la deuxième feuille de la spathe (Cf. la publication suivante sur le développement périodique des Iris bulbeux). Mais dès la fin d'octobre, le traitement de l'été a décidé de la possibilité que le bulbe suivra. Sous 3 à 8, on trouve les tableaux de bulbes de diverses mesures, les températures auxquelles ils ont été soumis, ainsi que le pourcentage de ceux qui fleurissent. Après 23° en été, les bulbes-limite produiront l'année suivante le plus haut pourcentage de fleurs; en ce qui concerne les plus gros bulbes, le souci de la plante future doit nous faire préférer une température de 17 à 20°. Les bulbes trop petits qui, même après 23°, donneront trop de plantes sans fleurs, ne se vendent pas. Ceux-ci auraient dû être traités de manière à fleurir le moins possible chez le cultivateur et à donner de nouveaux bulbes arrondis et assez gros pour fleurir une année plus tard sans difficultés après un traitement à 17—20°. (Les bulbes latéraux aplatis dans l'aisselle des feuilles à la base d'une plante ayant fleuri ont beaucoup plus de défauts pour le commerce que les bulbes ronds terminaux du même poids. — Cf. GRIFFITHS 1928, notre texte hollandais, et les prochaines publications). La marche à suivre pour obtenir des plantes ne fleurissant pas est donnée partiellement dans les Tabl. aux températures de 9 (et de 5) degrés. Mais ici, nous ne donnons pas encore d'instructions exactes: il nous reste à savoir si ces températures basses pourront faire grandir les bulbes dans une mesure satisfaisante. Nous reviendrons sur ce sujet dans les prochaines publications; d'autres expériences sont en cours. Nous espérons surtout rassembler des données sur le mode de réaction dans l'état remarquable où les organes de la plante se trouvent *sur la limite de l'aptitude à former fleurs.*

Mathematics. — *Electromagnetism, independent of metrical geometry. 3. Mass and motion.* By D. VAN DANTZIG. (Communicated by Prof. J. A. SCHOUTEN).

(Communicated at the meeting of October 27, 1934).

§ 1. *Introduction.*

In two previous papers [1]) I have shown that MAXWELL's equations may be brought into a form, which is independent of metrical geometry. In this paper I will show that the same can be done for the equations which determine the motion of a particle in an electromagnetic field. In the ordinary (relativistic) theory this is not the case, as $k_i\, ds = e/c \,.\, F_{ij}\, d\xi^j$ is proportional with the *covariant* differential belonging to the RIEMANNian

[1]) D. VAN DANTZIG Electromagnetism, independent of metrical geometry. 1. The foundations; 2. Variational principles and further generalisation of the theory, these Proceedings **37** (1934) 521—525; 526—531, abbreviated as EM 1, 2

connection, determined by g_{ij}, of the velocity $i_j = g_{jh}\, d\xi^h/ds$, which itself also depends on the g_{ij}.

To this end I give in § 2 a new form of the equations of dynamics, which is based on the LAGRANGian theory of *classical* dynamics and which might be of interest in itself. In § 3 it is shown that the vector of total momentum and energy is constant in the sense of LIE-derivation. § 4 contains the proof that also the equations of motion of *relativity* mechanics are a particular case of the general theory of § 2. It is remarkable that the *only* difference between classical and special relativistic dynamics results in the different values of the kinetic energy. In § 5 the mass-charge-relation of EDDINGTON is deduced by a very simple and evident identification which leads at once to the physical meaning of the φ_i in the interior of a particle, viz. c/e times the total momentum and energy. Finally in § 6 the relations between the mass and charge-density are studied in somewhat greater detail. It is shown that the total mass may be considered as an electromagnetic interaction-energy; and the equation of motion is interpreted in such a way that in a certain sense the well-known difficulty of the "non-MAXWELLian forces", which should keep together a particle, disappears.

