Mathematics. — *On relativistic gas theory.* By D. VAN DANTZIG. (Communicated by Prof. J. A. SCHOUTEN).

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1. *Introduction.*

In the foregoing paper [1]) relativistic thermodynamics was developed somewhat further than it had been done by previous authors (EINSTEIN, PLANCK, TOLMAN, e.a.), mainly by application of tensor-analysis to this part of physics. In the present paper the same method has been applied to relativistic gas theory. Without the use of tensor-analysis relativistic gas theory was mainly developed by JÜTTNER [2]) and by TOLMAN [3]) under MAXWELL—BOLTZMANN statistics. For the theory of radiation an analogous result had been reached already by VON MOSENGEIL [4]). For EINSTEIN—BOSE and FERMI—DIRAC statistics gas theory was worked out by JÜTTNER [5]) also. The method of dealing with arbitrary statistics was given for non-relativistic gas theory by FOWLER [6]) and by UHLENBECK [7]).

The main difference between the present paper and these older ones lies in the consequent symmetrical treatment of the time-coordinate with the other coordinates. By extending integrals over the interior of the characteristic surface (cf. § 2) instead of over this surface itself an important simplification can be reached. It is shown that only one integral need be directly computed, viz. the one representing the pressure; all other thermodynamic quantities can be derived from it.

An application is given to a controversial question concerning the relation between proper particle-density and proper energy-density, raised by EDDINGTON. EDDINGTON's proposal was recently criticised by SYNGE.

[1]) On relativistic thermodynamics, Proc. Kon. Ned. Akad. Wet., Amsterdam, **42**, 601, 1939, abbreviated for reference as Rel. Therm.

[2]) F. JÜTTNER, Das MAXWELLsche Gesetz der Geschwindigkeitsverteilung in der Relativtheorie. Ann. d. Ph. (4) **34**, 856—882 (1911); Die Dynamik eines bewegten Gases in der Relativtheorie, ibid. (4) **35**, 145—161 (1911).

[3]) R. C. TOLMAN, Relativity theory: the equipartition law in a system of particles. Phil. Mag. **28**, 583—600 (1914).

[4]) K. v. MOSENGEIL, Theorie der stationären Strahlung in einem gleichförmig bewegten Hohlraum. Ann. d. Ph. (5) **22**, 867—904 (1907).

[5]) F. JÜTTNER, Die relativistische Quantentheorie des idealen Gases. ZS. f. Ph. **47**, 542—566 (1928).

[6]) R. C. FOWLER, General forms of statistical mechanics etc. Proc. Roy. Soc. A **113**, 432—449 (1926).

[7]) G. E. UHLENBECK, Over statistische methoden in de theorie der quanta. Thesis, Leiden (1927).

It is shown here that EDDINGTON's criticism against SCHWARZSCHILD's proposal as well as SYNGE's criticism against EDDINGTON's proposal is correct, as was already indicated by the author in a more elementary way. [8]). The correct value of the proper particle-density is for an ideal gas intermediate between SCHWARZSCHILD's and EDDINGTON's values, but is in general independent of the stress-tensor. We can therefore not accept SYNGE's definition of incompressibility which is based upon the stress-tensor alone.

Finally it may be remarked that the fundamental tensor of space-time does not intervene except implicily in the function $\mathcal{H}$. Hence all laws which are independent of this function (e. g. BOYLE-GAY-LUSSAC's law, STEFAN-BOLTZMANN's law, WIEN's law, etc.) are *independent of metrical geometry*. In particular they are valid in classical as well as relativistic theory. More generally all relations occurring in this paper with the exception of (2), (3), § 8, § 9 and the end of § 10 are independent of metrical geometry.

2. *Phase-space; the characteristic surface.*

Let q^α, p_α be coordinates and corresponding momenta of a mechanical system ("molecule") with n degrees of freedom and let $H = H(p_\alpha, q^\alpha, t)$, be its HAMITONian. Following CARTAN [9]) and DIRAC [10]) we introduce two new variables p_0 and q^0 and let $\alpha, \beta, \gamma, \delta, \varepsilon$ and $\varkappa, \lambda, \mu, \nu$ run through the ranges $1, \ldots, n$ and $0, 1, \ldots, n$ respectively.

We suppose two functions $t(q^\varkappa)$ and $\mathcal{H}(p_\lambda, q^\varkappa)$ to be given such that:

A. *the equation* $t(q^\varkappa) = t$ *admits an unambiguous solution* $q^0 = q^0(t, q^\alpha)$ with $\frac{\partial q^0}{\partial t} > 0$;

B. *the equation* $\mathcal{H}(p_\lambda, q^\varkappa) = 0$ *is equivalent with* $p_0 + H(p_\alpha, q^\alpha, q^0) = 0$; *moreover* $\frac{\partial \mathcal{H}}{\partial p_0} < 0$.

Special and characteristic choices of t and $\mathcal{H}$ are $t = q^0$ and $\mathcal{H} = -p_0 - H(p_\alpha, q^\alpha, q^0)$. In order to avoid superfluous factors h^{-n} we put at once

$$p_\lambda = h\varkappa_\lambda, \quad \ldots\ldots\ldots\ldots \quad (1)$$

h being PLANCK's constant, and work with the $\varkappa_\lambda$ instead of the p_λ.

We have to consider the following manifolds [10a])

[8]) D. VAN DANTZIG, Stress tensor and particle-density in special relativity theory. Nature **143**, 855 (May 20, 1939).

[9]) E. CARTAN, Leçons sur les invariants intégraux. Paris, HERMANN, (1932).

[10]) P. A. M. DIRAC, Homogeneous variables in classical dynamics. Proc. Cambr. Phil. Soc. **9**, 389—400 (1933).

[10a]) The quantities between brackets refer to molecules with n "exterior" and f "interior" degrees of freedom. Cf. the end of § 4.

Manifold	Name	Number of dimensions	Symbol for elements	Coordinates	Subject to conditions:
Σ	coordinate-space-time	$n+1$	$\mathrm{d}\mathfrak{U}$	$q^\varkappa$	—
Γ	coordinate-space	n	$\mathrm{d}\mathfrak{B}_\lambda$	$q^\varkappa$	$t=\text{const.}$
Ω	$(\varkappa, q)$-space	$2n+2$ $(2n+2k+2)$	$\mathrm{d}\Omega$	$\varkappa_\lambda, q^\varkappa$ $(\varkappa_\lambda, \varkappa_\eta, q^\varkappa, q^\xi)$	—
Φ	phase-space	$2n$ $(2n+2k)$	$\mathrm{d}\Phi$	$\varkappa_\lambda, q^\varkappa$ $(\varkappa_\lambda, \varkappa_\eta, q^\varkappa, q^\xi)$	$\begin{cases} t=\text{const.} \\ \mathcal{H}=0 \end{cases}$
K	$\varkappa$-space	$n+1$ $(n+2k+1)$	$\mathrm{d}\mathfrak{k}$	$\varkappa_\lambda$ $(\varkappa_\lambda, \varkappa_\eta, q^\xi)$	$q^\varkappa=\text{const.}$
X	characteristic surface	n $(n+2k)$	$\mathrm{d}\mathfrak{g}^\varkappa$	$\varkappa_\lambda$ $(\varkappa_\lambda, \varkappa_\eta, q^\xi)$	$\begin{cases} q^\varkappa=\text{const.} \\ \mathcal{H}=0. \end{cases}$

Between these manifolds the following inclusions exist: $\Omega \supset \Phi \supset X$, $\Omega \supset K \supset X$, $\Sigma \supset \Gamma$, but not $\Omega \supset \Sigma$; each point of Σ is a $\varkappa$-space in Ω.

Because of condition A., Γ is a "cross-section" of coordinate-space-time, intersecting the worldline in Σ of every molecule exactly once.

