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A FORMALIZATION OF ESSENIN-VOLPIN'S PROOF THEORETICAL STUDIES BY MEANS OF NONSTANDARD ANALYSIS

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This paper constitutes a short report on the proof theory which developed in the course of an investigation of Essenin-Volpin's foundational studies (see his paper in ***Infinitistic methods***.) The ***Infinitistic methods*** paper is primarily concerned with proving the consistency of ZF. His arguments employ some very unusual ideas, especially that of the possibility of different length natural number series. The modal setting which he uses for these ideas and which is really all important is not reflected in the present paper. Only a simple representation of this idea is considered, namely, the use of $*N$, the nonstandard natural numbers, together with N, the standard natural numbers, to play the roles of a "long" and a "short" number series.

It will be useful to make a few general remarks before beginning on the technical details. These remarks are especially directed towards those people who have read Essenin-Volpin's work, the above-mentioned paper in particular and the one appearing in the proceedings of the *1968 Conference on Intuitionism and Proof Theory* at Buffalo, New York.

I want to draw a comparison between the ultra-intuitionistic position of Essenin-Volpin on the one hand with the intuitionistic and realistic positions on the foundations of mathematics on the other hand. Intuitionism makes a transition from the traditional mathematical position (realism) that the rules of mathematical reasoning must be constructed so as to preserve *truth* (as known by an omniscient being about "*the real world*" of abstract mathematical objects) to the position that the rules of mathematical reasoning must be in accord with the *process of knowing* (performed by the "ideal mathematician" or "creative subject") about the world of mental constructions. Ultra-intuitionism deepens and extends this transition to the position that the fundamental rules of mathematical reasoning must be in accord with processes of knowing accessible to human beings.

The realist, represented by Gödel, endeavors to reflect upon the (hypothetical) consciousness of a super mind, capable of grasping (knowing in entirety) infinite totalities. Brouwer reflected upon the consciousness of the idealized creative subject, not able to grasp the infinite, but yet capable of grasping any arbitrarily large finite totality, and understanding the infinite in terms of unending, potentially infinite processes. Essenin-Volpin considers the human mind capable of grasping only the small finite, but also able to understand the potentially infinite. Here the large finite may also be understood in a potential or modal sense.

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The transitions from realism to intuitionism to ultra-intuitionism has been accompanied by an enrichment of language and the ability to become sensitive to new aspects of mathematical phenomena. In particular, in ultra-intuitionism, the description of a process, the intensional component of a process and the consummation of a process all play explicit and separate although related roles. For example, a distinction is drawn between a finite process which *has been* completed and a process for which it has been proved that it *will* terminate. Feasible, possible and actual act as distinct modalities. There are temporal components to both intuitionism and ultra-intuitionism but these appear to be of very different natures.

As a final point of comparison let us consider various "obstacles" to the acceptance of the classical rules of reasoning. For the realist, of course, there are none. The intuitionist, on the other hand, must take into account that knowing is part of a temporal process from which we (and even the "creative subject") cannot remove ourselves. Because disjunction is understood in an effective sense (to *know* $A \vee B$ is to *know* A or to *know* B), $A \vee \neg A$ is accepted only if it can be established that A is decidable. The general Law of the Excluded Middle must be rejected because it is tantamount to asserting the existence of a decision procedure for all of mathematics.

In addition to this, the ultra-intuitionist recognizes certain further obstacles. These are connected with his understanding of the notion of potential infinity; the most outstanding of these is the hypothesis of the categoricity of "the" natural numbers. For him such an assumption cannot be justified and therefore there opens the possibility of different length natural number series. It may then happen (in an argument) that the hypothetical completion of one supposedly finite process (associated with the initial segment of a natural number series N_1) may require the completion of a related potentially infinite process (associated with the entire process of a "shorter" natural number series N_2). These are called *Zenonian situations* and must be accounted for and dealt with in the ultra-intuitionistic foundations of mathematics.

Having made these brief remarks and once again emphasising the very partial nature of the representation of Essenin-Volpin's work here reported upon, let us turn to the details.