The unification of electromagnetism with classical and relativistic dynamics is of a very intrinsic nature, and not only a formal one. It seems to be about the most intimate one [1]) which can be reached without introducing quantum-mechanics.

§ 2. *A new form of the dynamical equations.*

Let us consider a dynamical system with n degrees of freedom, determined by n independent dynamical coordinates q^α $(\alpha, \beta, \gamma, \delta, \varepsilon = 1, \ldots, n)$. We will suppose the system to be holonomic and derivable from a LAGRANGian function

$$L = T - P, \qquad (1)$$

where both T and P are allowed to depend on the q^α, the $\dot{q}^\alpha$ and t. We put $q^0 = t$, and suppose the indices $\varkappa, \ldots, \tau$ to run over the values $0, 1, \ldots, n$. The differential

$$d\Lambda = L\, dt \qquad (2)$$

is homogeneous of degree one (but not linear) in the $n+1$ differentials dq^λ. This is the *only* property of $d\Lambda$ we make use of. We put

$$j_\lambda = \partial\, T dt/\partial\, dq^\lambda \;,\quad f_\lambda = -\,\partial\, P dt/\partial\, dq^\lambda \;,\quad p_\lambda = \partial\, L\, dt/\partial\, dq^\lambda = j_\lambda + f_\lambda \,. \qquad (3)$$

[1]) Apart from *formal* simplifications, obtainable by the use of a five-dimensional formalism.

Hence the $p_\lambda, j_\lambda, f_\lambda$ are *homogeneous of degree zero in the differentials.* Especially if the mechanism is conservative $f_\alpha = 0$, $-f_0 = P =$ potential energy, $-j_0 = T =$ kinetic energy, $-p_0 = H =$ total energy, $p_\alpha = j_\alpha =$ momentum. However, in the general case also we will call $j_\lambda, f_\lambda, p_\lambda$ the kinetic, the potential and the total vector of momentum and energy. From EULER's theorem about homogeneous functions follows

$$d\Lambda = p_\lambda\, dq^\lambda = p_\alpha\, dq^\alpha - H\, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The equations of motion, found by variation of $\int d\Lambda$ are

$$d\frac{\partial\, d\Lambda}{\partial\, dq^\lambda} - \frac{\partial\, d\Lambda}{\partial\, q^\lambda} = 0\ ^{1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or

$$dp_\lambda - \partial\, d\Lambda / \partial\, q^\lambda = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now let us consider n independent first integrals which can be solved for the $\dot{q}^\alpha$: $\dot{q}^\alpha = \dot{q}^\alpha (q^\alpha, t, c_1, \ldots, c_n)$. For each system of constants we have in the $(n+1)$-dimensional space X_{n+1} of the q^λ a field of *directions*, the ratio's of the dq^λ being functions of the q^λ. If we divide the dq^λ by an arbitrary differential $d\tau$ [2]), the $dq^\lambda/d\tau$ are functions of the q^λ. As the differential $d\tau$ disappears automatically from all equations, we will omit it altogether and write simply

$$dq^\lambda = dq^\lambda (q^\mu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Then also the p_λ are functions of the q^λ. It is well known that the equations can be integrated by quadratures if the first integrals are in involution, viz. if

$$\partial_{[\mu}\, p_{\lambda]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Now let us consider (7) as defining n *unknown* functions of the q^λ, which we will substitute in p_λ and in the equation of motion (6). The first term in (6) becomes simply

$$dp_\lambda = (\partial_\mu\, p_\lambda)\, dq^\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where

$$\partial_\mu = \frac{\partial}{\partial q^\mu} + \frac{\partial\, dq^\rho}{\partial\, q^\mu} \cdot \frac{\partial}{\partial\, dq^\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

[1]) Cf. for this symmetrical form of LAGRANGE's equation P. A. M. DIRAC, Homogeneous variables in classical dynamics. Proc. Cambridge Philosophical Society **29** (1933) 389-400.