Moreover the form of the characteristic hypersurface X in K is of importance. In classical mechanics, where $H=K+P$ the kinetic energy K is homogeneous quadratic and positive definite in the $p_\alpha - f_\alpha$, whereas the vector-potential f_α and the potential energy P depend upon $q^\varkappa$ alone. Hence there X is a (hyper-) paraboloid open towards $p_0=-\infty$ and having its summit in $p_\alpha=f_\alpha$, $p_0=-P$. In relativistic mechanics of a charged particle the equation $\mathcal{H}=0$ is usually considered to be equivalent with

$$\overline{\mathcal{H}} \equiv g^{ij}(p_i-f_i)(p_j-f_j)-m^2c^2=0, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where x^h $(h, i, j, k=1, 2, 3, 4,\ x^4=ct=cx^0)$ take the place of our $q^\varkappa$. For constant x^h it is a 3-dimensional (hyper-) hyperboloid consisting of *two* blades. Solving for p_0 we obtain therefore two values, contrary to condition B. This two-valuedness is connected with many difficulties in relativity theory. In order to satisfy condition B. we have to take only *one* blade of the hyperboloid, viz. the one upon towards $p_0=-\infty$. This blade has a summit at $p_\alpha=f_\alpha$, $p_0=f_0-mc^2$.

Only by taking this blade *alone* the relativistic "correspondence-principle" is satisfied, stating that classical mechanics is obtained by passing to the limit for $c\to\infty$. It might seem that by imposing the condition $p_0-f_0<0$ the LORENTZ-invariance were disturbed. In fact

this is the case with respect to the *complete* group of all transformations leaving the light-cone invariant. The equations however remain invariant with respect to the sub-group leaving each half of the light-cone ("future" and "past") invariant and it is this invariance alone which is guaranteed by the experiments underlying relativity theory [10b]). Hence we have to take instead of (2) (if the coordinates are orthogonal)

$$\mathcal{H} \equiv -p_0+f_0-c\sqrt{m^2c^2-g^{\alpha\beta}(p_\alpha-f_\alpha)(p_\beta-f_\beta)}=0, \quad . \quad (3)$$

which is equivalent with

$$\overline{\mathcal{H}}=0,\ p_0-f_0<0.$$

Then both in classical and in relativistic mechanics the following conditions C., D. are satisfied, which are the sole properties of $\mathcal{H}$ which shall be used.

C. *For $q^\varkappa=$ const. and $\mathcal{H}=0$ $\varkappa_0$ has a finite upper boundary.*

D. *For $q^\varkappa=$ const., $\varkappa_0=$ const. and $\mathcal{H}=0$ $\underset{\alpha=1,\ldots,n}{Max}\ |p_\alpha|$ has a finite upper boundary.*

Conditions C., D. are supposed to be satisfied for every observer.

Then X roughly has the same form as a paraboloid or half a hyperboloid.

3. LIOUVILLE's *element.*

Let $\mathrm{d}\Phi$ be an element of Φ, $\mathrm{d}\mathfrak{g}^\varkappa$ its projection upon the $(n+1)$-dimensional sub-space $q^\varkappa=$ *const.* (i.e. K) and $\mathrm{d}\mathfrak{B}_\lambda$ the set of all $\varkappa$-spaces having with $\mathrm{d}\Phi$ a point in common (i.e. the "projection" of $\mathrm{d}\Phi$ upon Σ) [11]). Then $\mathrm{d}\mathfrak{g}^\varkappa$ and $\mathrm{d}\mathfrak{B}_\lambda$ are n-dimensional elements and

$$\mathrm{d}\Phi = \mathrm{d}\mathfrak{g}^\varkappa\,\mathrm{d}\mathfrak{B}_\varkappa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

is the generalisation [12]) of LIOUVILLE's element

$$\mathrm{d}\Phi = h^{-n}\,\mathrm{d}p_1 \ldots \mathrm{d}p_n\,\mathrm{d}q^1 \ldots \mathrm{d}q^n \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

10b) Formally this means that, t^h being an arbitrary time-like unit-vector pointing towards future, the equations of relativistic physics ought to be invariant under those transformations only which leave the fundamental differential form $ds^2=g_{ij}\,dx^i\,dxj$ and the discontinuous scalar $\iota\left(t_i\,dx^i-\sqrt{-ds^2+(t_i\,dx^i)^2}\right)$ invariant, $\iota(x)$ being the discontinuous factor (7). This scalar remains unaltered if t^h is replaced by another vector t'^h with the same properties, i.e. with $t'^i\,t'_i=+1$, $t'_i\,t^i>0$.

11) By means of the conditions A. and B. the projections of $\mathrm{d}\Phi$ are provided with exterior orientations, so that under transformations of the $q^\varkappa$ $\mathrm{d}q^\varkappa$ and $\mathrm{d}\mathfrak{B}_\varkappa$ are a contravariant W-vectordensity of weight $+1$ and a covariant W-vectordensity of weight -1 respectively. Therefore also $\mathfrak{N}^\varkappa$, $\mathfrak{S}^\varkappa$, $\mathfrak{Z}^\varkappa$, $\mathfrak{P}^\varkappa{}_\lambda$, $\mathfrak{p}$ and $\mathfrak{s}$ are W-densities of weight $+1$.

Cf. J. A. SCHOUTEN, Ueber die geometrische Deutung von gewöhnlichen p-Vektoren und W—p-Vektoren und den korrespondierenden Dichten. These Proc. **41**, 709—716 (1938). J. A. SCHOUTEN and D. VAN DANTZIG, On ordinary quantities and W-quantities. Classification and geometrical applications. To appear in Compositio Mathematica.

O. VEBLEN and J. H. C. WHITEHEAD, The Foundations of differential geometry, p. 55. (Cambridge, 1932).

12) CARTAN l.c. 10) in different notation.

As $\mathcal{H}=0$ on Φ, $\mathrm{d}\,\mathfrak{g}^\varkappa$ is an element of X, hence $\mathrm{d}\,\mathfrak{g}^\lambda :: \dfrac{\partial\,\mathcal{H}}{\partial\,\varkappa_\lambda} = h\dfrac{\partial\,\mathcal{H}}{\partial\,p_\lambda}$ [13]) (the "gradient" of $\mathcal{H}$ with respect to the $\varkappa_\lambda$). Moreover one set of HAMILTON's equations states that $\dfrac{\partial\,\mathcal{H}}{\partial\,p_\varkappa} :: dq^\varkappa$, where $dq^\varkappa$ is an element of the worldline of the molecule under consideration. Hence

$$\mathrm{d}\,\mathfrak{g}^\varkappa :: dq^\varkappa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is equivalent with $\mathrm{d}\,\mathfrak{g}^\varkappa = \dfrac{dq^\varkappa}{dq^0}\,\mathrm{d}\,\mathfrak{g}^0$.

4. *Integrals on phase-space.*

For a function $\tilde{f}=\tilde{f}(\varkappa_\alpha, q^\varkappa)$ in an infinity of ways a function $f=f(\varkappa_\lambda, q^\varkappa)$ can be found, reducing to $\tilde{f}$ for $\mathcal{H}=0$. We restrict ourselves to such functions which satisfy condition :

E. *For $q^\varkappa=$ const., $\varkappa_\alpha=$ const. and $\varkappa_0\to-\infty$, f tends exponentially towards zero.* In all practically important cases this condition is satisfied. We further suppose that a definite f reducing to $\tilde{f}$ for $\mathcal{H}=0$ and satisfying E. has been chosen.

