A Treatise on Associative Harmony

An Investigation into the Contrapuntal Possibilities of Chords related by Interval Content

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**1963-64;
revised 1987, 2004**

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Preface

This treatise exhibits certain important relationships that exist between 3- and 4-note chords sharing several intervals in common. When present in a composition, these common intervals are perceived as an "association" between those sonorities.

The structures that are elucidated are so fundamental to compositional technique, that it isn't necessary to give examples of compositions illustrating their use. It is important to emphasize that *this is not a theory of musical composition* . either diatonic, serial or 12-tone. Its relationship to musical composition may be considered analogous to that of a treatise on the phonemes of English to the writing of poetry.

Still, I believe that the analysis presented here, which generalizes techniques of suspension, anticipation, resolution, constructing cadences, etc., traditional to diatonic music, will be immediately perceived by composers as useful for their work.

The only prerequisites are the ability to read music, a general familiarity with the compositional techniques of European music, and some knowledge of simple abstract algebra, such as groups and modular arithmetic, in particular to the base 12.

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I. All-Interval Chords, Transpositions and Group Theory

An "All-Interval Chord" is one whose relative interval content includes all the intervals between the 12 notes of the chromatic scale. For certain tetrachords each interval occurs once and only once; obviously the tritone, though appearing twice, must be deemed equivalent to its inversion. For the purposes of

this treatise the phrase "all-interval chords" will mean these and no others. There are 4 distinct all-interval tetrachord sonorities:

Figure 1

A useful structural property of these sonorities : since every interval is present in the interval content once and only once, every transposition of a specific All-Interval Chord (save at the tritone), will have *one and only one note* in common with the original chord. The tritone transposition preserves the two notes of the tritone in the chord:

Figure 2

The unique note held in common between the original and the transposed chord may be called the intersection note of the two chords .

There are no other A.I. tetrachords; this will be demonstrated in the final section. Label each note by the number of semitones from C = 0 . In closed position, the chords are:

I = (0,4,6,7)

I* = (0,1,3,7) . This is the *inversion* of I .

T = (0,2,5,6) This is the "*circle of 5ths* " translation of I.

Multiply each number in the set (0,4,6,7) by 7 and take the residue (mod 12)

T* = (0,1,4,6) . This is both the inversion of T , and the "circle of fifths" transformation of I* .

Mathematically, the All-Interval chords combine the maximal "antigroups" (such as (4,6,7), with the identity (0), in the additive modular group Z_{12} .

Letting:

Q represent the action of inversion,

F the circle-of-fifths transformation

QF the result of applying both

e the identity

these transformations form a simple group of 4 elements acting on the set of All-Interval Chords.

$$Q^2 = F^2 = e$$

$$QF = FQ$$

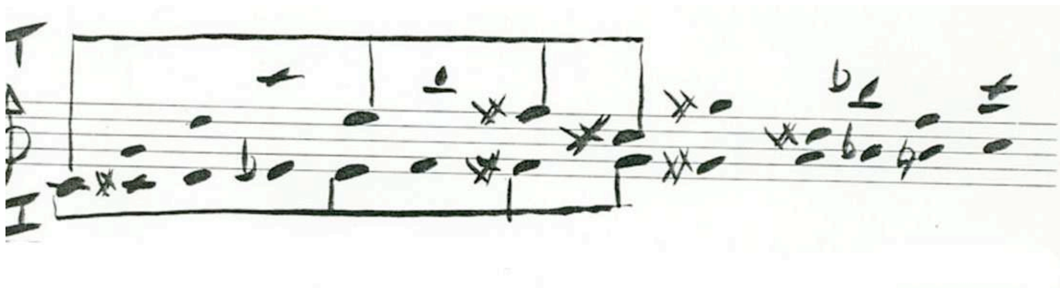

Figure 3

A few observations :

(a) The A.I. chords contain no "symmetric triads",(such as the augmented chord or the diminished or whole step chord, or any sub-chord with two identical intervals in its content.)

(b) The sonorities represented by the chords:

E = (0,4,5) and E* = (0,1,5) are not sub-chords of any of the all-interval chords. The combinatorial basis for this will be examined later.

(c) Every other 3-note sonority may be found as subchords in one or another of the all-interval chords. There are twelve of them all told:

Figure 4

One sees how the All Interval chords can be employed to unify the contrapuntal texture of music employing some or all of the distinct 3-note sonorities derivable from the chromatic scale. Observe that 8 of the 12 sonorities in the above chart are present in only one of the 4 A.L. chords, and the remaining four present in only two. The trichord sonorities can therefore be employed as "signals" indicating the presence of a full tetrachord, in the same way that the intervals of the 7th or tritone signal the presence of a dominant seventh chord, or even of an entire key. For example, as seen in Figure 4 , the major triad is uniquely associated with I, the minor triad with I*, etc.

