Mathematics. — *Electromagnetism, independent of metrical geometry.* 1. *The foundations.* By D. VAN DANTZIG. (Communicated by Prof. J. A. SCHOUTEN).

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§ 1. *Introduction.*

It has been detected by E. CUNNINGHAM [1]) and H. BATEMAN [1]), that MAXWELL's equations are invariant, not only under the LORENTZ-group, but also under the still wider group of conformal transformations of space-time. It has been shown recently by J. A. SCHOUTEN and J. HAANTJES [2]), that this fact may be understood in the following way: in order to write down the equations in a form, invariant under arbitrary transformations of coordinates in space-time, it is sufficient to adjoin, instead of the fundamental tensor g_{ij}, its density $\mathfrak{g}_{ij} = \mathfrak{g}^{-1/4} g_{ij}$, where $\mathfrak{g} = det\,(g_{ij})$. As a generalisation of the results of SCHOUTEN and HAANTJES I have found [3]) that the equations are independent of metric (either RIEMANNian or conformal) altogether.

This is to be understood in the following sense: in the ordinary theory we need: 1⁰. quantities characterising the metric (viz. g_{ij} or $\mathfrak{g}_{ij}$), 2⁰. quantities characterising matter (viz. permeability and dielectricity; in the general case of anisotropic matter they are given each by a tensor in three dimensions); it is possible, however, to avoid *the first* kind of quantities and to make use of quantities only, which characterise matter [4]) (of course aside the quantities, characterising the electromagnetical field itself). This possibility should be considered, not as a merely formal, but as a principal fact. Indeed, the notion of metric is a very complicated one: it requires measurements with clocks and scales, generally with *rigid* bodies, which are themselves things of extreme complexity. Hence it seems undesirable to take the notion of a metric as a fundament, also of phenomena which are much simpler and

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[1]) E. CUNNINGHAM, Proc. London Math. Soc. **8** (1910) 77—98.
H. BATEMAN, ibidem **8** (1910) 223—264; 469—488; **21** (1920) 256—270.
Compare also E. BESSEL—HAGEN, Ueber die Erhaltungssätze der Elektrodynamik, Math. Ann. **84** (1920) 258—276.

[2]) J. A. SCHOUTEN and J. HAANTJES, Ueber die konforminvariante Gestalt der MAXWELLschen Gleichungen und der elektromagnetischen Impuls-Energiegleichungen; Physica **1** (1934) 869—872.

[3]) D. VAN DANTZIG, The fundamental equations of electromagnetism, independent of metrical geometry. To appear in the Proceedings of the Cambridge Philosophical Society.

[4]) This word is to be understood in the sense of "environment", and will *include* the so-called free ether.

independent of it. I might state as a principle, or rather as a program: *to formulate the fundamental laws of physics* (not only of electromagnetism, but also of the material waves!) *in a form, independent of metrical geometry.* Metric should turn out finally to be a system of some statistical mean values of certain *physical* quantities. This point of view is entirely different from almost all recent theories which try to unify matter, gravitation and electromagnetism, which all take some kind of metric [1]) as a fundament and try to deduce electromagnetism from it [2]).

Nevertheless I believe that the point of view proposed above might be preferable to those other ones in more than one respect. Though, of course, an abstract (fourdimensional) space remains the mathematical background of the theory (at least as far as *classical* mechanics are concerned), each *special* kind of geometry has to become an a posteriori property of the physical quantities. Moreover metrical geometry has seduced mathematicians to introduce several kinds of connections.

A connection, however, is an always somewhat arbitrary method of linking up quantities in different points in space-time [3]); choosing a special connection in order to express the equations of physics in an invariant way is a process of quite the same nature as choosing a special system of coordinates, viz. a mathematical ("formal") act. Hence I think it to be quite important, that in our theory *all equations which occur are "naturally" invariant,* under *arbitrary* transformations of coordinates [4]), all differential operators being gradients of scalars, rotations of covariant

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1) Not necessarily RIEMANNian, but e.g. also projective.

