Nonstandard Analysis

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Rational Constructive Analysis

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Introduction

Using Alexander Yessenin-Volpin's theory of constructions [Volpin 1968][1], [Geiser 1975][2] in which the natural numbers are considered as an unfolding process over time, it is possible to develop a purely finitistic version of real analysis using only rational numbers. By employing an intuitionistic tense logic and a parameter z for a "large" natural number, infinitesimals can be introduced and the finitistic theory of real analysis developed along the lines of A. Robinson's nonstandard analysis [A. Robinson, 1966][3]. The notion of a "very large" natural number is defined relative to a particular finite set $\wp$ of proofs and calculations, so that z represents a natural number that has not yet arrived and is larger than any integer needed for the completion of the proofs and calculations in $\wp$. The notion "very large" is made precise axiomatically and in the Soundness Theorem of Section II of this paper.

I have called the resulting theory *Rational Constructive Analysis* (RCA). Even though RCA borrows some of the strategies of Robinson in the use of the concept of infinitesimals to do real analysis, it is entirely constructive and essentially finitistic. It also bares a strong relation to Mycielski's theory FIN [Mycielski, 1981][4] which he has proposed as a possible finitization of classical analysis. Taking off from Mycielski, Juha Ruokolainen [Ruokolainen 2004][5] has developed a very interesting, purely finitistic, constructive version of nonstandard analysis with a transfer principle permitting the elimination of special constant terms ∞_p, p a rational number, that act as proxies for infinitely large numbers whose inverses can act as infinitesimals ala Robinson.

The original development of RCA [Geiser 1981][6] was presented at the New Mexico State University Conference on Constructive Mathematics in 1980. The present paper is based on an extensive revision and expansion of that work.

In Section I, an intuitionistic theory of rational numbers including a tense operator Δ is presented, augmented by certain predicates that allow one to express a notion of infinitesimal numbers as the reciprocals of "very large" numbers. The logical framework of RCA is a first order intuitionistic tense logic. Section II provides a semantics for RCA synthesized from Kleene's recursive realizability [Kleene, 1945][7] and a version of Kripke [Kripke 1968][8] semantics for tense logic. The main result of this section is Theorem II.1, a conditional Soundness Theorem for RCA. Section III provides a representation of the continuum using infinitesimal rationals and begins the development of real analysis.

There are three features of the resulting mathematics worth noting at this point. First, only the (finite) natural numbers and the corresponding rational numbers are used. There is no reliance on infinitary non-standard models of arithmetic. Second, the occurrences of the tense operator Δ ("it will be the case that") in mathematical statements provides a specific bound on the computational resources underlying their numerical and logical content. The third feature concerns the natural number parameter z which acts as an uninterpreted constant term in RCA performing the role of a "large" natural number. In carrying out computations and proofs in RCA, one can assign a natural number value ζ to the parameter z in all of its occurrences forming a system RCA^{ζ}. If P is a proof in RCA, then P^{ζ} denotes the proof in RCA^{ζ} that results from the replacement of all occurrences of z by $\underline{\zeta}$, the constant in the language of RCA representing ζ. Actual computations and proofs in RCA thus transformed can be safely performed if the computations and proofs don't violate the conditions of the Soundness Theorem of Section II. Depending on the Δ embedding depth ($\Delta - depth$) of the computations and proofs to be carried out, the Soundness Theorem requires, in general, that, although finite, ζ must be extremely large. However, if the requirements of the Soundness Theorem are not observed, then for any particular natural number ζ assigned to z, one can easily describe how to construct a proof in RCA^{ζ} of a contradiction whose length is proportional to ζ*.

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* The contradiction in RCA^{ζ} can be established as follows. RCA contains a predicate K which is interpreted as a finite segment of "arrived" natural numbers. One can straight forwardly construct in RCA, for any natural number n, a proof P_n of $\Delta^n K(\underline{n})$ of Δ-$depth = n$ (the Δ embedding depth of the proof P_n.) Δ^n denotes the n-fold iteration of Δ and $\underline{n}$ is the constant term of RCA corresponding to n. Secondly, for each natural number n, one can construct a proof Q_n of $K(\underline{n}) \supset \underline{n} < z$ of $\Delta - depth = 0$. By applications of tense logic rules for Δ-Introduction and Δ-Distribution, for each natural number n, one can construct a proof R_n of $\Delta^n K(\underline{n}) \supset \Delta^n(\underline{n} < z)$ of $\Delta - depth = n$. Applying Modus Ponens to the end wffs of the proof P_n and R_n we get a proof S_n of $\Delta^n(\underline{n} < z)$ of $\Delta - depth = n$. If the parameter z is interpreted by a natural number ζ and z is replaced in these proofs by the constant $\underline{\zeta}$ then we can construct a proof S_{ζ}^{ζ} in RCA^{ζ} of $\Delta^{\zeta}\left(\underline{\zeta} < \underline{\zeta}\right)$ of $\Delta - depth = \zeta$. But in RCA we can also construct a proof P of $\neg\Delta^{\zeta}(z < z)$ of $\Delta - depth = \zeta$ so that P^{ζ} is a proof in RCA^{ζ} of $\neg\Delta^{\zeta}\left(\underline{\zeta} < \underline{\zeta}\right)$. This is related to the paradoxes of predicates like "X is bald", "n is not a large number", etc.

There are several things to note about this "argument". First of all, when the substitution of $\underline{\zeta}$ for z is performed, the resulting proofs, although they are in the language of RCA, they are no longer in the formal system RCA because, for example, $K(\underline{n}) \supset \underline{n} < \underline{\zeta}$ is not derivable from the axioms of RCA, for arbitrary natural numbers n. Secondly, the sketched proof of this contradiction does not meet the requirements of the Soundness Theorem (Section II). This theorem would require that the natural number ζ (greatly) exceed the $\Delta - depth$ of the proofs in question. The Soundness Theorem guarantees that the end-wff of a proof P has an interpretation if, among other things, the interpretation of z is greater than $F(\Delta(P))$ where $\Delta(P)$ is the $\Delta - depth$ of P and F is a specified, rapidly increasing function.

While the proofs in RCA can be converted into proofs in RCA^{ζ}, the reverse is easily shown not to be true. Putting aside the inconsistency of RCA^{ζ} for a moment, if the choice of ζ is an even integer, then we can provide a perfectly fine proof of $\exists x(N(x) \wedge \underline{\zeta} = 2 * x)$ in RCA^{ζ} (which includes Intuitionistic Peano Arithmetic) where N is a predicate denoting the natural numbers. But $\exists x(N(x) \wedge z = 2 * x)$ is not provable in RCA. This follows from the Soundness Theorem of Section II.

Part I The Formal Systems

I will introduce three formal systems. Rational Arithmetic (RA) is an intuitionistic first order theory of the rationals with an embedded theory of Arithmetic. The formal system Rational Constructive Analysis (RCA) extends RA to an intuitionistic first order tense logic together with a new predicate symbol $K(x)$ ("feasible" or "standard" natural number), a function $\mathcal{F}(x)$ under which K is "closed", and a parameter z denoting a "very large" natural number. It is in this system that I shall develop the mathematics of RCA. The third formal system, the RCA Modeling System (RCAMOD), provides the basis for the semantics for RCA that is developed in Part II.

I begin by presenting the logical axiom schemata and rules of inference for intuitionistic predicate calculus and tense logic. A, B, C denote arbitrary wffs, x, y any variables, and t and s any terms. A_t^x denotes the substitution of all free occurences of x in A by the term t. n, m, p, q will denote natural numbers. $\underline{n}$ will denote the standard representation of n as a rational (i.e., $(1+\cdots+1)/1$.)

Axioms for Intuitionistic Predicate Calculus (IPC)

Axiom Schemata for IPC

I1,2,3	$A \wedge B \supset A$, $A \wedge B \supset B$, $A \supset B \supset ((A \supset C) \supset (A \supset B \wedge C))$.
I4,5,6	$A \supset A \vee B$, $B \supset A \vee B$, $(B \supset A) \supset ((C \supset A) \supset ((B \vee C) \supset A))$.
I7,8,9	$A \supset A$, $A \supset (B \supset A)$, $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$.
I10,11,12	$A \supset (\neg A \supset B)$, $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$, $\big(A \supset (B \wedge \neg B)\big) \supset \neg A$.
I13	$\forall x A(x) \supset A(t)$, t is free* for x in A. (UI)
I14	$A(t) \supset \exists x A(x)$, t is free for x in A. (EG)
I15	$\forall x(A \supset B) \supset (\forall x A \supset \forall x B)$.
I16,17	$\forall x(A \supset B) \supset (A \supset \forall x B)$, $\exists x(A \supset B) \supset (A \supset \exists x B)$, x is not free in A.
I18	$t = s \supset (A(t) \supset A(s))$, s and t are free for x in $A(x)$.
I19	$\forall x(x = x)$.
I20	$\forall xy\, (x = y \vee \neg\, x = y)$.

* As usual, we define a term t to be free for x in wff A if no free occurrence of x in A is within the scope of a quantifier of a variable free in t. The expression "$A(t)$" designating the substitution of term t for variable x in $A(x)$ is equivalent to A_x^t in which each free occuence of x in A is replaced by the term t.

IPC Rules of Inference

MP Modus Ponens: $\frac{A \quad A \supset B}{B}$

UG Universal Generalization: $\frac{A(x)}{\forall x A(x)}$

IPC Derived rules of inference

$\supset Trans$ Transitivity of implication: $\frac{A \supset B \quad B \supset C}{A \supset C}$

UI Universal Instantiation: $\frac{\forall x A(x)}{A(t)}$, t is free for x in A.

$= Sub$ Substitution of equal terms: $\frac{A(s) \quad s=t}{A(t)}$, s and t are free for x in $A(x)$.

Axioms for Tense Logic (TL)

The expression ΔA is interpreted to mean that A will be true at the next stage of the construction of the natural numbers. Note that the "next stage of the construction" extends the current segment $0,1,\ldots,n$ to $0,1,\ldots,n,\ldots,\mathcal{F}(n)$ where $\mathcal{F}$ a specified, increasing, natural number valued function. (The behavior of $\mathcal{F}$ is specified in Axiom Group VI.)

Note: we use the expression $A \equiv \mathrm{B}$ as an abreviation for $(A \supset B) \wedge (B \supset A)$.

TL Axiom schemata

△1 $(\Delta A \wedge \Delta B) \equiv \Delta(A \wedge B)$

△2 $(\Delta A \vee \Delta B) \equiv \Delta(A \vee B)$

△3 $(\Delta A \supset \Delta B) \equiv \Delta(A \supset B)$

△4 $(\Delta \neg A) \equiv \neg \Delta A$

△5 $\Delta \forall x A(x) \equiv \forall x \Delta A(x)$

△6 $\Delta \exists x A(x) \equiv \exists x \Delta A(x)$

Axiom △5 and △6 can be interpreted as saying that the supply of terms to instantiate variables is not coupled to the temporal process under consideration. At the same time, a term doesn't necessarily denote a presently occurring object.

TL Rules of Inference

△I Δ introduction $\frac{A}{\Delta A}$

△E Δ elimination $\frac{\Delta A}{A}$

△I asserts the stability of correctness of proofs under passage into the (next stage of the) future. ΔE can be paraphrased as follows: "if a mathematical assertion will be true then it must already be true," or, "mathematical assertions are not contingent", as is an assertion like "it is raining".

The tense operator GA meaning "A will hence forth be true" is not part of the language we will use to express temporality in RCA. On the other hand, if $\vdash_{RCA} A$, that is, if we can proof A in RCA, then for every natural number n we can show that $\vdash_{RCA} \Delta^n A$ by n application of ΔI. However, in order for $\Delta^n A$ to be meaningful in the realization semantics developed in Part II, n must by an "arrived" natural number in a step-by-step production of natural numbers.

By a formal system F, I shall mean the specification of a first order predicate language L (possibly augmented with the temporal operator Δ), a class of axioms including logical axioms (IPC or IPC + TL) and non-logical axioms, and designated rules of inference (from IPC or from IPC + TL). Let $\mathcal{G}$ be a finite set of wffs of L. A formal proof P from $\mathcal{G}$ is a (finite) sequence $\{A_1, \ldots, A_n\}$* of wffs of L which are either axioms of F or members of $\mathcal{G}$ or follow from previous wffs of P by a rule of inference. If A is the last wff of P we write $P{:}\mathcal{G} \vdash_F A$ or simply $\mathcal{G} \vdash_F A$ if P is understood. $\mathcal{G}$ contains the assumptions of P but may contain other wffs as well. $\vdash_F A$ means there is a proof P of A from the axioms of F, that is, $P{:}\{\} \vdash_F A$. $P{:}\{\} \vdash_F A$ may also written as $P{:}\vdash_F A$.

Rules of Inference Schemata

If the last wff A of the proof P from assumptions G follows by modus ponens from B and $B \supset A$ we can extract two sub-proofs P_1 and P_2 and partition $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ so that $P_1{:}\mathcal{G}_1 \vdash_F B$ and $P_2{:}\mathcal{G}_2 \vdash_F B \supset A$. $\mathcal{G}_1$ and $\mathcal{G}_2$ are not necessarily disjoint. Alternately, if we have two proofs P_1 and P_2, $P_1{:}\mathcal{G}_1 \vdash_F B$ and $P_2{:}\mathcal{G}_2 \vdash_F B \supset A$ we can effectively combine them into a single proof P from $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ of A where A follows by modus ponens and we designate this with the schematic:

$P_1{:}\mathcal{G}_1 \vdash_F B \qquad P_2{:}\mathcal{G}_2 \vdash_F B \supset A$

---+-------------------MP

$P{:}\mathcal{G} \vdash_F B$

or simply

$\mathcal{G}_1 \vdash_F B \qquad \mathcal{G}_2 \vdash_F B \supset A$

--+---------------MP

$\mathcal{G} \vdash_F B$

where the presence of the needed proofs are understood.

* We use the curly bracket to denote (ordered) sequences as well a finite sets. In any case, the latter come with an implicit ordering. Thus $\{A_1, \ldots, A_n\}$ may denote a proof with n steps or may denote a set with n elements, as determined by context.

The other schematics that shall be used are:

Universal Generalization

$\vdash_F A(x)$
-------UG
$\vdash_F \forall x A(x)$

△ Elimination

$\vdash_F \Delta A$
----△E
$\vdash_F A$

△ Introduction

$\vdash_F A$
----△I
$\vdash_F \Delta A$

Derived Rules of Inference Schemata

Transitivity of implication

$\vdash_F A \supset B \qquad \vdash_F B \supset C$
---+--------------$\supset Trans$
$\vdash_F A \supset C$

Universal Instantiation

$\vdash_F \forall x A(x)$
---------UI (t is free for x in A)
$\vdash_F A(t)$

Substitution

$\vdash_F A(s) \qquad \vdash_F s = t$
---+-------------$= Sub$ (s and t are free for x in $A(x)$)
$\vdash_F A(t)$

In an obvious way, each proof schemata can be associated with an effectively calculable proof transform $\mathcal{T}$, that is an effective map from $\mathcal{P}_F \to \mathcal{P}_F$ or from $\mathcal{P}_F \times \mathcal{P}_F \to \mathcal{P}_F$, where $\mathcal{P}_F$ is the class of proofs in formal system F.

More formally, each schemata is associated with a rule $\mathcal{R}$ of inference (e.g., MP, UG, △I, etc.) and each such $\mathcal{R}$ is associated with an effectively calculable transformation $\mathcal{T}_{\mathcal{R}}: \mathcal{P}_F \to \mathcal{P}_F$ or $\mathcal{T}_{\mathcal{R}}: \mathcal{P}_F \times \mathcal{P}_F \to \mathcal{P}_F$. Then the association of proof schemata with proof transforms can be illustrated by:

$P: \vdash_F A$
------$\mathcal{R}$
$\mathcal{T}_{\mathcal{R}}(P): \vdash_F B$

and

$Q_1\colon \mathrm{G}_1 \vdash_F A \qquad Q_2\colon \mathrm{G}_2 \vdash_F B$

--------+---------$\mathcal{R}$

$\mathcal{T}_{\mathcal{R}}(Q_1, Q_2)\colon \mathrm{G}_1 \cup \mathrm{G}_2 \vdash_F C$

Typically we will leave out mention of the transform and write, for example:

$Q_1\colon \mathrm{G}_1 \vdash_F A \qquad Q_2\colon \mathrm{G}_2 \vdash_F A \supset B$

------+----------------MP

$P\colon \mathrm{G}_1 \cup \mathrm{G}_2 \vdash_F B$

where it is understood that $P = \mathcal{T}_{MP}(Q_1, Q_2)$.

The semantics provided in Part II is parameterized by two natural numbers: i and z. Informally, i designates the stage of a construction of the natural numbers and z designates a "very large" natural number. For a given choice of these parameters, a formal proof may or may not be interpretable. Thus traditional soundness gives way to a relative soundness. A proof P is sound (satisfaction in the interpretation passes from axioms down to conclusions) only if the complexity of P (to be specified in Part II) is bounded by the given parameters of the semantical interpretation. In this setting a formal proof is not necessarily meaningful (i.e. has an interpretation) even though the axioms are all valid and the rules of inference are accepted. The complexity must also be taken into consideration. I will now present the Axiom Groups that will specify the formal systems RA, RCA, and RCAMOD.

Axioms for Rational Arithmetic (RA)

The language L_{RA} is a first order predicate language formed as usual from

Variables	x, y, …
Constants	0, 1
Function	-x, $f_0, \ldots, f_n$, $int(x)$, $INT(x)$, $ABS(x)$
Operations	+, *, -, /
Predicate	$N(x)$
Relations	$=, <$
Connectives	$\neg$, $\wedge$, $\vee$, $\supset$ ($\equiv$ introduced as the usual abreviation)
Quantifiers	$\forall, \exists$

Rational numbers will be treated as signed order pairs $\pm< n,m >$ of non negative natural numbers and written in standard form of $\pm \mathrm{n}/\mathrm{m}$. Q denotes the class of rational numbers. The variables x,y, are intended to have rational numbers as values. f_0, $f_1 \ldots$ is a specified list of effectively calculable primitive recursive functions. f_0 plays a special role, defined in Axiom Group III. $int(x)$, the smallest integer $\geq x$, $INT(x)$, the largest integer $\leq x$, and $ABS(x)$, the absolute value, are also defined in Axiom Group III. We may write $ABS(x)$ as $|x|$. $N(x)$

selects out the class N of non-negative integers.

Define TERM to be the class of closed terms (i.e., without free variables) of L_{RA}.

It is convenient to introduce a special list of meta-variables k, l, n, m whose intended domain is N, and to use "$\forall n$" and "$\exists n$" to indicate abbreviations for corresponding bounded quantifiers:

$\forall n A(n)$ denotes $\forall x_n(N(x_n) \supset A(x_n))$

$\exists n A(n)$ denotes $\exists x_n(N(x_n) \land A(x_n))$

where x_n is a variable of L_{RA} associated with the meta-variable n.

We say that a term t of a formal System F containing RA is a *natural number term* iff there is a proof $P \vdash_F N(t)$.

We can formulate a natural number version of UI as the following derived rule:

Natural Number Universal Instantiation

$\forall n A(n)$
-----NNUI (where t is any natural number term)
$A(t)$

This is a contraction of the following schema:

$\forall x(N(x) \supset A(x))$
------------UG
$N(t) \supset A(t)$ $\quad$ $P: \vdash_F N(t)$
----------+--------MP
$A(t)$

Define $x \leq y$ to be the disjunction $x < y \lor x = y$.

Group I Axioms for a Rational Arithmetic

L1 $\quad \forall xy(x + y = y + x)$

L2 $\quad \forall xy(x * y = y * x)$

L3 $\quad \forall xyw(x + (y + w) = (x + y) + w)$

L4 $\quad \forall xyw(x * (y * w) = (x * y) * w)$

L5 $\quad \forall xyz(x * (y + z) = x * y + x * z)$

L6 $\quad \forall x(x + 0 = x)$

L7 $\quad \forall x(x * 1 = x)$

L8 $\quad \forall x \exists y(x + y = 0)$

L9 $\quad \forall x(x \neq 0 \supset \exists y(x * y = 1))$

L10 $\quad \forall xyw(x < y \land y < w \supset x < w)$

L11 $\quad \forall x(\neg x < x)$

L12 $\forall x(x < x + 1)$

L13 $\forall xy(x < y \lor y < x \lor x = y)$

L14 $\forall xyw(x < w \supset x + y < w + y)$

L15 $\forall xyw(0 < y \supset (x < w \supset x * y < w * y))$

L16 $\forall xy(x < y \supset -y < -x)$

L17 $\forall x(x + (-x) = 0)$,

L18 $\forall xy(x - y = (x + (-y)))$

L19 $\forall x(x \neq 0 \supset x * (1/x) = 1)$

L20 $\forall xy(y \neq 0 \supset x * (1/y) = x/y)$

Group II Axioms for N

N1 $N(0) \land \forall x(N(x) \supset x \geq 0)$

N2 $\forall x(N(x) \supset N(x + 1))$

N3 $\forall x(N(x) \land x > 0 \supset N(x - 1))$

N4 $\forall x(N(x) \land 0 \leq x \leq 1 \supset (x = 0 \lor x = 1)$

N5 $\forall x \exists nm(x = n/m \lor x = -n/m)$

Group III Definitional Axioms for $f_0, f_1, \ldots$, int, INT, ABS

int $\forall x \forall y(y = int(x) \equiv N(y) \land y \leq x \land \forall m(m > y \supset m > x))$

INT $\forall x \forall y(y = \mathrm{INT}(x) \equiv N(y) \land y \geq x \land \forall m(m < y \supset m < x))$

ABS $\forall x \forall y(y = ABS(x) \equiv ((x \geq 0 \land y = x) \lor (x < 0 \land y = -x)))$

Define $f_0, f_1, \ldots$ as specific, effectively computable, strictly increasing functions mapping N into N, selected to provide a sufficient rate of growth suitable for various semantical and proof theoretic purposes as developed in the Parts II and III. We assume that f_0 is chosen so that for all natural numbers n, $f_0(n) > n$.

DEF0 $\forall m(N(f_0(m)) \land f_0(m) > m \land \forall n(m < n \supset f_0(m) < f_0(n)))$

DEF1 $f_1(0) = 1 \land \forall n(f_1(n + 1) = f_0(f_1(n)))$

etc., as needed.

Group IV Induction Schema

IND(RA) $A(0) \land \forall n(A(n) \supset A(n + 1)) \supset \forall n A(n)$

where $A(x_n)$ is any wff in L_{RA} with free variable x_n.

Group V Derived Computational Axioms

C1 $\forall n(n + 0 = n) \land \forall nm(n + (m + 1) = (n + m) + 1)$

C2 $\forall n(n * 0 = n) \land \forall nm(n * (m + 1) = (n * m) + n)$

C3 $\forall n(n - 0 = n) \land \forall nm(n - (m + l) = (n - m) - l)$

C4 $\forall n((n + 1) - 1 = n) \land \forall nm(n - m = -(-n + m))$

C5 $\forall nmlk(m \neq 0 \land k \neq 0 \supset ((n/m) * (l/k) = (n * l)/m * k)$

C6 $\forall nmlk(m \neq 0 \wedge k \neq 0 \supset ((n/m) \pm (l/k) = ((n * k) \pm ((m * l))/(m * k))$

C7 $\forall nmlk(m \neq 0 \wedge k \neq 0 \wedge l \neq 0 \supset ((n/m)/(l/k) = (n * k)/(m * l))$

C8 $\forall nm(m \neq 0 \wedge n \neq 0 \supset 0 < n/m)$

C9 $\forall nmlk(m \neq 0 \wedge k \neq 0 \supset (n/m = l/k \equiv n * k = m * l)$

C10 $N(\underline{p})$ for any natural numbers p.

All of these axioms Cl – C10 are derivable from groups I - IV . They provide the usual calculation rules for arithmetic.

We can now specify RA as RA = IPC + Groups I - V.

Axioms for Rational Constructive Analysis (RCA)

The language L_{RCA} of RCA is a first order language with a modal tense operator Δ, which extends L_{RA} by adding

the predicate $K(x)$,

the functions $\mathcal{F}(x)$, $\mathcal{F}(x,y)$,

the parameter z,

the tense operator Δ.

$K(x)$ has the "presently available" natural numbers as its intended domain. Intuitively one imagines a process for generating natural numbers for the purpose of doing some computation or argument. At any particular stage of this process $K(t)$ holds if t is a term that denotes a natural number already produced, while $\Delta K(t)$ holds also of those terms t that will denote the natural numbers at the next stage of the process.

$\mathcal{F}(x)$ denotes an effectively calculable, increasing function (one of the f_0.....)under which $K(x)$ is closed in the sense $K(x) \supset \Delta K(\mathcal{F}(x))$. $\mathcal{F}(x,y)$ denotes $\mathcal{F}^{INT(x)}(y)$, the $INT(x)$-fold iteration of $\mathcal{F}$ applied to y. By convention $\mathcal{F}^0$ is the identity function and $\mathcal{F}^{n+1}(x) = \mathcal{F}(\mathcal{F}^n(x))$ for $n \geq 0$.

The parameter z, which is never quantified in a wff in L_{RCA}, will be interpreted by a "large" natural number. In particular, z is not a present number, nor will it be.

The closed terms (no free variables but with possible occurrences of the parameter z) constitute a class denoted by $TERM(z)$.

The tense operator Δ is a sentential operator for L_{RCA}: if A is a wff of L_{RCA} so is ΔA. $\Delta^P A$ shall denote $\Delta \ldots \Delta A$ where Δ has been applied p-fold.

The axioms for RCA include three additional groups.

Group VI Definitional Axioms for $\mathcal{F}(x)$ and $\mathcal{F}(x,y)$

F1 $\forall x(\mathcal{F}(x) = f_0(x))$

F2 $\forall mn(m < n \supset \mathcal{F}(m) < \mathcal{F}(n))$ (Note: F2 follows from F1 and DEF0 of Group III.)

F3 $\forall n(\mathcal{F}(0,n) = n)$

F4 $\forall nm(\mathcal{F}(n+1,m) = \mathcal{F}(\mathcal{F}(n,m)))$

Group VII Definitional Axioms for $K(x)$, z and p_0

K1 $K(1) \wedge \forall n(K(n) \wedge n > 0 \supset K(n-1))$

K2 $\forall x(K(x) \supset N(x))$

K3 $K(\underline{p_0}) \wedge \underline{p_0} > 0$ where p_0 is a fixed natural number greater than 0.

K4 $N(z)$

K5 $\forall n(K(n) \supset \Delta K(\mathcal{F}(n)))$

K6 $K(\underline{n}) \supset \Delta^n \forall m(K(m) \supset \mathcal{F}(\underline{n}, \mathrm{m}) < \mathrm{z})$ where n is any natural number.

K7 $K(\underline{\ell}) \supset \forall n(K(n) \supset \Delta^\ell \forall m(K(m) \supset \mathcal{F}(m,n) < \mathrm{z}))$ where ℓ is any natural number.

