Mathematics. — *Comments on Brouwer's Theorem on Essentially-negative predicates.* By D. VAN DANTZIG.

(Communicated at the meeting of October 29, 1949.)

1. In a recent note [1]) L. E. J. BROUWER constructed a real number ϱ, which he proved to be $\neq 0$ (i.e. he proved $\varrho = 0$ to be contradictory), without it being possible to prove either $\varrho > 0$ or $\varrho < 0$. This note, however, requires some comments because its validity depends 1° on the way in which some terms, not defined in the paper, are interpreted and 2° on some assumptions, partly of a psychological nature, which are not explicitly mentioned. Moreover the paper 1° is based on an idealistic philosophy, which certainly will not be accepted by all readers and thereby may invalidate it at least in appearance, and 2° is formulated in a terminology, fluctuating between "subjectivistic" and "objectivistic" [2]) statements — if only in appearance — which may make it difficult for some readers to form a judgment of their own on its validity.

In the present paper I shall try 1° to loosen the result from its philosophical origine, 2° to point out the assumptions underlying it and thereby 3° to make it better understandable to logicists. If some mathematicians and logicists — like myself — have been in doubt during some time whether the theorem were correct or not, I hope this paper may show them that, properly interpreted, it is. Moreover in § 1 I shall try to point out some peculiarities of Brouwer's terminology, and thereby, I hope, clarify it to some extent. In particular I shall try to show by some examples, how the paper can be "translated", either into a more consistently "subjectivistic", or into a more consistently "objectivistic" or "formal" terminology.

2. BROUWER defines the number ϱ in the following terms (translated from his Dutch text).

"Let α be a mathematical assertion [3]), which *cannot be "tested"*,

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[1]) L. E. J. BROUWER, Essentieel-negatieve eigenschappen, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **51**, 963—964 (1948); reprinted in Indagationes Mathematicae **10**, 322—323. Cf. also L. E. J. BROUWER, Consciousness, Philosophy and Mathematics, Proc. 10th International Congr. of Phil., 1948, vol. **2**, p. 1235—1249.

[2]) I do not go here into the meaning of the terms 'objectivistic' and 'subjectivistic', corresponding with G. MANNOURY's distinction between "physicalistic" (after Carnap) and "introspective" terminology and consequently drop the quotation-marks further on.

[3]) As in intuitionistic mathematics a statement is only asserted if it has been proved, the term 'proposition' might express the author's intention more clearly than the term 'assertion' does.

i.e. no method is known [4]), in order to deduce either its absurdity [5]) or the absurdity of its absurdity [6])".

"Then the creating subject can create in connexion with this assertion α an undefinitely proceeding sequence of rational numbers $a_1, a_2, a_3, \ldots$ according to the following instruction: As long as, during the choice of the a_n, neither the truth, nor the absurdity of α has become evident to the creating subject, each a_n is chosen $= 0$. As soon, however, as, between the choice of a_{r-1} and of a_r, the truth of α has become evident to the creating subject, a_r as well as $a_{r+\nu}$ for every natural ν is chosen $= 2^{-r}$. And as soon as, between the choice of a_{s-1} and of a_s the absurdity of α has become evident to the creating subject, a_s as well as $a_{s+\nu}$ for each natural ν is chosen $= -2^{-s}$.

"This undefinitely proceeding sequence $a_1, a_2, a_3, \ldots$ is positively convergent, hence it determines a real number ϱ."

§ 1. *Terminological remarks.*

3. In order not to deviate further from Brouwer's point of view than is strictly unavoidable, I shall admit here without criticism Brouwer's notion "unfinished infinite" (aftelbaar onaf"). [7])

4. It is not clear from Brouwer's paper, whether the term 'creative subject' is intended to denote:

1° the author himself,
2° an arbitrary human individual,
3° a human individual possessing some (which?) intellectual qualifications,
4° an „infinite" sequence of such individuals, successively performing the activities, ascribed to the creative subject,
5° a more or less definite *group* of human individuals, e.g. all mathematicians possessing some definite qualification,

or anything else.