[2]) A special choice of $d\tau$, which may sometimes be of use (Cf § 3), is $d\tau = d\Lambda$.

so that $\partial_\mu\, p_\lambda$ is the "total partial derivative" of p_λ, whereas $\partial p_\lambda / \partial q_\mu$ is only the "partial partial derivative", i.e. the partial derivative under the condition $dq^\lambda = \text{const.}$

The second term in (6) becomes by using (10):

$$\frac{\partial\, dA}{\partial\, q^\lambda} = \frac{\partial\, p_\mu\, dq^\mu}{\partial\, q^\lambda} = \frac{\partial\, p_\mu}{\partial\, q^\lambda}\, dq^\mu = (\partial_\lambda\, p_\mu)\, dq^\mu - \left(\frac{\partial\, dq^\rho}{\partial q^\lambda} \cdot \frac{\partial\, p_\mu}{\partial\, dq^\rho}\right) dq^\mu. \quad . \quad (11)$$

But the last term in (11) is zero, because

$$\frac{\partial\, p_\mu}{\partial\, dq^\rho}\, dq^\mu = \frac{\partial^2\, dA}{\partial\, dq^\rho\, \partial\, dq^\mu}\, dq^\mu = \frac{\partial\, p_\rho}{\partial\, dq^\mu}\, dq^\mu = 0, \quad . \quad . \quad . \quad (12)$$

p_μ being homogeneous of degree zero in the dq^μ (EULER's condition). Hence (11) becomes

$$\partial\, dA / \partial\, q^\lambda = (\partial_\lambda\, p_\mu)\, dq^\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and the equation of motion (6) is equivalent with

$$\boxed{2\, dq^\mu\, \partial_{[\mu}\, p_{\lambda]} = 0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

In (14) the p_λ are *given* functions of the q^λ, dq^λ, determined by (3), and the dq^λ are *unknown* functions of the q^λ, so that (14) is a set of $n+1$ partial differential equations of the first order for the n ratio's of the $n+1$ unknown functions $dq^\lambda\, (q^\mu)$. Indeed the $(n+1)$ equations (14) are dependent, the left side of (14) becoming zero by transvection with dq^λ. Evidently each solution of (8) satisfies (14), the operator ∂_μ, used in (8) i.e. $\partial / \partial q_\mu$ *after* first integration, being equivalent with the operator (10) *before* integration. But a solution of (14) will in general *not* satisfy (8).

§ 3. LIE-*derivative and energy-law.*

The equations of motion (14) admit of a very simple interpretation with the aid of the LIE-derivative, introduced by ŚLEBODZIŃSKI [1]), and studied in great detail by SCHOUTEN and VAN KAMPEN [2]).

---

[1]) W. ŚLEBODZIŃSKI, Sur les équations canoniques de Hamilton, Bull. Ac. Royale de Belgique (5) **17** (1931) 864—870.

[2]) J. A. SCHOUTEN & E. R. VAN KAMPEN, Beiträge zur Theorie der Deformation, Warszawa Prac Mat.-Fiz. **41** (1933) 1—19. The name „LIE-derivative" was proposed by D. VAN DANTZIG, Zur allgemeinen projektiven Differentialgeometrie. II. X_{n+1} mit eingliedriger Gruppe, Proc. Kon. Ak. **35** (1932) 535—542.