K, at any stage, denotes the set of "arrived" natural numbers. Kl - K3 say that $K(x)$ is *an* initial segment of $N(x)$ containing p_0. K4 says z is a natural number. K5 says K is closed under $\mathcal{F}$ (but you may have to wait until the next stage.) K6 says z is very "large": if you look any K-number n of stages into the future and consider any integer m then present, z is greater than $\mathcal{F}^n(m)$. K7 also asserts that z is very "large": if ℓ and n are any natural number presently in K and you look ℓ stages into the future and consider any integer m then present, z is greater than $\mathcal{F}^m(n)$.

We add an axiom schema that asserts the persistence of proved, quantifier free assertions.

Persistence of Atomic Truths

PAT $\forall x_{1.} \dots x_n(A \supset \Delta A)$

where A is a quantifier free wff in L_{RCA} not containing the predicate K, with free variables $x_{1.} \dots x_n$. A may contain the parameter z.

The RCA Continuum

The "continuum" in RCA, expressed in terms of the concepts of "large", "standard" and "infinitesimal" rationals, can now be introduced in terms of $K(x)$, $\mathcal{F}(n,m)$, and the parameter z by means of the following wffs.

$L(x)$: $\exists n(K(n) \wedge \mathcal{F}(\mathrm{n}, \mathrm{INT}(ABS(x))) \geq z)$

$S(x)$: $\exists \mathrm{n}(K(n) \wedge ABS(x) \leq n)$

$I(x)$: $x = 0 \vee L(1/x)$

$x\, EQ\, y$: $I(x-y)$

I shall use $L(x)$, $S(x)$, $I(x)$ and $xEQy$ as an abbreviation for the corresponding wff. $L(x)$ say that x is "large" if there is a K-number n of times that we may apply $\mathcal{F}$ to x and get a number $\geq z$, i.e. z is reachable from x via $\mathcal{F}$ in n-steps for some already arrived n. $S(x)$ say x is standard if its magnitude is bounded by an arrived natural number. $I(x)$ says x is infinitesimal if it is 0 or its reciprocal is large. Finally, $x\,EQ\,y$ says that x is to be identified with y if they differ by an infinitesimal. Note that all of these definitions require a "witness" n and this witness is required to be an already present natural number. (S, EQ) will play the role of the continuum in RCA.

Group VIII Induction for RCA

IND(RCA) $A(0) \wedge \forall n(A(n) \supset A(n+1)) \supset \forall n A(n)$

where $A(x)$ is any wff in L_{RCA}, with one free variable x.

Group IX Decidable Least Element Principle (LEP) and related schemata

LEP $\exists n A(n) \wedge \forall n\,(A\,(n)\ \vee \neg A\,(n)\,) \supset \exists n(A(n) \wedge \forall m(m < n \supset \neg A(m))$

where $A(x)$ is any wff of L_{RCA} with free variable x.

BLEP $\exists n(n \leq t \wedge A(n) \wedge \forall\, m(m\ \leq\ n \supset \big(A(m) \vee \neg A(m)\big))) \supset \exists n(n \leq t \wedge A(n) \wedge \forall m\big(m < n \supset \neg A(m)\big))$

where $A(x)$ any wff of L_{RCA} with free variable x and t is a closed term.

This is the Bounded Least Element Principle and is a direct consequence of LEP.

BMEP $\exists n(n \leq t \wedge A(n) \wedge \forall n(n \leq t \supset (A(n) \vee \neg A(n)))) \supset \exists n(n \leq t \wedge A(n) \wedge \forall m(m > n \wedge m \leq t \supset \neg A(m)))$

where $A(x)$ any wff of L_{RCA} with free variable x and t is a closed term.

This is the Bounded Maximum Element Principle and also is a direct consequence of LEP.

The formal system RCA can now be specified as

RCA = IPC + TL + RA + Groups VI – IX + PAT.

In the following theorem we gather together a few simple consequences of axiom Group VII, the definitional axioms for $K(x)$, z and p_0.

Theorem I.1: Basic properties of K, $\mathcal{F}$, and z.

The following are theorems of RCA of $\Delta - depth = 0$.

(1) $\vdash_{RCA} K(0)$.

(2) $\vdash_{RCA} K(1)$.

(3) $\vdash_{RCA} \forall l \forall m(l < m \supset \big(K(m) \supset K(l)\big))$.

(4) $\vdash_{RCA} \forall n(K(n) \supset \mathcal{F}\,(\mathrm{n},\mathrm{n}) < \mathrm{z})$, $\vdash_{RCA} \forall n(K(n) \supset \mathcal{F}\,(\mathrm{n}) < \mathrm{z})$.

(5) $\vdash_{RCA} 0 < \mathrm{z}$.

(6) $\vdash_{RCA} \forall n(n < \mathcal{F}(n))$.

(7) $\vdash_{RCA} \neg K(z)$

Proof: (1) and (2) are simple consequences of K1. (3) can be proved by induction using K1.

Proof of (4): from (1), (2), and from K7 with $l = 0$, we get

$\vdash_{RCA} K(0) \supset \forall \mathrm{n}(K(\mathrm{n}) \supset \forall m(K(m) \supset \mathcal{F}(m,\mathrm{n}) < \mathrm{z}))$ $\vdash_{RCA} K(0)$

---------------+-----------------------------------MP

$\vdash_{RCA} \forall \mathrm{n}(K(\mathrm{n}) \supset \forall m(K(m) \supset \mathcal{F}(m,\mathrm{n}) < \mathrm{z}))$

-------------+-----------------UI

$\vdash_{RCA} K(\mathrm{n}) \supset \forall m(K(m) \supset \mathcal{F}(m,\mathrm{n}) < \mathrm{z})$ $\vdash_{RCA} \forall m(K(m) \supset \mathcal{F}(m,\mathrm{n}) < \mathrm{z}) \supset (K(n) \supset \mathcal{F}(n,\mathrm{n}) < \mathrm{z})$

-------------+--(⊃ Trans)

$\vdash_{RCA} K(\mathrm{n}) \supset (K(n) \supset \mathcal{F}(n,\mathrm{n}) < \mathrm{z})$.

--------------------------Derived Rule: $\{(A \supset (A \supset B)\} \vdash (A \supset B)$

$\vdash_{RCA} K(\mathrm{n}) \supset \mathcal{F}(\mathrm{n},\mathrm{n}) < \mathrm{z})$

-----------------UG;

$\vdash_{RCA} \forall n(K(n) \supset \mathcal{F}(\mathrm{n},\mathrm{n}) < \mathrm{z})$.

The second part of (4) follows by showing that $\vdash_{RCA} \forall n(\mathcal{F}(n) \leq \mathcal{F}(n,\mathrm{n}))$.

Proof of (5): $\vdash_{RCA} 0 < \mathrm{z}$ follows from (1) and the instance $\vdash_{RCA} K(0) \supset \mathcal{F}(0) < \mathrm{z}$ of (4) and $\vdash_{RCA} 0 < \mathcal{F}(0)$ which follows from DEF0 and F1. None of these proofs involves ΔI or ΔE so their $\Delta - depth$ is 0.

Proof of (6): $\vdash_{RCA} n < \mathcal{F}(n)$. This follows directly from the definition DEF0 of f_0 which stipulates that $f_0(n) > n$ and from the definition F1 of $\mathcal{F}$ which stipulates that $\mathcal{F}(n) = f_0(n)$. The $\Delta - depth$ of this proof is 0.

Proof of (7): $\vdash_{RCA} \neg K(z)$. This can be derived from (4) and (6) as follows.

$\vdash_{RCA} \forall n(K(n) \supset \mathcal{F}(\mathrm{n}) < \mathrm{z})$ $\vdash_{RCA} \forall n(n < \mathcal{F}(n))$

--------------------UI ----------UI

$\vdash_{RCA} \mathrm{K}(z) \supset \mathcal{F}(\mathrm{z}) < \mathrm{z}$ $\vdash_{RCA} z < \mathcal{F}(z)$

-------------+------------------------(Derived Rule: $\{(A \supset B), C\} \vdash (A \supset B \wedge C)$)

$\vdash_{RCA} \mathrm{K}(z) \supset (\mathcal{F}(\mathrm{z}) < \mathrm{z} \wedge z < \mathcal{F}(z))$

--------+-----------------(Arithmetic)

$\vdash_{RCA} \mathrm{K}(z) \supset (\mathrm{z} < \mathrm{z})$ $\vdash_{RCA} \neg(\mathrm{z} < \mathrm{z})$

-----------+------------------(Derived Rule: $\{(A \supset B), C\} \vdash (A \supset B \wedge C)$)

$\vdash_{RCA} \mathrm{K}(z) \supset (\mathrm{z} < \mathrm{z}) \wedge \neg(\mathrm{z} < \mathrm{z})$ $\vdash (\mathrm{K}(z) \supset (\mathrm{z} < \mathrm{z}) \wedge \neg(\mathrm{z} < \mathrm{z})) \supset \neg\mathrm{K}(z)$

------+---MP

$\vdash_{RCA} \neg\mathrm{K}(z)$

[End of proof]

Axioms for The Interpretation System of RCA (*RCAMOD*)

The language L_{RCAMOD} is a first order language that extends L_{RA} by adding the predicate $K(x,y)$, the functions $\mathcal{F}(x)$, $\mathcal{F}(x,y)$, $\mathcal{k}(x)$, $\mathcal{m}(x)$.

Note that L_{RCAMOD} does not have the tense operator Δ, the predicate K, or the parameter z. The interpretation system provides a concept of "realizability" of a sentence A of L_{RCA} which includes a mapping of A into a sentence A^* in L_{RCAMOD}.

The axioms for RCAMOD consists of the axioms for IPC, RA , Group VI for $\mathcal{F}(x)$ and $\mathcal{F}(x,y)$, Group X below (also denoted MOD) for $\mathcal{k}(x)$, $\mathcal{m}(x)$ and $K(x,y)$, and Group XI below for Induction for RCAMOD.

$K(n,m)$ is to mean m occurs in K, the set of "arrived number", at stage n of the process that generates K.

Group X Modeling Axioms: Definitional axioms for $\mathcal{k}(x)$, $\mathcal{m}(x)$, and $K(x,y)$

MOD1 $\mathcal{k}(0) = \underline{p_0}$ where p_0 is the natural number in Axiom Group VII.

MOD2 $\forall n(\mathcal{k}(n+1) = \mathcal{F}(\mathcal{k}(n)))$

MOD3 $K(0,\underline{p_0}) \land \forall nm(K(n,m) \supset K(n+1,\mathcal{F}(m))$

MOD4 $\forall n(K(n,0) \land \forall \mathrm{m}\big(K(n,m) \supset \forall l(l < \mathrm{m} \supset K(n,l)\big))$

MOD5 $\forall xy(K(x,y) \supset (N(x) \land N(y)))$

MOD6 $\forall nml\ (K(n,m) \land l > n \supset K(l,m))$

MOD7 $\forall n\ (K(n,\mathcal{k}(n)) \land \forall m(m > \mathcal{k}(n) \supset \neg K(n,m))$

MOD8 $\forall n(\mathcal{m}(n) = \mathcal{F}(\mathcal{k}(n),\mathcal{k}(n)))$

MOD1 and MOD2 define $\mathcal{k}$ in terms of $\mathcal{F}$ and p_0, namely, $\mathcal{k}(n) = \mathcal{F}^n(p_o)$. MOD3-6 says that $K(x,y)$ is an indexed sequence of increasing initial segments of N all of which contain $\underline{p_0}$. $K(n,m)$ "says" that m is in the nth initial segment. MOD7 says $\mathcal{k}(p)$ is maximum element in the p^{th} segment $\{q|K(\underline{p},\underline{q})\}$. Finally, MOD1, MOD3 and MOD7 imply that p_0 is the maximum element in 0^{th} segment$\{q|K(0,\underline{q})\}$. MOD8 defines $\mathcal{m}(x)$ as a form of super-exponential diagonal function.

Group XI Induction for RCAMOD

IND(MOD) $A(0) \land \forall n(A(n) \supset A(n+1)) \supset \forall nA(n)$

where $A(x)$ is any wff in L_{RCAMOD} having x as its free variable.

We *can* now specify RCAMOD as IPC + RA + Groups VI, X, XI. In Part II we shall define an interpretation of RCA using RCAMOD to provide the underlying interpretation of K(x) and the evaluation of atomic sentences.

Summary of the formal systems

RA = IPC + I-V; Rules of inference: MP, UG

RCA = RA + TL + VI-IX + PAT; Rules of inference: MP, UG, ΔI, ΔE

RCAMOD = RA + VI + X + XI; Rules of inference: MP, UG

This finishes the presentation of the formal systems.

Part II An interpretation for RCA

In the following definitions e, a, b, d, i, j, ζ...will denote natural numbers, A, B will denote sentences and $A(x)$ a wff (with at most one free variable x) in L_{RCA}; P, Q, $Q1$, $Q2$ will denote formal proofs in RCA. $\mathcal{F}(x)$, $\mathscr{k}(x)$, $\mathscr{m}(x)$ will denote functions defined on natural numbers satisfying F1 and F2 of Axiom Group VI and MOD8. $\lceil P \rceil$, or equivalently, $gn(P)$, will denote the Gödel number of formal proof P under some specified Gödel numbering scheme. Similarly $\lceil A \rceil$(or $gn(A)$)and $\lceil t \rceil$ (or $gn(t)$) will denote the Gödel numbers of the wff A and term t. Given an indexed, enumeration of partial recursive functions, we shall use a modified Kleene notation $\mathrm{e}\{\mathrm{a}\}$ to denote the result of applying the partial recursive function φ_{e} with index e to the natural number a, that is, $\mathrm{e}\{\mathrm{a}\} = \varphi_{\mathrm{e}}(a)$. Using Church's lambda notation, we may write $e = gn(\lambda x[\varphi_{\mathrm{e}}(x)]) = \lceil \lambda x[\varphi_{\mathrm{e}}(x)] \rceil$.

$\mathrm{e}\{a_1, \dots, a_n\}$ is defined to be the composition $\mathrm{e}\{a_1\}\{a_2\}\dots\{a_n\}$, so that, for example, $\mathrm{e}\{\mathrm{a}, \mathrm{b}\} = \mathrm{e}\{\mathrm{a}\}\{\mathrm{b}\} = \varphi_{\varphi_{\mathrm{e}}(\mathrm{a})}(\mathrm{b})$.

<u>Lemma K:</u> Suppose f is a given recursive function of n variables. We can effectively find a Gödel number e such that for any natural numbers $a_1 \dots a_n$, $\mathrm{e}\{a_1, \dots, a_n\} = f(a_1, \dots, a_n)$.

<u>Proof:</u> Define a sequence of recursive functions $f_l(x_1, \dots, x_l)$, for $l = n, \dots, 1$ as $f_n(x_1, \dots, x_n) = f(x_1, \dots, x_n)$ and $f_l(x_1, \dots, x_l) = gn(\lambda x_{l+1}[f_{l+1}(x_1, \dots, x_{l+1})])$ for $l = n-1, \dots, 1$. Finally define $e = gn(\lambda x_1[f_1(x_1)])$. We can then show by recursion that for $l = 1, \dots, n-1$, $e\{a_1, \dots, a_l\} = gn(\lambda x_{l+1}[f_{l+1}(a_1, \dots, a_l, x_{l+1})])$. For $l = n-1$, this implies that $e\{a_1, \dots, a_{n-1}\} = gn(\lambda x_n[f_n(a_1, \dots, a_{n-1}, x_n)]) = gn(\lambda x_n[f(a_1, \dots, a_{n-1}, x_n)])$ and therefore $e\{a_1, \dots, a_n\} = \varphi_{e\{a_1, \dots, a_{n-1}\}}(a_n) = f(a_1, \dots, a_n)$.
[End of proof of Lemma K.]

Let pr be an effective, 1-1 onto, pairing function from N×N→N*. If $e = pr(n, m)$, we shall typically write n and m as e_0 and e_1, respectively, and write $e = \langle e_0, e_1 \rangle$. Using this notation, the maps $e \to e_0$ and $e \to e_1$ provide the first and second components of e considered as an ordered pair.

By convention we define φ_0 to be the constant 0 function and $\langle 0,0 \rangle = 0$. If t is in $TERM$ then $\underline{t}$ in $TERM$ denotes the value of t expressed in TERM as a rational number (n/m or $-n/m$) in the lowest reduced form. The natural number i is represented in TERM as $\underline{i}$, the rational (in lowest terms) whose value is i. For simplicity, for the natural numbers $0, 1, 2, \dots$ considered as rationals we will drop the under-bar. An integer ζ will used to interpret the parameter z of L_{RCA} and $\underline{\zeta}$ will denote the corresponding term in TERM. If $t \in TERM(z)$ then $t^{\zeta} \in$

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* The use of the symbol N×N→N does not presuppose the infinite set of natural numbers but is merely used as a shorthand to say that pr is a natural number valued function defined on pairs of natural numbers.

$TERM$ denotes the result of replacing every occurrence of z in t by $\underline{\zeta}$. Similarly if $A \in L_{RCA}$ then A^{ζ} is the wff that results by replacing all occurrences of z in A by $\underline{\zeta}$. If $t \in$ TERM then $\lfloor t \rfloor$ denotes the rational number equal to the value of t.

The complexity of a proof P in RCA is measured by the Δ-embedding in its wffs as well as the number of applications of Δ introduction and elimination.

Definition II.1 Let $\{A\}$ be a set consisting of a single wff A in L_{RCA}. Define $\Delta(\{A\})$, the $\Delta -$depth of A, recursively on the syntactic rank of A as follows:

1. If A is atomic then $\Delta(\{A\}) = 0$.
2. If A is $B \vee C$ or $B \wedge C$ or $B \supset C$ then $\Delta(\{A\}) = max\{\Delta(\{B\}), \Delta(\{C\})\}$.
3. It A is $\neg B$ or $\forall xB$ or $\exists xB$ then $\Delta(\{A\}) = \Delta(\{B\})$.
4. If A is ΔB then $\Delta(\{A\}) = 1 + \Delta(\{B\})$.

[End of definition]

Definition II.2 Let P be a formal proof in RCA. Define $\Delta(P)$ recursively as follows.

1. If P consists of a single wff A then $\Delta(P) = \Delta(\{A\})$.
2. If P: $\vdash_{RCA} A$ consists of two sub-proofs $Q1$: $\vdash_{RCA} B$ and $Q2$: $\vdash_{RCA} B \supset A$ followed by an application of MP than $\Delta(P) = max\{\Delta(Q1), \Delta(Q2)\}$.
3. If P: $\vdash_{RCA} \forall xA$ consists of a proof $Q \vdash_{RCA} A$ followed by an application of UG then $\Delta(P) = \Delta(Q)$.
4. If P: $\vdash_{RCA} \Delta A$ consists of a proof Q: $\vdash_{RCA} A$ followed by an application of ΔI then $\Delta(P) = 1 + \Delta(Q)$.
5. If P:$\vdash_{RCA} A$ consists of a proof Q:$\vdash_{RCA} \Delta A$ followed by an application of ΔE then $\Delta(P) = 1 + \Delta(Q)$.

For derived rules of inference define $\Delta(P)$ recursively as follows:

6. If P: $\vdash_{RCA} A \supset C$ consists of two subproofs $Q1$: $\vdash_{RCA} A \supset B$ and $Q2$: $\vdash_{RCA} B \supset C$ followed by an application of $\supset Trans$ than $\Delta(P) = max\{\Delta(Q1), \Delta(Q2)\}$.
7. If P: $\vdash_{RCA} A(t)$ consists of a proof $Q \vdash_{RCA} \forall xA(x)$ followed by an application of UI then $\Delta(P) = \Delta(Q)$.
8. If P: $\vdash_{RCA} A(t)$ consists of a proof $Q1 \vdash_{RCA} A(s)$ and $Q2 \vdash_{RCA} s = t$ followed by an application of $= Sub$ then $\Delta(P) = \max\{\Delta(Q1), \Delta(Q2)\}$.
9. If t is a natural number term and P: $\vdash_{RCA} A(t)$ consists of a proof $Q \vdash_{RCA} \forall nA(n)$ followed by an application of UI then $\Delta(P) = \max\{\Delta(Q), \Delta(R)\}$ where R:$\vdash_{RCA} N(t)$.

[End of definition]

Note: In Definition II.2, it must be assumed that a particular analysis of the justification of each line of P has been carried out; the $\Delta(P)$ is uniquely determined for the given analysis of the proof.

Kleene realizability and Kripke semantics are combined to handle the intuitionistic and tense logic aspects of RCA. A world, in Kripke's sense, indexed by i, will correspond to a stage i in the process that produces the natural numbers n satisfying $K(\underline{n})$. Furthermore, a value ζ for the parameter z must be selected before the semantics can be fully specified for the sentences of RCA. Finally, a consistent set T of sentences of L_{RCAMOD} containing $RCAMOD$ is selected which is used to decide atomic sentences of L_{RCAMOD}. Note that the parameter z is in L_{RCA} but not in L_{RCAMOD}. T has to have the disjunctive and existential property. That is, from a proof in T of $A \vee B$ one can effectively construct a proof of A or of B , and from a proof in T of $\exists x A(x)$ one can effectively construct a term t and a proof in T of $A(t)$. The notion <u>e realizes a sentence in L_{RCA} at world i</u> is defined relative to the choice of T and ζ.

<u>Definition II.3 REL:</u> (Realizability.) Let ζ be a natural number. Let e, a, and i denote natural numbers. (e,i,ζ) RE A is defined for sentences A of L_{RCA} (z may occur as a parameter in A) as follows.

Define $A^{i,\zeta}$ to be the sentence of L_{RCAMOD} that results from A by replacing all occurrences (if any) of $K(t)$ by $K(i,t)$ and replacing all occurrences of z by $\underline{\zeta}$.

REL1. Let A be an atomic sentence.

(e,i,ζ) RE A iff $e = 0$ and $\vdash_{RCAMOD} A^{i,\zeta}$.

REL2. (e,i,ζ) RE $A \wedge B$ iff (e_0,i,ζ) RE A and (e_1,i,ζ) RE B.

REL3. (e,i,ζ) RE $A \vee B$ iff ($e_0 = 0$ and (e_1,i,ζ) RE A) or ($e_0 \neq 0$ and (e_1,i,ζ) RE B).

REL4. (e,i,ζ) RE $A \supset B$ iff for all a, ((a,i,ζ) RE A implies $(e\{a\},i,\zeta)$ RE B).

REL5. (e,i,ζ) RE $\neg A$ iff for all a, (a,i,ζ) RE A is false.

REL6. (e,i,ζ) RE $\forall x A(x)$ iff for all $t \in TERM(z)$ $(e\{\lceil t \rceil\},i,\zeta)$ RE $A(t)$).

REL7. (e,i,ζ) RE $\exists x A(x)$ iff there is a $t \in TERM(z)$ such that (e_0,i,ζ) RE $A(t)$ and $e_1 = \lceil t \rceil$.

REL8. (e,i,ζ) RE ΔA iff $(e,i+1,\zeta)$ RE A.

The main result of Section II is the following Soundness Theorem: there is an effective, integer valued function $E(n,m)$ such that if P is a formal proof in RCA of a sentence A in L_{RCA} and $i \geq \Delta(P)$ and $\zeta > m(i + \Delta(P))$ then $(E(\lceil P \rceil,\zeta),i,\zeta)$ RE A.

Before considering the proof of this theorem note that the stage i at which one must begin in order to obtain a realization of A doesn't depend on the $\Delta - depth$ of A but rather on the $\Delta - depth$ of a proof of A, which may be far greater because of ΔE, ΔI, and MP.

In this regard, realizability is not so much a semantics for sentences as it is a semantics of provable sentences together with their formal proofs. This is in contrast with the metamathematical position underlying traditional semantics (like Tarski semantics [Tarski 1930][9]) that asserts that the "truth" of sentences is determined solely in terms of the independent "truth" of its parts. Our realizability is a proof-sensitive soundness.

In the following Lemmas supporting the proof of the Soundness Theorem, A denotes a sentence in L_{RCA} and may contain terms in $TERM(z)$.

Lemma II.1 Let $A(x)$ be a wff in L_{RCA} with no free variables other than x. If t, $s \in TERM(z)$ and have equal values with respect to z, i.e., $\lfloor \boldsymbol{t}^{\zeta} \rfloor = \lfloor s^{\zeta} \rfloor$, then (e,i,ζ) RE $A(t)$ iff $(e,i\,\zeta)$ RE $A(s)$.

Proof: Recall that $\boldsymbol{t}^{\zeta}$ denotes the constant term in which every occurrence of z in t is replaced by $\underline{\zeta}$, and $\underline{\boldsymbol{t}^{\zeta}}$ is the representation in lowest terms of the value of $\boldsymbol{t}^{\zeta}$ in RCA.

If $A(t)$ is atomic then $\vdash_T A(\boldsymbol{t}^{\zeta}) \equiv A(s^{\zeta})$ since $\lfloor \boldsymbol{t}^{\zeta} \rfloor = \lfloor s^{\zeta} \rfloor$ implies $\vdash_T \boldsymbol{t}^{\zeta} = s^{\zeta}$ and T contains the substitution of equals for equals schema I18. Consequently, by clause REL1 of the definition of realizability, $(e,i\,\zeta)$ RE $A(t)$ iff $\vdash_T A(\boldsymbol{t}^{\zeta})$ iff $\vdash_T A(s^{\zeta})$ iff $(e,i\,\zeta)$ RE $A(s)$.

This argument can be readily extended by induction on the syntactic rank of wffs to all wffs $A(x)$ L_{RCA} with a single free variable since the operation of substitution in wffs of terms in $TERM(z)$ for x commutes with all the syntactic operations: $\forall, \exists, \wedge, \vee, \supset, \neg, \Delta$. For example, suppose $A(x) = \forall y B(y,x)$ and for every term u in $TERM(z)$ the Lemma has been proved for $B(u,x)$. Then, (e,i,ζ) RE $\forall y B(y,t)$ iff for all u in $TERM(z)$, $(e\{\lceil u \rceil\},i,\zeta)$ RE $B(u,t)$. By the induction hypothesis, this implies that for all u in $TERM(z)$ $(e\{\lceil u \rceil\},i,\zeta)$ RE $B(u,s)$ and hence (e,i,ζ) RE $\forall y B(y,s)$. As another example, consider $A(x) = \Delta B(x)$. Since $(\Delta B(x))_x^t = \Delta B(x)_x^t$ then (e,i,ζ) RE $(\Delta B(x))_x^t$ iff $(e,i+1,\zeta)$ RE $B(x)_x^t$, which, by the induction hypothesis, is true iff $(e,i+1,\zeta)$ RE $B(x)_x^s$ iff (e,i,ζ) RE $(\Delta B(x))_x^s$. The rest of the syntactic operators are handled similarly.

[End of proof]

Lemma II.2 The set $\mathcal{S}$ of quantifier free sentences of L_{RCAMOD} is complete. Moreover, if A is in $\mathcal{S}$ then we can effectively construct a proof of A or of $\neg A$.