Clearly the interpretations 1°—3° leave open the possibility that the creative subject dies after having chosen a finite number of numbers a_n so that the sequence cannot be continued and the real number ϱ remains undetermined. If the groups of assertions which have "become evident" to the subject between two successive definitions a_n, a_{n+1} consist of all theorems published in some well-defined journals or books, the inter-

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[4]) BROUWER means: known to the creating subject.

[5]) The term 'absurd' as used by BROUWER, is meant as equivalent with "contrary to reason'; in Dutch it has not the meaning of 'nonsensical', it has in English.

[6]) "E.g. the assertion that 4 natural numbers $n > 2$, a, b and c exist, such that the relation $a^n + b^n = c^n$ holds, or the assertion that in the decimal development of π 10 successive digits occur, forming a sequence 0123456789." (Note by BROUWER.)

[7]) BROUWER, Over de grondslagen der wiskunde, thesis, 1907.

pretations 4° and 5° leave open the possibility that there is no unanimity between the mathematicians, as to whether a theorem has or has not been proved, in which case also the definition of the next a_n fails. Finally all these interpretations, apart from the concept of infinity itself, imply a semi-empirical hypothesis, viz. that there will "always" be a human being, willing and able to test the mathematical assertion α.

Perhaps the most consistently subjectivistic interpretation would be the first one, complemented by the admission that any individual reading the paper, or thinking about it, or having heard speaking about it, etc. may interpret it as denoting himself. As a subject does not die *subjectively*, this implies that any subject's concept of infinity corresponds with the concept of finiteness for any individual surviving him. This relativity of the meaning of the term 'infinite', which, in as far as I know, has never been admitted explicitly by BROUWER, was pointed out by G. MANNOURY and later by the present author to whom it seems to remain valid under any other interpretation of the term 'infinite' also. It also is not quite clear, what the author understands by a 'mathematical assertion'. I shall not go now into the possible interpretations of this term, nor into those of the expression 'has become evident to ...'.

5. Also Brouwer's use of the term 'absurd' needs some clarification, in particular as he intends it to be different from his use of the term 'not', although he is not always quite consistent in explicitly making this distinction [8]).

I think that the following interpretation of the term 'absurd' comes rather near to Brouwer's intention, if translated into objectivistic terminology:

> An assertion α is called 'absurd' by a subject S on a moment t, when he thinks he possesses on that moment a method, by which he can deduce a contradiction from any alleged demonstration of α whatsoever.

Here the term 'a contradiction' will probably be considered by BROUWER as being intuitively clear, by formalists as meaning 'a statement of the form '$A \wedge \neg A$' ', and by some other mathematicians and logicians as needing further clarification, and perhaps not being definable without recurrence to the notion of absurdity. Moreover BROUWER probably would omit the words 'he thinks' and would probably not accept an interpretation of 'any ... whatsoever' as 'any ... S can on the moment t imagine', which I should accept. Also the term 'deduce' would perhaps be interpreted by BROUWER in a more "absolutistic" sense than by me.

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[8]) E.g. '... then ϱ could not be < 0' is intended as '... then $\varrho < 0$ were absurd'; '... which is not the case' is intended as '... which contradicts the assumptions', i.e. as '... which, the assumptions supposed to be true, is absurd'; '... $\varrho > 0$ does not hold' is meant as '$\varrho > 0$ has not yet been proved'. The term 'contradictory' in '... $\varrho = 0$ is contradictory' is used as a synonym of 'absurd'.

In any case it seems impossible to define Brouwer's use of the term 'absurd' without using another negative term (like 'contradiction'), in other words it seems to be unavoidable to consider Brouwer's general form of the negation as an "entité primitive", irreducible to affirmative concepts. Therefore Brouwer's theorem can not invalidate the attempts of other mathematicians (in particular G. F. C. GRISS, Mrs. P. DESTOUCHES–FÉVRIER and the present author) to do without the concept of negation. On the contrary, it shows again, how unclear the concept of 'absurdity' is, and makes its avoidance appear more desirable. On the other hand I might point out that I consider affirmative mathematics as an interesting, but not as the only interesting, and still less as the only "justified" part of mathematics [9]).