We consider an infinitesimal point-transformation $q^\lambda \rightarrow q^\lambda + dq^\lambda$, determined by a vectorfield $v^\lambda = \varepsilon^{-1} dq^\lambda$. Under this transformation any object $X^{\cdot}_{\cdot}$ [1]), defined in a point q^λ can be "dragged along", i. e. we can consider the values q^λ as new coordinates of the point $q^\lambda + \varepsilon v^\lambda$ and consider the object $Y^{\cdot}_{\cdot}$ whose components in this *second* point with respect to these *new* coordinates are equal to the components of $X^{\cdot}_{\cdot}$ in the *first* point with respect to the *old* coordinates. The difference of $Y^{\cdot}_{\cdot}$ and the value $X^{\cdot}_{\cdot} + dX^{\cdot}_{\cdot}$ of $X^{\cdot}_{\cdot}$ in the point $(q^\lambda + dq^\lambda)$ divided by ε will be called the LIE-derivative of $X^{\cdot}_{\cdot}$ and denoted by $\underset{L}{D}X^{\cdot}_{\cdot}$. It determines a mode of *invariant* differentiation, which is independent, not only of the choice of arbitrary coordinates, but also of *any connection*. It follows the ordinary rules for derivation of sums, products and transvections. As examples we mention:

contravariant vector: $\underset{L}{D}\, u^\lambda = v^\mu \partial_\mu u^\lambda - u^\mu d_\mu v^\lambda$

density of weight $\mathfrak{k}$: $\underset{L}{D}\, \mathfrak{p} = \mathfrak{k}\mathfrak{p}^{(\mathfrak{k}-1)/\mathfrak{k}}\, \partial_\mu (v^\mu \mathfrak{p}^{1/\mathfrak{k}})$

covariant vector: $\underset{L}{D}\, w_\lambda = v^\mu \partial_\mu w_\lambda + w_\mu \partial_\lambda v^\mu =$

$$= 2 v^\mu \partial_{[\mu} w_{\lambda]} + \partial_\lambda (w_\mu v^\mu).$$

From the last example we see that especially

$$\underset{L}{D}\, w_\lambda = 2\, v^\mu \partial_{[\mu} w_{\lambda]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

if v^λ satisfies the condition $w_\mu v^\mu = \text{const.}$

Now the vectorfield v^λ is *not* uniquely determined by the infinitesimal displacement dq^λ. But, if dq^λ is the deplacement belonging to a mechanical system as considered in § 2, we can determine it uniquely by requiring

$$p_\lambda v^\lambda = 1\ ^{2)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

viz. by putting

$$v^\lambda = dq^\lambda / d\Lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

From (15), (16) we see that the equations of motion (14) can be written in the very simple form [3])

$$\boxed{\underset{L}{D}\, p_\lambda = 0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

[1]) The big points stand for arbitrary rows of indices.

[2]) Comp. note [2]) on page 645 and equation (4).

[3]) Equation (18) is equivalent with the following fact: if an arbitrary line-element $\underset{1}{dq^\lambda}$ is "dragged along" by the transformation (i.e. if $\underset{L}{D}\, \underset{1}{dq^\lambda} = 0$), $p_\lambda \underset{1}{dq^\lambda}$ is constant in the ordinary sense. Another equivalent form of this theorem was already found by HAMILTON (III. Supplement to the Theory of systems of rays). C.f. M. HERZBERGER Strahlenoptik, Berlin, J. SPRINGER (1931), p. 12.

and express the fact that the *vector of total momentum and energy is constant in the sense of the* LIE-*derivative during the motion*, under the assumption that v^λ be determined by (17) [1]).

§ 4. *Relativistic motion in an electromagnetic field.*

In this paragraph we will apply the general results of § 2 to the motion of a *material point*. Hence we will write ξ^a $(a, \ldots, g = 1, 2, 3)$ instead of q^α, $\xi^4 = ct$ instead of $q^0 = t$, and use the indices $h, i, \ldots, m = 1, 2, 3, 4$ instead of $\iota, \varkappa, \ldots, \tau = 0, 1, \ldots, n$.