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Fundamentals of Triadic Associative Harmony

Definition: **Two triads, or 3-tone sonorities, are** ***associated*** **or** ***related by association*** **if two distinct (non-inverted) intervals in their interval content are shared. Since the interval content of a trichord consists of 3 intervals less than or equal to the tritone, this means that associated chords differ by at most one interval (less than or equal to the tritone) . In Figure 5 we see all the triads associated with the C-major chord:**

Figure 5

Although these chords are associated with (CEG) = (0,4,7), they are not necessarily associated with each other. ***Musical association*** **is not** ***algebraically associative*** **! It is commutative however , and certainly reflexive (A chord is obviously associated to itself) . Because of the lack of associativity , "association" is not an equivalence relation, a good thing from the viewpoint of musical composition as it makes possible the construction of chains of associated chords moving through the entire catalogue of trichords.**

(With one exception: ***the trichord of the augmented third*** **, with a single interval in its interval content!)**

Obviously the major-minor association is an equivalence relation, since major and minor forms of a given sonority always

have exactly the same interval content. Counting the "symmetric" chords (excluding the chord of the augmented third) the total number of distinct trichord sonorities is 18 . A progression going through all of them by the principle of association might begin like this:

Figure 6

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Starting with a sequence of simple intervals (say minor third and fifth), one can construct groups of chords associated with them:

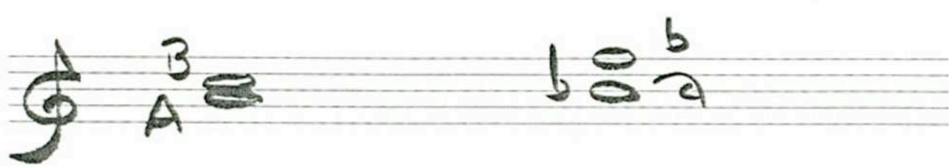

Figure 7

When this pair of dyads appears consecutively, or essentially in sequence, our habits of listening to diatonic music have induced the habit in us of interpreting both dyads as sub-intervals of the same sonority, in this case the major or minor triad, depending on context.

So that, if we hear minor and major thirds, an interval of a 4th or 5th, etc., we will automatically place them in major or minor diatonic triads depending on the key or the modulation between keys, and so on. Hearing seconds, sevenths and tritones, our musical imaginations conjure up seventh chords.

Using these two dyads as a base one can derive 4 associated chords which incorporate both of them and underlies the passage from dyad 1 to dyad 2 as a sonority preserving progression.

In the charts below this is worked out in detail for the example of Figure 7. The incorporation of the derived trichord sonorities into the all interval chords is also shown

Figure 8

The same analysis is presented for a pair of dyads in contrary motion, (a,c) to (g,d):

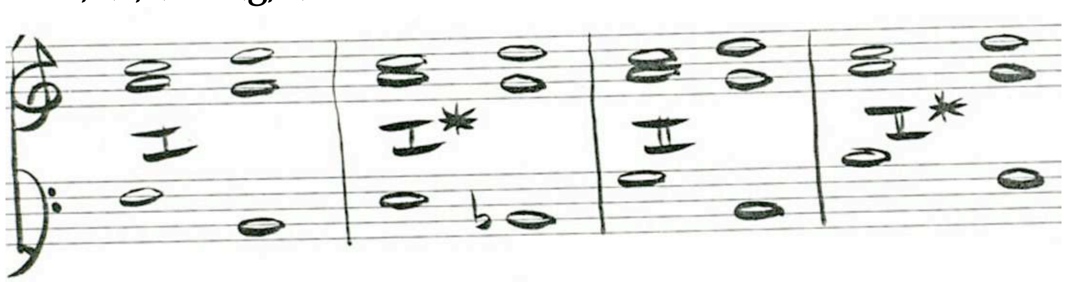

Figure 9

As I and I* are the triads from which the diatonic system is derived, one sees that, in a situation involving parallel motion, the uniqueness of the triad is undermined, which may have

something to do with the prohibition against parallel motion of unisons, fourths and fifths. Quite apart from its relationship to the diatonic system, this phenomenon is based on certain special *combinatorial* properties of the major triad, which we will come to presently.

We now allow the letter I to represent some arbitrary non-symmetrical trichord sonority. All of the trichords associated to I may be derived from the hexachord H which is generated by the 3 positions of I on a fixed base. This is best illustrated by an example.

