Mathematics. — *On conformal differential geometry.* I. *The conformal gradient.* By D. VAN DANTZIG. (Communicated by Prof. J. A. SCHOUTEN).

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1. It is generally known that the $n+2$ homogeneous coordinates $x^{\varkappa}$ of a (hyper)sphere are specially adapted to study problems of conformal geometry in n-dimensional space C_n. By means of these coordinates C_n is mapped on a quadric Q_n in a projective $(n+1)$-dimensional space P_{n+1}, and the conformal geometry of C_n is isomorphic with the projective geometry of Q_n in P_{n+1}.

In conformal *differential* geometry however the introduction of conformal coordinates leads to several rather exciting difficulties. It has not yet been possible to introduce them otherwise than in the *local* spaces. In the older theories of E. CARTAN and J. A. SCHOUTEN as well as in those of W. BLASCHKE and G. THOMSEN (which latter are restricted to Euclidean spaces), differentiation with respect to the conformal coordinates was not defined at all; only in 1932 T. Y. THOMAS succeeded in finding such a conformally invariant process of differentiation [1]). The differential operator he introduced however has not the character of an ordinary (or rather projective) gradient, in as much as the functions it generates, as well as those it operates on, are defined on Q_n only and not in the whole P_{n+1}.

Hence the problem arises, whether it is possible to define a conformal differential operator, which *really* is a gradient, or, what comes to the same, whether it is possible to extend an arbitrary field, defined for the *points* of the C_n to a field, defined for the *spheres* of the C_n. It is this question which I will investigate in this paper. I will restrict myself 1°. to *scalar*fields, 2°. to an ordinary ("conformally flat") conformal space C_n, the extension to "curved" conformal spaces being rather obvious.

2. If ξ^h $(h, i, j, k = 1, \ldots, n)$ are ordinary Cartesian coordinates in C_n and a_{ij}, a^{ij} is the fundamental tensor [2]), determined but for an arbitrary factor [3]), the conformal coordinates $x^{\varkappa}$ $(\varkappa, \lambda, \mu, \nu = 1, \ldots, n+2)$ of a sphere

[1]) T. Y. THOMAS, Conformal tensors, Proc. National Ac. of Sciences, **18** (1932) 103—112; 189—193.

[2]) It need *not* be positive definite, but is supposed not to be degenerated.

[3]) Instead of the fundamental tensor also its density $a_{ij} \,.\, \mathfrak{a}^{-1/n}$, $\mathfrak{a} = \mathrm{Det}\,(a_{ij})$ may be used. Comp. T. Y. THOMAS, Invariants of relative quadratic differential forms, Proc. Nat. Ac. of Sc. **11** (1925) 722—725.

with centre ξ^h and radius R may be given by

$$x^h : u : v = \xi^h : \tfrac{1}{2}(R^2 - r^2) : 1, \; r^2 = a_{ij}\,\xi^i\,\xi^j \text{ [4]}) \quad . \; . \; . \; . \quad (1)$$

The Q_n, i. e. the set of all spheres of radius 0, is the locus of all $x^\varkappa$ satisfying

$$G_{\lambda\mu}\, x^\lambda\, x^\mu = 0, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

were $G_{\lambda\mu} = G_{\mu\lambda}$ and

$$G_{ij} : G_{n+1,\,n+2} = a_{ij} : 1, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

all other components being zero. The expression

$$\omega^2 = G_{\lambda\mu}\, x^\lambda\, x^\mu \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

is invariant; $R^2 = \omega^2/v^2$.

3. Now let an arbitrary scalarfield $f = f(\xi^h)$ be given in C_n. We will try to find a function $F = F(x^\varkappa)$, satisfying the following conditions:

1. It is a "spotfunction", i. e. homogeneous of degree zero:

$$F(c\,x^\varkappa) = F(x^\varkappa), \; \therefore \; x^\lambda\, \partial_\lambda\, F = 0 \qquad (\partial_\lambda = \partial/\partial\, x^\lambda) \quad . \; . \; . \quad (5)$$

2. It is univalued.
3. On Q_n it is identical with $f(\xi^h)$:

$$F(\xi^h, -\tfrac{1}{2}\, r^2, 1) = f(\xi^h) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

4. Its definition is invariant under arbitrary conformal transformations, i. e. $F(x^\varkappa)$ is a projective *scalar*, depending only on f and its derivatives and on $G_{\lambda\mu}$.