If we split up p_i according to (3) into its kinetic and potential part:

$$p_i = j_i + f_i, \qquad (19)$$

the equation of motion (14) can be written:

$$2\, d\xi^j\, \partial_{[j}\, j_{i]} = -\, 2\, d\xi^j\, \partial_{[j}\, f_{i]} \qquad (20)$$

As this equation is valid for *any* form of the LAGRANGian, we can apply it in particular to the *relativistic* motion (of course by using the fundamental tensor!). To this end we put:

$$T\, dt = -\, m\, c\, ds, \qquad (21)$$

where m is the *invariant* mass of relativity and

$$ds = \sqrt{g_{ij}\, d\xi^i\, d\xi^j} \qquad (22)$$

is the RIEMANNian line-element of general relativity (signature $- - - +$). Then from (3) follows:

$$j_i = -\, m\, c\, g_{ih}\, i^h, \quad i^h = d\xi^h/ds, \qquad (23)$$

in particular $j_4 = -\, mc\,(1 - v^2/c^2)^{-1/2}$. Instead of the relation $cj_4 = -\,T$, which holds only in classical dynamics, we have here $cj_4 = T/(1 - v^2/c^2)$, whereas the j_a have the same values as in classical dynamics. Now equation (20) can be written with *covariant* derivatives $\nabla_j j_i = \partial_j j_i - \Gamma^h_{ji} j_h$, the RIEMANNian connection being symmetrical, so that $\nabla_{[j}\, j_{i]} = \partial_{[j}\, j_{i]}$. Hence the left side of (20) is equal to

$$-\, 2\, mc\, d\xi^j\, \nabla_{[j}\, i_{i]} = -\, mc\, ds\, i^j\, (\nabla_j\, i_i - \nabla_i\, i_j) \qquad (24)$$

But, the last term in (24) is zero because of $i^j\, \nabla_i\, i_j = \tfrac{1}{2}\, \nabla_i\, i^j\, i_j = 0$, $i^j\, i_j$ being constant $(= 1)$. Hence the second member of (24) is equal to

[1]) Moreover this is the *only* determination (but for a constant factor) for which the theorem holds.

$- mc\, \delta i_i$, where $\delta i_i = d\xi^j \nabla_j i_i = ds\, i^j \nabla_j i_i$ is the *covariant differential* of i_i, and we can write equation (20) in the well-known form

$$\delta j_i / ds = k_i, \quad \ldots \quad (25)$$

where k_i is the relativistic force-vector. It is equal to

$$k_i = (2\, \partial_{[i} f_{j]})\, d\xi^j / ds, \quad \ldots \quad (26)$$

which expression is equally well known in relativity: indeed, putting

$$f_i = e/c \,.\, \varphi_i, \quad \ldots \quad (27)$$

so that the action-differential is

$$d\Lambda = - mc\, ds + e/c \,.\, \varphi_i\, d\xi^i, \quad \ldots \quad (28)$$

(26) becomes:

$$k_i = e/c \,.\, F_{ij}\, d\xi^j / ds, \quad \ldots \quad (29)$$

which is the relativistic LORENTZ-force, working on a charge e. Hence we see that the form (29) or (26) of the force is not at all particularly special but is the most general one which is derivable from a kinetic potential. The only specialisation lies in the fact that φ_i *does not depend on the velocity.*

§ 5. *Mass, and the physical meaning of the* φ_i.

Now let us suppose that the φ_i may be expressed by means of the retardated potentials of special relativity, belonging to empty space:

$$\varphi_i(P) = \int g_{ij'}\, \gamma(P, P')\, \mathfrak{s}^{j'}(P')\, d\Sigma', \quad \ldots \quad (30)$$

where $\gamma(P, P')$ is the special operator, introduced in EM 2 (14):

$$\gamma(P, P') = - 1/4\pi r \,.\, \delta(ct - ct' - r(P, P')), \quad \ldots \quad (31)$$

and that only a *finite* number of particles have an appreciable influence on the field. Then the integral (30) may be replaced by a finite sum over all particles P':

$$\varphi_i = \sum_{P'} \overset{P'}{\varphi_i}(P) \quad , \quad \overset{P'}{\varphi_i}(P) = - e'/4\pi r \,.\, g_{ih'}\, i^{h'}, \quad \ldots \quad (32)$$