The integral of $\tilde{f}$ over Φ can be split up by (4):

$$\int_\Phi \tilde{f}(\varkappa_\alpha, q^\varkappa)\,\mathrm{d}\,\Phi = \int_\Gamma \mathfrak{f}^\varkappa\,\mathrm{d}\,\mathfrak{B}_\varkappa; \quad \mathfrak{f}^\varkappa = \int_X \tilde{f}\,\mathrm{d}\,\mathfrak{g}^\varkappa \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Moreover, introducing the function $\iota(x)$ of a real variable x:

$$\iota(x) = \begin{cases} 1 & x \geqq 0 \\ 0 & x < 0 \end{cases} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

we have

$$\left.\begin{aligned} \int_K \frac{\partial f}{\partial \varkappa_\varkappa}\,\iota(\mathcal{H})\,\mathrm{d}\,\mathfrak{k} = \int_{h\varkappa_0+H\leqq 0} \frac{\partial f}{\partial \varkappa_\varkappa}\,\mathrm{d}\,\mathfrak{k} = \\ = \int_X (f)_{\varkappa_0=-H/h}\,\mathrm{d}\,\varkappa_0 \ldots \mathrm{d}\,\varkappa_{\varkappa-1}\,\mathrm{d}\,\varkappa_{\varkappa+1}\ldots\mathrm{d}\,\varkappa_n = \int_X \tilde{f}\,\mathrm{d}\,\mathfrak{g}^\varkappa. \end{aligned}\right\} \quad . \quad . \quad (8)$$

Hence by (6)

$$\mathfrak{f}^\varkappa = \int_X \tilde{f}\,\mathrm{d}\,\mathfrak{g}^\varkappa = \int_K \frac{\partial f}{\partial \varkappa_\varkappa}\,\iota(\mathcal{H})\,\mathrm{d}\,\mathfrak{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The dynamic properties of the system are completely described by the function $\iota(\mathcal{H})$ which for constant $q^\varkappa$ is characterised by

$$\iota(\mathcal{H}) = \begin{cases} 1 & \text{inside and on } X, \\ 0 & \text{outside } X. \end{cases} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

[13]) :: means "is proportional with".

We could even take $\iota(\mathcal{H})$ as a definite specialisation of $\mathcal{H}$ if we would allow $\mathcal{H}$ not to be differentiable which we however prefer to do not.

We consider a gas consisting of molecules, each of which has $\mathcal{H}=0$ as its HAMILTONian equation, and without interactions, the only exchange of momentum and energy occurring by impacts. We suppose all elements $\mathrm{d}\,\Phi$ etc. to be sufficiently large to correspond with ("contain") many molecules [14]). If then $N^{\mathrm{d}\,\Phi}=\tilde{f}\,\mathrm{d}\,\Phi$ is the number of molecules in $\mathrm{d}\,\Phi$ (which we suppose to be described by means of a continuous differentiable function $\tilde{f}=\tilde{f}(\varkappa_\alpha, q^\varkappa)$ corresponding with just such an $f=f(\varkappa_\lambda, q^\varkappa)$, satisfying E.), we have by (6), (9), writing $\mathfrak{N}^\varkappa$ instead of $f^\varkappa$ [15]):

$$N^{\mathrm{d}\,V} = \mathfrak{N}^\varkappa\,\mathrm{d}\,\mathfrak{B}_\varkappa, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\mathfrak{N}^\varkappa = \int \tilde{f}\,\mathrm{d}\,\mathfrak{g}^\varkappa = \int \frac{\partial f}{\partial \varkappa_\varkappa}\,\iota(\mathcal{H})\,\mathrm{d}\,\mathfrak{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Here and further all integrals are taken over whole X or whole K respectively. (The $\sim$ indicating the value for $\mathcal{H}=0$ will further be omitted). The latter ones reduce by the factor $\iota(\mathcal{H})$ to integrals over *the interior of X.*

The total momentum and energy [16]) [17]) of all particles contained in $\mathrm{d}\,V$, the momentum and energy of an arbitrary particle being p_λ is

$$P_\lambda^{\mathrm{d}\,V} = \mathfrak{P}^\varkappa_{.\lambda}\,\mathrm{d}\,\mathfrak{B}_\varkappa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

with

$$\mathfrak{P}^\varkappa_{.\lambda} = \int p_\lambda f\,\mathrm{d}\,\mathfrak{g}^\varkappa = h\int\left(\frac{\partial}{\partial\,\varkappa_\varkappa} f\varkappa_\lambda\right)\iota(\mathcal{H})\,\mathrm{d}\,\mathfrak{k} \quad . \quad . \quad . \quad (12a)$$

Hence introducing

$$\boxed{\mathfrak{p} = h\int f\,\iota(\mathcal{H})\,\mathrm{d}\,\mathfrak{k}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

[14]) By $\mathrm{d}\,\Phi$ etc. domains in phase-space are denoted, the volumes of which are large compared with 1 and corresponding with ("containing") many particles, but such that for all functions under consideration (e.g. p_λ, f, $\mathfrak{g}^\varkappa$ etc.) the total variation on the element is small compared with the values the function takes on it. "Small" and "large" are to be understood in a practical sense, where e.g. $10^{-8} \ll 1$.

[15]) In expressions like $N^{\mathrm{d}\,\Phi}$, W^{Φ_n}, $P_\lambda^{\mathrm{d}\,\mathfrak{B}}$ etc. $\mathrm{d}\,\Phi$, Φ_n, $\mathrm{d}\,\mathfrak{B}$ are no exponents but arguments of which $N^{\mathrm{d}\,\Phi}$ etc. are functions. Only in the right hand of the explicit expressions for W_n on pag. 616, 617 Φ_n is an exponent.

[16]) The energy occurs always with a factor -1.

[17]) It would have been consequent to replace $P_\lambda^{\mathrm{d}\,V}$ also by $K_\lambda^{\mathrm{d}\,V}=h^{-1}\,P_\lambda^{\mathrm{d}\,V}$. This would change $\mathfrak{P}^\varkappa_{.\lambda}$, $\mathfrak{p}$, π_λ and $\mathfrak{g}^\varkappa$ by a factor h or h^{-1}. In order not to deviate too far from the current notation this however has not been done.

we obtain with $A_\lambda^\varkappa = \begin{cases} 1, & \varkappa = \lambda \\ 0, & \varkappa \neq \lambda \end{cases}$

$$\mathfrak{P}_{\cdot\lambda}^{\varkappa} = \mathfrak{p}\, A_\lambda^\varkappa + h \int \varkappa_\lambda \frac{\partial f}{\partial \varkappa_\varkappa}\, \iota(\mathcal{H})\, \mathrm{d}\mathfrak{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The quantity $\mathfrak{p}$ is the pressure of the gas, whereas $\mathfrak{P}_{\cdot\lambda}^{\varkappa}$ is the complete tensor-density of stress, momentum and energy.

[Added in proof, 8/8 '39. — These quantities are expressed here as functions of *all* coordinates. Particularly important is the case where the HAMILTONian $H = H_1 + H_2$, where H_1 depends upon one group of coordinates and their momenta alone, the *exterior* ones (to which also the time-coordinate q^0 belongs), whereas H_2 depends upon the other, the *interior* coordinates and their momenta alone. We suppose that for $H_2 =$ const. the "interior" coordinates and their momenta remain absolutely bounded, and that H_2 is an *even* function of its arguments. Finally let the function f be independent of the interior coordinates and their momenta. (Of course the function $\tilde{f}$ in (6), (9), i.e. f in the second member of (11), (12a) will depend upon them, as $\mathcal{H}$ does). Then the *ordinary* particle-current, pressure and stress-tensor, which are functions of the *exterior* coordinates alone, are obtained by integrating $\mathfrak{N}^\varkappa$, $\mathfrak{p}$ and $\mathfrak{P}_{\cdot\lambda}^{\varkappa}$ as defined above over the *whole* space of the *interior* coordinates.