Essenin-Volpin's first paper is concerned with the consistency of Zermelo-Fraenkel set theory (ZF). He investigates a proof theory corresponding to a structure $\mathscr{T}$ consisting of hereditarily finite sets over a set of urelements.[1] After suitable modifications of ZF to ZF^- (due to the lack of extentionality in $\mathscr{T}$), $\mathscr{T}$ is easily seen to be a model of ZF^- except for the axiom of *Infinity*. However, there are arbitrarily large (although always finite) sets in $\mathscr{T}$ which may play the role of an infinite set for a while. That is to say, if we set up a certain proof theory in which a term t, describing a large finite set is, axiomatically, asserted to be infinite then we can prove all the axioms of ZF^-; of course our deduction system is inconsistent. However, if we can show that the length of the shortest proof of $x \neq x$ is an increasing function of the size of the chosen term t, then we shall have proved the

[1] I should make it clear here that the metamathematical point of view that I adopt in this paper is that of the traditional mathematician, i.e., some limited form of set theory.

consistency of ZF^-. Essenin-Volpin's idea was to use different length number series, a long one (whose length is parameterized by the size of the term t) for constructing $\mathscr{T}$ and a shorter one which is unending and from whose point of view, so to speak, t appears to be infinite. As we have already said, the natural number series is used for the short series, the nonstandard number series $^*N$ is used for the long series and the term t is chosen to have a pseudofinite (but actually infinite) size.

To help limit the size of this paper we begin immediately to work within the framework of nonstandard analysis, after a few "orientation" remarks.

Proof theory and basic model theory can be carried out in a very limited portion of set theory. For example, in $\mathscr{V} = (V, \in)$, where V is the set of sets of rank ≤ 17 over a given set U of urelements. Assume that N (the set of natural numbers) is a subset of U. Let $j: \mathscr{V} < {}^*\mathscr{V}$ be a proper elementary embedding. ${}^*\mathscr{V} = ({}^*V, \in)$ will then be a collection of sets of rank ≤ 17 over the extended set $^*U$ of urelements; these are the so-called *admissible* or *internal sets*. The set N is the foremost example of a set of rank 1 over $^*U$ (we assume that $j \restriction U$ is the identity on U, i.e., $j(x) = x$ for all x in U) which is *not* internal. The sets of rank ≤ 17 over $^*U$ which are not internal are called *external*. In general we denote $j(x)$ by $^*x$. x in $^*V$ is *pseudofinite* (p.f.) iff ${}^*\mathscr{V} \models \text{FINITE}(x)$, where $\text{FINITE}(x)$ is a first-order sentence that says that x is isomorphic to some segment $\{1, 2, \cdots, n\}$ of $^*N$. *Choose* n_0 *in* $^*N - N$.

A notational technique that we shall make much use of is the following: Given an indexed collection $\{X_n \mid n \in N\}$ in V it may be "extended" to a collection, denoted by $^*X_n$, of internal objects indexed by $n \in {}^*N$. The sequence $^*X_n$ behaves like X_n with respect to first-order properties expressible in $\mathscr{V}$.

A final orientation remark. The nicest way to proceed in nonstandard analysis is simply to pretend that everything is normal, except that one must check, every once and a while, which sets are internal and which are not. Thus, for example, while not every bounded subset of $^*N$ has a least upper bound, (e.g., N) every bounded *internal* subset does. For more details see *Nonstandard analysis* by J. Geiser in ***Zeitschrift für mathematische Logik und Grundlagen der Mathematik*** vol. 16 (1970), pp. 297–318.

The model. Let $\{a_n \mid n \in N\}$ denote a subset of U. With each $n \in N$ we associated a structure $\mathscr{T}_n$ in V which is (isomorphic to) the set ofhereditarily finite sets over $\{a_1, \cdots, a_n\}$, with operations of singleton formation and union $x \cup y$, together with equality and a membership relation. In ${}^*\mathscr{V}$ we shall consider ${}^*\mathscr{T}_{n_0}$, the structure consisting of the hereditarily pseudofinite sets over the internal set $b = \{a_1, \cdots, a_{n_0}\}$. (*Note*. The membership relation of ${}^*\mathscr{T}_{n_0}$ is well founded from the point of view of ${}^*\mathscr{V}$ but is actually not well founded from our outside point of view.) We also consider the structure $\mathscr{T}_0$ obtained from ${}^*\mathscr{T}_{n_0}$ by removing the urelements a_k when k is infinite. From the point of view of $\mathscr{T}_0$, a_n is a member of b iff $n \in N$.