the $i^{j'}$ being taken at the time $t' = t - r/c$. Now the analogy between the right sides of equations (32) and (23) is very striking. Hence it seems natural to suppose that j_i, or rather (corresponding to (27))

$$\overline{\overset{0}{\varphi}}_i = c/e \,.\, j_i \quad \ldots \quad (33)$$

is the mean value of a covariant vector $\overset{0}{\varphi}_i$, which is of quite the same nature as the other φ_i, i.e. that $c/e \,.\, j_i$ is a *mean value of the potential-vector* $\overset{0}{\varphi}_i$ *of the particle P itself, taken over its own volume.* Then the

r in the denominator of (32), will have to be replaced by some mean value a, and we get $m \,.\, c/e = e/4\pi ca$, or

$$mc^2 = e^2/4\pi a.\ ^{1)} \qquad \ldots \ldots \ldots \quad (34)$$

Equation (34) was first deduced by EDDINGTON [2]) from a hypothesis which is equivalent with (33).

We may now write

$$c/e\,.\,p_i = \varphi_i = \overset{u}{\varphi}_i + \overset{0}{\varphi}_i, \qquad \ldots \ldots \ldots \quad (35)$$

φ_i, $\overset{u}{\varphi}_i$ and $\overset{0}{\varphi}_i$ being *the total, the exterior and the interior potential vector*, and we can say that their physical meaning is that *they are equal to e/c times the total, the potential and the kinetic momentum and energy* respectively. If we substitute (35) in (14), bring the factor e/c *before* the differential symbol and divide by $d\Sigma$, the equation of motion becomes: [3])

$$\boxed{\mathfrak{s}^j F_{ji} = 0}\,, \qquad \ldots \ldots \ldots \ldots \quad (36)$$

where again $F_{ji} = 2\partial_{[j}\,\varphi_{i]}$ is the *total* (exterior and interior) field. Though the first factor $\mathfrak{s}^j$ in (36) belongs only to the particle itself, it may be identified with the *total* current, the $\mathfrak{s}^i$ of the exterior field being zero on the place where the particle is.

It follows from (36) that for $\mathfrak{s}^k \neq 0$ the total bivector F_{ij} is simple:

$$F_{[ij}\,F_{kl]} = 0. \qquad \ldots \ldots \ldots \ldots \quad (37)$$

Hence it is geometrically represented by a (local) twodimensional cylindrical tube, envelopped by ∞^2 planes. Equation (36) expresses the fact that *the directions of the total current is always contained in the 2-direction of the tube, belonging to the total field-bivector.*

EDDINGTON deduces l.c. equation (34) from the hypothesis that a particle moves always in such a way that the total field F_{ij} is zero. This condition which is equivalent with equation (8) is, however, to strong, as it contains *six* independent equations for the three unknown ratio's of the $d\xi^i$, whereas our general equation (36) contains only *three* independent conditions. It is, however, easily seen, that EDDINGTON for his deduction only makes use of the weaker condition (36). Nevertheless it might be that EDDINGTON's condition $F_{ij} = 0$ were characteristic for *special kinds* of particles [4]), in the same way as *photons* are characterised by another trivial specialisation of (36), viz. $\mathfrak{s}^i = 0$, $F_{ij} \neq 0$.

---

[1]) We find for an electron with $m = 0.9 \times 10^{-27}$ g, $e = -4.77 \times 10^{-10} \sqrt{4\pi}$ $g^{1/2}\,\mathrm{cm}^{3/2}\,\mathrm{sec}^{-1}$ and $c = 3 \times 10^{10}$ cm sec^{-1}, $a = 2.8 \times 10^{-13}$ cm, which is about the radius of an electron, as it is usually supposed to be.

[2]) Sir A. S. EDDINGTON, The mathematical theory of relativity, Cambridge University Press (1923), § 80, Explanation of the mechanical force. EDDINGTON's explanation has been greatly inspiring to this part of my investigations.