2) Comp. e.g. the different unified field theories of
TH. KALUZA, Sitzungsber. Pr. Ak. (1921) 966—972; comp. also O. KLEIN, ZS. f. Ph. **37** (1926) 895—906; **46** (1927) 188—208; H. MANDEL, ZS. f. Ph. **39** (1926) 136—145; **45** (1927) 285—306; **49** (1928) 697—704.
H. WEYL, Raum, Zeit, Materie, Berlin, J. SPRINGER (1923);
D. J. STRUIK and O. WIENER, Jn. of Math. and Ph. **7** (1927) 1—23.
A. EINSTEIN, Sitzungsber. Pr. Ak. (1928) 217—221; 224—227; (1929) 2—7; 156—159. Comp. also T. Y. THOMAS, Proc. Nat. Ac. Sc. **16** (1930) 761—776; 830—835; **17** (1931) 48—58; 111—119; 199—210; 325—329.
O. VEBLEN and B. HOFFMANN, Phys. Review (2) **36** (1930) 810—822; comp. also O. VEBLEN, Projektive Relativitätstheorie, Berlin, J. SPRINGER (1933).
A. EINSTEIN and W. MAYER, Sitzungsber. Pr. Ak. (1931) 541—557; (1932) 130—137.
C. LANCZOS, ZS. f. Ph. **73** (1931) 147—168; Ph. Rev. (2) **39** (1932) 716—736.
J. A. SCHOUTEN and D. VAN DANTZIG, ZS. f. Ph. **78** (1932) 639—667; Ann. of Math. (2) **34** (1933) 271—312. Comp. also J. A. SCHOUTEN, Ann. de l'Inst. HENRI POINCARÉ (1934); W. PAULI, Ann. der Ph. **18** (1933) 305—372.

3) Comp. D. VAN DANTZIG, Die Wiederholung des MICHELSON-Versuchs und die Relativitätstheorie, Math. Ann. **96** (1926) 261—283, in particular § 7, Metrik und Physik.

4) We will, however, always suppose the coordinates to be holonomic; with respect to anholonomic coordinates the equations may be formulated with the aid of the "object of anholonomy" (Comp. e.g. J. A. SCHOUTEN and D J. STRUIK, Einf. in die neueren Meth. der Diff. Geom. 2e Aufl. Vol. **1**, Noordhoff Groningen (1935) which also is independent of any connection. The equations only become somewhat less simple.

p-vectors, divergences of contravariant p-vector-densities of weight $+1$, etc. [1] [2]. This is of special interest for the conservation-laws.

The ordinary "conservation-law" $\nabla_j\, T_i^{\cdot j} = 0$ is no such law at all, but only a formal generalisation of such a law. Indeed it cannot be expressed in an integral form, except by the aid of special coordinates [3].

§ 2. MAXWELL's *equations* [4].

An ordinary vector in ordinary Euclidian space can be interpreted not only as a covariant or as a contravariant vector, but also as a co- or contravariant bivector or as a co- or contravariant vector- or bivector-density. Now if we wish to write MAXWELL's equations, which we will use in their integral form

$$\left.\begin{array}{ll} \mathrm{I} \quad \int \overline{H}\,.\,\overline{ds} - \frac{1}{c}\iint \dot{\overline{D}}\,.\,\overline{d\sigma} = \frac{1}{c}\iint \overline{I}\,.\,\overline{d\sigma}\,; \; ^{*)} & \\ \qquad\qquad\qquad\qquad \iint \overline{D}\,.\,\overline{d\sigma} = \iiint \varrho\, d\mathfrak{S} & \\ \mathrm{II} \quad \int \overline{E}\,.\,\overline{ds} + \frac{1}{c}\iint \dot{\overline{B}}\,.\,\overline{d\sigma} = 0\,; \qquad \iint \overline{B}\,.\,\overline{d\sigma} = 0, & \end{array}\right\} \quad . \; . \quad (1)$$

without metric, all scalar products must become transvections. Moreover all integrands must be scalars, as a vector e.g. cannot be integrated without a connection existing. Now in any case the line-element $\overline{ds}$ is a contravariant vector $d\xi^a$ $(a, b, \ldots, g = 1, 2, 3\,;\; \xi^a = x, y, z)$, the surface-element $\overline{d\sigma}$ is a contravariant bivector $d\sigma^{ab} = 2\,\frac{\partial \xi^{[a}}{\partial u}\,\frac{\partial \xi^{b]}}{\partial v}\,du\,dv$ where u, v are arbitrary GAUSSian coordinates on the surface, and the volume-element $d\mathfrak{S}$ is a contravariant trivector dS^{abc}. Hence $\overline{H}$ and $\overline{E}$ *must* be covariant vectors H_a, E_a; $\overline{B}$, $\overline{D}$ and $\overline{I}$ covariant bivectors B_{ab}, D_{ab}, I_{ab} [5]) and ϱ a covariant trivector ϱ_{abc}. We might only by the aid of the trivector-densities of weight ± 1 $\mathfrak{e}^{abc}$, $\mathfrak{e}'_{abc}$, defined by $\mathfrak{e}^{123} = +1$, $\mathfrak{e}'_{123} = +1$, make from each covariant bivector a contravariant vector-density of weight $+1$, etc. We will use these (somewhat arbitrarily) to write $\overline{H}$ as a contravariant bivector-density, $\overline{D}$ and $\overline{I}$ as contravariant vector-densities, ϱ as

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[1]) It might be of interest that also LAPLACE's operator is such a natural invariant, viz. $\mathfrak{g}^{-1/2}\,\partial_j\,\mathfrak{g}^{ij}\,\partial_i$, where $\mathfrak{g}^{ij} = \mathfrak{g}^{1/2}\,g^{ij}$, if it works on a *scalar*.

