Mathematics. — *On the general projective differential geometry. III. Projective pointfield-algebra and -analysis* [1]). By D. VAN DANTZIG. (Communicated by Prof. J. A. SCHOUTEN).

(Communicated at the meeting of February 24, 1934).

1. This paper contains some remarks on the theory of *pointfields* $X^{\varkappa}(x^{\lambda})$ and other projectorfields in an ordinary projective space P_n [2]). It is shown that *every projectorfield has a geometrical meaning* (which is only clear a priori for projector-ideals [3]) [4])), i. e. *can be expressed in terms of (viz. as a directed pair of) projector-ideal-fields.* Some examples are given and geometrically interpreted. § 2 contains some remarks on the "pointfield-analysis" ("whereas § 1 belongs to the "pointfield-algebra") in P_n. [5])

§ 1. *Projective pointfield-algebra.*

2. Let $X^{\varkappa_1 \dots \varkappa_r}_{\cdot \cdot \cdot \cdot \cdot \lambda_1 \dots \lambda_s}$ be a projectorfield $\neq 0$, and suppose first $k = s - r > 0$. If a relation of the form

$$\lambda X^{\varkappa_1 \dots \varkappa_r}_{\cdot \cdot \cdot \cdot \cdot \lambda_1 \dots \lambda_r \mu_1 \dots \mu_k} x^{\nu_1} \dots x^{\nu_k} + \mu \mathfrak{A}^{\varkappa_1 \dots \varkappa_r \nu_1 \dots \nu_k}_{\lambda_1 \dots \lambda_r \mu_1 \dots \mu_k} = 0 \quad . \quad . \quad (1)$$

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[1]) The first two papers of this series appeared under the name "Zur allgemeinen proektiven Differentialgeometrie" in these Proceedings **35**, (1932), 524—534, 515 - 542 and will be quoted as *PD* I, II.

[2]) The results of § 1 are also valid in an arbitrary H_n (As to the notations and definitions comp. D. VAN DANTZIG, Theorie des projektiven Zusammenhangs n-dimensionaler Räume, Math. **106**, (1932) 400—454, quoted as *TPZ*, and *PD* I, II.

[3]) If $X^{\varkappa \dots \lambda}_{\cdot \cdot \cdot \cdot \cdot \mu \dots \nu}$ is a projector, the corresponding ideal is defined at the set of all projectors $\lambda X^{\varkappa \dots \lambda}_{\cdot \cdot \cdot \cdot \cdot \mu \dots \nu}$, where λ is an arbitrary function of the $x^{\varkappa}$, which may without restriction be supposed to be homogeneous of degree 0; this ideal will be denoted by $\llcorner X^{\varkappa \dots \lambda}_{\cdot \cdot \cdot \cdot \cdot \mu \dots \nu} \lrcorner$. Especially a spot $\llcorner X^{\varkappa} \lrcorner$ is a point-ideal.

[4]) That a projector-*ideal* has a geometrical meaning is clear: it determines and is determined uniquely by the equation

$$X^{\varkappa \dots \lambda}_{\cdot \cdot \cdot \cdot \cdot \mu \dots \nu} u_{\varkappa} \dots v_{\lambda} w^{\mu} \dots z^{\nu} = 0,$$

which expresses a definite relation between r hyperplanes $u_{\lambda} \dots v_{\lambda}$ and s spots $w^{\varkappa} \dots z^{\varkappa}$, which may be considered to be variables. F i. $\llcorner P^{\varkappa}_{\cdot \lambda} \lrcorner$ defines (and is defined by) a projective transformation, $\llcorner a_{(\lambda\mu)} \lrcorner$ a quadric, $\llcorner a_{[\lambda\mu]} \lrcorner$ a nullsystem, etc.

