The Completeness of Classical Arithmetic

Alexander S. Yessenin-Volpin

[Additions by JG are added in green font: March20, 2020]

$\widehat{\mathrm{T}}^{o}$ shall denote the formal system of classical arithmetic as developed, say, in [HB, 1934, vol. I], or [KL, 1952, IM] or [Ro, 1953 or 1978, L&M] whose logic, ℓo, is open, i.e., admits the use of open formulae or axioms or hypotheses of deduction; the language[1], $\mathrm{L}\widehat{\mathrm{T}}$, of $\widehat{\mathrm{T}}^{o}$ shall contain the logical symbols

$\supset, \&, \vee, \forall, \exists$

and the predicate symbols

$=, <, \leq$

as well as the n-numeroids[1] $0, 0', 0'', \ldots \;, 0^{\overbrace{\prime\prime\ldots\prime}^{\ell}}$ denoted here (respectively) by

$0, suc(0), suc\big(suc(0)\big), \ldots, suc^{\ell}(0), \ldots$

where ℓ denotes the 'length' (i.e., the number of all occurrences of the successor function symbol, or suc, in the (generic) n-numeroid), variables (denoted by the lower case Greek letters) and (unary, binary and tertiary) functions symbols, including the unary symbol suc, the binary symbols [page 2] $+$ and $\times$ for addition and multiplication. [Terms will be denoted by lower case letters r, s, t.] The unary function constant symbol

C_1

for the function whose only value is 1 ($= suc(0)$) as well as, for any unary function expressed in $\mathrm{L}\widehat{\mathrm{T}}$, say, by $\varphi(\theta)$, another function introduced by the operation of summation, specified in $\widehat{\mathrm{T}}^{o}$ by the axioms which define it as follows:

$def_1^{\sum_{\theta<0}^{\varphi}} \quad \sum_{\theta<0} \varphi(\theta) = 0.$

$def_2^{\sum_{\theta<0}^{\varphi}} \quad \sum_{\theta<\mathrm{suc}(\lambda)} \varphi(\theta) = \sum_{\theta<\lambda} \varphi(\theta) + \varphi(\lambda).$

Termoids, formulae, deductions (in particular, proofs) will be introduced in $\mathrm{L}\widehat{\mathrm{T}}$ in accordance with the 'formation' and 'transformation' rules as those accepted in the mentioned literature (where the current 'termoids' are systematically called 'terms'). The use of the summation provides the formation, in $\mathrm{L}\widehat{\mathrm{T}}$, of the [page 3] locutions $\sum_{\theta<r} C_1(\theta)$ which can be combined as the compact form of presentation of the Hartog's numeroids [Hartog's numeroids], or 'h-numeroids'

$0, 0+1, 0+1+1, \ldots, 0+\overbrace{1+1+\ldots+1}^{r}, \ldots$

where the constant term r indicates[2] the 'length' of the (generic) h-numeroid (i.e., the number of all occurrences of $+1$ in it). Certes, the generic numeroids

[1] A. S. Yessenin-Volpin takes the position that expressions in a formal language like $\mathrm{L}\widehat{\mathrm{T}}$ must be constructed and that such constructions unfold over time. A termoid is a term or description of a term in the language $\mathrm{L}\widehat{\mathrm{T}}$ without the requirement that the complete syntactical expression it denotes has already been constructed. Likewise a numeroid is a description of an expression of the form $suc^{n}(0)$ where n refers to a natural number without the requirement that the $n-$fold iteration $suc(suc(suc(\ldots.(0)\ldots.)$ has already been constructed.

[2] It is not clear if 'r' denotes a natural number or a constant term in $\mathrm{L}\widehat{\mathrm{T}}$, but see the beginning of [page 9] where, in the expression $suc^{r}(0)$, r is any constant termoid.

$$suc^{r}(0),\ \ 0+\overbrace{1+1+\ldots+1}^{\mathrm{r}}$$

belong not to the formal language – such as $\mathrm{L}\widehat{\mathrm{T}}$ – but to the metatheory[2], MT, to which also the '<u>generic equality'</u>,

$$0+\overbrace{1+1+\ldots+1}^{\mathrm{r}}=suc^{\mathrm{r}}(0)$$

belongs, while its <u>specific instances</u>,

$$0+\overbrace{1+1+\ldots+1}^{0}=suc^{0}(0),\ 0+\overbrace{1+1+\ldots+1}^{1}=suc^{1}(0),$$

$$0+\overbrace{1+1+\ldots+1}^{2}=suc^{2}(0),\ldots$$

i.e., $0=0, 0+1=suc(0), 0+1+1=suc\big(suc(0)\big)$, … belong to $\mathrm{L}\widehat{\mathrm{T}}$ and the provability in $\widehat{\mathrm{T}}^{o}$ of each of these instances can be proved in MT, with the aid of [~~the~~] induction on r [See <u>End Note 1</u>] – because each $\widehat{\mathrm{T}}^{o}$-proof (i.e., proofs in $\widehat{\mathrm{T}}^{o}$) of the equality [page 4]

$$0+\overbrace{1+1+\ldots+1}^{\mathrm{r}}=suc^{\mathrm{r}}(0)$$

can be continued can be continued in $\widehat{\mathrm{T}}^{o}$, i.e., the $\widehat{\mathrm{T}}^{o}$-proof between the first and the last termoids linked by the equalities of the string

$$0+\overbrace{1+1+\ldots+1}^{\mathrm{r+1}}=0+\overbrace{1+1+\ldots+1}^{\mathrm{r}}+1=suc^{\mathrm{r}}(0)+1=suc^{\mathrm{r}}(0)+suc(0)=$$
$$suc(suc^{\mathrm{r}}(0)+0)=suc(suc^{\mathrm{r}}(0))=suc^{\mathrm{r+1}}(0)$$

Using the summation $\sum_{\theta<r}C_1(\theta)$[,] these equalities can be considered, in MT, as providing the $\widehat{\mathrm{T}}^{o}$-provable equalities

$$\textstyle\sum_{\theta<\mathrm{r+1}}C_1(\theta)=0+\overbrace{1+1+\ldots+1}^{\mathrm{r+1}}=\cdots=suc^{\mathrm{r+1}}(0)$$

as soon as a $\widehat{\mathrm{T}}^{o}$-proof is obtained for the equality

$$\textstyle\sum_{\theta<\mathrm{r}}C_1(\theta)=suc^{\mathrm{r}}(0).$$

$\mathrm{L}\widehat{\mathrm{T}}$ will be extended to the language LT, of the open formal system, $\mathrm{T^o}$ by adjoining the '<u>descriptor's</u>' symbol, ι, to the list of logical symbols displayed in the first paragraph of this work; for each variable, λ, and formula, F, in LT, this language shall contain the termoid, $\iota\lambda F$, to be used on the level with other termoids in formation of formulae in LT (cf. [<u>Ro, 1953 or 1978</u>, p 182]). One of Rosser's axiom schemata for ι, the 'Axiom Scheme 11' (ibid, p185) shall be introduced with the aid of the abbreviation (cf. ibid, p167) [page 5]

~~$(\mathrm{E}_1\lambda)C(\lambda)$ for $\exists\lambda C(\lambda)\&\neg(\exists\lambda C(\lambda)\&\neg\exists\pi(\neg\lambda=\pi\&C(\pi)))$~~