The language. The appropriate "first-order" language for ${}^*\mathscr{T}_{n_0}$ is ${}^*\mathscr{L}_{n_0}$ with alphabet: constants $a_1, \cdots, a_{n_0}$; variables $x, y, \cdots$; [ ] (singleton operation), + (union operation); e (membership relation), $=$; $\neg$, $\vee$, $\forall$ (and $\wedge$, $\rightarrow$, $\exists$ defined from $\neg$, $\vee$, $\forall$ as usual). An example of a sentence of ${}^*\mathscr{L}_{n_0}$, which, incidentally, is true in ${}^*\mathscr{T}_{n_0}$, is

$$\forall x[x = a_1 \vee x = a_2 \vee \cdots \vee x = a_{n_0} \vee \exists y[y \mathrm{e} x]].$$

Let $C_{\mathrm{I}} = \{a_1, \cdots, a_{n_0}\}$ and let C_{II} denote the set of terms built up from $[a_1], \cdots, [a_{n_0}]$ by application of $[\ ]$ and $+$. Let $C = C_{\mathrm{I}} \cup C_{\mathrm{II}}$ and $C_0 = C - \{a_k \mid k$ is infinite$\}$. Let BASIC denote the set of atomic and the negation of atomic sentences true in $^*\mathscr{T}_{n_0}$. (*Note.* The image under j of the satisfaction relation as it occurs in $\mathscr{V}$ is used to define "true in $^*\mathscr{T}_{n_0}$.")

THEOREM 1. $^*\mathscr{T}_{n_0} \models^* \mathrm{ZF}^- - \{\mathit{Infinity}\}$, $^*\mathscr{T}_{n_0} \models^* \neg \mathit{Infinity}$.

REMARK. We shall specify later what axioms appear in ZF$^-$. The point here is that a nonextensional form of ZF "holds" in $^*\mathscr{T}_{n_0}$ except for the axiom of (Dedekind) *Infinity*. As remarked above $\models^*$ is just the image under j of the satisfaction relation which occurs in $\mathscr{V}$.

THEOREM 2. *$\mathscr{T}_0$ is a model of* ZF$^-$ *after Comprehension and Replacement have been removed. In particular, $\mathscr{T}_0$ is a model of Infinity.*

REMARK. Here we are considering $\mathscr{T}_0$ as an ordinary structure for which the usual satisfaction relation may be defined for the sentences of $^*\mathscr{L}_{n_0}$ which have constants in C_0 and have syntactical rank (number of connectives and quantifiers) in N. The set $b = \{a_1, \cdots, a_{n_0}\}$ (denoted in C_{II} by $\mathbf{b} = [a_1] + \cdots + [a_{n_0}]$) is "mapped" 1-1, properly into itself by the function

$$s = \{\{\{a_1\}, \{a_1, a_2\}\}, \cdots, \{\{a_{n_0-1}\}, \{a_{n_0-1}, a_{n_0}\}\}\}$$

denoted in C_{II} by $\mathbf{s} = [[[a_1]] + [[a_1] + [a_2]]] + \cdots + [[[a_{n_0-1}] + \cdots])$. *Comprehension* fails in $\mathscr{T}_0$, for example, in that there is no element of $\mathscr{T}_0$ consisting of the subsets of b which are finite within $\mathscr{T}_0$. (*Note.* The set b is our "large finite set.")

The deductive system. By a proof tree T we mean, in analogy with the standard definition, a partially ordered set T (T is to be an internal set) which is a tree to which a (pseudo) ordinal can be assigned according to the usual criteria together with an assignment of sentences of $^*\mathscr{L}_{n_0}$ to the nodes of T. The rules of inference (and hence the rules which govern the sentence assignment) are:

Weak rules. $\{D \vee A \vee B \vee C\} \vdash \{D \vee B \vee A \vee C\}$, $\{D \vee A \vee A \vee C\} \vdash \{D \vee A \vee C\}$, $\{A\} \vdash \{A\}$.