Proof: The functions of L_{RCAMOD} are all effectively computable so the constant terms of L_{RCAMOD} are effectively computable. Let n_t/m_t or $-n_t/m_t$ be the value of a constant term t in reduced form. Then establish that $\vdash_{RCAMOD} t = \underline{n_t}/\underline{m_t}$ or $\vdash_{RCAMOD} t = -\underline{n_t}/\underline{m_t}$. This can be used to construct the proof of the Lemma II.2 for the atomic sentences $s = t$ and $s < t$ or their negations by reducing the problem to proving in RCAMOD $\underline{n_s}/\underline{m_s} = \underline{n_t}/\underline{m_t}$ or $\underline{n_s}/\underline{m_s} < \underline{n_t}/\underline{m_t}$ or $\underline{n_t}/\underline{m_t} < \underline{n_s}/\underline{m_s}$. Having established completeness for atomic sentences, proceed by induction on syntactic rank.

Case 1. $A = B \wedge C$. Either $\vdash_{RCAMOD} B$ and $\vdash_{RCAMOD} C$ or $\vdash_{RCAMOD} \neg B$ or $\vdash_{RCAMOD} \neg C$. In the former case $\vdash_{RCAMOD} B \wedge C$ and in the later case $\vdash_{RCAMOD} \neg(B \wedge C)$ follows using the propositional tautology $\vdash_{RCAMOD} \neg B \supset \neg(B \wedge C)$ or $\vdash_{RCAMOD} \neg C \supset \neg(B \wedge C)$ as needed.

Case 2. $A = B \vee C$.

Case 2.1. If either $\vdash_{RCAMOD} B$ or $\vdash_{RCAMOD} C$ then $\vdash_{RCAMOD} B \vee C$.

Case 2.2. If neither $\vdash_{RCAMOD} B$ or $\vdash_{RCAMOD} C$. Then by the induction hypothesis, e know that $\vdash_{RCAMOD} \neg B$ and $\vdash_{RCAMOD} \neg C$. Use the tautologies

$\vdash_{RCAMOD} \neg B \supset (B \supset \neg(B \vee C))$ and $\vdash_{RCAMOD} \neg C \supset (C \supset \neg(B \vee C))$ to get

$\vdash_{RCAMOD} (B \supset \neg(B \vee C))$ and $\vdash_{RCAMOD} (C \supset \neg(B \vee C))$ from the supposition and MP.

Then apply the tautology $\vdash_{RCAMOD} ((B \supset \neg(B \vee C)) \supset (C \supset \neg(B \vee C)) \supset ((\mathrm{B} \vee \mathrm{C}) \supset \neg(\mathrm{B} \vee \mathrm{C}))$ to get $\vdash_{RCAMOD} (\mathrm{B} \vee \mathrm{C}) \supset \neg(\mathrm{B} \vee \mathrm{C})$ from which we can derive $\vdash_{RCAMOD} \neg(\mathrm{B} \vee \mathrm{C})$ from the tautology $\vdash_{RCAMOD} ((\mathrm{B} \vee \mathrm{C}) \supset \neg(\mathrm{B} \vee \mathrm{C})) \supset \neg(\mathrm{B} \vee \mathrm{C})$.

Case 3. $A = B \supset C$.

Case 3.1 Suppose $\vdash_{RCAMOD} C$. Then $\vdash_{RCAMOD} B \supset C$ follows by MP from the tautology $\vdash_{RCAMOD} (C \supset (B \supset C)$.

Case 3.2 Otherwise, by the induction hypothesis, $\vdash_{RCAMOD} \neg C$.

Case 3.2.1. Suppose $\vdash_{RCAMOD} B$. From the tautology $\vdash_{RCAMOD} B \supset ((B \supset C) \supset C)$ we get $\vdash_{RCAMOD} (B \supset C) \supset C$. From the tautology $\vdash_{RCAMOD} \neg C \supset ((B \supset C) \supset \neg C)$ we get $\vdash_{RCAMOD} (B \supset C) \supset \neg C$, so $\vdash_{RCAMOD} (B \supset C) \supset (C \wedge \neg C)$ from which we derive $\vdash_{RCAMOD} \neg(B \supset C)$.

Case 3.2.2. Otherwise $\vdash_{RCAMOD} \neg B$. But $\vdash_{RCAMOD} \neg B \supset (B \supset C)$ is a tautology so by MP we get $\vdash_{RCAMOD} (B \supset C)$.

Case 4. $A = \neg B$. By the induction hypothesis, $\vdash_{RCAMOD} B$ or $\vdash_{RCAMOD} \neg B$. It suffices to show that if $\vdash_{RCAMOD} B$ then $\vdash_{RCAMOD} \neg\neg B$ and this follows from the tautology $\vdash_{RCAMOD} B \supset \neg\neg B$.

[End of proof]

Lemma II.3 The set $\mathcal{S}$ of quantifier free sentences of L_{RCAMOD} satifies Tertium non Datur, that is, if A is in $\mathcal{S}$ then $\vdash_{\mathrm{RCAMOD}} A \vee \neg A$.

Proof: Lemma II.3 follows from Lemma II.2. For example, effectively enumerate the proofs of $RCAMOD$. Either a proof of A or of $\neg A$ will turn up from which $A \vee \neg A$ can be derived.

[End of proof]

Lemma II.4 Let $\mathcal{S}$ be the set of quantifier and Δ free sentence in L_{RCA}. Let $A \in \mathcal{S}$ and let T denote the theory $RCAMOD$. Then there is an effectively calculable function $\mathcal{B}: N \rightarrow N$, such that:

(1) (Adequacy for $\mathcal{S}$) For any natural numbers a, i, ζ, (a, i, ζ) RE A implies $\vdash_T A^{i,\zeta}$.

(2) (Soundness for $\mathcal{S}$) If $\vdash_T A^{i,\zeta}$ then $(\mathcal{B}(\lceil A \rceil), i, \zeta)$ RE A for all natural numbers i and ζ.

Proof: We prove (1) and (2) by induction on the syntactic complexity of A.

Case 1. Let A be atomic. REL1 of the definition of realizability states that

(a, i, ζ) RE A iff $a = 0$ and $\vdash_{RCAMOD} A^{i,\zeta}$.

Case 1.1. (1) follows directly from REL1.

Case 1.2. Suppose $\vdash_T A^{i,\zeta}$. Define $\mathcal{B}(\lceil A \rceil) = 0$. Then by REL1, $(\mathcal{B}(\lceil A \rceil), i, \zeta)$ RE A.

Case 2. Let $A = B \wedge C$ and assume (1) and (2) hold for B and C.

Case 2.1. Assume that (a, i, ζ) RE $B \wedge C$. Then (a_0, i, ζ) RE B and (a_1, i, ζ) RE C. By the induction hypothesis, it follows that $\vdash_T B^{i,\zeta}$ and $\vdash_T C^{i,\zeta}$ that in turn entails $\vdash_T B^{i,\zeta} \wedge C^{i,\zeta}$.

Case 2.2. Assume $\vdash_T B^{i,\zeta} \wedge C^{i,\zeta}$. By the induction hypothesis, there are numbers $a_0 = \mathcal{B}(\lceil \boldsymbol{A} \rceil)$ and $a_1 = \mathcal{B}(\lceil C \rceil)$ such that (a_0, i, ζ) RE B and (a_1, i, ζ) RE C. Define $\mathcal{B}(\lceil B \wedge C \rceil) = \langle a_0, a_1 \rangle$. Then $(\mathcal{B}(\lceil B \wedge C \rceil,), i, \zeta)$ RE $B \wedge C$.

Case 3. Let $A = B \vee C$ and assume (1) and (2) hold for B and C.

Case 3.1. Assume that (a, i, ζ) RE $B \vee C$. Then either $a = \langle 0, a_1 \rangle$ and (a_1, i, ζ) RE B which, by the induction hypothesis implies $\vdash_T C^{i,\zeta}$, or $a = \langle 1, a_1 \rangle$ and (a_1, i, ζ) RE C which, by the induction hypothesis implies $\vdash_T C^{i,\zeta}$. In either case we get $\vdash_T B^{i,\zeta} \vee C^{i,\zeta}$.

Case 3.2. Assume that $\vdash_T B^{i,\zeta} \vee C^{i,\zeta}$. We can apply Lemma II.2 to conclude that either $\vdash_T B^{i,\zeta}$ or $\vdash_T C^{i,\zeta}$, so, by the induction hypothesis, there is a $b = \mathcal{B}(\lceil B \rceil)$ such that either (b, i, ζ) RE B or there is a $c = \mathcal{B}(\lceil C \rceil)$ such that (c, i, ζ) RE C. In the former case, define $\mathcal{B}(\lceil B \vee C \rceil) = \langle 0, b \rangle$ and in the latter case define $\mathcal{B}(\lceil B \vee C \rceil) = \langle 1, c \rangle$. Then $(\mathcal{B}(\lceil B \vee C \rceil), i, \zeta)$ RE $B \vee C$.

Case 4. Let $A = B \supset C$ and assume (1) and (2) hold for B and C.

Case 4.1. Assume that (a, i, ζ) RE $B \supset C$. Then for any b, if (b, i, ζ) RE B then $(a\{b\}, i, \zeta)$ RE C.

Case 4.1.1. Suppose there is a b such that (b, i, ζ) RE B.that Therefore $(a\{b\}, i, \zeta)$ RE C and by the induction hypothesis, $\vdash_T C^{i,\zeta}$. Using the tautology $\vdash_T C^{i,\zeta} \supset (B^{i,\zeta} \supset C^{i,\zeta})$ we conclude $\vdash_T B^{i,\zeta} \supset C^{i,\zeta}$.

Case 4.1.2. Suppose that there is no b such that (b, i, ζ) RE B. That is, for all b, (b, i, ζ) RE $\neg B$. Claim: $\vdash_T \neg B^{i,\zeta}$. By Lemma II.2, if not $\vdash_T \neg B^{i,\zeta}$ then $\vdash_T B^{i,\zeta}$ but by the induction hypothesis this would imply that $(\mathcal{B}(\lceil B \rceil), i, \zeta)$ RE B, contradicting the assumption of 4.1.2. Then, combining $\vdash_T \neg B^{i,\zeta}$ with the tautology $\vdash_T \neg B^{i,\zeta} \supset (B^{i,\zeta} \supset C^{i,\zeta})$ we get $\vdash_T B^{i,\zeta} \supset C^{i,\zeta}$.

Case 4.2. Suppose that $\vdash_T (B^{i,\zeta} \supset C^{i,\zeta})$. Suppose that (b, i, ζ) RE B for some b. By the induction hypothesis it follows that $\vdash_T B^{i,\zeta}$ and therefore $\vdash_T C^{i,\zeta}$. Applying the induction hypothesis to $\vdash_T C^{i,\zeta}$ we conclude that $(\mathcal{B}(\lceil C \rceil), i, \zeta)$ RE C. Define

$\mathcal{B}(\lceil B \supset C \rceil)\{b\} = \mathcal{B}(\lceil C \rceil)$. It follows that $(\mathcal{B}(\lceil B \supset C \rceil), i, \zeta)$ RE $B \supset C$.

Case 5. Let $A = \neg B$.

Case 5.1. Suppose (a, i, ζ) RE $\neg B$. This implies that for any natural number b it is not the case that (b, i, ζ) RE B. If $\vdash_T B^{i,\zeta}$ were the case then, by the induction hypothesis, $(\mathcal{B}(\lceil B \rceil), i, \zeta)$ RE B which would contradict our assumption. By the completeness of $\mathcal{S}$ (Lemma II.2), it follows that $\vdash_T \neg B^{i,\zeta}$.

Case 5.2. Suppose that $\vdash_T \neg B^{i,\zeta}$. If there was a natural number a such that (a, i, ζ) RE B it would follow by the induction hypothesis that $\vdash_T B^{i,\zeta}$. But this would imply the inconsistency of T.* Therefore, if we define $\mathcal{B}(\lceil \neg B \rceil) = 0$, then $(\mathcal{B}(\lceil \neg B \rceil), i, \zeta)$ RE $\neg B$.

[End of proof of Lemma II.4]

<u>Lemma II.5</u> Let $T = RCAMOD$ Assume that $\vdash_T A$ where A is the AE sentence $\forall x_1 \dots \forall x_m \exists y_1 \dots \exists y_n B(x_1, \dots, x_m, y_1, \dots, y_n)$ in L_{RCA} without K, or Δ, or z, and B is quantifier free. There is an effectively computable function $\mathcal{E}(A)$ such that for any integers ζ and i, $(\mathcal{E}(\mathrm{A}), \mathrm{i}, \zeta)$ RE A.

<u>Proof:</u> Since T has the E-property there are n effectively computable functions $S_i: TERM^m \to TERM$, $i = 1, \dots, n$ such that for all $\mathrm{t}_1, \dots, \mathrm{t}_m \in \mathrm{TERM(z)}$,

$$\vdash_T \mathrm{B}\left(\mathrm{t}_1^\zeta, \dots, \mathrm{t}_m^\zeta, S_1\left(\mathrm{t}_1^\zeta, \dots, \mathrm{t}_m^\zeta\right), \dots, S_n\left(\mathrm{t}_1^\zeta, \dots, \mathrm{t}_m^\zeta\right)\right).$$

Let s_j denote $S_j(\mathrm{t}_1, \dots, \mathrm{t}_m)$ for $j = 1, \dots, n$, and let $F(\mathrm{t}_1, \dots, \mathrm{t}_m)$ denote $\mathrm{B}(\mathrm{t}_1, \dots, \mathrm{t}_m, s_1, \dots s_n)$.
By Lemma II.4, $(\mathcal{B}(\lceil F(\mathrm{t}_1, \dots, \mathrm{t}_m) \rceil), i, \zeta)$ RE $\mathrm{B}(\mathrm{t}_1, \dots, \mathrm{t}_m, s_1, \dots s_n)$.
By applying Definition REL7 n times, we successively get

(E_n) $(\langle \mathcal{B}(\lceil F(\mathrm{t}_1, \dots, \mathrm{t}_m) \rceil), \lceil \mathrm{s}_n \rceil \rangle, i, \zeta)$ RE $\exists y_n \mathrm{B}(\mathrm{t}_1, \dots, \mathrm{t}_m, s_1, \dots s_{n-1}, y_n)$,

(E_{n-1}) $(\langle\langle \mathcal{B}(\lceil F(\mathrm{t}_1, \dots, \mathrm{t}_m) \rceil), \lceil \mathrm{s}_n \rceil \rangle, \lceil \mathrm{s}_{n-1} \rceil \rangle, i, \zeta)$ RE $\exists y_{n-1} \exists y_n \mathrm{B}(\mathrm{t}_1, \dots, \mathrm{t}_m, s_1, \dots s_{n-2}, y_{n-1}, y_n)$,

…

(E_1) $(\langle \dots \langle\langle \mathcal{B}(\lceil F(\mathrm{t}_1, \dots, \mathrm{t}_m) \rceil), \lceil \mathrm{s}_n \rceil \rangle, \lceil \mathrm{s}_{n-1} \rceil \rangle \dots \lceil \mathrm{s}_1 \rceil \rangle, i, \zeta)$ RE $\exists y_1 \dots \exists y_n \mathrm{B}(\mathrm{t}_1, \dots, \mathrm{t}_m, y_1, \dots, y_n)$.

Define the function $\mathcal{E}(A)$ on $\mathrm{TERM(z)^m}$ as follows: for any $\mathrm{t}_1, \dots, \mathrm{t}_m \in \mathrm{TERM}(z)$,

$\mathcal{E}(A)\{\lceil \mathrm{t}_1 \rceil, \dots, \lceil \mathrm{t}_m \rceil\} = \langle \dots \langle\langle \mathcal{B}(\lceil F(\mathrm{t}_1, \dots, \mathrm{t}_m) \rceil), \lceil \mathrm{s}_n \rceil \rangle, \lceil \mathrm{s}_{n-1} \rceil \rangle \dots \lceil \mathrm{s}_1 \rceil \rangle$.

By (E_1),

(A_m) $((\mathcal{E}(A)\{\lceil \mathrm{t}_1 \rceil, \dots, \lceil \mathrm{t}_m \rceil\}, i, \zeta)$ RE $\exists y_1 \dots \exists y_n \mathrm{B}(\mathrm{t}_1, \dots, \mathrm{t}_m, y_1, \dots, y_n)$

and it follows by repeated applications REL6 that

(A_{m-1}) $((\mathcal{E}(A)\{\lceil \mathrm{t}_1 \rceil, \dots, \lceil \mathrm{t}_{m-1} \rceil\}, i, \zeta)$ RE $\forall x_m \exists y_1 \dots \exists y_n \mathrm{B}(\mathrm{t}_1, \dots, \mathrm{t}_{m-1}, \mathrm{x}_m, y_1, \dots, y_n))$,

(A_{m-2}) $((\mathcal{E}(A)\{\lceil \mathrm{t}_1 \rceil, \dots, \lceil \mathrm{t}_{m-2} \rceil\}, i, \zeta)$ RE $\forall x_{m-1} \forall x_m \exists y_1 \dots \exists y_n \mathrm{B}(\mathrm{t}_1, \dots, \mathrm{t}_{m-2}, \mathrm{x}_{m-1}, \mathrm{x}_m, y_1, \dots, y_n))$,

* Actually only the consistency of the quantifier and Δ free sentences of T (i.e. the computational assertions of T) are at issue.

…

(A_1) $((\mathcal{E}(A)\{\lceil t_1 \rceil\}, i, \zeta)$ RE $\forall x_2 \ldots \forall x_m \exists y_1 \ldots \exists y_n B(t_1, x_2, \ldots, x_{m-1}, x_m, y_1, .., y_n))$,

(A_0) $((\mathcal{E}(A), i, \zeta)$ RE $\forall x_1 \ldots \forall x_m \exists y_1 \ldots \exists y_n B(x_1, x_2, \ldots, x_{m-1}, x_m, y_1, .., y_n))$

[End of proof]

Theorem II.1: (Soundness of RCA) There is an integer valued function $E(n,m)$ such that if P is a formal proof in RCA of a sentence A and $i \geq \Delta(P)$ and $\zeta > m(i + \Delta(P))$ then $(E(\lceil P \rceil, \zeta), i, \zeta)$ RE A.

Proof: The proof proceeds by induction on the length of proof P of A. If P has length 1 then it consists of one of the axioms of RCA.

Case 1: Axiom Groups I, II, III, V, and VI.

If A is in Axiom Group I, II, III, V, or VI, then it is an AE sentence to which Lemma II.5 applies. For these cases define $E(\lceil P \rceil, \zeta) = \mathcal{E}(A)$.

Case 2: Axiom Groups IV and VIII

The induction schema IND(RCA) of Group VIII includes the induction schema IND(RA) of Group IV. Let A be the sentence $B(0) \wedge \forall n(B(n) \supset B(n+1)) \supset \forall n B(n)$, where $B(x)$ is any wff in L_{RCA} with free variable x.

Suppose (a, i, ζ) RE $B(0) \wedge \forall n(B(n) \supset B(n+1))$. By REL2, for $a = \langle a_0, a_1 \rangle$,

S_0: (a_0, i, ζ) RE $B(0)$ and

(a_1, i, ζ) RE $\forall n(B(n) \supset B(n+1))$.

That is,

(a_1, i, ζ) RE $\forall x(N(x) \supset (B(x) \supset B(x+1)))$.

From Axiom Group II, for any natural number p, it follows that $\vdash_T N(\underline{p})$, and, therefore by REL1,

$(0, i, \zeta)$ RE $N(\underline{p})$.

From REL6,

$(a_1\{\lceil \underline{p} \rceil\}, i, \zeta)$ RE $(N(\underline{p}) \supset (B(\underline{p}) \supset B(\underline{p}+1)))$

and therefore, from REL4,

S_{p+1}: $(a_1\{\lceil \underline{p} \rceil, 0\}, i, \zeta)$ RE $B(\underline{p}) \supset B(\underline{p}+1)$ for natural numbers $p \geq 0$.

Define by recursion the function

$\varphi(a, 0) = a_0$,

$\varphi(a, p+1) = a_1\{\lceil \underline{p} \rceil, 0, \varphi(a, p)\}$ for natural numbers $p \geq 0$.

Claim:

(E1) $(\varphi(a,p),i,\zeta)$ RE $B(\underline{p})$ for $p \geq 0$.

From S_0 we have

$(\varphi(a,0),i,\zeta)$ RE $B(0)$.

Now assume that

$(\varphi(a,p),i,\zeta)$RE $B(\underline{p})$.

From S_{p+1}, $(a_1\{\lceil \underline{p} \rceil,0\},i,\zeta)$ RE $B(\underline{p}) \supset B(\underline{p}+1)$ and therefore

$(a_1\{\lceil \underline{p} \rceil,0,\varphi(a,p)\},i,\zeta)$ RE $B(\underline{p}+1)$.

Now $\lfloor \underline{p}+1 \rfloor = \lfloor \underline{p+1} \rfloor$, so by Lemma II.1,

$(a_1\{\lceil \underline{p} \rceil,0,\varphi(a,p)\},i,\zeta)$ RE $B(\underline{p+1})$,

By the recursion definition of φ, $\varphi(a,p+1) = a_1\{\lceil \underline{p} \rceil,0,\varphi(a,p)\}$, so

$(\varphi(a,p+1),i,\zeta)$ RE $B(\underline{p+1})$.

Assertion E1 now follows by ordinary induction on p.

It then follows from E1 by Lemma II.1 that if $t \in TERM(z)$ and $\lfloor t^{\zeta} \rfloor$ is the natural number p, then

(E2) $(\varphi(a,p),i,\zeta)$ RE $B(t)$.

Define $E(\lceil P \rceil,\zeta)$ to be the number such that

$E(\lceil P \rceil,\zeta)\{a,\lceil t \rceil,b\} = \varphi(a,\lfloor t^{\zeta} \rfloor) = \varphi(a,\hbar(\lceil t \rceil,\zeta))$,

where $\hbar$ is an effectively computable function defined so that $\hbar(\lceil t \rceil,\zeta) = \lfloor t^{\zeta} \rfloor$. We now show that

$(E(\lceil P \rceil,\zeta),i,\zeta)$ RE $B(0) \wedge \forall n(B(n) \supset B(n+1)) \supset \forall nB(n)$.

It suffices to show that

$(E(\lceil P \rceil,\zeta)\{a\},i,\zeta)$ RE $\forall nB(n)$,

where (a,i,ζ) RE $B(0) \wedge \forall n(B(n) \supset B(n+1))$.

$\forall nB(n)$ is the sentence $\forall x(N(x) \supset B(x))$. Let $t \in TERM(z)$ and suppose (b,i,ζ) RE $N(t)$ for some natural number b. Then $\lfloor t^{\zeta} \rfloor$is a natural number, say p. By definition of $E(\lceil P \rceil,\zeta)$,

$(E(\lceil P \rceil,\zeta)\{a,\lceil t \rceil,b\},i,\zeta) = (\varphi(a,\lfloor t^{\zeta} \rfloor),i,\zeta) = (\varphi(a,p),i,\zeta)$

which, by assertion E2, realizes $B(t)$. Unwinding this through REL4 and REL6, we see that $(E(\lceil P \rceil,\zeta)\{a\},i,\zeta)$ RE $\forall nB(n)$ is established.

[End of proof of Case 2: Axiom Groups IV and VIII]

Case 3: Axiom Group VII

K1: $K(0) \wedge \forall n(K(n) \wedge n > 0 \supset K(n-1))$.

Define $E(\lceil P \rceil) = 0$.

Since $0 = \langle 0,0 \rangle$, we have to show that

(a) $(0, i, \zeta)$ RE $K(0)$ and

(b) $(0, i, \zeta)$ RE $\forall n(K(n) \wedge n > 0 \supset K(n-1))$, that is,

$(0, i, \zeta)$ RE $\forall x(N(x) \supset (K(x) \wedge x > 0 \supset K(x-1)))$.

Proof of (a): $\vdash_T K(\underline{i}, 0)$ follows from MOD4. (a) then follows from this and from REL1.

Proof of (b): Suppose $t \in TERM(z)$. To demonstrate (b) we must show that

$(0\{\lceil t \rceil\}, i, \zeta)$ RE $N(t) \supset (K(t) \wedge t > 0 \supset K(t-1))$, that is,

$(0, i, \zeta)$ RE $N(t) \supset (K(t) \wedge t > 0 \supset K(t-1))$.

Suppose

(c) (a, i, ζ) RE $N(t)$.

From REL1, it follows that $a = 0$ and $\vdash_T N(t^\zeta)$, so there is an integer $p = \lfloor t^\zeta \rfloor$. Since $0\{a\} = 0$ we must show that

$(0, i, \zeta)$ RE $K(t) \wedge t > 0 \supset K(t-1)$,

that is, if

(d) $(0, i, \zeta)$ RE $K(t) \wedge t > 0$

then

(e) $(0, i, \zeta)$ RE $K(t-1)$.

From REL2 and REL1, (d) implies $\vdash_T K(i, t^\zeta)$ and $\vdash_T t^\zeta > 0$, and consequently $\vdash_T K(i, t^\zeta - 1)$ from MOD4. (e) then follows by another application of REL1.

[End of proof for K1]

K2: $\forall x(K(x) \supset N(x))$

Define $E(\lceil P \rceil) = 0$ and let $t \in TERM(z)$. We must show

$(0\{\lceil t \rceil\}, i, \zeta)$ RE $K(t) \supset N(t)$, that is

(a) $(0, i, \zeta)$ RE $K(t) \supset N(t)$.

Let (a, i, ζ) RE $K(t)$ for some a. Then $\vdash_T K(i, t^\zeta)$. From MOD5 we can show $\vdash_T N(t^\zeta)$ from which it follows from REL1 that $(0\{\mathrm{a}\}, i, \zeta)$ RE $N(t)$ which in turn proves (a) and establishes the case for K2.

K3: $K(\underline{p_0})$ where p_0 is a given nalural number.

Define $E(\lceil P \rceil) = 0$. We must show that $(0, i, \zeta)$ RE $K\left(\underline{p_0}\right)$, that is, $\vdash_T K(i, \underline{p_0})$, but this is an follows from MOD3 and MOD6.

K4: $N(z)$.

ζ is a natural number so $\vdash_T N\left(\underline{\zeta}\right)$ follows from Axiom Group II. Apply REL1 to get $(0, i, \zeta)$ RE $N(z)$. Define $E(\lceil P \rceil) = 0$ and we are done with K4.

K5: $\forall n(K(n) \supset \Delta K(\mathcal{F}(n)))$

Define $E(\lceil P \rceil) = 0$. Need to show that

$(0, i, \zeta)$ RE $\forall x(N(x) \supset (K(x) \supset \Delta K(\mathcal{F}(x)))$.

Let $t \in TERM(z)$; we must show that

$(0\{\lceil t \rceil\}, i, \zeta)$ RE $N(t) \supset (K(t) \supset \Delta K(\mathcal{F}(t))$.

Let (a, i, ζ) RE $N(t)$ for some a. Then $\lfloor t^{\zeta} \rfloor$ is a natural number and a must be 0 by REL1. $0\{\lceil t \rceil\}\{0\} = 0$ so we only need to show that

$(0, i, \zeta)$ RE $K(t) \supset \Delta K(\mathcal{F}(t))$.