   The simplest way to satisfy condition 4 is to require F to be a solution of an invariant differential equation. If we require this equation to be *linear*, we may imply the condition.
5. F satisfies the differential equation

$$\square\, F = 0, \text{ [5]}) \qquad \square = G^{\lambda\mu}\, \partial_{\lambda\mu} = G^{\lambda\mu}\, \partial^2/\partial x^\lambda\, \partial x^\mu \quad . \; . \; . \quad (7)$$

   Hence we have to solve the boundary problem of equations (5), (7) with the boundary condition (6).

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[4]) We write shortly u and v instead of x^{n+1} and x^{n+2} resp.

[5]) The equation must be of the second order because equations of the first order, as f.i. $G^{\lambda\mu}(\partial_\lambda F)(\partial_\mu F) = 0$ do not allow F to take arbitrary values on Q_n, whereas equations of higher order allow too high a degree of freedom. Other invariant equations of the second order which suggest themselves are f.i. $\omega^2 \square F = F$ which however is not regular on Q_n, or $\square F = \Phi\, G^{\lambda\mu}(\partial_\lambda F)(\partial_\mu F)$, where Φ is a function of F, given once for all. If at least the equation is required to be *homogeneous* in F (instead of linear) Φ must be cF^{-1}, c being a constant. But then the last equation leads to a function of solutions (viz F^{1-c} or $\log F$) of the kind we will find. An other possible *non-linear* equation would however be f.i.

$$G^{\varkappa\lambda}\, G^{\mu\nu}\,(\partial_{\varkappa\mu} F)\,(\partial_{\lambda\nu} F) = 0.$$

4. Each solution of (5), satisfying condition 2, may be written as $F = F(\xi^i, \varrho)$, where we have put $\varrho = \frac{1}{2} R^2$. Hence from (1) we get:

$$\partial_\lambda F = (F_i + \xi_i F', F', -\xi^i F_i - \tfrac{1}{2}(R^2 + r^2) F') v^{-1}, \quad . \quad . \quad . \quad . \quad (8)$$

where we have written F_i for $\partial F/\partial \xi^i$, F' for $\partial F/\partial \varrho$. Hence (7) becomes

$$v^2 \square F = \triangle F + \frac{n-1}{R}\frac{\partial F}{\partial R} - \frac{\partial^2 F}{\partial R^2} = \triangle F + (n-2) F' - 2\varrho F'' = 0, \quad . \quad . \quad (9)$$

$$\triangle = a^{ij} \partial^2 / \partial \xi^i \partial \xi^j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now putting

$$F = \sum_{\nu=0}^{\infty} a_\nu \varrho^\nu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where the a_ν are functions of the ξ^h, (9) becomes

$$\sum_{\nu=1}^{\infty} \{ \triangle a_{\nu-1} + \nu (n - 2\nu) a_\nu \} \varrho^{\nu-1} = 0,$$

hence

$$\triangle a_{\nu-1} + \nu (n - 2\nu) a_\nu = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

From (12) a_ν can be calculated for $\nu \neq \frac{1}{2} n$, hence *always* if n is uneven. For $2\nu = n$ (12) gives $\triangle a_{1/2 n - 1} = 0$ hence for even values of $n = 2m$ $\triangle^m a_0 = 0$. Combining this result with (6) we find that *for even values of n the problem is only soluble if $f(\xi^h)$ satisfies the equation*

$$\triangle^{\frac{n}{2}} f = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If this condition is satisfied, *the solution exists but is not uniquely determined*, because in this case (12) does not allow to determine $a_{1/2 n}$. Especially for $n = 2$ and definite fundamental tensor f must be a *harmonic* function.

We now easily find from (12) the general solution of (5), (6), (7), viz.

$$F = \sum_{i=0}^{1/2 n - 1} \frac{(-1)^i (\frac{1}{2} n - i - 1)!}{2^{2i}\, i!\, (\frac{1}{2} n - 1)!} R^{2i} \triangle^i f + R^n \sum_{\nu=0}^{\infty} \frac{(\frac{1}{2} n)!}{2^{2\nu}\, \nu!\, (\nu + \frac{1}{2} n)!} R^{2\nu} \triangle^\nu g \quad (14)$$

($g = 2^{-1/2 n} a_{1/2 n}$) if n is even and f satisfies (13), and

$$F = \sum_{\nu=0}^{\infty} \frac{1}{2^{2\nu}\, \nu! \left(\nu - \frac{n}{2}\right)\left(\nu - \frac{n}{2} - 1\right) \ldots \frac{1}{2} \cdot (-\frac{1}{2}) \ldots \left(1 - \frac{n}{2}\right)} R^{2\nu} \triangle^\nu f \quad (15)$$

if n is uneven. In the latter case the second solution is an *uneven* function of R, which is excluded by condition 2, because the $x^\varkappa$ depend according to (1) only on R^2.