Strong rules. $\{D\} \vdash \{D \vee A\}$, $\{A \vee D\} \vdash \{\neg\neg A \vee D\}$, $\{\neg A \vee D, \neg B \vee D\} \vdash \{\neg(A \vee B) \vee D\}$, $\{A(t) \mid t \in C\} \vdash \{\forall x A(x)\}$ (this is called *Carnap's rule*).

Cut rule. $\{C \vee A, \neg A \vee D\} \vdash \{C \vee D\}$.

C and D are called the side formulae and do not have to be present except in $\{D\} \vdash \{D \vee A\}$ where obviously D must be present and in the Cut rule where one of C or D must be present. By Hyp(T) we mean the hypotheses of T, that is the set of sentences which are assigned to the initial points of T.

THEOREM 3. *For all sentences A in $^*\mathscr{L}_{n_0}$, $^*\mathscr{T}_{n_0} \models^* A$ iff there is a proof tree T of A such that* Hyp(T) $\subseteq$ BASIC.

In particular we see that $\neg$*Infinity* is provable, a situation that we shall try to avoid. We want our proof trees to pick and choose between the sentences true in $^*\mathscr{T}_{n_0}$ and $\mathscr{T}_0$ so as to come as close to ZF$^-$ as possible. Now we fail to be able to deduce from BASIC the sentence $A(\mathbf{s}, \mathbf{b})$ which asserts that $\mathbf{s}$ maps $\mathbf{b}$ 1-1, properly into itself just because we cannot prove $\mathbf{s}$ is defined at a_{n_0} in b, which in fact it is not. So we might introduce this as a further axiom. Actually this turns out to be equivalent to the addition of any contradiction (i.e., a sentence false in $^*\mathscr{T}_{n_0}$) to our set

BASIC. On the basis of convenience we set $\bot$ equal to a_{n_0} e a_{n_0} and choose it to be our added axiom. Then we can show that "sufficiently simple" proof trees using these hypotheses do not get into trouble although they are, nevertheless, capable of proving the axiom of *Infinity*. To devise the appropriate notion of "simplicity" we introduce a complexity measure called the Skolem function of a tree, and the notion of pre-EV trees.

Given a proof tree T we associate a sequence of subtrees $(T)_n$, $n \in^* N$, defined by the conditions: (1) the terminal node of T is in $(T)_n$; (2) if $x \in T$ and x has not been obtained by an application of a Carnap's rule then the immediate predecessors of x are in $(T)_n$; (3) if $x \in T$ has been obtained by an application of a Carnap's rule and y is an immediate predecessor of x corresponding to a term t in C_{II} or a term a_k where $k \leq n$, then y is in $(T)_n$. Let $\check{T} = \bigcup_{n \in N}(T)_n$. $\check{T}$ is called the *pruned tree* associated with T.

DEFINITION 1. The *Skolem function* s_T of a proof tree T is a function mapping $^*N$ into itself defined by the condition

$$s_T(n) = \max\{k \mid a_k \text{ occurs as a term in } (T)_n\}.$$

By the phrase "occurs as a term" we mean occurring not simply as a proper part of a term.

REMARK. Since T and $(T)_n$ are always internal sets, and $a_k \in T$ implies that $k \leqslant n_0$, it follows that $s_T(n)$ is always defined and has value $\leq n_0$. It is easy to see that s_T is nondecreasing and if T has a Carnap's rule $\{A(t) \mid t \in C\} \vdash \forall x A(x)$ where x occurs as a term in $A(x)$, then, for $k \leq n_0$, $s_T(k) \geq k$.

DEFINITION 2. T is *pre-EV* iff $s_T: N \to N$.