6. Finally I might point to Brouwer's hesitating between a rather "subjectivistic" and a more "objectivistic" terminology. Sometimes he refers his statements explicitly to a "creative subject" (here denoted by S), sometimes he omits this reference, although (or because) it can readily be added, sometimes, however, he formulates his sentences as if they did not refer to S at all. Although I believe that BROUWER intends the reference to S in the latter places also, a translation, either (A) into a more consistently "subjectivistic" or (B) into a more consistently "objectivistic" or "formal" terminology — which latter terminology, of course, does not correspond with Brouwer's views — might make the paper better understandable to many readers. I give some examples, where the consistency still is far from complete. The italics are mine.

1° '... a *mathematical* assertion'. Translations: A. '... an assertion to which S applies the predicate 'mathematical', such that S believes he will never be in doubt, whether it has or has not been proved by him — or to his satisfaction — nor whether it has or has not been "reduced to absurdity" ...'. B. 'a formula occurring in a formal system S — called 'mathematical' within some meta-system M —, formed in accordance with some rules, formulated in a meta-system of S — the "syntaxis" of S'.

2° '... which *cannot* be tested' Translation: A. '... by S on the moment under consideration'; B. '... such that the applications of a system of operations — called 'testing' —, formulated in a meta-system M of S — called the 'semantics of S' — neither results in a formula of the form $T[\neg\neg\, \alpha]$, nor in a formula of the form $T[\neg\, \alpha]$ — where $T[\beta]$ is a predicate in M, defined for elements of S as variables, subject to some rules formulated in M, — and called 'β is true' —'.

3° '... no method is *known*'. To add: 'to S on the moment t'.

4° 'S *can* ...'. Translation: either 'S expects he can ...' or 'I — BROUWER — expect that S can ...'.

[9]) Cf. D. VAN DANTZIG, Mathématique stable et mathématique affirmative, Congr. Intern. de Philos. des Sciences, Paris 1949; —, On the principles of affirmative and intuitionistic mathematics (1941), Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 918—929; 1092—1103 (1947).

5° 'This ... sequence ... *is* ... convergent ...'. To read: '... has been proved to S's — or to mine, Brouwer's — satisfaction to be convergent'. Translation with reference to an imagined super-intelligence — contrary to the general trend of Brouwer's ideas —: '... will be found by "posterity" to be ...' or '... is known by "God" to be ...'.

6° 'If ... $\varrho > 0$ ... did *hold* ...': 'If ... $\varrho > 0$ ... had been proved by S on or before the moment t' ...'. Here t' is *not* meant to be necessarily the same moment as t; it may be any later moment.

I hope that after these examples many a reader will find it not very difficult, but rather illuminating and facilitating to grasp Brouwer's intentions, to translate the paper either into a "subjectivistic" or into an "objectivistic" — or "formal" — terminology.

Resuming the content of this paragraph I might state that a consistently subjectivistic — which probably is the one most akin to Brouwer's intentions — interpretation of Brouwer's paper implies that the terms 'infinite', 'evident', 'absurd' (and other ones) should be used in a "subjectively absolute" way, which is untenable from an objectivistic point of view, hence also as soon as the subject recognises any other subject different from himself. The validity of Brouwer's theorem, however, is *not* restricted to this subjectivistic interpretation, as I shall show in the second part of this paper.

§ 2. *Interpretation of Brouwer's theorem.*

7. Although it is not completely clear, what the term 'creative subject' is intended to mean (cf. § 1, 4), it is clear that the *task* of this subject with regard to the present theorem is 1° to create the numbers a_n according to the definition, a task which is purely formal and which it can have performed by an appropriate machine, 2° in between two successive definitions of numbers a_{n-1}, a_n to prove — or controll proofs of — mathematical theorems, in particular of the proposition α. Let ω_n be the set of deductions it has performed or controlled between the definitions of a_{n-1} and a_n.