Suppose (b, i, ζ) RE $K(t)$ for some b. Then $b = 0$ and $\vdash_T K(i, t^{\zeta})$ by REL1, so, by MOD3, $\vdash_T K(i+1, \mathcal{F}(t^{\zeta}))$. Therefore $(0, i+1, \zeta)$ RE $K(\mathcal{F}(t))$, so by REL8, $(0, i, \zeta)$ RE $\Delta K(\mathcal{F}(t))$. Since $0\{0\} = 0$, it follows that $(0, i, \zeta)$ RE $K(t) \supset \Delta K(\mathcal{F}(t))$.

[End of proof for K5]

K6: $\mathrm{K}(\underline{n}) \supset \Delta^{\mathrm{n}} \forall m(K(m) \supset \mathcal{F}(\underline{n}, m) < \mathrm{z})$ for any natural number n.

The Δ-depth $\Delta(\{\mathrm{K6}\})$ is n. The hypothesis of the Soundness Theorem asserts that i and ζ are chosen so that $i \geq n$ and $\zeta > \mathscr{m}(i+n)$. Therefore

(1) $\vdash_T \underline{\zeta} > \mathscr{m}(\underline{i+n})$.

Define $E(\lceil P \rceil) = 0$.

Assume (a, i, ζ) RE $\mathrm{K}(\underline{\mathrm{n}})$. Then by REL1, $a = 0$ and $\vdash_T K(\underline{i}, \underline{n})$ so, from MOD2 and MOD7 we get

(2) $\vdash_T \underline{n} \leq \mathscr{k}(\underline{i}) \wedge \mathscr{k}(\underline{i}) \leq \mathscr{k}(\underline{i+n})$.

We have to show that $(0, i, \zeta)$ RE $\Delta^{\mathrm{n}} \forall m(K(m) \supset \mathcal{F}(\underline{n}, m) < \mathrm{z})$, that is, $(0, i+n, \zeta)$ RE $\forall m(K(m) \supset \mathcal{F}(\underline{n}, m) < \mathrm{z})$. It suffices to show that for any natural number l,

(3) $(0, i+n, \zeta)$ RE $K(\underline{l}) \supset \mathcal{F}(\underline{n}, \underline{l}) < \mathrm{z}$.

Suppose $(0, i+n, \zeta)$ RE $K(\underline{l})$. Then $\vdash_T K(\underline{i+n}, \underline{l})$, so by MOD7,

(4) $\vdash_T \underline{l} \leq \mathscr{k}(\underline{i+n})$.

$\mathcal{F}$ is increasing so, from (2) and (4) we get $\vdash_T \mathcal{F}(\underline{n}, \underline{l}) \leq \mathcal{F}(\mathscr{k}(\underline{i+n}), \mathscr{k}(\underline{i+n}))$ and by Mod8, $\vdash_T \mathcal{F}(\mathscr{k}(\underline{i+n}), \mathscr{k}(\underline{i+n})) = \mathscr{m}(\underline{i+n})$, so $\vdash_T \mathcal{F}(\underline{n}, \underline{l}) \leq \mathscr{m}(\underline{i+n})$ which, combined with (1) shows that $\vdash_T \mathcal{F}(\underline{n}, \underline{l}) < \underline{\zeta}$ and therefore $(0, i+n, \zeta)$ RE $\mathcal{F}(\underline{n}, \underline{l}) < \mathrm{z}$.

[End of proof for K6]

Note that no constraint is placed on n – it can be an arbitrary natural number. However, whatever the choice of n, under the hypothesis of the Soundness Theorem, n is an "arrived" number by stage i.

K7: $K(\underline{\ell}) \supset \forall n(K(n) \supset \Delta^{\ell}\, \forall m(K(m) \supset \mathcal{F}\,(m,n) < \mathrm{z}))$ where ℓ is any natural number.

The proof is similar to that of K6. Let P be a proof consisting of the axiom K7. Then $\Delta(P) = \ell$. Let e denote $E(\lceil P \rceil)$. The hypothesis of the Soundness Theorem asserts that i and ζ are choosen so that $i > \ell$ and $\zeta > m(i + \ell)$. Therefore

(1) $\vdash_T m(\underline{i + \ell}) < \underline{\zeta}$.

If (a,i,ζ) RE $K(\underline{\ell})$ then $a = 0$. We must have

$(e\{0\},i,\zeta)$ RE $\forall \mathrm{n}(K(\mathrm{n}) \supset \Delta^{\ell}\, \forall m(K(m) \supset \mathcal{F}\,(m,\mathrm{n}) < \mathrm{z}))$.

Let $s \in TERM(z)$ and assume that $(0,i,\zeta)$ RE $N(s)$.

We need to show that

$(e\{0,\lceil s \rceil,0\},i,\zeta)$ RE $(K(\mathrm{s}) \supset \Delta^{\ell}\, \forall m(K(m) \supset \mathcal{F}\,(m,\mathrm{s}) < \mathrm{z}))$.

Let (b,i,ζ) RE $K(\mathrm{s})$. Therefore $b = 0$ and $\vdash_T K(\underline{i},s^{\zeta})$ so

(2) $\vdash_T s^{\zeta} \le \mathscr{k}(\underline{i}) \wedge \mathscr{k}(\underline{i}) \le \mathscr{k}(\underline{i+\ell})$.

We now need to have

$(e\{0,\lceil s \rceil,0,0\},i,\zeta)$ RE $\Delta^{\ell}\, \forall m(K(m) \supset \mathcal{F}\,(m,\mathrm{s}) < \mathrm{z})$, that is,

$(e\{0,\lceil s \rceil,0,0\},i+\ell,\zeta)$ RE $\forall \mathrm{m}(K(m) \supset \mathcal{F}\,(m,\mathrm{s}) < \mathrm{z})$.

Let $t \in TERM(z)$. Now suppose $(0,i,\zeta)$ RE $N(t)$. We need

$(e\{0,\lceil s \rceil,0,0,\lceil t \rceil,0\},i+\ell,\zeta)$ RE $K(t) \supset \mathcal{F}\,(t,s) < \mathrm{z}$.

Let $(c,i+\ell,\zeta)$ RE $K(\mathrm{t})$. Therefore $c = 0$ and $\vdash_T K(\underline{i+\ell},t^{\zeta})$, and, consequently

(3) $\vdash_T t^{\zeta} \le \mathscr{k}(\underline{i+\ell})$.

Finally we need

$(e\{0,\lceil s \rceil,0,0,\lceil t \rceil,0,0\},i+\ell,\zeta)$ RE $\mathcal{F}\,(t,s) < \mathrm{z}$.

From the definition of m, $m(i + \ell) = \mathcal{F}(\mathscr{k}(i+\ell),\mathscr{k}(i+\ell))$ and therefore we have

(4) $\vdash_T m(\underline{i + \ell}) = \mathcal{F}\left(\mathscr{k}(\underline{i + \ell}),\mathscr{k}(\underline{i + \ell})\right)$.

And from (2), (3), and (4)

(5) $\vdash_T \mathcal{F}\left(t^{\zeta},s^{\zeta}\right) < m(\underline{i + \ell})$.

Combining (1) and (5) we get

(6) $\vdash_T \mathcal{F}\left(t^{\zeta},s^{\zeta}\right) < \underline{\zeta}$

It follows that if we define $E(\lceil P \rceil) = 0$, then $E(\lceil P \rceil)\{0,\lceil s \rceil,0,0,\lceil t \rceil,0,0\} = 0$, and hence

$(E(\lceil P \rceil),i+\ell,\zeta)$ RE $\mathcal{F}\,(t,s) < \mathrm{z}$ and hence

$(E(\lceil P \rceil),i,\zeta)$ RE $K(\underline{\ell}) \supset \forall n(K(n) \supset \Delta^{\ell}\, \forall m(K(m) \supset \mathcal{F}\,(m,n) < \mathrm{z}))$

as needed.

[End of proof for K7]

[End of proof of Case 3: Axiom Group VII]

Case 4: Axiom Group IX

BLEP and BMEP can be deduced from LEP. The proofs of these schemata have the same $\Delta - depth$ as the axioms themselves. One can therefore treat Case 4 as consequences of the Soundness Theorem once we have established the Soundness Theorem for LEP and the other axioms groups (TPC and TL) and that realization is preserved under the rules of inference (MP, EI, UI, ΔI and ΔE).

Proof of LEP: $\exists n A(n) \wedge \forall n\,(A\,(n)\ \vee \neg A\,(n)\,) \supset \exists n(A(n) \wedge \forall m(m < n \supset \neg A(m))$

Let $P = \langle \exists n A(n) \wedge \forall n\,(A\,(n)\ \vee \neg A\,(n)\,) \supset \exists n(A(n) \wedge \forall m(m < n \supset \neg A(m)) \rangle$. We will define $E(P,\zeta)$ such that for any i and ζ,

$(E(P,\zeta), i, \zeta)$ RE $\exists n A(n) \wedge \forall n\,(A\,(n)\ \vee \neg A\,(n)\,) \supset \exists n(A(n) \wedge \forall m(m < n \supset \neg A(m))$.

In the following realization statements, i and ζ will be understood so, for example, a RE S will stand for (a,i,ζ) RE S. Also let $e(\zeta)$ represent the value of $E(P,\zeta)$.

Assume

a RE $\exists n A(n) \wedge \forall n\,(A\,(n)\ \vee \neg A\,(n)$.

We must define $e(\zeta)$ so that

$e(\zeta)\{a\}$ RE $\exists n(A(n) \wedge \forall m(m < n \supset \neg A(m))$.

a RE $\exists n A(n) \wedge \forall n\,(A\,(n)\ \vee \neg A\,(n)$ implies that a_0 RE $\exists n A(n)$ and a_1 RE $\forall n\,(A\,(n) \vee \neg A\,(n))$.

So $a_0 = \langle a_{00}, a_{01} \rangle$ where $a_{01} = \lceil s \rceil$ for some numerical valued term s in $TERM(z)$ and a_{00} RE $N(s) \wedge A(s)$. Therefore $a_{00} = \langle 0, a_{001} \rangle$ where 0 RE $N(s)$ and a_{001} RE $A(s)$. 0 RE $N(s)$ implies that $\vdash_T N(s^\zeta)$; let σ_ζ be the natural number value of s^ζ. Then a_{001} RE $A(\underline{\sigma_\zeta})$.

Case 4.1: Suppose $\sigma_\zeta = 0$. It suffices to define $e(\zeta)\{a\} = \langle b, \lceil 0 \rceil \rangle$ where b is such that

b RE $N(0) \wedge (A(0) \wedge \forall m\big(m < 0 \supset \neg A(m)\big))$.

This requires that $b = \langle b_0, b_1 \rangle$ where $b_0 = 0$ and $b_1 = \langle a_{001}, c \rangle$ and

c RE $\forall m\big(m < 0 \supset \neg A(m)\big)$.

Claim: $c = 0$ realizes $\forall m\big(m < 0 \supset \neg A(m)\big)$.

That is, for any term $u \in TERM(z)$ $0\{\lceil u \rceil\}$ RE $\mathrm{N(u)} \supset \big(u < 0 \supset \neg A(u)\big)$. Note that if d RE $N(u)$ then $d = 0$ and $\vdash_T u^\zeta \geq 0$. Thus we must show that $0\{\lceil u \rceil, 0\}$ RE $u < 0 \supset \neg A(u)$. Since $\lfloor u^\zeta \rfloor \geq 0$, $u < 0$ is not realizable and hence $0\{\lceil u \rceil, 0\} = 0$ realizes $u < 0 \supset \neg A(u)$ by default, which established the claim. (Note that $0\{\lceil u \rceil, 0\} = 0$, since φ_0 is the constant 0 function and therefore $0\{\lceil u \rceil, 0\} = 0\{\lceil u \rceil\}\{0\} = 0\{0\} = 0$.)

So if we define $e(\zeta)\{a\} = \langle\langle 0, \langle a_{001}, 0 \rangle\rangle, \lceil 0 \rceil\rangle$, then $e(\zeta)\{a\}$ RE $\exists n(A(n) \wedge \forall m\big(m < n \supset \neg A(m)\big))$.

Case 4.2: Suppose $\sigma_\zeta > 0$. We are given that a_{001} RE $A(\underline{\sigma_\zeta})$ and a_1 RE $\forall n\,(A\,(n) \vee \neg A\,(n))$.

Therefore for any term $t \in TERM(z)$, $a_1\{\lceil t \rceil\}$ RE $\mathrm{N(t)} \supset A\,(t) \vee \neg A\,(t)$. Let 0 RE $\mathrm{N(t)}$. Then $a_1\{\lceil t \rceil, 0\}$ RE $A\,(t) \vee \neg A\,(t)$ and $\vdash_T N(t^\zeta)$.

Define $F_t(a,b) = \begin{cases} a_1\{\lceil t \rceil, 0\}\; if\; b = \lceil t \rceil \\ \quad 0\;\; if\; b \neq \lceil t \rceil \end{cases}$. Note that:

(a) if $F_t(a,\lceil t \rceil)_0 = 0$ then $F_t(a,\lceil t \rceil)_1$ RE $A\,(t)$ and

(b) if $F_t(a,\lceil t \rceil)_0 \neq 0$ then $F_t(a,\lceil t \rceil)_1$ RE $\neg A\,(t)$.

Case 4.2.1 Assume $F_{\underline{0}}\left(a,\left\lceil \underline{0} \right\rceil\right)_0 = 0$. Then $F_{\underline{0}}(a,\left\lceil \underline{0} \right\rceil)_1 = (a_1\{\left\lceil \underline{0} \right\rceil, \underline{0}\})_1$ RE $A\left(\underline{0}\right)$ and we can apply the above claim in Case 1 to conclude that $\langle F_{\underline{0}}(a,\left\lceil \underline{0} \right\rceil)_1, \underline{0}\rangle$ RE $A(\underline{0}) \wedge \forall m\left(m < \underline{0} \supset \neg A(m)\right)$. Therefore if we define $e(\zeta)\{a\} = \langle\langle F_{\underline{0}}(a,\left\lceil \underline{0} \right\rceil)_1, \underline{0}\rangle, \left\lceil \underline{0} \right\rceil\rangle$ then

$e(\zeta)\{a\}$ RE $\exists n(A(n) \wedge \forall m(m < n \supset \neg A(m))$.

Case 4.2.2 Assume $F_{\underline{0}}\left(a,\left\lceil \underline{0} \right\rceil\right)_0 \neq 0$. Let $\mathcal{S}(a,\zeta) = \{m \leq \sigma_\zeta \wedge \forall l(l \leq m \supset F_{\underline{l}}\left(a,\left\lceil \underline{l} \right\rceil\right)_0 \neq 0\,)\}$.

Since

a_{001} RE $A(\,\underline{\sigma}\,)$,

it is not the case that $F_{\underline{\sigma_\zeta}}(a,\left\lceil \underline{\sigma_\zeta} \right\rceil)_1$ RE $\neg A\left(\underline{\sigma_\zeta}\right)$ and therefore $F_{\underline{\sigma_\zeta}}(a,\left\lceil \underline{\sigma_\zeta} \right\rceil)_0 = 0$, that is, $\sigma_\zeta \notin \mathcal{S}(a,\zeta)$. By the assumption of Case 4.2.2, $0 \in \mathcal{S}(a,\zeta)$. Define the effectively calculable function:

$g(a,\zeta) = 1 + \max\,\{m\!: m \in\, \mathcal{S}(a,\zeta))\}$.

By the definition of $g(a,\zeta)$, $F_{\underline{g(a,\zeta)}}\left(a,\left\lceil \underline{g(a,\zeta)} \right\rceil\right)_0 = 0$ and therefore $F_{\underline{g(a,\zeta)}}\left(a,\left\lceil \underline{g(a,\zeta)} \right\rceil\right)_1$ RE $A\left(\underline{g(a,\zeta)}\right)$.

$e(\zeta)\{a\}$ will realize $\exists n(A(n) \wedge \forall m(m < n \supset \neg A(m))$ if we set

$e(\zeta)\{a\} = \langle\langle F_{\underline{g(a,\zeta)}}\left(a,\left\lceil \underline{g(a,\zeta)} \right\rceil\right)_1, b_{a,\zeta}\rangle, \left\lceil \underline{g(a,\zeta)} \right\rceil\rangle$

where $b_{a,\zeta}$ will be defined so that

(1) $b_{a,\zeta}$ RE $\forall x(\mathrm{N(x)} \supset (x < \underline{g(a,\zeta)} \supset \neg A(x)))$.

Let t be any term in $TERM(z)$. (1) holds iff

(2) $b_{a,\zeta}\{\lceil t \rceil\}$ RE $\mathrm{N(t)} \supset (t < \underline{g(a,\zeta)} \supset \neg A(t))$ which holds iff

(3) $b_{a,\zeta}\{\lceil t \rceil, 0\}$ RE $t < \underline{g(a,\zeta)} \supset \neg A(t)$ where 0 RE $\mathrm{N(t)}$ and therefore $\vdash_T \mathrm{N}(t^\zeta)$.

If d RE $t < \underline{g(a,\zeta)}$ then $d = 0$ and $\vdash_T t^\zeta < \underline{g(a,\zeta)}$, that is, $\lfloor t^\zeta \rfloor < g(a,\zeta)$.

From the definition of $g(a,\zeta)$, $\lfloor t^\zeta \rfloor < g(a,\zeta)$ implies that $F_t(a,\lceil t \rceil)_1$ RE $\neg A\,(t)$. Therefore, if we define $b_{a,\zeta}\{\lceil t \rceil, 0,0\} = F_t(a,\lceil t \rceil)_1$, then $b_{a,\zeta}\{\lceil t \rceil, 0,0\}$ RE $\neg A(t)$ and for such a $b_{a,\zeta}$ we have

$b_{a,\zeta}$ RE $\forall x(\mathrm{N}(\mathrm{x}) \supset (x < \underline{g(a,\zeta)} \supset \neg A(x)))$, as needed.

[End of proof Case 4: Axiom Group IX]

Case 5: Intuitionistic Predicate Calculus (IPC)

The axioms of IPC can all be handled as in Kleene (1952, §82). In the following arguments, the expression "(e,i,ζ) RE A" will be shortened to "e RE A" or "e realizes A" and the presence of i and ζ will be understood. We shall make extensive use of Lemma K.

Define $E(\lceil P \rceil)$ as follows.

I1 $P = \{A \wedge B \supset A\}$. Define $E(\lceil P \rceil)\{a\} = a_o$.

I2 $P = \{A \wedge B \supset B\}$. Define $E(\lceil P \rceil)\{a\} = a_1$.

I3 $P = \{(A \supset B) \supset ((A \supset C) \supset (A \supset B \wedge C)))\}$. Define $E(\lceil P \rceil)\{b,c,a\} = \langle b\{a\}, c\{a\}\rangle$.

To establish I3 let b realize $A \supset B$. Therefore, for any a that realizes A, $b\{a\}$ realizes B. For such a b, we must show that

$E(\lceil P \rceil)\{b\}$ RE $(A \supset C) \supset (A \supset B \wedge C)$.

Suppose c realizes $A \supset C$. Then for any a that realizes A, $c\{a\}$ must realize C. Finally we must show that for the selected b and c,

$E(\lceil P \rceil)\{b,c\}$ realizes $A \supset B \wedge C$.

Let a be any realization of A. Then $E(\lceil P \rceil)\{b,c,a\}$ which, by the above definition, equals $\langle b\{a\}, c\{a\}\rangle$, realizes $B \wedge C$ (by Definition REL2)and therefore $E(\lceil P \rceil)\{b,c\}$ realizes $A \supset B \wedge C$.

I4 $P = \{A \supset A \vee B\}$. Define $E(\lceil P \rceil)\{a\} = \langle 0, a\rangle$.

I5 $P = \{B \supset A \vee B\}$. Define $E(\lceil P \rceil)\{b\} = \langle 1, b\rangle$

I6 $P = \{(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))\}$. Define $E(\lceil P \rceil)\{a,b,c\} = \begin{cases} a\{c_1\}\ if\ c_0 = 0 \\ b\{c_1\}\ if\ c_0 \neq 0 \end{cases}$.

The argument proving I6 goes as follows. We must show that

$E(\lceil P \rceil)$ RE $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$.

Suppose a RE $A \supset C$. By Definition REL4, for any d that realizes A, $a\{d\}$ must realize C.

Now we must show that for such an a,

$E(\lceil P \rceil)\{a\}$ RE $(B \supset C) \supset ((A \vee B) \supset C)$.

Suppose b RE $B \supset C$. Again, by Definition REL4, for any d that realizes B, $b\{d\}$ must realize C.

So for such selected a and b, it suffices to show that

$$E(\lceil P \rceil)\{a, b\} \text{ RE } (A \vee B) \supset C.$$

Suppose c RE $A \vee B$. Then, if $c_0 = 0$, c_1 must realize A so by the choice of a, $a\{c_1\}$ realizes C. If $c_0 \neq 0$, c_1 must realize B so by the choice of b, $b\{c_1\}$ realizes C. By definition $E(\lceil P \rceil)\{a, b, c\} = \begin{cases} a\{c_1\} \, if \;\; c_0 = 0 \\ b\{c_1\} \, if \;\; c_0 \neq 0 \end{cases}$, so it follows that $E(\lceil P \rceil)\{a, b, c\}$ realizes C and, hence, $E(\lceil P \rceil)\{a, b\}$ RE $(A \vee B) \supset C$.

I7 $P = \{A \supset A\}$. Define $E(\lceil P \rceil)\{a\} = a$.

I8 $P = \{A \supset (B \supset A)\}$. Define $E(\lceil P \rceil)\{a, b\} = a$.

I9 $P = \{(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))\}$. Define $E(\lceil P \rceil)\{b, c, a\} = c\{a, b\{a\}\}$.

To demonstrate I9, let b be any realization of $A \supset B$ and choose any a that realizes A. Then $b\{a\}$ RE B. We must show that

$$E(\lceil P \rceil)\{b\} \text{ RE } (A \supset (B \supset C)) \supset (A \supset C).$$

Let c realize $A \supset (B \supset C)$. To complete the demonstration of I9, we must show that

$$E(\lceil P \rceil)\{b, c\} \text{ RE } (A \supset C).$$

By choice of c, for any a that realizes A, $c\{a\}$ realizes $B \supset C$. Then, for the above selected b, $c\{a\}\{b\{a\}\}$ must realize C since $b\{a\}$ realizes B. Therefore $E(\lceil P \rceil)\{b, c, a\}$, which is defined equal to $c\{a\}\{b\{a\}\}$, realizes C and, consequently, $E(\lceil P \rceil)\{b, c\}$ RE $(A \supset C)$.

I10 $P = \{A \supset (\neg A \supset B)\}$. Define $E(\lceil P \rceil)\{a, b\} = 0$.

Note that if a RE A then $E(\lceil P \rceil)\{a, b\}$ trivially realizes $\neg A \supset B$ since the hypothesis $\neg A$ is not realizable by Definition REL5.

I11 $P = \{(A \supset B) \supset ((A \supset \neg B) \supset \neg A)\}$. Define $E(\lceil P \rceil)\{a, b\} = 0$.

Suppose a RE $A \supset B$. (Note that if there is no such a then the axiom is trivially realized.) Suppose b RE $(A \supset \neg B)$. (Again, if there is no such b then $((A \supset \neg B) \supset \neg A)$ is trivialy realized.) Then, if there is a c which realizes A then $b\{c\}$ RE $\neg B$ while $a\{c\}$ RE B which is not possible. Therefore there is no c which realizes A so $E(\lceil P \rceil)\{a, b\} = 0$ RE $\neg A$.

I12 $A \supset ((B \wedge \neg B) \supset \neg A)$. Define $E(\lceil P \rceil)\{a\} = 0$. Note that 0 realizes $(B \wedge \neg B) \supset \neg A$ since $(B \wedge \neg B)$ is never realized.

I13 $P = \{\forall x A(x) \supset A(t)\}$, where t is in $TERM(z)$. Define $E(\lceil P \rceil)\{a\} = a\{\lceil t \rceil\}$.

By Definition REL6, a realize $\forall x A(x)$ implies $a\{\lceil t \rceil\}$ realizes $A(t)$ and hence $E(\lceil P \rceil)\{a\} = a\{\lceil t \rceil\}$ realizes $A(t)$.

I14 $P = \{A(t) \supset \exists x A(x)\}$, where t is a term in $TERM(z)$. Define $E(\lceil P \rceil)\{a\} = \langle a, \lceil t \rceil \rangle$. If a RE $A(t)$ then, by Definition REL7, $E(\lceil P \rceil)\{a\} = \langle a, \lceil t \rceil \rangle$ RE $\supset \exists x A(x)$.

I15 $P = \{\forall x (A \supset B) \supset (\forall x A \supset \forall x B)\}$. Define for any natural numbers a and b and for any $t \in TERM(z)$, $E(\lceil P \rceil)\{a, b, \lceil t \rceil\} = a\{\lceil t \rceil\}\{b\{\lceil t \rceil\}\}$.

Suppose a RE $\forall x(A \supset B)$. We must show that $E(\lceil P \rceil)\{a\}$ RE $(\forall xA \supset \forall xB)$. Suppose b RE $\forall xA$. We must show that $E(\lceil P \rceil)\{a,b\}$ RE $\forall xB$. Let t be any term in $TERM(z)$. For the selected a, $a\{\lceil t \rceil\}$ RE $A_t^x \supset B_t^x$. For the selected b, $b\{\lceil t \rceil\}$ RE A_t^x. Therefore $E(\lceil P \rceil)\{a,b,\lceil t \rceil\} = a\{\lceil t \rceil\}\{b\{\lceil t \rceil\}\}$ RE B_t^x, and hence, $E(\lceil P \rceil)\{a,b\}$ RE $\forall xB$.

I16 $P = \{\forall x(A \supset B) \supset (A \supset \forall xB)\}$ if x not free in A. Define $E(\lceil P \rceil)\{a,b,\lceil t \rceil\} = a\{\lceil t \rceil\}\{b\}$ for any natural numbers a and b and any term t int $TERM[z]$.

Let a RE $\forall x(A \supset B)$. Then for any t int $TERM(z)$, $a\{\lceil t \rceil\}$ RE $A \supset B_t^x$ since x not free in A. We must show that $E(\lceil P \rceil)\{a\}$ RE $A \supset \forall xB$, that is, for any natural number b, if b RE A, then $E(\lceil P \rceil)\{a,b\}$ RE $\forall xB$. For the selected a and b and any t in $TERM(z)$, $E(\lceil P \rceil)\{a,b,\lceil t \rceil\} = a\{\lceil t \rceil\}\{b\}$ RE B_t^x so $E(\lceil P \rceil)\{a,b\}$ RE $\forall xB$.

I17 $P = \{\exists x(A \supset B) \supset (A \supset \exists xB)\}$ if x not free in A. Define $E(\lceil P \rceil)\{a,b\} = \langle a_0\{b\}, a_1\rangle$ for any natural numbers a and b.