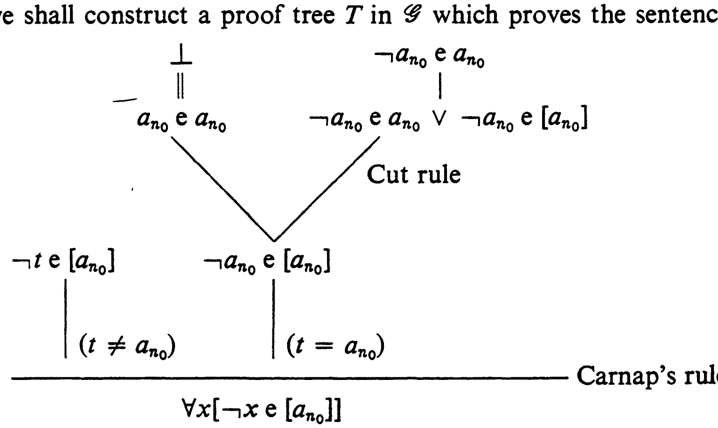
REMARK. Pre-EV trees are our "simple" trees. Also, pre-EV trees make sense for $\mathscr{T}_0$ if we consider the pruned part, while the general tree does not. It may be shown, for example, that if A has syntactical rank in N then A is true in $\mathscr{T}_0$ if it has a pre-EV proof from BASIC.

Use of contradictions. We define a collection of trees $\mathscr{G}$ by the conditions

$T \in \mathscr{G}$ iff (1) T is pre-EV,
(2) $\text{Hyp}(T) \subseteq \text{BASIC} \cup \{\bot\}$,
(3) $\check{T}$ is cut-free (i.e., contains no occurrences of the Cut rule).

As an example we shall construct a proof tree T in $\mathscr{G}$ which proves the sentence

$T =$



$\forall x[\neg x \mathrm{~e~} [a_{n_0}]]$. Of course this sentence is true in $\mathscr{T}_0$ and false in $^*\mathscr{T}_{n_0}$. According to Theorem 3 this sentence cannot be proved by a tree whose hypotheses lie just in BASIC. Observe that $\neg t \mathrm{~e~} [a_{n_0}] \in$ BASIC if $t \neq a_{n_0}$ and likewise $\neg a_{n_0} \mathrm{~e~} a_{n_0} \in$ BASIC. Notice that the three conditions for membership in $\mathscr{G}$ are met by T; in particular, $s_T(k) = k$ for $k \leq n_0$, so T is clearly pre-EV.

Results.

Consistency of the proof theory. If A has finite syntactical rank (in N) then not both A and $\neg A$ can be provable by a tree in $\mathscr{G}$.

(*Notation.* $\vdash_{\mathscr{G}} A$ means that there is a proof of A in $\mathscr{G}$.)

Cut-elimination. If A has finite rank then $\vdash_{\mathscr{G}} C \vee A$ and $\vdash_{\mathscr{G}} \neg A \vee D$ implies $\vdash_{\mathscr{G}} C \vee D$.

Logic. (1) $\vdash_{\mathscr{G}} A \vee B$ iff $\vdash_{\mathscr{G}} A$ or $\vdash_{\mathscr{G}} B$.

(2) $\vdash_{\mathscr{G}} A \wedge B$ iff $\vdash_{\mathscr{G}} A$ and $\vdash_{\mathscr{G}} B$.

(3) $\vdash_{\mathscr{G}} \neg\neg A$ iff $\vdash_{\mathscr{G}} A$.

(4) $\vdash_{\mathscr{G}} \exists x A(x)$ iff there is a term $t \in C_0$ such that $\vdash_{\mathscr{G}} A(t)$.

(5) $\vdash_{\mathscr{G}} \forall x A(x)$ implies for all terms $t \in C_0$, $\vdash_{\mathscr{G}} A(t)$, but not conversely.

(6) There is a sentence A with finite rank and containing no constant terms which is not decidable in $\mathscr{G}$, i.e., $A \vee \neg A$ has no proof in $\mathscr{G}$. Therefore the Law of the Excluded Middle fails for $\vdash_{\mathscr{G}}$.

