ \text{If } Sabc \text{ then } Scba; \\ \text{If } Rabc \text{ then, for some } x \in H, Scbx \text{ and} \\ a \leq x; \\ \text{If } Sabc \text{ then, for some } x \in C, Rbxc \text{ and} \\ x \leq a. \end{array} \right. \\
 \text{D3. } (A \rightarrow \sim A) \rightarrow \sim A \quad \text{s3} \quad \text{set} & \left\{ \begin{array}{l} \text{For some } x \in C, Raxa \text{ and } x \leq a; \\ \text{For some } x \in H, Saax \text{ and } a \leq x. \end{array} \right.
 \end{aligned}$$

Proof of soundness is straightforward if tedious. Proof of completeness is however far from straightforward, and appears to be ruled out entirely as regards conditions for B6.¹¹ Accordingly, the clean-up investigation has focussed on the closely related fourth method.

§ 8. Completing the American plan for R , T and E , through mateship

Worlds are mated, and cross-over occurs between mated worlds. A C a.m.s. with mateship relation M is a structure $M = \langle T, K, R, S, M \rangle$ which adds to a C a.m.s. a two-place relation M in K such that

- m1. If Mab then Mba , and
- m2. If Mab then $Racd$ iff $Sbcd$.

The valuation function τ is constrained by the condition

- m3. If Mab then

$$0 \in \tau(p, a) \text{ iff } 1 \in \tau(p, b), \text{ for every sentential parameter } p.$$

Conditions m1-m3 are imposed precisely in order to enable proof of the following cross-over result.

MATED CROSS-OVER LEMMA. *If Mab , then $0 \in \tau(A, a)$ iff $1 \notin \tau(A, b)$.*

COROLLARY. *If Mab , then $1 \in \tau(A, a)$ iff $0 \notin \tau(A, b)$.*

The corollary follows by symmetry of M .

PROOF of the lemma is by induction from the given basis. Suppose throughout Mab .

$$\text{ad } \sim. \quad 0 \in \tau(\sim B, a) \text{ iff } 1 \in \tau(B, a).$$

By induction hypothesis, since M is symmetric, and so Mba , $0 \in \tau(B, b)$ iff $1 \notin \tau(B, a)$ as in the corollary. Hence, as $1 \notin \tau(\sim B, b)$ iff $0 \notin \tau(B, b)$, $0 \in \tau(\sim B, a)$ iff $1 \notin \tau(\sim B, b)$.

¹¹Consider the final condition for B6. In virtue of the first two conditions which are unproblematic the supposition $Sabc$ can be replaced by Rba^*c . Given this supposition what has to be shown is that a^* can be extended to a complete theory x^* (not on its own a difficult feat) but without enlarging the class of consequences in c . Unfortunately this seems to be impossible. For suppose $p \vee \sim p \rightarrow C \in b$ but $p \vee \sim p \notin a^*$, then extending a^* to a complete theory does enlarge c .

ad &. $0 \in \tau(B \& C, a)$ iff $0 \in \tau(B, a)$ or $0 \in \tau(C, a)$
 iff $1 \notin \tau(B, b)$ or $1 \notin \tau(C, b)$
 iff $1 \notin \tau(B \& C, b)$

ad $\rightarrow$. $0 \in \tau(B \rightarrow C, a)$ iff, for some c and d , $Sacd$ and either $0 \in \tau(C, c)$ and $0 \notin \tau(B, d)$ or else $1 \in \tau(B, c)$ and $1 \notin \tau(C, d)$, i.e. (since given Mab , $Racd$ iff $Sacd$) iff, for some c and d , $Rbcd$ and either $1 \in \tau(B, c)$ and $1 \notin \tau(C, d)$ or else $0 \in \tau(C, c)$ and $0 \notin \tau(B, d)$, i.e. iff $1 \notin \tau(B \rightarrow C, b)$.

Adequacy results for system C and extensions already dealt with transpose unproblematically. For completeness, define Mab as $a = b^*$. Then m1-m3 are immediate. The sought extensions are modelled as follows:

B6. $A \rightarrow . A \rightarrow B \rightarrow B$ q6 set $\left\{ \begin{array}{l} \text{If } Rabc \text{ then } Rbac; \\ \text{If } Sabc \text{ then } Seba; \\ \text{If } Rabc \text{ then for some } x, Mxa \text{ and } Scbx; \\ \text{If } Sabc \text{ then for some } x, Max \text{ and } Rbxc. \end{array} \right.$

D3. $A \rightarrow \sim A \rightarrow . \sim A$ s3 set $\left\{ \begin{array}{l} \text{For some } x, Max \text{ and } Raxa; \\ \text{For some } x, Mxa \text{ and } Saax. \end{array} \right.$