Suppose a RE $\exists x(A \supset B)$. By REL7, there is a term t in $TERM(z)$, such that $a = \langle a_0, a_1\rangle$ where $a_1 = \lceil t \rceil$ and a_0 RE $A \supset B_t^x$. We must show that $E(\lceil P \rceil)\{a\}$ RE $A \supset \exists xB$, that is, we must show that if b RE A then $E(\lceil P \rceil)\{a,b\}$ RE $\exists xB$. By REL4 and the choice of a, $a_0\{b\}$ RE B_t^x so $E(\lceil P \rceil)\{a,b\} = \langle a_0\{b\}, a_1\rangle$ RE $\exists xB$.

I18 $P = \{t = s \supset (A(t) \supset A(s))\}$ where s and t are terms in $TERM(z)$ and s and t are free for x in $A(x)$. Define $E(\lceil P \rceil)\{a,b\} = b$.

Let (a,i,ζ) RE $t = s$. We must show that $(E(\lceil P \rceil)\{a\},i,\zeta)$ RE $A(t) \supset A(s)$. By REL1, $\vdash_T t^\zeta = s^\zeta$, so t^ζ and s^ζ have equal values. By Lemma II.1, for any natural number b, (b,i,ζ) RE $A(t)$ iff (b,i,ζ) RE $A(s)$. Consequently (b,i,ζ) RE $A(t)$ implies $(E(\lceil P \rceil)\{a,b\},i,\zeta) = (b,i,\zeta)$ RE $A(s)$, so $(E(\lceil P \rceil)\{a\},i,\zeta)$ RE $A(t) \supset A(s)$.

I19 $P = \{\forall x(x = x)\}$. Define $E(\lceil P \rceil, \zeta)\{\lceil t \rceil\} = 0$. I19 follows immediately from REL6 and REL1.

I20 $P = \{\forall xy(x = y \vee \neg x = y)\}$. For $t,s \in TERM(z)$, define

$$E(\lceil P \rceil)\{t,s\} = \begin{cases} \langle 0,0\rangle \; if \; \vdash_T t^\zeta = s^\zeta \\ \langle 1,0\rangle \; if \; \vdash_T \neg t^\zeta = s^\zeta \end{cases}.$$

We must show that for any t, $s \in TERM(z)$, $E(\lceil P \rceil)\{\lceil t \rceil, \lceil s \rceil\}$ RE $t = s \vee \neg t = s$. $t^\zeta = s^\zeta$ is decidable in T (see Lemma II.2.) If $\vdash_T t^\zeta = s^\zeta$ then $(E(\lceil P \rceil)\{\lceil t \rceil, \lceil s \rceil\})_0 = 0$ and $(E(\lceil P \rceil)\{\lceil t \rceil, \lceil s \rceil\})_1 = 0$ RE $t = s$, by REL1. Therefore $E(\lceil P \rceil)\{\lceil t \rceil, \lceil s \rceil\}$ RE $t = s \vee \neg t = s$ by Definition REL3. If $\vdash_T \neg\, t^\zeta = s^\zeta$ then $(E(\lceil P \rceil)\{\lceil t \rceil, \lceil s \rceil\})_0 = 1$. There is no a which realizes $t = s$ because if there were, by REL1, $t^\zeta = s^\zeta$ would be provable in T and T would be inconsistent. Therefore $(E(\lceil P \rceil)\{\lceil t \rceil, \lceil s \rceil\})_1 = 0$ RE $\neg t = s$ and hence $E(\lceil P \rceil)\{\lceil t \rceil, \lceil s \rceil\}$ RE $t = s \vee \neg t = s$ by Definition REL3.

[End of proof for Intuitionistic Predicate Calculus]

Case 6: Tense Logic(TL)

Δ1 $P = \{\Delta A \wedge \Delta B \equiv \Delta(A \wedge B)\}$. Define $E(\lceil P \rceil)\{a\} = a$. Note that

(a, i, ζ) RE $\Delta A \wedge \Delta B$

iff (a_0, i, ζ) RE ΔA and (a_1, i, ζ)RE ΔB

iff $(a_0, i+1, \zeta)$ RE A and $(a_1, i+1, \zeta)$RE B

iff $(a, i+1, \zeta)$ RE $A \wedge B$

iff (a, i, ζ)RE $\Delta(A \wedge B)$.

△2 $P = \{\Delta A \vee \Delta B \equiv \Delta(A \vee B)\}$. Define $E(\lceil P \rceil)\{a\} = a$. Note that

(a, i, ζ)RE $\Delta A \vee \Delta B$

iff $a_0 = 0$ and (a_1, i, ζ) RE ΔA, or $a_0 \neq 0$ and (a_1, i, ζ) RE ΔB

iff $a_0 = 0$ and $(a_1, i+1, \zeta)$ RE A, or $a_0 \neq 0$ and $(a_1, i+1, \zeta)$ RE B

iff $(a, i+1, \zeta)$ RE $A \vee B$

iff (a, i, ζ) RE $\Delta(A \vee B)$.

△3 $P\{(\Delta A \supset \Delta B) \equiv \Delta(A \supset B)\}$. Define $E(\lceil P \rceil)\{a\} = a$.

Assume (a, i, ζ) RE $\Delta A \supset \Delta B$. To show that (a, i, ζ) RE $\Delta(A \supset B)$ we must show that $(a, i+1, \zeta)$ RE $A \supset B$. Suppose $(b, i+1, \zeta)$ RE A. Then (b, i, ζ) RE ΔA so $(a\{b\}, i, \zeta)$ RE ΔB and therefore $(a\{b\}, i+1, \zeta)$ RE B. We conclude that $(a, i+1, \zeta)$ RE $(A \supset B)$, so (a, i, ζ) RE $\Delta(A \supset B)$. This argument is easily reversed.

△4 $P = \{(\Delta\neg A) \equiv \neg\Delta A\}$. Define $E(\lceil P \rceil)\{a\} = a$.

(a, i, ζ) RE $\Delta\neg A$

iff $(a, i+1, \zeta)$ RE $\neg A$ t

iff for all b, not $(b, i+1, \zeta)$ RE A

iff for all b, not (b, i, ζ) RE ΔA,

iff (a, i, ζ) RE $\neg\Delta A$.

△5 $P = \{\Delta\forall x A(x) \equiv \forall x \Delta A(x)\}$. Define $E(\lceil P \rceil)\{a\} = a$.

(a, i, ζ) RE $\Delta\forall x A(x)$

iff $(a, i+1, \zeta)$ RE $\forall x A(x)$

iff for any $t \in TERM(z)$, $(a\{\lceil t \rceil\}, i+1, \zeta)$ RE $A(t^{\zeta})$

iff for any $t \in TERM(z)$, $(a\{\lceil t \rceil\}, i, \zeta)$ RE $\Delta A(t^{\zeta})$

iff (a, i, ζ) RE $\forall x \Delta A(x)$.

△6 $P = \{\Delta\exists x A(x) \equiv \exists x \Delta A(x)\}$. Define $E(\lceil P \rceil)\{a\} = a$

(a, i, ζ) RE $\Delta\exists x A(x)$

iff $(a, i+1, \zeta)$ RE $\exists x A(x)$

iff for some $t \in TERM(z)$ $a = \langle a_0, \lceil t \rceil \rangle$ and $(a_0, i+1, \zeta)$ RE $A(t^{\zeta})$

iff for some $t \in TERM(z)$ $a = \langle a_0, \lceil t \rceil \rangle$ and (a_0, i, ζ) RE $\Delta A(t^{\zeta})$ so

iff (a, i, ζ) RE $\exists x \Delta A(x)$.

[End of proof for Tense Logic]

Case 7: Persistence of Atomic Truths schema (PAT).

Let B be the sentence $\forall x_{1.} \dots x_n(A \supset \Delta A)$ where A is a quantifier free wff in L_{RCA} not containing the predicate K and with free variables $x_{1.} \dots x_n$. Let $P = \{\forall x_{1.} \dots x_n(A \supset \Delta A)\}$.

$\Delta(P) = 1$.

Given any terms $t_{1,} \dots, t_n$ in $TERM(z)$, we must define $E(\lceil P \rceil$ so that

$$(E(\lceil P \rceil, \zeta)\{\lceil t_{1,} \rceil \dots, \lceil t_n \rceil\}, i, \zeta) \text{ RE } (A^{t_1,\dots,t_n}_{x_1.\dots x_n} \supset \Delta A^{t_1,\dots,t_n}_{x_1.\dots x_n}).$$

That is, we must show that for any a such that (a, i, ζ) RE $A^{t_1,\dots,t_n}_{x_1.\dots x_n}$ then

$$(E(\lceil P \rceil, \zeta)\{\lceil t_{1,} \rceil \dots, \lceil t_n \rceil, a\}, i, \zeta) \text{ RE } \Delta A^{t_1,\dots,t_n}_{x_1.\dots x_n}.$$

Now by Lemma II.4 part(2), (a, i, ζ) RE $A^{t_1,\dots,t_n}_{x_1.\dots x_n}$ implies $\vdash_T A^{t_1,\dots,t_n,i,\zeta}_{x_1.\dots x_n}$. Note that

(1) for any i and j, $A^{t_1,\dots,t_n,i,\zeta}_{x_1.\dots x_n} = A^{t_1,\dots,t_n,j,\zeta}_{x_1.\dots x_n}$

because A does not contain K. Therefore it follows that $\vdash_T A^{t_1,\dots,t_n,i+1,\zeta}_{x_1.\dots x_n}$.

By Lemma II.4 part (2),

$$(\mathcal{B}(\left\lceil A^{t_1,\dots,t_n,i+1,\zeta}_{x_1.\dots x_n} \right\rceil), i+1, \zeta) \text{ RE } A^{t_1,\dots,t_n}_{x_1.\dots x_n}.$$

Furthermore, because of (1) above, the value of $\mathcal{B}(\left\lceil A^{t_1,\dots,t_n,i+1,\zeta}_{x_1.\dots x_n} \right\rceil$ is independent of i.

Define $E(\lceil P \rceil, \zeta)\{\lceil t_1 \rceil, \dots, \lceil t_n \rceil, \mathrm{a}\} = \mathcal{B}(\left\lceil A^{t_1,\dots,t_n,i+1,\zeta}_{x_1.\dots x_n} \right\rceil)$. Then

$$(E(\lceil P \rceil, \zeta)\{\lceil t_1 \rceil, \dots, \lceil t_n \rceil, a\}, i+1, \zeta) \text{ RE } A^{t_1,\dots,t_n}_{x_1.\dots x_n}$$

and therefore

$$(E(\lceil P \rceil, \zeta)\{\lceil t_1 \rceil, \dots, \lceil t_n \rceil, a\}, i, \zeta) \text{ RE } \Delta A^{t_1,\dots,t_n}_{x_1.\dots x_n}$$

and so

$$(E(\lceil P \rceil, \zeta), i, \zeta) \text{ RE } \forall x_{1.} \dots x_n(A \supset \Delta A).$$

[End of proof for PAT][STOP]

Note: In the special case the A is a sentence not containing K or z, $E(\lceil P \rceil, \zeta)$ does not depend on ζ.

This finishes the proof of the Soundness Theorem for proofs of length 1. Suppose the length of proof P is greater than 1. We assume that the Soundness theorem has been proved for all proofs Q whose length is less than that of P.

The last statement A of P is either an axiom or follows from the previous statements by one of the rules of inference. If A is an Axiom then let Q be the proof $Q = \{A\}$. The $\Delta - depth(Q) \leq \Delta - depth(P)$. Since $i \geq \Delta - depth(P)$ and $\zeta > m(i + \Delta - depth(P))$, the same is true for Q. The Soundness Theorem has already been proved for Q so define $E(\lceil P \rceil, \zeta) = E(\lceil Q \rceil, \zeta)$.

It remains to consider the 4 rules of inference.

Modus Ponens Suppose $P \vdash_T A$ consists of two sub proofs $Q1 \vdash_T B$ and $Q2 \vdash_T B \supset A$ followed by an application of MP to obtain A. By the hypothesis of the Main Theorem, $i \geq \Delta(P)$ and $\zeta > m(i + \Delta(P)$ and, since the $\Delta(Q1)$ and $\Delta(Q2)$ are less than of equal to $\Delta(P)$, the Soundness Theorem induction hypothesis applies to *Q1* and *Q2* so $(E(\lceil Q1 \rceil, \zeta), i, \zeta)$ RE B and $(E(\lceil Q2 \rceil, \zeta), i, \zeta)$ RE $B \supset A$. Therefore $(E(\lceil Q2 \rceil, \zeta)\{E(\lceil Q1 \rceil, \zeta)\}, i, \zeta)$ RE A. Define $E(\lceil P \rceil, \zeta) = E(\lceil Q2 \rceil, \zeta)\{E(\lceil Q1 \rceil, \zeta)\}$.

Universal Generalization Suppose $P \vdash_T \forall x A(x)$ consists of a subproof $Q \vdash_T A(x)$ followed by an application of UG to obtain $\forall x A(x)$. $\Delta(P) \geq \Delta(Q)$. Let t be any term in $TERM(z)$. Let Q_t^x be the result of replacing all free occurrences of x in $Q \vdash_T A(x)$ by t.* Then $Q_t^x \vdash_T A(t)$ and $\Delta(P) \geq \Delta(Q_t^x)$.

If $i \geq \Delta(P)$ and $\zeta > m(i + \Delta(P))$ then $i \geq \Delta(Q_t^x)$ and $\zeta > m(i + \Delta(Q_t^x)$ so we can apply the Soundness Theorem induction hypothesis to Q_t^x, that is $(E(\lceil Q_t^x \rceil, \zeta), i, \zeta)$ RE $A(t)$. Define $E(\lceil P \rceil, \zeta)\{\lceil t \rceil\} = E(\lceil Q_t^x \rceil, \zeta)$. Since t was any term in $TERM(z)$, it follows from Definition REL6, $(E(\lceil P \rceil, \zeta), i, \zeta)$ RE $\forall x A(t)$.

Δ-Introduction Suppose $P: \vdash_T \Delta A$ contains a subproof $Q: \vdash_T A$ followed by an application of ΔI. That is:

$Q: \vdash_T A$
------ΔI
$P: \vdash_T \Delta A$

$\Delta(P) \geq \Delta(Q) + 1$. Assuming $i \geq \Delta(P)$ and $\zeta > m(i + \Delta(P))$ it follows that $i + 1 > \Delta(Q)$ and $\zeta > m(i + 1 + \Delta(Q))$ so we can apply the Soundness Theorem induction hypothesis to Q and conclude that $(E(\lceil Q \rceil, \zeta), i + 1, \zeta)$ RE A and therefore, if we define $E(\lceil P \rceil, \zeta) = E(\lceil Q \rceil, \zeta)$, it follows that $(E(\lceil P \rceil, \zeta), i, \zeta)$ RE ΔA.

---

* Two free occurrences O_1 and O_2 of a variable v in a proof Q are directly identified iff one occurs in the hypothesis of a rule of inference of Q and the other in the consequence of that rule of inference. Two free occurrences O_1 and O_n of a variable v in a proof Q are identified iff there is a sequence $O_1, O_2, \ldots, O_n$ of directly identified occurrence O_i, O_{i+1} of v in Q, $i = 1, .., n-1$. The expression Q_t^x represents the result of replacing with term t all free of occurrence of x in Q that can be identified with a free occurrence of x in the conclusion of Q.

Δ-Elimination. Suppose $P{:}\vdash_T A$ consists of a sub proof $Q{:}\vdash_T \Delta A$ followed by an application of ΔE. That is:

$Q{:}\vdash_T \Delta A$

------ΔE

$P{:}\vdash_T A$

Therefore $\Delta(P) \geq \Delta(Q) + 1$. Assume $i \geq \Delta(P)$ and $\zeta > m(i + \Delta(P))$. So $i - 1 \geq \Delta(P) - 1 \geq \Delta(Q)$ and $\zeta > m(i + \Delta(P)) \geq m(i + \Delta(Q) + 1)) > m(i - 1 + \Delta(Q))$. By the Soundness Theorem induction hypothesis, $(E(\lceil Q \rceil, \zeta), i - 1, \zeta)$ RE ΔA, and therefore by REL8, $(E(\lceil Q \rceil, \zeta), i, \zeta)$ RE A, so it suffices to define $E(\lceil P \rceil, \zeta) = E(\lceil Q \rceil, \zeta)$.

[End of proof of the Soundness Theorem]

Note: The dependency of $E(\lceil P \rceil, \zeta)$ on ζ is only required if the proof P uses the induction axiom or LEP or BMEP or MLEP ot PAT. As a result we get the immediate corollary:

Corollary II.1 (Soundness of RCA without IND, LEP, BMEP, MLEP, PAT) There is an integer valued function $E^{-}(n)$ such that if P is a formal proof without IND, LEP, BMEP, MLEP, PAT in RCA of a sentence A and $i \geq \Delta(P)$ and $\zeta > m(i + \Delta(P))$ then $(E^{-}(\lceil P \rceil), i, \zeta)$ RE A. As a simple consequence, $E^{-}(\lceil P \rceil)$ continues to realize A for any $i_1 \geq i$ and $\zeta_1 \geq \zeta$.

The undecidability of $K(x)$ in RCA follows from the consistency of $RCAMOD$.

Theorem II.2 Assuming $RCAMOD$ is consistent, there is no proof of $\forall n(K(n) \vee \neg K(n))$ in RCA.

Proof: Suppose $\vdash_{RCA} \forall n(K(n) \vee \neg K(n))$. Since $\vdash_{RCA} 0 \leq z \wedge K(0)$ (see Theorem I.1 (1),(5)) the hypothesis of BMEP for the predicate $K(x)$, namely

$(0 \leq z \wedge K(0) \wedge \forall n(n < z \supset K(n) \vee \neg K(n)))$,

would be provable in RCA. Then we could apply BMEP and conclude

(1) $\vdash_{RCA} \exists n\big(n \leq z \wedge K(n) \wedge (\neg K(n+1) \vee n + 1 > z)\big)$.

From Theorem I.1 (4) we know that, $\vdash_{RCA} \forall n(K(n) \supset n < z)$. Combining this with (1) we see that there would be a proof P of

(2) $\vdash_{RCA} \exists n(K(n) \wedge \neg K(n+1))$.

Choose $\zeta > m(1 + 2 * \Delta(P))$ and define $i = \Delta(P)$. Then, for $j = i$ and $j = i + 1$, j is greater than or equal to $\Delta(P)$ and $\zeta > m(j + \Delta(P))$. Therefore, by the Soundness Theorem,

$(E(\lceil P \rceil, \zeta), j, \zeta)$ RE $\exists n(K(n) \wedge \neg K(n+1))$ for $j = i$ and $j = i + 1$.

This means that for some term $t \in TERM(z)$, $E(\lceil P \rceil, \zeta) = \langle a, \lceil t \rceil \rangle$, where (a, j, ζ) RE $K(t) \wedge \neg K(t+1)$. From definition REL1, this implies

(3) $\vdash_{RCAMOD} K\big(\underline{i}, t^{\underline{\zeta}}\big) \wedge \neg K(\underline{i}, t^{\underline{\zeta}} + 1) \wedge K\big(\underline{i+1}, t^{\underline{\zeta}}\big) \wedge \neg K(\underline{i+1}, t^{\underline{\zeta}} + 1)$,

so in particular,

(4) $\vdash_{RCAMOD} K(\underline{i}, \underline{t^\zeta})$

and

(5) $\vdash_{RCAMOD} \neg K(\underline{i+1}, \underline{t^\zeta} + 1)$.

But $\vdash_{RCAMOD} K(\underline{i}, \underline{t^\zeta}) \supset K(\underline{i+1}, \mathcal{F}(\underline{t^\zeta}))$ from MOD3 and $\vdash_{RCAMOD} \mathcal{F}(\underline{t^\zeta}) \geq \underline{t^\zeta} + 1$ from Theorem I.1(6) so, applying MOD4, we can show that

(6) $\vdash_{RCAMOD} K(\underline{i}, \underline{t^\zeta}) \supset K(\underline{i+1}, \underline{t^\zeta} + 1)$

Combining (4) with (6) yields

(7) $\vdash_{RCAMOD} K(\underline{i+1}, \underline{t^\zeta} + 1)$.

which contradicts (5).

[End of proof]

The analogous argument for $\neg K(n)$ combined with the decidablity of $K(n,m)$ in $RCAMOD$, can be used to establish the undecidability of $\neg K(n)$.

Lemma II.6 For a given natural number $n \geq 1$ and a proof $Q: \mathcal{G} \vdash_{RCA} A \supset B$ with assumptions $\mathcal{G}$ we can construct a proof $\mathcal{R}(Q,n): \mathcal{G} \vdash_{RCA} \Delta^n A \supset \Delta^n B$ with $\Delta - depth$ $\Delta(\mathcal{R}(Q,n)) = n + \Delta(Q)$.

Proof: We first construct $\mathcal{R}(Q,1)$.

$Q: \mathcal{G} \vdash_{RCA} A \supset B$

----------------ΔI

$\mathcal{G} \vdash_{RCA} \Delta(A \supset B)$ $\quad\quad \vdash_{RCA} \Delta(A \supset B) \supset (\Delta A \supset \Delta B)$

--------+----------------------------MP

$\mathcal{R}(Q,1): \mathcal{G} \vdash_{RCA} \Delta A \supset \Delta B$

Note that $\Delta(\mathcal{R}(Q,1)) = 1 + \Delta(Q)$.

Given $\mathcal{R}(Q,n): \mathcal{G} \vdash_{RCA} \Delta^n A \supset \Delta^n B$ such that $\Delta(\mathcal{R}(Q,n)) = n + \Delta(Q)$, inductively define $\mathcal{R}(Q,n+1) = \mathcal{R}(\mathcal{R}(Q,n),1)): G \vdash_{RCA} \Delta^{n+1} A \supset \Delta^{n+1} B$. Then $\Delta(\mathcal{R}(Q,n+1)) = 1 + \Delta(\mathcal{R}(Q,n),1) = 1 + n + \Delta(Q)$.

The Lemma follows by induction.

[End of Proof]

Lemma II.7 For any natural number n and proof $Q_n: G \vdash_{RCA} \Delta^n A$ we can construct a proof $\mathcal{S}(Q_n, n): \mathcal{G} \vdash_{RCA} A$ of $\Delta - depth = n + \Delta(Q_n)$.

Proof: Just apply n applications of ΔE.

Theorem II.3 For any natural number n there is a proof $V_n: \vdash_{RCA} K(\underline{n})$ with $\Delta(V_0) = 0$ and for $n > 0$, $\Delta(V_n) = n + 1$.

Claim 1: For any natural number n, there is a proof $P_n: \vdash_{RCA} \underline{n} \leq \mathcal{F}^n(0)$ of $\Delta - depth =$ 0.

Proof: From Axiom Group III, DEF0 $\forall m(N(f_0(m)) \wedge f_0(m) > m \wedge \forall n(m < n \supset f_0(m) < f_0(n)))$

we can establish for any numerical term t in $TERM$, $\vdash_{RCA} f_0(t) > \mathcal{F}^0(t)$. By convention, $\mathcal{F}^0(0)$ denotes 0. Let P_0 denote the proof of $0 \leq \mathcal{F}^0(0)$.

Use the following proof schema to construct the proof P_n for $n \geq 0$.

```
P_0: ⊢_{RCA} 0 ≤ F^0(0)                              ⊢_{RCA} ∀m∀n(f_0(n) > m ⊃ f_0(n) ≥ m + 1)
     |      ⊢_{RCA} F^1(0) = f_0(F^0(0))             --------------------------------UI
     |        |      ⊢_{RCA} f_0(F^0(0)) > F^0(0)    ⊢_{RCA} f_0(F^0(0)) > F^0(0) ⊃ f_0(F^0(0)) ≥ F^0(0) + 1
     |        |      -----------------------+------------------------------------MP
     |        |                   ⊢_{RCA} f_0(F^0(0)) ≥ F^0(0) + 1
     |      --o--------------+---------------(Substitution)
     |                F^1(0) ≥ F^0(0) + 1
---o-----+-------------------(Arithmetic)
P_1 ⊢_{RCA} 1 ≤ F^1(0)                               ⊢_{RCA} ∀m∀n(f_0(n) > m ⊃ f_0(n) ≥ m + 1)
     |      ⊢_{RCA} F^2(0) = f_0(F^1(0))             --------------------------------UI
     |        |      ⊢_{RCA} f_0(F^1(0)) > F^1(0)    ⊢_{RCA} f_0(F^1(0)) > F^1(0)  ⊃ f_0(F^1(0)) ≥ F^1(0) + 1
     |        |      -----------------------+------------------------------------MP
     |        |                   ⊢_{RCA} f_0(F^1(0)) ≥ F^1(0) + 1
     |      --o-------------+----------------------(Substitution)
     |                F^2(0) ≥ F^1(0) + 1
 ---o------------------------------(Arithmetic)
    P_2 ⊢_{RCA} 2 ≤ F^2(0)
 .  .  .  .
 ---o----------------
   P_n ⊢_{RCA} n ≤ F^n(0)                             ⊢_{RCA} ∀m∀n(f_0(n) > m ⊃ f_0(n) ≥ m + 1)
     |      ⊢_{RCA} F^{n+1}(0) = f_0(F^n(0))          -----------------------------------UI
     |        |      ⊢_{RCA} f_0(F^n(0)) > F^n(0)    ⊢_{RCA} f_0(F^n(0)) > F^n(0)  ⊃ f_0(F^n(0)) ≥ F^n(0) + 1
     |        |      -----------------------+------------------------------------MP
     |        |                   ⊢_{RCA} f_0(F^n(0)) ≥ F^n(0) + 1
     |      --o-------------+----------------------(Substitution)
     |                F^{n+1}(0) ≥ F^n(0) + 1
 ---o------------------------------(Arithmetic)
    P_{n+1} ⊢_{RCA} n + 1 ≤ F^{n+1}(0)
```

Claim 1 follows by induction on n. No instance of ΔI or ΔE were used so $\Delta(P_n) = 0$.

<u>Claim 2</u>: For any natural number n, there is a proof Q_n:$\vdash_{RCA} K(\mathcal{F}^n(0))$ with $\Delta(Q_0) = 0$ and $\Delta(Q_n) = n + 1$ for $n > 0$.