Let I = (0,3,5) = (CE♭F) . Keeping C as the bottom note, write down the 3 positions for I:

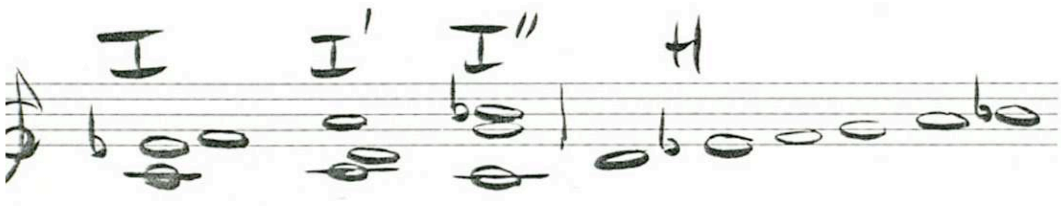

Figure 10

The top and middle voices themselves will be called '3-forms' , and designated as V and V' respectively:

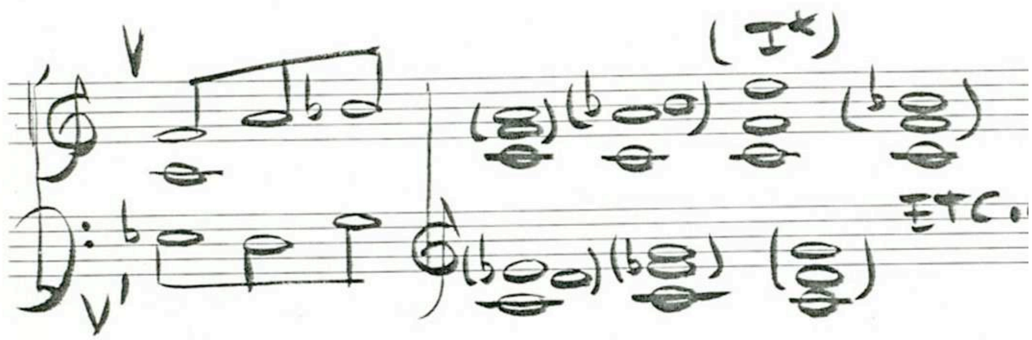

Figure 11

Take any pair of notes from V and set them against the bass note C. Doing this in all possible ways one derives the 3

associated chord, J, K, L . Likewise the notes of V' taken in pairs will generate their minors, J* K* and L* .

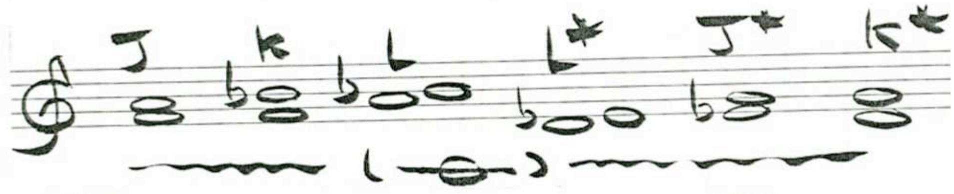

Figure 12

Generally speaking, take any trichord I = (0,a,b) (Mod 12) . Then:

I' = 1st Inversion = (0, b-a, 12 -a)

I'' = 2nd Inversion = (0. 12-b, 12 - b + a)

The 3-forms V and V' are therefore

V = (b, 12-a , 12 - b + a)

V' = (a, b-a , 12-b)

V and V' are inversions. Each V form is internally symmetrical relative to the bass note in the following : the sum of all 3 notes adds up to 0 (mod 12):

$$\boxed{\begin{array}{l} b+12-a+12-b+a \equiv 0(\mathrm{mod}\,12) \\ a+b-a+12-b \equiv 0(\mathrm{mod}\,12) \end{array}}$$

Therefore:

1. Given the first two notes of V and the root note (C in the example) , one can derive the 3rd note. One can also derive the form V'.

2. Given the generating chord I with root note C, and a V-form derived from the top notes of the 3 positions of I , then the forms V+4 , and V+8 , ***transpositions up a major third and up a minor sixth*** **will also be the V-forms of sonorities (differing in general from I) on the same root note.**

Proof : If $I = (0,a,b)$, then $V = (b, 12 - a , 12 +a-b)$, and $V^* = V+ 4 = (b+4 , 16 -a , 16 +a-b)$. Since

$$b + 4 + 16 -a + 16 + a - b = 36 = 0 \ (mod\ 12)$$

V^* is the V-form of the sonority $I^* = (0, a+4 , b-4)$, while $V^{**} = (b+8, 20-a, 20+a-b)$ (transposed to the appropriate octave) is the V-form of the sonority $I^{**} = (0,a +4 , b-4)$

We next show that the condition that the note values of a 3-form $U = (u,v, w)$ add up to a multiple of 3 is necessary and sufficient that U be the V-form of some trichord sonority I.