After the deductions ω_n and the determination γ_n of a_n the subject gives a discussion δ_n, based on the preceding performances, regarding the real number ϱ. We denote by δ_0 Brouwer's definition of ϱ, by σ_n the conjunction of the sets of deductions $\omega_1, \ldots, \omega_n$, by Ω_n the set $\omega_{n+1}, \omega_{n+2}, \ldots$, and by $\succ$ the predicate 'contains a deduction of'.

Putting $a_0 = 0$ the definition γ_n can then be formulated as follows: [10])

$$\begin{aligned}
&\text{if } a_{n-1} \neq 0 \text{ then } a_n = a_{n-1};\\
&\text{if } a_{n-1} = 0 \text{ then } {}^{11)}\ \text{`}\omega_n \succ \alpha\text{'} \equiv \text{`}a_n = 2^{-n}\text{'}\\
&\qquad\text{and } \text{`}\omega_n \succ \neg\, \alpha\text{'} \equiv \text{`}a_n = -2^{-n}\text{'}\\
&\qquad\text{and } \text{`}\neg\, \text{`}\omega_n \succ \alpha\text{'} \wedge \neg\, \text{`}\omega_n \succ \neg\, \alpha\text{''} \equiv \text{`}a_n = 0\text{'}.
\end{aligned}$$

[10]) It does not imply a restriction, if α is replaced by its double negation; then α is "stable" and everywhere $\neg\neg\, \alpha$ can be replaced by α.

It is tacitly assumed, that 1° the sequence of sets of deductions $\omega_1, \omega_2, \ldots$ does not contain a contradiction — e.g. that the subject is sufficiently educated in mathematics not to make errors —, 2° that for any n there is no doubt [12]), whether ω_n does or does not contain a deduction either of the assertion α or of its negation $\neg\, \alpha$.

Under these conditions a_n is uniquely determined by a_{n-1} and ω_n.

The question is left open, which further conditions are necessary in order to exclude the possibility that α by being dependent upon δ_0 becomes self-contradictory (e.g. if $\alpha \equiv$ '$\varrho < 0$') or by dependence upon the ω_n might become undecidable (e.g. $\alpha \equiv \bigcup_1^\infty \omega_n$).

It is a natural consequence of the principles of intuitionistic mathematics to require that each ω_n be a *finite* set of deductions, each application of complete or transfinite induction being counted as one deduction.

In connection with existing theories of deducibility, it is of importance to remark that '$\omega_n \succ \alpha$' does not mean that α is deduc*ible* from ω_n, but that a deduction of α *actually occurs* in ω_n ("α becomes evident to the subject"). In order to get a clear distinction between σ_n and the subsequent deductions δ_n it is therefore desirable to require that any deduction occurring in δ_n which does not depend on δ_0 occurs in σ_n already.

8. Hence ϱ is uniquely determined *as a function of* $\omega_1, \omega_2, \ldots$. Now, the most natural interpretation of Brouwer's statement that '$\varrho = 0$' is absurd, would be that this were so, *whatever* $\omega_1, \omega_2, \ldots$ were, i.e. *identically in the* ω_n. This, however, is not true. For if, e.g. all ω_n are empty, then *for these* ω_n $\varrho(\omega_1, \omega_2, \ldots) = 0$. This is the case, if the subject sleeps between every two successive γ_n, or also if he does not occupy himself with the parts of mathematics to which α belongs, etc.

On several occasions [13]) BROUWER has explicitly stated that the freedom of a subject to choose his own activity, according to his views, implies at any moment the freedom to restrict this freedom further on. Unless this is not admitted for its activity in general, which seems somewhat inconsistent, this decidedly leaves open the possibility that S decides on some moment nevermore to occupy himself with the assertion α, but nevertheless to go on with the construction of the numbers a_n. Under this decision Ω_n becomes

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[11]) Single inverted and upright upper commas are used as opening and closing brackets respectively. Pairs of commas have been omitted, when no misunderstanding was expected.