[$(\mathrm{E}_1\lambda)C(\lambda)$ for $\exists\lambda C(\lambda)\&\neg(\exists\lambda(C(\lambda)\&\exists\pi(\neg\lambda=\pi\&C(\pi))))$]

where π may be chosen as any variable which does not occur in $C(\lambda)$ (and $\neg$ is the negation sign, in MT: for each formula, G, in LT, $\neg G$ shall denote $G\supset 0=suc(0)$); $(E_1\lambda)C(\lambda)$ expresses the existence and uniqueness of such λ that $C(\lambda)$ holds. The axioms by the scheme [Axiom Scheme 11] shall be, in ℓo, all implications

$$(\mathrm{E}_1\lambda)C(\lambda)\supset\forall\lambda(\iota\lambda C(\lambda)=\lambda\sim C(\lambda)).$$

In particular, for any unary formula, $C(\theta)$, and variable, τ, which does not occur in $C(\theta)$, the disjunction

$$X_C(\tau,\theta): \qquad \tau = 0\&C(\theta).\vee.\tau = 1\&\neg C(\theta)$$

determines τ, for any value of θ, uniquely, and the antecedent

$$(\mathrm{E}_1\tau)X_C(\tau,\theta)$$

of the Rosser axiom [scheme 11] for X_C holds and will be provable in T^o; therefore, also the consequent of this axiom,

$$\forall\tau(\iota\tau X_C(\tau,\theta) = \tau\sim X_C(\tau,\theta)),$$

holds and will be provable in T^o, as well as its corollary,

$$\iota\tau X_C(\tau,\theta) = \iota\tau X_C(\tau,\theta)\sim X_C(\iota\tau X_C(\tau,\theta),\theta))$$

[page 6] (obtainable by substituting $\iota\tau X_C(\tau,\theta)$ of τ in the scope of $\forall\tau$ in this consequent) which entails the implication

$$\iota\tau X_C(\tau,\theta) = \iota\tau X_C(\tau,\theta)\sim X_C(\iota\tau X_C(\tau,\theta),\theta))$$

whose antecedent is, in T^o, an equality axiom and is, therefore, T^o-provable. It follows that also the consequent,

$$X_C(\iota\tau X_C(\tau,\theta),\theta),$$

of this T^o-provable implication, i.e., the disjunction

$$\iota\tau X_C(\tau,\theta) = 0\&C(\theta).\vee.\iota\tau X_C(\tau,\theta) = 1\&\neg C(\theta)$$

is T^o-provable. In MT, the termoid $\iota\tau X_C(\tau,\theta)$ – which is unary – will be abridged as $c(\theta)$, so that this T^o-provable disjunction can be abridged, in MT, as

$$c(\theta) = 0\&C(\theta).\vee.c(\theta) = 1\&\neg C(\theta).$$

Each member of this disjunction is a conjunction each of whose members entails the negation of the other member of the disjunction. It follows that each of the equivalences

$$c(\theta) = 0\sim C(\theta), c(\theta) = 1\sim\neg C(\theta)$$

holds and is T^o-provable. In MT, c can be considered as a [page7] unary characteristic function symbol 'representing' the unary predicate $\underline{C}(\theta)$, 'expressed' by the unary formula $C(\theta)$ (cf. [Kl, 1952, IM, §45). Here I shall use only the implication

$$\neg C(\theta) \supset c(\theta) = 1$$

which holds and - $c(\theta)$ being an abbreviation for the termoid $\iota\tau X_C(\tau,\theta)$, in LT – is T^o-provable. In $\widehat{\mathrm{T}}^{\mathrm{o}}$ – as well as in its extension, T^o – the unary function constant symbol C_1 will be introduced with the defining axiom

$$def_{C_1} \qquad \forall\theta(C_1(\theta) = 1)$$

which provides, for T^o, the possibility to replace the consequent $c(\theta) = 1$ of the last implication by $c(\theta) = C_1(\theta)$. Thus, the implication

$$\neg C(\theta) \supset c(\theta) = C_1(\theta)$$

shall hold and be T^o-provable. Here $C(\theta)$ is obtained from the unary formula $C(\lambda)$ by replacing, in the latter, its only free variable, λ, by the variable θ which does no occure in $C(\lambda)$; thus, also the formula $C(\lambda)$ is unary, and so is the last implication which entails in T^o, the binary implication [page 8]

$$\theta < \lambda \supset (\neg C(\theta) \supset c(\theta) = C_1(\theta))$$

and, further, the implication

$$(\theta < \lambda \supset \neg C(\theta)) \supset (\theta < \lambda \supset c(\theta) = C_1(\theta))$$

and its $\forall\theta$-generalization

$$\forall\theta[\big(\theta < \lambda \supset \neg C(\theta)\big) \supset \big(\theta < \lambda \supset c(\theta) = C_1(\theta)\big)]$$

and the implication

$$\forall\theta\big(\theta < \lambda \supset \neg C(\theta)\big) \supset \forall\theta\big(\theta < \lambda \supset c(\theta) = C_1(\theta)\big).$$

The implication

$$\forall\theta(\theta < \lambda \supset c(\theta) = C_1(\theta)) \supset \textstyle\sum_{\theta<\lambda} c(\theta) = \sum_{\theta<\lambda} C_1(\theta)$$

will be provable in T^o, with the aid of the induction on λ, and the last two implications entail, by virtue of the propositional rule of the chain inference, the next T^o-provable implication

$$\forall\theta(\theta < \lambda \supset \neg \mathrm{C}(\theta)) \supset \textstyle\sum_{\theta<\lambda} c(\theta) = \sum_{\theta<\lambda} C_1(\theta)$$

and also its $\forall\lambda$-closure

$$\forall\lambda[\forall\theta(\theta < \lambda \supset \neg \mathrm{C}(\theta)) \supset \textstyle\sum_{\theta<\lambda} c(\theta) = \sum_{\theta<\lambda} C_1(\theta)].$$

For any constant termoid, r, in LT, the substitution [page 9] of r for λ in the scope of $\forall\lambda$ in this T^o-provable $\forall\lambda$-closure shall give, in T^o, the implication

$$\forall\theta(\theta < r \supset \neg \mathrm{C}(\theta)) \supset \textstyle\sum_{\theta<r} c(\theta) = \sum_{\theta<r} C_1(\theta),$$

and with the aid of the T^o-provable equality $\sum_{\theta<r} C_1(\theta) = suc^r(0)$ (see page 4) this implication can be transformed into another T^o-provable implication,

$$\forall\theta(\theta < r \supset \neg \mathrm{C}(\theta)) \supset \textstyle\sum_{\theta<r} c(\theta) = suc^r(0).$$

Here $C(\theta)$ is obtained by replacing λ by θ in each free occurrence of λ in an arbitrarily chosen unary formula, $C(\lambda)$, in LT; as θ was chosen as a variable which does not occur in $C(\lambda)$ (see p7) also the formula $C(\theta)$ is unary, and $c(\theta)$, i.e., $\iota\tau X_C(\tau, \theta)$ (see p6) can be considered, in MT, as a unary characteristic function which represents the unary predicate, $\underline{C}(\theta)$, 'expressed' by the unary formula $C(\theta)$. The function symbol c is not included in LT because it is not p.r. – however, if the antecedent of the last [page 10] implication holds, then $c(\theta)$ shall be not merely p.r. but even a constant function on the 'segment'

$$[0, \ldots, pd(r)]$$

of the natural number series, at least if $\neg r = 0$, because on this segment the implication $\neg C(\theta) \supset c(\theta) = 1$ must hold (see p7). If $r = 0$ then this 'segment' is $\{0\}$ and the behavior of c on it is immaterial. The consequent of the last displayed implication holds because it is an equality whose left side equals 0 by virtue of the definition of $\sum_{\theta<0}$ and the right side coincides with 0.