If $u+v+w = 3k \ (mod\ 12)$, transpose U down an interval k, to obtain the 3-form $U' = (u',v',w') = (u-k , v-k, w-k)$. Without loss of generality we can drop the accents and assume that

$$u+v+w = 0 \ (mod\ 12)$$

Then, letting $u = b$, $v = 12-a$, one obtains the generating chord

$$I = (0,a,b) = (0, 12-v, u)$$

However, given the root note 0, and V-form (u,v,w), there are only 3 transpositions, namely V, $V^* = V+4$ and $V^{**} = V+8$, that imply trichord sonorities I, I^* and I^{**} that generate them. Then these combinations (I,V,V') , $(I^*,V^*,V^{*\prime})$ and $(I^{**}, V^{**}, V^{**\prime})$ will generate complete sets of associated chords.

Figure 12

To show this we start with a V-form $(b,12-a, 12+a-b)$

(from root "0") , and transpose it up an amount r which is not a multiple of 4 . If

$$\boxed{V + r = (b + r, 12 - a + r, 12 + r + a - b)}$$, **then**

$$\boxed{b + r + 12 - a + r + 12 + a - b \equiv 3r(\mathrm{mod}\,12)}$$

which is congruent to 0 only when r = 0, 4 or 8.

Observe that in practical terms what this means is that all the trichord sonorities generating the same pair of V-forms V and V' are obtained by fixing the root, dropping one of the remaining notes a major third, and raising the other note by a major third.

Summary

(1) Given a trichord sonority I (assuming, for convenience sake that I is not symmetric, i.e. has no repeated intervals in its interval content) , all of the trichord sonorities associated with I and its minor I* are gotten by taking the notes of the V-forms, V and V' in pairs against the root note.

(2) Given a V form above a root note, with sonority I, then V+4 and V+8 are also V-forms above that root note. The trichord sonorities associated with them are derivable from I by lowering one of the non-root notes by a major third and raising the other non-root note by a major third.

(3) Not every triple of notes is a V-form.

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The possible V-forms

Let I be a trichord sonority with root-note "0", I =(0,a,b) . We make the reasonable assumptions that

(i) b > a > 0

(ii) I is in "closed form" , which means that 0 < b ≤8

The V-form is

$$\boxed{(b, 12-a, 12+a-b) \equiv (b, -a, a-b)(\mathrm{mod}\,12)}$$

In the following table, the duplications b = a , b = -a , a-b = -a, a-b = b

have been eliminated :

a= 1 , -a = 11	b = 2,3,4,5,6,7,8	a-b = 10,9,8,7
a=2, -a = 10	b = 3,4,5,6, 7, 8	a-b = 11, 9, 8
a= 3, -a = 9	b = 4, 5, 6, 7,8	a-b = 11, 10, 8
a=4 , -a =8	b = 5,6,7	a-b = 11, 10, 9
a= 5, -a = 7	b = 6 , 8	a-b = 10, 9
a= 6 , -a = 6	b = 7,8	a-b = 11, 10
a= 7, -a = 5	b = 8	a-b = 11

Table I

The distinct forms which are not transpositions are :

V_1 = (0,1, 5) = (C$C^{\#}$F)

V_1*= (0,4,5) = (CEF)

V_2 = (0,5, 7) = (CFG) (≈ (CGD))

V3 = (0, 3, 6) = (C$E^{b}F^{\#}$)

V_4 = (0,2,4) = (CDE)

V5 = (0,1,2) = (C$C^{\#}$D)

Theorem : *The V-forms are the symmetric trichords and the major and minor forms of the exceptional chord E = (0,4,5) . Proof by inspection*

Corollary I : *With the exception of E, each V-form is identical , (up to permutation) to its minor* $V = V'$ *.*

Corollary 2: *The V-forms therefore, are precisely those trichord sonorities which are not subchords of any of the all-interval chords .*

That this is more than a simple coincidence will become clear in the ensuing discussion.

