Mathematics. — *A remark and a problem concerning the intuitionistic form of* CANTOR'S *intersection theorem.* By D. VAN DANTZIG. (Communicated by Prof. J. A. SCHOUTEN.)

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In classical mathematics the theorem proved in the preceding paper [1]) would imply PEANO'S theorem by an application of CANTOR'S theorem, which states that the intersection S of a decreasing sequence of compact non-empty subsets S_h is not empty [2]). An analogous application concerns the maxima of a real function $f(x)$, continuous for $a \leqq x \leqq b$. It is easy to construct for every natural h a set S_h consisting of a finite number > 0 of closed dyadic rational intervals, such that

A. $S_{h+1} \subset S_h$.
B. If $f(x) \leqq f(c)$ for all x with $a \leqq x \leqq b$, then $c \in S_h$ for every h.
C. If $c \in S_h$ for every h, then $f(x) \leqq f(c)$ for all x with $a \leqq x \leqq b$.

Denoting by $\forall$ the logical "all-symbol" and by $\exists$ the "existence-symbol" [2a]) (in the intuitionistic sense, hence requiring the possibility of explicit arbitrarily close approximation of the "existing" entity), the "strong interpretation" [3]) of CANTOR'S theorem would require

$$\exists x \, \forall h, x \in S_h, \quad \ldots \ldots \ldots \quad (1)$$

This, however, is certainly wrong, as in the case of the maximum of a real function is shown by a counter-example of BROUWER [4]) and in the case of PEANO'S theorem by the opening section of the preceding paper. At the other hand, denoting the negation-symbol by $\neg$, the strong interpretation of the negation of (1) would be

$$\forall x \, \exists h \neg, x \in S_h, \quad \ldots \ldots \ldots \quad (2)$$

A counter-example with this property, however, is not possible. For it would require the constructibility of a natural number h corresponding with *any* real number in the given interval. Hence, by BROUWER'S theorem [5]) such a number h could be determined simultaneously for all x, i.e.

$$\exists h \, \forall x \neg, x \in S_h, \quad \ldots \ldots \ldots \quad (3)$$

which is contradicted by the existence of at least one number x belonging to an arbitrarily given S_h.

1) D. VAN DANTZIG, On the affirmative content of PEANO'S theorem on differential equations. Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 367—373 (1942).

2) "Compact sets" may be interpreted here as "gecatalogiseerd compacte soorten" as defined by BROUWER, Proc. Kon. Akad. v. Wetensch., Amsterdam, **35**, 634—642, 677—678 (1927).

2a) Cf. A. HEYTING, Sber. Pr. Ak. v. Wiss. 1930, p. 42—71, 158—169. There brackets are used instead of the symbol $\forall$.

3) Cf. D. VAN DANTZIG, On the principles of intuitionistic and affirmative mathematics. This paper was written on bequest of the redaction of the Revista Matematica Hispano-Americana, and sent to the redaction in Febr./March 1941. Whether it has meanwhile appeared or not, could not be ascertained.

4) Cf. e.g. L. E. J. BROUWER, Wis- en Natuurk. Tijdschr. **2** (1923).

5) Proc. Kon. Akad. v. Wetensch., Amsterdam, **33**, 189—193 (1924).

Hence CANTOR'S theorem can only be valid in the "weak interpretation"

$$\neg \forall x \neg \forall h, x \in S_h, \quad \ldots \ldots \ldots \quad (4$$

whereas at the other hand a counter-example could only be possible also in the weak interpretation

$$\forall x \neg \forall h \neg \neg x \in S_h \ ^{6)} \quad \ldots \ldots \ldots \quad (5)$$

I have not succeeded, either in proving (4), nor in finding an example with (5). As it seems, the methods known to day in intuitionistic mathematics do not allow to decide between such purely negative statements like (4) and (5) [7]).

It is for this reason that I have hesitated so long to publish the preceding and the present paper. That I nevertheless have decided to publish them now has two reasons. The first of these is that if the difficulty is publicly signalised, other investigators might be induced to find general methods which allow the intuitionistic treatment of "stable statements" (i.e. statementts equivalent with their double negation) [8]), obtained by "weakening up" classical statements. This would be of importance, because it is the ideal of classical mathematicians to work with stable statements only, whereas in fact classical mathematics uses a peculiar mixture of negative and affirmative statements. The other reason is that the construction of the sets S_h mentioned above, as well as the corresponding construction given in the preceding paper is purely "affirmative" [9]), i.e. does not make use of any negations (nor of unrestricted existence statements either), and is in no way influenced by the proof of a purely negative statement like (4), nor even by its refutation (5). For (5) only causes that, starting with the successive approximation of a given real number x, the relation $x \in S_h$ can not be maintained always, without an upper limit being known for the number of the step at which the process is checked; the knowledge of such an upper limit can certainly not be ascertained for every x, as this would lead to (2).

6) As the statement $x \in S_h$ is "stable" (i.e. equivalent with its double negation), (5) can be replaced by $\forall x \neg \forall h, x \in S_h,$ which is the direct negation of (4), $\forall x \, \Omega[x]$ always being stable, if $\Omega[x]$ is, as it is the negation of $\exists x \neg \Omega[x]$.

7) With respect to the statements (4), (5) this was kindly confirmed to me by Dr. A. HEYTING.

8) Cf. l.c. [3]).

9) Cf. l.c. [3]).

10) Herein lies the difference between the proofs of PEANO'S theorem using CANTOR'S theorem (PEANO'S original proof, MIE), and those using BOLZANO–WEIERSTRASS' theorem (MONTEL, PERRON, a.o.): Though it is not possible in general to construct a point of the intersection, the decreasing sequence of compact sets itself, of which CANTOR'S theorem speaks, can be constructed. It is, however, in general impossible to construct not only the *limit* of a convergent subsequence, the existence of which is stated by BOLZANO–WEIERSTRASS, but even such a subsequence itself.