Below, r will be specified as the description whose value equals the least, if any, integer which satisfies $C(\lambda)$ and equals 0 if there is no such integer. This description shall be available in T^o, as soon as the classical disjunction

$\text{TND}_{\exists\lambda C(\lambda)}$: $\exists\lambda C(\lambda) \vee \neg\exists\lambda C(\lambda)$

[page 11] will be recognized as an axiom in T^o. As the logic, ℓo, of T^o shall be intuitionistic, this axiom must be a non-logical one.

The last T^o-provable implication used with such r shall provide, for each unary formula $C(\lambda)$ in LT, the truth of the equivalence

$$\exists\lambda C(\lambda) \sim C(suc^r(0))$$

whose T^o-provability will entail the completeness of the formal system T^o, in the sense that each closed formula, E, in the language of LT either is T^o-provable or has a negation, $\neg E$, which is T^o-provable.The connection with the completeness will result from the possibility, with the aid of these equivalences, to 'ban' all variables from E, i.e., to indicate – in MT – a formula E^* which contains no variable and for which the equivalence

$$E \sim E^*$$

[page 12] will be T^o-provable.

To start with, all descriptions in E can be banned in the way exposited in [HB, 1934, v.I], Ch. VIII] for the proof of the Theorem on the eliminability of definite descriptions. More specifically, each description in E can occur only via atomic subformulae of E. If an atomic formula, P, contains a definite description then it contains a leftmost occurrence of the symbol ι with which this description starts. Since P (in E) is a finite linearly ordered object, in this case, P contains the unique leftmost occurrence of ι in P; this description which starts with the occurrence, $\iota\delta F(\delta)$, must be contained in P; let P be redenoted as $P(\iota\delta F(\delta))$ in connection with this occurrence. It suffices to consider here only the cases when this description is 'normal'-so that δ occurs in $F(\delta)$ freely and the formula $(E_1\delta)F(\delta)$ is T^o-provable. Then this formula entails, in T^o, the equivalence

$$P(\iota\delta F(\delta)) \sim \exists\delta(F(\delta) \& P(\delta))$$

[page 13] and the formula E can be proved, in T^o, equivalent to the result, E^*, of replacing the part $P(\iota\delta F(\delta))$ – i.e., P – by $\exists\delta(F(\delta) \& P(\delta))$. The equivalence $E \sim E^*$ will be provable in T^o and E^* contains [only lesser][fewer] than E [number of] occurrences of ι. If E^* still contains occurrences of ι then let the formula E^{**} be obtained from E^* in exactly the same way that E^* was obtained from E. Then the equivalence $E^* \sim E^{**}$ will be T^o- provable, and the equivalences $E \sim E^*$, $E^* \sim E^{**}$ entail, in T^o, the equivalence $E \sim E^{**}$. E^{**} must contain [lesser][fewer] than E^* number of occurrences of i, and if it still contains an occurrence of ι then E^{***} can be obtained from E^{**} in the same way, and $E \sim E^{***}$ will be T^o-provable and the number of occurrences of ι in E^{***} will be lesser than such number of E^{**}, etc.

This procedure must terminate because E is finite and it can terminate only when a formula $E^\wedge$ without any [page 14] occurrences of ι – i.e., a formula, $E^\wedge$, in the language of $\mathrm{L}\widehat{\mathrm{T}}$ will be indicated and proved, in T^o, equivalent to E. As the equivalence $E \sim E^\wedge$ will be T^o-provable, also the equivalence $\neg E \sim \neg E^\wedge$ will be T^o-provable. Since $E^\wedge$ is obtainable from E by a definite string of replacing atomic parts, $P(\iota\delta F(\delta))$, by $\exists\delta(F(\delta) \& P(\delta))$ it is easy to prove in MT – using the course-of-values induction on the height of the construction tree for E – that the formula $(\neg E)^\wedge$ coincides with $\neg E^\wedge$; therefore, the T^o-provability of $\neg E \sim \neg E^\wedge$ means the same as the T^o-provability of $\neg E \sim (\neg E)^\wedge$. Both equations

$$E \sim E^\wedge \text{ and } \neg E \sim (\neg E)^\wedge$$

are $\mathrm{T^o}$-provable, both E and $(\neg E)^{\wedge}$ belong to $\mathrm{L\widehat{T}}$ and $(\neg E)^{\wedge}$ coincides with $\neg E^{\wedge}$. Therefore, if one of $E^{\wedge}$, $\neg E^{\wedge}$ is $\mathrm{T^o}$-provable then also one of E, $\neg E$ is $\mathrm{T^o}$-provable. If $\mathrm{\widehat{T}^o}$ is complete then one of $E^{\wedge}$, $\neg E^{\wedge}$ is $\mathrm{\widehat{T}^o}$-provable and, all the more, T^o-provable. It follows that either E, or $\neg E$ [page 15] is $\mathrm{T^o}$-provable. As E is chosen as any closed formula in LT, this means that – if $\mathrm{\widehat{T}^o}$ is complete then also T^o is complete.

(The scheme of this 'relative completeness' proof is very similar to the scheme of the relative consistency proof, for T^o w.r.t. $\mathrm{\widehat{T}}^o$.) If T^o is complete then any closed formula, G, in $\mathrm{\widehat{T}}^o$ is also a closed formula in T^o and also $\neg G$ is a closed formula in T^o; one of G, $\neg G$ is T^o-provable and, by virtue of the ι-eliminability theorem, this formula is also $\mathrm{\widehat{T}}^o$-provable. T^o is complete iff $\mathrm{\widehat{T}}^o$ is complete.

Now, in order to prove, in MT, that T^o is complete, it remains only to prove, in MT, that $\mathrm{\widehat{T}}^o$ is complete.

Let E be any closed formula in $\mathrm{\widehat{T}}^o$.If E contains a variable then it contains a quantifier with that variable. Each equivalence of the shape

$$\forall\rho H(\rho) \sim \neg\exists\rho\neg H(\rho)$$

[page 16] is provable in the classical arithmetic (see [Kl, 1952, IM, §35, Thm 17, *84º]; each implication $\neg\neg A \supset A$ is intuitionistically entailed by $A \vee \neg A$, i.e., by a TND). Therefore, E is equivalent, in $\mathrm{\widehat{T}}^o$, to a formula, $\tilde{E}$, obtainable from E by replacing each part $\forall\rho H(\rho)$ by $\neg\exists\rho\neg H(\rho)$; $\tilde{E}$ is closed and the equivalence $E \sim \tilde{E}$ is provable in $\mathrm{\widehat{T}}^o$. Now it is left only to consider the case when each variable in E occurs only via parts, $\exists\lambda C$, of E; this $\exists\lambda$ can and will be considered as binding, i.e., C may be considered as $C(\lambda)$ with free occurrence(s) of λ, because otherwise $\exists\lambda$ may simply be dropped. (See [Kl, 1952, IM, §35, Thm 17, *76]). As E is finite, if E contains occurrence(s) of parts $\exists\lambda C(\lambda)$ then it must contain a unique leftmost occurrence of $\exists\lambda$ with which such $\exists\lambda C(\lambda)$ starts. If this $C(\lambda)$ contains, besides λ, a variable, δ, distinct from λ, freely, then - as E is closed – there must be an occurrence of $\exists\delta$ in E to the left of $\exists\lambda$ and binding δ in $C(\lambda)$. This is not the case because [page 17] $\exists\lambda C(\lambda)$ was chosen in E with the leftmost, in E, occurrence of its starting quantifier. It follows that for the leftmost quantifier, $\exists\lambda$, in E, the scope, $C(\lambda)$, bound by this $\exists\lambda$, must be a unary formula in $\mathrm{L\widehat{T}}$, with λ as its only free variable. If for an integer, r, the equivalence