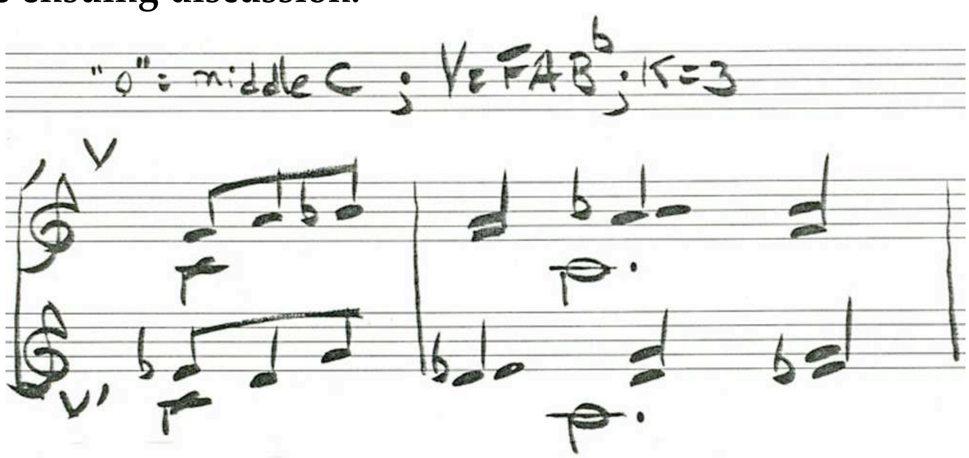

Figure 13

Given a symmetric V-form, what can be said about the generator trichord sonorities from which it is derived? Writing V as (b , -a, a-b) , V will be symmetric when either :

(c_1) $b - (a-b) = (a-b) + a \pmod{12}$, or

(c_2) $b - (a-b) = -a - b \pmod{12}$

The first condition is equivalent to

$3a = 3b \pmod{12}$ or (given that $a = b$ is ruled out)

$b = a\ (+/-)4$

The second condition reduces to

$3b = 0 \pmod{12}$, or $b = 4$ or 8 .

If the V-form of I is a symmetric form. then I contains a major third in its interval content . **In fact,**

Theorem :*The trichords which generate symmetric V-forms are precisely those which have a major third (minor sixth, etc.) in their interval content.*

Therefore, the exceptional forms E and E are generated by non-symmetric trichord sonorities which do not have a major third in their interval content.*

Figure 14

#17...

Figure 15

Further Transpositional Properties of V-forms and Roots

I. If an interval k be added to any note of a given V-form, and that same amount subtracted from another note, the resulting form will also be a V-form

2. If an interval k be added to 2-notes of a V-form, and the amount 2k subtracted from the third, the resulting form will again be a V-form

3. If any note of V be raised a minor third (k =+3) , and the root R be moved up a semitone, the new form will be a V-form over the new root.

These are all easily derivable by modular arithmetic. Briefly, everything adds up to 0, modulo 12 !