<u>Proof:</u> From Axiom K5 we can construct proofs U_n:$\vdash_{RCA} K(\mathcal{F}^n(0)) \supset \Delta K(\mathcal{F}^{n+1}(0))$ with $\Delta(U_n) = 1$ for $n \geq 0$, and from U_n and Theorem I.1(1) we can recursively construct the proofs Q_n by means of the following schema:

```
                    ⊢_{RCA} ∀n(K(n) ⊃ ΔK(F(n)))
                    --------------------UI
```

$Q_0 : \vdash_{RCA} K(0)$ $\quad$ $U_0 : \vdash_{RCA} K(0) \supset \Delta K(\mathcal{F}(0))$

-----+---------------------------------MP

$\vdash_{RCA} \Delta K(\mathcal{F}(0))$ $\quad$ $\vdash_{RCA} \forall n(K(n) \supset \Delta K(\mathcal{F}(n)))$

------------ΔE $\quad$ ---------------------UI

$Q_1 : \vdash_{RCA} K(\mathcal{F}(0))$ $\quad$ $U_1 : \vdash_{RCA} K(\mathcal{F}(0)) \supset \Delta K(\mathcal{F}^2(0))$

------+----------------------------------MP

$\vdash_{RCA} \Delta K(\mathcal{F}^2(0))$

------------ΔE

$Q_2 : \vdash_{RCA} K(\mathcal{F}^2(0))$

. . . $\quad$ $\vdash_{RCA} \forall n(K(n) \supset \Delta K(\mathcal{F}(n)))$

------------ΔE $\quad$ ---------------------UI

$Q_{n-1} : \vdash_{RCA} K(\mathcal{F}^{n-1}(0))$ $\quad$ $U_{n-1} : \vdash_{RCA} K(\mathcal{F}^{n-1}(0)) \supset \Delta K(\mathcal{F}^n(0))$

-------+-------------------------------------MP

$\vdash_{RCA} \Delta K(\mathcal{F}^n(0))$

--------------ΔE

$Q_n : \vdash_{RCA} K(\mathcal{F}^n(0))$

U_n is an instance of Axiom K5 and always has $\Delta - depth$ 1. We can see by inspection that

$\Delta(Q_0) = 0$,

$\Delta(Q_1) = 2$

and for $n > 1$, with the induction hypothesis that $\Delta(Q_{n-1}) = n$,

$\Delta(Q_n) = \max\{\Delta(Q_{n-1}), \Delta(U_{n-1})\} + 1 = \max\{\Delta(Q_{n-1}), 1\} + 1 = \Delta(Q_{n-1}) + 1 = n + 1$.

Claim 2 follows by induction.

Proof: (of Theorem II.3)

From Theorem I.1(3), namely that $\vdash_{RCA} \forall l \forall m(l < m \supset (K(m) \supset K(l)))$,

we can construct the following proof $R_n : \vdash_{RCA} \underline{n} < \mathcal{F}^n(0) \supset \left(K\left(\mathcal{F}^n(0)\right) \supset K\left(\underline{n}\right)\right)$

$\vdash_{RCA} \forall l \forall m(l < m \supset (K(m) \supset K(l)))$

-----------------------------UI

$\vdash_{RCA} \forall m(\underline{n} < m \supset \left(K(m) \supset K\left(\underline{n}\right)\right))$

-----------------------------UI

$R_n : \vdash_{RCA} \underline{n} < \mathcal{F}^n(0) \supset \left(K\left(\mathcal{F}^n(0)\right) \supset K\left(\underline{n}\right)\right)$

The $\Delta - depth$ of R_n is 0. Using this and the proof P_n of Claim 1 and proof Q_n of claim 2, we construct, for $n \geq 0$, a proof V_n as follows:

$P_n : \vdash_{RCA} \underline{n} \leq \mathcal{F}^n(0)$ $\quad$ $R_n : \vdash_{RCA} \underline{n} < \mathcal{F}^n(0) \supset \left(K\left(\mathcal{F}^n(0)\right) \supset K\left(\underline{n}\right)\right)$

---------+---MP

$\vdash_{RCA} K(\mathcal{F}^n(0)) \supset K(\underline{n})$ $Q_n: \vdash_{RCA} K(\mathcal{F}^n(0))$

-----+---------------------------------MP

$V_n: \vdash_{RCA} K(\underline{n})$

The $\Delta(V_n) = \max\{0, \Delta(Q_n)\} = \Delta(Q_n)$, since $\Delta(P_n) = \Delta(R_n) = 0$. Therefore $\Delta(V_0) = 0$ and for $n > 0$, $\Delta(V_n) = n + 1$.

[End of proof of Theorem II.3.]

On the other hand, not $\vdash_{RCA} \forall n K(n)$ since $\vdash_{RCA} \neg K(z)$ (Theorem I.1(7).)

Theorem II.4 The $\Delta - depth$ of formal proofs in RCA of $K(\underline{p})$ is unbounded as a function of p.

Proof: Suppose there is a bound n_0 such that for all natural numbers p, there is a proof $P \vdash_{RCA} K(\underline{p})$ whose $\Delta - depth$ is less than n_0. Fix $i = n_0$, $\zeta = m(i + n_0) + 1$ and choose $p = \zeta$. By our assumption we can find a proof P_0 of $K(\underline{\zeta})$ whose $\Delta - depth$ is less than n_0. By the Soundness Theorem, $(E(\lceil P_0 \rceil, \zeta), i, \zeta)$ RE $K(\underline{\zeta})$. By REL1, $\vdash_{RCAMOD} K(\underline{i}, \underline{\zeta})$ so $(E(\lceil P_0 \rceil, \zeta), i, \zeta)$ RE $K(z)$. Let $Q \vdash_{RCA} \neg K(z)$ with $\Delta(Q) = 0$ be the proof of $\neg K(z)$ constructed in Theorem I.1(7). Applying the Soundness Theorem again we conclude $(E(\lceil Q \rceil, \zeta), i, \zeta)$ RE $\neg K(z)$ and therefore $(\langle E(\lceil P_0 \rceil, \zeta), E(\lceil Q \rceil, \zeta) \rangle, i, \zeta)$ RE $K(z) \wedge \neg K(z)$ which is not possible. [End of proof]

Theorem II.5 If A is any sentence of L_{RCA} and $\mathcal{G}$ is a finite set of sentences of L_{RCA}. Then for any natural number p, $\mathcal{G} \vdash_{RCA} \Delta^p A$ iff $\mathcal{G} \vdash_{RCA} A$.

Proof: (⟹)apply ΔE p times.

(⟸)apply ΔI p times.

Note that both proofs have $\Delta - depth \geq p$.

[End of proof]

The following definitions of dependent wffs and constant variables in a proof are adopted from [Kleene 1967, p.95][10] and [Hakli & Negri, 2010][11].

Definition II.4 (Dependent wffs in a proof) Let P be a proof of B from a set $\mathcal{G}$ of wffs together with wff A, that is $P: \mathcal{G} \cup \{A\} \vdash_{RCA} B$. B is *dependent* on A iff either B is A or B follows by UG, ΔI or ΔE in P from a wff which is dependent on A or follows in P by MP from two wffs of P at least one of which is dependent on A.

Note: If $P: \mathcal{G} \cup \{A\} \vdash_{RCA} B$ and B in P is not dependent on A then we can effectively find a proof $Q: \mathcal{G} \vdash_{RCA} B$ which is a sub-sequence of P.

Definition II.5 (Constant variables in a proof) Let $A(x)$ with free variable x be a wff occurring in proof P. x is *constant* in P iff for any wff B of P which is dependent on $A(x)$, B does not follow in P from an application of UG applied to a variable which is identified with any free occurrence of x in $A(x)$.

Definition II.6 (Deduction Theorem Conditions) Let P be a proof from $\mathcal{G} \cup \{A\}$ in RCA. Then P and A satisfy the Deduction Theorem Conditions (DT Conditions) iff for every wff B in P one of the following conditions holds.

Condition 1: B is A or $B \in G$ or B ia an axiom in RCA.

Condition 2: $B = \forall x C(x)$ and follows in P from $C(x)$ by an application of UG and that if x is free in A, then x is constant in P.

Condition 3: B follows in P by MP from two previous wffs C and $C \supset B$ of P.

Condition 4: $B = \Delta C$ and follows in P from C by an application of ΔI and either C is not dependent on A or if C is dependent on A then one of condition 4.1-4.3 applies:

Condition 4.1: we can effectively find a proof $R{:}\mathcal{G} \vdash_{RCA} C \supset \Delta C$.

Condition 4.2: we can effectively find a proof $R{:}\mathcal{G} \vdash_{RCA} C$.

Condition 4.3. we can effectively find a proof $R{:}\mathcal{G} \vdash_{RCA} A \supset \Delta A$.

Condition 5: $B = C$ and follows in P from ΔC by an application of ΔE and we can effectively find a proof $R{:}\mathcal{G} \vdash_{RCA} \Delta A \supset A$.

Note: If P and A satisfies the DT conditions then so does every sub-proof of P.

Theorem II.6 (Deduction Theorem for RCA). Let $\mathcal{G}$ be a finite set of sentences in L_{RCA} and let P be a proof of B from $\mathcal{G} \cup \{A\}$ in RCA. If A and P satisfy the DT Conditions then we can effectively transform P into a proof $\mathcal{T}(P){:}\mathcal{G} \vdash_{RCA} A \supset B$.

Proof: The proof is by induction on the length of P. We proceed in parallel to the DT Conditions.

If P is length 1, i.e., $P = \langle B \rangle$, then Condition 1 applies, namely B is A or $B \in G$ or B ia an axiom in RCA.

If B is an axiom or $B \in \mathcal{G}$ then we have

$P{:}\mathcal{G} \vdash_{RCA} B$ $\qquad$ $\vdash_{RCA} B \supset (A \supset B)$

-------+-------------------MP

$\mathcal{T}(P){:}\mathcal{G} \vdash_{RCA} A \supset B$

If $B = A$, then, since $\vdash_{RCA} A \supset A$, we trivially have $\mathcal{T}(P){:}\mathcal{G} \vdash_{RCA} A \supset A$ where $\mathcal{T}(P) = \{A \supset A\}$.

Induction hypothesis: Assume that the Deduction theorem has been established for proofs of length n and that P is a proof of length n+1.

Condition 1: this is handled just as in the case of proofs of length 1.

Condition 2. $B = \forall x C(x)$ and follows in P from $C(x)$ by an application of UG. This occurrence of x is not constant in P because of the application of UG for this occurrence. Therefore, by the assumption of Condition 2, x is not free in A. We can effectively find a sub-proof Q of P of $C(x)$ from $\mathcal{G} \cup \{A\}$. That is

$Q: \mathcal{G} \cup \{A\} \vdash_{RCA} C(x)$

Since Q is shorter than P, we can apply the induction hypothesis and get $\mathcal{T}(Q): \mathcal{G} \vdash_{RCA} A \supset C(x)$. Then we have

$\mathcal{T}(Q): \mathcal{G} \vdash_{RCA} A \supset C(x)$
---------------UG
$\mathcal{G} \vdash_{RCA} \forall x(A \supset C(x))$ $\quad\quad$ $\vdash_{RCA} \forall x(A \supset C(x)) \supset (A \supset \forall x(C(x)))$
--------+--MP
$\mathcal{T}(P): \mathcal{G} \vdash_{RCA} A \supset \forall x C(x)$

Condition 3. B follows in P by MP from two previous wffs C and $C \supset B$ of P, That is, we can effectively find two sub-proofs Q_1 and Q_2 of P such that

$Q_1: \mathcal{G}_1 \cup \{A\} \vdash_{RCA} C$ $\quad\quad$ $Q_2: \mathcal{G}_2 \cup \{A\} \vdash_{RCA} C \supset B$
------+------------------------------MP
$\mathcal{G} \cup \{A\} \vdash_{RCA} B$

where $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$. We can apply the induction hypothesis to Q_1 and Q_2 and get $\mathcal{T}(Q_1): \mathcal{G}_1 \vdash_{RCA} A \supset C$ and $\mathcal{T}(Q_2): \mathcal{G}_2 \vdash_{RCA} A \supset (C \supset B)$. Combine these as follows:

$\mathcal{T}(Q_1): \mathcal{G}_1 \vdash_{RCA} A \supset C$ $\quad\quad$ $\vdash_{RCA} (A \supset C) \supset ((A \supset (C \supset B)) \supset (A \supset B))$
----------+-------------------------------------MP
$\mathcal{G}_1 \vdash_{RCA} ((A \supset (C \supset B)) \supset (A \supset B)$ $\quad\quad$ $\mathcal{T}(Q_2): \mathcal{G}_2 \vdash_{RCA} A \supset (C \supset B)$
-------+---MP
$\mathcal{T}(P): \mathcal{G} \vdash_{RCA} A \supset B$

Condition 4. We now consider the case where $B = \Delta C$ follows in P by ΔI. We can effectively find a sub-proof Q of P of C from $\mathcal{G} \cup \{A\}$, that is

$Q: \mathcal{G} \cup \{A\} \vdash_{RCA} C$

If C is not dependent on A then we can effectively find a sub-proof R of P of C from $\mathcal{G}$ without the additional assumption A, that is, $R: \mathcal{G} \vdash_{RCA} C$.

$R: \mathcal{G} \vdash_{RCA} C$
---------ΔI
$\mathcal{G} \vdash_{RCA} \Delta C$ $\quad\quad$ $\vdash_{RCA} \Delta C \supset (A \supset \Delta C)$
--------+------------------------MP
$\mathcal{T}(P): \mathcal{G} \vdash_{RCA} A \supset \Delta C$

Assume that C is dependent on A, There are three cases corresponding to conditions 4.1, 4.2, and 4.3. Since the sub-proof Q is shorter than P, we can apply the induction hypothesis and get $\mathcal{T}(Q): \mathcal{G} \vdash_{RCA} A \supset C$.

Case 4.1. Assume we can effectively find a proof $S{:}\mathcal{G} \vdash_{RCA} C \supset \Delta C$.

$\mathcal{T}(Q){:}\mathcal{G} \vdash_{RCA} A \supset C \qquad \vdash_{RCA} A \supset C \supset ((C \supset \Delta C) \supset (A \supset \Delta C))$

-----------+----------------------------------MP

$\mathcal{G} \vdash_{RCA} (C \supset \Delta C) \supset (A \supset \Delta C) \qquad S{:}\mathcal{G} \vdash_{RCA} C \supset \Delta C$

-------+------------------------------MP

$\mathcal{T}(P){:}\mathcal{G} \vdash_{RCA} (A \supset \Delta C)$

Case 4.2. Assume we can effectively find a proof $S{:}\mathcal{G} \vdash_{RCA} C$.

$S{:}\mathcal{G} \vdash_{RCA} C$

--------ΔI

$\mathcal{G} \vdash_{RCA} \Delta C \qquad \vdash_{RCA} \Delta C \supset (A \supset \Delta C)$

--------+------------------------MP

$\mathcal{T}(P){:}\mathcal{G} \vdash_{RCA} (A \supset \Delta C)$

Case 4.3. Assume we can effectively find a proof $S{:}\mathcal{G} \vdash_{RCA} A \supset \Delta A$. Apply the induction hypothesis to proof $Q{:}\mathcal{G} \vdash_{RCA} C$.

$\mathcal{T}(Q){:}\mathcal{G} \vdash_{RCA} A \supset C$

----------ΔI

$\mathcal{G} \vdash_{RCA} \Delta(A \supset C) \qquad \vdash_{RCA} \Delta(A \supset C) \supset (\Delta A \supset \Delta C)$

------+---------------------------------MP

$\mathcal{G} \vdash_{RCA} \Delta A \supset \Delta C \qquad \vdash_{RCA} (A\Delta \supset \Delta C) \supset ((A \supset \Delta A) \supset (A \supset \Delta C))$

------------------------------+---------------MP

$S{:}\mathcal{G} \vdash_{RCA} A \supset \Delta A \qquad \mathcal{G} \vdash_{RCA} (A \supset \Delta A) \supset (A \supset \Delta C)$

-------+-------------------------------MP

$\mathcal{T}(P){:}\mathcal{G} \vdash_{RCA} A \supset \Delta C$

Condition 5. $B = C$ and follows in P from ΔC by an application of ΔE and we can effectively find a proof $S{:}\mathcal{G} \vdash_{RCA} \Delta A \supset A$. Let Q be the (effectively determined) sub-proof of P of ΔC, that is

$Q{:}\mathcal{G} \cup \{A\} \vdash_{RCA} \Delta C$.

By the induction hypothesis, we can convert the sub-proof $Q{:}\mathcal{G} \vdash_{RCA} \Delta C$ to a proof $\mathcal{T}(Q){:}\mathcal{G} \vdash_{RCA} A \supset \Delta C$. Construct a proof of $A \supset C$ from G as follows.

$S{:}\mathcal{G} \vdash_{RCA} \Delta A \supset A \qquad \vdash_{RCA} (\Delta A \supset A) \supset ((A \supset \Delta C) \supset (\Delta A \supset \Delta C))$

----------+---------------------------------------MP

$\mathcal{G} \vdash_{RCA} (A \supset \Delta C) \supset (\Delta A \supset \Delta C) \qquad \mathcal{T}(Q){:}\mathcal{G} \vdash_{RCA} A \supset \Delta C$

-----+--------------------------------------MP

$\mathcal{G} \vdash_{RCA} \Delta A \supset \Delta C \qquad \vdash_{RCA} (\Delta A \supset \Delta C) \supset \Delta(A \supset C)$

-------+------------------------------MP

$\mathcal{G} \vdash_{RCA} \Delta(A \supset C)$

-------------ΔE

$\mathcal{T}(P){:}\mathcal{G} \vdash_{RCA} A \supset C$

Note: In Cases 4 and 5 the proof S might not be a sub-proof of P and might be longer than P nor is it assumed that the DT Conditions hold for R and A. Since the transformation $\mathcal{T}$ is not applied to proof R, this does not matter.

[End of proof of the Deduction Theorem]

Theorem II.7 (Proof by Contradiction for RCA). Let $\mathcal{G}$ be a set of sentences in L_{RCA} and let P be a proof of $B \wedge \neg B$ from $\mathcal{G} \cup \{A\}$ in RCA such that P and A satisfy the Deduction Theorem Conditions. Then $\mathcal{G} \vdash_{RCA} \neg A$.

Proof: This is a direct corollary of Theorem II.6.

PART III Rational Constructive Analysis

The formal system RCA shall provide the formal framework for the development of Rational Constructive Analysis. However, it is convenient to proceed in the usual informal style of mathematical arguments, supported by the possibility, should the need arise, of providing a complete formalization within RCA.

It is not possible in a few pages to present an extensive development of Rational Constructive Analysis. Instead I shall provide an excursion through the beginning parts of the theory.

The intuitive picture is this. I shall use rational numbers exclusively to do analysis. Two rationals are identified (we write $x\,EQ\,y$) iff their difference is 'infinitesimal'. Two rationals are distinguished (we write $x\,NE\,y$) if their difference is greater than $1/k$ where k is an 'arrived' natural number. With respect to this notion of equality, the *functionality* of a rule $x \to f(x)$ defined on rationals (i.e., $x\,EQ\,y \supset \Delta(f(x)\,EQ\,f(y))$ implies the continuity of the function defined by the rule. That is, as points get very close, the function values will do likewise. It is also required that, conversely, $f(x)\,NE\,f(y) \supset \Delta(x\,NE\,y)$ i.e. there are no 'jumps'. Thus the setting for analysis is configured. Interestingly, by allowing arbitrary rationals (i.e. no bound on the complexity of numerator and denominator) one gets something closer to the intuitionistic continuum of free choice sequences rather than the constructive continuum of recursive Reals.

Definition III.1 Standard, large, and infinitesimal number predicates:S, L, I.

$x \in K$: $K(x)$. "x is a present or standard positive integer."

$x \in L$: $(\exists n)(K(n) \wedge \mathcal{F}(n, INT(|x|)) \geq z)$. "$x$ is large."

$x \in S$: $(\exists n)\,(K(n) \wedge |x| \leq n)$. "$x$ is standard."

$x \in I$: $x = 0 \vee 1/x \in L$. "x is infinitesimal."

$x\,EQ\,y$: $x - y \in I$. "x equals y."

$x\,NE\,y$: $(\exists n)(K(n) \wedge (|x - y| > 1/n)$.. "$x$ is not equal to y."

$x\,LT\,y$: $(\exists n)(K(n) \wedge x + 1/n \leq y)$. "$x$ is strictly less than y."

$x\ LE\ y$: $(\exists e)\,(e\ \in I\ \wedge x \leq y\ +\ e)$. "$x$ is less than or equal to y."

$x\ GT\ y$: $(\exists n)(K(n)\ \wedge\ y + 1/n \leq x)$ "x is strictly greater than y."

$x\ GT\ 0$: $(\exists n)(K(n)\ \wedge\ 1/n \leq x)$ "x is strictly positive."

$x\ GE\ y$: $(\exists e)\,(e\ \in I\ \wedge x \geq y +\ e)$. "$x$ is greater than or equal to y."

Lemma III.1 z is large.

Proof: $0\ \in\ K$ and $\mathcal{F}(0, INT(|z|)) = \mathcal{F}(0,z) = z$ and $z \geq z$.

[End of proof]

Lemma III.2 If n is standard then $\mathcal{F}(n,n) < z$.

Proof: This can be derived from Axiom K7 of Group VII: $K(\underline{\ell}) \supset \forall n(\ K(n) \supset \Delta^{\ell}\ \forall m(K(m) \supset \mathcal{F}\ (m,n) < \mathrm{z}))$. With $\ell = 0$, and $\vdash_{RCA} K(0)$, we get $\vdash_{RCA} \forall n(\ K(n) \supset \forall m(K(m) \supset \mathcal{F}\ (m,n) < \mathrm{z}))$ from which we can derive

$\vdash_{RCA} \forall n(\ K(n) \supset \mathcal{F}\ (n,n) < \mathrm{z})$.

[End of proof]

Corollary III.1 If n is standard then $\mathcal{F}(n,n)$ is not large, that is, $\neg\mathcal{F}(n,n) \in L$.

Proof: From Lemma III.2 we have

(1) $\vdash_{RCA} \forall n(\ K(n) \supset \mathcal{F}\ (n,n) < \mathrm{z})$.

Let A be the sentence $\exists n(K(n) \wedge \mathcal{F}(n,n)\ \in\ L)$ that is,

$A = \exists n(K(n) \wedge \exists k(K(k) \wedge \mathcal{F}\big(k, \mathcal{F}(n,n)\big) \geq \mathrm{z}))$.

We shall show that $\vdash_{RCA} \neg A$.

We can show by induction on k that $\vdash_{RCA} \forall k \forall n(\mathcal{F}\big(k, \mathcal{F}(n,n)\big) = \mathcal{F}(n+k,n))$. Combining this with A we get

(3) $\vdash_{RCA} A \supset \exists nk(K(n) \wedge K(k) \wedge \mathcal{F}(k+n,n) \geq z)$.

Also we can show that

(4) $\vdash_{RCA} \forall nk(\mathcal{F}(k+n,n) \leq \mathcal{F}(k+n,k+n))$,

so

(5) $\vdash_{RCA} A \supset \exists nk(K(n) \wedge K(k) \wedge \mathcal{F}(k+n,k+n) \geq z)$.

Now $\vdash_{RCA} \forall nk(K(n) \wedge K(k) \supset \Delta K(k+n))$ so we can show from (5) that

(6) $\vdash_{RCA} A \supset \exists n(\Delta K(n) \wedge \mathcal{F}(n,n) \geq z)$.

By Axiom PAT, $\vdash_{RCA} \forall n(\mathcal{F}(n,n) \geq z \supset \Delta(\mathcal{F}(n,n) \geq z))$. Combining this with (6) we get $\vdash_{RCA} A \supset \exists n(\Delta(K(n) \wedge \Delta(\mathcal{F}(n,n) \geq z))$ and pulling out the Δ we get

(7) $\vdash_{RCA} A \supset \Delta\exists n(K(n) \wedge z \leq \mathcal{F}(n,n))$.

From (1) we get, applying ΔI,

(8) $\vdash_{RCA} \Delta\forall \mathrm{n}(\ K(\mathrm{n}) \supset \mathcal{F}\ (n,\mathrm{n}) < \mathrm{z})$.

From (7) and (8) we can derive

(9) $\vdash_{RCA} A \supset \Delta(z < z)$.

However, $\vdash_{RCA} \neg z < z$ and hence $\vdash_{RCA} \Delta(\neg z < z)$ by ΔI and then $\vdash_{RCA} \neg\Delta(z < z)$ follows by the Tense Logic axiom schemata Δ4. Using $\vdash_{RCA} \neg\Delta(z < z)$ and the tautology

$\vdash_{RCA}$ $((A \supset \Delta(z < z)) \wedge \neg\Delta(z < z)) \supset (A \supset \Delta(z < z) \wedge \neg\Delta(z < z))$,

(9) becomes

(10) $\vdash_{RCA} A \supset (\Delta(z < z) \wedge \neg\Delta(z < z)$ so $\vdash_{RCA} \neg A$.

[End of proof]

Corollary III.2 $x \in S$ implies $\neg x \in L$ and $x \in L$ implies $\neg x \in S$, that is, the sets of standard numbers and of large numbers are provably disjoint.

Proof: Suppose that $x \in L$. Then for some $n \in K$, $\mathcal{F}(\underline{n}, INT(|x|)) \geq z$. Suppose $x \in S$. Let $m = \max\{n, INT(|x|)\}$. Since both n and $INT(|x|)$ are in K, m is in K. By Lemma III.2, $\mathcal{F}(\underline{m}, \underline{m}) < z$. Since $\mathcal{F}$ is increasing in both variables, $z \leq \mathcal{F}(\underline{n}, INT(|x|)) \leq \mathcal{F}(\underline{m}, \underline{m}) < z$, a contradiction. Therefore $x \in L$ implies $\neg x \in S$.

Now suppose $x \in S$. If $x \in L$ then for some $n \in K$, $\mathcal{F}(\underline{n}, INT(|x|)) \geq z$. Let $m = \max\{n, INT(|x|)\}$. Arguing as before, m is in K, so again we get $z \leq \mathcal{F}(\underline{n}, INT(|x|)) \leq \mathcal{F}(\underline{m}, \underline{m}) < z$, a contradiction and therefore $\neg x \in L$.

[End of proof]

We can prove the following stronger result.

Corollary III.3 $K(\underline{n}) \supset \Delta^n(K \cap L = \phi)$.

Proof: The argument goes as follows.

Assume $K(\underline{l})$ and let $\mathscr{n}(x) = INT(|x|)$. Using the definitions of S and L, Axiom Group VII, ΔI and Δ-distribution, establish the following:

(1) $\Delta^l(x \in L) \supset \Delta^l \exists n(K(n) \supset \mathcal{F}(n, \mathscr{n}(x)) \geq z)$ (from definition of L, ΔI and Δ-distribution)

(2) $\Delta^l(x \in S) \supset \Delta^{2l} \forall n(K(n) \supset \mathcal{F}(n, \mathscr{n}(x)) < z)$ (from definition of S and ΔI and Δ-distribution)

(3) $\Delta^{2l} \forall n(K(n) \supset \mathcal{F}(n, \mathscr{n}(x)) < z) \supset \Delta^l \forall n(K(n) \supset \mathcal{F}(n, \mathscr{n}(x)) < z)$ ($\mathcal{F}$ is increasing and K grows over time)

(4) $\Delta^l(x \in S) \supset \Delta^l \forall n(K(n) \supset \mathcal{F}(n, \mathscr{n}(x)) < z)$ ((2) and (3))

(5) $\Delta^l(x \in L) \wedge \Delta^l(x \in S) \supset \Delta^l(z < z)$ ((1) and (4) and transitivity of $<$)

(6) $\Delta^l(x \in L) \supset (\Delta^l(x \in S) \supset \Delta^l(z < z))$ and $\Delta^l(x \in S) \supset (\Delta^l(x \in L) \supset \Delta^l(z < z))$

((5) and Tautologies $(A \wedge B \supset C) \supset (A \supset (B \supset C))$ and $(A \wedge B \supset C) \supset (B \supset (A \supset C))$

(7) $\Delta^l(\neg z < z)$ (Axiom L11 and ΔI)

(8) $\Delta^l(x \in L \supset (x \in S \supset (z < z \wedge \neg z < z)))$ and $\Delta^l(x \in S \supset (x \in L \supset (z < z \wedge \neg z < z)))$

((6) and (7) and Δ-distribution and the tautology $A \wedge (B \supset (C \supset D)) \supset (B \supset (C \supset D \wedge A))$

(9) $\Delta^l(x \in L \supset \neg x \in S)$ and $\Delta^l(x \in S \supset \neg x \in L)$ (Axioms for negation)

[End of sketch of proof]

As time proceeds, larger elements of K arrive, smaller elements of L arrive, and never the twain shall meet, if you're careful. What "careful" means here is that the choice of the interpretation for z and for $\mathcal{F}$ depend on the number of stages in the growth of K (temporal steps n represented by Δ^n) required by the various (finite number of) mathematical results of interest.