[5]) It is supposed throughout § 2 that *ordinary* homogeneous coordinates $x^{\varkappa}$ are used, i. e. that the equations of hyperplanes are *linear*. By replacing ∂_{μ} by ∇_{μ} the results remain valid with respect to *arbitrary* homogeneous coordinates (comp. *TPZ*, p. 405). The independant variables $x^{\varkappa}$ determine the "contactpoint" or "localisationpoint" which plays the same role as the radius-vector in ordinary vector-analysis. *All projectors are supposed to be of excess zero.*

exists, transvection with a covariant point $t_\lambda \neq 0$ with $(t\,x) = 0$ [6]) gives $\mu = 0$, hence $\lambda = 0$. Hence:

Theorem 1.[7]) *All projectorfields* $X^{\boldsymbol{\cdot}}_{\boldsymbol{\cdot}}$ [8]) *with* $s > r$ *are in* (1, 1)*-correspondence with all projector-ideal-fields of the forms* [9])

$$\left\lfloor X^{\varkappa_1 \dots \varkappa_r}_{\cdot \dots \cdot \lambda_1 \dots \lambda_s}\, x^{\nu_1} \dots x^{\nu_k} - \mathfrak{N}^{\varkappa_1 \dots \varkappa_r\, \nu_1 \dots \nu_k}_{\lambda_1 \dots \cdot \dots \lambda_s} \right\rfloor .$$

3. Now suppose $k = r - s \geqq 0$. If a relation of the form

$$\lambda\, X^{\varkappa_1 \dots \varkappa_s\, \nu_1 \dots \nu_k}_{\cdots\cdots\cdots\cdots \lambda_1 \dots \lambda_s} + \mu\, \mathfrak{N}^{\varkappa_1 \dots \varkappa_s}_{\lambda_1 \dots \lambda_s}\, x^{\nu_1} \dots x^{\nu_k} = 0 \quad . \quad . \quad . \quad (2)$$

exists, then either $\lambda = \mu = 0$, or $X^{\varkappa_1 \dots \varkappa_s\, \nu_1 \dots \nu_k}_{\cdots\cdots\cdots\cdots \lambda_1 \dots \lambda_s} :: \mathfrak{N}^{\varkappa_1 \dots \varkappa_s}_{\lambda_1 \dots \lambda_s}\, x^{\nu_1} \dots x^{\nu_k}$. [9a])
In the second case $X^{\boldsymbol{\cdot}}_{\boldsymbol{\cdot}}$ can be characterised uniquely by its "weight" $-\mu/\lambda$. Hence:

Theorem 2. *Projectorfields* $X^{\boldsymbol{\cdot}}_{\boldsymbol{\cdot}}$ *with* $s \leqq r$ *are in* (*1,1*)*-correspondence with directed* p a i r s *of projector-ideal-fields of the forms* $X^{\boldsymbol{\cdot}}_{\boldsymbol{\cdot}}$ *and* $\lfloor X^{\varkappa_1 \dots \varkappa_s\, \nu_1 \dots \nu_k}_{\cdots\cdots\cdots\cdots \lambda_1 \dots \lambda_s} + \mathfrak{N}^{\varkappa_1 \dots \varkappa_s}_{\lambda_1 \dots \lambda_s}\, x^{\nu_1} \dots x^{\nu_k} \rfloor$ (the *initial* and *final* ideal of $X^{\boldsymbol{\cdot}}_{\boldsymbol{\cdot}}$) *in the general case, or with their ideals together with their weigths, in the special* (*second*) *case mentionned above.*

After having stated the *possibility* of characterising each projectorfield in terms of ordinary projective geometry, the question arises, which relation must exist between two ideals $\lfloor X^{\boldsymbol{\cdot}}_{\boldsymbol{\cdot}} \rfloor$ and $\lfloor Y^{\boldsymbol{\cdot}}_{\boldsymbol{\cdot}} \rfloor$ (which of course must have the same valences r, s) in order that they may be the initial and final ideal of a projector, and which are the geometrical properties of such pairs. We will answer these questions in some special cases.