5. It is easily proved that the series occurring in (14), (15) will converge for sufficiently small values of R in all points of C_n where f is regular. Taking as an example the series (15), let us suppose that

$$f(\xi^i + \eta_i) = \Sigma \frac{1}{r_1! \dots r_n!} \eta_1^{r_1} \dots \eta_n^{r_n} f_{r_1 \dots r_n} \dots \quad (16)$$

converges for all η_i with $\Sigma \eta_i^2 \leqq K^2$. Then the function (16) is uniformly bounded; hence a positive number M exists[5a]), such that each term

$$\frac{1}{r_1! \dots r_n!} | \eta_1 |^{r_1} \dots | \eta_n |^{r_n} | f_{r_1 \dots r_n} | \leqq M \text{ for } \Sigma \eta_i^2 \leqq K^2 \quad \dots \quad (17)$$

Taking especially $\eta_1 = \dots = \eta_n = K/\sqrt{n}$ we find

$$| f_{r_1 \dots r_n} | \leqq r_1! \dots r_n! \, M (K/\sqrt{n})^{-(r_1 + \dots + r_n)} \quad \dots \quad (18)$$

At the other hand, taking orthogonal coordinates, we have $a_{ii} = \pm 1$, $a_{ij} = 0 \; (i \neq j)$, hence

$$\left.\begin{aligned} | \Delta^\nu f | &= | (\pm \partial_1^2 \pm \dots \pm \partial_n^2)^\nu f | = \left| \Sigma \pm \frac{\nu!}{r_1! \dots r_n!} f_{2r_1 \dots 2r_n} \right| \leqq \\ &\leqq \Sigma \, \nu! \frac{(2r_1)! \dots (2r_n)!}{r_1! \dots r_n!} M (K/\sqrt{n})^{-2\nu} = \\ &= \frac{\nu! \, 2^{2\nu} M}{(\Gamma \tfrac{1}{2})^n (K/\sqrt{n})^{2\nu}} \Sigma \, \Gamma(r_1 + \tfrac{1}{2}) \dots \Gamma(r_n + \tfrac{1}{2}) \leqq \\ &\leqq \frac{\nu! \, 2^{2\nu} M}{(K/\sqrt{n})^{2\nu}} \Sigma \, r_1! \dots r_n! \leqq M \frac{(\nu!)^2 2^{2\nu}}{(K^2/n)^\nu} n^\nu = (\nu!)^2 \left(\frac{2n}{K}\right)^{2\nu} M, \end{aligned}\right\} \quad \dots (19$$

where we have used the identity $(2r)! \, \Gamma(\tfrac{1}{2}) = 2^{2r} \, r! \, \Gamma(r + \tfrac{1}{2})$ and the inequalities $\Gamma(r + \tfrac{1}{2}) \leqq r! \, \Gamma(\tfrac{1}{2})$ and $r_1! \dots r_n! \leqq \nu!$ whereas the summations in (19) are to be extended on all non-negative integers $r_1, \dots, r_n$ with $r_1 + \dots + r_n = \nu$.

Hence the series (15) is uniformly majorated by the series

$$\left.\begin{aligned} &\frac{M \Gamma\left(1 - \frac{n}{2}\right)}{\Gamma(\frac{1}{2})} \sum_\nu \frac{R^{2\nu} \nu!}{2^{2\nu} \left(\nu - \frac{n}{2}\right) \dots \frac{1}{2}} \left(\frac{2n}{K}\right)^{2\nu} \leqq \\ &\leqq \frac{\Gamma\left(1 - \frac{n}{2}\right)}{\Gamma(\frac{1}{2})} 2M \sum_\nu \frac{\nu!}{\left(\nu - \frac{n+1}{2}\right)!} \left(\frac{Rn}{K}\right)^{2\nu}, \end{aligned}\right\} \quad \dots \quad (20)$$

which evidently is convergent for $R < K/n$.

5a) F.i. $M = Max \, | f |$ in the domain $\Sigma \eta_i^2 \leqq K^2$.

6. Resuming our results, we have proved the theorem:

A function f, defined and regular in some domain d of a conformal space C_n, where n is an u n e v e n number, may be extended in a way, invariant under conformal transformations to a function F, defined, regular and univalued in a domain D of the projective space P_{n+1} of all spheres of C_n, containing d, whereas F is homogeneous of degree 0, satisfies the equation $\square\, F = 0$ everywhere in D, and coincides with f everywhere in d. This extension is uniquely determined. If n is e v e n such an extension is neither always possible (viz: if and only if $\triangle^{\frac{n}{2}} f = 0$), nor uniquely determined if it is possible. The conformal derivatives, up to the order $\frac{1}{2}n-1$ however are determined on C_n.