We can obtain the same result in a more simple way which leaves all equations unaltered by slightly changing the notations. Let therefore $q^\varkappa$ denote the *exterior* coordinates *alone* (including q^0) and let q^ξ ($\xi, \eta = n+1, \ldots, n+f$) denote f *further* coordinates, the interior ones. Let the manifolds introduced in § 2 have the meaning indicated in the table by the quantities between brackets, so that Σ and Γ refer to the exterior coordinates alone, whereas Ω, Φ, K, X include the whole phase-space $\Phi_{(i)}$ of the *interior* coordinates and their momenta. Accordingly

$$\mathrm{d}\mathfrak{g}^\varkappa = \mathrm{d}\varkappa_0 \ldots \mathrm{d}\varkappa_{\varkappa-1}\, \mathrm{d}\varkappa_{\varkappa+1} \ldots \mathrm{d}\varkappa_n\, \mathrm{d}\Phi_{(i)},$$
$$\mathrm{d}\mathfrak{k} = \mathrm{d}\varkappa_0 \ldots \mathrm{d}\varkappa_n\, \mathrm{d}\Phi_{(i)},$$
$$\mathrm{d}\Phi_{(i)} = \mathrm{d}\varkappa_{n+1} \ldots \mathrm{d}\varkappa_{n+f}\, \mathrm{d}q^{n+1} \ldots \mathrm{d}q^{n+f}.$$

Then $\mathfrak{N}^\xi$, $\mathfrak{P}_{\cdot\eta}^{\xi}$, $\mathfrak{P}_{\cdot\lambda}^{\xi}$ vanish because of $\partial f/\partial \varkappa_\xi = 0$, and $\mathfrak{P}_{\cdot\eta}^{\xi} = 0$ as $\varkappa_\eta\, \iota(\mathcal{H})$ is an uneven function of the $\varkappa_\xi$. The only surviving components $\mathfrak{N}^\varkappa$, $\mathfrak{p}$, $\mathfrak{P}_{\cdot\lambda}^{\varkappa}$ are determined by (11), (13), (14) directly as functions of the exterior coordinates alone.

With this generalised meaning of $\mathrm{d}\mathfrak{g}^\varkappa$ and $\mathrm{d}\mathfrak{k}$ also the rest of the paper remains valid.]

5. *Probability and entropy.*

Let $\Phi_0, \Phi_1 \ldots$ etc. denote domains in phase-space Φ. We suppose

for each function $f_0(\varkappa_\lambda, q^\varkappa)$, satisfying E., and each domain Φ_0 a positive quantity $W^{\Phi_0}[f_0]$ to be given, being the relative probability that a function $f = f(\varkappa_\lambda, q^\varkappa)$ (always satisfying E.) coincides with f_0 for each point of Φ_0. We restrict it by condition:

F. If Φ_1 and Φ_2 have no point in common and $\Phi_1 + \Phi_2$ is their union, then

$$W^{\Phi_1}[f_0] \,.\, W^{\Phi_2}[f_0] = W^{\Phi_1+\Phi_2}[f_0]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

This means that "$f = f_0$ within Φ_1" and "$f = f_0$ within Φ_2" are independent conditions for f. Then (dropping further the suffix $_0$) $S^{\Phi_0}[f] = log\, W^{\Phi_0}[f]$ is an additive set-function, which moreover is a functional of f. It is called the *amount of entropy* of a gas with state-function f, contained in Φ_0.

We finally suppose:

G. *A continuous differentiable function $\sigma[f]$ of $p_\lambda, q^\varkappa$ (functional of f) exists, such that*

$$S^{\Phi_0}[f] = \int_{\Phi_0} \sigma[f]\, \mathrm{d}\Phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

or shortly, dropping the argument f altogether: $S^{\mathrm{d}\Phi} = \sigma\, \mathrm{d}\Phi$.

Again by (6), (9)

$$S^{\mathrm{d}V} = \mathfrak{S}^\varkappa\, \mathrm{d}\mathfrak{B}_\varkappa, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\mathfrak{S}^\varkappa = \int \sigma\, \mathrm{d}\mathfrak{g}^\varkappa = \int \frac{\partial \sigma}{\partial \varkappa_\varkappa}\, \iota(\mathcal{H})\, \mathrm{d}\mathfrak{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

6. *Statistical equilibrium.*

A function $f(\varkappa_\lambda, q^\varkappa)$ is called a *state of statistical equilibrium* if $\delta S^{\mathrm{d}V} = 0$ for all $\mathrm{d}V$ and all variations δf of the function f (under constant $\varkappa_\lambda$) satisfying the conditions $\delta N^{\mathrm{d}V} = 0$, $\delta P_\lambda^{\mathrm{d}V} = 0$ for each $\mathrm{d}V$. According to LAGRANGE's method multiplicators $\vartheta^\varkappa, \lambda$ exist, such that

$$\delta S^{\mathrm{d}V} + \vartheta^\lambda\, \delta P_\lambda^{\mathrm{d}V} + \lambda\, \delta N^{\mathrm{d}V} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

which is equivalent with

$$\delta \mathfrak{S}^\varkappa + \vartheta^\lambda\, \delta \mathfrak{P}_{\cdot\lambda}^{\varkappa} + \lambda\, \delta \mathfrak{N}^\varkappa = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and also, introducing

$$\beta = p_\lambda\, \vartheta^\lambda + \lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

with $\delta\sigma + \beta\, \delta f = 0$ [18]). Hence σ is a *function*, not only a functional, of f, satisfying

$$\frac{d\sigma}{df} = -\beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

[18]) As (19) must hold for *every* element $\mathrm{d}\Phi$.

Introducing

$$\zeta = \sigma + \beta f = \sigma - f\frac{d\sigma}{df}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$\mathfrak{Z}^x = \int \zeta \, d\mathfrak{g}^x = \mathfrak{S}^x + \vartheta^\lambda \mathfrak{P}^x_{.\lambda} + \lambda \mathfrak{N}^x, \quad . \quad . \quad . \quad . \quad (24)$$

we obtain

$$\delta\zeta = f\delta\beta = f\,.\,(p_\lambda\,\delta\vartheta^\lambda + \delta\lambda)$$

as the function f is varied under constant p_λ. Hence ζ is a function of β, hence also of f [19]) and

$$\frac{d\zeta}{d\beta} = f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$\sigma = \zeta - \beta f = \zeta - \beta\frac{d\zeta}{d\beta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Hence σ and ζ are related by a transformation of LEGENDRE. By (25), (21) we have

$$\frac{\partial\zeta}{\partial\lambda} = f, \qquad \frac{\partial\zeta}{\partial\vartheta^\lambda} = f p_\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

hence

$$\frac{\partial\mathfrak{Z}^x}{\partial\lambda} = \int\frac{\partial\zeta}{\partial\lambda}\,d\mathfrak{g}^x = \int f\,d\mathfrak{g}^x = \mathfrak{N}^x \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$\frac{\partial\mathfrak{Z}^x}{\partial\vartheta^\lambda} = \int\frac{\partial\zeta}{\partial\vartheta^\lambda}\,d\mathfrak{g}^x = \int f p_\lambda\,d\mathfrak{g}^x = \mathfrak{P}^x_{.\lambda}. \quad . \quad . \quad . \quad . \quad (29)$$

in accordance with Rel. Therm. (15).