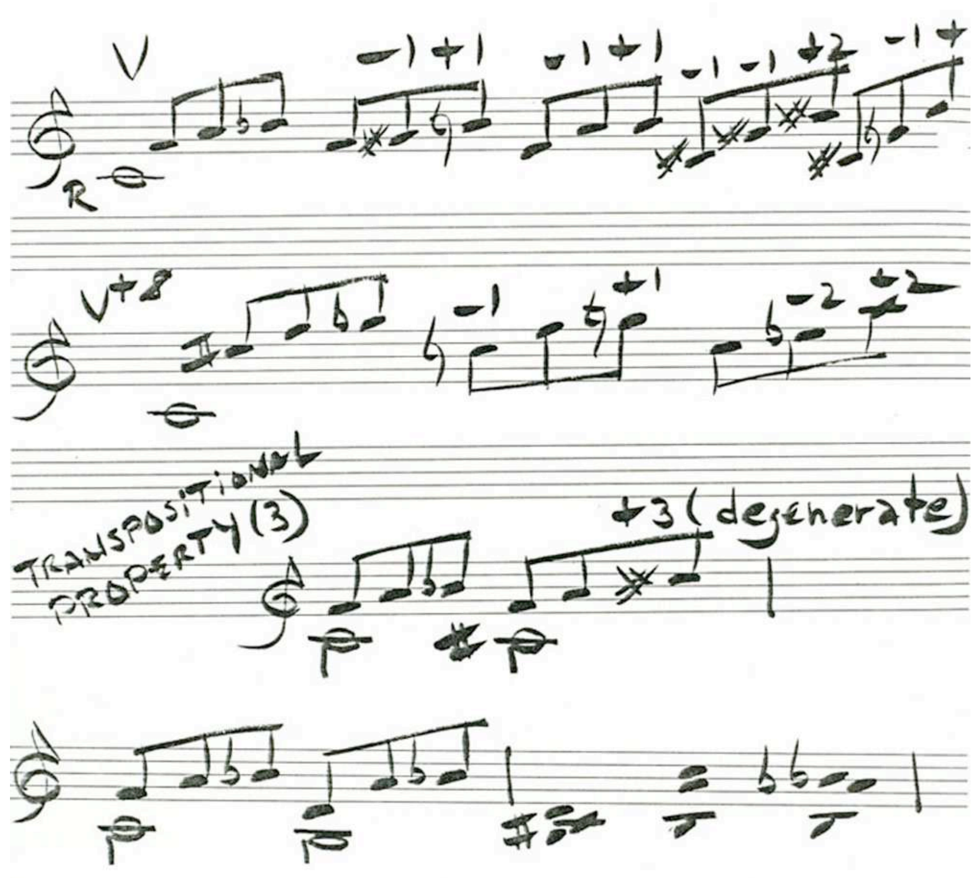

Figure 16

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Associational Techniques and Minor Forms

A few indications about the application of symmetric trichords in associative harmony: If I is a trichord sonority, I* its inversion, they will have the same interval content. Consequently the forms V* and V*' for I* will be permuted versions of those for I.

This suggests a compositional technique utilizing both symmetric and non-symmetric trichords one might call ***Combinatorial Switching*** **:**

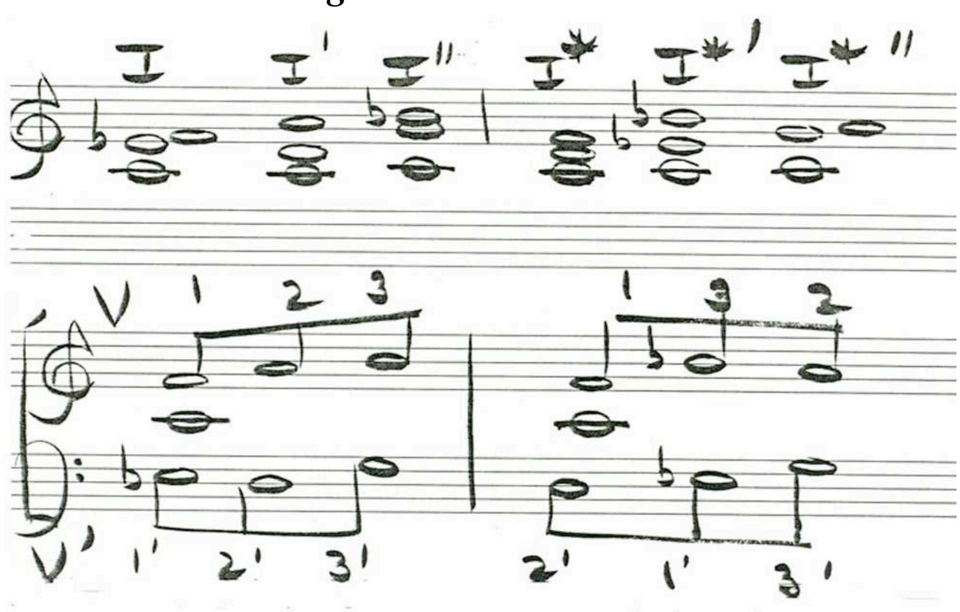

Figure 17

In the above example, the forms V , V' , V* and V*' have been drawn above and below their root notes of I = CE^{b}F and I* = CDF respectively .

If one arbitrarily switches a lower note with an upper note, the resultant 3-forms will either be V-forms, or will generate symmetric chords that are linked by association of 2-intervals to the other chords derived from them. This is best illustrated by example:

Figure 18

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Summary of Properties of Trichords and V-forms

A. The forms V and V' of a trichord sonority I with root r= "0", can generate the complete catalog of all sonorities associated to I by taking their notes in pairs in combination with the root note.

B. If intervals k,l,m are added to the notes of a V-form such that
k+l+m = 0 (mod 12), the result will be a new V-form. In particular if the numerical values of the notes of one V-form are added in sequence to those of another V-form, the result will be a third V-form.

C. Transposition of a V-form by a major third produces a new V form against the same root and a different sonority

D. The collection of V-forms consists of those trichords which are not sub-chords of any all interval chord.

E. Corresponding notes of V and V' , or V and V* , can be switched to give other forms, more general that V-forms, which generate associated chords, both symmetric and non-symmetric.