$$\exists\mu C(\lambda) \sim C(suc^r(0))$$

is $\hat{T}^o$-provable – as this was claimed on p. 11 for each unary formula $C(\lambda)$ in $\mathrm{L\widehat{T}}$ – then the replacing of this occurrence of $\exists\lambda C(\lambda)$ by the occurrence of $C(suc^r(0))$ shall change E into a closed formula, E_1, for which the equivalence

$$E \sim E_1$$

will be $\mathrm{\widehat{T}}^o$-provable. If also E_1 contains a variable then it can be changed into a closed formula, E_2, in $\mathrm{L\widehat{T}}$, related to E_1 as E_1 is related to E so that the equivalence [page 18]

$$E_1 \sim E_2$$

is $\mathrm{\widehat{T}}^o$- provable – and the $\mathrm{\widehat{T}}^o$-provability of the two equivalences,

$$E \sim E_1 \text{ and } E_1 \sim E_2$$

shall entail the $\mathrm{\widehat{T}}^o$-provability of the equivalence

$$E \sim E_2$$

where E_2 contains lesser than E_1 number of occurrences of ∃ which, in turn, is lesser than such number of E; if E_2 shall contain a variable then, in $\mathrm{L}\widehat{\mathrm{T}}$, a closed formula E_3 with still lesser number of quantifiers in it than E_2 can be indicated and the equivalences $E_2 \sim E_3$ and

$E \sim E_3$

will be $\widehat{\mathrm{T}}^o$-provable, etc. Since E is finite this procedure must stop and this can happen only when a closed formula, E', which contains no occurrence of a variable will be indicated and the equivalence [page 19]

$E \sim E'$

will be found $\widehat{\mathrm{T}}^o$-provable. The formula E', in $\mathrm{L}\widehat{\mathrm{T}}$, containing no occurrence of a variable, may be atomic, or constructed from atomic closed formulae in $\mathrm{L}\widehat{\mathrm{T}}$ with the aid of propositional operators only.

If each propositional part of E' is closed and 'solvable', i.e., provable or disprovable, in $\widehat{\mathrm{T}}^o$, then, as in [Kl, 1952, IM, §§29-30], also the formula E' is solvable. Therefore, it is left only to consider the case when E', in $\mathrm{L}\widehat{\mathrm{T}}$, is a closed atomic formula. In this case, E' is obtainable from n-numeroids with the aid of finitely many applications of function symbols in $\mathrm{L}\widehat{\mathrm{T}}$ and one application of $=, <, \text{or} \leq$. Since each of these symbols shall be p.r., E' must be 'solvable' in $\widehat{\mathrm{T}}^o$ also in this case.

(That is why the unary symbol c used above was not included in $\mathrm{L}\widehat{\mathrm{T}}$ – and still it works when interpreted [page 20] as above, i.e. $c(\theta)$ as $\iota X_C(\tau, \theta)$ – see p. 6.)

Now it is left only to indicate, in MT, for each unary formula $C(\lambda)$ such an integer r that the equivalence

$\exists\lambda \mathrm{C}(\lambda) \sim C(suc^r(0))$

is T^o-provable. This equivalence will be in $\mathrm{L}\widehat{\mathrm{T}}$ if $C(\lambda)$ is in $\mathrm{L}\widehat{\mathrm{T}}$; since the implication

$C(suc^r(0)) \supset \exists\lambda \mathrm{C}(\lambda)$

will be a Strong Bernays axiom for ∃, even regardless [to][of] the choice of r, it is left only to indicate, in MT, for each unary formula $C(\lambda)$ which contains λ freely such an integer, r, that the implication

$\exists\lambda \mathrm{C}(\lambda) \supset C(suc^r(0))$

is T^o-provable.

The 'least number principle' is expressed, in each of $\mathrm{L}\widehat{\mathrm{T}}$ and LT, by the implication [page 21]

$(lnp)\ \ \exists\lambda C(\lambda) \supset \exists\lambda(C(\lambda) \& \forall\theta(\theta < \lambda \supset \neg C(\theta))$

which can be deduced, in each of the formal systems $\widehat{\mathrm{T}}^o$ and T^o, from the formula

$(\mathrm{TND}_{C(\lambda)})\ \ \forall\lambda(C(\lambda) \vee \neg C(\lambda))$

(cf. [Kl, 1952, IM, §40, *149º]) which is postulated in these classical systems and is, therefore, provable in each of them and therefore also the implication (lnp) is provable in each of the formal systems $\widehat{\mathrm{T}}^o$ and T^o. In (lnp), λ, θ are any two distinct variables of which λ, but not θ occurs in the formula $C(\lambda)$ freely. So far, $C(\lambda)$ is not supposed [to be] unary though soon, when the length r of the n-numeroid $suc^r(0)$ will be defined $C(\lambda)$ will be considered as unary – just in order to make this length a definite integer. So far, definite or not, a/the integer r such that

$suc^r(0)$ is the 'shortest' n-numeroid for which $C(suc^r(0))$ holds, is considered, in MT, as existing provided [page 22] only that there exists a λ which satisfies $C(\lambda)$. Moreover, in accordance with the implication *174b, in [Kl, IM, §41]; the implication

$$\exists\lambda[C(\lambda)\&\forall\theta(\theta<\lambda\supset\neg C(\theta))]\supset\exists!\,\lambda[C(\lambda)\&\forall\theta(\theta<\lambda\supset\neg C(\theta))]$$

whose consequent expresses the uniqueness of λ whose existence is expressed by the antecedent is provable, even without using non-intuitionistic axioms in the arithmetic such as $\widehat{\mathrm{T}}^o$ and/or T^o. The antecedent of this implication is the same formula as the consequent of (lnp). Therefore, also the implication

$$(\exists!\,lnp)\quad \exists\lambda C(\lambda)\supset\exists!\,\lambda(C(\lambda)\&\forall\theta(\theta<\lambda\supset\neg C(\theta)))$$

is $\widehat{\mathrm{T}}^o$-provable. The consequent of this implication stands for the conjunction

$$\exists\lambda[C(\lambda)\&\forall\theta(\theta<\lambda\supset\neg C(\lambda))\&\forall\pi(C(\pi)\&\forall\theta(\theta<\pi\supset\neg C(\theta))\supset\lambda=\pi)$$

in which π denotes any variable which does not [page 23] occur (freely) in $C(\lambda)$ and also in $C(\theta)$, and is therefore distinct from each λ and θ. This conjunction does not coincide with the conjunction

$$(\mathrm{E}_1\lambda)(C(\lambda)\&\forall\theta(\theta<\lambda\supset\neg C(\theta)),$$

i.e.,

$$\exists\lambda\Big(C(\lambda)\ \&\ \forall\theta(\theta<\lambda\supset\neg C(\theta))\Big)\ \&$$

$$\neg\Bigg(\exists\lambda\Big(C(\lambda)\ \&\ \forall\theta(\theta<\lambda\supset\neg C(\theta))\Big)\ \&\ \exists\pi\Big(\neg\lambda=\pi\ \&\ \Big(C(\pi)\ \&\ \forall\theta(\theta<\pi\supset\neg C(\theta))\Big)\Big)\Bigg),$$

but both conjunctions are, in $\widehat{\mathrm{T}}^o$, and in T^o, equivalent (see [Ro, 1953, or 1978, LfM, ch VII, Thm VII.2.1, pp 167-169]).