Moreover

$$\mathfrak{Z}^x = \int\zeta\,d\mathfrak{g}^x = \int\frac{\partial\zeta}{\partial\varkappa_x}\,\iota(\mathcal{H})\,d\mathfrak{k} = h\int\frac{d\zeta}{d\beta}\frac{\partial\beta}{\partial p_x}\,\iota(\mathcal{H})\,d\mathfrak{k},$$

hence by (25), (13), (21),

$$\boxed{\mathfrak{Z}^x = \mathfrak{p}\,\vartheta^x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Then (28), (29) become

$$\mathfrak{N}^x = \frac{\partial\mathfrak{p}}{\partial\lambda}\,\vartheta^x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$\mathfrak{P}^x_{.\lambda} = \frac{\partial\mathfrak{p}}{\partial\vartheta^\lambda}\,\vartheta^x + \mathfrak{p}\,A^x_\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

[19]) The trivial and physically meaningless case that σ is linear in f is excluded. It would lead to $\delta\vartheta^x = \delta\lambda = 0$ for all variations of the state.

which is equivalent with the condition for a perfectly-perfect fluid [20]). Hence we have found:

All thermodynamic quantities can be obtained by differentiation and algebraic processes from the pressure $\mathfrak{p}$ given by (13). *The latter can be found by quadratures only, as soon as the two functions $\mathcal{H}(p_\lambda, q^x)$* (determining the dynamical properties of the molecules) and *$f(\varkappa_\lambda, q^x)$* (determining the kind of statistics) *are known.*

By (24), (30), (31), (32) we have

$$\mathfrak{S}^x = \mathfrak{s}\,\vartheta^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$\mathfrak{s} = -\left(\vartheta^\lambda\frac{\partial\mathfrak{p}}{\partial\vartheta^\lambda} + \lambda\frac{\partial\mathfrak{p}}{\partial\lambda}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

7. *Special statistics.*

M.B. (MAXWELL—BOLTZMANN-*statistics*). Let N particles be distributed over m domains Φ_u $(u = 1, \ldots, m)$ of phase-space, each Φ_u having a volume denoted by Φ_u also and containing N_u particles, so that $N = \Sigma N_u$. The probability of a definite distribution is $W = N!\prod_{u=1}^{m}\frac{\Phi_u^{N_u}}{N_u!}$. We drop the constant factor $N!$ (which means addition of a constant to the entropy). Then $W = \Pi W_u$ where, using STIRLING's formula in the form $N! \sim N^N e^{-N}$,

$$W_u = \frac{\Phi_u^{N_u}}{N_u!} \sim \left(\frac{\Phi_u\,e}{N_u}\right)^{N_u}.$$

For sufficiently small Φ_u (though not *too* small, as N_u must remain large! Cf. [14])) $N_u \sim f\Phi_u$, $W_u = W^{\Phi_u}[f]$, $S^{\Phi_u} = log\, W^{\Phi_u} \sim \sigma\,\Phi_u$, hence

$$\sigma = f - f\,log\,f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35, \text{M. B})$$

Hence $\beta = -\frac{d\sigma}{df} = log\,f$ and

$$f = e^\beta = e^{\vartheta^\lambda p_\lambda + \lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

which leads to MAXWELL's distribution law. (We obtain the latter for $\vartheta^\alpha = 0$, $p_0 = -\left(\frac{1}{2m}p^2 + P\right)$. Hence $\sigma = (1-\beta)\,e^\beta$, $\zeta = e^\beta = f$. Hence $\mathfrak{Z}^x = \mathfrak{N}^x$ or

$$\mathfrak{N}^x = \mathfrak{p}\,\vartheta^x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

which is BOYLE—GAY—LUSSAC's law [21]). Therefore this law is a consequence of the M.B.-statistics alone and is independent of the function $\mathcal{H}$. In particular this independence leads to the well known fact that

[20]) Cf. Rel. Therm. (4), (17) and D. VAN DANTZIG, On the phenomenological thermodynamics of moving matter, Physica **6**, p. 673—704 (1939), referred to as Phen. Therm., eq. (43), (64).

[21]) Cf. l.c. [20]) (75).

BOYLE—GAY—LUSSAC's law holds in relativity theory as well as in classical theory.

E.B. (EINSTEIN—BOSE). Here $W = \Pi W_u$, where W_u is the number of symmetrical products of N_u proper functions over Φ_u states. Hence as $\Phi_u \rangle\rangle 1$

$$W_u = \binom{N_u + \Phi_u - 1}{N_u} \sim \frac{(N_u + \Phi_u)^{N_u + \Phi_u}}{N_u^{N_u}\, \Phi_u^{\Phi_u}} \sim \left(\frac{(f+1)^{f+1}}{f^f}\right)^{\Phi_u}$$

and

$$\sigma = (1+f)\log(1+f) - f\log f. \quad . \quad . \quad . \quad (38, \text{E. B.})$$

F.D. (FERMI—DIRAC). Taking skew-symmetrical instead of symmetrical products we obtain

$$W_u = \binom{\Phi_u}{N_u} \sim \frac{\Phi_u^{\Phi_u}}{N_u^{N_u}\,(\Phi_u - N_u)^{\Phi_u - N_u}} \sim \left(\frac{1}{f^f (1-f)^{1-f}}\right)^{\Phi_u}$$

and

$$\sigma = -(1-f)\log(1-f) - f\log f. \quad . \quad . \quad . \quad (38, \text{F. D.})$$

We can gather the three equations (35, M.B.), (38, E.B., F.D.), into one viz.

$$\boxed{\sigma = (1+\varepsilon f)\frac{\log(1+\varepsilon f)}{\varepsilon} - f\log f}, \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where

$$\varepsilon = \begin{cases} +1 & \text{(EINSTEIN-BOSE)}, \\ 0 & \text{(MAXWELL-BOLTZMANN)}\ ^{22}), \\ -1 & \text{(FERMI-DIRAC)}. \end{cases} \quad . \quad . \quad . \quad . \quad (40)$$

By (40) $\beta = -\frac{d\sigma}{df} = \log\frac{f}{1+\varepsilon f}$, or [23])

$$\boxed{f = \frac{1}{e^{-\beta} - \varepsilon}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Finally we obtain

$$\zeta = -\frac{1}{\varepsilon}\log(1 - \varepsilon e^{\beta}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

[22]) Under M.B.-statistics we have always to take the *limit* for $\varepsilon \to 0$.

[23]) In order that σ and ζ remain real f must be > 0 (and < 1 for F.D.-statistics). For M.B.- and F.D.-statistics this is always the case, and for E.B.-statistics if and only if $\beta < 0$. We further suppose this condition to be satisfied.

With

$$\gamma_0 = \gamma_1 = 1,\ \gamma_n = \frac{1}{n!}(1+\varepsilon)(1+2\varepsilon)\ldots(1+(n-1)\varepsilon) \quad (n \geqq 2) \quad (43)$$

we have

$$e^{\zeta} = \sum_{n=0}^{\infty} \gamma_n e^{n\beta} = \begin{cases} \dfrac{1}{1-e^{\beta}} = \sum\limits_{n=0}^{\infty} e^{n\beta} & \text{(E. B.)} \\ e^{e^{\beta}} = \sum\limits_{n=0}^{\infty} \dfrac{1}{n!} e^{n\beta} & \text{(M. B.)} \\ 1 + e^{\beta} = \sum\limits_{n=0}^{\infty} \left[\dfrac{1}{n!}\right] e^{n\beta} & \text{(F. D.)}\ ^{24}) \end{cases} \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

The quantity ζ is closely related to the "partition-function" introduced by DARWIN and FOWLER.

In fact, if the quantities $x = e^{\lambda}$, $y_\alpha = e^{\vartheta^\alpha}$, $z = e^{-\vartheta^0}$ (which of course have no simple transformation law) are taken as independent variables, $e^{\zeta} = F(x\, y_1^{p_1}\, y_2^{p_2}\, y_3^{p_3}\, z^{\varepsilon})$ $(\varepsilon = -p_0)$ is exactly DARWIN and FOWLER's partition-function [25]) for the case under consideration. By our considerations it is seen that F is not only a purely formal function useful for obtaining the thermodynamic quantities, but is itself immediately connected with these quantities, in particular with the pressure. Moreover it is easily seen that also DARWIN and FOWLER's state-integrals are simplified if their independent variables are replaced by their logarithms, i.e. in this particular case by our λ and $\vartheta^\varkappa$.

8. *Relativistic gas theory.*

We specialise for the case $n = 3$ [26]), choose for $q^\varkappa$ locally orthogonal coordinates in space-time such that locally $\vartheta^1 = \vartheta^2 = \vartheta^3 = 0$ and suppose $\mathcal{H}$ to be given by (3). As f depends on $\beta = \lambda + \vartheta^\lambda p_\lambda$ alone we see that $\mathfrak{p}$ and therefore all thermodynamic quantities remain invariant under the transformation $p_\lambda \to p_\lambda - f_\lambda$, $\lambda \to \lambda + \vartheta^\lambda f_\lambda$. We can therefore without restriction suppose $f_\lambda = 0$. Then $\mathcal{H} = -p_0 - H$, $H = c\sqrt{m^2 c^2 + h^2 \varkappa^2}$, where $\varkappa^2 = \varkappa_1^2 + \varkappa_2^2 + \varkappa_3^2$.