[2]) They may all be expressed by means of "exterior derivatives". See E. CARTAN, Leçons sur les invariants intégraux, Paris, GAUTHIER-VILLARS, 1922.

[3]) Comp. e.g. W. PAULI, Relativitätstheorie, Enz. Math. Wiss. **5**, (2) Art. 19 (1920) 539—775, in particular Nr. **21** p. 611; A. E. EDDINGTON, The Mathematical Theory of Relativity, Cambridge 2nd ed. (1924) p. 135—136.

[4]) This § contains mainly a short extract of the paper, mentioned in note [3]) on page 521.

*) Vectors are denoted by letters with a line above them.

[5]) A covariant bivector is represented geometrically by a cylindrical tube with a sense of rotation around it; for B_{ab} these tubes form the well-known magnetic force-tubes.

a scalar-density (all of weight $+1$). In the *differential* equations corresponding to (1) all derivatives then become "natural" ones, i.e. generally covariant under arbitrary holonomic transformations and independent of any connection:

$$\left.\begin{array}{lll} \text{I} & \partial_b \mathfrak{H}^{ab} - \frac{1}{c}\dot{\mathfrak{D}}^a = \frac{1}{c}\mathfrak{J}^a ; & \partial_c \mathfrak{D}^c = \varrho ;\ ^{1)} \\ \text{II} & 2\partial_{[b} E_{a]} + \frac{1}{c}\dot{B}_{ba} = 0 ; & \partial_{[c} B_{ba]} = 0 \end{array}\right\} \quad \ldots\ldots \quad (2)$$

We may write the equations in fourdimensional form by putting

$$\left.\begin{array}{l} \overline{E} = (F_{14}, F_{24}, F_{34}) ; \quad \overline{B} = (F_{23}, F_{31}, F_{12}) ; \quad \overline{H} = (H_{14}, H_{24}, H_{34}) = \\ = (\mathfrak{H}^{23}, \mathfrak{H}^{31}, \mathfrak{H}^{12}) ; \quad -\overline{D} = (H_{23}, H_{31}, H_{12}) = (\mathfrak{H}^{14}, \mathfrak{H}^{24}, \mathfrak{H}^{34}) ; \\ \frac{1}{c}\overline{I} = (-s_{234}, -s_{314}, -s_{124}) = (\mathfrak{s}^1, \mathfrak{s}^2, \mathfrak{s}^3) ; \\ \varrho = s_{123} = \mathfrak{s}^4. \end{array}\right\} . \quad (3)$$

Here the indices have been raised and lowered by means of the quadrivector-densities of weight ± 1 $\mathfrak{E}^{hijk}$, $\mathfrak{E}'_{hijk}$, $(h, i, \ldots, m = 1, 2, 3, 4)$, defined by $\mathfrak{E}^{1234} = \mathfrak{E}'_{1234} = +1$; transvections are always performed over the *last* indices of $\mathfrak{E}'_{hijk}$ or over the *first* ones of $\mathfrak{E}^{hijk}$; p-fold transvections must always be preceded by a factor $\frac{1}{p!}$, e.g.

$$\left.\begin{array}{l} \mathfrak{H}^{ij} = \frac{1}{2!} H_{kl}\, \mathfrak{E}^{klij} ; \quad H_{ij} = \frac{1}{2!} \mathfrak{E}'_{ijkl}\, \mathfrak{H}^{kl}\ ^{2)} \\ \mathfrak{s}^i = \frac{1}{3!} s_{jkl}\, \mathfrak{E}^{jkli} ; \quad s_{ijk} = \mathfrak{E}'_{ijkl}\, \mathfrak{s}^l \end{array}\right\} \quad \ldots\ldots \quad (4)$$

Then the equations (2) become

$$\text{I}\ \ \partial_j \mathfrak{H}^{ij} = \mathfrak{s}^i ; \qquad \text{II}\ \ \partial_{[k} F_{ji]} = 0.\ ^{3)} \quad \ldots\ldots \quad (5)$$

§ 3. *The linking equations. Special case.*

It is well known that the linking equations of MAXWELL's theory

$$\overline{B} = \mu \overline{H} \quad , \quad \overline{D} = \varepsilon \overline{E}. \quad \ldots\ldots \quad (6)$$

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1) ∂_c is an abbreviation for $\partial / \partial \xi^c$, etc.

$$\partial_{[b} E_{a]} = \tfrac{1}{2}(\partial_b E_a - \partial_a E_b)$$

2) Note that in the ordinary theory $\mathfrak{F}^{ij}$ would be written for our $\mathfrak{H}^{ij}$ and vice-versa. We identify the bivector-density $\mathfrak{H}^{ij}$ with the bivector H_{ij}, because the relation between them is independent of metric or a unit of volume or anything; in the ordinary theory these quantities would be called orthogonal to each other.