Therefore, the implication $(\exists!\,lnp)$ is, in $\widehat{\mathrm{T}}^o$ and in T^o, equivalent to the implication

$$(\mathrm{E}_1 lnp)\quad \exists\lambda C(\lambda)\supset(\mathrm{E}_1\lambda)(C(\lambda)\ \&\ \forall\theta(\theta<\lambda\supset\neg C(\theta))).$$

As $(\exists!\,lnp)$ is T^o-provable, so is $(\mathrm{E}_1 lnp)$.

It is possible – with the aid of the disjunction

$$\mathrm{TND}_{\exists\lambda C(\lambda)}\quad \exists\lambda C(\lambda)\vee\neg\exists\lambda C(\lambda)$$

[page 24] which shall be postulated in each of the classical systems $\widehat{\mathrm{T}}^o$ and in T^o, to get rid of the antecedent of $\exists\lambda C(\lambda)$ in $(\mathrm{E}_1 lnp)$ if the scope

$$C(\lambda)\ \&\ \forall\theta(\theta<\lambda\supset\neg C(\theta))$$

of $(\mathrm{E}_1\lambda)$ in the consequent of this implication will be weakened by using it as the first member of the disjunction whose second member will be

$$\lambda=0\ \&\ \neg\exists\lambda C(\lambda).$$

More specifically let

$M(\lambda)$ denote $M_1(\lambda)\ \&\ M_2(\lambda)\ .\!\vee\!.\ M_3(\lambda)\ \&\ M_4$

where

$M_1(\lambda)$ denotes $C(\lambda)$,

$M_2(\lambda)$ denotes $\forall\theta(\theta < \lambda \supset \neg C(\theta))$,

$M_3(\lambda)$ denotes $\lambda = 0$,

M_4 denotes $\neg\exists\lambda C(\lambda)$.

Then, as soon as $C(\lambda) \supset \exists\lambda C(\lambda)$ shall be provable – and actually even postulated – in the logic, lo, of each [page 25] of the systems $\widehat{\mathrm{T}}^o$ and in T^o, the implication

$$\neg\exists\lambda C(\lambda) \supset \neg C(\lambda),$$

also shall be provable in the logic of each of these systems; in the current notations, this implication can be displayed as

$$M_4 \supset \neg M_1(\lambda)$$

so that the implication

$$M_3(\lambda)\ \&\ M_4 \supset \neg(M_1(\lambda)\ \&\ M_2(\lambda))$$

is provable in lo and $M(\lambda)$ is the disjunction of two incompatible conjunctions. The first of them is

$$M_1(\lambda)\ \&\ M_2(\lambda)$$

and occurs in the consequent of the T^o-provable implication $(\mathrm{E}_1 lnp)$ (see p. 23) which can be rewritten as

$$\exists\lambda C(\lambda) \supset (\mathrm{E}_1\lambda)(M_1(\lambda)\ \&\ M_2(\lambda)).$$

Thus, in T^o, the first member of the disjunction $\mathrm{TND}_{\exists\lambda C(\lambda)}$ entails the formula which expresses the existence and uniqueness of λ which satisfies $M_1(\lambda)\ \&\ M_2(\lambda)$ – [page 26] and therefore also of the disjunction $M(\lambda)$ (because $\exists\lambda C(\lambda)$ entails, in lo, the negation of $M_3(\lambda)\ \&\ M_4$). This passage can be easily formalized in T^o (and also in $\widehat{\mathrm{T}}^o$ if $C(\lambda)$ belongs to $\mathrm{L}\widehat{\mathrm{T}}$). Thus the implication

$$\exists\lambda C(\lambda) \supset (\mathrm{E}_1\lambda)M(\lambda)$$

is provable in T^o (and also in $\widehat{\mathrm{T}}^o$ if $C(\lambda)$ belongs to $\mathrm{L}\widehat{\mathrm{T}}$). On the other hand, the consequent of this implication is [also] entailed in T^o (and in $\widehat{\mathrm{T}}^o$ if $C(\lambda)$ belongs to $\mathrm{L}\widehat{\mathrm{T}}$) [~~also~~] by the second member of the disjunction $\mathrm{TND}_{\exists\lambda C(\lambda)}$[. This can be shown as follows. ~~because~~]

$$\neg\exists\lambda C(\lambda) \text{ entails } \neg\exists\lambda C(\lambda) \supset 0 = 0\ \&\ \neg\exists\lambda C(\lambda)$$

and, further,

$$\neg\exists\lambda C(\lambda) \supset M_1(\lambda)\ \&\ M_2(\lambda)\ .\vee.\ 0 = 0\ \&\ \neg\exists\lambda C(\lambda),$$

and the implication

$$M_1(\lambda)\ \&\ M_2(\lambda)\ .\vee.\ 0 = 0\ \&\ M_4 \supset \exists\lambda(M_1(\lambda)\ \&\ M_2(\lambda)\ .\vee.\ \lambda = 0\ \&\ M_4)$$

(where M_4 stands for $\neg\exists\lambda C(\lambda)$) is T^o-provable as an axiom of the substitution (of 0 for λ) in T^o[~~;~~. The] [~~the~~] chain inference step applied with [the] two last implications as [~~with~~] its premises [page 27] gives, in lo of T^o (and $\widehat{\mathrm{T}}^o$ if $C(\lambda)$ belongs to $\mathrm{L}\widehat{\mathrm{T}}$) the implication

$$\neg\exists\lambda C(\lambda) \supset \exists\lambda(M_1(\lambda)\ \&\ M_2(\lambda)\ .\vee.\ \lambda = 0\ \&\ M_4)\ [.]$$

[Its ~~whose~~] consequent can be strengthened by replacing is $\exists\lambda$ by $(\mathrm{E}_1\lambda)$ (see pp. 4-5) [which we can show as follows. For ~~because, for~~] any variable π which does not occur in $C(\lambda)$, [the] two disjunctions,

$M_1(\lambda)\ \&\ M_2(\lambda) .\vee. \lambda = 0\ \&\ M_4$ and

$M_1(\lambda)\ \&\ M_2(\lambda) .\vee. \pi = 0\ \&\ M_4$

entail $\lambda = \pi$ [. This is because ~~(because~~] $\lambda = 0$ and $\pi = 0$ entail $\lambda = \pi$ and thus the second members of these disjunctions entail $\lambda = \pi$; they entail also M_4 and therefore (see p.25) $\neg M_1(\lambda)$, and therefore also [entail] $\neg\big(M_1(\lambda)\ \&\ M_2(\lambda)\big)$~~[; thus~~ . Thus] the second member of each of these disjunctions entails the negation of their (common) first member, and the contradiction entails, in particular, $\lambda = \pi$; two first members of these disjunctions entail, respectively, $M_1(\lambda)$ and $M_1(\pi)$ and, further, $\exists\lambda M(\lambda)$; on pp. 21-23 it was [page 28] found that $\exists\lambda C(\lambda)$ entails