(7) The Deduction Theorem and the Rule of Contradiction fail for $\vdash_{\mathscr{G}}$. More precisely, $A \vdash_{\mathscr{G}} B$ means that there is a proof tree T satisfying conditions (1) and (3) of the definition of $\mathscr{G}$ and which also satisfies the condition that $\mathrm{Hyp}(T) \subseteq \mathrm{BASIC} \cup \{\bot\} \cup \{A\}$ and T is a proof of B. Then $A \vdash_{\mathscr{G}} B$ does not imply that $\vdash_{\mathscr{G}} A \to B$ and $A \vdash_{\mathscr{G}} \forall x[x \neq x]$ does not imply that $\vdash_{\mathscr{G}} \neg A$.

Set theory. $\vdash_{\mathscr{G}} (\mathrm{ZF}^-)^\sim$. Here $(\mathrm{ZF}^-)^\sim$ consists of the (appropriately modified versions of) *Pairing* axiom, *Sum set* axiom, *Power set* axiom, the axiom of (Dedekind) *Infinity*, and certain additionally modified versions of *Comprehension* and *Replacement*. For example, if $\vdash_{\mathscr{G}} \forall x[A(x) \vee \neg A(x)]$ then $\vdash_{\mathscr{G}} \forall x \exists y \forall z[z \mathrm{~e~} y \leftrightarrow z \mathrm{~e~} x \wedge A(z)]$. *Replacement* can take various forms such as if

(1) $\vdash_{\mathscr{G}} \forall x \forall y[F(x, y) \vee \neg F(x, y)]$,

(2) for every infinite $k \leq n_0$ if $^*\mathscr{T}_{n_0} \models^* F(t, a_k)$ then $t = a_{k'}$ for some infinite $k' \leq n_0$,

(3) $\vdash_{\mathscr{G}} \forall x \forall y \forall z[F(x, y) \wedge F(x, z) \to \forall w[w \mathrm{~e~} z \leftrightarrow w \mathrm{~e~} y]]$,

then $\vdash_{\mathscr{G}} \forall x \exists y \forall z[z \mathrm{~e~} y \leftrightarrow \exists w[w \mathrm{~e~} x \wedge F(w, z)]]$.

Needless to say, this set theory is quite weak. However, we can also show that if A is an instance of *Comprehension* or *Replacement* (in their ordinary formulation, modified for a nonextensional set theory) and if A has finite rank (in N) then $\neg A$ is not provable in $\mathscr{G}$. Because the Rule of Contradiction fails for $\vdash_{\mathscr{G}}$ this nice fact does not seem to yield any further consistency results.

Semantics. If A has finite rank (in N) then $\vdash_{\mathscr{G}} A$ implies that $\mathscr{T}_0 \models A$.

Arithmetic. We can code the language of arithmetic into $^*\mathscr{L}_{n_0}$ by relativizing the variables to **b**, and replacing atomic formulae such as $x + y = z$ by $(x, y, z) \mathrm{~e~} f$ where f is a term in C_{II} denoting the set of triples (a_k, a_m, a_n) such that $k, m, n \leq n_0$ and $k + m = n$ and where (x, y, z) is the open term $[[[x]] + [[x] + [y]]] + [[[[x]] + [[x] + [y]]] + [z]]$. The other atomic relations of arithmetic can be handled in a similar way and it can then be shown that all the true sentences of arithmetic when so translated are provable in $\vdash_{\mathscr{G}}$. In fact we can show that all the true Σ_1^1 sentences of second order arithmetic (suitably translated) are provable in $\vdash_{\mathscr{G}}$. However, there are simple Π_1^1 sentences of second-order arithmetic which are undecidable in $\vdash_{\mathscr{G}}$.

Final remarks. The details and proofs of the results summarized here will appear in a paper to be published as part of a Springer Lecture Series Notes on Strict Finitism, edited by E. Engeler. It may be of interest to "constructivize" the techniques used here. I mean constructivity in a weak sense where we might use nonstandard languages in the sense of Barwise rather than as we have done, using nonstandard analysis.

The Skolem function as defined in Definition 1 above is easily seen to make sense for standard proof trees as well as the nonstandard proof trees that we have been considering, and may be of interest to proof theorists as a complexity measure for proof trees with an ω-type rule. In fact many of the technical results established in the above-mentioned fuller presentation apply to standard proof trees. These are results concerned with cut-elimination and various proof reduction procedures.

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