4. Firstly consider a pointfield $X^\varkappa$. According to Th. 2 it is characterised by two spots $\lfloor X^\varkappa \rfloor$ and $\lfloor X^\varkappa + x^\varkappa \rfloor$, which are collinear with $\lfloor x^\varkappa \rfloor$, At the other hand each directed pair of different spots, which are collinear but not coincident with $\lfloor x_\varkappa \rfloor$ [10]) defines a pointfield $X^\varkappa$. For if $Y^\varkappa$, $Z^\varkappa$ are any points, determining these spots, a relation $x^\varkappa = \lambda Y^\varkappa + \mu Z^\varkappa$ exists. Hence we may put $X^\varkappa = -\lambda Y^\varkappa$, [11]) hence $\lfloor X^\varkappa \rfloor = \lfloor Y^\varkappa \rfloor$ and

6) For sake of brevity I write (tx) in stead of $t_\lambda\, x^\lambda$ etc.

7) When I had found the theorems 3,...., 8, the theorems I and 2 were suggested to me by Prof. SCHOUTEN. Moreover I am indebted to Prof. SCHOUTEN for several remarks which have improved the final version of the paper.

8) Sometimes rows of indices will be replaced by big points.

9) Evidently instead of the pair mentionned in the text also any two ideals of the form (2) with $\lambda = 1$ and arbitrary *definite* (but different) values of μ (f. i. $\mu_1 = +1$, $\mu_2 = -1$ may be taken).

9a) $::$ means "is proportional with".

10) We exclude the special case $X^\varkappa = a\; x^\varkappa$

11) If t_λ is an arbitrary hyperplane through $\lfloor Z^\varkappa \rfloor$ but not through $\lfloor Y^\varkappa \rfloor$, we have $X^\varkappa = -(tx)(tY)^{-1} Y^\varkappa$, which is homogeneous of degree zero in t_λ and $Y^\varkappa$ and of degree 1 in $x^\varkappa$ as it ought to be.

$\lfloor X^\varkappa + x^\varkappa \rfloor = \lfloor Z^\varkappa \rfloor$ [12]). The pointfield $X^\varkappa$ also defines a "translation" of the line $\lfloor x^\varkappa, X^\varkappa \rfloor$ [13]) into itself, i.e. a projective transformation, having only one invariant spot $\lfloor X^\varkappa \rfloor$ and transforming an arbitrary spot $\lfloor Y^\varkappa \rfloor$ with $Y^\varkappa = \lambda x^\varkappa + \mu x^\varkappa$ into $\lfloor Y^\varkappa + \lambda X^\varkappa \rfloor$, especially $\lfloor x^\varkappa \rfloor$ into $\lfloor X^\varkappa + x^\varkappa \rfloor$. Hence we have proved:

Theorem 3. All pointfields $X^\varkappa$ which do not coincide with $x^\varkappa$ are in (1, 1,)-correspondance 1°. with all directed pairs of non-coincident spots, 2°. with all projective transformations of straight lines through $\lfloor x^\varkappa \rfloor$ into themselves, which have only one invariant spot $\lfloor X^\varkappa \rfloor \neq \lfloor x^\varkappa \rfloor$.

Multiplication of $X^\varkappa$ with a scalar k means geometrically: leaving the initial spot unaltered and replacing the final spot by another one, which has with it a double ratio k with respect to $\lfloor X^\varkappa \rfloor, \lfloor x^\varkappa \rfloor$. [14])

If $X^\varkappa, Y^\varkappa$ are two points, represented by the spot-pairs $\lfloor X^\varkappa \rfloor, \lfloor U^\varkappa \rfloor$ and $\lfloor Y^\varkappa \rfloor, \lfloor V^\varkappa \rfloor$ respectively, then $X^\varkappa + Y^\varkappa$ will be represented by the spot-pair $\lfloor Z^\varkappa \rfloor, \lfloor W^\varkappa \rfloor$, where $\lfloor W^\varkappa \rfloor$ is the intersection of the lines $\lfloor X^\varkappa, V^\varkappa \rfloor$ and $\lfloor Y^\varkappa, U^\varkappa \rfloor$, whilst $\lfloor Z^\varkappa \rfloor$ is the intersection of the lines $\lfloor X^\varkappa, Y^\varkappa \rfloor$ and $\lfloor x^\varkappa, W^\varkappa \rfloor$