$(\mathrm{E}_1\lambda)(C(\lambda)\&\forall\theta\big(\theta < \lambda \supset \neg C(\theta)\big)$

and therefore $\exists\lambda C(\lambda)$ and $M_2(\lambda)$ entail $\lambda = \pi$ - in T^o (and, if $C(\lambda)$ and $M_2(\lambda)$ belong to $\mathrm{L}\widehat{\mathrm{T}}$, in $\widehat{\mathrm{T}}^o$). Thus these two disjunctions entail $\lambda = \pi$ and their parts $\lambda = 0$, $\pi = 0$ can be replaced, respectively, by $M_3(\lambda)$ and $M_3(\pi)$. Thus the uniqueness of λ such that $M(\lambda)$ is entailed, formally, not only by the first but also by the second member of the disjunction $\mathrm{TND}_{\exists\lambda C(\lambda)}$; by cases of this disjunction, the formula

$(\mathrm{E}_1\lambda)(M_1(\lambda)\ \&\ M_2(\lambda) .\vee.\ M_3(\lambda)\ \&\ M_4)$

is provable in T^o(see again[Ro, 1953, or 1978, LfM, ch VII, Thm VII.2.1, pp 169-170]. Shorter, this T^o-provable formula can be displayed as

$(\mathrm{E}_1\lambda)M(\lambda)$.

When $C(\lambda)$ is unary, this formula is closed and it can be used as the antecedent of the implication which is an axiom, in T^o, by the Rosser's scheme 11.

[Page 29] At this point I suggest that $C(\lambda)$ is a unary formula in LT whose only free variable is λ. Thus, the antecedent $(\mathrm{E}_1\lambda)M(\lambda)$ of the description axiom is closed, and so is also this axiom,

$(\mathrm{E}_1\lambda)M(\lambda) \supset \forall\lambda(\iota\lambda M(\lambda) = \lambda{\sim}M(\lambda))$.

The antecedent of this axiom – which is an implication – is T^o-provable – and so is its consequent,

$\forall\lambda(\iota\lambda M(\lambda) = \lambda{\sim}M(\lambda))$.

Since $\iota\lambda M(\lambda)$ occurs, this axiom and its consequent don't belong to $\mathrm{L}\widehat{\mathrm{T}}$, even when $C(\lambda)$ does, and further consideration will give as formalizable in T^o – not in $\widehat{\mathrm{T}}^o$, though the result, $C(suc^r(0)) \supset \exists\lambda C(\lambda)$ with some definite r – which will be, by the way, specified as $\iota\lambda M(\lambda)$ – will belong to $\mathrm{L}\widehat{\mathrm{T}}$ (see p. 20). By virtue of the ι-eliminability theorem, this result can be strengthened to the statement that this implication in provable in $\widehat{\mathrm{T}}^o$, but here I shall not explicitly present its $\widehat{\mathrm{T}}^o$-proof.

[Page 30] The conjunction $\iota\lambda M(\lambda) = 0\ \&\ \neg\exists\lambda C(\lambda)$ propositionally entails its second member, $\neg\exists\lambda C(\lambda)$, i.e., the implication

$\iota\lambda M(\lambda) = 0\ \&\ \neg\exists\lambda C(\lambda) \supset \neg\exists\lambda C(\lambda)$

is propositionally provable and, with the aid of the contraposition, it entails, propositionally,

$$\exists\lambda C(\lambda) \supset \neg(\iota\lambda M(\lambda) = 0 \ \& \ \neg\exists\lambda C(\lambda))$$

or shorter,

$$\exists\lambda C(\lambda) \supset \neg(M_3(\lambda) \ \& \ M_4).$$

Also the implication

$$M(\iota\lambda M(\lambda)) \supset (\neg(M_3(\lambda) \ \& \ M_4) \supset M_1(\iota\lambda M(\lambda)) \ \& \ M_2(\iota\lambda M(\lambda))))$$

is propositionally provable – and since its antecedent is T^o-provable, also its consequent,

$$\neg(M_3(\lambda) \ \& \ M_4) \supset M_1(\iota\lambda M(\lambda)) \ \& \ M_2(\iota\lambda M(\lambda))$$

is T^o-provable. It has the antecedent which coincides with the consequent of the propositionally – and therefore in T^o-provable implication $\exists\lambda C(\lambda) \supset \neg(M_3(\lambda) \ \& \ M_4)$, and the chain inference step using these two implications [page 31] as its premises has the T^o-provable conclusion

$$\exists\lambda(C\lambda) \supset M_1(\iota\lambda M_1(\lambda)) \ \& \ M_2(\iota\lambda M_2(\lambda))$$

whose consequent is a conjunction which can be used as the antecedent of each of two propositional axioms,

$$M_1(\iota\lambda M_1(\lambda)) \ \& \ M_2(\iota\lambda M_2(\lambda)) \supset M_1(\iota\lambda M_1(\lambda))$$

$$M_1(\iota\lambda M_1(\lambda)) \ \& \ M_2(\iota\lambda M_2(\lambda)) \supset M_2(\iota\lambda M_2(\lambda))$$

so that these axioms can be used, in T^o, as the right premise of the chain inference steps whose left premise has $\exists\lambda C(\lambda)$ as its antecedent. The conclusions

$$\exists\lambda(C\lambda) \supset M_1(\iota\lambda M_1(\lambda)),$$

$$\exists\lambda(C\lambda) \supset M_2(\iota\lambda M_2(\lambda))$$

of these chain inference steps are T^o-provable and they are, respectively (see p.24)

$$\exists\lambda(C\lambda) \supset M(\iota\lambda M(\lambda))$$

and

$$\exists\lambda C(\lambda) \supset \forall\theta(\theta < \iota\lambda M(\lambda) \supset \neg C(\theta)).$$

[Page 32] The $\iota\lambda M(\lambda)$-instance

$$\textstyle\sum_{\theta<\iota\lambda M(\lambda)} C_1(\theta) = suc^{\iota\lambda M(\lambda)}(0)$$

of the equality $\sum_{\theta<r} C_1(\theta) = suc^r(0)$ (of p.4) is $\widehat{\mathrm{T}}^o$-provable (see pp. 3-4) and has the left side which can be considered, in MT, as the compact form of the h-numeroid whose length equals $\iota\lambda M(\lambda)$. As the equality is provable in $\widehat{\mathrm{T}}^o$, it is T^o-provable.

On the other hand, the formula

$$\forall\lambda(\textstyle\sum_{\theta<\lambda} C_1(\theta) = \lambda)$$

can be proved, in $\widehat{\mathrm{T}}^o$, with the aid of the induction on λ (using the recursive definition of $\sum_{\theta<\lambda} C_1(\theta)$ given on p. 2 and the explicit definition of C_1 given on p. 7). It is thus provable in T^o, and also in $\widehat{\mathrm{T}}^{o}$; in T^{o} it can be used as the antecedent of the axiom of substitution of $\iota\lambda M(\lambda)$ for λ, and the consequent [page 33] of this axiom is T^{o}-provable.