The following picture illustrates how the infinitesimals, the standard rationals and the large rationals grow through time:

Infinitesimals   Positive Standard   Positive Large

← − − − − 0 − − − −→   ← − − − − − − − − −→   ← − − − − − − − − −

Lemma III.3 None of the predicates, K, L, S, I, or the relations EQ, LT, LE, GT, GE, or NE are decidable.

Proof: For example, to show the undecidability of L argue as follows. Assuming the decidability of L, use BMEP and $\neg L(0)$ to find an $m < z$ such that $\neg L(m)$ and $L(m+1)$. But $L(m+1)$ implies $\Delta L(m)$, by K5, while $\neg L(m)$ implies $\Delta(\neg L(m))$ and so, one obtains $\Delta(L(m) \wedge \neg L(m))$ from which we can deduce $\neg 0 = 0$, a contradiction. [End of proof]

Lemma III.4 K, L, I, LE, GE, LT, GT, EQ are preserved under Δ.

That is,

(a) $n \in K \supset \Delta(n \in K)$.

(b) $x \in L \supset \Delta(x \in L)$.

(c) $x \in I \supset \Delta(x \in I)$.

(d) $x\, LE\, y \supset \Delta(x\ LE\ y)$.

(e) $x\, GE\, y \supset \Delta(x\ GE\ y)$.

(f) $x\, LT\, y \supset \Delta(x\ LT\ y)$.

(g) $x\, GT\, y \supset \Delta(x\ GT\ y)$.

(h) $x\, EQ\, y \supset \Delta(x\ EQ\ y)$.

(i) $x\, EQ\, y \wedge y\, LE\, z \supset \Delta(x\, LE\, z)$.

(j) $x\, LE\, y \wedge y\, LT\, z \supset (x < z) \wedge \Delta(x\, LT\, z)$.

(k) $\forall n(n \in K \supset \neg\Delta(n \in L))$.

NOTE. Because of the Soundness Theorem, arguments iterating these results put constraints on the needed magnitude of z and the number of steps in the generation of K.

Proof: Let $n(x)$ denote $INT(ABS(x))$.

Case (a). K is increasing over time. In particular, from Axion K5, $n \in K \supset \Delta(\mathcal{F}(n) \in K)$ and $\mathcal{F}(n) > n$. Since K is closed under predecessor it follows that $n \in K \supset \Delta(n \in K)$.

Case (b). $x \in L$ impies that there is a $k \in K$ such that $\mathcal{F}(k, n(x)) \geq z$. From PAT $\Delta(\mathcal{F}(k, n(x)) \geq z)$ and from (a), $\Delta(k \in K)$. Therefore $\Delta(k \in K \wedge \mathcal{F}(k, n(x)) \geq z)$ and so $x \in L \supset \Delta(x \in L)$.

Case (c). $x \in I$ implies that $x = 0$ or there is an $1/x \in L$. From (b), $1/x \in L \supset \Delta(1/x \in L)$, so $x \in I \supset \Delta(x \in I)$.

Case (d). $x\, LE\, y$ implies there is an $e \in I$ such that $x \leq y + e$. From (c) it follows that $e \in I \supset \Delta(e \in I)$. Since $x \leq y + e \supset \Delta(x \leq y + e)$ it follows that $x\, LE\, y \supset \Delta(x\ LE\ y)$.

Case (e). The proof is similar to that of (d).

Case (f). $x\, LT\, y$ implies that there is $n \in K$ such that $x \leq y - 1/n$. By Case (a), $n \in K \supset \Delta(n \in K)$. $x \leq y - 1/n \supset \Delta(\, x \leq y - 1/n)$, so $x\, LT\, y \supset \Delta(x\ LT\ y)$.

Case (g). Similar to Case (f).

Case (h). $x\, EQ\, y$ implies that $|x - y| \in I$. From (c) it follows that

$$|x - y| \in I \supset \Delta(|x - y| \in I)$$

and hence, $x\, EQ\, y \supset \Delta(x\ EQ\ y)$.

Case (i) $x\, EQ\, y \wedge y\, LE\, z$ imply that there are $e, f \in I$ such that $x \leq y + e$ and $y \leq z + f$ so $x \leq z + e + f$. Now $e, f \in I \supset \Delta(e + f \in I)$, so $\Delta(x\, LE\, z)$.

Case (j). $w\, LE\, x \wedge x\, LT\, y$ implies that there is an $\varepsilon \in I$ and $n \in K$ such that $w \leq x + \varepsilon$ and $x + 1/n < y$, so $w + 1/n \leq x + 1/n + \varepsilon < y + \varepsilon$. Therefore $w < y + (\varepsilon - 1/n)$ and since ε is infinitesimal, $\varepsilon < 1/n$, it follows that $w\, LE\, x \wedge x\, LT\, y \supset w < y$. Moreover, $n \in K$ implies that $\Delta(2n \in K)$, so it follows that $\Delta(\varepsilon < 1/(2n))$. $w + 1/(2n) < y + \varepsilon - 1/2n$ and hence, $\Delta(2n \in K \wedge (w + 1/(2n) < y))$. Therefore $\Delta(w\, LT\, y)$.

Case (k). To prove: $\forall n(K(n) \supset \neg\Delta(n \in L))$.

This follows from the stronger result of Corollary III.3: $K(\underline{n}) \supset \Delta^n(K \cap L = \phi)$ and the fact that $K(1)$ and $\forall n\big(K(n) \supset \Delta K(n)\big)$ are theorems of RCA.

[8tarse End of Proof of Lemma III.4.]

The standard $x \in S$ shall form the 'continuum' under the identification relation EQ. EQ is an Δ-equivalence relation, that is, if $x\, EQ\, y$ and $y\, EQ\, z$ then $\Delta(x\, EQ\, z)$. This and other usual algebraic facts follow from Theorem III.1.

<u>Theorem III.1</u> Closure properties for RCA.

(a) $x, y \in S$ implies $\Delta(x + y \in S \wedge\, x * y \in S)$.

(b) $(x \in S) \wedge (x\, NE\, 0)$ implies $(1/x \in S) \wedge (1/x\, NE\, 0)$.

(c) $x, y \in I$ implies $(x * y \in I) \wedge \Delta(x + y \in I)$.

(d) $(x \in L) \wedge (0\, NE\, y) \wedge (y \in S)$ implies $\Delta(x/y \in L)$.

(e) $(x \in I) \wedge (y \in K)$ implies $\Delta(x * y \in I)$.

(f) $x * y \in L$ implies $\Delta(x \in L \vee y \in L)$.

(g) $x * y \in I$ implies $\Delta(x \in I \vee y \in I)$.

(h) $(n, m, k, l \in K \wedge n/m\ EQ\ k/l)$ implies $n/m = k/l$.

Proof: To prove these closure properties one must require that $\mathcal{F}(n)$ grow in excess of various rates.

Case(a). Let $w = max\{|x|, |y|\}$. Since $w \in S$, $|w| < n$ for some $n \in K$. By Axiom K5, $\Delta(\mathcal{F}(n) \in S)$, so, if $\mathcal{F}(n) > \max\{2n, n^2\}$ then $|x + y| \leq \mathcal{F}(n)$ and $|x * y| \leq \mathcal{F}(n)$ so $\Delta(x * y \in S)$.

Case (b). $(x \in S) \wedge (x\ NE\ 0)$ implies that there is an $n \in K$ such that $1/n < |x| < n$. Tacking recipricals we get $n > |1/x| > 1/n$ so $(1/x \in S) \wedge (1/x\ NE\ 0)$.

Case (c). Let $x, y \in I$. Therfore $1/x, 1/y \in L$ so is an $n \in K$ such that $\mathcal{F}(n, |1/x|) > z$ and $\mathcal{F}(n, |1/y|) > z$. Since $|1/x\,y|$ is greater than either $1/x$ or $1/y$ it follows that $\mathcal{F}(n, |1/xy|) > z$ and hence $xy \in I$. If $x + y = 0$ we are done. Without loss of generality, assume that $0 < |x| < |y|$. Then $|x + y| \leq |x| + |y|$ so $1/|x + y| \geq 1/(|x| + |y|) \geq 1/|2y|$. Since $y \in I$, there is an $n \in K$ such that $\mathcal{F}(n, |1/y|) > z$. If $\mathcal{F}(n) \geq 2n$ then

$$\mathcal{F}(n + 1, 1/|x + y|) = \mathcal{F}(n, \mathcal{F}(1/|x + y|)) \geq \mathcal{F}(n, \mathcal{F}(1/|2y|)) \geq \mathcal{F}(n, 1/|y|) > z$$

so $\mathcal{F}(n + 1, 1/|x + y|) > 1/\mathrm{z}$. Since $n \in K \supset \Delta(n + 1 \in K)$ it follows that $\Delta(\exists m(m \in K \wedge \mathcal{F}(m, 1/|x + y|) > 1/\mathrm{z})$ and therefore $\Delta(1/|x + y| \in L)$, that is, $\Delta(x + y \in I)$.

Case (d). Let $(x \in L) \wedge (0\ NE\ y) \wedge (y \in S)$. Then there exists $n_1, n_2, n_3 \in K$ such that $\mathcal{F}(n_1, |x|) > z$ and $1/n_2 < |y| < n_3$. Let $n = \max\{n_1, n_2, n_3\}$. Therefore $n \in K$ and $\mathcal{F}(n, |x|) > z$ and $n > 1/|y| > 1/n$. Assume $\mathcal{F}(n) > 2n$. $n \in K \supset \Delta(n + n \in K)$ and p0

$$\mathcal{F}(n + n, |x/y|) = \mathcal{F}(n, \mathcal{F}(n, |x/y|)) > \mathcal{F}(n, 2^n|x/y|)) > \mathcal{F}(n, 2^n|x|/n) > \mathcal{F}(n, |x|) > z.$$

So $\Delta(\exists \mathrm{m}(\mathrm{m} \in \mathrm{K} \wedge (\mathcal{F}(m, |x/y|) > z))$.

Case (e). Let $(x \in I) \wedge (y \in K)$. We must show that $\Delta(x * y \in I)$. We can assume that $|y| > 0$ since otherwise $xy = 0 \in I$. From the assumptions, there are $n_1, n_2 \in I$ such that $\mathcal{F}(n_1, 1/|x|) > z$ and $|y| < n_2$. Let $n = \max\{n_1, n_2\}$. Therefore $n \in K$ and $\mathcal{F}(n, 1/|x|) > z$ and $0 < |y| < n$. Assume $\mathcal{F}(n) > 2n$. Then $2^n > n > |y|$ and hence

$$\mathcal{F}(n + n, 1/|xy|) = \mathcal{F}(n, \mathcal{F}(n, 1/|xy|)) > \mathcal{F}(n, 2^n/|xy|)) > \mathcal{F}(n, 1/|x|) > z.$$

Now $n \in K \supset \Delta(n + n \in K)$ so $\Delta(\exists \mathrm{m}(\mathrm{m} \in \mathrm{K} \wedge (\mathcal{F}(m, 1/|xy|) > z))$ and we are done.

Case(f). Suppose that $x * y \in L$ and that $M = max\{|x|, |y|\}$. Then $M^2 \geq |x * y|$. $x * y \in L$ implies that $\mathcal{F}(k, |x * y|) > z$ for some $k \in K$ and therefore $\mathcal{F}(k, M^2) > z$. Assuming that $\forall n(\mathcal{F}(n) \geq n^2)$, it follows that $\mathcal{F}(k + 1, M) > z$. Therefore $\Delta(M \in L)$ and hence $\Delta(x \in L \vee y \in L)$.

Case (g). Follows directly from Case (f). $x * y \in I$ iff $1/(x * y) \in L$ which, by (f) implies that $1/x \in L \vee 1/y \in L$ which imlies that $x \in I \vee y \in I$.

Case (h). To prove: $(n, m, k, l \in K \wedge n/m\ EQ\ k/l)$ implies $n/m = k/l$. We are given that $|n/m - k/l| = |nl - mk|/|ml|$ is infinitesimal. Assume $n/m \neq k/l$. Then $|ml|/|nl - mk|$ is in L. Since n, m, k, l are integers, $|nl - mk| \geq 1$ and therefore $|ml|/|nl - mk| < |ml|$, so $ml \in L$. Without loss of generality we can assume that $|m| \geq |l|$ which, by previous Case (f) implies that $\Delta(m \in L)$. But $m \in K \supset \neg\Delta(m \in L)$ which follows from $m \in K \supset \Delta(m \in K)$ and $\Delta(K \cap L = \phi)$.(See Corollary III.3.) Therefore $n/m = k/l$.

[End of proof]

Corollary III.4 EQ is an Δ-equivalence relation. Furthermore, if n is in K and $x_i\ EQ\ x_{i+1}$ for $i = 1, \ldots, n$ then $\Delta(x_1\ EQ\ x_{n+1})$.

Proof: We are given that there are $N_1, \ldots, N_n \in L$ such that $|x_i - x_{i+1}| < 1/N_i$ for $i = 1..n$. Let N be the maximum of the N_i. Then $|x_1 - x_{n+1}| < 1/(N/n)$. Since $N \in L$, it follows from Theorem III.1 (d) that $\Delta(N/n \in L)$ and therefor $\Delta(x_1 EQ x_{n+1})$.

[End of proof]

Corollary III.5 $(S, EQ, NE, +, *, 0, 1)$ is a field.

Of course, any closure statements may require a tense operator and only standard $x\ NE\ 0$ have inverses. Strictly speaking this structure is not a field but might be called a "Δ-field".

Theorem III.2 If $x \in S$ and $x\ GE\ 0$ then x has a square root in S. That is, we can determine a $y \in S$ such that $y^2\ EQ\ x$.

Proof: If $x < 0$ then set $y = 0$. Since $x \geq -e$ for some positive infinitesimal e, $ABS(y * y - x) = -x \leq e$, and therefore $y * y\ EQ\ x$.

For x > 0 choose $c = max\{z * x, z\})$. The property $(k/z)^2 < x$ is decidable*. Note that $(c/z)^2 \geq x$ so we can use BMEP to define natural number $k \leq c$ such that $(k/z)^2 \leq x < ((k+1)/z)^2 = (k/z + 1/z)^2$. Set $y = k/z$. Then $y^2 \leq x < y^2 + 2y/z + 1/z^2$ so

$$ABS(y * y - x) < (2y + 1/z)/z.$$

$y^2 \leq x$ implies $y \geq 1 \wedge 1 \leq x \wedge y \leq x$ or $y < 1$ so, correspondingly,

$$ABS(y * y - x) < (2x + 1/z)/z < 3x/z \text{ or } ABS(y * y - x) < (2 + 1/z)/z < 3/z.$$

* That $(k/z)^2 < x$ is decidable follows from the axioms for linear ordering. Of course we can't determine which of $(k/z)^2 < x$ or $\neg((k/z)^2 < x)$ holds until particular natural numbers have been substituted in for k, x, and for the parameter z. What the proof of Theorem III.2 does is use BMEP to give a prescription for computing k and hence the candidate square root y of x, once x and z have been specified. And, of course, we guarantee that the result of Theorem III.2, that $y^2\ EQ\ x$ is valid, by choosing z to meet the conditions of the Soundness Theorem of Section II with respect to the proof of Theorem II.2.

$\Delta(3/z \in I)$ and, since x is in S, $\Delta^2(3x/z \in I)$, so either $y \geq 1$ and $\Delta^2(y^2\, EQ\, x)$ or $y < 1$ and $\Delta(y^2\, EQ\, x)$. These last two assertions are independent of the choice of $\mathcal{F}$.

By choosing a suitable $\mathcal{F}$ we can actually guarantee $y^2\, EQ\, x$. Chose n in K such that $x \leq n$ and $1 < n$. (This is possible since x is in S. If there is a finite set of standard positive rationals for which we want square roots, choose the largest such n.) Therefore

$ABS(y * y - x) < 3n/z$.

Now, if we assume that that $\mathcal{F}(n) > (3n)^2$, then

$$\mathcal{F}\big(INT(z/(3n))\big) > (3z/(3n))^2 = z * (z/n^2) > z\ .$$

The last inequality follows since $z/n^2 \geq 1$ for otherwise, $z \leq n^2 < \mathcal{F}(1,n)$. But, by K6, it will be the case that $\mathcal{F}(1,n) < z$, which is not possible. Therefore $z/(3n)$ is large and hense, $3n/z$ is in I, so $y^2\, EQ\, x$.

[End of proof]

As in Pythagoras' day, if n and m are standard and in reduced form then not $(n/m)^2\, EQ\, 2$. Simply apply Theorem III.1(h) and observe that $(n/m)^2 = 2$ is impossible by drawing the usual contradiction.

The predicate $A(x)$ $x < z$ is decidable in RCA even though z is a parameter. Without assigning a value to z, comparisons like '$3 < z$' or assertions like "z is even" can't be evaluated by numerical calculations. Nevertheless, $\forall x(A(x) \vee \neg A(x))$ is an immediate consequence of the decidability of $<$ in RA. One works informally in RCA as if a suitably large value for z had been chosen. In this conception, actual computations can be carried out. The expression "suitably large" means suitable for some particular finite set of calculations and/or arguments in accordance with the Soundness Theorem of Section II. Of course, no fixed choice if z is suitable for all calculations and arguments.

Theorem III.3 Let $p(x) = \sum_0^n p_i x^i$ with $p_n = 1$. Assume that $p(x)$ is an odd degree polynomial with $1 + degree(p)$ in S and all of its coefficients p_i in S Then $\Delta^2 \exists x(p(x)\, EQ\, 0 \wedge x \in S)$. Assume that $\mathcal{F}(n) > n^{n+2}$. This guarantees that $p(x)$ has has degree of continuity 2 (See Theorem III.7.)

Proof: Define M to be the Lagrange root bound $M = \max\{1, \sum_0^n |p_i|\}$. The zeros, if any, are within $[-M, M]$. Let $N = \max\{n+1, |p_0|, \dots, |p_n|\}$. Then $M \leq N * N$ so $\Delta M \in S$. Since $p(x)$ is an odd degree polynomial it must cross the x-axis in side $[-M, M]$. Therefore, applying the Intermediate Value Theorem (Theorem III.8) and the fact that $p(x)$ has degree of continuity 2, it follows that $\Delta^2 \exists x(p(x)\, EQ\, 0 \wedge x \in S)$.

[End of proof]

Corollary III.6 $(S, EQ, +, *, 0, 1)$ is a real closed field.*

Unlike ordinary constructive or intuitionistic analysis one cannot prove $x\, GE\, y$ from $\neg(x\, LT\, y)$. Specifically,

Theorem III.4 $\forall xy(\neg(x\, LT\, y) \supset x\, GE\, y)$ is not provable in RCA.

Proof: If this was a theorem of RCA then one could apply the Soundness Theorem of Part II and provide a class of models for this sentence. Suppose $P \vdash_{RCA} \forall xy(\neg(x\, LT\, y) \supset x\, GE\, y)$. Using the Soundness Theorem, if $i > \Delta - depth(P)$ and $\zeta > m(i + \Delta - depth(P))$ then there is a number $e = E(\lceil P \rceil, \zeta)$ such that (e, i, ζ) RE $\forall xy(\neg(x\, LT\, y) \supset x\, GE\, y)$. Within the realization model (e, i, ζ), K and L are defined. Define $n_0 = k(i) + 1$. $k(i)$ in the maximum member of K and n_0 strictly between the members of K and L. Let $t = 1$ and define $s = 1 - 1/n_0$. We can show that for all a, (a, i, ζ) does not realize $\underline{s}\, LT\, \underline{t}$ and therefore $(e\{\lceil \underline{s} \rceil, \lceil \underline{t} \rceil\}, i, \zeta)$ RE $\neg \underline{s}\, LT\, \underline{t}$. On the otherhand, $(e\{\lceil \underline{s} \rceil, \lceil \underline{t} \rceil\}, i, \zeta)$ does not realize $\underline{s}\, GE\, \underline{t}$ since this would require that $n_0 \in L$ which is false. Therefore (e, i, ζ) could not have realized $\forall xy(\neg(x\, LT\, y) \supset x\, GE\, y)$ and so $\forall xy(\neg(x\, LT\, y) \supset x\, GE\, y)$ is not provable in RCA.

[End of proof]

Intuitively there is a 'gap' between the x that are $LT\, 0$ and those that are $GE\, 0$.

------)………(-------

$x\, LT\, 0$ gap $x\, GE\, 0$

Such a gap is a necessary consequence of taking serious the idea that class K of 'standard' natural numbers is growing in time. This effects the completeness of 'closed' intervals. More about this point later.

STABLE SEQUENCES

Algebraic and transcendental numbers can be represented in RCA by means of "stable" sequences, the analogue of Cauchy sequences.

Definition III.2 A sequence $s: N \to Q$ is *stable* of degree d iff

(1) $n, m \in L$ implies $\Delta^d(s(n)\, EQ\, s(m))$;

(2) $s(n)\, NE\, s(m)$ implies $\Delta^d(n \in K \vee m \in K)$.

It follows from clause (1) that if s is a stable sequence, and n is large, then $s(n)$ acts as the 'limit' of the sequence.

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* Closure is expressed using the future tense operator Δ as in the sentence $x, y \in S \supset \Delta(x + y \in S)$.

Example III.1 Representing the square root of 2 by a stable sequence.

We can represent the square root of 2 by means of the following stable sequence of degree 0:

$$s(O) = 1,\ s(n+l) = s(n)/2 + 1/s(n) \text{ for } n \geq 1.$$

By induction one can show that $\forall n \forall m \left(m < n \supset \left(ABS\left(s(n) - s(m)\right) < 1/2^m\right)\right)$. If $n > m$ are in L, then 2^m is in L so $s(n)\, EQ\, s(m)$. On the other hand, if $s(n)\, NE\, s(m)$ for $m < n$ then there is a k in K such that $|s(n) - s(m)| \geq 1/k$ so $1/2^m > 1/k$ which implies $m < k$ so $m \in K$. Therefore $s(n)$ is a stable series of degree 0. Let $r = s(z)$. Then $s(z+1) = r/2 + 1/r$. Now $|r - (r/2 + 1/r)| < 1/2^z$ so $|r^2 - 2| < 1/2^{z-2} < 1/z$ (since $1 < r < 2$) so $r^2\, EQ\, 2$.

Example III.2 Regular sequences are stable.

A sequence $s(n)$ is regular if $ABS(s(n) - s(m)) \leq 1/n + 1/m$ for all n and $m > 0$. It is easy to check that regular sequences are stable of degree 1.

The relation between stable sequences and completeness is given in the next, theorem.

Definition III.3 Closed intervals $\lceil a,b \rceil$ and $|\lceil a,b \rceil|$.

Let a, b be in Q. Define

$$\lceil a,b \rceil = \{ x : a\ LE\ x \wedge x\ LE\ b\};$$

$$|\lceil a,b \rceil| = \{x : \neg(x\ LT\ a) \wedge \neg(x\ GT\ b)\}.$$

Note that $\neg(x\ LT\ y)$ is a weaker condition than $y\ LE\ x$ so $\lceil a,b \rceil$ is strictly contained in $|\lceil a,b \rceil|$.

Theorem III.5 Closure properties of RCA.

(1) S is complete. That is, if $s(n)$ is a stable sequence in S of degree $d \in K$, and $\forall n(n \in K \supset s(n) \in S)$, then $\Delta^{d+2} \forall n\, (s(n) \in S)$.

(2) $|\lceil a,b \rceil|$ is complete for any $a, b \in S$. That is, if $s(n)$ is a stable series of degree $d \in K$ and $\forall n(n \in K \supset \Delta(s(n) \in |\lceil a,b \rceil|))$ then $\Delta^{d+2} \forall n(s(n) \in |\lceil a,b \rceil|)$.

Proof of (1): Let n_0 be any natural number. We will prove that $\Delta^{d+2}(s(n_0) \in S)$. Assume that $ABS\left(s(n_0) - s(0)\right) < 1$. Then $ABS\left(s(n_0)\right) < ABS\left(s(0)\right) + 1$ which would imply that $\Delta(s(n_0) \in S)$ and we would be done.

Assume that $ABS\left(s(n_0) - s(0)\right) \geq 1$. Define $X_{n_0} = \{n : n \leq n_0 \wedge ABS(s(n_0) - s(n)) < 1\}$. X_{n_0} is decidable, $n_0 \in X_{n_0}$, so X_{n_0} is non-empty, and $0 \notin X_{n_0}$. Applying BLEP to X_{n_0} one can construct a $k \in X_{n_0}$ such that

$$0 < k \leq n_0 \text{ and } ABS(s(n_0) - s(k)) < 1 \leq ABS(s(n_0) - s(k-1)).$$

Therefore $s(n_0)\, NE\, s(k-1)$ so, s being stable of degree d, we must have $\Delta^d(k-1 \in K)$ and hence $\Delta^{d+1}(k \in K)$. By the hypothesis, $k \in K$ implies $s(k) \in S$. Therefore

$\Delta^{d+1}(\mathcal{s}(k) \in S)$. Since $ABS(\mathcal{s}(n_0)) < ABS(\mathcal{s}(k)) + 1$ it follows that $\Delta^{d+2}(\mathcal{s}(n_0) \in S)$.
[End of proof of Theorem III.5 (1)]

Proof of (2): It is required to show that $\neg(\mathcal{s}(n)\, LT\, a)$ and $\neg(b\, LT\, \mathcal{s}(n))$ for any n. Since the two cases are very similar, I shall just demonstrate the former.

Suppose $\mathcal{s}(n)\, LT\, a$ for some n. That is,

(a) $\mathcal{s}(n) \leq a - 1/k$, for some $k \in K$.

Let $X = \{m : m \leq n \wedge a - 1/2k < \mathcal{s}(m)\}$. X is decidable and $\neg n \in X$. However, $0 \in X$ for otherwise $\mathcal{s}(0) \leq a - 1/2k$ which would imply that $\Delta(\mathcal{s}(0)\, LT\, a)$ contradicting the hypothesis of (2). BMEP can be applied to effectively obtain $m \in X$ such that

(b) $\mathcal{s}(m+1) \leq a - 1/2k < \mathcal{s}(m)$.