5. Secondly consider a covariant pointfield $t_\lambda \neq 0$. According to Theorem 1 it is characterised by the ideal of

$$T^{\varkappa}_{.\lambda} = \mathfrak{A}^{\varkappa}_{\lambda} - x^\varkappa t_\lambda . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This ideal defines a projective transformation

$$X^\varkappa \rightarrow T^{\varkappa}_{.\lambda} X^\lambda = X^\varkappa - (t X)\, x^\varkappa \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

of the P_n into itself, which is a *central collineation*, having $\lfloor x^\varkappa \rfloor$ as invariant spot and $\lfloor t_\lambda \rfloor$ as invariant hyperplane. The constant double ratio of the four spots of $X^\varkappa, x^\varkappa, T^{\varkappa}_{.\lambda} X^\lambda$ and the intersection of the straight line joining these three with $\lfloor t_\lambda \rfloor$ (shortly (X, x, TX, t)) is equal to the *weight* $\tau = (t x)$ of the covariant point t_λ. For $\tau \neq 0$ this covariant point is determined uniquely by its hyperplane, together with its weight: if $u_\lambda :: t_\lambda$ is given, then $t_\lambda = \tau\, (u x)^{-1} u_\lambda$. If $\tau = 1$, the collineation degenerates into a projection of the whole P_n out of $\lfloor x^\varkappa \rfloor$ upon $\lfloor t_\lambda \rfloor$; in

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[12]) According to [9]) the point $X^\varkappa$ may also be characterised by the pair of spots $\lfloor X^\varkappa + x^\varkappa \rfloor, \lfloor X^\varkappa - x^\varkappa \rfloor$. Then the spot $\lfloor X^\varkappa \rfloor$ is the fourth harmonic of $\lfloor x^\varkappa \rfloor$ with respect to this pair. This interpretation was suggested to me by Prof. SCHOUTEN and is perhaps still more elegant than my own one.

[13]) I.e. the straight line, joining the spots $\lfloor x^\varkappa \rfloor$ and $\lfloor X^\varkappa \rfloor$.

[14]) Correspondance with affine geometry: if $X^\varkappa$ is a vector, $\lfloor X^\varkappa \rfloor$ is its direction, $\lfloor x^\varkappa \rfloor$ its initial and $\lfloor X^\varkappa + x^\varkappa \rfloor$ its final spot.

this case the point t_λ evidently is characterised by its hyperplane alone. Hence:

Theorem 4. All covariant points are in (1,1)-correspondance with all central collineations with $\llcorner x^\varkappa \lrcorner$ *as invariant spot. The hyperplane of the covariant point is the invariant hyperplane of the collineation; its weight is the constant double ratio*[14a]). *If the latter is* $\neq 0$, *the point is determined by its hyperplane and its weight. In any case it is characterised by its hyperplane and any general directed pair of corresponding spots. If* $\tau = 0$, *the collineation is special (a "translation"); in this case it transforms the two spots of a contravariant point into the spots of its image-point.*

6. Finally consider a projector of the form $P^{\varkappa}_{\cdot\lambda}$. According to Theorem 2 it is determined by two projective transformations $\llcorner P^{\varkappa}_{\cdot\lambda} \lrcorner$, $\llcorner Q^{\varkappa}_{\cdot\lambda} \lrcorner$ of the P_n into itself, where

$$Q^{\varkappa}_{\cdot\lambda} = P^{\varkappa}_{\cdot\lambda} + A^{\varkappa}_{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