Thus, two equalities,

$$\sum_{\theta<\iota\lambda M(\lambda)} C_1(\theta) = \iota\lambda M(\lambda)$$

and

$$\sum_{\theta<\iota\lambda M(\lambda)} C_1(\theta) = suc^{\iota\lambda M(\lambda)}(0)$$

are T^{o}-provable, and they have the same left sides. Therefore, in T^{o}, they entail the equality

$$\iota\lambda M(\lambda) = suc^{\iota\lambda M(\lambda)}(0)$$

which is T^{o}-provable and can be used, in T^{o}, as the antecedent of the 'general Leibniz equality Axiom' [g lea]:

$$\iota\lambda M(\lambda) = suc^{\iota\lambda M(\lambda)}(0) \supset (C(\iota\lambda M(\lambda)) \supset C(suc^{\iota\lambda M(\lambda)}(0)))$$

(which is applicable because the termoids $\iota\lambda M(\lambda)$ and $suc^{\iota\lambda M(\lambda)}(0)$ are constant). It follows also the consequent,

$$C(\iota\lambda M(\lambda)) \supset C(suc^{\iota\lambda M(\lambda)}(0)),$$

of this axiom is T^{o}-provable. Its antecedent coincides with the consequent of the last but one implication, [page 34]

$$\exists\lambda C(\lambda) \supset C(\iota\lambda M(\lambda)),$$

found T^{o}-provable on p.31, and the chain inference step having these two implications as its premises has the T^{o}-provable conclusion,

$$\exists\lambda C(\lambda) \supset C(suc^{\iota\lambda M(\lambda)}).$$

As the integer $\iota\lambda M(\lambda)$ – i.e.,

$$\iota\lambda(C(\lambda)\ \&\ \forall\theta(\theta < \lambda \supset \neg C(\theta).\vee.\lambda = 0\&\neg\exists\lambda C(\lambda)) -$$

is considered in MT as determined by the unary formula $C(\lambda)$ in LT, the T^{o}-provability of this implication, for any such $C(\lambda)$, can be used in the argument given on pp. 11-20 as a proof, in MT, of the completeness of T^{o} and (see p.15) of $\widehat{\mathrm{T}}^{o}$.

Of course, this MT-proof is not constructive because it used $\mathrm{TND}_{C(\lambda)}$ and $\mathrm{TND}_{\exists\lambda C(\lambda)}$ – for each $\exists\lambda C(\lambda)$ with unary $C(\lambda)$ which occurs in the arbitrarily given closed formula, E, in $\mathrm{L}\widehat{\mathrm{T}}$ – not only formally, but also intuitively, because the length's [page 35] superscript, $\iota\lambda M(\lambda)$, occurs in the presentation of the n-numeroid $suc^{\iota\lambda M(\lambda)}(0)$ only at the place which is not specified by the grammar of LT. This does not conflict with the concept of completeness as this concept is applied to $\widehat{\mathrm{T}}^{o}$ or T^{o} because the 'existence' of the integer equal to $\iota\lambda M(\lambda)$ does not presuppose a possibility to calculate it.

Such a possibility, even when non-constructively established, would provide some positive approach to solution of the decision problem. Here I don't claim much in this direction. Still it seems worth mentioning that each proof of completeness of T^{o} or $\widehat{\mathrm{T}}^{o}$ gives a way of systematic search of proof of one of the closed formulae, E or $\neg E$, in LT or, respectively, int $\mathrm{L}\widehat{\mathrm{T}}$. for the T^{o}- or $\widehat{\mathrm{T}}^{o}$-proof can be enumerated – say, in the Gödelian way [Gö 1931]. Thus all proofs – say in T^{o} – can be considered, in MT, [page 36] as arranged in a definite sequence

$P_0, P_1, \ldots, P_\lambda, \ldots$

and for each of them, P_λ, its root formula, R_λ, can be definitely indicated. As T^o is complete, for each closed formula, E, there must be a λ for which R_λ coincides with one of the formulae, E, $\neg E$. The least such lambda for a given closed E can be denoted by λ_E in order to solve the decision problem for E, and if a way of such finding is indicated, for all $E's$, constructively, then the decision problem can be considered as constructively solved. The proof given above for completeness of T^o gives nothing in the direction of such constructivity.

This 'proof' of solvability of either E or $\neg E$ by a single way for all closed $E's$, in LT and T^o, uses a Gödel style enumeration of proofs – without using such enumeration of formulae in LT. The use of enumeration [page 37] for both has lead Gödel [Gö, 1931] to his famous result conflicting with the current proof of completeness of T^o. Why not? K. Gödel did not use the term of 'incompleteness', his theorem was called simply 'Satz VI' ['Sentence VI']. Closed formulae were enumerated so that each of them, E, got a definite 'G-number', $\underline{\nu}_E$. The enumeration was introduced in [Gö 1] on the whole formal system including formulae and their parts and deductions and proofs, and in the well known way the metatheory of the system was 'arithmeticized'. It was assumed that each closed E gets exactly one $\underline{\nu}_E$, in all its occurrences the same. The 'truth' or 'falsehood' of E was considered as an i[n]ternal property of E determined by its syntactical structure. This could be a sound ideas so far as the content of the formulae was always mathematical – but the subject of the mathematics was extended with arithmetization of the metatheories. [Page 38] A certain closed formula, G, was formed to be 'self-referential' – viz., it meant that G is not provable in the formal system, say, T^0. Thus two occurrences of G arose, one being mentioned by the other, and mentioned in a negative way, as in the ancient 'Liar paradox'. The phrase 'I am lying' is both, true and false, if it must have exactly one sense – and this paradox disappears as soon as one starts to differ its 'real' and 'mentioned' occurrences. The Gödel's proof about G disappears when 'this' formula is recognized as splitting in (at least) two occurrences, each of which has to be given it's own 'G-number'.

Philosophically, this should be clear already then, about 1931. A few authors [E. Wette, I. Smirnoff] have noticed this and similar 'contradictions' in the metatheories during the '60-'90 – and I have criticized [page 39] some fundamental questions even before those authors and only about '99-'02 came to a relatively concise proofs of the paradoxicalities of metatheories.

It is easy to achieve the consistency of any formalized theory with arithmetized theory simply by narrowing the concept of formal proof by restricting the use of modus ponens rule

$$\text{(MP)} \qquad \frac{E \quad E \supset F}{F}$$

to cases when, in the proof none of the premises explicitly contradicts to a formula already proved (as any A, $\neg A$ explicitly contradict[ory] to each other). In cases when E is G, F is f (i.e., $0 = 1$) f can be deduced – but not used as a premise in continuation of [the] proof (if f is recognized as explicitly contradicting each axiom). Look for better solutions for consistency problems. Of course, this approach requires a revision [page 40] of [the] interrelation between deductions and proofs, if the consistency restrictions are imposed only on the proofs.

The completeness of $\widehat{\mathrm{T}}^o$ was proved above using only the following TND-axiom schemata in which $C(\lambda)$ denotes any unary formula in $\mathrm{L}\widehat{\mathrm{T}}$ whose free variable, λ, is arbitrary:

$\mathrm{TND}_{C(\lambda)}$ $\forall\lambda(C(\lambda) \vee \neg C(\lambda))$,

$\mathrm{TND}_{\exists\lambda C(\lambda)}$ $\exists\lambda(C(\lambda) \vee \neg\exists\lambda C(\lambda))$.