$k \in K$ implies that $\Delta(2k \in K)$ and therefore (a) and (b) imply that $\Delta(\mathcal{s}(n)\, \mathrm{NE}\, \mathcal{s}(m))$. Since $\mathcal{s}$ is stable of degree d and $m \leq n$ it follows that $\Delta^{d+1}(m \in K)$ and therefore

(c) $\Delta^{d+2}((m+1) \in K \wedge 2k \in K \wedge (\mathcal{s}(m+1) < a - 1/2k))$.

This would imply that

(d) $\Delta^{d+2} \exists k(k \in K \wedge \mathcal{s}(k)\, LT\, a)$.

From the hypothesis of (2) we get

(e) $\Delta^{d+2} \forall n(n \in K \supset \neg \mathcal{s}(n)\, LT\, a)$

Since (e) contradicts (d) we conclude that $\forall n(\neg(\mathcal{s}(n)\, LT\, a)$. Using a similar argument we can prove that $\forall n(\neg(\mathcal{s}(n)\, GT\, b)$.
[End of proof of Theorem III.5 (2)]

Continuous Functions

Definition III.4 Continuous RCA functions of degree $d \in K$.

Let A be a subset of S and let f be a rational valued function mapping A into S. Let $d \in K$.

Define f to be an RCA function with degree of functionality d iff

(1) for all $x, y \in A$, $x\, EQ\, y$ implies $\Delta^d(f(x)\, EQ\, f(y))$.

Define f to be an *RCA continuous function with degree of continuity d* iff f is an RCA function with degree of functionality d and

(2) for all $x, y \in A$, $f(x)\, NE\, f(y)$ implies $\Delta^d(x\, NE\, y)$.

(1) is just the basic assertion of the functionality of $f(x)$ with respect to EQ. (2) says that there are no "jumps". (1) and (2) imply that RCA continuous functions preserve stable sequences in the sense of the following theorem.

Theorem III.6 Let f be an RCA function of continuity degree d with domain $A \subseteq S$. If $\mathcal{s}(n)$, n = 0,1,… is a stable sequence in A of degree e, then $f(\mathcal{s}(n))$ is a stable sequence of degree $d+e$.

Proof: Since $\mathcal{s}$ is stable sequence of degree e,

(1) $n, m \in L$ implies $\Delta^e(\mathcal{s}(n)\, EQ\, \mathcal{s}(m))$;

(2) $\mathcal{s}(n)\, NE\, \mathcal{s}(m)$ implies $\Delta^e(n \in K \vee m \in K)$.

From condition (1) it follows that $\Delta^{d+e}(f(\mathcal{s}(n)) EQ\, f(\mathcal{s}(m)))$ for all $n, m \in L$.

From condition (2) it follows that $f(\mathcal{s}(n))\, NE\, f(\mathcal{s}(m)) \supset \Delta^d(\mathcal{s}(n) NE\, \mathcal{s}(m))) \supset \Delta^{d+e}(n \in K \vee m \in K)$.

[End of proof]

Theorem III.7 Let $p(x)$ be a polynomial with degree in K and coefficients in S. p is an RCA continuous function defined on S (assuming $\mathcal{F}(n) > n^{n+2}$) with degree of continuity 2.

Proof: Let $x, y \in S$. To prove the theorem use the following estimate:

> $ABS(p(x) - p(y)) < ABS(x-y) * m^{(m+2)}$ where $m \in K$ is an upper bound to the absolute values of x, y, and the degree and coefficients of $p(x)$.

Note that $m \in K$ and this implies $\Delta(m^{m+2} \in K)$ and if $e = |x-y| \in I$ then $ABS(p(x) - p(y)) < e * m^{m+2}$ and, by Lemma III.1(e), $\Delta^2(e * m^{m+2} \in I)$, so $\Delta^2(p(x) EQ\, p(y))$. On the other hand, if $1/k < ABS(p(x) - p(y))$ for $k \in K$ then $1/k * 1/m^{m+2} < ABS(x-y)$. $\Delta(m^{m+2} \in K)$ so by Lemma III.1(a), $\Delta^2(k * m^{m+2} \in K)$, and hence $\Delta^2(x\, NE\, y)$.

[End of proof]

Note: in general we will drop the RCA from "RCA continuous" and understand that by the expression "continuous function" we mean RCA continuous.

Theorem III.8 (Intermediate value Theorem.) Let $a, b \in S$ and let f be continuous on $[a, b]$ with degree of continuity d and suppose that $f(a) < f(b)$. There is a rational valued function $c(y)$ defined on $[f(a), f(b)]$ such that if $y \in [f(a), f(b)]$ then $\Delta^{d+1}(f(c(y))\, EQ\, y)$.

Proof: Assume y is in $[f(a), f(b)]$. If $y < f(a)$ define $c(y) = a$. Since $y\, GE\, f(a)$ it follows that $y\, EQ\, f(a)$ so $\Delta^{d+1}(y\, EQ\, f(c(y))$. Similarly, if $y > f(b)$ define $c(y) = b$ and we can show that $y\, EQ\, f(b)$ so $\Delta^{d+1}(y\, EQ\, f(c(y))$.

Now assume $f(a) \leq y \leq f(b)$ and define the finite set

$$A(y) = \{m : m \leq (b-a) * z \wedge (\forall k)(k \leq m \supset f(a + k/z) \leq y)\}.$$

$A(y)$ is non-empty since 0 is in it. It is decidable and bounded above. Apply BMEP to get the maximum number $m \in A(y)$ and define $c(y) = a + m/z$. By the definition of m, $f(c(y)) \leq y$ and $m \leq (b-a)*z$. Therefore $m/z \leq (b-a)$ and, hence $0 \leq b-(a+m/z)$. Claim that $\Delta^d(f(c(y))EQ\ y)$.

Case 1. $m+1 \leq (b-a)*z$. Then $a+m/z$ and $a+(m+1)/z$ are in $[a,b]$ and

$$f\ (a+m/z)\ \leq y\ <\ f\ (a+(m+1)/z).$$

This implies that

$$ABS(f(c(y))\ -\ y)\ <\ ABS(f(a+m/z)\ -f(a+m/z+1/z))$$

and since f is continuous on $[a,b]$ with degree of continuity d and $1/z \in I$, it follows that $\Delta^d(ABS(f(a+m/z)\ -f(a+m/z+1/z)) \in I)$ and therefore $\Delta^d(f(c(y))EQ\ y)$, and so $\Delta^{d+1}(f(c(y))EQ\ y)$.

Case 2. $m+1\ >\ (b-a)*z$. Therefore $0 \leq b-(a+m/z) < 1/z$ so $c(y)\ EQ\ b$. From the d-continuity of f it follows that $\Delta^d(f(c(y))EQ\ f(b))$. $y\ LE\ f(b)$ since $y\ \in [f(a), f(b)]$, so $\Delta^d(y\ LE\ f(b))$. Therefore $\Delta^d(f(c(y))EQ\ f(b) \wedge y\ LE\ f(b))$ so $\Delta^{d+1}(y\ LE\ f(c(y)))$. But $f\ (c(y))\ \leq\ y$ so it follows that $\Delta^{d+1}(f\ (c(y))\ EQ\ y)$.

[End of proof]

I shall finish Part III with the Uniform Continuity theorem for RCA. This kind of result can be established in intuitionistic analysis but can't be proved (without directly building it in to the definition of continuity) in constructive analysis (see [Beeson, 1979][12]).

<u>Lemma III.5</u> Let a and b be in S, $a\ LT\ b$, and let $f(x)$ be a rational valued function continuous in $[a,b]$ of degree d. Let $e\ >\ 0$ and let x be any standard rational such that $a\ \leq\ x\ \leq\ b$. Define the predicate

$$\mathrm{A(k,e,x)}\ \equiv\ 0 \leq k\ \leq z \wedge ((a \leq x+k/z \leq\ b) \supset \forall m\ (m\ \leq\ k \supset ABS(f\ (x) -\ f\ (x+m/z))\ <\ e).$$

and define $g(e,x)\ =\ \max\{k/z|A(k,e,x)\}$. By definition, $a \leq x+g(e,x)\ \leq\ b$.

Claim: $g(e,x)$ is well defined for $e > 0$ and $a\ \leq\ x\ \leq\ b$, and, if $e\ GT\ 0$ then $\Delta^{d+1}\ (g(e,x)\ GT\ 0)$. Note that by the definition of $g(e,x)$, if $m/z \leq\ g(e,x)$ then $|f\ (x) -\ f\ (x+m/z)| < e$.

<u>Proof:</u> Note that $A(0,e,x)$ always holds true and $A(n,e,x)$ implies $n\ \leq\ z$. Furthermore, $A(n,e,x)$ is decidable. Use of BMEP guarantees that $g(e,x)\ = \max\{n/z|A(n,e,x)\}$ is well defined. By definition of $g(e,x)$, if $m/z \leq\ g(e,x)$ and $a\ \leq x+m/z\ \leq\ b$ then $ABS(f(x)\ -\ f(x+m/z))\ <\ e$.

Suppose $e\ GT\ 0$.
By the definition of $g(e,x)$, there is an n, $0 \leq n \leq z$ such that $g(e,x) = n/z$.
Assume $n = z$. Then $g(e,x) = 1$ which is $GT\ 0$ and we are done.

Assume $n\ <\ z$. If $a\ \leq\ x+n/z\ \leq\ b$ then, since n is maximal for $A(n,x,e)$, and $x+(n+1)/z\ LE\ b$ and f is defined on $[a,b]$, it follows that

$$ABS(f(x)\ -\ f(x+n/z))\ <\ e < ABS(f(x)\ -\ f\ (x+(n+1)/z)\ ).$$

If $e\ GT\ 0$ then $f(x)\ NE\ f(x+(n+1)/z)$, which, because f is continuous of degree d, implies $\Delta^d((n+1)/z\, GT\, 0)$. $(n+1)/z\, GT\, 0$ implies $\Delta(n/z\ GT\ 0)$ and therefore $\Delta^d((n+1)/z\, GT\, 0)$ implies $\Delta^{d+1}(n/z\ GT\ 0)$, that is, $\Delta^{d+1}(g(e,x)\ GT\ 0)$.

[End of proof of Lemma III.5.]

Lemma III.6 Let a and b be in S, $a\, LT\, b$, and let $f(x)$ be a rational valued function continuous in $[a,b]$ of degree d. Use the function $g(e,x)$ defined in Lemma III.5 to define for $e > 0$

$$g(e) = \min\{g(e/2, a + k/z) | a \le a + k/z \le b\}.$$

Claim (1): $e\, GT\, 0$ implies $\Delta^{d+2}(g(e)\, GT\, 0)$.

Claim (2): Let $a \le x \le b$ and $m \ge 0$. If $e\, GT\, 0$ and $m/z \le g(e)$ and $a \le x + m/z \le b$ then $\Delta^{d+2}\big(ABS\big(f(x) - f(x+m/z)\big) < 2e/3\big)$.

Proof: From Lemma III.5, $m/z \le g(e)$ implies that $a \le x + m/z \le b$. Define

$n_e = z * g(e)$ so that $n_e/z = g(e)$. Note that $a \le a + n_e/z \le b$.

Let $e\, GT\, 0$ and let x be any standard rational such that $a \le x \le b$.

By Lemma III.5, $e/2\, GT\, 0$ implies that $\Delta^{d+1}(g(e/2, a+n_e/z)\, GT\, 0)$. Since $e\, GT\, 0$ implies that $\Delta(e/2\, GT\, 0)$ and since $g(e) = g(e/2, a+n_e/z)$ it follows that $e\, GT\, 0$ implies $\Delta^{d+2}(g(e)\, \mathrm{GT}\, 0)$, which proves Claim (1).

We now prove Claim (2). The set $\{k | 0 \le k \wedge a + k/z \le x\}$ is non-empty and bounded above since $a + k/z \le x \le b$ implies that $k \le (b-a)/z$. Let $k_0 = max\{k | a + k/z \le x\}$. Then

$a + k_0/z \le x < a + (k_0+1)/z$ and hence $|(a + k_0/z) - x| < 1/z$.

Now

$$ABS(f(x) - f(x+m/z)) \le$$
$$ABS(f(x) - f(a+k_0/z))$$
$$+ABS\big(f(a+k_0/z+m/z) - f(x+m/z)\big)$$
$$+ ABS(f(a+k_0/z) - f(a+k_0/z+m/z)).$$

Since x and $a + k_0/z$ differ by less than $1/z$, it follows that $x\, EQ\, (a+k_0/z)$. Therefore, by the d-continuity of f,

$\Delta^d(ABS(f(x) - f(a+k_0/z)) \in I)$ and

$\Delta^d(ABS(f(a+k_0/z+m/z) - f(x+m/z)) \in I)$.

Since $e\, GT\, 0$, it follows that $\Delta(e/12\ GT\ 0)$ and therefore

$\Delta^{d+1}(ABS\big(f(x) - f(a+k_0/z)\big) < e/12)$ and

$\Delta^{d+1}\big(ABS\big(f(a+k_0/z+m/z) - f(x+m/z)\big) < e/12\big)$.

Also, $a \le a + k_0/z + m/z \le x + m/z \le b$ and $m/z \le g(e) \le g(e/2, a+k_0/z)$ so, by Lemma III.5,

$$\Delta^{d+2}(ABS(f(a+k_0/z) - f(a+k_0/z+m/z)) < e/2)).$$

Therefore the sum of the three terms is less than $2e/3$ so $\Delta^{d+2}\big(ABS\big(f(x) - f(x+m/z)\big) < 2e/3\big)$ which establishes Claim (2).

[End of proof of Lemma III.6]

<u>Theorem III.9</u> (Uniform Continuity Principle.) Let a and b be in S, $a\,LT\,b$, and let $f(x)$ be a rational valued function continuous in $[a,b]$ of degree d. There is a function $\mathcal{g}(e)$ mapping the positive rationals into $[0,1]$ such that

(1) $e\,GT\,0$ implies $\Delta^{d+2}(\mathcal{g}(e)\,GT\,0)$.

(2) $e\,GT\,0$ and $0 < r \leq \mathcal{g}(e)$ and $x, x+r \in [a,b]$ implies $\Delta^{d+2}(ABS(f(x) - f(x+r)) < e)$.

<u>Proof:</u> Let $\mathcal{g}(e)$ be the function defined in Lemma III.6.

Part (1) follows directly from Lemma III.6.

To prove part (2) let x be in $[a,b]$ and $e\,GT\,0$. Divide this into three cases:

Case 1: $x > b$.

Case 2: $a \leq x \leq b$.

Case 3: $x < a$.

Case 1. The case where $x > b$ is trivial, since if $x, x+r \in [a,b]$ then r is infinitesimal so $\Delta^d(ABS(f(x) - f(x+r))\,EQ\,0)$. We are given that $0\,LT\,e$, so $\Delta(0\,LT\,e/2)$ (assuming that $\mathcal{F}(n) \geq 2 * n$.) If $d \leq 1$, then $\Delta(ABS(f(x) - f(x+r))\,EQ\,0 \wedge 0\,LT\,e/2)$ and, therefore $\Delta^2(ABS(f(x) - f(x+r))\,LT\,e/2)$ and so $\Delta^{d+2}(ABS(f(x) - f(x+r) < e))$. If $d > 1$ then $\Delta^d(ABS(f(x) - f(x+r))\,EQ\,0)$ and $\Delta^d(0\,LT\,e/2)$ so

$$\Delta^d(ABS(f(x) - f(x+r))\,EQ\,0 \wedge 0\,LT\,e/2)$$

and hence

$$\Delta^{d+1}(ABS(f(x) - f(x+r))\,LT\,e/2)$$

which implies $\Delta^{d+2}(ABS(f(x) - f(x+r))\,LT\, < e)$.

Case 2. $a \leq x \leq b$. Choose $e\,GT\,0$ and any positive $r \leq \mathcal{g}(e)$ such that $x+r$ is in $[a,b]$. We will show that

$$\Delta^{d+2}(ABS(f(x) - f(x+r)) < 5e/6)$$

and therefore $\Delta^{d+2}(ABS(f(x) - f(x+\mathrm{r})) < e)$.

Assume $x + r \leq b$. Define m such that $m/z \leq r \leq (m+1)/z$. Note that $r\,EQ\,m/z$. Now,

$$ABS\big(f(x) - f(x+r)\big) \leq$$
$$ABS\big(f(x) - f(x+m/z)\big)$$
$$+ABS(f(x+m/z) - f(x+r)).$$

Since $x + m/z \leq b$ and $m/z \leq \mathcal{g}(e)$ we can apply Lemma III.6 and conclude that $\Delta^{d+2}\big(ABS\big(f(x) - f(x+m/z)\big) < 2e/3\big)$. Since f is d-continuous, $\Delta^d(ABS(f(x+m/z) - f(x+r)) \in I)$. Therefore $\Delta^{d+1}(ABS(f(x+m/z) - f(x+r)) < e/6)$ since $0\,LT\,e$. Consequently $\Delta^{d+2}\big(ABS\big(f(x) - f(x+\mathrm{r})\big) < 5e/6\big)$.

On the other hand, suppose $x + r > b$. Then $x + r\,EQ\,b$. Define m such that $x + m/z \leq b < x + (m+1)/z$. Then $x + m/z\,EQ\,b$ and therefore $\Delta(x + m/z\,EQ\,x+r)$.

$ABS\big(f(x)\text{-}f(x+r)\big) \leq$

$ABS\big(f(x)-f(x+m/z)\big)$

$+ABS(f(x+m/z)-f(x+r))$.

Once again we can apply Lemma III.6 to the first term and conclude that $\Delta^{d+2}\big(ABS\big(f(x)-\ f(x+m/z)\big) < 2e/3\big)$. From the d-continuity of f, we get $\Delta^{d+1}(ABS(f(x+m/z)-f(x+r)) \in\ I)$. Therefore $\Delta^{d+2}(ABS(f(x+m/z)-f(x+r)) < e/6)$ and we can conclude that $\Delta^{d+2}\big(ABS\big(f(x)-f(x+\mathrm{r})\big) < 5e/6\big)$.

Case 3. $x < a$. Choose $e\,GT\,0$ and any positive $r \leq g(e)$ such that $x+r$ is in $[a,b]$. Since $x \in [a,b]$, it follows that $x\,EQ\,a$ and therefore $0 < \varepsilon = a-x \in I$. Moreover, $x+r$ is in $[a,b]$ implies $x+r \leq b+\delta$ for some non-negative $\delta \in I$.

Consider the term $a+\mathrm{r}-\delta-\varepsilon$. It's equal to $\mathrm{x}+\mathrm{r}-\delta$ which, by definition, is less than or equal to b.

If $a+\mathrm{r}-\delta-\varepsilon < \mathrm{a}$ then $\mathrm{r}-\delta-\varepsilon < 0$ so $0 < \mathrm{r} \leq \delta+\varepsilon$ and hence $\Delta(r \in I)$. In this case we directly have $\Delta^{d+1}(ABS(f(x)\text{-}f(x+r)) \in I)$ because of the d-continuity of f, and hence $\Delta^{d+2}(ABS(f(x)\text{-}f(x+r)) < e)$.

On the other hand, if $a+\mathrm{r}-\delta-\varepsilon \geq a$, then a and $a+\mathrm{r}-\delta-\varepsilon$ are in $[a,b]$.

Now

$ABS\big(f(x)\text{-}f(x+r)\big) \leq$

$+ABS(f(x)-f(a))$

$+ABS\big(f(a)-f(a+\mathrm{r}-\delta-\varepsilon)\big)$

$+ABS(f(a+\mathrm{r}-\delta-\varepsilon)-f(x+r))$.

Since f is d-continuous, $\Delta^{d}(ABS(f(x)-f(a)) \in\ I)$ and therefore

$\Delta^{d+1}(ABS(f(x)-f(a)) < e/12)$.

We can apply Case 2 to the second term and conclude that

$\Delta^{d+2}\big(ABS\big(f(a)-f(a+\mathrm{r}-\delta-\varepsilon)\big) < 5e/6\big)$

Finally, $ABS\big(f(a+\mathrm{r}-\delta-\varepsilon)-f(x+r)\big) = ABS(f(x+\mathrm{r}-\varepsilon)-f(x+r))$, so, since f is d-continous, $\Delta^{d}\big(ABS\big(f(a+\mathrm{r}-\delta-\varepsilon)-f(x+r)\big) \in I\big)$ and therefore

$\Delta^{d+1}\big(ABS\big(f(a+\mathrm{r}-\delta-\varepsilon)-f(x+r)\big) < e/12\big)$.

Putting these three inequalities together we get $\Delta^{d+2}(ABS(f(x)\text{-}f(x+r)) < e)$.

[End of proof of Theorem III.9]

Final Remark. Since RCA is restricted to considering rational valued functions, irrational and transcendental functions can be defined, for example, by making use of limit equalities, finitely iterated numerical methods algorithms or finite Taylor series, and performing z steps of the computation.

For example, consider the classic limit $e^x = \lim_{n\to\infty}(1 + x/n)^n$ and define the rational function $e(x) = (1 + x/z)^z$. Then we can find a computable rational valued function $\varepsilon_z(x)$ with values in I for standard x, so that $|e^x - e(x)| < \varepsilon_z(x)$. This is a statement of classical analysis. The relation $|e^x - e(x)| < \varepsilon_z(x)$ can be used to demonstrate the usual exponential properties such as $\Delta(e(x) * e(y) EQ\; e(x + y))$* (assuming that $\mathcal{F}(n) > 3n^2$) and so forth. These relationships can also be proved algebraically, directly from the definition of $e(x)$ without referencing the classical e^x function.

As an example using iteration consider square root function $a^{1/2}$ for $a > 1$. Choose x_1 as any estimate of the square root† of a such that $2a > {x_1}^2 > a$ and $0 < x_1 < a$. Then iteratively compute $x_{n+1} = (x_n + a/x_n)/2$ and define the rational function $sqrt(a) = x_z$. The error term $e_n = ({x_n}^2/a) - 1$ satisfies the recursion formula $e_{n+1} = {e_n}^2/(4e_n + 4)$. Since $0 \le x_1 < a$, it follows that e_1 is standard and since $2a > {x_1}^2 > a$, it follows that $1 > e_1 > 0$. Therefore $0 < e_{n+1} = {e_n}^2/(4e_n + 4) < {e_n}^2/4 < e_n/4$ for $n \ge 1$ and so e_n converges rapidly. In particular $0 < e_z < e_1(1/4)^{z-1}$ so $|({x_z}^2/a) - 1| < (1/4)^{z-1} < 1/z^2$ (for any $z > 2$) and hence $|({x_z}^2) - a| < (a/z)(1/z) < 1/z$ for standard a. This shows that for standard a, $sqrt(a)^2\; EQ\; a$.

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* Just observe that $e(x) * e(y) = (e^x + a)(e^y + b) = e^x e^y + a * e^x + b * e^y + a * b$ for some infinitesimals a and b.

† A standard estimate is $x_1 = (1 + 6a + a^2)/\big(4(a + 1)\big)$.

Footnotes

[1] [Volpin, 1968] For an early work of Yessenin-Volpin in English see "The ultra-intuitionistic criticism and the antitraditional program for foundations of mathematics", in "Intuitionism and proof theory", Edited by Kino, A., Myhill, J., Vesley, R. E. (Proc. Conf., Buffalo, N.Y., 1968), Studies in Logic and the Foundations of Mathematics, Amsterdam: North-Holland, pp. 3–45.

[2] [GEISER 1975] Review: "A. S. Yessenin-Volpin, The Ultra-Intuitionistic Criticism and the Anti-traditional Program for Foundations of Mathematics", J. Symbolic Logic 40 (1975), no. 1, pp.95-97.

[3] [A. Robinson, 1966] "Nonstandard Analysis", North-Holland, Amsterdam. (Second, revised edition, 1974). One can consider a nonstandard model of the rationals and do analysis therein. This of course would not be constructive. Furthermore, there are certain key strategies of non-standard analysis that can't be directly applied in RCA. If they could, there would be no gap between S and L.

[4] [Mycielski, 1981] "Analysis Without Actual Infinity", The Journal of Symbolic Logic, Vol. 46, No. 3 (Sep., 1981), pp. 625-633. Although this is an explicitly classical approach it shares certain computational parallels with RCA. Δ corresponds to the requirement of choosing a larger rational index p on ω_p in order to complete some computation.

[5] [Ruokolainen 2004] "Constructive Nonstandard Analysis Without Actual Infinity", Academic dissertation, University of Helsinki, Faculty of Science Department of Mathematics and Statistic, 2004. The dissertation is available through the link http://ethesis.helsinki.fi/julkaisut/mat/matem/vk/ruokolainen/construc.pdf.

[6] [Geiser 1981] "Rational Constructive Analysis", Lecture Notes in Mathematics #873, Edited by A. Dold and B. Eckmann, Springer Verlag, 1981. The Lecture notes are simply a compilation of the working papers read at the State University Conference on Constructive Mathematics held at Las Cruses, New Mexico, in the Summer of 1980.

[7] [Kleene 1945] "On the Interpretation of Intuitionistic Number Theory", The Journal of Symbolic Logic, Vol. 10, No. 4 (Dec., 1945), pp. 109-124

[8] [Kripke 1968] For a presentation of Kripke semantics the reader is referred to Hughes and Cresswell, "An Introduction to Modal Logic", chpt.15, Methuen and Co. Ltd., 1968.

[9] [Tarski 1930]) "The Concept of Truth in Formalized Languages in Logic, Semantics, Metamathematics", Hackett, 1983, pp. 152-268.

[10] [Kleene, 1967] "Introduction to Metamathematics" , §82 , Van Nostrum. See pp. 502-503 for the definition of realizability. Corresponding to the Soundness theorem of part II is Kleene's Theorem 62, p. 504.

[11] [Hakli & Negri, 2010] "Does the deduction theorem fail for model logic?", Synthese Vol. 187, No. 3,(August 2012), pp. 849-867. See especially Section 3: "A proof of the deduction theorem." The model system HK with model operator ☐ (necessitation) is examined in this section and the inference *from* A *infer* $\Box A$ is restricted to the cases where A is a theorem of HK. Ofcourse, ☐ is only very partially analogous to the temporal operator. For example, *from* A *infer* ΔA is a permitted rule of inference. But $\Box A$ means "true in all worlds", whereas our ΔA mean will be true in the next stage of constructing the natural numbers and not necessarily true at "all future times". In HK ☐ distributes over implication ($\Box(A \supset B) \supset (\Box A \supset \Box B)$) while in our tense logic, Δ distributes and factors over implication ($\Delta(A \supset B) \supset (\Delta A \supset \Delta B)$ and $(\Delta A \supset \Delta B) \supset \Delta(A \supset B)$.)

[12] [Beeson, 1979] Logic Col. 1978 p. 36 North Holland. Beeson exhibits a (non-constructive) example of a real valued function defined on the Cantor subset of the recursive Reals that is not uniformly continuous. In the setting of RCA, the corresponding example is provably not a function.