3) $u_{[i}\, v_j\, w_{k]}$ is an abbreviation for $\frac{1}{6}\,(u_i v_j w_k + u_j v_k w_i + u_k v_i w_j - u_k v_j w_i - u_j v_i w_k - u_i v_k w_j)$, etc. ("alternating part" of $u_i v_j w_k$).

may be brought into a fourdimensional invariant form, viz.

$$\left.\begin{array}{l} v_j\,(\mathfrak{H}^{ij} - \varepsilon\, \mathfrak{g}^{ik}\, \mathfrak{g}^{jl}\, F_{kl}) = 0 \\ v_{[h}\,(F_{ji]} - \mu\, \mathfrak{g}_{j|k|}\, \mathfrak{g}_{i]l}\, \mathfrak{H}^{kl}) = 0\ ^{1)} \end{array}\right\} \quad \ldots \ldots \quad (7)$$

This form, however, expresses only a formal invariance, not an independence of the motion of matter which is expressed by the vector v_i; they hold only 1. if all points of the moving matter have the *same* velocity v_i, 2. if matter is isotropic in its restsystem. Now, isotropy is a metrical property and cannot occur in our theory. Hence we *must* formulate the equations (6) for anisotropic bodies also. Especially we cannot accept LORENTZ' theory which requires the equations (6) to hold in the free ether with $\mu = \varepsilon = 1$; these conditions depend essentially on metric [2]).

Then they express the fact that $\overline{B}$ and $\overline{E}$ are linearly dependent of $\overline{H}$ and $\overline{D}$ respectively

$$\mathfrak{B}^a = \mu^{ab}\, H_b\ ,\ E_a = \eta_{ab}\, \mathfrak{D}^b\ . \quad \ldots \ldots \quad (8)$$

Hence for moving bodies $\overline{B}$ and $\overline{E}$ will depend linearly on $\overline{H}$ and $\overline{D}$ together:

$$F_{ij} = \tfrac{1}{2}\, \eta_{ijkl}\, \mathfrak{H}^{kl}\ . \quad \ldots \ldots \quad (9)$$

so that "matter" is characterised by the bivectortensor-density of weight -1 η_{ijkl}. From the well-known property of symmetry of μ^{ab} and η_{ab} follow the identities

$$\left.\begin{array}{ll} \eta_{ijkl} = \eta_{[ij]kl} = \eta_{ij[kl]} & \\ \eta_{[ijk]l} = 0 & \eta_{ijkl} = \eta_{klij} \end{array}\right\} \quad \ldots \ldots \quad (10)$$

In the special case, when matter is isotropic, η_{ijkl} splits up into the system of minors of degree 2 of a tensor-density $\mathfrak{m}_{ij}$:

$$\eta_{ijkl} = \mathfrak{m}_{ik}\, \mathfrak{m}_{jl} - \mathfrak{m}_{il}\, \mathfrak{m}_{jk}\ . \quad \ldots \ldots \quad (11)$$

where $\mathfrak{m}_{ab} = \sqrt{\mu}\; a_{ab}\, (-\mathfrak{a})^{-1/2}$, $\mathfrak{m}_{44} = 1/(-\mathfrak{a})^{1/2}\, \varepsilon \sqrt{\mu}$; a_{ab} is the fundamental tensor of *space* and is taken *negative* definite; $\mathfrak{a} = det\,(a_{ab})$. Especially in free ether $\mathfrak{m}_{ij} = \mathfrak{g}_{ij}$. Of course equation (11) is not *sufficient* for isotropy, a notion which cannot be formulated without the aid of $\mathfrak{g}_{ij}$; it expresses only the property that the ellipsoids of permeability and of dielectricity have parallel and proportional axes.

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[1]) The $|\ |$ on both sides of the index k mean, that this index does not participate in the alternating process.

[2]) The idea of making a principal (not only phenomenological) difference between the $\overline{B}$, $\overline{E}$ and $\overline{H}$, $\overline{D}$ was proposed by G. MIE, Ann. der Ph. **37** (1912) 511—534; **39** (1912) 1—40; **40** (1913) 1—66.

Recently M. BORN has constructed a theory reposing on the same idea (though it starts from quite an other principle than ours, viz. avoiding the infinite self-energy of an electron, and makes use of RIEMANNian metrics). Comp. M. BORN and L. INFELD, Foundations of the new field theory, Proc. Royal Soc. A **144** (1934) 425—451.