[12]) This consistent neglect of the possibility of doubt as to whether a construction has or has not been performed by the subject, is the most important formal element in intuitionistic mathematics.

[13]) E.g. L. E. J. BROUWER, De non-aequivalentie van de constructieve en de negatieve orderelatie in het continuum, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **52**, 122—124 (1949) (Indag. Math. **11**, 37—39 (1949): "... which have retained their complete freedom of continuation, inclusive their freedom of later restriction of freedom" (translated from BROUWER's Dutch text). Here, however, only choices of numbers for the construction of points of the continuum, not of other activities, are considered.

empty and $\varrho = 0$ becomes true instead of contradictory. Brouwer's theorem can therefore not be interpreted as: for every choice of the ω_n it is absurd that $\varrho = 0$, but only as: it is absurd that $\varrho = 0$ for every choice of the ω_n. This distinction is deciding for the validity of Brouwer's theorem.

Brouwer's proof can be formulated thus:

If $\delta_n \succ `\varrho \leqq 0'$, then not only for no $m$ $\omega_m \succ \alpha$, but this must follow already from $\sigma_n$, i.e.[14]) $\sigma_n \succ `\forall_m \neg `\omega_m \succ \alpha''$. Herefrom BROUWER concludes $\sigma_n \succ \neg \alpha$ (cf. 9). Hence:

$$`\delta_n \succ `\varrho \leqq 0'' \supset `\sigma_n \succ \neg \alpha' \quad \ldots\ldots \quad (1)$$

and in the same way

$$`\delta_n \succ `\varrho \geqq 0'' \supset `\sigma_n \succ \neg\neg \alpha'. \quad \ldots\ldots \quad (2)$$

Hence

$$`\delta_n \succ `\varrho = 0'' \supset `\sigma_n \succ `\neg \alpha \wedge \neg\neg \alpha'' \quad \ldots\ldots \quad (3)$$

hence

$$\neg `\delta_n \succ `\varrho = 0'' \quad \ldots\ldots \quad (4)$$

as otherwise σ_n would contain a contradiction.

9. The transition from $\neg `\Omega_n \succ \alpha'$ to $\neg \alpha$ requires that the former statement be true *for any* $\Omega_n$. Then it is a consequence of Brouwer's concept of truth. An assertion $\alpha$ is only then admitted to be true, if a demonstration of it exists: $\exists_\Omega `\Omega \succ \alpha'$. Hence if $\forall_\Omega \neg `\Omega \succ \alpha'$, i.e. if a deduction is obtained, which from *any* alleged deduction Ω of α deduces a contradiction, α can not be true, i.e. $\neg \alpha$ is true.

For the correct understanding of this conclusion it is of importance to remark that *in intuitionistic mathematics an assertion α can not be — i.e. be proven to be — undecidable.* If it were, then we should have a deduction of a contradiction from any alleged proof of $`\alpha \vee \neg \alpha'$, in particular also of α, from which $\neg \alpha$ would follow. In the same way $\neg\neg \alpha$ would follow, so that an alleged proof of the undecidability of α leads to the contradiction $\neg \alpha \wedge \neg\neg \alpha$. Formally, as A. HEYTING showed, the classical implication $`\neg `\alpha \vee \beta'' \supset `\neg \alpha \wedge \neg \beta'$ remains valid in intuitionistic logic, (contrary to $`\neg `\alpha \wedge \beta'' \supset `\neg \alpha \vee \neg \beta'$), from which the contradiction follows when β is replaced by $\neg \alpha$.

The non-existence of intuitionistically undecidable propositions together with the existence of propositions which are undecidable within a definite formal system is difficult to understand by means of the formal distinction between a system and its semantic meta-system. The paper under discussion, as well as Brouwer's previous paper on the continuity of everywhere defined functions, based on the theory of demonstration, make it

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14) When using the quantifiers $\forall$ und $\exists$, we omit for brevity the statements that m and n denote natural numbers, Ω subsets of Σ, and $\omega_1, \omega_2, \ldots$ finite subsets of Ω.

appear dubious whether it will be possible on the long run to maintain the strict distinction between formal systems and their semantic meta-systems against Brouwer's opinion that this is not possible.