[3]) $\mathfrak{s}^i$ is the current-density. Hence if $\varrho = \mathfrak{s}^4$ is the charge-density, $\mathfrak{s}^i d\Sigma = \varrho/c\,.\,d\xi^i/dt\,.\,d\Sigma = \varrho\, d\mathfrak{S}/c\,.\,d\xi^i$, which becomes $e/c\,.\,d\xi^i$ after integration over the volume of the particle.

[4]) Probably these would be particles without spin.

§ 6. *Mass as interaction-energy.*

The formula for the mass of a charged particle may be brought into another and more expressive form. Such a particle is a set of ∞^3 paths in space-time, filling up the interior of a threedimensional tube. Its charge is

$$e = c \int^{3} \mathfrak{s}^i \, d\mathfrak{S}_i \,, \quad \ldots \ldots \ldots \quad (38)$$

integrated over any threedimensional space X_3, which cuts each streamline once. It follows from the equation of continuity $\partial_i \mathfrak{s}^i = 0$ that e is the same for each such X_3.

Now the HAMILTONian, belonging to the action

$$\mathfrak{W} = \tfrac{1}{4} F_{ij} \mathfrak{H}^{ij} - \tfrac{1}{2} \varphi_i \mathfrak{s}^i \quad \ldots \ldots \ldots \quad (39)$$

(comp. EM 2 (7)) is $\mathfrak{H}$ if

$$\left. \begin{aligned} -\mathfrak{H} = \mathfrak{W} - \varphi_i \frac{\partial \mathfrak{W}}{\partial \varphi_i} = \tfrac{1}{2} \mathfrak{H}^{ij} \frac{\partial \mathfrak{W}}{\partial \mathfrak{H}^{ij}} - \mathfrak{W} = \tfrac{1}{4} F_{ij} \mathfrak{H}^{ij} + \tfrac{1}{2} \varphi_i \mathfrak{s}^i = \\ = \varphi_i \mathfrak{s}^i + \tfrac{1}{2} \partial_i (\mathfrak{H}^{ij} \varphi_j). \end{aligned} \right\} \quad (40)$$

Hence, if the boundary-integral vanishes:

$$H = \int^{4} \mathfrak{H} \, d\Sigma = -\int^{4} \varphi_i \mathfrak{s}^i \, d\Sigma \quad \ldots \ldots \quad (41)$$

We will suppose the *exterior* field to be zero: $\varphi_i = \overset{0}{\varphi}_i$ and we will extend the integral (41) over some finite length of the tube. If this is filled up arbitrarily with $\infty^1 X_3$, each cutting each streamline once, we may decompose each fourdimensional volume-element $d\Sigma$ into a threedimensional volume-element $d\mathfrak{S}_i$ of the X_3 passing through it and a line-element $d\xi^i$, in the direction of $\mathfrak{s}^i$:

$$d\xi^i = \mathfrak{s}^i \, d\mathfrak{X}, \quad d\Sigma = d\xi^i \, d\mathfrak{S}_i \,. \quad \ldots \ldots \ldots \quad (42)$$

Then (41) becomes:

$$-H = \int \varphi_i \mathfrak{s}^i \mathfrak{s}^j \, d\mathfrak{X} d\mathfrak{S}_j = e/c \,. \int \bar{\varphi}_i \, d\xi^i \,, \quad \ldots \ldots \quad (43)$$

where $\bar{\varphi}_i$ is a mean-value of φ_i over the X_3 in the interior of the tube. *Now we suppose* $\bar{\varphi}_i$ *to be identical with the* $\overline{\overset{0}{\varphi}}_i$ *of (33).* Then (43) becomes

$$H = -\int j_i \, d\xi^i = \int mc \, ds \quad \ldots \ldots \quad (44)$$

Comparing (44) with (41) we find:

$$m \, ds/dt \stackrel{*}{=} -\int \varphi_i \mathfrak{s}^i \, d\mathfrak{S}_4 \stackrel{*}{=} -\int d\mathfrak{S}_4 \int d\Sigma' \, \gamma_{ij'} \mathfrak{s}^i \mathfrak{s}^{j'} \quad \ldots \quad (45)$$