(excepted in the case when both are the identical transformation i. e. $P^{\varkappa}_{\cdot\lambda} :: Q^{\varkappa}_{\cdot\lambda} :: A^{\varkappa}_{\lambda}$, which we will exclude [15])). These two transformations are *collinear*, i. e. they map each spot $\llcorner X^\varkappa \lrcorner$ on two spots $\llcorner P^{\varkappa}_{\cdot\lambda} X^\lambda \lrcorner$ and $\llcorner Q^{\varkappa}_{\cdot\lambda} X^\lambda \lrcorner$, which are collinear with $\llcorner X^\varkappa \lrcorner$, whereas the inverse transformations transform each hyperplane $\llcorner t_\lambda \lrcorner$ into two hyperplanes $\llcorner t_\varkappa P^{\varkappa}_{\cdot\lambda} \lrcorner$ and $\llcorner t_\varkappa Q^{\varkappa}_{\cdot\lambda} \lrcorner$ which intersect on $\llcorner t_\lambda \lrcorner$. For $n > 1$ the condition of collinearity is also *sufficient*. To prove this, let $\llcorner U^{\varkappa}_{\cdot\lambda} \lrcorner$, $\llcorner V^{\varkappa}_{\cdot\lambda} \lrcorner$ be two arbitrary collinear transformations. Then for each spot $\llcorner X^\varkappa \lrcorner$ a relation of the form

$$X^\varkappa = a\, U^{\varkappa}_{\cdot\lambda} X^\lambda + b\, V^{\varkappa}_{\cdot\lambda} X^\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

must hold, where a and b may be *functions* of $X^\varkappa$, homogeneous of degree 0. Now, if $\llcorner u_\lambda \lrcorner$ is a hyperplane, then for all $X^\varkappa$, such that $\llcorner UX \lrcorner$ lies in $\llcorner u_\lambda \lrcorner$, b is a broken linear function of $X^\varkappa$, viz. $(u X)(u V X)^{-1}$. Hence (as is easily shown) b (and also a) is *for all* $X^\varkappa$ a broken linear function of $X^\varkappa$, i. e. (6) can be brought into the form

$$X^\varkappa = (p X)(q X)^{-1}\, U^{\varkappa}_{\cdot\lambda} X^\lambda + (r X)(s X)^{-1}\, V^{\varkappa}_{\cdot\lambda} X^\lambda \quad . \quad . \quad . \quad (7)$$

Now, if p_λ coincides with q_λ and r_λ with s_λ, then a and b are *constants*; hence by putting $P^{\varkappa}_{\cdot\lambda} = -a\, U^{\varkappa}_{\cdot\lambda}$, $Q^{\varkappa}_{\cdot\lambda} = b\, V^{\varkappa}_{\cdot\lambda}$, equation (5) is obtained. If either p_λ and q_λ, or r_λ and s_λ do *not* coincide, then it is easily seen that relations of the forms

$$\left.\begin{aligned} U^{\varkappa}_{\cdot\lambda} &= u\, A^{\varkappa}_{\lambda} + u^\varkappa y_\lambda, \\ V^{\varkappa}_{\cdot\lambda} &= v\, A^{\varkappa}_{\lambda} + v^\varkappa z_\lambda \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

14a) The double ratio ordinarily considered is $(TX, X, t, x) = 1 - \tau$.

15) In this case it is determined by its weight, i.e. its spur divided by $n + 1$.

exist. Then from (6) follows $X^{[\varkappa}\, u^{\lambda}\, v^{\mu]}\, (y\, X)\, (z\, X) = 0$ for each $X^{\varkappa}$, hence ($y_\lambda = 0$ or $z_\lambda = 0$ giving the excluded case) $u^{[\lambda}\, v^{\mu]} = 0$. Applying the same reasoning to hyperplanes instead of spots we get $y_{[\lambda}\, z_{\mu]} = 0$, hence $s\, v^{\varkappa}\, z_\lambda = r\, u^{\varkappa}\, y_\lambda$, and $r\, U^{\varkappa}_{.\lambda} - s\, V^{\varkappa}_{.\lambda} = (u\, r - v\, s)\, \mathscr{A}^{\varkappa}_{\lambda}$ whether again (5) is easily obtained.