Hence Brouwer's conclusion requires that $\varrho \leqq 0$ is proved in δ_n, whatever Ω_n be. Indicating explicitly the dependence of ϱ upon Ω_n by writing $\varrho_n(\Omega_n)$ instead of $\varrho = \varrho(\omega_1, \omega_2, \ldots)$, with $\Omega_n = (\omega_{n+1}, \ldots)$, we have therefore:

$$\text{`}\delta_n \succ \forall_\Omega \text{`}\varrho_n(\Omega) \leqq 0\text{''} \supset \text{`}\sigma_n \succ \text{`}\forall_\Omega \neg \text{`}\Omega \succ \alpha\text{'''} \supset \text{`}\sigma_n \succ \neg \alpha\text{'}$$

Hence Brouwer's statement that $\varrho = 0$ is contradictory must be meant in the sense:

$$\forall_n \neg \text{`}\delta_n \succ \forall_\Omega \text{`}\varrho_n(\Omega) = 0\text{''}$$

i.e. for no n can $\varrho_n(\Omega) = 0$ (*identically in* Ω) be proven from $\omega_1, \ldots, \omega_n$.

Writing this for a moment shortly as $\neg \forall_\Omega \text{`}\varrho_n(\Omega) = 0\text{'}$, the decisive point mentioned above consists in distinguishing this assertion from $\forall_\Omega \neg \text{`}\varrho_n(\Omega) = 0\text{'}$, which is not true. As long as the dependence of ϱ upon $\omega_1, \omega_2, \ldots$ is not explicitly mentioned, both statements are symbolised by $\neg \text{`}\varrho = 0\text{'}$ and the distinction remains impossible.

Further Brouwer argues: if $\neg \text{`}\varrho_n(\Omega) \geqq 0\text{'}$ [15]) for some $\Omega$, then this implies that $\neg \alpha$ is true. Hence, if $\delta_n \succ \exists_\Omega \neg \text{`}\varrho_n(\Omega) \geqq 0\text{'}$, then the truth of $\neg \alpha$ must have been concluded from σ_n already:

$$\text{`}\delta_n \succ \exists_\Omega \neg \text{`}\varrho_n(\Omega) \geqq 0\text{''} \supset \text{`}\sigma_n \succ \neg \alpha\text{'}$$

and in the same way:

$$\text{`}\delta_n \succ \exists_\Omega \neg \text{`}\varrho_n(\Omega) \leqq 0\text{''} \supset \text{`}\sigma_n \succ \neg\neg \alpha\text{'}.$$

Hence in this case we can draw in δ_n not only the conclusion $\exists_\Omega \neg \text{`}\varrho_n(\Omega) \geqq 0\text{'}$ (or $\leqq 0$), but even

$$\forall_{\omega_{n+1}} \forall_{\omega_{n+2}} \ldots \text{`}\varrho_n(\omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots) \leqq -2^{-n}\text{'}$$

(or $\geqq +2^{-n}$ respectively).

10. We can eliminate the concept of 'creating subject' altogether, by replacing this subject by the produce of its activity. Brouwer's theorem can then be given the following form, which, I think, is acceptable to logicists as well as — if the restrictions 'according to ...' in 1° and 2° are omitted — to intuitionists.

Hypotheses.

1° A is a set of formulae, representing mathematical assertions, according to the rules of a given syntactical meta-system.

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15) BROUWER's notation for $\neg \text{`}x \geqq 0\text{'}$ is: $x < 0$.

2° Σ is a set of deductions of formulae belonging to A, according to the rules of a given semantical metasystem.

3° Σ is free from contradiction.

4° $\omega_1, \omega_2, \omega_3, \ldots$ are finite sets of deductions belonging to Σ, (each application of complete or transfinite induction being counted as one deduction).

5° $\sigma_n = \bigcup_1^n \omega_i$.