Figure 19

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The Chord Equation
Double Tonics
Invariant Dyads

For our present purposes we identify sonorities with equivalence classes on the basis of:

(i) Octave equivalence

(ii) Transpositions

(iii) Positions (a combination of ii and iii)

Fix some note "0", and label intervals by the number of semitones from this root. Thereby one can identify any chord by its "trope" of numbers C = (a,b,c,d ...) , modulo 12. The equivalence conditions can be restated as:

(i) For any note n , n ≈n+12

(ii) For any C, and integer k C ≈C+k

(iii) C is a trope, which means that

C= (a,b,c,d,...,h) is equivalent to any rearrangement C'

All 3 conditions can be brought together as :

$$\boxed{A = (a_1, a_2, \ldots a_n) \sim B = (b_1, b_2, \ldots, b_n)}$$

if there is some rearrangement of the a's

$$\boxed{A' = (a_{j_1}, a_{j_2}, \ldots, a_{j_n}) = (a_1', a_2', \ldots., a_n')}$$

such that

$$\boxed{a_i' - a_1' \equiv b_i - b_1 (\mathrm{mod}\, 12)}$$

Problem : *Given dyads (a,b) and (c,d) above a root note "0", when is it possible to find a note x such that (x,a,b) and (x,c,d) are transpositions of the same sonority, and how does one compute x ?* **Such a note x will be called an "implied tonic" .**

Figure 20

In the above example the dyads (a,b) and (d, e^{b}) are sub-intervals of the same sonority (c,a,b) = (c,d,e^{b}) . Likewise, (a,d) and (b,d$^{\#}$) are sub-intervals of (b^{b}, a, d) = (b^{b}, b, d$^{\#}$) . For the dyad pair (a,d#) and (c,d) their is no such bass note x , as one can see through trial and error . The ability to find such a common note figures into the formation of ***musical suspensions*** **and** ***anticipations*** **in associative harmony.**

In terms of modular arithmetic one is being asked to solve the equation (*a,b,c and d are of course arbitrary "constants", not the customary notes in the chromatic scale* **) :**

$$\boxed{(x,a,b) = (x+k, c+k, d+k)(\mathrm{mod}\,12)}$$

for unknowns x and k, for some rearrangement of x, a and b. We call this ***the chord equation*** **. If k = 0, c and d will be equal to a and b , or b and a (mod 12) . This is the trivial case, so one can require that x and x +k be distinct.**

$$\boxed{x \equiv c+k; a \equiv x+k; b \equiv d+k(\mathrm{mod}\,12)}$$

These modular equations can be solved in a straightforward manner . The solutions depend on the relationship between a,b,c, and d :

$$\boxed{\begin{array}{l} I.x \equiv \frac{a+c}{2}; d-b \equiv \frac{c-a}{2} \pmod{12} \\ II.x \equiv 6+\frac{a+c}{2}; d-b \equiv 6+\frac{c-a}{2} \pmod{12} \end{array}}$$

In both cases (c-a)/2 must be an integer, which implies that the notes c and a must be separated by a certain number of whole steps. It also means that the interval between the notes c and a is double the interval between the notes d and b (modulo 12) .

Figure 21

The above chart is based on the cross-matching of notes according to the example given above. Usually the implied tonic is unique, however it turns out that in certain cases it is possible to compute 2 *notes* , x and y, which can function as implied tonics of a dyads (a,b) and (c,d)

To find these we start once more from the chord equation:

$$\boxed{(x,a,b) = (x+k,c+k,d+k)(\mathrm{mod}\,12)}$$

This time we will look at *all* possible ways in which the notes of the left and right sides of the equation can be crossed matched:

$$\boxed{\begin{array}{l} I.x_1 \equiv c+k_1; a \equiv d+k_1; b \equiv x_1+k_1 \\ II.x_2 \equiv c+k_2; b \equiv d+k_2; a \equiv x_2+k_2 \\ III.x_3 \equiv d+k_3; a \equiv c+k_3; b \equiv x_3+k_3 \\ IV.x_4 \equiv d+k_4; a \equiv x_4+k_4; b \equiv c+k_4 \end{array}}$$

From these 4 sets of 3 equations in 2 unknowns we can eliminate the k's . The equations in the x's simplify to:

$$\boxed{\begin{array}{l} I.2x_1 \equiv c+b; a-b \equiv d-x_1 \\ II.2x_2 \equiv c+a; a-b = x_2-d \\ III.2x_3 \equiv d+b; a-b \equiv c-x_3 \\ IV.2x_4 \equiv d+a; a-b \equiv x_4-c \end{array}}$$

There are conditions on these equations: a and b must be distinct, as must c and d. Also, none of the x's can equal any of the notes a,b, c or d. Under these conditions it can be shown that although there are 4 equations, there are only two solutions. Any pair of equations can be solved, while the others will have either no solution or solutions identical to these.