Hence by (13) and (41)

$$\left.\begin{aligned} \mathfrak{p} &= h \iiint\limits_{\mathcal{H} \geqq 0} \frac{d\varkappa_0\, d\varkappa_1\, d\varkappa_2\, d\varkappa_3}{e^{-\beta} - \varepsilon} = 4\pi h \int_0^{\infty} \varkappa^2\, d\varkappa \int_{-\infty}^{-H/h} \frac{d\varkappa_0}{e^{-\lambda - h\vartheta^0 \varkappa_0} - \varepsilon} = \\ &= -\frac{4\pi}{\varepsilon \vartheta^0} \int_0^{\infty} \varkappa^2 \ln\left(1 - \varepsilon\, e^{\lambda - \vartheta^0 c\sqrt{m^2 c^2 + h^2 \varkappa^2}}\right) d\varkappa, \end{aligned}\right\} \quad (45^*)$$

[24]) $[x]$ denotes the integer part of x.

[25]) Cf. e.g. R. H. FOWLER, Statistical Mechanics, Cambridge, 2nd Ed. (1938).

[26]) And $f = 0$, i.e. mono-atomic gases.

or

$$\mathfrak{p} = a\,F(\lambda, \tau, \varepsilon) \ldots\ldots\ldots\ldots (45)$$

where

$$a = 4\pi\, m^4\, c^5\, h^{-3}, \ldots\ldots\ldots\ldots (46)$$

$$\tau = m\,c^2\,\vartheta^0 = m\,c^2/k\,T, \ldots\ldots\ldots\ldots (47)$$

(T is here equal to the proper temperature T_0 as the gas is at rest) and [27])

$$\left.\begin{aligned} F(\lambda, \tau, \varepsilon) &= -\frac{1}{\varepsilon\tau}\int_0^\infty x^2\, ln\,(1-\varepsilon\, e^{\lambda-\tau\sqrt{1+x^2}})\,dx = \\ &= \frac{1}{3}\int_0^\infty \frac{sh^4\,u}{e^{-\lambda+\tau\,ch\,u}-\varepsilon}\,du \end{aligned}\right\} \ldots (48)$$

The expression (45), (48) for $\mathfrak{p}$ has been found by F. JÜTTNER [28]) who also gave expansions for the function $F(\lambda, \tau, \varepsilon)$ and its derivatives [29]). For M.B.-statistics ($\varepsilon = 0$) we have in particular

$$\left.\begin{aligned} F(\lambda, \tau, 0) &= \tfrac{1}{3}\, e^\lambda \int_0^\infty sh^4\,u\, e^{-\tau\,ch\,u}\,du = e^\lambda\,\tau^{-2}\int_0^\infty e^{-\tau\,ch\,u}\,ch\,2u\,du = \\ &= e^\lambda\,\tau^{-3}\int_0^\infty e^{-\tau\,ch\,u}\,(\tau + 2\,ch\,u)\,du = \frac{e^\lambda}{\tau^2}\,\frac{\pi}{2}\,\frac{H_2^{(1)}(i\,\tau)}{i}, \end{aligned}\right\} (49)$$

where $H_2^{(1)}$ is HANKEL's cylinder function of the first kind and the second order [29]).

Except in the hottest stars τ is always very large [30]). In that case $-i\,H_2^{(1)}(i\,\tau)$ becomes $e^{-\tau}\,(\tfrac{1}{2}\,\pi\tau)^{-\frac{1}{2}}\,g(\tau)$, where $g(\tau) = 1 + \dfrac{15}{8\,\tau} + \ldots$ is a semiconvergent

[27]) The condition $\beta < 0$ for E.B.-statistics (Cf. [23])) is satisfied for all $\varkappa_\lambda$ if and only if $\lambda < \tau$.

[28]) L.c. [5]) In order to compare JÜTTNER's notations with ours we mention that our λ, τ, ε, $g(\tau)$, $\varepsilon\,\tau\,F(\lambda, \tau, \varepsilon)$ are denoted by J. as $-\alpha = log\,\varepsilon$, β or γ, $-\eta$, $S_2(2\beta)$, $Q(\alpha, \gamma; \eta)$ respectively. It is further useful to mention that JÜTTNER's $M(\alpha, \gamma; \eta)$ and $L(\alpha, \gamma; \eta)$ are simply our $\tau\dfrac{\partial F}{\partial\lambda}$ and $-\left(F + \tau\dfrac{\partial F}{\partial\tau}\right)$, i.e. using his notations $-\eta^{-1}\dfrac{\partial Q}{\partial\alpha}$ and $-\eta^{-1}\dfrac{\partial Q}{\partial\gamma}$ respectively, as can be seen from our relations (28), (29).

[29]) Cf. E. JAHNKE and F. EMDE, Funktionentafeln, and JÜTTNER l.c. [2]), § 6—8.

[30]) E.g. for *He* $\tau \sim 4{,}32 \times 10^{13}\,T-1$ (Cf. l.c. [2])); moreover $a \sim 2{,}05 \times 10^{39}$. Hence in (50a) the first term is very large, e.g. for *He* at $T = 300$, $\tau = 1{,}44 \times 10^{11}$, whereas the second term at this temperature and for $\mathfrak{p} = 1$ atm $= 1{,}013 \times 10^6$ dn/cm^2 is $\sim -12{,}7$. The relativistic corrections, in particular the factor $g(\tau)$ may be of importance in the interior of the white dwarfs.

power series in τ^{-1} [29]). Hence we have then with a high degree of approximation, taking $g(\tau) \sim 1$:

$$\mathfrak{p} \sim a\sqrt{\tfrac{1}{2}\pi}\; e^{\lambda-\tau}\,\tau^{-\frac{5}{2}} = (2\pi m/h^2)^{\frac{3}{2}}\,(k\,T_0)^{+\frac{5}{2}}\,e^{\lambda-\frac{mc^2}{kT_0}}. \quad .. (50)$$

or

$$\lambda \sim \frac{m\,c^2}{k\,T_0} + log\,\frac{\mathfrak{p}\,h^3}{(2\pi m)^{\frac{3}{2}}\,(kT_0)^{\frac{5}{2}}} \ldots\ldots (50a)$$

Reintroducing the factor $g(\tau)$, the exterior field and arbitrary coordinates x^h in space-time, (50) becomes

$$\mathfrak{p} = (-\mathfrak{g})^{\frac{1}{2}}\,(2\pi m/h^2)^{\frac{3}{2}}\,(\vartheta^i\,\vartheta_i)^{-\frac{5}{4}}\,e^{\lambda+\vartheta^i f_i - mc^2\sqrt{\vartheta^i\,\vartheta_i}}\,g\,(mc^2\sqrt{\vartheta^i\,\vartheta_i}), \ldots (51)$$

where $\mathfrak{g} = det\,(g_{ij})$.