The details required for adequacy go as follows:

ad B6. It is enough to show 1) if $1 \in \tau(A, a)$ then $1 \in \tau(A \rightarrow B \rightarrow B, a)$ and 2) if $0 \in \tau(A \rightarrow B \rightarrow B, a)$ then $0 \in \tau(A, a)$. For 1) suppose $1 \in \tau(A, a)$ but $1 \notin \tau(A \rightarrow B \rightarrow B, a)$. Then for some b and c in K such that $Rabc$ either 1.1) $1 \in \tau(A \rightarrow B, b)$ and $1 \notin \tau(B, c)$ or 1.2) $0 \in \tau(B, b)$ and $0 \notin \tau(A \rightarrow B, c)$. Option 1.1) is readily ruled out by the first modelling condition which permutes a and b . For 1.2) apply the third condition. Since $Rabc$, for some x , $Scbx$ and Mxa . As $0 \notin \tau(A \rightarrow B, c)$ and $0 \in \tau(B, b)$, $0 \in \tau(A, x)$. Hence as Mxc , by Mated Cross-over $1 \notin \tau(A, a)$ which is impossible. Under case 2) there are two more rather similar cases.

ad q6 set. The first condition follows directly using B6, as under the Australian theory (see [7] 4). Suppose, for the second, $Sabc$. Then $Ra^*c^*b^*$, so by the first condition $Rc^*a^*b^*$, whence $Seba$. For the third, suppose $Rabc$. Then permuting, $Rbac$, Rbc^*a^* , Rc^*ba^* . So setting $x = a^*$, Rc^*bx , i.e. Mxa and $Scbx$. The fourth is similarly established.

ad D3. It is enough to show 1) if $1 \in \tau(A \rightarrow \sim A, a)$ then $1 \in \tau(\sim A, a)$ and 2) if $0 \in \tau(\sim A, a)$ then $0 \in \tau(A \rightarrow \sim A, a)$. As to 1) suppose otherwise $1 \in \tau(A \rightarrow \sim A, a)$ but $1 \notin \tau(\sim A, a)$, i.e. $0 \notin \tau(A, a)$. By the first modelling condition for some x Max and $Raxa$. Hence if $1 \in \tau(A, x)$ then $1 \in \tau(\sim A, a)$, whence $1 \notin \tau(A, x)$. So by the cross-over lemma $0 \in \tau(A, a)$, which is impossible. For 2) suppose $0 \in \tau(\sim A, a)$, i.e. $1 \in \tau(A, a)$, but $0 \notin \tau(A \rightarrow \sim A, a)$. By the second modelling condition, for some x , Mxa and $Saax$. So as $0 \in \tau(\sim A, a)$, $0 \in \tau(A, x)$. Therefore by the cross-over lemma, $1 \notin \tau(A, a)$, which is impossible.

ad s3 set. The Australian modelling condition for D3 is Raa^*a , the adequacy of which is readily established (see [7] 4). Set $x = a^*$. Then Max and $Raxa$. Similarly $Ra^*x^*a^*$, so $Saax$.

The *Strong Adequacy Theorem for R and T* under the American plan with mateship now follows. Statement and other details of proof are like those for *C*. The Theorem similarly follows for *E* upon complicating the structures by class *O* and conditions theorem for BR1. Add to mated m.s. a class *O* of (regular) worlds included in *K* and containing *T*. Then BR1 is modelled as follows:

$$\text{BR1. } A/A \rightarrow B \rightarrow B \text{ br1 set } \begin{cases} \text{For some } x \text{ in } 0, Raxa \\ \text{For some } x \text{ in } 0 \text{ and some } z, Mxz \text{ and} \\ Saaz. \end{cases}$$

Corollaries such as compactness follow in the usual way.

The mateship relation, though it suggests the star operation, does not enable the definition of star, since neither functionality condition is guaranteed. Nor, for a similar reason, does it permit contraction of model structures by definition of *S*. The addition of a functionality requirement,

m4. If Mac and Mbc then $a = b$,

does allow definitional elimination of *S* thus: $Sabc =_{df} (Px) (Rxbc \ \& \ Mxa)$. But condition m4 is now required to establish m2. Suppose for m2, Mab . If further $Racd$ then $Sbcd$, by quantificational logic alone. For the converse suppose further that for some z $Rzcd$ and $Mzcb$. Since Mab and $Mzcb$, $a = z$, so $Racd$. (Strictly an ordering relation $a \leq z$ would suffice, but symmetry of *M* converts this to what is effectively an identity relation.)

The penalty for the natural and obvious elimination of relation *S* is the definitional restoration of star. Since *M* is symmetric it supplies an operation, * say, on worlds, defined through the connection $b = a^*$ iff Mab . But then the star rule follows and values can be deflated from four to two.

STAR RULE LEMMA. $1 \in \tau(\sim A, a) \text{ iff } 1 \notin \tau(A, a^*)$.

PROOF. $1 \in \tau(\sim A, a) \text{ iff } 0 \in \tau(A, a)$, i.e. by Mated Cross-Over, iff $1 \in \tau(A, b)$ where Mab , i.e. iff $1 \notin \tau(A, a^*)$.

The American plan is, in the end, when duly enriched with the Australian mateship component, little more than a very circuitous and cumbersome way of avoiding the much more intuitive Australian plan.

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 UNIVERSITY OF VICTORIA, CANADA

Received June 1980