For example, consider equations I and IV, with solutions x and y. Write them as:

$$\boxed{\begin{array}{l} 2x = c + b \\ d - x = a - b \\ 2y = d + a \\ a - b = y - c \end{array}}$$

Eliminating x and y from this system one obtains the relation:

$$\boxed{\begin{array}{l} c - b \equiv 4(c - b), or \\ 3(c - b) \equiv 0(\mathrm{mod}12) \end{array}}$$

Therefore, the interval between c and b must be 4 , that is to say, a major third. The same relationship pertains to d and a.

Combining these restrictions with equations II and/or III, one finds that a,b,c,d must all be separated by intervals of a major third, which is impossible without at least two of them being equal. If a is the same note as c, and b is the same note as d (both major thirds), then *every* note is an implied tonic of that progression !

There is one more possibility, namely:

a = C ; b = E ; c = E ; d = G$^{\#}$

Then there is no implied tonic. However the chords F#CE and F#EG# are inversions. This situation may be treated by the same methods that we have been using . The chart in Figure 22 depicts all those situations in which a "double root" may be found beneath the sequence of a minor second followed by a major third:

Figure 22

These solutions can be strung together to form a cycle:

Figure 23

In terms of V- forms this is:

Figure 24

One uncovers In the symmetries of the V and V' forms one uncovers the structures underlying this cycle. Cycles can be calculated for every sequence in which either the first or second dyad is a major third and the other dyad is not an even number of semitones. Putting together all cycles produces the grand cycle which is depicted in Figure 25 :

#29...

Figure 25

Table 2

The Associative Cycle of the Major Third

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Conclusion.

Exceptional Chords and Final Observations

The interval content of A. I. tetrachords is unique. However the association of trichords require that they have two non-inverted intervals in common. ***These requirements taken together***

imply that none of the 3-note subchords of a single all-interval chord can be associated.

Therefore none of the trichords contained in an all-interval chord can be associated. Also, the sum of the numbers of their notes cannot be a multiple of 3. The quickest way to prove this is to try out the possible combinations for the chord I = (0,4,6,7) , then use the fact that I* has the same interval content, and J is obtained from I through multiplication by 7 , a prime (mod 12) .

Since the sums of the numbers of the notes in a V form must add up to a multiple of 3, *it follows that the trichord sonorities which are not subchords of all-interval chords are precisely those whose notes add up to a multiple of 3, that is to say, the V-forms.*

There is another way of exhibiting these relationships which may also be useful compositionally:

Let D_1 , D_2 , D_3 represent the 3 diminished seventh chords:

$D_1 = CE^{b}F^{\#}A$

$D_2 = C^{\#}EGB^{b}$

$D_3 = DF^{\#}G^{\#}B$

Choose a tritone from any one of these chords, and one of the minor thirds in another, then put them together as a tetrachord. The result will always be an all-interval chord. The proof is a simple exercise in modular arithmetic.

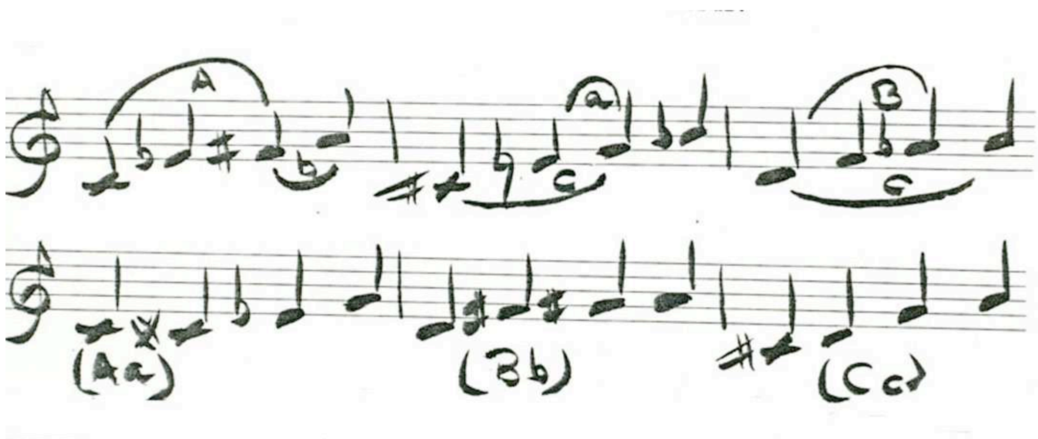

Figure 26

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❄❄❄❄❄❄❄❄❄❄❄❄❄❄

#33...