Hence we obtain for $\pi_i = \dfrac{\partial\,\mathfrak{p}}{\partial\,\vartheta^i}\Big/\dfrac{\partial\,\mathfrak{p}}{\partial\,\lambda} = \dfrac{1}{\mathfrak{p}}\,\dfrac{\partial\,\mathfrak{p}}{\partial\,\vartheta^i}$ [31]):

$$\pi_i = -mc^2\left(1 + \frac{5}{2\tau}\right) i_i + f_i + \frac{g'}{g}\,mc^2\,i_i, \ldots\ldots (52)$$

where i^h is a unit-vector along ϑ^h whereas $g' = \dfrac{dg}{d\tau}$, so that the last term in (52) is very small. Hence the entropy pro molecule [31]) is

$$\eta = \tfrac{5}{2} + \tau - f_i\,\vartheta^i - \lambda - \frac{g'\,\tau}{g}. \ldots\ldots\ldots (53)$$

and the proper heat-content pro molecule [31]) $\varepsilon \sim \dfrac{E_0 + \mathfrak{p}\,V}{N_0}$ if no exterior field is present and $g(\tau) - 1$ and g' are neglected:

$$\varepsilon = -\sqrt{\pi_i\,\pi^i} = mc^2 + \tfrac{5}{2}\,k\,T_0. \ldots\ldots (54)$$

Eliminating λ between (51), (53) we obtain generally:

$$\eta = \tfrac{5}{2} - log\,\frac{\mathfrak{p}\,h^3}{(2\pi m)^{\frac{3}{2}}\,(kT_0)^{\frac{5}{2}}} + log\,g - \frac{g'\,\tau}{g} =$$

$$= \left(\tfrac{5}{2} + log\,\frac{(2\pi m)^{\frac{3}{2}}\,k^{\frac{5}{2}}}{h^3}\right) - log\,\frac{\mathfrak{p}}{T^{\frac{5}{2}}} +$$

$$+ \left\{\tfrac{5}{4}\left(3\,\frac{kT}{mc^2} - \frac{u^2}{c^2}\right) - \tfrac{5}{16}\left(9\left(\frac{kT}{mc^2}\right)^2 - 6\,\frac{kT}{mc^2}\cdot\frac{u^2}{c^2} + 2\,\frac{u^4}{c^4}\right) + \ldots\right\},$$

where the first term is a characteristic constant for each (mono-atomic) gas (viz. the sum of $c_\mathfrak{p}$ at $T = 0$ and the chemical constant i), the second one is the principal term of ordinary thermodynamics and the last one, expanded in powers of $\tau^{-1} = kT/mc^2$ and u^2/c^2, is the relativistic correction, viz. $-\tfrac{5}{4}\,log\,(1 - u^2/c^2) + log\,g(\tau) - \tau g'(\tau)/g(\tau)$.

[31]) Cf. Phen. Therm. (65), (70); Rel. Therm. (4), (19), (17).

9. *The relativistic stress-tensor.*

By (51), (52) we find the stress-tensor

$$\mathfrak{P}^h_{.i} = -\mathfrak{p}\,\{(\tau + {}^5/_2 - g'\tau/g)\,i_i - f_i\tau/mc^2\}\,i^h + \mathfrak{p}\,A^h_i. \quad . \quad . \quad (55)$$

Comparing this expression for $f_i = 0$ with the well known expression

$$\mathfrak{P}^h_{.i} = -(\varrho + \mathfrak{p})\,i_i\,i^h + \mathfrak{p}\,A^h_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

we see that $\varrho + \mathfrak{p} = \mathfrak{p}\,(\tau + \frac{5}{2} - g'\tau/g)$ and

$$\varrho = \mathfrak{p}\left(\tau + \tfrac{3}{2} - \frac{g'\tau}{g}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

Here $\varrho = -\pi_i\,\mathfrak{P}^i_{.h}\,\vartheta^h/\pi_j\,\vartheta^j = -(\pi_i\,\mathfrak{N}^i + \mathfrak{p}) = -\mathfrak{p}\,(\pi_i\,\vartheta^i + 1)$ is SCHWARZSCHILD's value for the proper energy-density (except for a factor c^2 equal to EDDINGTON's ϱ_{00}). EDDINGTON [32]) has argued that ϱ/mc^2 can not be the proper particle-density, and that the latter must be ϱ_0/mc^2. where

$$\varrho_0 = -\mathfrak{P}^i_{.i} = \varrho - 3\,\mathfrak{p} = \mathfrak{p}\left(\tau - \tfrac{3}{2} - \frac{g'\tau}{g}\right) . . \quad . \quad . \quad . \quad (58)$$

We see that EDDINGTON's latter conclusion is incorrect, as has been remarked already by SYNGE [33]), who however reinstalled ϱ/mc^2 as the proper particle-density. But this also is incorrect, so that EDDINGTON's first conclusion was correct. In fact the proper particle-density, say $\bar{\varrho}/mc^2$, is

$$\bar{\varrho}/mc^2 = \mathfrak{N}^h\,i_h = \mathfrak{p}\,\vartheta^h\,i_h = \mathfrak{p}\,\tau/mc^2,$$

so that we have

$$\bar{\varrho} = \mathfrak{p}\,\tau. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Comparing this value with (57) and (58) we find

$$\bar{\varrho} = \tfrac{1}{2}\,(\varrho + \varrho_0) + \frac{g'\tau}{g}\,\mathfrak{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

or, to a first approximation, valid for not excessively high temperatures

$$\bar{\varrho} \sim \tfrac{1}{2}\,(\varrho + \varrho_0),$$

i.e. *the correct value of the particle-density of an ideal gas lies approximately in the middle between* SCHWARZSCHILD's *and* EDDINGTON's *values* [34]).

[32]) A. R. EDDINGTON, The mathematical theory of relativity, p. 122 (Cambridge 1924).

[33]) J. L. SYNGE, The energy tensor of a continuous medium. Trans. Roy. Soc. Canada (3) III, **28**, 127—171 (1934) § 10; Relativistic hydrodynamics. Proc. London Math. Soc. (2) **43**, 376—416 (1937).

[34]) The relative differences of ϱ, ϱ_0 and $\bar{\varrho}$ are of the order of τ^{-1}, and therefore are only of importance in the interior of the hottest stars. Moreover for radiation (where $\tau = 0$) $\bar{\varrho} = \varrho_0 = 0$, $\varrho > 0$. The quantities $\varrho = -\mathfrak{p} - \tau\,\partial\mathfrak{p}/\partial\tau$ and $\varrho_0 = 4\mathfrak{p} - \tau\,\partial\mathfrak{p}/\partial\tau$ are equal to the products of the constant a (46) and the integrals obtained from (48) by dropping the factor $\frac{1}{3}$ and replacing the numerator $sh^4\,u$ by $sh^2\,u\;ch^2\,u$ and $sh^2\,u$ respectively.

For higher temperatures however $\bar{\varrho}$ can not be deduced from $\mathfrak{P}^h_{.i}$ at all, as this would imply that $\mathfrak{N}^h = \mathfrak{p}\,\vartheta^h$ hence ϑ^h, hence the proper temperature were determined by the stress-tensor alone, which evidently is not the case.

10. *Black radiation.*

The principal properties of black radiation can immediately be deduced from our formulae as radiation can be considered as a fluid consisting of "photons" instead of molecules. The blackness of the radiation is equivalent with the perfectly-perfectness of the fluid.

It is mainly characterised by two properties, viz.

H. *The total number of photons in a volume of V is defined* [34a]).

I. *Photons move along zero-lines* [35]).

The first property implies that in § 6 $\int S^{dV}$ has to be varied without the condition $\delta \int N^{dV} = 0$, which loses its meaning. Hence (19) remains valid if we put

$$\lambda = 0 . . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

The quantities $\mathfrak{N}^\varkappa$, $\frac{\partial \mathfrak{p}}{\partial \lambda}$, $\pi_\mathfrak{p}$ and η become undetermined. The forms (32), (33) for the stress-tensor and the entropy-current and all relations in which none of the before-mentioned quantities occur however remain valid.