From (44) and (41) follows also that the mass-density $\mathfrak{m}$ is determined by

$$\mathfrak{m} c = \mathfrak{H} \,. \quad \ldots \ldots \ldots \ldots \quad (45^a)$$

If the particle is in rest with respect to a GALILEIan frame (hence $\mathfrak{s}^a \overset{*}{=} 0$), and has spherical symmetry, its charge-density being given by $\mathfrak{s}^4 \overset{*}{=} \varrho = \varrho(r)$, its potential is

$$\varphi(r) \overset{*}{=} \frac{1}{r}\int_0^r \varrho(r')\, r'^2\, dr' + \int_r^\infty \varrho(r')\, r'\, dr', \quad . \quad . \quad . \quad . \qquad (46)$$

provided that we may take the ordinary COULOMB-potential. Its charge and mass are according to (38) and (45):

$$e \overset{*}{=} 4\pi \int_0^\infty \varrho(r)\, r^2\, dr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (47)$$

$$mc \overset{*}{=} 4\pi \int_0^\infty \varrho(r)\, \varphi(r)\, r^2\, dr. \text{ [1]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (48)$$

From the fact that $\varrho(r)$ occurs (after substitution of (46) in (48)) in a *quadratic* way in m, it follows that the invariant mass of a *system* of particles is *not* equal to the sum of the invariant masses, even if the particles are in rest relative to each other. However, if the mutual distances of the particles are large compared with each a, the cross-terms of the form $\overset{1}{\varrho}\, \overset{2}{\varphi}\, d\mathfrak{S}_4$ will be small, $\overset{2}{\varphi}$ being small when $\overset{1}{\varrho}$ is great and vice versa, so that in that case the invariant mass is at least *approximately* additive.

As another consequence from the fact that ϱ occurs quadratically in m, we remark that *to a given value of m belong two particles of equal but opposite charges.*

Equation (45) expresses the fact that *the total mass of a particle is of the kind of an electromagnetic interaction-energy*. From this point of view the equation of motion (36) may be interpreted as follows: *a charged particle moves as if it were composed of* ∞^3 *point-charges, leading to a resulting current-density* $\mathfrak{s}^i$, *each of which has a mass zero, and moves in such a way, that the total* LORENTZ-*force which works on it is zero* [2]. Hence in a certain sense each point-charge is a *cause* of the electromagnetic field, but is not *influenced* by it, so that no non-MAXWELLian forces are needed to keep them together.

---

[1]) If we take as an example $\varrho = \varrho_0\, e^{-r/b}$ where b is some given length, we find $e = 8\pi \varrho_0 b^3$, $\varphi = e/4\pi r \{1 - (1 + r/2b) e^{-r/b}\} \sim e/4\pi r$ for $r \gg b$, $mc = 5e^2/64\pi b$ and $a = {}^{16}/_5\, b$. With this density, however, a much to great part of the charge would ly outside of a sphere of radius a; a density as e.g. $\varrho = \varrho_0\, e^{-r^2/b^2}$ would fit much better. Ir. L. KOSTEN attires my attention to the fact that a pure surface-charge [i.e. $\varrho(r) = \varrho_0\, \delta(r - b)$] leads exactly to $a = b$.

[2]) Mathematically this means that the splitting up of $d\Lambda$ into $L dt$ and $-P dt$, which is rather arbitrary (because the secondterm also depends on the differentials), is effectuated by taking $T = 0$ and $L = -P$. This is trivial (and can be done for any mechanism), but the possibility of interpreting mass as interaction-energy (which depends on (30)) is not.