7. For $n=0$ the impossibility of extending an *arbitrary* function f in *any* invariant way (Comp. [5])) is easily proved. In fact, in this case P_{n+1} is a straight line, Q_n a pair of non-coinciding points A, B on this line; f has given values in A and B. But by a projective transformation of the line which leaves A and B invariant, *any* point $\neq A$ and B can be transformed into any other. Hence F must be a *constant*. But this is only possible if $f(A) = f(B)$, a condition taking the place of (13) which looses its sense for $n=0$.

For $n=1$ each "sphere" consists of two points on a line C_1 with coordinates $\xi + R$ and $\xi - R$. The simplest manner of extending a function $f(\xi)$, given on this line, consists in taking the mean value of f in these two points:

$$F(\xi, R) = \tfrac{1}{2}\{f(\xi+R) + f(\xi-R)\} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This is exactly the extension (15) which we have found above.

For other values of n the solution (15) is *not* equal to the mean value of $f(\xi^h)$ over the sphere (ξ^h, R); moreover this mean value is not a conformal invariant for $n > 1$. The *integral* of $f(\xi^h)$ however over the *interior* of this sphere (at least when the fundamental tensor is positive definite), though it neither is a conformal invariant, is always a solution of equations (7), (5), viz.

$$\frac{(\Gamma\frac{1}{2})^n}{\Gamma(\frac{1}{2}n+1)} R^n \sum_{\nu=0}^{\infty} \frac{\Gamma(\frac{1}{2}n+1)}{\nu!\,\Gamma(\nu+\frac{1}{2}n+1)} \left(\frac{R^2}{4}\right)^{\nu} \triangle^{\nu} f \quad . \quad . \quad . \quad (22)$$

The infinite sum in (22) itself is equal to the *mean value* of f over the *interior* of the sphere [6]). For *even* values of n the solution (22) corresponds with the second term in (14), with $g.\ (\Gamma\frac{1}{2})^n/\Gamma(\frac{1}{2}n+1)$ instead of f; for *uneven* n it is the solution we have discarded by requiring F to be univalued.

[6]) This follows from a theorem which will be proved in the "Wiskundige opgaven met oplossingen", **16**, (1935) problem N⁰. 100.

8. We might have found the solutions (15), (22) by remarking that the equation (9) may be easily transformed into an equation of BESSEL. The two solutions of (9) may (but for constant factors) be symbolically represented by

$$\left.\begin{array}{l} R^{\frac{n}{2}} I_{\frac{n}{2}} (R\sqrt{-\triangle})\, f \\ \\ R^{\frac{n}{2}} I_{-\frac{n}{2}} (R\sqrt{-\triangle})\, f, \end{array}\right\} \quad \ldots\ldots\ldots \quad (23)$$

because the differential operator $\triangle$ may be treated as a symbolical *constant*, as it commutes with $\partial/\partial R$. The first solution (23) corresponds with (22) and exists always, but becomes zero on Q_n. The second solution (23) is only different from the first one (but for the sign) if n is *uneven*, in which case it corresponds with our solution (15). [7])

It is curious that the first solution in (14), the *finite* sum, has no analogon in the theory of ordinary BESSEL equations, because an *ordinary* constant $\triangle$ would be zero if this were the case with its $(\frac{1}{2}n)^{\text{th}}$ power (condition (13)).

[7]) NEUMANN's function cannot be used as a solution, because it is singular for $R=0$.

Astronomy. — *Viscosity and steady states of the disc constituting the embryo of the planetary system.* By H. P. BERLAGE Jr., (Meteorological Observatory Batavia.) (Communicated by Prof. H. A. KRAMERS).

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In a previous paper [1]) I have made an erroneous statement concerning the influence of viscosity in the gaseous disc which we have recognized as the embryo of the planetary system. In formula (21) l.c. I naïvely assumed the force with which viscosity acts on the rotating matter of the disc, to be proportional to the density of the gas. It is independent of it. Fortunately the consequences can be proved to be anything but serious.

The right way of dealing with the influence of viscosity on the disc is the following. Every element of an infinitesimal ring with radius r, expressed in cylindrical coordinates

$$r\, da\, dr\, dh \quad \ldots\ldots\ldots\ldots \quad (1)$$

receives, in consequence of the permanent interchange of molecules with

[1]) These Proceedings **35**, 1932, p. 554. See also **33**, 1930, p. 614 and **33**, 1930, p. 719.