For $n = 1$ the condition is *not* sufficient, all transformations being collinear. In this case however transformations satisfying (5) can be characterised by their property of having the *same invariant* spots (where a *singular* spot, i. e. a spot $\lfloor X^{\varkappa} \rfloor$ with f. i. $U^{\varkappa}_{.\lambda}\, X^{\lambda} = 0$ is considered as invariant). [16]) In fact, any two (non coincident) transformations with the same invariant spots can for $n = 1$ be brought into the form

$$U^{\varkappa}_{.\lambda} = p\, \mathscr{A}^{\varkappa}_{\lambda} + q\, a^{\varkappa}\, b_\lambda, \qquad V^{\varkappa}_{.\lambda} = r\, \mathscr{A}^{\varkappa}_{\lambda} + s\, a^{\varkappa}\, b_\lambda, \quad . \quad . \quad . \quad (9)$$

with $\triangle = p\, s - q\, r \neq 0$. Hence, by putting $P^{\varkappa}_{.\lambda} = - s\, \triangle^{-1}\, U^{\varkappa}_{.\lambda}$, $Q^{\varkappa}_{.\lambda} = - q\, \triangle^{-1}\, V^{\varkappa}_{.\lambda}$ (5) is satisfied.

Hence we have proved:

Theorem 5. All projectors $P^{\varkappa}_{.\lambda}$ (non coincident with $\mathscr{A}^{\varkappa}_{\lambda}$) are in (1, 1)-correspondance with all directed pairs of non-coincident projective transformations, which for $n > 1$ are collinear and for $n = 1$ have the same invariant spots. [17])

§ 2. *The projective gradient.*

7. Consider a scalar field $\varphi(x)$, homogeneous of degree 0. Its projective gradient

$$t_\mu = \partial_\mu\, \varphi = \partial \varphi / \partial x^\mu \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

is a covariant point of degree -1 and weight 0; hence according to theorem 4 it is represented by a special central collineation, given by (3), (4). Now the most important property of the partial derivatives of a function φ is that, their values as well as the value of φ itself being given in some spot $\lfloor x^{\varkappa} \rfloor$, the value of φ in any spot $\lfloor y^{\varkappa} \rfloor$ within a small neighbourhood of $\lfloor x^{\varkappa} \rfloor$ may be determined to a first approximation. We will show that this may be done *geometrically* by means of the collineation, representing $\lfloor t_\mu \rfloor$. Hence we may solve f. i. equations of the form $\partial_\mu\, \varphi = t_\mu$ where $t_\mu(x)$ is given by giving the collineations $T^{\varkappa}_{.\lambda}$ (belonging to all spots $\lfloor x^{\varkappa} \rfloor$) "graphically", in a way, invariant under *projective* transformations.

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16) For $n > 1$ this condition is *not* sufficient, as is seen from the example of two translations in different directions. Not even the condition that the two trans ormations have the same invariant linear submanifolds for each dimension is sufficient, as is seen from the example given by $U^{0}_{.0} = U^{1}_{.1} = U^{2}_{.2} = U^{0}_{.1} = U^{1}_{.2} = 1$, $V^{0}_{.0} = V^{1}_{.1} = V^{2}_{.2} = V^{0}_{.1} = - V^{1}_{.2} = 1$ (all other components being zero).

17) If the two invariant spots coincide, we have $b_0 : b_1 = - a^1 : a^0$.

In fact, let $\llcorner y^\varkappa \lrcorner$ be a spot near to $\llcorner x^\varkappa \lrcorner$, and let u_λ and $v_\lambda :: u_\lambda - (ux)\, t_\lambda$ determine two corresponding hyperplanes with respect to the collineation (N.B. $\llcorner u_\lambda \lrcorner$ is the image of $\llcorner v_\lambda \lrcorner$, not $\llcorner v_\lambda \lrcorner$ of $\llcorner u_\lambda \lrcorner$!), where $\llcorner u_\lambda \lrcorner$ does *not* pass through $\llcorner y^\varkappa \lrcorner$. Let further $Y^\varkappa :: x^\varkappa - (u\,x)\,(u\,y)^{-1}\, y^\varkappa$ and $Z^\varkappa :: x^\varkappa - (v\,x)\,(v\,y)^{-1}\, y^\varkappa$ be two points, determining the intersections of $\llcorner u_\lambda \lrcorner$ and $\llcorner v_\lambda \lrcorner$ resp. with the straight line $\llcorner x^\varkappa, y^\varkappa \lrcorner$. Then the double ratio $(x, Y, y, Z) = (x, u, y, v)$ is equal to