6° If '$\vartheta \succ \beta$' denotes for any $\beta \in A$ and any $\vartheta \subset \Omega$, that a deduction of β occurs in ϑ, then '$\vartheta \succ \beta$' $\vee \neg$ '$\vartheta \succ \beta$'.

7° $\delta_0\,(\alpha; \omega_1, \omega_2, \ldots)$ (shortly δ_0) for any $\alpha \in A$, and any sequence $\omega_n \subset \Omega$ is the conjunction of the following definitions (1)—(5) *not* contained in A:

$$\text{'}\mathbf{a}_0\,(\alpha) = 0\text{'} \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$\text{'}\forall_{n \geqq 1}\ \text{''}\neg\ \text{'}\sigma_n \succ \alpha\text{'} \wedge \neg\ \text{'}\sigma_n \succ \neg\ \alpha\text{''} \supset \text{'}\mathbf{a}_n\,(\alpha; \omega_1, \ldots, \omega_n) = 0\text{'''} \ . \quad (2)$$

$$\text{'''}\sigma_n \succ \alpha\text{'} \wedge \text{'}m = \min_{\sigma_k \succ \neg\neg\ \alpha} k\text{''} \supset \text{'}a_n\,(\alpha; \omega_1, \ldots, \omega_n) = 2^{-m}\text{''} \ . \quad (3)$$

$$\text{'''}\sigma_n \succ \neg\ \alpha\text{'} \wedge \text{'}m = \min_{\sigma_k \succ \neg\ \alpha} k\text{''} \supset \text{'}a_n\,(\alpha; \omega_1, \ldots, \omega_n) = -2^{-m}\text{''} \quad (4)$$

$$\text{'}\varrho\,(\alpha; \omega_1, \omega_2, \ldots) = \lim_{n \to \infty} \mathbf{a}_n\,(\alpha; \omega_1, \ldots, \omega_n)\text{'}. \quad \ldots \quad (5)$$

8° δ_n is a finite set of deductions which can be drawn from δ_0 and σ_n [16]).

9° If a deduction of an assertion not depending on δ_0 is contained in δ_n then it is contained in σ_n already.

Then Brouwer's theorem is aequivalent with the following statement: For all natural $n \geqq 1$:

$$1^\circ \qquad \neg\ \forall^{\Omega}_{\omega_{n+1}}\ \forall^{\Omega}_{\omega_{n+2}} \ldots \text{'}\varrho\,(\alpha; \omega_1, \omega_2, \ldots) = 0\text{'}$$

$$2^\circ \quad \text{'}\sigma_n \succ \neg\ \alpha\text{'} \equiv \delta_n \succ \text{'}\exists^{\Omega}_{\omega_{n+1}}\ \exists^{\Omega}_{\omega_{n+2}} \ldots \neg\ \text{'}\varrho\,(\alpha; \omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots) \geqq 0\text{''} \equiv$$

$$\equiv \delta_n \succ \text{'}\forall^{\Omega}_{\omega_{n+1}}\ \forall^{\Omega}_{\omega_{n+2}} \ldots \text{'}\varrho\,(\alpha; \omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots) \leqq -2^{-n}\text{''}$$

$$3^\circ \quad \text{'}\sigma_n \succ \neg\neg\ \alpha\text{'} \equiv \delta_n \succ \text{'}\exists^{\Omega}_{\omega_{n+1}} \exists^{\Omega}_{\omega_{n+2}} \ldots \neg\ \text{'}\varrho\,(\alpha; \omega_1, \ldots \omega_n, \omega_{n+1}, \ldots) \leqq 0\text{''} \equiv$$

$$\equiv \delta_n \succ \text{'}\forall^{\Omega}_{\omega_{n+1}}\ \forall^{\Omega}_{\omega_{n+2}} \ldots \text{'}\varrho\,(\alpha; \omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots) \geqq +2^{-n}\text{''}$$

With the corresponding interpretations Brouwer's demonstration remains valid.

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[16]) Formally the specification of the deductions which are admitted in δ_n needs some more precision than is given in 8° and 9°.