$\mathrm{TND}_{C(\lambda)}$ is used for $\neg\exists\lambda\neg C(\lambda) \supset \forall\lambda C(\lambda)$ (the inverse implication is intuitionistic – see [Kl, 1952, IM, §33, Thm 17, *84a] and for $(\mathrm{E}_1\lambda)X_C(\tau,\lambda)$; $\mathrm{TND}_{\exists\lambda C(\lambda)}$ – for $(\mathrm{E}_1\lambda)M(\lambda)$.

In these cases $C(\lambda)$ contains λ freely (and λ is denoted[).]

For and arbitrary closed formula, E, in $\mathrm{L}\widehat{\mathrm{T}}^o$ (or $\mathrm{L}\widehat{\mathrm{T}}$) the disjunction

TND_E $E \vee \neg E$

is provable in $\widehat{\mathrm{T}}^o$ (respectively, in T^o) because one of E, $\neg E$ is provable [page 41] in the system. In other words, the closed TND_E is entailed by the formulae $\mathrm{TND}_{C(\lambda)}$, $\mathrm{TND}_{\exists\lambda C(\lambda)}$ whose $C(\lambda)$ is a subformula of E.

Bibliography

HB 1934, GM, vol. I – D. Hilbert and P. Bernays, "Grundlagen der Mathematic", Berlin, 1934.

Kl, 1952, IM – S. Kleene, "Introduction to Metamathematics", Amsterdam, Groningen, New York, Toronto, 1952.

Ro, 1953 or 1978, LfM – J. Rosser, "Logic for Mathematicians", New York, 1953 (1st ed.) or 1978 (2nd ed.).

Gö, 1931 – K. Gödel, "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I", Vienna ["On Formally Undecidable Propositions of Principia Mathematica and Related Systems I" – a Gödel approved translation by Jean van Heijenoort appears in the collection *From Frege to Gödel: A Source Book in Mathematical Logic* (van Heijenoort 1967).]

E. Wette [Reference not specified]

I. Smirnoff [Reference not specified]

End Notes

---

[1] **Notational Conventions**: Alex considers two theories, $\widehat{\mathrm{T}}^{o}$ and T^{o}, both of which permit open axioms and open hypotheses in derivations. They differ in that the language of T^{o} (which Alex denotes by LT) extends the language of $\widehat{\mathrm{T}}^{o}$ (which Alex denotes by $\mathrm{L}\widehat{\mathrm{T}}$) by adding the iota symbol ι to the alphabet of $\mathrm{L}\widehat{\mathrm{T}}$ together with the terms $\iota\lambda F$ for any formulae in $\mathrm{L}\widehat{\mathrm{T}}$.

He does not denote the language of $\widehat{\mathrm{T}}^{o}$ by $\mathrm{L}\widehat{\mathrm{T}}^{o}$ (or the language of T^{o} by LT^{o}) because the superscript 'o' (= 'open') in the designation of a language is meaningless – languages are neither open nor closed – they always contain all formulae, open and closed.

One would normally used the ˆ applied to L to denote an extension of L, so you might reasonably expect that $\mathrm{L}\widehat{\mathrm{T}}$ extends LT, but Alex does this the other way around.

[2] **MT induction**: A number of arguments in this paper make use of induction in the metatheory which can be formulated as follows.

Let $\mathcal{A}(\lambda)$ denote a metatheory assertion about T, with free variable λ, and let r denote any closed term in LT.

MT Induction rule:

$\dfrac{\mathcal{A}(0) \quad \mathcal{A}(r)\supset\mathcal{A}(r+1)}{\forall r\mathcal{A}(r)}$ where the metatheory quantifier $\forall r$ means for all closed terms in LT.

An example of this is the MT demonstration of the following MT equation. Let r range over constant terms.

Theorem: $\forall r \vdash_{\mathrm{T}} \sum_{\theta<\mathrm{r}} C_1(\theta) = 0 + \overbrace{1 + 1 + \ldots +1}^{\mathrm{r}}$. (See page 1 for the definition of $\sum_{\theta<r}$.)

(Note: Expressions such as $0 + \overbrace{1 + 1 + \ldots +1}^{\mathrm{r}}$ and $suc^{\mathrm{r}}(0)$, where r is any closed term in LT, belong to MT. In this case $\mathcal{A}(r)$ is $\vdash_{\mathrm{T}} \sum_{\theta<r} C_1(\theta) = 0 + \overbrace{1 + 1 + \ldots +1}^{r}$.)

Derivation:

1	$\vdash_{\mathrm{T}} \sum_{\theta<0} C_1(\theta) = 0 = 0 + \overbrace{1 + 1 + \ldots +1}^{0}$	$[def_1^{\sum_{\theta<0}^{C_1}}]$
2	$\vdash_{\mathrm{T}} \sum_{\theta<\mathrm{r}} C_1(\theta) = 0 + \overbrace{1 + 1 + \ldots +1}^{\mathrm{r}}$	[assume]
3	$\vdash_{\mathrm{T}} \sum_{\theta<\mathrm{r}+1} C_1(\theta) = \sum_{\theta<\mathrm{r}} C_1(\theta) + C_1(r)$	$[def_2^{\sum_{\theta<0}^{C_1}}]$
4	$\vdash_{\mathrm{T}} \sum_{\theta<\mathrm{r}} C_1(\theta) + C_1(r) = 0 + \overbrace{1 + 1 + \ldots +1}^{\mathrm{r}} + 1 = 0 + \overbrace{1 + 1 + \ldots +1}^{\mathrm{r}+1}$	[2, def of C_1]
5	$\vdash_{\mathrm{T}} \sum_{\theta<\mathrm{r}+1} C_1(\theta) = 0 + \overbrace{1 + 1 + \ldots +1}^{\mathrm{r}+1}$	[3, 4; transitivity of =]
6	$\vdash_{\mathrm{T}} \sum_{\theta<\mathrm{r}} C_1(\theta) = 0 + \overbrace{1 + 1 + \ldots +1}^{\mathrm{r}} \supset \sum_{\theta<r+1} C_1(\theta) = 0 + \overbrace{1 + 1 + \ldots +1}^{\mathrm{r}+1}$	[discharge 2]
7	$\forall r \vdash_{\mathrm{T}} \sum_{\theta<r} C_1(\theta) = 0 + \overbrace{1 + 1 + \ldots +1}^{r}$	[1, 6; MT induction]

MT Induction can be formulated somewhat differently as

MT Numeral-wise Induction: $\frac{\mathcal{A}(0) \quad \mathcal{A}(n) \supset \mathcal{A}(n+1)}{\forall r \mathcal{A}(r)}$

where 'n' denotes a numeral and 'r' denotes ant constant term in the language.

The understanding of numeral-wise induction step is that it "provides a proof of $\mathcal{A}(n+1)$ as soon as a proof of $\mathcal{A}(n)$ appears", and the justification of $\forall r \mathcal{A}(r)$ is then based on the assertion "every constant term r has a numeral value $\underline{n}$, i.e., the equality $r = \underline{n}$ is true". Note that this is different from asserting that there is a proof in T of the equality $r = \underline{n}$. It what ways is this different from ordinary induction: $\frac{\mathcal{A}(0) \quad \mathcal{A}(\lambda) \supset \mathcal{A}(\lambda+1)}{\forall \lambda \mathcal{A}(\lambda)}$ since $\forall \lambda \mathcal{A}(\lambda) \supset \mathcal{A}(r)$ for any constant term r.