$$1 - \frac{(ux)\,(vy)}{(uy)\,(vx)} = 1 - \frac{(ux)}{(u\,y)} \cdot \frac{(uy) - (ux)\,(ty)}{(ux)} = \frac{(ux)}{(u\,y)}(ty) \quad . \;.\;. \quad (11)$$

Hence we have to a first approximation, if $\varepsilon^\varkappa = (u\,x)\,(u\,y)^{-1}\, y^\varkappa - x^\varkappa$ is small:

$$\varphi\,(y) = \varphi\,(x + \varepsilon) \sim \varphi\,(x) + \varepsilon^\mu\, \partial_\mu\, \varphi = \varphi\,(x) + (ux)\,(uy)^{-1}\,(ty) \quad . \quad (12)$$

Hence we have proved.

Theorem 6. The difference of the values of a homogeneous function of degree 0 in two neighboring spots $\llcorner y^\varkappa \lrcorner$, $\llcorner x^\varkappa \lrcorner$ *is equal to the double ratio* (x, u, y, v), *where* $\llcorner v_\lambda \lrcorner$ *is an arbitrary general hyperplane and* $\llcorner u_\lambda \lrcorner$ *its image under the central collineation, representing the projective gradient of* φ *in* $\llcorner x^\varkappa \lrcorner$.

8. Finally consider a pointfield $X^\varkappa\,(x)$ and let

$$P^\varkappa_{\cdot\,\mu} = \partial_\mu\, X^\varkappa \quad . \;.\;.\;.\;.\;.\;.\;.\;.\;. \quad (13)$$

be its projective gradient in $\llcorner x^\varkappa \lrcorner$. If $\varepsilon^\varkappa = y^\varkappa - x^\varkappa$ is small, we have as a first approximation, using EULER's equation

$$P^\varkappa_{\cdot\,\mu}\, x^\mu = X^\varkappa\,, \quad . \;.\;.\;.\;.\;.\;.\;.\;. \quad (14)$$

$$X^\varkappa\,(y) = X^\varkappa\,(x + \varepsilon) \sim X^\varkappa\,(x) + \varepsilon^\mu\, \partial_\mu\, X^\varkappa = P^\varkappa_{\cdot\,\mu}\, y^\mu, \quad . \;. \quad (15)$$

$$X^\varkappa\,(y) + y^\varkappa \sim (P^\varkappa_{\cdot\,\mu} + \mathcal{A}^\varkappa_\mu)\, y^\mu \quad . \;.\;.\;.\;.\;. \quad (16)$$

As it should be, (15) and (16) are homogeneous of degree 1 in $y^\varkappa$ and of degree 0 in $x^\varkappa$. These two equations prove:

Theorem 7. The two spots, belonging (according to theorem 3) to the value of a pointfield $X^\varkappa$ *in a spot* $\llcorner y^\varkappa \lrcorner$ *lying within a small neighbourhood of* $\llcorner x^\varkappa \lrcorner$, *are obtained from* $\llcorner Y^\varkappa \lrcorner$ *to a first approximation by means of the two collinear transformations, belonging (according to theorem 5) to the projective gradient of* $X^\varkappa$, *taken in* $\llcorner x^\varkappa \lrcorner$.

It is remarkable, that the value of $X^\varkappa$ in $\llcorner x^\varkappa \lrcorner$ itself is *not* needed, but may (because of EULER's equation (14)) be obtained in the same way.