The second condition states that

$$\varkappa_\lambda\,dq^\lambda = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \quad (62)$$

along the rays of the radiation. By the equations of motion $dq^\lambda :: \frac{\partial \mathcal{H}}{\partial \varkappa_\lambda}$ [36]), hence (62) implies $\varkappa_\lambda \frac{\partial \mathcal{H}}{\partial \varkappa_\lambda} = 0$. By (5) (62) is equivalent with $\varkappa_\lambda\,\mathrm{d}\,\mathfrak{g}^\lambda = 0$ and therefore by the definition of $\mathfrak{P}^\varkappa_{.\lambda}$ it immediately leads to the well-known relation

$$\mathfrak{P}^\lambda_{.\lambda} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

[34a]) If $\mathfrak{p}'$ is *defined* by $\mathfrak{p}' = h \int f'\,(\beta)\,\epsilon\,(\mathcal{H})\,\mathrm{d}\,\mathfrak{k}$ (Cf. (13)), $\mathfrak{N}^h$ by $\mathfrak{N}^h = \mathfrak{p}'\,\vartheta^h$, etc. these quantities remain finite for radiation. No physical meaning however is to be seen which can be attributed to the formally possible interpretation of $\mathfrak{N}^h$ as "photonic current", of η as average entropy pro photon, etc.

[35]) On the basis of *wave-optics* this assumption of course is meaningless, as a photon has neither a definite place nor a definite motion. The present paper however is entirely based upon *classical* (relativistic) mechanics, completed with quantum statistics. This corresponds with *geometric* optics, completed also with quantum statistics. As we see here, this is sufficient for the deduction of all fundamental properties of radiation. (Wave-optics and wave-mechanics will be dealt with in another paper). In particular the dissolution of the radiation field into a system of harmonic oscillators is entirely irrelevant here.

[36]) More precisely: ($dq^\varkappa$ being an arbitrary element along the rays in space-time) $dq^\varkappa :: g^{\varkappa\lambda}\,\varkappa_\lambda$. This relation however need not yet be used here. It follows from (67) and is only necessary for PLANCK's law.

which here is deduced without making use of the form of the function $\mathcal{H}$.

Inserting (63) into (32) we obtain at once

$$\vartheta^\lambda \frac{\partial \mathfrak{p}}{\partial \vartheta^\lambda} = -(n+1)\,\mathfrak{p}, \quad \ldots \ldots \quad (64)$$

i.e. $\mathfrak{p}$ is homogeneous of degree $-n-1$ with respect to $\vartheta^\varkappa$. For $n=3$ therefore $\mathfrak{p}$ *is proportional with* $(T_0)^4$, which is STEFAN—BOLTZMANN*'s law*.

The energy-density with respect to the direction of $\vartheta^\varkappa$ as time-axis and that of $\frac{\partial \mathfrak{p}}{\partial \vartheta^\lambda}$ as momentary space is

$$\varrho = -\frac{\partial \mathfrak{p}}{\partial \vartheta_\varkappa}\,\mathfrak{P}^\varkappa_{.\lambda}\,\vartheta^\lambda\,\vartheta^\mu \Big/ \frac{\partial \mathfrak{p}}{\partial \vartheta^\mu} = -\left(\vartheta^\lambda \frac{\partial \mathfrak{p}}{\partial \vartheta^\lambda} + \mathfrak{p}\right).$$

By (63) this becomes

$$\varrho = n\,\mathfrak{p} \quad \ldots \ldots \quad (65)$$

i.e. for $n=3$ the well-known relation $\varrho = 3\,\mathfrak{p}$ [37]). Finally by (34), (61), (64)

$$\mathfrak{s} = -\vartheta^\lambda \frac{\partial \mathfrak{p}}{\partial \lambda} = (n+1)\,\mathfrak{p} = 4\,\mathfrak{p}. \quad \ldots \ldots \quad (66)$$

Equation (22) implies that f is a function of β alone. But here $\beta = \vartheta^\lambda p_\lambda = h\,\vartheta^\lambda \varkappa_\lambda = h\,\vartheta^0 \varkappa_0 (1 - u^\alpha \bar{u}_\alpha) = -\frac{h\nu}{kT}(1 - u^\alpha \bar{u}_\alpha)$, where $\nu = -\varkappa_0$ is the frequency, $u^\alpha = \frac{\vartheta^\alpha}{\vartheta^0}$ the average velocity and $\bar{u}_\alpha = -\frac{\varkappa_\alpha}{\varkappa_0}$ the wave-vector [38]). Hence f, which is a function of β alone, is a function of ν/T. This is WIEN*'s displacement-law*. Like STEFAN—BOLTZMANN's law and the relation $\varrho = 3\,\mathfrak{p}$ it is independent 1^0. of the function $\mathcal{H}$, i.e. of the form of the worldlines (rays) and 2^0. of the form of the function f, i.e. of the kind of statistics.

For unpolarised radiation the factor h in (13) has to be replaced by $2h$. Then (41) with $\varepsilon = +1$ (E.B.-statistics), $n=3$ and

$$\mathcal{H} = -h\,(\varkappa_0 + \sqrt{\varkappa_1^2 + \varkappa_2^2 + \varkappa_3^2}). \quad \ldots \ldots \quad (67)$$

(with respect to orthogonal coordinates) leads immediately to PLANCK's distribution law.

[37]) It is remarkable that the analogous relation $\mathfrak{u} = \frac{3}{2}\,\mathfrak{p}$ for the energy-density of an ideal gas on the contrary is only valid under very special conditions. Its deduction depends essentially upon the function f (M.B.-statistics) as well as $\mathcal{H}$ (non-relativistic mechanics) and even then it holds only if the potential energy as well as the mass-energy are substracted. In fact $\mathfrak{u}$ is the second term in (57), the first term being the sum of the proper energies of the molecules and the third term the relativistic correction. Analogous restrictions of course exist for the equality of the temperature and the average vis viva per degree of freedom and for the equipartition-law.

[38]) More precisely: with respect to orthogonal coordinates $\bar{u}_\alpha = -g_{\alpha\beta}\,u^\beta$, but this relation is not needed here.

In fact, in (45*), or also in the equivalent form

$$\mathfrak{p} = \tfrac{4}{3}\pi h^2 c \int_0^\infty \frac{\varkappa^4 (m^2c^2 + h^2\varkappa^2)^{-\frac{1}{2}}}{e^{-\lambda + \vartheta_0 c\sqrt{m^2c^2 + h^2\varkappa^2}} - \varepsilon}\,d\varkappa \quad \ldots \ldots \quad (68)$$

we can substitute at once $\lambda = 0$, $m = 0$, $\varepsilon = 1$. After insertion of the polarisation factor 2 we obtain:

$$\mathfrak{p} = \tfrac{8}{3}\pi h c \int_0^\infty \frac{\varkappa^3\,d\varkappa}{e^{\vartheta_0 h c \varkappa} - 1}\,d\varkappa \quad \ldots \ldots \quad (69)$$

which with $\varkappa c = \nu$ and $\varrho = 3\,\mathfrak{p}$ is equivalent with PLANCK's radiation formula.

Finally we draw attention to the fact that black radiation is characterised by a *vector* $\vartheta^\varkappa$ and therefore determines a definite direction in space-time, unless $\vartheta^\varkappa$ is zero or infinite.

Hence *black radiation of positive and finite temperature has a definite average velocity*. It has therefore an unambiguous meaning to speak of black radiation in rest or in motion. This is of importance because the radiation in the interstellar space is usually considered to have a positive temperature. This leads to the somewhat baffling conclusion that *notwithstanding the relativity principle the velocity of the earth with respect to "empty space" (i.e. the radiation field) has on every moment a definite value.*

The analogous relation with respect to the average velocity of the matter in the universe is well-known. This vector has only a cosmological meaning and can only be obtained by averageing over enormous spaces. Our $\vartheta^\varkappa$ however is defined locally. Its direction is the one for which the apparent temperature T is maximal. In order to determine it with respect to the motion of the earth, it would be necessary to determine the temperature of the radiation surrounding the earth with a relative error of about 10^{-8}. Apart from the fact that this value is much too small, it is rather doubtful whether the temperature of interplanetary radiation, even if it could be defined unambiguously [39]), were sufficiently constant in order to make it significant.

[39]) Cf. E. C. WIERSMA, Eenige punten uit de ontwikkeling van het temperatuurbegrip, 'sGravenhage 1936, p. 10. We have here to do with the temperature determined by the intensity of the radiation alone, which is about 3.2 °K.